Biomedical Imaging FS 2019

Signals and Systems

Use Matlab for programming and display.

Please prepare solutions in pdf format and upload them to the Moodle platform (https://moodle-app2.let.ethz.ch/).

Exercises

1. Plane waves

A plane wave in two dimensions is described by $f(x,y) = e^{i(k_x x + k_y y)}$ with $k_x, k_y \in \mathbb{R}$

- What happens to the waves?

- Write a program that calculates 2D plane waves for any given k_x, k_y.
- Display and examine Re(f), Im(f), Abs(f), Phase(f) for a selection of k_x, k_y.
- What determines the direction of the wave?

  The vector $\begin{pmatrix} k_x \\ k_y \end{pmatrix}$ points in the direction of the wave. It is orthogonal to the wavefront.

- What is the wavelength as a function of k_x, k_y?

  Define a vector in the wave direction, scaled to length 1:

  $\vec{d} = |\vec{k}|^{-1} \begin{pmatrix} k_x \\ k_y \end{pmatrix}$ with $|\vec{k}| = \sqrt{k_x^2 + k_y^2}$

  A shift in this direction by the wavelength λ,

  $\begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = \lambda\, \vec{d},$ (1)

  increments the argument of the exponential by one cycle:

  $k_x \Delta x + k_y \Delta y \;=\; 2\pi$ (2)

  Substitute (1) into (2) and solve for λ:

  $\lambda = 2\pi\, |\vec{k}|^{-1}$

- Think of some linear, shift-invariant (LSI) operation and apply it.

  Any linear and shift-invariant operator is suitable. Differential operators are a common example. The sample code uses the Laplace operator $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$.

– What happens to the waves?

They undergo mere scaling and phase shift, corresponding to global multiplication with a complex number. This reflects the fact that they are eigenfunctions of the operator used. The complex scaling factor is the respective eigenvalue.

For the Laplace operator the eigenvalues are $-|\vec{k}|^2$. This is easy to verify by formal differentiation and by using the demo code.

2. Fast Fourier Transform (FFT)

This exercise studies the behavior of FFT, using a rectangle input as an example.

The prepared code works with a 1D input vector of length 256, with a rectangle of length 16 at the center. Run the code and answer the following questions. Along with the questions, review the related parts of the code.

a) The code first calculates and displays the FFT of the input vector straightforwardly (Figure 1).

Why is the sinc-shaped Fourier transform split in two halves?

As discussed in the lecture, the vector calculated by FFT represents the Fourier transform starting from zero frequency ($k_0 = 0$) and covering only positive frequencies. Due to the transform's periodicity, the second half of the vector is the same as the negative frequency range.

b) The code then performs 'fftshift' on the transform, which swaps its first and second halves (Figure 2). Why does the phase of the transform oscillate rapidly? Isn't the rectangle symmetric so that the transform should be purely real?

Since FFT equally assumes the input vector to start at zero in the original domain the rectangle is not at the origin there but shifted to the right, resulting in a linear phase increase in the Fourier domain (c.f. Fourier shift theorem).

c) The code now includes phase correction, shifting the origin in the original domain to the center of the rectangle (Figure 3). Review the related lecture notes.

d) Next the code attempts to shift the origin simply by fftshift instead, similar to what it does in the Fourier domain (Figure 4). As you see, this works only almost. Why doesn't it work fully?

Because the center of the rectangle does not coincide with a sample in the input vector but sits between two samples. In this case, only phase correction as done in c) can fully center the rectangle at $x = 0$. With a rectangle of odd length in terms of samples, a discrete cyclical shift as done by fftshift could suffice, though.

3. Building a Comb

Build a 1D comb function by starting with a single point impulse in the center and successively adding impulses on the left and right

- At each stage, calculate and examine the Fourier transform. What happens?

  The transform approaches the reciprocal comb.

- What happens when you vary Δx of the original comb?

  The spacing of the transform comb varies reciprocally.

Questions?

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