

$F: \mathcal{C} \rightarrow \mathcal{D}$, $G: \mathcal{D} \rightarrow \mathcal{E}$ monoidal functor.

Def $GF: \mathcal{C} \rightarrow \mathcal{E}$ with.

$$(GF)_0 = G(F_0)G_0, \quad (GF)_2(X, Y) = G(F_2(X, Y))G_2(F(X), F(Y)).$$

$$(GF)_0: 1 \xrightarrow{G_0} G(1) \xrightarrow{G(F_0)} GF(1)$$

$$(GF)_2(X, Y): \underline{GF(X)} \otimes \underline{GF(Y)} \xrightarrow{G_2(F(X), F(Y))} \underline{G(F(X) \otimes F(Y))} \xrightarrow{G(F_2(X, Y))} \underline{GF(X \otimes Y)}.$$

$F: \mathcal{C} \rightarrow \mathcal{D}$ strong monoidal functor

Def: $F^{\text{rev}}: \mathcal{C}^{\text{rev}} \rightarrow \mathcal{D}^{\text{rev}}$, with.

$$(F^{\text{rev}})_0 = F_0^{-1}, \quad (F^{\text{rev}})_2(X, Y) = F_2(Y, X)^{-1}.$$

\mathcal{C} pivotal category, double dual functor $?^{**}: \mathcal{C} \rightarrow \mathcal{C}$.

$$\text{with } X \mapsto X^{**} = (X^*)^*, \quad f \mapsto f^{**} = (f^*)^*.$$

Recall: $?^*: \mathcal{C}^{\text{rev}} \rightarrow \mathcal{C}$ strong monoidal functor

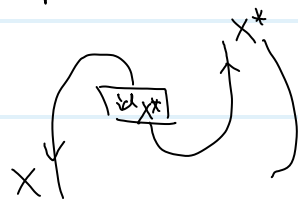
$$(?^*)^{\text{rev}}: (\mathcal{C}^{\text{rev}})^{\text{rev}} = \mathcal{C} \rightarrow \mathcal{C}^{\text{rev}},$$

Then $?^{**} = ?^* \circ (?^*)^{\text{rev}}: \mathcal{C} \rightarrow \mathcal{C}$ is strong monoidal functor.

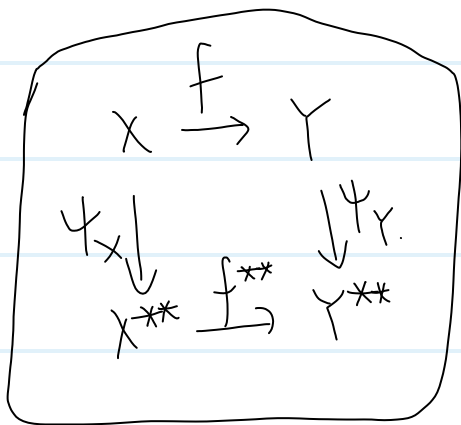
Then: The pivotal structure $\psi: 1_{\mathcal{C}} \rightarrow ?^{**}$ is monoidal.

$$1\psi_x: X \rightarrow X^{**}$$

natural isomorphism.



natural



The top diagram illustrates the naturality of the adjunction. It shows a commutative square with nodes f^{**} , η_X^* , f , and η_Y^* . The edges are η_X^* , f , η_Y^* , and f^{**} . The naturality condition is expressed as $f^{**} = \eta_X^* \circ f \circ \eta_Y^*$.

The bottom diagram shows the composition $\eta_X^* \circ f \circ \eta_Y^*$ and its naturality condition. It shows a commutative square with nodes η_X^* , f , η_Y^* , and $\eta_X^* \circ f \circ \eta_Y^*$. The edges are η_X^* , f , η_Y^* , and $\eta_X^* \circ f \circ \eta_Y^*$. The naturality condition is expressed as $\eta_X^* \circ f \circ \eta_Y^* = \eta_X^* \circ f \circ \eta_Y^*$.

$$(R^*)_{\circ} = ((R^*)^*)^{\dagger} (R^*)_{\circ}$$

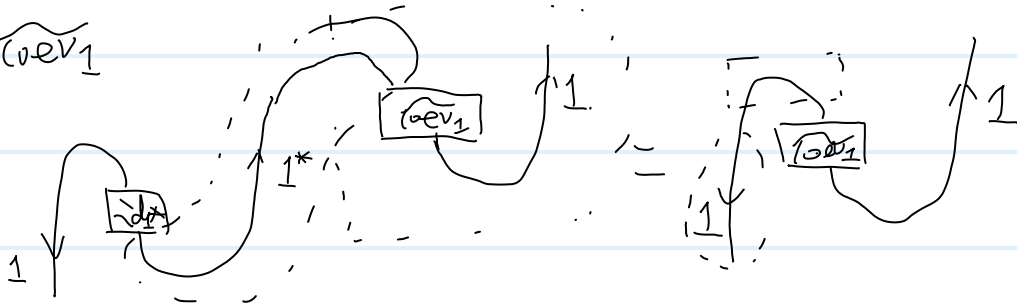
$$\begin{pmatrix} ?^{**} \end{pmatrix}_2 (X, Y) = \begin{pmatrix} ?^* \end{pmatrix}_2 (Y, X)^* \begin{pmatrix} ?^* \end{pmatrix}_2 (X^*, Y^*)$$

$$(17) \quad (P^*)^* \mid_1 = (P^*)_0$$

$$(2) \quad (f^*)_2 (Y, X)^* \varphi_{X \otimes Y} = (f^*)_2 (X^*, Y^*) (\varphi_X \otimes \varphi_Y)$$

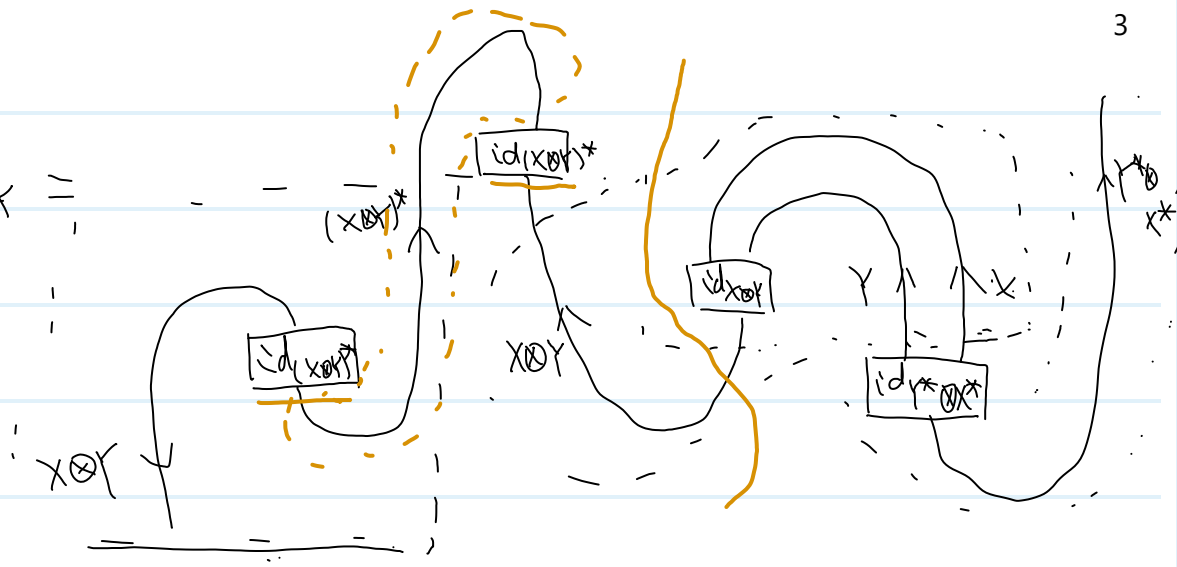
$$(Z^*)_0 = \text{vev}_1 = \widetilde{\text{vev}}_1$$

$$(3^*)^* \begin{array}{c} 4 \\ 1 \end{array} =$$

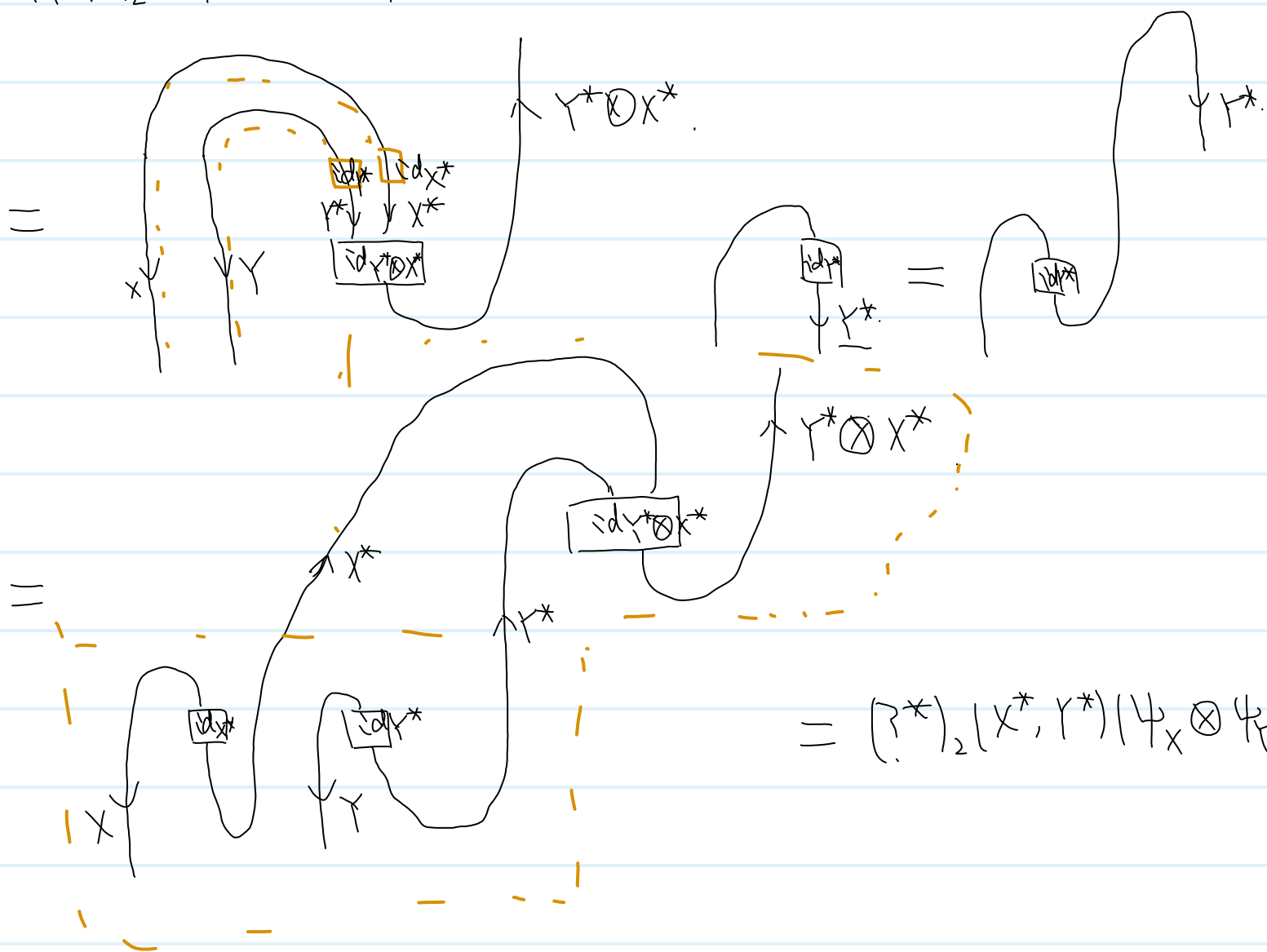


$$= \text{Diagram 1} = \text{Diagram 2} = \overline{\text{Diagram 3}} = (\text{Diagram 4})_0$$

$$(\tau^*)_2(Y, X)^* \psi_{X \otimes Y} =$$



$$((\tau^*)_2(Y, X))^* = Y^* \otimes X^* \rightarrow (X \otimes Y)^*$$



$$= (\tau^*)_2(X^*, Y^*)(\psi_X \otimes \psi_Y)$$

\mathcal{C} pivotal category.

For $X \in \text{ob}(\mathcal{C})$, $\varepsilon \in \{+, -\}$, def $X^\varepsilon = \begin{cases} X & \varepsilon = + \\ X^* & \varepsilon = - \end{cases}$.

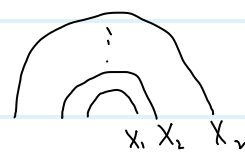
For $S = (X_1, \varepsilon_1, \dots, X_n, \varepsilon_n)$, $X_i \in \text{ob}(\mathcal{C})$, $\varepsilon_i \in \{+, -\}$


def $X_S = X_1^{\varepsilon_1} \otimes \dots \otimes X_n^{\varepsilon_n} \in \text{ob}(\mathcal{C})$

For $S = \emptyset$, set $X_\emptyset = 1$

def $S^* = (X_1, -\varepsilon_1, \dots, X_n, -\varepsilon_n)$, $\emptyset^* = \emptyset$.


Set

$$\underline{\text{ev}}_S = \text{diagram} \quad X_{S^*} \otimes X_S \rightarrow 1$$



$$\underline{\text{coev}}_S = \text{diagram} \quad 1 \rightarrow X_S \otimes X_{S^*}$$


Here the arc labeled with X_i is oriented toward

right endpoint if $\varepsilon_i = +$ and toward the left endpoint if $\varepsilon_i = -$.

$$\text{ev}_{(X, +)} = \text{diagram} = \text{ev}_X$$


$$\text{coev}_{(X, +)} = \text{coev}_X$$

$$\text{ev}_{(X, -)} = \text{diagram} = \overline{\text{ev}}_X$$


$$\text{coev}_{(X, -)} = \overline{\text{coev}}_X$$

$$\begin{array}{c} \boxed{\text{ev}_S} \\ \downarrow \\ X_S^* \end{array} \quad \begin{array}{c} X_S \\ \downarrow \\ \boxed{\text{coev}_S} \end{array} = \text{diagram of two nested arcs} = \text{id}_{X_S^*}$$

$$\begin{array}{c} \boxed{\text{ev}_S} \\ \downarrow \\ X_S \end{array} \quad \begin{array}{c} X_S^* \\ \downarrow \\ \boxed{\text{coev}_S} \end{array} = \text{id}_{X_S}$$

ev_S is non-degenerate with inverse coev_S .

$$X_S \rightarrow (X_{S^*})^*$$

$$\begin{array}{c} \boxed{\text{id}_{X_{\phi=1}}} \\ \downarrow \\ X_{\phi=1} \end{array}$$

Set $\bar{\psi}_{\phi} = \text{coev}_1 = \widetilde{\text{coev}}_1 = 1 \rightarrow 1^*$.

$$S = \phi \quad \phi^* = \phi$$

For $S = ((X_1, \varepsilon_1), \dots, (X_n, \varepsilon_n))$ Def

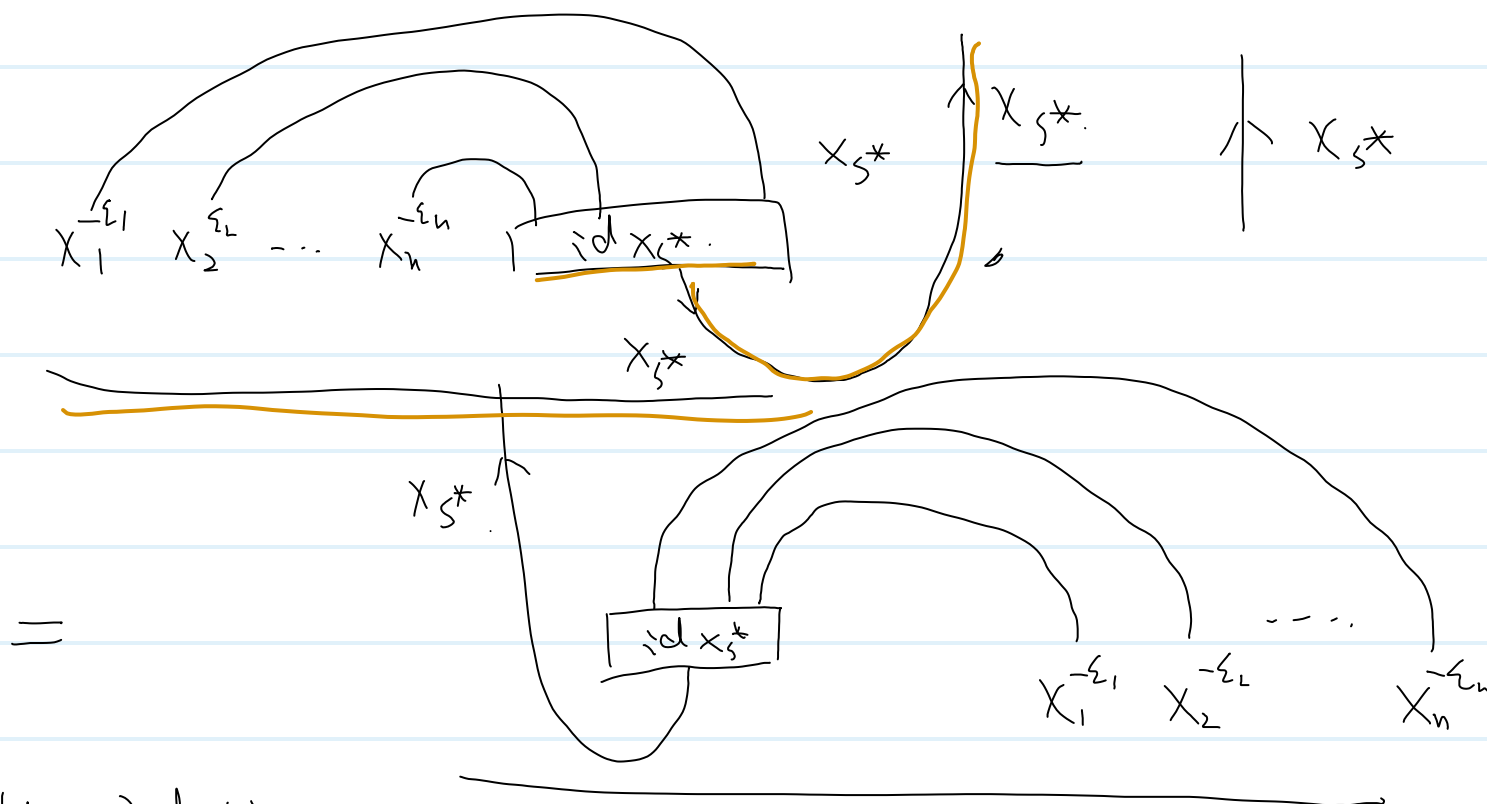
$$\bar{\psi}_S : X_S \rightarrow (X_{S^*})^* \text{ as composition of two maps}$$

$$1. X_S = X_1^{\varepsilon_1} \otimes \dots \otimes X_n^{\varepsilon_n} \rightarrow (X_1^{-\varepsilon_1})^* \otimes \dots \otimes (X_n^{-\varepsilon_n})^*$$

$$n=1, \varepsilon_1 = + \quad X_1 \rightarrow X_1^{**}$$

$$\begin{array}{c} \boxed{\text{id}_{X_1^{\varepsilon_1}}} \\ \downarrow \\ X_1 \end{array} \quad \begin{array}{c} \boxed{\text{id}_{X_2^{\varepsilon_2}}} \\ \downarrow \\ X_2 \end{array} \quad \dots \quad \begin{array}{c} \boxed{\text{id}_{X_n^{\varepsilon_n}}} \\ \downarrow \\ X_n \end{array} = \text{diagram of two nested arcs}$$

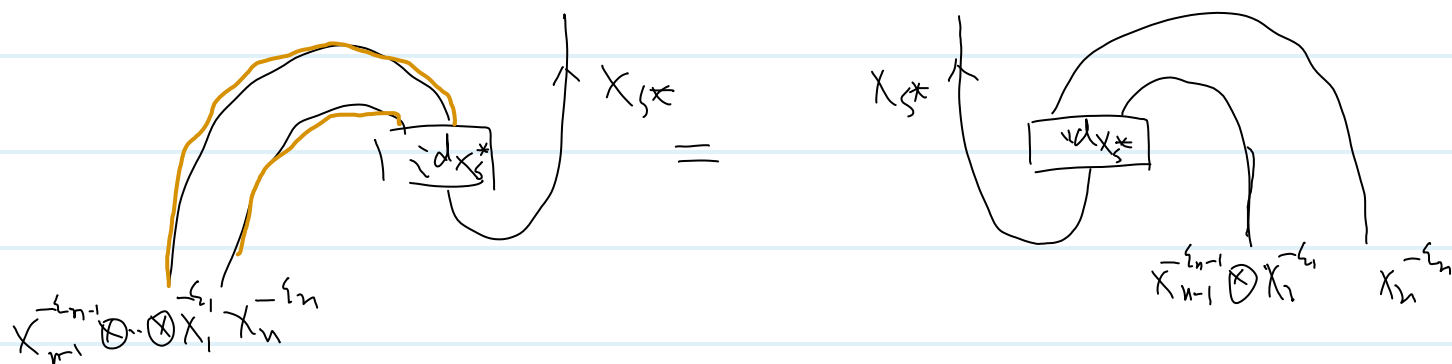
$$2. (X_1^{-\varepsilon_1})^* \otimes \dots \otimes (X_n^{-\varepsilon_n})^* \rightarrow (X_n^{-\varepsilon_n} \otimes \dots) \otimes (X_1^{-\varepsilon_1})^* = \underline{(X_{S^*})^*}$$



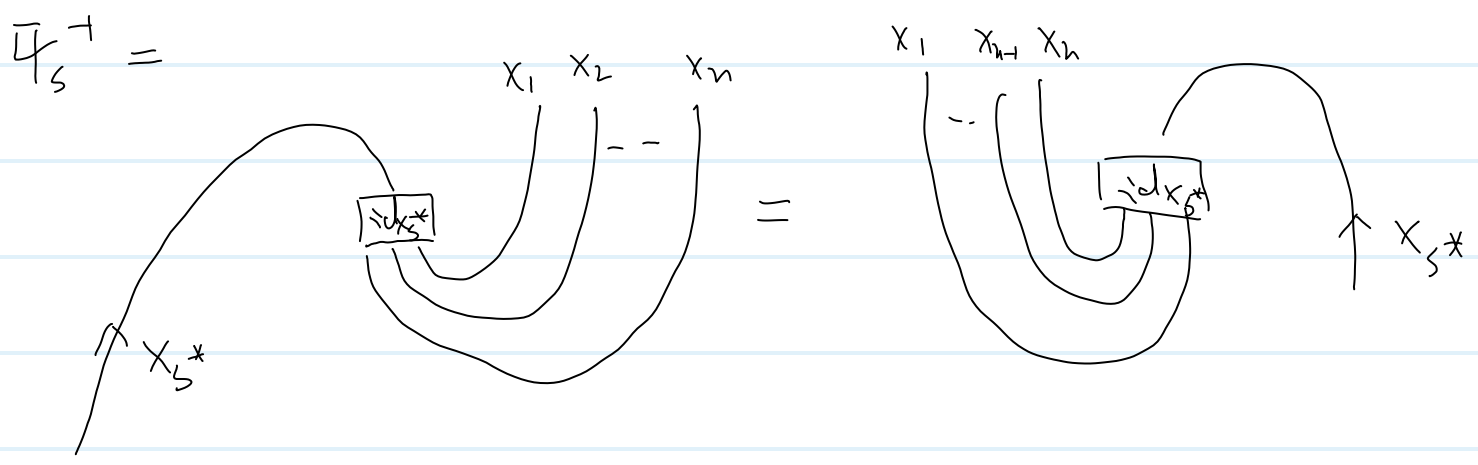
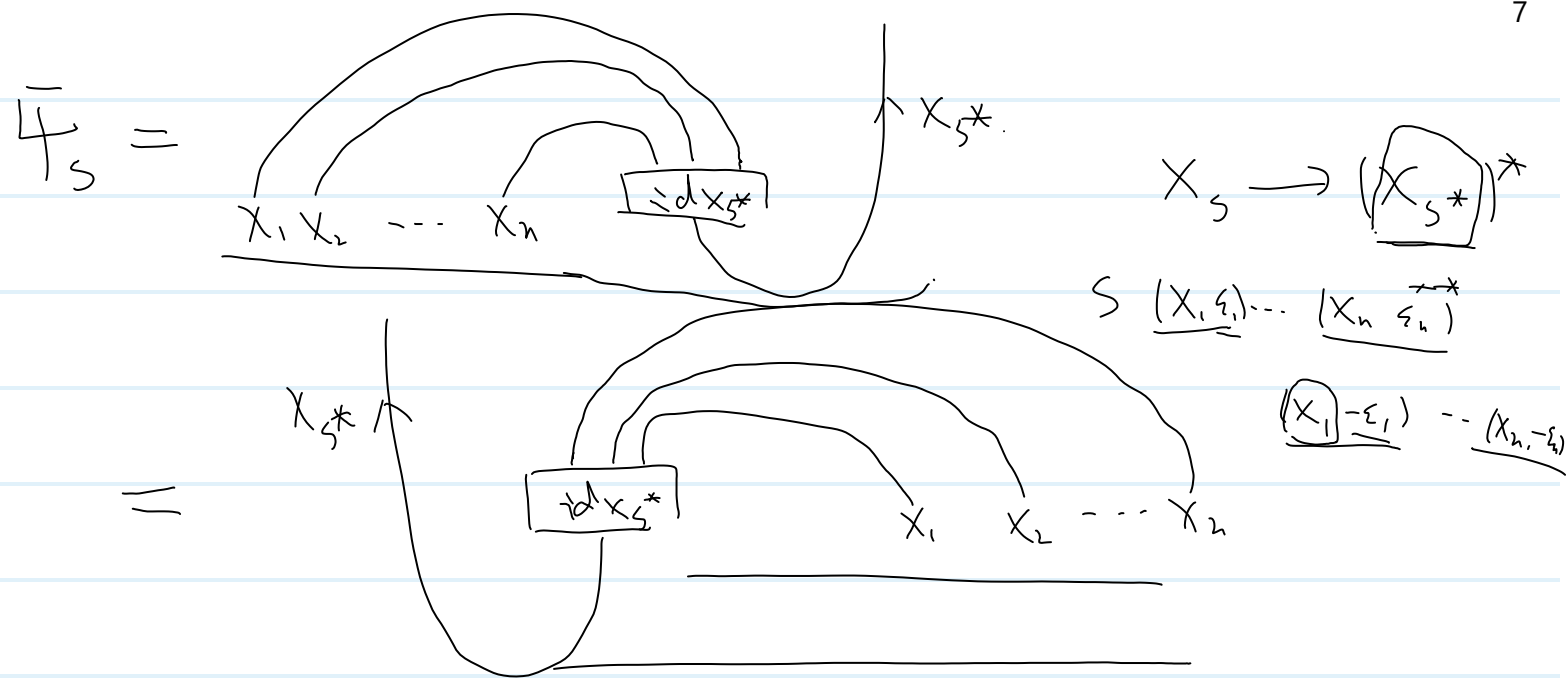
Use induction on n

$n=1, 2$ ✓

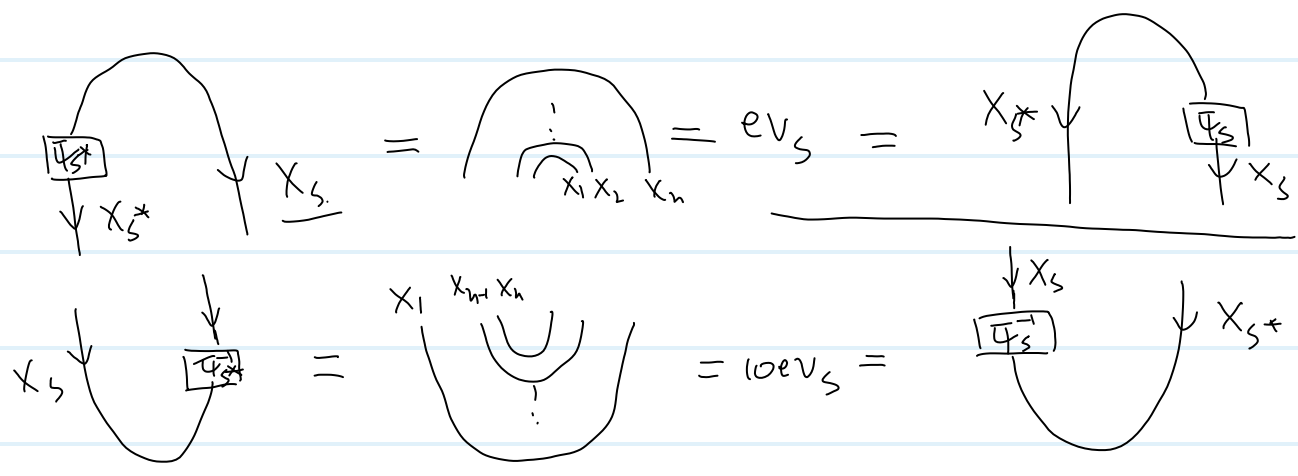
$$(X_{n-1}^{-\varepsilon_{n-1}} \otimes \dots \otimes X_1^{-\varepsilon_1})^* \otimes (X_n^{-\varepsilon_n})^* \rightarrow (X_{S^*})^*$$



↑



Lem: $\bar{\Psi}_S: X_S \rightarrow (X_S^*)^*$, $\bar{\Psi}_S^*: X_S^* \rightarrow (X_S)^*$.



Lem: For any $S = ((X_1, \varepsilon_1) \dots (X_m, \varepsilon_m))$
 $T = ((Y_1, \nu_1) \dots (Y_n, \nu_n))$

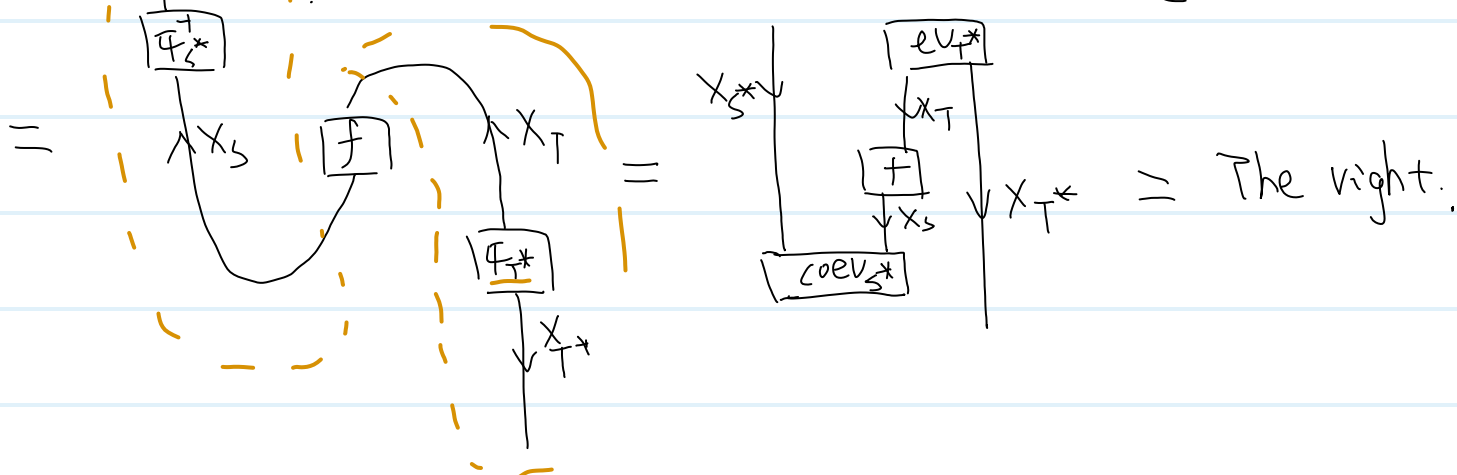
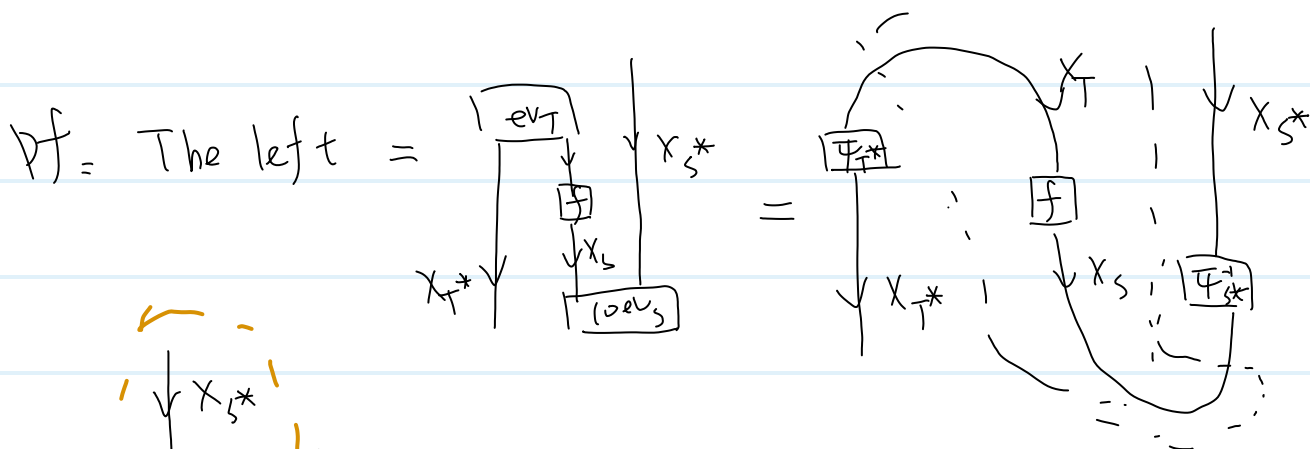
$$f: X_S \rightarrow X_T$$

Then



$$X_T^* \rightarrow X_S^*$$

$$X_T^* \rightarrow X_S^*$$



\mathcal{C} pivotal category, $X \in \text{ob}(\mathcal{C})$, $f \in \text{End}_{\mathcal{C}}(X)$.

Def left trace of f $\text{tr}_l(f) = X \uparrow \boxed{f} \in \text{End}_{\mathcal{C}}(1)$

right trace of f $\text{tr}_r(f) = \boxed{f} \uparrow X \in \text{End}_{\mathcal{C}}(1)$

Prop: $\forall p: X \rightarrow Y$, $q: Y \rightarrow X$, then

$$\text{tr}_l(pq) = \text{tr}_l(qp), \quad \text{tr}_r(pq) = \text{tr}_r(qp)$$

Pf:

$$\text{tr}_l(pq) = Y \uparrow \boxed{p} \downarrow \boxed{q} X \quad = \quad \begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \end{array} \quad \begin{array}{c} \boxed{p} \\ \downarrow \\ \boxed{q} \end{array} \quad = \quad \begin{array}{c} \boxed{p^*} \\ \downarrow \\ \boxed{q} \end{array}$$

$$\stackrel{\text{dual}}{=} X \uparrow \boxed{q} \downarrow \boxed{p} = \text{tr}_l(qp).$$

$\forall \alpha \in \text{End}_{\mathcal{C}}(1)$, $f \in \text{End}_{\mathcal{C}}(X)$, $g \in \text{End}_{\mathcal{C}}(Y)$. Then

$$\text{tr}_l(\alpha) = \text{tr}_r(\alpha) = \alpha, \quad \text{tr}_l(f \cdot \alpha) = \alpha \text{tr}_l(f), \quad \text{tr}_r(\alpha \cdot f) = \alpha \text{tr}_r(f).$$

$$\text{tr}_l(f \otimes g) = \text{tr}_l(\text{tr}_l(f) \cdot g), \quad \text{tr}_l(f) = \text{tr}_r(f^*)$$

$$\text{tr}_r(f \otimes g) = \text{tr}_r(f \cdot \text{tr}_r(g)), \quad \text{tr}_r(f) = \text{tr}_l(f^*)$$

Pf: $\text{tr}_l(\alpha) = 1 \uparrow \boxed{\alpha} = \begin{array}{c} \boxed{\text{ev}_1} \\ \downarrow 1 \\ \boxed{\text{coev}_1} \end{array} \boxed{\alpha} = \boxed{\begin{array}{c} \boxed{\text{ev}_1} \\ 1 \uparrow \\ \boxed{\text{coev}_1} \end{array}} \boxed{\alpha} = \boxed{\alpha}.$

$\text{tr}_l(f \cdot \alpha) = \uparrow \boxed{f} \boxed{\alpha} = \text{tr}_l(f) \cdot \alpha = \alpha \text{tr}_l(f).$

$\text{tr}_l(f \otimes g) = \otimes \boxed{f \otimes g} = \uparrow \boxed{f} \boxed{g} = \text{tr}_l(\text{tr}_l(f) \cdot g)$

$\text{tr}_l(f) = X \uparrow \boxed{f} \stackrel{\text{dual}}{=} \boxed{f^*} \downarrow X = \begin{array}{c} \boxed{f^*} \\ \downarrow X \\ \boxed{f} \\ \uparrow X \end{array} = \boxed{f^*} \downarrow X \uparrow X^* = \boxed{f^*} \downarrow X^*.$

$\psi_X: \begin{array}{c} \uparrow X^* \\ \boxed{\text{id}_{X^*}} \\ \downarrow X \end{array} \quad \psi_X^{-1}$

$\text{tr}_l(f) = \text{tr}_l(f^{**}), \quad \text{tr}_r(f) = \text{tr}_r(f^{**})$

For $f: X^* \rightarrow X^*.$

$\text{tr}_l(f) = X \downarrow \boxed{f}$

$\text{tr}_r(f) = \boxed{f} \downarrow X.$

Def left dimension
right

$\dim_l(X) = \text{tr}_l(\text{id}_X)$

$X \in \text{ob}(\mathcal{C})$

$\dim_r(X) = \text{tr}_r(\text{id}_X)$

$X \in \text{ob}(\mathcal{C})$

A pivotal category is spherical, if $\text{tr}_r(f) = \text{tr}_l(f)$, $\forall f \in \text{End}_\ell(x)$.
 Def trace of f : $\text{tr}(f) = \text{tr}_l(f) = \text{tr}_r(f)$. □