

A Game-Theoretic Approach for Optimal Time-of-Use Electricity Pricing

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Abstract—Demand for electricity varies throughout the day, increasing the average cost of power supply. Time-of-use (TOU) pricing has been proposed as a demand-side management (DSM) method to influence user demands. In this paper, we describe a game-theoretic approach to optimize TOU pricing strategies (GT-TOU). We propose models of costs to utility companies arising from user demand fluctuations, and models of user satisfaction with the difference between the nominal demand and the actual consumption. We design utility functions for the company and the users, and obtain a Nash equilibrium using backward induction. In addition to a single-user-type scenario, we also consider a scenario with multiple types of users, each of whom responds differently to time-dependent prices. Numerical examples show that our method is effective in leveling the user demand by setting optimal TOU prices, potentially decreasing costs for the utility companies, and increasing user benefits. An increase in social welfare measure indicates improved market efficiency through TOU pricing.

Index Terms—Electricity price, game theory, optimization, smart grid, time-of-use.

I. INTRODUCTION

THE fluctuation of electricity demand throughout the day has long been a problem for utility companies. During peak hours, the utility companies face significant pressure to provide users with enough electricity, and may even have to ration the electricity supply of certain areas when the gap between demand and generation is too large. During off-peak hours, only a small number of generators are needed to provide sufficient electricity to meet user demand, and the idle generators result in a waste of generation capacity. For greatest efficiency, the utility companies wish to operate the power system on a base load, for which the system is optimized. The base load is not the highest load that a unit can provide, but operation far from base load is not cost efficient, and may harm the stability of the power system. Therefore, utility companies wish the user demand to remain relatively “constant” during the day, so that they can design and build generation units according to the constant demand.

Time-of-use (TOU) pricing is an efficient method of demand-side management (DSM) [1]–[4] that utility companies can employ to influence user behavior. By setting different prices during the day, the utility company can encourage customers to shift their demand to off-peak hours, resulting in a more level demand curve. In [5], Caves *et al.* provide an econometric analysis of a TOU pricing experiment in Wisconsin, showing that short-term electricity demand is not inelastic, and that peak and off-peak electricity are partial substitutes. In [6], Hartway *et al.* demonstrated experimentally that TOU is profitable to a utility company, and that in general, the customers are satisfied with the TOU price option. California’s Statewide Pricing Pilot showed that residential and small-to-medium commercial and industrial customers are willing to reduce their peak-period energy use as a result of time-varying pricing [7]. In recent years, TOU pricing and real-time pricing have attracted growing attention both in academia and in industry [8]–[14], especially with the emerging development of a smart grid [15], which enables the implementation of time-dependent pricing.

In this paper we propose a game-theoretic optimal TOU pricing strategy for smart grids (GT-TOU). A day is divided into N periods, and the price is optimized for each time period. The goal is to influence user behavior through TOU pricing, so that the load throughout the day is leveled. Because utility companies seek to maximize profits under regulations while users seek minimized costs and assured supply, we consider a game between utility companies and users (single type and multiple types) using a multi-stage game model. In this model, the utility company sets the electricity prices, and the customers respond to the price by adjusting the amount of electricity they use. Utility functions are designed for both the company and the users, in which we take into account the cost of fluctuating demands to the utility company, and the satisfaction costs of users. Our pricing strategy is different from the real-time pricing in [13] and [14]. In a real-time pricing scenario, the prices are often updated hourly, and thus there is a strict requirement for fast user response. As described in [13], users are not well prepared to respond to time-varying prices. Therefore, price prediction is often required to implement real-time pricing [13], and an energy-management controller may be needed to help users manage their power usage [14]. In contrast, in our model, users are informed of the price ahead of time, and the TOU prices remain stable during a relatively long time unless there is significant change in the characteristics of user demands or generation cost. We believe these features would make our model easier to implement in practice.

A novel model for efficient TOU pricing schemes was proposed recently in [11]. The authors divided a day into four time

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periods with three price levels, following the pricing pattern of Ontario, Canada. The objective was to minimize the discounted total operating costs of multiple facilities by setting efficient TOU prices under certain constraints. Our pricing strategy is relatively more flexible, as we can either set hourly prices or divide a day into multiple time blocks with a constant price in each time block. The latter is a special case of the former by adding a new constraint, which we will show in Section III. In addition, we take into consideration user satisfaction and costs due to fluctuations of user demands.

The remainder of this paper is organized as follows: Section II introduces notations and the game model for single user type; Section III solves the equilibrium of the game; Section IV extends the model to a scenario with multiple types of users, and solves for the optimal prices and demands; Section V presents numerical examples with a single type of user and multiple types of users; and the paper is concluded in Section VI.

II. GAME MODEL FOR SINGLE USER TYPE

In this section we formulate the model with a single type of user. We divide a day into N periods, where N depends on the scenario of the application. For hourly-based pricing, $N = 24$. The notations are listed below.

- c_k : marginal cost of electricity
- p_k : unit sales price of electricity
- g_k : electricity generation
- d_k : nominal user demand
- l_k : actual user load in response to the price

The subscript $k = 1, \dots, N$ denotes the corresponding time period. For simplicity, we also use the notations $\mathbf{p} = [p_1, p_2, \dots, p_N]^T$ and $\{p_k\}_{k=1}^N$ in this paper.

We model the profit of the company as

$$P = \sum_{k=1}^N p_k l_k - \sum_{k=1}^N c_k g_k - f(\mathbf{g}) \quad (1)$$

where $f(\mathbf{g})$ corresponds to the cost caused by the variation of user demand during the day. We model this cost using the sum of squared generation deviations from the mean, multiplied by a coefficient μ , i.e.,

$$f(\mathbf{g}) = \mu \sum_{k=1}^N (g_k - \bar{g})^2 \quad (2)$$

where \bar{g} is the average electricity generation during the day. The cost function C of electricity users includes the money they pay for the electricity and their satisfaction with the service, i.e.,

$$C = \sum_{k=1}^N p_k l_k + \sum_{k=1}^N s_k(l_k, d_k) \quad (3)$$

where $s_k(l_k, d_k)$ denotes the user satisfaction function. The user satisfaction function quantitatively models the user satisfaction as a function of the difference between nominal user demand and actual consumption (we will use “load” to refer to the actual consumption in the rest of this paper). If the actual load is smaller than the demand, the function value is positive, meaning the users are not satisfied, which results in a loss of satisfaction that comes at a cost to the end-user. The value of the function

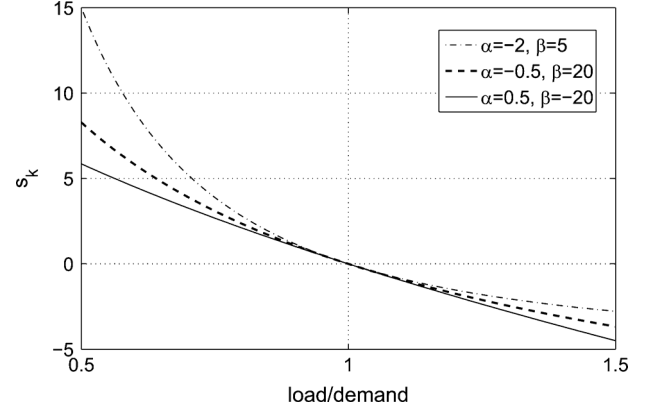


Fig. 1. s_k with different parameters α_k and β_k (with unit d_k).

increases faster as the actual load decreases. If the actual load is greater than the user demand, the function value is negative, meaning the users are satisfied. The decrease of the function value, however, slows down as the actual load continues to increase, because the users will not be “infinitely” more satisfied when they use more electricity. When the actual load equals the user demand, the function value is zero. Therefore the satisfaction function $s_k(l_k, d_k)$ should meet the following conditions:

- 1) If $l_k = d_k$:

$$s_k(l_k, d_k) = 0.$$

- 2) If $l_k > d_k$:

$$s_k(l_k, d_k) < 0, \quad \frac{\partial s_k}{\partial l_k} < 0, \quad \frac{\partial^2 s_k}{\partial l_k^2} > 0.$$

- 3) If $l_k < d_k$:

$$s_k(l_k, d_k) > 0, \quad \frac{\partial s_k}{\partial l_k} < 0, \quad \frac{\partial^2 s_k}{\partial l_k^2} > 0.$$

These conditions are similar to the conditions for utility functions proposed in [12], but are different because the conditions here are used to model the satisfaction with the difference between demand and actual load. In this paper we select $s_k(l_k, d_k)$ as

$$s_k(l_k, d_k) = d_k \beta_k \left[\left(\frac{l_k}{d_k} \right)^{\alpha_k} - 1 \right] \quad (4)$$

where $\alpha_k < 1$ and $\alpha_k \beta_k < 0$. This function satisfies all three conditions listed above. An illustration of s_k with different parameters α_k and β_k is shown in Fig. 1. From the example we can see that by adjusting the parameters α_k and β_k , (4) can be used to characterize different types of users. Other proper satisfaction functions can also be used based on the nature and behavior of users.

The utility company wishes to maximize the profit, as well as fulfill its obligation to serve the public and satisfy the electricity users. Therefore the utility function of the company is its profit minus the satisfaction cost of the users, i.e.,

$$u_1 = \sum_{k=1}^N p_k l_k - \sum_{k=1}^N c_k g_k - \sum_{k=1}^N s_k(l_k, d_k) - f(\mathbf{g}). \quad (5)$$

The utility function of users is the negative of the cost function, i.e.,

$$u_2 = -C = -\sum_{k=1}^N p_k l_k - \sum_{k=1}^N s_k(l_k, d_k). \quad (6)$$

The goal is to maximize the utility functions u_1 and u_2 under certain constraints. The optimization problem is formulated as

$$\begin{aligned} (\mathbf{p}^*, \mathbf{g}^*) &= \arg \max_{\mathbf{p}, \mathbf{g}} u_1 = \sum_{k=1}^N (p_k l_k - c_k g_k - s_k) \\ &\quad - f(\mathbf{g}) \\ \mathbf{l}^* &= \arg \max_{\mathbf{l}} u_2 = -\sum_{k=1}^N (p_k l_k + s_k) \\ \text{subject to} \quad &l_{k,\min} \leq l_k \leq g_k \\ &\leq \min(d_{k,\max}, g_{k,\max}), \\ &c_k \leq p_k, \quad k = 1, 2, \dots, N. \end{aligned}$$

The constraints are used to regulate the activity of the utility company and the users. The first constraint is on the actual user load. Due to the nature of electricity markets, electricity prices have to be regulated [16]. In order to guarantee the minimum load $l_{k,\min}$ required by users, the utility company should not set the price too high. Also, the actual load cannot exceed $\min\{d_{k,\max}, g_{k,\max}\}$, which is the minimum between the maximum possible user load $d_{k,\max}$ at time period k and the maximum generation limit $g_{k,\max}$. This constraint means that even though users want to increase their use of electricity at a specific time period k when the price is low, there is a limit on how much they could increase it. Also, the actual user load always has to be less than the generation capacity of the system. The second constraint guarantees that the sales price is always greater than or equal to the marginal generation cost.

In actual power systems, the total generation should match the user load at all times, which is controlled by the system operator. Therefore we can simplify the problem by letting $l_k = g_k$. Let $l_{k,\max} = \min\{d_{k,\max}, g_{k,\max}\}$, the problem can then be rewritten as

$$\begin{aligned} \mathbf{p}^* &= \arg \max_{\mathbf{p}} u_1 = \sum_{k=1}^N (p_k l_k - c_k l_k - s_k) - f(\mathbf{l}) \quad (7) \\ \mathbf{l}^* &= \arg \max_{\mathbf{l}} u_2 = -\sum_{k=1}^N (p_k l_k + s_k) \\ \text{subject to} \quad &l_{k,\min} \leq l_k \leq l_{k,\max}, \quad k = 1, 2, \dots, N, \\ &c_k \leq p_k, \quad k = 1, 2, \dots, N. \end{aligned} \quad (8)$$

In this game model, the utility company decides the TOU price \mathbf{p} , and the electricity users decide the actual consumption of electricity \mathbf{l} according to the price. Their utility functions depend on both \mathbf{p} and \mathbf{l} . Let \mathcal{P} denote the strategy set of the utility company, which is all the possible TOU prices the company can set. Let \mathcal{L} denote the strategy set of the users, which is all the possible load responses from which the users can choose. The strategy sets can be defined as follows:

$$\begin{aligned} \mathcal{P} &= \{\mathbf{p} \mid \mathbf{p} \in \mathbb{R}^N, l_{\min} \leq \mathbf{l}(\mathbf{p}) \leq \mathbf{l}_{\max}, \mathbf{p} \geq \mathbf{c}\}, \\ \mathcal{L} &= \{\mathbf{l} \mid \mathbf{l} \in \mathbb{R}^N, l_{\min} \leq \mathbf{l} \leq \mathbf{l}_{\max}\}. \end{aligned}$$

Note that in the definition of \mathcal{P} we write \mathbf{l} as a function of \mathbf{p} , because the actual user load is dependent on the prices. We aim to find the optimal price $\mathbf{p}^* \in \mathcal{P}$ and optimal load response $\mathbf{l}^* \in \mathcal{L}$ such that a Nash equilibrium $(\mathbf{p}^*, \mathbf{l}^*) \in \mathcal{P} \times \mathcal{L}$ is achieved between the utility company and electricity users. A strategy profile is called a Nash equilibrium [17] if any unilateral change of strategy by a single agent does not increase its utility function. In this problem, a Nash equilibrium is achieved when the following conditions are satisfied:

$$\forall \mathbf{p} \in \mathcal{P}, \mathbf{p} \neq \mathbf{p}^* : u_1(\mathbf{p}^*, \mathbf{l}^*) \geq u_1(\mathbf{p}, \mathbf{l}^*), \quad (9)$$

$$\forall \mathbf{l} \in \mathcal{L}, \mathbf{l} \neq \mathbf{l}^* : u_2(\mathbf{p}^*, \mathbf{l}^*) \geq u_2(\mathbf{p}^*, \mathbf{l}). \quad (10)$$

III. OPTIMIZING UTILITY FUNCTIONS

Since this is a multi-stage game, we use backward induction [17] to solve for the equilibrium. The utility company takes action first by setting the electricity price, and then customers adjust the amount of electricity they use. Therefore, according to the backward induction principle, we first maximize u_2 with respect to $\{l_k\}_{k=1}^N$, and then plug the optimal load response $\mathbf{l}^*(\mathbf{p})$ into u_1 and optimize u_1 with respect to $\{p_k\}_{k=1}^N$.

A. Optimal Demand Response to Price

In order to find a user's optimal demand response to the price set by the utility company, we consider the electricity prices of different time periods $\{p_k\}_{k=1}^N$ as given, and take the first-order derivatives of u_2 with respect to $\{l_k\}_{k=1}^N$:

$$\frac{\partial u_2}{\partial l_k} = -p_k - \alpha_k \beta_k \left(\frac{l_k}{d_k} \right)^{\alpha_k - 1}. \quad (11)$$

Setting (11) equal to zero, we determine that

$$l_k^* = \left(-\frac{p_k}{\alpha_k \beta_k} \right)^{1/(\alpha_k - 1)} d_k. \quad (12)$$

The second-order derivative of u_2 is

$$\frac{\partial^2 u_2}{\partial l_k \partial l_i} = \begin{cases} -\alpha_k \beta_k (\alpha_k - 1) \frac{l_k^{\alpha_k - 2}}{d_k^{\alpha_k - 1}} & \text{when } k = i \\ 0 & \text{when } k \neq i. \end{cases} \quad (13)$$

Since $\alpha_k < 1$ and $\alpha_k \beta_k < 0$, the diagonal elements of the Hessian matrix are all negative, and the off-diagonal elements are all zero. The Hessian matrix is negative definite, meaning that $\{l_k^*\}_{k=1}^N$ is the optimal user load given price \mathbf{p} . Let

$$\epsilon_k = \frac{1}{\alpha_k - 1} < 0, \quad k = 1, 2, \dots, N \quad (14)$$

and

$$\eta_k = -\alpha_k \beta_k > 0, \quad k = 1, 2, \dots, N \quad (15)$$

we can then rewrite (12) as

$$l_k^* = \left(\frac{p_k}{\eta_k} \right)^{\epsilon_k} d_k, \quad k = 1, 2, \dots, N. \quad (16)$$

For simplicity of notation, we will use (16) instead of (12) in the rest of this paper. We need to clarify that the optimal response l_k^* is not always in the form of (16), if a different satisfaction function is chosen based on user characteristics.

B. Optimal Pricing Based on User Response

In Section III-A we obtained the optimal response of users to electricity prices. In this section, we will maximize the utility function of companies by finding the optimal pricing strategy based on the user response. Plugging (16) into (5), we obtain u_1 as a function of \mathbf{p} as follows:

$$u_1(\mathbf{p}) = \sum_{k=1}^N \{p_k l_k^*(p_k) - c_k l_k^*(p_k) - s_k [l_k^*(p_k), d_k]\} - f[l^*(\mathbf{p})]. \quad (17)$$

Given the optimal user load as a function of the electricity price, we can rewrite the constraints on user loads as constraints on the prices. From (16) we obtain

$$p_k = \left(\frac{l_k}{d_k} \right)^{1/\epsilon_k} \eta_k. \quad (18)$$

Since (18) is a decreasing function of l_k , the constraints on prices can be written as

$$p_{k,\min} \leq p_k \leq p_{k,\max} \quad (19)$$

where $p_{k,\min} = \max \left\{ c_k, (l_{k,\max}/d_k)^{1/\epsilon_k} \eta_k \right\}$ and $p_{k,\max} = (l_{k,\min}/d_k)^{1/\epsilon_k} \eta_k$. The optimization of u_1 with respect to the prices \mathbf{p} now becomes

$$\begin{aligned} & \max_{\mathbf{p}} \quad u_1(\mathbf{p}) \\ & \text{subject to} \quad \mathbf{p}_{\min} \leq \mathbf{p} \leq \mathbf{p}_{\max}. \end{aligned}$$

The constraints of this optimization problem are linear. To ensure that the solution is the optimum, we need to check the negative-definiteness of the Hessian matrix. In this problem, the negative-definiteness of the Hessian matrix of u_1 is parameter dependent. In the Appendix we derive the conditions under which the Hessian matrix is negative definite. In a traditional TOU pricing strategy, a day is divided into multiple blocks of hours, and each block is considered as “peak”, “semi-peak”, or “off-peak” hours. The price is constant in each time block. We can add an additional constraint to (20) to fit our proposed pricing strategy with the traditional TOU pricing pattern. An intuitive interpretation of the constraint is to enforce constant price within each block by adding linear constraints of the form $p_i = p_j$, when i and j are time periods in the same block. The constraint can be written as

$$\mathbf{A} \cdot \mathbf{p} = \mathbf{0} \quad (20)$$

where \mathbf{A} is an $n \times n$ matrix with $A_{i,j} \in \{-1, 0, 1\}$ and $\mathbf{A} \cdot \mathbf{1} = \mathbf{0}$. Here $\mathbf{1}$ denotes an all-one vector of dimension $N \times 1$, and $\mathbf{0}$ denotes an all-zero vector of dimension $N \times 1$. Assume we choose $N = 24$, and set the beginning of each hour as the beginning of that time period. Then for a flat pricing strategy with a fixed price throughout the day, matrix \mathbf{A} is set to be the matrix shown in Fig. 2(a). For the example time-block division in Table I, the corresponding matrix \mathbf{A} is shown in Fig. 2(b). In this case a day is divided into four time blocks with three different price levels.

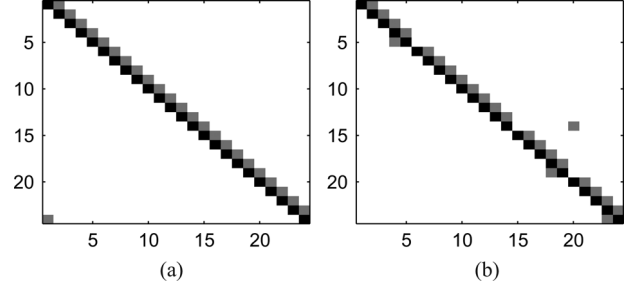


Fig. 2. Illustration of the matrix \mathbf{A} . Black denotes 1, light grey denotes -1 , and white denotes 0. (a) Matrix \mathbf{A} for flat pricing. (b) Matrix \mathbf{A} for TOU block pricing.

TABLE I
EXAMPLE OF TIME BLOCK DIVISION FOR TOU PRICING

Period	Hours
Peak	2:00 p.m. - 7:00 p.m.
Semi-peak	5:00 a.m. - 2:00 p.m., 7:00 p.m.-12:00 a.m.
Off-peak	12:00 a.m. - 5:00 a.m.

IV. MODEL WITH MULTIPLE USER TYPES

Since not all users respond to a price change in the same way, and their demands during the day are significantly different, we extend the model in Section II to a scenario with multiple types of users. We consider three types of users: residential users (R), commercial users (B), and small industrial users (F). These users have different price response characteristics.

• Residential users

Users in residential areas are generally sensitive to price change, and they would like to adjust their consumption of electricity according to the time-varying prices. The flexibility of residential users is relatively low, as they have limited ability to reduce or increase their *total* use of electricity. But they would be willing to reschedule their use of electricity to reduce their electricity bills.

• Commercial users

During office hours, the demands for electricity in business districts are high, and commercial users do not want to reduce the use of electricity which may affect their business. Energy conservation methods, however, can be used to save electricity if the electricity price is high. Part of the less time-urgent work can also be scheduled to other times of day. After office hours, the demand for electricity is lower, but there is more flexibility as the high-energy-consumption jobs left over from the day can be rescheduled to these hours.

• Industrial users

Industrial users, especially those with high energy consumption facilities, use a lot of electricity. Therefore they are more sensitive to electricity prices. They can reschedule their production times to minimize the cost of electricity. For example, procedures with high energy requirements can be shifted to the hours when electricity prices are lower. These users have relatively higher flexibility in adjusting and rescheduling their consumption of electricity.

Considering the different responses of multiple types of users, we can set different prices for each type throughout the day. The

goal is to level the total load of all users instead of just one type of user. The company and each type of user would have a utility function reflecting its overall profit/cost. The utility functions are listed below, where u_1 is for the utility company, u_{2R} , u_{2B} , and u_{2F} are for residential, commercial, and industrial users, respectively:

$$u_1 = \sum_{k=1}^N (p_{Rk}l_{Rk} + p_{Bk}l_{Bk} + p_{Fk}l_{Fk}) - \sum_{k=1}^N c_k l_k - \sum_{k=1}^N (s_{Rk} + s_{Bk} + s_{Fk}) - f(\mathbf{l}), \quad (21)$$

$$u_{2R} = - \sum_{k=1}^N p_{Rk}l_{Rk} - \sum_{k=1}^N s_{Rk}, \quad (22)$$

$$u_{2B} = - \sum_{k=1}^N p_{Bk}l_{Bk} - \sum_{k=1}^N s_{Bk}, \quad (23)$$

$$u_{2F} = - \sum_{k=1}^N p_{Fk}l_{Fk} - \sum_{k=1}^N s_{Fk}. \quad (24)$$

Here l_k is the sum of the loads of all types of users, i.e.,

$$l_k = l_{Rk} + l_{Bk} + l_{Fk}, \quad k = 1, 2, \dots, N \quad (25)$$

and $f(\mathbf{l})$ is the cost due to fluctuation of total user loads. s_{Rk} , s_{Bk} , and s_{Fk} are satisfaction functions for residential, commercial and industrial users, respectively. These functions have different parameters based on the characteristics of the users, and may take different forms other than (4) if necessary. In this paper, we employ the same satisfaction function as in the single-user-type case with different parameters, and the users' optimal responses to prices are in a form similar to (16), with different parameters ϵ and η . The optimal responses of different types of users can be obtained as follows:

$$l_{Rk}^* = \left(\frac{p_{Rk}^*}{\eta_{Rk}} \right)^{\epsilon_{Rk}} d_{Rk}, \quad (26)$$

$$l_{Bk}^* = \left(\frac{p_{Bk}^*}{\eta_{Bk}} \right)^{\epsilon_{Bk}} d_{Bk}, \quad (27)$$

$$l_{Fk}^* = \left(\frac{p_{Fk}^*}{\eta_{Fk}} \right)^{\epsilon_{Fk}} d_{Fk}. \quad (28)$$

We then find the optimal prices by solving the optimization problem similar to (20) as follows:

$$\begin{aligned} & \min_{\mathbf{p}_R, \mathbf{p}_B, \mathbf{p}_F} u_1(\mathbf{p}_R, \mathbf{p}_B, \mathbf{p}_F) \\ & \text{subject to} \quad \mathbf{p}_{R,\min} \leq \mathbf{p}_R \leq \mathbf{p}_{R,\max}, \\ & \quad \mathbf{p}_{B,\min} \leq \mathbf{p}_B \leq \mathbf{p}_{B,\max}, \\ & \quad \mathbf{p}_{F,\min} \leq \mathbf{p}_F \leq \mathbf{p}_{F,\max}. \end{aligned} \quad (29)$$

V. NUMERICAL EXAMPLES

A. Example With a Single Type of User

In this numerical example, we adopt hourly-based pricing by dividing a day into $N = 24$ equal time periods, and also

time-block based pricing by dividing a day into four time blocks (as shown in Table I) with three price levels. Given the nominal user demands $\{d_k\}_{k=1}^N$ and marginal electricity generation cost $\{c_k\}_{k=1}^N$ (both shown in Fig. 3), and parameters of the satisfaction function, we used the game-theoretic model to find the optimal pricing strategy and user load. The data on user demand and marginal electricity cost are obtained from [18]. We observe from (12) that parameter ϵ_k has a similar property to the price elasticity [19], and parameter η_k can be considered as the nominal price corresponding to the nominal demand d_k . This provides us an intuitive way to select parameters for this illustrative example.¹ In real applications, these parameters should be carefully estimated based on historical data and surveys of users. There have been several studies such as [5], [20], and [21] that estimate electricity price elasticities. Ameren Illinois recently released a report on its hourly electricity pricing experiment with 11 000 residential customers over four years [22]. They found out that short run elasticity ranged from -0.21 in the hour from 3 p.m. to 4 p.m. to -0.89 in the hours between noon to 2 p.m. We select ϵ_k based on this, and the detailed values of ϵ_k can be found in the first sub-figure of Fig. 6. η_k is selected to be 10, and μ is selected to be 1. The total user load is defined by

$$l_{\text{total}} = \sum_{k=1}^N l_k \quad (30)$$

and the average unit price is defined by

$$\bar{p} = \frac{\sum_{k=1}^N p_k l_k}{\sum_{k=1}^N l_k}. \quad (31)$$

We use the function `fmincon` from Matlab® Optimization Toolbox to solve the optimization problem (20).

Fig. 3 shows a comparison of prices and loads using flat and GT-TOU pricing. The flat pricing is obtained by constraining prices to be equal at different time periods (as mentioned in Section III). When using GT-TOU pricing, the price is set higher in peak hours and lower in off-peak hours. Therefore, in peak hours, users try to reduce their consumption of electricity in order to reduce cost. This can be done by either rescheduling the use of some electric appliances to off-peak hours, or reducing the load of some appliances, e.g., by adjusting the temperature settings of air conditioners. In off-peak hours, users will use more electricity in order to finish the tasks left over from peak-hours. Also, since the price is much lower during the night, users may increase the use of electricity by replacing home appliances which used other sources of energy with electrically powered substitutes. With the increasing penetration of plug-in hybrid electrical vehicles (PHEV) [23], residential users will have more flexibility to adjust their use of electricity during off-peak hours. In Fig. 3 we also show the TOU prices and load

¹The relationship between change of price and change of demand can be described as $(\partial d/d)/(\partial p/p) = \epsilon$, where ϵ is the price elasticity. Assume ϵ to be constant. After integration on both sides and simple calculation, we obtain that $\bar{d} = (\bar{p}/p_0)^\epsilon d_0$, where d_0 and p_0 are the original demand and price, respectively, and \bar{d} is the new demand corresponding to the new price \bar{p} .

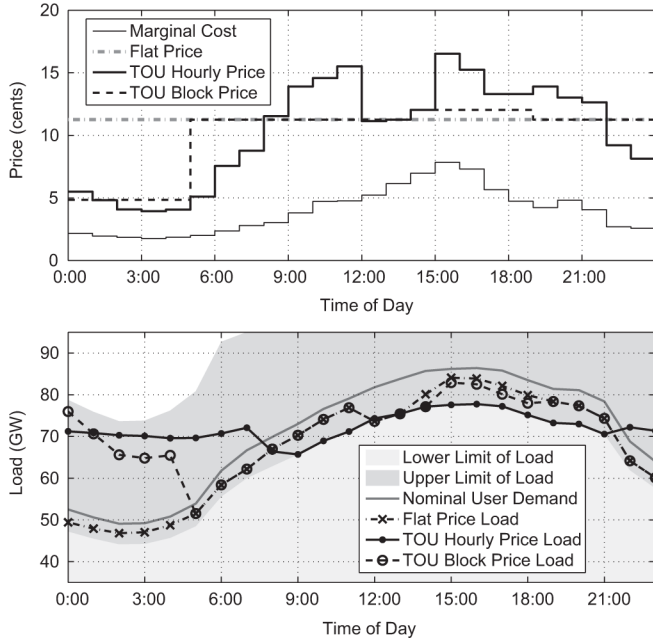


Fig. 3. Comparison of prices and loads between flat pricing and GT-TOU pricing.

response when applying the time-block TOU pricing. We observe that the fluctuation of demands is higher than when hourly TOU pricing is applied, especially at 5:00AM when there is a sudden increase in price.

Table II shows a comparison of results from flat pricing and GT-TOU pricing. When the prices are set to be the same throughout the day, users will reduce their use of electricity because the price is relatively high. When GT-TOU pricing is applied, users are encouraged to reschedule their electricity consumption and are incentivized to use more electricity during off-peak hours. Therefore their total use of electricity is similar to their nominal demand. We also conclude from the table that the profit of utility company increases, the average unit price of electricity decreases, and user utility increases under GT-TOU pricing. The results of block TOU prices are also shown in Table II. When applying this pricing strategy, users actually benefit more than when hourly TOU pricing is applied. In this case the prices within each time block are fixed, and the utility company has to satisfy the minimum user demand. Therefore the price cannot be set too high, meaning that the strategy set of utility companies is further restricted. The decrease in company profit results in an increase in user utility. We also calculate the social welfare measure (S.W.), which is the sum of the company profit and the user utility. Note that the social welfare measure takes negative value, because we defined user utility as negative of their monetary and satisfaction cost. Higher social welfare is preferable as it indicates higher market efficiency. We observe that TOU hourly pricing achieves the highest social welfare, followed by the relatively more constrained TOU block pricing. Flat pricing achieves the lowest social welfare among these three strategies.

Fig. 4 shows how the gross profit of the utility company and user utility (u_2) change with the parameter μ . As μ increases,

TABLE II
COMPARISON OF THREE PRICING STRATEGIES IN SINGLE-USER-TYPE SCENARIO. TOTAL NOMINAL DEMAND IS 1700.10 GWh

Pricing Strategy	Total load (GWh)	\bar{p} (¢/kWh)	Profit (M \$)	u_2 (M \$)	S.W. (M \$)
Flat	1613.48	11.26	71.68	-190.82	-119.14
Hourly	1731.75	10.45	108.42	-183.48	-75.06
Block	1706.90	10.15	86.64	-175.93	-89.29

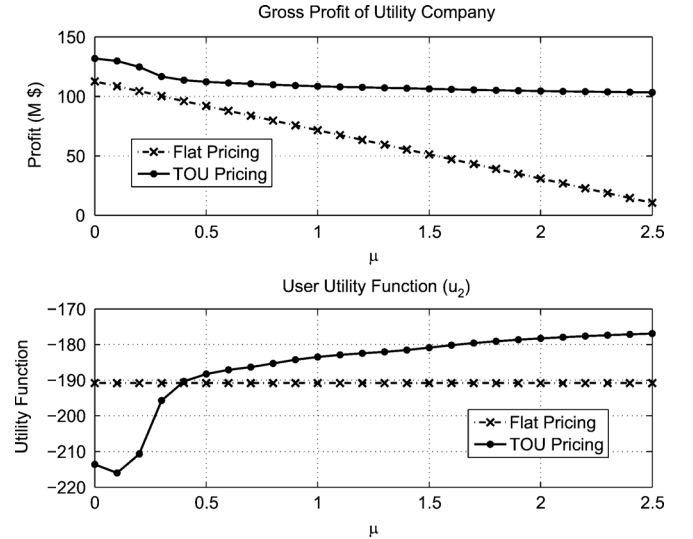


Fig. 4. Comparison of company profit and user benefit in the scenarios of flat pricing and GT-TOU pricing as μ changes.

the profit of the company decreases due to increased cost resulting from fluctuation of user loads. When applying GT-TOU, the users are encouraged to reschedule their use of electricity, and thus the loads are relatively leveled compared to the scenario when flat pricing is applied. This results in less cost due to load fluctuations, and the decrease of profit is not significant. For users, when μ is greater than a threshold, the user utility function is higher than that under flat pricing: when μ is low, the cost due to fluctuating user demands is low, and users do not benefit much from time-varying prices. When μ is high, both users and the utility company benefit from TOU pricing because it significantly reduces the cost due to load fluctuations.

Fig. 5 shows how the daily total user load and average unit price of electricity change with the parameter μ . We observe that when GT-TOU pricing is applied, users have the flexibility to reschedule their use of electricity, and are encouraged to use more electricity during off-peak hours. Therefore the total daily load is slightly higher than that when TOU pricing is applied (the increase is not significant compared to the nominal user demand). Also, users will benefit from the reduced average unit cost of electricity.

B. Example With Multiple Types of Users

For a more complex example, we consider a scenario with three types of users: residential users, commercial users, and industrial users. These three types of users respond differently to price changes, as described in Section IV. In general, the commercial user demand is more inelastic, whereas the industrial user demand is more elastic. The parameter ϵ 's are set to the values shown in Fig. 6. For residential users, we choose

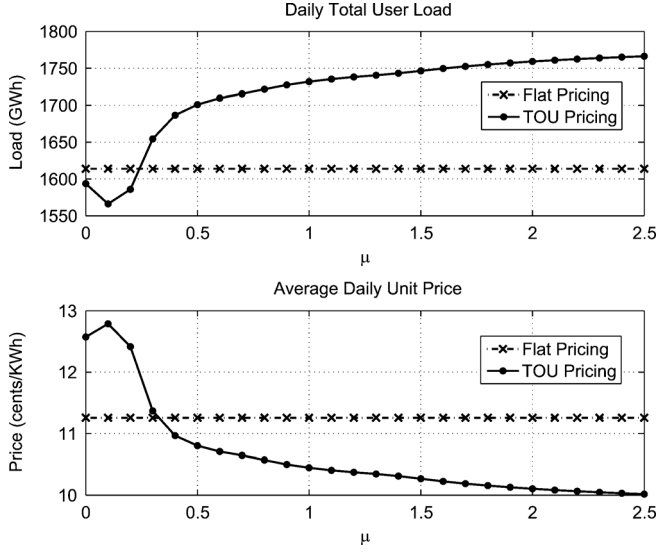


Fig. 5. Comparison of total load and average unit price in the scenarios of flat pricing and GT-TOU pricing as μ changes.

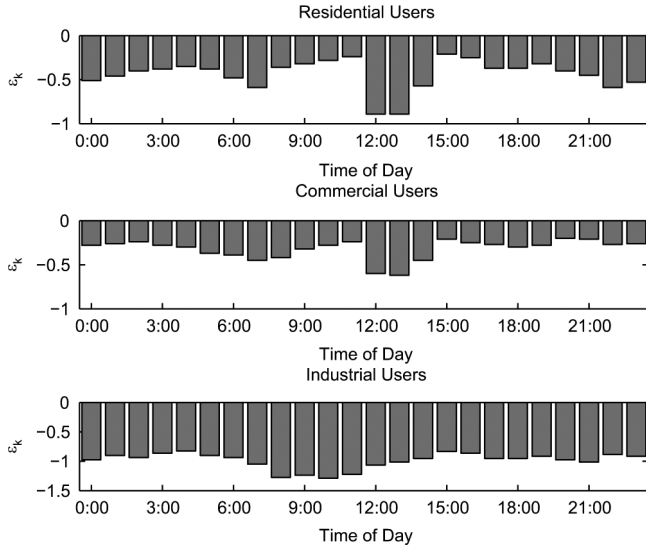


Fig. 6. Parameter ϵ_k for different types of users used in the numerical examples.

$-0.89 \leq \epsilon_R \leq -0.21$ with mean -0.44 ; for commercial users, we choose $-0.62 \leq \epsilon_B \leq -0.20$ with mean -0.32 ; and for industrial users, we choose $-1.29 \leq \epsilon_F \leq -0.82$ with mean -0.98 . Similar to the single-user-type case, we select $\eta_k = 10$ and $\mu = 1$ in this example. We set the minimum load and maximum load of each type of customer as shown in Table III. The numbers are percentages with respect to the nominal demands. In this illustrative example, we set them to the same value throughout the day, for simplicity. In practical use, the parameters should be obtained by analyzing users' load profiles. The system capacity is set to be 110% of the peak hour demand. The flat prices are simulated by solving (29), with additional constraints such that for each type of user, the price is fixed throughout the day.

Fig. 7 shows the comparison of GT-TOU prices and constant prices, and Fig. 8 shows the significantly different load

TABLE III
MINIMUM AND MAXIMUM LOAD USED IN NUMERICAL
EXAMPLE WITH MULTIPLE TYPES OF USERS

User Type	Residential	Commercial	Industrial
l_{\min}	90%	95%	70%
l_{\max}	125%	120%	150%

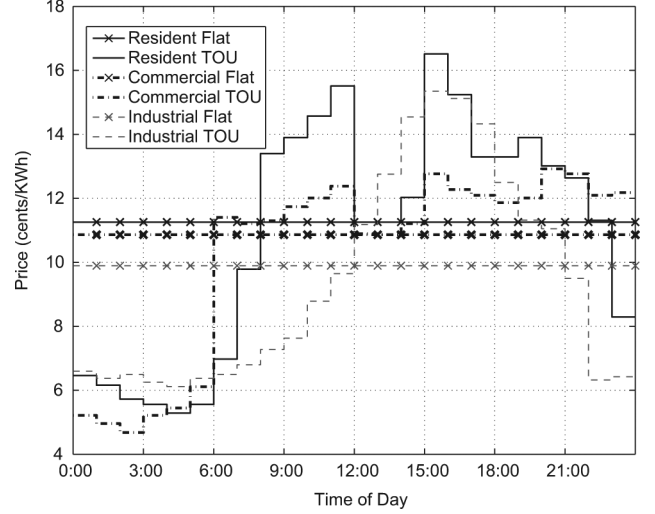


Fig. 7. Comparison of GT-TOU prices with flat prices for multiple user types.

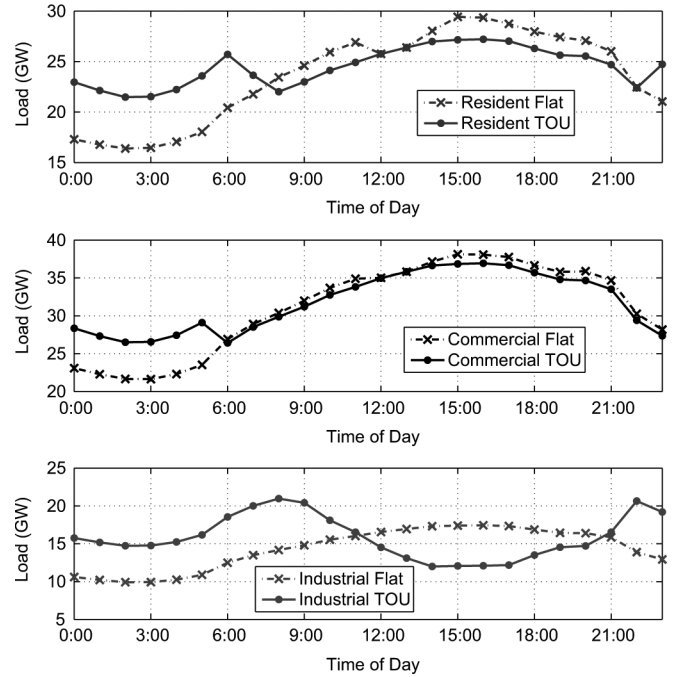


Fig. 8. Loads of different types of users before and after applying GT-TOU pricing.

response of the three types of users. To reduce their electricity bills, residential users use more electricity in off-peak hours and less electricity in peak hours. Business users do not adjust their consumption much during office hours, as their need for electricity is relatively "inelastic". The factory users are the most flexible. They use much more electricity in off-peak hours, and significantly reduce electricity consumption during peak hours. This can be done by rescheduling their production, and shifting

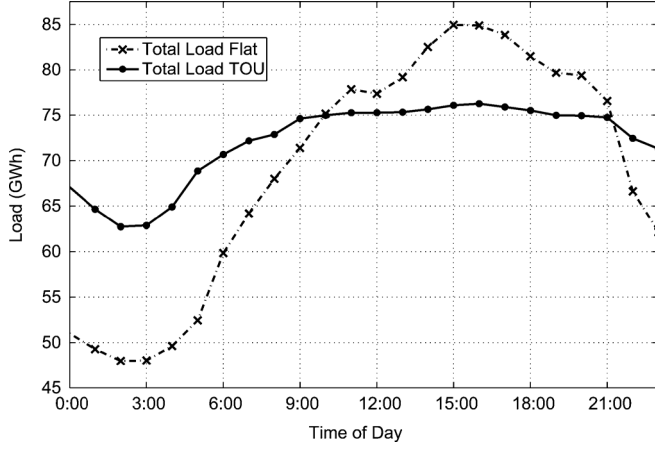


Fig. 9. Comparison of total load before and after GT-TOU pricing.

TABLE IV
COMPARISON OF COMPANY PROFIT AND USER UTILITY
LEVEL IN A MULTIPLE-USER-TYPE SCENARIO

Pricing Strategy	Gross Profit (M \$)	u_{2R} (M \$)	u_{2B} (M \$)	u_{2F} (M \$)
Flat	65.45	-63.71	-80.96	-33.97
TOU	103.62	-64.58	-79.65	-34.12

TABLE V
COMPARISON OF USER LOAD AND PRICE IN A MULTIPLE-USER-TYPE
SCENARIO. TOTAL NOMINAL DEMAND IS 1700.10 GWh

	Strategy	R	B	F	Overall
Daily Load (GWh)	Flat	564.83	744.73	343.70	1653.26
	TOU	587.21	761.23	381.48	1729.92
Unit Price (£/kWh)	Flat	11.26	10.86	9.89	10.80
	TOU	11.04	10.46	8.97	10.33

high-energy-consumption procedures to hours when the electricity price is low. In this way they can substantially reduce the average unit electricity price. As illustrated in Fig. 9, after applying the GT-TOU pricing strategy, the total electricity consumption of these three types of users is much more leveled. The peak total load is reduced by about 10.24%.

We also compare company profit and user utility levels in Table IV, and loads and unit electricity prices in Table V, respectively. Table IV shows that the utility company significantly increases their profit when GT-TOU pricing is applied, whereas all three types of users maintain similar utility levels. Although user utility does not increase significantly, the users still benefit from this pricing strategy, because they actually use more electricity for a similar amount of money. From Table V we observe a slight increase in user consumption of electricity, and a decrease in average unit electricity prices.

VI. CONCLUSIONS

We proposed an optimal game-theoretic TOU electricity pricing strategy (GT-TOU). We designed utility functions for both utility companies and users, and solved for a Nash equilibrium, which provides optimal prices and user responses. In practice, the parameters of the model can be estimated using historical data from utility companies and surveys of electricity

users. The utility functions can be modified according to the nature of utility companies and users. We can also incorporate different types of users into this model, and optimize the prices for each type to achieve a leveled total user load. The pricing strategy is flexible, as the model is suitable for multiple pricing patterns, including hourly pricing and time-block TOU pricing. Simulation results illustrate that our strategy can level user demand, increase the profits of the utility companies, reduce unit prices for electricity users, and ensure overall user benefit. The leveled user load also potentially helps ensure a more stable power system.

The basic framework proposed in this paper can be extended in multiple ways. We can employ a more realistic model for the costs to utility companies, and a more accurate satisfaction function to characterize the benefit of users. In a more accurate satisfaction model, the user satisfaction will not only depend on the use of electricity at the current time period, but also depend on the use at other time periods. In addition, we plan to consider the scenario with integration of renewable distributed generators [24], in which utility company is not the only electricity source. Users can partially depend on electricity generated by their on-site generators, and can also sell the extra generation into the grid, thus forming a new market paradigm. We can extend the proposed framework to this scenario, and find the optimal strategies for the company and users in this new market model.

APPENDIX

Let $\hat{u}_1 = -u_1$. We need to check the positive-definiteness of the Hessian matrix of \hat{u}_1 .

The second order derivatives of \hat{u}_1 with respect to \mathbf{p} are

$$\frac{\partial^2 \hat{u}_1}{\partial p_k^2} = -3 \frac{\partial l_k}{\partial p_k} + [2\mu(l_k - \bar{l}) + c_k - 2p_k] \frac{\partial^2 l_k}{\partial p_k^2} + 2\mu \left(\frac{\partial l_k}{\partial p_k} \right)^2 - \frac{2\mu}{N} \left(\frac{\partial l_k}{\partial p_k} \right)^2 \quad (32)$$

and

$$\frac{\partial^2 \hat{u}_1}{\partial p_k \partial p_i} = -\frac{2\mu}{N} \frac{\partial l_k}{\partial p_k} \frac{\partial l_i}{\partial p_i}, \quad k \neq i. \quad (33)$$

We observe that the Hessian matrix \mathbf{H} of \hat{u}_1 can be decomposed into a diagonal matrix and a rank 1 matrix

$$\mathbf{H} = \mathbf{D} - \mathbf{u}\mathbf{u}^T. \quad (34)$$

The diagonal elements of \mathbf{D} are

$$D_{kk} = -3 \frac{\partial l_k}{\partial p_k} + 2\mu \left(\frac{\partial l_k}{\partial p_k} \right)^2 + [2\mu(l_k - \bar{l}) + c_k - 2p_k] \frac{\partial^2 l_k}{\partial p_k^2} \quad (35)$$

and

$$\mathbf{u} = \sqrt{\frac{2\mu}{N}} \left(\frac{\partial l_1}{\partial p_1}, \frac{\partial l_2}{\partial p_2}, \dots, \frac{\partial l_N}{\partial p_N} \right)^T. \quad (36)$$

Construct matrix

$$\mathbf{M} = \begin{bmatrix} \mathbf{D} & \mathbf{u} \\ \mathbf{u}^T & 1 \end{bmatrix}. \quad (37)$$

Since $1 > 0$, $\mathbf{M} \succ 0$ iff the Schur-complement $\mathbf{D} - \mathbf{u}\mathbf{u}^T \succ 0$ [25]. Also, we know that $\mathbf{M} \succ 0$ iff $\mathbf{D} \succ 0$ and $1 - \mathbf{u}^T \mathbf{D}^{-1} \mathbf{u} > 0$. Therefore $\mathbf{H} = \mathbf{D} - \mathbf{u}\mathbf{u}^T$ is positive definite if the following conditions are satisfied:

$$D_{kk} > 0, \text{ for } k = 1, 2, \dots, N, \quad (38)$$

$$1 - \mathbf{u}^T \mathbf{D}^{-1} \mathbf{u} = 1 - \sum_{k=1}^N D_{kk}^{-1} u_k^2 > 0. \quad (39)$$

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