Let's define a sequence $b_i$ such that $b_0 = 2$ and for all non-negative integers $n$

$$b_{n+1} = b_n^2 - b_n + 1.$$

We will now prove that

$$3^{2^{n-1}} < b_{n+1} - 1 < 3^{2^n}$$

is satisfied for all positive integers $n$. We know that $3^{2^n} = (3^{2^{n-1}})^2$ and $b_{n+1} - 1 = b_n^2 - b_n$. Let's break this inequality into two parts.

We will first prove $3^{2^{n-1}} < b_{n+1} - 1$ for all positive integers $n$. Base case: If $n = 1$, then $3^{2^0} = 3$ and $b_2 - 1 = 7 - 1 = 6$. Thus, the inequality is satisfied for $n = 1$ since $3 < 6$. For the inductive step, assume that $3^{2^{n-1}} < b_{n+1} - 1$ is satisifed for a positive integer $n$. We will now prove that this inequality is still satisified for $n + 1$. Plugging in $n + 1$, we have $3^{2^n} = 3^{2^{n-1}} \cdot 3^{2^{n-1}}$ and $b_{n+2} - 1 = b_{n+1}^2 - b_{n+1} = (b_{n+1} - 1)(b_{n+1})$.
Thus, we want to prove

$$3^{2^{n-1}} \cdot 3^{2^{n-1}} < (b_{n+1} - 1)(b_{n+1}).$$

Since we know that $3^{2^{n-1}} < b_{n+1} - 1$, $3^{2^{n-1}} < b_{n+1}$ is also true. Since $n$ is positive, $2^{2^{n-1}}, b_{n+1}, b_{n+1}$ are all positive. Thus, we can multiply the two inequalities together and get

$$3^{2^{n-1}} \cdot 3^{2^{n-1}} < (b_{n+1} - 1)(b_{n+1})$$

which is what we wanted to prove. Thus, we know $3^{2^{n-1}} < b_{n+1} - 1$ is satisfied for all positive integers $n$.

We will now prove $b_{n+1} - 1 < 3^{2^n}$ for all positive integers $n$. Base case: If $n = 1$, then $3^{2^1} = 9$ and $b_2 - 1 = 7 - 1 = 6$. Thus, the inequality is satisfied for $n = 1$ since $6 < 9$. For the inductive step, assume that $b_{n+1} - 1 < 3^{2^n}$ is satisifed for a positive integer $n$. We will now prove that this inequality is still satisified for $n + 1$. Plugging in $n + 1$, we have $3^{2^{n+1}} = 3^{2^n} \cdot 3^{2^n}$ and $b_{n+2} - 1 = b_{n+1}^2 - b_{n+1} = (b_{n+1} - 1)(b_{n+1})$.
Thus, we want to prove

$$(b_{n+1}-1)(b_{n+1}) < 3^{2^n} \cdot 3^{2^n}.$$

We know that $b_{n+1}-1 < 3^{2^n}$. Thus, $b_{n+1} \leq 3^{2^n}$ since $b_{n+1}$ and $3^{2^n}$ are both integers and thus differ by no less than 1. Since $n$ is positive, $2^{2^{n-1}}, b_{n+1}, b_{n+1}$ are all positive. If $b_{n+1} < 3^{2^n}$, then we can multiply the two inequalities together to get

$$(b_{n+1}-1)(b_{n+1}) < 3^{2^n} \cdot 3^{2^n}.$$

If $b_{n+1} = 3^{2^n}$, then multplying left hand side by $b_{n+1}-1$ and the right hand side by $3^{2^n}$ will guarentee

$$(b_{n+1}-1)(b_{n+1}) < 3^{2^n} \cdot 3^{2^n}$$

since $b_{n+1}-1 < 3^{2^n}$. Thus, we have proven both cases, which means $b_{n+1}-1 < 3^{2^n}$ is satisfied for all positive integers $n$.

Since both parts were proved, we have proved that

$$3^{2^{n-1}} < b_{n+1}-1 < 3^{2^n}$$

is satisfied for all positive integers $n$. For every expression in this inequality, we will now take it's inverse, reverse its sign, then add 1. We then get

$$1-\frac{1}{3^{2^{n-1}}} < 1-\frac{1}{b_{n+1}-1} < 1-\frac{1}{3^{2^n}} \$\$ issatisfiedforallpositiveintegers \$n\$. Wecanre$$

\begin{align}
& \frac{1}{bn} = \frac{(b_n-1)^2}{(b_n-1)^2b_n} = \frac{b_n^2 - b_n}{(b_n-1)^2b_n} - \frac{b_n - 1}{(b_n-1)^2b_n}\ &= \frac{1}{b_n-1} - \frac{1}{b_n^2-b_n} = \frac{1}{b_n-1}-\frac{1}{b{n+1}-1}.
\end{align}

$$Bytelescoping, weget$$

\begin{align}
& \sum{i=0}^{n}\frac{1}{b_i} = \frac{1}{b_0-1}-\frac{1}{b{1}-1} + \frac{1}{b1-1}-\frac{1}{b{2}-1} + \cdots + \frac{1}{bn-1}-\frac{1}{b{n+1}-1} \ &= \frac{1}{b0-1} - \frac{1}{b{n+1}-1} = \frac{1}{2-1} - \frac{1}{b{n+1}-1} = 1 - \frac{1}{b{n+1}-1}.
\end{align}

$$There fore, wecanreplace\$1-\frac{1}{b_{n+1}-1}\$inourinequalitywith\$\sum_{i=0}^{n}\frac{1}{b_i}\$sothat$$

1− \frac{1}{3^{2^{n−1}}}<1 - \sum_{i=0}^{n}\frac{1}{b_i}<1− \frac{1}{3^{2^{n}}}
\$\$ is satisfied for all positive integers $n$.

From whats given in the problem, we have

$$a_{n+1}=\frac{a_n^2}{a_n^2-a_n+1}=\frac{1}{1-\frac{1}{a_n}+\frac{1}{a_n^2}}.$$

Lets define $a_0=\frac{1}{2}$. We will now prove $a_n=\frac{1}{b_n}$ for all non-negative integers $n$. The base case is already satisfied, as $a_0=\frac{1}{2}=\frac{1}{b_0}$ since we've defined $b_0=2$. The inductive step is as follows: Assume that $a_n=\frac{1}{b_n}$ is true for a positive integer $n$. We will now prove that this equation is satisifed for $n+1$. We have

$$a_{n+1}=\frac{1}{1-\frac{1}{a_n}+\frac{1}{a_n^2}}=\frac{1}{1-b_n+b_n^2}=\frac{1}{b_{n+1}}.$$

This is based off the fact that $\frac{1}{a_n}=b_n$ and $b_{n+1}=b_n^2-b_n+1$. Thus, we have

$$\frac{1}{2}+a_1+a_2+\cdots+a_n=\frac{1}{b_0}+\frac{1}{b_1}+\frac{1}{b_2}+\cdots+\frac{1}{b_n}=\sum_{i=0}^{n}\frac{1}{b_i}.$$

We can now rewrite our inequality so that

$$1-\frac{1}{3^{2^{n-1}}}<1-\frac{1}{2}+a_1+a_2+\cdots+a_n<1-\frac{1}{3^{2^n}}\$\$issatisfiedforallpositiveintegers$$

\frac{1}{2} − \frac{1}{3^{2^{n−1}}}<a_1+a_2+\cdots+a_n<\frac{1}{2}− \frac{1}
{3^{2^{n}}}

$$whichissatisfiedforallpositiveintegers\$n\$.$$