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ARTICLE

Multiple list coloring of 3-choice critical graphs

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Abstract

A graph G is called 3-choice critical if G is not 2-choosable but any proper subgraph is 2-choosable. A characterization of 3-choice critical graphs was given by Voigt in 1998. Voigt conjectured that if G is a bipartite 3-choice critical graph, then G is $(4m, 2m)$-choosable for every integer m. This conjecture was disproved by Meng et al. in 2017. They showed that if $G = \Theta_{r,s,t}$ where r, s, t have the same parity and $\min\{r, s, t\} \geq 3$, or $G = \Theta_{2,2,2,2p}$ with $p \geq 2$, then G is bipartite 3-choice critical, but not (4,2)-choosable. On the other hand, all the other bipartite 3-choice critical graphs are (4,2)-choosable. This paper strengthens the result of Meng, Puleo and Zhu and show that all the other bipartite 3-choice critical graphs are $(4m, 2m)$-choosable for every integer m.

KEYWORDS

choice critical graphs, bipartite graphs, multiple list coloring

1 | INTRODUCTION

An *a-list assignment* of a graph G is a mapping L which assigns to each vertex v of G a set $L(v)$ of a colors. A *b-fold coloring* of G is a mapping ϕ which assigns to each vertex v of G a set $\phi(v)$ of b colors such that for every edge uv, $\phi(u) \cap \phi(v) = \varnothing$. An *$(L, b)$-coloring* of G is a b-fold coloring ϕ of G such that $\phi(v) \subseteq L(v)$ for each vertex v. We say G is *(a, b)-choosable* if for any a-list assignment L of G, there is an (L, b)-coloring of G. We say G is *a-choosable* if G is $(a, 1)$-choosable. The concept of list coloring of graphs was introduced independently by Erdős et al. [1] and Vizing [9] in the 1970s. Since then, list coloring of graphs has attracted considerable attention and becomes an important branch of chromatic graph theory.

Erdős et al. [1] characterized all the 2-choosable graphs. Given a graph G, the *core* of G is obtained from G by repeatedly removing degree 1 vertices. Denote by $\Theta_{k_1,k_2,\ldots,k_q}$ the graph consisting of internally vertex-disjoint paths of lengths $k_1, k_2, \ldots, k_q$ connecting two vertices u and v. Erdős et al. proved that a graph G is 2-choosable if and only if the core of G is K_1 or an even cycle or $\Theta_{2,2,2p}$ for some positive integer p.

We say a graph G is 3-*choice-critical* if G is not 2-choosable and any proper subgraph of G is 2-choosable. In 1998, Voigt characterized all the 3-choice-critical graphs.

Theorem 1 (Vogit [8]). *A graph is* 3-*choice-critical if and only if it is one of the following:*

1. *An odd cycle.*
2. *Two vertex-disjoint even cycles joined by a path.*
3. *Two even cycles with one vertex in common.*
4. $\Theta_{2r,2s,2t}$ *with* $r \geq 1$, *and* $s, t > 1$, *or* $\Theta_{2r+1,2s+1,2t+1}$ *with* $r \geq 0, s, t > 0$.
5. $\Theta_{2,2,2,2t}$ *graph with* $t \geq 1$.

Except the odd cycle, all the other 3-choice-critical graphs are bipartite. In [8], Voigt conjectured that every bipartite 3-choice -critical graph G is $(2m, m)$-choosable for every even integer m. This conjecture is true if $G = \Theta_{2,2,2,2}$ [7]. However, Meng, Puleo, and Zhu [5] proved that if $min\{r, s, t\} \geq 3$, r, s, t have the same parity, then $\Theta_{r,s,t}$ is not $(4, 2)$-choosable, and if $t \geq 2$, then $\Theta_{2,2,2,2t}$ is not $(4, 2)$-choosable. Nevertheless, the other bipartite 3-choice-critical graphs are $(4, 2)$-choosable [5]. It was conjectured by Erdős et al. [1] that every (a, b)-choosable graph is (am, bm)-choosable. This conjecture was refuted recently by Dvořák et al. [2] who proved that for any integer $k \geq 4$, there exists a k-choosable graph which is not $(2k, 2)$-choosable. On the other hand, it was proved by Tuza and Voigt [6] that if G is 2-choosable, then G is $(2m, m)$-choosable for any positive integer m. A natural question is whether all the $(4, 2)$-choosable 3-choice critical graphs are $(4m, 2m)$-choosable for all integer m. In this paper, we answer this question in affirmative.

Theorem 2. *If G consists of two vertex-disjoint even cycles joined by a path or two even cycles intersecting at a single vertex, or $G = \Theta_{r,s,t}$ and $r \leq 2, s, t > 2$ and r, s, t have the same parity, then G is $(4m, 2m)$-choosable for every integer m.*

The *strong fractional choice number* of a graph G studied in [3, 4, 10] is defined as

$$ch_f^*(G) = \inf\{r \in \mathbb{R}\colon G \text{ is } (a, b)\text{–choosable for any } a, b \text{ for which } a/b \geq r\}.$$

As a consequence of Theorem 2, every $(4, 2)$-choosable 3-choice critical graph G has $ch_f^*(G) = 2$. It remains an open problem whether every bipartite 3-choice critical graph G has $ch_f^*(G) = 2$.

2 | PROOF OF THEOREM 2

The idea of the proof of Theorem 2 is the following: Assume G is a graph as in Theorem 2 and L is a $4m$-list assignment of G. Let H be the set of vertices of G of degree at least 3. Then $G - H$ is the disjoint union of a family of two or three paths, where each end vertex of these paths has exactly one neighbor in H unless the path consists of a single vertex w, in which case w has two neighbors in H, and other vertices of the paths have no neighbor in H.

We shall assign a set of $2m$ colors to each vertex in H. Then extend this pre-coloring of H to an $(L, 2m)$-coloring of the remaining vertices of G, that consists of two or three paths.

The extension to the paths are independent to each other. The question in concern becomes the following: Assume P is a path with vertices $v_1, v_2, \ldots, v_n$ in order and L is a $4m$-list assignment on P. Assume S, T are the $2m$-sets of colors assigned to the neighbors of v_1 and v_n in H, respectively (note that the neighbors of v_1 and v_n maybe the same, in that case, $S = T$). Under what condition, we can find an $(L, 2m)$-coloring of P so that the end vertices of P avoid the colors from S and T, respectively? A sufficient condition for the existence of such an extension to a $2m$-fold coloring of the whole path was given in [5]. We shall use this condition to show that there exists appropriate $(L, 2m)$-coloring of H so that the coloring can be extended to all the paths in $G - H$. This is the same idea used in [5].

Definition 3. Assume P is an n-vertex path with vertices $v_1, v_2, \ldots, v_n$ in order. For a list assignment L of P, Let

$$X_1 = L(v_1),$$
$$X_i = L(v_i) - X_{i-1}, i \in \{2, 3, \ldots, n\},$$
$$S_L(P) = \sum_{i=1}^{n} |X_i| \ .$$

The following lemma was proved in [5].

Lemma 4 (Meng et al. [5]). *Let P be an n-vertex path and let L be a list assignment on P. If $|L(v_1)|, |L(v_n)| \geq 2m$ and $|L(v_i)| = 4m$ for $i \in \{2, 3, \ldots, n-1\}$, then path P is $(L, 2m)$-colorable if and only if $S_L(P) \geq 2nm$.*

Definition 5. Assume L is a $4m$-list assignment on P and S, T are two color sets. Let $L \ominus (S, T)$ be the list assignment obtained from L by deleting all colors in S from $L(v_1)$, all colors in T from $L(v_n)$, and leaving all other lists unchanged. The damage of (S, T) with respect to L and P is defined as

$$dam_{L,P}(S, T) = S_L(P) - S_{L \ominus (S,T)}(P).$$

So to prove that P has an $2m$-fold $L \ominus (S, T)$-coloring, it suffices to show that

$$S_L(P) - dam_{L,P}(S, T) \geq 2nm.$$

For this purpose, a few lemmas were proved in [5] that give lower bounds for $S_L(P)$ and upper bounds for $dam_{L,P}(S, T)$.

Definition 6 (Meng et al. [5]). Assume n is an odd integer, P is an n-vertex path with vertices $v_1, v_2, \ldots, v_n$ in order, and L is a list assignment on P. Let

$$\Lambda = \bigcap_{x \in V(P)} L(x),$$
$$\hat{X}_1 = \{c \in L(v_1) - \Lambda : \text{ the smallest index } i \text{ for which } c \notin L(v_i) \text{ is even}\},$$
$$\hat{X}_n = \{c \in L(v_n) - \Lambda : \text{ the largest index } i \text{ for which } c \notin L(v_i) \text{ is even}\}.$$

Lemma 7 (Meng et al. [5]). *Let L be a list assignment on an n-vertex path P, where n is odd. For any sets of colors S, T,*

$$dam_{L,P}(S, T) = |\hat{X}_1 \cap S| + |\hat{X}_n \cap T| + |\Lambda \cap (S \cup T)|.$$

Lemma 8 (Meng et al. [5]). *If L is a list assignment on an n-vertex path P, where n is odd and $|L(v_i)| = 4m$ for all i, then*

$$S_L(P) \geq \max\{2(n-1)m + |\hat{X}_1| + |\hat{X}_n| + |\Lambda|, 2(n+1)m\}.$$

The following is a key lemma for the proof in this paper.

Lemma 9. *Let m, ℓ and τ be fixed integers, where $m \geq 1, 0 \leq \ell \leq 4m, 0 \leq \tau \leq 2m-2, \ell + \tau \geq 2m+2$, and both ℓ and τ are even. Assume x, y are non-negative integers with $x + y \leq \ell$. Let*

$$F(x, y) = \sum \binom{x}{a}\binom{y}{b}\binom{\ell - x - y}{2m - \tau - a - b},$$

where the summation is over all non-negative integer pairs (a, b) for which $0 \leq a \leq x, 0 \leq b \leq y, a + b \leq 2m - \tau$ and $2a + b \geq \max\{2x + y + 2m + 1 - \ell - \tau, 2m + 1 - \tau\}$. Then

$$F(x, y) \leq \frac{1}{2}\binom{\ell}{2m - \tau} - 1.$$

Note that when $a > x$ or $b > y$, then $\binom{x}{a}\binom{y}{b} = 0$. Also $a + b \leq 2m - \tau$ and $2a + b \geq 2x + y + 2m - \tau + 1 - \ell$ implies that $2x + y \leq \ell + \tau - 1 + 2a + b \leq \ell + 2m - \tau - 1$. Thus the summation can be restricted to $0 \leq a \leq x, 0 \leq b \leq y, a + b \leq 2m - \tau$ and $2x + y \leq \ell + 2m - \tau - 1$.

The proof of Lemma 9 will be given in next section. In the rest of the section, we will prove Theorem 2. In Section 2.1, we will prove the first half of Theorem 2: If G is a graph consisting of two edge-disjoint even cycles E and F connected by a path Q (possibly Q is a single vertex path), then G is $(4m, 2m)$-choosable for all positive m. In Section 2.2, we prove the second half of Theorem 2: $\Theta_{r,s,t}$ is $(4m, 2m)$-choosable if r, s, t have the same parity and $r \leq 2, s, t > 2$.

2.1 | Proof of the first part of Theorem 2

Definition 10. Assume P is a path and L is a $4m$-list assignment for P, S, T are two $2m$-sets of colors. We say S is bad with respect to (L, P) if $dam_{L,P}(S, S) > S_L(P) - 2nm$.

Lemma 11. *Assume P is a path with an odd number of vertices, L is a $4m$-list assignment on P, and W is a set of $4m$ colors. Then W has less than $\frac{1}{2}\binom{4m}{2m}$ bad $2m$-subsets with respect to (L, P).*

Proof. Let $\Lambda, \hat{X}_1, \hat{X}_n$ be calculated for P as in Definition 6. Let $X = \hat{X}_1 \cap \hat{X}_n \cap W$, $Y = [(\hat{X}_1 \Delta \hat{X}_n) \cup \Lambda] \cap W$ (Δ means symmetric difference) and $Z = W - Y - X$.

Assume S is a bad subset of W with respect to (L, P).

Let $A = S \cap X, B = S \cap Y$ and $C = S \cap Z$, see Figure 1. Let $|X| = x, |Y| = y, |Z| = z, |A| = a, |B| = b$ and $|C| = c$.

Since W is the disjoint union of X, Y, Z and S is the disjoint union of A, B, C, we have

$$x + y + z = 4m, \quad a + b + c = 2m.$$

As

$$\begin{aligned} |\hat{X}_1| + |\hat{X}_n| + |\Lambda| &= |\hat{X}_1 \cup \hat{X}_n| + |\hat{X}_1 \cap \hat{X}_n| + |\Lambda| \\ &\geq |(\hat{X}_1 \cup \hat{X}_n) \cap W| + |(\hat{X}_1 \cap \hat{X}_n) \cap W| + |\Lambda \cap W| \\ &= |(\hat{X}_1 \Delta \hat{X}_n) \cap W| + 2|(\hat{X}_1 \cap \hat{X}_n) \cap W| + |\Lambda \cap W| \\ &= 2x + y, \end{aligned}$$

by Lemma 8, we have $S_L(P) \geq \max\{2x + y + 2(n-1)m, 2(n+1)m\}$. By Lemma 7,

$$\begin{aligned} dam_{L,P}(S, S) = |\hat{X}_1 \cap S| + |\hat{X}_n \cap S| + |\Lambda \cap S| = |(\hat{X}_1 \Delta \hat{X}_n) \cap S| + 2|(\hat{X}_1 \cap \hat{X}_n) \cap S| \\ + |\Lambda \cap S| = 2a + b. \end{aligned}$$

As $dam_{L,P}(S, S) > S_L(P) - 2nm$, we conclude that

$$2a + b \geq \max\{2x + y - 2m + 1, 2m + 1\}.$$

Thus the number of bad $2m$-subsets of W with respect to (L, P) is at most

$$F(x, y) = \sum \binom{x}{a}\binom{y}{b}\binom{4m - x - y}{2m - a - b},$$

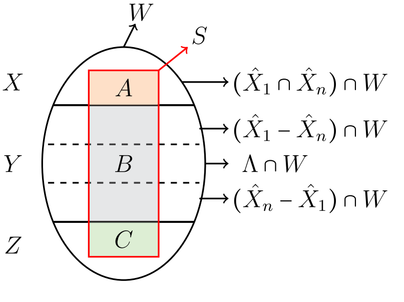

FIGURE 1 W, X, Y, Z and S, A, B, C [Color figure can be viewed at wileyonlinelibrary.com

where the summation is over all pairs of integers (a, b) with $0 \le a \le x, 0 \le b \le y, a + b \le 2m$ and $2a + b \ge \max\{2x + y - 2m + 1, 2m + 1\}$. By the special case with $\ell = 4m$ and $\tau = 0$ of Lemma 9, $F(x, y) < \frac{1}{2}\binom{4m}{2m}$.

This completes the proof of Lemma 11. □

If G consists of two even cycles intersecting at a single vertex v, then $G - v$ is consists of two odd paths P_1 and P_2. It follows from Lemma 11 that there is a $2m$-subset S of $L(v)$ such that S is not bad with respect to both (L, P_1) and (L, P_2). Thus we can color v by S and extend it to an $(L, 2m)$-coloring of G.

Assume G consists of two even cycles E and F joined by a path Q, and let u, v be the end vertices of Q, as shown in Figure 2.

Observe that there is an injective function $h: \binom{L(u)}{2m} \to \binom{L(v)}{2m}$ such that for all $S \in \binom{L(u)}{2m}$, the precoloring $\phi(u) = S, \phi(v) = h(S)$ extends to all of Q. Indeed, if Q consists of a single vertex v, then $u = v$ and $h(S) = S$. Otherwise, for each $S \in \binom{L(u)}{2m}$, let $\phi(u) = S$, extend ϕ to a $2m$-fold L-coloring ϕ of Q. We simply let $h(S) = \phi(v)$.

By Lemma 11, $L(u)$ has less than $\frac{1}{2}\binom{4m}{2m}$ bad $2m$-subsets with respect to (L, P), and $L(v)$ has less than $\frac{1}{2}\binom{4m}{2m}$ bad subsets of size $2m$ with respect to (L, R). So there exists some S such that S is not bad with respect to (L, P) and $h(S)$ is not bad with respect to (L, R). Therefore the precoloring ϕ of u, v defined as $\phi(u) = S$ and $\phi(v) = h(S)$ extends to an $(L, 2m)$-coloring of G. This completes the first half of Theorem 2.

2.2 | Proof of the second part of Theorem 2

The following lemma was proved in [5].

Lemma 12 (Meng et al. [5]). *Let G be a graph, let $v \in V(G)$, and let G' be obtained from G by deleting v and merging its neighbors. If G is $(4m, 2m)$-choosable, then G' is $(4m, 2m)$-choosable.*

Corollary 13. *If $\Theta_{2,2r,2s}$ is $(4m, 2m)$-choosable, then $\Theta_{1,2r-1,2s-1}$ is $(4m, 2m)$-choosable.*

So it suffices to show that $\Theta_{2,2r,2s}$ is $(4m, 2m)$-choosable, where $r, s \ge 2$. Instead of proving it directly, we prove the following stronger result.

Theorem 14. *Assume $G = \Theta_{2,2r,2s}$, where $r, s \ge 2$. Let u, v be the two vertices of degree 3. Let P^0, P^1, P^2 be the three paths of $G - \{u, v\}$. Assume $P^i = \left(v_1^i, v_2^i, \ldots, v_{n_i}^i\right)$, $|V(P^0)| = 1$, v_1^i is adjacent to u and $v_{n_i}^i$ is adjacent to v. Assume ℓ, τ are non-negative even integers and L is a list assignment for G satisfying the following:*

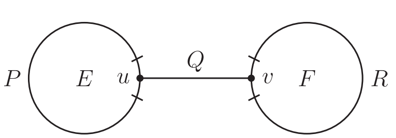

FIGURE 2 Decomposing G into P, Q, R

(*C*1) $\tau \le 2m$ *and* $\ell + \tau \ge 2m$.
(*C*2) $|L(u)| = |L(v)| = \ell \ge 0$.
(*C*3) *For each* $i \in \{0, 1, 2\}$, $|L(v_1^i)|, |L(v_{n_i}^i)| \ge 4m - \tau$.
(*C*4) $|L(w)| = 4m$ *for* $w \ne u, v, v_1^i, v_{n_i}^i$.
(*C*5) *For* $i = 0, 1, 2$, $S_L(P^i) \ge 2m|V(P^i)| + 2m - \tau$, *and*

$$dam_{L,P^i}(L(u), L(v)) \le S_L(P^i) - 2m|V(P^i)| + \ell - 2m + \tau.$$

Then there exists a set $S \subset L(u)$ *and a set* $T \subset L(v)$ *satisfying* $|S| = |T| = 2m - \tau$ *such that for each i,*

$$dam_{L,P^i}(S, T) \le S_L(P^i) - 2m|V(P^i)|.$$

Theorem 2 follows from Theorem 14 by setting $\ell = 4m$ and $\tau = 0$. Indeed, by Lemma 8, $S_L(P^i) \ge 2m|V(P^i)| + 2m$, and $S_L(P) - 2m|V(P^i)| + 2m \ge |\hat{X}_1^i| + |\hat{X}_n^i| + |\Lambda^i| \ge dam_{L,P^i}(L(u), L(v))$ (The last inequality holds by Lemma 7). So there exist two sets $S \subset L(u)$, $T \subset L(v)$ such that $|S| = |T| = 2m$ and $dam_{L,P^i}(S, T) \le S_L(P^i) - 2m|V(P^i)|$, which implies that G is $(4m, 2m)$-choosable.

Let $L(u) = \{c_1, c_2, ..., c_\ell\}$ and $L(v) = \{c_1', c_2', ..., c_\ell'\}$ be indexed in such a way that $c_j = c_j'$ whenever $c_j \in L(u) \cap L(v)$. In other words, $\{c_i, c_i'\} \cap \{c_j, c_j'\} = \emptyset$ whenever $i \ne j$.

Definition 15. For a fixed indexing of $L(u)$ and $L(v)$, a *couple* is a tuple of the form (c_j, c_j') for $j \in \{1, 2, ..., \ell\}$. When we write a couple, we suppress the parentheses and simply write $c_j c_j'$. A *pair* is a tuple (S, T) with $S \subset L(u)$, $T \subset L(v)$, and $|S| = |T|$, and we define the *size* of a pair as $|S|$. A pair (S, T) is *bad with respect to* (L, P) if $dam_{L,P}(S, T) > S_L(P) - 2m|V(P)|$. A *simple pair* is a pair (S, T) such that S, T have the same index set.

By Lemma 7, we know that if (S, T) is a simple pair, then

$$dam_{L,P}(S, T) = \sum_{c_j \in S} dam_{L,P}(\{c_j\}, \{c_j'\}). \tag{1}$$

In the following, we may write $dam_{L,P}(c, c')$ for $dam_{L,P}(\{c\}, \{c'\})$. The following observation follows from Lemma 7.

Observation 16. For any couple cc', the following hold:

1. $dam_{L,P}(c, c') = 2$ if $c \in \hat{X}_1 \cup \Lambda$ and $c' \in \hat{X}_n \cup \Lambda$, and moreover if $c = c'$, then $c \notin \Lambda$;
2. $dam_{L,P}(c, c') = 1$ if $c \in \hat{X}_1 \cup \Lambda$ or $c' \in \hat{X}_n \cup \Lambda$ but not both unless $c = c' \in \Lambda$;
3. $dam_{L,P}(c, c') = 0$ if $c \notin \hat{X}_1 \cup \Lambda$ and $c' \notin \hat{X}_n \cup \Lambda$.

Definition 17. Assume $c_j c_j'$ is a couple.

- $c_j c_j'$ is *heavy* for the internal path P if $dam_{L,P}(c_j, c_j') = 2$;
- $c_j c_j'$ is *light* for the internal path P if $dam_{L,P}(c_j, c_j') = 1$;
- $c_j c_j'$ is *safe* for the internal path P if $dam_{L,P}(c_j, c_j') = 0$.

For each $i \in \{0, 1, 2\}$, let $x^{(i)}, y^{(i)}, z^{(i)}$ denote the number of heavy, light couples and safe couples for P^i, respectively. For a simple pair (S, T) which is of size $2m - \tau$, let $a^{(i)}(S, T), b^{(i)}(S, T), c^{(i)}(S, T)$ denote the number of heavy, light and safe couples for P^i in (S, T), respectively.

It follows from the definition that $x^{(i)} + y^{(i)} + z^{(i)} = \ell$, and $a^{(i)}(S, T) + b^{(i)}(S, T) + c^{(i)}(S, T) = 2m - \tau$. Thus by Equality (1), $dam_{L,P^i}(S, T) = 2a^{(i)}(S, T) + b^{(i)}(S, T)$. Let $\beta(P^i)$ denote the number of bad simple pairs of size $2m - \tau$ with respect to (L, P^i). We write $\hat{X}_1^i, \hat{X}_n^i$ and Λ^i for the sets $\hat{X}_1, \hat{X}_n, \Lambda$ calculated for P^i.

Proof of Theorem 14. First we observe that the conclusion of Theorem 14 is equivalent to the statement that there exists a pair (S, T) which is not a bad pair for any of the paths P^0, P^1, P^2.

The proof is by induction on $2\ell + \tau$. First assume that $2\ell + \tau = 2m$. Since both ℓ and τ are non-negative, and $\ell + \tau \geq 2m$, we have that $\ell = 0$ and $\tau = 2m$. By assumption, for each $i \in \{0, 1, 2\}$,

$$S_L(P^i) - 2m|V(P^i)| \geq dam_{L,P^i}(L(u), L(v)) = 0,$$

so $S_L(P^i) \geq 2m|V(P^i)|$. Then let $S = L(u) = \emptyset, T = L(v) = \emptyset$, and we are done. This finishes the basic step of the induction.

Thus in the sequel, we assume that $2\ell + \tau \geq 2m + 2$. If $\ell + \tau = 2m$, then we let $S = L(u), T = L(v)$, and we are done. Hence we assume that $\ell + \tau \geq 2m + 2$. If $\tau = 2m$, then the statement holds obviously, since we can just take $S = T = \emptyset$ and we are done. So we assume that $\tau \leq 2m - 2$. Assume that Theorem 14 is not true for L. □

Claim 1. There does not exist a simple pair (D_u, D_v) such that $|D_u| = |D_v| = d \leq \ell - 2m + \tau$ is even, and $dam_{L,P^i}(D_u, D_v) \geq d$ for each $i \in \{0, 1, 2\}$.

Proof. Assume (D_u, D_v) is such a simple pair. Let L' be a new list assignment for G with $L'(u) = L(u) - D_u, L'(v) = L(v) - D_v, L'(w) = L(w)$ for $w \in V(G) \backslash \{u, v\}$.

(C1) to (C4) of Theorem 14 are easily seen to be satisfied by L', with $\ell' = \ell - d$ and $\tau' = \tau$. As

$$\begin{aligned} dam_{L',P^i}(L'(u), L'(v)) &= dam_{L,P^i}(L(u), L(v)) - dam_{L,P^i}(D_u, D_v) \\ &\leq S_L(P^i) - 2m|V(P^i)| + \ell - 2m + \tau - d \\ &= S_{L'}(P^i) - 2m|V(P^i)| + \ell' + \tau' - 2m, \end{aligned}$$

(C5) is also satisfied by L'. By induction, there exists a pair (S, T), where $|S| = |T| = 2m - \tau$ such that for each $i \in \{0, 1, 2\}$,

$$dam_{L,P^i}(S, T) \leq S_L(P^i) - 2m|V(P^i)|.$$

This completes the proof of this claim. □

Claim 2. There does not exist a simple pair (D_u, D_v) such that $|D_u| = |D_v| = d \leq 2m - \tau$ is even, and $dam_{L,P^i}(D_u, D_v) \leq d$ for each $i \in \{0, 1, 2\}$.

Proof. Assume (D_u, D_v) is such a simple pair. Let L' be a new list assignment for G with $L'(u) = L(u) - D_u, L'(v) = L(v) - D_v$, for $i = 1, 2, L'\left(v_1^i\right) = L\left(v_1^i\right) - D_u, L'\left(v_{n_i}^i\right) = L\left(v_{n_i}^i\right) - D_v, L'\left(v_j^i\right) = L\left(v_j^i\right)$ where $1 < j < n_i$, and $L'\left(v_1^0\right) = L\left(v_1^0\right) - D_u \cup D_v$.

Note that $dam_{L,P^0}(D_u, D_v) \leq d$ implies that $|L(v_1^0) - D_u \cup D_v| \geq |L(v_1^0)| - d$. So (C1) to (C4) of Theorem 14 are satisfied by L', with $\ell' = \ell - d$ and $\tau' = \tau + d$. As $S_L(P^i) = S_{L'}(P) + dam_{L,P^i}(D_u, D_v)$, we have $S_L^{'}(P^i) = S_L(P^i) - dam_{L,P^i}(D_u, D_v) \geq 2m|V(P^i)| + 2m - \tau - d$, and also by the second part of (C5), it follows that

$$
\begin{aligned}
dam_{L',P^i}(L'(u), L'(v)) &\leq dam_{L,P^i}(L(u), L(v)) - d \\
&\leq S_L(P^i) - 2m|V(P^i)| + \ell - 2m + \tau - d \\
&= S_{L'}(P^i) + d - 2m|V(P^i)| + \ell - 2m + \tau - d \\
&= S_{L'}(P^i) - 2m|V(P^i)| + (\ell - d) + (\tau + d) - 2m.
\end{aligned}
$$

So (C5) is also satisfied by L'. By induction, there exists a pair (S', T'), where $|S'| = |T'| = 2m - \tau' = 2m - \tau - d$ such that for every i,

$$dam_{L',P^i}(S', T') \leq S_{L'}(P^i) - 2m|V(P^i)|.$$

Let $S = S' \cup D_u$ and $T = T' \cup D_v$. We have $|S| = |T| = 2m - \tau$ and

$$
\begin{aligned}
dam_{L,P^i}(S, T) &= dam_{L,P^i}(D_u, D_v) + dam_{L,P^i}(S', T') \\
&\leq dam_{L,P^i}(D_u, D_v) + S_{L'}(P^i) - 2m|V(P^i)| \\
&= S_L(P^i) - 2m|V(P^i)|.
\end{aligned}
$$

This completes the proof of Claim 2. □

The following claim gives a necessary condition for a simple pair of size $2m - \tau$ being bad with respect to (L, P^i).

Claim 3. If (S, T) is a bad simple pair of size $2m - \tau$ with respect to (L, P^i), then $dam_{L,P^i}(S, T) = 2a^{(i)}(S, T) + b^{(i)}(S, T) \geq \max\{2x^{(i)} + y^{(i)} + 2m + 1 - \ell - \tau, 2m + 1 - \tau\}$.

Proof. By Equality (1), $dam_{L,P^i}(L(u), L(v)) = 2x^{(i)} + y^{(i)}$. By (C5),

$$S_L(P^i) \geq \max\{2m|V(P^i)| + 2x^{(i)} + y^{(i)} + 2m - \ell - \tau, 2m|V(P^i)| + 2m - \tau\}.$$

If (S, T) is a bad simple pair of size $2m - \tau$ with respect to (L, P^i), then by Definition 15 and above inequality,

$$
\begin{aligned}
dam_{L,P^i}(S, T) &= 2a^{(i)}(S, T) + b^{(i)}(S, T) \\
&\geq S_L(P^i) - 2m|V(P^i)| + 1 \\
&\geq \max\{2x^{(i)} + y^{(i)} + 2m + 1 - \ell - \tau, 2m + 1 - \tau\}.
\end{aligned}
$$

Thus we proved this claim. □

The following claim gives an upper bound and a lower bound of the number of bad simple pairs of size $2m-\tau$ with respect to (L, P^i).

Claim 4. For each $i \in \{0, 1, 2\}$, $2 \le \beta(P^i) \le \frac{1}{2}\binom{\ell}{2m-\tau} - 1$.

Proof. If a simple pair (S, T) of size $2m-\tau$ is bad with respect to (L, P^i), then by Claim 3, $dam_{L,P^i}(S, T) \ge \max\{2x^{(i)} + y^{(i)} + 2m + 1 - \ell - \tau, 2m + 1 - \tau\}$. Note that $a^{(i)}(S, T) + b^{(i)}(S, T) + c^{(i)}(S, T) = 2m - \tau$, it follows from Lemma 9 that $\beta(P^i) \le \frac{1}{2}\binom{\ell}{2m-\tau} - 1$.

If $\beta(P^i) \le 1$ for some i, then $\beta(P^0) + \beta(P^1) + \beta(P^2) \le \binom{\ell}{2m-\tau} - 1$. So there exists a simple pair (S, T) of size $2m-\tau$ which is not bad with respect to any (L, P^i), a contradiction to the assumption. □

Claim 5. For each $i \in \{0, 1, 2\}$, $2x^{(i)} + y^{(i)} \le \ell + 2m - \tau - 1$, and $x^{(i)}, z^{(i)} \ge 1$.

Proof. If $S_L(P^i) \ge 2m|V(P^i)| + 2m - \tau$, then for any simple pair (S, T) of size $2m-\tau$, $S_L(P^i) - dam_{L,P^i}(S, T) \ge 2m|V(P^i)|$ (as $dam_{L,P^i}(S, T) \le 2m - \tau$), and hence (S, T) is not bad with respect to (L, P^i) which means that $\beta(P^i) = 0$, a contradiction. Thus we may assume that $S_L(P^i) \le 2m|V(P^i)| + 4m - 1 - 2\tau$. By (C5),

$$\begin{aligned} 2x^{(i)} + y^{(i)} = dam_{L,P^i}(L(u), L(v)) &\le S_L(P^i) - 2m|V(P^i)| + \ell + \tau - 2m \\ &\le \ell + 2m - 1 - \tau. \end{aligned}$$

Assume $x^{(i)} = 0$ for some $i \in \{0, 1, 2\}$, say $x^{(0)} = 0$, then for every simple pair (S, T) of size $2m-\tau$, $dam_{L,P^0}(S, T) \le 2m - \tau$. As $S_L(P^i) - 2m|V(P^i)| \ge 2m - \tau$ (by (C5)), (S, T) is not bad with respect to (L, P^i), so $\beta(P^i) = 0$, in contrary to Claim 4. Thus $x^{(i)} \ge 1$.

Assume $z^{(0)} = 0$, then $x^{(0)} + y^{(0)} = \ell$ and for any simple pair (S, T) of size $2m-\tau$, $a^{(0)}(S, T) + b^{(0)}(S, T) = 2m - \tau$. By Claim 3, we have

$$\begin{aligned} 2a^{(0)}(S, T) + b^{(0)}(S, T) &= a^{(0)}(S, T) + 2m - \tau \\ &\ge 2x^{(0)} + y^{(0)} + 2m + 1 - \ell - \tau \\ &= x^{(0)} + \ell + 2m + 1 - \ell - \tau \\ &= x^{(0)} + 1 + 2m - \tau. \end{aligned}$$

This implies that $a^{(0)}(S, T) \ge x^{(0)} + 1$, in contrary to the fact that $a^{(0)}(S, T) \le x^{(0)}$. □

Claim 6. Every couple is heavy (respectively, safe) for at most one internal path. There is at most one couple which is light for all internal paths. If there exists a couple which is light for at least two internal paths, then it is light for all internal paths.

Proof. Assume to the contrary, $c_j c_j'$ is heavy for two paths, say for both P_0 and P_1. If $c_j c_j'$ is also heavy for P^2, then for any other couple $c_k c_k'$, we know that $(\{c_j, c_k\}, \{c_j', c_k'\})$ is a simple pair of size 2 contradicting to Claim 1. Thus $c_j c_j'$ is not heavy for P^2. Note that $x^{(2)} \ge 1$, there exists a couple $c_k c_k'$ which is heavy for P^2. It follows that $(\{c_j, c_k\}, \{c_j', c_k'\})$ is a simple pair of size 2 contradicting to Claim 1. Similarly, we can prove that no couple is safe for at least two internal paths.

If there are two couples which are light for all internal paths, then two such couples comprise a simple pair of size 2 which contradicts to Claim 2.

Assume the last sentence of this claim is not true, say c_jc_j' is light for P^0 and P^1 but not light for P^2. Note that by Claim 5, $z^{(2)} \geq 1$, so if c_jc_j' is heavy for P^2, then there exists a distinct couple c_kc_k' which is safe for P^2. By the first part of this claim, c_kc_k' is safe for neither P^0 nor P^1. Then $(\{c_j, c_k\}, \{c_j', c_k'\})$ is a simple pair of size 2 contradicting to Claim 1. If c_jc_j' is safe for P^2, then since $x^{(2)} \geq 1$ (by Claim 5), there exists a distinct couple $c_kc'_k$ which is heavy for P^2. By the first part of this claim, c_kc_k' is heavy for neither P^0 nor P^1. Then $(\{c_j, c_k\}, \{c_j', c_k'\})$ is a simple pair of size 2 contradicting to Claim 2. This completes the proof of Claim 6. □

Without loss of generality, we may assume that c_0c_0' is heavy for P^0, light for P^1 and safe for P^2.

Claim 7. For any couple c_jc_j',

- if it is heavy for P^0, then it is light for P^1, safe for P^2;
- if it is light for P^0, then it is safe for P^1, heavy for P^2;
- if it is safe for P^0, then it is heavy for P^1, light for P^2.

Consequently, $x^{(0)} = y^{(1)} = z^{(2)}, y^{(0)} = z^{(1)} = x^{(2)}$ *and* $z^{(0)} = x^{(1)} = y^{(2)}$.

Proof. If c_jc_j' is safe for P^0, then c_jc_j' is light for P^2, for otherwise by Claim 6, c_jc_j' is heavy for P^2, and light for P^1. Then $(\{c_0, c_j\}, \{c_0', c_j'\})$ is a simple pair of size 2 which contradicts to Claim 2. By Claim 6, c_jc_j' is heavy for P^1.

By Claim 5, $z^{(0)} \geq 1$, there is a couple c_ic_i', which is safe for P^0. Hence c_ic_i' is light for P^2 and heavy for P^1.

Also by Claim 5, $z^{(1)} \geq 1$, there exists a couple c_kc_k' which is safe for P^1. Then c_kc_k' is light for P^0, otherwise by Claim 6, c_kc_k' is heavy for P^0, and light for P^2. Then $(\{c_i, c_k\}, \{c_i', c_k'\})$ is a simple pair of size 2 which contradicts to Claim 2. By Claim 6, c_kc_k' is heavy for P^2.

If c_jc_j' is heavy for P^0, then it is light for P^1, for otherwise, it is safe for P^1, light for P^2, but then $(\{c_i, c_j\}, \{c_i', c_j'\})$ is a simple pair of size 2 which contradicts to Claim 2.

Next we show that no couple is light for all internal paths. Assume that there exist a couple cc' which is light for all internal paths. If $\tau \leq 2m - 4$, then $2m - \tau \geq 4$, so $(\{c_0, c_i, c_k, c\}, \{c_0', c_i', c_k', c'\})$ is a simple pair of size 4 contradicting to Claim 2.

Assume $\tau = 2m - 2$. Recall that $\ell + \tau \geq 2m + 2$, so if $\ell \neq 4$, then $\ell \geq 6$ and $\ell - 2m + \tau \geq 4$. Thus $(\{c_i, c_j, c_k, c\}, \{c_i', c_j', c_k', c'\})$ is a simple pair of size 4 which contradicts to Claim 1.

Assume $\tau = 2m - 2$ and $\ell = 4$. Since $|V(P^0)| = 1$, we know that $\hat{X}_1^0 = \hat{X}_n^0 = \emptyset$. By Observation 16, $c_0, c_0' \in \Lambda^0$ and $c_0 \neq c_0'$. As c_0c_0' is light for P^1, by Observation 16, $c_0 \notin \hat{X}_1^1 \cup \Lambda^1$ or $c_0' \notin \hat{X}_n^1 \cup \Lambda^1$. By symmetric, we may assume that $c_0 \notin \hat{X}_1^1 \cup \Lambda^1$. Then for $S = \{c_0, c\}$ and $T = \{c_i', c'\}$, the conclusion of Theorem 14 holds.

If c_jc_j' is light for P^0, as c_jc_j' is not light for all internal paths, we conclude that it must be safe for P^1, for otherwise, c_jc_j' is heavy for P^1 and $(\{c_k, c_j\}, \{c_k', c_j'\})$ is a simple pair of size 2 contradicting to Claim 2. This implies that c_jc_j' heavy for P^2. □

Claim 8. For each $i \in \{0, 1, 2\}$, $x^{(i)}, y^{(i)}, z^{(i)} \geq 2$.

Proof. Assume to the contrary that $x^{(0)} = 1$. As $x^{(0)} + y^{(0)} + z^{(0)} = \ell$, we have $z^{(0)} = \ell - y^{(0)} - 1$. By Claim 5 and Claim 7, $2y^{(0)} + z^{(0)} = 2x^{(2)} + y^{(2)} \leq \ell + 2m - \tau - 1$. So $2y^{(0)} + (\ell - 1 - y^{(0)}) \leq \ell + 2m - \tau - 1$, which implies that $y^{(0)} \leq 2m - \tau$.

If $y^{(0)} \leq 2m - \tau - 2$, then for any simple pair (S, T),

$$2a^{(0)}(S, T) + b^{(0)}(S, T) \leq 2x^{(0)} + y^{(0)} \leq 2 + 2m - \tau - 2 = 2m - \tau.$$

This implies that (S, T) is not bad with respect to (L, P^0). Hence $\beta(P^0) = 0$, in contrary to Claim 4. So we have $y^{(0)} \geq 2m - \tau - 1$, thus $y^{(0)} = 2m - \tau - 1$ or $y^{(0)} = 2m - \tau$.

If $y^{(0)} = 2m - \tau - 1$, then a bad simple pair with respect to (L, P^0) consists of the unique couple which is heavy for P^0 and the exactly $2m - \tau - 1$ couples which are light for P^0. So $\beta(P^0) = 1$, in contrary to Claim 4.

Assume $y^{(0)} = 2m - \tau$. Suppose (S, T) is a bad simple pair with respect to (L, P^2). By Claim 3,

$$2a^{(2)}(S, T) + b^{(2)}(S, T) \geq \max\{2x^{(2)} + y^{(2)} + 2m - \tau + 1 - \ell, 2m - \tau + 1\}.$$

By Claim 7, $x^{(2)} = y^{(0)}$ and $y^{(2)} = z^{(0)} = \ell - x^{(0)} - y^{(0)} = \ell - 1 - y^{(0)}$. Hence,

$$2a^{(2)}(S, T) + b^{(2)}(S, T) \geq y^{(0)} + 2m - \tau = 4m - 2\tau. \tag{2}$$

As $a^{(2)}(S, T) + b^{(2)}(S, T) = 2m - \tau - c^{(2)}(S, T) \leq 2m - \tau$, we have $2a^{(2)}(S, T) + b^{(2)}(S, T) \leq 2x^{(2)} + (2m - \tau - x^{(2)}) = x^{(2)} + 2m - \tau = 4m - 2\tau$. Together with Inequality (2), we have $2a^{(2)}(S, T) + b^{(2)}(S, T) = 4m - 2\tau$ and hence $a^{(2)}(S, T) = x^{(2)} = 2m - \tau$, that is, a bad simple with respect to (L, P^2) consists of exactly the $2m - \tau$ couples which are heavy for P^2. So $\beta(P^2) = 1$, in contrary to Claim 4. □

Without loss of generality, we assume that

- c_0c_0' and c_1c_1' are heavy for P^0 (and hence light for P^1 and safe for P^2 by Claim 7).
- c_2c_2' and c_3c_3' are light for P^0 (and hence safe for P^1 heavy for P^2).
- c_4c_4' and c_5c_5' are safe for P^0 (and hence heavy for P^1 and light for P^2).

As $|V(P^0)| = 1$, we know that $\hat{X}_1^0 = \hat{X}_n^0 = \emptyset$. By Observation 16, $c_0 \neq c_0'$ and $c_1 \neq c_1'$, and we may assume that $c_0 \notin \hat{X}_1^1 \cup \Lambda^1$.

Let $S_1 = \{c_0, c_4\}$, $T_1 = \{c_2', c_4'\}$, and let $S_3 = \{c_0, c_1, ..., c_5\}$, $T_3 = \{c_0', c_1', ..., c_5'\}$. If $c_1 \notin \hat{X}_1^1 \cup \Lambda^1$, then let $S_2 = \{c_0, c_1, c_4, c_5\}$, $T_2 = \{c_2', c_3', c_4', c_5'\}$. Otherwise, $c_1' \notin \hat{X}_n^1 \cup \Lambda^1$ (again by Observation 16), we let $S_2 = \{c_0, c_2, c_4, c_5\}$, $T_2 = \{c_1', c_3', c_4', c_5'\}$.

Now we show that for each $i \in \{0, 1, 2\}, j \in \{1, 2, 3\}$, $dam_{L,P^i}(S_j, T_j) \leq 2j$.

For each $i \in \{0, 1, 2\}$, among the six couples $c_0c_0', ..., c_5c_5'$, two are light, two are safe and two are heavy for P^i. Therefore, $dam_{L,P^i}(S_3, T_3) = 6$.

As c_4c_4' is safe for P^0, $dam_{L,P^0}(c_4, c_4') = 0$. Hence $dam_{L,P^0}(S_1, T_1) = dam_{L,P^0}(c_0, c_2') \leq 2$. As $c_0 \notin \hat{X}_1^1 \cup \Lambda^1$ and c_2c_2' is safe for P^1, we have $dam_{L,P^1}(c_0, c_2') = 0$. Hence $dam_{L,P^1}(S_1, T_1)$

$= dam_{L,P^1}(c_4, c_4') = 2$. As c_0c_0' is safe for P^2 and c_4c_4' is light for P^2, we have $dam_{L,P^2}(c_0, c_2') = 1$ and $dam_{L,P^2}(c_4, c_4') = 1$. Thus $dam_{L,P^2}(S_1, T_1) = 2$.

Consider the case that $c_1 \notin \hat{X}_1^1 \cup \Lambda^1$. As c_4c_4', c_5c_5' are safe for P^0, we conclude that $dam_{L,P^0}(S_2, T_2) = dam_{L,P^0}(c_0, c_2') + dam_{L,P^0}(c_1, c_3') \leq 4$. As $c_0, c_1 \notin \hat{X}_1^1 \cup \Lambda^1$ and c_2c_2', c_3c_3' are safe for P^1, it follows that $dam_{L,P^1}(S_2, T_2) = dam_{L,P^1}(c_4, c_4') + dam_{L,P^1}(c_5, c_5') = 4$. As c_0c_0', c_1c_1' are safe for P^2, it follows that $dam_{L,P^2}(c_0, c_2') + dam_{L,P^2}(c_1, c_3') \leq 2$. As c_4c_4', c_5c_5' are light for P^2, we have $dam_{L,P^2}(S_2, T_2) \leq 2 + dam_{L,P^2}(c_4, c_4') + dam_{L,P^2}(c_5, c_5') = 4$. The other case is verified similarly and the details are omitted.

If $\tau \leq 2m - 6$, then we know that (S_3, T_3) is a simple pair of size 6 which contradicts to Claim 2. So we have that $\tau \in \{2m - 4, 2m - 2\}$. If $\tau = 2m - 4$, then we let $S = S_2$ and $T = T_2$ and if $\tau = 2m - 2$, then we let $S = S_1$ and $T = T_1$. In each case, $|S| = |T| = 2m - \tau$ and by the arguments above, $dam_{L,P^i}(S, T) \leq 2m - \tau$ for $i = 0, 1, 2$. By (C5), for $i = 0, 1, 2$,

$$dam_{L,P^i}(S, T) \leq 2m - \tau \leq S_L(P^i) - 2m|V(P^i)|.$$

This completes the proof of Theorem 14.

3 | PROOF OF LEMMA 9

This section proves Lemma 9. That is,

$$F(x, y) = \sum \binom{x}{a}\binom{y}{b}\binom{\ell - x - y}{2m - \tau - a - b} \leq \frac{1}{2}\binom{\ell}{2m - \tau} - 1, \tag{3}$$

where $x + y \leq \ell$, $2x + y \leq \ell + 2m - \tau - 1$, $m \geq 1$, $0 \leq \ell \leq 4m$, $0 \leq \tau \leq 2m - 2$, $\ell + \tau \geq 2m + 2$, ℓ and τ are both even, and the summation is over non-negative integer pairs (a, b) for which $0 \leq a \leq x$, $0 \leq b \leq y$, $a + b \leq 2m - \tau$ and $2a + b \geq \max\{2x + y + 2m - \tau + 1 - \ell, 2m - \tau + 1\}$.

Note that $a + b \leq 2m - \tau$ and $2a + b \geq 2m - \tau + 1$ implies that $a \geq 1$.

In the sequel, we define

$$\binom{p}{q}_+ = \begin{cases} \binom{p}{q} & \text{if} \quad p \geq q \geq 0, \\ 0 & \text{if} \quad q < 0 \quad \text{or} \quad p < q. \end{cases}$$

For convenience, we allow $p < q$ or $q < 0$ in the binomial coefficient in the summations below. It is easy to check in these cases, either the pair (a, b) does not lie in the range of the summation, and hence contributes 0 to the summations, or by extending the equality $\binom{p+1}{q} = \binom{p}{q} + \binom{p}{q-1}$ to $q = 0$. For the readability, we suppress the index "+".

First, we analyze the monotonicity of $F(x, y)$ about y when x is fixed, say $x = x_0$. For convenience, we let $2k = 2m - \tau$.

Lemma 18. *Assume $x = x_0$ is fixed.*

- *If $y \geq \ell - 2x_0$, then $F(x_0, y + 1) \leq F(x_0, y)$.*
- *If $y < \ell - 2x_0$, then $F(x_0, y) \leq F(x_0, y + 1)$.*

Proof. In the following, let $z = \ell - x_0 - y$.

Case 1: $\ell - 2x_0 \le y \le \ell + 2m - \tau - 1 - 2x_0 = \ell + 2k - 1 - 2x_0$.

In this case, $\max\{2x_0 + y + 2m - \tau + 1 - \ell, 2m - \tau + 1\} = 2x_0 + y + 2m - \tau + 1 - \ell = 2x_0 + y + 2k + 1 - \ell$. For brevity, let

$$t(y) = 2x_0 + y + 1 - \ell, \quad r(y) = t(y) + 2k.$$

As $2a + b \ge r(y)$ and $a + b \le 2m - \tau = 2k$, we have $a \ge r(y) - 2k = t(y)$ and $r(y) - 2a \le b \le 2k - a$. So

$$F(x_0, y) = \sum_{i=t(y)}^{x_0} \sum_{j=r(y)-2i}^{2k-i} \binom{x_0}{i}\binom{y}{j}\binom{z}{2k-i-j}.$$

Note that $r(y+1) = r(y) + 1$ and $t(y+1) = t(y) + 1$. Let $\Delta = F(x_0, y) - F(x_0, y+1)$ and for $t(y+1) \le i \le x_0$, let

$$\Delta_i = \sum_{j=r(y)-2i}^{2k-i} \binom{y}{j}\binom{z}{2k-i-j} - \sum_{j=r(y)+1-2i}^{2k-i} \binom{y+1}{j}\binom{z-1}{2k-i-j}. \tag{4}$$

Note that $2k - t(y) = r(y) - 2t(y)$, we have

$$\Delta \ge \sum_{j=r(y)-2t(y)}^{2k-t(y)} \binom{x_0}{t(y)}\binom{y}{j}\binom{z}{2k-t(y)-j} + \sum_{i=t(y+1)}^{x_0} \binom{x_0}{i}\Delta_i = \binom{x_0}{t(y)}\binom{y}{2k-t(y)} + \sum_{i=t(y+1)}^{x_0} \binom{x_0}{i}\Delta_i.$$

Therefore, to prove that $F(x_0, y) \ge F(x_0, y+1)$, it suffices to prove that $\Delta_i \ge 0$ for $t(y) + 1 \le i \le x_0$. Using equalities $\binom{z}{2k-i-j} = \binom{z-1}{2k-i-j} + \binom{z-1}{2k-i-j-1}$ (in the first sum of Equality (4)) and $\binom{y+1}{j} = \binom{y}{j} + \binom{y}{j-1}$ (in the second sum of Equality (4)), and cancel the term $\sum_{j=r(y)+1-2i}^{2k-i} \binom{y}{j}\binom{z-1}{2k-i-j}$, we have

$$\Delta_i = \sum_{j=r(y)+1-2i}^{2k-i} \binom{y}{j}\binom{z-1}{2k-1-i-j} - \sum_{j=r(y)+1-2i}^{2k-i} \binom{y}{j-1}\binom{z-1}{2k-i-j} + \binom{y}{r(y)-2i}\binom{z}{2k+i-r(y)}.$$

When $j = 2k - i$ in the first sum, we have $\binom{z-1}{-1} = 0$. Writing the second sum in the equality as $\sum_{j=r(y)-2i}^{2k-1-i} \binom{y}{j}\binom{z-1}{2k-1-i-j}$, we have

$$\Delta_i = \binom{y}{r(y)-2i}\left[-\binom{z-1}{2k+i-r(y)-1}+\binom{z}{2k+i-r(y)}\right]$$
$$=\binom{y}{r(y)-2i}\binom{z-1}{2k+i-r(y)} \geq 0.$$

Case 2: $0 \leq y < \ell - 2x_0$.

In this case, $2x_0 + y \leq \ell$, thus we have that $2a+b \geq \max\{2x_0 + y + 2k + 1 - \ell, 2k+1\} = 2k+1$. Let $s(y) = \max\left\{\lceil \frac{2k+1-y}{2} \rceil, 1\right\}$. As $2a+b \geq 2k+1$ and $b \leq y$, we have $a \geq \lceil \frac{2k+1-y}{2} \rceil$. We have observed already that $a \geq 1$. So $a \geq s(y)$. and

$$F(x_0, y) = \sum_{i=s(y)}^{x_0} \sum_{j=2k+1-2i}^{2k-i} \binom{x_0}{i}\binom{y}{j}\binom{z}{2k-i-j}.$$

Note that $s(y+1) \leq s(y)$ and the equality holds when y is odd. Let $\Delta = F(x_0, y+1) - F(x_0, y)$ and for $s(y) \leq i \leq x_0$, let

$$\Delta_i = \sum_{j=2k+1-2i}^{2k-i} \binom{y+1}{j}\binom{z-1}{2k-i-j} - \sum_{j=2k+1-2i}^{2k-i} \binom{y}{j}\binom{z}{2k-i-j}. \tag{5}$$

Then $\Delta \geq \sum_{i=s(y)}^{x_0} \binom{x_0}{i}\Delta_i$. To prove that $F(x_0, y+1) \geq F(x_0, y)$, it suffices to prove that for each $i, \Delta_i \geq 0$. Using equalities $\binom{y+1}{j} = \binom{y}{j} + \binom{y}{j-1}$ and $\binom{z}{2k-i-j} = \binom{z-1}{2k-i-j} + \binom{z-1}{2k-i-j-1}$ in Equality (5) and cancel the term $\sum_{j=2k+1-2i}^{2k-i}\binom{y}{j}\binom{z-1}{2k-i-j}$, we have

$$\Delta_i = \sum_{j=2k+1-2i}^{2k-i} \binom{y}{j-1}\binom{z-1}{2k-i-j} - \sum_{j=2k+1-2i}^{2k-i} \binom{y}{j}\binom{z-1}{2k-1-i-j}.$$

When $j = 2k - i$ in the second sum, we have $\binom{z-1}{-1} = 0$. Writing the first sum in the equality above as $\sum_{j=2k-2i}^{2k-1-i}\binom{y}{j}\binom{z-1}{2k-1-i-j}$, we have

$$\Delta_i = \binom{y}{2k-2i}\binom{z-1}{i-1} \geq 0.$$

□

Now, we continue with the proof of Lemma 9. First assume that $x < \frac{\ell}{2}$. By Lemma 18, $F(x, y) \leq F(x, \ell - 2x)$. So it suffices to show that $F(x, \ell - 2x) \leq \frac{1}{2}\binom{\ell}{2k} - 1$. Recall that (by Equality (3))

$$F(x, \ell - 2x) = \sum_{t=2k+1}^{4k} \sum_{2a+b=t} \binom{x}{a}\binom{\ell - 2x}{b}\binom{x}{2k-a-b} = \sum_{t=2k+1}^{4k} C(t, x),$$

where

$$C(t,x)=\sum_{2a+b=t}\binom{x}{a}\binom{\ell-2x}{b}\binom{x}{2k-a-b}=\sum_{2a\le t}\binom{x}{a}\binom{\ell-2x}{t-2a}\binom{x}{2k+a-t}.$$

Then $\sum_{t=0}^{4k}C(t,x)=\binom{\ell}{2k}$. Since for any $0\le t\le 2k$,

$$C(t,x)=\sum_{2a\le t}\binom{x}{a}\binom{\ell-2x}{t-2a}\binom{x}{2k+a-t}=\sum_{2a'\le 4k-t}\binom{x}{a'}\binom{\ell-2x}{4k-t-a'}\binom{x}{a'+t-2k}$$
$$=C(4k-t,x),$$

where $a'=2k+a-t$. So

$$F(x,\ell-2x)=\sum_{t=2k+1}^{4k}C(t,x)=\frac{\binom{\ell}{2k}-C(2k,x)}{2}\le\frac{1}{2}\binom{\ell}{2k}-1.$$

Here we used the fact that $C(2k,x)\ge 2$ when $1\le x<\frac{\ell}{2}$. Indeed, if $x\ge k$,

$$C(2k,x)\ge\binom{x}{k}^2\binom{\ell-2x}{0}+\binom{x}{k-1}^2\binom{\ell-2x}{2}\ge 1+1=2.$$

If $1\le x\le k-1$, then

$$C(2k,x)\ge\binom{x}{x}^2\binom{\ell-2x}{2k-2x}+\binom{x}{x-1}^2\binom{\ell-2x}{2k-2x+2}\ge 1+1=2.$$

Now we assume that $x\ge\frac{\ell}{2}$. It follows that $y\ge 0\ge\ell-2x$. Hence, by the first part of Lemma 18, $F(x,y)\le F(x,0)$. So it suffices to prove that $F(x,0)\le\frac{1}{2}\binom{\ell}{2k}-1$.

Note that in this case, $b=y=0$ and $2x+y+2k+1-\ell\ge 2k+1$, so $2a+b=2a\ge 2x+y+2k+1-\ell=2x+2k+1-\ell$, which implies that $a\ge x+k-\ell'+1$, thus

$$F(x,0)=\sum_{i=x+k-\ell'+1}^{2k}\binom{x}{i}\binom{\ell-x}{2k-i}.$$

We first prove that when $x\ge\frac{\ell}{2}=\ell'$, $F(x,0)>F(x+1,0)$. Let $\Delta=F(x,0)-F(x+1,0)$, then

$$\Delta=\sum_{i=x+1+k-\ell'}^{2k}\binom{x}{i}\binom{\ell-x}{2k-i}-\sum_{i=x+2+k-\ell'}^{2k}\binom{x+1}{i}\binom{\ell-1-x}{2k-i}.\tag{6}$$

Using equalities $\binom{x+1}{i}=\binom{x}{i}+\binom{x}{i-1}$ and $\binom{\ell-x}{2k-i}=\binom{\ell-x-1}{2k-i}+\binom{\ell-x-1}{2k-i-1}$, and cancel the term $\sum_{j=x+k+2-\ell'}^{2k-1}\binom{x}{i}\binom{\ell-x-1}{2k-i}$, we have

$$\Delta=\binom{x}{x+1+k-\ell'}\binom{\ell-x}{k+\ell'-x-1}+\sum_{i=x+2+k-\ell'}^{2k}\binom{x}{i}\binom{\ell-1-x}{2k-1-i}$$
$$-\sum_{i=x+2+k-\ell'}^{2k}\binom{x}{i-1}\binom{\ell-1-x}{2k-i}.$$

When $i = 2k$ in the first sum above, we have $\binom{\ell-1-x}{-1} = 0$. Writing the last sum in the equality above as $\sum_{i=x+1+k-\ell'}^{2k-1} \binom{x}{i}\binom{\ell-1-x}{2k-1-i}$, we have

$$\begin{aligned}\Delta &= \binom{x}{x+1+k-\ell'}\binom{\ell-x}{k+\ell'-x-1} - \binom{x}{x+1+k-\ell'}\binom{\ell-x-1}{k+\ell'-x-2}\\ &= \binom{x}{x+1+k-\ell'}\binom{\ell-x-1}{k+\ell'-x-1} \geq 0.\end{aligned}$$

So, $F(x, y) \leq F(\ell', 0)$. Note that $\sum_{i=k+1}^{2k} \binom{\ell'}{i}\binom{\ell'}{2k-i} = \sum_{i=1}^{k-1} \binom{\ell'}{2k-i}\binom{\ell'}{i}$ and

$$\binom{\ell}{2k} = \sum_{i=k+1}^{2k} \binom{\ell'}{i}\binom{\ell'}{2k-i} + \sum_{i=1}^{k-1} \binom{\ell'}{2k-i}\binom{\ell'}{i} + \binom{\ell'}{k}^2.$$

As $\ell' = \frac{\ell}{2} \geq k + 1 \geq 2$, we have

$$F(x, y) \leq F(\ell', 0) = \sum_{i=k+1}^{2k} \binom{\ell'}{i}\binom{\ell'}{2k-i} = \frac{\binom{\ell}{2k} - \binom{\ell'}{k}^2}{2} < \frac{1}{2}\binom{\ell}{2k} - 1.$$

This completes the proof of Lemma 9.

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