

$\mathcal{C}$  monoidal category.

Pairing between  $X, Y \in \text{ob}(\mathcal{C})$  is morphism  $X \otimes Y \rightarrow 1$ .

Pairing  $w: X \otimes Y \rightarrow 1$  is non-degenerate, if  $\exists \Omega: 1 \rightarrow Y \otimes X$  s.t.

$$\begin{array}{c} Y \\ | \\ \boxed{w} \\ | \\ X \\ | \\ \boxed{\Omega} \\ | \\ Y \end{array} = \begin{array}{c} | \\ Y \end{array} \quad \text{and} \quad \begin{array}{c} \boxed{w} \\ | \\ X \\ | \\ \boxed{\Omega} \\ | \\ X \end{array} = \begin{array}{c} | \\ X \end{array}$$

$\Omega$  is called inverse of  $w$ .

Prop: Pairing  $w: X \otimes Y \rightarrow 1$  its inverse  $\Omega: 1 \rightarrow Y \otimes X$ .

Then  $\text{Hom}_{\mathcal{C}}(Z, T \otimes X) \rightarrow \text{Hom}_{\mathcal{C}}(Z \otimes Y, T)$  is bijection.

$$\begin{array}{c} \alpha \mapsto \begin{array}{c} T \\ | \\ \boxed{w} \\ | \\ X \\ | \\ \boxed{\alpha} \\ | \\ Z \\ | \\ Y \end{array} \quad \text{and} \quad \begin{array}{c} T \\ | \\ \boxed{\beta} \\ | \\ Z \\ | \\ Y \end{array} \quad \leftarrow \beta. \\ \alpha \mapsto \begin{array}{c} T \\ | \\ \boxed{w} \\ | \\ X \\ | \\ \boxed{\alpha} \\ | \\ Z \\ | \\ Y \end{array} = \begin{array}{c} T \\ | \\ \boxed{\beta} \\ | \\ Z \\ | \\ Y \end{array} = \begin{array}{c} T \\ | \\ \boxed{\alpha} \\ | \\ Z \end{array} = \alpha. \end{array}$$

$\text{Hom}_{\mathcal{C}}(Z, Y \otimes T) \rightarrow \text{Hom}_{\mathcal{C}}(X \otimes Z, T)$  is bijection.

$$\alpha \mapsto \begin{array}{c} \boxed{w} \\ | \\ Y \\ | \\ \boxed{\alpha} \\ | \\ X \\ | \\ Z \end{array} \quad \text{and} \quad \begin{array}{c} Y \\ | \\ \boxed{\beta} \\ | \\ X \\ | \\ \boxed{\Omega} \\ | \\ Z \end{array} \quad \leftarrow \beta$$

Lem:  $F: \mathcal{C} \rightarrow \mathcal{D}$  strong monoidal functor.  $(F_0, F_2, F)$ .

Pairing  $\omega: X \otimes Y \rightarrow 1$  its inverse  $\Omega: 1 \rightarrow Y \otimes X$  in  $\mathcal{C}$ .

Then we have.

Pairing  $\omega^F: F(X) \otimes F(Y) \xrightarrow{F_2(X,Y)} F(X \otimes Y) \xrightarrow{F(\omega)} F(1) \xrightarrow{F_0^{-1}} 1$ .  
 its inverse  $\Omega^F: 1 \xrightarrow{F_0} F(1) \xrightarrow{F(\Omega)} F(Y \otimes X) \xrightarrow{F_2(Y,X)^{-1}} F(Y) \otimes F(X)$

Pf: show that.

$$\begin{array}{c} \boxed{\omega^F} \\ | \\ F(X) \end{array} \begin{array}{c} | \\ \boxed{F(Y)} \\ | \\ \boxed{\Omega^F} \end{array} \begin{array}{c} | \\ F(X) \end{array} = \begin{array}{c} | \\ F(X) \end{array} \quad \text{and}$$

know  
we  $\checkmark$  that.

$$\begin{array}{c} \boxed{\omega} \\ | \\ X \end{array} \begin{array}{c} | \\ X \\ | \\ \boxed{\Omega} \end{array} \begin{array}{c} | \\ X \end{array} = \begin{array}{c} | \\ X \end{array} \quad \text{and}$$

$$\text{id}_X = X \rightarrow X \otimes 1 \xrightarrow{\text{id}_X \otimes \Omega} X \otimes (Y \otimes X) \rightarrow (X \otimes Y) \otimes X \xrightarrow{\omega \otimes \text{id}_X} 1 \otimes X \rightarrow X.$$

$\square$

$$\text{id}_{F(X)} =$$

$$\begin{array}{c}
 F(X) \xleftarrow{r_{F(X)}^{-1}} F(X) \otimes 1 \\
 \downarrow F(r_X)^{-1} \quad \quad \downarrow \text{id}_{F(X)} \otimes F_0 \\
 F(X \otimes 1) \xleftarrow{F_2(X, 1)} F(X) \otimes F(1) \\
 \downarrow F(\text{id}_X \otimes \Omega) \quad \quad \downarrow \text{id}_{F(X)} \otimes F(\Omega) \\
 F(X \otimes (Y \otimes X)) \xleftarrow{F_2(X, Y \otimes X)} F(X) \otimes F(Y \otimes X) \xrightarrow{\text{id}_{F(X)} \otimes F_2(Y, X)^{-1}} F(X) \otimes (F(Y) \otimes F(X)) \\
 \downarrow F(\alpha_{X, Y, X})^{-1} \quad \quad \quad \downarrow a_{F(X), F(Y), F(X)}^{-1} \\
 F((X \otimes Y) \otimes X) \xrightarrow{F_2(X, Y) \otimes \text{id}_{F(X)}} F(X \otimes Y) \otimes F(X) \xleftarrow{\text{id}_{F(X)} \otimes F_2(Y, X)^{-1}} F(X) \otimes (F(Y) \otimes F(X)) \\
 \downarrow F(\omega \otimes \text{id}_X) \quad \quad \downarrow F(\omega) \otimes \text{id}_{F(X)} \\
 F(1 \otimes X) \xrightarrow{F_2(1, X)^{-1}} F(1) \otimes F(X) \\
 \downarrow F(l_X) \quad \quad \downarrow F_0^{-1} \otimes \text{id}_{F(X)} \\
 F(X) \xrightarrow{l_{F(X)}} 1 \otimes F(X)
 \end{array}$$

□

Left dual of  $X \in \text{ob}(\mathcal{C})$  is  ${}^{\vee}X \in \text{ob}(\mathcal{C})$  and.

Pairing  $\text{ev}_X : {}^{\vee}X \otimes X \rightarrow 1$  its inverse  $\text{coev}_X : 1 \rightarrow X \otimes {}^{\vee}X$ .

Right dual of  $X \in \text{ob}(\mathcal{C})$  is  $X^{\vee} \in \text{ob}(\mathcal{C})$  and.

Pairing  $\widetilde{\text{ev}}_X : X \otimes X^{\vee} \rightarrow 1$  its inverse  $\widetilde{\text{coev}}_X : 1 \rightarrow X^{\vee} \otimes X$ .

Def: Monoidal cat-  $\mathcal{C}$  is left(right) rigid, if

$\forall X \in \text{ob}(\mathcal{C})$  has left(right) dual.

is rigid, if  $\forall X \in \text{ob}(\mathcal{C})$  has left and right dual.

$\mathcal{C}$  category, Def  $\mathcal{C}^{op}$ .

$$ob(\mathcal{C}^{op}) = ob(\mathcal{C}), \quad Hom_{\mathcal{C}^{op}}(X, Y) = Hom_{\mathcal{C}}(Y, X)$$

$\mathcal{C}$  monoidal category, Def  $\mathcal{C}^{rev}$

$$||(\mathcal{C}, 1, \otimes, a, l, r) = ||(\mathcal{C}^{op}, \otimes^{op}, 1, a^{rev}, l^{rev}, r^{rev}).$$

where.  $X \otimes^{op} Y = Y \otimes X$ ,  $(a^{rev})_{X, Y, Z} = a_{Z, Y, X}$ .

$$(l^{rev})_X = r_X^{-1}, \quad (r^{rev})_X = l_X^{-1}.$$

$\mathcal{C}$  left rigid category, def left dual functor.

$$V_? : \mathcal{C}^{rev} \rightarrow \mathcal{C}.$$

1)  $X \in ob(\mathcal{C}^{rev}) \mapsto {}^v X$ .

2)  $f : X \rightarrow Y$  in  $\mathcal{C}$ .

$${}^v Y \rightarrow {}^v X.$$

$${}^v f = \begin{array}{c} \boxed{ev_Y} \\ \downarrow \\ \boxed{f} \\ \downarrow \\ \boxed{X} \\ \boxed{coev_X} \end{array} \begin{array}{c} {}^v Y \\ \downarrow \\ {}^v X \end{array} = {}^v Y \rightarrow {}^v X.$$

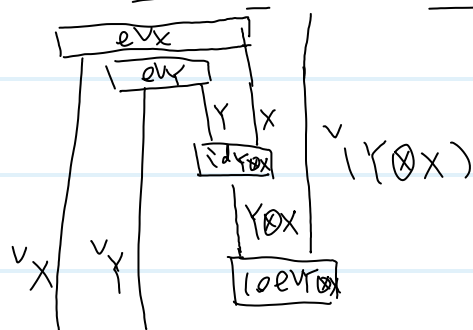
$V_?$  is strong monoidal functor.

$$(V_?)_0 = coev_1 : 1 \rightarrow {}^v 1$$

$$1 \xrightarrow{coev_1} 1 \otimes {}^v 1 \rightarrow {}^v 1.$$

$$\forall X, Y \in \text{ob}(\mathcal{C}^{\text{rev}}).$$

$$\nu_2(X, Y) : \nu X \otimes \nu Y \rightarrow \nu(Y \otimes X)$$



$$\text{Pf: } \nu_{\text{id}_X} = \begin{array}{c} \boxed{\text{ev}_X} \\ \swarrow \quad \searrow \\ \nu_X \quad \boxed{\text{roev}_X} \end{array} \nu_X = \nu_X = \text{id}_{\nu_X}.$$

$$X \xrightarrow{f} Y \xrightarrow{g} Z, \quad \nu(gf) = (\nu f)(\nu g).$$

$$\begin{aligned} \nu(fg) &= \begin{array}{c} \boxed{\text{ev}_Z} \\ \downarrow \\ \begin{array}{c} \boxed{g} \\ \downarrow \\ \boxed{f} \\ \downarrow \\ \boxed{\text{roev}_X} \end{array} \end{array} \nu_X \\ &= \begin{array}{c} \boxed{\text{ev}_Z} \\ \downarrow \\ \begin{array}{c} \boxed{g} \\ \downarrow \\ \boxed{\text{ev}_Y} \\ \downarrow \\ \boxed{\text{roev}_Y} \end{array} \end{array} \nu_Z \quad \begin{array}{c} \boxed{\text{ev}_Y} \\ \downarrow \\ \boxed{f} \\ \downarrow \\ \boxed{\text{roev}_X} \end{array} \nu_X \\ &= \begin{array}{c} \boxed{\text{ev}_Z} \\ \downarrow \\ \begin{array}{c} \boxed{g} \\ \downarrow \\ \boxed{\text{roev}_Y} \end{array} \end{array} \nu_Z \quad \begin{array}{c} \boxed{\text{ev}_Y} \\ \downarrow \\ \boxed{f} \\ \downarrow \\ \boxed{\text{roev}_X} \end{array} \nu_X \\ &= (\nu f)(\nu g). \end{aligned}$$

$$\text{coev}_1: 1 \rightarrow 1 \otimes {}^v 1 \rightarrow {}^v 1, \quad \underline{(\text{coev}_1)^T = \text{ev}_1}: \underline{{}^v 1} \rightarrow \underline{1}$$

$${}^v 1 \rightarrow {}^v 1 \otimes 1 \xrightarrow{\text{ev}_1} 1$$

$$\begin{array}{c} \boxed{\text{ev}_1} \\ | \\ {}^v 1 \end{array} \begin{array}{c} | \\ \boxed{1} \\ | \\ \boxed{\text{coev}_1} \end{array} {}^v 1 = \begin{array}{c} | \\ {}^v 1 \end{array} \Rightarrow \begin{array}{c} {}^v 1 \\ | \\ \boxed{\text{ev}_1} \\ | \\ {}^v 1 \end{array} = \begin{array}{c} | \\ {}^v 1 \end{array}$$

$$\begin{array}{c} 1 \\ | \\ {}^v 1 \end{array} \begin{array}{c} \boxed{\text{ev}_1} \\ | \\ \boxed{\text{coev}_1} \end{array} 1 = \begin{array}{c} | \\ 1 \end{array} \Rightarrow \begin{array}{c} \boxed{\text{ev}_1} \\ | \\ {}^v 1 \\ | \\ \boxed{\text{coev}_1} \end{array} = \begin{array}{c} | \\ 1 \end{array}$$

$$({}^v ?_2(X, Y))^T =$$

$${}^v X \otimes {}^v Y \rightarrow (Y \otimes X)^v$$

$$?_2(X, Y)$$

$$\underline{{}^v(Y \otimes X) \rightarrow {}^v X \otimes {}^v Y}$$

$$\square \quad {}^v X \quad | \quad {}^v Y$$

$\mathcal{C}$  right rigid category def right dual functor

$$?^v: \mathcal{C}^{\text{rev}} \rightarrow \mathcal{C}$$

$$(1) \quad X \mapsto X^v$$

$$(2) \quad f: X \rightarrow Y$$

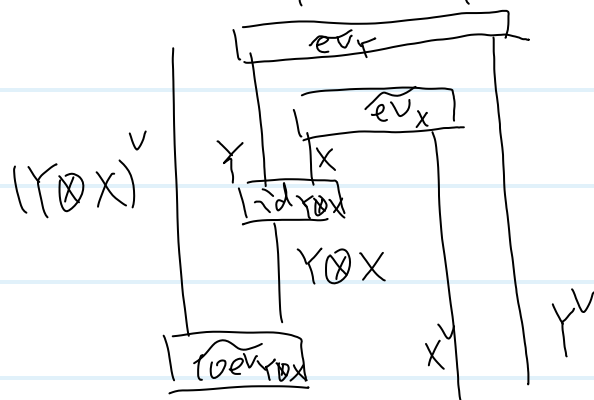
$$f^v =$$

$$Y^v: Y^v \rightarrow X^v$$

$\mathcal{P}^V$  is strong monoidal functor.

$$(\mathcal{P}^V)_0 = \widetilde{\text{coev}}_1, \quad 1 \rightarrow 1^V$$

$$\mathcal{P}_2^V(X, Y) : X^V \otimes Y^V \rightarrow (Y \otimes X)^V.$$



□

Def: A pivotal category  $\mathcal{C}$  is rigid category and  $\mathcal{P}^V = \mathcal{P}^V$  as monoidal functor.

$\forall X \in \text{ob}(\mathcal{C}), X^* = {}^V X = X^V$  is called dual of  $X$ .

$$\text{ev}_X : X^* \otimes X \rightarrow 1 \quad \text{coev}_X : 1 \rightarrow X \otimes X^*$$

$$\widehat{\text{ev}}_X : X \otimes X^* \rightarrow 1 \quad \widetilde{\text{coev}}_X : 1 \rightarrow X^* \otimes X.$$

Penrose diagram in pivotal category.

$$\text{id}_X = \downarrow_X, \quad \text{id}_{X^*} = \uparrow_X = \downarrow_{X^*}.$$

$$\text{ev}_X = \text{X} \uparrow \text{cap}$$

$$\text{coev}_X = \text{X} \downarrow \text{cup}$$

$$\widehat{\text{ev}}_X = \text{X} \downarrow \text{cap}$$

$$\widetilde{\text{coev}}_X = \text{X} \uparrow \text{cup}$$

$$\begin{array}{c} \downarrow \\ \text{X} \end{array} \curvearrowright = \downarrow \text{X} = \curvearrowright \begin{array}{c} \downarrow \\ \text{X} \end{array}, \quad \begin{array}{c} \uparrow \\ \text{X} \end{array} \curvearrowright = \uparrow \text{X} = \curvearrowright \begin{array}{c} \uparrow \\ \text{X} \end{array}$$

$$\forall f = X \rightarrow Y \text{ in } \mathcal{C}.$$

$$f^* = {}^v f = f^v : Y^* \rightarrow X^*$$

$$\begin{array}{c} \text{X} \uparrow \\ \curvearrowright \\ \boxed{\oplus} \\ \uparrow \text{Y} \end{array} = \begin{array}{c} \uparrow \text{Y} \\ \curvearrowright \\ \boxed{\oplus} \\ \uparrow \text{X} \end{array}$$

$$\text{coev}_1 = \widetilde{\text{ev}}_1 : 1 \rightarrow 1^* \Rightarrow \text{ev}_1 = \widehat{\text{ev}}_1 : 1^* \rightarrow 1.$$

$$X^* \otimes Y^* \longrightarrow (Y \otimes X)^*, \quad X, Y \in \text{ob}(\mathcal{C})$$

$$\begin{array}{c} \text{X} \uparrow \quad \text{Y} \uparrow \\ \curvearrowright \\ \boxed{\text{id}_{Y \otimes X}} \\ \uparrow \text{Y} \otimes \text{X} \end{array} = \begin{array}{c} \text{Y} \otimes \text{X} \uparrow \\ \curvearrowright \\ \boxed{\text{id}_{Y \otimes X}} \\ \downarrow \text{X} \quad \text{Y} \downarrow \end{array}$$

$$(Y \otimes X)^* \longrightarrow X^* \otimes Y^*, \quad X, Y \in \text{ob}(\mathcal{C})$$

$$\begin{array}{c} \text{X} \uparrow \quad \text{Y} \uparrow \\ \curvearrowright \\ \boxed{\text{id}_{Y \otimes X}} \\ \uparrow (Y \otimes X)^* \end{array} = \begin{array}{c} \text{X} \uparrow \quad \text{Y} \uparrow \\ \curvearrowright \\ \boxed{\text{id}_{Y \otimes X}} \\ \uparrow (Y \otimes X)^* \end{array}$$



dual morphism identities

$$f: X \rightarrow Y, \quad f^*: Y^* \rightarrow X^*$$

$$\begin{array}{c} \text{Box } f \text{ with } Y \text{ in, } X \text{ out} \\ \hline \text{Box } f \text{ with } X \text{ in, } Y \text{ out} \end{array}, \quad \begin{array}{c} \text{Box } f \text{ with } X \text{ in, } Y \text{ out} \\ \hline \text{Box } f \text{ with } Y \text{ in, } X \text{ out} \end{array}$$

||

$$\text{Box } f \text{ with } Y \text{ in, } X \text{ out} = \text{Box } f \text{ with } Y \text{ in, } X \text{ out}$$

$$\begin{array}{c} \text{Box } f \text{ with } Y \text{ in, } X \text{ out} \\ \hline \text{Box } f \text{ with } X \text{ in, } Y \text{ out} \end{array}, \quad \begin{array}{c} \text{Box } f \text{ with } X \text{ in, } Y \text{ out} \\ \hline \text{Box } f \text{ with } Y \text{ in, } X \text{ out} \end{array}$$

$\mathcal{C}$  pivotal category.

Double dual.  $\forall X \in \text{ob}(\mathcal{C})$ , set  $X^{**} = (X^*)^*$ , define.

$$\psi_X = \text{Box } \psi_X \text{ with } X \text{ in, } X^{**} \text{ out}$$

$$X \rightarrow X^{**}$$

Lem.  $\forall X \in \text{ob}(\mathcal{C})$ ,  $\psi_X$  is invertible set.

$$\underline{\phi}_X = \text{diagram}, \quad \underline{\theta}_X = \text{diagram}, \quad \underline{\nu}_X = \text{diagram}$$

Diagrams for  $\phi_X$ ,  $\theta_X$ , and  $\nu_X$  are shown, each involving a box labeled  $\text{id}_X^*$  and arrows representing the objects  $X$  and  $X^*$ .

Then  $\psi_X = \phi_X$ ,  $\psi_X^{-1} = \theta_X = \nu_X$ .

pf:  $\underline{\theta}_X \psi_X = \text{id}_X = \nu_X \phi_X$ ,  $\psi_X \theta_X = \text{id}_{X^{**}} = \phi_X \nu_X$ .

$\Rightarrow \psi_X^{-1} = \theta_X$ ,  $\phi_X^{-1} = \nu_X$ , Need to show that  $\underline{\theta}_X = \underline{\nu}_X$ .

$\Leftrightarrow$  show that.

$$\text{diagram} = \text{diagram}$$

Diagrammatic equation showing the equality of two expressions involving  $\theta_X$  and  $X^*$ .

$$X^* \otimes X^{**} \rightarrow 1$$

Diagrammatic representation of the evaluation map  $X^* \otimes X^{**} \rightarrow 1$ .

Consider

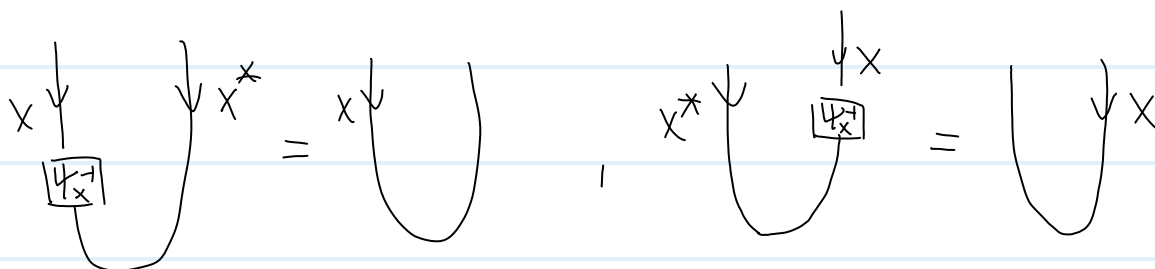
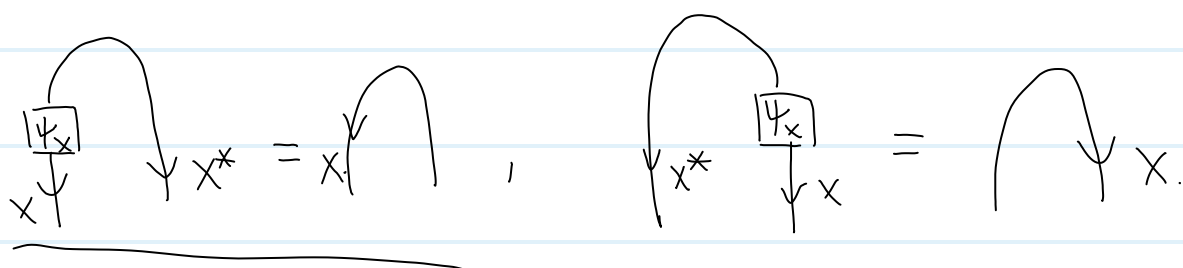
$$X^* \otimes X^{**} \xrightarrow{?^*(X, X^*)} (X^* \otimes X)^* \xrightarrow{(\text{coev}_X)^*} 1^* \xrightarrow{\text{ev}_1 = \widehat{\text{ev}}_1} 1$$

$$\text{diagram} = \text{diagram} = \text{diagram}$$

Complex diagrammatic proof showing the equality of three expressions involving  $\text{id}_X^*$ ,  $\text{coev}_X$ ,  $\text{ev}_1$ , and  $\text{coev}_1$ .



Lemma.  $\forall X \in \text{ob}(\mathcal{C})$ .



$\square$