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TWO NEW LOWER BOUNDS FOR THE SMALLEST SINGULAR VALUE

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Abstract. In this paper, we obtain two new lower bounds for the smallest singular value of nonsingular matrices which is better than the bound presented by Zou [1], Lin and Xie [2] under certain circumstances.

1. Introduction

Let M_n $(n \geqslant 2)$ be the space of $n \times n$ complex matrices. Let σ_i $(i = 1, \cdots, n)$ be the singular values of $A \in M_n$ which is nonsingular and suppose that $\sigma_1 \geqslant \sigma_2 \geqslant \cdots \geqslant \sigma_{n-1} \geqslant \sigma_n > 0$. For $A = [a_{ij}] \in M_n$, the Frobenius norm of A is defined by

$$\|A\|_F = \left(\sum_{i,j=1}^{n} \left| a_{ij} \right|^2 \right)^{1/2} = \operatorname{tr}\left(A^H A\right)^{\frac{1}{2}}$$

where A^H is the conjugate transpose of A. The relationship between the Frobenius norm and singular values is

$$\|A\|_F^2 = \sigma_1^2 + \sigma_2^2 + \cdots + \sigma_n^2.$$

It is well known that lower bounds for the smallest singular value σ_n of a nonsingular matrix $A \in M_n$ have many potential theoretical and practical applications [3, 4]. Yu and Gu [5] obtained a lower bound for σ_n as follows:

$$\sigma_n \geqslant |\det A| \cdot \left(\frac{n-1}{\|A\|_F^2} \right)^{(n-1)/2} = l > 0.$$

The above inequality is also shown in [6]. In [1], Zou improved the above inequality by showing that

$$\sigma_n \geqslant |\det A| \left(\frac{n-1}{\|A\|_F^2 - l^2} \right)^{(n-1)/2} = l_0.$$

In [2], Lin, Minghua and Xie, Mengyan improve a lower bound for smallest singular value of matrices by showing that a is the smallest positive solution to the equation

$$x^2 \left(\|A\|_F^2 - x^2 \right)^{n-1} = |\det A|^2 (n-1)^{n-1}$$

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and $\sigma \geqslant a > l_0$.

In this paper, we obtain two new lower bounds for the smallest singular value of nonsingular matrices. We give some numerical examples which will show that our result is better than l_0 and a under certain circumstances.

2. Main results

LEMMA 1. *Let*

$$l_0 = |\det A| \left(\frac{n-1}{\|A\|_F^2 - l^2} \right)^{(n-1)/2}$$

then $\sigma_n > l_0$.

Proof. In [1], we have

$$\sigma_n \geqslant |\det A| \left(\frac{n-1}{\|A\|_F^2 - \sigma_n^2} \right)^{(n-1)/2}$$

since $\sigma_n \geqslant l_0 > l$, thus

$$\begin{aligned} \sigma &\geqslant |\det A| \left(\frac{n-1}{\|A\|_F^2 - \sigma_n^2} \right)^{(n-1)/2} \\ &\geqslant |\det A| \left(\frac{n-1}{\|A\|_F^2 - l_0^2} \right)^{(n-1)/2} \\ &> |\det A| \left(\frac{n-1}{\|A\|_F^2 - l^2} \right)^{(n-1)/2} = l_0 \end{aligned}$$

so $\sigma_n > l_0$. □

THEOREM 1. *Let $A \in M_n$ be nonsingular. Then*

$$\left(l_0^2 + |\det(l_0^2 I_n - A^H A)| \left(\frac{n-1}{\|A\|_F^2 - n l_0^2} \right)^{n-1} \right)^{1/2} = l_1$$

then $\sigma_n \geqslant l_1$, *where*

$$l = |\det A| \left(\frac{n-1}{\|A\|_F^2} \right)^{\frac{n-1}{2}}, \quad l_0 = |\det A| \left(\frac{n-1}{\|A\|_F^2 - l^2} \right)^{\frac{n-1}{2}}.$$

Proof. Let $0 < \lambda < \sigma_n^2$, then

$$\left| (\lambda - \sigma_1^2)(\lambda - \sigma_2^2) \cdots (\lambda - \sigma_{n-1}^2) \right| \leqslant \left(\frac{\sigma_1^2 + \cdots + \sigma_{n-1}^2 - (n-1)\lambda}{n-1} \right)^{n-1}.$$

Since

$$\left|\left(\lambda-\sigma_1^2\right)\left(\lambda-\sigma_2^2\right)\cdots\left(\lambda-\sigma_{n-1}^2\right)\right| = \frac{\left|\left(\lambda-\sigma_1^2\right)\left(\lambda-\sigma_2^2\right)\cdots\left(\lambda-\sigma_n^2\right)\right|}{\sigma_n^2-\lambda} = \frac{|\det(\lambda I_n - A^H A)|}{\sigma_n^2-\lambda}$$

then

$$\frac{|\det(\lambda I_n - A^H A)|}{\sigma_n^2-\lambda} \leqslant \left(\frac{\sigma_1^2+\cdots+\sigma_{n-1}^2-(n-1)\lambda}{n-1}\right)^{n-1}$$

$$\sigma_n^2 \geqslant \lambda + |\det(\lambda I_n - A^H A)| \left(\frac{n-1}{\sigma_1^2+\cdots+\sigma_{n-1}^2-(n-1)\lambda}\right)^{n-1}$$

$$\sigma_n \geqslant \left(\lambda + |\det(\lambda I_n - A^H A)| \left(\frac{n-1}{\sigma_1^2+\cdots+\sigma_{n-1}^2-(n-1)\lambda}\right)^{n-1}\right)^{1/2}.$$

By Lemma 1, $l_0 < \sigma_n$, $l_0^2 < \sigma_n^2$, let $\lambda = l_0^2$, then

$$\sigma_n \geqslant \left(l_0^2 + |\det(l_0^2 I_n - A^H A)| \left(\frac{n-1}{\|A\|_F^2-\sigma_n^2-(n-1)l_0^2}\right)^{n-1}\right)^{1/2}. \tag{1}$$

Therefore

$$\sigma_n \geqslant \left(l_0^2 + |\det(l_0^2 I_n - A^H A)| \left(\frac{n-1}{\|A\|_F^2-nl_0^2}\right)^{n-1}\right)^{1/2}. \quad \square$$

THEOREM 2. *Let $A \in M_n$ be nonsingular. Let*

$$b_{k+1} = \left(l_0^2 + |\det(l_0^2 I_n - A^H A)| \left(\frac{n-1}{\|A\|_F^2-(n-1)l_0^2-b_k^2}\right)^{n-1}\right)^{1/2}, \quad k=1,2,\cdots$$

with $l = |\det A|\left(\frac{n-1}{\|A\|_F^2}\right)^{\frac{n-1}{2}}$, $l_0 = |\det A|\left(\frac{n-1}{\|A\|_F^2-l^2}\right)^{\frac{n-1}{2}}$

$$b_1 = \left(l_0^2 + |\det(l_0^2 I_n - A^H A)| \left(\frac{n-1}{\|A\|_F^2-(n-1)l_0^2}\right)^{n-1}\right)^{1/2}$$

then $0 < b_k < b_{k+1} \leqslant \sigma_n$, $k=1,2,\cdots$, $\lim_{k\to\infty} b_k$ *exists.*

Proof. We show by induction on k that

$$\sigma_n \geqslant b_{k+1} > b_k > 0.$$

By (1), we have

$$\begin{aligned}\sigma_n &\geqslant \left(l_0^2 + |\det(l_0^2 I_n - A^H A)| \left(\frac{n-1}{\|A\|_F^2 - \sigma_n^2 - (n-1)l_0^2}\right)^{n-1}\right)^{1/2} \\ &\geqslant \left(l_0^2 + |\det(l_0^2 I_n - A^H A)| \left(\frac{n-1}{\|A\|_F^2 - (n-1)l_0^2}\right)^{n-1}\right)^{1/2} = b_1\end{aligned}$$

so $\sigma_n \geqslant b_1$, then

$$\begin{aligned}\sigma_n &\geqslant \left(l_0^2 + |\det(l_0^2 I_n - A^H A)| \left(\frac{n-1}{\|A\|_F^2 - \sigma_n^2 - (n-1)l_0^2}\right)^{n-1}\right)^{1/2} \\ &\geqslant \left(l_0^2 + |\det(l_0^2 I_n - A^H A)| \left(\frac{n-1}{\|A\|_F^2 - (n-1)l_0^2 - b_1^2}\right)^{n-1}\right)^{1/2} = b_2 \\ &> \left(l_0^2 + |\det(l_0^2 I_n - A^H A)| \left(\frac{n-1}{\|A\|_F^2 - (n-1)l_0^2}\right)^{n-1}\right)^{1/2} = b_1 > 0.\end{aligned}$$

When $k=1$, we have

$$\sigma_n \geqslant b_2 > b_1 > 0.$$

Assume that our claim is true for $k=m$, that is $\sigma_n \geqslant b_{m+1} > b_m > 0$. Now we consider the case when $k=m+1$. By (1), we have

$$\begin{aligned}\sigma_n &\geqslant \left(l_0^2 + |\det(l_0^2 I_n - A^H A)| \left(\frac{n-1}{\|A\|_F^2 - \sigma_n^2 - (n-1)l_0^2}\right)^{n-1}\right)^{1/2} \\ &\geqslant \left(l_0^2 + |\det(l_0^2 I_n - A^H A)| \left(\frac{n-1}{\|A\|_F^2 - b_{m+1}^2 - (n-1)l_0^2}\right)^{n-1}\right)^{1/2} = b_{m+2} \\ &> \left(l_0^2 + |\det(l_0^2 I_n - A^H A)| \left(\frac{n-1}{\|A\|_F^2 - b_m^2 - (n-1)l_0^2}\right)^{n-1}\right)^{1/2} = b_{m+1} > 0.\end{aligned}$$

Hence $\sigma_n \geqslant b_{m+2} > b_{m+1} > 0$. This proves $\sigma_n \geqslant b_{k+1} > b_k > 0$, $k = 1, 2, \cdots$. By the well known monotone convergence theorem, $\lim_{k\to\infty} b_k$ exists. $\square$

THEOREM 3. *Let* $b = \lim_{k\to\infty} b_k$,

$$f(x) = \left(l_0^2 + |\det(l_0^2 I_n - A^H A)| \left(\frac{n-1}{\|A\|_F^2 - x^2 - (n-1)l_0^2}\right)^{n-1}\right)^{1/2}$$

then b *is the smallest positive solution to the equation* $x = f(x)$, *and* $\sigma_n \geqslant b$.

Proof. Let x_0 is the smallest positive solution to the equation $x = f(x)$, we show by induction on k that $x_0 > b_k$, $k = 1,2,\cdots$. When $k = 1$

$$x_0 = \left(l_0^2 + |\det(l_0^2 I_n - A^H A)| \left(\frac{n-1}{\|A\|_F^2 - x_0^2 - (n-1)l_0^2}\right)^{n-1}\right)^{1/2}$$
$$> \left(l_0^2 + |\det(l_0^2 I_n - A^H A)| \left(\frac{n-1}{\|A\|_F^2 - (n-1)l_0^2}\right)^{n-1}\right)^{1/2} = b_1.$$

Assume that our claim is true for $k = m$, that is $\sigma_n > b_m$. Now we consider the case when $k = m+1$.

$$x_0 = \left(l_0^2 + |\det(l_0^2 I_n - A^H A)| \left(\frac{n-1}{\|A\|_F^2 - x_0^2 - (n-1)l_0^2}\right)^{n-1}\right)^{1/2}$$
$$> \left(l_0^2 + |\det(l_0^2 I_n - A^H A)| \left(\frac{n-1}{\|A\|_F^2 - b_m^2 - (n-1)l_0^2}\right)^{n-1}\right)^{1/2} = b_{m+1}.$$

Hence $x_0 > b_{m+1}$. This proves $x_0 > b_k$, $k = 1,2,\cdots$. Since b is a positive solution to the equation $x = f(x)$ and $x_0 > b_k$, $k = 1,2,\cdots$, then $b = x_0$. Therefore b is the smallest positive solution to the equation $x = f(x)$ and $\sigma_n \geqslant b$. $\square$

Therefore we obtain two new lower bounds l_1 and b for the smallest singular value of nonsingular matrices.

3. Numerical examples

We use Examples 1 and Example 2 to compare the values of l, l_0, l_1.

EXAMPLE 1. Let

$$A = \begin{bmatrix} 4 & -4 & -3 \\ 3 & 4 & 2 \\ 4 & 1 & 0 \end{bmatrix}.$$

Then $\sigma_{\min} = 0.0231$, and

$$l = 0.0229885$$

$$l_0 = 0.0229886.$$

Our result:

$$l_1 = 0.0230691.$$

EXAMPLE 2. Let

$$A = \begin{bmatrix} 4 & 0 & 0 \\ -1 & 5 & 0 \\ 0 & 5 & 4 \end{bmatrix}.$$

Then

$$l = 1.92771$$

$$l_0 = 2.01806.$$

Our result:

$$l_1 = 2.31515.$$

Next we use the following example to compare the values of a, b, l_1.

EXAMPLE 3. Let

$$A = \begin{bmatrix} 3 & 2 & 0 \\ 1 & 9 & 5 \\ 0 & 5 & 7 \end{bmatrix}.$$

Then

$$a = 1.0367.$$

Our result:

$$l_1 = 1.3434$$

$$b = 1.3455.$$

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