Practical Assignment № 2

Adaptive and Robust Control

Design of adaptive and robust control systems for disturbed objects

variant number : 16

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A.Problem statement

Consider the plant, represented by a model of the form

$$\dot{x} = \theta x + u + \delta, \qquad (2.1)$$

where δ is the unknown bounded disturbance such that $|\delta(t)| \leqslant \bar{\delta}$. As in the work $\mathrm{N}^{\circ}1, x$ is the state variable (coincides with system output), u is the the control, θ is the unknown constant parameter.

The objective is to design a control that will provide the boundness of all the signals in the closed-loop system and ensure the inequality:

$$|x_m(t) - x(t)| \leqslant \Delta, \forall t \geqslant T, \qquad (2.2)$$

where Δ, T are the maximum steady-state error and transient time, $x_m(t)$ is the output of reference model (1.2). It is assumed that the constants Δ and T can be regulated by the parameters of controller.

B.Theoretical background

- **Adaptive but no robust Controller**

From the previous theoretical background we can obtain the Adaptive Controller

as follows:

$$\begin{aligned} &\dot{x}_m = -\lambda x_m + \lambda g \\ &\varepsilon = x_m - x \\ &\dot{\hat{\theta}} = -\gamma x \varepsilon \\ &u = -\hat{\theta} x - \lambda x + \lambda g. \end{aligned} \qquad (2.0)$$

Let us consider the possibility of using the controller (2.0) as a solution to the formulated problem. We define the error model $\varepsilon = x_m - x$

$$\dot{\varepsilon} = -\lambda \varepsilon - \tilde{\theta} x - \delta. \qquad (2.3)$$

Next, we analyze the stability of a closed system using the Lyapunov function (1.8). Given the last expression and the adaptation algorithm (1.8), we obtain the following derivative of the Lyapunov function:

$$\dot{V} = \frac{1}{2}2\varepsilon\dot{\varepsilon} + \frac{1}{2\gamma}2\tilde{\theta}\dot{\tilde{\theta}} = \varepsilon(-\lambda\varepsilon - \tilde{\theta}x - \delta) - \frac{1}{\gamma}\tilde{\theta}\dot{\hat{\theta}} = -\lambda\varepsilon^2 - \tilde{\theta}x\varepsilon - \delta\varepsilon + \tilde{\theta}\frac{1}{\gamma}\gamma x\varepsilon =$$

$$= -\lambda\varepsilon^2 - \delta\varepsilon = -\frac{\lambda}{2}\varepsilon^2 - \frac{\lambda}{2}\varepsilon^2 - \delta\varepsilon \pm \frac{1}{2\lambda}\delta^2 = -\frac{\lambda}{2}\varepsilon^2 - \left(\sqrt{\frac{\lambda}{2}}\varepsilon + \sqrt{\frac{1}{2\lambda}}\delta\right)^2 + \frac{1}{2\lambda}\delta^2 \leqslant$$

$$\leqslant -\frac{\lambda}{2}\varepsilon^2 + \frac{1}{2\lambda}\delta^2 \leqslant -\frac{\lambda}{2}\varepsilon^2 + \frac{1}{2\lambda}\delta^2.$$

The resulting inequality implies an asymptotic convergence of the error ε to some bounded set defined by the upper bound of the perturbation signal $\bar{\delta}$ and the parameter λ. In this case, the accuracy of the control system can be increased by increasing λ. However, the above analysis does not imply that the signal $\hat{\theta}$ is limited. If we continue the analysis and consider a special case, where the variable x and the error ε converge towards non-zero constant values due to the influence of perturbation, then

$$\dot{\hat{\theta}} = -\gamma x_{\text{stable}}\, \varepsilon_{\text{stable}} = C = \text{ const,}$$

which implies

$$\hat{\theta} = Ct,$$

and the unlimited growth $\hat{\theta}$ over time. This phenomenon is called unlimited parametric drift.

In this case, the controller (1.6) and (1.9) does not ensure boundedness of all signals in general and is not robust to external perturbations.

The proposed approach is not practically applicable and requires the control algorithm to be modified. Let us look at two possible solutions.

- **Solution N°1（Sapplying a nonlinear static feedback）**

Solution $N°I$. Solution obtained by applying a nonlinear static feedback

$$\hat{\theta} = -\gamma x\varepsilon \tag{2.4}$$

in (1.6) instead of adaptation algorithm (1.9). Substituting (2.4) for $\hat{\theta}$ in (1.6), we obtain the robust control law

$$u = \gamma x^2\varepsilon - \lambda x + \lambda g \tag{2.5}$$

This algorithm is both static, since it does not contain an integral feedback, and nonlinear because of the equation member $\gamma x^2\varepsilon$.

Let us show that the proposed control algorithm (2.5) boundedness of the signals ε and $\hat{\theta}$. For this we choose the Lyapunov function

$$V = \frac{1}{2}\varepsilon^2 \tag{2.6}$$

and calculate its derivative. Considering (2.5) and the error model (2.3), we perform the following algebraic operations:

$$\begin{aligned}
\dot{V} &= \frac{1}{2}2\varepsilon\dot{\varepsilon} = -\lambda\varepsilon^2 - \tilde{\theta}x\varepsilon - \delta\varepsilon = -\frac{\lambda}{2}\varepsilon^2 - \frac{\lambda}{2}\varepsilon^2 - (\theta - \hat{\theta})x\varepsilon - \delta\varepsilon = \\
&= -\frac{\lambda}{2}\varepsilon^2 - \frac{\lambda}{2}\varepsilon^2 - \delta\varepsilon \pm \frac{1}{\lambda}\delta^2 - (\theta + \gamma x\varepsilon)x\varepsilon = \\
&= -\frac{\lambda}{2}\varepsilon^2 - \left(\sqrt{\frac{\lambda}{2}}\varepsilon + \sqrt{\frac{1}{2\lambda}}\delta\right)^2 + \frac{1}{2\lambda}\delta^2 \pm \frac{\theta^2}{4\gamma} - \theta x\varepsilon - \gamma x^2\varepsilon^2 = \\
&= -\frac{\lambda}{2}\varepsilon^2 - \left(\sqrt{\frac{\lambda}{2}}\varepsilon + \sqrt{\frac{1}{2\lambda}}\delta\right)^2 + \frac{1}{2\lambda}\delta^2 - \left(\frac{\theta}{2\sqrt{\gamma}} + \sqrt{\gamma}x\varepsilon\right)^2 + \frac{\theta^2}{4\gamma} \leqslant \\
&\leqslant -\frac{\lambda}{2}\varepsilon^2 + \frac{1}{2\lambda}\bar{\delta}^2 + \frac{\theta^2}{4\gamma} = -\lambda V + \frac{1}{2\lambda}\bar{\delta}^2 + \frac{\theta^2}{4\gamma} = -\lambda V + \bar{\Delta}
\end{aligned}$$

where $\bar{\Delta} = \bar{\delta}^2/2\lambda + \theta^2/4\gamma$ are constant values. After solving the resulting differential inequality, we obtain:

$$V(\varepsilon(t)) \leqslant e^{-\lambda t}V(0) + \frac{\bar{\Delta}}{\lambda} - \frac{\bar{\Delta}}{\lambda}e^{-\lambda t},$$

and by taking into account (2.6), then

$$\frac{1}{2}\varepsilon^2 \leqslant e^{-\lambda t}V(0) + \frac{\bar{\Delta}}{\lambda} - \frac{\bar{\Delta}}{\lambda}e^{-\lambda t}$$

or

$$|\varepsilon(t)| \leqslant \sqrt{2\left(e^{-\lambda t}V(0) + \frac{\bar{\Delta}}{\lambda} - \frac{\bar{\Delta}}{\lambda}e^{-\lambda t}\right)} \tag{2.7}$$

The last inequality implies exponential convergence of the control error ε towards a bounded set with boundary $\Delta = \sqrt{2\bar{\Delta}/\lambda}$. In this case it is possible to decrease the value of Δ by increasing the coefficients λ and γ. As a result $\hat{\theta}$ becomes bounded.

That way the control algorithm (2.5) provides stability in the closed-loop system and is robust to external disturbances. At the same time, this algorithm has the following disadvantages:

- even in the absence of perturbation, the established error $\varepsilon(t)$ can be different from zero, as can be seen from the inequality (2.7);
- the control is proportional to the value x^2. Therefore, when increasing x the control amplitude increases four folds, and therefore the practical applicability of such a law (1.6) has significant limitations.

Let us now consider a solution devoid of the disadvantages of algorithms (1.6), (1.9) and (2.4) by endowing a new control algorithm with adaptive and robust properties.

● Solution N°2（applying the leakage factor (static linear feedback)）

Solution № 2. We modify the adaptation algorithm (1.6) by applying the leakage factor (static linear feedback):

$$\dot{\hat{\theta}} = -\sigma\hat{\theta} - \gamma x\varepsilon, \tag{2.8}$$

where σ is a constant positive value. Then use this adaptive robust modification together with adjustable control (1.6).

We analyze the stability of the closed system represented by the plant (2.1), the controller (1.6) the adaptation algorithm (2.8) with the help of the Lyapunov

function (1.8). We take its derivative and perform the following operations:

$$\begin{aligned}
\dot{V} &= \frac{1}{2}2\varepsilon\dot{\varepsilon} + \frac{1}{2\gamma}2\tilde{\theta}\dot{\tilde{\theta}} = \varepsilon(-\lambda\varepsilon - \tilde{\theta}x - \delta) - \frac{1}{\gamma}\tilde{\theta}\dot{\hat{\theta}} = -\lambda\varepsilon^2 - \tilde{\theta}x\varepsilon - \delta\varepsilon - \frac{\tilde{\theta}}{\gamma}(-\sigma\hat{\theta} - \gamma x\varepsilon) = \\
&= -\lambda\varepsilon^2 - \delta\varepsilon + \frac{\sigma}{\gamma}\tilde{\theta}\hat{\theta} = -\lambda\varepsilon^2 - \delta\varepsilon + \frac{\sigma}{\gamma}\tilde{\theta}(-\tilde{\theta} + \theta) = -\lambda\varepsilon^2 - \delta\varepsilon - \frac{\sigma}{\gamma}\tilde{\theta}^2 + \frac{\sigma}{\gamma}\tilde{\theta}\theta = \\
&= -\frac{\lambda}{2}\varepsilon^2 - \frac{\lambda}{2}\varepsilon^2 - \delta\varepsilon \pm \frac{1}{2\lambda}\delta^2 - \frac{\sigma}{2\gamma}\tilde{\theta}^2 - \frac{\sigma}{2\gamma}\tilde{\theta}^2 + \frac{\sigma}{\gamma}\tilde{\theta}\theta \pm \frac{\sigma}{2\gamma}\theta^2 = \\
&= -\frac{\lambda}{2}\varepsilon^2 - \frac{\sigma}{2\gamma}\tilde{\theta}^2 - \left(\sqrt{\frac{\lambda}{2}}\varepsilon + \sqrt{\frac{1}{2\lambda}}\delta\right)^2 - \frac{\sigma}{\gamma}\left(\sqrt{\frac{1}{2}}\tilde{\theta} + \sqrt{\frac{1}{2}}\theta\right)^2 + \frac{1}{2\lambda}\delta^2 + \frac{\sigma}{2\gamma}\theta^2 \leqslant \\
&\leqslant -\frac{\lambda}{2}\varepsilon^2 - \frac{\sigma}{2\gamma}\tilde{\theta}^2 + \frac{1}{2\lambda}\delta^2 + \frac{\sigma}{2\gamma}\theta^2 \leqslant -\frac{\lambda}{2}\varepsilon^2 - \frac{\sigma}{2\gamma}\tilde{\theta}^2 + \frac{1}{2\lambda}\bar{\delta}^2 + \frac{\sigma}{2\gamma}\theta^2.
\end{aligned}$$

We introduce the notation $\kappa = \min\{\lambda, \sigma\}$. Then, considering λ, σ to be positive, we have:

$$\dot{V} \leqslant -\kappa\left(\frac{1}{2}\varepsilon^2 - \frac{1}{2\gamma}\tilde{\theta}^2\right) + \frac{1}{2\lambda}\bar{\delta}^2 + \frac{\sigma}{2\gamma}\theta^2$$

or

$$\dot{V} \leqslant -\kappa V + \bar{\Delta},$$

where $\bar{\Delta} = \bar{\delta}^2/2\lambda + \sigma\theta^2/2\gamma$ is a constant value. Further, by solving this differential inequality, we obtain:

$$V(\varepsilon(t)) \leqslant e^{-\kappa t}V(0) + \frac{\bar{\Delta}}{\kappa} - \frac{\bar{\Delta}}{\kappa}e^{-\kappa}$$

from which

$$|\varepsilon(t)| \leqslant \sqrt{2\left(e^{-\kappa}\ V(0) + \frac{\bar{\Delta}}{\kappa} - \frac{\bar{\Delta}}{\kappa}e^{-\kappa t}\right)}.$$

From the last inequality follows the exponential convergence of the control error to a bounded set with boundary $\Delta = \sqrt{2\bar{\Delta}/\kappa}$.

The control algorithm (1.6), based on the adaptation algorithm (2.8), also provides stability of the closed system and is robust to external disturbances. At the same time algorithm $(1.6), (2.8)$ allows to fend off the shortcomings of the

robust control algorithm (2.5). So, in the absence of an external disturbance or with its insignificant influence, the upper bound $\bar{\Delta}$ can be reduced to zero by defining the coefficient σ to be equal to zero. Also, in order to decrease Δ there is no need to significantly increase γ, which entails an increase in the amplitude of the control action. It is possible to decrease Δ by decreasing σ.

C. Experimental part

- **Experimental parameters（Group 16）**

Task No.	Plant parameter θ	Reference model parameter λ	Reference $g(t)$
16	7	1	$\cos t + 3$

From the previous theoretical background we can obtain the equation of state of the system as follows：

$$\dot{x} = \theta x + u + \delta, \qquad (3.1)$$

where δ is the unknown bounded disturbance such that $|\delta(t)| \leqslant \bar{\delta}$. As in the work $\mathrm{N}^{\circ}1$, x is the state variable (coincides with system output), u is the the control, θ is the unknown constant parameter.

- Adaptive but no robust Controller

1. **design the Adaptive but no robust Controller**

From the the work $\mathrm{N}^{\circ}1$we can obtain the Adaptive but no robust Controller controller

as follows:

$$
\begin{aligned}
&\dot{x}_m = -\lambda x_m + \lambda g \\
&\varepsilon = x_m - x \\
&\dot{\hat{\theta}} = -\gamma x \varepsilon \\
&u = -\hat{\theta} x - \lambda x + \lambda g.
\end{aligned}
\tag{3.2}
$$

The simulink scheme is constructed from (3.1) and (3.2) as follows:

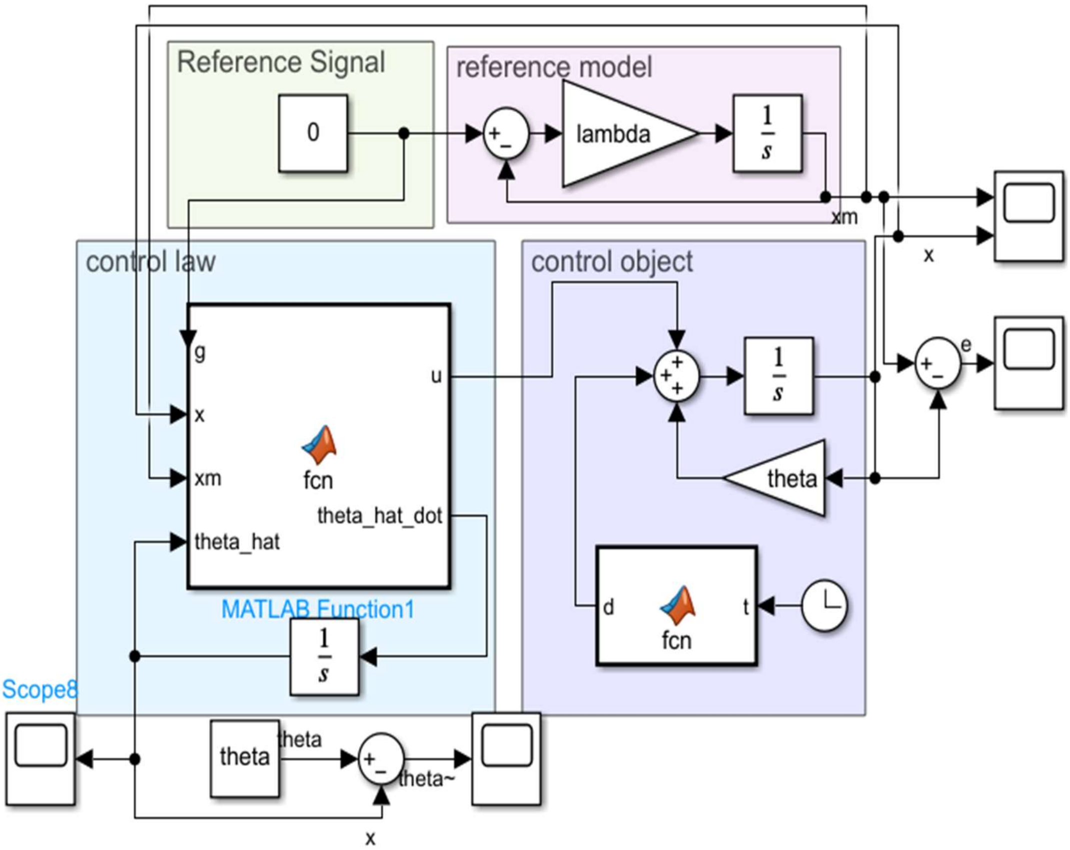

Figure 1 : Adaptive but no robust Controller simulation scheme

2. **Simulation**

Make a simulation experiment,for the plant (2.1) disturbed by the signal:

$$\delta(t) = (1+t)^{-1/8}\left[1 - \theta(1+t)^{-1/4} - \frac{3}{8}(1+t)^{-5/4}\right]$$

The simulation results are shown in the following figure:

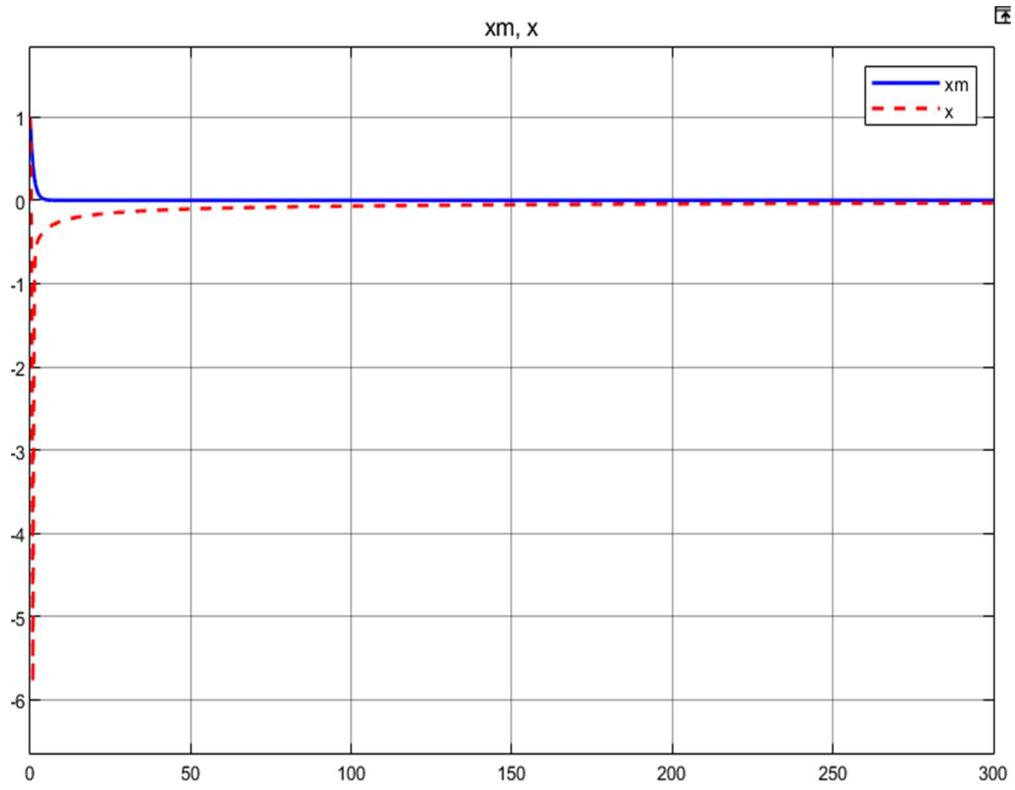

Figure 2： Adaptive but no robust Controller simulation result （$x(t), x_m(t)$）

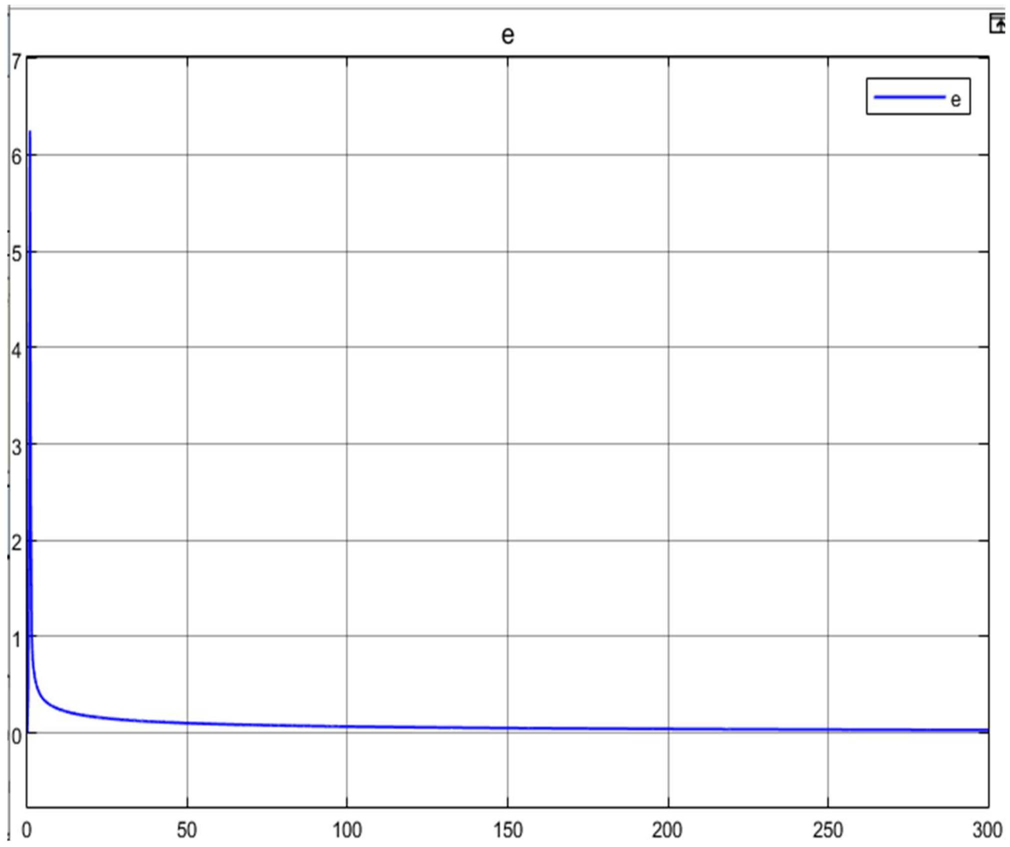

Figure 3： Adaptive but no robust Controller simulation result （$\epsilon(t)$）

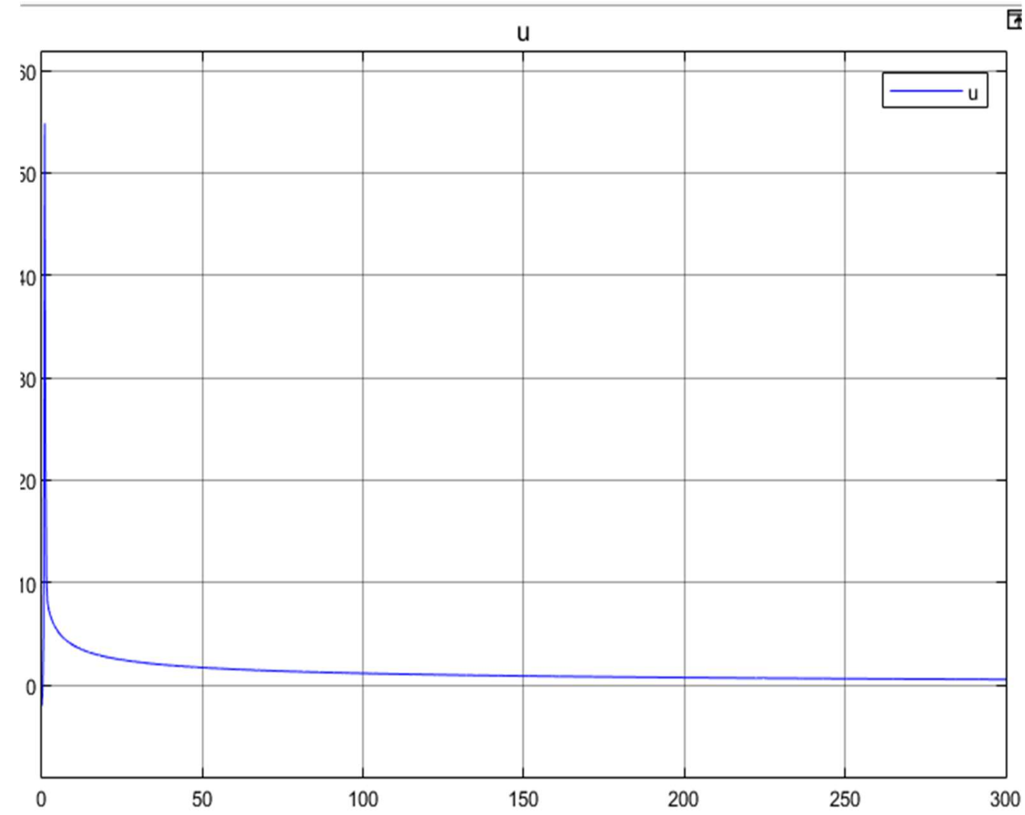

Figure 4： Adaptive but no robust Controller simulation result （$u(t)$）

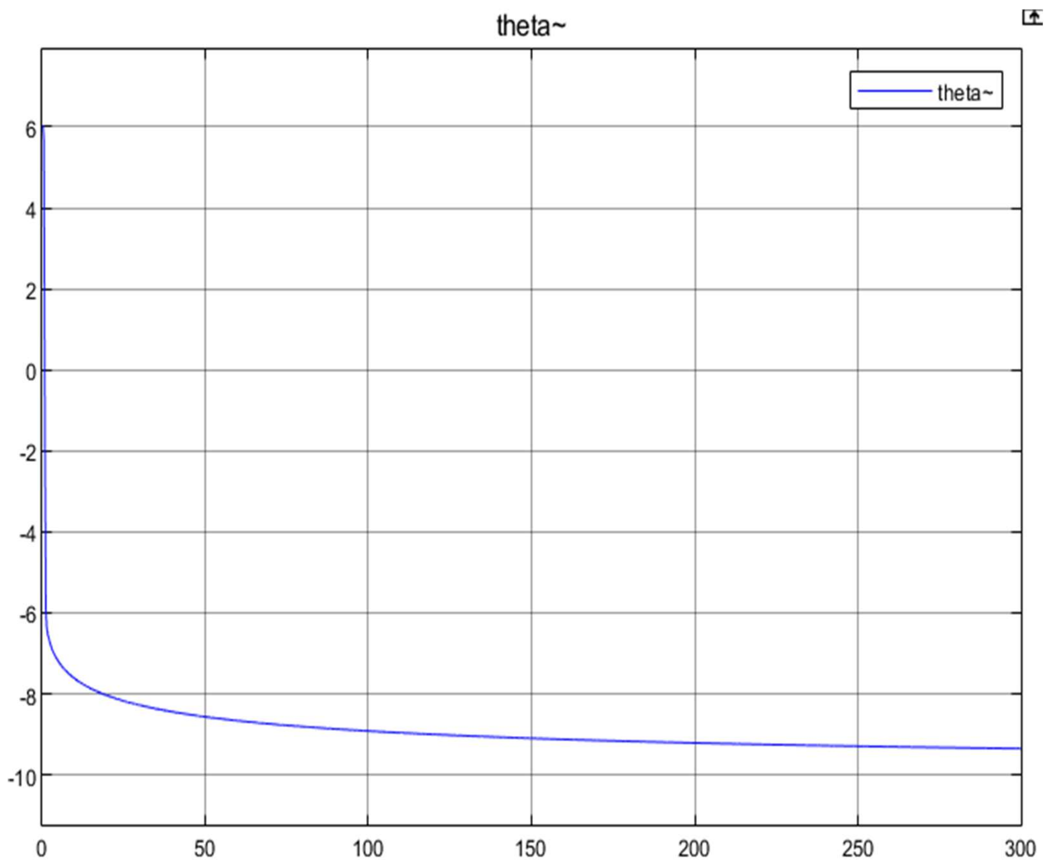

Figure 5：Adaptive but no robust Controller simulation result （$\tilde{\theta}(t)$）

3. Conclusion

1) It can be seen from Figure 3:
    a) $\epsilon(t)$ tends to be close to 0 as time tends to infinity,is bounded
    b) which indicates that the adaptive but non-robust controller can still achieve the control effect of making the system and the reference model output error tend to be close to 0
2) From Figure 5, it can be seen that:
    a) $\tilde{\theta}(t)$tends to negative infinity as time tends to approach 0, which indicates that $\hat{\theta}(t)$ tends to infinity with time, without bounds, and parameter shift occurs;
    b) this phenomenon indicates that the system controlled by an adaptive but non-robust controller loses robustness

● Solution N°1（applying a nonlinear static feedback)

1. design the Robust Controller 1（applying a nonlinear static feedback)

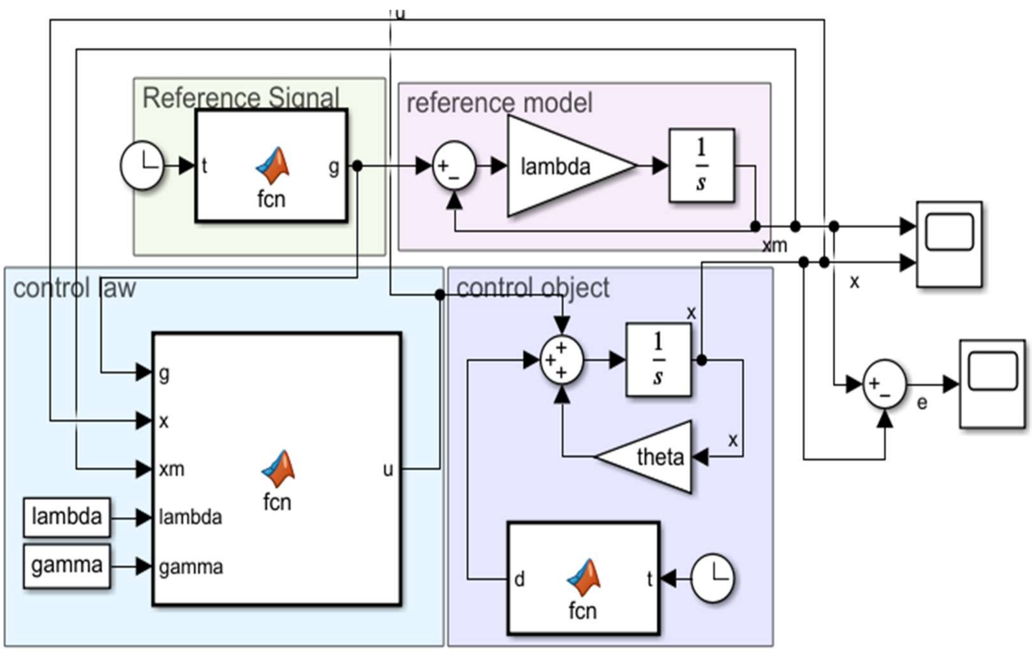

Figure 6：Robust Controller（applying a nonlinear static feedback） simulation scheme

2. Simulation

- Small gain $\gamma = 0.01$

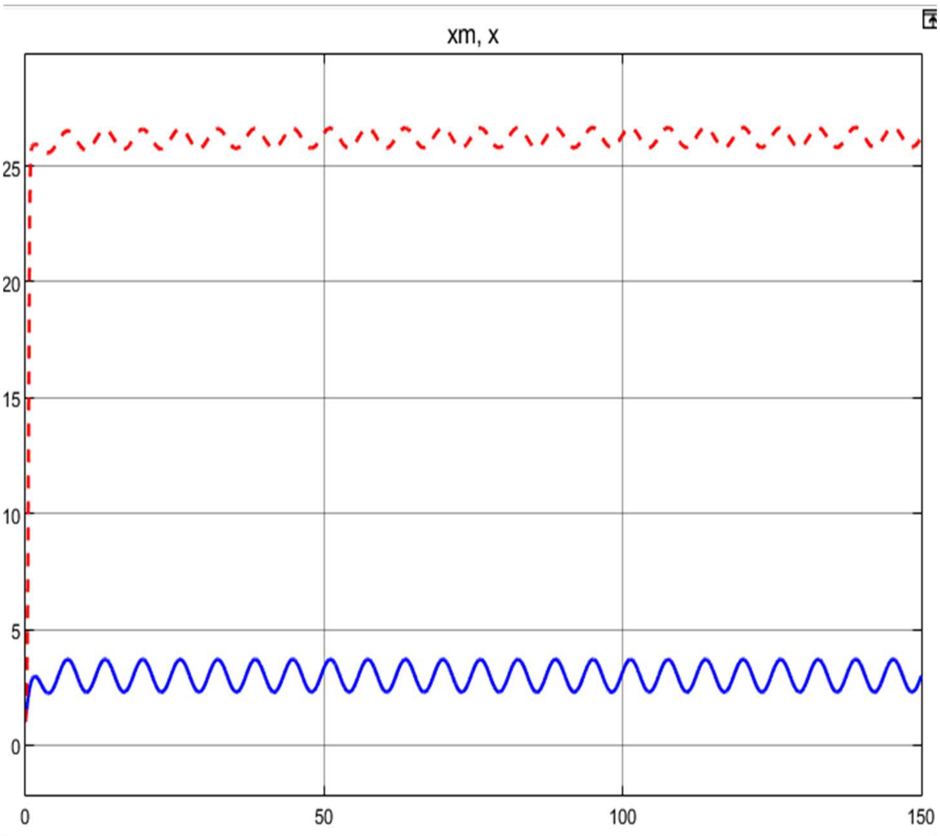

Figure 7： Robust Controller 1 simulation result （$x(t), x_m(t)$， $\gamma = 0.01$）

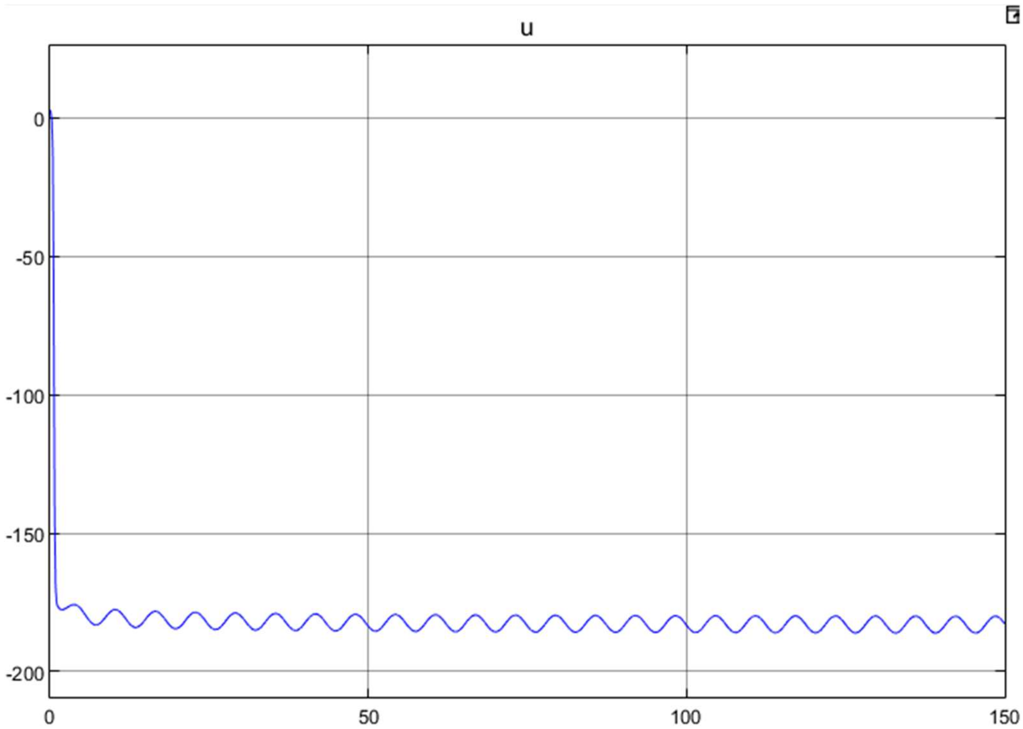

Figure 8： Robust Controller 1 simulation result （$u(t)$， $\gamma = 0.01$）

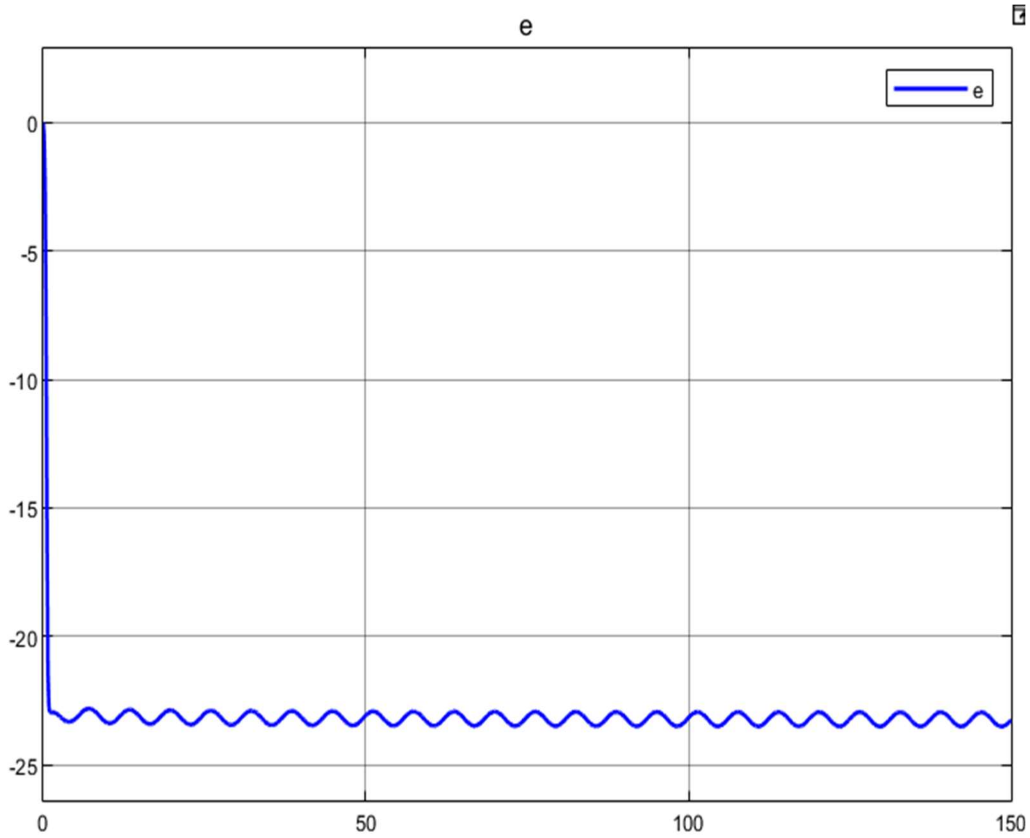

Figure 9： Robust Controller 1 simulation result with small gain （$\epsilon(t)$， $\gamma = 0.01$）

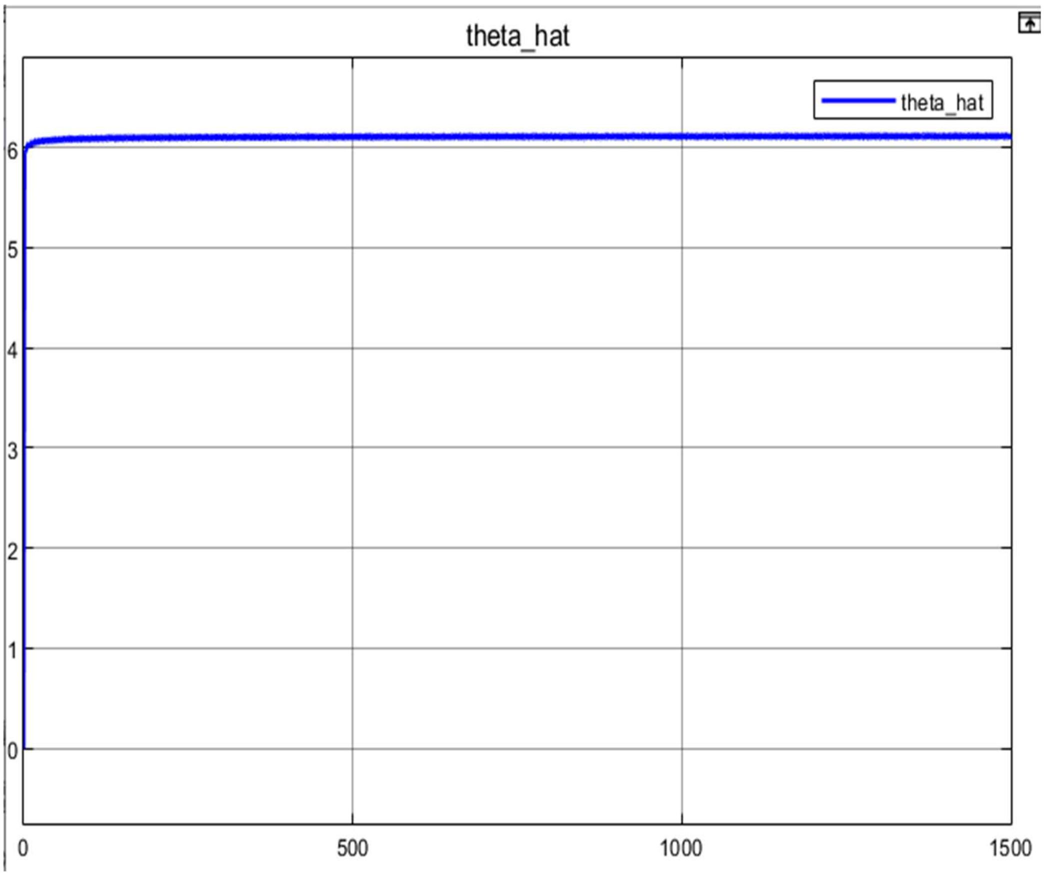

Figure 10： Robust Controller 1 simulation result with small gain （$\hat{\theta}(t)$， $\gamma = 0.01$）

- **Medium gain $\gamma = 2$**

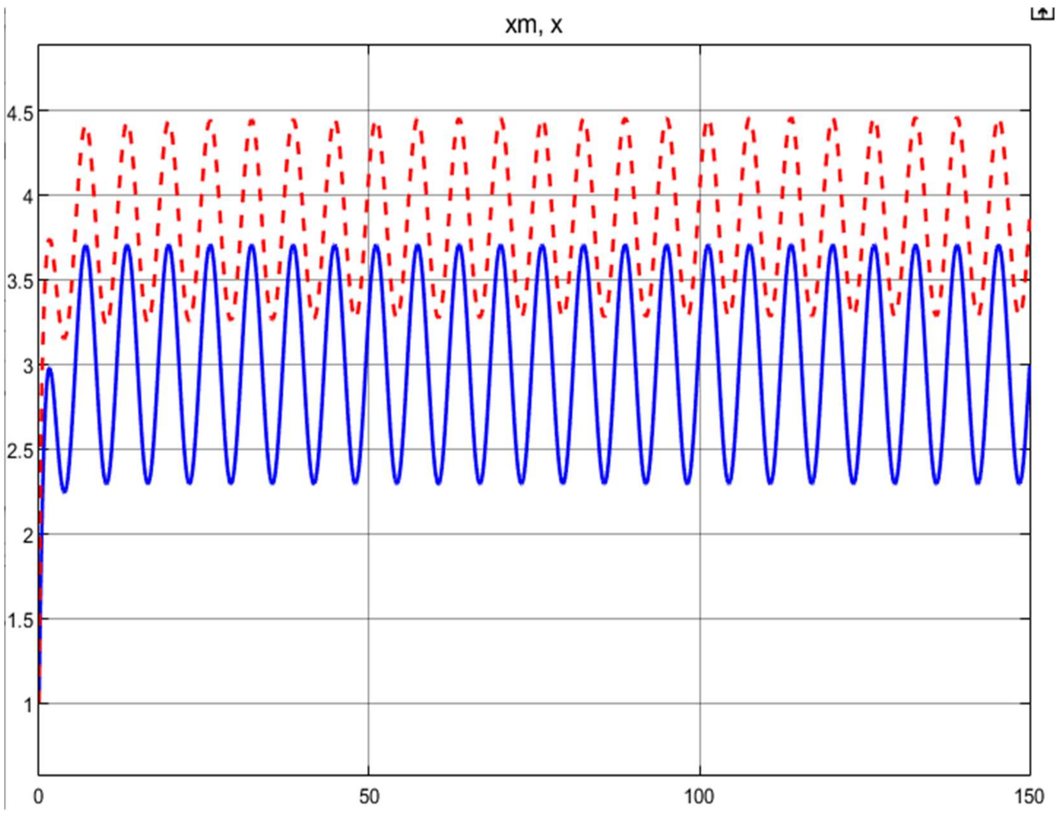

Figure 11： Robust Controller 1 simulation result（$x(t), x_m(t)$，$\gamma = 2$）

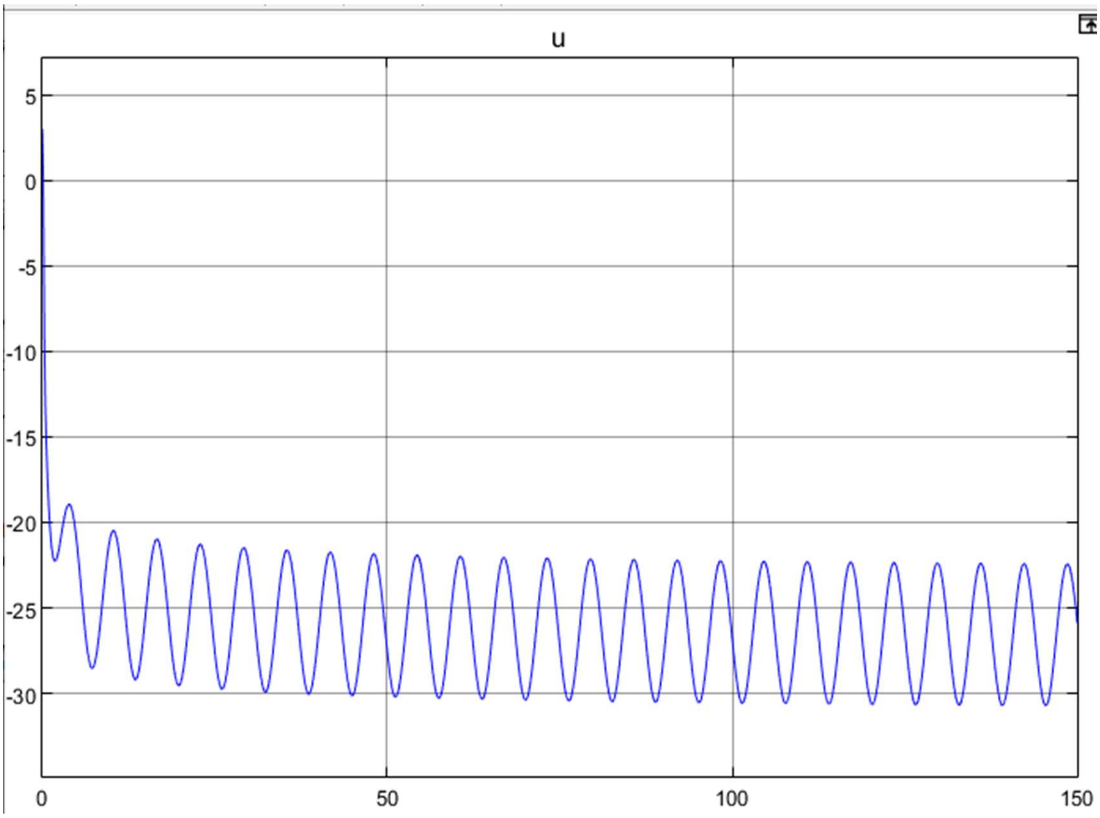

Figure 12： Robust Controller 1 simulation result（$u(t)$，$\gamma = 2$）

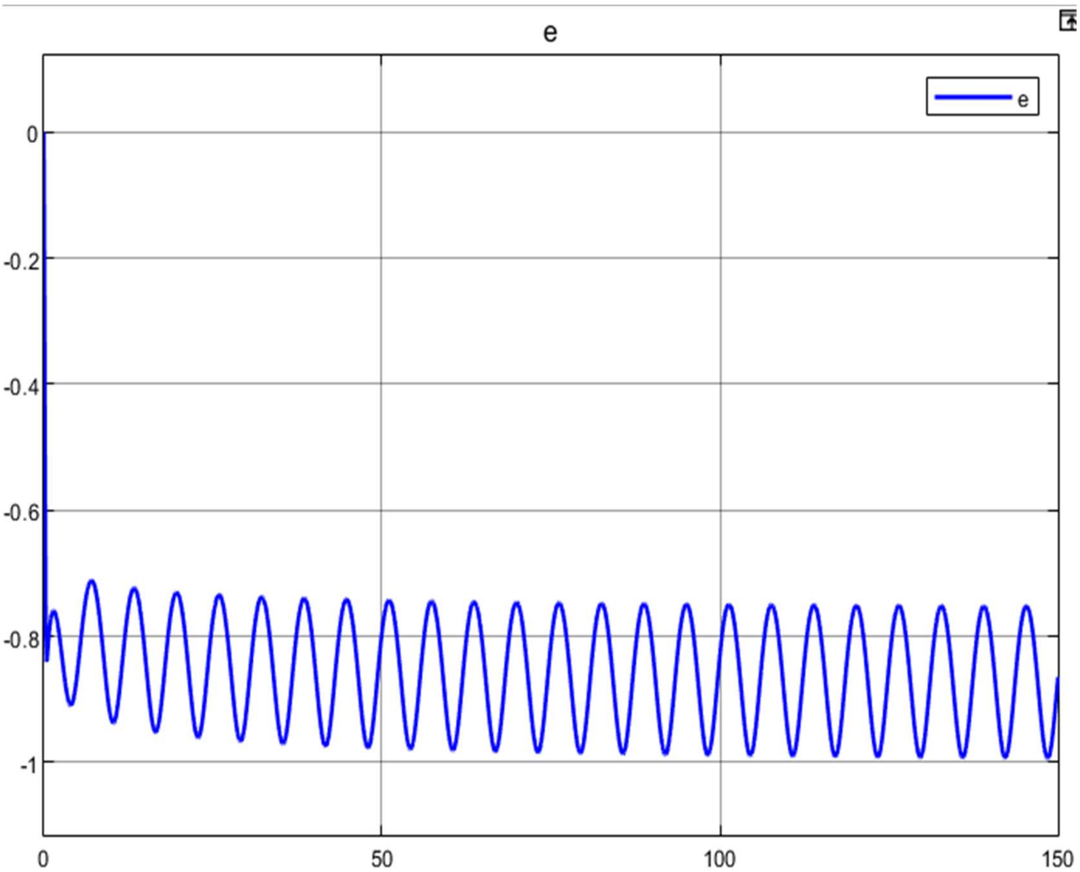

Figure 13： Robust Controller 1 simulation result with small gain（$\epsilon(t)$, $\gamma = 2$）

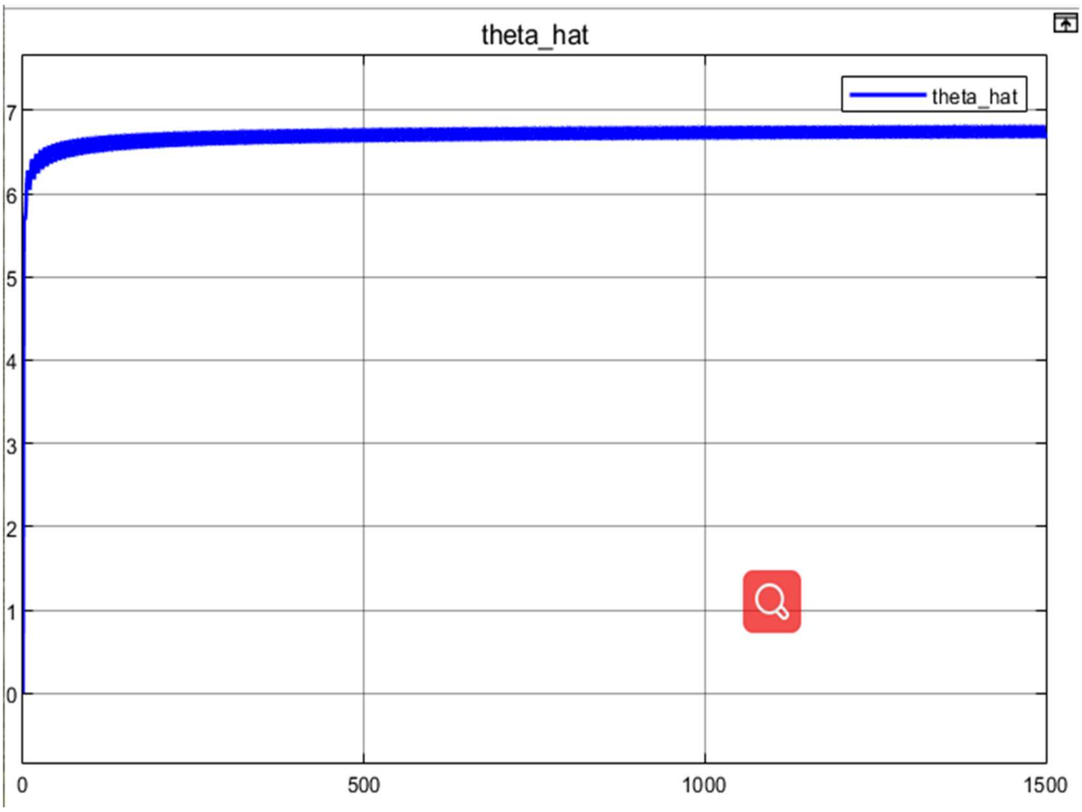

Figure 14： Robust Controller 1 simulation result with small gain（$\hat{\theta}(t)$, $\gamma = 2$）

- **Large gain** $\gamma = 10$

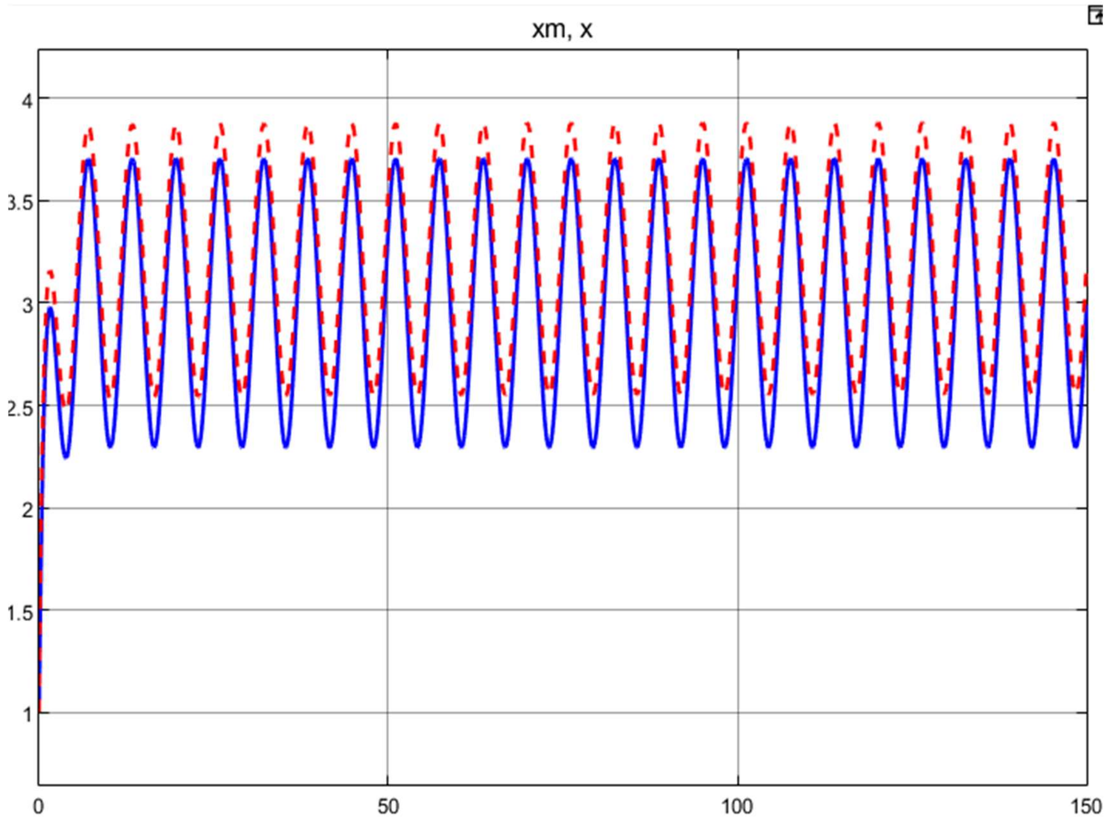

Figure 15： Robust Controller 1 simulation result（$x(t), x_m(t)$， $\gamma = 10$）

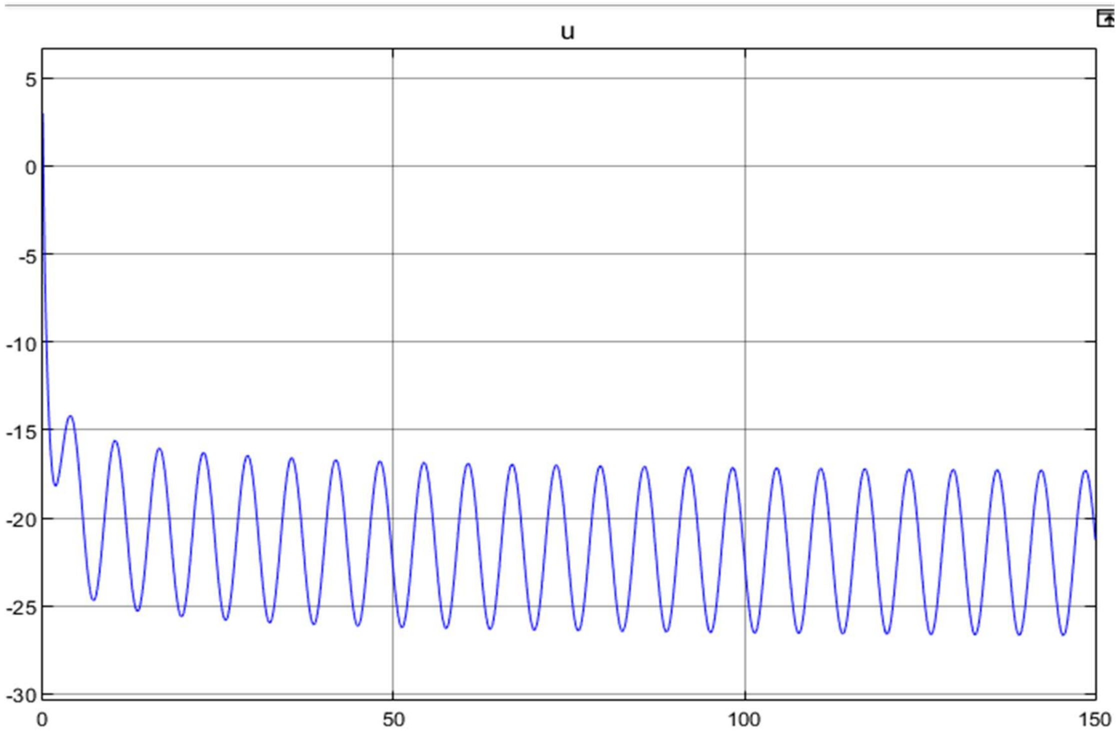

Figure 16： Robust Controller 1 simulation result（$u(t)$， $\gamma = 10$）

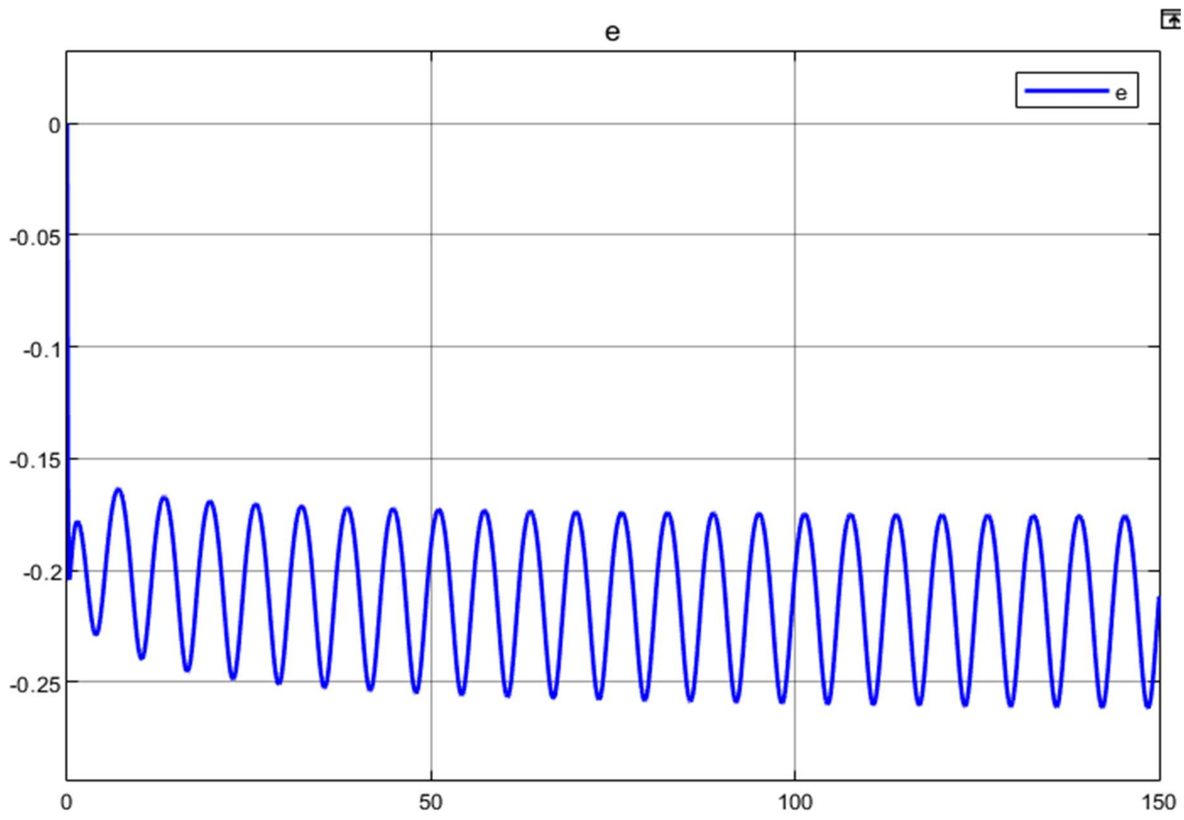

Figure 17： Robust Controller 1 simulation result with small gain （$\epsilon(t)$， $\gamma = 10$）

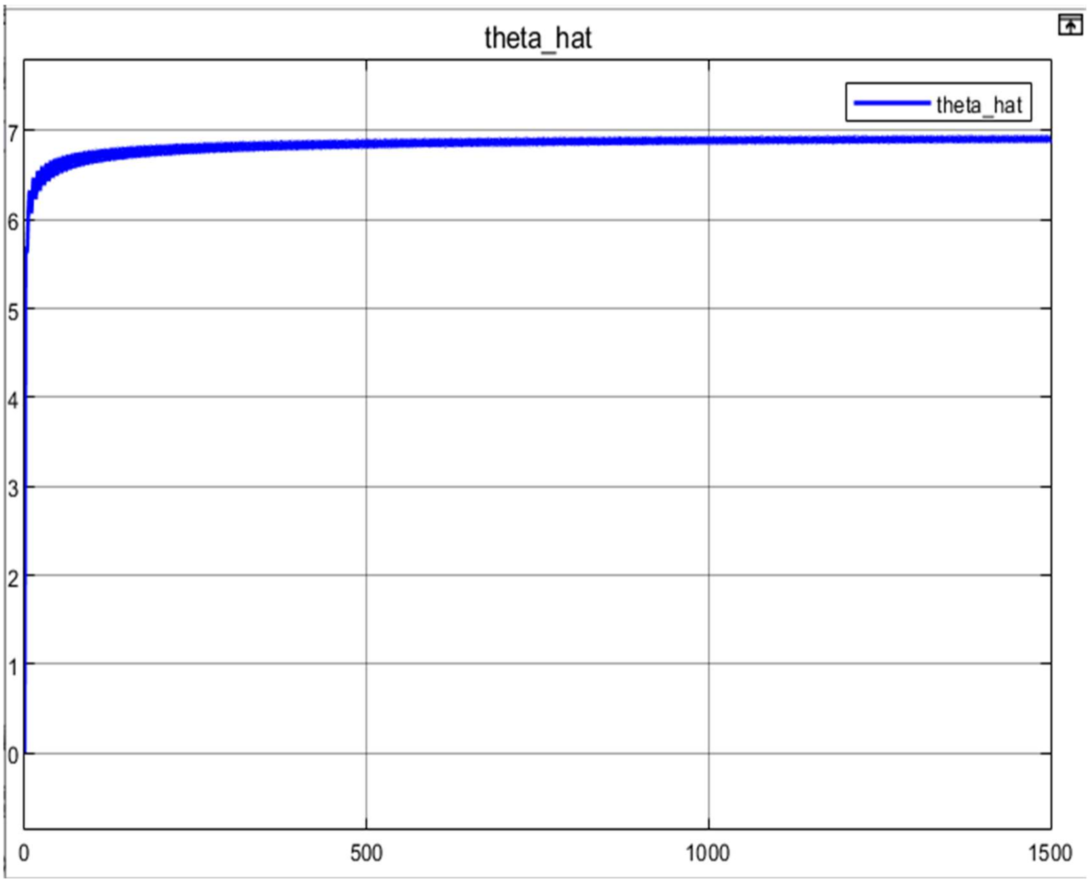

Figure 18： Robust Controller 1 simulation result with small gain （$\hat{\theta}(t)$， $\gamma = 10$）

3. Conclusion

1) Comparing Figs. 7, 9, 11, 13, 15, 17 we can see that:
   a) the error $\epsilon(t)$ at steady state decreases with increasing
   b) But it is clear from the theoretical background that: even in the absence of perturbation, the established error ε(t) can be different from zero, as can be seen from the inequality (2.7);
2) From Figures 10, 14, 18 we can see that:
   a) the parameter $\hat{\theta}(t)$ is bounded in robust controller 1, i.e., we solve the problem of parameter shift in the previous controller

- **Solution N°2 (applying the leakage factor (static linear feedback))**

1. design the Robust Controller 2 (applying the leakage factor (static linear feedback))

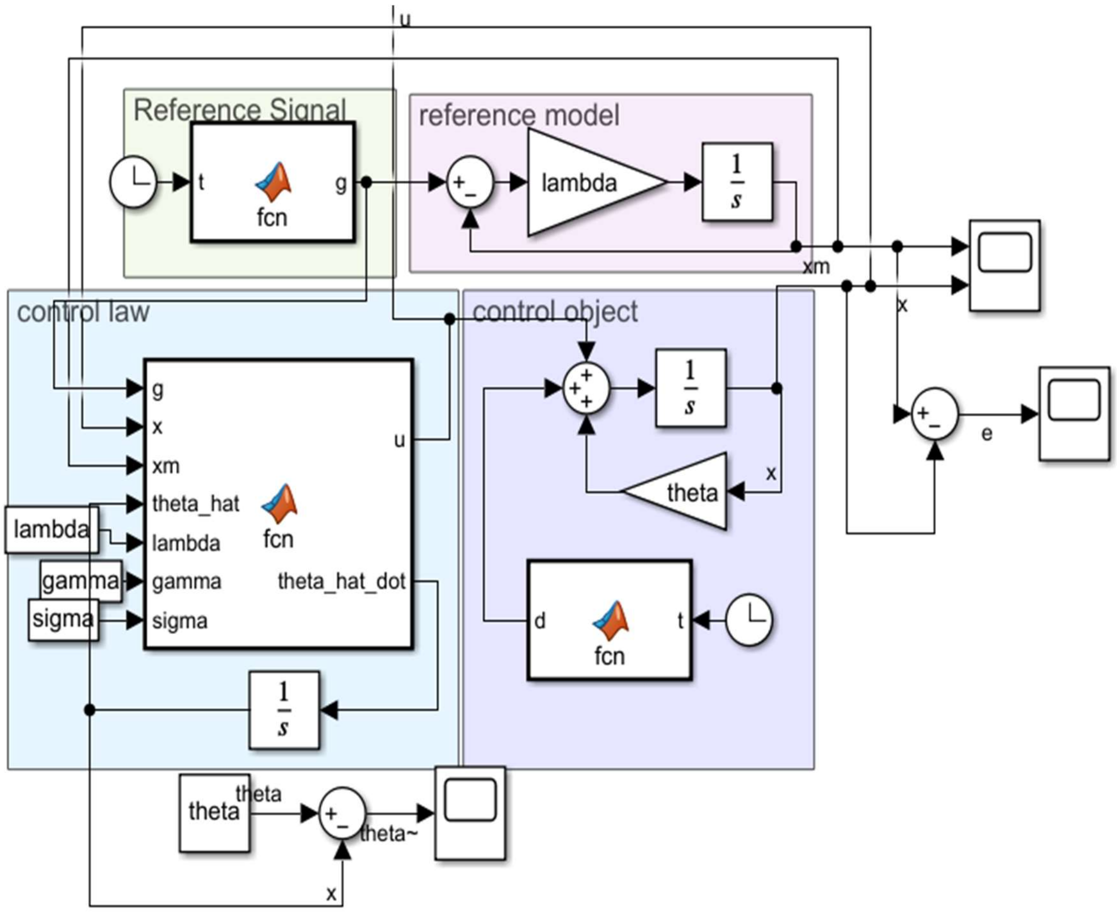

Figure 19: Robust Controller 2 (applying the leakage factor) simulation scheme

2. Simulation

- Small gain $\sigma = 10$

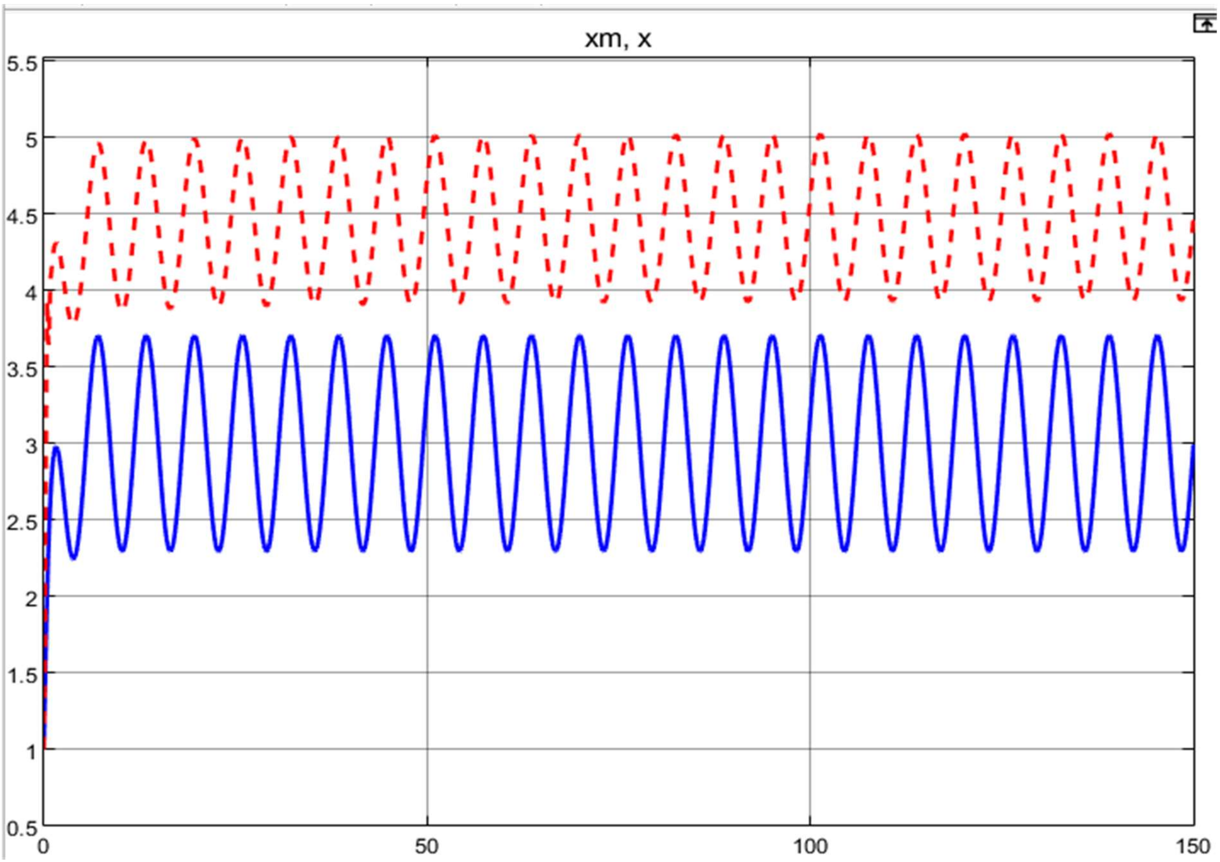

Figure 20： Robust Controller 2 simulation result （$x(t), x_m(t)$， $\gamma = 10$,$\sigma = 10$）

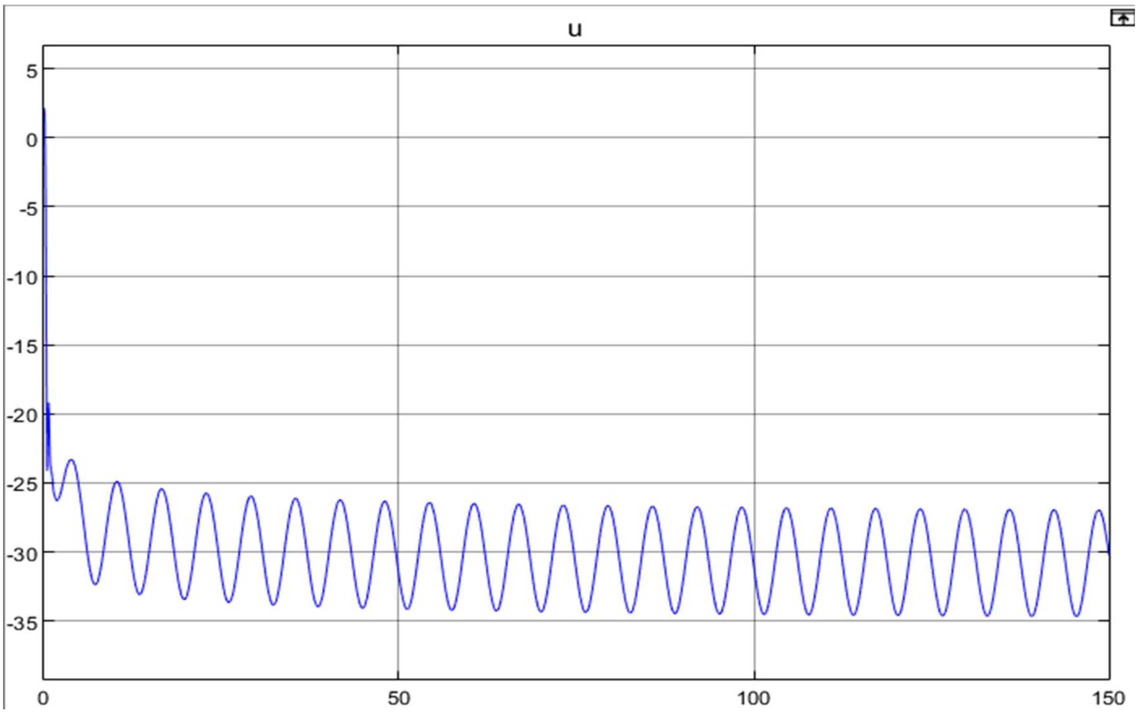

Figure 21： Robust Controller 2 simulation result （$u(t)$， $\gamma = 10$,$\sigma = 10$）

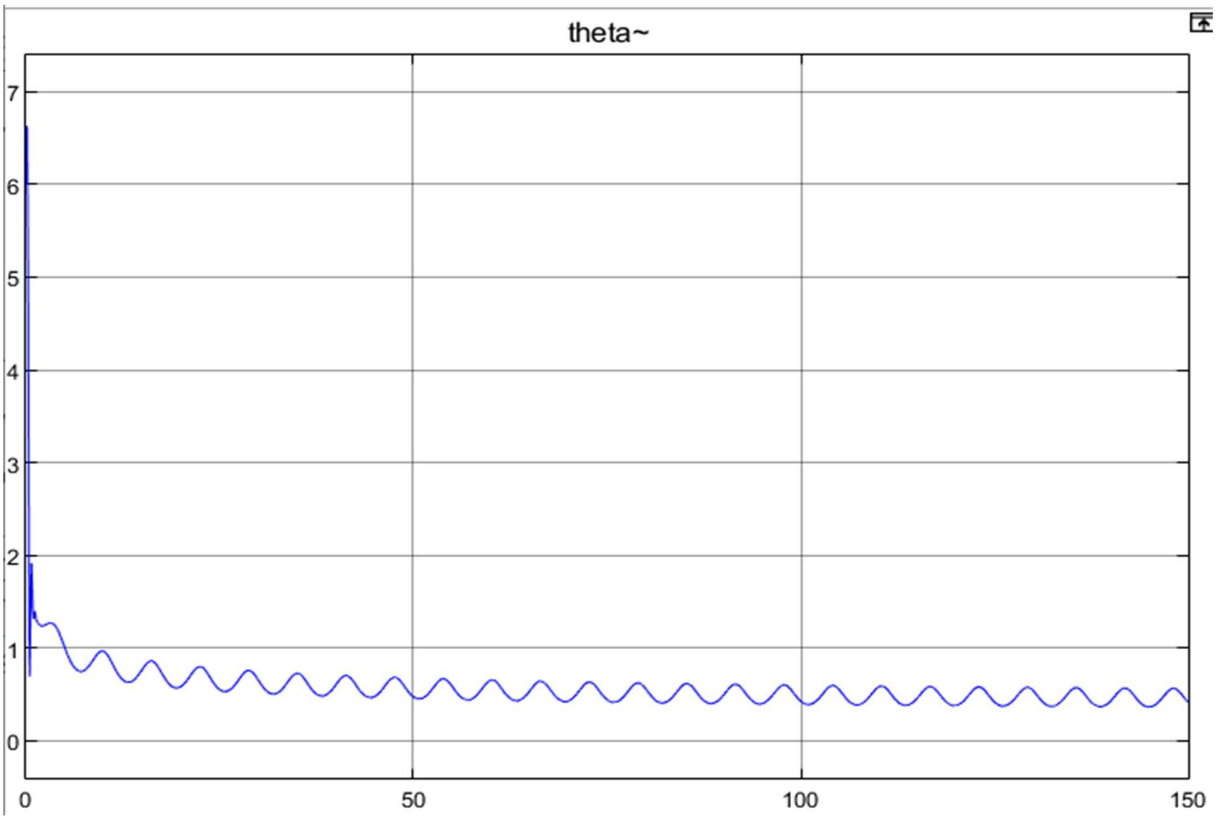

Figure 22: Robust Controller 2 simulation result with small gain ($\tilde{\theta}(t)\gamma = 10$, $\sigma = 10$)

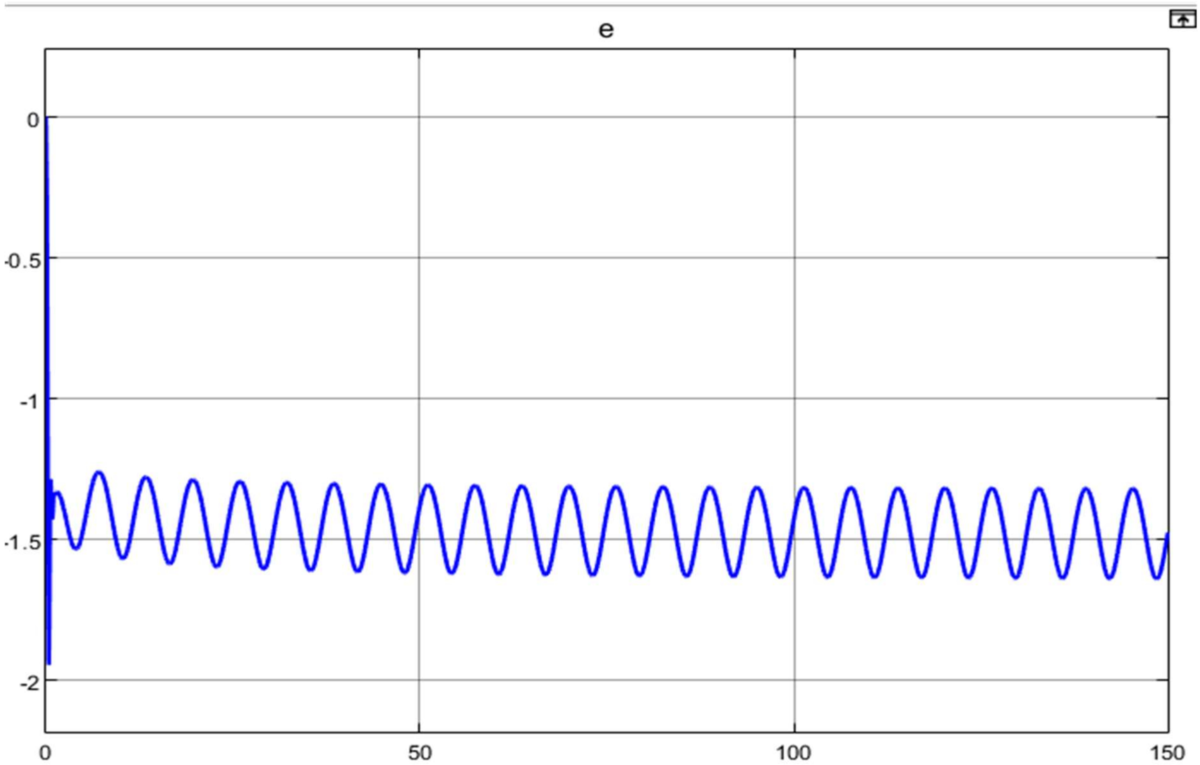

Figure 23 : Robust Controller 2 simulation result with small gain ($\epsilon(t)$, $\gamma = 10$ $\sigma = 10$)

- **Optimal gain** $\sigma = 1$

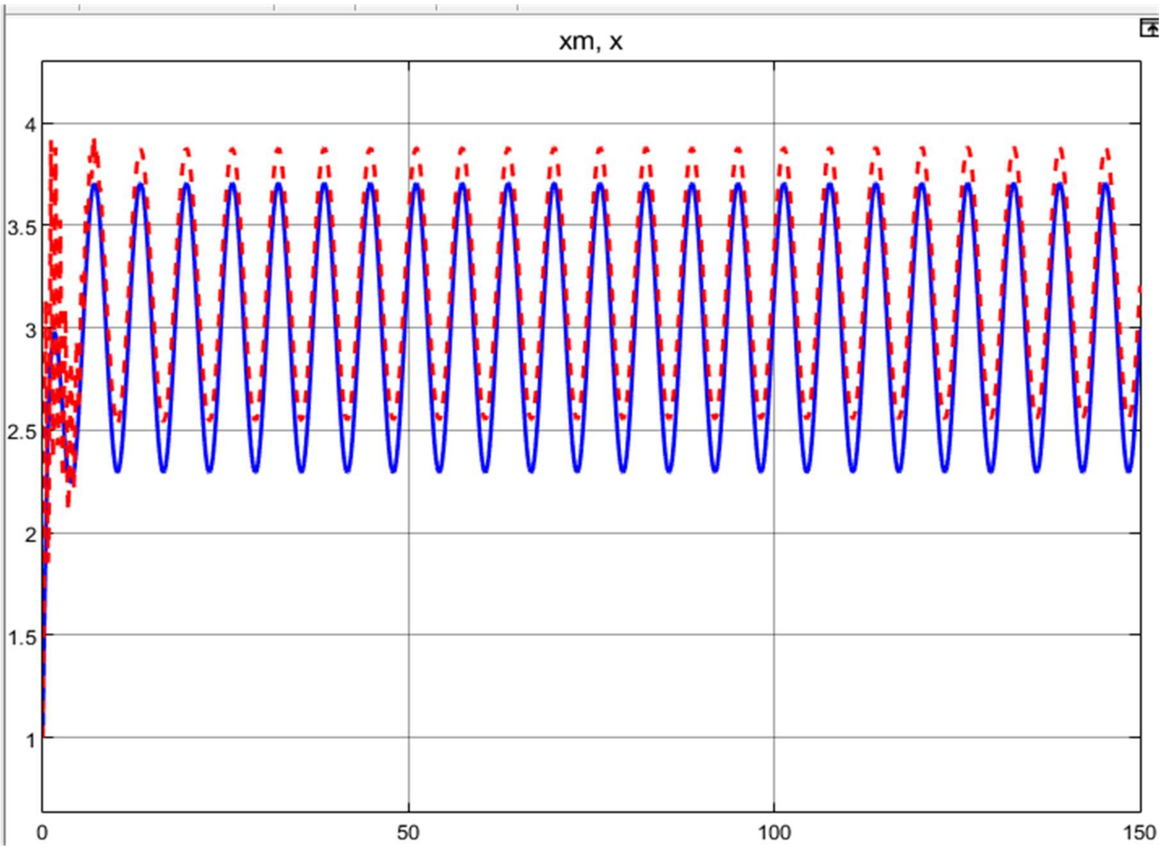

Figure 24： Robust Controller 2 simulation result（$x(t), x_m(t)$， $\gamma = 10$,$\sigma = 1$）

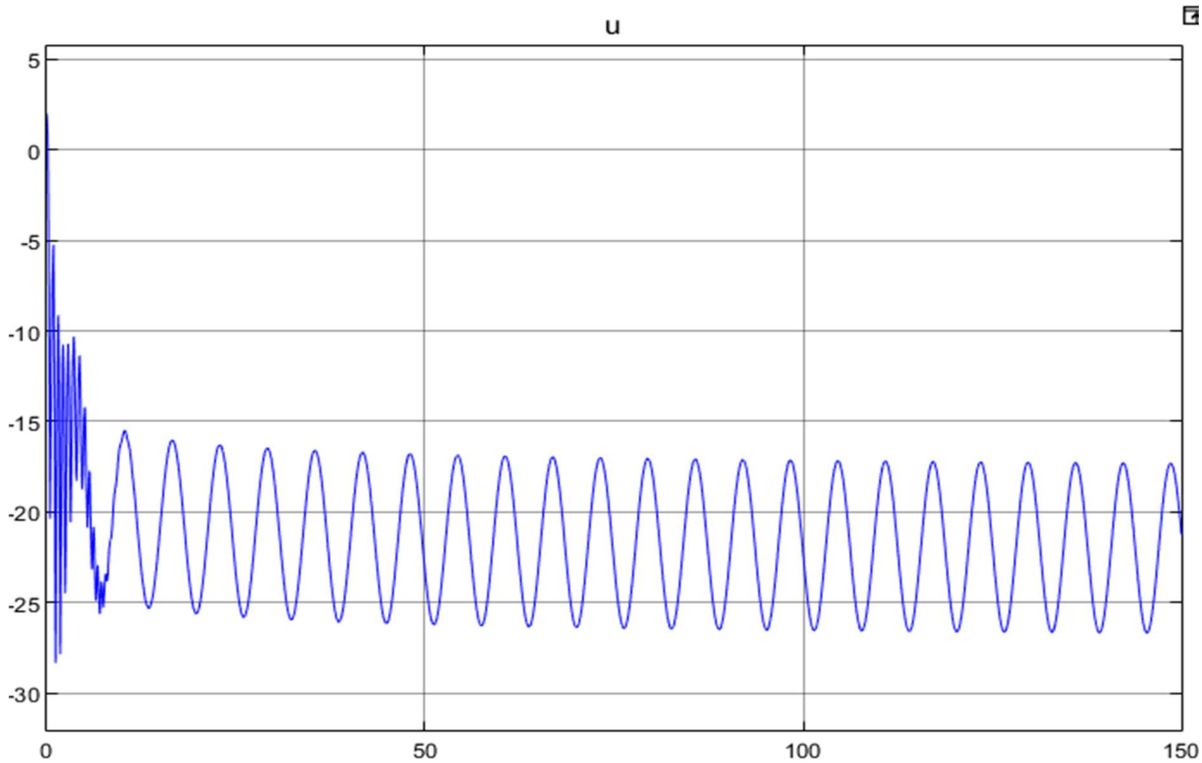

Figure 25： Robust Controller 2 simulation result（$u(t)$， $\gamma = 10$,$\sigma = 1$）

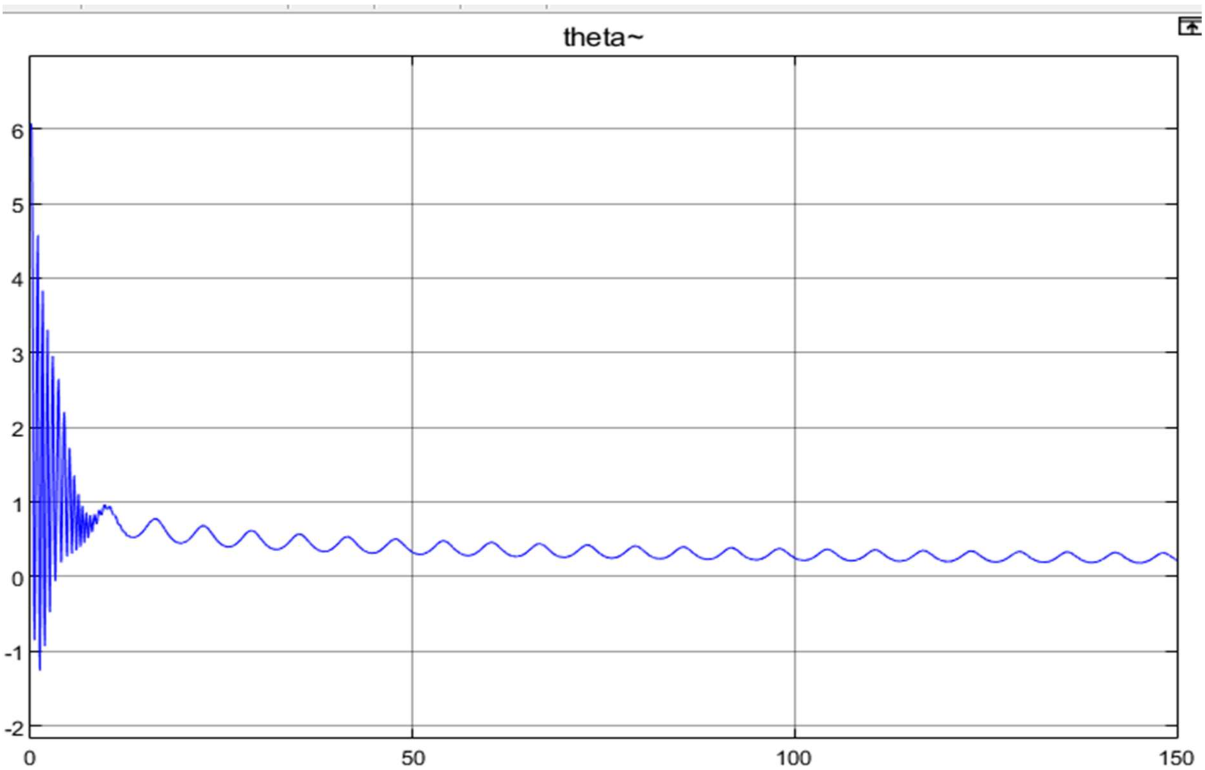

Figure 26: Robust Controller 2 simulation result with small gain ($\tilde{\theta}(t)$, $\gamma = 10$, $\sigma = 1$)

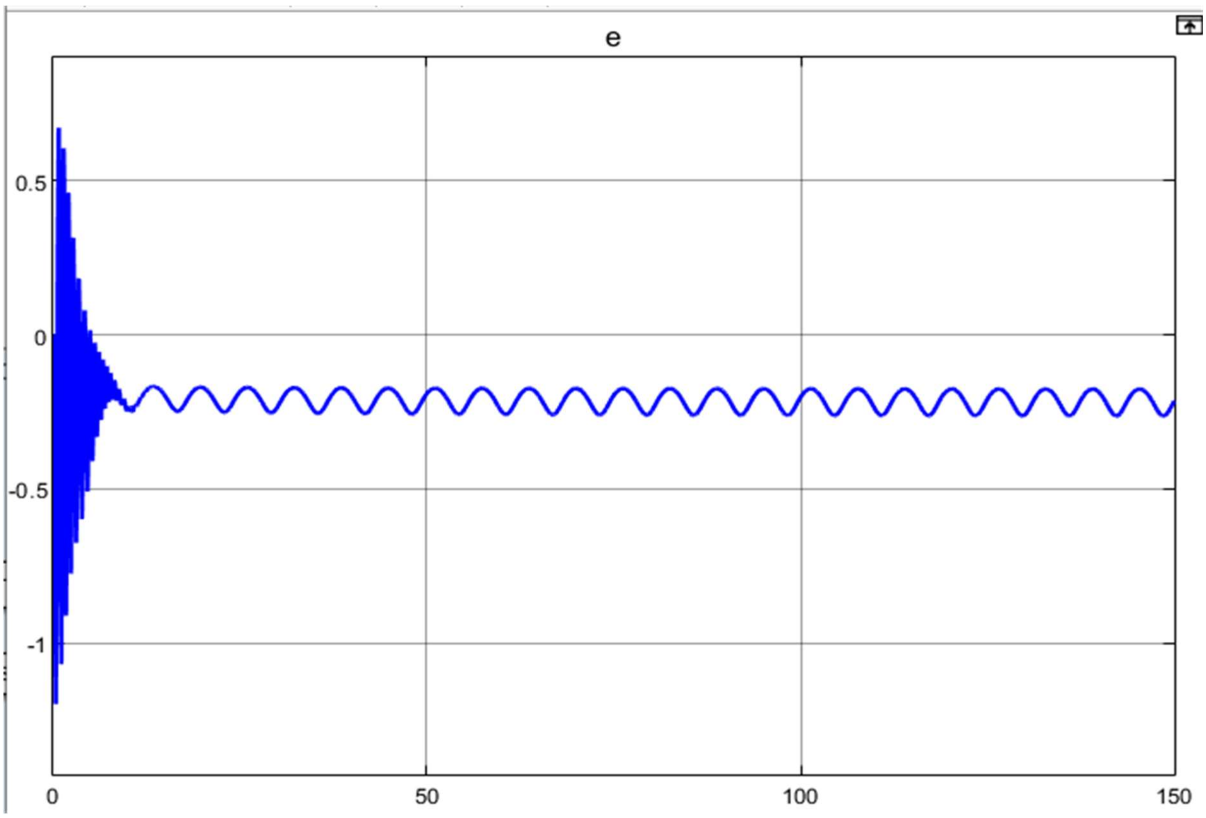

Figure 27 : Robust Controller 2 simulation result with small gain ($\epsilon(t)$, $\gamma = 10$, $\sigma = 1$)

- **Large gain** $\sigma = 0.01$

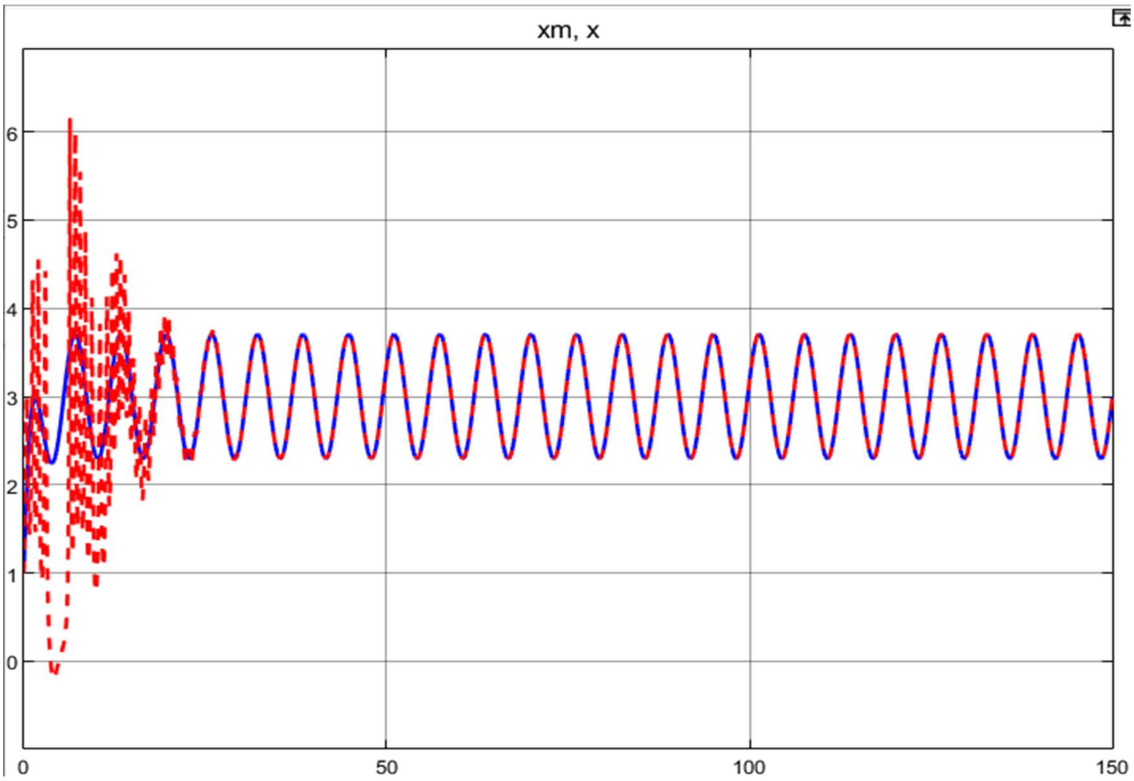

Figure 28： Robust Controller 2 simulation result（$x(t), x_m(t)$， $\gamma = 10$，$\sigma = 0.01$）

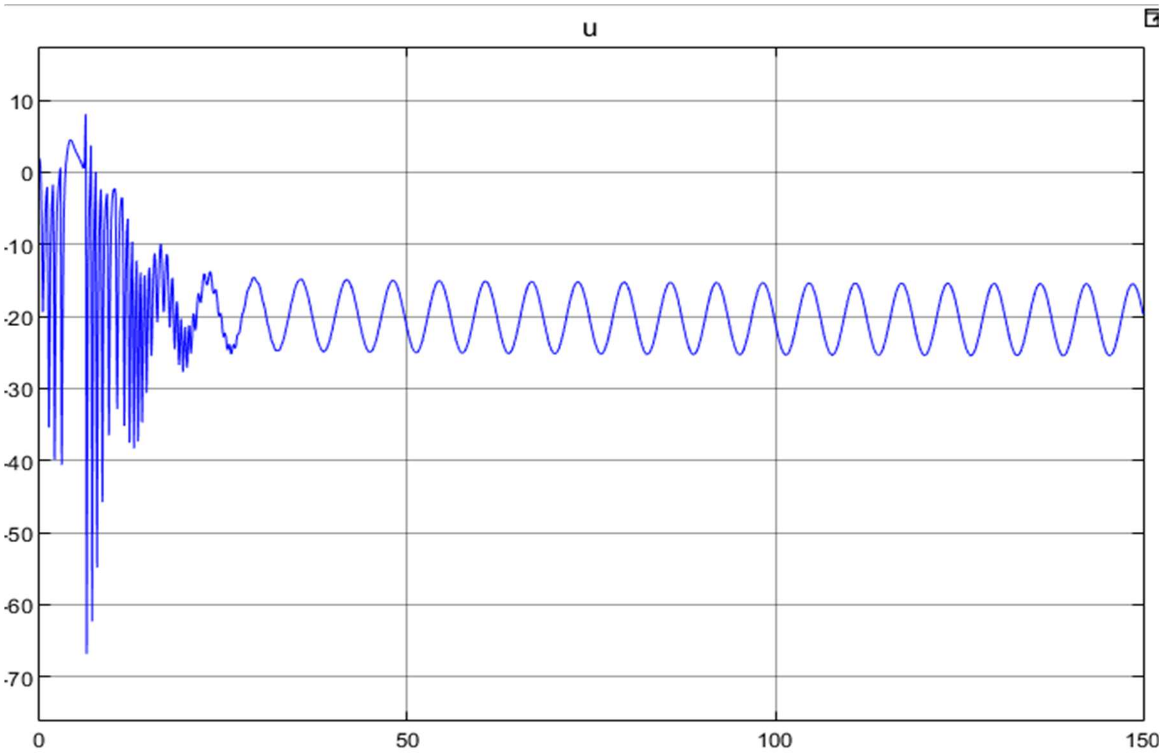

Figure 29： Robust Controller 2 simulation result（$u(t)$， $\gamma = 10$，$\sigma = 0.01$）

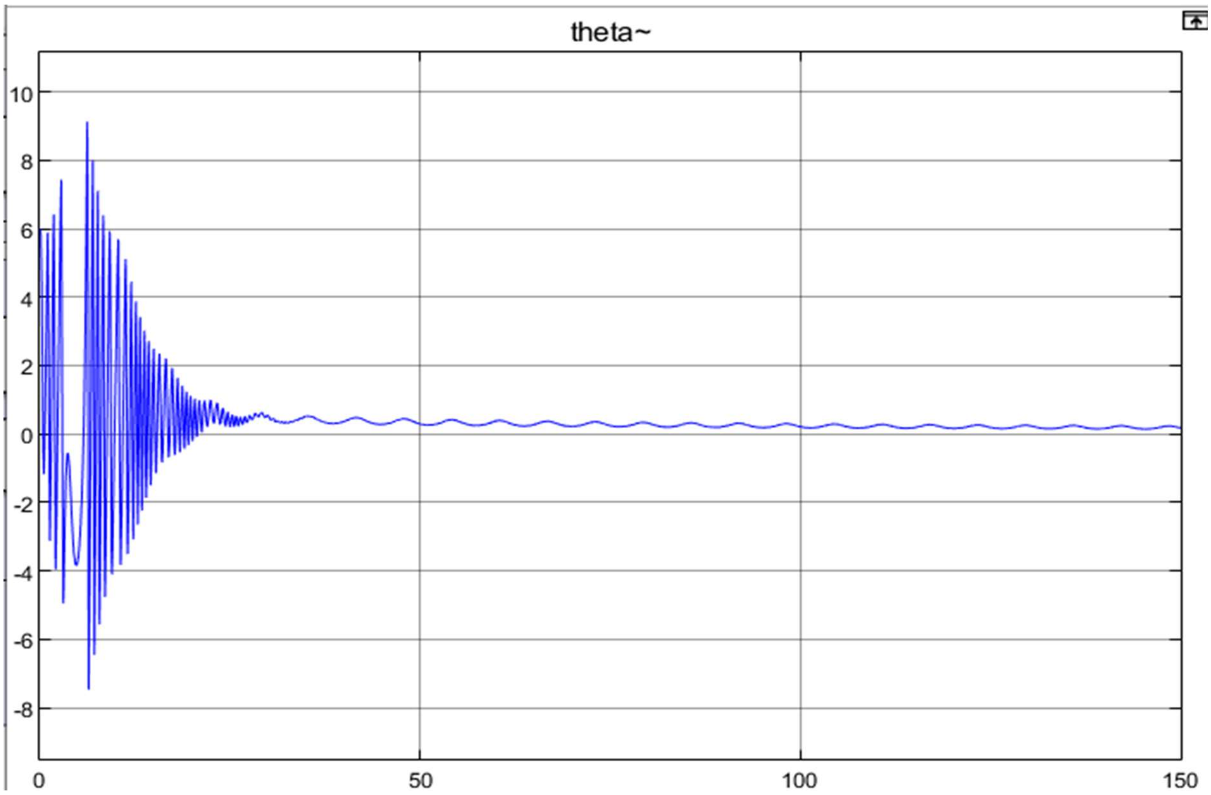

Figure 30: Robust Controller 2 simulation result with small gain ($\tilde{\theta}(t)\gamma = 10$

$\sigma = 0.01$)

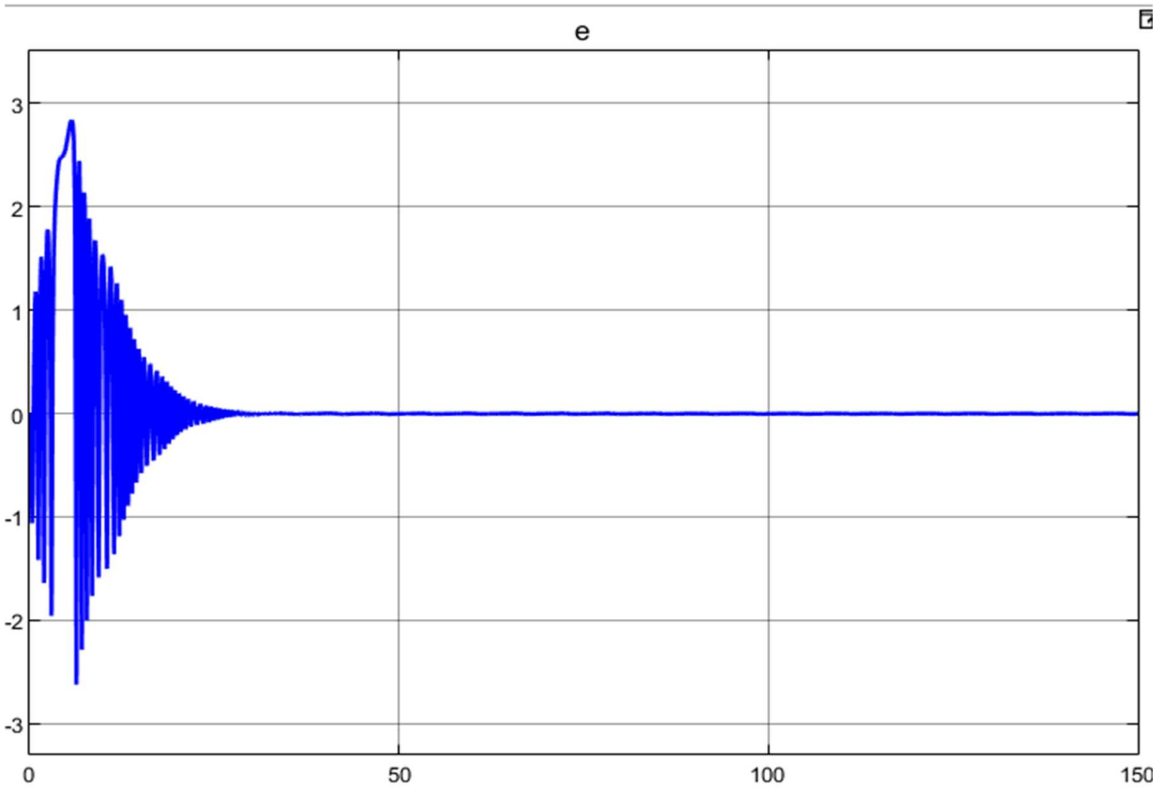

Figure 31 : Robust Controller 2 simulation result with small gain ($\epsilon(t)\gamma = 10$

$\sigma = 0.01$)

3. Conclusion

1) Comparing Figs. 23, 27, 31 we can see that:
   a) the average value of the steady-state value of $\epsilon(t)$ decreases with σ
   b) A larger σ may lead to a larger $\epsilon(t)$;
   c) however, from Figure 31 we can see that a larger σ leads to a more severe oscillation
2) From Figures 22, 23, 26, 27, 30, 31, it can be seen that:
   a) $\hat{\theta}(t)$ and $\epsilon(t)$ converge to zero and are bounded as time tends to infinity under the control of robust controller 2,
   b) and the upper bound on $\epsilon(t)$ can be reduced by decreasing σ

D. Conclusions

- Adaptive but non-robust controllers suffer from parameter shift and loss of robustness in the face of a system with disturbances
- We can build robust controllers in two ways：
  - applying a nonlinear static feedback
    - even in the absence of perturbation, the established error $\epsilon(t)$ can be different from zero
    - the control is proportional to the value x^2. Therefore, when increasing x the control amplitude increases four folds, and therefore the practical applicability of such a law has significant limitations
  - applying the leakage factor (static linear feedback)
    - Also, in order to decrease Δ there is no need to significantly increase γ, which entails an increase in the amplitude of the control action. It is possible to decrease Δ by decreasing σ.