# Practical Assignment № 6

Adaptive and Robust Control

*SYNTHESIS OF AN ADAPTIVE OBSERVER OF THE STATE OF A LINEAR PLANT*

variant number: 16

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# A. Problem statement

Consider the asymptotically stable plant:

$$\begin{aligned} \dot{x} &= Ax + bu,\ x(0) \\ y &= c^\top x \end{aligned}$$

where $x$ is the unmeasurable state vector, $u, y$ is the measurable input and output respectively. Let the plant matrices be presented in canonical (observable) form

$$A = \begin{bmatrix} -a_{n-1} & 1 & \cdots & 0 \\ -a_{n-2} & 0 & & 0 \\ \vdots & & \ddots & 1 \\ -a_0 & 0 & & 0 \end{bmatrix},\ b = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ b_m \\ \vdots \\ b_0 \end{bmatrix},\ c^\top = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}$$

where $a_i, i = \overline{0, n-1}, b_j, j = \overline{0, m}$ are the unknown coefficients.

The problem is to design an observer providing the boundaries of all the signals in the closed-loop system and generating $\hat{x}$ such that under the persistent excitation condition (3.14) the following equality holds

$$\lim_{t \to \infty} \| x(t) - \hat{x}(t) \| = 0.$$

The designed adaptive observer must simultaneously estimate the unknown parameters of the plant $\theta$ and generates $\hat{x}$.

Note that in this problem the class of plant to which (6.1) belongs is limited by the following assumption.

Assumption (matching condition). For some $n$-dimensional vector $\bar{\theta}$ the matrices $A, c^\top$ and $A_0$ are linked via the following equality:

$$A_0 = A - \bar{\theta} c^\top.$$

It can be shown (cf. work №5), that for the plant considered

$$\bar{\theta} = \begin{bmatrix} k_{n-1} - a_{n-1} \\ k_{n-2} - a_{n-2} \\ \vdots \\ k_0 - a_0 \end{bmatrix}.$$

# B. Theoretical background

Problem statement:

Consider the asymptotically stable plant:

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where $a_i, i = \overline{0, n-1}, b_j, j = \overline{0, m}$ are the unknown coefficients.

The problem is to design an observer providing the boundaries of all the signals in the closed-loop system and generating $\hat{x}$ such that under the persistent excitation condition (3.14) the following equality holds

$$\lim_{t\to\infty} \parallel x(t) - \hat{x}(t) \parallel = 0.$$

The designed adaptive observer must simultaneously estimate the unknown parameters of the plant $\theta$ and generates $\hat{x}$.

Note that in this problem the class of plant to which (6.1) belongs is limited by the following assumption.

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Problem solution:

The problem is resolved in two steps. First step is the identification of the plant parameters. The second step is to design the observer in which the unknown parameters will be replaced by their estimates.

Starting from the parameters identification, we parameterize the output variable (5.4) and the state vector (5.8). By replacing the parameters $\theta$ with their estimates $\hat{\theta}$ we design the adjustable model of the plant:

$$\hat{y} = \hat{\theta}^\top \omega,$$

where $\hat{y}$ is the estimate of $y$. We introduce the identification error

$$\varepsilon = y - \hat{y}.$$

By taking into account (5.4) and (6.3) we get:

$$\varepsilon = \tilde{\theta}^\top \omega,$$

where $\tilde{\theta} = \theta - \hat{\theta}$ is the vector of parametric errors. The last equality represents the static error model that can be used to design the following adaptation algorithm based on standard Lyapunov function $V = \tilde{\theta}^\top \tilde{\theta}/2\gamma$

$$\dot{\hat{\theta}} = \gamma \omega \varepsilon,$$

where $\gamma > 0$ is the adaptation gain.

Indeed, calculating $\dot{V}$ yields

$$\dot{V} = \frac{1}{\gamma}\tilde{\theta}^{\top}\dot{\tilde{\theta}} = -\frac{1}{\gamma}\tilde{\theta}^{\top}\dot{\hat{\theta}}.$$

When choosing the structure of the adaptation algorithm (6.5) we get:

$$\dot{V} = -\frac{1}{\gamma}\tilde{\theta}^{\top}\gamma\omega\varepsilon = -\varepsilon^{2} < 0.$$

From the last inequality, provided that the function $\omega$ and its first derivative $\dot{\omega}$ are bounded (the condition is imposed on the input signal $u$ ), follow the next properties of the system, consisting of (6.4), (6.5):

P.1. all signals in the observer are bounded;

P.2. the identification $\varepsilon$ approaches zero asymptotically;

P.3. the parametric error $\tilde{\theta}$ approaches zero exponentially fast, if the vector $\omega$ satisfies the persistent excitation condition (3.14) (see [4,5]). This condition (3.14) depends on the number of harmonics (spectral lines) in the signal $u$;

P.4. if the error $\tilde{\theta}$ converges towards zero, then the state vector $\hat{x}$ estimation also converges towards $x$.

After replacing in (5.8) the unknown parameters $\theta$ with their estimates $\hat{\theta}$ generated by we obtain the adjustable observer:

$$\hat{x} = \sum_{i=0}^{n-1} \hat{\theta}_{i+1}(sI - A_0)^{-1}e_{n-i}[y] + \sum_{j=0}^{m} \hat{\theta}_{j+1+n}(sI - A_0)^{-1}e_{m-j}[u].$$

Thus, the adaptive observer, which ensures that the condition (6.2) if met (if the condition of persistent excitation is met (3.14)), consists of the adjustable parameterized model (6.2), adaptation algorithm (6.5) and an algorithm for evaluating the state vector (6.6).

## C. Experimental part

- Experimental parameters (Group 16)

| Var. | Coefficients of the plant's transfer function ($n = 2$) | | | | Filter coefficients | |
|---|---|---|---|---|---|---|
| | $a_1$ | $a_0$ | $b_1$ | $b_0$ | $k_1$ | $k_0$ |
| 16 | 1 | 4 | 3 | 4 | $\sqrt{2}$ | 1 |

1. Based on the result obtained in work N°5, build and simulate adaptive observer for the state vector of the plant (6.3), (6.5), (6.6). The adaptation gain $\gamma$ is found experimentally. Plot two graphs. On the first display the state variables of the norm $\| x - \hat{x} \| = \sqrt{(x-\hat{x})^\top (x-\hat{x})}$. On the second — the parametric error $\tilde{\theta}$.

- simulation scheme: $(\gamma = 10)$

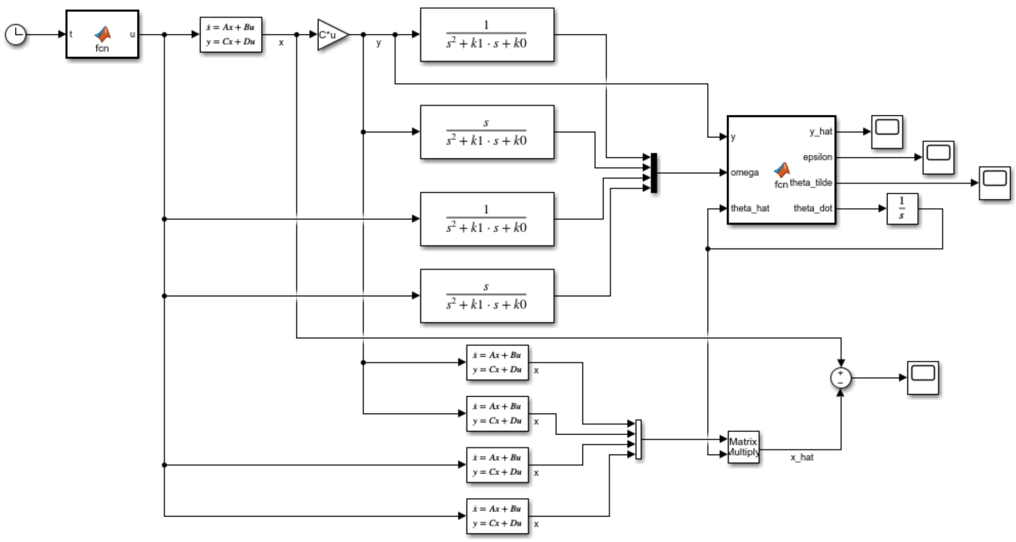


*Figure 1: simulation scheme of adaptive observer*

- simulation result:

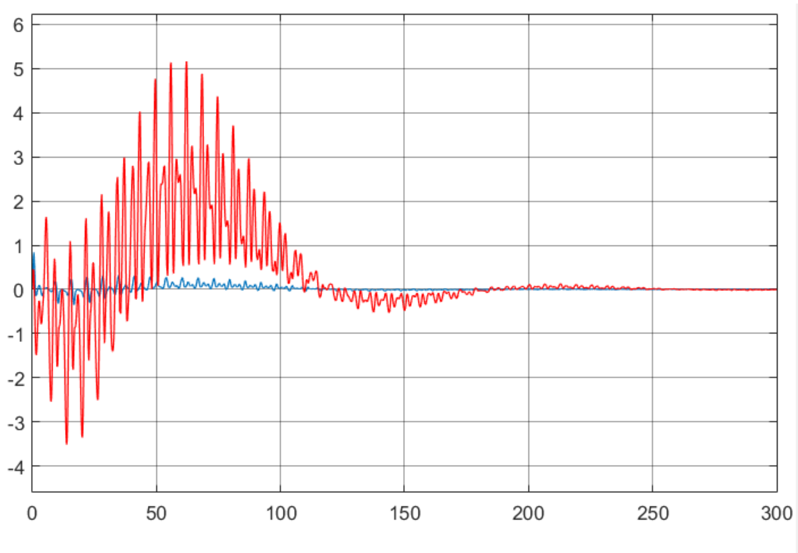

*Figure 2: state variables of the norm* $\| x-\hat{x} \| = \sqrt{(x-\hat{x})^{\top}(x-\hat{x})}$

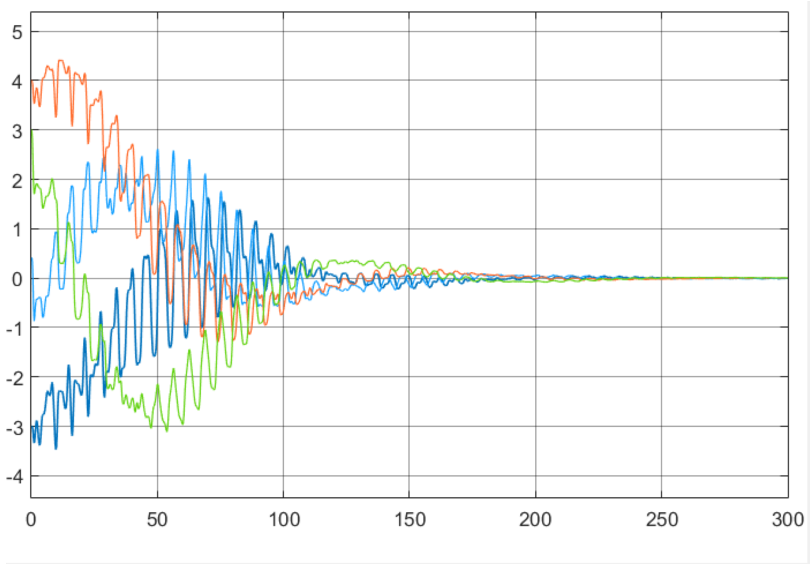


*Figure 3: Parametric errors* $\tilde{\theta}$

## 2. Repeat the experiment for $u = 10\sin\ t + 5\cos\ 2t + 4\cos\ 4t + 3\cos\ 8t$.

- simulation result:

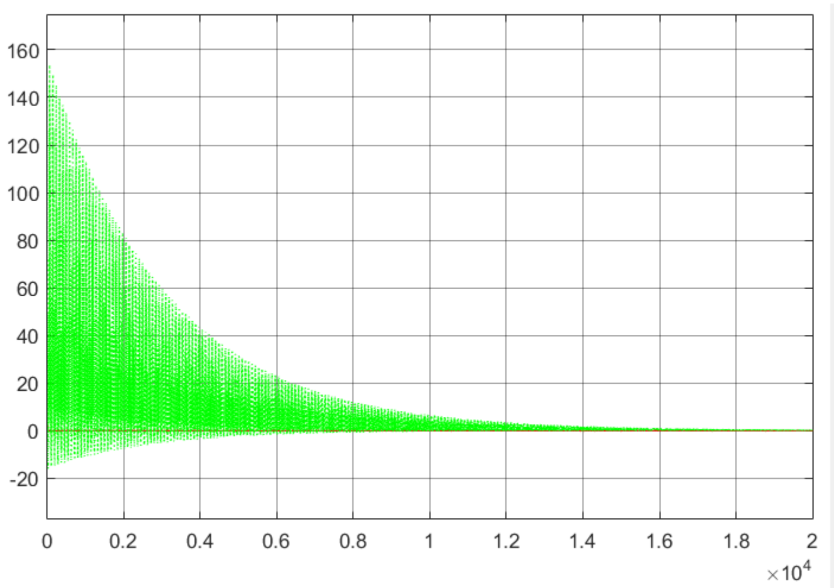


*Figure 4: State variables of the norm* $\| x-\hat{x} \| = \sqrt{(x-\hat{x})^{\top}(x-\hat{x})}$

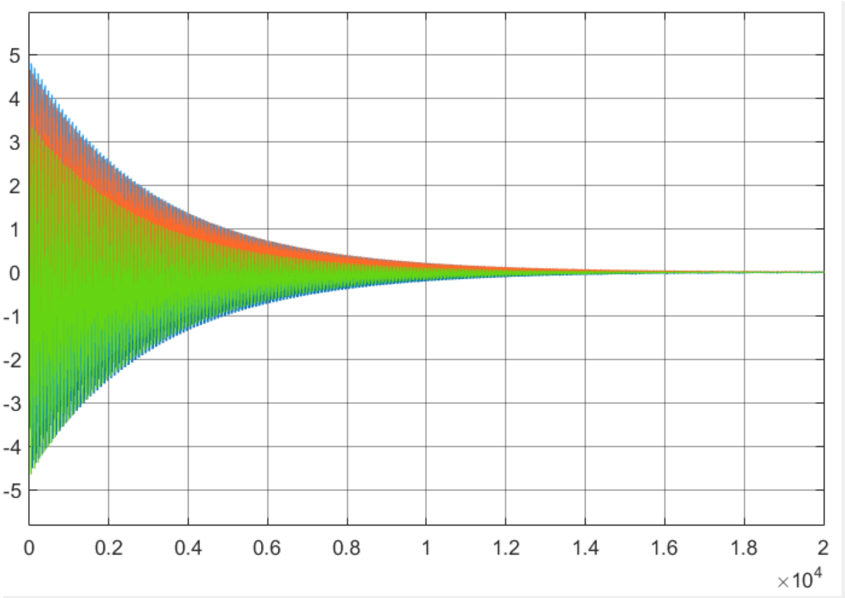


*Figure 5: Parametric errors $\tilde{\theta}$*

## 3. Draw conclusions from the simulation results.

- All signals in the observer are bounded and identification $\epsilon$ approaches to zero asymptotically.
- The parametric errors $\tilde{\theta}$ converges towards zero, and the state vector $\hat{x}$ also converges towards $x$ which is showed by the norm $\| x - \hat{x} \|$ converged towards 0.
- After increasing the number of harmonics in the input signal $u$, the convergence speed of parametric error $\tilde{\theta}$ becomes significantly slow.