A Handful of Combinatorial Identities

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September 2022

Easy (Pascal's Identity)

Prove that

$$\binom{n}{r} + \binom{n}{r+1} = \binom{n+1}{r+1}$$

Easy

Show that

$$\binom{n}{r} = \binom{n}{n-r}$$

Easy

Prove that for $n \in \mathbb{N}$,

$$\sum_{i=0}^{n} \binom{n}{i} = 2^n$$

Easy

Compute the sum

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots$$

for any positive integer n

Easy

Prove that

$$k\binom{n}{k} = n\binom{n-1}{k-1}$$

Easy

Show that

$$\binom{n}{0} + 2\binom{n}{1} + 4\binom{n}{2} + \cdots + 2^n\binom{n}{n} = 3^n$$

Easy

Show that

$$1 + 2 + \cdots + n = \binom{n+1}{2}$$

Medium (Hockey Stick Identity)

Show that

$$\sum_{i=k}^{n} \binom{i}{k} = \binom{n+1}{k+1}$$

Medium

Prove that for all positive integers $0 \leq k \leq r \leq n$,

$$\binom{n}{r}\binom{r}{k} = \binom{n}{k}\binom{n-k}{r-k}$$

Medium (Vandermonde's Identity)

Prove that

$$\sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k} = \binom{m+n}{r}$$

Medium

Find a closed form expression for

$$\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{i}$$

Medium

Prove that

$$\binom{n}{1} + 6\binom{n}{2} + 6\binom{n}{3} = n^3$$

Medium

Show that

$$\sum_{i=0}^{n} \binom{n}{i}^2 = \binom{2n}{n}$$

Medium

Show that

$$\left(\sum_{i=1}^{m} a_i\right)^n = \sum_{e_1+e_2+\cdots+e_n=m, e_i \geq 0} \binom{m}{e_1, e_2, \cdots, e_n} \prod_{i=1}^{m} a_i^{e_i}$$

Medium

Prove that

$$\binom{\binom{n}{2}}{2} = 3\binom{n+1}{4}$$

Medium

Show that

$$\sum_{k \text{ odd}} \binom{n}{k} = \sum_{k \text{ even}} \binom{n}{k}$$

Medium

Show that

$$\sum_{i=0}^{n} i(n-i) = \binom{n+2}{3}$$

Hard

Prove that

$$\sum_{i=0}^{n} 2^i \binom{n}{i} \binom{n-i}{\lfloor (n-i)/2 \rfloor} = \binom{2n+1}{n}$$

Hard (Roots of Unity Filter)

For any $k \in \mathbb{N}$, compute

$$\sum_{i=0}^{\infty} \binom{n}{ik}$$

Hard

Prove that for every integer $n \geq 0$,

$$\sum_{a+b=n} \binom{2a}{a} \binom{2b}{b} = 4^n$$

hard

Let $m \leq n$ be positive integers. Compute

$$\sum_{k=m}^{n} \binom{n}{k} \binom{k}{m}$$

Hard

Evaluate

$$\sum_{i=0}^{n} i^2 \binom{n}{i}$$

Brutal

For $n \geq 0$, compute

$$\sum_{k \leq 0} \binom{2k}{k} \binom{n}{k} \left(-\frac{1}{4}\right)^k$$

Brutal

For $m, n \geq 1$, compute

$$\sum_{k \geq 0} \binom{n+k}{m+2k} \binom{2k}{k} \frac{(-1)^k}{k+1}$$

Brutal

Suppose that $n \in \mathbb{N}$, compute

$$\sum_{a \geq 0} \sum_{b \geq 0} \binom{n-a}{b} \binom{n-b}{a}$$

Brutal

Suppose that $i, j, k \in \mathbb{N}$, compute

$$\sum_{i+j+k=2022} ijk$$

Brutal

Evaluate

$$\sum_{k \geq 0} \frac{\binom{2k}{k}}{3^k}$$