Practice 5

Student Information

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> k = 7

★ Note :The Equilibrium Points in the report is calculated in practice 4

1)

$$\begin{aligned}\dot{x}_1 &= -x_1 + 14x_1^3 + x_2 \\ \dot{x}_2 &= -x_1 - 7x_2\end{aligned} \tag{40}$$

1.1. find the equilibrium points

Equilibrium points :

$$x^{\star} = (0;0)\ ,(\frac{2}{7};-\frac{2}{49})\ ,(-\frac{2}{7};\frac{2}{49}) \tag{41}$$

2. Linearize the system (using the Jacobian matrix)

Jacobian matrix :

$$\begin{gathered} J = \frac{\partial f}{\partial x} = \begin{bmatrix} -1+42x_1^2 & 1 \\ -1 & -7 \end{bmatrix} \\ \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} \approx \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}_{|x_1=a,x_2=b} \begin{bmatrix} x_1 - a \\ x_2 - b \end{bmatrix} \end{gathered} \tag{42}$$

3. investigate stability and behaviour & compare the phase portraits

2.1 at $x^{1\star} = (0;0)$

$$\begin{gathered} A_1 = \begin{bmatrix} -1 & 1 \\ -1 & -7 \end{bmatrix} \\ \dot{x} = A_1 \cdot x \end{gathered} \tag{43}$$

$$\begin{gathered} \lambda_1 = -4 + 2\sqrt{2} \approx -1.1716 < 0 \\ \lambda_2 = -4 - 2\sqrt{2} \approx -6.8284 < 0 \end{gathered} \tag{44}$$

Equilibrium point type : Stable node asymptotically stable

Phase plot 1 (Phase diagram of the original system)

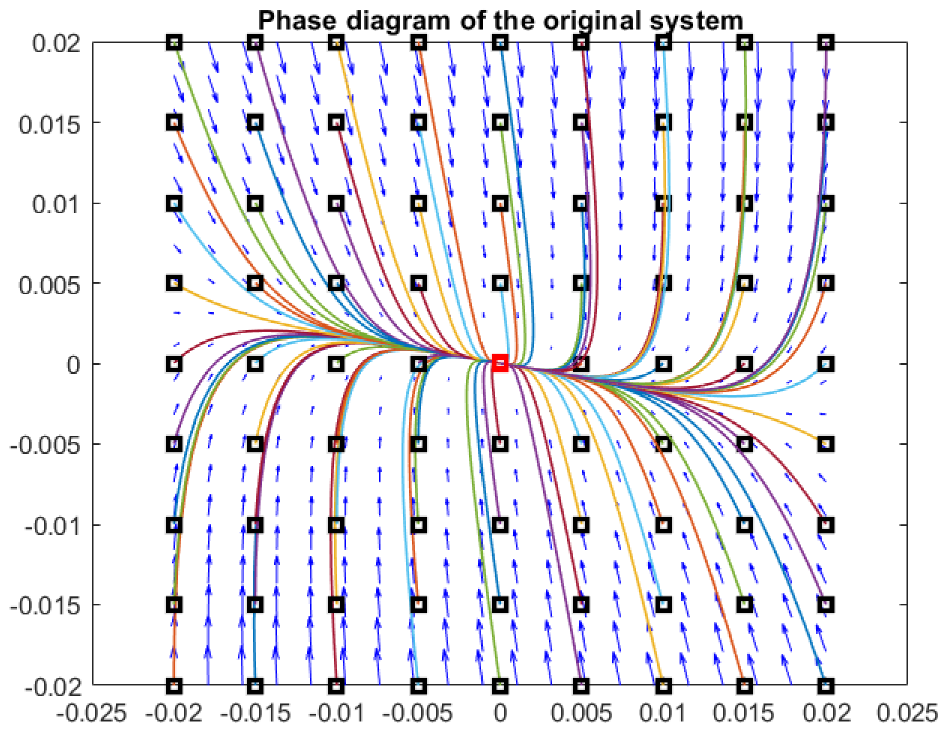

Fig.1.1 Phase diagram of the original system

Phase plot 2 (Phase diagram of the linearized system)

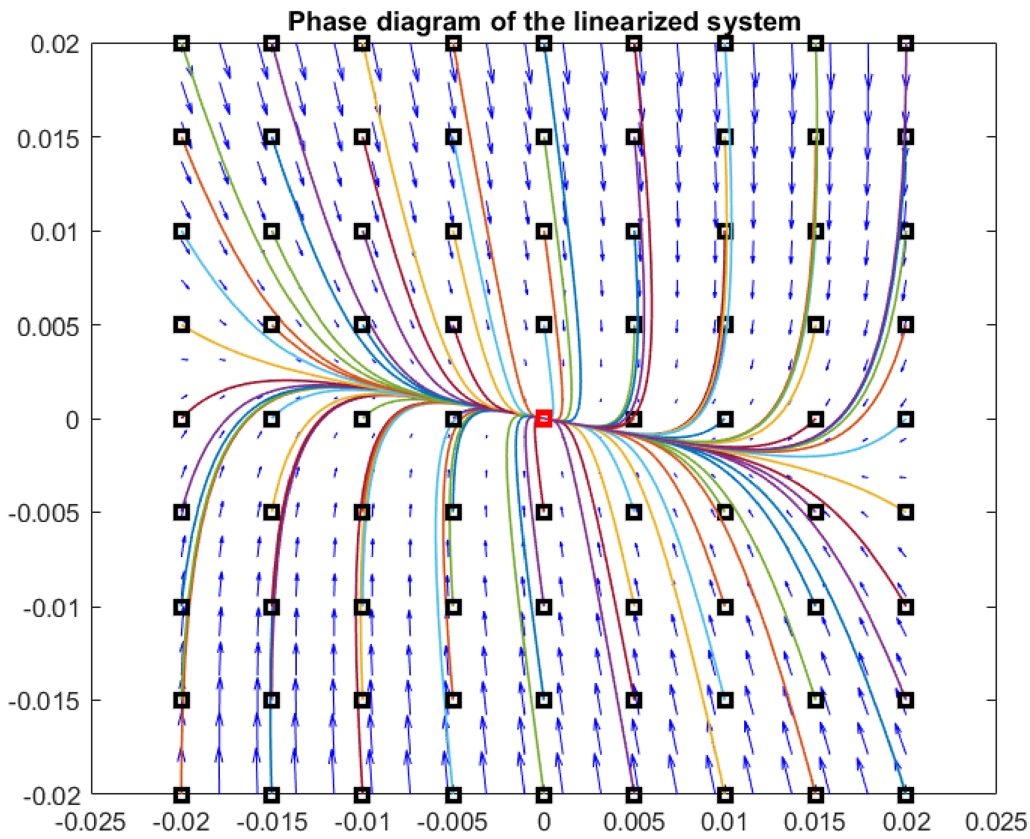

Fig.1.2 Phase diagram of the linearized system (0 ; 0)

2.2 at $x^{2\star} = (\frac{2}{7}; -\frac{2}{49})$ or$x^{3\star} = (-\frac{2}{7}; \frac{2}{49})$

$$A_2 = \begin{bmatrix} \frac{17}{7} & 1 \\ -1 & -7 \end{bmatrix}$$
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} \approx A_2 \cdot \begin{bmatrix} x_1 - \frac{2}{7} \\ x_2 + \frac{2}{49} \end{bmatrix} \text{or } A_2 \cdot \begin{bmatrix} x_1 + \frac{2}{7} \\ x_2 - \frac{2}{49} \end{bmatrix} \tag{45}$$

$$\lambda_1 \approx 0.2743 > 0$$
$$\lambda_2 \approx -6.8625 < 0 \tag{46}$$

Equilibrium point type : Saddle

Phase plot 1 (Phase diagram of the original system)$x^{2\star} = (\frac{2}{7}; -\frac{2}{49})$

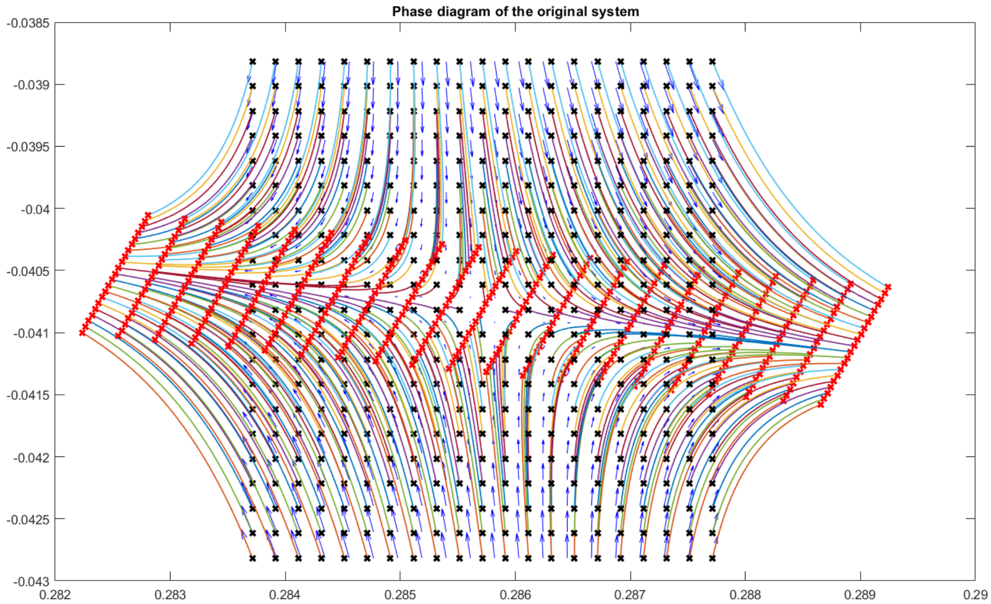

Fig.1.3 Phase diagram of the original system

Phase plot 2 (Phase diagram of the linearized system)$x^{2\star} = (\frac{2}{7}; -\frac{2}{49})$

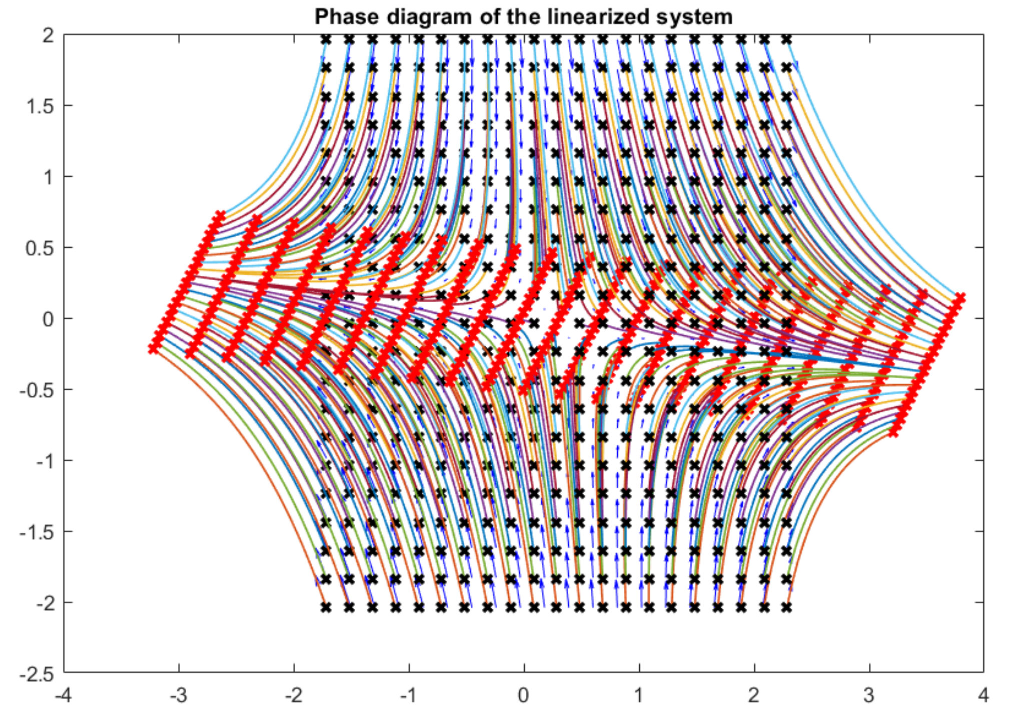

Fig.1.4 Phase diagram of the linearized system

Phase plot 3 (Phase diagram of the original system)$x^{3\star} = (-\frac{2}{7}; \frac{2}{49})$

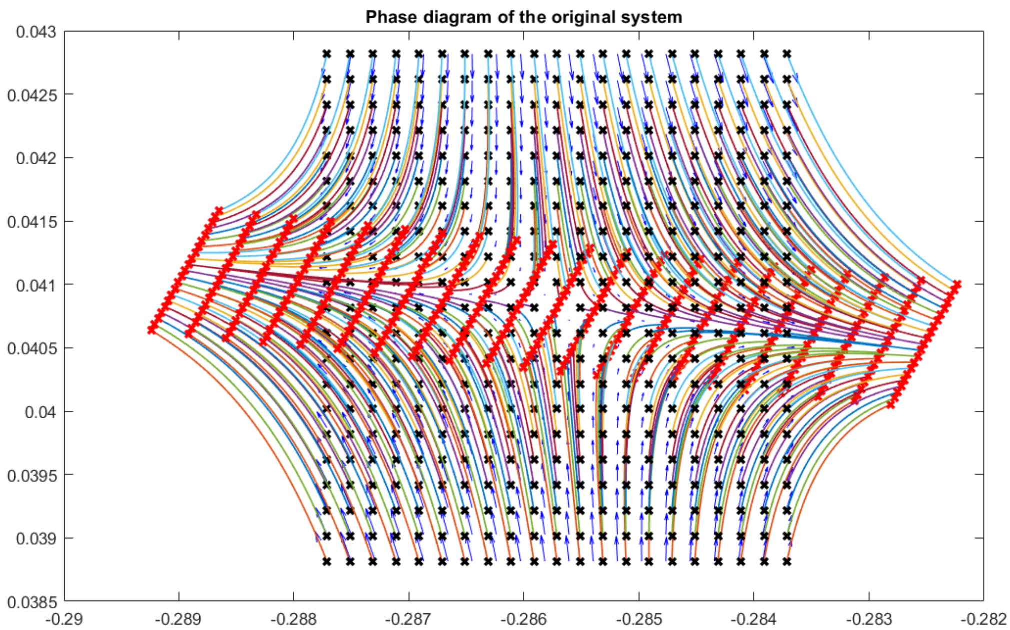

Fig.1.5 Phase diagram of the original system

Phase plot 4 (Phase diagram of the linearized system)$x^{3\star} = (-\frac{2}{7}; \frac{2}{49})$

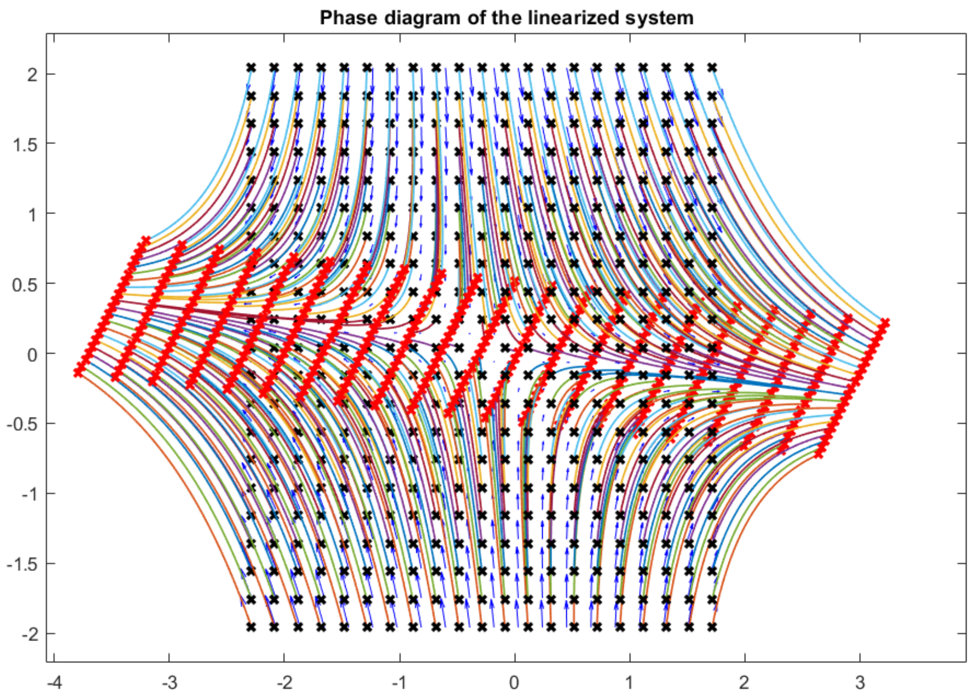

Fig.1.6 Phase diagram of the linearized system

2)

$$\dot{x}_1 = 7x_1 + x_1x_2$$
$$\dot{x}_2 = -x_2 + x_2^2 + x_1x_2 - x_1^3 \tag{47}$$

1. find the equilibrium points

Equilibrium points :

$$(0;0)\ ,(0;1)\ ,(3.2219;-7) \tag{48}$$

2. Linearize the system (using the Jacobian matrix)

Jacobian matrix :

$$J = \frac{\partial f}{\partial x} = \begin{bmatrix} 7+x_2 & x_1 \\ x_2 - 3x_1^2 & -1+2x_2+x_1 \end{bmatrix} \tag{49}$$

3. investigate stability and behaviour & compare the phase portraits

2.1 at $x^{1\star} = (0;0)$

$$A_1 = \begin{bmatrix} 7 & 0 \\ 0 & -1 \end{bmatrix}$$
$$dx = A_1 \cdot x \tag{50}$$

$$\lambda_1 = -1 - 1.1716 < 0$$
$$\lambda_2 = 7 > 0 \tag{51}$$

Equilibrium point type : Saddle

Phase plot 1 (Phase diagram of the original system)

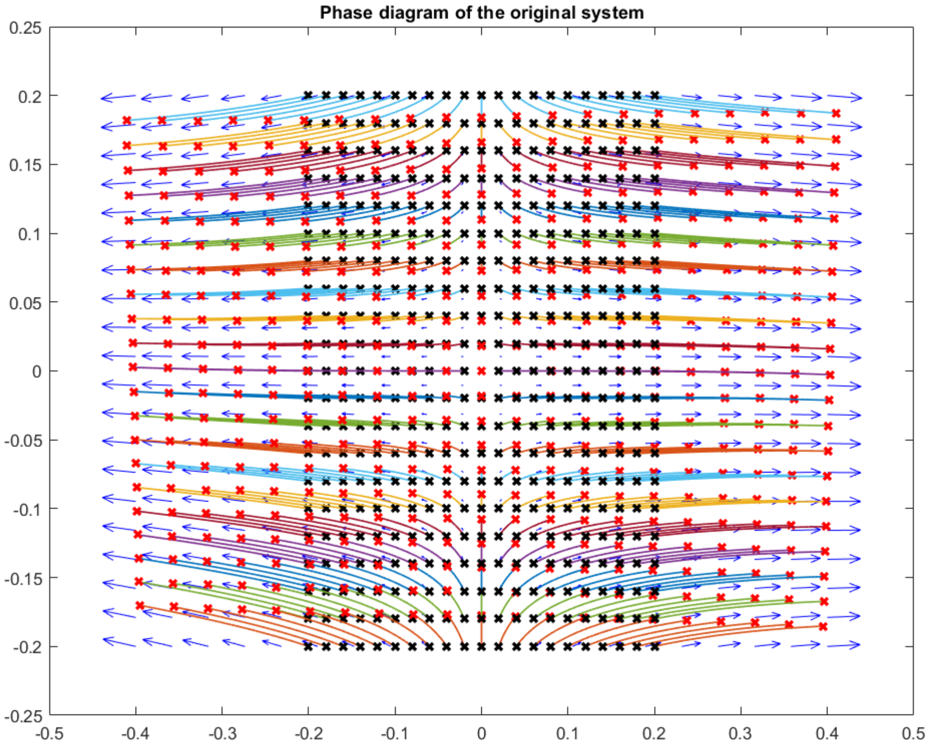

Fig.2.1 Phase diagram of the original system

Phase plot 2 (Phase diagram of the linearized system)

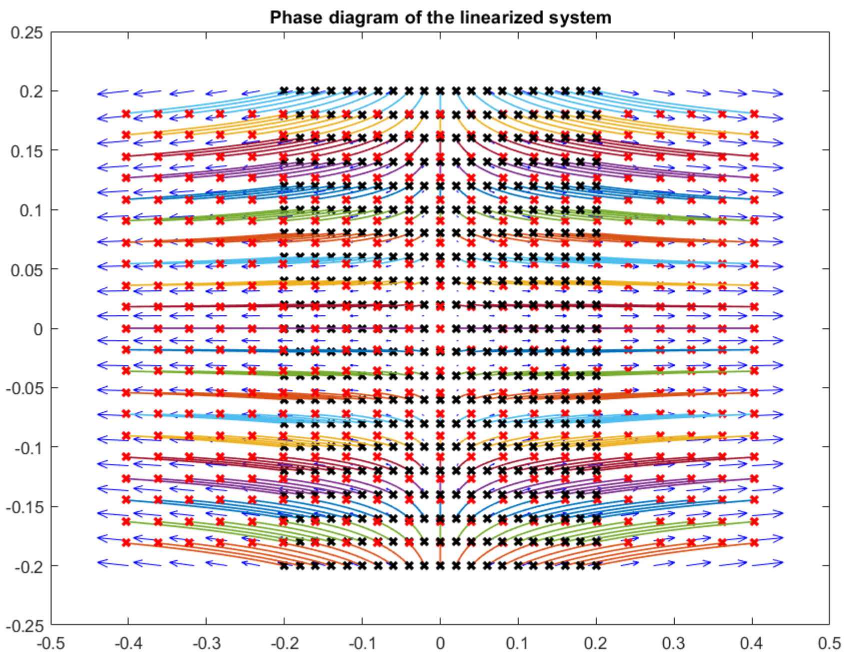

Fig.2.2 Phase diagram of the linearized system

2.2 at $x^{2\star} = (0;1)$

$$
\begin{gathered}
A_2 = \begin{bmatrix} 8 & 0 \\ 1 & 1 \end{bmatrix} \\
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} \approx A_2 \cdot \begin{bmatrix} x_1 \\ x_2 - 1 \end{bmatrix}
\end{gathered} \tag{52}
$$

$$
\begin{gathered}
\lambda_1 = 1 > 0 \\
\lambda_2 = 8 > 0
\end{gathered} \tag{53}
$$

Equilibrium point type : Unstable node

Phase plot 1 (Phase diagram of the original system)

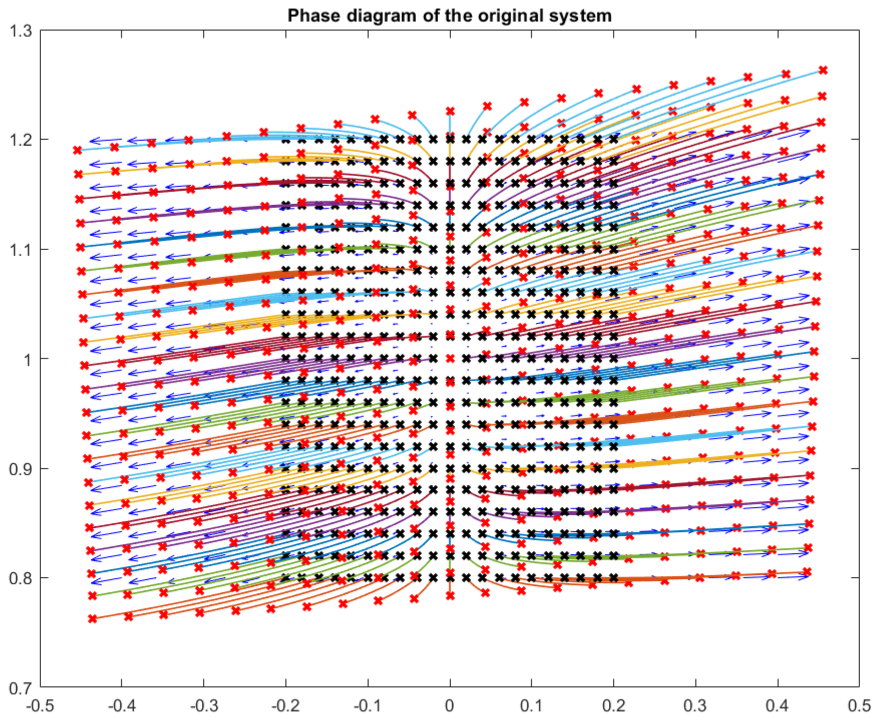

Fig.2.3 Phase diagram of the original system

Phase plot 2 (Phase diagram of the linearized system)

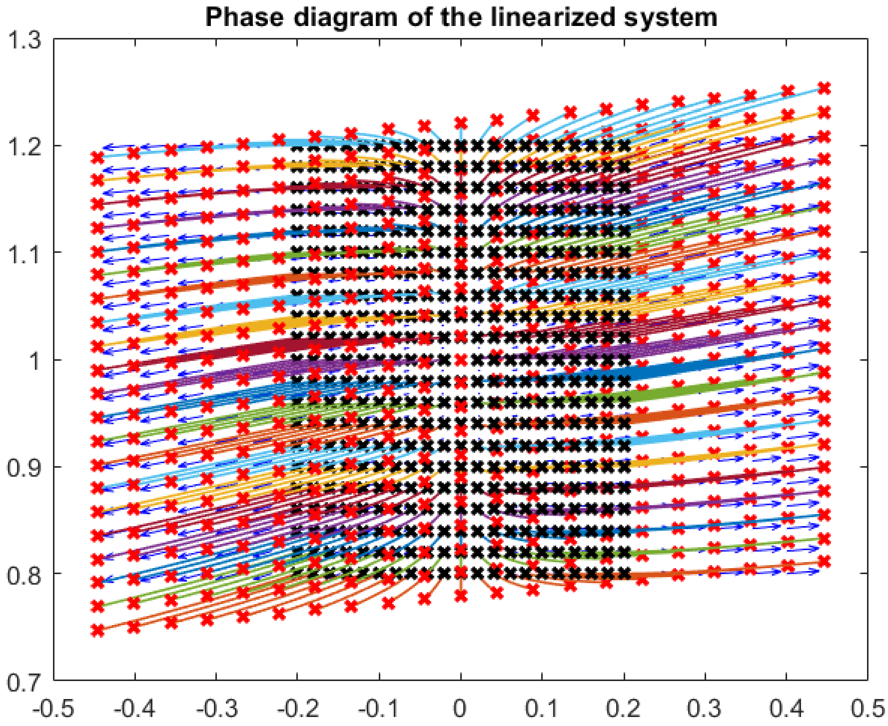

Fig.2.4 Phase diagram of the linearized system

2.3 at $x^{3\star} = (3.2219; -7)$

$$
\begin{aligned}
A_3 &= \begin{bmatrix} 0 & 3.2219 \\ -38.1419 & -11.7781 \end{bmatrix} \\
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &\approx A_3 \cdot \begin{bmatrix} x_1 - 3.2219 \\ x_2 + 7 \end{bmatrix}
\end{aligned} \tag{54}
$$

$$
\begin{aligned}
\lambda_1 &\approx -5.8890 + 9.3919i \\
\lambda_2 &\approx -5.8890 - 9.3919i
\end{aligned} \tag{55}
$$

Equilibrium point type : Stable focus

Phase plot 1 (Phase diagram of the original system)$x^{3\star} = (3.2219; -7)$

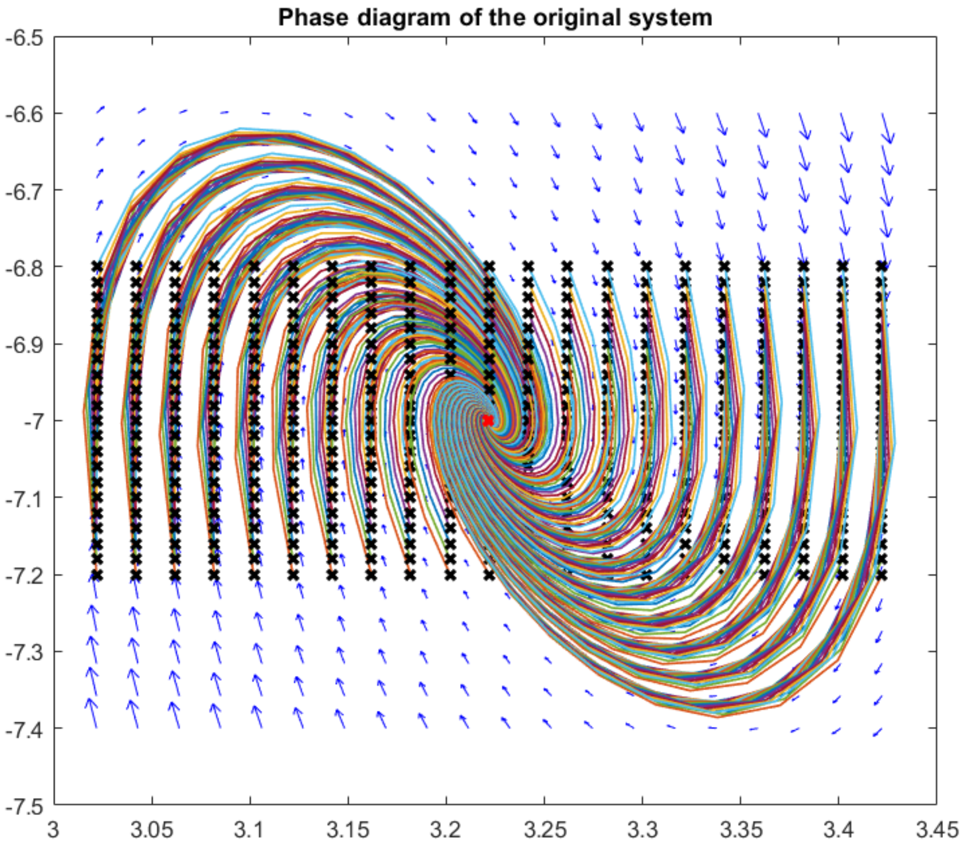

Fig.2.5 Phase diagram of the original system

Phase plot 2 (Phase diagram of the linearized system) $x^{3\star} = (3.2219; -7)$

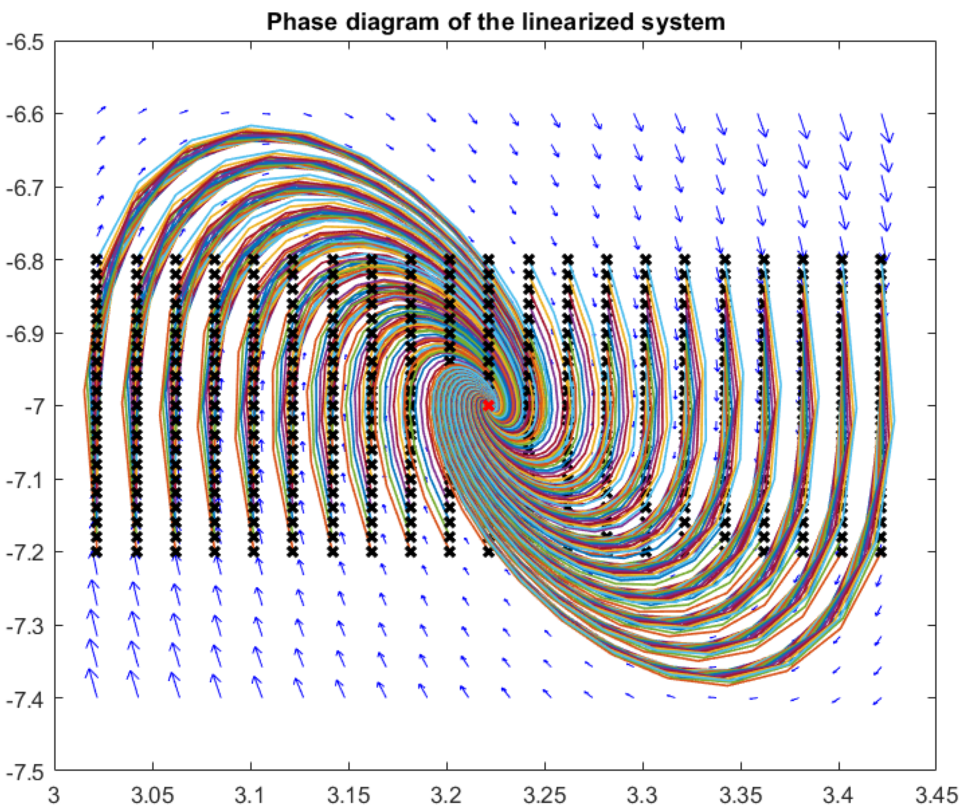

Fig.2.6 Phase diagram of the linearized system

3)

$$\begin{aligned}\dot{x}_1 &= 7x_2 \\ \dot{x}_2 &= -x_1 + x_2\left(1 - x_1^2 + 0.7x_1^4\right)\end{aligned} \tag{56}$$

1. find the equilibrium points

Equilibrium points :

$$(0;0) \tag{57}$$

2. Linearize the system (using the Jacobian matrix)

Jacobian matrix :

$$J = \frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 7 \\ -1 - 2x_1x_2 + 2.8x_1^3x_2 & 1 - x_1^2 + 0.7x_1^4 \end{bmatrix} \tag{58}$$

3. investigate stability and behaviour & compare the phase portraits

2.1 at $x^{1\star} = (0;0)$

$$\begin{gathered} A_1 = \begin{bmatrix} 0 & 7 \\ -1 & 1 \end{bmatrix} \\ dx = A_1 \cdot x \end{gathered} \tag{59}$$

$$\begin{aligned}\lambda_1 &= 0.5000 + 2.5981i \\ \lambda_2 &= 0.5000 - 2.5981i\end{aligned} \tag{60}$$

Equilibrium point type : Unstable focus

Phase plot 1 (Phase diagram of the original system)

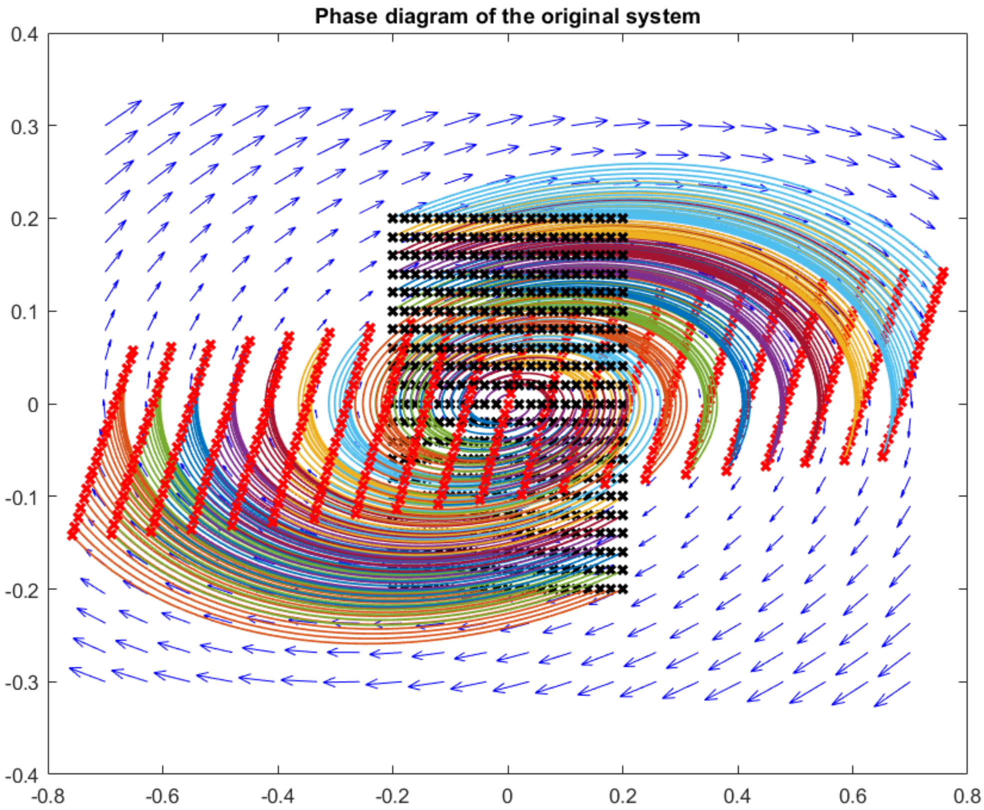

Fig.3.1 Phase diagram of the original system

Phase plot 2 (Phase diagram of the linearized system)

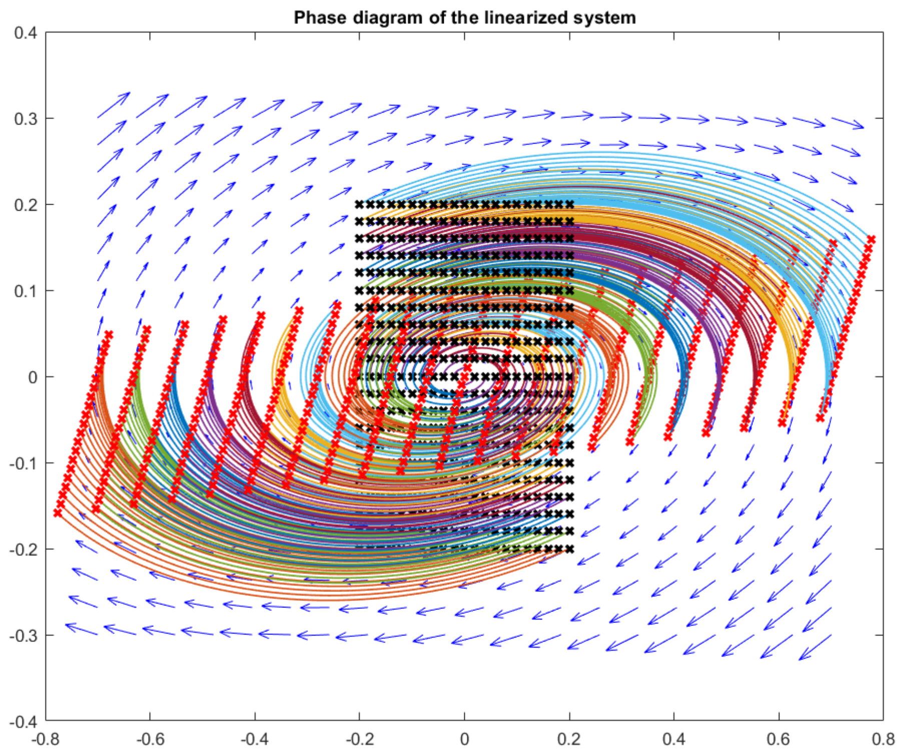

Fig.3.2 Phase diagram of the linearized system

4)

$$\begin{aligned}\dot{x}_1 &= 7\,(x_1 - x_2)\left(1 - x_1^2 - x_2^2\right)\\ \dot{x}_2 &= (x_1 + x_2)\left(1 - x_1^2 - x_2^2\right)\end{aligned} \tag{61}$$

1. find the equilibrium points

Equilibrium points :

1.

$$\text{All points on the circle :} \qquad x_1^2 + x_2^2 = 1 \tag{62}$$

2.

$$(0;0) \tag{63}$$

2. Linearize the system (using the Jacobian matrix)

Jacobian matrix :

$$J = \frac{\partial f}{\partial x} = \begin{bmatrix} 7\left(-3x_1^2 - x_2^2 + 2x_1x_2 + 1\right) & 7\left(x_1^2 + 3x_2^2 - 2x_1x_2 - 1\right) \\ -3x_1^2 - x_2^2 - 2x_1x_2 + 1 & -x_1^2 - 3x_2^2 - 2x_1x_2 + 1 \end{bmatrix} \tag{64}$$

3. investigate stability and behaviour & compare the phase portraits

2.1 at $x^{1\star} = (0;0)$

$$\begin{aligned} A_1 &= \begin{bmatrix} 7 & -7 \\ 1 & 1 \end{bmatrix} \\ dx &= A_1 \cdot x \end{aligned} \tag{65}$$

$$\begin{aligned} \lambda_1 &= 5.4142 > 0 \\ \lambda_2 &= 2.5858 > 0 \end{aligned} \tag{66}$$

Equilibrium point type : Unstable node

Phase plot 1 (Phase diagram of the original system)

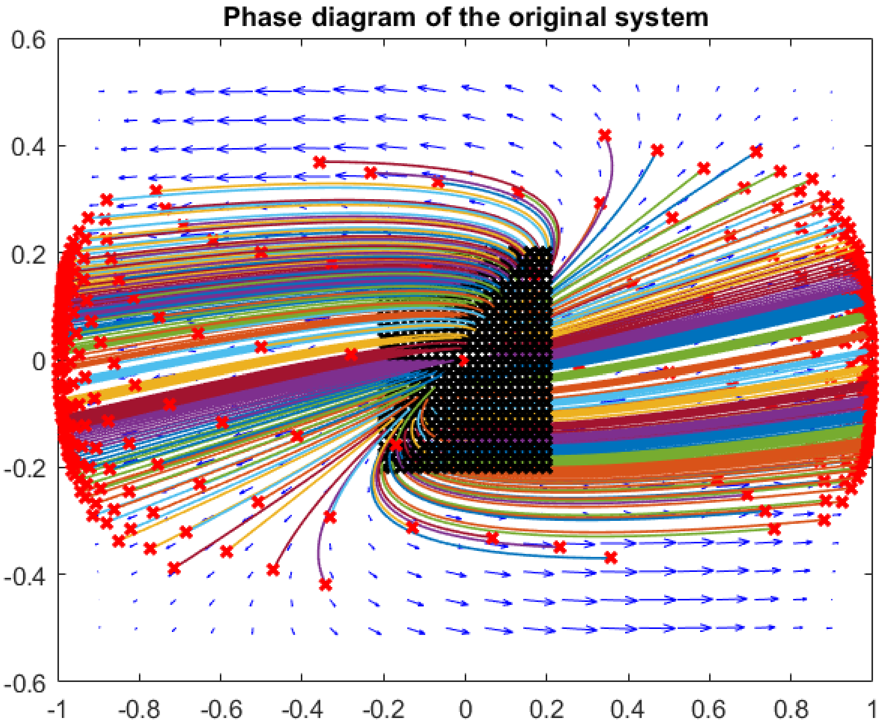

Fig.4.1 Phase diagram of the original system

Phase plot 2 (Phase diagram of the linearized system)

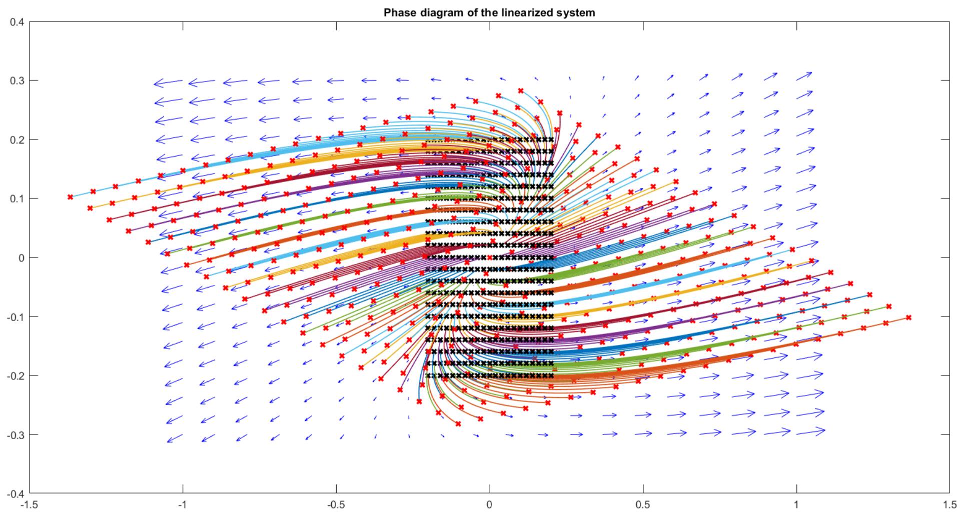

Fig.4.2 Phase diagram of the linearized system

2.1 at $x^{1\star} = (x_1^2 + x_2^2 = 1)$ any point on the circle

$$A_2 = \begin{bmatrix} 7\left(-3x_1^2 - x_2^2 + 2x_1x_2 + 1\right) & 7\left(x_1^2 + 3x_2^2 - 2x_1x_2 - 1\right) \\ -3x_1^2 - x_2^2 - 2x_1x_2 + 1 & -x_1^2 - 3x_2^2 - 2x_1x_2 + 1 \end{bmatrix} \tag{67}$$
$$x_1^2 + x_2^2 = 1$$

There are countless equilibrium points on the ring, and the linearized matrix A_i eigenvalues of each point are different.

For the calculation of the eigenvalues of all equilibrium points, draw as shown in the following figure:

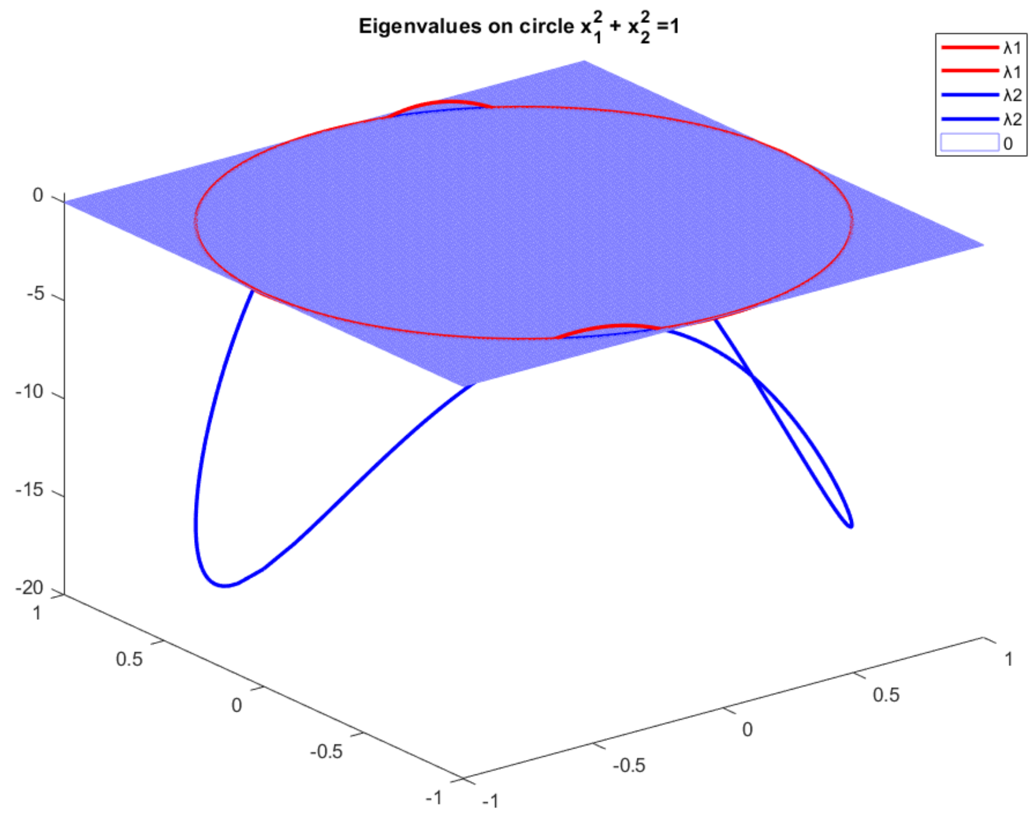

Fig.4.3 eigenvalues of all equilibrium points

As can be seen from the above figure, there are two cases of eigenvalues：

1. One eigenvalue is 0, the other >0 **It is not possible to tell what type of balance point it is by the linearization method**
2. One eigenvalue is 0, the other < 0 **It is not possible to tell what type of balance point it is by the linearization method**

We take one of the points$(x_1 = \frac{\sqrt{2}}{2}; x_2 = \frac{\sqrt{2}}{2})$ as an example to linearize

$$A_3 = \begin{bmatrix} 0 & 0 \\ -2 & -2 \end{bmatrix}$$
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} \approx A_3 \cdot \begin{bmatrix} x_1 - \frac{\sqrt{2}}{2} \\ x_2 - \frac{\sqrt{2}}{2} \end{bmatrix} \tag{68}$$

$$\lambda_1 = 0$$
$$\lambda_2 < 0 \tag{69}$$

Because only one eigenvalue is 0, we can't tell what type of balance point it is， It is between the center point and the stable point.

Plot the phase diagram of the linearized system as shown below:

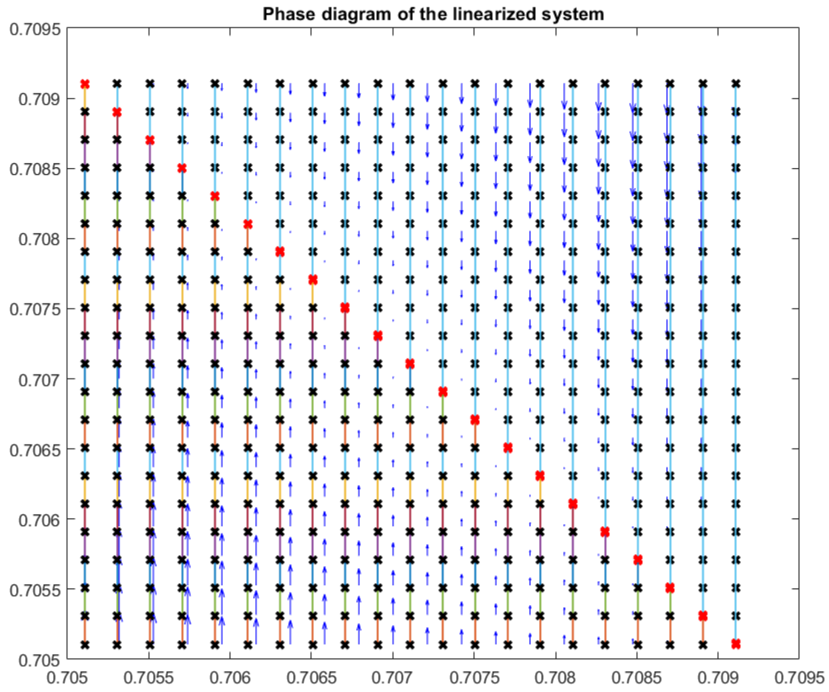

Fig.4.4 Phase diagram of the linearized system

From the phase diagram, we cannot judge the characteristics of this linearized equilibrium point either.

Draw a phase diagram of the original image to observe its properties:

Phase plot 1 (Phase diagram of the original system)

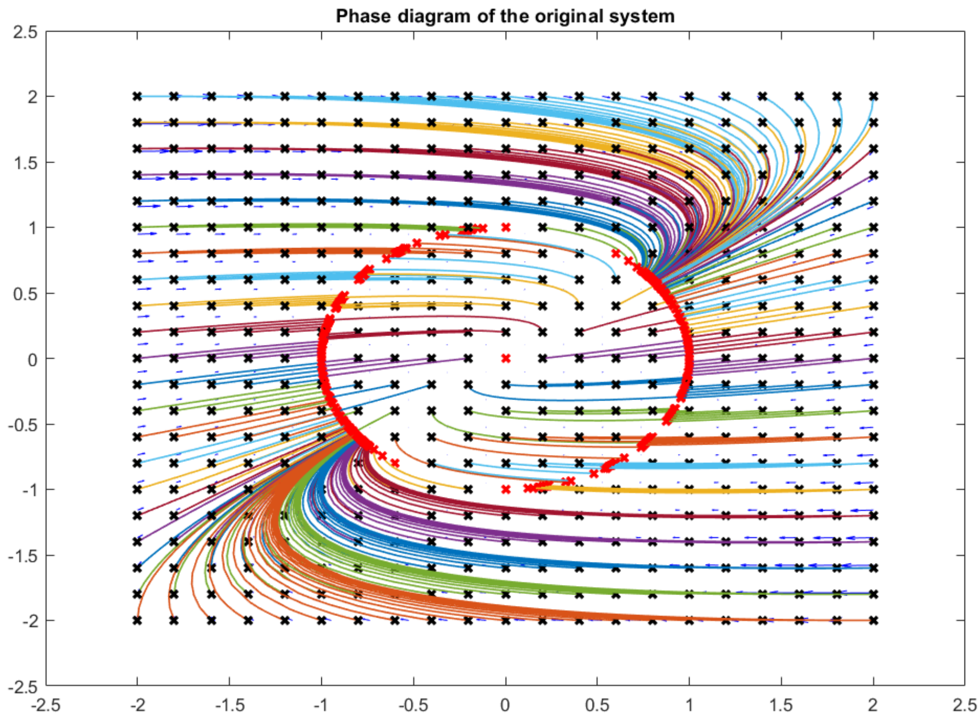

Fig.4.5 Phase diagram of the original system

We can see that the points within a certain range converge to part of the circle, and the point at the center of the circle remains balanced.

5)

$$\dot{x}_1 = -x_1^3 + 7x_2$$
$$\dot{x}_2 = 7x_1 - x_2^3 \quad (70)$$

1. find the equilibrium points

Equilibrium points :

$$(0;0)\ ,(\sqrt{7};\sqrt{7})\ ,(-\sqrt{7};-\sqrt{7}) \quad (71)$$

2. Linearize the system (using the Jacobian matrix)

Jacobian matrix :

$$J = \frac{\partial f}{\partial x} = \begin{bmatrix} -3x_1^2 & 7 \\ 7 & -3x_2^2 \end{bmatrix} \quad (72)$$

3. investigate stability and behaviour & compare the phase portraits

2.1 at $x^{1\star} = (0;0)$

$$A_1 = \begin{bmatrix} 0 & 7 \\ 7 & 0 \end{bmatrix} \quad (73)$$
$$dx = A_1 \cdot x$$

$$\lambda_1 = -7 < 0$$
$$\lambda_2 = 7 > 0 \quad (74)$$

Equilibrium point type : Saddle

Phase plot 1 (Phase diagram of the original system)

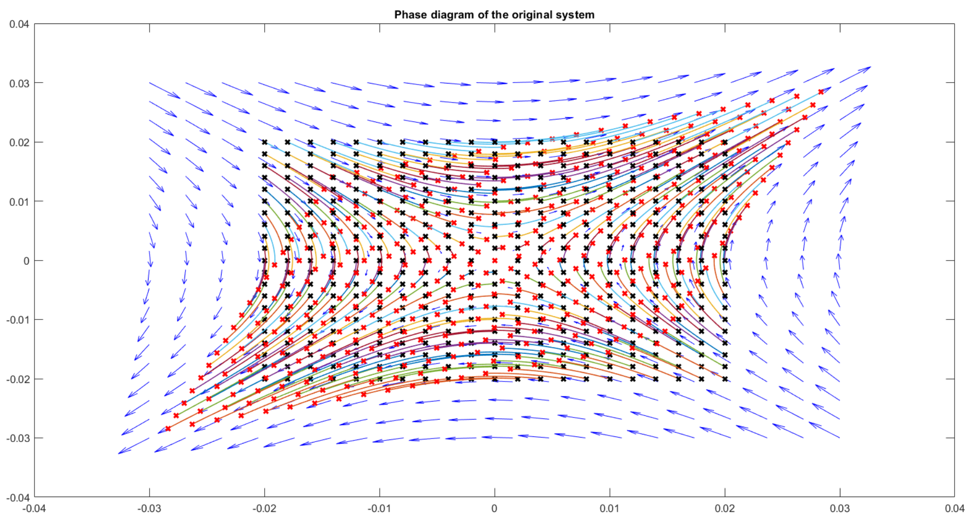

Fig.5.1 Phase diagram of the original system

Phase plot 2 (Phase diagram of the linearized system)

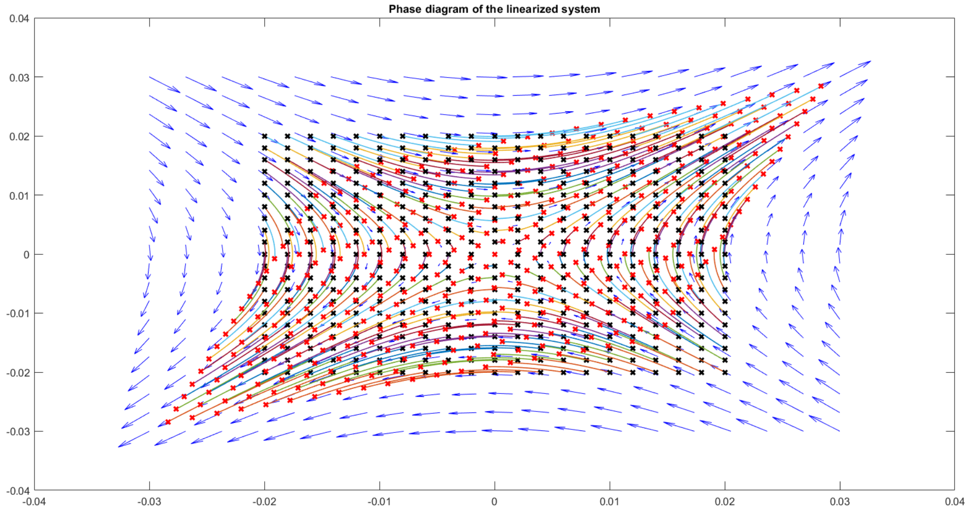

Fig.5.2 Phase diagram of the linearized system

2.2 at $x^{2\star} = (\sqrt{7}; \sqrt{7})$

$$A_2 = \begin{bmatrix} -21 & 7 \\ 7 & -21 \end{bmatrix}$$
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} \approx A_2 \cdot \begin{bmatrix} x_1 - \sqrt{7} \\ x_2 - \sqrt{7} \end{bmatrix} \tag{75}$$

$$\begin{aligned} \lambda_1 &= -28 < 0 \\ \lambda_2 &= -14 < 0 \end{aligned} \tag{76}$$

Equilibrium point type : Stable node

Phase plot 1 (Phase diagram of the original system) $x^{2\star} = (\sqrt{7}; \sqrt{7})$

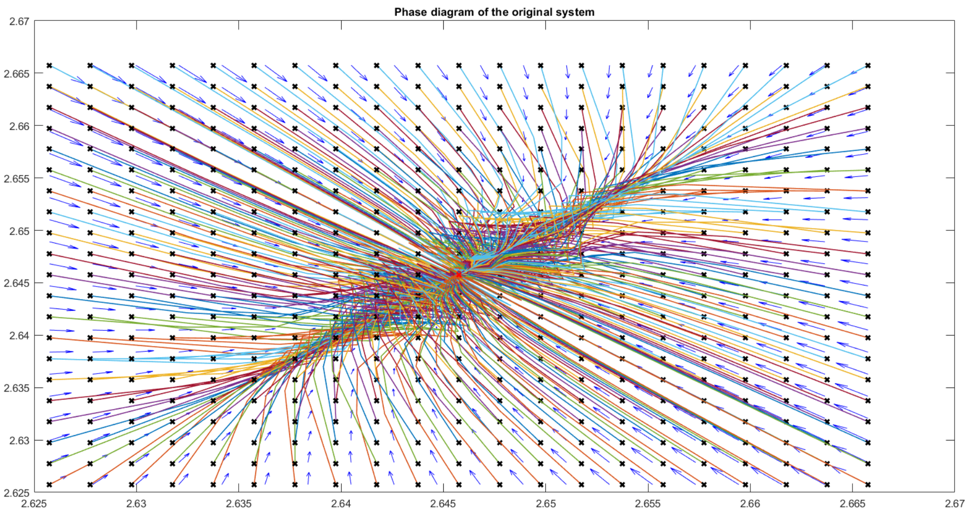

Fig.5.3 Phase diagram of the original system

Phase plot 2 (Phase diagram of the linearized system) $x^{2\star} = (\sqrt{7}; \sqrt{7})$

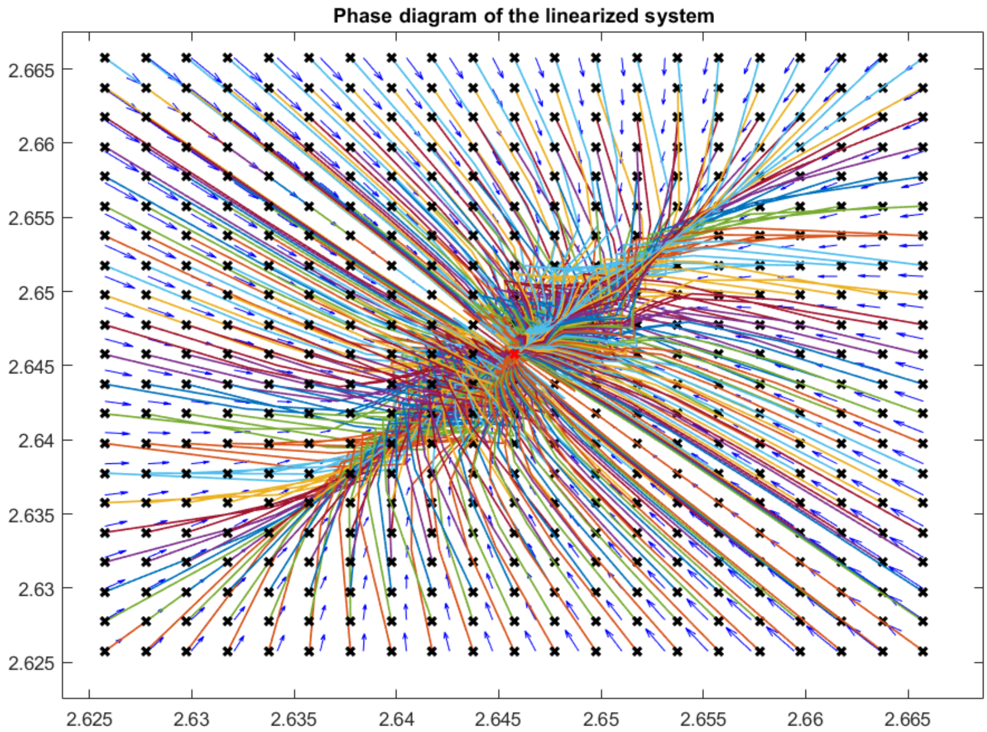

Fig.5.4 Phase diagram of the linearized system

2.3 at $x^{3\star} = (-\sqrt{7}; -\sqrt{7})$

$$A_3 = \begin{bmatrix} -21 & 7 \\ 7 & -21 \end{bmatrix}$$
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} \approx A_3 \cdot \begin{bmatrix} x_1 + \sqrt{7} \\ x_2 + \sqrt{7} \end{bmatrix} \tag{77}$$

$$\lambda_1 = -28 < 0$$
$$\lambda_2 = -14 < 0 \tag{78}$$

Equilibrium point type : Unstable node

Phase plot 1 (Phase diagram of the original system)$x^{2\star} = (-\sqrt{7}; -\sqrt{7})$

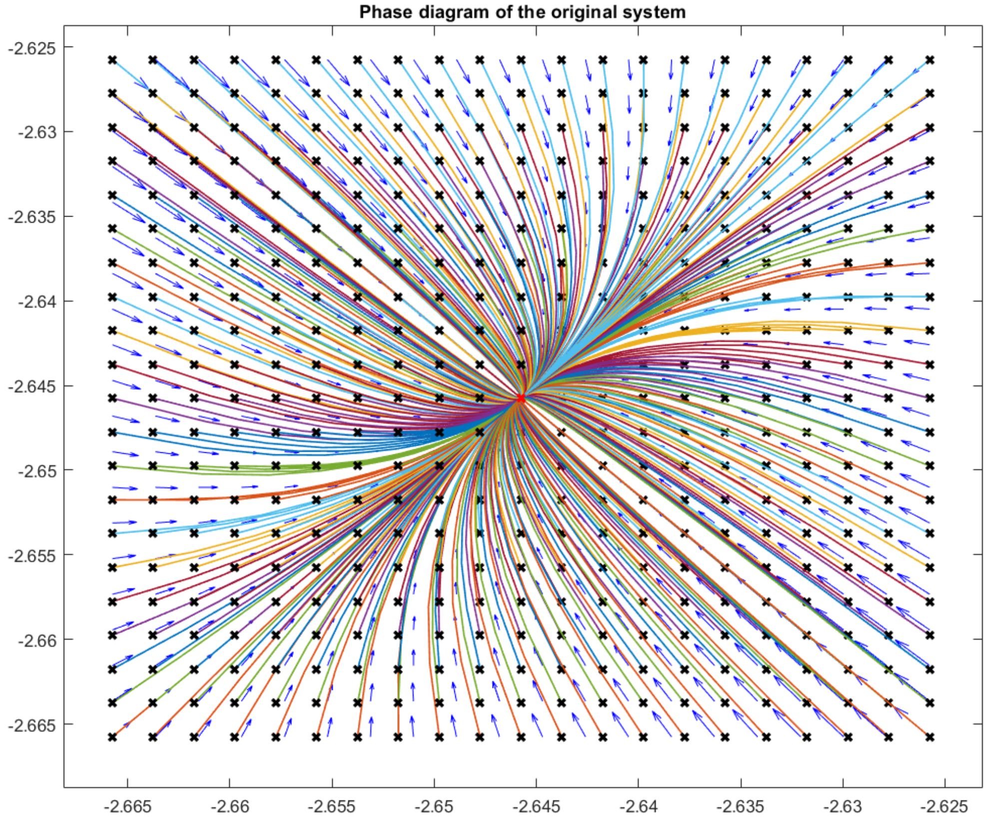

Fig.5.5 Phase diagram of the original system

Phase plot 2 (Phase diagram of the linearized system)$x^{2\star} = (-\sqrt{7}; -\sqrt{7})$

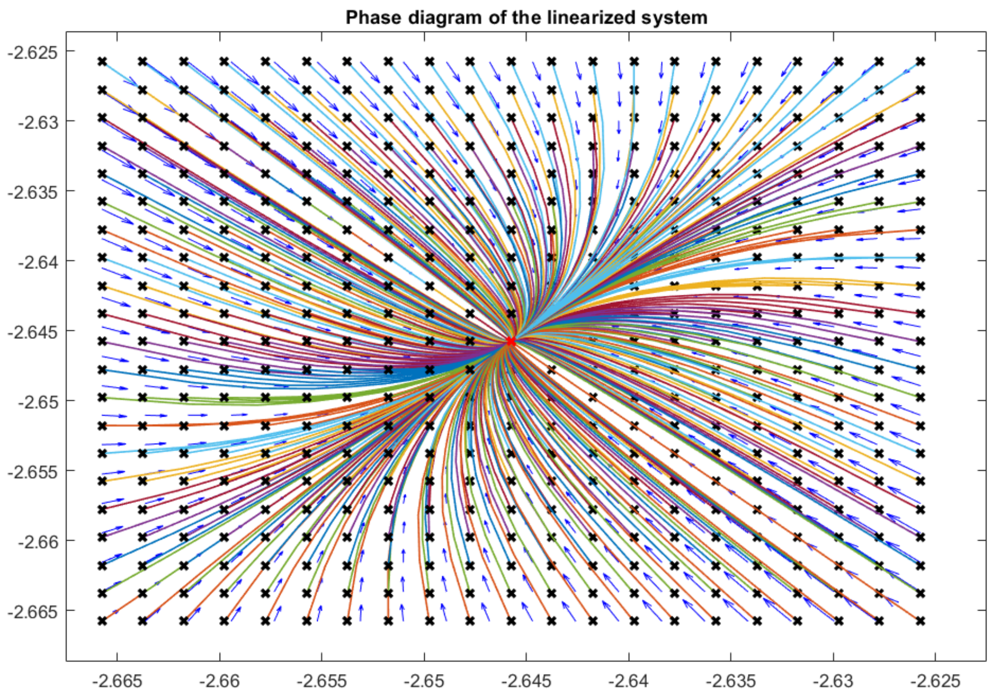

Fig.5.6 Phase diagram of the linearized system

Conclusion

From the experimental results we can draw the following conclusions:

Why do we use the Jacobian matrix to linearize the equilibrium point?

1. Analyzing linear systems is **much simpler** than analyzing nonlinear systems
2. From the experimental results, for **most** of the **characteristics** and **stability** near the equilibrium point are the **same as** the linearized system

(Fig.1.1,Fig.1.2),(Fig.1.3,Fig.1.4),(Fig.1.5,Fig.1.6),(Fig.2.1,Fig.2.2),(Fig.2.5,Fig.2.6),(Fig.3.1,Fig.3.2),(Fig.5.1,Fig.5.2),(Fig.5.3,Fig.5.4),(Fig.5.5,Fig.5.6)

Are there any caveats (or limitations) we need to be aware of using this approach?

1. Nonlinear systems are far **more complex** than linear systems
2. It can be seen from some experimental results that the performance at the equilibrium point of the linearized system and the original nonlinear system may be **similar** but **not all are exactly the same** (and within **a certain small range need to be found**).

Meaning, this method is **not very suitable** for **high precision** and **large scale** analysis.

(Fig.2.3,Fig.2.4),(Fig.4.1,Fig.4.2)

3. When the calculated **eigenvalues do not belong to the regular linear system eigenvalues** (eg: $\lambda_1 = 0, \lambda_2 > 0 \text{or} \lambda_2 < 0$). We **cannot analyze stability** at equilibrium in this way

(Fig.4.3,Fig.4.4,Fig.4.5)