Course Work

Student Information

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>
> k = 7

Experimental data

$$\begin{array}{|c|c|c|c|c|c|}\hline k & m_1 & m_2 & m_3 & l_1 & l_2 \\ \hline 7 & m_1 = 3 & m_2 = 1 & m_3 = 1 & l_1 = 1.25 & l_2 = 1.5 \\ \hline \end{array} \quad (1)$$

0.Physical Model

Consider a double inverted pendulum system (shown in Fig. 1), where

Symbol	Meaning
q_1	Cart position
q_2	Angle of the lower pendulum
q_3	Angle of the upper pendulum
u	Applied force (control variable)
m_1	Mass of the cart
m_2	Mass of the lower pendulum
m_3	Mass of the upper pendulum
l_1	Length of the lower pendulum
l_2	Length of the upper pendulum
g	Gravitational acceleration

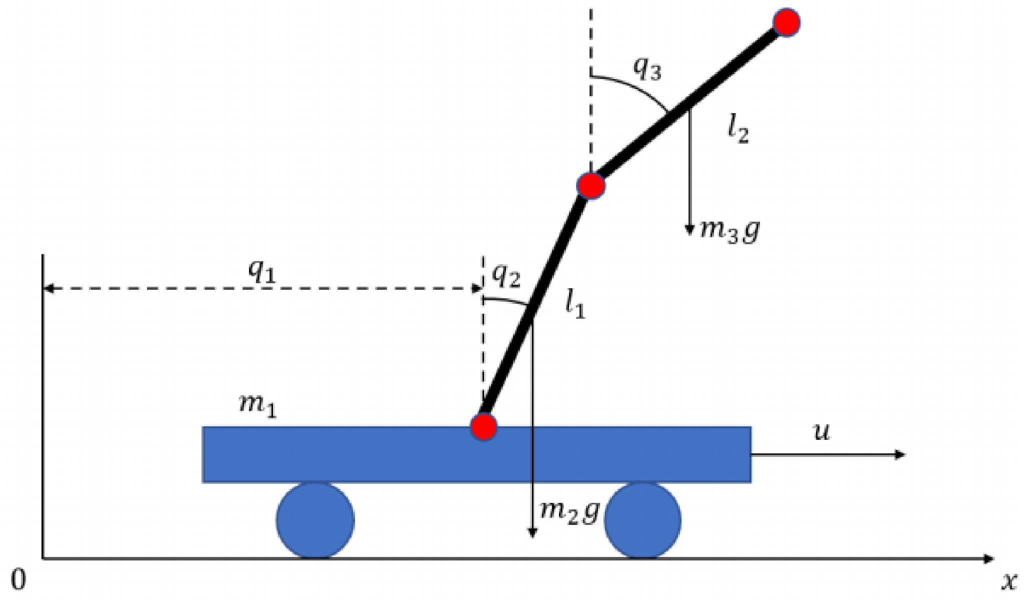


Fig.0.1 Double inverted pendulum on a cart

The dynamics of this system can be described in the following standard form:

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = Hu \tag{2}$$

where：

$$\begin{aligned}
q = [q_1, q_2, q_3]^T \quad & \text{Generalized joint coordinates} \\
M(q) \quad & \text{Regular mass matrix} \\
C(q,\dot{q}) \quad & \text{Centrifugal and Coriolis forces} \\
G(q) \quad & \text{Gravity force} \\
H \quad & \text{Control matrix} \\
M(q) = & \begin{bmatrix} a_1 & a_2\cos q_2 & a_3\cos q_3 \\ a_2\cos q_2 & a_4 & a_5\cos(q_2 - q_3) \\ a_3\cos q_3 & a_5\cos(q_2 - q_3) & a_6 \end{bmatrix}, \\
C(q,\dot{q}) = & \begin{bmatrix} 0 & -a_2\sin q_2\dot{q}_2 & -a_3\sin q_3\dot{q}_3 \\ 0 & 0 & a_5\sin(q_2 - q_3)\dot{q}_3 \\ 0 & -a_5\sin(q_2 - q_3)\dot{q}_2 & 0 \end{bmatrix}, \\
G(q) = & \begin{bmatrix} 0 \\ -g_1\sin q_2 \\ -g_2\sin q_3 \end{bmatrix}, \quad H = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \\
a_1 = m_1 + m_2 + m_3, \quad & a_5 = \tfrac{1}{2}m_3 l_1 l_2, \\
a_2 = \left(\tfrac{1}{2}m_2 + m_3\right)l_1 \quad & a_6 = \tfrac{1}{3}m_3 l_2^2, \\
a_3 = \tfrac{1}{2}m_3 l_2, \quad & g_1 = \left(\tfrac{1}{2}m_2 + m_3\right)l_1 g, \\
a_4 = \left(\tfrac{1}{3}m_2 + m_3\right)l_1^2, \quad & g_2 = \tfrac{1}{2}m_3 l_2 g
\end{aligned} \tag{3}$$

1. state-space form

Choosing the state vector as $x = [q_1 \quad q_2 \quad q_3 \quad \dot{q}_1 \quad \dot{q}_2 \quad \dot{q}_3]^T$, represent the system in the state-space form.

First, from (1) we get：

$$\ddot{q} = -M^{-1}(q)C(q,\dot{q})\dot{q} - M^{-1}(q)G(q) + M^{-1}(q)Hu \tag{4}$$

Therefore, we can get the state-space form of the system state-space form

$$\dot{x}=\begin{bmatrix}\dot{q}\\ \ddot{q}\end{bmatrix}=\begin{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\cdot\dot{q}\\ -M^{-1}(q)C(q,\dot{q})\dot{q}-M^{-1}(q)G(q)\end{bmatrix}+\begin{bmatrix}0\\0\\0\\M^{-1}(q)H\end{bmatrix}\cdot u \tag{5}$$

which is :

$$\dot{x}=f(x)+h(x)u \tag{6}$$

where :

$$f(x)=\begin{bmatrix}I(3,3)\cdot\dot{q}\\ -M^{-1}(q)C(q,\dot{q})\dot{q}-M^{-1}(q)G(q)\end{bmatrix} \tag{7}$$

$$h(x)=\begin{bmatrix}0\\0\\0\\M^{-1}(q)H\end{bmatrix} \tag{8}$$

2. Simulation—Original System

2.1 Simulink Mathematical Model

Use (3) to build a model in simulink as shown below:

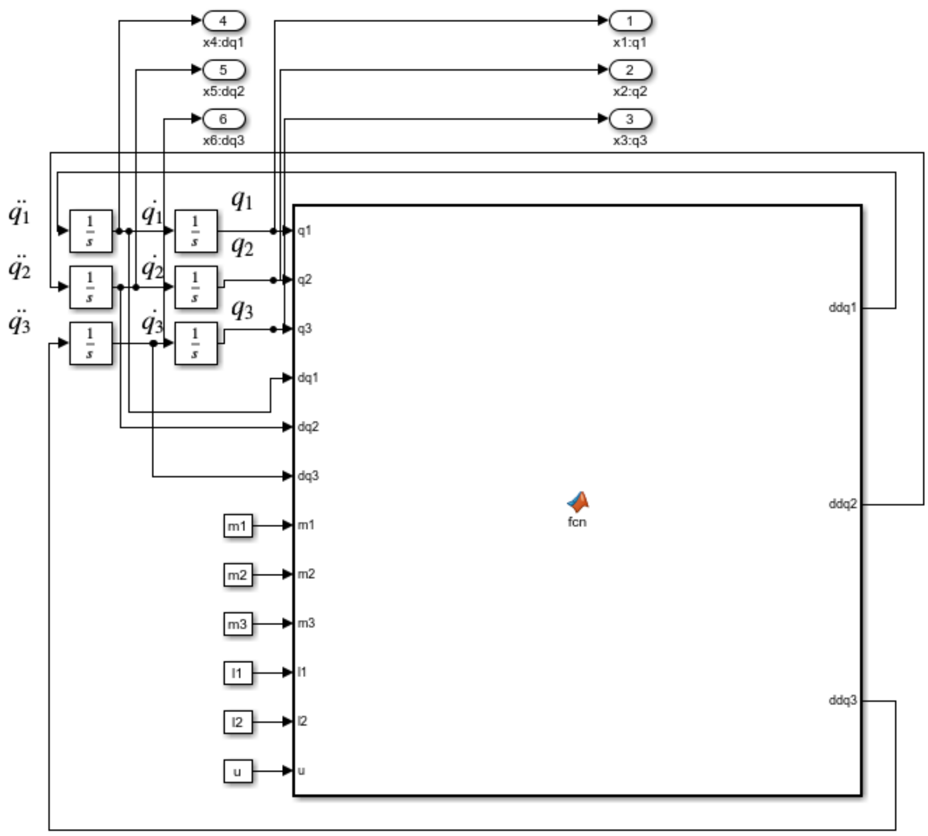


Fig.2.1 Simulink Mathematical Model

p2_model ▶ Subsystem ▶ MATLAB Function

```
function [ddq1, ddq2, ddq3] = fcn(q1, q2, q3, dq1, dq2, dq3, m1, m2, m3, l1, l2, u)
g = 9.80665;
a1 = m1 +m2 +m3;
a2 = (1/2*m2+m3)*l1;
a3 = 1/2*m3*l2;
a4 = (1/3*m2 + m3)*l1^2;
a5 = 1/2*m3*l1*l2;
a6 = 1/3*m3*l2^2;
g1 = (1/2*m2+m3)*l1*g;
g2 = 1/2*m3*l2*g;
M = [a1 a2*cos(q2) a3*cos(q3);
     a2*cos(q2) a4 a5*cos(q2-q3);
     a3*cos(q3) a5*cos(q2-q3) a6];
C = [0 -1*a2*sin(q2)*dq2 -1*a3*sin(q3)*dq3;
     0 0 a5*sin(q2-q3)*dq3;
     0 -1*a5*sin(q2-q3)*dq2 0];
G = [0;-1*g1*sin(q2);-1*g2*sin(q3)];
H = [1;0;0];
DQ = [dq1;dq2;dq3];
ddq = inv(M)*(-1*C*DQ-G+H*u );
ddq1 =ddq(1);
ddq2 =ddq(2);
ddq3 =ddq(3);
```

Fig.2.2 Simulink Mathematical Model(Fcn)

2.2 Simulink Physical Model

In order to verify the correctness of the model, the physical model is established in simulink as shown in the following figure:

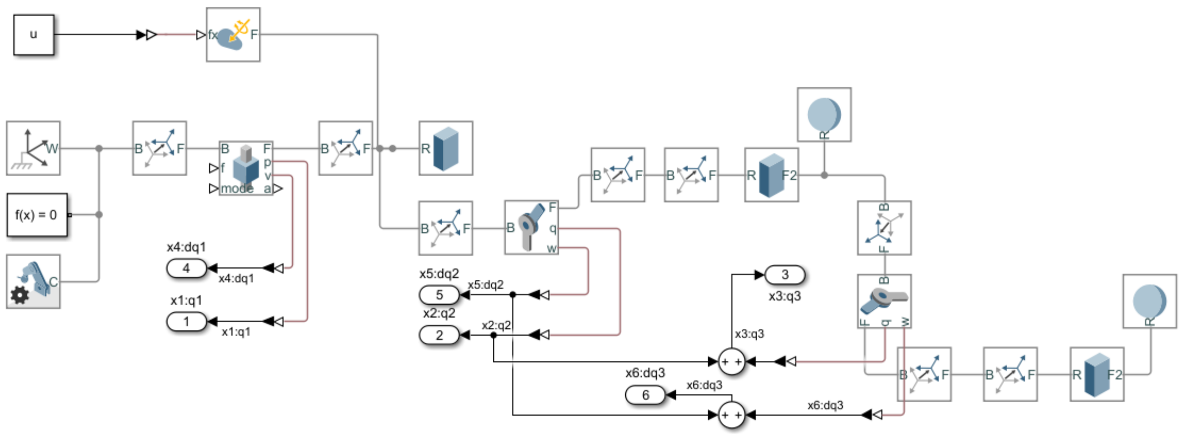


Fig.2.3 Simulink Physical Model

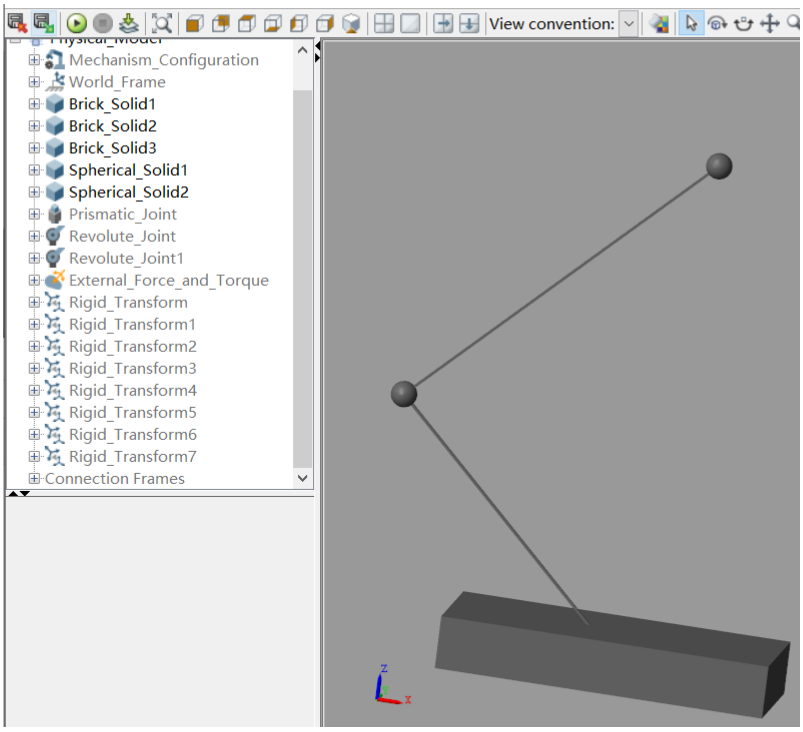


Fig.2.4 Simulink Physical Model-Screenshot of the running process

2.3 Simulation Results

To sum up 2.1 and 2.2, use the input $u = 1$ to simulate the inverted pendulum system, and the results are shown in the following figure :

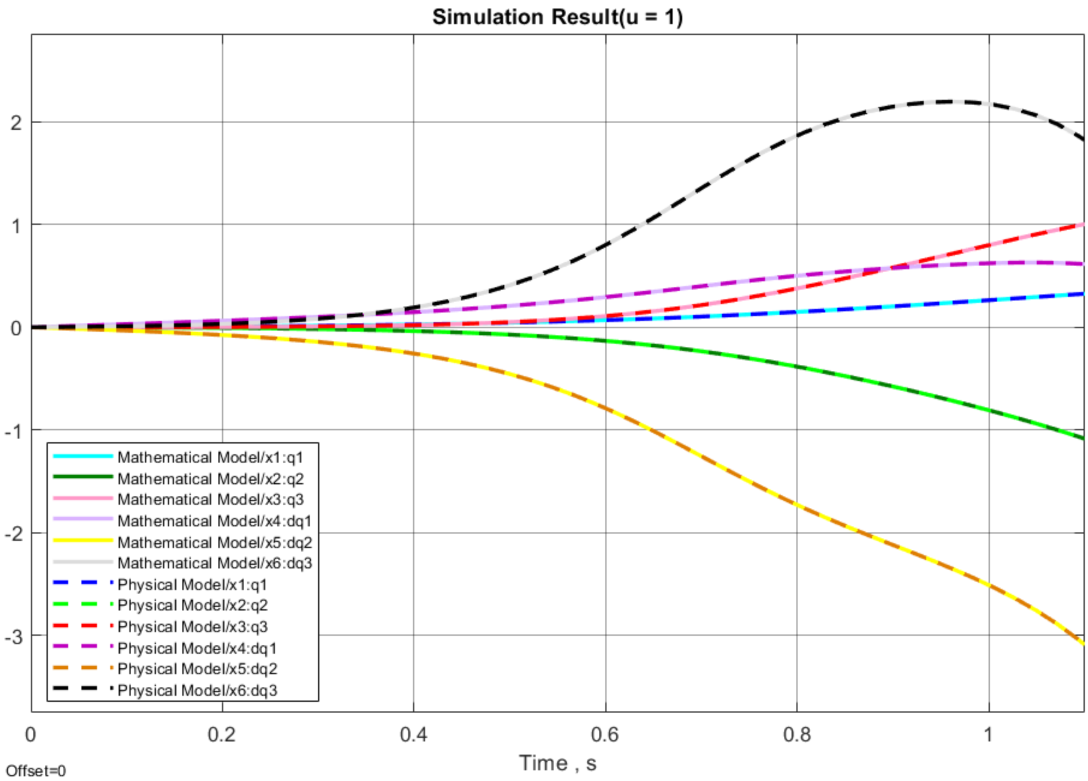


Fig.2.5 Simulation results (u = 1)

It can be seen from the simulation results that the physical simulink model and the mathematical simulink model are completely coincident, which proves that the two models are established **correctly**. Since the running of the physical model consumes more computing resources, subsequent experiments will only be performed using the mathematical model.

3. Linearize the system

3.1 Calculation Process

Bringing (3) into (7), (8) we get:

$$
\begin{aligned}
f(x) &= \begin{bmatrix} I(3,3)\cdot\dot{q} \\ -M^{-1}(q)C(q,\dot{q})\dot{q} - M^{-1}(q)G(q) \end{bmatrix} \\
&= \begin{bmatrix} x_4 \\ x_5 \\ x_6 \\ -M^{-1}(q)C(q,\dot{q})\dot{q} - M^{-1}(q)G(q) \end{bmatrix}
\end{aligned} \tag{9}
$$

$$
\frac{\partial f(x)}{\partial x} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \multicolumn{6}{l}{\dfrac{\partial M^{-1}}{\partial x}(C\dot{q} - G) + M^{-1}\left(\dfrac{\partial C}{\partial x}\dot{q} + C\dfrac{\partial \dot{q}}{\partial x} - \dfrac{\partial G}{\partial x}\right)} \end{bmatrix} \tag{10}
$$

the point $x_{eq} = 0$:

Bringing $x_{eq} = 0$ into (3) we can easily get:

$$
C = 0\ ,\ \dot{q} = 0\ ,\ G = 0 \tag{11}
$$

$$
\begin{aligned}
&\left(\frac{\partial M^{-1}}{\partial x}(C\dot{q} - G) + M^{-1}\left(\frac{\partial C}{\partial x}\dot{q} + C\frac{\partial \dot{q}}{\partial x} - \frac{\partial G}{\partial x}\right)\right)\Bigg|_{x=x_{eq}} \\
&= \left(-M^{-1}\cdot\frac{\partial G}{\partial x}\right)\Bigg|_{x=x_{eq}}
\end{aligned} \tag{12}
$$

Using MATLAB calculate we get:

$$
\begin{aligned}
&\left(-M^{-1}\cdot\frac{\partial G}{\partial x}\right)\Bigg|_{x=x_{eq}} \\
&= \begin{bmatrix} 0 & \frac{g_1(a_2a_6 - a_3a_5)}{q} & \frac{-g_2(a_2a_5 - a_3a_4)}{q} & 0 & 0 & 0 \\ 0 & \frac{g_1(a_3^2 - a_1a_6)}{q} & \frac{g_2(a_1a_5 - a_2a_3)}{q} & 0 & 0 & 0 \\ 0 & \frac{g_1(a_1a_5 - a_2a_3)}{q} & \frac{g_2(a_2^2 - a_1a_4)}{q} & 0 & 0 & 0 \end{bmatrix}
\end{aligned} \tag{13}
$$

where:

$$
q = a_6a_2^2 - 2a_2a_3a_5 + a_4a_3^2 + a_1a_5^2 - a_1a_4a_6 \tag{14}
$$

In summary, we can get：

$$
\begin{aligned}
A &= \frac{\partial f(x)}{\partial x}\Bigg|_{x=x_{eq}} \\
&= \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & \frac{g_1(a_2a_6 - a_3a_5)}{q} & \frac{-g_2(a_2a_5 - a_3a_4)}{q} & 0 & 0 & 0 \\ 0 & \frac{g_1(a_3^2 - a_1a_6)}{q} & \frac{g_2(a_1a_5 - a_2a_3)}{q} & 0 & 0 & 0 \\ 0 & \frac{g_1(a_1a_5 - a_2a_3)}{q} & \frac{g_2(a_2^2 - a_1a_4)}{q} & 0 & 0 & 0 \end{bmatrix}
\end{aligned} \tag{15}
$$

Next, use matlab to calculate matrix B

$$B = h(x)|_{x=x_{eq}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ M^{-1}(q)H \end{bmatrix}\Bigg|_{x=x_{eq}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{a_5^2-a_4a_6}{q} \\ \frac{(a_2a_6-a_3a_5)}{q} \\ \frac{-(a_2a_5-a_3a_4)}{q} \end{bmatrix} \tag{16}$$

3.2 Result

Finally we get the system after linearization at the equilibrium point $x_{eq} = 0$ as follows:

$$\dot{x} = Ax + Bu \tag{17}$$

where:

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & \frac{g_1(a_2a_6-a_3a_5)}{q} & \frac{-g_2(a_2a_5-a_3a_4)}{q} & 0 & 0 & 0 \\ 0 & \frac{g_1(a_3^2-a_1a_6)}{q} & \frac{g_2(a_1a_5-a_2a_3)}{q} & 0 & 0 & 0 \\ 0 & \frac{g_1(a_1a_5-a_2a_3)}{q} & \frac{g_2(a_2^2-a_1a_4)}{q} & 0 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{a_5^2-a_4a_6}{q} \\ \frac{(a_2a_6-a_3a_5)}{q} \\ \frac{-(a_2a_5-a_3a_4)}{q} \end{bmatrix} \tag{18}$$

where:

$$q = a_6a_2^2 - 2a_2a_3a_5 + a_4a_3^2 + a_1a_5^2 - a_1a_4a_6 \tag{19}$$

4. Design a linear feedback control

4.1 Calculation of Modal Control Coefficients

Calculate feedback control$u(x) = -Kx$ parameters K using modal control

1. Determine whether the system is stable now:

Find the eigenvalues of matrix A:

$$\begin{aligned} \lambda_1 &= 0 & \lambda_2 &= 0 \\ \lambda_3 &= 6.434 & \lambda_4 &= 2.6949 \\ \lambda_5 &= -6.434 & \lambda_6 &= -2.6949 \end{aligned} \tag{20}$$

The system is now **unstable**

2. checking system for complete controllability

$$\begin{gathered} N_c = [B \quad AB \quad A^2B \quad \cdots \quad A^{n-1}B] \\ \text{rank}(N_c) = 6 \end{gathered} \tag{21}$$

system is completely controllable

3. Form reference model

Newton polynomial : $\sum_{i=1}^{n} C_n^i \omega_0^i s^{n-i}$

$$\begin{array}{|l|l|l|l|}\hline \text{System order} & \text{Newton polynomial} & \text{Overshooting } \sigma,\% & \text{Settling time } t_s^*,\ \text{sec} \\ \hline n=6 & s^6+6\omega_0 s^5+15\omega_0^2 s^4+20\omega_0^3 s^3 + 15\omega_0^4 s^2+6\omega_0^5 s+\omega_0^6 & 0 & 10.5 \\ \hline \end{array} \tag{22}$$

n= 6 Newton polynomial

let $t_S = 1$

$$\omega = \frac{t_s^*}{t_S} = 10.5$$

$$\Gamma = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -\omega^6 & -6\omega^5 & -15\omega^4 & -20\omega^3 & -15\omega^2 & -6\omega \end{bmatrix} \tag{23}$$

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

4. Finding matrix transfoemation M

The solution of the Sylvester matrix equation with respect to the matrix M:

```
M = sylv(-A,gamma,-B*H)
```

5. Calculation of matrix K

$$K = HM^{-1} \tag{24}$$

6. Form control signal

$$v(t) = -\mathrm{K}x(t) \tag{25}$$

5. Simulation—linear system &&Modal control

5.1 Modelling

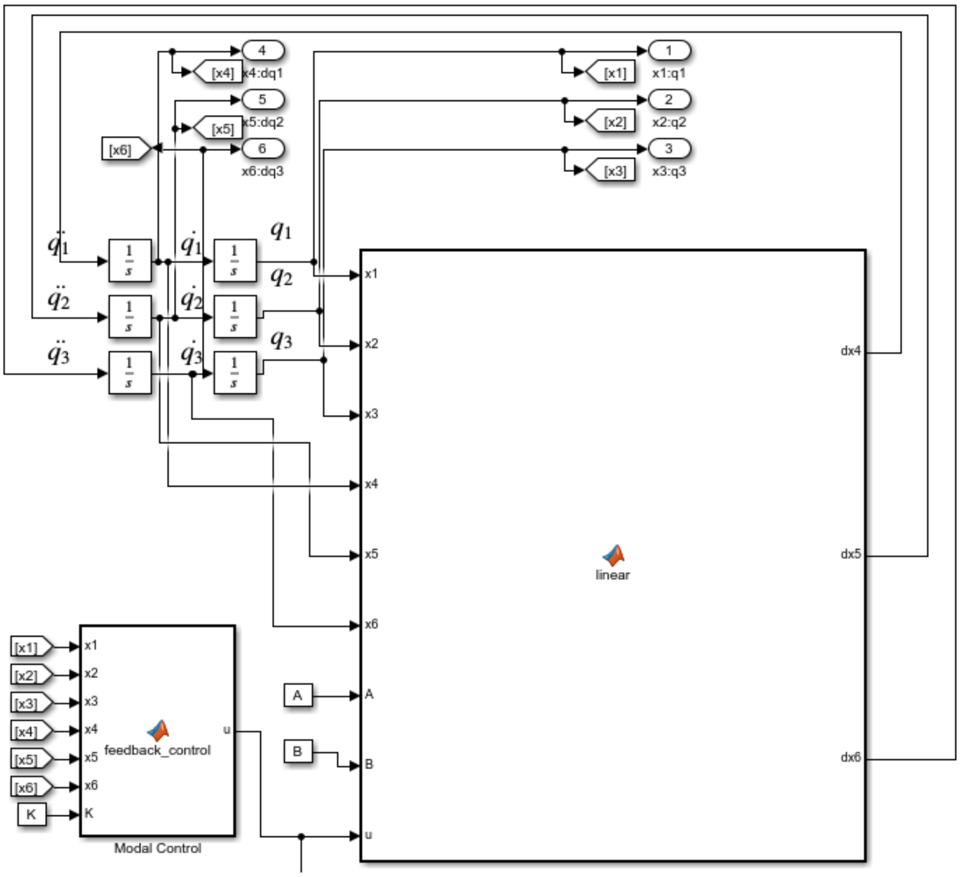


Fig.2.5 Linear + Feedback control Model

p5_model ▸ Linear Model(LQR control) ▸ MATLAB Function

```
function [dx4,dx5,dx6]=linear(x1,x2,x3,x4,x5,x6,A,B,u)
X=[x1;x2;x3;x4;x5;x6];
DX=A*X+B*u;
dx4 = DX(4);
dx5 = DX(5);
dx6 = DX(6);
```

Fig.2.5 Linear Model Fcn

p5_model ▸ Linear Model(LQR control) ▸ MATLAB Function

```
function [dx4,dx5,dx6]=linear(x1,x2,x3,x4,x5,x6,A,B,u)
X=[x1;x2;x3;x4;x5;x6];
DX=A*X+B*u;
dx4 = DX(4);
dx5 = DX(5);
dx6 = DX(6);
```

Fig.2.5Feedback control Model Fcn

5.2 Simulation

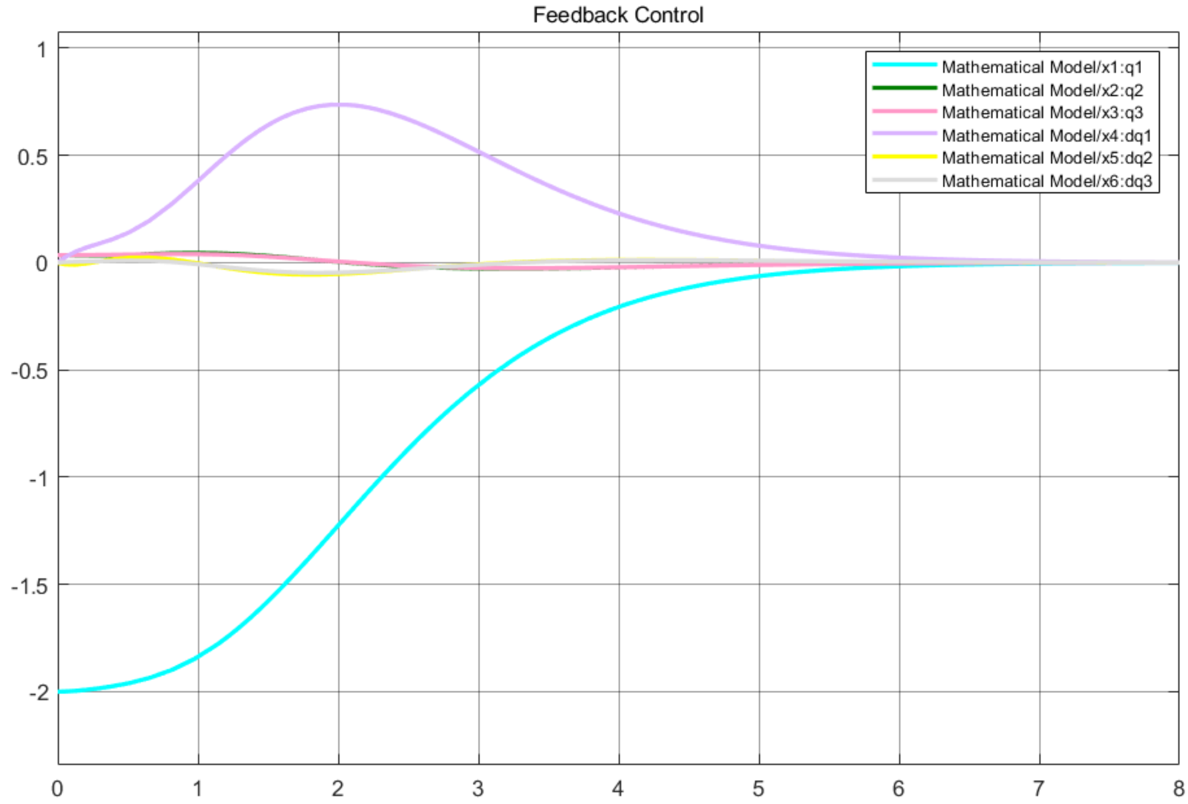


Fig.2.5 Simulation results (Modal control)

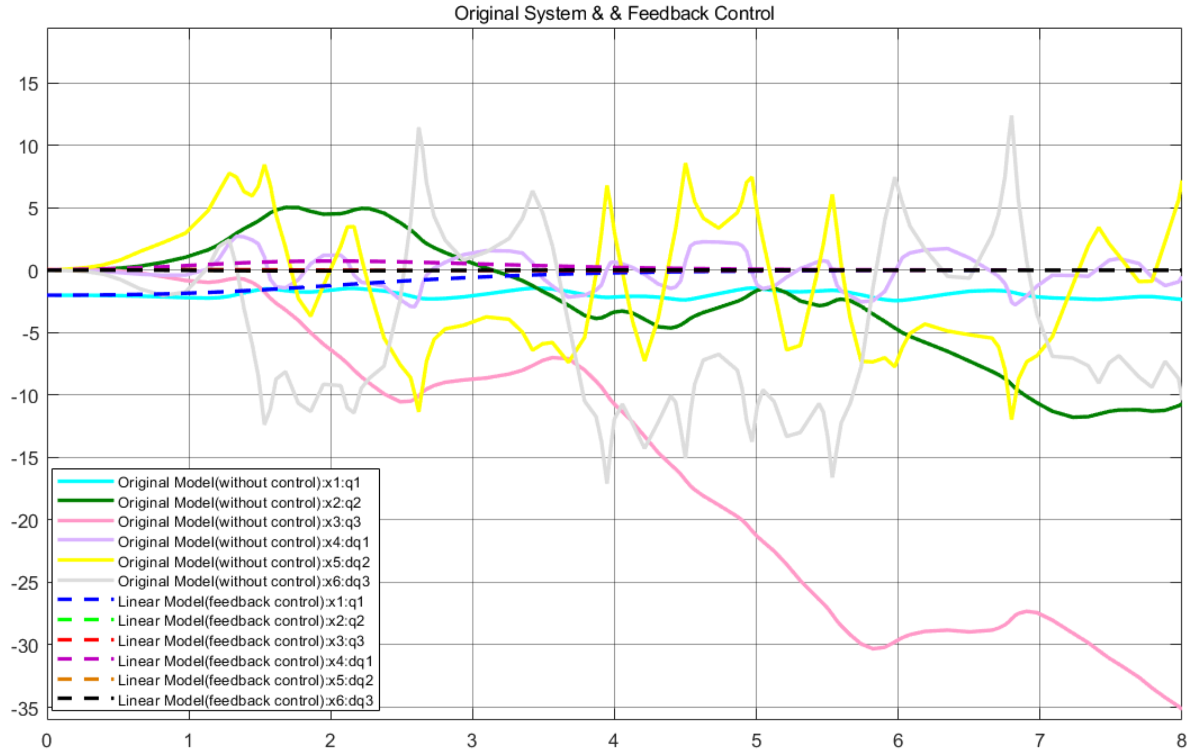


Fig.2.5 Simulation results (Modal control)Compare with original

6. Analytical construct of a linear-quadratic regulator (LQR)

6.1 Theory && Algorithm

algebraic Riccati equation (ARE)

$$A^TP + PA + Q - PBR^{-1}B^TP = 0 \tag{26}$$

The design procedure for finding the LQR feedback K

The design procedure for finding the LQR feedback K is:

- Select design parameter matrices Q and R
- Solve the algebraic Riccati equation for P
- Find the SVFB using $K = R^{-1}B^TP$

Note : There are very | good numerical procedures for solving the ARE. The MATLAB routine that performs this is named $\mathrm{lqr}(A, B, Q, R)$.

6.2 Calculation process

Because nonlinear systems are generally relatively unstable, I choose the cheap control scheme to obtain a relatively stable control scheme, in which:

$$Q = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{27}$$
$$R = [0.01]$$

Then use MATLAB to calculate the feedback gain

```
[K1,P] = lqr(A,B,Q,R);
```

7. Simulation—linear system && LQR control

7.1 Modelling

Because they are all feedback control, this part of the simulink model is the same as the model in part 5, only the feedback parameter K is different.

7.2 Simulation

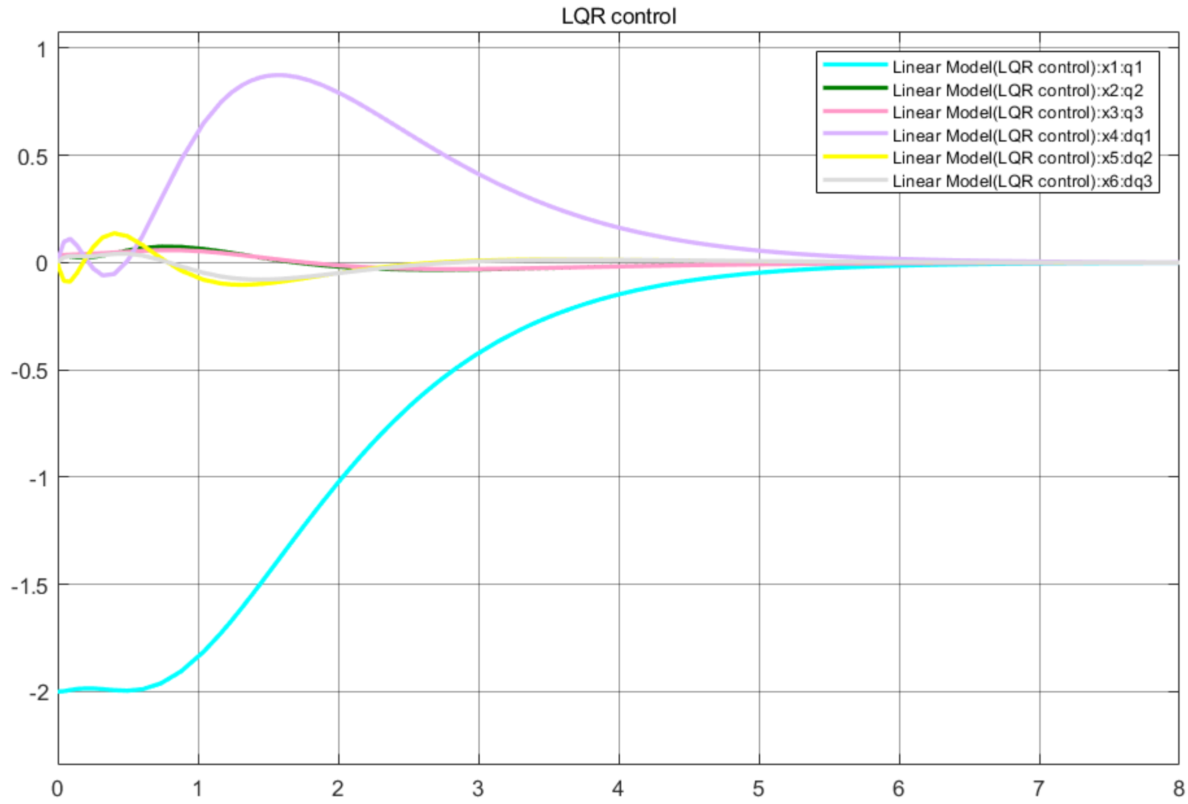


Fig.2.5 Simulation results (LQR control)

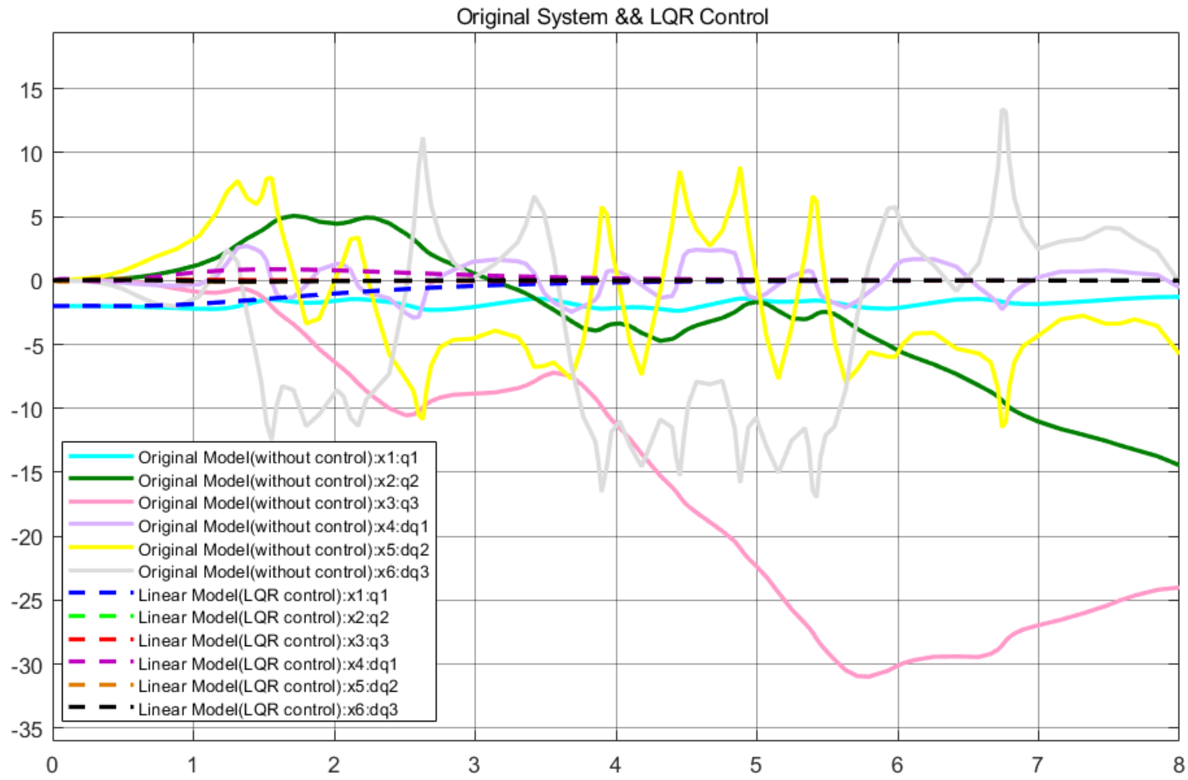
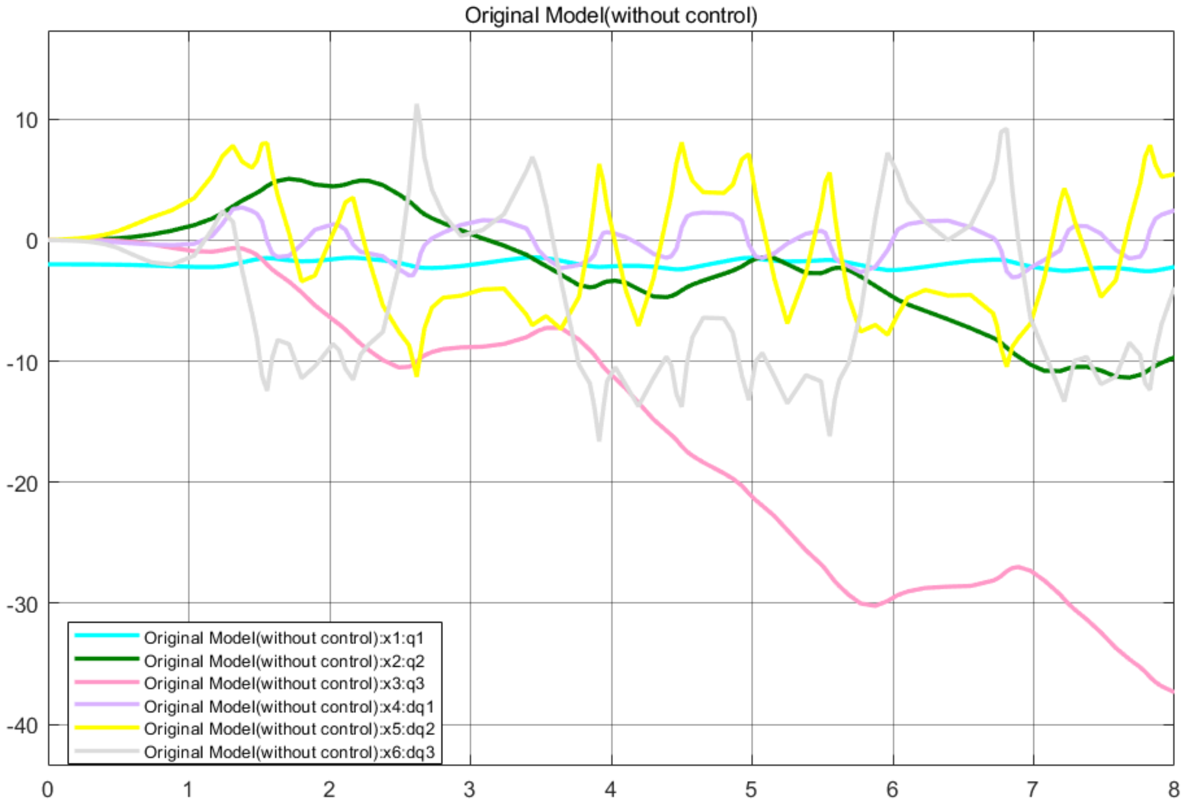


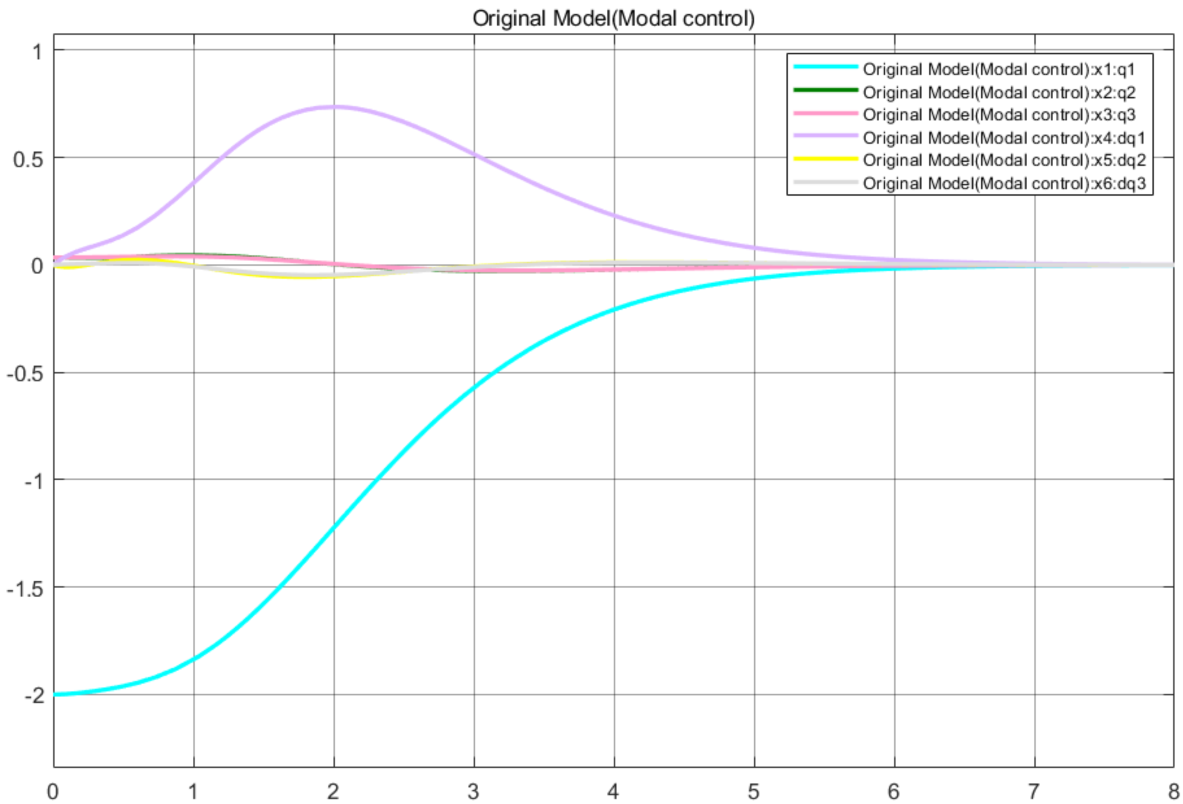
Fig.2.5 Simulation results (LQR control)Compare with original

8. 5. Simulation—original system &&Modal control &&LQR control

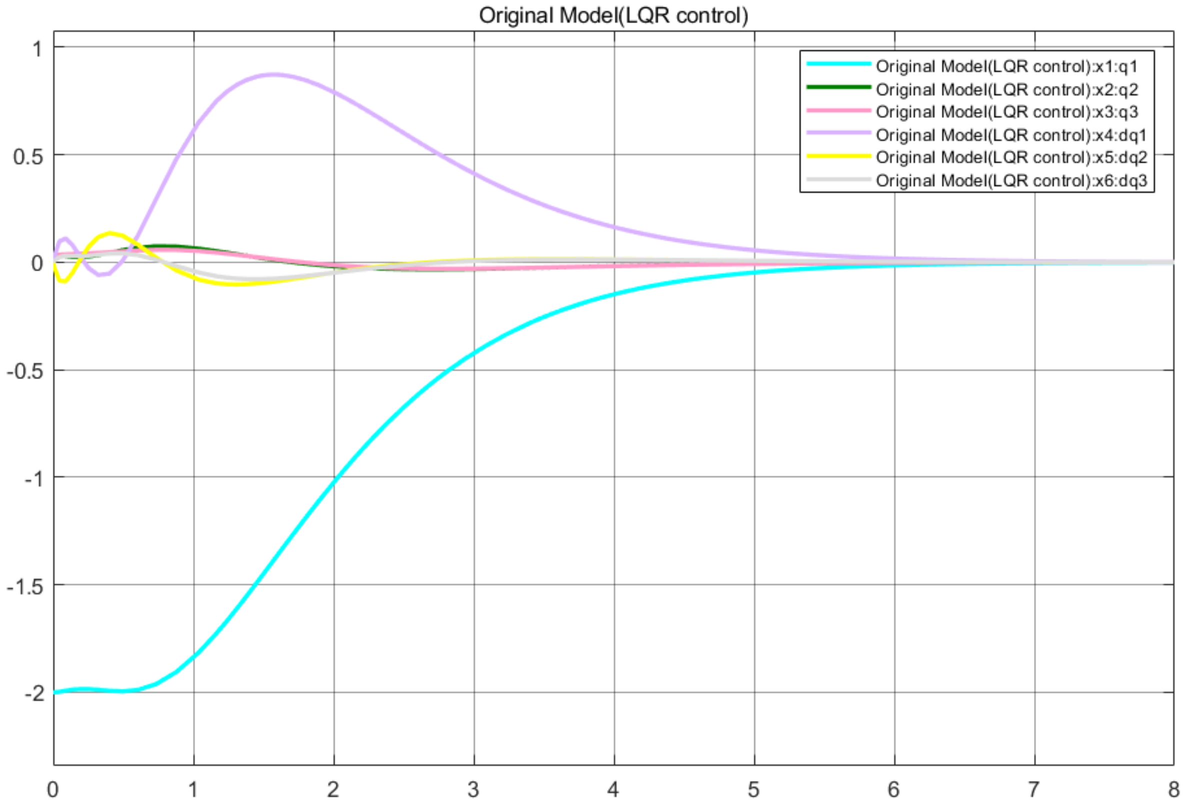
Original system (without control)



Original system (Modal control)



Original system (LQR control)



10. conclusion

> 1. Linear systems have many well-established control laws and are easy to analyze and computerized (using matrices).
> 2. After we use a certain method to linearize the nonlinear system, we can design its control scheme through many kinds of linear system control schemes, so the linearization of the nonlinear system is very important and very beneficial.
> 3. From the experimental results, (the initial value and control method and system used in the experiment), the control scheme designed after using the linearization method can be applied to the original system to a certain extent, and better control results can be obtained.
> 4. **Note:** However, it is also found in the experiment (due to the space problem, not shown in the report), for some initial values, the control scheme designed for the linear system cannot make the original system reach stability.
>
>   **Therefore,** in some cases we can use the linearized system to design the control scheme because it is easy for us to analyze and manipulate.
>
>   **However,** before practical application, the control scheme must be applied to the original nonlinear system to verify that it can be extended to nonlinear systems