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# THE STRONG FRACTIONAL CHOICE NUMBER AND THE STRONG FRACTIONAL PAINT NUMBER OF GRAPHS*

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**Abstract.** This paper studies the strong fractional choice number $ch^s_f(G)$ and the strong fractional paint number $pt^s_f(G)$ of a graph $G$. We prove that these parameters of any finite graph are rational numbers. On the other hand, for any positive integers $p,q$ satisfying $2 \le \frac{2p}{2q+1} \le \lfloor \frac{p}{q} \rfloor$, we construct a graph $G$ with $ch^s_f(G) = pt^s_f(G) = \frac{p}{q}$. The relationship between $pt^s_f(G)$ and $ch^s_f(G)$ is explored. We prove that the gap $pt^s_f(G) - ch^s_f(G)$ can be arbitrarily large. The strong fractional choice number of a family $\mathcal{G}$ of graphs is the supremum of the strong fractional choice numbers of graphs in $\mathcal{G}$. Let $\mathcal{P}$ denote the class of planar graphs and $\mathcal{P}_{k_1,\ldots,k_q}$ denote the class of planar graphs without $k_i$-cycles for $i = 1,\ldots,q$. We prove that $3 + \frac{1}{2} \le ch^s_f(\mathcal{P}_4) \le 4$, $ch^s_f(\mathcal{P}_k) = 4$ for $k \in \{5,6\}$, $3 + \frac{1}{12} \le ch^s_f(\mathcal{P}_{4,5}) \le 4$, and $ch^s_f(\mathcal{P}) \ge 4 + \frac{1}{3}$. The last result improves the lower bound $4 + \frac{2}{9}$ in [Zhu, *J. Combin. Theory Ser. B*, 122 (2017), pp. 794–799].

**Key words.** list coloring, strong fractional choice number, painting game

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## 1. Introduction.

Suppose $G$ is a graph, $f$ and $g$ are two functions from $V(G)$ to $\mathbb{N}$, with $f(v) \ge g(v)$ for every $v \in V(G)$. An *$f$-list assignment* of $G$ is a mapping $L$ which assigns to each vertex $v$ of $G$ a set $L(v)$ of $f(v)$ integers as *permissible colors*. A *$g$-fold coloring* of $G$ is a mapping $S$ which assigns to each vertex $v$ of $G$ a set $S(v)$ of $g(v)$ colors such that for any two adjacent vertices $u$ and $v$, $S(u) \cap S(v) = \varnothing$.

Given a list assignment $L$ of $G$, an $(L,g)$-coloring of $G$ is a $g$-fold coloring $S$ of $G$ such that for each $v$, $S(v) \subseteq L(v)$. We say that $G$ is *$(L,g)$-colorable* if there exists an $(L,g)$-coloring of $G$. For a positive integer $a$, we write $f \equiv a$ if $f$ is the constant function with $f(v) = a$ for every vertex $v$. If $g \equiv b$, then $(L,g)$-colorable is denoted by $(L,b)$-colorable. If $L(v) = \{1,2,\ldots,a\}$ for each $v \in V(G)$, then $(L,b)$-colorable is called $(a,b)$-colorable. The *$b$-fold chromatic number* $\chi_b(G)$ of $G$ is the least $k$ such that $G$ is $(k,b)$-colorable. The 1-fold chromatic number of $G$ is also called the chromatic number of $G$ and denoted by $\chi(G)$. The *fractional chromatic number* of $G$ is defined as $\chi_f(G) = \inf\{\frac{a}{b} : G \text{ is } (a,b)\text{-colorable}\}$.

Similarly, we say $G$ is *$(f,g)$-choosable* if for every $f$-list assignment $L$, $G$ is $(L,g)$-colorable.

- If $f \equiv a$ and $g \equiv b$, then $(f,g)$-choosable is called $(a,b)$-choosable.
- If $b = 1$, then $(f,1)$-choosable is called $f$-choosable.
- $(a,1)$-choosable is also called $a$-choosable.

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The *choice number* $ch(G)$ of $G$ is the minimum $k$ such that $G$ is $k$-choosable. The $b$-*fold choice number* $ch_b(G)$ of $G$ is the minimum $k$ such that $G$ is $(k,b)$-choosable. The *fractional choice number* of $G$ is defined as $ch_f(G) = \inf\{\frac{a}{b} : G \text{ is } (a,b)\text{-choosable}\}$.

List coloring of graphs was introduced in the 1970s by Vizing [24] and independently by Erdős, Rubin, and Taylor [8]. The subject offers a large number of challenging problems and has attracted an increasing attention since 1990. Readers are referred to [21] for a comprehensive survey on results and open problems.

The paint number of a graph is a variation of the choice number of a graph. Given two functions $f$ and $g$ from $V(G)$ to $N$, with $f(v) \geq g(v)$ for all $v \in V(G)$, the $(f,g)$-*painting game on* $G$ is played by two players: Lister and Painter. Initially, each vertex $v$ is given $f(v)$ tokens and is uncolored. On the $i$th round, Lister selects a set $U$ of vertices and removes one token from each selected vertex. Painter chooses an independent subset $X$ of $U$ and all the vertices in $X$ receive color $i$. If at the end of some round, there is a vertex $v$ with no tokens left and has received less than $g(v)$ colors, then the game ends and Lister wins the game. If at the end of some round, each vertex $v$ is colored with $g(v)$ colors, then the game ends and Painter wins the game. We say $G$ is $(f,g)$-*paintable* if Painter has a winning strategy for the $(f,g)$-painting game.

- If $f \equiv a$ and $g \equiv b$, then $(f,g)$-paintable is called $(a,b)$-paintable.
- If $b = 1$, then $(f,b)$-paintable is called $f$-paintable.
- $(a,1)$-paintable is also called $a$-paintable.

The $b$-*fold paint number* $pt_b(G)$ is the minimum $k$ such that $G$ is $(k,b)$-paintable, and $pt_1(G)$ is called the *paint number* (or the *online choice number*) of $G$ and is denoted by $pt(G)$. The *fractional paint number* of $G$ is defined as $pt_f(G) = \inf\{\frac{a}{b} : G \text{ is } (a,b)\text{-paintable}\}$.

It follows from the definition that for any graph $G$, $\chi_f(G) \leq ch_f(G) \leq pt_f(G)$. It was proved in [1] that $\chi_f(G) = ch_f(G)$ for every graph $G$ and proved in [10] that $\chi_f(G) = pt_f(G)$ for every graph $G$. So the fractional chromatic number, the fractional choice number, and the fractional paint number of a graph are the same invariant. As a variation of the fractional choice number, the concept of strong fractional choice number of a graph was introduced in [29].

DEFINITION 1.1. *Assume $G$ is a graph and $r$ is a real number. We say $G$ is* strongly fractional $r$-choosable *(respectively,* strongly fractional $r$-paintable *or* strongly fractional $r$-colorable *) if $G$ is $(a,b)$-choosable (respectively, $(a,b)$-paintable or $(a,b)$-colorable) for any $(a,b)$ for which $\frac{a}{b} \geq r$. The* strong fractional choice number *of $G$ is defined as*

$$ch_f^s(G) = \inf\{r \in \mathbf{R} : G \textit{ is strongly fractional } r\textit{-choosable}\}.$$

*The* strong fractional paint number *of $G$ is defined as*

$$pt_f^s(G) = \inf\{r \in \mathbf{R} : G \textit{ is strongly fractional } r\textit{-paintable}\}.$$

*We also define the* strong fractional chromatic number *of $G$ as*

$$\chi_f^s(G) = \inf\{r \in \mathbf{R} : G \textit{ is strongly fractional } r\textit{-colorable}\}.$$

*The strong fractional choice number, the strong fractional paint number, and the strong fractional chromatic number of a class $\mathcal{G}$ of graphs is defined as*

$$ch_f^s(\mathcal{G}) = \sup\{ch_f^s(G) : G \in \mathcal{G}\},\ pt_f^s(\mathcal{G}) = \sup\{pt_f^s(G) : G \in \mathcal{G}\},\ \chi_f^s(\mathcal{G}) = \sup\{\chi_f^s(G) : G \in \mathcal{G}\}.$$

This paper studies basic properties of these parameters and upper and lower bounds for these parameters for special families of graphs. In section 2, we show that both $ch^s_f(G)$ and $pt^s_f(G)$ are rational numbers. In section 3, we study the problem that which rational numbers are the strong fractional choice number and strong fractional paint number of graphs. We conjecture that, for every rational $r \geq 2$, there exists a graph $G$ with $ch^s_f(G) = r$ and a graph $G$ with $pt^s_f(G) = r$ and prove that for any positive integers $p, q$, where $p \geq 2q$ satisfying $\frac{2p}{2q+1} \leq \lfloor \frac{p}{q} \rfloor$, there exists a graph $G$ with $ch^s_f(G) = pt^s_f(G) = \frac{p}{q}$. In section 4, we show that the gap $pt^s_f(G) - ch^s_f(G)$ can be arbitrarily large. In section 5, we study upper and lower bounds for the strong fractional choice number of planar graphs. Let $\mathcal{P}$ denote the family of planar graphs and for positive integers $k_1, k_2, \ldots, k_q$, and let $\mathcal{P}_{k_1,\ldots,k_q}$ be the family of planar graphs without cycles of lengths $k_i$ for $i = 1, \ldots, q$. It was proved in [29] that $ch^s_f(\mathcal{P}) \geq 4+\frac{2}{9}$. We improve this result and prove that $ch^s_f(\mathcal{P}) \geq 4+\frac{1}{3}$. We also prove that $3 + \frac{1}{2} \leq ch^s_f(\mathcal{P}_4) \leq 4$, $ch^s_f(\mathcal{P}_k) = 4$ for $k \in \{5, 6\}$, and $3 + \frac{1}{12} \leq ch^s_f(\mathcal{P}_{4,5}) \leq 4$. Some open problems are posed in section 6.

## 2. Basic properties.

Lemma 2.1 gives an alternative definitions of $ch^s_f(G)$ and $pt^s_f(G)$.

LEMMA 2.1. *For any graph $G$,*

$$ch^s_f(G) = \sup\left\{\frac{ch_k(G)-1}{k} : k \in \mathbf{N}\right\},\ pt^s_f(G) = \sup\left\{\frac{pt_k(G)-1}{k} : k \in \mathbf{N}\right\}.$$

*Proof.* Let $r = \sup\{\frac{ch_k(G)-1}{k} : k \in \mathbf{N}\}$. Then for any $\epsilon > 0$, there is an integer $k$ such that $(r-\epsilon)k < ch_k(G) - 1$. Thus $\lceil (r-\epsilon)k \rceil < ch_k(G)$ and $G$ is not $(\lceil (r-\epsilon)k \rceil, k)$-choosable. Therefore, $ch^s_f(G) \geq r - \epsilon$ for any $\epsilon > 0$, which implies that $ch^s_f(G) \geq r$. On the other hand, for any $\epsilon > 0$, for any integer $k$, $\lceil (r+\epsilon)k \rceil \geq ch_k(G)$. Hence $G$ is $(\lceil (r+\epsilon)k \rceil, k)$-choosable. So $ch^s_f(G) \leq r + \epsilon$ for any $\epsilon > 0$, which implies that $ch^s_f(G) \leq r$. Therefore $ch^s_f(G) = r$. The other part of Lemma 2.1 is proved similarly. ∎

The following lemma was proved in [10]. For the completeness of this paper, we present a short proof.

LEMMA 2.2. *Assume $G$ is a finite graph. Then for any $\epsilon > 0$, there is a constant $k_0$ such that for any $k \geq k_0$, $\frac{ch_k(G)}{k} \leq \frac{pt_k(G)}{k} \leq \chi_f(G) + \epsilon$.*

*Proof.* Assume $\chi_f(G) = a/b$ and $\phi$ is a $b$-fold coloring of $G$ using colors $\{1, 2, \ldots, a\}$ ($a, b$ need not be coprime). Assume $k > \frac{a2^{|V(G)|}}{\epsilon}$, and let $m = k(\frac{a}{b} + \epsilon)$ (for simplicity, we may choose $\epsilon$ so that $k(\frac{a}{b} + \epsilon)$ is an integer). It suffices to show that Painter has a winning strategy for the $(m, k)$-painting game on $G$.

For $i = 1, 2, \ldots$, assume the set chosen by Lister at the $i$th round is $U_i$. Let

$$t_i = |\{j \leq i : U_j = U_i\}|,$$

and let $\tau_i \in \{1, 2, \ldots, a\}$ be the unique integer for which $\tau_i \cong t_i \mod a$. Let $\phi^{-1}(\tau_i) = \{v \in V(G) : \tau_i \in \phi(v)\}$. Painter's strategy is to color all the vertices in the set $X_i = \phi^{-1}(\tau_i) \cap U_i$ in the $i$th round. As $\phi^{-1}(\tau_i)$ is an independent set, $X_i$ is an independent set.

We shall show that this is a winning strategy for Painter; i.e., when the game ends, every vertex will be colored by at least $k$ colors.

Assume to the contrary that at the end of some round, say, at the end of the $i$th round, a vertex $v$ has no token left (hence $v$ has been chosen $m = k(\frac{a}{b} + \epsilon)$ times by Lister) and is colored in $k(v) < k$ rounds.

For each subset $U$ of $V(G)$ and for each $t \in \{1, 2, \ldots, a\}$, let

$$(U, t) = \{j \le i : U_j = U, \tau_j = t\}.$$

By the strategy, for each $j \le i$, if $j \in (U, t)$, $v \in U$, and $t \in \phi(v)$, then $v$ is colored in round $j$. Therefore,

$$k(v) = \sum_{v \in U, t \in \phi(v)} |(U, t)|.$$

For each subset $U$ of $V(G)$, let $t_U = |\{j \le i : U_j = U\}|$. It follows from the choice of color $\tau_j$ that for any color $t$, either $|(U, t)| = \lfloor \frac{t_U}{a} \rfloor$ or $|(U, t)| = \lceil \frac{t_U}{a} \rceil$. Therefore

$$|(U, t)| \ge \frac{t_U}{a} - 1.$$

Note that $\sum_{v \in U} t_U = m$ is the total number of times vertex $v$ is chosen by Lister. Since $\phi(v)$ is a $b$-subset of $\{1, 2, \ldots, a\}$, we conclude that

$$k(v) = \sum_{v \in U, t \in \phi(v)} |(U, t)| \ge b \sum_{v \in U} \left( \frac{t_U}{a} - 1 \right) \ge \frac{bm}{a} - b2^{|V(G)|} = \frac{k(a/b + \epsilon)b}{a} - b2^{|V(G)|} \ge k,$$

which is a contradiction. ◻

THEOREM 2.3. *For any finite graph $G$, $ch_f^s(G)$ and $pt_f^s(G)$ are rational numbers.*

*Proof.* If $\frac{pt_k(G)-1}{k} \le \chi_f(G)$ for every positive integer $k$, then it follows from Lemma 2.1 that $pt_f^s(G) \le \chi_f(G)$. Since $\chi_f(G) \le pt_f^s(G)$, we conclude that $pt_f^s(G) = \chi_f(G)$, which is a rational number.

Assume there is an integer $k_0$ such that $\frac{pt_{k_0}(G)-1}{k_0} > \chi_f(G)$. Let $\epsilon = \frac{pt_{k_0}(G)-1}{k_0} - \chi_f(G) > 0$. By Lemma 2.2, there is a constant $k_1 \ge k_0$ such that for $k \ge k_1$, $\frac{pt_k(G)}{k} \le \chi_f(G) + \epsilon$. Hence

$$\sup\{\frac{pt_k(G) - 1}{k} : k \in \mathbf{N}, k \ge k_1\} \le \frac{pt_{k_0}(G) - 1}{k_0}.$$

Therefore

$$pt_f^s(G) = \sup\{\frac{pt_k(G) - 1}{k} : k \in \mathbf{N}\} = \max\{\frac{pt_k(G) - 1}{k} : 1 \le k \le k_1\}$$

is a rational number. Moreover, the supremum in Lemma 2.1 is attained.

The part of the lemma concerning $ch_f^s(G)$ is proved similarly. ◻

Lemma 2.1 gives an alternate definition of $ch_f^s(G)$ and $pt_f^s(G)$. It follows from the proof of Theorem 2.3 that either $ch_f^s(G) = \chi_f(G)$ or the supremum in the definition $ch_f^s(G) = \sup\{\frac{ch_k(G)-1}{k} : k \in \mathbf{N}\}$ is attained. However, the infimum in the definition $ch_f^s(G) = \inf\{r : G \text{ is strongly fractional } r\text{-choosable}\}$ may be not attained even if $ch_f^s(G) \ne \chi_f(G)$. For example, Tuza and Voigt [23] showed that $K_{2,4}$ is $(4m, 2m)$-choosable for each positive integer $m$, hence $ch_f^s(K_{2,4}) = 2$, but it is not $(2m, m)$-choosable for any odd $m$, hence it is not strongly fractional 2-choosable. Similarly, either $pt_f^s(G) = \chi_f(G)$ or the sumpremum in the definition $pt_f^s(G) = \sup\{\frac{pt_k(G)-1}{k} : k \in \mathbf{N}\}$ is attained. But the infimum in the definition $pt_f^s(G) = \inf\{r : G \text{ is strongly fractional } r\text{-paintable}\}$ may be not attained even if $pt_f^s(G) \ne \chi_f(G)$.

**3. Constructing graphs with given $ch^s_f(G)$ and $pt^s_f(G)$.** By Theorem 2.3, for any finite graph $G$, $ch^s_f(G)$ and $\chi^s_{f,P}(G)$ are rational numbers. A natural question is whether every rational number $r \geq 2$ is the strong fractional choice (paint) number of a graph. We conjecture that the answer is yes. In this section, for some rational numbers $p/q$, we construct graphs $G$ with $pt^s_f(G) = ch^s_f(G) = p/q$.

Given a graph $G$, a subset $S$ of $G$ and two graphs $H_1$ and $H_2$, let $G[S : H_1, H_2]$ be the graph with

$$V(G[S : H_1, H_2]) = \{(u, v) : u \in S \text{ and } \ v \in V(H_1), \text{ or } u \in V(G) \smallsetminus S \text{ and } \ v \in V(H_2)\},$$
$$E(G[S : H_1, H_2]) = \{(u, v)(u', v') : uu' \in E(G), \ \text{ or } u = u' \in S, vv' \in E(H_1), \text{ or } u = u' \in V(G) \smallsetminus S, vv' \in E(H_2)\}.$$

Note that if $S = V(G)$ or $H_1 = H_2$, then $G[S : H_1, H_2] = G[H_1]$ is the *lexicographic product* of $G$ and $H_1$. In the rest of this section, we let $G_{n,m,k}$ denote the graph $C_{2k+1}[I : K_n, K_m]$, where $I$ is a maximum independent set of $C_{2k+1}$; see Figure 1 for the example of $G_{6,4,3}$.

THEOREM 3.1. *For any positive integer $n, m, k$ with $n \geq m$,*

$$ch^s_f(G_{n,m,k}) = pt^s_f(G_{n,m,k}) = \chi_f(G_{n,m,k}) = n + m + \frac{m}{k}.$$

*Proof.* Assume the vertices of $C_{2k+1}$ are $(v_0, v_1, \ldots v_{2k})$ in this cyclic order, and assume that $I = \{v_1, v_3, \ldots, v_{2k-1}\}$ and $G_{n,m,k} = C_{2k+1}[I : K_n, K_m]$.

For $s \in \{0, 1, \ldots, 2k\}$, let

$$V_s = \{(x, y) \in V(G_{n,m,k}) : x = v_s\}.$$

For any vertex set $S \subseteq V(G_{n,m,k})$, let

$$\partial(S) = \{s : S \cap V_s \neq \varnothing\}.$$

It is clear that $\alpha(G_{n,m,k}) = \alpha(C_{2k+1}) = k$. It is well known that $\chi_f(G) \geq \frac{|V(G)|}{\alpha(G)}$ for any graph $G$, so we have

$$\chi_f(G_{n,m,k}) \geq \frac{nk + m(k+1)}{k} = n + m + \frac{m}{k}.$$

Since $\chi_f(G) \leq ch^s_f(G) \leq pt^s_f(G)$ for any graph $G$, it suffices to show that $pt^s_f(G_{n,m,k}) \leq n + m + \frac{m}{k}$. For this purpose, we will show that for any $\frac{a}{b} \geq n + m + \frac{m}{k}$,

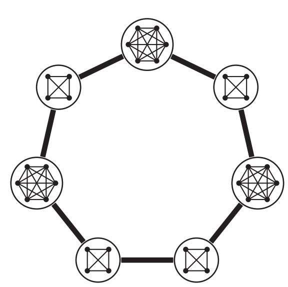


FIG. 1. $G_{6,4,3}$.

$G_{n,m,k}$ is $(a,b)$-paintable. In the following, we present a winning strategy for Painter in the $(a,b)$-painting game on $G_{n,m,k}$.

For simplicity, we assume that if a vertex $v$ has been colored with $b$ colors, then Lister will not choose $v$ in later moves.

For $i = 1, 2, \ldots$, we denote by $U_i$ the set of vertices chosen by Lister, and by $X_i$ the independent set contained in $U_i$, colored by Painter at the $i$th round.

Assume Lister has chosen $U_i$. We describe a strategy for Painter to choose the independent set $X_i$.

We consider two cases.

*Case* 1. $\partial(U_i) = \{0, 1, 2, \ldots, 2k\}$.

Let $t_i = |\{j : j \le i \text{ and } \partial(U_j) = \{0, 1, \ldots, 2k\}\}|$. Let $\tau_i$ be the unique integer in $\{0, 1, \ldots, 2k\}$ which is congruent to $t_i$ modulo $2k+1$. Then $X_i$ is any independent set contained in $U_i$ with $\partial(X_i) = \{\tau_i, \tau_i + 2, \ldots, \tau_i + 2k - 2\}$ (the summations are modulo $2k+1$).

*Case* 2. $\partial(U_i) \ne \{0, 1, \ldots, 2k\}$.

Painter traverses the sets $V_0, V_1, \ldots, V_{2k}$ of $G_{n,m,k}$ one by one in cyclic order along the clockwise direction, starting at an arbitrary set $V_s$ for which $s \notin \partial(U_i)$, and chooses an independent set $X_i$ as follows: Initially, $X_i = \varnothing$ and vertices will be added to $X_i$ in the process. When Painter traverse to $V_j$, if $U_i \cap V_j \ne \varnothing$ and $X_i \cap V_{j-1} = \varnothing$, then add a vertex from $V_j \cap U_i$ to $X_i$. Otherwise, do not add any vertex from $V_j$ into $X_i$ (again the calculation in the indices are modulo $2k+1$).

It follows from the definition that in both cases, the set $X_i$ is an independent set of $G$ contained in $U_i$. We shall show that this is a winning strategy for Painter. First we have the following claim.

CLAIM 3.1. *Case* 1 *happens at most* $\frac{mb(2k+1)}{k}$ *times.*

*Proof.* Every $2k+1$ times Case 1 happens, $k$ vertices (not necessarily distinct) in $V_0$ will be colored. However, all the vertices from $V_0$ need to be colored $mb$ times in total. Thus the claim holds. ∎

Assume to the contrary that at the end of some round, say, at the end of the $r$th round, a vertex $v \in V_j$ has no token left and is colored at most $b-1$ times.

Let $B_r(v) = \{i \le r : v \in U_r - X_r\}$, which is the collection of rounds of the game that $v$ is chosen by Lister but not colored by Painter at the end of $r$th round. Then $|B_r(v)| \ge a - b + 1$. It follows from the strategy that for each $i \in B_r(v)$, one of the following holds:

- $V_{j-1} \cap X_i \ne \varnothing$, i.e., some vertex in $V_{j-1}$ is colored in this round.
- $\partial(U_i) = \{0, 1, \ldots, 2k\}$ and $\tau_i = j+1$.
- $V_j \cap X_i \ne \varnothing$ and $v \notin X_i$, i.e., some other vertex from $V_j$ is colored at this round.

Since $|V_{j-1} \cup (V_j \smallsetminus v)| \le m + n - 1$ and each vertex in $V_{j-1} \cup (V_j \smallsetminus v)$ is colored at most $b$ times, we conclude that

$$\begin{aligned} |\{i \le r : \partial(U_i) = \{0, 1, \ldots, 2k\}, \tau_i = j+1| &\ge a - b + 1 - (m+n-1)b \\ &= a - (m+n)b + 1. \end{aligned} \tag{3.1}$$

It follows from the definition of $\tau_i$ that

$$|\{i \le r : \partial(U_i) = \{0, 1, \ldots, 2k\}, \tau_i = j+1| \le \left\lceil \frac{t_i}{2k+1} \right\rceil. \tag{3.2}$$

By Claim 3.1, $t_i \le \frac{(2k+1)mb}{k}$. Hence

$$\left\lceil \frac{t_i}{2k+1} \right\rceil \le \left\lceil \frac{(2k+1)mb}{k(2k+1)} \right\rceil = \left\lceil \frac{mb}{k} \right\rceil,$$

Combining the inequality above with inequalities (3.1) and (3.2), we have

$$\frac{mb}{k} > a - (m+n)b,$$

that is,

$$\frac{a}{b} < m + n + \frac{m}{k},$$

which is contrary to our assumption. ∎

By setting $m = n = 1$ in Theorem 3.1, we have the following.

COROLLARY 3.2. $\chi_f(C_{2q+1}) = ch_f^s(C_{2q+1}) = pt_f^s(C_{2q+1}) = 2 + \frac{1}{q}$.

PROPOSITION 3.3. *For any positive integers $p, q$ with $p \ge 2q$ and $\frac{2p}{2q+1} \le \lfloor \frac{p}{q} \rfloor$, there exists a graph $G$ such that*

$$ch_f^s(G) = pt_f^s(G) = \chi_f(G) = \frac{p}{q}.$$

*Proof.* Let $a = \lfloor \frac{p}{q} \rfloor (q+1) - p$ and $b = p - q\lfloor \frac{p}{q} \rfloor$. As $p \ge 2q$, we first have that $a, b > 0$. On the other hand, note that by the condition $\frac{2p}{2q+1} \le \lfloor \frac{p}{q} \rfloor$, we have that

$$a - b = (2q+1)\lfloor \frac{p}{q} \rfloor - 2p \ge 0.$$

So by setting $a = n$ and $b = m$ in Theorem 3.1, we have that

$$ch_f^s(G_{a,b,q}) = pt_f^s(G_{a,b,q}) = a + b + \frac{b}{q} = \frac{p}{q}.$$

This completes the proof. ∎

According to Proposition 3.3, if $q \le 2$, then for any $p \ge 2q$, there exists a graph $G$ with $ch_f^s(G) = pt_f^s(G) = p/q$. If $q = 3$, the only cases $p \ge 2q$ unknown are $p = 8$ and $p = 11$.

**4. Relation among $\chi_f^s(G)$, $ch_f^s(G)$ and $pt_f^s(G)$.** It follows from the definitions that for any graph $G$,

$$\chi_f^s(G) \le ch_f^s(G) \le pt_f^s(G).$$

The gap $ch_f^s(G) - \chi_f^s(G)$ can be arbitrarily large; as for complete bipartite graphs, we have $\chi_f^s(K_{n,n}) = 2$ and $ch_f^s(K_{n,n}) \ge ch(K_{n,n}) - 1 \ge \log_2 n - (2 + o(1)) \log_2 \log_2 n$.

In the following we show that the difference $pt_f^s(K_{n,n}) - ch_f^s(K_{n,n})$ also goes to infinity with $n$. It was proved in [4] that for $n \ge 2^{k+3}$, the graph $K_{n,n}$ is not $k$-paintable, so $pt(K_{n,n}) \ge \log_2 n - 4$. Therefore, $pt_f^s(K_{n,n}) \ge pt(K_{n,n}) - 1 \ge \log_2 n - 5$. So it suffices to show that $\log_2 n - ch_f^s(K_{n,n})$ goes to infinity with $n$.

A *$k$-uniform hypergraph* $H = (V, E)$ consists of a vertex set $V$ and an edge set $E$, where each edge $e \in E$ is a $k$-subset of $V$. A proper $c$-coloring of $H$ is a mapping $\phi : V \mapsto \{1, 2, \ldots, c\}$ such that no edge is monochromatic. Hypergraph 2-colorability, which is an alternate formulation of list coloring of complete bipartite graphs, is a central problem in combinatorics that has been studied in many papers (see [2, 7, 17], etc.) Corresponding to $b$-fold list coloring of complete bipartite graphs, we define a *$b$-proper 2-coloring* of a hypergraph $H$ as a mapping $\phi : V(H) \to \{1,2\}$ such that for each edge $e$ of $H$, for each $i \in \{1,2\}$, $|\phi^{-1}(i) \cap e| \geq b$; i.e., each edge contains at least $b$ vertices of each color. We say that $H$ *is $b$-proper 2-colorable* if $H$ has a $b$-proper 2-coloring. Let $m(k,b)$ denote the minimum possible number of edges of a $kb$-uniform hypergraph which is not $b$-proper 2-colorable.

LEMMA 4.1. *Every $p$-uniform hypergraph with $m$ edges satisfying $m\sum_{i=0}^{b-1}\binom{p}{i}\frac{1}{2^{p-1}} < 1$ has a $b$-proper 2-coloring. As a result,*

$$m(k,b) \geq \left(\sum_{i=0}^{b-1}\binom{ks}{i}\right)^{-1} 2^{kb-1}.$$

*Proof.* Let $H = (V,E)$ be a $p$-uniform hypergraph satisfying the condition.

Color the vertices of $H$ randomly by two colors with equal probability. We say an edge $e$ is *bad* if one color is used on less than $b$ vertices in $e$. For each edge $e$, let $A_e$ be the event that $e$ is bad. Then

$$\Pr(A_e) = 2\sum_{i=0}^{b-1}\binom{p}{i}\frac{1}{2^p} = \sum_{i=0}^{b-1}\binom{p}{i}\frac{1}{2^{p-1}}.$$

Therefore,

$$\Pr(\bigvee_{e\in E} A_e) \leq \sum_{e\in E}\Pr(A_e) = m\Pr(A_e) < 1.$$

So there exists a coloring such that there is no bad edge. ☐

LEMMA 4.2. *Let $G$ be a bipartite graph with $n$ vertices. When $n$ is big enough, the $b$-fold choice number $ch_b(G)$ satisfies the following:*

$$\frac{ch_b(G)}{b} < \frac{1}{b}\log_2 n + \left(1 - \frac{1}{b}\right)\log_2\log_2 n + O(1).$$

*Proof.* Let $k = \frac{1}{b}\log_2 n + (1-\frac{1}{b})\log_2\log_2 n + C$; for some constant $C$, we shall prove that $G$ is $(kb,b)$-choosable for any $b \geq 2$. For convenience, let $t = \log_2\log_2 n$.

Assume $G = (X \cup Y, E)$ is a bipartite graph with $X$ and $Y$ being the two parts, and $L$ is a list assignment of $G$ with $|L(v)| = kb$ for each $v \in V(G)$. We construct a $kb$-uniform hypergraph $H$ with $V(H) = \bigcup_{v\in V(G)} L(v)$, and $E(H) = \{L(v) : v \in V(G)\}$. So $|E(H)| = |V(G)| = n$.

Observe that if $H$ has a $b$-proper 2-coloring, then $G$ is $(L,b)$-colorable. Indeed, each vertex is either labeled with red or blue in the $b$-proper 2-coloring of $H$. Then for each vertex $v \in V(G)$, we can choose $b$ colors with label red for $v$ if $v \in X$, and choose $b$ colors with label blue for it if $v \in Y$.

Now, it suffices to prove that $H$ satisfies the condition in Lemma 4.1, so we only need to verify that

$$n\sum_{i=0}^{b-1}\binom{kb}{i}\frac{1}{2^{kb-1}} < 1.$$

The case that $b = 1$ was proved in [7]. If $b = 2$, then we have $kb = \log_2 n + \log_2 \log_2 n + 2C$, so,

$$
\begin{aligned}
n \sum_{i=0}^{b-1} \binom{kb}{i} \frac{1}{2^{kb-1}} &= n(\log_2 n + \log_2 \log_2 n + 2C + 1) \frac{1}{2^{\log_2 n + \log_2 \log_2 n + 2C - 1}} \\
&= (\log_2 n + \log_2 \log_2 n + 2C + 1) \frac{1}{2^{\log_2 \log_2 n + 2C - 1}} \\
&= \frac{2^t + t + 2C + 1}{2^{t+2C-1}}.
\end{aligned}
$$

When $n$ is large enough and hence $t$ is large enough and $C \geq 1$, $n \sum_{i=0}^{b-1} \binom{kb}{i} \frac{1}{2^{kb-1}} < 1$.

Similarly, if $b = 3$, then we have $kb = \log_2 n + 2\log_2 \log_2 n + 3C$, so,

$$
\begin{aligned}
n \sum_{i=0}^{b-1} \binom{kb}{i} \frac{1}{2^{kb-1}} &= n(k^2 b^2 + kb + 2) \frac{1}{2^{kb}} \\
&= (\log_2 n + \log_2 \log_2 n + 2C + 1) \frac{1}{2^{2\log_2 \log_2 n + 3C}} \\
&= \frac{2^{2t} + o(2^t)}{2^{2t+3C}}.
\end{aligned}
$$

So when $n$ is large enough and hence $t$ is large enough, $n \sum_{i=0}^{b-1} \binom{kb}{i} \frac{1}{2^{kb-1}} < 1$.

Assume $b \geq 4$. Using the inequality $\binom{n}{k} < (\frac{en}{k})^k$, we have

$$
\begin{aligned}
n \sum_{i=0}^{b-1} \binom{kb}{i} \frac{1}{2^{kb-1}} &\leq nb \left( \frac{e(\log_2 n + (b-1)\log_2 \log_2 n + bC)}{b-1} \right)^{b-1} \frac{1}{2^{\log_2 n + (b-1)\log_2 \log_2 n + bC - 1}} \\
&= b \left( \frac{e(\log_2 n + (b-1)\log_2 \log_2 n + bC)}{b-1} \right)^{b-1} \frac{1}{2^{(b-1)\log_2 \log_2 n + bC - 1}} \\
&= \frac{b}{2^{bC-1}} \left( \frac{e(\log_2 n + (b-1)\log_2 \log_2 n + bC)}{(b-1)\log_2 n} \right)^{b-1}.
\end{aligned}
$$

Note that in this case, $b-1 > e$, so when $n$ is large enough, we have $n \sum_{i=0}^{b-1} \binom{kb}{i} \frac{1}{2^{kb-1}} < 1$. This finishes the proof of the lemma. ∎

In 2000, Radhakrishnan and Srinivasan [17] actually gave a better lower bound for $m(k,1)$, who showed that $m(k,1) = \Omega(2^k \sqrt{\frac{k}{\ln k}})$. This implies that $\frac{ch_1(G)}{1} = ch(G) \leq \log_2 n - (\frac{1}{2} - o(1)) \log_2 \log_2 n$ if $G$ is a complete bipartite graph with $n$ vertices. Combining this fact and Lemma 4.2 and Lemma 2.1, we have the following corollary.

COROLLARY 4.3. *Let $G$ be a bipartite graph with $n$ vertices. When $n$ is big enough,*

$$ch_f^s(G) \leq \log_2 n - \left(\frac{1}{2} - o(1)\right) \log_2 \log_2 n.$$

*Consequently, $pt_f^s(K_{n,n}) - ch_f^s(K_{n,n})$ can be arbitrarily large.*

Although the gaps $ch_f^s(G) - \chi_f^s(G)$ and $pt_f(G) - ch_f^s(G)$ can be arbitrarily large, there are also many graphs $G$ for which the equality $ch_f^s(G) = \chi_f^s(G)$ and/or $pt_f(G) = ch_f^s(G)$ hold. Recall that a graph $G$ is called *chromatic-choosable* if $\chi(G) = ch(G)$. The study of chromatic choosable graphs attracted a lot of attention. The well-known list coloring conjecture asserts that line graphs are chromatic-choosable. This conjecture remains largely open; however, it was shown by Galvin [9] the the line graphs of bipartite graphs are chromatic-choosable. This result extends to strong fractional choice number and strong fractional paint number.

THEOREM 4.4. *If $G = L(H)$ is the line graph of a bipartite graph $H$, then*

$$\chi_f^s(G) = ch_f^s(G) = pt_f^s(G) = \Delta(H).$$

*Proof.* It is well-known that $\chi(G) = \omega(G) = \Delta(H)$. Hence $\chi_f^s(G) \geq \Delta(H)$. It remains to show that $pt_f(G) \leq \Delta(H)$.

An orientation $D$ of $G$ is *kernel perfect* if any subset $U$ of $V(D)$ contains an independent set $X$ such that for any $v \in U - X$, $N_D^+(v) \cap X \neq \varnothing$. Here $N_D^+(v)$ is the set of out-neighbors of $v$. We set $N_D^+[v] = N_D^+(v) \cup \{v\}$. It was proved in [9] that $G$ has a kernel perfect orientation $D$ with $\Delta^+(D) = \Delta(H)$. On the other hand, for any kernel perfect orientation $D$ of $G$, for any $f, g : V(D) \to \mathbb{N}$, if $f(v) \geq \sum_{u \in N_D^+[v]} g(u)$ for every vertex $v$, then it is easy to show by induction on $\sum_{v \in V(D)} f(v)$ that Painter has a winning strategy in the $(f, g)$-painting game.

Indeed, if Lister chooses a subset $U$ of $V(G)$ in a round, then Painter chooses an independent set $X$ of $U$ for which $N_D^+(v) \cap X \neq \varnothing$ for all $v \in U - X$. Let

$$f'(v) = \begin{cases} f(v) - 1 & \text{if } v \in U - X, \\ f(v), & \text{otherwise,} \end{cases}$$

and

$$g'(v) = \begin{cases} g(v) - 1 & \text{if } v \in X, \\ g(v), & \text{otherwise.} \end{cases}$$

It follows from the definition that for any vertex $v$, we still have $f'(v) \geq \sum'_{u \in N_D^+[v]}$ $(u)$. By induction hypothesis, Painter has a winning strategy in the $(f', g')$-painting game on $G$. Therefore Painter has a winning strategy for the $(f, g)$-painting game on $G$.

For any positive integer $m$, by letting $f(v) = m(d_D^+(v) + 1) \leq \Delta(H)m$ and $g(v) = m$ for each vertex $v$, we have that $G$ is $(\Delta(H)m, m)$-paintable. Hence $pt_f^s(G) \leq \Delta(H)$. ∎

## 5. $ch_f^s(G)$ for planar graphs.

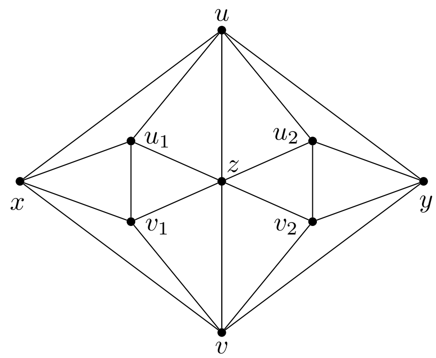
In this section, we study the strong fractional choice number of planar graphs. Let $\mathcal{P}$ denote the class of planar graphs and for integers $k_1, \ldots, k_q \geq 3$, let $\mathcal{P}_{k_1,\ldots,k_q}$ denote the class of planar graphs without $k_i$-cycles for $i = 1, \ldots, q$. For example, $\mathcal{P}_{3,4,5}$ denotes planar graphs with girth 6.

It was shown in [29] that $4 + \frac{2}{9} \leq ch_f^s(\mathcal{P}) \leq 5$. The following result improves the lower bound.

PROPOSITION 5.1. *For each positive integer $m$, there is a planar graph $G$ which is not $(4m + \lfloor \frac{m-1}{3} \rfloor, m)$-choosable. Consequently, $ch_f^s(\mathcal{P}) \geq 4 + \frac{1}{3}$.*

*Proof.* Let $T$ be the graph as shown in Figure 2, $\epsilon$ be a real number such that $\epsilon m = \lfloor \frac{m-1}{3} \rfloor$. Assume $A, B, C, D, E, F$ are pairwise disjoint sets of colors with $|A| = |B| = |C| = |D| = m$, $|E| = \epsilon m$ and $|F| = 2m$. For any disjoint sets $A$ and $B$, we define a list assignment $L_{A,B}$ (when $A, B$ are clear, and there is no confusion, we write $L$ in short) of $T$ as in Figure 2.

Now we show that there is no $m$-fold $L$-coloring of $T$. Suppose to the contrary, $\phi$ is an $m$-fold $L$-coloring of $T$. Then $\phi(u) = A$ and $\phi(v) = B$. Note that $u_1 v_1 x$ is a clique, so each color in $E \cup F$ can be used at most once in $u_1, v_1$ and $x$. As altogether, we use $3m$ distinct colors in these three vertices, at least $(1 - \epsilon)m$ colors in $C$ are




- $L(u) = A$ and $L(v) = B$.
- $L(x) = L(y) = A \cup B \cup F \cup E$.
- $L(u_1) = A \cup C \cup F \cup E$ and $L(v_1) = B \cup C \cup F \cup E$.
- $L(u_2) = A \cup D \cup F \cup E$. and $L(v_2) = B \cup D \cup F \cup E$.
- $L(z) = A \cup B \cup C \cup D \cup E$.

FIG. 2. *The gadget graph $T$ for $\mathcal{P}$ and the list assignment $L_{A,B}$.*

used on vertex $u_1$ and $v_1$, which implies that at most $\epsilon m$ colors in $C$ can be used at vertex $z$. By symmetry, at most $\epsilon m$ colors in $D$ can be used at vertex $z$. Recall that $|E| = \epsilon m$, so for the vertex $z$,

$$m = |\phi(z)| = |\phi(z) \cap C| + |\phi(z) \cap D| + |\phi(z) \cap E| \leq 3\epsilon m < m,$$

which is a contradiction.

Let $p = \binom{(4+\epsilon)m}{m}^2$. Let $G$ be obtained from the disjoint union of $p$ copies of $T$, by identifying all the copies of $u$ into a single vertex, also named $u$, and identifying all the copies of $v$ into a single vertex named $v$. Let $L$ be the $(4+\epsilon)m$-list assignment of $G$ defined as follows: Let $L(u) = X$ and $L(v) = Y$, where $X, Y$ are two disjoint set of size $(4+\epsilon)m$. For each pair of $m$-sets $(A, B)$, where $A \subseteq X$ and $B \subseteq Y$, we associate a copy of $T_{A,B}$ of $T$ so that the lists of the vertices of this copy of $T_{A,B}$ is as given above. Then $G$ is not $m$-fold $L$-colorable; otherwise, $u$ is colored with an $m$-subset $A$ of $X$, $v$ is colored with an $m$-subset $B$ of $Y$. However, by the argument above, $T_{A,B}$ has no $m$-fold $L$-coloring. ∎

Next we consider the family $\mathcal{P}_4$.

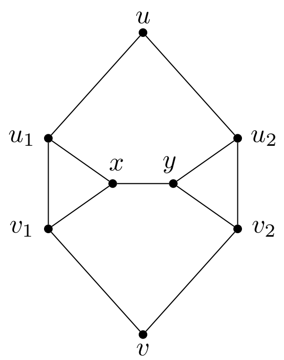
PROPOSITION 5.2. *For each positive integer $m$, there is a planar graph $G$ without 4-cycle, which is not $(3m + \lfloor \frac{m-1}{2} \rfloor, m)$-choosable. Consequently, $3 + \frac{1}{2} \leq ch^s_f(\mathcal{P}_4) \leq 4$.*

*Proof.* Let $T$ be labeled as shown in Figure 3, $\epsilon$ be a real number such that $\epsilon m = \lfloor \frac{m-1}{2} \rfloor$. Assume $A, B, C, D, E$ are pairwise disjoint sets of colors with $|A| = |B| = |D| = m$, $|C| = 2m$, and $|E| = \epsilon m$. For any disjoint sets $A$ and $B$, we define a list assignment $L_{A,B}$ of $T$ as in Figure 3.

By the same argument as in the proof of Theorem 2, it suffices to show that there is no $m$-fold $L$-coloring of $T$. Before showing this, we observe that the shortest path between $u, v$ in $T$ has length 3, so gluing the copies of $T$ in the construction of $G$ will not create 4-cycles. Suppose to the contrary $\phi$ is an $m$-fold $L$-coloring of $G$. Then $\phi(u) = A$ and $\phi(v) = B$. Note that $u_1v_1x$ is a clique, and we use $2m$ colors in $C \cup E$ on $u_1$ and $v_1$. Therefore, only $\epsilon m$ colors in $C \cup E$ can be used at $x$. By symmetry, only $\epsilon m$ colors in $C \cup E$ can be used at $y$. Note that $|D| = m$, so we have

$$2m = |\phi(x)| + |\phi(y)| \leq 2\epsilon m + |D| = 2\epsilon m + m < 2m,$$

which is a contradiction.




- $L(u) = A$ and $L(v) = B$.
- $L(u_1) = L(u_2) = A \cup C \cup E$.
- $L(v_1) = L(v_2) = B \cup C \cup E$.
- $L(x) = L(y) = C \cup D \cup E$.

FIG. 3. *The gadget graph $T$ for $\mathcal{P}_4$ and the list assignment $L_{A,B}$.*

It was proved in [12] that planar graphs without 4-cycles are $(4m, m)$-choosable for any positive integer $m$. So $ch_f^s(\mathcal{P}_4) \leq 4$. ∎

Observe that $K_4$ is a planar graph without $k$-cycle for any $k \geq 5$, and $ch_f^s(K_4) = 4$. On the other hand, it was shown in [12] that every graph without $k$-cycle is $(4m, m)$-choosable, where $k \in \{4, 5, 6\}$. We have the following.

OBSERVATION 5.3. *For any $k \geq 5$, $ch_f^s(\mathcal{P}_k) \geq 4$. In particular, $ch_f^s(\mathcal{P}_k) = 4$ when $k \in \{5, 6\}$.*

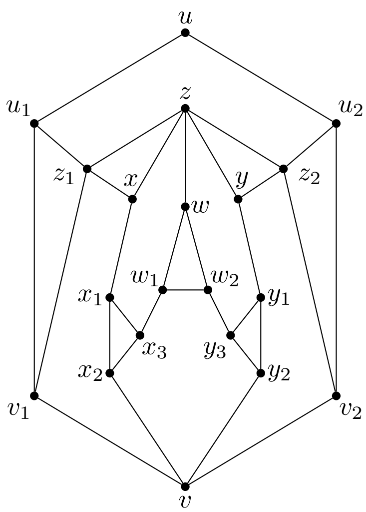
The construction in Proposition 5.1 does not contain $k$-cycle for $k \geq 17$, which means that $ch_f^s(\mathcal{P}_k) > 4 + \frac{1}{3}$ for $k \geq 17$. It remains an open question as what is the smallest $k$ such that $ch_f^s(\mathcal{P}_k) > 4$?

The family of planar graphs without 4-, 5-cycles has been studied extensively in the literature, because of the well-known Steinberg's conjecture see [18]. The conjecture asserts that every planar graph contains neither 4-cycle nor 5-cycle is 3-colorable. This conjecture was disproved [3]. The list version of this conjecture was disproved earlier by Voigt [26], who first constructed a non-3-choosable planar graph without 4-, 5-cycle with 344 vertices. A non-3-choosable planar graph without 4-, 5-cycle with 209 vertices was given by Montassier [15].

However, the counterexample graph to Steinberg's conjecture given in [3] is $(6, 2)$-colorable (see the appendix); hence it has fractional chromatic number exactly 3. (The graph contains a triangle, so the lower bound is 3.) Therefore, it is $(3m, m)$-choosable for some $m$ by the main result in [1]. On the other hand, it is easy to check that all the non-3-choosable examples constructed in [15, 16, 26, 27] mentioned above are 3-colorable; hence they are also $(3m, m)$-choosable for some $m$. So before the present paper, it was unknown whether or not for every positive integer $m$, there is a planar graph without 4- and 5-cycles which is not $(3m, m)$-choosable. In the following, for each positive integer $m$ we construct a planar graph without cycles of length 4 and 5 which is not $(3m + \lfloor \frac{m-1}{12} \rfloor, m)$-choosable. When $m = 1$, the graph has 164 vertices, which is smaller than the counterexample graph found by Montassier in [15].

PROPOSITION 5.4. *For each positive integer $m$, there is a planar graph $G$ without 4-cycle and 5-cycle, which is not $(3m + \lfloor \frac{m-1}{12} \rfloor, m)$-choosable. Consequently, $ch_f^s(\mathcal{P}_{4,5}) \geq 3 + \frac{1}{12}$.*

*Proof.* Let $T$ be the graph depicted in Figure 4, $\epsilon$ be a real number such that $\epsilon m = \lfloor \frac{m-1}{12} \rfloor$. Assume $A, B, C, D, E$ are pairwise disjoint sets of colors with $|A| = |B| =$




- $L(u) = A$, $L(v) = B$ and $L(w) = C \cup D \cup E$.
- $L(u_1) = L(u_2) = L(w_1) = L(w_2) = L(x_3) = L(y_1) = L(y_3) = L(z_1) = A \cup D \cup E$.
- $L(v_1) = L(v_2) = L(x_1) = L(x_2) = L(y_2) = L(z_2) = B \cup D \cup E$.
- $L(x) = L(y) = L(z) = A \cup B \cup C \cup E$.

FIG. 4. *The gadget graph $T$ for $\mathcal{P}_{4,5}$ and the list assignment $L_{A,B}$.*

$|C| = m$, $|D| = 2m$, and $|E| = \epsilon m$. For any disjoint sets $A$ and $B$, we define a list assignment $L_{A,B}$ of $T$ as in Figure 4.

By the same argument as in the proof of Theorem 2, it suffices to show that there is no $m$-fold $L$-coloring of $G$. Before showing this, we observe that the shortest path between $u, v$ in $T$ has length 3, so gluing copies of $T$ in the construction of $G$ will not create 4-, 5-cycles. Suppose to the contrary, $\phi$ is an $m$-fold $L$-coloring of $G$. Then $\phi(u) = A$ and $\phi(v) = B$. Note that $u_1v_1z_1$ is a clique, so each color in $D \cup E$ can be used only once in these vertices, which means that

$$|\phi(z_1) \cap A| \geq 3m - |D \cup E| = m - \epsilon m.$$

Similarly, $|\phi(z_2) \cap B| \geq m - \epsilon m$. So $|\phi(z) \cap A| \leq \epsilon m$, and $|\phi(z) \cap B| \leq \epsilon m$. We assume that $|\phi(z) \cap A| = \alpha m$, $|\phi(z) \cap B| = \beta m$, and $|\phi(z) \cap E| = \gamma m$; it is clear that $\alpha, \beta, \gamma \leq \epsilon$, and $|\phi(z) \cap C| = (1 - (\alpha + \beta + \gamma))m$.

As $zz_1x$ is a clique, each color in $A$ can be used once in these three vertices, so

$$|\phi(x) \cap A| \leq |A| - |\phi(z_1) \cap A| - |\phi(z) \cap A| \leq (\epsilon - \alpha)m.$$

Similarly, by considering the edge $xz$, we have that $|\phi(x) \cap C| \leq (\alpha + \beta + \gamma)m$ and $|\phi(x) \cap E| \leq (\epsilon - \gamma)m$. Note that $\phi(z_1) \cap E$ might be empty. So we only have $|\phi(x) \cap E| \leq |E| - |E \cap \phi(z)|$. Therefore, we have

$$|\phi(x) \cap B| \geq m - |\phi(x) \cap A| - |\phi(x) \cap C| - |\phi(x) \cap E| \geq m - (2\epsilon + \beta)m.$$

This implies that $|\phi(x_1) \cap B| \leq (2\epsilon + \beta)m$.

Since $x_1x_2x_3$ is a clique, each color in $D \cup E$ can be used at most once on these three vertices, but we need $3m$ colors for these vertices, so

$$|\phi(x_3) \cap A| \geq 3m - |\phi(x_1) \cap B| - |(\phi(x_1) \cup \phi(x_2) \cup \phi(x_3)) \cap (D \cup E)| \geq m - (3\epsilon + \beta)m.$$

Hence, $|\phi(w_1) \cap A| \leq (3\epsilon + \beta)m$.

By symmetry, $|\phi(w_2)\cap A| \le (3\epsilon+\alpha)m$. On the other hand, $|\phi(w)\cap C| \le m - |\phi(z)\cap C| \le (\alpha+\beta+\gamma)m$, so we have

$$\begin{aligned}
3m &= |\phi(w)| + |\phi(w_1)| + |\phi(w_2)| \\
&\le |\phi(w_1)\cap A| + |\phi(w_2)\cap A| + |\phi(w)\cap C| + |(\phi(w)\cup\phi(w_1)\cup\phi(w_2))\cap(D\cup E)| \\
&\le 2m + 7\epsilon m + 2(\alpha+\beta)m + \gamma m \\
&\le 2m + 12\epsilon m < 3m,
\end{aligned}$$

which is a contradiction. ◻

It was proved in [13] that the strong fractional choice number of $K_4$-minor-free graphs with girth at least $g$ is $2 + \frac{1}{\lfloor (g+1)/4 \rfloor}$. Thus the strong fractional choice number of the family of planar graphs of girth 5 or 6 is at least 3, i.e., $ch_f^s(\mathcal{P}_{3,4}) \ge 3$. On the other hand, extending the proofs in [19, 20], Voigt [25] proved that every planar graph with girth 5 is $(3m, m)$-choosable, so the family of planar graphs of girth 5 or 6 has strong fractional choice number at most 3.

PROPOSITION 5.5. *$ch_f^s(\mathcal{P}_{3,4}) = ch_f^s(\mathcal{P}_{3,4,5}) = 3$.*

For the case of $\mathcal{P}_3$, the best known upper and lower bounds for their strong fractional chromatic number was obtained [11]: $3 + \frac{1}{17} \le ch_f^s(\mathcal{P}_3) \le 4$.

**6. Open problems.** One basic unsolved problem concerning the strong fractional choice number is whether every rational $r \ge 2$ is the strong fractional choice number of a graph. We conjecture an affirmative answer.

CONJECTURE 6.1. *For any rational number $r \ge 2$, there exists a graph $G$ such that $ch_f^s(G) = r$ and a graph $G'$ with $pt_f^s(G') = r$.*

Erdős, Rubin, and Taylor [8] characterized all the 2-choosable graphs. However, it seems to be a difficult problem to characterize all graphs $G$ with $ch_f^s(G) = 2$. In a companion paper [28], we proved that every 3-choice critical bipartite graph $G$ (i.e., $G$ is not 2-choosable, but every proper subgraph of $G$ is 2-choosable) has strong fractional choice number 2.

QUESTION 6.2. *Give a characterization of the class of graphs whose strong fractional choice number equals 2.*

It was asked by Erdős, Rubin, and Taylor [8] whether every $(a,b)$-choosable graph is also $(am, bm)$-choosable for any positive integer $m$. The case $(a,b) = (2,1)$ was affirmed by Tuza and Voigt [22], but the case $a \ge 4$ and $b = 1$ was resolved to be false by Dvořák, Hu, and Sereni [6] recently. For a relaxed and possibly correct version, we ask the following question.

QUESTION 6.3. *Is it true that $ch_f^s(G) \le ch(G)$ for any graph $G$?*

Similarly, it was conjectured by Mahoney, Meng, and Zhu [14] that every $(a,b)$-paintable graph is also $(am, bm)$-paintable for any positive integer $m$. We also ask the following weaker problem.

QUESTION 6.4. *Is it true that $pt_f^s(G) \le pt(G)$ for any graph $G$?*

Planar graph coloring is a central problem with respect to many coloring concepts. This is also the case for the strong fractional choice number of graphs.

QUESTION 6.5. *What is the exact value of $ch_f^s(\mathcal{P})$? Is it true that $ch_f^s(\mathcal{P}) < 5$?*

QUESTION 6.6. *What is the exact value of $ch_f^s(\mathcal{P}_3)$? Is it true that $ch_f^s(\mathcal{P}_3) < 4$?*

QUESTION 6.7. *What is the exact value of $ch_f^s(\mathcal{P}_{4,5})$? Is it true that $ch_f^s(\mathcal{P}_{4,5}) < 4$?*

Although Steinberg's conjecture is false, the fractional chromatic number and the strong fractional chromatic number of graphs in $\mathcal{P}_{4,5}$ is open. It was proved in [5] that for any $G \in \mathcal{P}_{4,5}$, $\chi_f(G) \le 11/3$. The following question remains open.

QUESTION 6.8. *Is it true that every graph $G \in \mathcal{P}_{4,5}$ has $\chi_f(G) \le 3$ or even has $\chi_f^s(G) \le 3$?*

**Appendix A.** In this part, we give a $(6,2)$-coloring $\phi$ of the counterexample to Steinberg's conjecture presented in [3]. The counterexample constructed in [3] is the graph depicted in Figure 5, where $G_2$ is depicted as in Figure 6. We first precolor

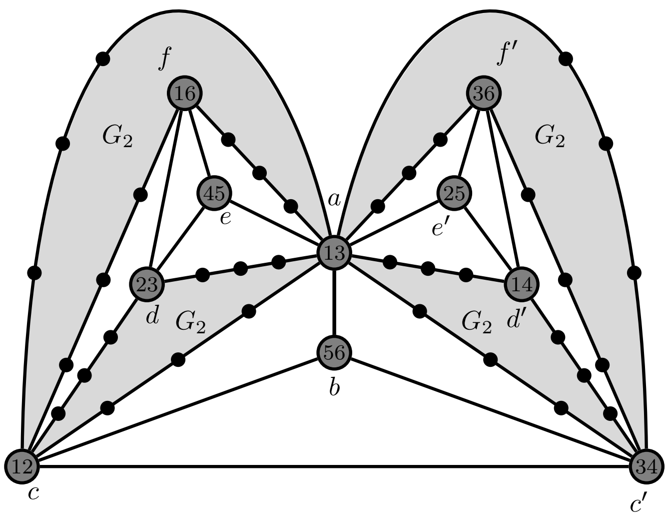



FIG. 5. *The counterexample to Steinberg's conjecture in* [3].

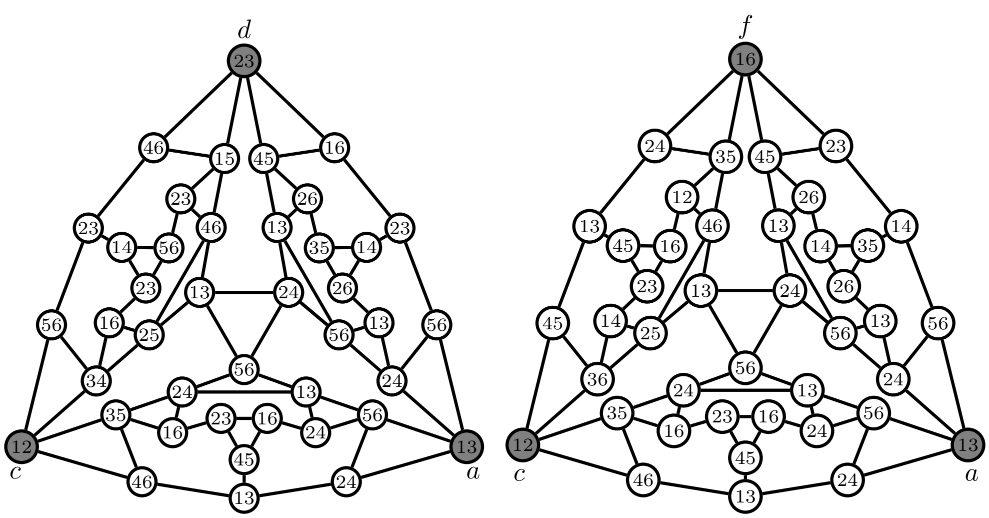



FIG. 6. $(6,2)$*-colorings of the left two copies of* $G_2$.

part of the graph in Figure 5 as follows: $\phi(a) = \{1,3\}$, $\phi(b) = \{5,6\}$, $\phi(c) = \{1,2\}$, $\phi(d) = \{2,3\}$, $\phi(e) = \{4,5\}$, $\phi(f) = \{1,6\}$, $\phi(c') = \{3,4\}$, $\phi(d') = \{1,4\}$, $\phi(e') = \{2,5\}$, and $\phi(f') = \{3,6\}$.

We shall show that this partial coloring can be extended to a 2-fold 6-coloring of the whole graph. By symmetry, it suffices to extend the partial coloring to the left two copies of $G_2$, which is given in Figure 6.

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