

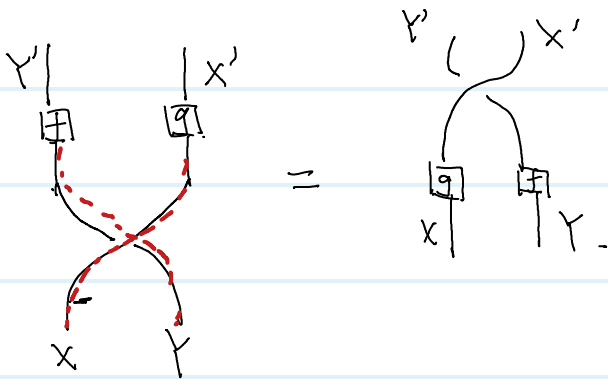


$$\tau = \{ \tau_{x,Y} : x \otimes Y \rightarrow Y \otimes x \}_{x,Y \in \text{ob}(\mathcal{C})}$$

$\tau_{X,Y} =$ 
 $\tau_{Y,X} =$ 

$$\forall f: X \rightarrow Y, g: X \rightarrow X'$$


A braided functor between braided categories (\mathcal{C}, τ) and (\mathcal{C}', τ') is a monoidal functor $F: \mathcal{C} \rightarrow \mathcal{C}'$ s.t.

$$F_2(x, Y) \tau'_{F(x), F(Y)} = F(\tau_{x, Y}) F_2(x, Y), \quad \forall x, Y \in \text{Obj}(\mathcal{C})$$

A braid category is symmetric, if $\begin{array}{c} \diagup \\ x \end{array} \begin{array}{c} \diagdown \\ y \end{array} = \begin{array}{c} \diagdown \\ x \end{array} \begin{array}{c} \diagup \\ y \end{array}, \forall x, y \in \text{Obj}(\mathcal{C})$

$$\tau_{y,x}^{-1} = \tau_{x,y}$$

Lemma: $\begin{array}{c} \diagup \\ 1 \end{array} \begin{array}{c} \diagdown \\ x \end{array} = \begin{array}{c} | \\ x \end{array}, \begin{array}{c} \diagdown \\ x \end{array} \begin{array}{c} \diagup \\ 1 \end{array} = \begin{array}{c} | \\ x \end{array}$

Pf: $\tau_{1,x} = \text{id}_x$, $r_x \tau_{1,x} = l_x$, $\text{id}_1 \otimes$

$$\begin{array}{c}
 \begin{array}{ccccc}
 & & (1 \otimes 1) \otimes x & \xrightarrow{\tau_{1 \otimes 1, x}} & x \otimes (1 \otimes 1) \\
 & \nearrow a_{1,1,x}^{-1} & \downarrow r_1 \otimes \text{id}_x & \downarrow \text{id}_x \otimes l_1 = \text{id}_x \otimes r_1 & \searrow a_{x,1,1}^{-1} \\
 1 \otimes (1 \otimes x) & \xrightarrow{\text{id}_1 \otimes l_1} & 1 \otimes x & \xrightarrow{\tau_{1,x}} & x \otimes 1 \xleftarrow{r_x \otimes 1} x \otimes 1 \otimes 1 \\
 \downarrow \text{id}_1 \otimes \tau_{1,x} & \boxed{2} & \uparrow \text{id}_1 \otimes r_x & \nearrow r_{1 \otimes x} & \downarrow \tau_{1,x} \otimes \text{id}_1 \\
 & & 1 \otimes (x \otimes 1) & \xrightarrow{a_{1,x,1}^+} & (1 \otimes x) \otimes 1
 \end{array}
 \end{array}$$

$$\begin{array}{c}
 x \\
 | \\
 \boxed{f} \\
 | \\
 x
 \end{array}
 \begin{array}{c}
 x' \\
 | \\
 \boxed{g} \\
 | \\
 y
 \end{array}
 =$$

$$\begin{array}{c}
 \begin{array}{c} | \\ \boxed{g} \\ | \\ 1 \otimes 1 \end{array}
 \begin{array}{c} | \\ \boxed{f} \\ | \\ x \end{array}
 =
 \begin{array}{c}
 \begin{array}{c} \diagup \\ \boxed{f} \end{array}
 \begin{array}{c} \diagdown \end{array}
 \end{array}$$

Lem: braid category is pure.

$$\boxtimes \mid = \mid \boxtimes, \quad \alpha \in \text{End}_{\mathcal{C}}(1).$$

Pf:

$$\boxtimes \mid_x = \boxtimes \begin{array}{c} \text{ } \\ \diagup \quad \diagdown \\ 1 \quad x \end{array} = \begin{array}{c} \text{ } \\ \diagup \quad \diagdown \\ \boxtimes \quad \text{ } \\ 1 \quad x \end{array} = \begin{array}{c} \text{ } \\ \diagup \quad \diagdown \\ \text{ } \quad \boxtimes \\ 1 \quad x \end{array} \stackrel{\alpha}{=} \begin{array}{c} \text{ } \\ \diagup \quad \diagdown \\ 1 \quad x \end{array} \boxtimes \mid_x = \mid_x \boxtimes.$$

Lem: \mathcal{C} braided pivotal category. Then.

$$(1) \quad \begin{array}{c} \text{ } \\ \diagup \quad \diagdown \\ x \quad y \end{array} = \begin{array}{c} \text{ } \\ \mid \quad \mid \\ x \quad y \end{array}, \quad \tau_{x,y}^{-1} \cdot \tau_{x,y} = \text{id}_{x \otimes y}$$

$$(2) \quad \begin{array}{c} y_1 \quad \dots \quad y_m \\ \boxed{f} \\ \vdots \\ x_1 \quad x_n \end{array} = \begin{array}{c} y_1 \quad \dots \quad y_m \\ \boxed{f} \\ \vdots \\ x_1 \quad x_n \end{array}$$

The left.

$$\begin{array}{c} y_1 \otimes \dots \otimes y_m \\ \boxed{\boxplus} \\ \vdots \\ x_1 \otimes \dots \otimes x_n \end{array} = \begin{array}{c} \text{ } \\ \mid \\ \boxed{\boxplus} \\ \text{ } \end{array} = \begin{array}{c} y_1 \quad y_m \\ \boxed{f} \\ \vdots \\ x_1 \quad x_n \end{array}.$$

(3)

(4)

Lemma: \mathcal{C} braided pivotal category.

(1).

Pf:

(2)

(3)

pf

$$\text{cup} = \text{cup}$$

$$(4) \quad \cap' = \cap, \quad \cup' = \cup, \quad \cap \times = \cap.$$

\mathcal{C} braided pivotal category.

def: left twist of $X \in \text{obj}(\mathcal{C})$.

$$\theta_X^L = \text{cup}_X : X \rightarrow X.$$

right twist $\theta_X^R = \text{cup}_X : X \rightarrow X.$

lem: The twists θ^L and θ^R are natural isomorphisms ($1_{\mathcal{C}} \rightarrow 1_{\mathcal{C}}$).

$$\underline{(\theta_X^L)^{-1}} = \text{cup}_X, \quad \underline{(\theta_X^R)^{-1}} = \text{cup}_X.$$

Moreover, $\underline{(\theta_X^R)^*} = \theta_{X^*}^L$, $\underline{(\theta_X^L)^*} = \theta_{X^*}^R$ and.

$$\underline{\theta_{X^*}^L} = \text{cup}_X, \quad \theta_{X^*}^R = \text{cup}_X, \quad (\theta_{X^*}^L)^{-1} = \text{cup}_X, \quad (\theta_{X^*}^R)^{-1} = \text{cup}_X.$$

pf: naturality of θ^L .

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \theta_X^L \downarrow & \text{2.} & \downarrow \theta_Y^L \\ X & \xrightarrow{f} & Y \end{array}$$

$$f \theta_x^L = \text{diagram} = \text{diagram} = \text{diagram} = \text{diagram} = \text{diagram} = \theta_Y^L f.$$

Diagram 1: A vertical line with a box labeled f and a loop labeled γ on the left.

Diagram 2: A loop labeled γ with a box labeled f on the right, and a vertical line labeled x on the left.

Diagram 3: A loop labeled γ with a box labeled f on the right, and a vertical line labeled x on the left. The loop is labeled ev. .

Diagram 4: A loop labeled γ with a box labeled f on the right, and a vertical line labeled x on the left.

Diagram 5: A loop labeled γ with a box labeled f on the right, and a vertical line labeled x on the left.

Set $v_x^L = \text{diagram}$

Diagram: A vertical line labeled x with a loop labeled γ on the left.

$$\theta_x^L v_x^L = \text{diagram} = \text{diagram} = \text{diagram} = \text{diagram} = \text{diagram}$$

Diagram 1: A vertical line labeled x with a loop labeled γ on the left. The loop is highlighted with a dashed blue box.

Diagram 2: A loop labeled γ with a box labeled f on the right, and a vertical line labeled x on the left. The loop is labeled co-ev. .

Diagram 3: A loop labeled γ with a box labeled f on the right, and a vertical line labeled x on the left. The loop is labeled ev. .

Diagram 4: A loop labeled γ with a box labeled f on the right, and a vertical line labeled x on the left.

Diagram 5: A loop labeled γ with a box labeled f on the right, and a vertical line labeled x on the left.

$$= \text{diagram}, \quad v_x^L \theta_x^L = \theta_x^L v_x^L = \text{id}_x. \quad \text{by naturality of } \theta^L.$$

Diagram: A vertical line labeled x with a loop labeled γ on the left.

$$\theta_{X^*}^L = \text{diagram} = \text{diagram} = \text{diagram}$$

Diagram 1: A vertical line labeled x^* with a loop labeled γ on the left. The loop is highlighted with a dashed blue box.

Diagram 2: A loop labeled γ with a box labeled ψ_x on the right, and a vertical line labeled x^* on the left.

Diagram 3: A loop labeled γ with a box labeled ψ_x on the right, and a vertical line labeled x^* on the left.

Diagram 4: A loop labeled γ with a box labeled ψ_x on the right, and a vertical line labeled x^* on the left.

$$\psi_x : X \rightarrow X^{**}.$$

$$= \text{diagram} = \text{diagram} \quad \text{diagram} \quad \text{diagram}$$

Diagram 1: A loop labeled γ with a box labeled ψ_x on the right, and a vertical line labeled x^* on the left.

Diagram 2: A loop labeled γ with a box labeled ψ_x on the right, and a vertical line labeled x^* on the left.

Diagram 3: A loop labeled γ with a box labeled ψ_x on the right, and a vertical line labeled x^* on the left.

Diagram 4: A loop labeled γ with a box labeled ψ_x on the right, and a vertical line labeled x^* on the left.

$$(\theta_x^r)^* = \text{diagram with a loop and a crossing} \xrightarrow{\text{ev.}} \text{diagram with a loop and a crossing} = \text{diagram with a loop and a crossing}$$

$$\simeq \text{diagram with a loop and a crossing} = \theta_{x^*}^l.$$

$$\text{tr}_r(\theta_x^l) = x \text{ loop} = \text{tr}_l(\theta_x^r)$$

Def: A braided pivotal category is ribbon, if $\theta^l = \theta^r$.
Twist of \mathcal{C} is $\theta = \theta^l = \theta^r$.

$$(\theta_x)^* = \theta_{x^*}.$$

Lem: \mathcal{C} ribbon category, then $\text{diagram} = \text{diagram}$

$$\text{pf: } \text{diagram} = \theta_x^r (\theta_{x^*}^l)^{-1} = \text{id}_x, \quad \text{diagram} = \theta_{x^*}^r (\theta_x^l)^{-1} = \text{id}_{x^*} \quad \square$$

Cor: Ribbon category is spherical.

$$\text{pf: } \text{tr}_r(f) = \text{diagram} = \text{diagram} = \text{diagram} = \text{diagram} = \text{diagram} = \text{diagram} = \text{tr}_l(f).$$

lem: (\mathcal{C}, τ) symmetric category, a left duality $\{({}^v X, ev_x)\}_{x \in \text{ob}(\mathcal{C})}$

$ev_x : {}^v X \otimes X \rightarrow 1$, its inverse $coev_x : 1 \rightarrow X \otimes {}^v X$.

Set $X^* = {}^v X$.

$$\widetilde{ev}_x = \begin{array}{c} \boxed{ev_x} \\ \swarrow \searrow \\ X \quad X^* \end{array} : X \otimes X^* \rightarrow 1, \quad \widetilde{coev}_x = \begin{array}{c} X^* \quad X \\ \searrow \swarrow \\ \boxed{coev_x} \end{array} : 1 \rightarrow X^* \otimes X.$$

1) \mathcal{C} is pivotal category.

$$\underline{ev}_x = \downarrow_x, \quad \underline{coev}_x = \uparrow_x, \quad \uparrow_x \cap = \downarrow_x, \quad \downarrow_x \cap = \uparrow_x.$$

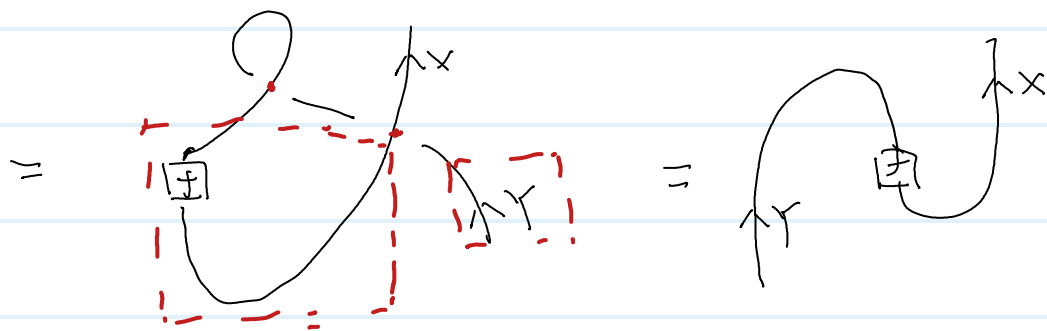
$$\begin{array}{c} \boxed{ev_x} \\ \downarrow X \\ \downarrow X^* \\ \boxed{coev_x} \end{array} = \begin{array}{c} \text{blue box} \\ \downarrow X \quad \uparrow X^* \end{array} = \begin{array}{c} \text{red box} \\ \downarrow X^* \quad \uparrow X \end{array} = \begin{array}{c} \text{loop} \\ \downarrow X \end{array} = \begin{array}{c} \text{loop} \\ \downarrow X \end{array} = \begin{array}{c} \text{loop} \\ \downarrow X \end{array}$$

$$\begin{array}{c} X^* \\ \downarrow \\ \boxed{ev_x} \\ \downarrow X \\ \boxed{coev_x} \\ \downarrow X^* \end{array} = \downarrow_{X^*}.$$

$${}^v f = f^v, \quad f : X \rightarrow Y.$$

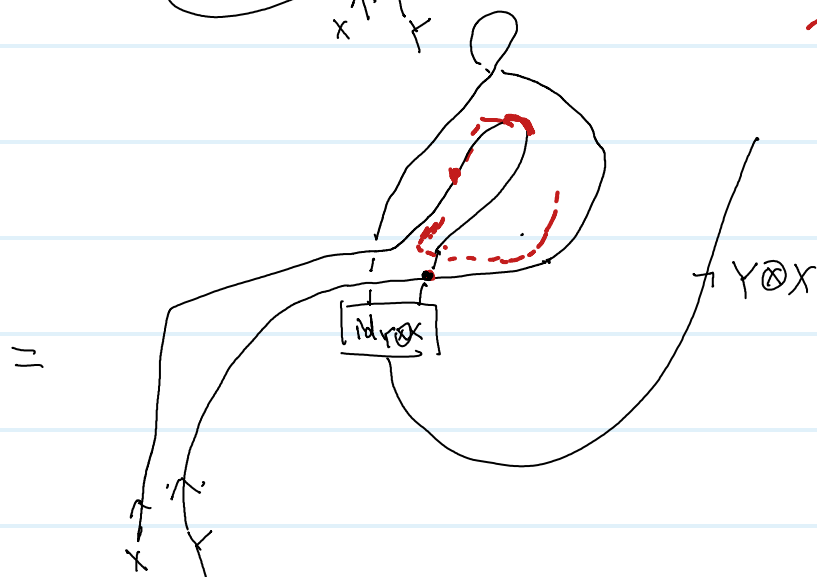
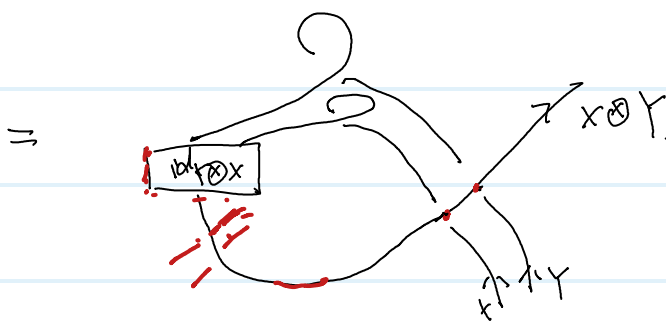
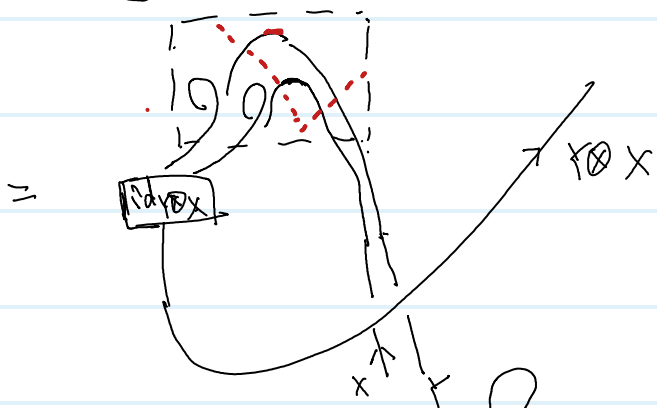
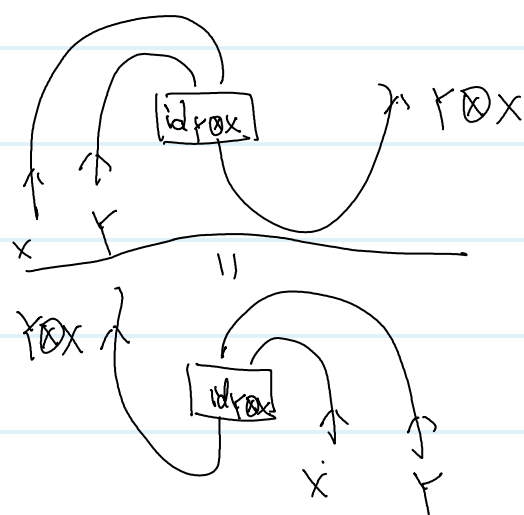
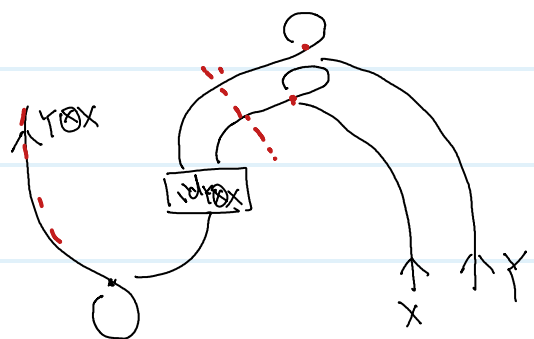
$$\begin{array}{c} \text{red box} \\ \downarrow X \quad \uparrow Y \end{array} = \begin{array}{c} \text{red box} \\ \downarrow X \quad \uparrow Y \end{array} = \begin{array}{c} \text{red box} \\ \downarrow X \quad \uparrow Y \end{array}$$

$$\uparrow_Y \cap \boxed{f} \cap \downarrow_X.$$



$$(\epsilon^v)_0 = (\epsilon^v)_0, \quad \widetilde{\text{coev}}_1 = \begin{array}{c} \downarrow^* \\ \text{coev}_1 \end{array} = \begin{array}{c} \downarrow^* \\ \text{coev}_1 \end{array} = \text{coev}_1$$

$$(\epsilon^v)_2 = (\epsilon^v)_2, \quad X^* \otimes Y^* \rightarrow (Y \otimes X)^*$$



\otimes - multi

$$=$$

$$=$$

□

(2) (\mathcal{C}, τ) is a ribbon category, twist $\theta_X = \text{id}_X$, $\forall X \in \text{Obj}(\mathcal{C})$.

pf: $\theta_X^r =$

$$\theta_X^l = \text{id}_X.$$

(3). The traces of endomorphisms in \mathcal{C} do not depend on the choice of left duality in \mathcal{C} .

Pf: $f: X \rightarrow X$. \check{X} .

let $\underline{w}: Y \otimes X \rightarrow 1$ another left dual.

its inverse $\underline{\Omega}: 1 \rightarrow X \otimes Y$.

def trace of f using w and Ω .

