Machine Learning for Official Statistics and SDGs

Regression

UNITED NATIONS siap
Statistical Institute for Asia and the Pacific

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with possibly many regressors x_j

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Except: α_0 is the mean of y **if all x_j are equal their mean**

[Linear Regression: Scaling]

We can also scale each variable by its own standard deviation to obtain

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- The goal is to have variables and coefficients that are comparable

[Linear Regression: Example]

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- Example on a regression model with few variables

	Est.	S.E.	t val.	p
(Intercept)	4.390	0.136	32.204	0.000
Trans	-0.002	0.001	-1.603	0.110
HighTrans	0.013	0.003	3.945	0.000
Checks	0.008	0.005	1.534	0.126
Years	0.096	0.008	11.579	0.000

Initial regression with original values

[Linear Regression: Example]

- Example on a regression model with few variables

	Est.	S.E.	t val.	p
(Intercept)	5.927	0.039	150.243	0.000
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Regression with centred variables

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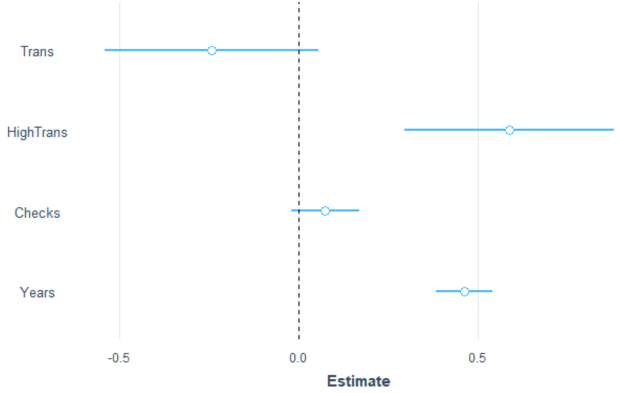
	Est.	S.E.	t val.	p
(Intercept)	5.927	0.039	150.243	0.000
Trans	-0.243	0.152	-1.603	0.110
HighTrans	0.586	0.149	3.945	0.000
Checks	0.074	0.048	1.534	0.126
Years	0.462	0.040	11.579	0.000

Regression with scaled variables

[Linear Regression: Example]

- Example on a regression model with few variables
  The goal is to have *comparable* effects (same range)



Visual regression with scaled variables

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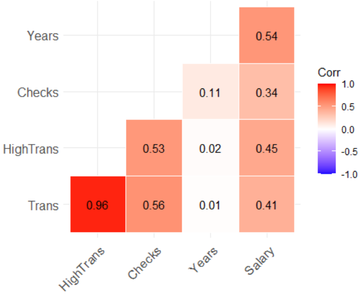
$\hookrightarrow$ Correlation plot



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$VIF_j = 1$ indicates no collinearity; a $VIF \geq 10$ is considered as large and problematic

[Solutions to multi-collinearity]

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- Remove some variables

[Using *VIF* to remove collinearity]

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Computing VIFs for all x_js

[USING *VIF* TO REMOVE COLLINEARITY]

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	Est.	S.E.	t val.	p	VIF
(Intercept)	5.93	0.04	150.24	0.00	NA
Trans	-0.24	0.15	-1.60	0.11	14.71
HighTrans	0.59	0.15	3.94	0.00	14.15
Checks	0.07	0.05	1.53	0.13	1.47
Years	0.46	0.04	11.58	0.00	1.02

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- *Trans* has the highest VIF

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Checks	0.06	0.05	1.24	0.22	1.41
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- does not change the coefficients of the uncorrelated regressors
- reduces all the *VIF*s

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In real life, one may have many variables

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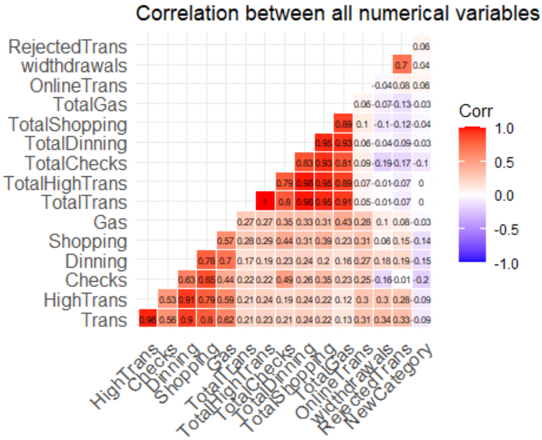

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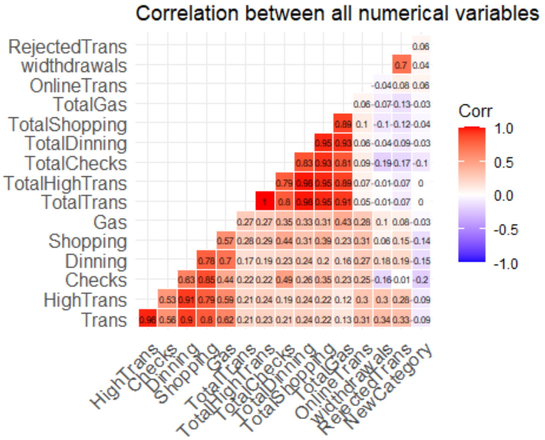

↪ *Automatic selection of regressors*

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- Compute the optimal nb of regressors before testing which regressors to include with Cross Validation

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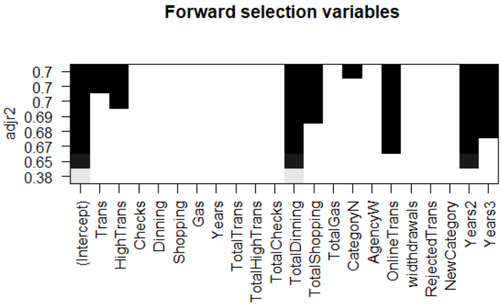

Visual representation of variables used (Forward)

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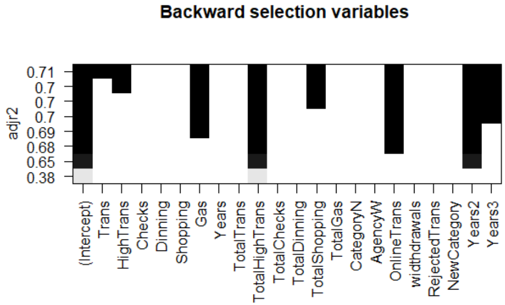

Visual representation of variables used (Backward)

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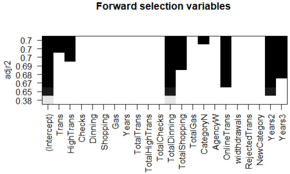

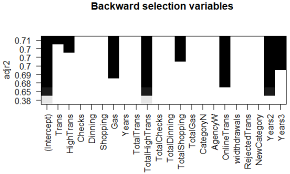

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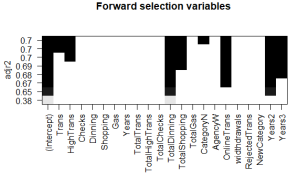

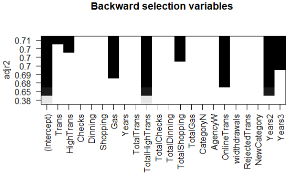

- *Great Need for Criteria*

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$\hookrightarrow$ *Mean Squared Error of Prediction* or MSEP:

$$MSEP = n^{-1}E\|y_{new} - X_p\widehat{\beta}_p\|^2$$

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The MSEP can be decomposed to help define criterions:

$$\begin{aligned} MSEP &= n^{-1} E\|y_{new} - X_p \widehat{\beta}_p\|^2 \\ &= n^{-1}\left\{ E\|y_{new} - X\beta\|^2 + E\|X\beta - X_p\widehat{\beta}_p\|^2 \right\} \\ &= (1 + (p+1)/n)\sigma^2 + (1/n)\beta' X' M_p X\beta \end{aligned}$$

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- Mallow's **Cp** $= \frac{RSS_p}{n} + \frac{2(p+1)}{n}\frac{RSS_k}{n-k-1}$

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- Bayesian Information Criterion (**BIC**):

$$BIC \propto \frac{RSS_p}{n} + \frac{(p+1)\log n}{n}\frac{RSS_k}{n-k-1}$$

[Application on the Example]

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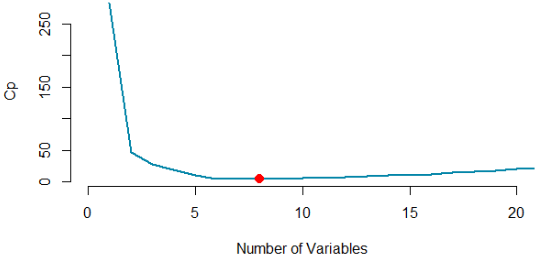
Selection using Mallow's C_p ($\to$ 8 variables)



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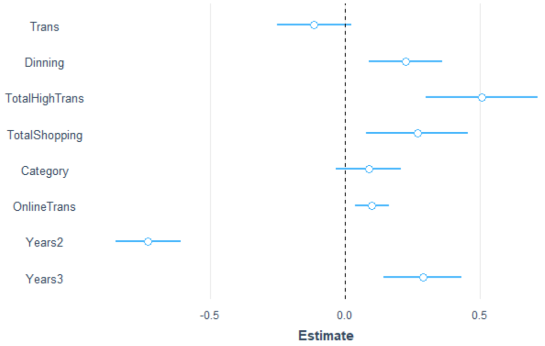

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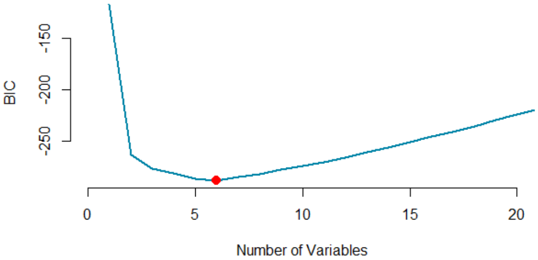
Selection using BIC ($\rightarrow$ 6 variables)



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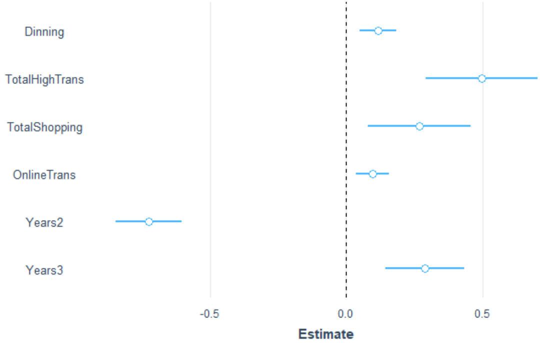

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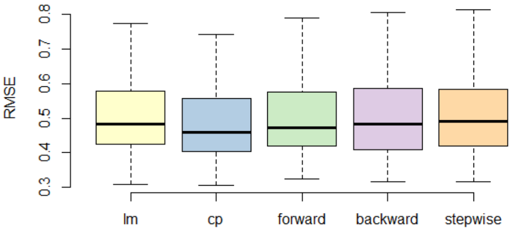

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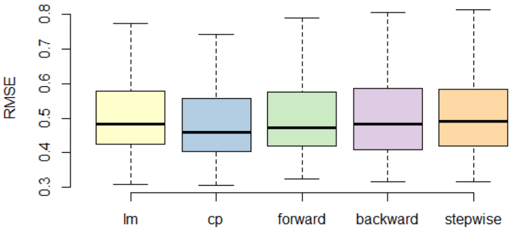

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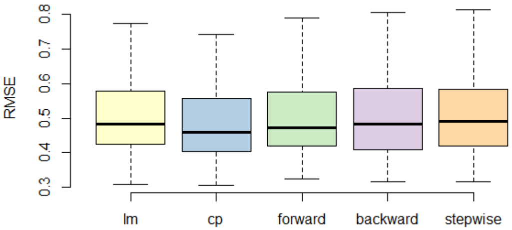

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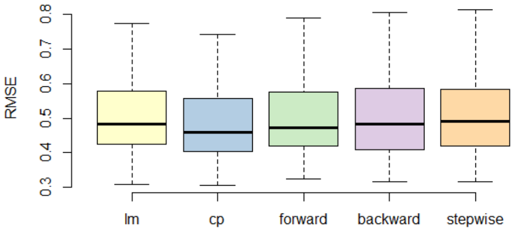

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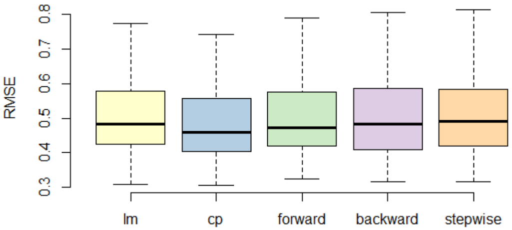

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NB: *If* $\lambda = 0$ *the regression is just the classic OLS estimator*

[Penalization methods: Ridge regression]

Ridge regression is the solution of:

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$\hookrightarrow$ When there is collinearity, $X'X$ cannot be inverted, while if $\lambda > 0$ the matrix $(X'X + \lambda I)$ is invertible

[Ridge regression in Practice]

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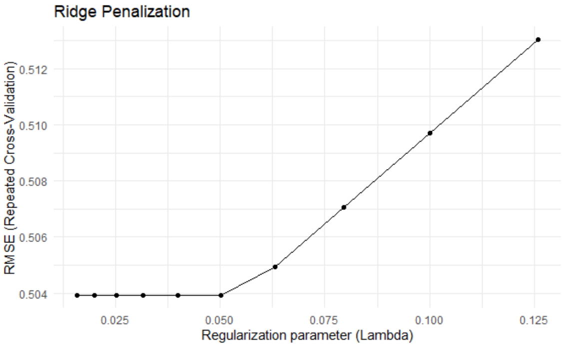

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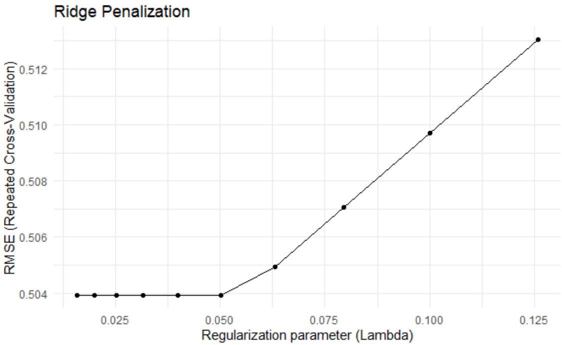

Remark: λ is typically small, and we usually select it on the log scale.

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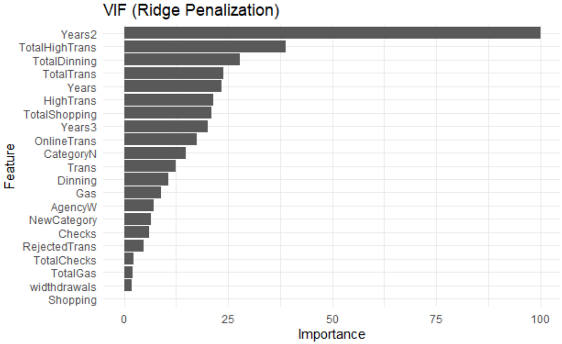

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LASSO or *Least Absolute Shrinkage and Selection Operator*, is another common penalization method, is the solution of:

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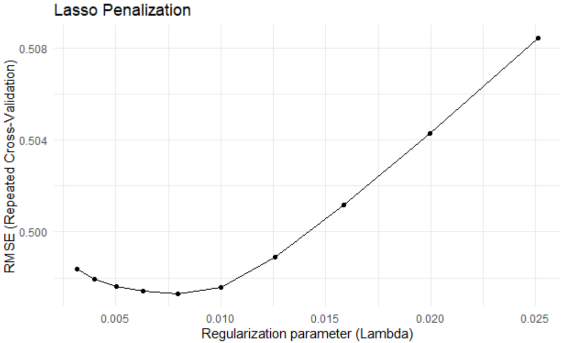

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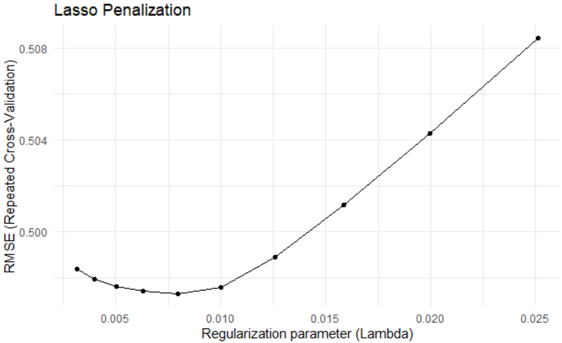

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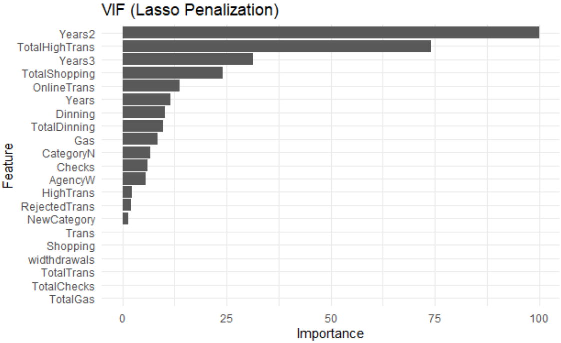

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- Through α we balance variable selection (Lasso) and coefficient reduction (Ridge)

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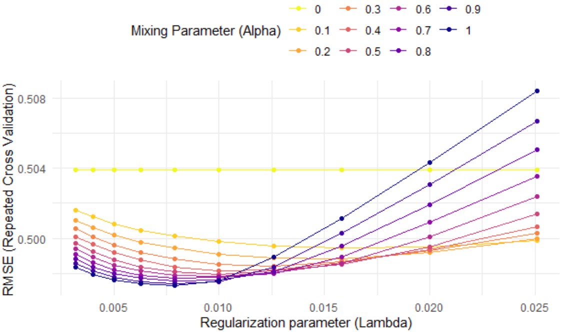

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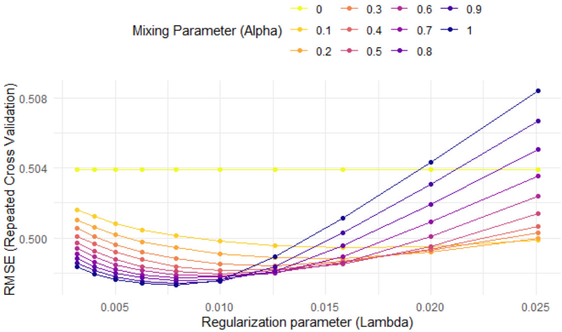

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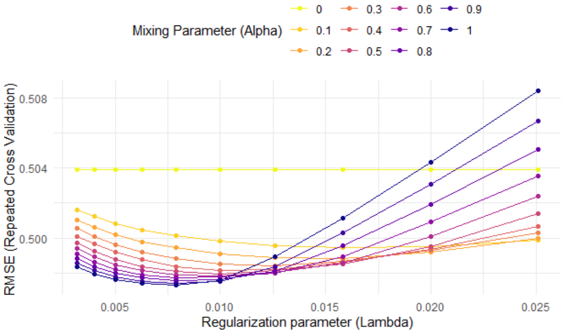

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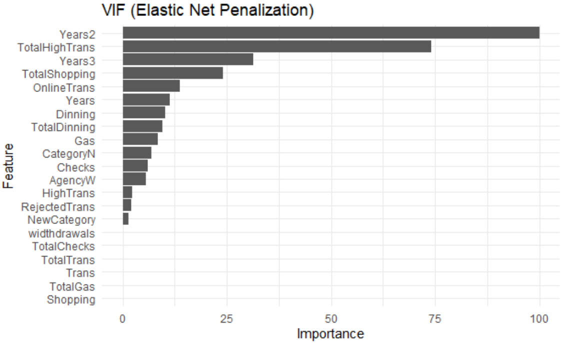

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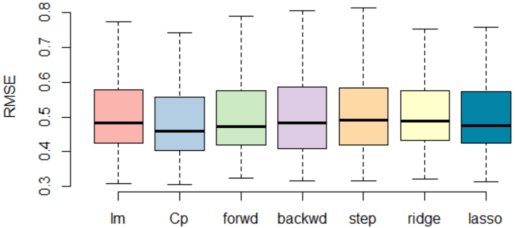

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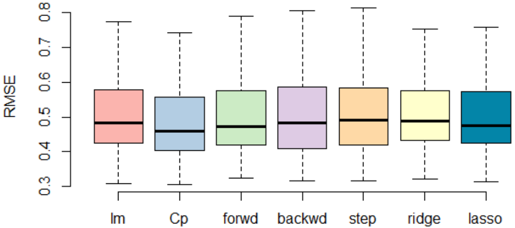

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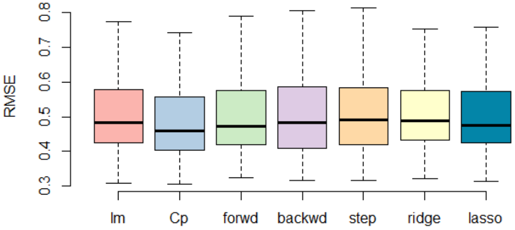

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