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Bounded Greedy Nim ☆

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ABSTRACT

This paper introduces a new variant of the game Nim–Bounded Greedy Nim. This game is a combination of Bounded Nim and Greedy Nim. We present a complete solution to this game, which generalizes the solution of Greedy Nim.

1. Introduction

In the well-known game Nim, one or more heaps of stones are provided. Two players take turns removing at least one stone from any one heap. The player who takes away the last stone wins the game. Bouton [3] developed a complete mathematical theory for this game in 1902. Since then, many variants of Nim have been studied in the literature. For example, Nim_k, introduced by Moore [8] in 1910, is a variation of Nim in which the players are allowed to remove stones from up to k heaps, where k is a fixed integer. Bounded Nim, introduced by Schwartz [9] in 1971, is a variation of Nim in which the number of stones removed in each turn is no more than a given constant. *Greedy Nim*, introduced by Albert and Nowakowski [2] in 2004, is a variation of Nim in which the players always remove stones from the largest heap. Other Nim-type games can be found in [1], [4], [5], [6], [7], [10], [11].

In this paper, we propose a new game, *Bounded Greedy Nim*, which is a combination of Bounded Nim and Greedy Nim. The game is played as in ordinary Nim except that each player can only remove stones from the largest heap, and the number of removed stones is no more than a given constant.

Let k be a fixed positive integer. The game *k-bounded greedy nim* is played by two players, P_1 and P_2, who make moves in turn. There is a collection of heaps of stones. The player making the current move chooses a number $1 \le t \le k$ and takes away t stones from the largest heap (so t must be no larger than the number of stones in the largest heap). The player who takes away the last stone wins the game.

We denote a collection of heaps of stones by a sequence $\mathcal{S} = (x_1, x_2, \dots, x_n)$ of nonnegative integers in non-decreasing order, which means that there are n heaps, and the ith heap has x_i stones. For convenience, we shall always assume that $n \ge 3$ and allow $x_i = 0$ i.e., a heap may have 0 stones. If $x_1 = 0$, then we view the sequence $\mathcal{S} = (x_1, x_2, \dots, x_n)$ to be equivalent to the sequence $(x_2, \dots, x_n)$. We call such a sequence a *position*. A position is called an *N-position* if the

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next player has a winning strategy. Otherwise the previous player has a winning strategy and it is called a *P-position*. By convention, the position $(0, 0, \ldots, 0)$ is a P-position.

In the remainder of the paper, we denote by P_1 the next player, and by P_2 the previous player.

Example 1. Consider the case that $k = 3$. Assume $\mathcal{S} = (1, 1, 1, 5)$. We show that $\mathcal{S}$ is a P-position. We shall show that P_2 has a winning strategy. In the first round, P_1 removes t stones from the last heap, where $1 \le t \le 3$. After P_1's move, the position becomes $(1, 1, 1, 5 - t)$. Then P_2 removes $4 - t$ stones from the last heap, so the position becomes $(1, 1, 1, 1)$. The game will end in two more rounds, with P_2 removing the last stone and hence wins the game. If $\mathcal{S} = (1, 1, 2, 5)$, then this is an N-position i.e., P_1 has a winning strategy. In his first move, P_1 removes 3 stones from the last heap, so the position becomes $(1, 1, 2, 2)$. Then the game will end in two or three more rounds, with P_1 removing the last stone and wins the game.

2. Solution for bounded greedy nim

If $k = 1$, then the P-positions are precisely those in which the total number of all the stones is even, as each move takes away exactly one stone.

In the following, let $k \ge 2$ be a fixed integer. We denote by $\mathbb{P}$ the set of all P-positions. In this section, we give a characterization of all the P-positions (hence all the N-positions) for this game.

To prove that a position $\mathcal{S}$ is a P-position, it suffices to show that for any legal move from $\mathcal{S}$, the resulting position $\mathcal{S}'$ is not a P-position i.e., $\mathcal{S}' \notin \mathbb{P}$. To prove that a position is an N-position, we need to show that P_1 has a move, so that after his move, the resulting position $\mathcal{S}'$ is a P-position i.e., $\mathcal{S}' \in \mathbb{P}$. In the following, we shall denote by $\mathcal{S}$ the current position, and by $\mathcal{S}'$ the next position.

For positive integers a and m, let $R_a(m)$ be the remainder after division of m by a i.e., $R_a(m) = m - a\lfloor m/a \rfloor$.

First we consider the case $x_{n-2} = 0$ i.e., there are at most two heaps with a positive number of stones.

Theorem 2. *Suppose $\mathcal{S} = (x_1, x_2, \ldots, x_n)$ is a position for k-bounded greedy nim, where $x_i = 0$ for $1 \le i \le n-2$. Then $\mathcal{S}$ is a P-position if and only if $R_{k+1}(x_n - x_{n-1}) = 0$.*

Proof. The proof is by induction on the total number $x = \sum_{i=1}^{n} x_i$ of the stones. If $x = 0$, then $R_{k+1}(x_n - x_{n-1}) = 0$ and by our convention, $\mathcal{S}$ is a P-position.

Let $x_n - x_{n-1} = m(k + 1) + \ell$, $m \ge 0$, $0 \le \ell \le k$.

Necessity. Assume that $R_{k+1}(x_n - x_{n-1}) \ne 0$, so $1 \le \ell \le k$. P_1 removes ℓ stones from x_n, then $\mathcal{S}' = (x'_1, x'_2, \ldots, x'_n)$ with $x'_j = x_j$ for all $1 \le j \le n - 1$, and $x'_n = x_n - \ell$. As $R_{k+1}(x'_n - x'_{n-1}) = 0$ and $x'_{n-2} = x_{n-2} = 0$, by the induction hypothesis, $\mathcal{S}'$ is a P-position.

Sufficiency. Assume that $R_{k+1}(x_n - x_{n-1}) = 0$ i.e., $\ell = 0$, and P_1 removes t stones from the heap of size x_n, where $1 \le t \le min\{x_n, k\}$. For the resulting position $\mathcal{S}' = (x'_1, x'_2, \ldots, x'_n)$, we have $x'_{n-2} = 0$ and $R_{k+1}(x'_n - x'_{n-1}) = R_{k+1}(|x_n - x_{n-1} - t|) = k + 1 - t \ne 0$. So $\mathcal{S}'$ is not a P-position (and hence is an N-position). $\square$

In the remainder of this section, we assume that $x_{n-2} > 0$.

Definition 3. Suppose $\mathcal{S} = (x_1, x_2, \ldots, x_n)$ is a position with $x_{n-2} > 0$. For $1 \le i \le n$, let

$$\beta(\mathcal{S}) = n - 1 - min\{j : x_j = x_{n-2}\},$$

which is the number of repetitions of the value of x_{n-2} in the sequence $(x_1, x_2, \ldots, x_{n-2})$.

Definition 4. Suppose a, b, c, k are four positive integers, where $a \le b \le c$ and $k \ge 2$. We say the triple (a, b, c) is *k-good* if one of the following holds:

(i) $R_{k+1}(b - a) = 0$ and $R_{k+1}(c - b) = k$;
(ii) $1 \le R_{k+1}(b - a) \le k - 1$ and $R_{k+1}(c - b) = 0$;
(iii) $R_{k+1}(b - a) = k$ and $R_{k+1}(c - b) = 1$.

As shown in Theorem 6 below, for a position $\mathcal{S} = (x_1, x_2, \ldots, x_n)$, if $\beta(\mathcal{S})$ is odd, then what matters is whether (x_{n-2}, x_{n-1}, x_n) is k-good; if $\beta(\mathcal{S})$ is even, then what matters is whether $R_{k+1}(x_n - x_{n-1}) = 0$.

Example 5. Consider the case that $k = 3$. If $\mathcal{S}_1 = (5, 5, 5, 5, 8)$, then $\beta(\mathcal{S}_1) = 3$ is odd, and $(5, 5, 8)$ satisfies (i) of the definition of k-good. Similarly, for $\mathcal{S}_2 = (5, 5, 5, 5, 9)$, $\beta(\mathcal{S}_2) = 3$ is odd, and $(5, 5, 9)$ is not k-good. For $\mathcal{S}_3 = (1, 1, 5, 6)$, $\beta(\mathcal{S}_3) = 2$ is even, $R_4(6 - 5) \ne 0$. For $\mathcal{S}_4 = (1, 1, 5, 5)$, $\beta(\mathcal{S}_4) = 2$ is even and $R_4(5 - 5) = 0$.

Theorem 6. *Suppose $\mathcal{S} = (x_1, x_2, \ldots, x_n)$ is a k-bounded greedy nim position, and $x_{n-2} > 0$. If $\beta(\mathcal{S})$ is even, then $\mathcal{S}$ is a P-position if and only if $R_{k+1}(x_n - x_{n-1}) = 0$. If $\beta(\mathcal{S})$ is odd, then $\mathcal{S}$ is a P-position if and only if (x_{n-2}, x_{n-1}, x_n) is k-good.*

Proof. The proof is by induction on the total number of the stones. Since $x_{n-2} > 0$, there are at least three stones. If there are exactly three stones, then the conclusion is easily verified. Assume the total number of stones is $\sum_{i=1}^{n} x_i \geq 4$, and the theorem holds if there are less than $\sum_{i=1}^{n} x_i$ stones.

Claim 1. *For any position $\mathcal{S}^* = (x_1^*, x_2^*, \ldots, x_n^*)$ with $\sum_{i=1}^{n} x_i^* < \sum_{i=1}^{n} x_i$, if the triple $(x_{n-2}^*, x_{n-1}^*, x_n^*)$ satisfies (ii) of the definition of k-good, then $\mathcal{S}^*$ is a P-position.*

Proof. Assume $\mathcal{S}^* = (x_1^*, x_2^*, \ldots, x_n^*)$ satisfies (ii) of the definition of k-good. Then $R_{k+1}(x_n^* - x_{n-1}^*) = 0$. Whether $\beta(\mathcal{S}^*)$ is even or odd, $\mathcal{S}^*$ satisfies the condition of Theorem 6 for being a P-position. Thus by induction hypothesis, $\mathcal{S}^*$ is a P-position. □

Assume $x_n - x_{n-1} = m(k+1) + \ell$, $m \geq 0$, $0 \leq \ell \leq k$. Assume P_1 removes t stones from the largest heap. In general, the heap of $x_n - t$ stones may not be the largest heap in the resulting position $\mathcal{S}'$. However, if $m > 0$, then $x_n - t$ remains the largest heap.

Claim 2. *If $m > 0$, then $\mathcal{S}' = (x_1, x_2, \ldots, x_{n-1}, x_n - t)$. So $\beta(\mathcal{S}) = \beta(\mathcal{S}')$. Since $R_{k+1}(x_n - t - x_{n-1}) \neq R_{k+1}(x_n - x_{n-1})$, if $\mathcal{S}$ satisfies the conditions of Theorem 6 for being a P-position, then $\mathcal{S}'$ does not satisfy the condition for being a P-position.*

We prove the theorem by considering two cases.

Case 1: $\beta(\mathcal{S})$ is even.

Necessity. Assume $\ell = R_{k+1}(x_n - x_{n-1}) \neq 0$. We need to show that $\mathcal{S}$ is an N-position i.e., there exists a legal move from $\mathcal{S}$ such that $\mathcal{S}'$ is a P-position.

Let P_1 remove ℓ stones from x_n. Then $\mathcal{S}' = (x_1', x_2', \ldots, x_n')$ with $x_j' = x_j$ for all $1 \leq j \leq n-1$, and $x_n' = x_n - \ell \geq x_{n-1}'$. Then $R_{k+1}(x_n' - x_{n-1}') = 0$ and $\beta(\mathcal{S}') = \beta(\mathcal{S})$ is even. By induction hypothesis, $\mathcal{S}' \in \mathbb{P}$.

Sufficiency. Assume $R_{k+1}(x_n - x_{n-1}) = 0$. We need to show that $\mathcal{S}$ is a P-position i.e., any legal move from $\mathcal{S}$ leads to a position $\mathcal{S}' \notin \mathbb{P}$.

By Claim 2, we only need to consider the case that $m = 0$ i.e., $x_n = x_{n-1}$. There are two possibilities:

- $x_n - t \geq x_{n-2}$. Now $\mathcal{S}' = (x_1', x_2', \ldots, x_n')$ with $x_j' = x_j$ for $1 \leq j \leq n-2$, $x_{n-1}' = x_n - t$ and $x_n' = x_{n-1}$. Then $\beta(\mathcal{S}') = \beta(\mathcal{S})$ is even, and $R_{k+1}(x_n' - x_{n-1}') = R_{k+1}(x_{n-1} - x_n + t) = t \neq 0$. By induction hypothesis, $\mathcal{S}' \notin \mathbb{P}$.
- $x_n - t < x_{n-2}$. As $\beta(\mathcal{S}) \geq 2$, we have $x_{n-3} = x_{n-2}$. Then $\mathcal{S}' = (x_1', x_2', \ldots, x_n')$ with $x_{n-2}' = x_{n-3}$, $x_{n-1}' = x_{n-2}$, $x_n' = x_{n-1}$ and $\beta(\mathcal{S}') = \beta(\mathcal{S}) - 1$ is odd. Note that $R_{k+1}(x_{n-1}' - x_{n-2}') = R_{k+1}(x_{n-2} - x_{n-3}) = 0$ and $x_n' - x_{n-1}' = x_{n-1} - x_{n-2} = x_n - x_{n-2} < t \leq k$. Therefore $x_n' - x_{n-1}' < k$ and hence $R_{k+1}(x_n' - x_{n-1}') \neq k$. Hence $(x_{n-2}', x_{n-1}', x_n')$ is not k-good. By induction hypothesis, $\mathcal{S}' \notin \mathbb{P}$.

Case 2: $\beta(\mathcal{S})$ is odd.

Necessity. Assume (x_{n-2}, x_{n-1}, x_n) is not k-good. We shall prove that $\mathcal{S}$ is an N-position, by showing that there exists a legal move $x_n \to x_n - t$ so that the resulting position $\mathcal{S}' = (x_1', x_2', \ldots, x_n')$ is a P-position.

The number t is determined as follows:

(i) If $R_{k+1}(x_{n-1} - x_{n-2}) = 0$, let $t = \ell + 1$. As $\ell \neq k$, the move is legal.
(ii) If $1 \leq R_{k+1}(x_{n-1} - x_{n-2}) \leq k-1$, let $t = \ell$. As $\ell \neq 0$ the move is legal.
(iii) If $R_{k+1}(x_{n-1} - x_{n-2}) = k$, let $t = R_{k+1}(\ell - 1)$. As $\ell \neq 1$ the move is legal.

Now we shall verify that in each of these cases, $\mathcal{S}' \in \mathbb{P}$.

(1) $R_{k+1}(x_{n-1} - x_{n-2}) = 0$.
There are three possibilities:
- $m > 0$.
Then $x_j' = x_j$ for $1 \leq j \leq n-1$ and $x_n' = x_n - t = x_n - \ell - 1 > x_{n-1}'$. As $\beta(\mathcal{S}') = \beta(\mathcal{S})$ is odd, and $R_{k+1}(x_n' - x_{n-1}') = R_{k+1}(x_n - x_{n-1} - \ell - 1) = k$ and $R_{k+1}(x_{n-1}' - x_{n-2}') = R_{k+1}(x_{n-1} - x_{n-2}) = 0$, $(x_{n-2}', x_{n-1}', x_n')$ is k-good. By induction hypothesis, $\mathcal{S}'$ is in $\mathbb{P}$.

- $m = 0$ and $x_{n-1} > x_{n-2}$.
  Then $x'_j = x_j$ for $1 \le j \le n-2$, $x'_{n-1} = x_n - t = x_n - \ell - 1 = x_{n-1} - 1$, and $x'_n = x_{n-1}$. So $\beta(\mathcal{S}') = \beta(\mathcal{S})$ is odd, and $R_{k+1}(x'_n - x'_{n-1}) = 1$ and $R_{k+1}(x'_{n-1} - x'_{n-2}) = R_{k+1}(x_{n-1} - x_{n-2} - 1) = k$. Therefore $(x'_{n-2}, x'_{n-1}, x'_n)$ is k-good. By induction hypothesis, $\mathcal{S}'$ is in $\mathbb{P}$.
- $m = 0$ and $x_{n-1} = x_{n-2}$.
  - $\beta(\mathcal{S}) = 1$ and $x_n - (\ell + 1) = 0$.
    Since $x_n - \ell - 1 = x_{n-1} - 1 = x_{n-2} - 1$, we conclude that $x_{n-3} = 0$ and hence $x'_j = 0$ for $1 \le j \le n-2$. As $R_{k+1}(x'_n - x'_{n-1}) = x_{n-1} - x_{n-2} = 0$, by Theorem 2, $\mathcal{S}'$ is in $\mathbb{P}$.
  - $\beta(\mathcal{S}) = 1$ and $x_n - (\ell + 1) > 0$.
    Then $x'_j = x_j$ for $1 \le j \le n-3$, and $x'_{n-2} = x_n - t = x_{n-2} - 1$, $x'_{n-1} = x'_n = x_{n-2}$. So $(x'_{n-2}, x'_{n-1}, x'_n)$ satisfies (ii) of the definition of k-good. By Claim 1, $\mathcal{S}'$ is in $\mathbb{P}$.
  - $\beta(\mathcal{S}) \ge 3$.
    Then $\beta(\mathcal{S}') = \beta(\mathcal{S}) - 1$ is even. As $x'_{n-2} = x_{n-3} = x_{n-2} > 0$ and $x'_{n-1} = x'_n$, by induction hypothesis, $\mathcal{S}'$ is in $\mathbb{P}$.

(2) $1 \le R_{k+1}(x_{n-1} - x_{n-2}) \le k - 1$.
Then $x'_j = x_j$ for $1 \le j \le n-1$, $x'_n = x_n - \ell$. As $R_{k+1}(x'_n - x'_{n-1}) = R_{k+1}(x_n - x_{n-1}) - \ell = 0$, $(x'_{n-2}, x'_{n-1}, x'_n)$ satisfies (ii) of the definition of k-good. By Claim 1, $\mathcal{S}'$ is in $\mathbb{P}$.

(3) $R_{k+1}(x_{n-1} - x_{n-2}) = k$.
As $x_n - t \ge x_{n-2}$, we have $x'_j = x_j$ for $1 \le j \le n-2$. So $x'_{n-2} > 0$ and $\beta(\mathcal{S}') = \beta(\mathcal{S})$ is odd.

- If $x_n - x_{n-1} \ge 2$, then $x'_{n-1} = x_{n-1}$ and $x'_n = x_n - R_{k+1}(\ell - 1)$. As $R_{k+1}(x'_n - x'_{n-1}) = R_{k+1}(x_n - x_{n-1} - R_{k+1}(\ell - 1)) = 1$, and $R_{k+1}(x'_{n-1} - x'_{n-2}) = R_{k+1}(x'_{n-1} - x'_{n-2}) = k$, $(x'_{n-2}, x'_{n-1}, x'_n)$ is k-good. By induction hypothesis, $\mathcal{S}'$ is in $\mathbb{P}$.
- If $x_n = x_{n-1}$, then $x'_{n-1} = x_n - k$, $x'_n = x_{n-1}$. As $R_{k+1}(x'_{n-1} - x'_{n-2}) = R_{k+1}(x_n - x_{n-2} - k) = 0$, and $R_{k+1}(x'_n - x'_{n-1}) = R_{k+1}(x_{n-1} - x_{n-2}) = k$, so $(x'_{n-2}, x'_{n-1}, x'_n)$ is k-good. By induction hypothesis, $\mathcal{S}'$ is in $\mathbb{P}$.

Sufficiency. Assume (x_{n-2}, x_{n-1}, x_n) is k-good. We shall prove that $\mathcal{S}$ is a P-position by showing that for any legal move $x_n \to x_n - t$ with $1 \le t \le min\{k, x_n\}$, the resulting position $\mathcal{S}' = (x'_1, x'_2, \ldots, x'_n)$ is not in $\mathbb{P}$.

By Claim 2, we only need to consider the case that $m = 0$ i.e., $x_n - x_{n-1} = \ell$, where $0 \le \ell \le k$. We consider three subcases.

Case (i): $R_{k+1}(x_{n-1} - x_{n-2}) = 0$ and $x_n - x_{n-1} = k$.

Then $x_n - t \ge x_{n-1}$. So $x'_j = x_j$ for $1 \le j \le n-1$, $x'_n = x_n - t$. Thus $\beta(\mathcal{S}') = \beta(\mathcal{S})$ is odd. As $R_{k+1}(x'_{n-1} - x'_{n-2}) = R_{k+1}(x_{n-1} - x_{n-2}) = 0$ and $R_{k+1}(x'_n - x'_{n-1}) = R_{k+1}(x_n - x_{n-1} - t) \neq k$, $(x'_{n-2}, x'_{n-1}, x'_n)$ is not k-good. So $\mathcal{S}' \notin \mathbb{P}$.

Case (ii): $1 \le R_{k+1}(x_{n-1} - x_{n-2}) \le k - 1$ and $x_n - x_{n-1} = 0$.

There are three possibilities:

(1) $x_n - t \ge x_{n-2}$.
Then $x'_j = x_j$ for $1 \le j \le n-2$, $x'_{n-1} = x_n - t$, $x'_n = x_{n-1} = x_n$. So $\beta(\mathcal{S}') = \beta(\mathcal{S})$ is odd. It remains to show that $(x'_{n-2}, x'_{n-1}, x'_n)$ is not k-good. Assume to the contrary that $(x'_{n-2}, x'_{n-1}, x'_n)$ is k-good. By definition, $R_{k+1}(x'_n - x'_{n-1}) = 0, 1$ or k. Note that $R_{k+1}(x'_n - x'_{n-1}) = t \neq 0$.

- If $R_{k+1}(x'_n - x'_{n-1}) = t = k$, then since $1 \le R_{k+1}(x_{n-1} - x_{n-2}) \le k - 1$, $R_{k+1}(x'_{n-1} - x'_{n-2}) = R_{k+1}(x_{n-1} - x_{n-2} - k) \neq 0$, a contradiction.
- If $R_{k+1}(x'_n - x'_{n-1}) = t = 1$, then since $1 \le R_{k+1}(x_{n-1} - x_{n-2}) \le k - 1$, $R_{k+1}(x'_{n-1} - x'_{n-2}) = R_{k+1}(x_{n-1} - x_{n-2} - 1) \neq k$, a contradiction.

(2) $0 < x_n - t < x_{n-2}$ or $x_n - t = 0, x_{n-3} > 0$.
Then $R_{k+1}(x'_n - x'_{n-1}) = R_{k+1}(x_{n-1} - x_{n-2}) \neq 0$. If $\beta(\mathcal{S}')$ is even, then $\mathcal{S}'$ is not in $\mathbb{P}$. Assume $\beta(\mathcal{S}')$ is odd. Note that $x'_{n-2} \ge max\{x_n - t, x_{n-3}\} > 0$. It remains to show that $(x'_{n-2}, x'_{n-1}, x'_n)$ is not k-good. First we observe that $\beta(\mathcal{S}) = 1$, for otherwise $\beta(\mathcal{S}') = \beta(\mathcal{S}) - 1$ is even, contrary to the assumption that $\beta(\mathcal{S}')$ is odd. Assume to the contrary that $(x'_{n-2}, x'_{n-1}, x'_n)$ is k-good. As $R_{k+1}(x'_n - x'_{n-1}) = R_{k+1}(x_{n-1} - x_{n-2}) \in [1, k-1]$, we must have $R_{k+1}(x'_n - x'_{n-1}) = 1$ and $R_{k+1}(x'_{n-1} - x'_{n-2}) = k$. However,

$$0 \le x'_{n-1} - x'_{n-2} = x_{n-2} - max\{x_{n-3}, x_n - t\} \le x_{n-2} - (x_n - t) = t - 1 < k,$$

a contradiction.

(3) $x_n - t = 0$ and $x_{n-3} = 0$.
Then $x'_{n-2} = 0$. As $R_{k+1}(x'_n - x'_{n-1}) = R_{k+1}(x_{n-1} - x_{n-2}) \neq 0$, by Theorem 2, $\mathcal{S}'$ is not in $\mathbb{P}$.

Case (iii): $R_{k+1}(x_{n-1} - x_{n-2}) = k$ and $x_n - x_{n-1} = 1$.

Then $x_n - x_{n-2} \ge k + 1$, and hence $x_n - t > x_{n-2}$. Therefore $x'_j = x_j$ for $1 \le j \le n-2$. As $\beta(\mathcal{S}')$ is odd, it remains to show that $(x'_{n-2}, x'_{n-1}, x'_n)$ is not k-good. There are two possibilities:

- $t = 1$.
  Then $x'_{n-1} = x'_n = x_{n-1}$. Since $R_{k+1}(x'_{n-1} - x'_{n-2}) = R_{k+1}(x_{n-1} - x_{n-2}) = k$, $(x'_{n-2}, x'_{n-1}, x'_n)$ is not k-good.
- $2 \le t \le max\{x_n, k\}$.

Then $x'_{n-1} = x_n - t$ and $x'_n = x_{n-1}$. Since $R_{k+1}(x'_{n-1} - x'_{n-2}) = R_{k+1}(x_n - t - x_{n-2}) \in [1, k-1]$ and $R_{k+1}(x'_n - x'_{n-1}) = R_{k+1}(x_{n-1} - x_n + t) = t - 1 \neq 0$, $(x'_{n-2}, x'_{n-1}, x'_n)$ is not k-good.

This completes the proof of Theorem 6. □

3. Comparison

The rules of Bounded Greedy Nim are a combination of the rules of Bounded Nim and Greedy Nim. Nevertheless, the solution for Bounded Greedy Nim is much more complicated than the solutions of Bounded Nim and Greedy Nim.

Each of the Bounded Nim and the Greedy Nim has a simple solution.

Solution for k-Bounded Nim [9]: For a position $\mathcal{S} = (x_1, x_2, \dots, x_n)$, let $\mathcal{S}' = (x'_1, x'_2, \dots, x'_n)$, where $x'_i = R_{k+1}(x_i)$. Then $\mathcal{S}$ is a P-position in Bounded Nim if and only if $\mathcal{S}'$ is a P-position in Nim.

Solution for Greedy Nim [2]: The P-positions for Greedy Nim are precisely those in which the number of equal largest heaps is even.

Example 7. Assume $n = 7$, $k = 3$. Let

$$\mathcal{S}_1 = (2,3,5,8,8),\ \mathcal{S}_2 = (3,3,5,8,8),\ \mathcal{S}_3 = (2,3,5,5,6),$$
$$\mathcal{S}_4 = (2,3,5,5,5),\ \mathcal{S}_5 = (2,3,5,6,6),\ \mathcal{S}_6 = (1,3,5,6,6),$$
$$\mathcal{S}_7 = (2,2,5,8,9),\ \mathcal{S}_8 = (2,5,5,5,9).$$

The following table shows whether each of these positions is a P-position or an N-position in each of the three games.

The game	Position	$\mathcal{S}_1$	$\mathcal{S}_2$	$\mathcal{S}_3$	$\mathcal{S}_4$	$\mathcal{S}_5$	$\mathcal{S}_6$	$\mathcal{S}_7$	$\mathcal{S}_8$
Bounded Greedy Nim	N-position	√	√	√	√				
	P-position					√	√	√	√
Bounded Nim	N-position		√	√			√		√
	P-position	√			√	√		√	
Greedy Nim	N-position			√	√			√	√
	P-position	√	√			√	√		

From this table, we see that the outcome of a Bounded Greedy Nim position is not easily computed from the outcomes of the position under the other rulesets. However, if $k \geq x_n$, then k-bounded Greedy Nim on $\mathcal{S}$ is the same as Greedy Nim on $\mathcal{S}$. Indeed, if $k \geq x_n$, then $R_{k+1}(x_n - x_{n-1}) = x_n - x_{n-1}$, $R_{k+1}(x_{n-1} - x_{n-2}) = x_{n-1} - x_{n-2}$. If $x_{n-2} = 0$, then by Theorem 2, $\mathcal{S}$ is a P-position if and only if $x_{n-1} - x_n = 0$. If $x_{n-2} > 0$, then by Theorem 6, $\mathcal{S}$ is a P-position if and only if one of the following holds,

- $\beta(\mathcal{S})$ is even and $x_n - x_{n-1} = 0$;
- $\beta(\mathcal{S})$ is odd and (x_{n-2}, x_{n-1}, x_n) satisfies (ii) of the definition of k-good.

Thus the P-positions for k-bounded greedy nim with $k \geq x_n$ are precisely those in which the number of equal largest heaps is even. This coincides with the result for Greedy Nim in [2].

References

[1] M.H. Albert, R.J. Nowakowski, The game of End–Nim, Electron. J. Combin. 8 (2) (2001), Research Paper 1, 12 pp.
[2] M.H. Albert, R.J. Nowakowski, Nim restrictions, Integers, Electron. J. Comb. Number Theory 4 (2004), # G 01.
[3] C.L. Bouton, Nim, a game with a complete mathematical theory, Ann. of Math. 3 (2) (1902) 35–39.
[4] M. Fukuyama, A Nim game played on graphs, Theoret. Comput. Sci. (2003).
[5] S.-Y.R. Li, N-person Nim and N-person Moore's games, Internat. J. Game Theory 7 (1978) 31–36.
[6] C.-W. Lim, Partial Nim, Integers, Electron. J. Comb. Number Theory 5 (2005), # G 02.
[7] R.M. Low, W.H. Chan, An Atlas of N- and P-positions in 'Nim with a pass', Integers 15 (2015), # G2.
[8] E.H. Moore, A generalization of a game called Nim, Ann. of Math. 11 (1910) 93–94.
[9] B.L. Schwartz, Some extensions of Nim, Math. Mag. 44 (5) (Nov. 1971) 252–257.
[10] W. Wythoff, A modification of the game of Nim, Nieuw Arch. Wiskd. 7 (1907) 199–202.
[11] X. Lv, R. Xu, X. Zhu, Greedy Nim$_k$ Game, J. Comb. Optim. 35 (4) (2018) 1241–1249.