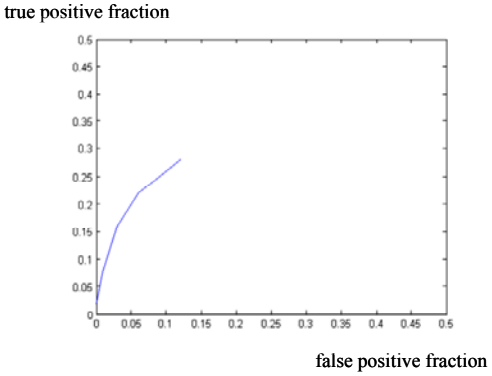
Solutions to the exercises.

1.1 In a patient study for a new test for multiple sclerosis (MS), thirty-two of the one hundred patients studied actually have MS. For the data given below, complete the two-by-two matrices and construct an ROC. The number of lesions (50, 40, 30, 20 or 10) corresponds to the threshold value for designating MS as the diagnosis.

50 lesions		40 lesions		30 lesions		20 lesions		10 lesions	
2	0	8	1	16	3	22	6	28	12

Solution. The sum of the left hand column (true positives plus false negatives) must add up to thirty-two. Therefore the bottom left square is filled in this way. This leaves only the bottom right which has to fulfill the requirement that the total number of patients is one hundred. So the completed tables and corresponding ROC are:

50 lesions		40 lesions		30 lesions		20 lesions		10 lesions	
2	0	8	1	16	3	22	6	28	12
30	68	24	67	16	65	10	62	4	56



The ROC lies above the random line, but clearly needs some extra points corresponding to a smaller number of lesions for full characterization.

1.2 Choose a medical condition and suggest a clinical test which would have:
(a) High sensitivity but low specificity,
(b) Low sensitivity but high specificity.

Solution. (a) the sensitivity is defined as the number of true positives divided by the sum of the true positives and the false negatives, whereas the specificity is the

number of true negatives divided by the sum of the true negatives and false positives. So a high sensitivity but low specificity suggests a diagnosis with a very low number of false negatives but significant number of false positives, i.e. a diagnosis which doesn't often miss the disease but often suggests that there is a disease present whereas in fact there isn't. An example might be mammography in which small lesions are very well visualized, but often subsequent biopsies result in the lesions being found not to be malignant.

(b) A low sensitivity but high specificity implies a relatively high number of false negatives and low number of false positives, i.e. a diagnosis that often misses the disease but almost never gives a false impression that the disease is present when it isn't. An example might be cognitive and behavioural tests in the early progression of neurodegenerative diseases such as Alzheimers. Signs are often missed, but very poor test results are very certain indicators of brain disfunction.

1.3. What does an ROC curve that lies below the random line suggest? Could this be diagnostically useful?

Solution. This suggests that the criteria that are being used to provide a clinical assessment are being used to support a hypothesis that is directly opposite to the truth. For example, cardiac disease is being diagnosed due to a low heart rate, whereas in fact the low heart rate occurs due to the patients being more fit and therefore these patients have a lower level of cardiac disease. Since the ROC curve is not random, there is useful information in the analysis, but the interpretation needs to be reversed with respect to the original hypothesis in order to take advantage of this information.

1.4 For the one-dimensional objects and PSF's shown in Figure 1.29, draw the resulting projections I(x). Write down whether each object contains high, low, or both high and low spatial frequencies, and which projection is affected most by h(x).

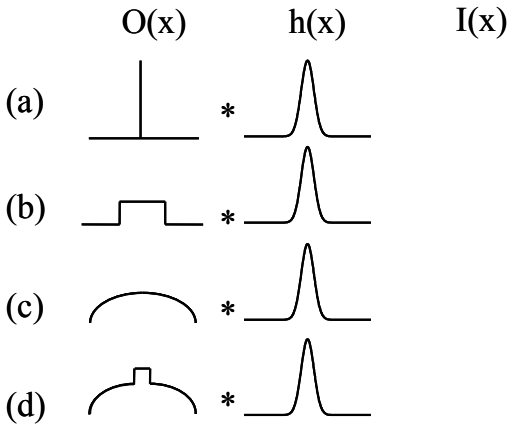
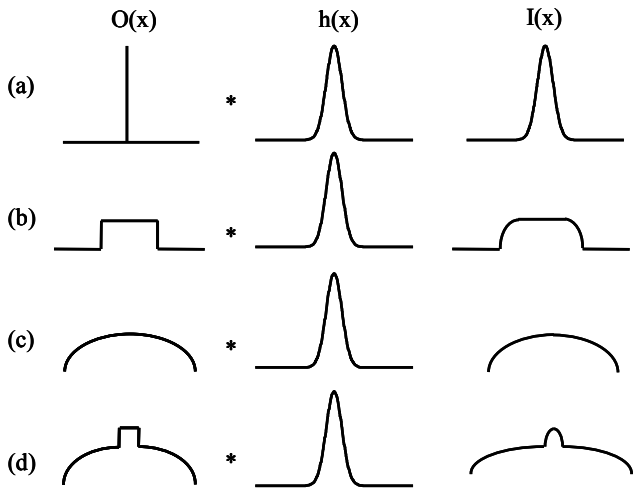


Figure 1.29.

Solution.



In (a) the object contains only very high spatial frequencies, (b) has both very high (at the edges) and very low (the flat parts of the profile) spatial frequencies, (c) has predominantly low spatial frequencies, and (d) has both low and high spatial frequencies. Since h(x) contains mainly low spatial frequencies, it will affect the object with the greatest proportion of high spatial frequencies, i.e. (a), to the greatest degree

1.5 Show mathematically that the FWHM of a gaussian function is given by:

$$\text{FWHM} = \left(2\sqrt{2\ln 2}\right)\sigma \cong 2.36\sigma$$

Solution. The equation for a gaussian function is given by:

$$h(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x - x_0)^2}{2\sigma^2}\right)$$

The maximum value of h(x) occurs when $x=x_0$, and the exponent equals unity. Therefore, the FWHM corresponds to h(x)=0.5. Solving for the exponential term:

$$\exp\left(-\frac{(x-x_0)^2}{2\sigma^2}\right)=\frac{1}{2}$$

$$\Rightarrow -\frac{(x-x_0)^2}{2\sigma^2}=-\ln 2$$

$$\Rightarrow x-x_0=\pm\sigma\sqrt{2\ln 2}$$

Since the gaussian function is two-sided and symmetric, the FWHM is given by:

$$\text{FWHM}=\left(2\sqrt{2\ln 2}\right)\sigma$$

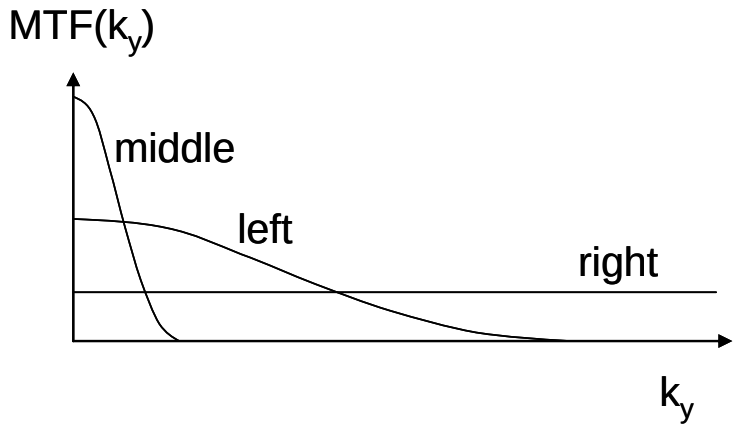
1.6 Plot the MTF on a single graph for each of the convolution filters shown below.

1	1	1
1	4	1
1	1	1

1	1	1
1	12	1
1	1	1

1	1	1
1	1	1
1	1	1

Solution. The filter on the left is a low-pass filter, the one in the middle is also low-pass with a greater degree of smoothing than the one on the left, and the filter on the right leaves the image unchanged. The MTF's are sketched below.



1.7 What type of filter is represented by the following kernel?

1	0	-1
1	0	-1
1	0	-1

Solution. This is a one-dimensional edge filter in the left/right direction which effectively produces a one-dimensional derivative of the image, i.e. it will emphasize the edges in the image. This is possible to see since the positive and negative numbers in the horizontal direction will emphasize any differences around the central pixel.

1.8 Using the filter in exercise 1.7 calculate the filtered image using the original image from Figure 1.11.

original image

1	5	3	5	4	6
4	3	32	5	6	9
6	10	4	8	8	7

$*$

filter

1	0	-1
1	0	-1
1	0	-1

$=$

filtered image

1	5	3	5	4	6
4	a	b	c	d	9
6	10	4	8	8	7

$$\begin{cases} a=(1)(1)+(5)(0)+(3)(-1)+(4)(1)+(3)(0)+(32)(-1)+(6)(1)+(10)(0)+(4)(-1)=-28 \\ b=(5)(1)+(3)(0)+(5)(-1)+(3)(1)+(32)(0)+(5)(-1)+(10)(1)+(4)(0)+(8)(-1)=0 \\ c=(3)(1)+(5)(0)+(4)(-1)+(32)(1)+(5)(0)+(6)(-1)+(4)(1)+(8)(0)+(8)(-1)=21 \\ d=(5)(1)+(4)(0)+(6)(-1)+(5)(1)+(6)(0)+(9)(-1)+(8)(1)+(8)(0)+(7)(-1)=-4 \end{cases}$$

1	5	3	5	4	6
4	-28	0	21	-4	9
6	10	4	8	8	7

filtered image

1.9 An ultrasound signal is digitized using a 16-bit ADC at a sampling rate of 3 MHz. If the image takes 20 ms to acquire, how much data (in Mbytes) are there in each ultrasound image. If images are acquired for 20 seconds continuously, what is the total data output of the scan?

Solution. Since 8-bits are equivalent to 1 byte, a 16-bit ADC produces 2 bytes of data every time that it samples. Sampling at 3 MHz produces 6 megabytes every second. Each image therefore contains 0.12 megabytes. For the continuous imaging 120 megabytes are produced over 20 seconds.

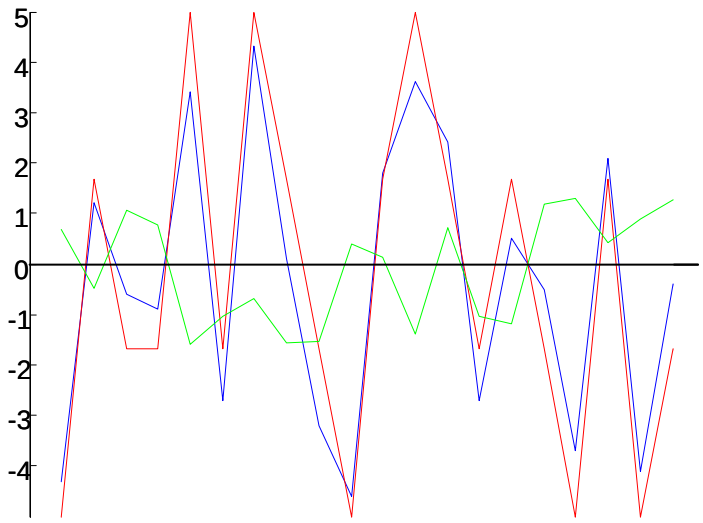
1.10 If a signal is digitized at a sampling rate of 20 kHz centred at 10 kHz, at what frequency would a signal at 22 kHz appear?

Solution. The frequency span is 0 to 20 kHz, centred at 10 kHz, and so a signal at 22 kHz is identical to one at 2 kHz, which is the frequency at which it appears in the spectrum.

1.11 A signal is sampled every 1 ms for 20 ms, with the following "true" values of the analogue voltage at successive sampling times. Plot the values of the voltage recorded by a 5 volt, 4-bit ADC assuming that the noise level is much lower than the signal and so can be neglected. On the same graph, plot the quantization error.

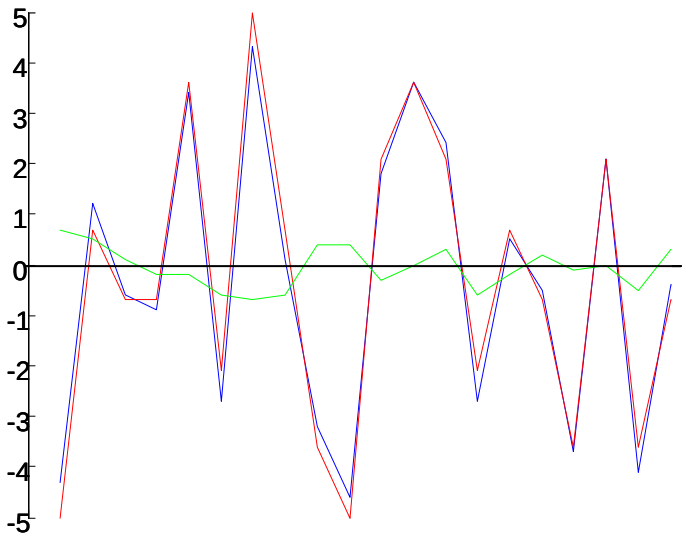
Signal (volts) = -4.3, +1.2, -0.6, -0.9, +3.4, -2.7, +4.3, +0.1, -3.2, -4.6, +1.8, +3.6, +2.4, -2.7, +0.5, -0.5, -3.7, +2.1, -4.1, -0.4

Solution. A 5 volt ADC has a range of -5 to +5 volts. Since it is a 4-bit, it can record values of -5, -1.67, + 1.67 and +5 volts. Each voltage is rounded either up or down to the nearest of these four values. The graph is shown below (blue)-true values, (red) 4-bit ADC, (green) quantization error.



1.12 Using the same signal as in exercise 1.11, plot the values of the voltage and the quantization error recorded by a 5 volt, 8-bit ADC.

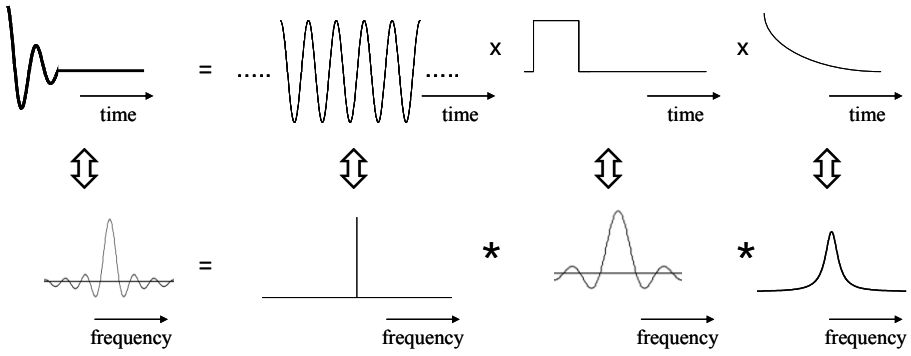
Solution. The recorded values can now be: -5, -3.6, -2.1, -0.7, +0.7, +2.1, +3.6 and +5. The graph is shown below (blue)-true values, (red) 8-bit ADC, (green) quantization error.



Fourier transforms

1.13 In Figure 1.20, plot the time and frequency domain signals for the case that the sampling time becomes very short.

Solution. If the sampling time is short, then the sinc function in the frequency domain becomes very broad, and the signal in the time domain is truncated. This leads to so-called "ringing artifacts" in the frequency spectrum as shown below.



1.14 Figure 1.30 shows two different two-dimensional PSFs in the (x,y) spatial domain. Draw the corresponding two-dimensional MTFs in the (k_x, k_y) spatial

frequency dimension, and plot separately the one-dimensional MTF vs. k_x and MTF vs. k_y.

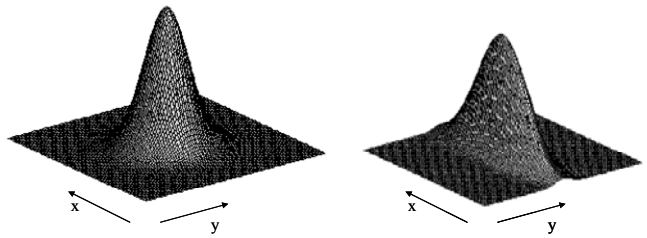
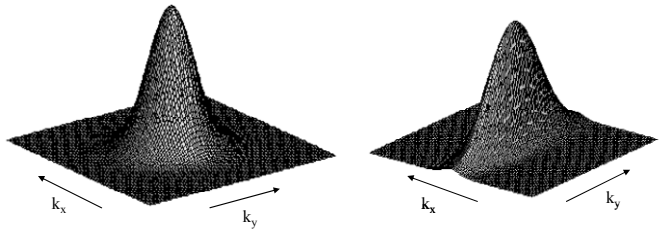
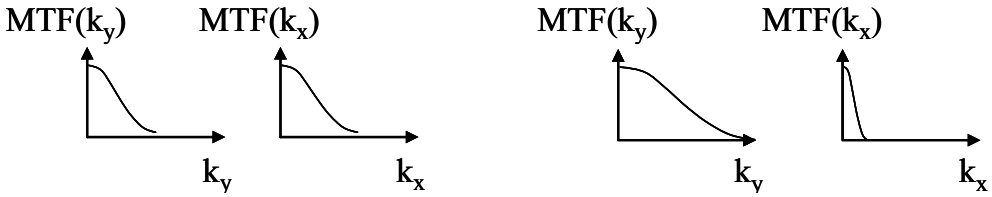


Figure 1.30.

Solution. Since a broad function in the spatial domain (PSF(x,y)) corresponds to a narrow function in the spatial frequency domain (MTF(k_x,k_y)), the plots in the spatial frequency domain are:

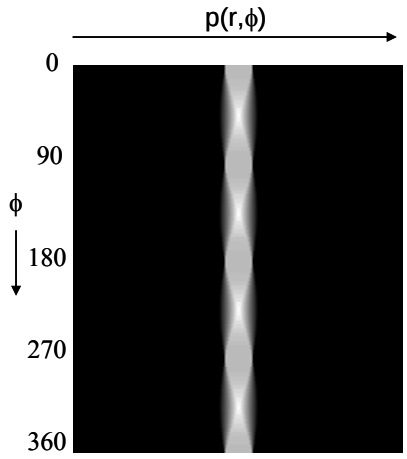


The corresponding one-dimensional MTFs are shown below:



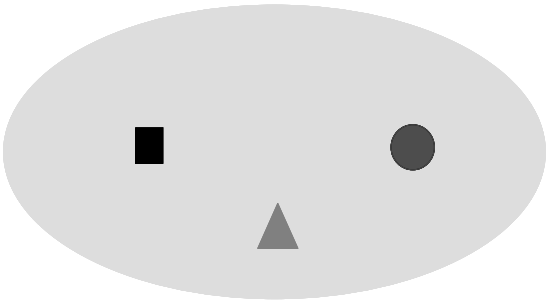
Backprojection

1.15 For Figure 1.25, suggest one possible shape that could have produced the sinogram.

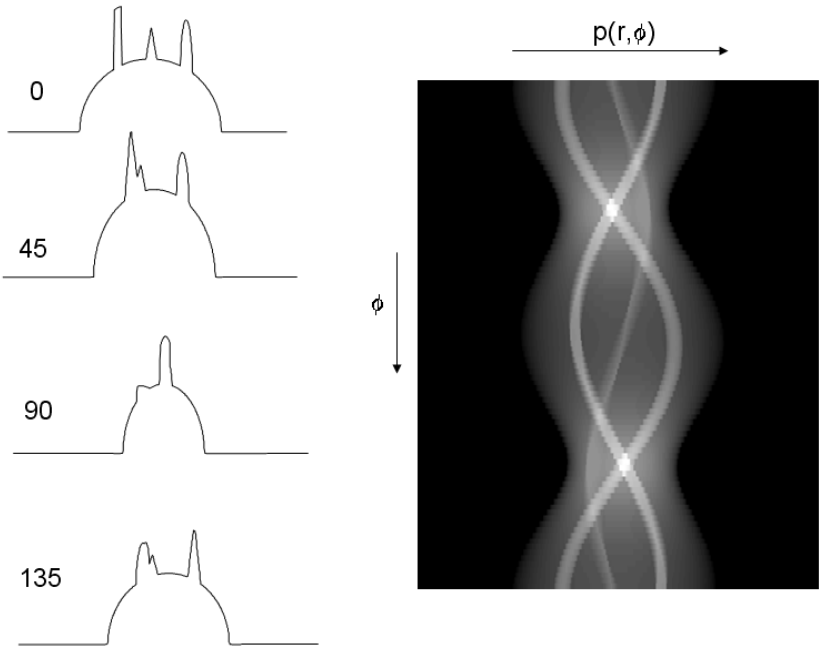


Solution. There are four repeating structures in the sinogram as the angle goes through 360 degrees, suggesting two-fold symmetry. One possible solution, therefore, is a square.

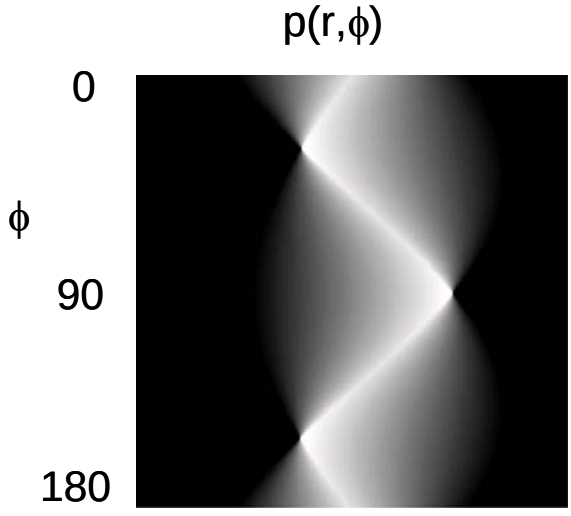
1.16 For the object shown in Figure 1.31: (a) draw the projections at angles of 0, 45, 90, and 135^{o}, and (b) draw the sinogram from the object. Assume that a dark area corresponds to an area of high signal.



Solution.



1.17 A scan is taken of a patient, and an area of radioactivity is found. The sinogram is shown in Figure 1.32. Assuming that the radioactivity is uniform within the targeted area, what is the possible shape of the area of radioactivity?

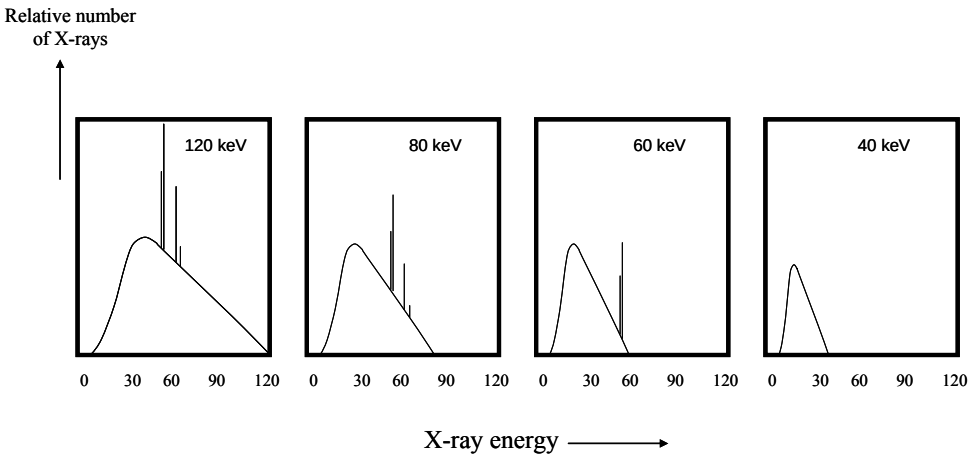


Solution. There are three bright "apices" which suggest that the object is triangular in shape. The three maxima occur at ~30, 90 and 150 degrees and so the triangle must be equilateral to produce these equally spaced angles.

Chapter 2.

2.1 Plot the energy spectra from a tungsten tube with the following kVp values: 120 keV, 80 keV, 60 keV and 40 keV.

Solution. The general shape of the curve becomes narrower, and also the characteristic lines disappear as the kVp value falls below the K-edge of tungsten.



2.2 (a) Calculate the total number of electrons bombarding the target of an X-ray tube operated at 100 mA for 0.1 seconds.
(b) Calculate the maximum energy and minimum wavelength for an X-ray beam generated at 110 kVp.

Solution. (a) $Q = Nq = tI$

$$N = \frac{tI}{q} = \frac{0.1 \times 100 \times 10^{-3}}{1.6 \times 10^{-19}} = 6.25 \times 10^{16}$$

(b) $E_{\max} = kV_p = 110\ keV$

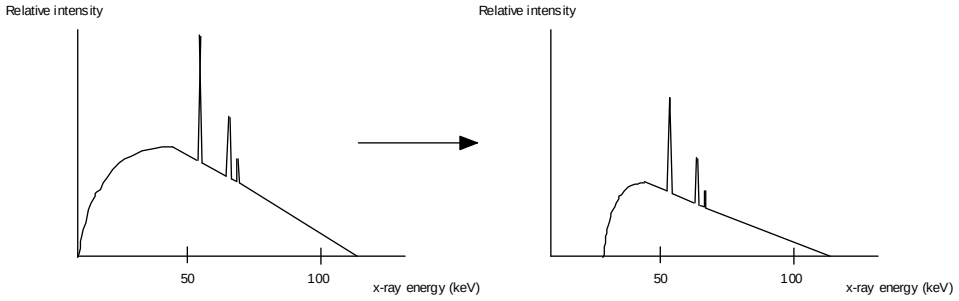
$$E = h\nu$$

$$\nu = \frac{E}{h} = \frac{110 \times 10^3 \times 1.6 \times 10^{-19}}{6.63 \times 10^{-34}} = 2.7 \times 10^{19}\ Hz$$

$$\lambda = \frac{c}{\nu} = 11.2 \times 10^{-3}\ nm$$

2.3 The spectrum of X-ray energies changes as the X-rays pass through tissue due to the energy dependence of the linear attenuation coefficient: this is a phenomenon known as beam hardening. A typical energy distribution of the beam from the X-ray source is shown in Figure 2.6. Sketch the energy spectrum after the beam has passed through the body.

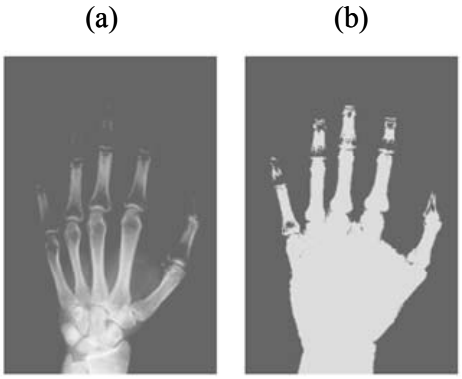
Solution. Since higher energy X-rays are attenuated less than lower energy X-rays as they pass through tissue, the energy spectrum will shift to higher values. Very low energy X-rays will be almost completely attenuated, as shown below.



2.4 Look up the exact binding energies of the L and M shell electrons in tungsten. Plot the exact distribution of characteristic energy lines that are produced based upon these binding energies.

Solution. Tungsten has full K, L and M shells (2, 8 and 18 electrons, respectively). The binding energies of the L-shell electrons are ~12.1, 11.5 and 10.2 keV, and the M-shell electrons are 2.8, 2.6, 2.3, 1.9 and 1.8 keV. Transitions between the L and K shells therefore have energy ~59 keV, and M and K shells ~67 keV. The two K-shell electrons have slightly different energies which makes two peaks at 59.32 and 57.98 keV.

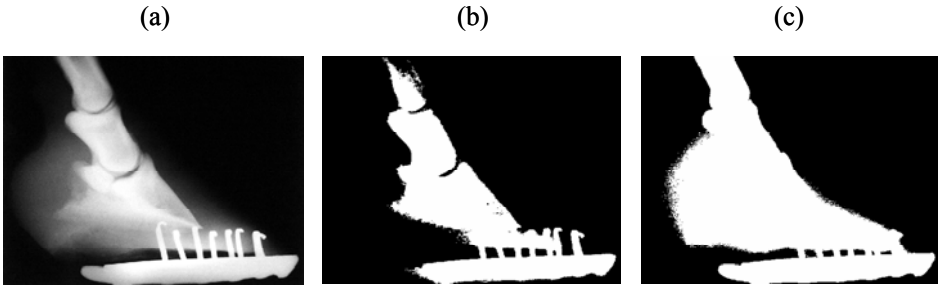
2.5 Two X-ray images of the hand are shown in Figure 2.39. One corresponds to an X-ray beam with an effective energy of 140 keV and the other to an effective energy of 50 keV. Explain which is which, and the reasons for the differences in image contrast and signal intensity.



Solution. The contrast between bone and soft tissue in (a) is much higher than for (b), suggesting that it has been obtained at 50 keV where the photoelectric interactions dominate. The image in (b) has almost no contrast, indicating that

Compton scattering is dominant. Image (b) also looks as if it has been over-exposed to give such poor contrast.

2.6 Planar X-ray of the horse's leg are shown in Figure 2.40. In (a), the kVp and tube current (mA) are chosen to give high contrast between the bone and surrounding tissue. Explain how the kVp and mA settings would have to be changed (increased, decreased, remain constant) to give the images shown in figures (b) and (c).



Solution. First note that in these particular images a high signal intensity (white) corresponds to a high X-ray attenuation, since bone appears bright. In image (b) there is much higher contrast between bone and muscle than in image (a) meaning that the kVp must have been decreased. The signal from bone is higher than in (a), so that the total number of X-rays being transmitted through the bone is lower, and so the mA must have been reduced. In image (c) there is no contrast between bone and muscle and so the kVp has been increased. The increase in intensity means that the mA has been decreased.

2.7 Starting from equation [2.4] derive both equations [2.5] and [2.6].

The step from equation (2.4) to (2.5) simply involves de Broglie's relationship:

$$E_{X,inc} = hf_{X,inc} = \frac{hc}{\lambda_{X,inc}}, E_{X,scat} = hf_{X,scat} = \frac{hc}{\lambda_{X,scat}}$$

and therefore

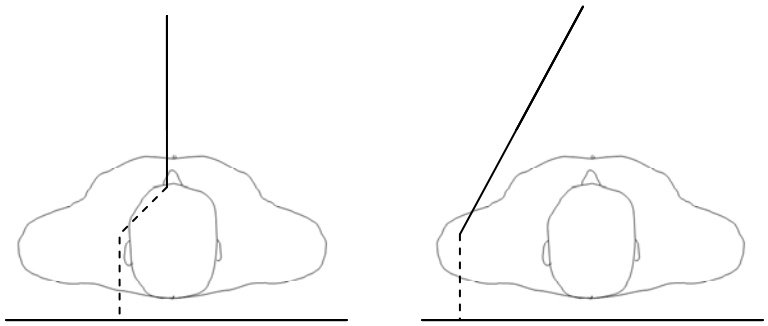
$$\Delta E = E_{X,inc} - E_{X,scat} = \frac{hc}{\lambda_{inc}} - \frac{hc}{\lambda_{scat}}$$

From here we can write:

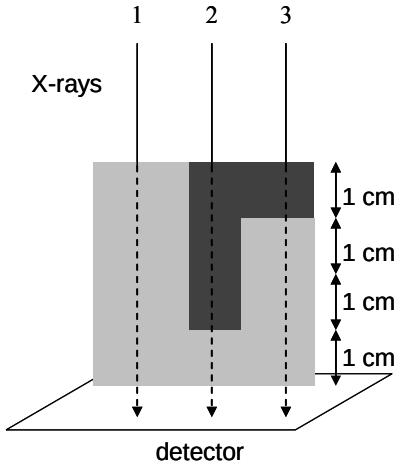
$$E_{X,scat} = \frac{hc}{\lambda_{scat}} = \frac{hc}{\lambda_{inc} + \frac{h}{m_0 c}(1-\cos\theta)} = \frac{hc/\lambda_{inc}}{1 + \frac{h}{\lambda_{inc} m_0 c}(1-\cos\theta)} = \frac{E_{X,inc}}{1 + \frac{E_{X,inc}}{m_0 c^2}(1-\cos\theta)}$$

2.8 Show two mechanisms involving scattering in the body which allow Compton scattered X-rays to be detected despite the presence of an anti-scattering grid.

Solution. In addition to the simple case of a small angle of scattering, a couple of possible mechanisms are shown below. On the left, two successive Compton scattering events take place, resulting in an X-ray emerging from the body parallel to the anti-scatter grid. On the right, a part of the body at the extremity (in this case the arm) produces a large angle scattering.



2.9 In Figure 2.41 calculate the X-ray intensity, as a function of the incident intensity I_0, that reaches the detector for each of the three X-ray beams. The dark-shaded area represents bone, and the light-shaded area represents tissue. The linear attenutation coefficients at the effective X-ray energy of 68 keV are 10 cm^{-1} and 1 cm^{-1} for bone and tissue, respectively.



Solution.

Beam 3

$$I = I_0 e^{-(10\times1+1\times3)} = I_0 e^{-13} = 2.26\times10^{-6} I_0$$

Beam 2

$$I = I_0 e^{-(10\times3+1\times1)} = I_0 e^{-31} = 3.44\times10^{-14} I_0$$

Beam 1

$$I = I_0 e^{-(1\times4)} = I_0 e^{-4} = 1.83\times10^{-2} I_0$$

2.10 The linear attenuation coefficient of a gadolinium-based phosphor used for detection of X-rays is 560 cm^{-1} at an X-ray energy of 150 keV. What percentage of X-rays are detected by phosphor layers of 10 μm, 25 μm, and 50 μm thickness? What are the trade-offs in terms of spatial resolution?

Solution. The percentage of X-rays detected is the percentage that are absorbed by the phosphor layers. Therefore, the three relevant numbers are:

10 μm thickness: percentage =100*(1-exp(-0.0001*560)) = 5.4%
25 μm thickness: percentage =100*(1-exp(-0.00025*560) = 13.1%
50 μm thickness: percentage =100*(1-exp(-0.0005*560) = 24.4%

The trade-off for increased SNR due to screen thickness is poorer spatial resolution due to the diffusion of light in the screen.

2.11 An X-ray with energy 60 keV strikes a gadolinium-based phosphor on a CR plate producing photons at a wavelength of 415 nm. The energy conversion coefficient for this process is 20%. How many photons are produced for each incident X-ray? (Planck's constant=$6.63\text{x}10^{34}$ Js, 1 eV=$1.602\text{x}10^{-19}$J).

Solution. The energy of each produced photon, at 415 nm, is:

$$E = h\nu = \frac{hc}{\lambda} = \frac{6.63\times10^{-34}\times3\times10^{8}}{415\times10^{-9}} = 4.79\times10^{-19}\,J = 2.99eV$$

Therefore, the number of photons produced per incident X-ray is:

$$\frac{60\times10^{3}\times20\%}{2.99} = 4,013$$

2.12 An X-ray is Compton scattered by an angle θ at a distance z from the top of the anti-scatter grid. Using simple geometry, calculate the maximum value of θ for a given grid ratio h/d that results in the X-ray being detected.

Solution. The answer is given by simple trigonometry:

$$\theta_{\max} = \sin^{-1}\left(\frac{d}{h+z}\right)$$

2.13 The width of the electron beam in the X-ray tube is 6 mm, and the anode bevel angle is 10^{o}. The patient is placed 180 cm from the X-ray tube, directly on top of the flat panel detector. Calculate the geometric unsharpness for a very small lesion at the front of the abdomen and at the back of the abdomen, assuming that the tissue thickness of the body at this point is 25 cm.

Solution. The effective focal spot size is given by (equation 2.1):

$$f = 6\sin 10 \approx 1 \; mm$$

The size of the penumbra region at the front of the abdomen is given by:

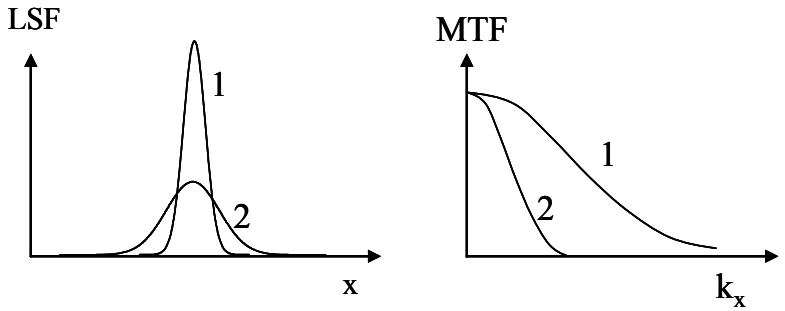
$$P = \frac{1(205-180)}{180} = 0.13 \text{ mm}$$

For the lesion at the back of the abdomen, since the patient lies directly on top of the FPD, there is no geometrical unsharpness.

2.14 If the average number of X-rays striking the detector is 100 per pixel per second, what is the probability that only 90 strike in any one second? If the X-ray dose is increased by a factor of ten, what is the probability of 900 striking?

Solution. Using the formula for the Poisson distribution, equation (2.10), the probability of 90 X-rays is 0.025. If the dose is increased by a factor of ten, then an average of 1000 strike per second, and the corresponding probability of only 900 striking is 0.00008.

2.15 (a) Match up a thick and thin phosphor layer in computed radiography with its associated LSF (1 or 2) in Figure 2.41.
(b) Similarly, match up a small and large X-ray focal spot with its MTF (1 or 2). Provide a brief explanation for each choice.



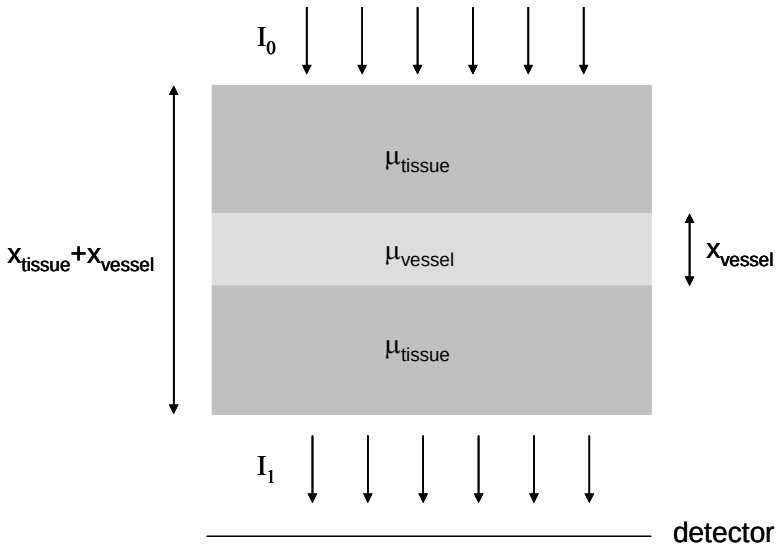
Solution. (a) A thick phosphor layer has high SNR but poor spatial resolution due to the broad light spread function, and so corresponds to a broad LSF, namely number 1.
(b) A small X-ray focal spot has a high spatial resolution since it reduces the geometric unsharpness, and therefore has a broad MTF (corresponding to a narrow LSF), MTF number 1.

2.16 In mammographic examinations, the breast is compressed between two plates. Answer the following with a brief explanation.
(a) Is the geometric unsharpness increased or decreased by compression?
(b) Why is the image contrast improved by this procedure?
(c) Is the required X-ray dose for a given image SNR higher or lower with compression?

Solution. (a) Compression of the breast reduces the object to detector distance (S_1-S_0), but also the object to source distance (S_0). However, the former is reduced to a greater degree. Therefore the size of the penumbra will decrease, and the geometric unsharpness becomes less.
(b) The image contrast is degraded by Compton scattered X-rays originating in the breast. Compression makes the object thinner in the direction of X-ray transmission, and therefore the number of Compton scattered X-rays decreases, and image contrast increases.
(c) The reduction in Compton scattered X-rays decreases the overall SNR of the image, but increases the contrast as described in part (b), so the required dose increases (though only very slightly). For small lesions the image contrast is the most important criterion which is why compression is used.

2.17 In digital subtraction angiography, two images are acquired, the first before injection of the contrast agent, and the other post-injection.
(a) Write an expression for the X-ray intensity (I_1) in the first scan in terms of I_0, μ_{tissue}, x_{tissue}, μ_{blood} and x_{vessel}, where x_{tissue} and x_{vessel} are the dimensions of the respective organs in the direction of X-ray propagation.
(b) Write a corresponding expression for the X-ray intensity (I_2) for the second scan, replacing μ_{blood} with $\mu_{constrast}$
(c) Is the image signal intensity from static tissue removed by subtracting the two images?
(d) Show that the signal from static tissue is removed by computing the quantity $\log(I_2)$-$\log(I_1)$.

Solution. The diagram below shows one possible set up for the solution:



(a) Before the contrast agent is injected, the X-ray intensity is given by:

$$I_1 = I_0 \, \mathrm{e}^{-\left[\mu_{tissue} x_{tissue} + \mu_{vessel} x_{vessel}\right]}$$

(b) After injection of the contrast agent, and assuming that $\mu_{contrast} >> \mu_{blood}$

$$I_2 = I_0 \, \mathrm{e}^{-\left[\mu_{tissue} x_{tissue} + \mu_{contrast} x_{vessel}\right]}$$

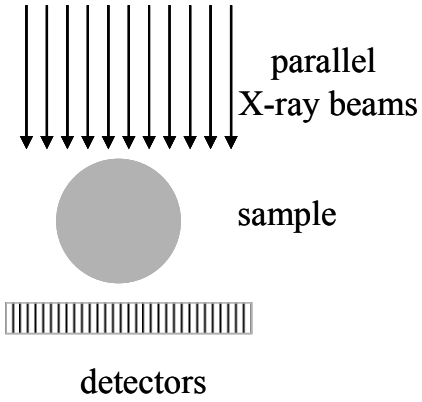
(c) I_1-I_2 results in an expression which still has a contribution from the tissue

(d) Subtracting the log of the two signal intensities removes the tissue contribution:

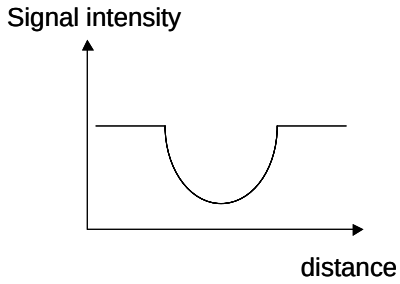
$$\log I_2 - \log I_1 = x_{vessel} \left(\mu_{contrast} - \mu_{vessel} \right)$$

In which there is now no contribution from the static tissue surrounding the vessel.

2.18 Draw the CT projection obtained from the setup shown in Figure 2.42. Assume that the spherical sample has a uniform attenuation coefficient throughout its volume.

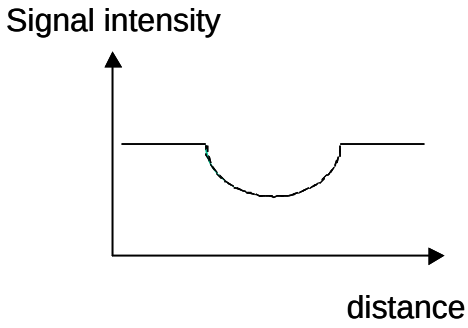


Solution. In the plot shown below the signal intensity corresponds to the number of X-rays which are detected.



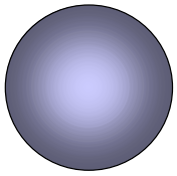
2.19 Considering the effects of beam hardening, draw the actual CT projection that would be obtained from the sample in Figure 2.42. Sketch the final image that would be formed from filtered backprojection of all of the projections acquired in a full scan of the sample in Exercise 2.18.

Solution. Since the effective energy of the X-ray beam increases as it passes through the sample, due to the preferential attenuation of lower X-ray energies, and the attenuation coefficient decreases with energy, the net effect of beam hardening is to reduce the amount of attenuation in the profile with the greatest reduction occurring through the central part of the object.

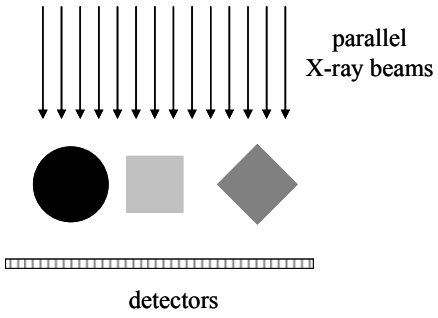


The effect is that the apparent value of the attenuation coefficient is lower in the centre of the object. As a result, the ratio of attenuation to its "original" value is

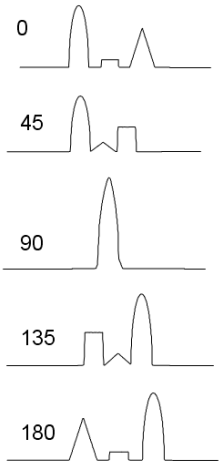
dependent on its position and the closer to the center, the smaller is the ratio. Therefore, the resulting image has the same shape and size but a ring-type artifact with the lowest value of μ in the middle.

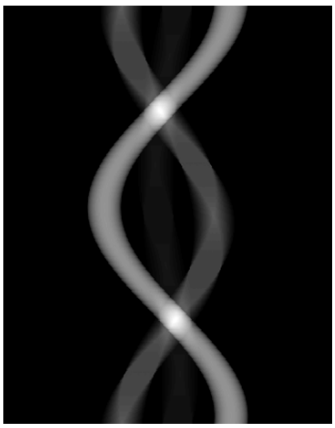


2.20 For the object shown in Figure 2.43, draw the projections that would be acquired at angles ϕ=0, 45, 90, 135 and 180^o (ignore beam hardening). Sketch the sinogram for values of ϕ from 0 to 360^o.



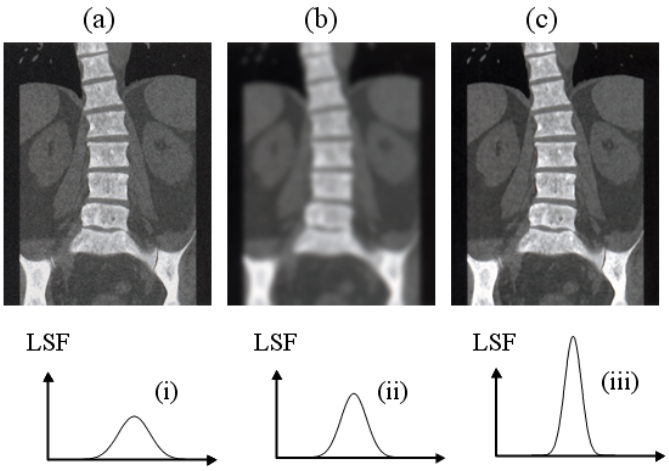
Solution





2.21 Shown below are three CT images: one represents an image with no spatial filtering, one has been high-pass filtered and the other has been low-pass filtered.

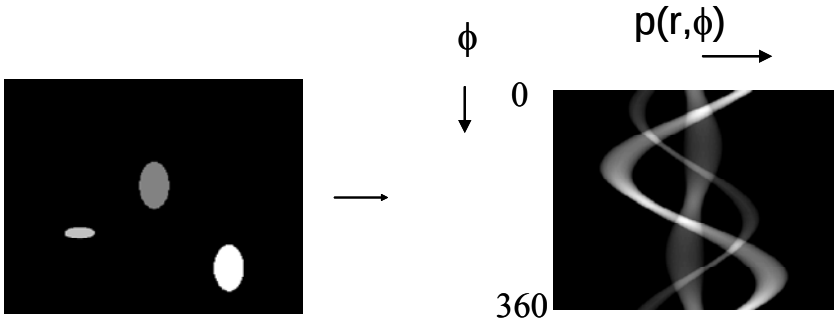
For each image, (a), (b) and (c), state which form of filtering has been used, and the effects on signal-to-noise, spatial resolution, and contrast-to-noise of the filtering. Finally, match the image to one of the line spread functions, (i), (ii), or (iii).



Solution. Of the three images, image (b) is clearly the most blurred with the highest SNR, image (c) is intermediate in both measures, and image (a) is the sharpest with the lowest SNR. This means that (c) must be the original image which has been low-pass filtered to give (b), and high-pass filtered to give (a). in terms of the LSFs, (i) is the broadest and so corresponds to image (b), (ii) is intermediate and relates to (c), and (iii) is the sharpest and so corresponds to (a).

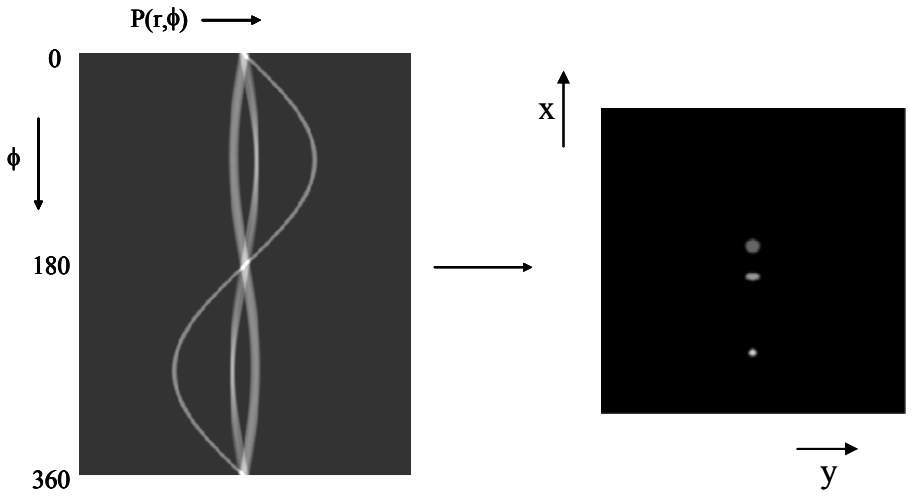
2.22 For the CT image shown in Figure 2.45(a), sketch the sinogram that produced such an image.

Solution



2.23 For the sinogram in Figure 2.45(b), sketch the filtered backprojection image that would result.

Solution.



Chapter 3

3.1 (a) In a sample of 20,000 atoms, if 400 decay in 8 seconds what is the radioactivity, measured in mCi, of the sample?
(b) In order to produce a level of radioactivity of 1 mCi, how many nuclei of ^{99m}Tc ($\lambda = 3.22 \times 10^{-5}\ s^{-1}$) must be present? What mass of the radiotracer does this correspond to? (Avogadro's number is 6.02×10^{23}).
(c) A radioactive sample of ^{99m}Tc contains 10 mCi activity at 9 am. What is the radioactivity of the sample at 12 pm on the same day?

Solution.

(a) $Q = \frac{dN}{dt} = \frac{400}{8} = 50$ disintegrations per second.

$$Q = \frac{50}{3.7 \times 10^{10}} = 1.35 \times 10^{-6}\, mCi$$

(b) Using the formula for radioactivity, and converting Curies into disintegrations/sec:

$$-3.7 \times 10^{7} = -3.22 \times 10^{5} N$$
$$\Rightarrow N = 1.149 \times 10^{12}\, nuclei$$

In order to convert this into mass:

$$m = \frac{1.149 \times 10^{12}}{6.02 \times 10^{23}} \times 99 = 1.89 \times 10^{-10}\, g \approx 0.19 ng$$

(c) $$N = N_0 e^{-\lambda t} = N_0 e^{-(3.22 x 10^{-5})(3600 x 3)} = 0.71 N_0$$

Since $Q = -\lambda N$, the radioactivity is ~7.1 mCi.

3.2 In a nuclear medicine scan using ^{99m}Tc, the image SNR for a 30 minute scan was 25:1 for an injected radioactive dose of 1 mCi. Imaging began immediately after injection.
(a) If the injected dose were tripled to 3 mCi, what would be the image SNR for a 30 minute scan?
(b) If the scan time were doubled to 60 minutes with an initial dose of 1 mCi, what would be the image SNR?

Solution.

(a) Since the SNR is proportional to the square root of the number of counts, tripling the injected dose increases the SNR by the square root of 3 to give a value of 43.3:1.

(b) After 30 minutes the number of nuclei is reduced to (exp–3.22x10^{-5}*30*60)= 94.4% of the original number. After 60 minutes, the number is reduced to 89.1%. The total number of disintegrations during the second 30 minutes is approximately 94% (89.1/94.4) that of the first 30 minutes. Therefore the S/N is 25*sqrt(1.94)=34.8:1.

3.3 A dose of 1 mCi of ^{99m}Tc is administered to a patient. Calculate the total dose to the patient if the biological half-life of the radiotracer in the body is:
(a) 2 years,
(b) 6 hours,
(c) 2 minutes.

Solution. Using equation (3.4) the effective half-lives are 6 hrs, 3 hrs and 2 minutes, respectively, which gives λ_{eff}=3.2x10^{-5}, 6.4x10^{-5} and 5.8x10^{-3} sec^{-1}. The overall amount of radioactivity is given by the time integral:

$$\int_{t=0}^{\tau} Q dt = -\lambda N_0 \int_{t=0}^{\tau} e^{-\lambda t} dt = +N_0 \left[e^{-\lambda \tau} - 1 \right]$$

This gives: (a) 1 mCi, (b) 0.3 mCi and

3.4 In the technetium generator, show mathematically that if $\lambda_2 >> \lambda_1$, that the radioactivities of the parent and daughter nuclei become equal in value at long times.

Solution. Since $\lambda_2 >> \lambda_1$, Equation (3.11)

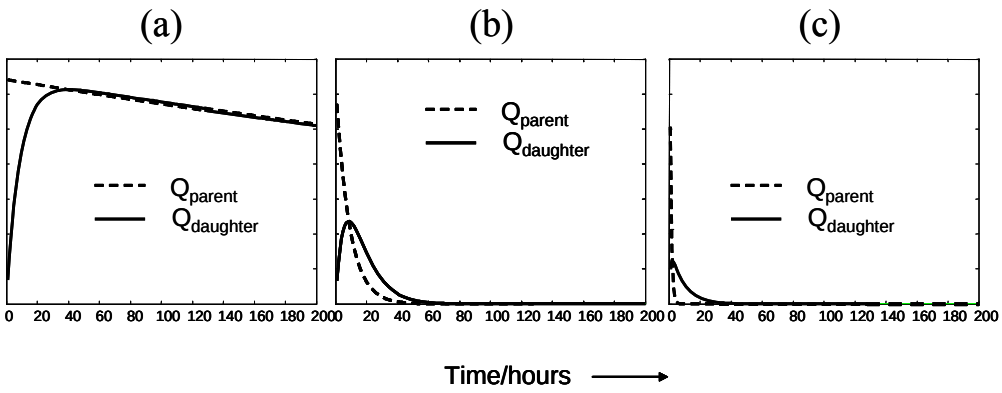
$$Q_2 = \frac{\lambda_1 \lambda_2 N_0}{\lambda_2 - \lambda_1} \left(e^{-\lambda_1 t} - e^{-\lambda_2 t} \right)$$

can be rewritten as

$$Q_2 \xrightarrow{\lambda_2 \gg \lambda_1} \lambda_1 N_0 \left(e^{-\lambda_1 t} - e^{-\lambda_2 t} \right) \xrightarrow{\lambda_2 \gg \lambda_1} \lambda_1 N_0 e^{-\lambda_1 t} = \lambda_1 N_1 = Q_1$$

3.5 Using the equations derived in the analysis of the technetium generator, plot graphs of the activity of parent and daughter nuclei for the following cases:
(a) $\tau_{1/2}$(parent) = 600 hours, $\tau_{1/2}$ (daughter) = 6 hours
(b) $\tau_{1/2}$ (parent) = 6.1 hours, $\tau_{1/2}$ (daughter) = 6 hours
(c) $\tau_{1/2}$ (parent) = 0.6 hours, $\tau_{1/2}$ (daughter) = 6 hours

Solution.



3.6. Calculate the exact time at which the first three "milkings" of the technetium cow should be performed.

Solution. For the Q_2 to be the maximum value, equation (3.11) is differentiated and set equal to zero. So ignoring the constant terms:

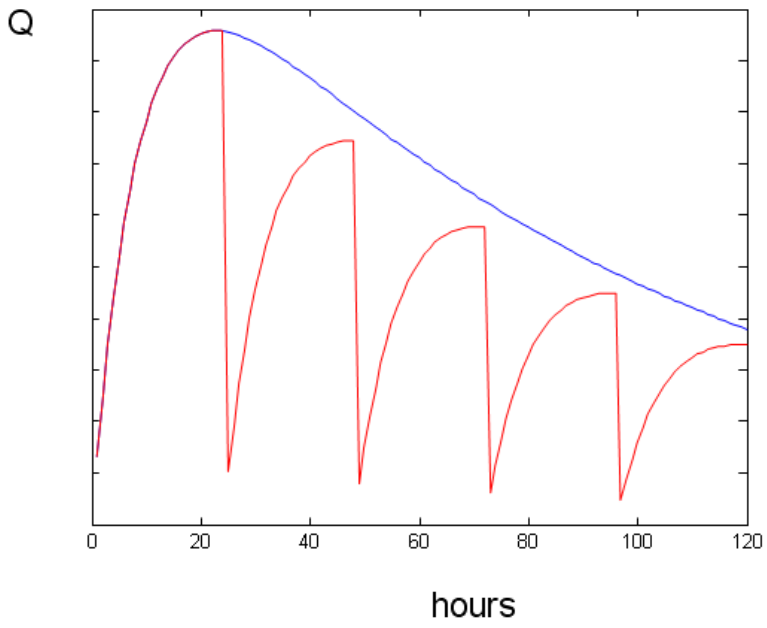
$$\frac{d}{dt}\left[e^{-\lambda_1 t} - e^{-\lambda_2 t}\right] = 0 \Rightarrow -\lambda_1 e^{-\lambda_1 t} - \lambda_2 e^{-\lambda_2 t} = 0$$

$$\Rightarrow t = \frac{\ln \lambda_1 - \ln \lambda_2}{\lambda_1 - \lambda_2} = 22.7 hrs$$

Therefore the second two times will be at 45.4 hours and 68.1 hours.

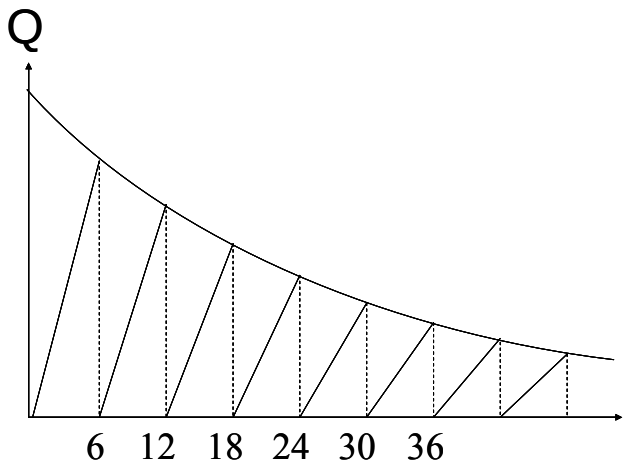
3.7. Do the tops of the curves in Figure 3.4 lie at the same values that would have been obtained if the technetium cow were not milked at all?

Solution. Intuitively one would say not, since after milking it will take time for the radioactivity to build up again. Plotting this mathematically gives the graph below with the blue line representing with no milking, and the red with milking.

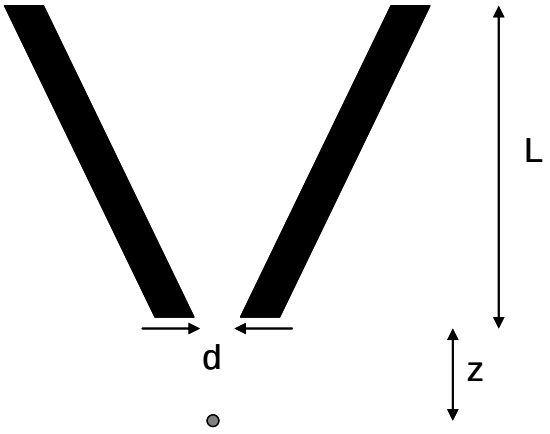


3.8 Rather than waiting 24 hours, only 6 hours are left between milkings. Plot the graph of radioactivity over the first two days.

Solution. By only waiting 6 hours, the plotof radioactivity vs. time is quite linear, and so the plots are quite similar to Figure 3.4 but do not show the non-linear behaviour.



3.9 Calculate the magnification factor for the pinhole camera shown in Figure 3.32(a). What implications does this have for image distortions?



Solution. If we make the assumption that d is very small (a true pinhole) then using simple trigonometry the magnification factor is given by L/z. This means that the degree to which the image is magnified depends upon the depth of the radioactivity within the patient, and deeper structures appear larger. If one considers the effect of d, then there is a magnification factor given by d(L+z)/z.

3.10 If the acceptance window for a planar nuclear medicine scan is set to 15%, what is the maximum angle that a γ-ray could be Compton scattered and still be accepted if it strikes the scintillation crystal?

Solution. The acceptance window of 15% means that the lowest energy is 129.5 keV. Using equation (2.6):

$$129.5 = \frac{140}{1+\left(\frac{140}{mc^2}\right)(1-\cos\vartheta)}$$

where the rest mass energy of the electron, mc^2, is 511 keV. Solving gives an angle of ~45°.

3.11 What is the energy of a γ-ray which has been Compton scattered at an angle of 30° in the body?

Solution. Equation (2.6) derived for Compton scattered X-rays can be used.

$$E_{\gamma,scat} = \frac{140}{1+\left(\frac{140}{mc^2}\right)(1-\cos 30)} \approx 135 \text{ keV}$$

3.12 (i) The thickness of the lead septa is chosen to ensure that only 5% of γ-rays penetrate from one collimator hole to the adjacent one. Using Figure 3.32(b) show that the thickness is given by [6d/μ]/[L-3/μ].

Solution. From the figure below, the minimum path distance (x) for a γ-ray to pass through the collimator and be detected is related to l, t and d by:

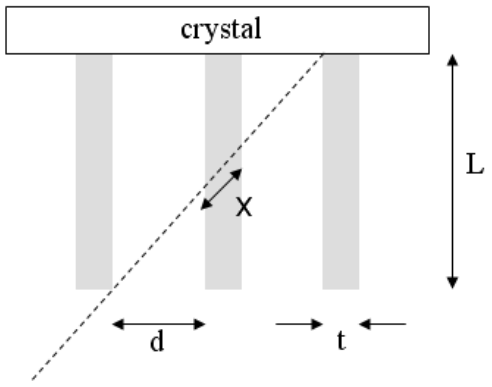
$$x = \frac{tL}{2d + t}$$

Since only 5% of the γ-rays can penetrate:

$$e^{-\mu x} = 0.05 \Rightarrow x = \frac{3}{\mu}$$

Rearranging the above two equations gives:

$$t = \frac{6\,d/\mu}{L - 3/\mu}$$



(ii) Calculate the septal thickness required for γ-rays of 140 keV for lead collimators with a hole diameter of 0.1 cm and a length of 2.5 cm. The attenuation coefficient for lead is 30 cm^{-1} at 140 keV.

Solution

$$t = \frac{6\left(0.1/30\right)}{2.5 - 3/30} = 0.008 \text{ cm}$$

3.13 Assuming that the body is circular with a diameter of 30 cm, calculate the spatial resolution (FWHM) for a parallel hole collimator with length 2.5 cm, for two sources of radioactivity, one very close to the detector (z=0) and one at the other side of the body (z~30).

Solution. The effective length is given by:

$$L_{eff} = 2.5 - \frac{2}{30} = 2.43 \text{ cm}$$

$$R_{coll} = \frac{d\left(L_{eff} + z\right)}{L_{eff}}$$

$$\text{For z=0: } R_{coll} = \frac{0.1\left(2.43\right)}{2.43} = 0.1 \text{ cm}$$

$$\text{For z=30: } R_{coll} = \frac{0.1\left(2.43 + 30\right)}{2.43} = 1.33 \text{ cm}$$

3.14 Stating any assumptions that you make, show that for the parallel collimator there is an approximate relationship between collimator efficiency and spatial resolution given by:

$$g \propto R_{coll}^2$$

Solution

Starting from equation [3.14] for geometric efficiency:

$$g \approx K^2 \frac{d^2}{L_{eff}^2} \frac{d^2}{\left(d + t\right)^2}$$

For a long, thin collimator, t<<d and therefore:

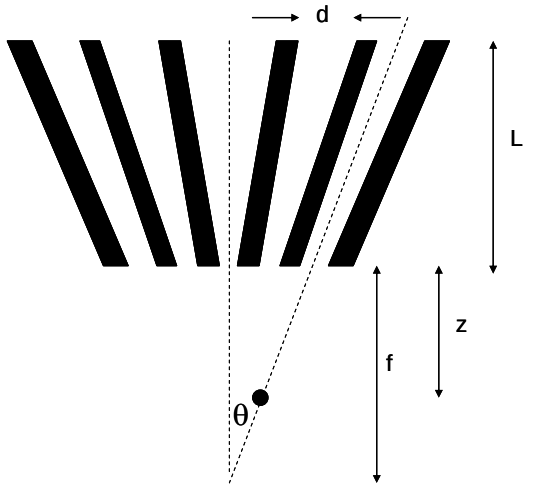
$$g \propto \frac{d^2}{L_{eff}^2}$$

Similarly for equation [3.13] for the spatial resolution, if the depth of the radioactivity is greater than the effective length of the collimator, then for a given depth z:

$$R \propto \frac{d}{L_{eff}}$$

Therefore, given these assumptions the collimator efficiency is proportional to the square of the collimator spatial resolution.

3.15 For the converging collimator shown in Figure 3.33 describe qualitatively (without mathematical proof) (i) whether the efficiency increases or decreases as a function of z, and as a function of θ, and (ii) whether the resolution increases or decreases as a function of θ, and as a function of z.

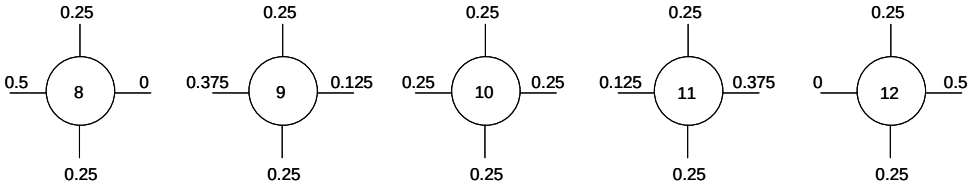


Solution. The efficiency is defined as the percentage of γ-rays emerging from the body which are detected. As the value of z increases it is clear that the efficiency decreases, similarly if θ becomes larger then the efficiency increases. For the spatial resolution, the larger the value of z then the better the spatial resolution, and for a larger value of θ then the spatial resolution becomes poorer.

3.16 Given the following resistor values for the Anger network, show that the output is linear in X.

PMT	R_{X+} (Ω)	R_{X-} (Ω)	R_{Y+} (Ω)	R_{Y-} (Ω)
8	Infinite	14.3	28.6	28.6
9	57.1	19.0	28.6	28.6
10	28.6	28.6	28.6	28.6
11	19.0	57.1	28.6	28.6
12	14.3	Infinite	28.6	28.6

Solution. The current in each channel is inversely proportional to the resistance by Ohm's law. By considering unit current produced in each PMT, and splitting it accordingly between the four resistors, the following outputs are obtained.



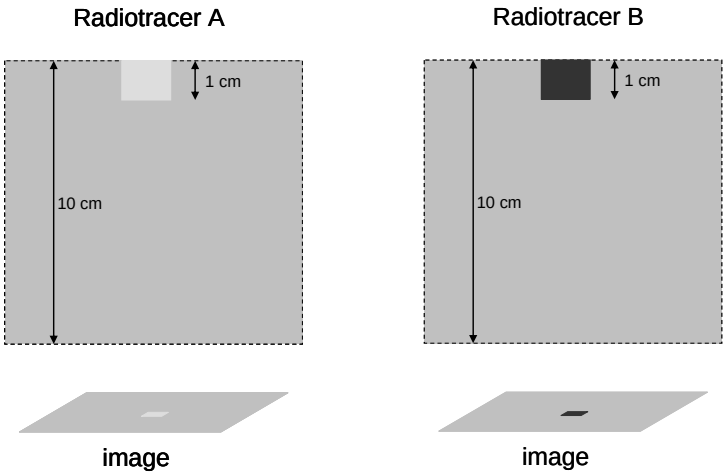
which shows a linear output as a function of position in x (defined in the left/right direction).

3.17 Three parameters which affect the image SNR in nuclear medicine are the thickness of the detector crystal, the length of the lead septa in the anti-scatter grid, and the FWHM of the energy window centered around 140 keV. For each parameter, does an increase in the value of the particular parameter increase or decrease the image SNR? In each case, name one other image characteristic (e.g. CNR, spatial resolution) that is affected, and explain whether this image characteristic is improved or degraded.

Solution. Increases in the detector crystal thickness increases the signal-to-noise since a larger proporation of the γ-rays are detected, but decreases the spatial resolution due to the increased light spread function. Increasing the length of the lead septa decreases the signal-to-noise since more γ-rays strike the collimator and are absorbed before reaching the detector, but increases the contrast since more Compton scattered γ-rays are absorbed and also the spatial resolution since the solid angle of acceptance is lower. Increasing the width of the energy window increases the signal-to-noise since more detected γ-rays are accepted as having not been scattered, but decreases the contrast since some of these have, in fact, been scattered.

3.18 Suppose that two radiotracers could be given to a patient. Radiotracer A is taken up in a tumour ten times higher than in the surrounding tissue, whereas B suppresses uptake by a factor of ten. Assume the tumour is 1 cm thick and the surrounding tissue is 10 cm thick. Ignoring attenuation effects, what is the CNR generated from each tumour. Assume that the uptake in normal tissue gives a rate of 10 counts per minute per square centimeter of tissue, and that the imaging time is 1 minute.

Solution. The figure below shows the distribution and images from the two cases.



For radiotracer A, the number of counts from the voxels corresponding to normal tissue is 10x10=100, whereas the number of counts from the voxels directly in line with the tumour is 190 (100 from the tumour + 90 from tissue). Therefore, the respective SNR values are 10 and 13.8, which gives a CNR of 3.8. For radiotracer B the respective numbers are 100 and 91 for the number of counts, 10 and 9.5 for the SNR, and 0.5 for the CNR.

3.19 Isosensitive imaging is a technique that acquires nuclear medicine scans from opposite sides of the patient, and then combines the signals to remove the depth dependence of the signal intensity. By considering the attenuation of γ-rays in the patient, show how this technique works, and what mathematical processing of the two scans is necessary.

Solution. Suppose that there are two sources of radioactivity, labeled 1 and 2 in the figure below. If only one scan is taken, shown on the left, then the intensity of γ-rays detected are:

$$I_1 = I_{1,0}e^{-\mu x_1}, \quad I_2 = I_{2,0}e^{-\mu x_2}$$

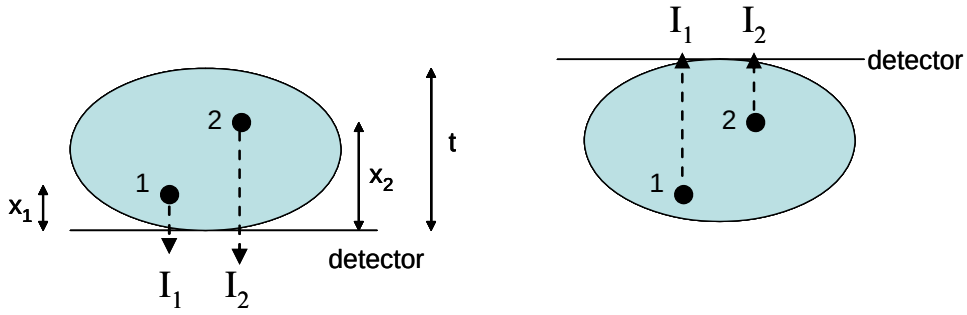
If we denote the thickness of the body as t, then the intensities from the second scan shown on the right:

$$I_1 = I_{1,0}e^{-\mu(t-x_1)}, \quad I_2 = I_{2,0}e^{-\mu(t-x_2)}$$

If the signals are multiplied together, and the square root of the product taken, to give I_1' and I_2' then these have values:

$$I_1' = I_{1,0}e^{-\mu t/2}, \quad I_2' = I_{2,0}e^{-\mu t/2}$$

This shows that the depth-dependence of the signals has been eliminated.



3.20 For a 64 x 64 data matrix, how many total counts are necessary for a 10% pixel-by-pixel uniformity level in a SPECT image?

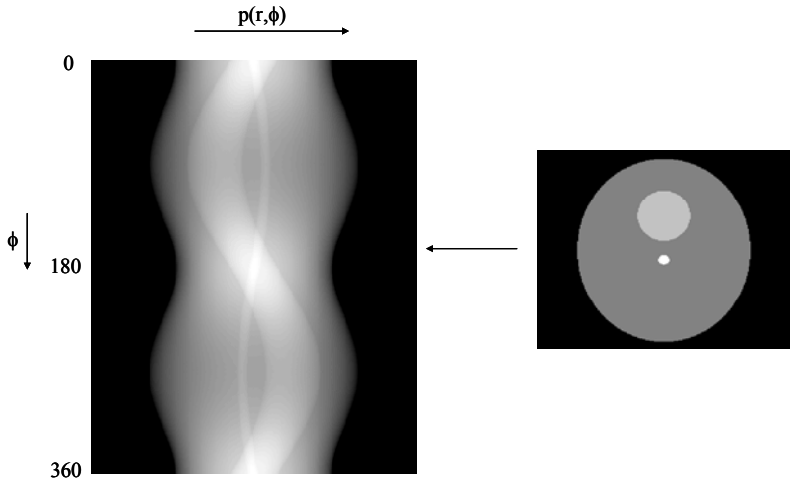
Solution. A 10% uniformity due to quantum mottle requires 100 counts. If we make the assumption that the signal intensity is uniformly distributed across the 64 x 64 pixels (clearly not the case in a real image), then the number of counts per pixel is 100 and the total number of counts is 64 x 64 x 100 = 409,600.

3.21 Answer true or false with a couple of sentences of explanation. If a uniform attenuation correction is applied to a SPECT scan, a tumour positioned close to bone appears to have a lower radioactivity than is actually the true situation.

Solution. The γ-rays originating from the tumour undergo more attenuation than estimated by the correction algorithm. Therefore the "corrected image" underestimates the level of radioactivity present in the tumour: the statement is true.

3.22 A SPECT scan is taken of a patient, and areas of radioactivity are found. The sinogram from the SPECT scan is shown in Figure 3.34. What are the shapes of the areas of radioactivity?

Solution.



3.23 Calculate the maximum angle and corresponding energy of Compton scattered γ-rays accepted for energy resolutions of 5, 15 and 25%.

Solution. For a 5% energy resolution, the limits are 136.5 to 143.5 keV. The lower energy corresponds to a scatter angle determined by:

$$136.5 = \frac{140}{1+\left(\frac{140}{511}\right)(1-\cos\theta)}$$

Solving gives $\theta=25^{o}$
For a 15% energy resolution, the limits are 129.5 to 150.5 keV. The value of θ is 54^{o}. For a 25% energy resolution, the limits are 122.5 to 157.5 keV. The value of θ is 61^{o}.

3.24 Using the rest mass of the electron, show that the energies of the two γ-rays produced by the annihilation of an electron with a positron are 511 keV.

Solution. The rest mass of an electron, and also a positron is 9.11×10^{-31} kg. Using $E=mc^2$:

$$E = (18.22 \times 10^{-31})(3 \times 10^{8})^2 = 16.4 \times 10^{-14} \text{ kg m}^2 \text{ s}^{-2} = 16.4 \times 10^{-14} \text{ J}$$

Since two γ-rays are produced for each annihilation, $E(\gamma\text{-ray}) = 8.2 \times 10^{-14}$ J. Converting this into eV ($1 \text{ eV} = 1.6 \times 10^{-19}$ J), gives an energy of 511 keV.

3.25 PET scans often show an artificially high level of radioactivity in the lungs. Suggest one mechanism by which this might occur.

Solution. The lungs attenuate γ-rays to a very low extent. If a standard attenuation correction algorithm is used (assuming uniform attenuation through the chest),

then it overestimates the attenuation in the lungs and produces a higher value of radioactivity than is actually there after correction.

3.26 For an ^{15}O PET scan, if an initial dose of 1 mCi is injected, calculate the total number of γ-rays that are produced over a scan time of four minutes, assuming that scanning starts immediately after injection.

Solution. During the four minute scan the total number of disintegrations is given by:

$$N_0\left(1-e^{-240\lambda}\right)$$

The value of λ is derived from the half-life and is equal to 0.0056 sec^{-1}. The value of N_0 can be calculated from the initial radioactivity of 1 mCi, equal to 3.7×10^7 disintegrations per second.

$$3.7x10^7 = 0.0056N_0$$

which gives a value of N_0 of 6.63×10^9. Therefore the total number of disintegrations is 4.9×10^9, and the number of gamma rays produced is double that, namely 9.8×10^9.

3.27 What timing resolution would be necessary to obtain a position resolution of 5 mm in TOF PET based only upon time-of-flight considerations?

Solution. Applying equation (3.21):

$$0.005 = \frac{3x10^8 \Delta t}{2} \Rightarrow \Delta t = 33\,ps$$

This is well below the resolving power of any current PET detectors.

3.28 If the brain is assumed to be a sphere with diameter 20 cm, and the largest dimension of the body to be 40 cm, what are the respective values of the timing resolution necessary to reduce the noise in TOF PET compared to conventional PET?

Solution. For the brain:

$$\frac{2D}{c\Delta t} > 1 \Rightarrow \Delta t < \frac{2(0.2)}{3x10^8} \Rightarrow \Delta t < 1.3 \text{ ps}$$

and for the body:

$$\frac{2D}{c\Delta t} > 1 \Rightarrow \Delta t < \frac{2(0.4)}{3x10^8} \Rightarrow \Delta t < 2.6 \text{ ps}$$

3.29 Suggest why a PET/CT scanner operating in 2D mode has a relatively uniform axial sensitivity profile, whereas in 3D mode the sensitivity is much higher at the centre of the scanner.

Solution. In two-dimensional mode (with the septa in place) the PET signal at each crystal ring comes only from a very well-defined region in the z-direction which is adjacent to that ring. In three-dimensional mode, on the other hand, as shown in Figure 3.26, the PET signal detected by each crystal ring can originate from anywhere within the body. Since the central ring in the z-direction detects γ-rays that have been attenuated in the body to a lower degree than for the outer rings, due to a shorter pathway through tissue, the sensitivity will be higher. The central ring also receives signal from tissue lying either side of it, whereas the outer rings only receive signal from one side.

3.30 Suggest why a curvilinear region of low signal intensity is often seen on PET/CT scans of the thorax and abdomen, which parallels the dome of the diaphragm.

Solution. The CT image corresponds to one position within the breathing cycle since data acquisition is so fast. This means that the attenuation map represents a snap-shot unless it is artifically blurred. If the diaphragm is fully contracted when the CT is acquired, then the attenuation coefficient will be underestimated compared to it's average position during the PET scan, and therefore the attenuation correction algorithm will not compensate sufficiently, and there will be a region of low signal intensity of similar shape to the dome of the diaphragm.

Chapter 4

4.1 Calculate the intensity transmission coefficient, T_I, for the following interfaces, assuming that the ultrasound beam is exactly perpendicular to the interface:
(i) muscle/kidney,
(ii) air/muscle, and
(iii) bone/muscle.
Discuss briefly the implications of these values of T_I for ultrasound imaging.

Solution. Using the intensity transmission formulae gives values of T_I for muscle/kidney of 0.99, air/muscle 0.001, and bone/muscle 0.59. The implications for clinical diagnosis are as follows: for the muscle/kidney interface there will only be a small reflected signal to indicate the presence of the boundary, but the situation for imaging structures behind this boundary is good. The exact converse applies for the air/muscle boundary. An intermediate case is the bone/muscle boundary, from which a large reflected signal will be detected, and through which a relatively large proportion of the ultrasound wave will be transmitted.

4.2 Repeat the calculations in exercise 4.1 with the angle of incidence of the ultrasound beam now being 45^o.

Solution. The first step is to calculate the angle of transmission.

For muscle/kidney $\dfrac{\sin 45}{\sin \theta_t} = \dfrac{1590}{1560} \Rightarrow \theta_t = 44^0$

For air/muscle $\dfrac{\sin 45}{\sin \theta_t} = \dfrac{330}{1590} \Rightarrow \theta_t$ is undefined,

There is total internal reflection and so no signal is detected.

For bone/muscle $\dfrac{\sin 45}{\sin \theta_t} = \dfrac{4000}{1590} \Rightarrow \theta_t = 16.3^0$

$$T_{I,muscle/kidney} = \frac{4(1.7)(1.62)\cos^2 45}{(1.62\cos 45 + 1.7\cos 44)^2} = 0.98$$

$$T_{I,bone/muscle} = \frac{4(1.7)(1.62)\cos^2 45}{(1.62\cos 45 + 1.7\cos 16)^2} = 0.71$$

4.3 Within tissue lies a strongly reflecting boundary, which backscatters 70% of the intensity of the ultrasound beam. Given a 100 dB receiver dynamic range, and an

operating frequency of 3 MHz, what is the maximum depth within tissue at which this boundary can be detected?

Solution. Since the frequency dependence of the attenuation coefficient for soft tissue is 1 dB/cm/MHz., the attenuation coefficient at 3 MHz is 3 dB/cm. 70% of the energy is backscattered, which means that the loss is 10 log(0.7) = -1.6 dB. Given the limit of 100 dB attenuation, the maximum travel path for the ultrasound beam corresponds to the distance that results in 98.4 dB of loss. The total path is therefore given by 98.4/3 = 32.8 cm. Knowing that the beam traveling length is twice as long as the reflection depth, the maximum detectable depth of the reflector is 16.4 cm.

4.4 Calculate the distance at which the intensity of a 1 MHz and 5 MHz ultrasound beam will be reduced by half traveling through (a) bone, (b) air, and (c) muscle.

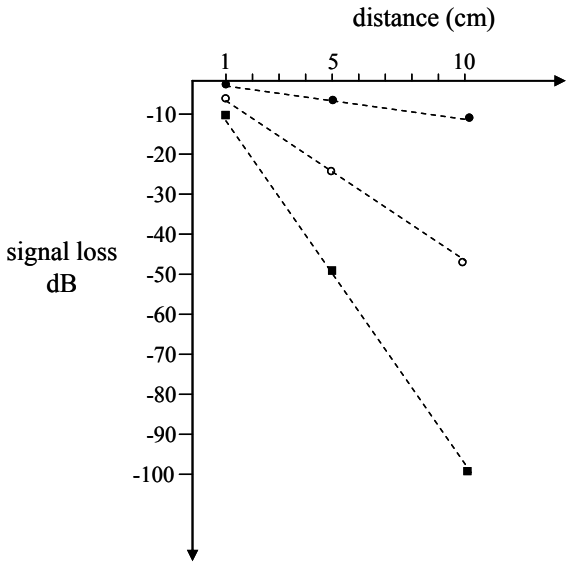
Solution. As stated in the text the values of μ for muscle, bone and air are 1, 8.7 and 45 dB cm^{-1} MHz^{-1}, respectively. For the intensity to be reduced by half, the value of (μx) must be ~3 dB. Therefore at 1 MHz, the half value distance is 0.34 cm for bone, 0.067 cm for air and 3 cm for muscle. At 5 MHz, the distances are one-fifth those at 1 MHz, i.e. 0.068 cm for bone, 0.013 cm for air and 0.6 cm for muscle.

4.5 Explain why a very fast or very slow tissue relaxation time results in a very small amount of energy being lost due to absorption.

Solution. If the relaxation time is very fast compared to the period of the wave, then as it passes through the tissue the wave essentially "sees" the tissue at its equilibrium physical position throughout its passage and the energy lost and gained is effectively averaged to a value close to zero. If the relaxation time is very slow compare to the passage, then the physical displacement of the molecules is essentially constant throughout the passage of the wave, and again there is very little restoring force to be counteracted.

4.6 Plot the attenuation of the ultrasound beam for 1, 5 and 10 MHz at depths within tissue of 1 cm, 5 cm, and 10 cm. For each depth calculate the fraction decrease in transmitted power, and the absolute power assuming an output power from the transducer of 100 mW/cm^2.

The respective attenuation coefficients are 1 dB/cm, 5 dB/cm and 10 dB/cm for the three frequencies.



4.7 In order to improve the efficiency of a given transducer, the amount of energy reflected by the skin directly under the transducer must be minimized. A layer of material with an acoustic impedance $Z_{\text{matching layer}}$ is placed between the transducer and the skin. If the acoustic impedance of the skin is denoted by Z_{skin}, and that of the transducer crystal Z_{PZT}, show mathematically that the value of $Z_{\text{matching layer}}$ which minimizes the energy of the reflected wave is given by:

$$Z_{matching\ layer} = \sqrt{Z_{PZT} Z_{skin}}$$

Solution. We want to maximize the intensity of the ultrasound that is transmitted through the skin into the body. This value is given by the product of the intensity transmitted through the crystal/matching layer boundary and the matching layer/skin boundary.

$$T_I = \frac{4Z_c Z_{ML}}{\left(Z_c + Z_{ML}\right)^2} \frac{4Z_s Z_{ML}}{\left(Z_s + Z_{ML}\right)^2}$$

To solve for the value of Z_{ML}, simply solve for:

$$\frac{\partial T_I}{\partial Z_{ML}} = 0$$

Using the division rule for differentiation:

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]=\frac{f'(x)g(x)-f(x)g'(x)}{g(x)^2}$$

and setting the numerator equal to zero leads to the desired result.

4.8 Given values of Z_{PZT} and Z_{skin} of $30\text{x}10^5$ $gcm^{-2}s^{-1}$ and $1.7\text{x}10^5$ $gcm^{-2}s^{-1}$, respectively, calculate what fraction of the energy from the transducer is actually transmitted into the patient.

Solution. The value of Z_{ML} is given by:

$$Z_{ML}=\sqrt{\left(30x10^5\right)\left(1.7x10^5\right)}=7.14x10^5\,gcm^{-2}s^{-1}$$

The value of T_I is now calculated:

$$T_I=\frac{4(30)(7.14)}{(30+7.14)^2}\frac{4(1.7)(7.14)}{(1.7+7.14)^2}=0.39$$

4.9 If two matching layers are used instead of one, and the respective acoustic impedances are given by the analogues of the equation above, then calculate the increase in efficiency in transmitting power into the patient.

Solution. If the two matching layers have Z values Z_{ML1} and Z_{ML2}, then two equations can be written to solve for these values:

$$Z_{ML1}=\sqrt{30.Z_{ML2}}$$
$$Z_{ML2}=\sqrt{1.7\sqrt{30.Z_{ML2}}}$$

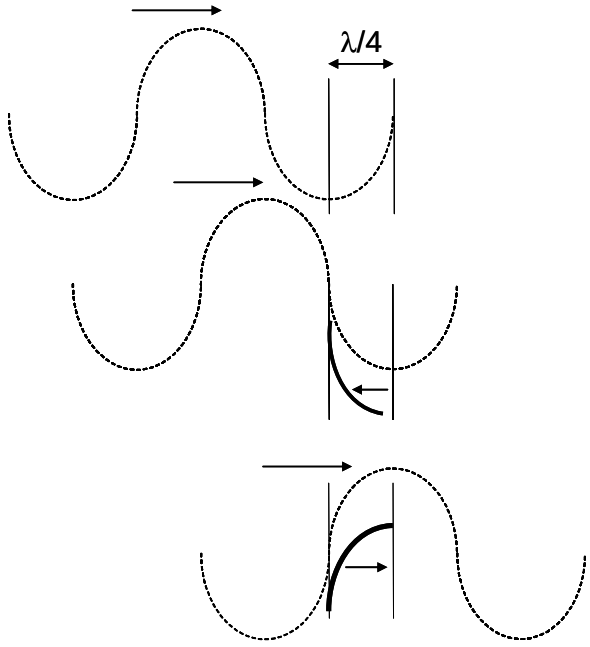
These equations can be solved easily to give Z_{ML1}=11.5 and Z_{ML2}=4.4. Calculating TI across the two boundaries gives:

$$T_I=\frac{4(30)(11.5)}{(30+11.5)^2}\frac{4(11.5)(4.4)}{(11.5+4.4)^2}\frac{4(4.4)(1.7)}{(4.4+1.7)^2}=0.51$$

4.10 Consider a transducer which has a thickness given by equation (4.22). A matching layer is used to maximize the energy transferred from the transducer to the body.

Show that the thickness of this matching layer should be one-quarter of the ultrasound wavelength.

Solution. Even though the mathing layer is optimized in terms of its characteriztic impedance, there is still quite significant reflection from the transducer/skin interface. The wave that is reflected from the matching-layer/skin boundary and then again from the matching-layer/crystal boundary is in phase with successive cycles of the transmitted pulse when the matching layer is $\lambda/4$ thick as shown in the figure below. This increases the ultrasound beam intensity going through skin. Note that there is a pressure inversion at the second reflection boundary due to $Z_{ML}<Z_c$.

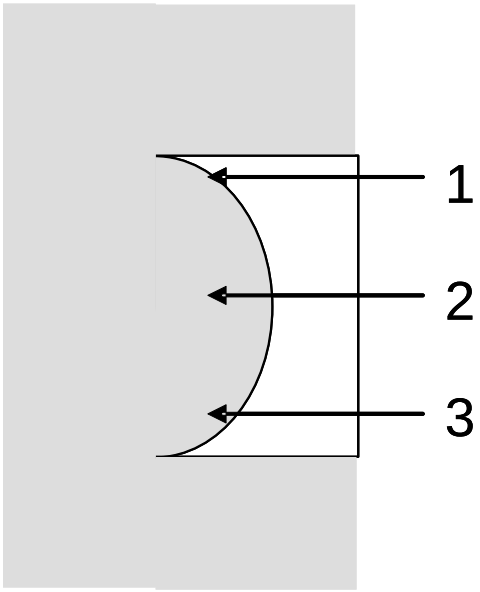


4.11 Why is the thickness of the PZT element set to one-half wavelength?

Solution. The crystal has a natural resonance frequency, which means that the output is maximum at this thickness. If this is a half-wavelength, then any energy which bounces backwards and forwards due to reflections at the boundaries will add constructively to increase the intensity of the beam.

4.12 For a concave lens to focus a beam, should the speed of sound in the lens be greater than or less than in tissue?

Solution. If we consider the situation of three rays travelling through the lens and tissue as shown below, for a concave lens to focus it is clear that waves 1 and 3 should be ahead of 2, and therefore the speed of sound in the lens should be higher than in the surrounding tissue.



4.13 How does the frequency profile of the ultrasound beam change as it passes through tissue? How does this affect the lateral resolution.

Solution. Since the attenuation coefficient of tissue is linearly dependent on frequency (in units of dB/cm), the higher frequencies will be preferentially attenuated (see also exercise 4.16). This means that the contributions of the lower frequencies to the overall frequency profile increase as the wave passes through tissue. Since the lateral resolution is given by $1.22\lambda/D$, this means that the effective wavelength increases and the lateral resolution decreases.

4.14 Consider a focused transducer with a radius of curvature of 10 cm and a diameter of 4 cm. This transducer operates at a frequency of 3.5 MHz, and transmits a pulse of duration 0.857 μs. What is the axial and lateral resolution at the focal point of the transducer?

Solution. The axial resolution is defined as half of the pulse length:
Pulse length = $1540 \times 0.857 \times 10^{-6} = 1.32 \times 10^{-3}$ m = 1.32 mm
So the axial resolution is 0.66 mm.

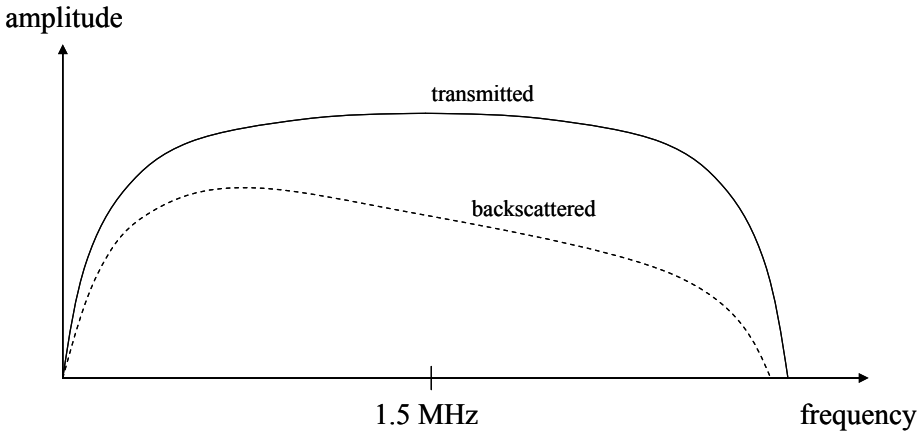
The lateral resolution is given by $\lambda F/D$. One can approximate the value of F by the radius of curvature, 10 cm. The wavelength in tissue is given by $1540/3.5 \times 10^{6}$ = 0.00044 m = 0.44 mm. Therefore, the lateral resolution is 0.44*100/40 = 1.1 mm

4.15 If the axial resolution was to be improved by a factor-of-two from that calculated in Exercise 4.14, what physical or operating parameters could be changed?

Solution. In order to decrease the length of the pulse, the transducer damping could be increased or the frequency of operation could be increased.

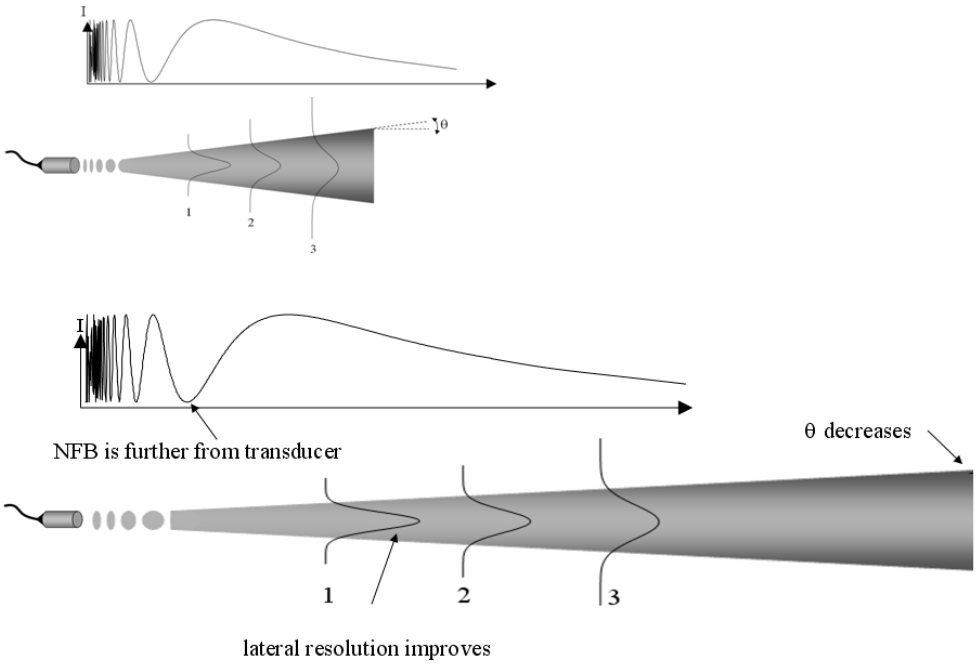
4.16 Plot the transmitted frequency spectrum of an ultrasound beam from a transducer operating at a central frequency of 1.5 MHz. Assume that the transducer is damped. Repeat the plot for the beam returning to the transducer after having passed through tissue and been backscattered.

Solution. In the plot shown below, the central frequency is 1.5 MHz, and the bandwidth of the transducer is very large corresponding to a low Q (you can draw the bandwidth to be whatever you want of course). For the backscattered energy, the intensity is lower, and the higher frequencies are preferentially attenuated, meaning that the frequency spectrum becomes skewed towards the lower frequencies.



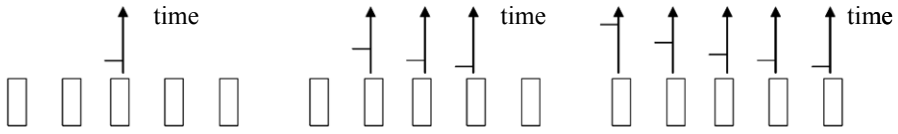
4.17 Draw the corresponding beam pattern to that shown in Figure 4.10 for a transducer operating at double the frequency. Note all of the frequency-dependent changes in the beam pattern.

Solution. In the figure shown at the bottom, three features are shown. Since the wavelength is halved for a transducer operating at double the frequency, the near field boundary is further from the transducer, and the angle of deviation (θ) is smaller. The lateral resolution also improves.



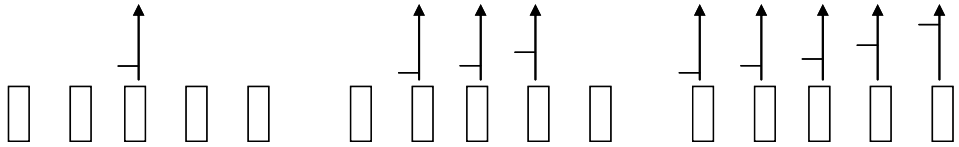
4.18 Show the required timing for simultaneous steering and dynamic focusing a phased array. For simplicity, sketch the general scheme using a small number (for example five) of elements.

Solution. Three successive time points are shown, the first fires a single element, the second three elements and the third five elements. The timings steer the beam towards the left and also focus the beam.



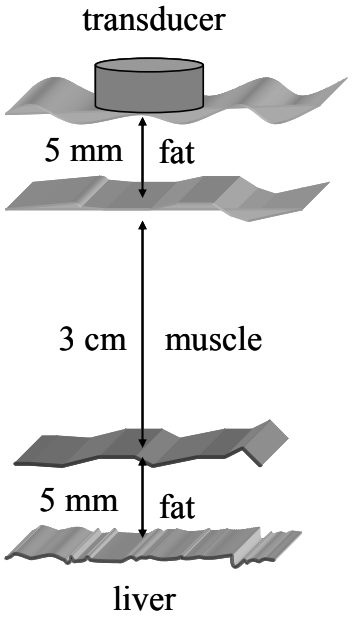
4.19 Sketch the corresponding delays required for dynamic beam-forming during signal reception.

Solution. These are effectively the reverse of the dynamic focussing delays.



4.20 Use the following data to sketch the A-mode scan from Figure 4.40(a). The amplitude axis should be on a dB scale, and the time axis in microseconds. Ignore

any reflected signal from the transducer/fat interface, and assume that a signal of 0 dB enters the body. At a transducer frequency of 5 MHz, the linear attenuation coefficient for muscle and liver is 5 dB cm^{-1}, and for fat is 7 dB cm^{-1}. Relevant values of the characteristic acoustic impedance and speed of sound can be found in Table 4.1.



Solution. The ultrasound beam is first attenuated through the fat layer, and then partially backscattered from the fat/muscle boundary. The backscattered signal is further attenuated by the fat layer during its transit to the transducer, where it produces the first echo. The fraction of the beam that is transmitted through the fat/muscle boundary is then attenuated by the muscle layer before it reaches the muscle/fat boundary, where a second process of partial reflection and partial transmission happens. The same analysis applies to the entire path. Due to high loss, second order reflection, i.e. waves bouncing back and forth between layers can be neglected because of the very low signal intensity.
The intensity of the signal at the fat/muscle boundary due to attenuation is (-0.5x7) = -3.5 dB. The intensity reflection coefficient at this boundary is given by:

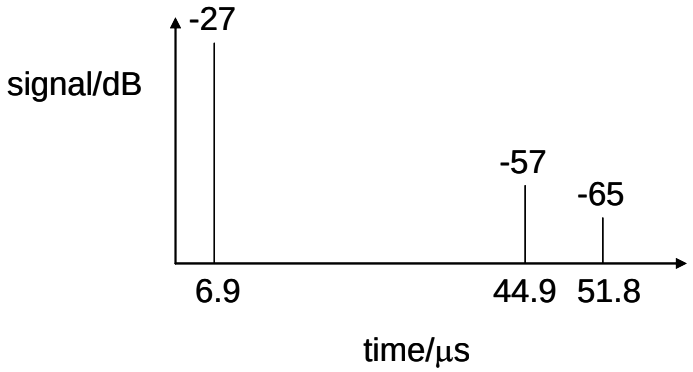
$$R_I = \frac{\left(1.38x10^5 - 1.7x10^5\right)^2}{\left(1.38x10^5 + 1.7x10^5\right)^2} \approx 0.01$$

This is equivalent to a reflection "loss" of R(dB) = 10 log 0.01 = -20 dB. Another attenuation of -3.5 dB occurs on the return path through the fat layer. Therefore the transducer detects a signal with an intensity of -27 dB. It occurs at a time 6.9 μs (round trip path 0.01 m, speed 1450 $m\text{-}s^{-1}$) after the pulse is transmitted.

The other signal strengths are calculated using the same method. One-way attenuation within the fat layer is -3.5 dB;One-way attenuation within the muscle layer is -15 dB; Intensity reflection coefficient at the fat/muscle boundary is -20 dB; Intensity reflection coefficient at the fat/liver boundary is -21 dB. One-way traveling time of signal within the fat layer is 0.005/1450 = 3.45 μs; One-way traveling time of signal within the muscle layer is 0.03/1580 = 19 μs;

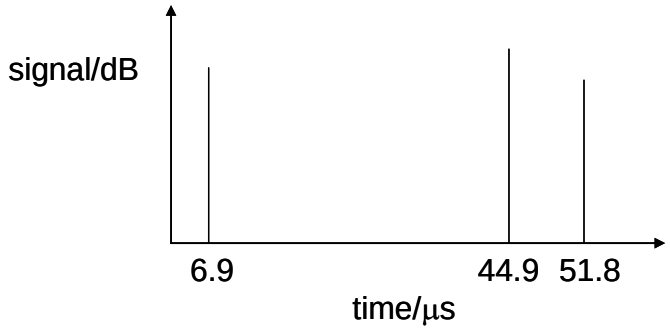
The second echo occurs at (3.45+19)x2 = 44.9 μs. Its intensity is 0-3.5-15-20-15-3.5 = -57 dB.

The third echo occurs at (3.45+19+3.45)x2 = 51.8 μs. Its intensity is 0-3.5-15-3.5-21-3.5-15-3.5 = -65 dB.

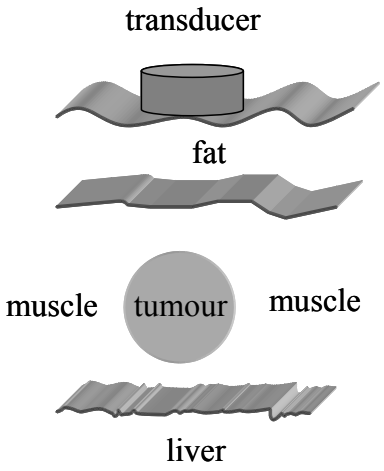


4.21 Determine and sketch the A-mode scan using the same parameters as above, but with a time gain compensation of 0.8 dB μs-1.

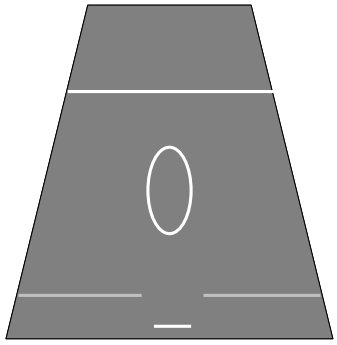
Solution. Using a time gain compensation of 0.8 dB μs^{-1} increases the three peaks above to value of: -27+(0.8x6.9)= -21.4 dB, -57+(0.8x44.9)=-21.1 dB and –65+(0.8*51.8)=-23.6 dB.



4.22 For the object shown in Figure 4.40(b), qualitatively sketch the B-mode ultrasound image. Ignore speckle or scatter and only consider signals backscattered from the tissue boundaries. Acoustic impedances: muscle 1.61, tumor 1.52 (x 10^5 g/cm^2s). Attenuation coefficients: muscle 1.0, tumour 0.4 (dB/cm/MHz). Speeds of sound: muscle 1540, tumour 750 (m/s).



Solution. The general characteristics of the B-mode scan are shown below. Backscattered signals appear as white, the gray areas represent general speckle throughout the image. The tumor appears elongated in the direction of ultrasound propagation since the speed of sound in the tumor is much slower than in the surrounding muscle. As a result the section of the liver directly behind the tumor appears to lie at a greater depth than the two sections either side of it. Finally, since attenuation in the tumor is much lower than tissue, there is acoustic enhancement for that part of the liver/muscle boundary behind the tumour.

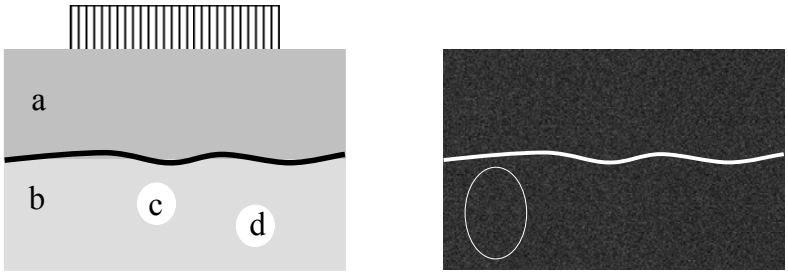


4.23 In a particular real-time imaging application the transducer moves through a 90^o sector with a frame rate of 30 frames per second, acquiring 128 lines of data per frame. If the image is acquired up to a depth of 20 cm, and the lateral resolution of the beam width at this depth is 5 mm, calculate the effect of transducer motion on overall image blurring, i.e., is it the dominant factor?

Solution. Consider the sector scan to be part of a circle of radius r = 20 cm. Since an angle of 90^o corresponds to 128 lines, the distance between adjacent lines at a depth of 20 cm is given by $2\pi r/(4x128)$ or 2.5 mm. This represents the blurring of the image due to continuous motion of the ultrasound transducer. Since the lateral

resolution is 5 mm, there is only a small degree of blurring due to motion of the transducer.

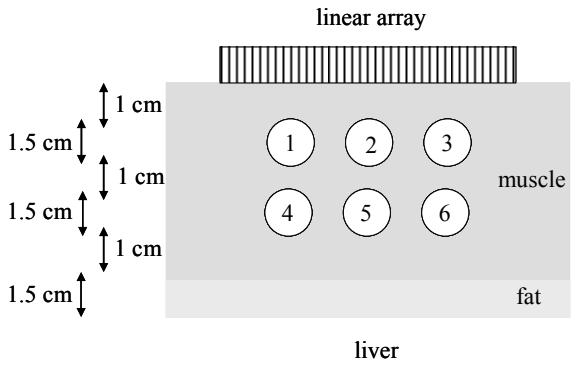
4.24 A B-mode scan is taken of the object in Figure 4.41 with a linear array. There are four tissue components, a and b with a boundary in-between and two spherical tumors c and d. Given the corresponding ultrasound image shown on the right what can you deduce about the acoustic characteristics of components a, b, c and d?



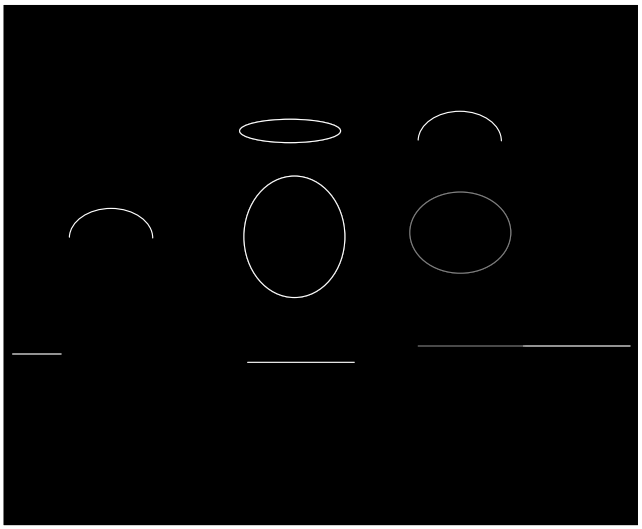
Solution. There are five different things that can be determined. (i) Since the boundary between a and b is visible, $Z_a \neq Z_b$. (ii) Since tumour d is not visible, $Z_b = Z_d$. (iii) Since tumour c is visible, $Z_c \neq Z_b$. (iv) Since tumour c is elongated, $c_c < c_b$.(v) Since tumour C is shifted to the left via refraction, $c_a \neq c_b$.

4.25 Given the ultrasound data in the table below, sketch the B-mode scan that would be obtained from the linear sequential array in Figure 4.42 (a quantitative analysis is NOT needed, ignore any refraction of the ultrasound beam).

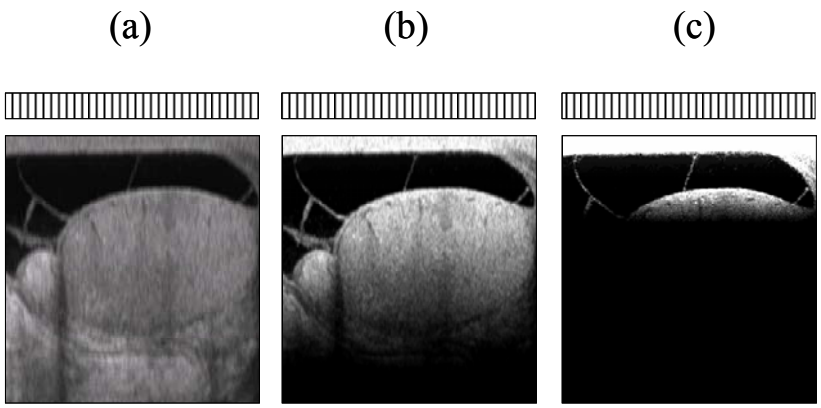
Tissue	Z ($\times 10^5$)	c (m/s)	μ (dB/cm)
Muscle	1.7	1540	1
Fat	1.4	1450	1.7
Liver	1.6	1570	1.5
Tumour 1	1.7	1540	1
Tumour 2	1.9	3080	1
Tumour 3	10	1540	5
Tumour 4	1.9	1540	20
Tumour 5	1.9	770	1
Tumour 6	1.9	1540	1



Solution. Tumour 1 is not seen since the Z value is the same as the surrounding muscle. Tumour 4 has strong attenuation so the front edge is brighter than the back edge, and there is no signal from behind it. Tumour 2 appears "squashed" due to the high speed-of-sound, and tumour 5 is shifted up as a result and also elongated since the speed of sound is relatively slow. The muscle/fat and liver/fat boundaries below tumour 5 are shifted down compared to their actual positions. Tumour 3 has a very bright front edge, and shadows everything behind it, so Tumour 6 is hardly seen at all.

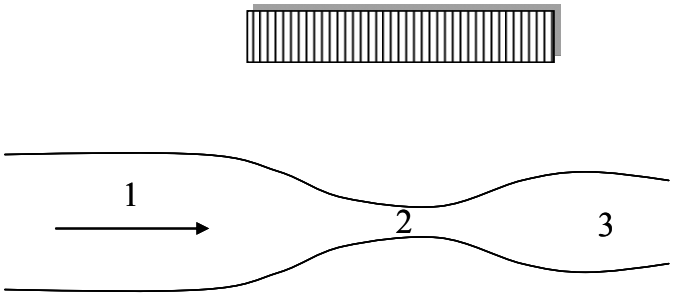


4.26 The three ultrasound images in Figure 4.43 are of the same object. Explain which single operating parameter changes from image (a) to image (b) to image (c).

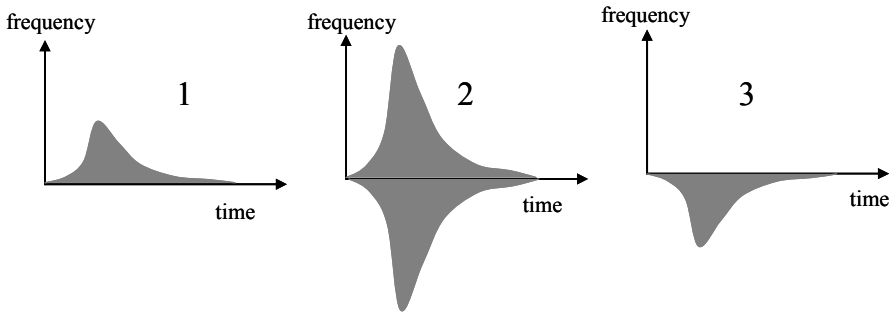


Solution. There are three image features to note: the penetration depth decreases from A to B to C, the signal-to-noise increases from A to B to C, and the axial spatial resolution increases from A to B to C. The only operating parameter that could give rise to all three of these effects is an increase in the ultrasound frequency.

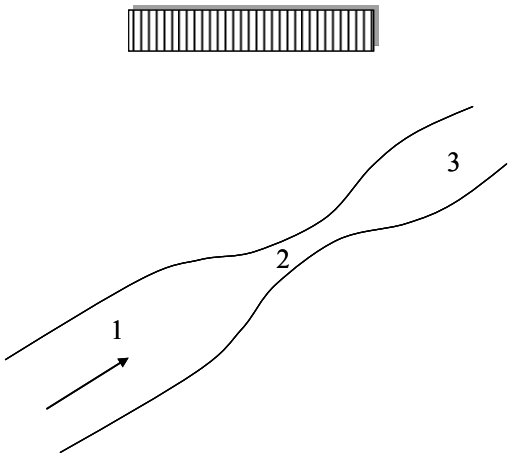
4.27 Sketch the Doppler spectral patterns at points 1, 2, and 3 below in a stenotic artery, shown in Figure 4.44 (a).



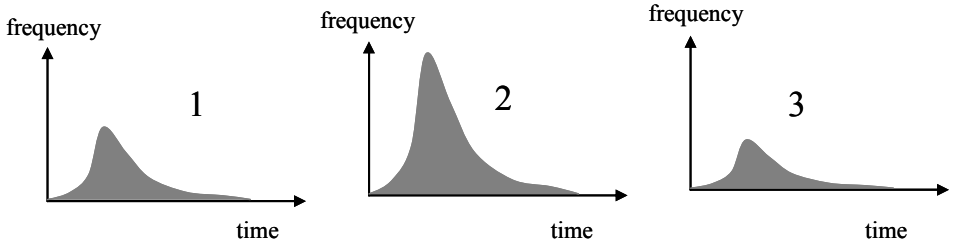
Solution. All of the plots are made over one cardiac cycle. At position 1 the flow is characterized by a range of relatively low velocities predominantly flowing towards the transducer. At 2, since the vessel narrows, the velocities become much higher. However, there are equal contributions from flow towards and away from the transducer, and so equal positive and negative frequencies. At 3, there will be a broad range of velocities, probably including turbulent flow. The Doppler spectral patterns will have the general appearance shown below.



4.28 On the same scale as for exercise 4.27, sketch the Doppler spectral plots for the situation in Figure 4.44 (b) in which the angle between the artery and the phased array transducer is altered.



Solution. The major difference between this situation and 4.28 is that the angle of blood flow at points 1, 2 and 3 is significantly different. At point 3 the angle is much lower than at 1 and 2, and so the apparent velocity will decrease. So the plot changes to something similar to below.



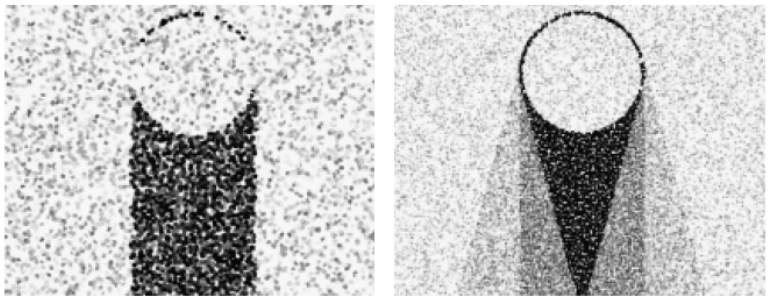
4.29 Show that the effects of a fixed error in the estimated angle between the transducer and the direction of flow in Doppler imaging are minimized by using a small value of the angle.

Solution. If there is a fixed error Δθ, then the effects of this error on the estimated flow are minimized by differentiating and setting equal to zero:

$$\frac{d}{d\theta}\left(\cos\frac{\Delta\theta}{\theta}\right)=0\Rightarrow\frac{1}{\theta}\sin\frac{\Delta\theta}{\theta}=0\Rightarrow\theta=0$$

4.30 Sketch the shape of the acoustic shadowing artifact produced from compound scanning.

Solution. The conventional shadowing artifact, shown on the left, is modified to a more triangular shape using compound scanning, as shown on the right.



4.31 The well-known reverberation artifact occurs when a strongly reflecting boundary within tissue is close to the transducer. Assume that there is a 2 cm thickness of muscle in front of the ribs. $Z_{crystal}$ is $33\text{x}10^5$, Z_{muscle} is $1.7\text{x}10^5$ and Z_{bone} is $7.8\text{x}10^5$ $gcm^{-2}s^{-1}$, the speed of sound in muscle is 1540 m/s, and the attenuation coefficient of muscle is 1 dB/cm, calculate the time gain compensation (units of dB/microsecond) that must be used to make the intensity of each of the reverberation signals the same.

Solution. Each successive echo undergoes 4 dB of attenuation, and reflection losses of:

$$R_{I,muscle/crystal}=10\log\left(\frac{1.7-33}{1.7+33}\right)^2\sim-0.9\ dB$$

$$R_{I,muscle/bone}=10\log\left(\frac{1.7-7.8}{1.7+7.8}\right)^2\sim-3.8\ dB$$

Overall the los is ~8.7 dB. The time between successive reverberation echoes is 0.04/1540 = 26 μs. Therefore, the TGC is 8.7/26 = 0.33 dB/μs.

Chapter 5

5.1 Assuming that there are 6.7 x 10^{22} protons in a cubic centimeter of water, what is the magnetization contained within this volume at a magnetic field strength of 3 Tesla?

Solution. The net magnetization is defined as the sum of magnetic moment of nuclei involved. It is determined by the population difference of nuclei occupying the two energy levels, multiplied by the z component of the magnetic moment for each nucleus

$$M_0 = \mu_z \left(N_{parallel} - N_{anti-parallel} \right) = \frac{\gamma^2 h^2 B_0 N_s}{16\pi^2 kT}$$

At room temperature,

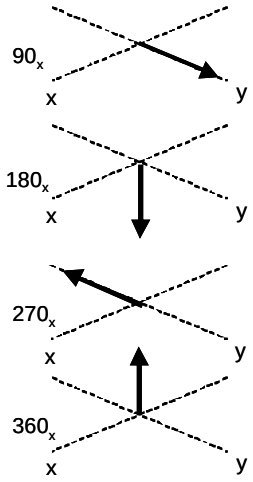
$$M_0 = \frac{\left(42.58\times10^6 \times 2\pi\right)^2 \left(6.63\times10^{-34}\right)^2 \times 3 \times 6.7\times10^{22}}{4\times4\pi^2 \times 1.38\times10^{-23} \times 300} = 9.68\times10^{-9}\, J/Tesla$$

5.2 Using classical mechanics, show that the effect of an RF pulse applied around the x-axis is to rotate z-magnetization towards the y-axis.

Solution. As covered in Section 5.2.2 the effect of a B-field is to create a torque C which is tangential both to the direction of the magnetic moment and the B-field. Since the net magnetization is in the z-direction, if the B-field is applied around the x-axis, then the torque must act to rotate the magnetization towards the y-axis.

5.3 Show schematically the separate effects of: (i) a 90_x^o, (ii) a 180_x^o, (iii) a 270_x^o, and (iv) a 360_x^o pulse on thermal equilibrium magnetization using the vector model.

Solution. The net effect is to produce rotation of the magnetization about the x-axis towards the y-axis.



5.4 What is the effect of changing the orientation of the RF coil so that it produces a pulse about the z-axis?

Solution. If the RF coil produces a pulse about the z-axis, then it does not produce any net magnetization and therefore no signal.

5.5 Calculate the effects of the following pulse sequences on thermal equilibrium magnetization. The final answer should include x-, y-, and z-components of magnetization.
a) 90^{o}_{x} (a pulse with tip angle 90^{o}, applied about the x-axis)
b) 80^{o}_{x}
c) $90^{o}_{x}\ 90^{o}_{y}$ (the second 90^{o} pulse is applied immediately after the first)

Solution.
(a) $M_z = 0$, $M_y = M0$, $M_x = 0$
(b) $M_z = M_0\cos 80 = 0.17\ M_0$, $M_y = M_0\sin 80 = 0.985\ M_0$, $M_x = 0$
(c) $M_z = 0$, $M_y = M_0$, $M_x = 0$

5.6 Answer true or false with 1-2 sentences of explanation:
a) recovery of magnetization along the z-axis after a 90^{o} pulse does not necessarily result in loss of magnetization from the xy-plane.
b) a static magnetic field B_0 that is homogeneous results in a free induction decay which persists for a long time.
c) a short tissue T_1 indicates a slow spin-lattice relaxation process.

Solution.
(a) False. The total magnetization, given by $\sqrt{(M_x^2+M_y^2+M_z^2)}$ must be less or equal to M_0. After a 90^{o} pulse, the value of M_z is zero, but this increases as a function of time due to T_1 relaxation. Therefore, the transverse magnetization must decrease as a function of time.

(b) True. If the B_0 field is inhomogeneous, nuclei at different positions will process at different rates and hence the spin system dephases faster, which results in a shorter free induction decay. So the more homogeneous the B_0 field the longer the FID lasts.
(c) False. A shorter T_1 means that it takes shorter time for the spin system resumes to its thermal equilibrium state, which corresponds to faster relaxation.

5.7 Write an expression for the M_z magnetization as a function of time after a 180_x^o pulse. After what time is the M_z component zero? Plot the magnetization after applying a 135_x^o pulse instead of a 180_x^o .

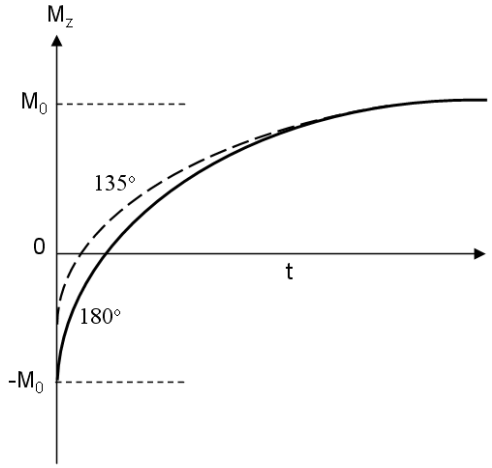
Solution. The z-component of magnetization as a function of τ after the RF pulse is given by:

$$M_z(\tau) = M_z(\tau = 0)\left(1 - 2e^{-\tau/T_1}\right)$$

By setting the left hand side equal to zero, and solving for τ we obtain:

$$\tau = \frac{\ln 2}{T_1}$$

If, instead of a 180_x^o pulse, a 135_x^o pulse is applied, then the initial component of z magnetization is given by $M_0\cos(135)$. The return to equilibrium, M_0, follows an identical curve to that immediately after a 180_x^o pulse, except shifted with respect to τ.



5.8 The hydrogen nuclei in the body are found mainly in lipid and water. The T_2 value of lipid was measured to be 100 ms, and that of water to be 500 ms. In a spin-echo experiment, calculate the delay between the 90^o pulse and the 180^o pulse which maximizes the difference in signal intensities between the lipid and

water. Assume that the total number of lipid protons is the same as the total number of water protons.

Solution. From the main text the signal intensities from lipid and water using a spin-echo sequence are given by

$$I_{lipid} \propto \rho_{lipid}\left(1-e^{-TR/T_{1,lipid}}\right)e^{-TE/T_{2,lipid}}, I_{water} \propto \rho_{water}\left(1-e^{-TR/T_{1,water}}\right)e^{-TE/T_{2,water}}$$

Since $\rho_{lipid}=\rho_{water}$, and we can assume that $TR>>T_1$ for both species since we are going to differentiate based upon the different T_2 values (a T_2-weighted sequence):

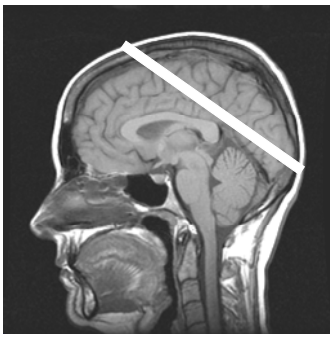
$$\Delta I = I_{water} - I_{lipid} \propto \left(e^{-TE/500} - e^{-TE/100}\right)$$

To maximize the contrast (difference of signal), set

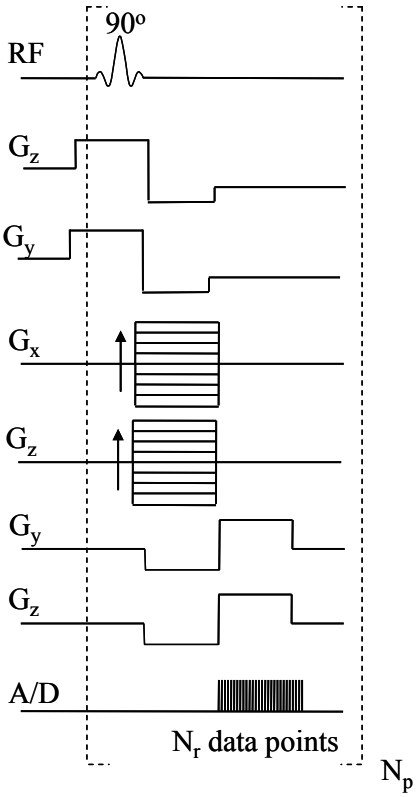
$$\frac{\partial(\Delta I)}{\partial(TE)} = 0$$

Solving for TE gives a value of 86.6 ms, and therefore the time between the 90 and 180 degree pulses is approximately 43.3 ms.

5.9 The operator wishes to acquire an oblique slice shown by the orientation of the white bar in Figure 5.56 (a). Draw the gradient echo imaging sequence that would be run to acquire such an image.



Solution. The oblique slice is in the head/foot (z) and anterior/posterior (y) planes, and therefore both gradients must be applied in the slice select direction, with approximately equal strengths since the slice is at an angle of ~45^o. Since the phase and frequency encoding gradients are both applied orthogonal to the slice selection gradient, both phase and frequency encoding mus also have contributions from two gradients. So the sequence is (note that the phase and frequency axes can be interchanged).



5.10 A multi-slice spin-echo imaging sequence is run with the following parameters: a 256 x 256 data matrix, TR is 2 seconds, TE is 20 ms, and ten slices are acquired. How long does the MRI scan take to complete?

Solution. The total MRI scan takes TR x no. phase encoding steps, so 512 sec. Provided that no.slices x TE < TR then the number of slices has no effect on the total imaging time.

5.11 Derive the value of the Ernst angle given in equation (5.35).

Solution. In the gradient-echo sequence, the M_z magnetization reaches a steady state. This means that the reduction in M_z from a tip angle of α degrees, is exactly balanced by the gain in M_z due to T_1 relaxation during the TR interval, i.e.

$$M_z = M_z \cos\alpha + \left(M_0 - M_z \cos\alpha\right)\left(1 - e^{-TR/T_1}\right)$$

Re-arranging and solving for the steady-state value of M_z gives:

$$M_z = M_0 \frac{1 - e^{-TR/T_1}}{1 - \cos\alpha e^{-TR/T_1}}$$

Therefore the steady state signal intensity is given by:

$$M_y = \frac{M_0 \sin a\left(1 - e^{-TR/T_1}\right)}{1 - \cos\alpha e^{-TR/T_1}}$$

In order to determine the optimum value of α for a given value of TR, set

$$\frac{\partial M_y}{\partial \alpha} = 0$$

which leads directly to:

$$\alpha = \cos^{-1}\left(e^{-TR/T_1}\right)$$

5.12 Given a maximum magnetic field gradient of 40 mT/m, how homogenous must the magnet be (in parts per million) to enable a spatial resolution of 1 mm to be acquired.

Solution. A gradient value of 40 mT/m means that over 1 mm the field variation is 40 μT. For a 3T magnet, for example, this corresponds to a homogeneity of 1.33×10^{-5} or 13 ppm.

5.13 Assume that the correct incremental phase-encoding gradient values are $G_n = n\Delta G_0$. What will happen to the reconstruced image if the gradient system malfunctions such that the effective gradients applied are: $G_n = n(\Delta G_0/2)$, i.e. one-half of the correct values.

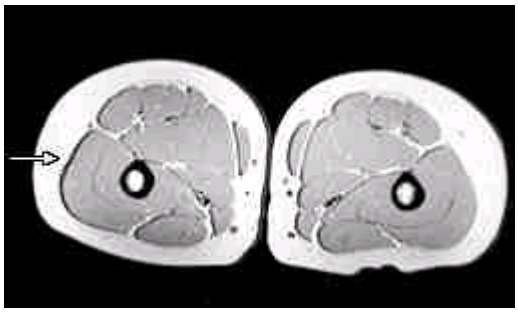
Solution. We know that the field-of-view is given by the inverse of the increment in k-space, Δk_y. If the gradient increment halves, then the field-of-view is twice the desired value, and so the spatial resolution of the image will halve.

5.14 In the image shown in Figure 5.56 (b), acquired using a standard spin-echo sequence, the bright signal corresponds to lipid and the lower intensity signal to water. The lipid and water signals appear spatially shifted with respect to one another.

(i) Given the facts above, which of the left/right or up/down dimensions corresponds to the frequency encoding direction, and which to the phase encoding direction. Explain your answer fully.

(ii) The image is acquired at a field strength of 3 Tesla, and the black band in the image is 3 pixels wide. If the total image data size is 256 x 256, what is the overall data acquisition bandwidth? The image field-of-view is 5 x 5 cm: what is the strength of the frequency encoding gradient?

(iii) If the frequency encoding gradient were increased by a factor of three, what effect would this have on the imaging artifact?



Solution. (i) We know that fat and water resonate at different frequencies, with a difference of approximately 3 ppm, corresponding to ~390 Hz at 3 tesla. Since spatial dimensions are represented by different resonant frequencies in the presence of a frequency-encoding gradient, the images from fat and water will be displaced slightly. This causes signal "pile-up" on one side and a signal void (shown as a black line) on the other. So the left/right direction is the frequency encoding and up/down the phase encoding direction in the image.

(ii) Since the black band is 3 pixels wide and corresponds to a shift of 390 Hz, the full image width of 256 pixels corresponds to a data acqusition bandwidth of 256*(390/3)= 33.28 kHz. Since the field-of-view is 5 cm, the gradient strength is 33280/5=6656 Hz/cm.

(iii) Increasing the frequency encoding gradient by a factor of three to ~20 kHz/cm would increase the bandwidth to ~100 kHz. The frequency shift between fat and water is unchanged at 390 Hz, but this now corresponds to a shift of only one, rather than three, pixels. Therefore, the image artifact would become less severe. However, the signal to noise of the image would decrease by a factor of 1.73, since the bandwidth has increased by a factor of three.

5.15 Use the concept of signal aliasing to derive the image field-of-view in terms of the incremental phase encoding gradient step and the amount of time that the gradient is applied for.

Solution. The phase difference at every spatial location induced by successive phase encoding gradients must be less than $\pm 180^{o}$, otherwise it is not possible to determine whether it represents a positive or negative change in phase. For example a 240^{o} difference is the same as a -120^{o} difference and so the two spatial locations would fall on top of one another in the resulting image.

The phase shift between successive phase encoding steps G_1 and G_2 is given by:

$$\Delta\phi = \phi_1 - \phi_2 = \gamma G_1 y \tau_{pe} - \gamma G_2 y \tau_{pe} = \gamma \Delta G y \tau_{pe}$$

The value of the spatial position furthest from the centre of the image is given by y_{max} = FOV/2, which must correspond to a phase difference between successive phase encoding steps of 180^o or π radians, and so:

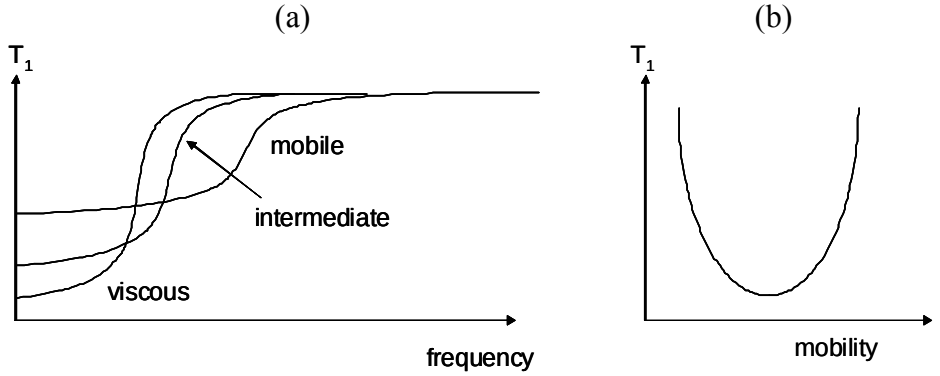
$$\pi = \gamma \Delta G \frac{FOV}{2} \tau_{pe} \Rightarrow FOV = \frac{2\pi}{\gamma \Delta G \tau_{pe}}$$

In k-space this corresponds to:

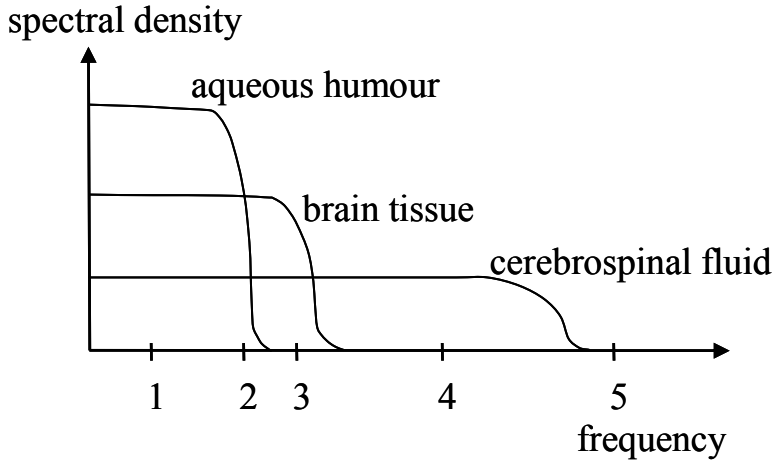
$$\Delta k = \frac{\gamma}{2\pi} \Delta G \tau_{pe} = \frac{1}{FOV}$$

5.16 (a) Plot qualitatively the dependence of T_1 on the strength of the applied magnetic field for a mobile, intermediate, and viscous liquid.
(b) Plot qualitatively the variation in T_1 as a function of the mobility of a liquid.

Solution. (a) Since the T_1 is inversely proportional to the spectral density (a high spectral density means a short or low value of T_1, and vice-versa), the graph of T_1 vs frequency is essentially the inverse of that of J(ω) vs frequency.
(b) At very high values of the mobility, then the T_1 is very long since J(ω) is very low. At very low mobilities, then J(ω) is also very low, and so T_1 is also very long. Therefore, the graph of T_1 vs. mobility is U-shaped as shown below.



5.17 For the five frequencies (1-5) shown in Figure 5.57, state the order of the T_1 values, e.g., T_1(brain)>T_1 (CSF)>T_1 (aqueous humour). Where possible, do the same for the T_2 values.



Solution. Since the T_1 value is essentially inversely proportional to the spectral density it follows that :

At frequency 1: T_1(aqueous humour) < T_1(brain) < T_1(CSF).
At frequency 2: T_1(aqueous humour) = T_1(brain) < T_1(CSF).
At frequency 3: T_1(brain) < T_1(CSF) < T_1(aqueous humour).
At frequency 4: T_1(CSF)< T_1(aqueous humour) ≈T_1(brain).
At frequency 5: T_1(brain) ≈ T_1(aqueous humour) ≈ T_1(CSF).

The relative T_2 values can only be absolutely determined at frequencies where the spectral densities of two tissues are the same, and then one can look at the contribution of the zero frequency J(0) component.

At frequency 2: T_2(aqueous humour)>T_2(brain)

5.18 Choose the correct option from (a)-(e) and explain why this is your choice.
The maximum MR signal is obtained by using:

(a) 90^o RF pulse, short TE, and short TR
(b) 45^o RF pulse, short TE, and short TR
(c) 90^o RF pulse, short TE, and long TR
(d) 90^o RF pulse, long TE, and short TR
(e) 45^o RF pulse, long TE, and short TR

Solution. The correct solution is (c). In order to maximize the signal, the tip angle should be 90^o which requires a long TR for the magnetization to recover to M_0 for full T_1 relaxation. T_2 relaxation decreases the signal, and so should be minimized by choosing a short value of TE.

5.19 Choose the correct option from (a)-(e) and explain why this is your choice.
Water in tendons is bound very strongly and cannot diffuse freely. It produces very low MR signal intensity because:

(a) T_1 is very short
(b) T_2 is very short
(c) T_2^* is very long
(d) T_2 is longer than T_1
(e) T_2^* is longer than T_2

Solution. Very tightly bound tissue has a very short T_2 and T_2* which produces a low signal intensity. Therefore, (b) is the only possibility that is correct. T_2 can never be longer than T_1 and so (d) is incorrect, and T_2* can never be longer than T_2 so (e) is incorrect. A short T_1 gives a high MR signal, as does a long T_2*, so both (a) and (c) are wrong.

5.20 A brain tumour has a lower concentration of water than surrounding healthy tissue. The T_1 value of the protons in the tumour is shorter than that of the protons in healthy tissue, but the T_2 value of the tumour protons is longer. Which kind of weighting should be introduced into the imaging sequence in order to ensure that there is contrast between the tumour and healthy tissue. If a large concentration of superparamagnetic contrast agent is injected and accumulates in the tumour only, which kind of weighting would now be optimal?

Solution. We can construct a simple table to show the relative signal intensities from the influence of proton density, T_1 and T_2.

	ρ	T_1	T_2
tumour	low	high	high
healthy	high	low	low

For a proton-density weighted scan, the signal depends only on the proton density, whereas for both T_1- and T_2-weighted images the signal intensity depends on the product of the proton density and the T_1- and T_2-dependent terms, respectively. Therefore, in both T_1- and T_2-weighted scans the higher signal intensity arising from the respective relaxation time effects (higher signal from the table) could cancel out the effects of the proton density. Therefore, to ensure contrast a proton-density weighted image should be acquired.

Since the paramagnetic agent accumulates in the tumour, then it will shorten the T_2 value in the tumour and now a T_2-weighted sequence would produce more signal from the healthy tissue than the tumour. Therefore, a T_2-weighted sequence should be run to produce even more contrast than a proton-density weighted one.

5.21 A region of the brain to be imaged contains areas corresponding to tumour, normal brain and lipid. The relevant MRI parameters are:

$$\rho(\text{tumour}) = \rho(\text{lipid}) > \rho(\text{brain})$$
$$T_1(\text{lipid}) > T_1(\text{tumour}) > T_1(\text{brain})$$
$$T_2(\text{lipid}) > T_2(\text{tumour}) > T_2(\text{brain})$$

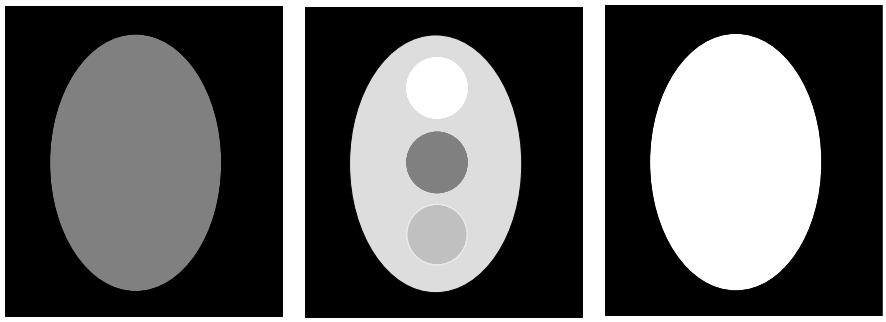
Which type of weighted spin-echo sequence should be run in order to get contrast between the three different tissues. Explain your reasoning, including why the other two types of weighting would not work.

Solution. Again we can draw up a table relating the image signal intensity to the three parameters.

	ρ	T_1	T_2
tumour	high	medium	medium
lipid	high	low	high
brain	low	high	low

Obviously a proton-density weighted sequence would not work, since tumour and lipid would have equal signals. If we use a T_2-weighted sequence, then we are guaranteed that brain will have the lowest signal due to the combined effects of proton density and short T_2. If we were to use a T_1-weighted sequence there is a chance that the higher signal from brain due to a short T_1 would cancel out the lower proton density and produce no contrast. Therefore, T_2-weighting should be used.

5.22 Three MRIs of the brain are acquired using identical parameters except for the TR and TE times. Three tumours (upper, middle and lower) are seen in one of the images but not in the other two, as shown in Figure 5.58. If the T_1 values for all the tissues (tumours and brain) are less than 2 seconds, and the T_2 values are all greater than 80 ms, describe the *relative* values of proton density, T_1 and T_2 of brain tissue and the three tumours.

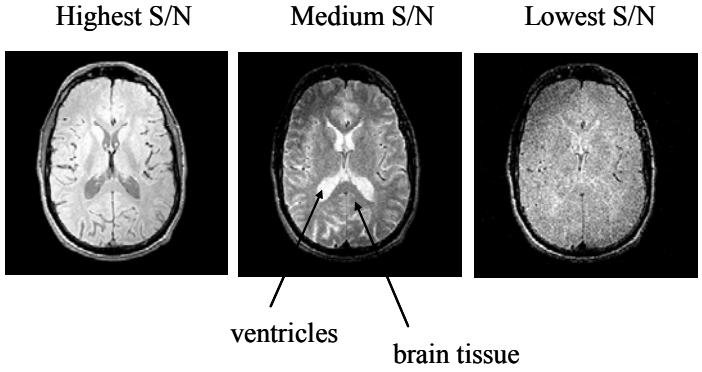


TR=0.5 s, TE=80 ms TR=0.5 s, TE=1 ms TR=10 s, TE=1 ms

Solution. The image on the right is proton-density weighted due to the long TR and short TE. This shows that the proton density of the three tumours and the brain tissue are the same. Since only the TE changes from the image in the centre to the one on the left, the associated signal changes are related to the relative T_2 values. The top tumour signal decreases by the most with the increased TE meaning that it has the shortest T_2, followed by the brain, the bottom tumour and finally the central tumour which has the longest T2. Comparing the middle and right images, the changes in signal intensity as a function of the TR changing can be related to the relative T_1 values. The top tumour changes signal the least, meaning that it has the shortest T_1 value, followed by the brain, bottom tumour and middle tumour which has the longest T_1.

5.23 Three images are shown in Figure 5.59: the scaling in each image is different and is normalized to the same maximum value. The imaging parameters are TR=2000 ms, TE= 20 ms for one image, TR=750 ms, TE=80 ms for another, and TR=2000 ms, TE =80 ms for the final one.

(i) Assign each image to the appropriate TR and TE values.
(ii) Based on your answer do the ventricles have a higher or lower T_1 value than brain tissue? What is the corresponding answer for T_2?



Solution. (i) The highest S/N corresponds to a long TR and short TE, and so the image on the left has TR 2000 ms, TE 20 ms. The lowest S/N has the shortest TR and longest TE, and so the image on the right has TR 750 ms, TE 80 ms.

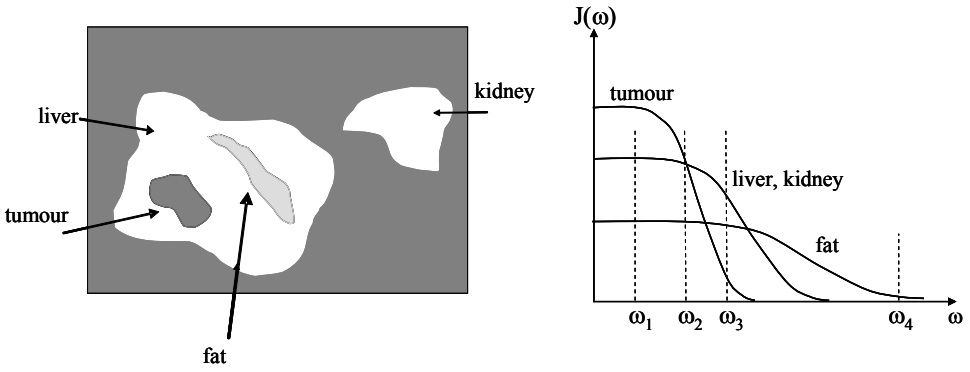
(ii) For relative T_1 values we compare images with different TR values, the middle and right images. Increasing the TR takes the ventricles from being isointense with brain tissue to much higher value, meaning that the T_1 of the ventricles is higher than that of the brain. For relative T_2 values we compare images with different TE values, the left and middle images. Increasing the TE takes the ventricles from being darker than brain tissue to brighter, and therefore the ventricles have a longer T_2 than brain tissue.

5.24 The T_1-weighted image in Figure 5.60 represents a slice through the abdomen, with signals coming from liver, kidney, tumour and lipid. Given the following information:

$$\rho(\text{liver})=\rho(\text{kidney}) < \rho(\text{lipid}) < \rho(\text{tumour})$$

and the spectral density plot shown below:

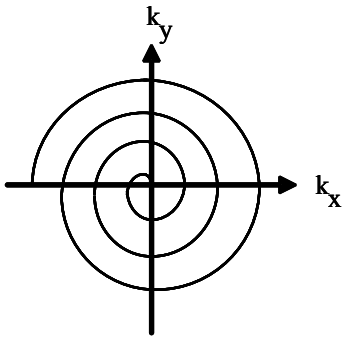
(a) determine at which frequency (ω_1, ω_2, ω_3 or ω_4) the image was acquired, together with your reasons.
(b) At which of these frequencies (ω_1, ω_2, ω_3 or ω_4) would the relative signal intensities of the four tissues be reversed, i.e., the highest becomes lowest and vice-versa?



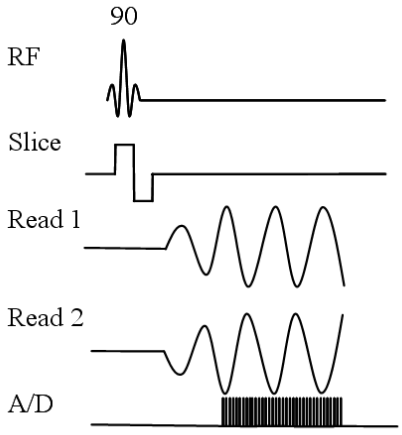
Solution. (a) The relative signal intensities are liver=kidney>fat>tumour. Since the proton densities of liver and kidney are equal, and a T_1-weighted scan was run we can say that the T_1 values of liver and kidney are also equal. Since the proton density of tumour is greater than lipid and liver/kidney, but the signal intensity of the tumour is lower than the other tissues, we can say that the T_1 value of the tumour is the highest, followed by lipid and then liver/kidney. This corresponds to position ω_3 on the spectral density plot.

(b) For the relative signal intensities to be reversed we require the T_1 values of all the tissues to be the same, since then the proton densities will be the only relevant parameter. This corresponds to an image acquired at ω_4.

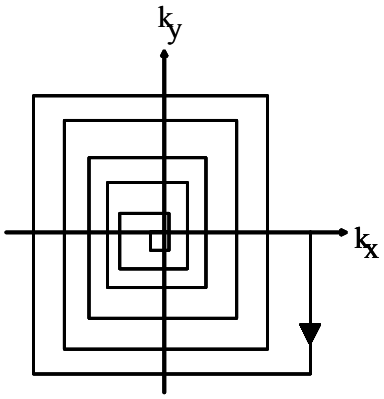
5.25 For a spiral k-space trajectory shown in Figure 5.61 (a), draw the gradient waveforms for the x- and y-gradients.



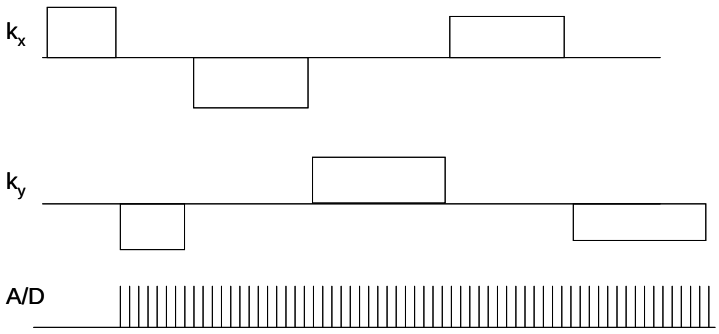
Solution. Assume that the trajectory starts from kx=ky=0 and spirals outwards to the edge of k-space (it can also do the reverse, in which case the gradient waveforms are also reversed in the answer). First ky is positive and kx negative, then both are negative, kx is positive and ky negative etc. The two gradient waveforms are shown below.



5.26 Design an EPI pulse sequence that gives the square spiral k-space trajectory shown in Figure 5.61 (b).



Solution. The trajectory starts at k_x maximum positive and k_y zero. The first part of the gradient sequence is shown below, with the remaining gradient values being equally spaced but gradually decreasing in amplitude.



5.27 For Gd-DTPA the value of α_1 is approximately 5 mM^{-1} s^{-1}. Assuming that a small white matter lesion has a T_1 of 1.2 seconds at 3 Tesla, and that the concentration of Gd-chelate inside the lesion is 2 mM, what is the T_1 of the lesion post-administration of the contrast agent. If a T_1-weighted gradient echo sequence is run with a TR of 100 ms and a tip angle of 10^o, how much is the signal increased post-administration?

Solution. From equation (5.46):

$$\frac{1}{T_1^{CA}} = \frac{1}{1.2} + (5)(2) \Rightarrow T_1^{CA} = 0.09s$$

The steady-state signals pre- and post-administration are given by:

$$S_{pre} \propto \frac{\sin(10)\left(1-e^{-\frac{0.1}{1.2}}\right)}{1-\cos(10)e^{-\frac{0.1}{1.2}}} = 0.15, \quad S_{post} \propto \frac{\sin(10)\left(1-e^{-\frac{0.1}{0.09}}\right)}{1-\cos(10)e^{-\frac{0.1}{0.09}}} = 0.17$$

5.28 Assuming a T_1 value of tissue and blood of 1 second, calculate the $T_{1(eff)}$ for a blood velocity of 5 cm/s and a slice thickness of 5 mm. Using a TR of 50 ms and a tip angle of 60^o, what are the relative signal intensities from blood and stationary tissue? If a contrast agent is added to the blood so that the T_1 of blood is reduced to 200 ms, what are the relative values now?

Solution. From equation (5.45):

$$\frac{1}{T_{1(eff)}} = \frac{1}{1} + \frac{5}{0.5} \Rightarrow T_{1(eff)} = 0.09s$$

The relative signal intensities are given by:

$$S_{tissue} \propto \frac{\sin(60)\left(1 - e^{-\frac{0.05}{1}}\right)}{1 - \cos(60)e^{-\frac{0.05}{1}}} = 0.08, \quad S_{blood} \propto \frac{\sin(60)\left(1 - e^{-\frac{0.05}{0.09}}\right)}{1 - \cos(60)e^{-\frac{0.05}{0.09}}} = 0.52$$

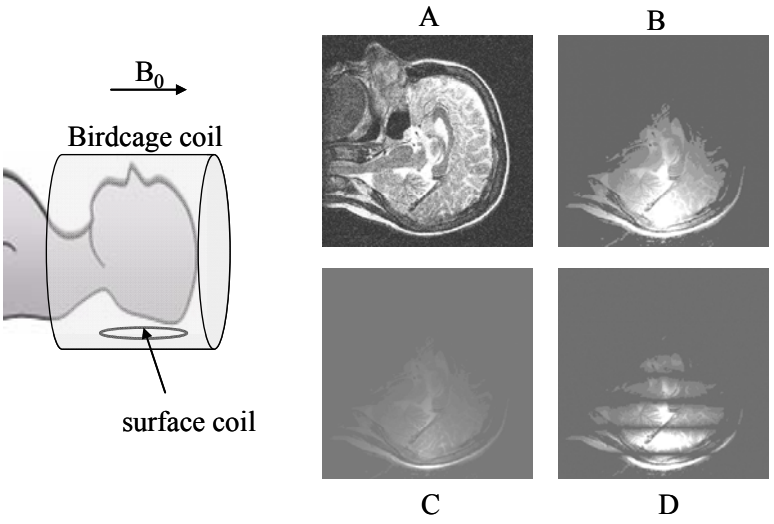
If the blood T_1 is reduced to 200 ms, then the effective T_1 is also reduced:

$$\frac{1}{T_{1(eff)}} = \frac{1}{0.2} + \frac{5}{0.5} \Rightarrow T_{1(eff)} = 0.067s$$

The relative signal intensities are now given by:

$$S_{tissue} \propto \frac{\sin(60)\left(1 - e^{-\frac{0.05}{1}}\right)}{1 - \cos(60)e^{-\frac{0.05}{1}}} = 0.08, \quad S_{blood} \propto \frac{\sin(60)\left(1 - e^{-\frac{0.05}{0.067}}\right)}{1 - \cos(60)e^{-\frac{0.05}{0.067}}} = 0.6$$

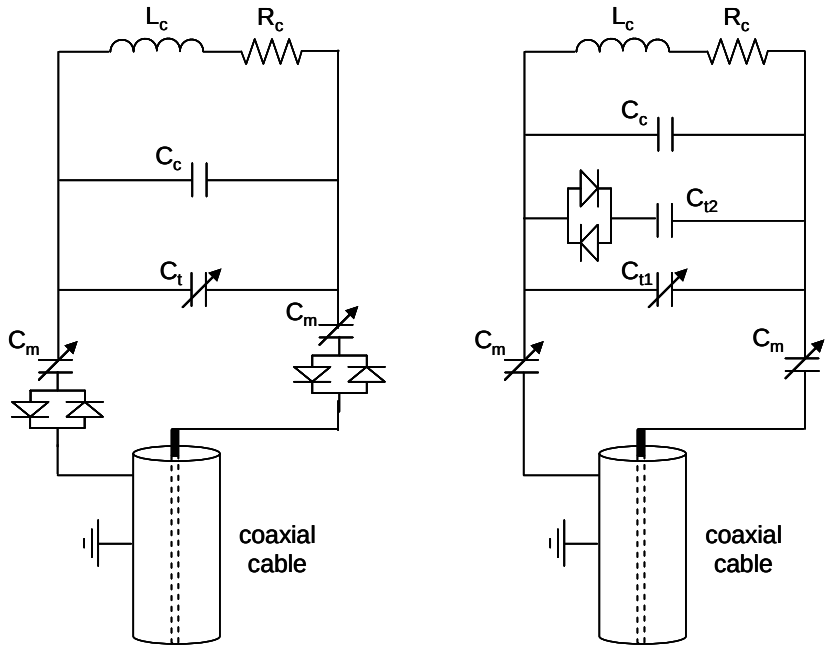
5.29 A brain MRI is being performed on a patient. Two RF coils are available to obtain the images, a birdcage coil and a surface coil as shown in Figure 5.62. Each can be used to transmit the RF pulses, receive the signal, or both transmit and receive. Images A to D correspond to different combinations of the two coils used in transmit, receive or both transmit and receive mode. State, with reasons, which combination of transmit and receive (i.e. Image A, surface coil transmit, birdcage coil receive) are used for the four images A-D.



Solution. Image A has a uniform signal intensity, and therefore the birdcage coil is used for transmission and reception. Figures B and C show drop off in signal as a function of depth, with the effect much greater in B with a higher signal close to the surface. This suggests that the birdcage was used to transmit and the surface coil to receive in B, and the surface coil was used to transmit with the birdcage receiving in C. Image D shows many image artifacts, which can arise if the surface coil is used to transmit and receive, but too much power is used so that the tip angle is a multiple of 90^o at the surface, with the dark lines corresponding to multiples of 180^o which give no signal.

5.30 Draw two electrical circuits which could be used as analogues to Figure 5.37 for the transmit RF coil.

Solution. The circuits should be designed so that the RF coil is tuned during transmit and detuned during signal acquisition. If we maintain the same signals as considered in Figure 5.37 in which the diodes conduct during transmission and do not conduct during reception (in practice the DC values to the transmit coil can be different from those to the receive coil and so this condition does not necessarily hold), then there are many possible circuits, two of which are shown below.



5.31 If the lipid T_1 value is 360 ms and the value for muscle is 1420 ms, calculate the inversion time necessary to null the signal from lipid. What percentage of the signal from tissue is observed using this inversion time?

Solution. After a 180° pulse, the inversion null time for lipid is given by $\ln 2 * T_1 = 250$ ms. At this time the percentage signal from muscle is given by:

$$100\left(1-2e^{-\frac{250}{1420}}\right)=-68\%$$

So 68% of the muscle signal can be produced using a 90° pulse at this point.

5.32 Explain why inversion recovery sequences for lipid suppression are not a good choice after positive contrast agent administration.

Solution. A high concentration of positive contrast agent can reduce the T_1 value of tissue in which it has accumulated, for example in tumours, so that the value is close to that of lipid. An inversion recovery sequence with an inversion time optimized for lipid (see previous exercise) would also essentially null out the signal from the tissue with contrast agent.

5.33 Explain whether each of the following statements is true or false with one/two sentences of explanation.

(a) Protons precessing at a faster frequency relax back to equilibrium faster and so have a shorter T_1 relaxation time.
(b) A higher strength of the RF field from the RF coil means that the duration of the 90° pulse decreases.

(c) A longer T_1 relaxation time means that the voltage induced in the RF coil lasts longer and so a larger MRI signal is achieved.
(d) One line of k-space data acquired for each value of the phase encoding gradient corresponds to one line of the image.
(e) All other parameters being equal, a longer RF pulse in a spin-echo imaging sequence results in a lower signal-to-noise.
(f) At very high magnetic fields the T_1 values of all tissues approach the same value.

Solution.
(a) False. If the protons precess faster, it means that the magnetic field is higher, and the higher the field the longer the T_1 relaxation time.
(b) True. For a given tip angle, the stronger the field the faster the spins are tipped by the field and so the shorter the RF pulse lasts.
(c) False. The length of time that the voltage lasts depends on the T_2 relaxation time and not the T_1 value.
(d) False. Every point and line in k-space produces an effect on the entire image, since the k-space data are inverse Fourier transformed to produce the image.
(e) True. A longer RF pulse will produce a thinner slice since its bandwidth is narrower. If the gradient strength and TE are kept the same, then the signal-to-noise will indeed be lower.
(f) True. The plot of spectral density vs. frequency shows that the spectral density decreases asymptotically towards zero, and so the T_1 values also approach the same value.