Medical image analysis: Refresher

Just pointers and reminders

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Outline

- Basic notation
- Probabilistic modeling
- Optimization, cost function and regularization
- Linear basis models and function parameterizations
- Spatial transformations
- Derivative approximations

Basic notation

- **Continuous version**
  A volumetric grayscale image is a function
  $I : \Omega \rightarrow \mathbb{R}$, $\Omega \subset \mathbb{R}^3$ is the image domain
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- For dynamic images with temporal information
  $I : \Omega x \mathbb{R}^+ \to \mathbb{R}^N$
  $I(x, t)$ is the intensity at point x and time t

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  - PDF, CDF, PMF
  - Histogram of an image
  - Conditionals and the Bayes' Rule
  - Posterior distribution, MAP and MLE
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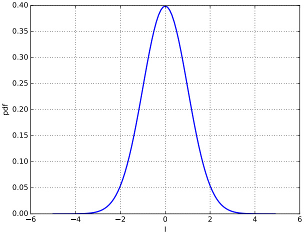

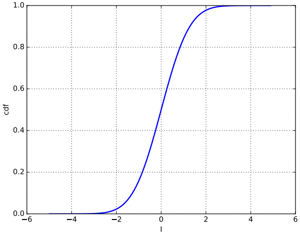

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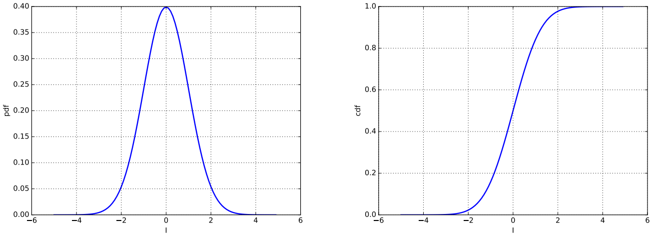
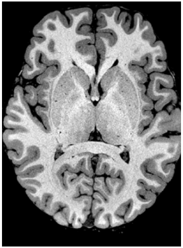


- For discrete random variables, L, we define
$p(l) = p(L = l)$ as its *probability mass function (PMF)*
PMF can be seen as a PDF is often named as PDF in scientific articles

Histogram of an image



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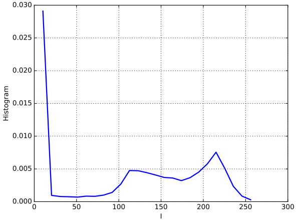

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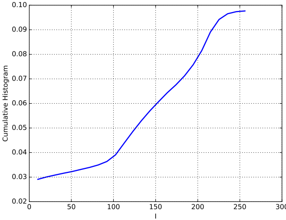

Cumulative histogram

If we consider each pixel intensity as an independent realization of the random variable I then the histogram is an approximation to the PDF and cumulative histogram is an approximation to the CDF

Conditionals and the Bayes' Rule

- When we have two variables such as I and L
  $p(i,l)$: joint distribution
  $p(i) = \sum_{l=0}^{N} p(i,l)$ and $p(l) = \int_{-\infty}^{\infty} p(i,l)di$: marginal distributions
  $p(i|l)$ and $p(l|i)$: conditional distributions
  $p(i,l) = p(i|l)p(l) = p(l|i)p(i)$

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- The conditionals are linked with the Bayes' rule

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- In a large variety of problems one of them is observed and the other not. Assume i is observed in that case
  $p(i|l)$: likelihood
  $p(l)$: prior distribution
  $p(l|i)$: posterior distribution

Posterior distribution, MAP and MLE

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- MLE is the same as MAP when the prior is uniform, i.e. $p(I) = c, \ \forall I$

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Data term and regularization

- Most problems in medical image analysis are formulated as optimization problems

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  - Image registration: Determine T between images I and J

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- The MAP estimate is also in the same form

$$\arg_\theta \max \log p(i|\theta) + \log p(\theta) = \arg_\theta \min - \log p(i|\theta) - \log p(\theta)$$

Regularizers can be thought of as priors with $-\log p(\theta) \propto \mathcal{R}(\theta)$

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- Some references
  - Convex Optimization, Boyd and Vandenberghe, Cambridge University Press, 2004.
  - Nonlinear Programming, Bertsekas, Athena Scientific, 2016.

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- Reference: The Calculus of Variations, Bruce van Brunt, Springer, 2004.

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    - Basics
    - Function parameterizations with linear basis models
    - Kernel-based parameterization
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- In finite vector spaces

$$\vec{v} = a_1 \vec{b}_1 + a_2 \vec{b}_2 + \cdots + a_d \vec{b}_d = \mathbf{B}\vec{a}$$

$\vec{b}_i$ are the basis functions and a_i coefficients.
if $\vec{b}_i$ are orthogonal, i.e. $\vec{b}_i^T \vec{b}_j = 0, \ \forall i \neq j$ then $a_i = \vec{v}^T \vec{b}_i / \|\vec{b}_i\|_2$.
if not then Ordinary Least Square (OLS) regression must be performed

$$\arg_{\vec{a}} \min \|\vec{v} - \mathbf{B}\vec{a}\|_2^2 = \left(\mathbf{B}^T \mathbf{B}\right)^{-1} \mathbf{B}^T \vec{v}$$

For known $\vec{b}_i$, $\vec{a}$ can be a parameterization for $\vec{v}$

Function parameterizations with linear basis models

Global parameterizations

$$f(x) = \sum_{i}^{d} a_i b_i(x)$$

where $b_i(x)$ are smooth basis functions.

- The parameterization is $\vec{a}$
- To determine $\vec{a}$, the space is discretized and a **B** matrix is formed.
- Note that this parameterization cannot represent all functions.
- Examples:
  - polynomials: bias-field correction in MRI
  - splines: non-linear registration

Kernel-based parameterization

$$f(x) = \sum_{i=1}^{d} a_i K(x, x_i)$$

where x_i are the control points and $K(x, x_i)$ is a kernel function

- Radial basis functions are often used: $K(x, x_i) = K(\|x - x_i\|_2)$
- Popular radial basis functions:
  Gaussian
  $$K(\|x - x_i\|_2) = \exp - \frac{\|x - x_i\|_2^2}{\sigma^2}$$
  Thin-plate spline
  $$K(\|x - x_i\|_2) = \|x - x_i\|_2^2 \ln(\|x - x_i\|_2)$$
- Example: landmark based registration, linear/non-linear registration, kernel density estimation

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  - 3 dof in 2D - 2 translation and 1 rotation
  - 6 dof in 3D - 3 translation and 3 rotation
  - Intra-subject registration, multi-modal intra-subject registration

Linear transformations

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- Similarity transformation
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  - 7 dof in 3D - Rigid + scale
  - Used for coarse alignment in inter-subject. Initialization for non-linear

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- Affine transformation
  - 6 dof in 2D
  - 12 dof in 3D
  - Often not used in full dof
  - Instead, 9 dof transformation (3 translation + 3 rotation + 3 scale) may be preferred.

$$\mathbf{T} = \underbrace{\begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & s_z \end{bmatrix}}_{\text{scale}} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & cos(\theta) & -sin(\theta) \\ 0 & sin(\theta) & cos(\theta) \end{bmatrix}}_{\text{rotation around x-axis}} \underbrace{\begin{bmatrix} cos(\phi) & 0 & sin(\phi) \\ 0 & 1 & 0 \\ -sin(\phi) & 0 & cos(\theta) \end{bmatrix}}_{\text{rotation around y-axis}} \underbrace{\begin{bmatrix} cos(\gamma) & sin(\gamma) & 0 \\ -sin(\gamma) & cos(\gamma) & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{rotation around z-axis}} + \underbrace{\begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix}}_{\text{translation}}$$

Non-linear transformations

$$T(\vec{x}) = \vec{x} + \underbrace{\vec{u}(\vec{x})}_{\text{displacement field}}$$

- Used in non-linear registration.
- Different models are used.
- Linear basis function models - additive
- More advanced techniques based on composition of transformations

$$T(\vec{x}) = T_n \circ T_{n-1} \circ \cdots \circ T_1(\vec{x})$$

where $\circ$ is basic function composition
Allows modeling diffeomorphisms
see Computational Anatomy

Transformation related identities

$$T(\vec{x}) = [T_1(\vec{x}), T_2(\vec{x}), T_3(\vec{x})]^T$$

- Jacobian

$$\mathbf{J} = \begin{bmatrix} \frac{\partial T_1}{\partial x_1} & \frac{\partial T_1}{\partial x_2} & \frac{\partial T_1}{\partial x_3} \\ \frac{\partial T_2}{\partial x_1} & \frac{\partial T_2}{\partial x_2} & \frac{\partial T_2}{\partial x_3} \\ \frac{\partial T_3}{\partial x_1} & \frac{\partial T_3}{\partial x_2} & \frac{\partial T_3}{\partial x_3} \end{bmatrix}$$

  Jacobian determinant quantifies local volumetric change.
  $\det(\mathbf{J}) = 1$: no change, $\det(\mathbf{J}) < 1$: compression, $\det(\mathbf{J}) > 1$: expansion
- Divergence: $\nabla \cdot T$ - trace of the Jacobian.
  Gives information about the amount of compression and expansion as well.
- Curl - $\nabla \times T$ - information on the amount of infinitesimal rotation

Outline

- Basic notation
- Probabilistic modeling
- Optimization, cost function and regularization
- Linear basis models and function parameterizations
- Spatial transformations
- Derivative approximations

Derivative approximations

- Finite difference approximations is used the most often
- First order derivative at a grid point x_0 with grid spacing Δx

$$\begin{aligned}\frac{df}{dx}\Big|_{x_0} &\approx \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}\\ &\approx \frac{f(x_0)-f(x_0-\Delta x)}{\Delta x}\\ &\approx \frac{f(x_0+\Delta x)-f(x_0-\Delta x)}{2\Delta x}\end{aligned}$$

- Second order derivative at a grid point x_0 with grid spacing Δx

$$\frac{d^2 f}{dx^2}\Big|_{x_0} \approx \frac{f(x_0+\Delta x)-2f(x_0)+f(x_0-\Delta x)}{\Delta x^2}$$

- Many different approximations exist