Practical Assignment № 3

Adaptive and Robust Control

Design of adaptive control systems for undisturbed objects

variant number : 16

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A.Problem statement

Consider the plant presented in the "state-space" form

$$\dot{x} = Ax + bu,\ x(0) \tag{3.1}$$

$$y = c^T x \tag{3.2}$$

where $x \in R^n$ is the state vector, $u \in R$ is the control variable, $y \in R$ is the regulated variable,

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix}, b = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ b_0 \end{bmatrix}, c = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$a_i, i = \overline{0, n-1}$ are unknown parameters, b_0 is the coefficient which is known for the sake of simplicity.

B.Theoretical background

Reference Model

The problem is to compensate for the parametric uncertainty and provide the limiting equality

$$\lim_{t\to\infty} \|x_M(t) - x(t)\| = \lim_{t\to\infty} \| e(t) \| = 0, \tag{3.3}$$

where $e = x_M - x$ is the vector of control errors, $x_M \in R^n$ is the state vector of reference model

$$\dot{x}_M = A_M x_M + b_M g, \tag{3.4}$$

$$y_M = c_M^T x_M \tag{3.5}$$

with piece-wise continuous reference signal $g(t)$ and matrices

$$A_M = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_{M0} & -a_{M1} & -a_{M2} & \cdots & -a_{Mn-1} \end{bmatrix}, b_M = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ a_{M0} \end{bmatrix}, c_M = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

The parameters $a_{Mi}, i = \overline{1, n-1}$ are calculated based on the poleplacement approach [1,2] to provide the desired transient performance for the closed-loop system after adaptation process.

● NonAdaptive Controller

Note that in general case, the class of plants is restricted by the following assumption.

Assumption (Matching conditions). There is an n-dimensional vector θ and a scalar κ such that the matrices A, b, A_M and b_M are related via equalities

$$A_M = A + b\theta^T,\ b = \kappa b_M \tag{3.6}$$

Called matching conditions.

This assumption permits to apply certainty equivalence principle and design an error model used for derivation of an adaptation algorithm.

Problem solution. Assuming the plant parameters known we design a control providing condition (3.3) with desired transient performance determined by transient time t_n and the overshooting $\bar{\sigma}$.

To this end we define the control error as $e = x_M - x$ and calculate its time derivative in view of $(3.1), (3.4)$ and (3.6) :

$$\begin{aligned} \dot{e} &= \dot{x}_M - \dot{x} = A_M x_M + b_M g - Ax - bu = \\ &= \underline{A_M x_M} + \frac{1}{\kappa}\underline{\underline{b}} - \left(\underline{A_M} - \underline{\underline{b}}\theta^T\right)x - \underline{\underline{b}}u = A_M e + b\left(\theta^T x - u + \frac{1}{\kappa}g\right), \end{aligned} \tag{3.7}$$

where $\theta^T = [\theta_1, \theta_2, \dots, \theta_n]$ is the vector of constant parameters defined by the difference between matrices A and A_M :

$$\theta_1 = \frac{-a_{M0} + a_0}{b_0}, \theta_2 = \frac{-a_{M1} + a_1}{b_0}, \dots, \theta_n = \frac{-a_{Mn-1} + a_{n-1}}{b_0}, \kappa = \frac{b_0}{a_{M0}}.$$

Expression (3.7) is reduced to the equality

$$\dot{e} = A_M e + b\left(\theta^T x - u + \frac{1}{\kappa}g\right) \tag{3.8}$$

which can be used for design of nonadaptive control

$$u = \theta^T x + \frac{1}{\kappa}g. \tag{3.9}$$

Adaptive Controller

Replacing (3.9) in (3.8) we obtain the control error model ensuring exponential convergence of the error e to zero:

$$\dot{e} = A_M e.$$

Since initially the parameters of the matrix A are unknown, the control law (3.9) is not implementable. Therefore replace the unknown parameters θ by their estimates $\hat{\theta}$ in (3.9) and obtain the adjustable control law

$$u = \hat{\theta}^T x + \frac{1}{\kappa} g. \tag{3.10}$$

Substitute the last equality in (3.8) and get the control error model

$$\dot{e} = A_M e + b\tilde{\theta}^T x, \tag{3.11}$$

where $\tilde{\theta} = \theta - \hat{\theta}$ is the vector of parametric errors.

Generalizing solution presented in the work #1 for multivariable case we select the Lyapunov function candidate

$$V = \frac{1}{2} e^T P e + \frac{1}{2\gamma} \tilde{\theta}^T \tilde{\theta},$$

where $P = P^T \succ 0$ is the solution of the Lyapunov equation

$$A_M^T P + P A_M = -Q \tag{3.12}$$

with preselected positively defined constant matrix Q. Calculating time derivative of V in view of (3.11) we obtain:

$$\dot{V} = -\frac{1}{2} e^T Q e + \tilde{\theta}^T x b^T P e + \frac{1}{\gamma} \tilde{\theta}^T \dot{\tilde{\theta}}.$$

Now, if we select the following structure for the adaptation algorithm

$$\dot{\hat{\theta}} = \gamma x b^T P e,\ \hat{\theta}(0) = 0 \tag{3.13}$$

then the Lyapunov function derivative will be nonpositive, i.e.

$$\dot{V} = -\frac{1}{2} e^T Q e \le 0.$$

As a result, the choice (3.13) ensures the objective (3.3).

Note that the coefficient $\gamma > 0$ in (3.13) is called adaptation gain responsible for the rate of parametric convergence (3.10).

Thus the algorithm of adaptive control consists of adjustable control law (3.10) and adaptation algorithm (3.13), in which the matrix P is calculated using (3.12).

For any initial conditions in the plant and the adaptation algorithm $x(0), \hat{\theta}(0)$ and any bounded g adaptive control (3.10), (3.13) ensures $[6,7]$:
C.1. the boundedness of all the signals in the closed-loop system;

C. 2. asymptotic convergence of the control error e to zero;

C.3. exponential convergence of the vector $\hat{\theta}$ to the vector θ if x satisfies the persistent excitation condition

$$\int_{t}^{t+T} x(\tau)x^{T}(\tau)d\tau > \alpha I \qquad (3.14)$$

for some constant values $\alpha > 0, T > 0$.

Condition (3.14) is equivalent to existence at least $(n+1)/2$ harmonics (spectral lines) in the vector x. Note that in the framework of this problem the frequency properties of x are dependent from the external signal g. As a result the persistent excitation condition can be reformulated in terms of signal g;
C.4. if the vector x satisfies the condition (3.14), then there is an optimal coefficient γ, for which the rate of parametric convergence of $\tilde{\theta}$ to zero is maximum.

C. Experimental part

● Experimental parameters（Group 16）

Var.	Matrix A	Transmission coef. b_0	Transient time, t_{tr}	Overshoot $\bar{\sigma}$,%	Input signal $g(t)$
16	$\begin{bmatrix} 0 & 1 \\ -7 & 6 \end{bmatrix}$	7	0.5	15	$3sign\left(\cos 0.2t\right)+3$

➢ state of the system

From the previous theoretical background we can obtain the equation of state of the system as follows：

$$\begin{aligned} \dot{x} &= Ax + bu \\ y &= c^{\top} x \end{aligned} \tag{4.1}$$

where $x \in R^n$ is the state vector, u is the control, $y \in R$ is the adjustable variable,

➢ Reference Model

1. design the Reference Model

The maximum overshooting is not equal to zero, So we use Butterworth's polynomial

$$s^2 + \sqrt{2}\omega_0 s + \omega_0^2$$

where ω_0 is the radius of the poles distribution calculated as

$$\omega_0 = \frac{t_n'}{t_n} = \frac{4.5}{0.5} = 9$$

So we can get the matrix A_M and the vector $b_M(n=2)$ of the reference model:

$$A_M = \begin{bmatrix} 0 & 1 \\ -a_{M0} & -a_{M1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & -\sqrt{2}\omega_0 \end{bmatrix},$$

$$b_M = \begin{bmatrix} 0 \\ a_{M0} \end{bmatrix} = \begin{bmatrix} 0 \\ \omega_0^2 \end{bmatrix}, c_M = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

In order to verify the accuracy of the reference model, we build the simulation schema as shown below：

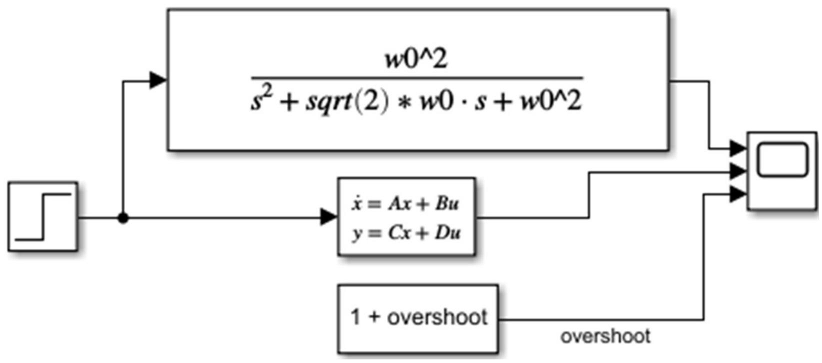

Figure 1 ： Reference Model simulation scheme

2. Simulation

Make a simulation experiment, The simulation results are shown in the following figure:

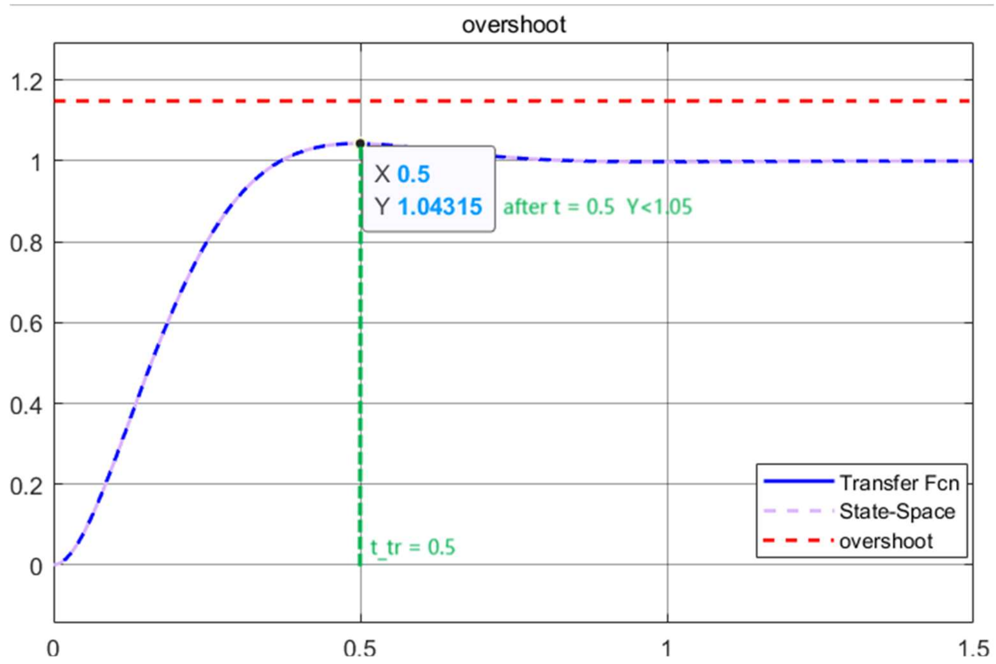

Figure 2 : Reference Model simulation result (with overshooting and transient time)

3. Conclusion

As can be seen in Figure 2, the output of our reference model at step input satisfies the requirements in Table 1 ($t_{tr} = 0.5, \sigma < 15\%$), which proves that our reference model is calculated correctly.

➢ closed-loop system with non-adaptive control(plant parameters are known)

1. design the NonAdaptive Controller

From the theoretical background we can obtain the nonadaptive controller as follows:

$$\begin{aligned}
&\dot{x}_M = A_M x_M + b_M g, \\
&y_M = c_M^\top x_M \\
&A_M = A + b\theta^\top, b = \kappa b_M \\
&\theta^\top = \left[\frac{-a_{M0} + a_0}{b_0}, \frac{-a_{M1} + a_1}{b_0}\right] \\
&u = \theta^\top x + \frac{1}{\kappa} g
\end{aligned} \tag{4.2}$$

The simulink scheme is constructed from (4.1) and (4.2) as follows:

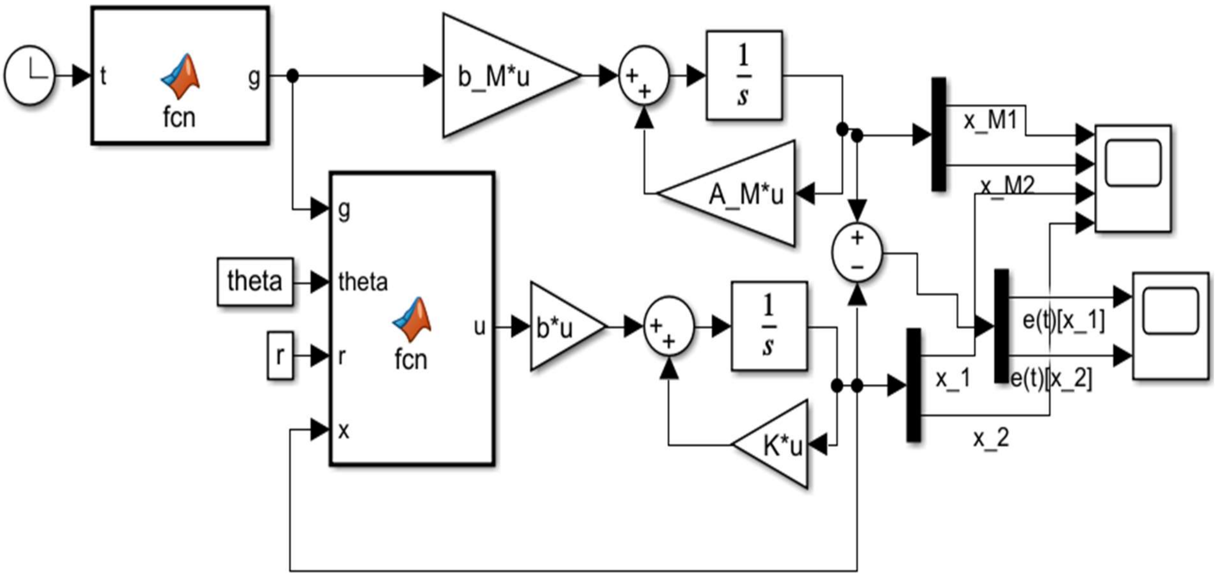

Figure 3: NonAdaptive Controller simulation scheme

2. Simulation

Make three experiments with three sets of plant parameters. The simulation results are shown in the following figure:

1) a_0 and a_1

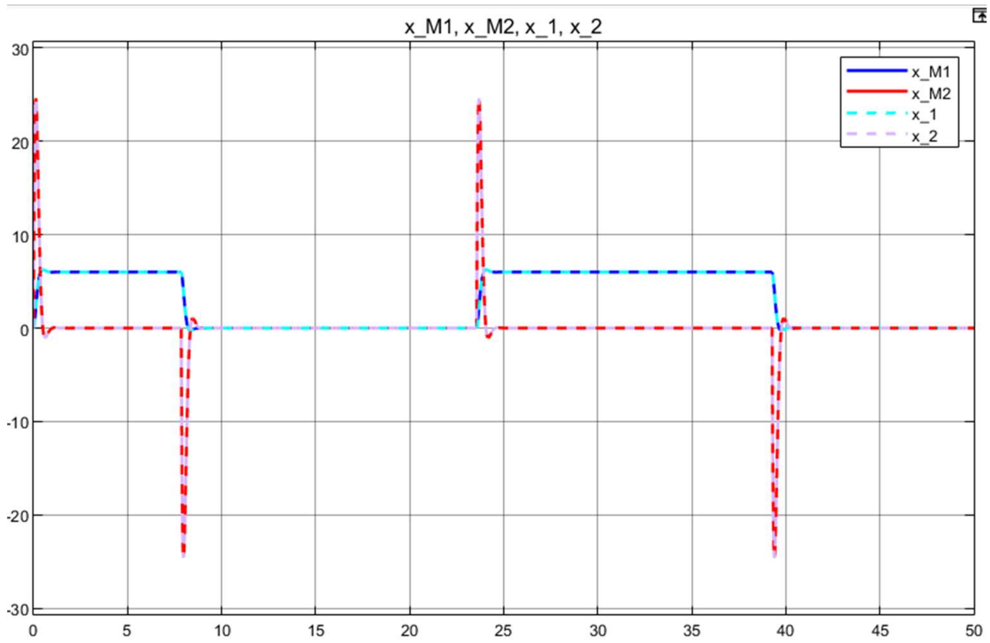

Figure 4： NonAdaptive Controller simulation result（$x(t), x_m(t)$）

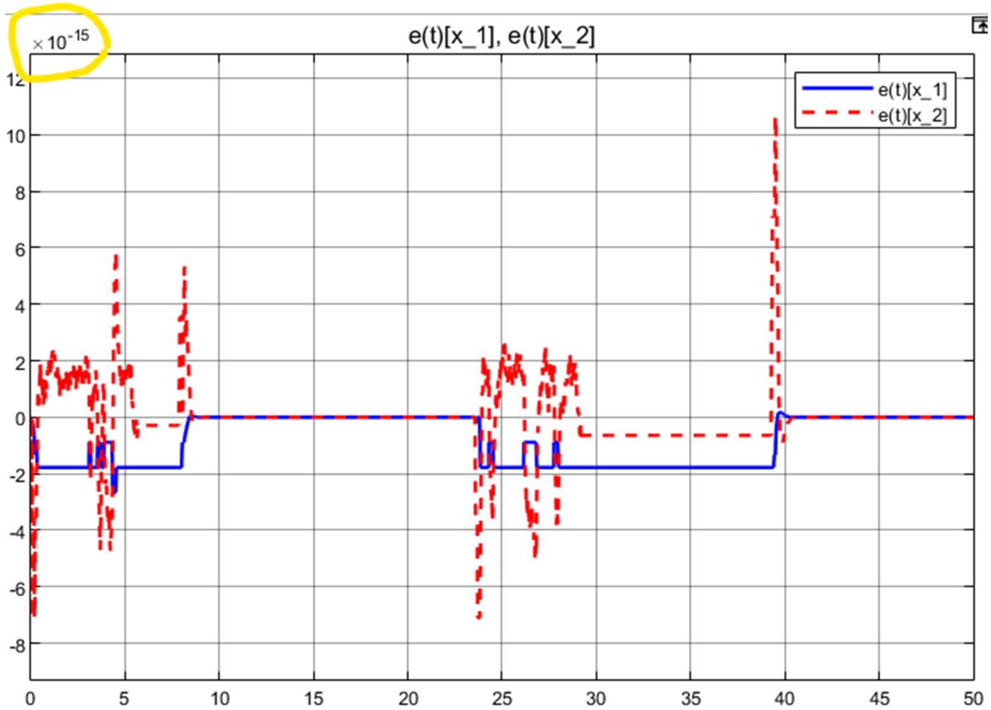

Figure 5： NonAdaptive Controller simulation result（$\epsilon(t)$）

2) negligibly change a_0 and a_1 (system preserves the stability);

We change the parameters as shown below for the experiment:

$$A \rightarrow A + \Delta A, \Delta A = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$$

The simulation results are shown in the following figures:

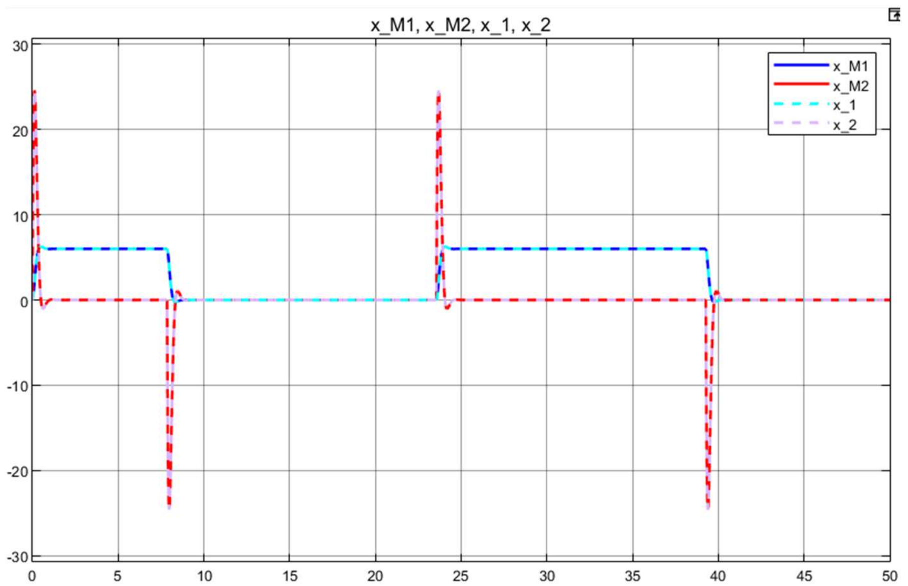

Figure 7: NonAdaptive Controller simulation result ($x(t), x_m(t)$)

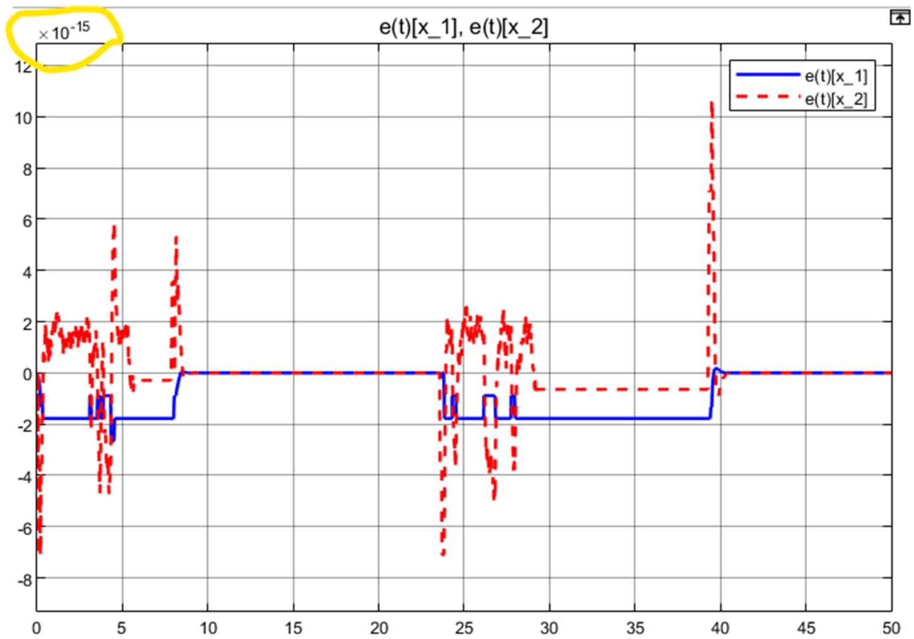

Figure 8: NonAdaptive Controller simulation result ($\epsilon(t)$)

3) change the parameters (the system looses the stability)

We change the parameters as shown below for the experiment:

$$A \to A + \Delta A, \Delta A = \begin{bmatrix} 0 & 0 \\ 13 & 13 \end{bmatrix}$$

The simulation results are shown in the following figures:

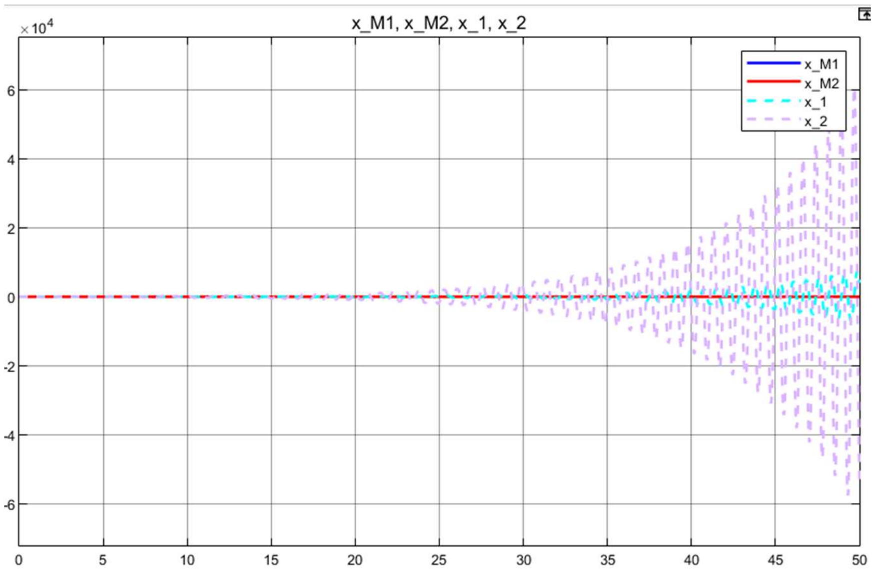

Figure 10： NonAdaptive Controller simulation result（$x(t), x_m(t)$）

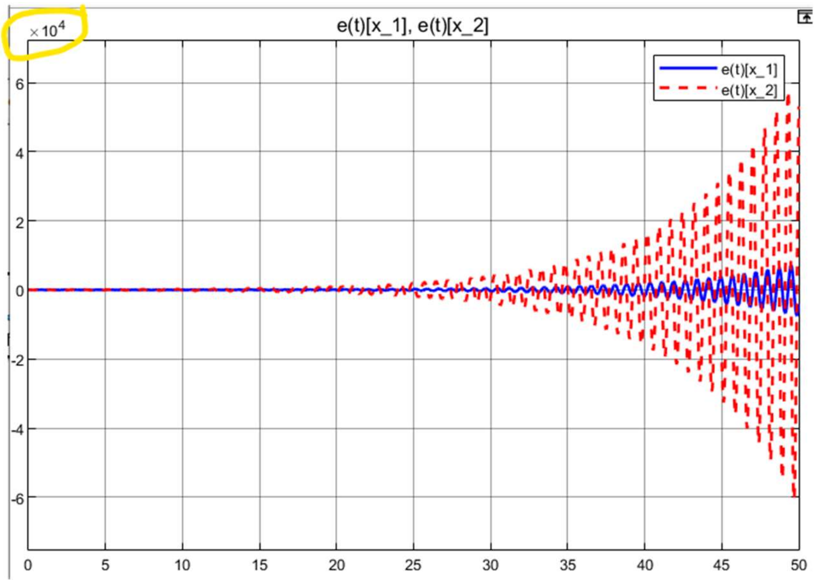

Figure 11： NonAdaptive Controller simulation result（$\epsilon(t)$）

3. Conclusion

- When the plant parameters are known and do not change, the designed non-adaptive controller can achieve the control objectives faster (Figures 4, 5)
- The designed non-adaptive controller can also achieve the control objective of maintaining system stability when slight changes in plant parameters occur (Figures 6, 7)
- When the plant parameters change greatly, the designed non-adaptive controller cannot achieve the control goal and the system loses stability, which indicates that the controller is not adaptive and cannot meet our control needs (Figures 8, 9)

➢ closed-loop system with adaptive controller (plant parameters are unknown)

1. design the Adaptive Controller

From the previous theoretical background we can obtain the Adaptive Controller as follows:

$$
\begin{aligned}
&\dot{x}_M = A_M x_M + b_M g, \\
&y_M = c_M^{\top} x_M \\
&A_M = A + b\theta^{\top}, b = \kappa b_M \\
&A_M^T P + P A_M = -Q \\
&\dot{\hat{\theta}} = \gamma x b^T P e, \quad \hat{\theta}(0) = 0 \\
&u = \hat{\theta}^T x + \frac{1}{\kappa} g.
\end{aligned}
\tag{4.3}
$$

The simulink scheme is constructed from (4.1) and (4.3) as follows:

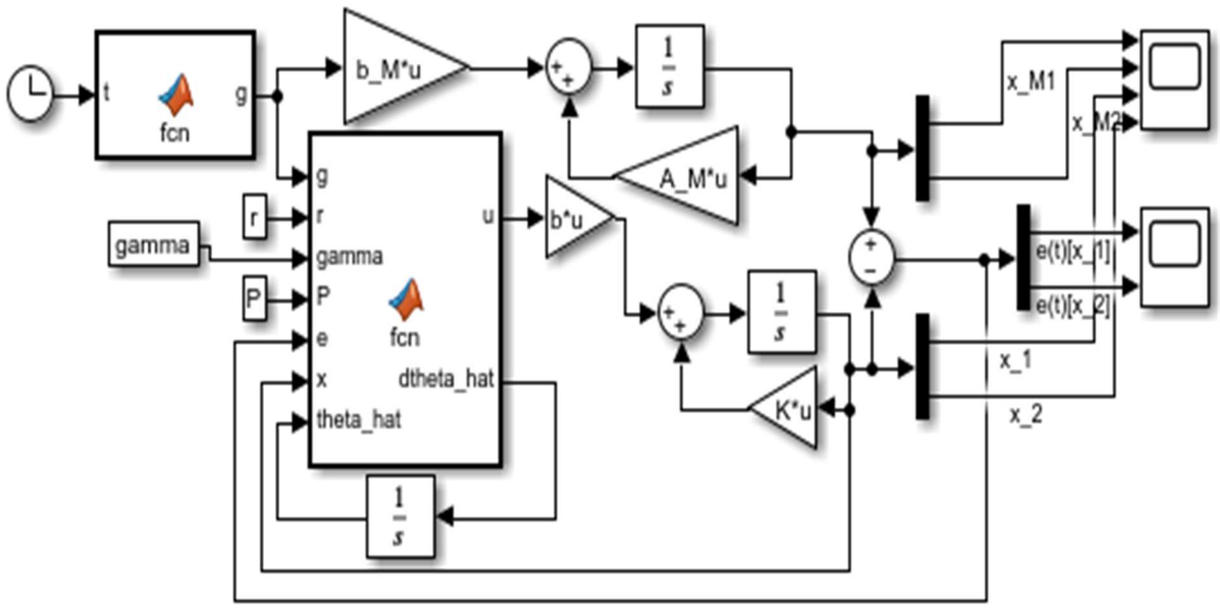

Figure 12: Adaptive Controller simulation scheme

2. Simulation

A. Repeat this triple of experiments with adaptive controller(4.3)

1) a_0 and $a_1\gamma = 10$

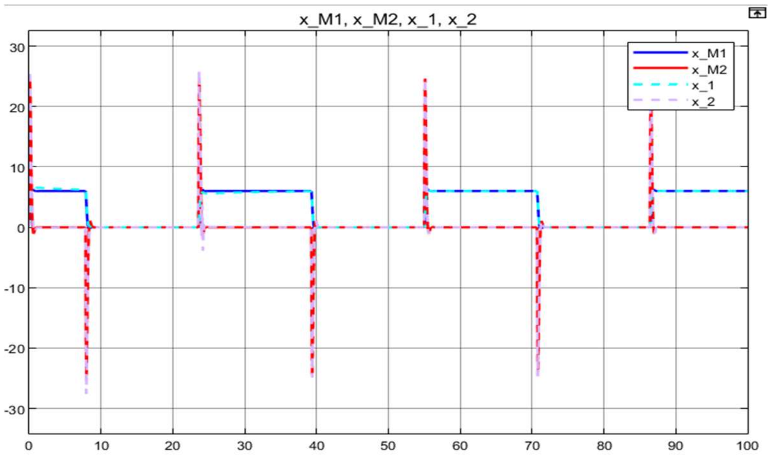

Figure 13： Adaptive Controller simulation result（$x(t), x_m(t)$）

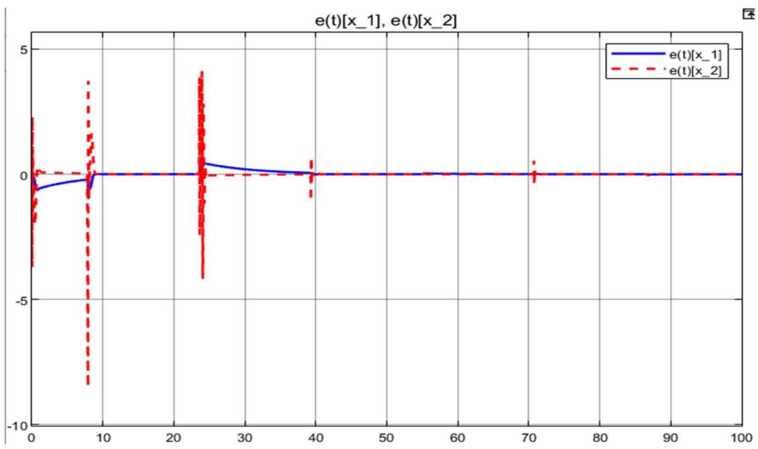

Figure 14： Adaptive Controller simulation result（$\epsilon(t)$）

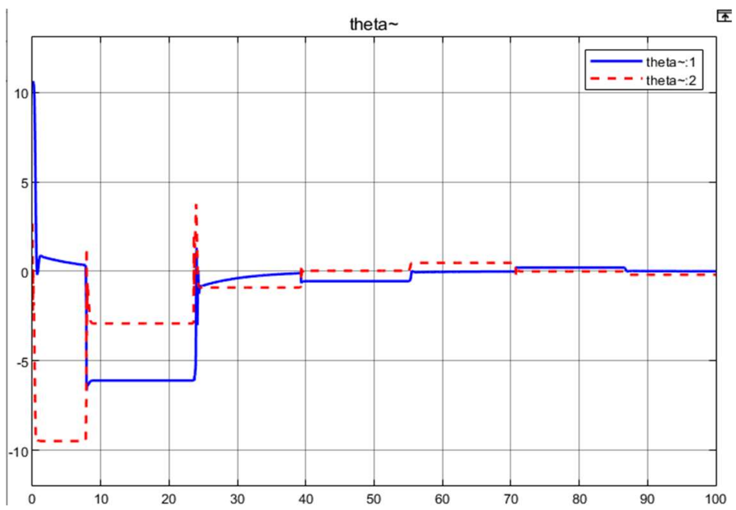

Figure 15：Adaptive Controller simulation result（$\tilde{\theta}(t)$）

2) negligibly change a_0 and a_1 (system preserves the stability); $\gamma = 10$

We change the parameters as shown below for the experiment:

$$A \rightarrow A + \Delta A, \Delta A = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$$

The simulation results are shown in the following figures:

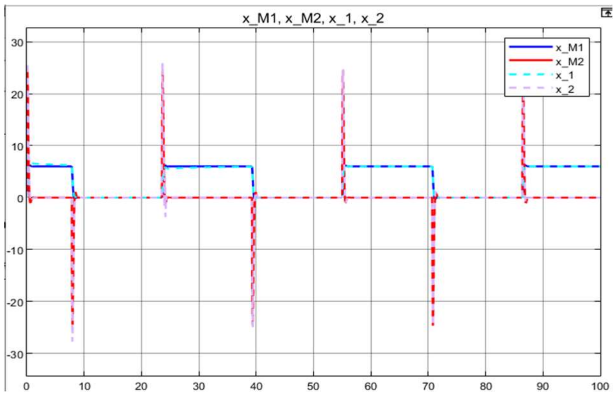

Figure 16 : Adaptive Controller simulation result ($x(t), x_m(t)$)

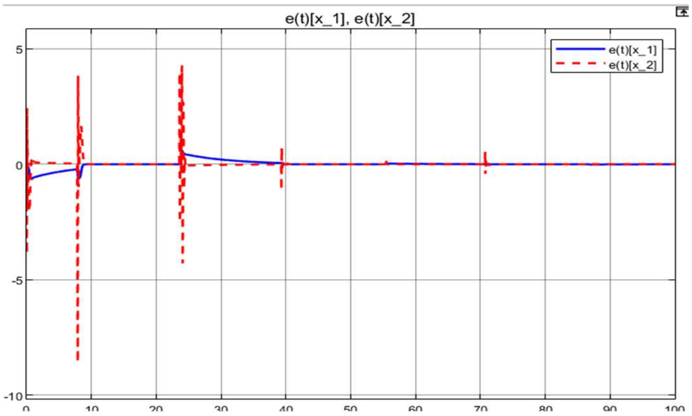

Figure 17 : Adaptive Controller simulation result ($\epsilon(t)$)

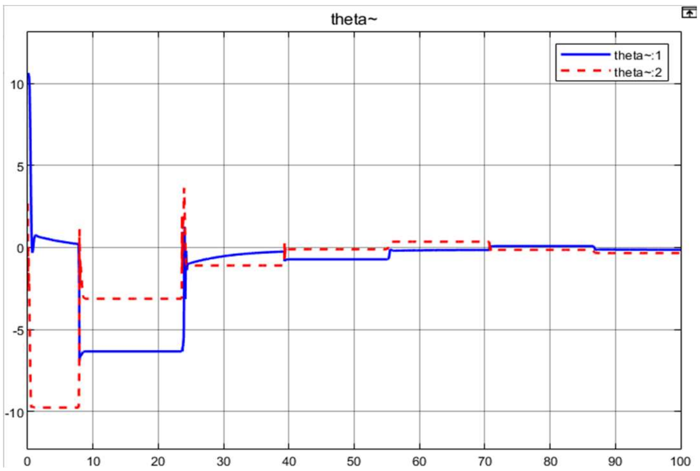

Figure 18: Adaptive Controller simulation result ($\tilde{\theta}(t)$)

3) change the parameters (the system looses the stability) $\gamma = 10$

We change the parameters as shown below for the experiment:

$$A \to A + \Delta A, \Delta A = \begin{bmatrix} 0 & 0 \\ 13 & 13 \end{bmatrix}$$

The simulation results are shown in the following figures:

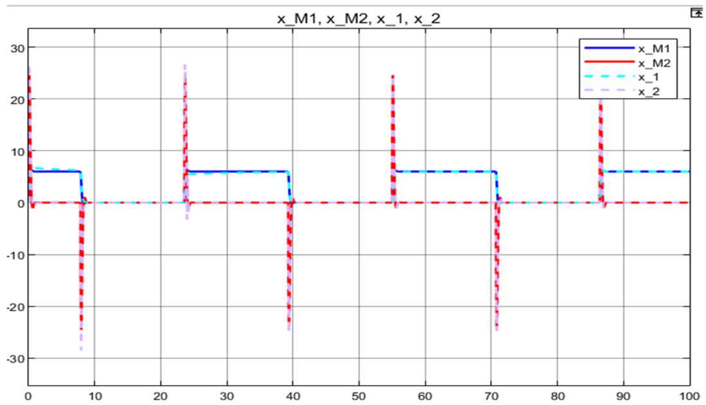

Figure 19: Adaptive Controller simulation result ($x(t), x_m(t)$)

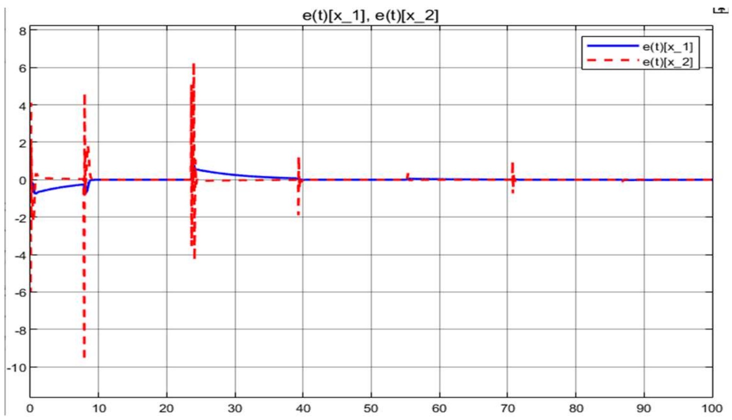

Figure 20: Adaptive Controller simulation result ($\epsilon(t)$)

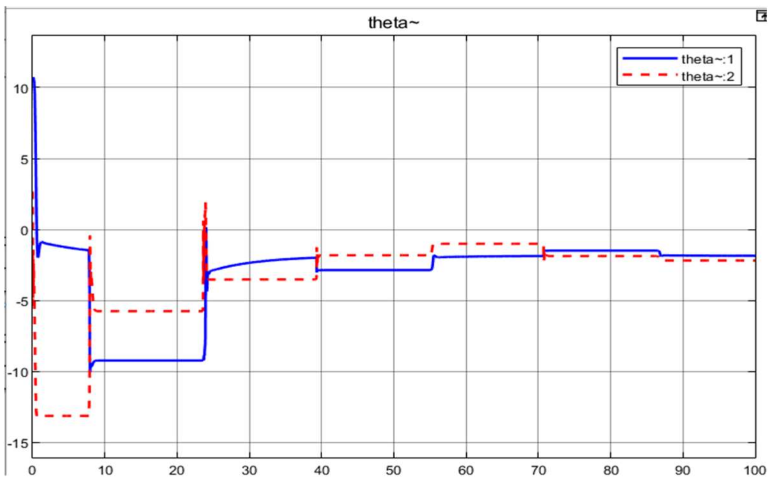

Figure 21: Adaptive Controller simulation result ($\tilde{\theta}(t)$)

B. Make experiment with original parameters of the plant and three values of γ

1) a_0 and $a_1 \gamma = 0.01$

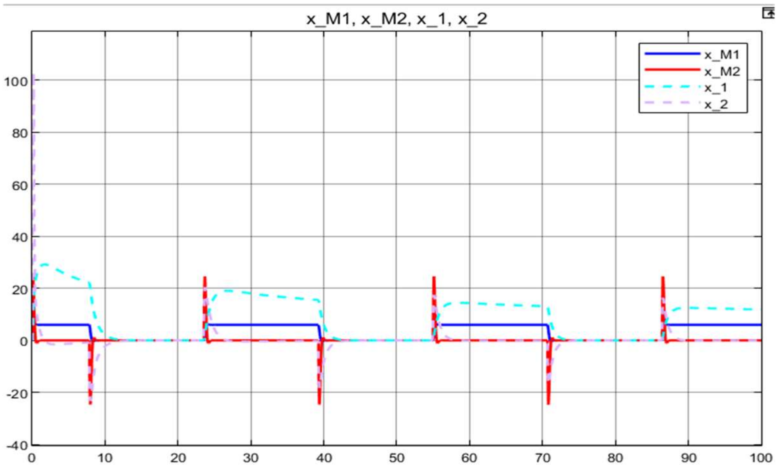

Figure 22： Adaptive Controller simulation result（$x(t), x_m(t)$）

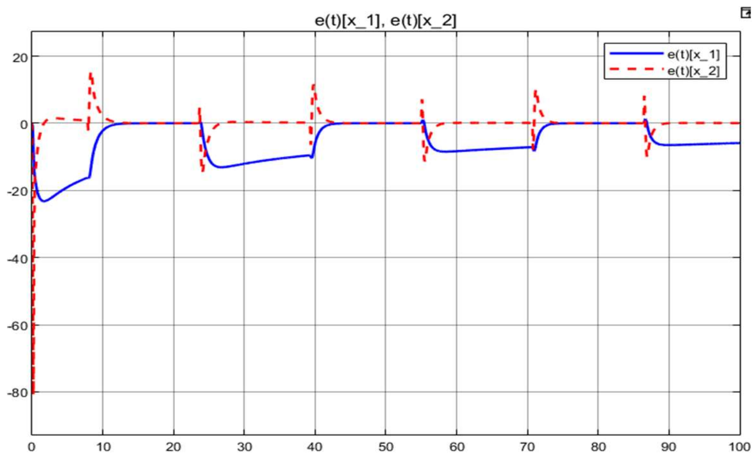

Figure 23： Adaptive Controller simulation result（$\epsilon(t)$）

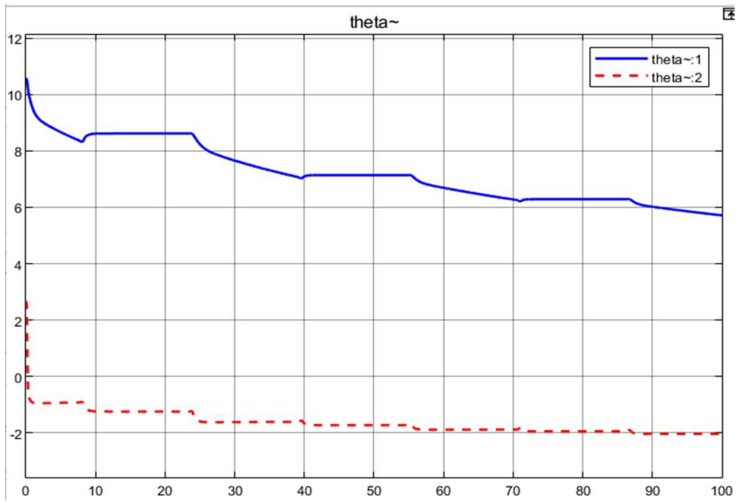

Figure 24: Adaptive Controller simulation result（$\tilde{\theta}(t)$）

2) a_0 and $a_1\gamma = 0.1$

The simulation results are shown in the following figures:

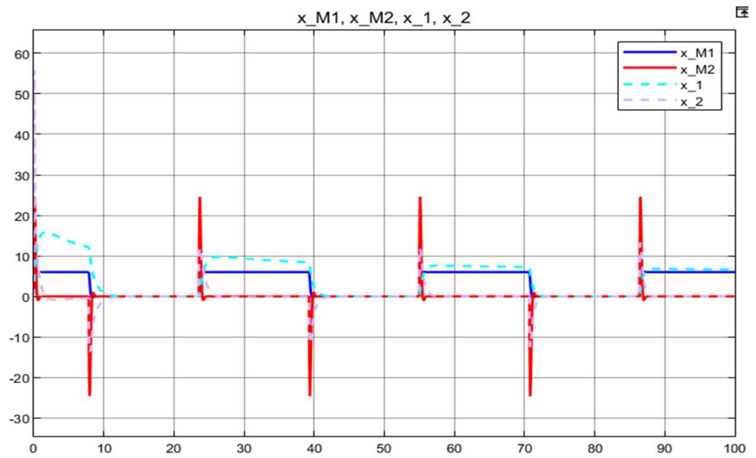

Figure 25： Adaptive Controller simulation result（$x(t), x_m(t)$）

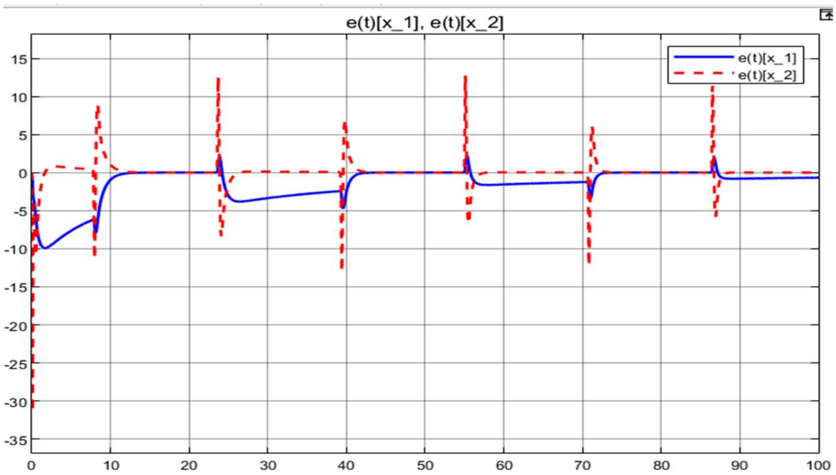

Figure 26： Adaptive Controller simulation result（$\epsilon(t)$）

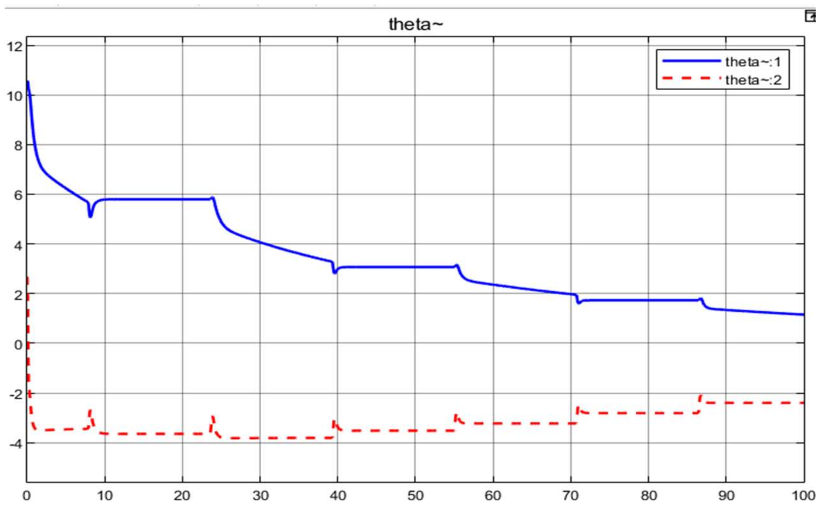

Figure 27：Adaptive Controller simulation result（$\tilde{\theta}(t)$）

3) a_0 and $a_1\gamma = 10$

The simulation results are shown in the following figures:

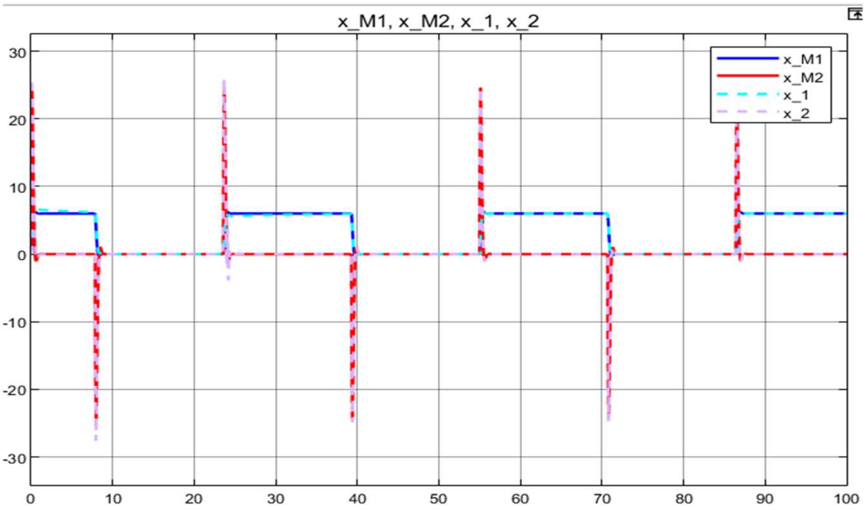

Figure 28： Adaptive Controller simulation result （$x(t), x_m(t)$）

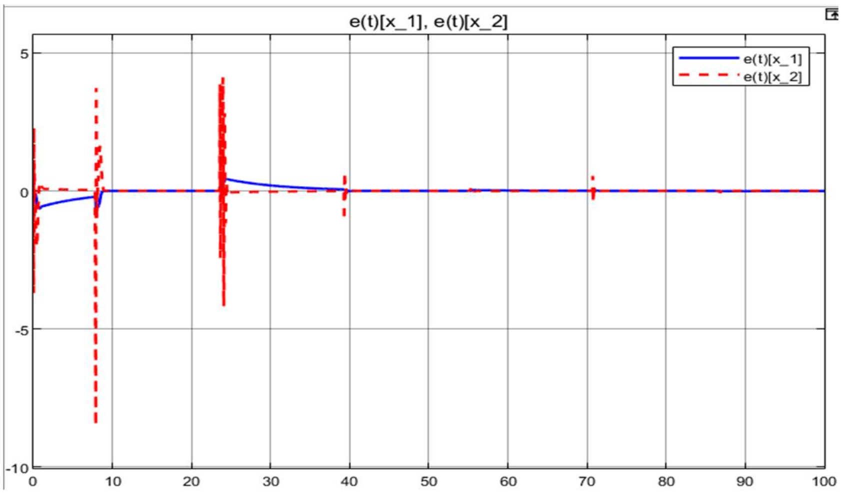

Figure 29： Adaptive Controller simulation result （$\epsilon(t)$）

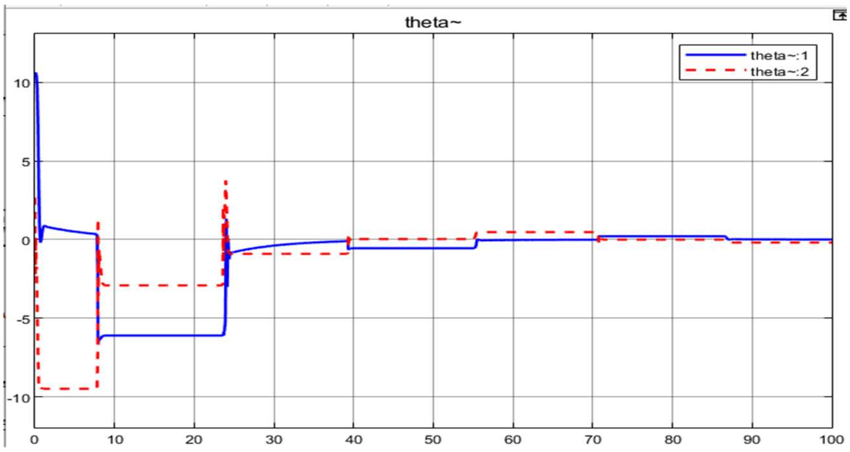

Figure 30：Adaptive Controller simulation result （$\tilde{\theta}(t)$）

C. Repeat the experiment for g(t)=1.

1) a_0 and $a_1 \gamma = 0.01$

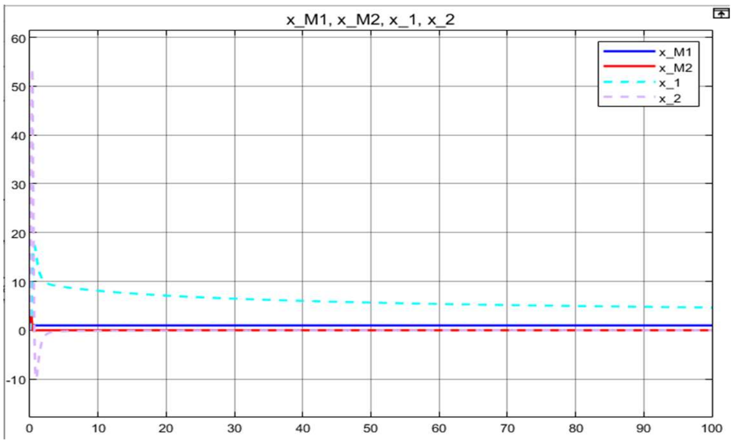

Figure 31： Adaptive Controller simulation result ($x(t), x_m(t)$)

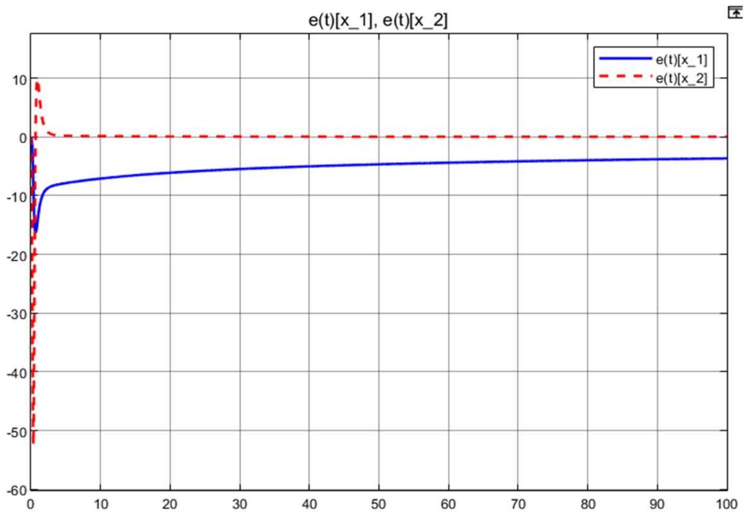

Figure 32： Adaptive Controller simulation result ($\epsilon(t)$)

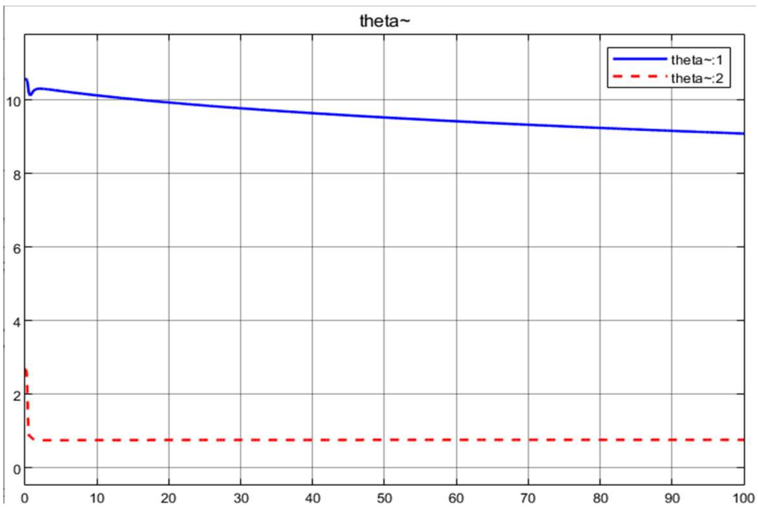

Figure 33: Adaptive Controller simulation result ($\tilde{\theta}(t)$)

2) a_0 and $a_1 \gamma = 0.1$

The simulation results are shown in the following figures:

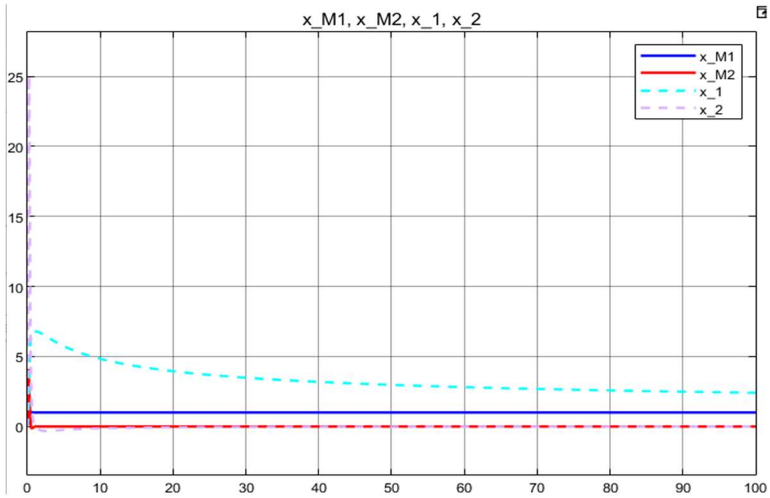

Figure 34： Adaptive Controller simulation result（$x(t), x_m(t)$）

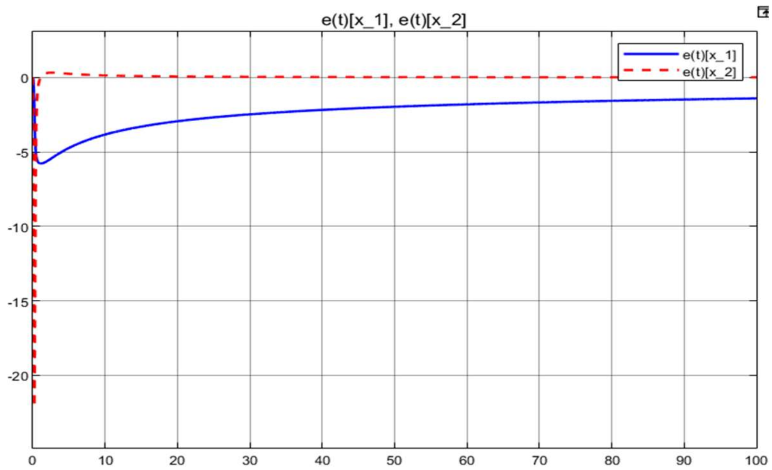

Figure 35： Adaptive Controller simulation result（$\epsilon(t)$）

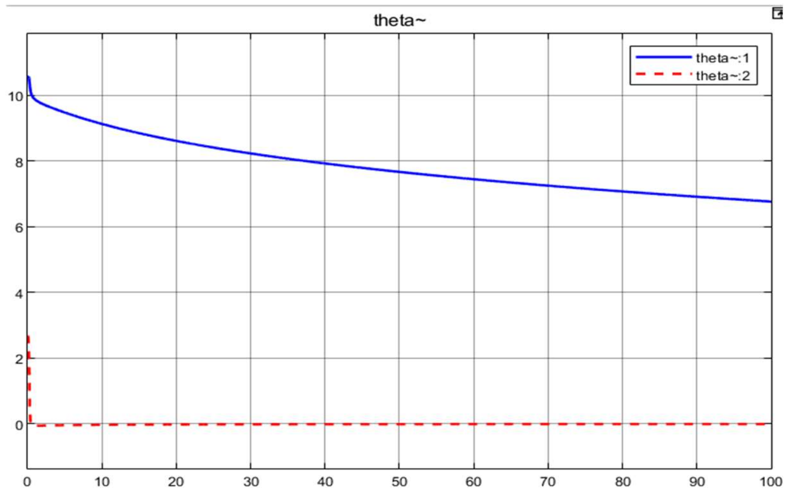

Figure 36：Adaptive Controller simulation result（$\tilde{\theta}(t)$）

3) a_0 and $a_1\gamma = 10$

The simulation results are shown in the following figures:

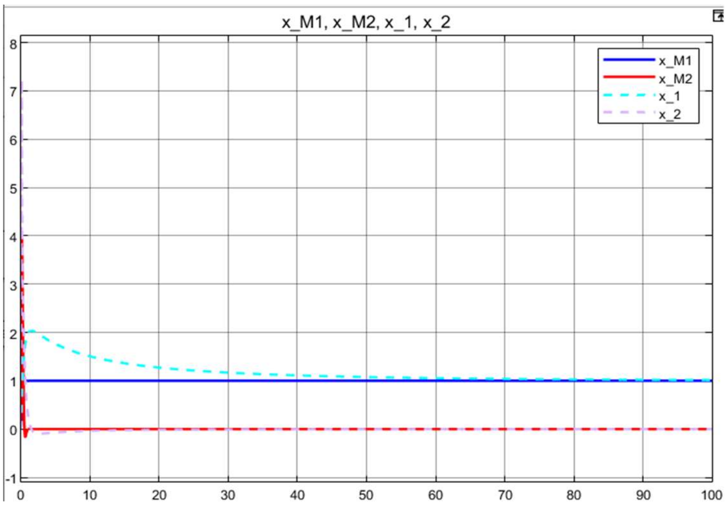

Figure 37： Adaptive Controller simulation result（$x(t), x_m(t)$）

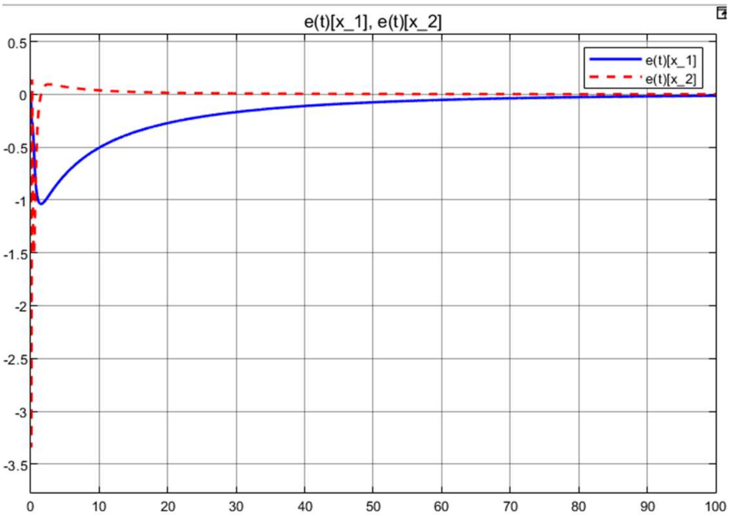

Figure 38： Adaptive Controller simulation result（$\epsilon(t)$）

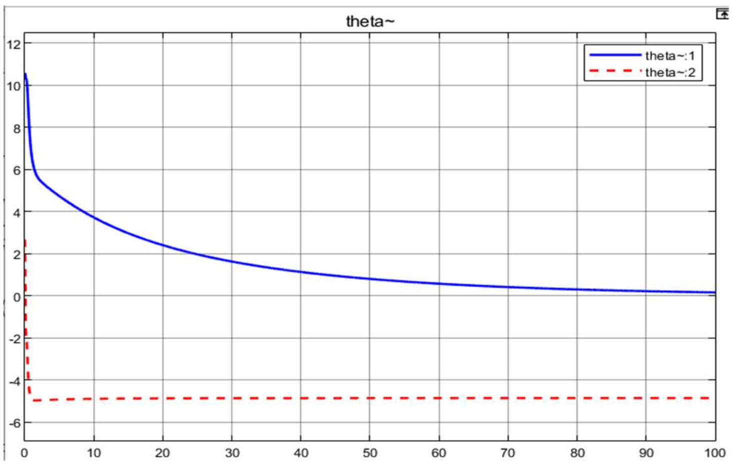

Figure 39：Adaptive Controller simulation result（$\tilde{\theta}(t)$）

D. Conclusions

From the simulation results we can show the following properties of the adaptive algorithm：

- Therefore, when facing a system with unknown parameters, the adaptive controller can be designed using the method of this experiment.
- from the experimental results, the designed adaptive controller satisfies the technical specification requirements.
- it is worth noting that: the adaptive gain γ needs experiments to determine the optimal γ, for which the rate of convergence of $\tilde{\theta}$ to zero is maximum. Small γ lead to slow aperiodic parameters convergence, while for large γ one can observe the oscillations of parameters tuning .