Biomedical Imaging FS 2015

II. Signals and Systems

Signal = some quantity, function of parameter(s)

E.g. $\mu(\vec{r})$ attenuation coefficient

$M(\vec{r})$ magnetization

$U(t)$ detector voltage

$U(\omega)$ voltage spectrum

Image g ⟵ $\boxed{\text{System}}$ ⟵ Original signal f

2D, space: $g(x,y) = S[f](x,y)$

"input-output equation"

What makes a good imaging system?

Imaging emulates vision!

→ different features should be depicted independently

$$S[f_1 + f_2] = S[f_1] + S[f_2]$$

②

$$S\left[\sum_i a_i f_i\right] = \sum_i a_i S[f_i]$$

S should be <u>linear</u>.

→ depiction of a feature should be <u>independent on position</u>

$$f'(x,y) = f(x+\Delta x, y+\Delta y)$$

$$S[f'](x,y) = S[f](x+\Delta x, y+\Delta y)$$

S should be <u>shift-invariant</u>

Linear shift-invariant (LSI) systems

Key property: Complex exponentials are eigenfunctions

For $k_x, k_y \in \mathbb{C}$, $f(x,y) = e^{i(k_x \cdot x + k_y \cdot y)}$

$$S[f] = H(k_x, k_y) \cdot f$$

↑ eigenvalue $\in \mathbb{C}$

Proof: Consider shifted version

$$f'(x,y) = e^{i(k_x(x+\Delta x) + k_y(y+\Delta y))}$$

$$= e^{i(k_x \cdot \Delta x + k_y \cdot \Delta y)} \cdot f(x,y)$$

Linearity:

$$S[f'] = e^{i(k_x \cdot \Delta x + k_y \cdot \Delta y)} S[f]$$

Shift-invariance:

$$S[f'](x,y) = S[f](x+\Delta x, y+\Delta y)$$

$$\Rightarrow S[f](x+\Delta x, y+\Delta y) = e^{i(k_x \Delta x + k_y \cdot \Delta y)} S[f]$$

Set $x=0$, $y=0$:

$$S[f](\Delta x, \Delta y) = e^{i(k_x \cdot \Delta x + k_y \cdot \Delta y)} S[f](0,0)$$

Rename $\Delta x \to x$, $\Delta y \to y$:

$$S[f] = \underbrace{S[f](0,0)}_{\text{eigenvalue } H(k_x, k_y)} \cdot f \qquad \square$$

LSI multiplies exponential by eigenvalue.
But what about other inputs?

Fourier Transform

Subset of exponentials with $k_x, k_y \in \mathbb{R}$ (plane waves) form a orthogonal basis.

So we can expand f as:

$$f(x,y) = \frac{1}{2\pi} \iint \mathcal{F}[f](k_x,k_y)\, e^{i(k_x x + k_y \cdot y)}\, dk_x\, dk_y$$

(Inverse Fourier transform)

$$\mathcal{F}[f](k_x,k_y) = \frac{1}{2\pi} \iint f(x,y) \cdot e^{-i(k_x x + k_y \cdot y)}\, dx\, dy$$

(Fourier transform)

Image domain	One-to-one mapping	Frequency domain ("k-space")
$f(x,y)$	$\xrightarrow{\mathcal{F}}$ $\xleftarrow[\mathcal{F}^{-1}]{}$	$\mathcal{F}[f](k_x,k_y)$
coordinates units: [mm]		spatial frequencies units: [mm^{-1}]

S is linear, so it transfers each plane wave independently:

$$\mathcal{F}[S[f]](k_x,k_y) = H(k_x,k_y) \cdot \mathcal{F}[f](k_x,k_y)$$

Transfer function

Ok, but what happens in the image domain?

$$S[f] = \mathcal{F}^{-1}[H \cdot \mathcal{F}[f]]$$

Convolution theorem:

$$S[f] = \mathcal{F}^{-1}[H] * f$$

$$S[f](x,y) = \iint \mathcal{F}^{-1}[H](x-x', y-y')\, f(x',y')\, dx'dy'$$

LSI system performs a <u>convolution</u>.

<u>Point spread function</u>

Consider a point signal as input:

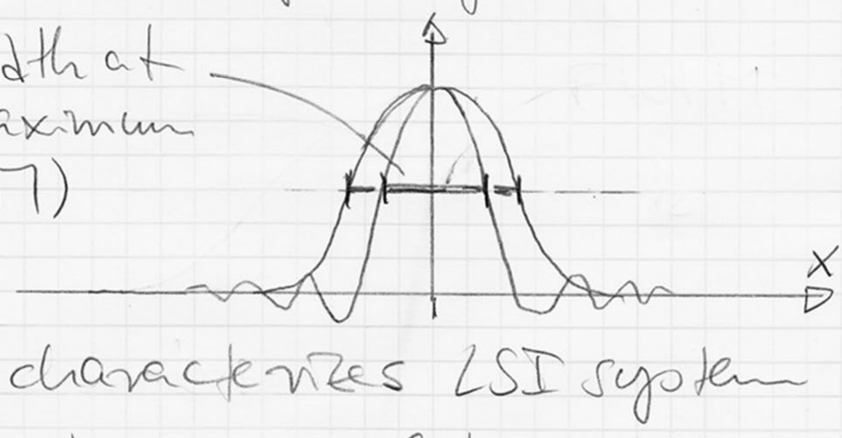
$$\delta(x,y) \begin{cases} \delta(x,y) = 0 & \text{for } (x,y) \neq (0,0) \\ \iint a(x,y)\cdot\delta(x,y)\, dx\, dy = a(0,0) & \end{cases}$$

$$S[\delta](x,y) = \mathcal{F}^{-1}[H](x,y)$$

= image of a point

"point spread function" (PSF)

full width at half maximum (FWHM)



PSF fully characterizes LSI system

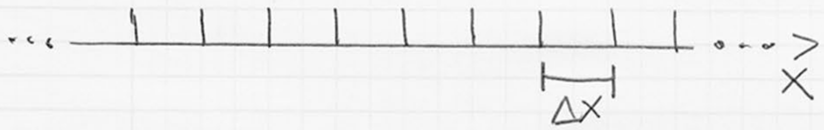
Derived metric: resolution (FWHM)

Sampling and Discrete Signals

- IT is digital
- Data acquisition, processing, storage, display take resources

$\Rightarrow$ Discretize

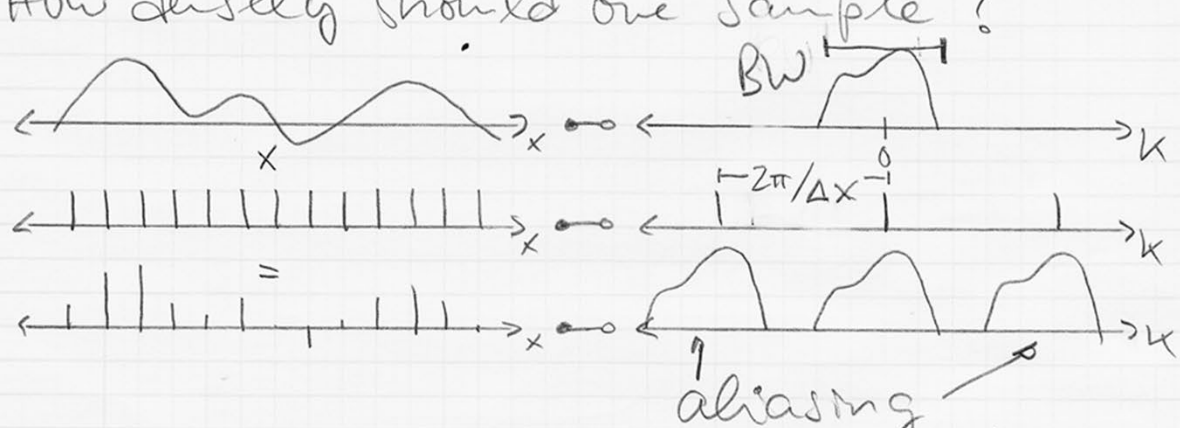
Comb function (shah function)



$$Ш(x;\Delta x) = \sum_{n=-\infty}^{\infty} \delta(x - n\cdot\Delta x) \qquad (1D)$$

$$Ш(x,y;\Delta x,\Delta y) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(x - m\cdot\Delta x, y - n\cdot\Delta y)$$

How densely should one sample?



Prevent aliasing: $\frac{2\pi}{\Delta x} > BW \Rightarrow \Delta x \overset{!}{<} \frac{2\pi}{BW}$

angular frequency terms

Sampling theorem (Kotelnikov 1933, Shannon 1948)

Similarly for sampling k-space:

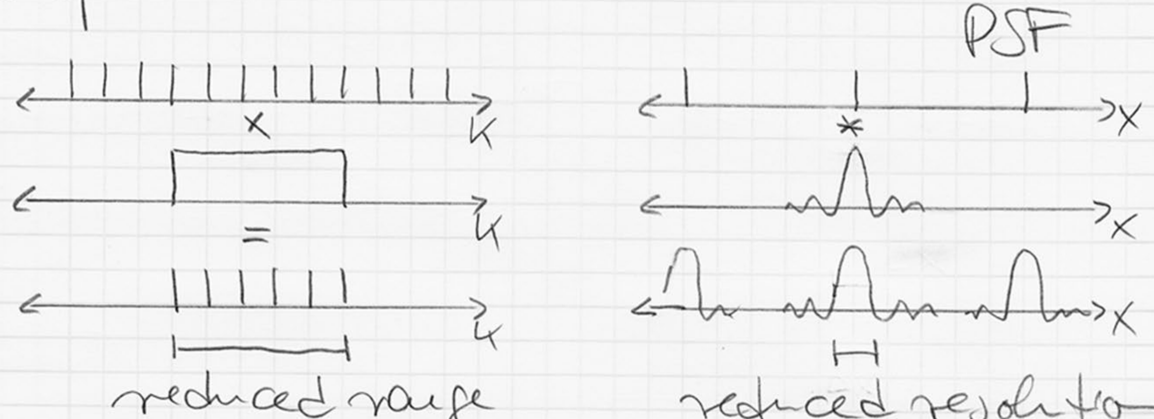
Field of view (FOV) ≙ extent of support in image domain

$$\Delta k \overset{!}{<} \frac{2\pi}{FOV}$$

Which range should one sample?

Image domain: Region of interest

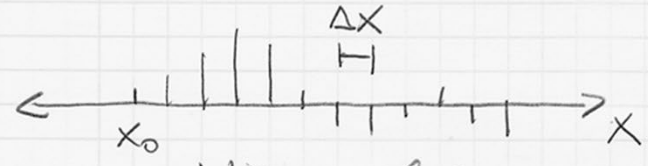
k-space:



⇒ Choose sampling range according to desired resolution

Discrete Fourier Transform (DFT)

Task: Calculate $\mathcal{F}[f]$ for



N samples

$$\mathcal{F}[f](k) = \sum_{n=0}^{N-1} f(x_0 + n \cdot \Delta x)\, e^{-ik(x_0 + n\Delta x)}$$

$$= e^{-ikx_0} \underbrace{\sum_{n=0}^{N-1} f(x_0 + n\Delta x)\, e^{-ik \cdot n \cdot \Delta x}}$$

periodic, $p = 2\pi/\Delta x$

can be fully sampled at

$\Delta k = \frac{2\pi}{N \cdot \Delta x}$ (sampling theorem)

$\Rightarrow$ need $\frac{p}{\Delta k} = N$ samples

$$\left.\begin{array}{l} k_m = m \cdot \Delta k \\ x_n = x_0 + n \cdot \Delta x \end{array}\right\} m, n = 0, \ldots, N-1$$

$$\mathcal{F}[f](k_m) = e^{-ikx_0} \underbrace{\sum_{n=0}^{N-1} f(x_n)\, e^{-i\frac{2\pi m \cdot n}{N}}}_{\text{Discrete Fourier Transform}}$$

input, output = vectors of length N

Number of operations:

$\propto N^2$ naively

$\propto N \log N$ with FFT

Noise

In practice: $g(x,y) = S[f](x,y) + n(x,y)$

arises from
- physics of detection
- electronic noise
- digitization error
- ...

Statistics vary based on mechanism.
Most common:

Gaussian: $p_n(\eta) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{\eta^2}{2\sigma^2}}$

Results when many independent mechanisms sum up

Poisson: $p_n(N) = \frac{\sigma^{2N}}{N!} e^{-\sigma^2}$

Occurs when noise is related to discrete events (e.g. photon counting)

Variance $\sigma_n^2 = \overline{|n-\mu|^2}$ 11

For image quality, the ratio of signal and noise is critical:

$$SNR = \frac{|S[f](x,y)|}{\sigma_n}$$

So, what makes a good imaging system?

- linearity
- shift invariance
- large spatial frequency range
  → high resolution
- low noise