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目 录(Catalog)

引言(Introduction)

Determinant, in fact, is a function in mathematic. We usually using |A| or det(A) to represent the value of the determinant A and it is a scalar. The definition of determinant first came into being in the late 17^{th} century when some mathematicians studied how to solve the system of the linear equations. In 1683, the Japanese mathematician Guanxiaohe(关孝和) first introduce the "determinant" (at that time, this name have not formed yet) in one of his works, which was used to solve high order equations. 10 years later, German mathematician Leibnitz(莱布尼茨) obtained some search findings about determinant but that were unknown at that time. In 1771, French mathematician Vandermonde(范德蒙德) first separate determinant from equations, which was regarded the beginning of studying determinant itself. In 1812, French mathematician Augustin Louis Cauchy(柯西) first used the word "determinant" to represent determinant. After approximately 300 years of development, nowadays, determinant is a greatly essential tool to study linear algebra. What is more, it also has numerous uses in many other branches of math and other subjects such as social economic, mechanics and so on. Therefore, searching some efficient algorithm to compute the value of a determinant is imperative and meaningful. Firstly, we must know some basic knowledge about determinant including definition, nature etc. Then we can explore good algorithm. As the old saying goes, "more preparation may quicken the speed in doing work".

行列式(Determinant)

定义(Definition)

We call

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1,n-1} & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2,n-1} & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3,n-1} & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1,1} & a_{n-1,2} & a_{n-1,3} & \cdots & a_{n-1,n-1} & a_{n-1,n} \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{n,n-1} & a_{nn} \end{vmatrix}$$

n-order determinant and usually use following symbol to represent.

$$A=\det(a_{ij})=|a_{ij}|_n \ (i, j=1, 2, 3\ldots n-1, n)$$

Where a_{ij} is called the element of A

性质(Nature)

In the following statement, A_k represents line k of A and C_k represents column k of A. (k=1, 2, 3 … n)

I. $|A|=|A^T|$, that is

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1,n-1} & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2,n-1} & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3,n-1} & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1,1} & a_{n-1,2} & a_{n-1,3} & \cdots & a_{n-1,n-1} & a_{n-1,n} \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{n,n-1} & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{21} & a_{31} & \cdots & a_{n-1,1} & a_{n1} \\ a_{12} & a_{22} & a_{32} & \cdots & a_{n-1,2} & a_{n2} \\ a_{13} & a_{23} & a_{33} & \cdots & a_{n-1,3} & a_{n3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{1,n-1} & a_{2,n-1} & a_{3,n-1} & \cdots & a_{n-1,n-1} & a_{n,n-1} \\ a_{1n} & a_{2n} & a_{3n} & \cdots & a_{n-1,n} & a_{nn} \end{vmatrix}$$

For example, if A is

$$\begin{vmatrix} 3 & 2 \\ 1 & 4 \end{vmatrix}$$

Then A^T is

$$\begin{vmatrix} 3 & 1 \\ 2 & 4 \end{vmatrix}$$

Obviously, $|A|=3\times4-2\times1=10=3\times4-1\times2=|A^T|$

II. If let A_i add c times of A_j and get a new determinant B, then the value of B equals A. ($i, j=1, 2, 3\ldots n-1, n$) For example, if A is

$$\begin{vmatrix} 3 & 2 \\ 1 & 4 \end{vmatrix}$$

If let $A_1+(-3)A_2$ and get a new determinant B, which is

$$\begin{vmatrix} 0 & -10 \\ 1 & 4 \end{vmatrix}$$

Then $|B|=0\times4-1\times(-10)=10=|A|$.

If let $C_2+(-4)C_1$ and get a new determinant B, which is

$$\begin{vmatrix} 3 & -10 \\ 1 & 0 \end{vmatrix}$$

Then $|B|=3\times0-(-10)\times1=10=|A|$.

III. If all elements in A_k(or C_k) have a common factor c, then we can extract c to get a new determinant B. ($k=1, 2\ldots n-1, n$) In addition, we have

$$|A|=c|B|$$

For example, if A is

$$\begin{vmatrix} 12 & 6 \\ 4 & 3 \end{vmatrix}$$

Obviously, $|A|=12\times3-4\times6=12$. If extract 6 from A_1 and get a new determinant B, which is

$$\begin{vmatrix} 2 & 1 \\ 4 & 3 \end{vmatrix}$$

Then $|B|=2\times3-1\times4=2$, and $|A|=6|B|=12$.

If extract 4 from C_1 and get a new determinant B, which is

$$\begin{vmatrix} 3 & 6 \\ 1 & 3 \end{vmatrix}$$

Then $|B|=3\times3-1\times6=3$, and $|A|=4|B|=12$.

Ⅳ. If swap A_i and A_j (or swap C_i and C_j) and get a new determinant B, then the value of B is -1 times of A. (i, j=1, 2, 3…n-1, n) Again for example, if A is

$$\begin{vmatrix} 12 & 6 \\ 4 & 3 \end{vmatrix}$$

If swap A_1 and A_2 and get a new determinant B, which is

$$\begin{vmatrix} 4 & 3 \\ 12 & 6 \end{vmatrix}$$

Then |B|=4×6-3×12=-12=-|A|

If swap C_1 and C_2 and get a new determinant B, which is

$$\begin{vmatrix} 6 & 12 \\ 3 & 4 \end{vmatrix}$$

Then |B|=6×4-12×3=-12=-|A|

特殊行列式(Special determinant)

1. Upper (triangle) determinant

When A (or A^T) is like

$$\begin{vmatrix} a_{11} & a_{21} & a_{31} & \cdots & a_{n-1,1} & a_{n1} \\ 0 & a_{22} & a_{32} & \cdots & a_{n-1,2} & a_{n2} \\ 0 & 0 & a_{33} & \cdots & a_{n-1,3} & a_{n3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{n-1,n-1} & a_{n,n-1} \\ 0 & 0 & 0 & \cdots & 0 & a_{nn} \end{vmatrix}$$

That equals

$$a_{ij}=0 \text{ (i, j=1, 2, 3…n-1, n \& i>j)}$$

Then A is called upper (triangle) determinant and the value of A is

$$|A|=\prod_{k=1}^{n} a_{kk} \text{ (k=1, 2, 3…n-1, n)}$$

2. Lower (triangle) determinant

When A (or A^T) is like

$$\begin{vmatrix} 0 & 0 & 0 & \cdots & 0 & a_{n1} \\ 0 & 0 & 0 & \cdots & a_{n-1,2} & a_{n2} \\ 0 & 0 & 0 & \cdots & a_{n-1,3} & a_{n3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & a_{n-1,2} & a_{n-1,3} & \cdots & a_{n-1,n-1} & a_{n,n-1} \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{n,n-1} & a_{nn} \end{vmatrix}$$

That equals

$$a_{ij}=0 \text{ (i, j=1, 2, 3…n-1, n \& i+j} \leqslant \text{n)}$$

Then A is called lower (triangle) determinant and the value of A is

$$|A|=(-1)^{\frac{n(n-1)}{2}}\prod_{k=1}^{n} a_{k,n+1-k} \text{ (k=1, 2, 3…n-1, n)}$$

拉普拉斯定理(Theorem of Laplace)

Supposing A=det (a_{ij}) is an n-order determinant. Choose r lines from A_i and r columns from C_j to form a new r-order determinant B. The other n-r lines and n-r columns also form a new (n-r)-order determinant C. B and C is respectively called r-order minor and (n-r)-order minor of A. Supposing the r lines are l_k and the r columns are c_k, then let

$$M= (-1)^N|C|,\ N=\sum_{k=1}^{r}(c_k+l_k) \text{ (i, j=1, 2, 3…n-1, n \& 1} \leqslant \text{r<n)}$$

M is called the algebraic complement of B.

Choose r lines (or columns) of A at will, then |A| equals the sum of all r-order minor in the r lines (or columns) multiply corresponding algebraic complement. This is the theorem of Laplace. If using S_k to represent r-order of A and M_k to represent corresponding algebraic complement, then

$$|A|=\sum_{k=1}^{N}|S_k|M_k,\ N=\binom{n}{r}(1\le r<n)$$

The theorem of Laplace is a very powerful tool to calculate the value of a determinant. I mainly design two algorithm, one of which is relevant with this theorem.

算法设计和分析(Design and analysis of algorithm)

Supposing A is an n-order determinant that we want to compute, whose element is a_{ij}. (i, j=1, 2, 3…n-1, n)

直接法(Direct)

If the order of A is 1 or 2, we can directly get its value.

When A is

$$|a_{11}|$$

$$|A|=a_{11}$$

When A is

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$|A|=a_{11}a_{22}-a_{12}a_{21}$$

This method is relevant to the function compute of the class Determinant.

展开法(Laplace)

1. According to the theorem of Laplace, particularly, if we choose only one line to compute |A| such as line i (1⩽i⩽n), then

$$|A|=\sum_{k=1}^{n}a_{ik}M_k$$

Where M_k is the algebraic complement of element a_{ik}, and it is

$$(-1)^{i+k}\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1,k-1} & a_{1,k+1} & \cdots & a_{1,n-1} & a_{1,n} \\ a_{21} & a_{22} & \cdots & a_{2,k-1} & a_{2,k+1} & \cdots & a_{2,n-1} & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{i-1,1} & a_{i-1,2} & \cdots & a_{i-1,k-1} & a_{i-1,k+1} & \cdots & a_{i-1,n-1} & a_{i-1,n} \\ a_{i+1,1} & a_{i+1,2} & \cdots & a_{i+1,k-1} & a_{i+1,k+1} & \cdots & a_{i+1,n-1} & a_{i+1,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1,1} & a_{n-1,2} & \cdots & a_{n-1,k-1} & a_{n-1,k+1} & \cdots & a_{n-1,n-1} & a_{n-1,n} \\ a_{n,1} & a_{n,2} & \cdots & a_{n,k-1} & a_{n,k+1} & \cdots & a_{n,n-1} & a_{n,n} \end{vmatrix}$$

As for M_k, in a similar way, we choose only one line to compute its value. Finally, we could get the value of A.

This method is relevant to the function Laplace1 of the class Determinant.

2. Although we can choose 1 line to compute |A|, its efficiency is less than satisfactory (later I will analysis its complexity). Therefore, we can choose two lines to accelerate the computing speed. Supposing choose line k and line l to compute |A| ($1<l\leqslant n$, $1\leqslant k<l$), then

$$|A|=\sum_{i=1}^{n}\sum_{j=i+1}^{n}|S_{ij}|M_{ij}$$

Where S_{ij} is 2-order minor of A that choose column i and column j, and it is

$$\begin{vmatrix} a_{ki} & a_{kj} \\ a_{li} & a_{lj} \end{vmatrix}$$

$$|S_{ij}|=a_{ki}a_{lj}-a_{kj}a_{li}$$

M_{ij} is the algebraic complement of S_{ij}, and it is

$$(-1)^N\begin{vmatrix} a_{11} & \cdots & a_{1,i-1} & a_{1,i+1} & \cdots & a_{1,j-1} & a_{1,j+1} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{k-1,1} & \cdots & a_{k-1,i-1} & a_{k-1,i+1} & \cdots & a_{k-1,j-1} & a_{k-1,j+1} & \cdots & a_{k-1,n} \\ a_{k+1,1} & \cdots & a_{k+1,i-1} & a_{k+1,i+1} & \cdots & a_{k+1,j-1} & a_{k+1,j+1} & \cdots & a_{k+1,n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{l-1,1} & \cdots & a_{l-1,i-1} & a_{l-1,i+1} & \cdots & a_{l-1,j-1} & a_{l-1,j+1} & \cdots & a_{l-1,n} \\ a_{l+1,1} & \cdots & a_{l+1,i-1} & a_{l+1,i+1} & \cdots & a_{l+1,j-1} & a_{l+1,j+1} & \cdots & a_{l+1,n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{n,i-1} & a_{n,i+1} & \cdots & a_{n,j-1} & a_{n,j+1} & \cdots & a_{nn} \end{vmatrix} (N=k+l+i+j)$$

As for M_{ij}, in a similar way, we choose two lines to compute its value. Finally, we could get the value of A.

This method is relevant to the function Laplace2 of the class Determinant.

变换法(Transform)

If A is an upper determinant, with the help of the computer, we could quickly get its value. Because in "special determinant" part we know that the value of an upper determinant equals the product of elements in A's principal diagonal. However, the reality is always cruel. A is not an upper determinant more often. Therefore, the main problem now becomes how to transform a non-upper determinant into an upper determinant. Maybe we can realize this in the light of following steps.

① $A_i=A_i+(-\frac{a_{i1}}{a_{11}})A_1$ (i=2, 3, 4…n-1, n).

② $A_j=A_j+(-\frac{a_{ji}}{a_{ii}})A_i$ (i=2, 3, 4…n-1, j=i+1, i+2…n-1, n).

After above 2 steps, we can get an upper determinant U. According to the nature of determinant (II), we know that

$$|A|=|U|$$

However, in above steps, we may miss one point. Supposing

$$a_{kk}=0 \ (k=1, 2, 3\ldots n-1, n),$$

how can we get the final upper determinant? Don't worry, according to the nature of determinant (IV), we can improve above steps to solve this problem. The new steps are as follows.

① e=0, turn to ②.

② If $a_{11}=0$, swap A_1 & A_i ($a_{i1}\neq0$ & i=2, 3, 4…n-1, n), e=e+1, turn to ③;

If $a_{11}=0$ & $a_{i1}=0$, then |A|=0, end.

③ If $a_{i1}\neq0$, let $c=-\frac{a_{i1}}{a_{11}}$, $A_i=A_i+cA_1$ (i=2, 3, 4…n-1, n), turn to ④.

④ If $a_{ii}=0$, swap A_i & A_j ($a_{ji}\neq0$ & i=2, 3, 4…n-1, j=i+1, i+2…n-1, n), e=e+1, turn to ⑤;

If $a_{ii}=0$ & $a_{ji}=0$, then |A|=0, end.

⑤ If $a_{ji}\neq0$, let $c=-\frac{a_{ji}}{a_{ii}}$, $A_j=A_j+cA_i$ (=2, 3, 4…n-1, j=i+1, i+2…n-1, n)

After above *5* steps, if |A| dose not equal 0, we finally can get an upper determinant U. According to the nature of determinant (II & IV), we know that

$$|A|=(-1)^e|U|$$

Where e is the number of swap times.

This method is relevant to the function transform of the class Determinant.

Until now, I have mainly introduced two methods to calculate A, but which one is better? The answer is the “Transform” method, which is quicker than “Laplace” method in most cases. Next, I will analysis their complexity to prove this.

复杂度分析(Analysis of complexity)

I use the number of times that doing addition, subtraction, multiplication and division to measure the complexity of every methods. That is not accurate but can surely tell us which method is quicker. Using symbol f(n) to represent the complexity of computing a n-order determinant.

直接法(Direct)

$f(1)=O(1)$, $f(2)=O(3)$.

展开法(Laplace)

1. Multiplication: $N_1=n$ (compute $a_{ik}M_k$).

Addition: $N_2=n-1$ (compute the sum).

Minors: $N_3=nf(n-1)$.

Therefore, $f(n)=N_1+N_2+N_3=2n-1+nf(n-1)=O(n)+nf(n-1)\approx O(n!)$.

2. Multiplication: $N_1=\binom{n}{2}=\frac{n(n-1)}{2}$ (compute $S_{ij}M_{ij}$).

Addition: $N_2=N_1-1$ (compute the sum).

Minors: $N_3=3N_1f(n-2)$.

Therefore, $f(n)=N_1+N_2+N_3=n^2-n-1+1.5n(n-1)f(n-2)=O(n^2)+1.5n(n-1)f(n-2)\approx O(n!)$.

变换法(Transform)

Division: $N_1=(n-1)+(n-2)+\ldots+2+1=\frac{n}{2}(n-1)$(compute c).

Multiplication: $N_2=nN_1$(compute cA_k)+n(compute $(-1)^e|U|$)$=\frac{n}{2}(n^2-n+2)$.

Addition: $N_3=N_1$(compute $A_j=A_j+cA_i$)+(n-1)(compute e)$=\frac{n}{2}(n+1)-1$

Therefore, $f(n)=N_1+N_2+N_3=\frac{n}{2}(n^2+n+2)-1=O(n^3)$.

With calculation, we know

Order/n	3	4	5	6	7	8	9	10
n!	6	24	120	720	5040	40320	362880	3628800
n^3	27	64	125	216	343	512	729	1000

According to above table, we know when $2<n<6$, "Laplace" method may be more efficient; however, if $n\geq 6$, "Transform" method is much faster (when n=10, it is about 3628 times faster than "Laplace"!). Therefore, at most case, "Transform" is a wise choose to compute high order determinant!

结束语(Conclusion)

Although I provide two methods to compute the value of a determinant, there are surely many other methods to compute. For example, in "Laplace" method, why don't choose 1 or 2 columns to compute? Why don't choose more lines or columns? What's more, in "Transform" method, we can also consider transforming a non-lower determinant into a lower determinant. In addition, the two methods that I designed must have some shortcomings.

However, using them to calculate some easy determinants or low order determinants is OK!

文件说明(File explanation)

Math.h: some basic math functions, which may be useful.

Determinant.h: the class Determinant, includes its definition and body. Aside from the functions that compute determinant, there are some other functions. You can read it by yourself. (I have written some explanation, though it is very easy)

main.cpp: the main function, that’s all.

ReaderMe: the file you are reading now.

Data: there are some specific determinant with its correct value and some formula with derivation.