Test Data

Provided by XuFanY, Nankai University, 2019.2.13

行列式(DETERMINANT)

3
1 2 3
2 3 1
3 1 2

Value: -18

4
1 2 3 4
2 3 4 1
3 4 1 2
4 1 2 3

Value: 160

4
3 1 -1 2
-5 1 3 -4
2 0 1 -1
1 -5 3 -3

Value: 40

4
3.5 1.5 1.5 1.5
1.5 3.5 1.5 1.5
1.5 1.5 3.5 1.5
1.5 1.5 1.5 3.5

Value: 64

4
1 2 3 4
1 0 1 2
3 -1 -1 0
1 2 0 -5

Value: -24

4
1 2 0 0
3 4 0 0
2 1 -1 3
1 7 5 1

Value: 32

4
2 0 -4 -1
3 6 1 1
3 -13 12 -1
2 3 3 1

Value: 50

4
3 1 1 1
1 3 1 1
1 1 3 1
1 1 1 3

Value: 48

4
1 2 3 4
1 0 1 2
3 -1 -1 0
1 2 0 -5

Value: -24

5
1 2 3 4 5
2 3 4 5 1
3 4 5 1 2
4 5 1 2 3
5 1 2 3 4

Value: 1875

5
-2.2 1.8 1.8 1.8 1.8
-1.2 -2.2 1.8 1.8 1.8
-1.2 -1.2 -2.2 1.8 1.8
-1.2 -1.2 -1.2 -2.2 1.8
-1.2 -1.2 -1.2 -1.2 -2.2

Value: -410.2

5
2 1 0 0 0
1 2 1 0 0
0 1 2 1 0
0 0 1 2 1
0 0 0 1 2

Value: 6

5
1 1 1 1 1
2 3 4 5 6
4 9 16 25 36
8 27 64 125 216
16 81 256 625 1296

Value: 288

6
1 2 3 4 5 6
2 3 4 5 6 1
3 4 5 6 1 2
4 5 6 1 2 3
5 6 1 2 3 4
6 1 2 3 4 5

Value: -27216

7
1 2 3 4 5 6 7
2 3 4 5 6 7 1
3 4 5 6 7 1 2
4 5 6 7 1 2 3
5 6 7 1 2 3 4
6 7 1 2 3 4 5
7 1 2 3 4 5 6

Value: -470596

8
1 2 3 4 5 6 7 8
2 3 4 5 6 7 8 1
3 4 5 6 7 8 1 2
4 5 6 7 8 1 2 3
5 6 7 8 1 2 3 4
6 7 8 1 2 3 4 5
7 8 1 2 3 4 5 6
8 1 2 3 4 5 6 7

Value: 9437184

Appendix-partial formula derivation

Supposing A is an n-order determinant. When A is one of special determinant as below, then no matter how big the order is, we could easily get A's value through corresponding formula. Now I will verify these formulas. A_k represents line k of A and C_k represents column k of A. (k=1, 2, 3 … n)

1. When A is like

$$\begin{vmatrix} 1 & 2 & 3 & \dots & n-1 & n \\ 2 & 3 & 4 & \dots & n & 1 \\ 3 & 4 & 5 & \dots & 1 & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ n-1 & n & 1 & \dots & n-3 & n-2 \\ n & 1 & 2 & \dots & n-2 & n-1 \end{vmatrix}$$

The value of A is

$$|A|= (-1)^\wedge\frac{n(n-1)}{2}\cdot\frac{(n+1)n^\wedge(n-1)}{2}(n=1, 2 \dots)$$

(1) A_j-A_{j-1} (j=n, n-1…3, 2), then A is transformed into

$$\begin{vmatrix} 1 & 2 & 3 & \dots & n-1 & n \\ 1 & 1 & 1 & \dots & 1 & 1-n \\ 1 & 1 & 1 & \dots & 1-n & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1-n & \dots & 1 & 1 \\ 1 & 1-n & 1 & \dots & 1 & 1 \end{vmatrix}$$

(2) C_1+C_j (j=2, 3…n-1, n), then A is transformed into

$$\begin{vmatrix} \frac{n(n-1)}{2} & 2 & 3 & \dots & n-1 & n \\ 0 & 1 & 1 & \dots & 1 & 1-n \\ 0 & 1 & 1 & \dots & 1-n & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & 1-n & \dots & 1 & 1 \\ 0 & 1-n & 1 & \dots & 1 & 1 \end{vmatrix}$$

Therefore, according to the theorem of Laplace(拉普拉斯定理),

$$|A|=\frac{n(n-1)}{2}|B|$$

Where B is a determinant like

$$\begin{vmatrix} 1 & 1 & \dots & 1 & 1-n \\ 1 & 1 & \dots & 1-n & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1-n & \dots & 1 & 1 \\ 1-n & 1 & \dots & 1 & 1 \end{vmatrix}$$

Obviously, the order of B is n-1.

(3) B_j-B_1 (j=2, 3…n-2, n-1), then B is transformed into

$$\begin{vmatrix} 1 & 1 & \dots & 1 & 1-n \\ 0 & 0 & \dots & -n & n \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & -n & \dots & 0 & n \\ -n & 0 & \dots & 0 & n \end{vmatrix}$$

(4) C_{n-1}+C_j (j=1, 2…n-3, n-2), then B is transformed into

$$\begin{vmatrix} 1 & 1 & \dots & 1 & -1 \\ 0 & 0 & \dots & -n & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & -n & \dots & 0 & 0 \\ -n & 0 & \dots & 0 & 0 \end{vmatrix}$$

(5) C_j+C_{n-1} (j=1, 2…n-3, n-2), then B is transformed into

$$\begin{vmatrix} 0 & 0 & \dots & 0 & -1 \\ 0 & 0 & \dots & -n & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & -n & \dots & 0 & 0 \\ -n & 0 & \dots & 0 & 0 \end{vmatrix}$$

Again, according to the theorem of Laplace, it is easy to know the value of B, which is

$$|B|= (-1)^\wedge\frac{n(n-1)}{2}.n^\wedge(n-2)$$

Then, we get the value of A, which is

$$|A|= (-1)^\wedge\frac{n(n-1)}{2}.\frac{(n+1)n^\wedge(n-1)}{2}(n=1, 2 \dots)$$

2. When A is like

$$\begin{vmatrix} a & c & c & \dots & c & c \\ b & a & c & \dots & c & c \\ b & b & a & \dots & c & c \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ b & b & b & \dots & a & c \\ b & b & b & \dots & b & a \end{vmatrix}$$

The value of A is

$$|A|= [b(a-c)^n - c(a-b)^n]/(b-c) \quad (b \neq c,\ n=1, 2 \dots)$$

(1) A_j-A_{j+1} (j=1, 2…n-2, n-1), then A is transformed into

$$\begin{vmatrix} a-b & c-a & 0 & \dots & 0 & 0 \\ 0 & a-b & c-a & \dots & 0 & 0 \\ 0 & 0 & a-b & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & a-b & c-a \\ b & b & b & \dots & b & a \end{vmatrix}$$

Using symbol D_n to represent the above determinant, then D_{n-1} is

$$\begin{vmatrix} a-b & c-a & \dots & 0 & 0 \\ 0 & a-b & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & a-b & c-a \\ b & b & \dots & b & a \end{vmatrix}$$

According to the theorem of Laplace, we know

$$|A|=|D_n|=(a-b)|D_{n-1}|+b(-1)^{(n+1)}(c-a)^{(n-1)}=(a-b)|D_{n-1}|+b(a-c)^{(n-1)} \quad ①$$

(2) Let $B=A^T$, then B is

$$\begin{vmatrix} a & b & b & \dots & b & b \\ c & a & b & \dots & b & b \\ c & c & a & \dots & b & b \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ c & c & c & \dots & a & b \\ c & c & c & \dots & c & a \end{vmatrix}$$

B_j-B_{j+1} (j=1, 2…n-2, n-1), then B is transformed into

$$\begin{vmatrix} a-c & b-a & 0 & \dots & 0 & 0 \\ 0 & a-c & b-a & \dots & 0 & 0 \\ 0 & 0 & a-c & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & a-c & b-a \\ c & c & c & \dots & c & a \end{vmatrix}$$

Using symbol E_n to represent the above determinant, then E_{n-1} is

$$\begin{vmatrix} a-c & b-a & \dots & 0 & 0 \\ 0 & a-c & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & a-c & b-a \\ c & c & \dots & c & a \end{vmatrix}$$

Again, according to the theorem of Laplace, we know

$$|B|=|E_n|=(a-c)|E_{n-1}|+c(-1)^{(n+1)}(b-a)^{(n-1)}=(a-c)|E_{n-1}|+c(a-b)^{(n-1)} \quad ②$$

(3) It is obvious that,

$$|A|=|B|,\ |D_{n-1}|=|E_{n-1}|$$

Therefore, with formula ① and ②, we can get the follow formula,

$$|A|= [b(a-c)^n - c(a-b)^n]/(b-c) \quad (b\neq c,\ n=1, 2 \dots)$$

3. in the second part (2.), if b equals c, which means A is like

$$\begin{vmatrix} a & b & b & \dots & b & b \\ b & a & b & \dots & b & b \\ b & b & a & \dots & b & b \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ b & b & b & \dots & a & b \\ b & b & b & \dots & b & a \end{vmatrix}$$

The value of A is

$$|A|= [a+(n-1)b](a-b)^{n-1} \quad (n=1, 2 \dots)$$

(1) A_j-A_{j+1} (j=1, 2…n-2, n-1), then A is transformed into

$$\begin{vmatrix} a-b & b-a & 0 & \dots & 0 & 0 \\ 0 & a-b & b-a & \dots & 0 & 0 \\ 0 & 0 & a-b & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & a-b & b-a \\ b & b & b & \dots & b & a \end{vmatrix}$$

(2) C_j+C_{j-1}(j=2, 3…n-1, n), then A is transformed into

$$\begin{vmatrix} a-b & 0 & 0 & \dots & 0 & 0 \\ 0 & a-b & 0 & \dots & 0 & 0 \\ 0 & 0 & a-b & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & a-b & 0 \\ b & 2b & 3b & \dots & (n-1)b & a+(n-1)b \end{vmatrix}$$

Then the value of A is

$$|A|= [a+(n-1)b](a-b)^{n-1}\ (n=1, 2\ \dots)$$

4. When A is a Vandermonde determinant (usually using V_n to represent), which is like

$$\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ x_1 & x_2 & x_3 & \cdots & x_{n-1} & x_n \\ x_1^2 & x_2^2 & x_3^2 & \cdots & x_{n-1}^2 & x_n^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ x_1^{n-2} & x_2^{n-2} & x_3^{n-2} & \cdots & x_{n-1}^{n-2} & x_n^{n-2} \\ x_1^{n-1} & x_2^{n-1} & x_3^{n-1} & \cdots & x_{n-1}^{n-1} & x_n^{n-1} \end{vmatrix}$$

The value of A is

$$|A|= \prod_{\substack{0<i<n \\ i<j\leq n}} (x_j - x_i)\ (n=2, 3\ \dots)$$

(1) $A_j-x_1A_{j-1}$ (j=n, n-1…3, 2), then A is transformed into

$$\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 0 & x_2-x_1 & x_3-x_1 & \cdots & x_{n-1}-x_1 & x_n-x_1 \\ 0 & x_2(x_2-x_1) & x_3(x_3-x_1) & \cdots & x_{n-1}(x_{n-1}-x_1) & x_n(x_n-x_1) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & x_2^{n-3}(x_2-x_1) & x_3^{n-3}(x_3-x_1) & \cdots & x_{n-1}^{n-3}(x_{n-1}-x_1) & x_n^{n-3}(x_n-x_1) \\ 0 & x_2^{n-2}(x_2-x_1) & x_3^{n-2}(x_3-x_1) & \cdots & x_{n-1}^{n-2}(x_{n-1}-x_1) & x_n^{n-2}(x_n-x_1) \end{vmatrix}$$

According to the theorem of Laplace and the nature of determinant(III), we know that,

$$|A|=|V_n|=\coprod_{1<j\le n}(x_j-x_1)\,|V_{n-1}|$$

Which V_{n-1} is

$$\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ x_2 & x_3 & x_4 & \cdots & x_{n-1} & x_n \\ x_2^2 & x_3^2 & x_4^2 & \cdots & x_{n-1}^2 & x_n^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ x_2^{n-2} & x_3^{n-2} & x_4^{n-2} & \cdots & x_{n-1}^{n-2} & x_n^{n-2} \\ x_2^{n-1} & x_3^{n-1} & x_4^{n-1} & \cdots & x_{n-1}^{n-1} & x_n^{n-1} \end{vmatrix}$$

It is an n-1 order Vandermonde determinant.

(2) In a similar way, for V_{n-1}, it is easy to know that,

$$|V_{n-1}|=\coprod_{2<j\le n}(x_j-x_2)\,|V_{n-2}|$$

Then for V_{n-2}, we have

$$|V_{n-2}|=\coprod_{3<j\le n}(x_j-x_3)\,|V_{n-3}|$$

$$\cdots$$

$$|V_4|=\coprod_{n-3<j\le n}(x_j-x_{n-3})\,|V_3|$$

$$|V_3|=\coprod_{n-2<j\le n}(x_j-x_{n-2})\,|V_2|$$

Obviously, V_2 is

$$\begin{vmatrix} 1 & 1 \\ x_{n-1} & x_n \end{vmatrix}$$

And

$$|V_2| = x_n - x_{n-1}$$

Therefore, the value of A is

$$|A| = \coprod_{\substack{0<i<n \\ i<j\le n}} (x_j - x_i) \text{ (n=2, 3 …)}$$