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The strong fractional choice number of 3-choice-critical graphs

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Abstract

A graph G is called 3-choice-critical if G is not 2-choosable but any proper subgraph of G is 2-choosable. A graph G is strongly fractional r-choosable if G is (a, b)-choosable for all positive integers a, b for which $a/b \geq r$. The strong fractional choice number of G is $ch_f^s(G) = \inf\{r : G \text{ is strongly fractional } r\text{-choosable}\}$. This paper determines the strong fractional choice number of all 3-choice-critical graphs.

KEYWORDS

choice critical graphs, multiple list colouring, strong fractional choice number

1 | INTRODUCTION

An *a-list assignment* of a graph G is a mapping L which assigns to each vertex v of G a set $L(v)$ of a colours. A *b-fold colouring* of G is a mapping ϕ which assigns to each vertex v of G a set $\phi(v)$ of b colours such that for every edge uv, $\phi(u) \cap \phi(v) = \emptyset$. An *$(L, b)$-colouring* of G is a b-fold colouring ϕ of G such that $\phi(v) \subseteq L(v)$ for each vertex v. We say G is *(a, b)-choosable* if for any a-list assignment L of G, there is an (L, b)-colouring of G, and G is (a, b)-colourable if there is a b-fold colouring ϕ of G such that $\phi(v) \subseteq \{1, 2, ..., a\}$ for each vertex v. We say G is *a-choosable* (respectively, a-colourable) if G is $(a, 1)$-choosable (respectively, $(a, 1)$-colourable). The *choice number* $ch(G)$ of G is the minimum integer a such that G is a-choosable, and the *chromatic number* $\chi(G)$ of G is the minimum integer a such that G is a-colourable. The concept of list colouring of graphs was introduced independently by Erdős, Rubin and Taylor [2] and Vizing [8] in the 1970s, and has been studied extensively in the literature.

The *fractional chromatic number* $\chi_f(G)$ of a graph G is defined as

$$\chi_f(G) = \inf\left\{\frac{a}{b} : G \text{ is } (a, b) - \text{ colourable.}\right\}$$

The *fractional choice number* $ch_f(G)$ of a graph G is defined as

$$ch_f(G) = \inf\left\{\frac{a}{b} : G \text{ is } (a, b) - \text{ choosable.}\right\}$$

It follows from the definition that for any graph G, $\chi(G) \leq ch(G)$ and $\chi_f(G) \leq ch_f(G)$. It is known that there are bipartite graphs with arbitrary large choice number. On the other hand, it was proved by Alon, Tuza and Voigt [1] that $ch_f(G) = \chi_f(G)$ for every graph G. So $ch_f(G)$ is not really a new graph parameter. In particular, $ch_f(G) = 2$ for all bipartite graphs with at least one edge.

The concept of strong fractional choice number of a graph was introduced in [12]. Given a real number r, we say a graph G is *strongly fractional r-choosable* if G is (a, b)-choosable for any a, b for which $\frac{a}{b} \geq r$. The *strong fractional choice number* $ch_f^s(G)$ of G is defined as

$$ch_f^s(G) = \inf\{r : G \text{ is strongly fractional } r - \text{choosable}\}.$$

It follows from the definition that $ch_f^s(G) \geq ch(G) - 1$. It was proved in [11] that for any finite graph G, $ch_f^s(G)$ is a rational number. Moreover, if $ch_f^s(G) \neq \chi_f(G)$, then the infimum in the definition is attained and hence can be replaced by the minimum. However, if $ch_f^s(G) = \chi_f(G)$, then the infimum in the definition may be not attained. The parameter $ch_f^s(G)$ may serve as a refinement for the choice number of G and has been studied in a few papers [3, 4]. However, it remains an open question whether $ch_f^s(G) \leq ch(G)$ for every graph G.

For any graph G, we have $ch_f^s(G) \geq \chi_f(G)$, and $ch_f^s(G) \geq 2$ for every graph with at least one edge. It seems to be a difficult problem to characterize all graphs G with $ch_f^s(G) = 2$.

Erdős, Rubin and Taylor [2] characterized all the 2-choosable graphs. Given a graph G, the *core* of G is obtained from G by repeatedly removing degree 1 vertices. Denote by $\Theta_{k_1,k_2,\ldots,k_q}$ the graph consisting of internally vertex disjoint paths of lengths $k_1, k_2, \ldots, k_q$ connecting two vertices u and v. Erdős, Rubin and Taylor proved that a connected graph G is 2-choosable if and only if the core of G is K_1 or an even cycle or $\Theta_{2,2,2p}$ for some positive integer p.

We say a graph G is *3-choice-critical* if G is not 2-choosable but any proper subgraph of G is 2-choosable. Voigt characterized all the 3-choice-critical graphs.

Theorem 1.1 (Voigt [9]). *A graph is 3-choice-critical if and only if it is one of the following*:

1. *An odd cycle.*
2. *Two vertex-disjoint even cycles joined by a path.*
3 *Two even cycles with one vertex in common.*
4. $\Theta_{2r,2s,2t}$ *graph with* $r \geq 1$, *and* $s, t > 1$, *or* $\Theta_{2r+1,2s+1,2t+1}$ *with* $r \geq 0$, $s, t > 0$.
5. $\Theta_{2,2,2,2t}$ *graph with* $t \geq 1$.

The strong fractional choice numbers of odd cycles are easily determined.

Proposition 1.2. *For odd cycle* C_{2k+1}, $ch_f^s(C_{2k+1}) = 2 + \frac{1}{k}$.

Proof. It is well-known that $\chi_f(C_{2k+1}) = 2 + \frac{1}{k}$. As $ch_f^s(G) \geq \chi_f(G)$ for any graph G, it suffices to show that for any $a/b \geq 2 + 1/k$, C_{2k+1} is (a, b)-choosable.

Assume the vertices of C_{2k+1} are $(v_0, v_1, \ldots, v_{2k})$ in this cyclic order, $a/b \geq 2 + 1/k$ and L is an a-list assignment of C_{2k+1}. Assume $\bigcup_{i=0}^{2k} L(v_i) = \{c_1, c_2, \ldots, c_p\}$. By permuting colours, we may assume that $\bigcap_{i=0}^{2k} L(v_i) = \{c_1, c_2, \ldots, c_q\}$, where $0 \leq q \leq a \leq p$. (Note that $q = 0$ when $\bigcap_{i=0}^{2k} L(v_i) = \emptyset$). For $i = q + 1, q + 2, \ldots, p$, let s_i be an arbitrary index such that $c_i \notin L(v_{s_i})$. We recursively assign colours $c_1, c_2, \ldots, c_p$ to vertices of C_{2k+1}. Assume colours $c_1, c_2, \ldots, c_{i-1}$ have been assigned to vertices of C_{2k+1} already. We assign colour c_i to vertices of C_{2k+1} as follows:

If $i \leq q$, then assign colour c_i to vertices in the set $\{v_i, v_{i+2}, \ldots, v_{i+2k-2}\}$, where the summations in the indices of vertices are modulo $2k + 1$.

If $i \geq q + 1$, then traverse the vertices of C_{2k+1} one by one in the order $v_{s_i}, v_{s_i+1}, \ldots, v_{s_i+2k}$, and assign colour c_i to vertex v_j provided the following hold:

- $c_i \in L(v_j)$ and c_i is not assigned to v_{j-1}.
- v_j has received less than b colours from $c_1, c_2, \ldots, c_{i-1}$.

It follows from the construction that each colour class is an independent set and each vertex v_j is assigned at most b colours and all the colours assigned to v_j are from $L(v_j)$. Now we show that each vertex is assigned exactly b colours.

Assume to the contrary that v_j is assigned at most $b - 1$ colours. Assume $c_i \in L(v_j)$ and c_i is not assigned to v_j. It follows from the colouring procedure that one of the following holds:

1. c_i is assigned to v_{j-1}.
2. $i \leq q$ and $j = i + 2k$.

The first case occurs at most b times as v_{j-1} receives at most b colours, and the second case occurs at most $\left\lceil \frac{q}{2k+1} \right\rceil \leq \left\lceil \frac{a}{2k+1} \right\rceil$ times. Therefore,

$$a = |L(v_j)| \leq b - 1 + b + \left\lceil \frac{a}{2k + 1} \right\rceil < 2b + \frac{a}{2k + 1}$$

and hence $a/b < 2 + 1/k$, contrary to our assumption. $\square$

The main result of this paper is that every bipartite 3-choice-critical graph has strong fractional choice number 2. It suffices to show that every bipartite 3-choice-critical graph is $(2m + 1, m)$-choosable for any positive integer m.

It is known [6, 9] that for an odd integer m, a graph G is $(2m, m)$-choosable if and only if G is 2-choosable. In [9], Voigt conjectured that every bipartite 3-choice-critical graph G is $(2m, m)$-choosable for every even integer m. Note that if a graph G is $(4m, 2m)$-choosable, then it is $(4m - 1, 2m - 1)$-choosable: if L is a $(4m - 1)$-list assignment, then let c be a new colour, and let $L'(v) = L(v) \cup \{c\}$, we obtain a $4m$-list assignment. Let f be a $2m$-fold L'-colouring of G, and let $g(v) = f(v) - \{c\}$ if $c \in f(v)$ and $g(v) = f(v) - \{c'\}$ if $c \notin f(v)$, where c' is an arbitrary colour in $f(v)$. Then g is a $(2m - 1)$-fold L-colouring of G. So if Voigt's conjecture were true, then all bipartite 3-choice-critical graphs have strong fractional choice number 2. Voigt's conjecture was verified for

$G = \Theta_{2,2,2,2}$ [7]. However, Meng, Puleo and Zhu [5] proved that if $\min\{r, s, t\} \geq 3$, and r, s, t have the same parity, then $\Theta_{r,s,t}$ is not (4, 2)-choosable, and if $t \geq 2$, then $\Theta_{2,2,2,2t}$ is not (4, 2)-choosable. Nevertheless, the other bipartite 3-choice-critical graphs, that is, two vertex-disjoint even cycles joined by a path, two even cycles with one vertex in common, $\Theta_{2,2s,2t}$ with $s, t > 1$, and $\Theta_{1,2s+1,2t+1}$ with $s, t > 0$, are (4, 2)-choosable [5]. Xu and Zhu [10] strengthened these results and proved that these graphs are also $(4m, 2m)$-choosable for all integer m. So $ch_f^s(G) = 2$ if one of the following holds:

1. G is two vertex-disjoint even cycles joined by a path, two even cycles with one vertex in common.
2. $G = \Theta_{2,2s,2t}$ with $s, t > 1$.
4. $G = \Theta_{1,2s+1,2t+1}$ with $s, t \geq 1$.
5. $G = \Theta_{2,2,2,2}$.

In this paper, we prove the following result.

Theorem 1.3. *If $G = \Theta_{2r,2s,2t}$ with $r, s, t > 1$, or $G = \Theta_{2r+1,2s+1,2t+1}$ with $r > 0, s, t > 0$, or $G = \Theta_{2,2,2,2t}$ graph with $t \geq 1$, then G is $(2m + 1, m)$-choosable for any positive integer m.*

Thus if G is a bipartite 3-choice-critical graph, then for any $r > 2$, G is strongly fractional r-choosable. Hence we have the following corollary.

Corollary 1.4. *Every bipartite 3-choice-critical graph G has $ch_f^s(G) = 2$.*

2 | PRELIMINARIES

The proof of Theorem 1.3 uses the idea in [5, 10]: Assume G is a graph as in Theorem 1.3 and L is a $(2m + 1)$-list assignment of G. Let u, v be the two vertices of G of degree at least 3. Then $G - \{u, v\}$ is the disjoint union of a family of three or four paths, where each end vertex of these paths has exactly one neighbour in $\{u, v\}$ unless the path consists of a single vertex w, in which case w is adjacent to both u and v. Other vertices of the paths are not adjacent to u or v.

We shall find appropriate m-sets $S \subseteq L(u)$ and $T \subseteq L(v)$, assign S to u and T to v. Then extend this pre-colouring of u, v to an (L, m)-colouring of the remaining vertices of G, that form three or four paths. The extension to the paths are independent to each other. The difficulties lie in proving the existence of such m-sets S and T.

Assume P is a path with vertices $v_1, v_2, ..., v_n$ in order and L is a $(2m + 1)$-list assignment on P, with v_1 adjacent to u and v_n adjacent to v. Assume S, T are the m-sets of colours assigned to u, v respectively. A necessary and sufficient condition was given in [5] under which P has an (L, m)-colouring so that v_1 and v_n avoid the colours from S and T.

Definition 2.1. Assume P is an n-vertex path with vertices $v_1, v_2, ..., v_n$ in order. For a list assignment L of P, let

$$\begin{aligned} X_1 &= L(v_1), \\ X_i &= L(v_i) \setminus X_{i-1}, \text{ for } i \in \{2, 3, ..., n\}, \\ S_L(P) &= \sum_{i=1}^{n} |X_i|. \end{aligned}$$

The following lemma was proved in [5] (the statement is slightly different, but it does not affect the proof).

Lemma 2.2. *Let P be an n-vertex path and let L be a list assignment on P. If $|L(v_1)|$, $|L(v_n)| \geq m$ and $|L(v_i)| \geq 2m$ for $i \in \{2, 3, ..., n-1\}$, then path P is (L, m)-colourable if and only if $S_L(P) \geq nm$.*

Definition 2.3. Assume n is an odd integer, P is an n-vertex path with vertices $v_1, v_2, ..., v_n$ in order, and L is a list assignment on P. Let

$$\Lambda = \bigcap_{x \in V(P)} L(x),$$
$$\hat{X}_1 = \{c \in L(v_1) - \Lambda\text{: the smallest index } i \text{ for which } c \notin L(v_i) \text{ is even}\},$$
$$\hat{X}_n = \{c \in L(v_n) - \Lambda\text{: the largest index } i \text{ for which } c \notin L(v_i) \text{ is even}\}.$$

Definition 2.4. Assume L is a list assignment on P and S, T are two colour sets. Let $L \ominus (S, T)$ be the list assignment obtained from L by deleting all colours in S from $L(v_1)$, all colours in T from $L(v_n)$, and leaving all other lists unchanged. The damage of (S, T) with respect to L and P is defined as

$$dam_{L,P}(S, T) = S_L(P) - S_{L\ominus(S,T)}(P).$$

The following lemma was proved in [5].

Lemma 2.5 [Meng, Puleo, and Zhu 5]. *Let L be a list assignment on an n-vertex path P, where n is odd. For any sets of colours S, T,*

$$S_{L\ominus(S,T)}(P) = S_L(P) - (|(\Lambda \cup \hat{X}_1) \cap S| + |(\Lambda \cup \hat{X}_n) \cap T| - |\Lambda \cap S \cap T|),$$

and

$$dam_{L,P}(S, T) = |\hat{X}_1 \cap S| + |\hat{X}_n \cap T| + |\Lambda \cap (S \cup T)|. \tag{1}$$

Lemma 2.6. *Let L be a list assignment on an n-vertex path P with vertices $v_1, v_2, ..., v_n$, where n is odd, and $|L(v_i)| = l$ for all $i \geq 2$ if $n \geq 3$. Then*

$$S_L(P) = \frac{n-1}{2}l + \sum_{\substack{k \text{ is even} \\ k<n}} |X_{k-1} - L(v_k)| + |X_n|.$$

Proof. We use induction on n. If $n = 1$, $S_L(P) = |L(v_1)| = |X_1|$. Assume $n \geq 3$. Apply induction hypothesis to $P' = P - \{v_{n-1}, v_n\}$, we have

$$S_L(P) = \frac{n-3}{2}l + \sum_{\substack{k \text{ is even} \\ k<n-2}} |X_{k-1} - L(v_k)| + |X_{n-2}| + |X_{n-1}| + |X_n|.$$

As

$$|X_{n-1}| = |L(v_{n-1}) - X_{n-2}| = |L(v_{n-1})| - |X_{n-2}| + |X_{n-2} - L(v_{n-1})| = l - |X_{n-2}| + |X_{n-2} - L(v_{n-1})|,$$

the equality in the lemma holds. □

Lemma 2.7. *Let L be a list assignment on an n-vertex path P with vertices $v_1, v_2, \ldots, v_n$, where n is odd, and $|L(v_i)| = l$ for all $i \geq 2$ if $n \geq 3$. Then*

$$S_L(P) \geq \frac{n-1}{2} l + |\hat{X}_1| + |\hat{X}_n| + |\Lambda|.$$

Proof. By the definition of $\hat{X}_1$, every element of $\hat{X}_1$ appears in a set of the form $X_{k-1} - L(v_k)$ where k is even. By Lemma 2.6 and the fact that $|X_n| = |\hat{X}_n| + |\Lambda|$, the lemma holds. □

Lemma 2.8. *Let L be a list assignment on an n-vertex path P with vertices $v_1, v_2, \ldots, v_n$, where n is odd, $|L(v_1)| = l_1$, and $|L(v_i)| = l_2$ for all $i \geq 2$ if $n \geq 3$. Then $S_L(P) \geq l_1 + \frac{n-1}{2} l_2$.*

Proof. Since $|X_1| = |L(v_1)| = l_1$ and $|X_i| + |X_{i+1}| \geq l_2$ for each even $i \geq 2$ when $n \geq 3$, $S_L(P) = \sum_{i=1}^{n} |X_i| \geq l_1 + \frac{n-1}{2} l_2$. □

The following is a key lemma for the proof in this paper.

Lemma 2.9. *Let ℓ and k be fixed integers, where $k \geq 1$, $\ell > k$, $0 \leq k \leq m$. Assume x, y are non-negative integers with $x + y \leq \ell$. Let*

$$F(x, y) = \sum \binom{x}{a} \binom{y}{b} \binom{\ell - x - y}{k - a - b},$$

where the summation is over all non-negative integer pairs (a, b) for which $0 \leq a \leq x$, $0 \leq b \leq y$, $a + b \leq k$ and $2a + b \geq \max\{2x + y + k + 1 - \ell, k + 1\}$. Then

$$F(x, y) \leq \frac{1}{2} \binom{\ell}{k}.$$

Moreover, the equality holds if and only if ℓ is even, k is odd, $x = \frac{\ell}{2}$ and $y = 0$.

Note that when $a > x$ or $b > y$, then $\binom{x}{a}\binom{y}{b} = 0$. Also $a + b \leq k$ and $2a + b \geq 2x + y + k + 1 - \ell$ implies that $2x + y \leq \ell - k - 1 + 2a + b \leq \ell - k - 1 + 2(a + b) \leq \ell + k - 1$. Thus the summation can be restricted to $0 \leq a \leq x, 0 \leq b \leq y$, $a + b \leq k$, $2a + b \geq \max\{2x + y + k + 1 - \ell, k + 1\}$ and $2x + y \leq \ell + k - 1$. The proof of Lemma 2.9 will be given in Section 5.

Observation 2.10. If the restriction on $2a + b$ is replaced by $2a + b > \max\{2x + y + k + 1 - \ell, k + 1\}$ in Lemma 2.9, then we have $F(x, y) < \frac{1}{2}\binom{\ell}{k}$.

Proof. By the 'moreover' part of Lemma 2.9, it suffice to prove the observation holds when ℓ is even, k is odd, $x = \frac{\ell}{2}$ and $y = 0$: Let $H(x, 0)$ be the new function which is same as $F(x, 0)$ except that $2a + b = 2a \geq \max\{2x + y + k + 2 - \ell, k + 2\} = k + 2$. So $H(x, 0) = F(x, 0) - \binom{\frac{\ell}{2}}{\frac{k+1}{2}}\binom{\frac{\ell}{2}}{\frac{k-1}{2}} < \frac{1}{2}\binom{\ell}{k}$. □

3 | PROOF OF THEOREM 1.3 FOR $\Theta_{2r,2s,2t}$ and $\Theta_{2r+1,2s+1,2t+1}$

Let $G = \Theta_{2r,2s,2t}$, where $r, s, t > 1$. Let u, v be the two degree 3 vertices. Let P^0, P^1, P^2 be the paths in $G - \{u, v\}$, where $P^i = (v_1^i, v_2^i, \ldots, v_{n_i}^i)$, v_1^i is adjacent to u and $v_{n_i}^i$ is adjacent to v.

For the purpose of using induction, instead of proving Theorem 1.3 directly, we shall prove a stronger result: Theorem 3.8. In this theorem, the list assignment L does not assign $2m + 1$ colours to every vertex. In particular, $|L(u)| = |L(v)| = \ell$, where $0 \leq \ell \leq 2m$. We need do some preparation before stating Theorem 3.8.

Definition 3.1. A pair is a tuple (S, T) with $S \subset L(u)$, $T \subset L(v)$, and $|S| = |T|$. We define the *size* of a pair as $|S|$. A pair (S, T) is bad with respect to (L, P) if $dam_{L,P}(S, T) > S_L(P) - m|V(P)|$.

Definition 3.2. An indexing of colours in $L(u)$ and $L(v)$ as $L(u) = \{c_1, c_2, \ldots, c_\ell\}$ and $L(v) = \{c'_1, c'_2, \ldots, c'_\ell\}$ is *consistent* if $c_j = c'_j$ whenever $c_j \in L(u) \cap L(v)$. In other words, $\{c_i, c'_i\} \cap \{c_j, c'_j\} = \emptyset$ whenever $i \neq j$. A simple pair with respect to (L, P) and a consistent indexing of $(L(u), L(v))$ is a pair (S, T) such that $T = \{c'_j : c_j \in S\}$. A semi-simple pair is a pair (S, T) such that $S \cap (L(v) - T) \cap \Lambda = \emptyset$ and $T \cap (L(u) - S) \cap \Lambda = \emptyset$.

Note that a simple pair with respect to any consistent indexing of $(L(u), L(v))$ is a semi-simple pair. On the other hand, a semi-simple is not based on a consistent indexing of $(L(u), L(v))$.

The following lemma follows directly from the definition.

Lemma 3.3. *If (S_1, T_1) and (S_2, T_2) are two semi-simple pairs such that $S_1 \cap (S_2 \cup T_2) = \emptyset$ and $T_1 \cap (S_2 \cup T_2) = \emptyset$, then $(S_1 \cup S_2, T_1 \cup T_2)$ is a semi-simple pair and*

$$dam_{L,P}(S_1 \cup S_2, T_1 \cup T_2) = dam_{L,P}(S_1, T_1) + dam_{L,P}(S_2, T_2).$$

Note that if (S, T) is a semi-simple pair, then $(L(u) - S, L(v) - T)$ is a semi-simple pair. Hence we have the following corollary.

Corollary 3.4. *If (S, T) is a semi-simple pair with respect to (L, P), then*

$$dam_{L,P}(L(u), L(v)) = dam_{L,P}(S, T) + dam_{L,P}(L(u) - S, L(v) - T).$$

For convenience, in the sequel, whenever $L(u)$ and $L(v)$ are given, a consistent indexing of colours in $L(u)$ and $L(v)$ are fixed. All simple pairs are with respect to this indexing. A *couple* is a tuple of the form (c_j, c_j') for $j \in \{1, 2, ..., \ell\}$. When we write a couple, we suppress the parentheses and simply write $c_j c_j'$. In the sequel, we may write $dam_{L,P(c,c')}$ for $dam_{L,P(\{c\},\{c'\})}$.

Observation 3.5. If (S, T) is a simple pair, then

$$dam_{L,P}(S, T) = \sum_{c_j \in S} dam_{L,P(\{c_j\},\{c'^j\})}. \tag{2}$$

If $c_1 c_1'$ and $c_2 c_2'$ are two couples satisfying $\{c_1\} \cap \{c_1'\} \cap \Lambda = \emptyset$ and $\{c_2\} \cap \{c_2'\} \cap \Lambda = \emptyset$, then both (c_1, c_2') and (c_2, c_1') are semi-simple pairs.

The following observation follows from Lemma 2.5.

Observation 3.6. For any couple cc' and $P = (v_1, v_2, ..., v_n)$, where n is odd, the following hold:

1. $dam_{L,P(c,c')} = 2$ if $c \in \hat{X}_1 \cup \Lambda$ and $c' \in \hat{X}_n \cup \Lambda$, and moreover if $c = c'$, then $c \notin \Lambda$;
2. $dam_{L,P(c,c')} = 1$ if $c \in \hat{X}_1 \cup \Lambda$ or $c' \in \hat{X}_n \cup \Lambda$ but not both unless $c = c' \in \Lambda$;
3. $dam_{L,P(c,c')} = 0$ if $c \notin \hat{X}_1 \cup \Lambda$ and $c' \notin \hat{X}_n \cup \Lambda$.

In particular, if $dam_{L,P(c,c')} = 2$ *and* $n = 1$, *then* $c \neq c'$.

Definition 3.7. Assume $c_j c_j'$ is a couple.

- $c_j c_j'$ is heavy for the internal path P if $dam_{L,P(c_j,c_j')} = 2$;
- $c_j c_j'$ is light for the internal path P if $dam_{L,P(c_j,c_j')} = 1$;
- $c_j c_j'$ is safe for the internal path P if $dam_{L,P(c_j,c_j')} = 0$.

For each path P^i, let $x^{(i)}, y^{(i)}, z^{(i)}$ denote the number of heavy, light and safe couples for P^i, respectively. Then for $i = 0, 1, 2$,

$$x^{(i)} + y^{(i)} + z^{(i)} = \ell, \text{ and } dam_{L,P^i}(L(u), L(v)) = 2x^{(i)} + y^{(i)}.$$

Assume $m \geq \tau$ are non-negative integers, (S, T) is a simple pair of size $m - \tau$. Let $a^{(i)}(S, T), b^{(i)}(S, T), c^{(i)}(S, T)$ denote the number of heavy, light and safe couples for P^i in (S, T), respectively. Then for $i = 0, 1, 2$, by Observation 3.5,

$$a^{(i)}(S, T) + b^{(i)}(S, T) + c^{(i)}(S, T) = m - \tau \text{ and } dam_{L,P}(S, T) = 2a^{(i)}(S, T) + b^{(i)}(S, T).$$

Let $\beta(P^i)$ denote the number of bad simple pairs of size $m - \tau$ with respect to (L, P^i) We write $\hat{X}_1^i$, $\hat{X}_{n_i}^i$ and Λ^i for the sets $\hat{X}_1, \hat{X}_n, \Lambda$ calculated for $P = P^i$.

Theorem 3.8. *Assume ℓ and τ are non-negative even integers, L is a list assignment for G satisfying the following:*

(C1) $\tau \leq 2\left\lfloor \frac{m}{2} \right\rfloor$ *and* $\ell + \tau \geq 2\left\lceil \frac{m}{2} \right\rceil$.

(C2) $|L(u)| = |L(v)| = \ell$.
(C3) *For each* $i \in \{0, 1, 2\}$, $|L(v_1^i)| \geq 2m - \tau$ *and* $|L(v_{n_i}^i)| \geq 2m + 1 - \tau$.
(C4) $|L(w)| \geq 2m + 1$ *for* $w \neq u, v, v_1^i, v_{n_i}^i$.

(C5) *For* $i = 0, 1, 2$,

$$S_L(P^i) - n_i m \geq \max\{m + \frac{n_i - 1}{2} + dam_{L,P^i}(L(u), L(v)) - \ell - \tau, m + \frac{n_i - 1}{2} - \tau\}.$$

Then there exists a set $S \subset L(u)$ and a set $T \subset L(v)$ satisfying $|S| = |T| = m - \tau$ such that for each i,

$$dam_{L,P^i}(S, T) \leq S_L(P^i) - n_i m.$$

Proof. We prove the theorem by induction on $2\ell + \tau$. First assume that $2\ell + \tau = 2\left\lceil \frac{m}{2} \right\rceil$. Since ℓ and τ are non-negative, and $\ell + \tau \geq 2\left\lceil \frac{m}{2} \right\rceil$, we have $\ell = 0$ and $\tau = 2\left\lceil \frac{m}{2} \right\rceil$. Note that by (C1), $\tau \leq 2\left\lfloor \frac{m}{2} \right\rfloor$, so m is even and $\tau = m$. By (C5), for each $i \in \{0, 1, 2\}$, $S_L(P^i) - n_i m \geq \frac{n_i - 1}{2} + m - \tau > m - \tau = 0$. By choosing $S = T = \emptyset$, we are done.

Thus we assume that $2\ell + \tau > 2\left\lceil \frac{m}{2} \right\rceil$ in the sequel. Assume to the contrary, Theorem 3.8 is not true for L.

The following claim gives a necessary condition for a simple pair of size $m - \tau$ being bad with respect to (L, P^i). Recall that $dam_{L,P^i}(L(u), L(v)) = 2x^{(i)} + y^{(i)}$. Claim 3.1 follows from (C5) and the definition of a bad pair directly.

Claim 3.1. If (S, T) is a bad simple pair of size $m - \tau$ with respect to (L, P^i), then

$$\begin{aligned} dam_{L,P^i}(S, T) &= 2a^{(i)}(S, T) + b^{(i)}(S, T) \\ &\geq \max\{2x^{(i)} + y^{(i)} + m + \frac{n_i + 1}{2} - \ell - \tau, m + \frac{n_i + 1}{2} - \tau\} \\ &\geq \max\{2x^{(i)} + y^{(i)} + m + 2 - \ell - \tau, m + 2 - \tau\}. \end{aligned}$$

The last inequality above uses the fact that $n_i \geq 3$.

The following claim gives an upper bound and a lower bound of the number $\beta(P^i)$ of bad simple pairs of size $m - \tau$ with respect to (L, P^i).

Claim 3.2. For each $i \in \{0, 1, 2\}$, $0 < \beta(P^i) < \frac{1}{2}\binom{\ell}{m - \tau}$.

Proof. If a simple pair (S, T) of size $m - \tau$ is bad with respect to (L, P^i), then by Claim 3.1, $dam_{L,P^i}(S, T) \geq \max\{2x^{(i)} + y^{(i)} + m + 2 - \ell - \tau, m + 2 - \tau\}$. Note that $a^{(i)}(S, T) + b^{(i)}(S, T) + c^{(i)}(S, T) = m - \tau$, so by Observation 2.10 (setting $m - \tau = k$) and Claim 3.1, we have that $\beta(P^i) < \frac{1}{2}\binom{\ell}{m-\tau}$.

If $\beta(P^i) = 0$ for some i, then $\beta(P^0) + \beta(P^1) + \beta(P^2) \leq \binom{\ell}{m-\tau} - 1$. So there exists a simple pair (S, T) of size $m - \tau$ which is not bad with respect to any (L, P^i), a contradiction to the assumption. □

Claim 3.3. For each $i \in \{0, 1, 2\}$, $x^{(i)} \geq 2$ and $z^{(i)} \geq 1$.

Proof. Assume that $x^{(i)} \leq 1$ for some $i \in \{0, 1, 2\}$. Then for every simple pair (S, T) of size $m - \tau$, $dam_{L,P^i}(S, T) = 2a^{(i)}(S, T) + b^{(i)}(S, T) \leq 2x^{(i)} + y^{(i)} \leq 2 \times 1 + (m - \tau - 1)$.

$= m - \tau + 1$

By Claim 3.1, it is not bad with respect to (L, P^i), which implies that $\beta(P^i) = 0$, a contradiction to Claim 3.2.

Assume that $z^{(i)} = 0$. Then $x^{(i)} + y^{(i)} = \ell$ and for any simple pair (S, T) of size $m - \tau$, $a^{(i)}(S, T) + b^{(i)}(S, T) = m - \tau$. By Claim 3.1, we have

$$\begin{aligned} a^{(i)}(S, T) + m - \tau &= 2a^{(i)}(S, T) + b^{(i)}(S, T) \\ &\geq 2x^{(i)} + y^{(i)} + m + 1 - \ell - \tau \\ &= x^{(i)} + 1 + m - \tau. \end{aligned}$$

This implies that $a^{(i)}(S, T) \geq x^{(i)} + 1$, contrary to the fact that $a^{(i)}(S, T) \leq x^{(i)}$. □

Claim 3.4. $\ell + \tau \geq m + 2$ and $\tau \leq 2\left\lfloor \frac{m}{2} \right\rfloor - 2 \leq m - 2$.

Proof. Suppose to the contrary, $\ell + \tau \leq m + 1$. By (C5) and the assumption that $n_i \geq 3$, we have that for $i \in \{0, 1, 2\}$,

$$\begin{aligned} S_L(P^i) - n_i m &\geq m + \frac{n_i - 1}{2} + 2x^{(i)} + y^{(i)} - \ell - \tau \\ &\geq 2x^{(i)} + y^{(i)} = dam_{L,P^i}(L(u), L(v)). \end{aligned}$$

As for any pair (S, T), $dam_{L^i,P}(S, T) \leq dam_{L^i,P}(L(u), L(v))$, we know that any pair (S, T) with $|S| = m - \tau$ satisfies Theorem 3.8, a contradiction.

Assume to the contrary that $\tau > 2\left\lfloor \frac{m}{2} \right\rfloor - 2$. As τ is even and $\tau \leq 2\left\lfloor \frac{m}{2} \right\rfloor$, by (C1), we have $\tau = 2\left\lfloor \frac{m}{2} \right\rfloor$. If m is even, then $\tau = m$ and we take $S = T = \emptyset$. By (C5), $dam_{L,P^i}(S, T) = 0 < S_L(P^i) - n_i m$ for $i = 0, 1, 2$, a contradiction.

Assume that m is odd. Then we have $\tau = m - 1$. By (C5), $S_L(P^i) - n_i m \geq m + 1 - \tau \geq 2$. Let $S = \{c\}$ and $T = \{c'\}$ for any couple cc'. Then $dam_{L,P^i}(S, T) \leq 2 \leq S_L(P^i) - n_i m$ for $i = 0, 1, 2$, a contradiction. □

Claim 3.5. There does not exist a semi-simple pair (D_u, D_v) such that $|D_u| = |D_v| = d \leq \ell - m + \tau$ and $|D_u|$ is even, and $dam_{L,P^i}(D_u, D_v) \geq d$ for each $i \in \{0, 1, 2\}$.

Proof. Assume (D_u, D_v) is such a semi-simple pair. Let L' be a new list assignment for G with $L'(u) = L(u) - D_u$, $L'(v) = L(v) - D_v$, $L'(w) = L(w)$ for $w \in V(G)\setminus\{u, v\}$.

(C1)-(C4) of Theorem 3.8 are easily seen to be satisfied by L', with $\ell' = \ell - d$ and $\tau' = \tau$.

As $L'(w) = L(w)$ for $w \in V(G)\setminus\{u, v\}$, for each $i \in \{0, 1, 2\}$, $dam_{L',P^i}(L'(u), L'(v)) = dam_{L,P^i}(L'(u), L'(v))$ and $S_L(P^i) = S_{L'}(P^i)$. Therefore, by Corollary 3.4,

$$dam_{L,P^i}(L(u), L(v)) = dam_{L',P^i}(L'(u), L'(v)) + dam_{L,P^i}(D_u, D_v) \geq dam_{L',P^i}(L'(u), L'(v)) + d,$$

and

$$\begin{aligned} S_{L'}(P^i) - n_i m &= S_L(P^i) - n_i m \\ &\geq \max\left\{m + \frac{n_i - 1}{2} + dam_{L,P^i}(L(u), L(v)) - \ell - \tau, m + \frac{n_i - 1}{2} - \tau\right\} \\ &\geq \max\left\{m + \frac{n_i - 1}{2} + dam_{L',P^i}(L'(u), L'(v)) - \ell' - \tau, m + \frac{n_i - 1}{2} - \tau\right\}. \end{aligned}$$

That is, (C5) is also satisfied by L'. By induction hypothesis, there exists a pair (S, T), with $|S| = |T| = m - \tau$, $S \subseteq L'(u) \subseteq L(u)$, $T \subseteq L'(v) \subseteq L(v)$, such that for each $i \in \{0, 1, 2\}$, $dam_{L,P^i}(S, T) = dam_{L',P^i}(S, T) \leq S_L(P^i) - n_i m$.

This completes the proof of this claim. □

Claim 3.6. There does not exist a semi-simple pair (D_u, D_v) such that $0 < |D_u| = |D_v| = d \leq m - \tau$ and $|D_u|$is even, and $dam_{L,P^i}(D_u, D_v) \leq d$ for each $i \in \{0, 1, 2\}$.

Proof. Assume (D_u, D_v) is such a semi-simple pair. Let L' be a new list assignment for G with $L'(u) = L(u) - D_u$, $L'(v) = L(v) - D_v$, for each i, $L'(v_1^i) = L(v_1^i) - D_u$, $L'(v_{n_i}^i) = L(v_{n_i}^i) - D_v$, $L'(v_j^i) = L(v_j^i)$ where $1 < j < n_i$.

Observe that (C1)-(C4) of Theorem 3.8 are satisfied by L', with $\ell' = \ell - d$ and $\tau' = \tau + d$. Note that $S_{L'}(P^i) = S_L(P^i) - dam_{L,P^i}(D_u, D_v) \geq S_L(P^i) - d$. As $S_L(P^i) - n_i m \geq m + \frac{n_i - 1}{2} - \tau$, we have $S_{L'}(P^i) - n_i m \geq m + \frac{n_i - 1}{2} - \tau'$. On the other hand, as $dam_{L',P^i}(L'(u), L'(v)) = dam_{L,P^i}(L'(u), L'(v))$, by Corollary 3.4,

$$dam_{L',P^i}(L'(u), L'(v)) = dam_{L,P^i}(L(u), L(v)) - dam_{L,P^i}(D_u, D_v).$$

So

$$\begin{aligned} S_{L'}(P^i) - n_i m &= S_L(P^i) - n_i m - dam_{L,P^i}(D_u, D_v) \\ &\geq m + \frac{n_i - 1}{2} + dam_{L,P^i}(L(u), L(v)) - dam_{L,P^i}(D_u, D_v) - \ell - \tau, \\ &= m + \frac{n_i - 1}{2} + dam_{L',P^i}(L'(u), L'(v)) - \ell' - \tau. \end{aligned}$$

Therefore, (C5) is also satisfied by L'.

By induction, there exists a pair (S', T'), where $|S'| = |T'| = m - \tau' = m - \tau - d$ such that for every i, $dam_{L',P^i}(S', T') \leq S_{L'}(P^i) - n_i m$. Let $S = S' \cup D_u$ and $T = T' \cup D_v$. As $S' \cap D_u = \emptyset$ and $T' \cap D_v = \emptyset$, $dam_{L',P^i}(S', T') = dam_{L,P^i}(S', T')$. Thus we have $|S| = |T| = m - \tau$ and

$$\begin{aligned} dam_{L,P^i}(S, T) &\leq dam_{L,P^i}(D_u, D_v) + dam_{L,P^i}(S', T') \\ &\leq dam_{L,P^i}(D_u, D_v) + S_{L'}(P^i) - n_i m \\ &= S_L(P^i) - n_i m. \end{aligned}$$

This completes the proof of Claim 3.6. □

By Claim 3.4, $\ell + \tau \geq m + 2$ and $m - \tau \geq 2$, so we can freely use Claim 3.5 and Claim 3.6 with $d = 2$ in the following, and we do not mention this again.

Claim 3.7. The following hold:

(1) Every couple is safe (respectively, heavy) for at most one internal path.
(2) No couple is light for exactly two internal paths. Moreover, there is at most one couple which is light for all internal paths.

Proof. (1). Assume to the contrary, $c_j c_j'$ is safe for two paths, say for both P^0 and P^1. If $c_j c_j'$ is also safe for P^2, then for any other couple $c_k c_k'$, we know that $(\{c_j, c_k\}, \{c_j', c_k'\})$ is a simple pair of size 2 contradicting Claim 3.6. Thus $c_j c_j'$ is not safe for P^2. As $z^{(2)} \geq 1$, there exists a couple $c_k c_k'$ which is safe for P^2. It follows that $(\{c_j, c_k\}, \{c_j', c_k'\})$ is a simple pair of size 2 contradicting Claim 3.6.

Next, we shall prove that every couple is heavy for at most one path. Assume $c_j c_j'$ is a couple which is heavy for at least two internal paths, say P^0 and P^1. If $c_j c_j'$ is also heavy for P^2, then for any other couple $c_k c_k'$, $(\{c_j, c_k\}, \{c_j', c_k'\})$ is a simple pair of size 2 contradicting Claim 3.5. So $c_j c_j'$ is not heavy for P^2. As $x^{(2)} \geq 2$, there exists a couple $c_k c_k'$ which is heavy for P^2. This implies that $(\{c_j, c_k\}, \{c_j', c_k'\})$ is also a simple pair of size 2 contradicting Claim 3.5.

(2). Assume to the contrary that there is a couple $c_j c_j'$ which is light for exactly two internal paths, say P^0 and P^1, and $c_j c_j'$ is either heavy or safe for P^2.

First assume that $c_j c_j'$ is heavy for P^2. Note that by Claim 3.3, $z^{(2)} \geq 1$, and hence there exists a distinct couple $c_k c_k'$ which is safe for P^2. By the fact that no couple is safe for two internal paths (by (1) of this claim), $c_k c_k'$ is safe for neither P^0 nor P^1. This implies that $(\{c_j, c_k\}, \{c_j', c_k'\})$ is a simple pair of size 2 contradicting Claim 3.5.

So $c_j c_j'$ is safe for P^2. As $x^{(2)} \geq 2$ (by Claim 3.3), there exists a distinct couple $c_k c_k'$ which is heavy for P^2. By the fact that no couple is heavy for two internal paths, $c_k c_k'$ is heavy for neither P^0 nor P^1. Then $(\{c_j, c_k\}, \{c_j', c_k'\})$ is a simple pair of size 2 contradicting Claim 3.6.

For the 'moreover' part, if there are two couples which are light for all internal paths, then two such couples comprise a simple pair of size 2 which contradicts Claim 3.6. This completes the proof of (2) as well as the proof of Claim 3.7. □

Since $x^{(i)} \geq 2$ for $i = 0, 1, 2$ (by Claim 3.3) and no couple is heavy for two internal paths (by Claim 3.7(1)), there exist distinct couples $c_i c_i'$ for $i = 0, 1, ..., 5$ such that

- $c_0 c_0'$ and $c_1 c_1'$ are heavy for P^0.
- $c_2 c_2'$ and $c_3 c_3'$ are heavy for P^1.
- $c_4 c_4'$ and $c_5 c_5'$ are heavy for P^2.

Without loss of generality, we may assume that $c_0 c_0'$ is light for P^1 and safe for P^2 (by Claim 3.7(2), $c_0 c_0'$ cannot be light for both P^1 and P^2, and by Claim 3.7(1), $c_0 c_0'$ cannot be safe for both P^1 and P^2). Then both $c_4 c_4'$ and $c_5 c_5'$ are light for P^0, for otherwise, $(\{c_0, c_4\}, \{c_0', c_4'\})$ or $(\{c_0, c_5\}, \{c_0', c_5'\})$ is a simple pair of size 2 which contradicts Claim 3.6. Consequently, by Claim 3.7, both $c_4 c_4'$ and $c_5 c_5'$ are safe for P^1.

Also, both $c_2 c_2'$ and $c_3 c_3'$ are light for P^2, and safe for P^0, as otherwise, $(\{c_2, c_4\}, \{c_2', c_4'\})$ or $(\{c_3, c_4\}, \{c_3', c_4'\})$ is a simple pair of size 2 which contradicts Claim 3.6.

Similarly, $c_1 c_1'$ is light for P^1, safe for P^2, since otherwise, $(\{c_1, c_2\}, \{c_1', c_2'\})$ is a simple pair of size 2 which contradicts Claim 3.6. See Table 1.

If $\tau \leq 2\left\lfloor \frac{m}{2} \right\rfloor - 6$, then $(\{c_0, c_1, ..., c_5\}, \{c_0', c_1', ..., c_5'\})$ is a simple pair of size 6 which contradicts Claim 3.6.

Assume $\tau = 2\left\lfloor \frac{m}{2} \right\rfloor - 4$. If m is even, then $m - \tau = 4$. By (C5), $S_L(P^i) - n_i m \geq \frac{n_i - 1}{2} + m - \tau \geq m - \tau + 1 = 5$. Let $S = \{c_0, c_1, c_2, c_4\}$ and $T = \{c_0', c_1', c_2', c_4'\}$. Then $dam_{L,P^i}(S, T) \leq 5$, we are done. If m is odd, then $m - \tau = 5$ and $S_L(P^i) - n_i m \geq 6$. Let $S = \{c_0, c_1, c_2, c_3, c_4\}$ and $T = \{c_0', c_1', c_2', c_3', c_4'\}$, and we have $dam_{L,P^i}(S, T) \leq 6$, we are also done.

Assume $\tau = 2\left\lfloor \frac{m}{2} \right\rfloor - 2$. If m is even, then $m - \tau = 2$, and $S_L(P^i) - n_i m \geq 3$. Let $S = \{c_0, c_2\}$ and $T = \{c_0', c_2'\}$. Then $dam_{L,P^i}(S, T) \leq 3$ for each i, so we are done. If m is odd, $m - \tau = 3$, and $S_L(P^i) - n_i m \geq 4$. Let $S = \{c_0, c_2, c_4\}$ and $T = \{c_0', c_2', c_4'\}$. Then $dam_{L,P^i}(S, T) \leq 3$ for each i, and we are also done.

This completes the proof of Theorem 3.8. □

Corollary 3.9. *Suppose $G = \Theta_{2r,2s,2t}$ with $r, s, t \geq 2$, u, v are the two vertices of degree 3, L is a list assignment of G with $|L(x)| = 2m$ if $x \in \{u, v\} \cup N_G(u)$ and $|L(x)| \geq 2m + 1$ for the other vertices. Then G is (L, m)-colourable.*

Proof. Let $\ell = 2m$ and $\tau = 0$. By Lemma 2.5, $|\hat{X}_1^i| + |\hat{X}_n^i| + |\Lambda^i| \geq |\hat{X}_1^i \cap L(u)| + |\hat{X}_n^i \cap L(v)| + |\Lambda^i \cap (L(u) \cup L(v))| = dam_{L,P^i}(L(u), L(v))$. Setting $l = 2m + 1$, by

TABLE 1 $dam_{L,P^i}(c_i, c_i')$

$L(u)$	c_0	c_1	c_2	c_3	c_4	c_5	⋯
P^0	Heavy	Heavy	Safe	Safe	Light	Light	⋯
P^1	Light	Light	Heavy	Heavy	Safe	Safe	⋯
P^2	Safe	Safe	Light	Light	Heavy	Heavy	⋯
$L(v)$	c_0'	c_1'	c_2'	c_3'	c_4'	c_5'	⋯

Lemma 2.7, $S_L(P^i) - n_i m \geq \frac{n_i - 1}{2} - m + |\hat{X}_1^i| + |\hat{X}_n^i| + |\Lambda^i| \geq \frac{n_i - 1}{2} + m - \ell - \tau + dam_{L,P^i}(L(u), L(v))$. On the other hand, setting $l_1 = 2m$ and $l_2 = 2m + 1$, by Lemma 2.8, $S_L(P^i) \geq l_1 + \frac{n_i - 1}{2} l_2 = n_i m + m + \frac{n_i - 1}{2} - \tau$. So (C5) holds. Observe that L, ℓ, τ also satisfies (C1)-(C4). By Theorem 3.8, there exist $S \subset L(u)$, $T \subset L(v)$ such that $|S| = |T| = m$ and $dam_{L,P^i}(S, T) \leq S_L(P^i) - n_i m$, which implies that G is (L, m)-colourable. □

Corollary 3.10. *Suppose $G = \Theta_{2r+1,2s+1,2t+1}$ with $r, s, t \geq 1$, u, v are the two vertices of degree 3, L is a list assignment of G with $|L(x)| = 2m$ if $x \in \{u, v\}$ and $|L(x)| \geq 2m + 1$ for the other vertices. Then G is (L, m)-colourable.*

Proof. Let $G' = \Theta_{2r+2,2s+2,2t+2}$ be obtained from G by splitting u into three vertices u_1, u_2, u_3 of degree 1 (each adjacent to one neighbour of u), adding a vertex u' adjacent to u_1, u_2, u_3. Let L' be a list assignment of G' with $L'(x) = L(u)$ if $x \in \{u', u_1, u_2, u_3\}$, and $L'(x) = L(x)$ for other vertices. By Corollary 3.9, G' is (L', m)-colourable and assume ϕ' is such an (L', m)-colouring of G'. Observe that for each $x \in \{u_1, u_2, u_3\}$, $\phi'(x) = L'(u') - \phi'(u)$. Now let ϕ be an (L, m)-colouring of G as follows: $\phi(u) = \phi'(u_1)$, and $\phi(x) = \phi'(x)$ for $x \in V(G) - \{u\}$. It is clear that ϕ is a proper (L, m)-colouring of G. □

4 | PROOF OF THEOREM 1.3 FOR $\Theta_{2,2,2,2p}$

In this section, $G = \Theta_{2,2,2,2p}$ with $p \geq 1$, u, v are the two vertices of degree 4, and P^0, P^1, P^2, P^3 are the four paths of $G - \{u, v\}$. Similarly, assume $P^i = (v_1^i, v_2^i, ..., v_{n_i}^i)$, v_1^i is adjacent to u and $v_{n_i}^i$ is adjacent to v, where $n_0 = n_1 = n_2 = 1$ and $n_3 \geq 1$. We shall use the notation introduced in Section 3.

Similarly, instead of proving directly that G is $(2m + 1, m)$-choosable, we prove the following stronger and more technical result. One may find that Theorem 4.1 is similar to Theorem 3.8. However, besides these two theorems refer to different graphs, there is another subtle difference: ℓ and τ are allowed to be odd in Theorem 4.1.

Theorem 4.1. *Assume ℓ and τ are non-negative integer, L is a list assignment for G satisfying the following:*

(T1) $\tau \leq m$ *and* $\ell + \tau \geq m$.
(T2) $|L(u)| = |L(v)| = \ell \geq 0$.
(T3) *For each* $i \in \{0, 1, 2, 3\}$, $|L(v_1^i)| \geq 2m + 1 - \tau$. *If* $n_3 \geq 3$, *then* $\left|L\left(v_{n_3}^3\right)\right| \geq 2m + 1 - \tau$.
(T4) $|L(w)| \geq 2m + 1$ *for* $w \neq u, v, v_1^i, v_{n_i}^i$.
(T5) *For* $i = 0, 1, 2, 3$,

$$S_L(P^i) - n_i m \geq \max\left\{\frac{n_i + 1}{2} + m - \ell - \tau + dam_{L,P^i}(L(u), L(v)), \frac{n_i + 1}{2} + m - \tau\right\}.$$

Then there exists a set $S \subset L(u)$ and a set $T \subset L(v)$ satisfying $|S| = |T| = m - \tau$ such that for each i,

$$dam_{L,P^i}(S, T) \leq S_L(P^i) - n_i m.$$

Proof. The proof is by induction on $2\ell + \tau$. First assume that $2\ell + \tau = m$. Since $\ell + \tau \geq m$ and ℓ, τ are non-negative, we have that $\ell = 0$ and $\tau = m$. By (T5), for each $i \in \{0, 1, 2, 3\}$, $S_L(P^i) - n_i m \geq \frac{n_i+1}{2} + m - \tau \geq 1$. By choosing $S = L(u) = \emptyset$, $T = L(v) = \emptyset$, we are done.

Assume that $2\ell + \tau \geq m + 1$. If $\ell + \tau = m$, then let $S = L(u)$, $T = L(v)$. If $\ell + \tau = m + 1$, then we let (S, T) be an arbitrary simple pair of size $(\ell - 1)$. In either case, for $i = 0, 1, 2, 3$,

$$\begin{aligned}
dam_{L,P^i}(S, T) &\leq dam_{L,P^i}(L(u), L(v)) \\
&\leq S_L(P^i) - n_i m - \frac{n_i + 1}{2} - m + \ell + \tau \\
&\leq S_L(P^i) - n_i m - \frac{n_i + 1}{2} - m + (m + 1) \\
&= S_L(P^i) - n_i m - \frac{n_i - 1}{2} \\
&\leq S_L(P^i) - n_i m.
\end{aligned}$$

So we are done. Thus we assume that $\ell + \tau \geq m + 2$.

If $\tau = m$, then let $S = T = \emptyset$ and we are done. If $\tau = m - 1$, then let (S, T) be any simple pair of size 1, we have $dam_{L,P^i}(S, T) \leq 2 \leq S_L(P^i) - n_i m$ by (T5).

In the sequel, we assume $\tau \leq m - 2$. Assume to the contrary that Theorem 4.1 is not true for L.

Claim 4.1. There is no semi-simple pair (D_u, D_v) such that $|D_u| = |D_v| = d \leq \ell - m + \tau$, and for each $i \in \{0, 1, 2, 3\}$, $x^{(i)} = 0$ or $dam_{L,P^i}(D_u, D_v) \geq d$.

Proof. Assume that (D_u, D_v) is such a pair. Let L' be a new list assignment for G with $L'(u) = L(u) \setminus D_u$, $L'(v) = L(v) \setminus D_v$, $L'(w) = L(w)$ for $w \in V(G) \setminus \{u, v\}$.

(T1)-(T4) of Theorem 4.1 are easily seen to be satisfied by L', with $\ell' = \ell - d$ and $\tau' = \tau$. Note that $S_{L'}(P^i) - n_i m = S_L(P^i) - n_i m \geq m + \frac{n_i+1}{2} - \tau = m + \frac{n_i+1}{2} - \tau'$. On the other hand, note that $dam_{L',P^i}(L'(u), L'(v)) = dam_{L,P^i}(L'(u), L'(v))$. So if $dam_{L,P^i}(D_u, D_v) \geq d$, then by Corollary 3.4, $dam_{L,P^i}(L(u), L(v)) = dam_{L',P^i}(L'(u), L'(v)) + dam_{L,P^i}(D_u, D_v) \geq dam_{L',P^i}(L'(u), L'(v)) + d$. So

$$\begin{aligned}
S_{L'}(P^i) - n_i m &= S_L(P^i) - n_i m \\
&\geq m + \frac{n_i + 1}{2} + dam_{L,P^i}(L(u), L(v)) - \ell - \tau, \\
&\geq m + \frac{n_i + 1}{2} + dam_{L',P^i}(L'(u), L'(v)) - \ell' - \tau.
\end{aligned}$$

If $x^{(i)} = 0$, then for every couple cc', $dam_{L,P^i}(c, c') \leq 1$, so

$$dam_{L',P^i}(L'(u), L'(v)) \leq \ell - d = \ell' \leq \ell' + (S_{L'}(P^i) - n_j m - \frac{n_i + 1}{2} - m + \tau'),$$

which implies that

$$S_{L'}(P^i) - n_i m \geq \frac{n_i + 1}{2} + m + dam_{L',P^i}(L'(u), L'(v)) - \ell' - \tau'.$$

Hence, (T5) is satisfied by L'. By induction hypothesis, there exists a pair (S, T), where $|S| = |T| = m - \tau$ such that for each $i \in \{0, 1, 2, 3\}$, $dam_{L,P^i}(S, T) \leq S_L(P^i) - n_i m$.

This completes the proof of this claim. □

Claim 4.2. There does not exist a semi-simple pair (D_u, D_v) such that $|D_u| = |D_v| = d \leq m - \tau$, and for each $i \in \{0, 1, 2, 3\}$, $z^{(i)} = 0$ or $dam_{L,P^i}(D_u, D_v) \leq d$.

Proof. Assume (D_u, D_v) is such a semi-simple pair. Let L' be a new list assignment for G with $L'(u) = L(u)\backslash D_u$, $L'(v) = L(v)\backslash D_v$, $L'(v_1^i) = L(v_1^i)\backslash(D_u \cup D_v)$ for $i = 0, 1, 2$. If $n_3 = 1$, then $L'(v_1^3) = L(v_1^3)\backslash(D_u \cup D_v)$. Otherwise, $L'(v_1^3) = L(v_1^3)\backslash D_u$, $L'(v_{n_3}^3) = L(v_{n_3}^3)\backslash D_v$, $L'(v_j^3) = L(v_j^3)$, where $1 < j < n_3$.

(T1)-(T2) and (T4) of Theorem 4.1 are easily seen to be satisfied by L', with $\ell' = \ell - d$ and $\tau' = \tau + d$.

It is obvious that (T3) is satisfied by (L', P^i) when $n_i \geq 3$. Now we show that (T3) is also satisfied when $n_i = 1$. Assume $i \in \{0, 1, 2, 3\}$ and $n_i = 1$. If $dam_{L,P^i}(D_u, D_v) \leq d$, then $|L(v_1^i)\backslash(D_u \cup D_v)| \geq |L(v_1^i)| - d$, so (T3) is satisfied by L'. Assume $z^{(i)} = 0$. Then for any couple cc', $dam_{L,P^i}(c, c') \geq 1$, which implies that $dam_{L,P^i}(L(u), L(v)) \geq dam_{L,P^i}(D_u, D_v) + \ell - d$ (using Corollary 3.4). By (T5),

$$\begin{aligned} S_L(P^i) &\geq \frac{n_i + 1}{2} + m - \ell - \tau + dam_{L,P^i}(L(u), L(v)) + n_i m \\ &\geq 2m + 1 - (\tau + d) + dam_{L,P^i}(D_u, D_v) \\ &= 2m + 1 - \tau' + dam_{L,P^i}(D_u, D_v). \end{aligned}$$

As $n_i = 1$ implies that $|L(v_1^i)| = S_L(P^i)$, $|L(v_1^i) - D_u \cup D_v| = |L(v_1^i)| - dam_{L,P^i}(D_u, D_v) = S_L(P^i) - dam_{L,P^i}(D_u, D_v) \geq 2m + 1 - \tau'$. So (T3) is also satisfied by L' in this case.

Next, we show that (T5) is satisfied by L'. By Corollary 3.4, $dam_{L,P^i}(L(u), L(v)) = dam_{L',P^i}(L'(u), L'(v)) + dam_{L,P^i}(D_u, D_v)$, so

$$\begin{aligned} S_{L'}(P^i) - n_i m &= S_L(P^i) - n_i m - dam_{L,P^i}(D_u, D_v) \\ &\geq m + \frac{n_i + 1}{2} + dam_{L,P^i}(L(u), L(v)) - \ell - \tau - dam_{L,P^i}(D_u, D_v) \end{aligned} \tag{3}$$

$$= m + \frac{n_i + 1}{2} + dam_{L',P^i}(L'(u), L'(v)) - \ell' - \tau'. \tag{4}$$

Now it suffices to prove that $S_{L'}(P^i) - n_i m \geq m + \frac{n_i+1}{2} - \tau'$. Indeed, if $z^{(i)} = 0$, then for each couple cc', $dam_{L',P^i}(c, c') \geq 1$, hence, by Observation 3.4, $dam_{L',P^i}(L'(u), L'(v)) \geq \ell - d = \ell'$. By Inequality (4), we are done. Assume that $dam_{L,P^i}(D_u, D_v) \leq d$. By Equality (3),

$$S_{L'}(P^i) - n_i m \geq \frac{n_i + 1}{2} + m - \tau - d = \frac{n_i + 1}{2} + m - \tau'.$$

Therefore, (T5) is satisfied by L'.

By induction, there exists a pair (S', T'), where $|S'| = |T'| = m - \tau' = m - \tau - d$ such that for every i,

$$dam_{L',P^i}(S', T') \leq S_{L'}(P^i) - n_i m.$$

Let $S = S' \cup D_u$ and $T = T' \cup D_v$. As $S' \cap D_u = \emptyset$ and $T' \cap D_v = \emptyset$, $dam_{L',P^i}(S', T')$)= $dam_{L,P^i}(S', T')$. So we have $|S| = |T| = m - \tau$ and

$$\begin{aligned} dam_{L,P^i}(S, T) &\leq dam_{L,P^i}(D_u, D_v) + dam_{L,P^i}(S', T') \\ &\leq dam_{L,P^i}(D_u, D_v) + S_{L'}(P^i) - n_i m \\ &= S_L(P^i) - n_i m. \end{aligned}$$

This completes the proof of Claim 4.2. □

Claim 4.3 follows directly from the definitions and (T5).

Claim 4.3. If (S, T) is a bad simple pair of size $m - \tau$ with respect to (L, P^i), then $dam_{L,P^i}(S, T) = 2a^{(i)}(S, T) + b^{(i)}(S, T) \geq \max\{2x^{(i)} + y^{(i)} + m + \frac{n_i+3}{2} - \ell - \tau, m + \frac{n_i+3}{2} - \tau\}$.

Claim 4.4. There is no simple pair (S_0, T_0) of size 3 such that $dam_{L,P^i}(S_0, T_0) \leq 3$ for each $i \in \{0, 1, 2, 3\}$.

Proof. Assume the claim is not true, and assume that (S_0, T_0) of size 3 for which $dam_{L,P^i}(S_0, T_0) \leq 3$ for each $i \in \{0, 1, 2, 3\}$. By Claim 4.2, if $m - \tau \geq 3$, then there is a contradiction. Thus assume that $m - \tau \leq 2$. Recall that in the beginning of the proof of Theorem 4.1, we argued that $m - \tau \geq 2$, so $m - \tau = 2$. By (T5), $S_L(P^i) - n_i m \geq \frac{n_i+1}{2} + m - \tau \geq m - \tau + 1 \geq 3$. Then any simple pair (S, T) of size 2 with $S \subseteq S_0$, $T \subseteq T_0$ satisfies the theorem, a contradiction. □

Claim 4.5. For each $i \in \{0, 1, 2, 3\}$, $x^{(i)} = 0$ or $z^{(i)} = 0$ implies that $\beta(P^i) = 0$.

Proof. If $x^{(i)} = 0$, then $dam_{L,P^i}(S, T) \leq m - \tau$ for any simple pair (S, T) of size $m - \tau$. By (T5), $S_L(P^i) - n_i m \geq \frac{n_i+1}{2} + m - \tau \geq m - \tau + 1$. So (S, T) is not bad with respect to (L, P^i), hence $\beta(P^i) = 0$.

If $z^{(i)} = 0$, then $x^{(i)} + y^{(i)} = \ell$ and for any simple pair (S, T) of size $m - \tau$, $a^{(i)}(S, T) + b^{(i)}(S, T) = m - \tau$. If (S, T) is bad with respect to (L, P^i), then by Claim 4.3, we have

$$\begin{aligned} a^{(i)}(S, T) + m - \tau &= 2a^{(i)}(S, T) + b^{(i)}(S, T) \\ &\geq 2x^{(i)} + y^{(i)} + m + \frac{n_i + 3}{2} - \ell - \tau \\ &\geq x^{(i)} + \ell + m + \frac{n_i + 3}{2} - \ell - \tau \\ &= x^{(i)} + m - \tau + \frac{n_i + 3}{2}. \end{aligned}$$

This implies that $a^{(i)}(S, T) \geq x^{(i)} + 2$, in contrary to that $a^{(i)}(S, T) \leq x^{(i)}$. So any simple pair (S, T) of size $m - \tau$ is not bad with respect to (L, P^i), hence $\beta(P^i) = 0$. □

Claim 4.6. For each $j \in \{0, 1, 2, 3\}$, $x^{(j)}, z^{(j)} \geq 1$.

Proof. Assume to the contrary, $x^{(j)} = 0$ or $z^{(j)} = 0$ for some j. Then $\beta(P^j) = 0$ by Claim 4.5. For convenience, we let $j = 0$ below, but do not use the fact that $n_1 = 0$ so that the argument also works for $j = 1, 2, 3$.

Note that there is no simple pair (D_u, D_v) of size $d \leq 2$ such that $dam_{L,P^i}(D_u, D_v) \geq d$ for all $i \in \{1, 2, 3\}$. As for $i = 0$, we have either $x^{(0)} = 0$, or $z^{(0)} = 0$ which implies that $dam_{L,P^0}(D_u, D_v) \geq d$, a contradiction to Claim 4.1. Similarly, there does not exists a simple pair (D_u, D_v) of size $d \leq 2$ such that $dam_{L,P^i}(D_u, D_v) \leq d$ for all $i \in \{1, 2, 3\}$. As for $i = 0$, we have either $z^{(0)} = 0$, or $x^{(0)} = 0$ which implies that $dam_{L,P^0}(D_u, D_v) \leq d$, a contradiction to Claim 4.2.

We first show that $x^{(i)}, z^{(i)} \geq 1$ for $i \neq 0$. Indeed, if this fails for some i, then by Claim 4.5, $\beta(P^i) = 0$. Thus by Claim 4.3 and Observation 2.10 (setting $m - \tau = k$), $\sum_{k=0}^{3} \beta(P^k) < \binom{\ell}{m-\tau}$. So there exists a simple pair of size $m - \tau$ which is not bad with respect to any (L, P^k) for $k = 0, 1, 2, 3$, a contradiction.

Next we show that every couple is heavy (respectively, safe, light) for at most one of P^1, P^2, P^3.

Assume to the contrary, $c_j c_j'$ is heavy for two paths, say for both P^1 and P^2. By the second paragraph of this proof, $c_j c_j'$ is not heavy for P^3. As $x^{(3)} \geq 1$, there exists a couple $c_k c_k'$ which is heavy for P^3. Then for $D_u = \{c_j, c_k\}, D_v = \{c_j', c_k'\}$, we have $dam_{L,P^i}(D_u, D_v) \geq 2$ for $i = 1, 2, 3$, and for $i = 0$, either $x^{(0)} = 0$, or $z^{(0)} = 0$ which means that $dam_{L,P^0}(D_u, D_v) \geq 2$. In either case, it contradicts Claim 4.1.

Similarly, if $c_j c_j'$ is safe for P^1 and P^2, then by the second paragraph of this proof, $c_j c_j'$ is not safe for P^3. As $z^{(3)} \geq 1$, so there exists a couple $c_k c_k'$ which is safe for P^3. Then for $D_u = \{c_j, c_k\}, D_v = \{c_j', c_k'\}$, we have $dam_{L,P^i}(D_u, D_v) \leq 2$ for $i = 1, 2, 3$, and for $i = 0$, either $z^{(0)} = 0$, or $x^{(0)} = 0$ which means that $dam_{L,P^0}(D_u, D_v) \leq 2$. In either case, it contradicts Claim 4.2.

Assume $c_j c_j'$ is light for P^1 and P^2. If $c_j c_j'$ is safe for P^3, then $c_j c_j'$ is a simple pair contradicting Claim 4.2. Otherwise, $c_j c_j'$ is a simple pair contradicting Claim 4.1.

Without loss of generality, assume $c_1 c_1'$ is heavy for P^1, light for P^2 and safe for P^3. Assume that $c_2 c_2'$ is heavy for P^2, $c_3 c_3'$ is heavy for P^3. Then $c_3 c_3'$ is light for P^1 and safe for

P^2, for otherwise, $(\{c_1, c_3\}, \{c_1', c_3'\})$ is a simple pair of size 2 contradicting Claim 4.2. Similarly, c_2c_2' is light for P^3 and safe for P^1, for otherwise, $(\{c_2, c_3\}, \{c_2', c_3'\})$ is a simple pair of size 2 contradicting Claim 4.2.

If $x^{(0)} = 0$, then $(\{c_1, c_2, c_3\}, \{c_1', c_2', c_3'\})$ is a simple pair of size 3 contradicting Claim 4.4.

Assume that $x^{(0)} \geq 1$ and $z^{(0)} = 0$. If $m - \tau \geq 3$, then $(\{c_1, c_2, c_3\}, \{c_1', c_2', c_3'\})$ is a simple pair of size 3 contradicting Claim 4.2.

Assume $m - \tau = 2$. By (T5),

$$S_L(P^i) - n_i m \geq \max\left\{dam_{L,P^i}(L(u), L(v)) - \ell + 3, 3\right\} \geq 3.$$

Note that $x^{(0)} \geq 1$ and $z^{(0)} = 0$ implies that $dam_{L,P^0}(L(u), L(v)) \geq 2 + (\ell - 1) = \ell + 1$. Therefore, $S_L(P^0) - n_0 m \geq 4$. Let $S = \{c_1, c_2\}$, $T = \{c_1', c_2'\}$. Then (S, T) is a pair of size $m - \tau = 2$ satisfying Theorem 4.1.

This completes the proof of this claim. □

Claim 4.7. Every couple is heavy (respectively, light, safe) for at most two internal paths.

Proof. If there exists a couple c_jc_j' which is light for at least three internal paths, then c_jc_j' is counterexample with $d = 1$ to either Claim 4.1 or Claim 4.2.

By Claim 4.1, every couple is heavy for at most three internal paths. Assume c_jc_j' is heavy for all the internal paths except P^i for some $i \in \{0, 1, 2, 3\}$. By Claim 4.6, $x^{(i)} \geq 1$, there exists a heavy couple c_kc_k' for P^i. Then $(\{c_j, c_k\}, \{c_j', c_k'\})$ is a simple pair of size 2 contradicting Claim 4.1. Thus every couple is heavy for at most two internal paths. Similarly, we can prove that every couple is safe for at most two internal paths. □

Claim 4.8. If a couple is heavy for exactly two internal paths, then it is safe for the other two paths.

Proof. Assume the claim is not true and c_0c_0' is heavy for two internal paths, and light for at least one internal path. If c_0c_0' is light for two internal paths, then c_0c_0' is a simple pair that contradicts Claim 4.1. So c_0c_0' is light for one internal path P^i and safe for one internal path P^j. Without loss of generality, assume $c_0 \in \hat{X}_1^i \cup \Lambda^i$.

As $x^{(j)} \geq 1$, there is a couple c_1c_1' which is heavy for P^j. Note that c_0c_0' is heavy for at least one internal path with only one vertex. So $c_0 \neq c_0'$. If $c_1 \neq c_1'$, then by Observation 3.5, (c_0, c_1') is a semi-simple pair. But $dam_{L,P^i}(c_0, c_1') \geq 1$ for each $i \in \{0, 1, 2, 3\}$, contrary to Claim 4.1. Thus $c_1 = c_1'$. By Observation 3.6, c_1c_1' cannot be heavy for an internal path with only one vertex. So $j = 3$ and $n_3 \geq 3$. Thus we may assume that c_0c_0' is heavy for P^0 and P^1, light for P^2 and safe for P^3, that is, $i = 2$.

Then c_1c_1' is safe for P^2, for otherwise $(\{c_0, c_1\}, \{c_0', c_1'\})$ is a simple pair of size 2 contradicting Claim 4.1.

By Claim 4.7, we may assume that c_1c_1' is light for P^0, and is either light for P^1 or safe for P^1 (Recall that c_1c_1' cannot be heavy for an internal path with only one vertex).

By Claim 4.6, $x^{(2)} \geq 1$. Let c_2c_2' be a couple which is heavy for P^2. Then $c_2 \neq c_2'$.

We claim that c_2c_2' is safe for P^3. Otherwise c_2c_2' is heavy or light for P^3. Without loss of generality, assume that $c_2 \in \hat{X}_1^3 \cup \Lambda^3$. By Observation 3.5, (c_2, c_0') is a semi-simple pair of size 1 and $dam_{L,P^i}(c_2, c_0') \geq 1$, a contradiction to Claim 4.1.

Recall that we assumed that $c_0 \in \hat{X}_1^2 \cup \Lambda^2$. Hence $dam_{L,P^i}(c_0, c_2') \geq 1$ for $i = 0, 1$, $dam_{L,P^2}(c_0, c_2') = 2$ and $dam_{L,P^3}(c_0, c_2') = 0$. So if c_1c_1' is light for P^1, then $(\{c_0, c_1\}, \{c_1', c_2'\})$ is a semi-simple pair (by Observation 3.5) of size 2 contradicting Claim 4.1. Therefore, c_1c_1' is safe for P^1.

Then c_2c_2' must be heavy for P^0, for otherwise $(\{c_1, c_2\}, \{c_1', c_2'\})$ is a simple pair of size 2 contradicting Claim 4.2.

Thus c_2c_2' is either light for P^1 or safe for P^1.

By Claim 4.6, $z^{(0)} \geq 1$. Let c_3c_3' be a couple which is safe for P^0.

We claim that c_3c_3' is heavy for at least one of P^1 and P^2. Otherwise c_3c_3' is heavy for P^3 by Claim 4.2. If c_3c_3' is safe for P^2, then $(\{c_2, c_3\}, \{c_2', c_3'\})$ is a simple pair of size 2 contradicting Claim 4.2.

If c_3c_3' is safe for P^1, then $(\{c_0, c_3\}, \{c_0', c_3'\})$ is a simple pair which contradicts Claim 4.2. So c_3c_3' is light for P^1 and P^2.

Note that now (c_0, c_2') is a semi-simple pair satisfying that $dam_{L,P^i}(c_0, c_2') = 2$ for $i \in \{0, 2\}$, $dam_{L,P^1}(c_0, c_2') \geq 1$ and $dam_{L,P^3}(c_0, c_2') = 0$. Thus $(\{c_0, c_3\}, \{c_2', c_3'\})$ is a semi-simple pair of size 2 contradicting Claim 4.1.

This completes the proof of the claim that c_3c_3' is heavy for at least one of P^1 and P^2. Hence $c_3 \neq c_3'$. If c_3c_3' is safe for P^3, then $(\{c_1, c_3\}, \{c_1', c_3'\})$ is a simple pair of size 2 contradicting Claim 4.2. So c_3c_3' is not safe for P^3. Thus $c_3 \in \hat{X}_1^3 \cup \Lambda^3$ or $c_3' \in \hat{X}_{n_3}^3 \cup \Lambda^3$ (or both). If $c_3' \in \hat{X}_{n_3}^3 \cup \Lambda^3$, then (c_0, c_3') is a semi-simple pair of size 1 contradicting Claim 4.1. So c_3c_3' is light for P^3 and $c_3 \in \hat{X}_1^3 \cup \Lambda^3$.

If c_3c_3' is heavy for P^1, then (c_3, c_2') is a semi-simple pair of size 1 contradicting Claim 4.1. If c_3c_3' is heavy for P^2, then (c_3, c_0') is a semi-simple pair of size 1 contradicting Claim 4.1.

This completes the proof of Claim 4.8. □

Claim 4.9. No couple is heavy for two internal paths with one being P^3.

Proof. Assume to the contrary that c_0c_0' is heavy for P^0 and P^3. As $n_0 = 1$, by Observation 3.6, $c_0 \neq c_0'$. By Claim 4.8, c_0c_0' is safe for P^1 and P^2.

By Claim 4.6, $x^{(1)} \geq 1$. Let c_1c_1' be a couple which is heavy for P^1. Then $c_1 \neq c_1'$. Observe that c_1c_1' is safe for P^2, for otherwise, we may assume $c_1 \in \hat{X}_1^2 \cup \Lambda^2$, and hence (c_1, c_0') is a semi-simple pair (by Observation 3.5) of size 1 contradicting Claim 4.1. Similarly, there exists a couple c_2c_2', which is heavy for P^2 and safe for P^1, and $c_2 \neq c_2'$.

At least one of c_1c_1' and c_2c_2' is heavy for P^0 or P^3, for otherwise, $(\{c_2, c_3\}, \{c_2', c_3'\})$ is a simple pair of size 2 contradicting Claim 4.2. Without loss of generality, assume that c_1c_1' is heavy for P^0 or P^3. For convenience, assume that c_1c_1' is heavy for P^0, and we will not use the fact that $n_1 = 1$. By Claim 4.8, c_1c_1' is safe for P^3.

As $c_2 \neq c_2'$, c_2c_2' is safe for P^3, since otherwise, without loss of generality, we assume $c_2 \in \hat{X}_1^3 \cup \Lambda^3$. Then (c_2, c_1') is a semi-simple pair (by Observation 3.5) of size 1 contradicting Claim 4.1. Similarly, we have (c_2, c_2') is heavy for P^0, for otherwise, without loss of generality, we assume $c_2 \notin \hat{X}_1^0 \cup \Lambda^0$. Then (c_2, c_1') is a semi-simple pair (by Observation 3.5) of size 1 contradicting Claim 4.2.

By Claim 4.6, $z^{(0)} \geq 1$. Let c_3c_3' be a couple which is safe for P^0. Note that (c_1, c_2') is a semi-simple pair of size 1 which is heavy for P^0, light for P^1 and P^2, and safe for P^3. Therefore, if c_3c_3' is neither heavy for P^1 nor for P^2, then $(\{c_1, c_3\}, \{c_2', c_3'\})$ is a semi-simple pair of size 2 contradicting Claim 4.2. Thus without loss of generality, assume that c_3c_3' is heavy for P^1, so $c_3 \neq c_3'$ by Observation 3.6. If c_3c_3' is also heavy for P^2, then by Claim 4.8, c_3c_3' is safe for P^3. But then $(\{c_0, c_3\}, \{c_0', c_3'\})$ is a simple pair of size 2 contradicting Claim 4.1. So c_3c_3' is not heavy for P^2. If c_3c_3' is safe for P^2, then $(\{c_2, c_3\}, \{c_2', c_3'\})$ is a simple pair of size 2 contradicting Claim 4.2. Thus c_3c_3' is light for P^2. Without loss of generality, assume that $c_3 \notin \hat{X}_1^2 \cup \Lambda^2$. But then (c_3, c_2') is a semi-simple pair of size 1 contradicting Claim 4.2.

This completes the proof of Claim 4.9. □

Claim 4.10. Every couple is heavy for exactly one internal path.

Proof. By Claim 4.2, every couple is heavy for at least one internal path. Suppose to the contrary, c_0c_0' is heavy for two internal paths. By Claim 4.9, we may assume that c_0c_0' is heavy for P^0 and P^1. By Claim 4.8, c_0c_0' is safe for both P^2 and P^3.

By Claim 4.6, $x^{(2)} \geq 1$. Let c_1c_1' be a couple which is heavy for P^2. Then $c_1 \neq c_1'$. Note that c_1c_1' must be safe for P^3, for otherwise, we assume that $c_1 \in \hat{X}_1^3 \cup \Lambda^3$. Then (c_1, c_0') is a semi-simple pair (By Observation 3.5) of size 1 contradicting Claim 4.1.

As $x^{(3)} \geq 1$, there exists a couple c_2c_2' which is heavy for P^3. By Claim 4.9, we know that c_2c_2' is not heavy for any of P^0, P^1 and P^2.

We first claim that c_1c_1' is heavy for exactly one of P^0 and P^1. Suppose this is not true. By Claim 4.7, c_1c_1' cannot be safe for three paths, so c_1c_1' is light for at least one of P^0 and P^1, without loss of generality, say P^0, and assume that $c_1 \notin \hat{X}_1^0 \cup \Lambda^0$. If c_1c_1' is safe for P^1, then (c_1, c_0') is a semi-simple pair of size 1 contradicting Claim 4.2. Thus c_1c_1' is also light for P^1. Recall that c_2c_2' is heavy for none of P^0, P^1 and P^2. If c_2c_2' is safe for P^2, then $(\{c_1, c_2\}, \{c_1', c_2'\})$ is a simple pair of size 2 contradicting Claim 4.2. So c_2c_2' is light for P^2. By Claim 4.1, c_2c_2' is safe for at least one of P^0, P^1. Assume that it is safe for P^1. Then c_2c_2' is light for P^0, for otherwise $(\{c_0, c_2\}, \{c_0', c_2'\})$ is a simple pair of size 2 contradicting Claim 4.2. Recall that $c_1 \neq c_1'$, and we assumed that $c_1 \notin \hat{X}_1^0 \cup \Lambda^0$. So (c_1, c_0') is a semi-simple pair of size 1 such that it is light for P^0 and P^2, safe for P^3. Hence $(\{c_1, c_2\}, \{c_0', c_2'\})$ is a semi-simple pair of size 2 contradicting Claim 4.2.

Without loss of generality, we assume that c_1c_1' is heavy for P^0, by Claim 4.8, c_1c_1' is safe for P^1. Note that (c_0, c_1') is a semi-simple pair of size 1 which is heavy for P^0, light for P^1 and P^2, and safe for P^3. If c_2c_2' is safe for P^0, then $(\{c_0, c_2\}, \{c_1', c_2'\})$ is a semi-simple pair of size 2 contradicting Claim 4.2. Hence c_2c_2' is light for P^0. On the other hand, by Claim 4.1, c_2c_2' is safe for some P^i. Without loss of generality, we assume that c_2c_2' is safe for P^1.

By Claim 4.6, $z^{(0)} \geq 1$, we may assume that c_3c_3' is safe for P^0. Note that c_3c_3' is heavy for at least one of P^1 or P^2, for otherwise, $(\{c_0, c_3\}, \{c_1', c_3'\})$ is a semi-simple pair of size 2 contradicting Claim 4.2. So $c_3 \neq c_3'$, and by Claim 4.9, c_3c_3' is not heavy for P^3.

First assume that c_3c_3' is heavy for P^1. If c_3c_3' is also heavy for P^2, then by Claim 4.8, c_3c_3' is safe for P^3. But then $(\{c_0, c_2\}, \{c_2', c_3'\})$ is a semi-simple pair of size 2 contradicting

Claim 4.2 (recall that by Claim 4.9, c_2c_2' is not heavy for P^2 as it is already heavy for P^3). So c_3c_3' is not heavy for P^2. Thus without loss of generality, we may assume that $c_3 \notin \hat{X}_1^2 \cup \Lambda^2$. Then (c_3, c_1') is a semi-simple pair of size 1 contradicting Claim 4.2. So c_3c_3' is not heavy for P^1.

Therefore, c_3c_3' is heavy for P^2 but not heavy for P^1. Thus without loss of generality, assume that $c_3 \notin \hat{X}_1^1 \cup \Lambda^1$. But then we have that (c_3, c_0') is a semi-simple pair of size 1 which is not heavy for any internal paths, a contradiction to Claim 4.2.

This completes the proof of Claim 4.10. □

Claim 4.11. Every couple is safe for exactly one internal path.

Proof. By Claim 4.1, we know that every couple is safe for at least one internal path.

Assume to the contrary that c_0c_0' is safe for P^2 and P^3. As every couple is heavy for exactly one internal path, we may assume that c_0c_0' is heavy for P^0, light for P^1. Note that path P^3 may be different from the other paths, as $n_i = 1$ for $i = 0, 1, 2$ and n_3 can be greater than 1. However, the argument below does not use this difference.

By Claim 4.6, $x^{(2)} \geq 1$, $x^{(3)} \geq 1$, and by Claim 4.10, every couple is heavy for exactly one path, thus we assume that c_1c_1' is heavy for P^2 and c_2c_2' is heavy for P^3.

If c_1c_1' is safe for P^3 and c_2c_2' is safe for P^2, then $(\{c_1, c_2\}, \{c_1', c_2'\})$ is a simple pair of size 2 contradicting Claim 4.2. So without loss of generality, we assume that c_1c_1' is light for P^3.

By Claim 4.2, both c_1c_1' and c_2c_2' are light for P^0 by considering $(\{c_0, c_1\}, \{c_0', c_1'\})$ and $(\{c_0, c_2\}, \{c_0', c_2'\})$, respectively. Consequently, c_1c_1' is safe for P^1 since otherwise c_1c_1' is not safe for any internal path, a contradiction.

By Claim 4.6, $x^{(1)} \geq 1$, there exists a couple, say c_3c_3', which is heavy for P^1. By Claim 4.10, c_3c_3' is not heavy for any other path. Thus c_3c_3' is light for P^2, for otherwise, $(\{c_1, c_3\}, \{c_1', c_3'\})$ is a simple pair of size 2 contradicting Claim 4.2.

If c_3c_3' is safe for P^0, then $(\{c_0, c_1, c_3\}, \{c_0', c_1', c_3'\})$ is a simple pair of size 3 which contradicts Claim 4.4. So c_3c_3' is light for P^0, which implies that c_3c_3' is safe for P^3. Similarly, c_2c_2' is light for P^2, as otherwise $(\{c_1, c_2, c_3\}, \{c_1', c_2', c_3'\})$ is a simple pair of size 3 which contradicts Claim 4.4. This implies that c_2c_2' is safe for P^1. But then $(\{c_2, c_3\}, \{c_2', c_3'\})$ is a simple pair of size 2 which contradicts Claim 4.2.

This completes the proof of Claim 4.11. □

By Claim 4.6, $x^{(i)} \geq 1$ for each $i \in \{0, 1, 2, 3\}$. Without loss of generality, assume that c_0c_0' is heavy for P^3, light for P^1 and P^2, safe for P^0. Also, we assume that c_1c_1' is heavy for P^0, c_2c_2' is heavy for P^2, c_3c_3' is heavy for P^1. Observe that c_1c_1' must be light for P^3, since otherwise, c_0c_0' and c_1c_1' comprise a simple pair of size 2 contradicting Claim 4.2. By Claim 4.11, c_1c_1' must be safe for exactly one of P^1 and P^2, say P^2, and then it is light for P^1. This implies that c_2c_2' is light for P^0, for otherwise c_1c_1' and c_2c_2' comprise a simple pair of size 2 contradicting Claim 4.2. Similarly, c_2c_2' is light for P^3 by considering Claim 4.4 and the three couples c_0c_0', c_1c_1' and c_2c_2'. Consequently, c_2c_2' is safe for P^1. Again by these techniques, c_3c_3' is light for P^2 by considering Claim 4.2 and the two couples c_2c_2', c_3c_3', and light for P^0 by considering Claim 4.4 and the three couples c_1c_1', c_2c_2' and c_3c_3'. So c_3c_3' is safe for P^3. See Table 2.

TABLE 2 $dam_{L,P^i}(c_i, c_i')$

$L(u)$	c_0	c_1	c_2	c_3	$\cdots$
P^0	Safe	Heavy	Light	Light	$\cdots$
P^1	Light	Light	Safe	Heavy	$\cdots$
P^2	Light	Safe	Heavy	Light	$\cdots$
P^3	Heavy	Light	Light	Safe	$\cdots$
$L(v)$	c_0'	c_1'	c_2'	c_3'	$\cdots$

If $m - \tau \geq 4$, then $(\{c_0, c_1, c_2, c_3\}, \{c_0', c_1', c_2', c_3'\})$ is a simple pair of size 4 contradicting Claim 4.2. So $m - \tau \leq 3$. Recall that $m - \tau \geq 2$, so $m - \tau = 2$ or $m - \tau = 3$.

By (T5), $S_L(P^i) - n_i m \geq \frac{n_i+1}{2} + m - \tau \geq m - \tau + 1$. If $m - \tau = 2$, then $S_L(P^i) - n_i m \geq 3$, we let $S = \{c_0, c_1\}$ and $T = \{c_0', c_1'\}$; If $m - \tau = 3$, then $S_L(P^i) - n_i m \geq 4$, we let $S = \{c_0, c_1, c_2\}$ and $T = \{c_0', c_1', c_2'\}$. In either case, we find a simple pair of size $m - \tau$ which satisfies the theorem, a contradiction.

This finishes the proof of Theorem 4.1. □

Corollary 4.2. *$\Theta_{2,2,2,2p}$ is $(2m+1, m)$-choosable*

Proof. By setting $\ell = 2m + 1$ and $\tau = 0$ in Theorem 4.1, we know that $\Theta_{2,2,2,2p}$ is $(2m+1, m)$-choosable. Indeed, assume L is a $(2m+1)$-list assignment of $G = \Theta_{2,2,2,2p}$. By Lemma 2.8, $S_L(P^i) \geq \frac{n_i+1}{2}(2m+1)$, namely, $S_L(P^i) - n_i m \geq \frac{n_i+1}{2} + m = \frac{n_i+1}{2} + m - \tau$. By Lemma 2.7, $S_L(P^i) - \frac{n_i+1}{2}(2m+1) + (2m+1) \geq |\hat{X}_1^i| + |\hat{X}_n^i| + |\Lambda^i| \geq dam_{L,P^i}(L(u), L(v))$, so $S_L(P^i) - n_i m \geq \frac{n_i+1}{2} + m + dam_{L,P^i}(L(u), L(v)) - (2m+1) = \frac{n_i+1}{2} + m + dam_{L,P^i}(L(u), L(v)) - \ell - \tau$. So (T5) holds. Observe that L, ℓ, τ also satisfies (T1)-(T4). Therefore, there exist two sets $S \subset L(u)$, $T \subset L(v)$ such that $|S| = |T| = m$ and $dam_{L,P^i}(S, T) \leq S_L(P^i) - n_i m$ for $i = 0, 1, 2, 3$. Hence G is $(2m+1, m)$-choosable. □

5 | PROOF OF LEMMA 2.9

This section proves Lemma 2.9. That is,

$$F(x, y) = \sum \binom{x}{a}\binom{y}{b}\binom{\ell - x - y}{k - a - b} \leq \frac{1}{2}\binom{\ell}{k}, \tag{5}$$

where $x + y \leq \ell, 2x + y \leq \ell + k - 1$, $k \geq 1$, $\ell \geq k + 1$, and the summation is over non-negative integer pairs (a, b) for which $0 \leq a \leq x$, $0 \leq b \leq y$, $a + b \leq k$ and $2a + b \geq \max\{2x + y + k + 1 - \ell, k + 1\}$. Moreover, we will show that the equality holds if and only if ℓ is even, k is odd, and $x = \frac{\ell}{2}$, $y = 0$.

Note that $a + b \leq k$ and $2a + b \geq k + 1$ implies that $a \geq 1$.

In the sequel, we define

$$\binom{p}{q}_+ = \begin{cases} \binom{p}{q} & \text{if } p \geq q \geq 0, \\ 0 & \text{if } q < 0 \text{ or } p < q. \end{cases} \tag{6}$$

For convenience, we allow $p < q$ or $q < 0$ in the binomial coefficient in the summations below. It is easy to check that in these cases, either the pair (a, b) does not lie in the range of the summation, and hence contributes 0 to the summations, or we can extend the equality $\binom{p+1}{q} = \binom{p}{q} + \binom{p}{q-1}$ to $q = 0$. For the readability, we suppress the index '+'.

The following lemma is proved in [10] (Lemma 18 in [10], where the parameter $2k$ is changed to k, but the proof still works).

Lemma 5.1. *Assume $x = x_0$ is fixed.*

(1) *If $y \geq \ell - 2x_0$, then $F(x_0, y + 1) \leq F(x_0, y)$.*
(2) *If $y < \ell - 2x_0$, then $F(x_0, y) \leq F(x_0, y + 1)$.*

We consider two cases.

Case 1. $x \leq \left\lfloor \frac{\ell}{2} \right\rfloor$.

By Lemma 5.1, $F(x, y) \leq F(x, \ell - 2x)$. So it suffices to show that $F(x, \ell - 2x) \leq \frac{1}{2}\binom{\ell}{k}$. Recall that (by Equality (5))

$$F(x, \ell - 2x) = \sum_{t=k+1}^{2k} \sum_{2a+b=t} \binom{x}{a}\binom{\ell - 2x}{b}\binom{x}{k - a - b} = \sum_{t=k+1}^{2k} C(t, x),$$

where

$$C(t, x) = \sum_{2a+b=t} \binom{x}{a}\binom{\ell - 2x}{b}\binom{x}{k - a - b} = \sum_{2a \leq t} \binom{x}{a}\binom{\ell - 2x}{t - 2a}\binom{x}{k + a - t}.$$

Note that for any $0 \leq x \leq \frac{\ell}{2}$, $\sum_{t=0}^{2k} C(t, x) = \binom{\ell}{k}$.

For $0 \leq t \leq k$,

$$\begin{aligned} C(t, x) &= \sum_{2a \leq t} \binom{x}{a}\binom{\ell - 2x}{t - 2a}\binom{x}{k + a - t} = \sum_{2a' \leq 2k-t} \binom{x}{a'}\binom{\ell - 2x}{2k - t - 2a'}\binom{x}{a' + t - k} \\ &= C(2k - t, x), \end{aligned}$$

where $a' = k + a - t$.

When $1 \leq x \leq \left\lfloor \frac{\ell}{2} \right\rfloor$,

$$F(x, \ell - 2x) = \sum_{t=k+1}^{2k} C(t, x) = \frac{\binom{\ell}{k} - C(k, x)}{2}. \tag{7}$$

Lemma 5.2. *$C(k,x) \geq 0$ when $1 \leq x \leq \left\lfloor \frac{\ell}{2} \right\rfloor$ and the equality holds if and only if $\ell = 2x$ and k is odd.*

Proof. If $x = \frac{\ell}{2}$ and k is odd, then $y = \ell - 2x = 0$, which implies that $b = 0$ as $0 \leq b \leq y$. Therefore, as $2a$ is even, we have $C(k,x) = \sum_{2a=k} \binom{x}{a}^2 = 0$.

Assume $\ell \neq 2x$ or $\ell = 2x$ and k is even.

First assume that $x \geq \left\lfloor \frac{k}{2} \right\rfloor$. As $x \leq \left\lfloor \frac{\ell}{2} \right\rfloor$, we have $\ell - 2x \geq 0$. Note that

$$k - 2\left\lfloor \frac{k}{2} \right\rfloor = \begin{cases} 0 & \text{if } k \text{ is even}, \\ 1 & \text{if } k \text{ is odd}. \end{cases}$$

By Equality (6), $\binom{\ell - 2x}{k - 2\left\lfloor \frac{k}{2} \right\rfloor} = 0$ if and only if $\ell - 2x < k - 2\left\lfloor \frac{k}{2} \right\rfloor$, that is $\ell - 2x = 0$ and $k - 2\left\lfloor \frac{k}{2} \right\rfloor = 1$, which means that $\ell = 2x$ and k is odd. So $\binom{\ell - 2x}{k - 2\left\lfloor \frac{k}{2} \right\rfloor} > 0$ and

$$C(k,x) \geq \binom{x}{\left\lfloor \frac{k}{2} \right\rfloor}^2 \binom{\ell - 2x}{k - 2\left\lfloor \frac{k}{2} \right\rfloor} \geq 1.$$

Next assume that $1 \leq x \leq \left\lfloor \frac{k}{2} \right\rfloor - 1$, then $k - 2x > 0$. Recall that $\ell \geq k + 1$, so $\ell - 2x > k - 2x$. Hence,

$$C(k,x) \geq \binom{x}{x}^2 \binom{\ell - 2x}{k - 2x} \geq 1.$$

This completes the proof of Lemma 5.2. □

Lemma 5.2 and Inequality (7) imply that when $1 \leq x \leq \left\lfloor \frac{\ell}{2} \right\rfloor$, $F(x,y) \leq \frac{1}{2}\binom{\ell}{k}$, and the equality holds if and only if $\ell = 2x$ and k is odd.

Case 2. $x \geq \left\lceil \frac{\ell}{2} \right\rceil$.

In this case, $y \geq 0 \geq \ell - 2x$. By Lemma 5.1(1), $F(x,y) \leq F(x,0)$.

Note that in this case, $b = y = 0$ and $2x + y + k + 1 - \ell \geq k + 1$, so $2a + b = 2a \geq 2x + y + k + 1 - \ell = 2x + k + 1 - \ell$. For brevity, let $p(x) = x + \left\lceil \frac{k+1-\ell}{2} \right\rceil$. Then $a \geq p(x)$ and

$$F(x,0) = \sum_{i=p(x)}^{k} \binom{x}{i}\binom{\ell - x}{k - i}. \tag{8}$$

Lemma 5.3. *$F(x,y) \le F\left(\left\lceil \frac{\ell}{2}\right\rceil, 0\right)$ whenever $x \ge \left\lceil \frac{\ell}{2}\right\rceil$.*

Proof. We first prove that $F(x,0) \ge F(x+1,0)$. Let $\Delta = F(x,0) - F(x+1,0)$, then

$$\Delta = \sum_{i=p(x)}^{k} \binom{x}{i}\binom{\ell - x}{k-i} - \sum_{i=p(x+1)}^{k} \binom{x+1}{i}\binom{\ell-1-x}{k-i}. \tag{9}$$

Note that $p(x+1) = p(x)+1$. Using equalities $\binom{x+1}{i} = \binom{x}{i} + \binom{x}{i-1}$ and $\binom{\ell-x}{k-i} = \binom{\ell-x-1}{k-i} + \binom{\ell-x-1}{k-i-1}$, and cancelling the term $\sum_{j=p(x)+1}^{k-1}\binom{x}{i}\binom{\ell-x-1}{k-i}$, we have

$$\Delta = \binom{x}{p(x)}\binom{\ell-x}{k-p(x)} + \sum_{i=p(x)+1}^{k}\binom{x}{i}\binom{\ell-1-x}{k-1-i} - \sum_{i=p(x)+1}^{k}\binom{x}{i-1}\binom{\ell-1-x}{k-i}.$$

When $i=k$ in the first sum above, we have $\binom{\ell-1-x}{-1} = 0$. Writing the last sum in the equality above as $\sum_{i=p(x)}^{k-1}\binom{x}{i}\binom{\ell-1-x}{k-1-i}$, we have

$$\Delta = \binom{x}{p(x)}\binom{\ell-x}{k-p(x)} - \binom{x}{p(x)}\binom{\ell-1-x}{k-p(x)-1} = \binom{x}{p(x)}\binom{\ell-1-x}{k-p(x)} \ge 0.$$

So, $F(x,y) \le F\left(\left\lceil \frac{\ell}{2}\right\rceil, 0\right)$. ☐

If ℓ is even, then by Lemma 5.1 and the case that $x \le \left\lfloor \frac{\ell}{2}\right\rfloor$, we have $F\left(\left\lceil \frac{\ell}{2}\right\rceil, 0\right) = F\left(\left\lfloor \frac{\ell}{2}\right\rfloor, \ell - 2\left\lfloor \frac{\ell}{2}\right\rfloor\right) \le \frac{1}{2}\binom{\ell}{k}$, the equality holds if and only if $\ell = 2x$ and k is odd.

In the rest of the proof, we assume that ℓ is odd, and we prove that $F\left(\left\lceil \frac{\ell}{2}\right\rceil, 0\right) \le F\left(\left\lfloor \frac{\ell}{2}\right\rfloor, 1\right)$, that is, $F\left(\frac{\ell+1}{2}, 0\right) \le F\left(\frac{\ell-1}{2}, 1\right)$, which implies that Lemma 2.9 holds when $x \ge \left\lceil \frac{\ell}{2}\right\rceil$, as by the first case, $F\left(\frac{\ell-1}{2}, 1\right) < \frac{1}{2}\binom{\ell}{k}$.

Note that in $F(\frac{\ell-1}{2}, 1)$, the summation is over $b = \in 0, 1$. Using $\binom{\frac{\ell-1}{2}}{-1} = 0$ and writing $\sum_{i=\lceil\frac{k}{2}\rceil}^{k-1}\binom{\frac{\ell-1}{2}}{i}\binom{\frac{\ell-1}{2}}{k-1-i}$ as $\sum_{i=\lceil\frac{k}{2}\rceil+1}^{k}\binom{\frac{\ell-1}{2}}{i-1}\binom{\frac{\ell-1}{2}}{k-i}$, we have

$$
\begin{aligned}
F(\frac{\ell-1}{2},1) &= \sum_{i=\left\lceil\frac{k}{2}\right\rceil}^{k}\binom{\frac{\ell-1}{2}}{i}\binom{1}{1}\binom{\frac{\ell-1}{2}}{k-1-i}+\sum_{i=\left\lceil\frac{k+1}{2}\right\rceil}^{k}\binom{\frac{\ell-1}{2}}{i}\binom{1}{0}\binom{\frac{\ell-1}{2}}{k-i}\\
&= \sum_{i=\left\lceil\frac{k}{2}\right\rceil+1}^{k}\binom{\frac{\ell-1}{2}}{i-1}\binom{\frac{\ell-1}{2}}{k-i}+\sum_{i=\left\lceil\frac{k+1}{2}\right\rceil}^{k}\binom{\frac{\ell-1}{2}}{i}\binom{\frac{\ell-1}{2}}{k-i}\\
&\geq \sum_{i=\left\lceil\frac{k}{2}\right\rceil+1}^{k}\binom{\frac{\ell-1}{2}}{i-1}\binom{\frac{\ell-1}{2}}{k-i}+\sum_{i=\left\lceil\frac{k}{2}\right\rceil+1}^{k}\binom{\frac{\ell-1}{2}}{i}\binom{\frac{\ell-1}{2}}{k-i}.
\end{aligned}
$$

On the other hand, by Equality (8), $F(\frac{\ell+1}{2},0)=\sum_{i=\left\lceil\frac{k}{2}\right\rceil+1}^{k}\binom{\frac{\ell+1}{2}}{i}\binom{\frac{\ell-1}{2}}{k-i}$. Therefore, using equalities $\binom{\frac{\ell+1}{2}}{i}=\binom{\frac{\ell-1}{2}}{i}+\binom{\frac{\ell-1}{2}}{i-1}$, we have

$$
\begin{aligned}
F\left(\frac{\ell-1}{2},1\right)-F\left(\frac{\ell+1}{2},0\right) \geq{}& \sum_{i=\left\lceil\frac{k}{2}\right\rceil+1}^{k}\binom{\frac{\ell-1}{2}}{i-1}\binom{\frac{\ell-1}{2}}{k-i}\\
&+\left(\sum_{i=\left\lceil\frac{k}{2}\right\rceil+1}^{k}\binom{\frac{\ell-1}{2}}{i}\binom{\frac{\ell-1}{2}}{k-i}-\sum_{i=\left\lceil\frac{k}{2}\right\rceil+1}^{k}\binom{\frac{\ell+1}{2}}{i}\binom{\frac{\ell-1}{2}}{k-i}\right)=0.
\end{aligned}
$$

This completes the proof of Lemma 2.9.

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DATA AVAILABILITY STATEMENT

Data sharing is not applicable to this article as no new data were created or analysed in this study.

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