# C2922 Economics
# Utility Functions

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## 1 Introduction

"Utility" refers to the perceived value of a good and utility theory spans mathematics, economics and psychology. For example, if you prefer vanilla ice-cream to chocolate, you would assign greater utility to vanilla ice-cream than to the same quantity of chocolate ice-cream. The fact that different agents have different utilities for goods is the basis of all markets.

In the context of actuarial science, the focus is on the utility of money, the study of the utility of money started in the early 1700's with the St Petersburg Paradox.

#### 1.0.1 The St Petersburg Paradox

> A player pays some fee to a casino in order to play a game where a fair coin will be tossed repeatedly until a "tail" first appears, ending the game. The payoff of the game is $2^{N-1}$ where $N$ is the number of times until the first tail appears.
>
> How much should the casino charge to play this game, how much should a player expect to play the game?
>
> This game was played in 18th century St Petersburg and numerous mathematicians, including Bernoulli were interested in it, since the *expected* payoff of the game is infinite
>
> $$\mathbb{E}[X] = \sum_{k=1}^{\infty} 2^{k-1} \frac{1}{2^k} = \sum_{k=1}^{\infty} \frac{1}{2}$$

In 1728, a Swiss mathematician, Gabriel Cramer, observed that

> "the mathematicians estimate money in proportion to its quantity, and men of good sense in proportion to the usage that they may make of it".

Modern psychology explains utility in terms of how we perceive wealth, for example

- A beggar will value a £1 much more than a millionaire. A pound may double the wealth of the beggar where as a millionaire would never notice the loss of a pound.
- The expectation of an uncertain outcome is worth less than a certain outcome.
- Future earnings are perceived as being worth less than current earnings, you do not know what your future situation will be (i.e. dead).
- Money in your pocket is worth more than money invested. Money in your pocket has immediate utility, it can be spent. Money invested cannot be spent immediately.

The third of these examples are usually handled by discounting (time value of money), if the investment in the forth example is risk free, it would be handled simply by discounting as well. In actuarial mathematics we are particularly interested in uncertain outcomes, which treats utility of wealth (the perceived value of wealth) as a random variable.

# 2 Utility functions

Mathematically we need a function to map between the physical measure of money and the perceived value of money. Such functions are called utility functions, and in the context of wealth being a random variable on a probability space, they need to be measurable functions on that space, and hence, utility functions are random variables.

Bernoulli was the first to suggest a utility function in 1738 as an solution to the St Petersburg Paradox. The theory was developed in its modern form by von Neumann and Morgenstern in 1944. They developed the axioms underlying utility theory, in a synthesis of economics and probability, as

**Independence** of different utility functions (associated with the fact that utility functions are random variables).

**Completeness** all outcomes are assigned a utility.

**Transitivity** if $A$ is preferred to $B$, and $B$ is preferred to $C$, then $A$ is preferred to $C$.

**Continuity** of utility (if wealth is continuous).

However, the basic attribute of a utility function is that it an increasing function, everyone would value more money over less money, so

$$u'(x) > 0.$$

The fact that $u'(x) \neq 0$ means that there is *non-satiation*, i.e. the agent never becomes completely satisfied and will always prefer more to less.

In the example of the beggar and the millionaire, we can see that as wealth increases, each additional £1 has a lower perceived value. This is not surprising and is known as *decreasing marginal utility*, that is

$$u''(x) \leq 0.$$

Consider Figure 1.

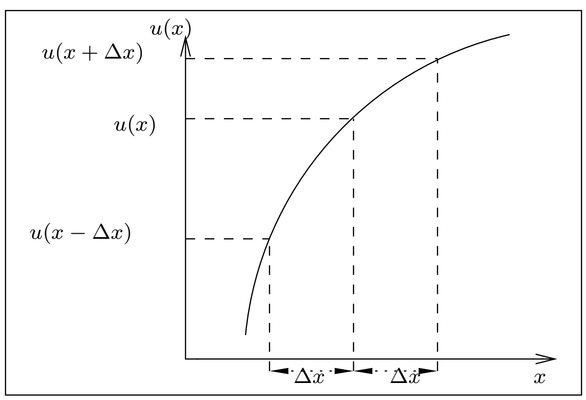


Figure 1.

It is clear that if $u''(x) \leq 0$ then $u(x + \Delta x) - u(x) \leq u(x) - u(x - \Delta x)$, and so each additional unit of wealth increases utility by a smaller amount, the marginal utility, $u(x + \Delta x) - u(x)$ decreases.

A consequence of decreasing marginal utility is *risk aversion*, and an agent with decreasing marginal utility would value certain outcomes over uncertain ones, this effect will be demonstrated in the section on Risk Aversion.

## 2.1 Examples

The most common utility functions are

- Exponential $u(x) = -e^{-\alpha x}, \quad \alpha > 0$ (or if you want positive utility $u(x) = 1 - e^{-\alpha x}, \quad \alpha > 0$.
- Log, $u(x) = \log x$
- Power,

$$u(x) = \frac{x^\alpha - 1}{\gamma}, \quad \gamma < 1$$

- Iso-elastic

$$u(x) = \frac{x^{1-\rho}}{1-\rho}, \quad \rho < 1$$

It is important to note that utility functions, in the context of finance, are relative. Hence, in utility based decision making we must always have at least two alternatives to choose from, often the alternatives will be to consider a portfolio with or without a particular investment.

It is notoriously difficult to establish what an agent's utility function actually is. This means the choice often comes down to identifying a function that helps in the calculations, which makes exponential utility a particular favourite, since calculating expected utility with an exponential utility function reduces to calculating the moment generating function of the random wealth distribution. For example, consider an agents wealth as a random variable, $W$, and the agent has exponential utility, $u(w) = -e^{-\alpha x}$, then

$$\mathbb{E}[u(W)] = -\mathbb{E}[e^{-\alpha W}] = -M_W(\alpha).$$

## 2.2 Expected Utility

Modern financial and economic theory states that if an agent is faced with uncertainty, they should base their decision making on expected utility. That is, integrating (or summing) the utility of wealth over the probability of that outcome. This is natural if you bear in mind that a utility function is a random variable and the expectation of a random variable is a number.

## 2.3 Exercises

1. What does a utility function do?
2. How are non-satiation, diminishing marinal utility and risk aversion related?

# 3 Stochastic dominance

Utility functions are random variables that map the payoffs of a gamble from the real values to perceived values. Note that two random variables, on their own, cannot be compared, if $X$ and $Y$ are random variables the statement $X < Y$ does not make sense on its own. The statement $X(\omega) < Y(\omega)$ for all $\omega \in \Omega$ does. One way of comparing random variables is calculate their expectations, which is a number. The process of taking an expectation involves synthesising the random variable and the probability of the random variable taking on different values. Since it is difficult to identify an agent's utility function, a way of comparing projects (which should be done using expected utility) without actually defining a utility function, might be useful.

Stochastic dominance provides a mechanism for doing this by measuring the relative riskiness of two probability distributions.

## 3.1 First-Order Stochastic Dominance

Given an uncertain gain from and investment (or a gamble), denoted by $W$, a probability distribution, $F_W$, yields a higher pay-off under each contingency and/or attaches a higher probability to higher pay-offs than another probability function, $G_W$, then $F_W$ should be preferred to $G_W$.

**Definition 3.1 (First-Order Stochastic Dominance)** *A probability distribution function $F_W$ dominates another probability distribution function, $G_W$, according to first-order stochastic dominance (FSD) if*

$$F_W(x) \leq G_W(x), \quad \textit{for all } x. \tag{1}$$

Recalling that a distribution function of a random variable $W$ is defined as

$$F_W(c) = \mathbb{P}(W \leq c)$$

The definition of FSD means that, for a given value of $x$, $G_W$ is higher than $F_W$ and so the cumulative probability of $x$ is higher under $G_W$ than $F_W$, implying there is a greater probability of exceeding $x$ under $F_W$ than under $G_W$.

**Example 3.1** *Consider the distribution functions*

$$F_W(x) = \begin{cases} 0, & \text{if } x < 2.5, \\ 0.4 & \text{if } 2.5 \leq x < 3.5, \\ 1, & \text{if } 3.5 \leq x, \end{cases}$$

*and*

$$G_W(x) = \begin{cases} 0, & \text{if } x < 2, \\ 0.5 & \text{if } 2 \leq x < 3, \\ 1, & \text{if } 3 \leq x. \end{cases}$$

*Then $F_W$ dominates $G_W$ by FSD.*

Using the distributions in Example 3.1 and an arbitrary utility function, we have that

$$\begin{aligned} \mathbb{E}_{F_W}[u(W)] &= 0.4 \times u(2.5) + 0.6 \times u(3.5) \\ &\geq 0.4 \times u(2.0) + 0.6 \times u(3.0) \\ &= u(2.0) + 0.6 \times (u(3.0) - u(2.0)) \\ &> u(2.0) + 0.5 \times (u(3.0) - u(2.0)) \\ &= \mathbb{E}_{G_W}[u(W)]. \end{aligned}$$

In fact, there is the following lemma

**Lemma 3.1** *For all* strictly increasing *utility functions, $u$, $F_W$ dominates $G_W$ by FSD if and only if*

$$\mathbb{E}_{F_W}[u(X)] \geq \mathbb{E}_{G_W}[u(X)].$$

An expected utility maximising agent who prefers more to less will always prefers probability distributions that exhibit First Order Stochastic Dominance.

## 3.2 Second-Order Stochastic Dominance

If the condition (1) does not hold, is there an alternative condition? The answer is provided by second -order stochastic dominance.

**Definition 3.2 (Second-Order Stochastic Dominance)** *A probability function $F$ dominates another probability function, $G$, according to second-order stochastic dominance (SSD) if*

$$\int_{-\infty}^{x} F(y)dy \leq \int_{-\infty}^{x} G(y)dy, \quad \textit{for all } x. \tag{2}$$

SSD requires the *area* under the dominant distribution is less than the area under the subordinate distribution. We have the following lemma, which corresponds to Lemma 3.1

**Lemma 3.2** *For all* strictly increasing and concave *utility functions, $u$, $F$ dominates $G$ by SSD if and only if*

$$\mathbb{E}_F[u(X)] \geq \mathbb{E}_G[u(X)].$$

## 3.3 Summary

Given that we should base our decision making on expected utility, stochastic dominance enables us to choose between two gambles (lotteries) independently of utility functions. The decision can be made simply based on the probability distributions of the payoffs of the gambles. This is useful since it is generally very difficult to identify an agent's utility function.

## 3.4 Exercises

1. Which distributions are dominant in the following pairs.

   (a)

$$F(x) = \begin{cases} 0, & \text{if } x < 0, \\ 1 - e^{-2x}, & \text{if } 0 \leq x, \end{cases}$$

   and

$$G(x) = \begin{cases} 0, & \text{if } x < 0, \\ 1 - e^{-x}, & \text{if } 0 \leq x, \end{cases}$$

(b)

$$F(x) = \begin{cases} 0, & \text{if } x < 1, \\ (x-1) & \text{if } 1 \leq x < 2, \\ 1, & \text{if } 2 \leq x, \end{cases}$$

and

$$G(x) = \begin{cases} 0, & \text{if } x < 0, \\ \frac{x}{3} & \text{if } 0 \leq x < 3, \\ 1, & \text{if } 3 \leq x. \end{cases}$$

# 4 Risk aversion

We start by showing why concavity of the utility function (that is $u''(x) \leq 0$ ) leads to risk aversion.

Consider an expected utility maximising investor who has the opportunity to participate in a risky investment. The investment will either offer a gain of $X_A$ with probability $p_A$ or $X_B > X_A$ with probability $(1-p_A)$. To participate in this investment the investor would have to pay $\bar{X} = p_A X_A + (1 - p_A)X_B$, i.e. the investment opportunity is 'fair'. If the utility function is concave, Figure 2 shows that the utility of $\bar{X}$ is greater than the utility of the expected value of the investment, $p_A u(X_A) + (1 - p_A)u(X_B)$

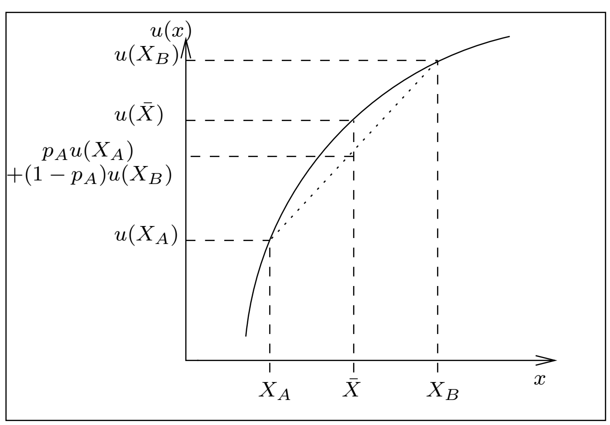


Figure 2

This means that the investor would prefer to *keep* $\bar{X}$ rather than the expected pay-off of the investment. This result is a consequence of *Jensen's inequality*.

**Definition 4.1 (Jensen's inequlity)** *For any random variable $X$ and any function $g$ that is strictly concave, that is $g''(x) < 0$, then*

$$\mathbb{E}[g(X)] \leq g(\mathbb{E}[X])$$

*Equality holds if and only if $X$ is constant.*

Jensen's inequality tells us that, if the utility is concave, the expected utility is less than the utility of the expected value. If the gamble is fair, that is the cost of participating in the gamble is equal to its expected outcome, the utility associated with this cost is greater than the expected utility of the winnings. Therefore the loss of utility, associated with buying the gamble, is more than the expected gain in utility provided by the possible winning.

## 4.1 Measures of Risk Aversion

From Figure 2 you may be able to see that the curvature of a utility function, the way the marginal utility changes, is important since it determines how an agent acts in the face of uncertainty, and is known as *risk aversion.* There are two ways to measure the level of risk aversion, *absolute risk aversion* (ARA)

$$R_a(x) = -\frac{u''(x)}{u'(x)}$$

and *relative risk aversion*, (RRA)

$$R_r(x) = -x\frac{u''(x)}{u'(x)}.$$

Absolute risk aversion measures risk aversion to a loss in absolute terms where as relative risk aversion measures aversion to a loss relative to agents wealth.

An agent's allocation of their wealth to risky assets depends on the risk aversion characteristics of their utility function in the following way.

- If an agent has increasing absolute risk aversion (IARA), then as wealth increases they will hold fewer pounds in risky assets.

- If an agent has constant absolute risk aversion (CARA), as wealth increases they will the same number of pounds in risky assets.

- If an agent has decreasing absolute risk aversion (DARA), then as wealth increases they will hold more pounds in risky assets.

- If an agent has increasing relative risk aversion (IRRA), then as wealth increases they will hold a lower percentage of their wealth in risky assets.
- If an agent has constant relative risk aversion (CRRA), as wealth increases they will the same percentage of their wealth in risky assets.
- If an agent has decreasing relative risk aversion (DRRA), then as wealth increases they will hold a higher percentage of their wealth in risky assets.

CRRA matches empirical observations more than CARA, for example as your wealth increases would you expect to invest the same absolute value or the same proportion in the risky asset? Generally, we would expect rational agent's to invest a greater absolute amount in the risky asset as they become richer.

Exponential utility is unique is exhibiting constant absolute risk aversion (CARA), while log utility exhibits DARA, decreasing absolute risk aversion (which is consistent with experimental data). You could also construct functions that exhibit increasing absolute risk aversion.

The "iso-elastic" utility function exhibits constant relative risk aversion (CRRA) with $R_r(x) = \rho$. If $\rho = 1$, then the iso-elastic utility function is equivalent to log utility function.

Traditionally risk aversion was considered to be always rational, however Kahneman and Tversky, in 1979, showed that when faced with losses a risk-seeking utility function would be rational. To see this, observe that when faced with a choice of being *given* £100 with certainty or a lottery to *win* £1000 with a probability of 0.1 and nothing with probability 0.9, it is rational to take the £100. However, when faced with a loss of £100 with certainty or a lottery to *lose* £1000 with a probability of 0.1 and nothing with probability 0.9, it is rational to take the gamble. This is consistent with *increasing marginal utility*, a loss of £1,001 is not that different from a loss of £1,000, but a loss of £2 is very different to a loss of £1.

There are now cognitive explanations for increasing and decreasing marginal utility of wealth, which will lead to risk tolerant or risk seeking behaviour.

### 4.2 Summary

Utility functions map the "real world" value of money onto an agent's perceived value. They are increasing functions, and if you are considering gains, the marginal utility is decreasing.

### 4.3 Exercises

1. Show that

   (a) exponential utility exhibits CARA and IRRA.

   (b) log utility exhibits DARA and CRRA.

   What utility function would be better at reflecting normal behaviour?

2. What type of absolute risk aversion do the utility functions

$$u(x) = \frac{2\alpha}{1 - 2\alpha x} \quad \text{and} \quad u(x) = \beta x$$

   exhibit.

## 5 Insurance

Consider a risk averse agent (that is the utility function is concave), with initial wealth $x$, who faces a random loss $L$, which will occur with probability $p_L$, and they can insure against this loss by buying insurance for a premium $P < L$. Under what conditions will the risk averse agent purchase insurance?

The agent should base their decision on expected utility, in particular if the expected utility of paying for insurance and not suffering the loss is more than the expected utility of bearing any losses, they should insure,

$$\mathbb{E}[U(x - p)] > \mathbb{E}[U(x - L)]$$

and since $x$ and $p$ are deterministic, this reduces to

$$U(x - p) > \mathbb{E}[U(x - L)].$$

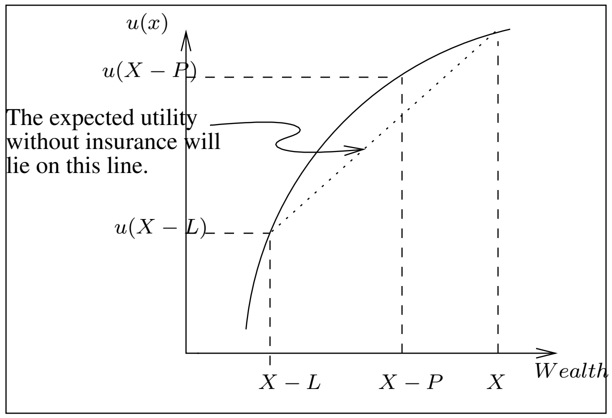


Figure 3

The expected utility without insurance, $\mathbb{E}[U(x-L)]$, will lie along the line indicated in Figure 3, its position being determined by $p_L$ (the higher $p_L$ the closer expected utility without insurance will be to $U(x-L)$). The maximum premium the agent will pay, $p_{max}$ is given by

$$U(x - p_{max}) = \mathbb{E}[U(x-L)]$$

Given Jensen's inequality(the expected utility is less than the utility of the expected value), for the rhs we have

$$\mathbb{E}[U(x-L)] \leq U(\mathbb{E}[x-L])$$

and so, given $x$ is deterministic,

$$U(x - p_{max}) \leq U(x - \mathbb{E}[L]).$$

Since $u$ is increasing, this implies that $p_{max} \geq \mathbb{E}[L]$.

**Remark 5.1** *A risk averse agent will always be happy to pay an insurance premium* greater *than the expected loss covered by the insurance.*

To calculate $p_{max}$ you first find the expected utility without insurance, $\mathbb{E}[U(x-L)]$. This gives you $U(x-p_{max})$, and inverting you have $x - p_{max}$, this procedure is shown in Figure 4.

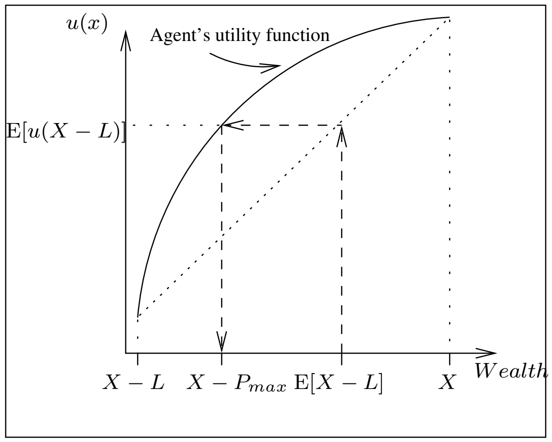


Figure 4

(note that since $p_{max}$ and $x$ are known and deterministic $\mathbb{E}[x - p_{max}] = x - p_{max}$)

The insurer, with wealth $w$, faces a similar problem. They calculate the *minimum* premium, $p_{min}$ that they would accept as

$$\mathbb{E}[U(w + p_{min} - L)] = U(w).$$

Using Jensen's inequality (the expected utility is less than the utility of the expected value), we calculate that

$$\begin{aligned} U(w) = \mathbb{E}[U(w + p_{min} - L)] &\leq U(w + p_{min} - \mathbb{E}[L]) \\ \Rightarrow \quad U(w + p_{min} - \mathbb{E}[L]) &\geq U(w) \end{aligned}$$

Since $u$ is strictly increasing, this implies that $p_{min} - \mathbb{E}[L] \geq 0$, and so

$$p_{min} \geq \mathbb{E}[L].$$

Competition in insurance markets implies that insures should charge the minimum principle, this is called the *zero utility principle.*

**Example 5.1** *A risk averse agent, whose utility is given by* $U(x) = \ln x$ *and wealth is* £$50,000$ *is faced with a potential loss of* £$10,000$ *with a probability of* $p_l = 0.1$*. What is the maximum premium they would be willing to pay to protect themselves against this loss?*

*What is the minimum premium that an insurer, with the same utility function and with wealth* £$1,000,000$*, be willing to charge to cover this loss?*

The agent's expected wealth without insurance is

$$\begin{aligned}\mathbb{E}[W_L] &= (50,000 \times 0.9) + (40,000 \times 0.1) \\ &= 49,000\end{aligned}$$

while their expected utility of wealth without insurance is

$$\begin{aligned}\mathbb{E}[U(W_I)] &= (\ln(50,000) \times 0.9) + (\ln(40,000) \times 0.1) \\ &= 10.797463...\end{aligned}$$

To identify the maximum premium we need to equate these two,

$$\ln(50,000 - p_{max}) = 10.797463..$$

or

$$\begin{aligned}p_{max} &= 50,000 - e^{10.797463} \\ &= 50,000 - e^{10.797463} \\ &= 50,000 - 48,869.64 \\ &= 1,103.39\end{aligned}$$

Note that $p_{max} = 1,103.39 > 1,000$, the expected loss.

The insurer will set $p_{min}$ by equating

$$\begin{aligned}U(w) &= \mathbb{E}[U(w + p_{min} - L)] \\ \ln(1,000,000) &= 0.9 \times \ln(1,000,000 + p_{min}) + 0.1 \times \ln(1,000,000 + p_{min} - 10,000) \\ \Rightarrow \quad p_{min} &\approx 1,004.52\end{aligned}$$

Observe that the insurer had initial wealth £10,000,000 then $p_{min} \approx 1,000.45$, showing that the wealthier the insurer, the better they are able to compete for business.