4.29 Homework

Student Information

Name : Xu Miao

ITMO number : 293687

HDU number : 19322103

Theory

Lagrangian function:

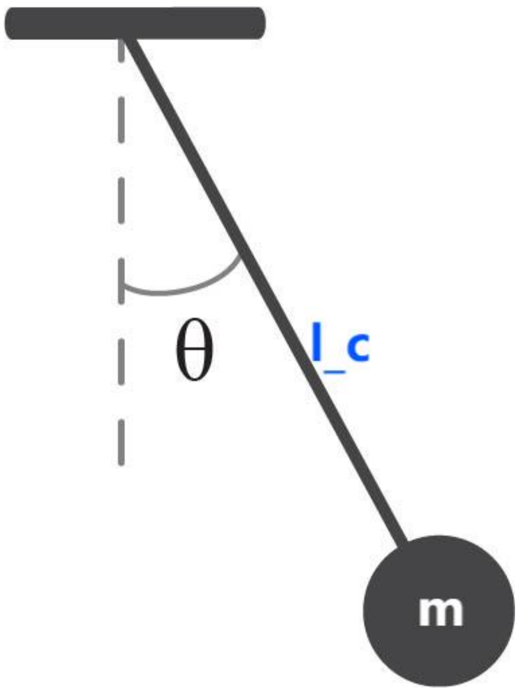
$$\mathcal{L}(q,\dot{q}) = \mathcal{T}(q,\dot{q}) - \mathcal{V}(q) \tag{1}$$

Euler-Lagrange equation:

$$\frac{d}{dt}\frac{\partial \mathcal{L}(q,\dot{q})}{\partial \dot{q}} - \frac{\partial \mathcal{L}(q,\dot{q})}{\partial q} = f \tag{2}$$

1-DOF Pendulum

1.1 schematic diagram



1.2 Mathematical model

Consider each energy separatly:

$$K = \frac{1}{2}(J_c + ml_c^2)\dot{\theta}^2$$
$$P = mgl_c(1 - \cos(\theta)) \tag{6}$$

Write Lagrangian function:

$$\mathcal{L} = K - P = \frac{1}{2}(ml_c^2)\dot{\theta}^2 - mgl_c(1 - \cos(\theta)) \tag{7}$$

Substitute Lagrangian into the Euler-Lagrange equation:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \dot{\theta}} &= (J_c + ml_c^2)\dot{\theta} \qquad \frac{\mathrm{d}}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right] (J_c + ml_c^2)\ddot{\theta} \\ \frac{\partial \mathcal{L}}{\partial \theta} &= -mgl_c \sin(\theta) \end{aligned} \tag{8}$$

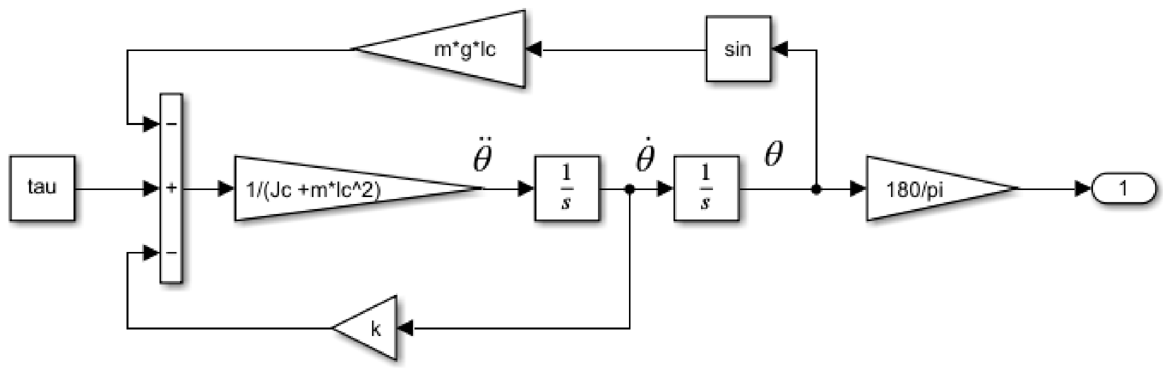
result:

$$(J_c + ml_c^2)\ddot{\theta} + mgl_c \sin(\theta) = \tau - k\dot{\theta} \tag{9}$$

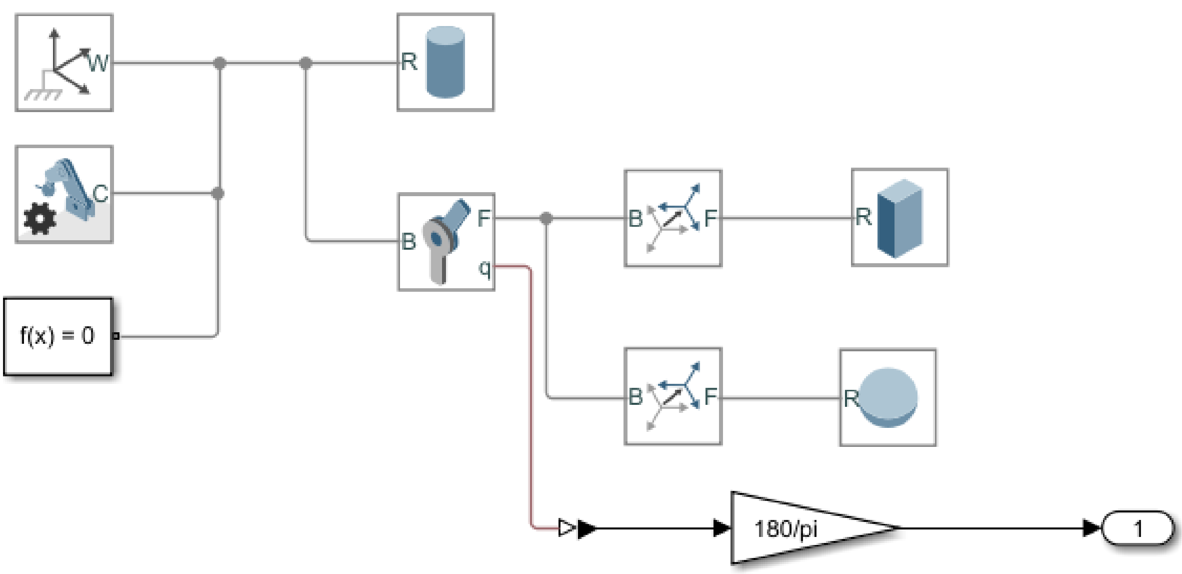
Simplify (6) to get:

$$\ddot{\theta} = -\frac{mgl_c}{J_c + ml_c^2}\sin(\theta) + \frac{1}{J_c + ml_c^2}(\tau - k\dot{\theta}) \tag{10}$$

1.3 Simulink model



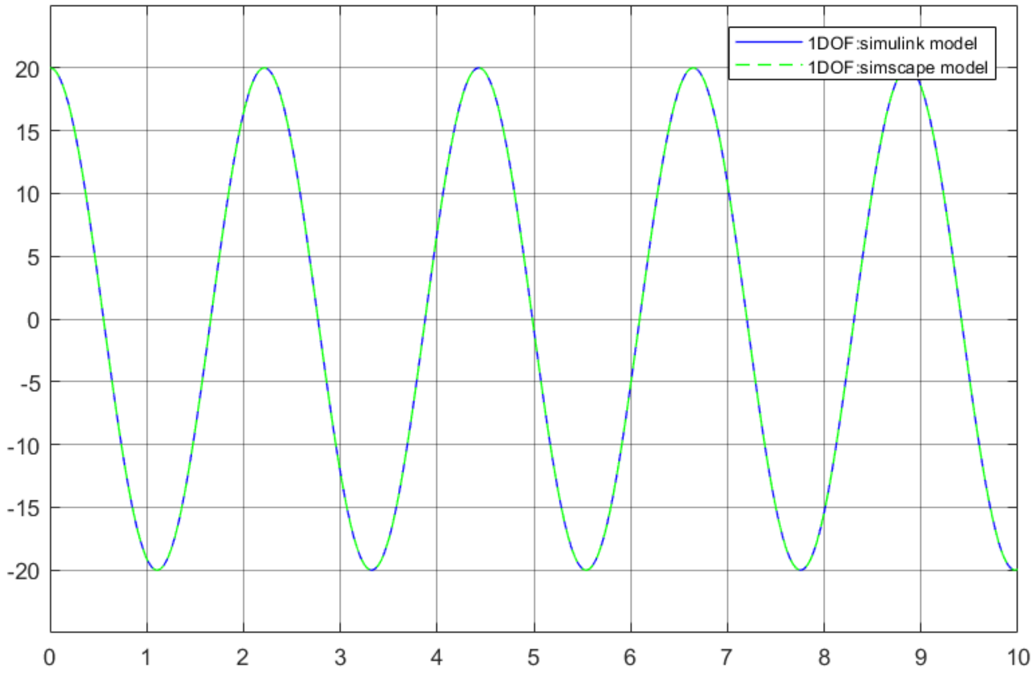
1.4 Simscape model



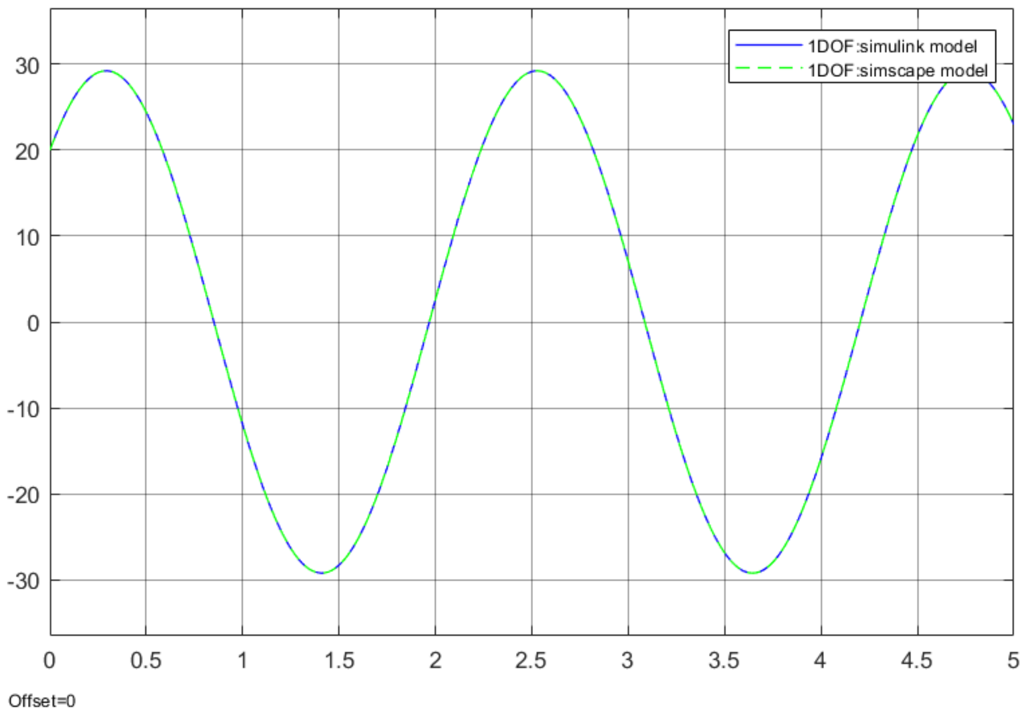
1.5 simulation && compare

without friction

$\tau = 0, k = 0, \theta(0) = 20\,\deg, \dot{\theta}(0) = 0$

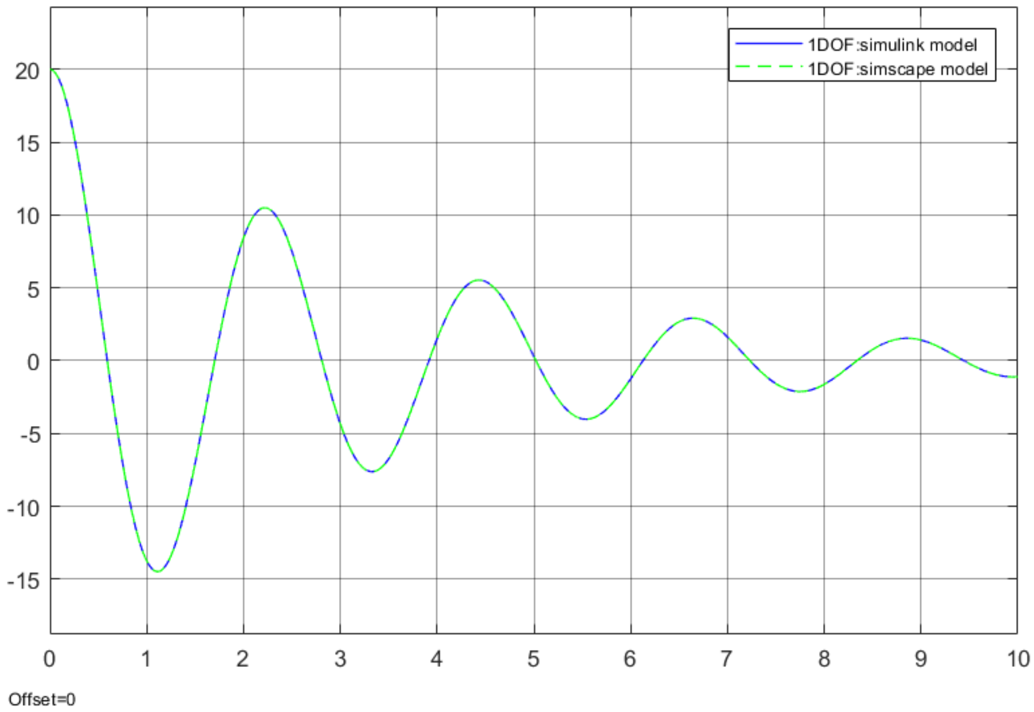


$\tau = 0, k = 0, \theta(0) = 20, \deg\,\dot{\theta}(0) = 60\,\deg/s$

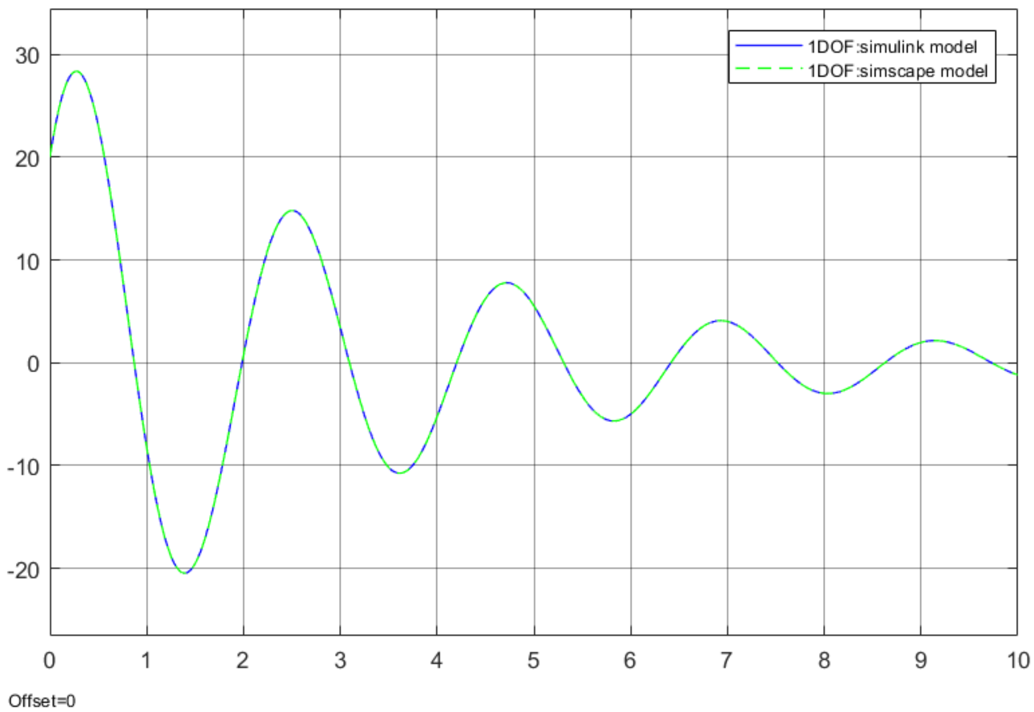


with friction

$\tau = 0, k = 3, \theta(0) = 20\,\deg, \dot{\theta}(0) = 0$

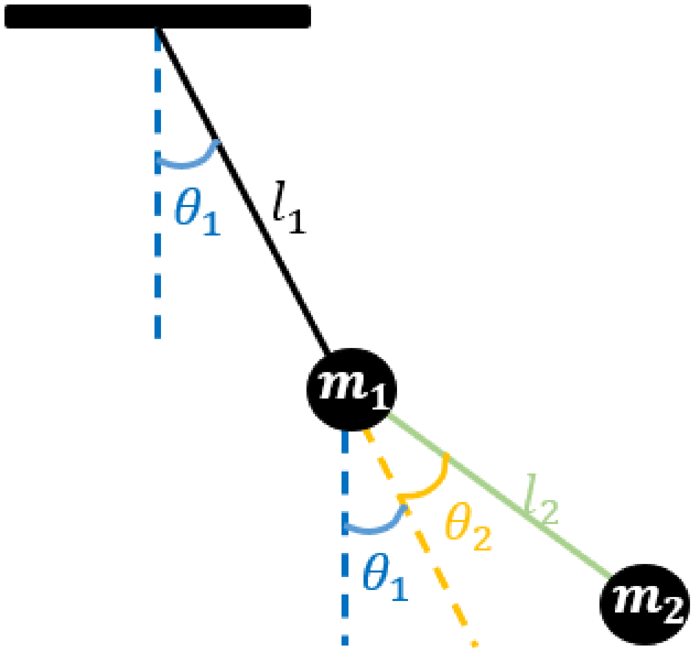


$\tau = 0, k = 3, \theta(0) = 20, \deg \dot{\theta}(0) = 60 \deg / s$



2-DOF Pendulum

2.1 schematic diagram



2.2 Mathematical model

2.2.1kinetic energy

Consider center of the mass for first and second link:

$$
\begin{aligned}
c_1 &= \begin{bmatrix} l_1 \cdot \sin\theta_1 \\ -l_1 \cdot \cos\theta_1 \end{bmatrix} \\
c_2 &= \begin{bmatrix} l_1 \sin\theta_1 + l_2 \sin(\theta_1 + \theta_2) \\ -(l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2)) \end{bmatrix}
\end{aligned} \tag{11}
$$

Find the derivatives:

$$
\begin{aligned}
v_1 &= \begin{bmatrix} l_1 \cos\theta_1 \\ l_1 \sin\theta_1 \end{bmatrix} \dot{\theta}_1 \\
v_2 &= \begin{bmatrix} l_1 \cos\theta_1 \dot{\theta}_1 + l_2 \cos(\theta_1 + \theta_2) \left(\dot{\theta}_1 + \dot{\theta}_2\right) \\ l_1 \sin\theta_1 \dot{\theta}_1 + l_2 \sin(\theta_1 + \theta_2) \left(\dot{\theta}_1 + \dot{\theta}_2\right) \end{bmatrix}
\end{aligned} \tag{12}
$$

Consider each energy separatly:

$$
\begin{aligned}
K &= K_1 + K_2 \\
K_1 &= \frac{1}{2} m_1 \left(v_1^T v_1\right) + \frac{1}{2} J_{1,c} \dot{\theta}_1^2 \\
K_2 &= \frac{1}{2} m_2 \left(v_2^T v_2\right) + \frac{1}{2} J_{2,c} \left(\dot{\theta}_1 + \dot{\theta}_2\right)^2
\end{aligned} \tag{13}
$$

Each kinetic energy:

$$
\begin{aligned}
K_1 &= \frac{1}{2} (m_1 l_1^2 + J_{1,c}) \dot{\theta}_1^2 \\
K_2 &= \frac{1}{2} m_2 \left[l_1^2 \dot{\theta}_1^2 + l_2^2 \left(\dot{\theta}_1 + \dot{\theta}_2\right)^2 + 2 l_1 l_2 \dot{\theta}_1 \left(\dot{\theta}_1 + \dot{\theta}_2\right) \cos\theta_2 \right] + \frac{1}{2} J_{2,c} \left(\dot{\theta}_1 + \dot{\theta}_2\right)^2
\end{aligned} \tag{14}
$$

2.2.2 Potential energy:

$$
\begin{aligned}
&P_1 = -m_1 g l_1 \cdot \cos\theta_1 \\
&P_2 = -m_2 g\left[l_1 \cos\theta_1 + l_2 \cdot \cos\left(\theta_1+\theta_2\right)\right] \\
&P = P_1 + P_2 = -(m_1+m_2) g l_1 \cdot \cos\theta_1 - m_2 g l_2 \cdot \cos\left(\theta_1+\theta_2\right)
\end{aligned} \tag{15}
$$

2.2.3 Eule-Lagrange equation:

Consider Eule-Lagrange equation:

$$
\begin{aligned}
&\frac{\mathrm{d}}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right] - \frac{\partial \mathcal{L}}{\partial \theta} = \frac{\mathrm{d}}{dt}\left[\frac{\partial (K-P)}{\partial \dot{\theta}}\right] - \frac{\partial (K-P)}{\partial \theta} = \tau - k\dot{\theta} = \begin{bmatrix} \tau_1 - k\dot{\theta}_1 \\ \tau_2 - k\dot{\theta}_2 \end{bmatrix} \\
&\frac{\mathrm{d}}{dt}\left[\frac{\partial (K-P)}{\partial \dot{\theta}}\right] - \frac{\partial (K-P)}{\partial \theta} = \underbrace{\frac{\mathrm{d}}{dt}\left[\frac{\partial K}{\partial \dot{\theta}}\right] - \frac{\partial K}{\partial \theta}}_{\text{part kinetic}} - \underbrace{\left(\frac{\mathrm{d}}{dt}\left[\frac{\partial P}{\partial \dot{\theta}}\right] - \frac{\partial P}{\partial \theta}\right)}_{\text{part potential}} \\
&= \underbrace{\frac{\mathrm{d}}{dt}\left[\frac{\partial K_1}{\partial \dot{\theta}}\right] - \frac{\partial K_1}{\partial \theta}}_{\text{part 1}} + \underbrace{\frac{\mathrm{d}}{dt}\left[\frac{\partial K_2}{\partial \dot{\theta}}\right] - \frac{\partial K_2}{\partial \theta}}_{\text{part 2}} \\
&- \underbrace{\left(\frac{\mathrm{d}}{dt}\left[\frac{\partial P_1}{\partial \dot{\theta}}\right] - \frac{\partial P_1}{\partial \theta}\right)}_{\text{part 3}} - \underbrace{\left(\frac{\mathrm{d}}{dt}\left[\frac{\partial P_2}{\partial \dot{\theta}}\right] - \frac{\partial P_2}{\partial \theta}\right)}_{\text{part 4}}
\end{aligned} \tag{16}
$$

Part 1

$$
\mathbf{P1} = \begin{bmatrix} \left(m_1 l_1^2 + J_{1,c}\right)\ddot{\theta}_1 \\ 0 \end{bmatrix} \tag{17}
$$

Part 2

$$
\begin{aligned}
\mathbf{P2} = &\begin{bmatrix} \left[m_2\left(l_1^2 + l_2^2 + 2 l_1 l_2 \cos\theta_2\right) + J_{2,c}\right]\ddot{\theta}_1 + \left[m_2 \cdot \left(l_2^2 + l_1 l_2 \cos\theta_2\right) + J_{2,c}\right]\ddot{\theta}_2 \\ \left[m_2\left(l_2^2 + l_1 l_2 \cos\theta_2\right) + J_{2,c}\right]\ddot{\theta}_1 + \left(m_2 l_2^2 + J_{2,c}\right)\ddot{\theta}_2 \end{bmatrix} \\
&+ \begin{bmatrix} -m_2 l_1 l_2 \sin\left(\theta_2\right) \cdot \dot{\theta}_2 \cdot \left(2\dot{\theta}_1 + \dot{\theta}_2\right) \\ m_2 l_1 l_2 \sin\theta_2 \cdot \dot{\theta}_1^2 \end{bmatrix}
\end{aligned} \tag{18}
$$

Part 3

$$
\mathbf{P3} = \begin{bmatrix} -m_1 g l_1 \sin\left(\theta_1\right) \\ 0 \end{bmatrix} \tag{19}
$$

Part 4

$$
\mathbf{P4} = \begin{bmatrix} -m_2 g\left(l_1 \sin\theta_1 + l_2 \sin\left(\theta_1+\theta_2\right)\right) \\ -m_2 g \cdot l_2 \cdot \sin\left(\theta_1+\theta_2\right) \end{bmatrix} \tag{20}
$$

$$
P_1 + P_2 - P_3 - P_4 = \tau - k\dot{\theta} = \begin{bmatrix} \tau_1 - k\dot{\theta}_1 \\ \tau_2 - k\dot{\theta}_2 \end{bmatrix} \tag{21}
$$

Bring (14), (15), (16), (17) into (18),and Rewrite dynamics in matrix form.we can get:

$$
D(\theta)\ddot{\theta} + C(\dot{\theta},\theta)\dot{\theta} + G(\theta) = \tau - k\dot{\theta} \tag{22}
$$

where:

$$
D(\theta) = \begin{bmatrix} m_1 l_1^2 + m_2\left(l_1^2 + l_2^2 + 2 l_1 l_2 \cos\theta_2\right) + J_{1,c} + J_{2,c} & m_2\left(l_2^2 + l_1 l_2 \cos\theta_2\right) + J_{2,c} \\ m_2\left(l_2^2 + l_1 l_2 \cos\theta_2\right) + J_{2,c} & m_2 l_2^2 + J_{2,c} \end{bmatrix} \tag{23}
$$

$$
\ddot{\theta} = \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} \tag{24}
$$

$$C(\dot{\theta},\theta)=\begin{bmatrix}-2m_2l_1l_2\sin\theta_2\dot{\theta}_2 & -m_2l_1l_2\sin\theta_2\dot{\theta}_2\\ m_2l_1l_2\sin\theta_2\cdot\dot{\theta}_1 & 0\end{bmatrix} \tag{25}$$

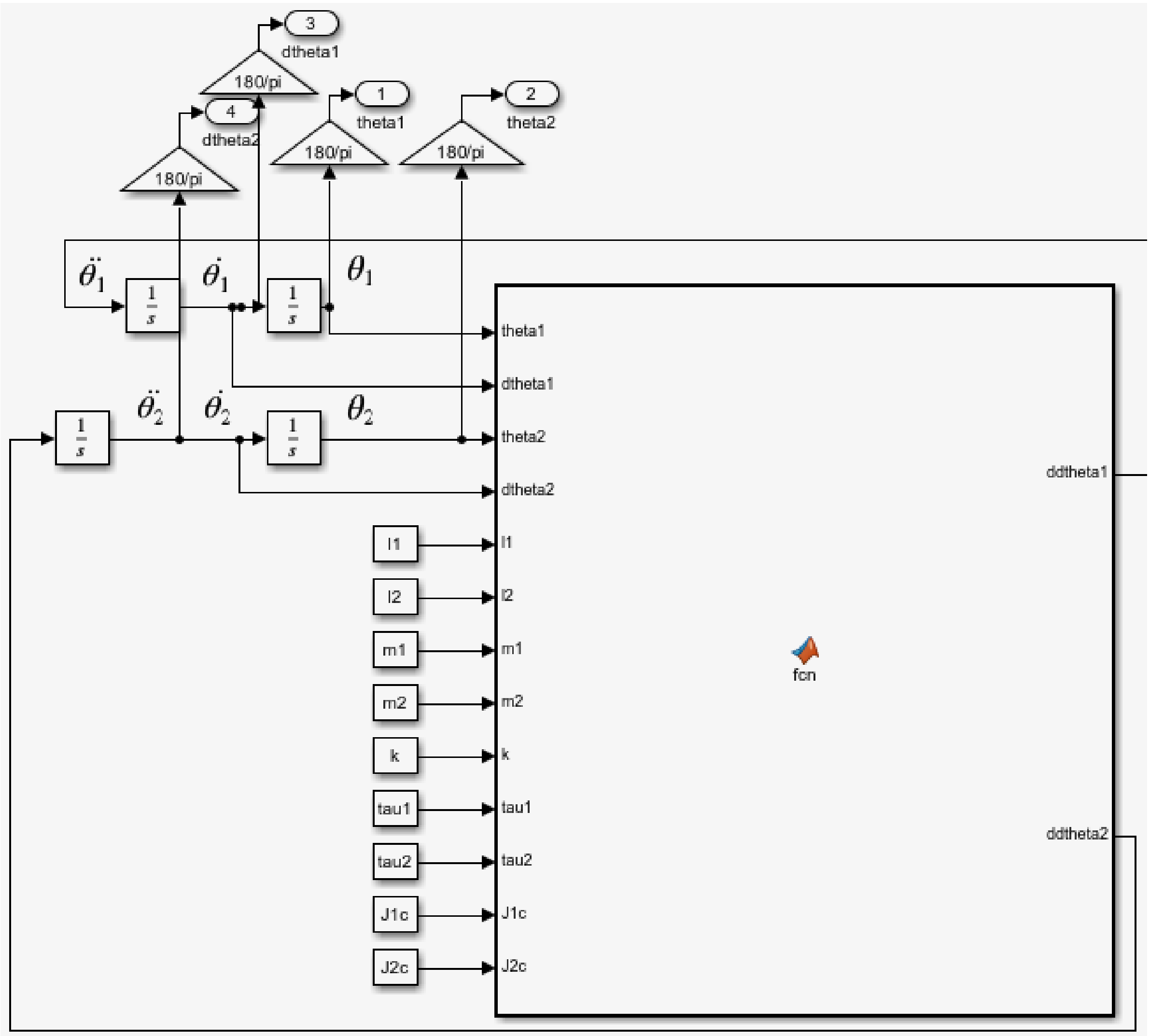
$$G(\theta)=\begin{bmatrix}(m_1+m_2)gl_1\sin\theta_1+m_2gl_2\sin{(\theta_1+\theta_2)}\\ m_2gl_2\sin{(\theta_1+\theta_2)}\end{bmatrix} \tag{26}$$

inverse $D(\theta)$ matirx:

$$\ddot{\theta}=D^{-1}(\theta)(\tau-k\dot{\theta}-C(\dot{\theta},\theta)\dot{\theta}-G(\theta)) \tag{27}$$

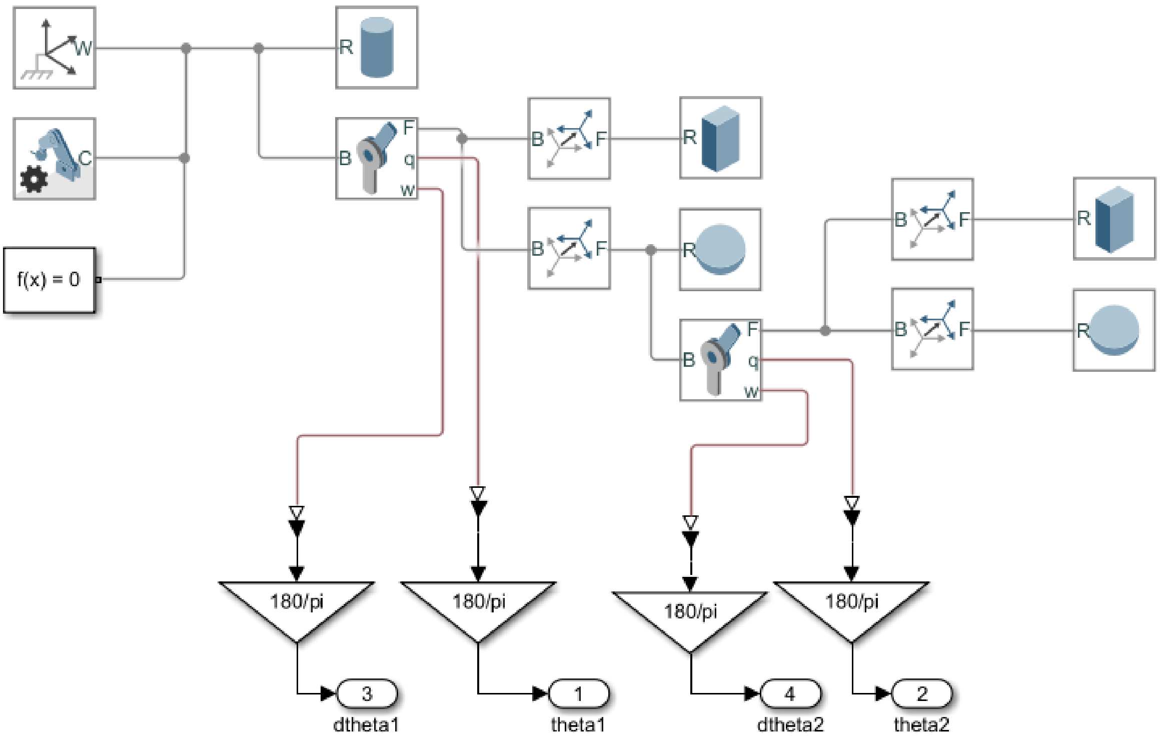
2.3 Simulink model

Use (20)-(24) to build the model as shown below:



```
function [ddtheta1, ddtheta2] = fcn(theta1, dtheta1, theta2, dtheta2, l1, l2, m1, m2, k, tau1, tau2)
g = 9.8;

D = [m1*l1^2+m2*(l1^2+l2^2+2*l1*l2*cos(theta2))+J1c+J2c,m2*(l2^2+l1*l2*cos(theta2))+J2c;
    m2*(l2^2+l1*l2*cos(theta2))+J2c , m2*l2^2+J2c];

C = [-2*m2*l1*l2*sin(theta2)*dtheta2 , -1*m2*l1*l2*sin(theta2)*dtheta2;
    m2*l1*l2*sin(theta2)*dtheta1,0];

G = [(m1+m2)*g*l1*sin(theta1)+m2*g*l2*sin(theta1+theta2);
    m2*g*l2*sin(theta1+theta2)];

Tau = [tau1;tau2];

dtheta = [dtheta1;dtheta2];

ddtheta = inv(D)*(Tau-k*dtheta-C*dtheta -G);

ddtheta1 = ddtheta(1,:);
ddtheta2 = ddtheta(2,:);
```

2.4 Simscape model



2.5 simulation && compare

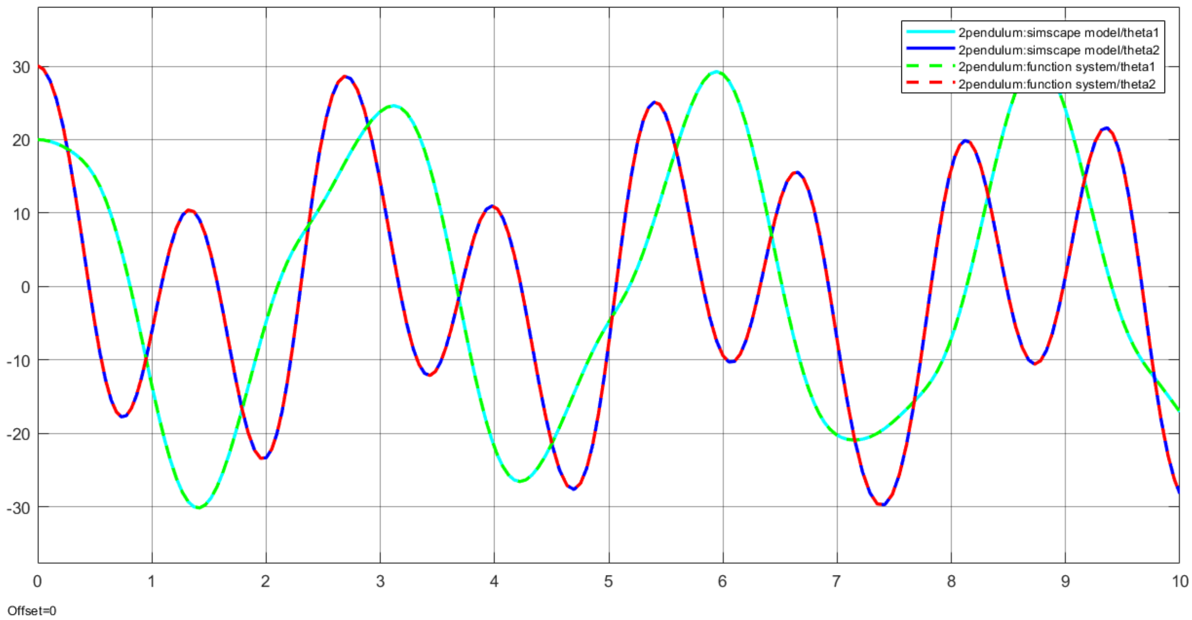
2.5.1 Experimental data:

$$\begin{cases} l_1 = 1.5 \text{ m} \\ l_2 = 1 \text{ m} \\ m_1 = 1.5 \text{ kg} \\ m_2 = 1 \text{ kg} \end{cases} \tag{28}$$

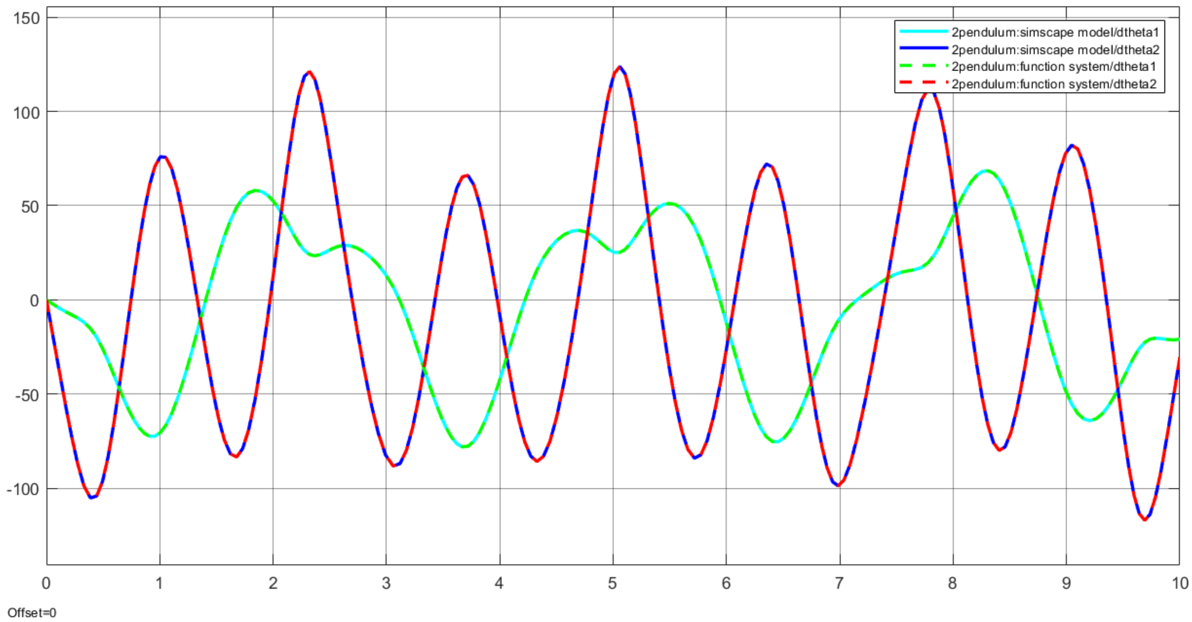
2.5.2 Simulation Condition 1(without viscous friction):

$$\begin{cases} \theta_1(0) = 20\text{deg} \\ \theta_2(0) = 30\text{deg} \\ \dot{\theta}_1(0) = 0\text{deg/s} \\ \dot{\theta}_2(0) = 0\text{deg/s} \end{cases} \quad \begin{cases} \tau_1 = 0 \\ \tau_2 = 0 \\ k = 0 \end{cases} \tag{29}$$

Simulation Result:θ



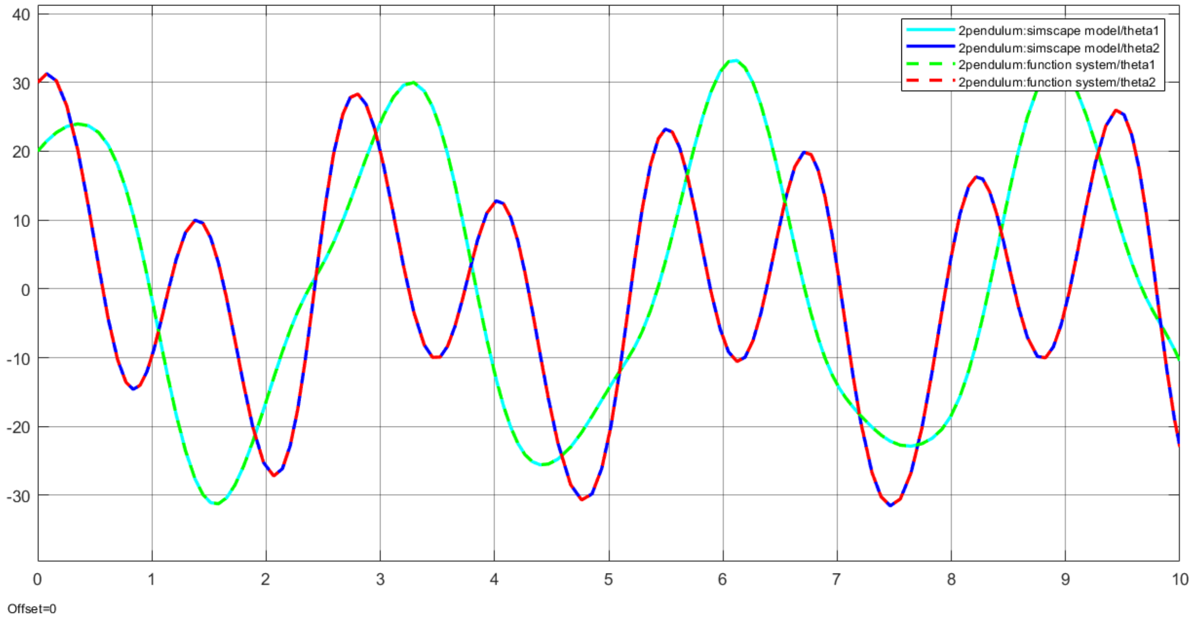
Simulation Result:$\dot{\theta}$



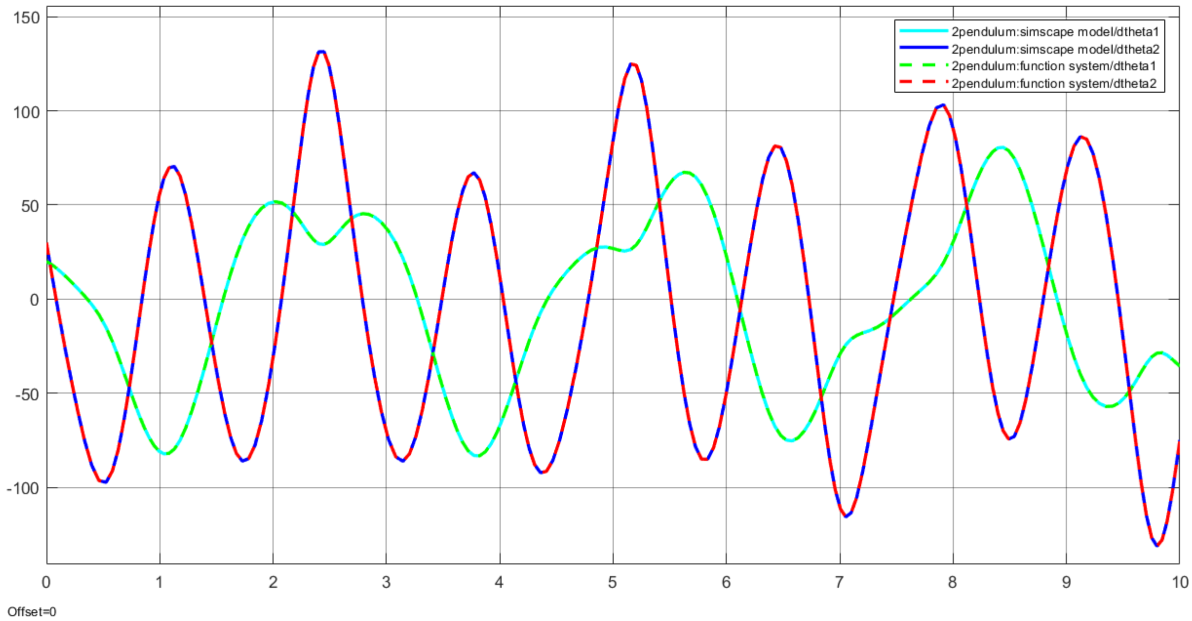
2.5.3 Simulation Condition 2(without viscous friction):

$$\begin{cases} \theta_1(0) = 20\mathrm{deg} \\ \theta_2(0) = 30\mathrm{deg} \\ \dot{\theta}_1(0) = 20\mathrm{deg/s} \\ \dot{\theta}_2(0) = 30\mathrm{deg/s} \end{cases} \quad \begin{cases} \tau_1 = 0 \\ \tau_2 = 0 \\ k = 0 \end{cases} \tag{30}$$

Simulation Result:θ



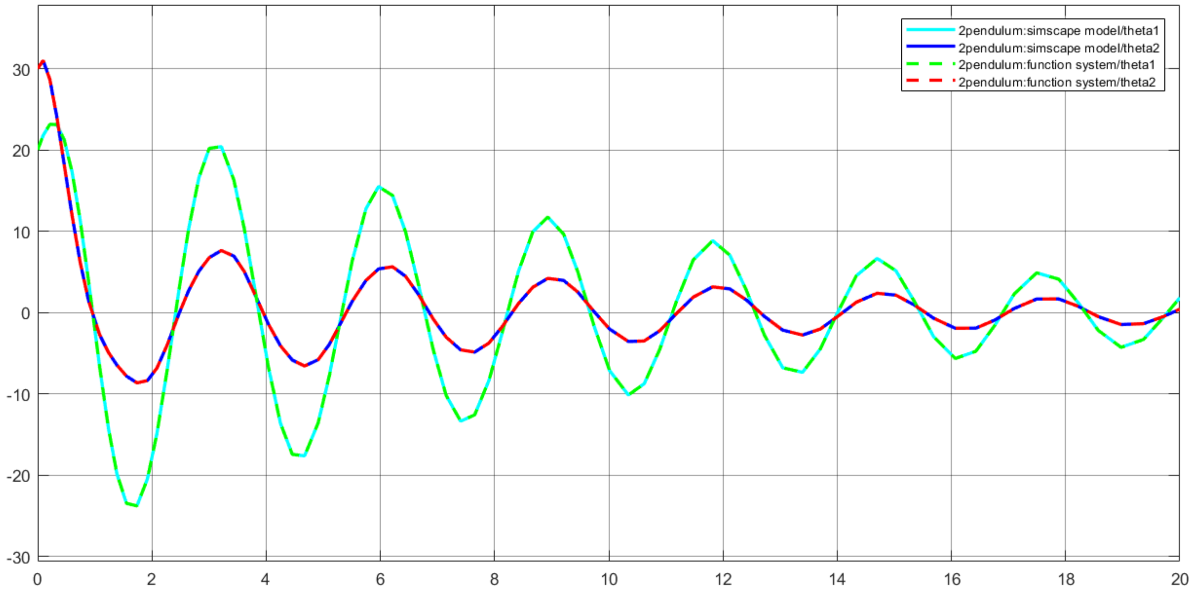
Simulation Result:$\dot{\theta}$



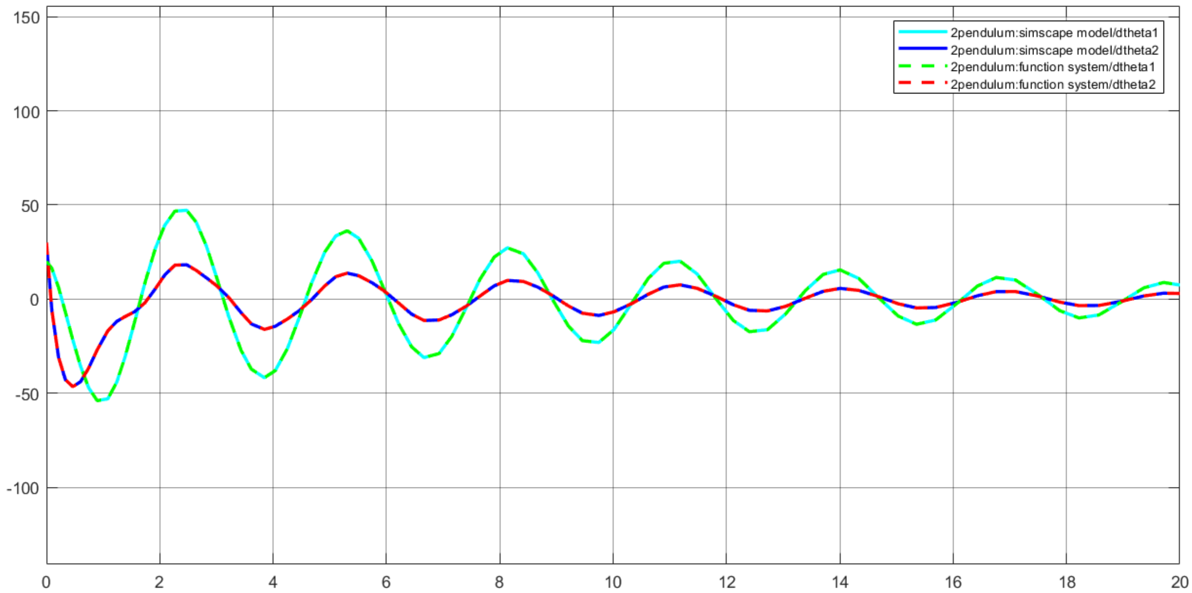
2.5.3 Simulation Condition 3:(with viscous friction)

$$\begin{cases}\theta_1(0)=20\text{deg}\\ \theta_2(0)=30\text{deg}\\ \dot{\theta}_1(0)=20\text{deg/s}\\ \dot{\theta}_2(0)=30\text{deg/s}\end{cases}\quad\begin{cases}\tau_1=0\\ \tau_2=0\\ k=2\end{cases}\tag{31}$$

Simulation Result:θ



Simulation Result:$\dot{\theta}$



Conclusion

1. From the comparison of experimental results, it can be seen that the oscillation of the single-pendulum/double-pendulum system with viscous friction is gradually attenuated, while the system without viscous friction exhibits periodic oscillation