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Greedy Nim_k Game

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Abstract This paper introduces a new variant of Nim game, the Greedy Nim_k Game. We present a complete solution for this game.

Keywords Nim game · Greedy Nim game · Greedy Nim_k game

1 Introduction

In the well-known Nim game, one or more heaps of stones are provided. Two players take turns removing any number (at least one) stones from any one heap. The player takes away the last stone wins the game. Bouton (1902) developed a complete mathematical theory for this game in 1902. Since then, many variants of Nim game have been studied in the literature. For example, Nim_k, introduced by Moore (1910), is a variation of Nim in which the players are allowed to remove stones from up to k heaps, where k is a fixed integer. Bounded Nim Schwartz (1971) is a variation of Nim in which the number of stones removed in each turn is no more than a given constant. *Greedy Nim*, introduced by Albert and Nowakowski (2004), is a variation of Nim in which the players always remove stones from the largest heap. k-bounded greedy Nim, introduced in Xu and Zhu (2018), is a combination of bounded Nim and greedy Nim, in which the player removes at most k stones from the largest heap. The

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Euclid's game, introduced in Cole and Davie (1969), in which there are two heaps of stones, and the player alwys takes away number of stones from the larger heap that is a muplity of the number of stones in the smaller heap. Other Nim-type games can be found in Albert and Nowakowski (2001), Fukuyama (2003), Li (1978), Lim (2005), Low and Chan (2015),Wythoff (1907) and Xu and Zhu (2018).

In this paper, we proposed a new game, *Greedy* Nim_k , which is a combination of $\mathrm{Nim_k}$ and Greedy Nim. The game is played as in ordinary Nim except that each the player is allowed to remove any number of stones (at least one) from up to k heaps, however, all the heaps from which stones are removed must have the maximum size. A complete solution for this game is presented.

In this paper, $k \geq 2$ is a fixed positive integer. The *greedy* Nim_k *game* is a game played by two players, P_1 and P_2, who make moves in turn. There is a collection of heaps of stones. In each turn, the player chooses a number of heaps of maximum size, and removes an arbitrary number (but at last one) stones from the chosen heaps. The number of chosen heaps must be at least one and at most k.

We denote a collection of heaps of stones by a sequence $\mathcal{S} = (x_1, x_2, \ldots, x_n)$ of positive integers in non-decreasing order, which means that there are n heaps, and the ith heap has x_i stones. For convenience, we shall always assume that $n \geq 3$ and allow $x_i = 0$ i.e., a heap may have 0 stones. If $x_1 = 0$, then we view the sequence $\mathcal{S} = (x_1, x_2, \ldots, x_n)$ to be equivalent to the sequence $(x_2, \ldots, x_n)$. We call such a sequence a *position*. A position is called an *N-position* if the next player has a winning strategy. Otherwise the previous player has a winning strategy and it is called a *P-position*. By convention, the position $(0, 0, \ldots, 0)$ is a P-position.

In the remainder of the paper, we denote by P_1 the next player, and by P_2 the previously player.

Example 1 Consider the case that $k = 3$. Assume $\mathcal{S} = (1, 1, 1, 5, 5)$. We show that $\mathcal{S}$ is an N-position i.e. P_2 has a winning strategy. In the first round, Player P_1 removes 5 stones from one heap of size 5 and 4 stones from the other heap of size 5. After P_1's move, the position becomes $(1, 1, 1, 1)$. Then Player P_2 removes 1 stones from at least one heap at most 3heaps, leaving at most three heaps of size 1, so P_1 removes the last stone and hence wins the game. I.e., $(1, 1, 1, 1)$ is a P-position.

In Sect. 2, we give a characterization of all the P-positions (hence all the N-positions) for this game.

2 Solution for Greedy $\mathrm{Nim_k}$

We denote by $\mathbb{P}$ the set of all P-positions. To prove that a position $\mathcal{S}$ is a P-position, it suffices to show that for any legal move from $\mathcal{S}$, the resulting position $\mathcal{S}'$ is not a P-position i.e., $\mathcal{S}' \notin \mathbb{P}$. To prove that a position is an N-position, we need to show that P_1 has a move, so that after his move, the resulting position $\mathcal{S}'$ is a P-position i.e., $\mathcal{S}' \in \mathbb{P}$. In the following, we shall denote by $\mathcal{S}$ the current position, and by $\mathcal{S}'$ the next position.

For positive integers a and m, let $R_a(m)$ be the remainder after division of m by a i.e., $R_a(m) = m - a\lfloor m/a \rfloor$.

We first give all the P-positions for $n \le 2$.

Theorem 2 *Suppose Greedy Nim_k is played on $\mathcal{S} = (x_1, \ldots, x_n)$, where $n \le 2$. If $n = 1$, then $\mathcal{S}$ is an N-position. If $n = 2$, then $\mathcal{S}$ is a P-position if and only if $x_1 = 2s - 1$ and $x_2 = 2s$ for some positive integer s.*

Proof If $n = 1$, then P_1 removes all the stones from the heap and wins.

Assume $n = 2$. We use induction on the total number of all the stones.

Basis step ($s = 1$): After P_1's move, $\mathcal{S}' = (1, 1)$ or $\mathcal{S}' = (1)$, both of the two cases, P_2 can remove all the stones and win the game.

Induction step ($s \ge 2$): After P_1's move, if $\mathcal{S}' = (2s-1, 2s-1)$ or $\mathcal{S}' = (2s-1)$, then P_2 removes all stones and wins. If $\mathcal{S}' = (2t-1, 2s-1)$, where $1 \le t < s$, then P_2 removes $2s - 2t - 1$ stones from the heap of size $2s - 1$. If $\mathcal{S}' = (2t, 2s - 1)$, where $1 \le t < s$, then P_2 removes $2s - 2t$ stones from the heap of size $2s - 1$. In each case, P_2 leaves a P-position for P_1. □

In the following, we assume $n \ge 3$. The following theorem characterizes all the P-positions for the case $x_{n-2} = 1$.

Theorem 3 *If $\mathcal{S} = (x_1, x_2, \ldots, x_n)$ and $x_{n-2} = 1$, then $\mathcal{S}$ is a P-position if and only if one of the following holds:*

(i) $x_{n-1} = x_n = 1$, *and* $R_{k+1}(n) = 0$.
(ii) $x_{n-1} = 1$, $x_n = 2$, *and* $R_{k+1}(n) \ge 2$.
(iii) $x_{n-1} = 2s - 1$, $x_n = 2s$, *where* $s \ge 2$, *and* $R_{k+1}(n) \ne 1$.
(iv) $x_{n-1} = 2s$, $x_n = 2s + 1$, *where* $s \ge 1$, *and* $R_{k+1}(n) = 1$.

Proof The proof is by induction on the total number of stones. If $n = 3$ and $x_1 = x_2 = x_3 = 1$, then obviously, it is an N-position if $k \ge 3$ and a P-position if $k = 2$. So the theorem holds. Assume the total number of stones is more than 3 and the theorem is true for a position with fewer stones.

Necessity We shall prove that if $\mathcal{S}$ is none of the four cases, then it is an N-position. It suffices to prove that P_1 has a legal move so that $\mathcal{S}' \in \mathbb{P}$.

Case 1 $R_{k+1}(n) = 0$. The legal move is determined as follows:

1. If $x_{n-1} = 1$, then P_1 removes $x_n - 1$ stones from the heap of size x_n.
2. If $x_{n-1} = 2$, then P_1 removes the entire heap of size x_n.
3. If $x_{n-1} \ge 3$ and $x_n = x_{n-1}$, then P_1 removes $x_n - 1$ stones from both heaps.
4. If $x_{n-1} \ge 3$ and $x_n > x_{n-1}$, where x_{n-1} is even, then P_1 removes $x_n - x_{n-1} + 1$ stones from x_n.
5. If $x_{n-1} \ge 3$ and $x_n > x_{n-1}$, where x_{n-1} is odd, then P_1 removes $x_n - x_{n-1} - 1$ stones from x_n.

Case 2 $R_{k+1}(n) = 1$. The legal move is determined as follows:

1. If $x_{n-1} = 1$, then P_1 removes the entire heap of size x_n.
2. If $x_{n-1} \ge 2$ and $x_n = x_{n-1}$, then P_1 removes the entire heap of size x_n, and also removes $x_{n-1} - 1$ stones from the heap of size x_{n-1}.
3. If $x_{n-1} \ge 2$ and $x_n > x_{n-1}$ where x_{n-1} is even, then $x_n \ne x_{n-1} + 1$, i.e., $x_n \ge x_{n-1} + 2$, then P_1 removes $x_n - x_{n-1} - 1$ stones from x_n.

4. If $x_{n-1} \geq 2$ and $x_n > x_{n-1}$ where x_{n-1} is odd, then $x_{n-1} \geq 3$, P_1 removes $x_n - x_{n-1} + 1$ stones from x_n.

Case 3 $R_{k+1}(n) \geq 2$. The legal move is determined as follows:

1. If $x_n = x_{n-1} = 1$, then P_1 removes $R_{k+1}(n)$ entire heaps.
2. If $x_n = x_{n-1} \geq 2$, then P_1 removes the $x_n - 2$ stones from x_n, and removes $x_{n-1} - 1$ stones from the heap of size x_{n-1}.
3. If $x_n > x_{n-1}$, where x_{n-1} is even, then P_1 removes $x_n - x_{n-1} + 1$ stones from x_n.
4. If $x_n > x_{n-1}$, where x_{n-1} is odd, then P_1 removes $x_n - x_{n-1} - 1$ stones from x_n.

It is not difficult to verify that in each of these cases, the move is legal and $\mathcal{S}' \in \mathbb{P}$.

Sufficiency

(i) In this case, there are n heaps of size 1. Assume P_1 removes t heaps of stones. Since $1 \leq t \leq k$ then $R_{k+1}(n-t) \neq 0$, hence $\mathcal{S}' \notin \mathbb{P}$.

(ii) Supppose P_2 removes t stones from the heap of size $x_n = 2$, then each heap of the remaining position is of size 1. Assume there are m heaps. Then $m = n$ or $n-1$. Since $R_{k+1}(n) \geq 2$, we have $R_{k+1}(m) \geq 1$, so $\mathcal{S}' \notin \mathbb{P}$.

(iii) Suppose P_1 removes t stones from the heap of size $x_n = 2s$. There are two possibilities:
- If $t = 2$, then $\mathcal{S}' = (x'_1, x'_2, \ldots, x'_n)$ with $x'_i = x_i$ for $1 \leq i \leq n-2$, and $x'_{n-1} = 2s - 2$, $x'_n = 2s - 1$ for some $s \geq 2$. Note that there are still n heaps of stones and $R_{k+1}(n) \neq 1$, by (iv), $\mathcal{S}'$ is out of $\mathbb{P}$.
- If $t \neq 2$, then $\mathcal{S}' = (x'_1, x'_2, \ldots, x'_m)$ with $x'_i = x_i = 1$ for $1 \leq i \leq m-2$, $x'_{m-1} = 1 \neq x'_m$, $x'_{m-1} \neq 2s - 2$ and $x'_m = 2s - 1$, where $m \in \{n-1, n\}$. Hence (x'_{m-1}, x'_m) of $\mathcal{S}'$ satisfies none of the conditions listed in the theorem, so $\mathcal{S}'$ is out of $\mathbb{P}$.

(iv) The proof of this case is the same as that of (iii).

□

In the remainder of this paper, we assume that $n \geq 3$ and $x_{n-2} \geq 2$.

Definition 4 Suppose $\mathcal{S} = (x_1, x_2, \ldots, x_n)$. Let i be the smallest index such that $x_i = x_{n-2}$. We call $(x_i, x_{i+1}, \ldots, x_{n-2}, x_{n-1}, x_n)$ the *effective sequence* of $\mathcal{S}$ and denote it by $\mathcal{S}_e$ (i.e. $x_i = x_{i+1} = \cdots = x_{n-2}$). Let n_e be the size of $\mathcal{S}_e$.

Example 5 Consider the case $n = 10$. Assume $\mathcal{S} = (1, 1, 2, 3, 4, 6, 6, 6, 7, 8)$, then $\mathcal{S}_e = (6, 6, 6, 7, 8)$, $n_e = 5$.

It turns out that whether $\mathcal{S}$ is a P-position or an N-position depends only on $\mathcal{S}_e$.

Definition 6 Suppose $2 \leq a \leq b \leq c$ are three positive integers. We say (a, b, c) is a *good triple* if a and b have the same parity and c=b+1.

The following definition partitions all the positions into four types.

Definition 7 Suppose $\mathcal{S} = (x_1, x_2, \ldots, x_n)$ is a position i.e., a sequence of positive integers in non-decreasing order, where $n \geq 3$, $x_{n-2} \geq 2$.

(1) Assume (x_{n-2}, x_{n-1}, x_n) is a good triple. If one of the following holds:
  (i) $R_{k+1}(n_e) = 1$ and $x_{n-2} = x_{n-1}$,
  (ii) $R_{k+1}(n_e) = 2$,
  then $\mathcal{S}$ is of Type I. Otherwise, it is of Type II.
(2) Assume (x_{n-2}, x_{n-1}, x_n) is not a good triple. If one of the following holds:
  (i) $R_{k+1}(n_e) = 1$, $x_{n-2} = x_{n-1} = x_n$,
  (ii) $R_{k+1}(n_e) = 2$, x_{n-1} and x_{n-2} have different parities, and $x_n = x_{n-1} + 1$,
  then we say $\mathcal{S}$ is of Type III. Otherwise, $\mathcal{S}$ is of Type IV.

Theorem 8 *A position $\mathcal{S}$ is a P-position if and only if it is of Type II or Type III.*

Proof The proof is by induction on the total number of stones. The smallest number of stones in a position for which $n \geq 3$ and $x_{n-2} \geq 2$ is the case that $n = 3$ and $x_1 = x_2 = x_3 = 2$. If $k \geq 3$, then P_1 removes all the stones. If $k = 2$, then P_1 removes one entire heap and one stone from another heap. In any case, P_1 wins the game. Hence it is an N-position. Assume the total number of stones is more than 6, and the Theorem holds for a position with smaller number of stones. We shall denote by x_i', n_e' the corresponding parameters for $\mathcal{S}'$.
Necessity We shall prove that if $\mathcal{S}$ is of Type I or Type IV, then there is a legal move for P_1 such that $\mathcal{S}' \in \mathbb{P}$. First assume $\mathcal{S}$ is of Type I. Then the legal move for P_1 is determined as follows:

1. If $R_{k+1}(n_e) = 1$ and $x_{n-1} = x_{n-2}$, then P_1 removes one stone from the heap of size x_n. As $x_n = x_{n-1} + 1$, $\mathcal{S}' = (x_1', x_2', \ldots, x_n')$ with $R_{k+1}(n_e') = R_{k+1}(n_e)$, and $x_n' = x_n - 1 = x_{n-1}' = x_{n-2}'$, so $\mathcal{S}'$ is of Type III.
2. If $R_{k+1}(n_e) = 2$, then P_1 removes 2 stones from the heap of size x_n. If $x_{n-1} = x_{n-2}$, then $x_n - 2 = x_{n-1} + 1 - 2 < x_{n-2}$, hence $R_{k+1}(n_e') = R_{k+1}(n_e) - 1 = 1$, and $x_n' = x_{n-1}' = x_{n-2}' \geq 2$, so $\mathcal{S}'$ satisfies (2)-(i) of Definition 7. If $x_{n-1} > x_{n-2}$, then $x_{n-2}' = x_{n-2}$, $x_{n-1}' = x_n - 2$ and $x_n' = x_{n-1} = x_n - 1$. Hence $R_{k+1}(n_e') = R_{k+1}(n_e) = 2$, x_{n-1}' and x_{n-2}' have different parities, and $x_n' = x_{n-1}' + 1$. So $\mathcal{S}'$ satisfies (2)-(ii) of Definition 7. Thus $\mathcal{S}'$ is of Type III and hence out of $\mathbb{P}$.

Now assume $\mathcal{S}$ is of Type IV. Let

$$\alpha = \begin{cases} \max\{i : x_i < x_{n-2}\}, & \text{if} \quad n > n_e, \\ 0, & \text{otherwise.} \end{cases}$$

$$\beta = \begin{cases} |\{j : x_j = x_\alpha\}|, & \text{if} \quad \alpha > 0, \\ 0, & \text{otherwise.} \end{cases}$$

We first assume that x_{n-2} and x_{n-1} have different parities. Then $P_1's$ move is determined as follows:

1. If $R_{k+1}(n_e) = 0$, then P_1 removes $x_n - x_{n-1} + 1$ stones from the heap of size x_n. Then $(x_{n-2}', x_{n-1}', x_n') = (x_{n-2}, x_{n-1} - 1, x_{n-1})$ is a good triple, and $R_{k+1}(n_e') = 0$, $\mathcal{S}'$ is of Type II,

2. $R_{k+1}(n_e) = 1$. If $x_n = x_{n-1}$, then P_1 removes $x_n - x_{n-2}$ stones from each of x_n and x_{n-1}. Now $\mathcal{S}'$ is of Type III, by (2)-(i) of Definition 7. If $x_n \neq x_{n-1}$. there are two possibilities
   - If $x_{n-1} - x_{n-2} = 1$, then P_1 removes the entire heap of size x_n. Now $R_{k+1}(n'_e) = 0$, $x'_{n-2} = x_{n-3} \geq 2$, and $(x'_{n-2}, x'_{n-1}, x'_n)$ is a good triple, so $\mathcal{S}'$ is of Type II.
   - If $x_{n-1} - x_{n-2} \neq 1$, then P_1 removes $x_n - x_{n-1} + 1$ stones from the heap of size x_n. Now $R_{k+1}(n'_e) = 1$, $(x'_{n-2}, x'_{n-1}, x'_n)$ is a good triple, but $x'_{n-1} = x_{n-1} - 1 > x'_{n-2} = x_{n-2}$, thus $\mathcal{S}'$ is of Type II.
3. $R_{k+1}(n_e) = 2$. If $x_n = x_{n-1}$, then P_1 removes the entire heap of size x_n and removes $x_{n-1} - x_{n-2}$ stones from the heap of size x_{n-1}. Then $\mathcal{S}'$ is of Type III, by (2)-(i) of Definition 7. Otherwise, since $\mathcal{S}$ is not of Type III, we have $x_n \geq x_{n-1}+2$. In this case, P_1 removes $x_n - x_{n-1} - 1$ stones from the heap of size x_n. Then $\mathcal{S}'$ is of Type III, by (2)-(ii) of Definition 7.
4. If $3 \leq R_{k+1}(n_e) \leq k$, then P_1 removes $x_n - x_{n-1} + 1$ stones from the heap of size x_n, $\mathcal{S}'$ is of Type II.

It is not difficult to verify that for each of the cases listed above, we have $\mathcal{S}' \in \mathbb{P}$.

Now assume that x_{n-2} and x_{n-1} have the same parity. As (x_{n-2}, x_{n-1}, x_n) is not a good triple, we know that $x_n \neq x_{n-1} + 1$.

Case 1 $R_{k+1}(n_e) = 0$.

If $x_n - x_{n-1} > 1$, then P_1 removes $x_n - x_{n-1} - 1$ stones from the heap of size x_n. Since $(x'_{n-2}, x'_{n-1}, x'_n)$ is a good triple, and $R_{k+1}(n'_e) = R_{k+1}(n_e) = 0$, $\mathcal{S}'$ is of Type II.

Assume $x_n = x_{n-1}$.

First we consider the case that $x_{n-1} > x_{n-2}$. In this case, P_1 removes $x_{n-1} - x_{n-2}$ stones from the heap of size x_{n-1} and $x_n - x_{n-2} - 1$ stones from the heap of size x_n. Then $\mathcal{S}'$ is of Type II since $(x'_{n-2}, x'_{n-1}, x'_n)$ is a good triple, and $R_{k+1}(n'_e) = R_{k+1}(n_e) = 0$.

Next we consider the case that $x_{n-1} = x_{n-2}$. In this case, we have $x_{n-2} = x_{n-1} = x_n$.

First assume $n_e > k + 1$, then P_1 removes k entire heaps of size x_n, $\mathcal{S}'$ is of Type III, by (2)-(i) of Definition 7.

Now assume $n_e = k + 1$. $P'_1 s$ move is determined as follows:

1. $0 \leq x_\alpha < x_{n-2} - 1$.
   - If $x_{n-2} = 2$, then P_1 removes one stone from each of $k - 1$ heaps of size 2 and also removes one entire heap of size 2.
   - If $x_{n-2} \geq 3$, then P_1 removes one stone from each of k heaps of size x_{n-2}.
2. $x_\alpha = x_{n-2} - 1$ and $x_{n-2} = 2$.
   - If $R_{k+1}(\beta) = 0$, then P_1 removes $k - 1$ entire heaps of size x_{n-2}, and also removes one stone from one heap of size x_{n-2}.
   - If $1 \leq R_{k+1}(\beta) \leq k - 1$, then P_1 removes k entire heaps of size x_{n-2}.
   - If $R_{k+1}(\beta) = k$, then P_1 removes one stone from each of k heaps of size x_{n-2}.
3. $x_\alpha = x_{n-2} - 1$ and $x_{n-2} \geq 3$. P_1 removes one stone from each of $k - R_{k+1}(\beta)$ heaps of size x_{n-2}, and also removes $R_{k+1}(\beta)$ entire heaps of size x_{n-2}.

It is not difficult to verify that for each of the cases listed above, we have $\mathcal{S}' \in \mathbb{P}$.

Case 2 $R_{k+1}(n_e) = 1$. $P'_1 s$ move is determined as follows:

1. If $x_{n-1} = x_{n-2}$, then $x_n \geq x_{n-2} + 2$, and P_1 removes $x_n - x_{n-2}$ stones from the heap of size x_n. Then $\mathcal{S}'$ is of Type III by (2)-(i) of Definition 7.
2. $x_{n-1} \geq x_{n-2} + 2$. If $x_n \geq x_{n-1} + 2$, then P_1 removes $x_n - x_{n-1} - 1$ stones from the heap of size x_n. Then $\mathcal{S}'$ is of Type II. If $x_n = x_{n-1}$, then P_1 removes $x_n - x_{n-2}$ stones from each of x_n and x_{n-1}. Then $\mathcal{S}'$ is of Type III, by (2)-(i) of Definition 7.

Case 3 $R_{k+1}(n_e) = 2$.

1. If $x_{n-1} = x_{n-2}$, then P_1 removes the entire heap of size x_n. Then $\mathcal{S}'$ is of Type III by (2)-(i) of Definition 7.
2. If $x_{n-1} > x_{n-2}$, then P_1 removes $x_n - x_{n-1} + 1$ stones from the heap of size x_n. Then $\mathcal{S}'$ is of Type III by (2)-(ii) of Definition 7.

Case 4 $3 \leq R_{k+1}(n_e) \leq k$.

Note that in this case, $k \geq 3$.

1. $x_{n-1} > x_{n-2}$. If $x_n = x_{n-1}$, then P_1 removes $x_n - x_{n-2}$ stones from the heap of size x_n and $x_{n-1} - x_{n-2} - 1$ stones from the heap of size x_{n-1}. Otherwise, $x_n \geq x_{n-1} + 2$. In this case, P_1 removes $x_n - x_{n-1} - 1$ stones from the heap of size x_n. Either of which leaves a position of Type II.
2. $x_{n-1} = x_{n-2}$. If $x_n > x_{n-1} + 1$, then P_1 removes $x_n - x_{n-1} - 1$ stones from the heap of size x_n. Then $\mathcal{S}'$ is of Type II. Assume $x_n = x_{n-1}$. If $n_e \geq k + 1$, then P_1 removes $R_{k+1}(n_e) - 1$ entire heaps of size x_{n-2}. Then $\mathcal{S}'$ is of Type III, by (2)-(i) of Definition 7.

   Thus assume $3 \leq n_e \leq k$. There are two cases:
   - If $x_\alpha < x_{n-2} - 1$, then P_1 removes one stone from each of $n_e - 1$ heaps of size x_{n-2}, then $\mathcal{S}'$ is a P-position. Indeed, if $x_{n-2} = 2$, then $\mathcal{S} = \mathcal{S}_e$, and $x'_{n-2} = 1$, by Theorem 3, this is a P-position. If $x_{n-2} \geq 3$, then $3 \leq R_{k+1}(n'_e) \leq k$, $x'_{n-2} \geq 2$, so $\mathcal{S}'$ is of Type II.
   - If $x_\alpha = x_{n-2} - 1$, then P_1 removes appropriate stones from $n_e - 1$ heaps of size x_{n-2} so that $\mathcal{S}'$ is a P-position. More specifically, Player $P_1's$ move is determined as follows:
     - If $0 \leq R_{k+1}(\beta) \leq 2$, then P_1 removes one stone from each of $2 - R_{k+1}(\beta)$ heaps of size x_{n-2}, and also removes $n_e + R_{k+1}(\beta) - 3$ entire heaps of size x_{n-2}.
     - If $3 \leq R_{k+1}(\beta) \leq k - 1$, or $R_{k+1}(\beta) = k$ and $x_{n-2} \geq 3$, then P_1 removes $n_e - 1$ entire heaps of size x_{n-2}.
     - If $R_{k+1}(\beta) = k$ and $x_{n-2} = 2$, then P_1 removes one stone from 2 heaps of size x_{n-2}, and also removes $n_e - 3$ entire heaps of size x_{n-2}.

It is not difficult to verify that for either of the cases listed above, $\mathcal{S}' \in \mathbb{P}$.

Sufficiency We shall prove that if $\mathcal{S}$ is of Type II or Type III, then for any legal move, $\mathcal{S}' \notin \mathbb{P}$. First assume that $\mathcal{S}$ is of Type II.

As (x_{n-2}, x_{n-1}, x_n) is a good triple, x_n is the only heap of maximum size. Assume P_1 removes t stones from this heap.

First we consider the case that $n = n_e = 3$. There are three possibilities:

1. If $t = x_3$, then $\mathcal{S}' = (x_1, x_2)$. Since x_1 and x_2 have the same parity, by Theorem 2, $\mathcal{S}' \notin \mathbb{P}$.

2. If $t = x_3 - 1$, then $\mathcal{S}' = (1, x_2', x_3')$. Note $x_2' = x_1$ and $x_3' = x_2$ have the same parity, then by Theorem 3, $\mathcal{S}' \notin \mathbb{P}$.
3. If $1 \le t < x_3 - 1$, then $\mathcal{S}' = (x_1', x_2', x_3')$ with $x_1' \ge 2$ and either x_1' and x_3' or x_2' and x_3' have the same parity, so (x_1', x_2', x_3') is not a good triple. Also, we have

$$R_{k+1}(n_e') = \begin{cases} 0, & k = 2 \\ 3, & k \ge 3. \end{cases}$$

So $\mathcal{S}'$ is of Type IV i.e. $\mathcal{S}' \notin \mathbb{P}$.

Next, we consider the case that $n > n_e = 3$. There are two possibilities:

1. If $x_n - t \ge x_{n-2}$, then $n_e' = n_e = 3$, and $x_{n-2}' = x_{n-2} \ge 2$. Note x_{n-2}' and $x_n' = x_{n-1}$ have the same parity, so $(x_{n-2}', x_{n-1}', x_n')$ is not a good triple. Besides, we have
$$R_{k+1}(n_e') = \begin{cases} 0, & k = 2 \\ 3, & k \ge 3. \end{cases}$$
Thus $\mathcal{S}'$ is of Type IV, $\mathcal{S}' \notin \mathbb{P}$.
2. If $x_n - t < x_{n-2}$, since $n > n_e$ and $1 \le x_{n-3} < x_{n-2}$, then $\mathcal{S}' = (x_1', x_2', \ldots, x_n')$ with $x_{n-2}' = max\{x_{n-3}, x_n - t\} < x_{n-1}' = x_{n-2}$. Note that $x_n' = x_{n-1}$ and x_{n-1}' have the same parity. So if $x_{n-2}' = 1$, then by Theorem 3, $\mathcal{S}' \notin \mathbb{P}$. If $x_{n-2}' \ge 2$, then $\mathcal{S}'$ is of Type IV, and hence out of $\mathbb{P}$.

Now we consider the case that $n_e \ge 4$. So $x_{n-2}' = x_{n-3} = x_{n-2} \ge 2$. We shall prove that $\mathcal{S}'$ is always of Type IV, and hence is out of $\mathbb{P}$. Note that since x_{n-2}' and x_n' have the same parity, $(x_{n-2}', x_{n-1}', x_n')$ is not a good triple. So in order to prove that $\mathcal{S}'$ is of Type IV, it suffices to show that $\mathcal{S}'$ is not of Type III, there are three possibilities:

1. $R_{k+1}(n_e) = 0$. If $x_n - x_{n-2} < t \le x_n$, then $R_{k+1}(n_e') = k \ge 2$. If $1 \le t \le x_n - x_{n-2}$, then $R_{k+1}(n_e') = 0$, $x_{n-2}' = x_{n-2} \ge 2$.
2. $R_{k+1}(n_e) = 1$. As $\mathcal{S}$ is of Type II, $x_{n-1} > x_{n-2}$. If $x_n - x_{n-2} < t \le x_n$, then $R_{k+1}(n_e') = 0$. If $1 \le t \le x_n - x_{n-2}$, then $R_{k+1}(n_e') = 1$, and $x_n' = x_{n-1} > x_{n-2} = x_{n-2}'$.
3. $3 \le R_{k+1}(n_e) \le k$ (implying that $k \ge 3$). If $R_{k+1}(n_e) > 3$, then $R_{k+1}(n_e') \ge 3$. If $R_{k+1}(n_e) = 3$, then $R_{k+1}(n_e') \ge 2$. Furthermore, if $x_n - x_{n-2} < t \le x_n$, then $R_{k+1}(n_e') = 2$, x_{n-1}' and x_n' have the same parity; if $1 \le t \le x_n - x_{n-2}$, then $R_{k+1}(n_e') \ge 3$.

Note that either $x_{n-2}' = x_{n-3} \ge 2$ or $x_{n-2}' = x_{n-2} \ge 2$. Thus for either of cases listed above, $\mathcal{S}'$ is of Type IV, and hence $\mathcal{S}' \notin \mathbb{P}$.

Now we assume that $\mathcal{S}$ is of Type III. There are two cases.

Case 1 $R_{k+1}(n_e) = 1$, $x_{n-2} = x_{n-1} = x_n$.

In this case, there are $n_e \ge 3$ heaps of size x_n. As $R_{k+1}(n_e) = 1$, we conclude that $n_e \ge k + 2$. Since P_1 removes stones from at most k heaps, after P_1's move, there are m heaps of size x_n for some $m \ge n_e - k \ge 2$. So $x_{n-1}' = x_n'$ and $R_{k+1}(m) \ne R_{k+1}(n_e) = 1$. If $n' = 2$ or $x_{n-2}' = 1$, then by Theorem 2 or 3, $\mathcal{S}'$ is an N-position.

Thus assume that $x'_{n-2} \geq 2$. If $R_{k+1}(n'_e) \neq 1$, then as $x'_{n-1} = x'_n$, $\mathcal{S}'$ is of Type IV. If $R_{k+1}(n'_e) = 1$, then $x'_{n-2} < x_{n-2} = x_n = x'_{n-1}$, $\mathcal{S}'$ is of Type IV.

Case 2 $R_{k+1}(n_e) = 2$, $x_n = x_{n-1} + 1$, x_{n-1} and x_{n-2} have different parities.

In this case, x_n is the only heap of maximum size. Assume P_1 removes t stones from this heap. First we consider the case that $x_n - t < x_{n-2}$. In this case, $n'_e = n_e - 1$, and hence $R_{k+1}(n'_e) = 1$. If $x_{n-1} = x_{n-2} + 1$, since $x'_{n-2} = x_{n-3} = x_{n-2} = x'_{n-1} \geq 2$, then $(x'_{n-2}, x'_{n-1}, x'_n)$ is a good triple. Hence $\mathcal{S}'$ is of Type I, by (1)-(i) of Definition 7. If $x_{n-1} > x_{n-2} + 1$, then $x'_n > x'_{n-1} + 1$. Hence $\mathcal{S}'$ is of Type IV.

Next we consider the case that $x_n - t \geq x_{n-2}$. In this case, $x'_{n-2} = x_{n-2} \geq 2$ and $n'_e = n_e$. So $R_{k+1}(n'_e) = 2$. If $x_n - t = x_{n-1} - 1$, then $(x'_{n-2}, x'_{n-1}, x'_n)$ is a good triple. Then $\mathcal{S}'$ is of Type I, by (1)-(ii) of Definition 7. If $x_n - t \neq x_{n-1} - 1$, then $(x'_{n-2}, x'_{n-1}, x'_n)$ is not a good triple. Since x'_{n-2} and x'_n have different parities, $\mathcal{S}'$ is of Type IV. Both of the two cases imply that $\mathcal{S}' \notin \mathbb{P}$. □

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