A Recap on Some Mathematical Subjects

or "How to stay fresh all the time!"

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Topics Covered

Linear Algebra

Multivariate Analysis

Probability Theory

Outline

Linear Algebra

Multivariate Analysis

Probability Theory

Matrix Multiplication

- Let $A \in \mathbb{R}^{m \times n}$, $x \in \mathbb{R}^n$, and $b \in \mathbb{R}^m$.
  $Ax = b \iff b$ is a linear combination of columns of A.

$$b = \sum_{j=1}^{n} x_j \, a_j$$

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- **Outer product.** Let $u \in \mathbb{R}^n, v \in \mathbb{R}^m$. We call $uv^\top$ the outer product of u and v:

$$uv^\top = \begin{bmatrix} v_1 u & v_2 u & \cdots & v_m u \end{bmatrix}$$

Range, Kernel and Rank

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$$\mathrm{null}(A) = \{\mathbf{0}\} \iff A \text{ is injective.}$$

- **Rank** of a matrix A is the dimension of its range. It's equal to dim(col space) = dim(row space).

$$\dim \mathrm{null}(A) + \mathrm{rank}(A) = \#\text{cols of A}$$

Inverse

- A is *invertible* or *nonsingular* iff it is square and full rank. Equivalently, having $\det(A) \neq 0$, or $\ker(A) = \{\mathbf{0}\}$.

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- A is *invertible* or *nonsingular* iff it is square and full rank. Equivalently, having $\det(A) \neq 0$, or $\ker(A) = \{\mathbf{0}\}$.
- Multiplication by A^{-1} is a change of basis:

$$\begin{tikzcd}
\boxed{\begin{array}{c}\text{Coefficients of}\\ \text{expansion of } b\\ \text{in } \{e_1, \ldots, e_n\}\end{array}} \arrow[rr, bend left, "\times A^{-1}"] & & \boxed{\begin{array}{c}\text{Coefficients of}\\ \text{expansion of } b\\ \text{in } \{a_1, \ldots, a_n\}\end{array}} \arrow[ll, bend left, "\times A"]
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Orthogonality

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- Two vectors are *orthogonal* if their inner product is zero.
- Let $\{q_1, \ldots, q_n\}$ be a set of pairwise orthogonal unit vectors in $\mathbb{R}^n$. Then

$$\forall v \in \mathbb{R}^n : \quad v = \sum_{i=1}^{n} (q_i^\top v) q_i = \sum_{i=1}^{n} (q_i q_i^\top) v$$

  Note: $q_i q_i^\top$ is orthogonal projection onto direction q_i, which is a rank-one operator.

Unitary Matrices

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- If U is unitary, then it preserves angles,

  $$\langle Ux, Uy \rangle = \langle x, y \rangle \, ,$$

  as well as lengths,

  $$\|Ux\|_2 = \|x\|_2.$$

  If $\det(U) = 1$, then U is a rigid rotation, and if $\det(U) = -1$, then U includes a reflection.

Problem

By considering what space is spanned by the first n columns of R, show that if R is a nonsingular $m \times m$ upper-triangular matrix, then R^{-1} is also upper-triangular. (The analogous result also holds for lower-triangular matrices.)

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Problem

Show that if a matrix is both triangular and unitary, then it is diagonal.

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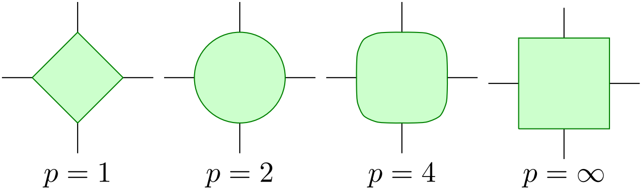
- Unit balls:



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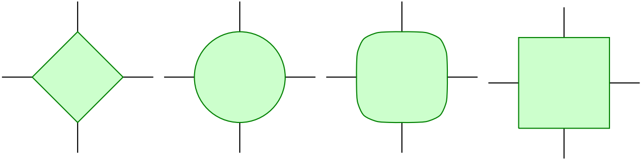

$p = 1$ $\quad p = 2$ $\quad p = 4$ $\quad p = \infty$

- The *Hölder inequality*. (Case $p = q = 2$ is known as *Cauchy-Schwartz* inequality)

$$\langle x, y \rangle \le \|x\|_p \|y\|_q, \text{ for } 1/p + 1/q = 1.$$

Matrix Norms

- We can view a matrix as a *linear operator*, and we can define norms on the space of linear operators. A famous norm is the **operator norm** of a matrix A. Let $A : (\mathbb{R}^n, \|\cdot\|_p) \to (\mathbb{R}^m, \|\cdot\|_q)$. Then we define

$$\|A\|_{(p,q)} := \sup_{x \in \mathbb{R}^n \setminus \{0\}} \frac{\|Ax\|_q}{\|x\|_p} = \sup_{\substack{x \in \mathbb{R}^n \\ \|x\|_p = 1}} \|Ax\|_q$$

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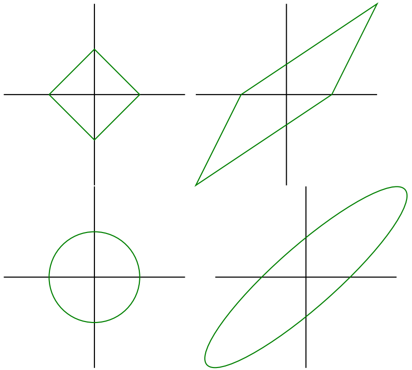
- Defines the maximum *stretch* of the unit ball.
- When $p = q$ we just write $\|A\|_p$.
  e.g. $\|A\|_2$ is the largest *singular value* of A.

Matrix Norms (Example)

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$$



$$\|A\|_1 = 4$$

$$\|A\|_2 \approx 2.92$$

Norms

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- All norms in finite-dimensional vector spaces are *equivalent*, that is, there exists positive constants C_1, C_2, such that

$$C_1 \cdot \|x\|_2 \leq \|x\|_1 \leq C_2 \cdot \|x\|_2$$

  Or, one can squeeze ones unit ball inside the other one.

Notes on Inner Products and Norms

One may introduce different inner products on $\mathbb{R}^n$. Formally, any bilinear[†] symmetric function which satisfies $\langle x, x\rangle \geq 0$ (and $\langle x, x\rangle = 0$ if and only if $x = 0$) could be an inner product. In $\mathbb{R}^n$, for example, it is enough to define the inner product of all $\langle e_i, e_j\rangle := a_{i,j}$. By bilinearity, it will be defined over all $\mathbb{R}^n$. Also, it is assumed that the inner product has values in $\mathbb{R}$. The more general case is when it takes values in $\mathbb{C}$. In that case, we should have $\langle x, y\rangle = \overline{\langle y, x\rangle}$.

Problem

The only p-norm that is induced by some inner product is only the 2-norm: (a) Prove that if the norm $\|\cdot\|$ comes from an inner product, then it should satisfy the parallelogram law:

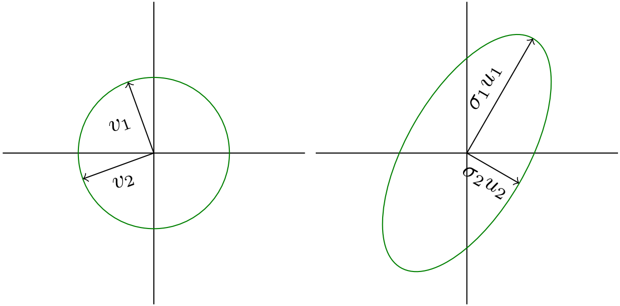
$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2).$$

(b) Show that this could happen if and only if $p = 2$.

[†] = linear in both arguments

Singular Values and Singular Vectors

- **Theorem.** The image of the unit sphere under a linear transform is always a hyperellipse.



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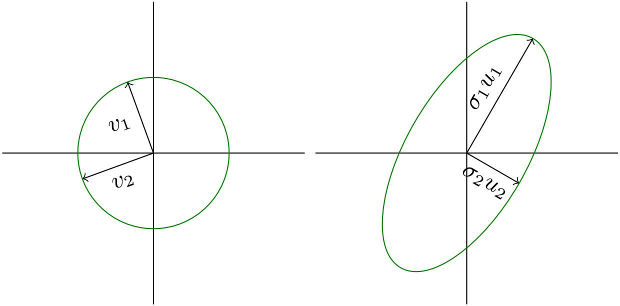

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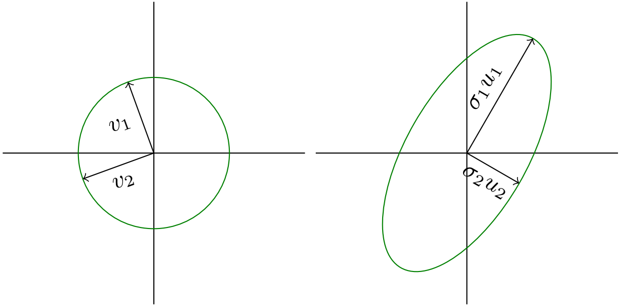

- We have $Av_i = \sigma_i u_i$.
- v_1, v_2 are called *right singular vectors* and u_1, u_2 the *left singular vectors*. Also σ_1, σ_2 are *singular values*.

Singular Value Decomposition (SVD)

- We can decompose any matrix A in the form

$$A = U\Sigma V^\top,$$

where U and V are unitary and Σ is a diagonal matrix, *i.e.*

$$U = \begin{bmatrix} u_1 & \cdots & u_n \end{bmatrix},$$

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- This is the **Full SVD**. There is also a **Reduced SVD**…

Eigenvalues and Eigenvectors

- If for some vector $v \neq \mathbf{0}$ we have $Av = \lambda v$ then v is an **eigenvector** of A associated to the **eigenvalue** λ. In this case we have $(A - \lambda I)v = \mathbf{0}$, and this can only happen when $\det(A - \lambda I) = 0$.

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- $\sigma_i^2(A) = \lambda_i(A^\top A) = \lambda_i(AA^\top)$.

Problem

Let $A \in \mathbb{C}^{m \times m}$ *be hermitian.*[‡] *(a) Prove that all eigenvalues of* A *are real. (b) Prove that if* x *and* y *are eigenvectors corresponding to distinct eigenvalues, then* x *and* y *are orthogonal.*

[‡] The *adjoint* of an $m \times n$ matrix A, written A^*, is the $n \times m$ matrix whose i, j entry is the complex conjugate of j, i entry of A. If $A = A^*$, A is *hermitian*.

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Problem

If u and v are m-vectors, the matrix $A = I + uv^*$ is known as a rank-one perturbation of the identity. Show that if A is nonsingular, then its inverse has the form $A^{-1} = I + \alpha uv^*$ for some scalar α, and give an expression for α. For what u and v is A singular? If it is singular, what is $\mathrm{null}(A)$*?*

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Outline

Notion of the Derivative

- As you may recall, for real differentiable functions $f : \Omega \subseteq \mathbb{R} \to \mathbb{R}$, the derivative $f'(x) = \frac{df}{dx}(x)$ is the *slope* of the **tangent** line at the point x. This notion is geometrically plausible, but unfortunately hard to generalize.

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- A better notion for derivative would be *the best linear approximation* of a function near a point x.
- Formally, let $f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^m$ and $x_0 \in \Omega$. We call f to be differentiable at x_0 iff there is a *linear* function $Df(x_0) : \mathbb{R}^n \to \mathbb{R}^m$, for which we have

$$\lim_{\|h\| \to 0} \frac{|f(x_0 + h) - f(x_0) - Df(x_0)(h)|}{\|h\|} = 0$$

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- We call $Df(x_0)$ the derivative of f at the point x_0.

Derivative

- Take $f(x) = (f_1(x), \ldots, f_m(x))$. We call f_i the **components** of f. If the derivative exists, then we have

$$Df(x_0) = \left[\frac{\partial f_i}{\partial x_j}(x_0)\right]_{1 \le i \le m, 1 \le j \le n},$$

where $\frac{\partial f_i}{\partial x_j}(x_0)$ is the **partial derivative** of f_i w.r.t. x_j at the point x_0, namely

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- If for a function f, all partial derivatives exist and *are continuous* at the point x_0 then f is continuously differentiable at x_0 and its derivative would be the matrix $Df(x_0)$ above.

Notion of the Gradient

- Let $f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$ be a real-valued differentiable function. Then we have

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  - If f attains a local minimum (or maximum) at some point x_0, then $\nabla f(x_0) = \mathbf{0}$. (The **first-order condition**)
  - We have the following (first-order) approximation for x sufficiently close to x_0:

$$f(x) = f(x_0) + \nabla f(x_0)^\top (x - x_0) + o(\|x - x_0\|)$$

Chain Rule

Let $f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^m$ and $g : \Omega' \subseteq \mathbb{R}^p \to \mathbb{R}^n$. Assume g is differentiable at x_0 and $g(x_0) \in \Omega$ and f is differentiable at $g(x_0)$. Then $f \circ g : \Omega' \to \mathbb{R}^m$ is differntiable at x_0 and we have

$$D(f \circ g)(x_0) = Df(g(x_0)) \circ Dg(x_0)$$

A good example is the directional derivatives. Assume $f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$. Let $u \in \mathbb{R}^n$. We want to find the rate of change of f in the direction of u, *i.e.* $\frac{d}{dt} f(x_0 + tu)$ for $t = 0$. Define $g(t) = x_0 + tu$. We have

$$D(f \circ g)(0) = Df(g(0)) \circ Dg(0) = \nabla f(x_0)^\top u.$$

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Second-Order Approximation

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- In the single-variable regime, we can use quadratic polynomials to approximate a function in some neighborhood.
- We need to understand what are quadratic functions in multi-dimensional case and try to approximate our function.
- We expect our new approximation's error has a faster convergence to 0 than $\|x - x_0\|^2$.

Multidimensional Quadratic Functions

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- Note that the quadratic form is a weighted sum of all possible second degree terms, *e.g.* $x_i x_j$ or x_i^2.

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$$f \ : \ x \in \mathbb{R}^n \mapsto x^\top A x.$$

- Note that the quadratic form is a weighted sum of all possible second degree terms, *e.g.* $x_i x_j$ or x_i^2.
- We call A a **positive (negative) definite** matrix, iff all eigenvalues of A are positive (negative). We say A is a **positive semi-definite** or p.s.d., if all eigenvalues are nonnegative.

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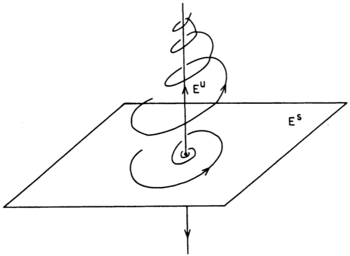
- We call A a **positive (negative) definite** matrix, iff all eigenvalues of A are positive (negative). We say A is a **positive semi-definite** or p.s.d., if all eigenvalues are nonnegative.
- If A is p.s.d., then the contour levels of f are concentric ellipsoids.
- One can prove that $\nabla f(x) = 2Ax$.

A Reminder from ODE Theory*

Consider the linear ODE $\dot{x} = Ax$, with A having real nonzero eigenvalues $\lambda_1, \dots, \lambda_n$ and eigenvectors $v_1, \dots, v_n$. Define

$$E^s = \mathrm{Span}\{v_j \mid \lambda_j < 0\},$$
$$E^u = \mathrm{Span}\{v_j \mid \lambda_j > 0\},$$

which are *stable*, and *unstable* subspaces.



Explore. Consider the ODE $\dot{x} = -\nabla f(x)$, when f is a quadratic form. Try to understand the shape of the function using the stable and unstable subspaces.

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- **Cholskey Decomposition.** A can be decomposed as

$$A = LL^\top,$$

  where L is a lower triangular matrix with positive diagonal entries (if A is p.s.d., nonnegative entries).

Example

Let $A \in \mathbb{R}^{d \times d}$ be a symmetric p.d. matrix, and set $c \in \mathbb{R}^d$. How does the set

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Answer.

Let $A = LL^\top$ be the Cholskey decomposition of A. Thus $A^{-1} = L^{-\top}L^{-1}$. By a change of variable $y = L^{-1}(x-c)$, we get

$$E = \{Ly + c \mid \|y\|_2^2 = y^\top y \le 1\}.$$

Thus, E is the result of an affine transformation applied to the unit Euclidean ball, which is an **ellipsoid**. $\square$

The Hessian

- Assume $f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$ is twice-differentiable at x_0. Then there exists some symmetric matrix $D^2 f(x_0)$ which we call the **Hessian** of f at x_0, with

$$D^2 f(x_0) = \left[\frac{\partial^2 f}{\partial x_i \partial x_j}(x_0)\right]_{1 \leq i,j \leq n},$$

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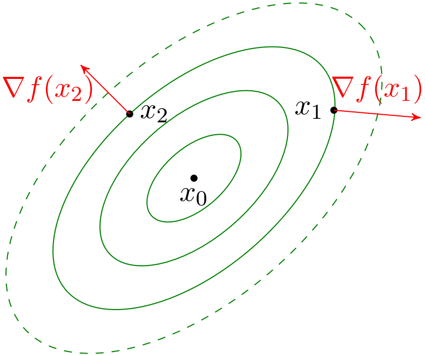
  where $\frac{\partial^2 f}{\partial x_i \partial x_j}(x_0) = \frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_j}\right)(x_0)$.
- We have the following (second-order) approximation for x sufficiently close to x_0:

$$\begin{aligned} f(x) = f(x_0) &+ \nabla f(x_0)^\top (x - x_0) \\ &+ \frac{1}{2}(x - x_0)^\top D^2 f(x_0)(x - x_0) + o(\|x - x_0\|^2) \end{aligned}$$

All-in-one Picture

In this example, we assume that $f : \mathbb{R}^2 \to \mathbb{R}$ is twice-differentiable, having a local minimum at x_0. This picture demonstrates a (sufficiently small) neighborhood of x_0:



Note that around the local minimum, the Hessian of f is positive semi-definite, thus the elliptic contour lines.

KKT Optimality Conditions

Let $f, g_i, h_j : \mathbb{R}^d \to \mathbb{R}$ be continuously differentiable convex functions for $i = 1, \ldots, k$ and $j = 1, \ldots, l$. Consider the optimization problem

$$\begin{aligned} & \text{minimize}\, f(x) \\ \text{s.t.} \quad & g_i(x) \leq 0 \\ & h_j(x) = 0. \end{aligned}$$

Assuming the Slater's Condition, if $x^\star$ is a local minimum of the optimization problem, then there exists constants λ_i and μ_j such that

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- $\lambda_i \cdot g_i(x^\star) = 0$ for $i = 1, \ldots, k$. (Complementary Slackness)

Outline

Linear Algebra

Multivariate Analysis

Probability Theory

Basic Notations

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- A family $\mathcal{F}$ of subsets of Ω, called the **events**, are also interesting for us. Rather than asking whether a certain outcome has happened, we want to ask harder questions. For example, if we want to ask whether the score is higher than 60 or not, we are asking about the event $\{80, 100\}$.

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- We also want to apply rules of logic. Taking "and" is translated to intersection of events, "or" is union, and "not" is complements. So we desire our family of events $\mathcal{F}$ to be closed under these operations.

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  - $\Omega = [0, 1]$, $\mathcal{F} = \mathcal{B}([0, 1])$, $\mathbb{P}((a, b]) = b - a$.

Random Variables

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$$X(\omega) = \text{distance of } \omega \text{ to the center of dartboard.}$$

  Now we can ask, whether the throw was further than 0.3 cm, via asking about the event $\{\omega : X(\omega) \geq 0.3\}$. For brevity we write this event as $\{X \geq 0.3\}$.

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- Link between X and Ω is the **inverse image** X^{-1}.
- We can only ask a question $A \in \mathcal{B}(\mathbb{R})$ from X if the inverse image of A is already inside $\mathcal{F}$, *i.e.* $X^{-1}(A) \in \mathcal{F}$.

On Measurability*

Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a function $X : \Omega \to \mathbb{R}$ is called *measurable* if it respects the structure of $\mathcal{F}$. That is, for all $t \in \mathbb{R}$ one should have

$$\{X \le t\} = X^{-1}((-\infty, t]) \in \mathcal{F}.$$

An example of a non-measurable function is as follows: Take $\Omega = \{\epsdice{1}, \epsdice{2}, \dots, \epsdice{6}\}$ to be the different faces of a die. Define $\mathcal{F} = \{\emptyset, \Omega, \{\epsdice{1}, \epsdice{2}, \epsdice{3}\}, \{\epsdice{4}, \epsdice{5}, \epsdice{6}\}\}$ to be the *information available about the experiment*. Then, the function $X = \mathbf{1}_{\{\epsdice{1}, \epsdice{2}\}}$ is *not* a random variable, as the inverse image $X^{-1}(\{1\}) = \{\epsdice{1}, \epsdice{2}\}$ is not a member of $\mathcal{F}$.
If you are interested, see more about Borel sets, Lebesgue measure, and Measurable functions.

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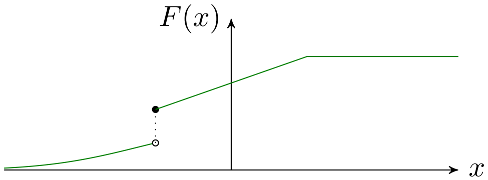

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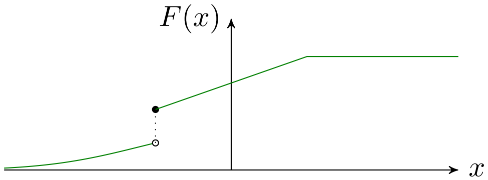

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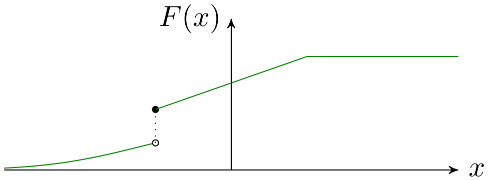

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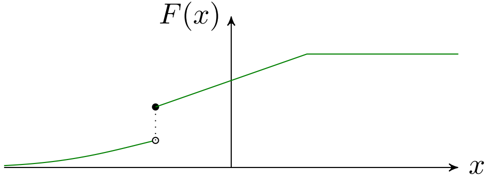
    - We have $\lim_{x\to\infty} F(x) = 1$, $\lim_{x\to-\infty} F(x) = 0$.
    - $\mathbb{P}(a < X \leq b) = F(b) - F(a)$.



Densities

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$$f(x) := \lim_{\substack{|I| \to 0 \\ x \in I}} \frac{\mathbb{P}(X \in I)}{|I|},$$

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- The value of $f(x)$ can be used to estimate probabilities, *e.g.* if $f(x) = 2$, then for a small interval I of size ϵ around x, we know that $\mathbb{P}(X \in I) \approx 2\epsilon$.

Motto!

With random variables and their densities (or distributions) we can even forget about Ω and just look at the probability space that is defined on $\mathbb{R}$ via X. So the following holds:

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>
> *If we know* $F(x)$ *or* $f(x)$*, we can build a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *and a random variable* X*, such that the distribution of* X *is exactly* $F(x)$*.*

Joint Distribution and Marginals

- Let X, Y be two random variables over the same probability space. Then we can define the **joint distribution** as

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- Given the joint distribution, one can find the distribution of each of variables by **marginalizing**:

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy, \quad F_X(x) = F_{X,Y}(x,\infty)$$

Independence

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- A sequence of n RVs $X_1, \ldots, X_n$ are said to be independent, iff their joint distribution factorizes. Note that if X_i are pairwise independent, it does *not* follow that they are independent.

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- **Chain Rule.** Let $A_1, \ldots, A_n$ be arbitrary events. We have

$$\begin{aligned}\mathbb{P}(A_1, \ldots, A_n) = &\mathbb{P}(A_1)\times \\ &\mathbb{P}(A_2|A_1)\times \\ &\mathbb{P}(A_3|A_1, A_2) \times \cdots \times \mathbb{P}(A_n|A_1, \ldots, A_{n-1})\end{aligned}$$

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- Expected value is linear! $\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y]$, even when X and Y are not independent. For a distribution, we usually use μ to represent its mean.

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- For a distribution, we show its variance by σ^2.

Law of Large Numbers

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- SLLN states that

$$\mathbb{P}\left(\lim_{n\to\infty} \tfrac{S_n}{n} = \mu\right) = 1$$

Central Limit Theorem(s)

- If $X_1, X_2, \ldots$ is an iid seq. of RVs, having mean μ and variance σ^2, we have the following:

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- Good for creating *approximate confidence intervals*
- Caveat! Speed of convergence, Uniform convergence, regularity conditions…

Normal Distribution

- The density of the multivariate normal distribution is

$$f(x) = \frac{1}{\sqrt{(2\pi)^d |\Sigma|}} \exp\left\{-\frac{1}{2}(x-\mu)^\top \Sigma^{-1}(x-\mu)\right\}$$

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- Any linear combination of normal random vectors, and also the image of a normal random vector under a linear transformation would be a normal random vector (with maybe different mean and covariance matrix).

Much more things to say, but no time!

Hope You Enjoy the Course!