Practical Assignment № 1

Optimal Control

variant number : 6

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Finding the minima using static optimization

1. experimental values（Group 6）

- experimental values

№	$J(x,u)$	$c(x,u)$
6	$J(x,u)=6x^2+2u^2+5xu-3x-2u-12$	$4x^2+u+2$

2. Finding the global minimum $J(x,u)$ based on necessary and sufficient conditions

2.1. without constraints

- analytical method

Organize J(x,y) into the following equation:

$$6x^2+2u^2+5xu-3x-2u-12-J=0 \tag{1}$$

The partial derivative of equation (1) gives :

$$\begin{cases} 12x+5u-3-J'_x=0 \\ 4u+5x-2-J'_u=0 \end{cases} \tag{2}$$

let $J'_x=0, J'_u=0$:

$$\Downarrow$$

$$\begin{cases} x=\frac{2}{23} \\ u=\frac{9}{23} \end{cases}$$

Thus we can obtain **the stationary point:**

$$P:\left(\frac{2}{23},\frac{9}{23}\right)$$

$$A=J''_{xx}\big|_p=12 \quad B=J''_{xu}\big|_p=5 \quad C=J''_{uu}\big|_p=4$$

$$B^2-AC=-23<0 \Rightarrow \text{ Extreme values exist}$$

$$A=12>0 \Rightarrow \text{ The extreme value is the minimal value}$$

In summary, we can obtain the unconstrained extremum of the function as:

$$J_{min} = J\left(\frac{2}{23}, \frac{9}{23}\right) \approx J(0.0869, 0.3913) = -\frac{288}{23} \approx -12.5217$$

2.2. under equality constraint $c(x,u) = 0$

$$\begin{aligned} J_A(x) &= J(x) + \lambda c(x) \\ &= 6x^2 + 2u^2 + 5xu - 3x - 2u - 12 + \lambda\left(4x^2 + u + 2\right) \end{aligned}$$

$$\begin{cases} 12x^* + 5u^* - 3 + 8\lambda^* x^* = 0 \\ 4u^* + 5x^* - 2 + \lambda^* = 0 \\ 4x^{*2} + u^* + 2 = 0 \end{cases}$$

Solving the above equation we get：

$$(x^*, u^*, \lambda^*) \approx (0.1514, -2.0917, 9.6097)$$

$$x^* \approx \begin{bmatrix} 0.1514 \\ -2.0917 \end{bmatrix}, J_{\text{extreme}}(x^*) \approx -0.9664$$

Code

```
clear all
syms x u l
eq1 = 12*x+5*u-3+8*l*x == 0;
eq2 = 4*u+5*x-2+l == 0;
eq3 = 4*x^2 + u + 2 == 0;
[x,u,l] = solve(eq1,eq2,eq3,x,u,l)
x=vpa(x);u=vpa(u);l=vpa(l);
x=x(3);u=u(3);l=l(3);
J = 6*x^2+2*u^2 +5*x*u-3*x-2*u-12
```

2.3. under inequality constraint $c(x,u) \le 0$

Calculate whether the unconstrained extremum obtained in 2.1 is consistent with the inequality:

$$J_{min} = J\left(\frac{2}{23}, \frac{9}{23}\right) \approx J(0.0869, 0.3913) = -\frac{288}{23} \approx -12.5217$$

$$c\left(\frac{2}{23}, \frac{9}{23}\right) = \frac{1281}{529} > 0 \qquad \text{Inequality constraints are not satisfied}$$

By the **Kuhn-Tucker condition**: the extremum is **on the boundary**:

That is, the result in 2.2:

$$x^* \approx \begin{bmatrix} 0.1514 \\ -2.0917 \end{bmatrix}, J(x^*) \approx -0.9664$$

3. Gradient descent methods.

$$J_1(x,u) = J(x,u)$$

3.1. Using Newton-Raphson method (in optimization) find the extremum with step-by-step calculation.

$$J(x,u) = 6x^2 + 2u^2 + 5xu - 3x - 2u - 12$$

Let $X = \begin{bmatrix} x \\ u \end{bmatrix}$,we can get :

$$J(X) = \frac{1}{2} X^T Q X - X^T b, Q = \begin{bmatrix} 12 & 5 \\ 5 & 4 \end{bmatrix}, b = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

- **Newton (one step convergence):**

The following formula was used for the calculation：

$$x(k+1) = x(k) - H^{-1}(x)\Big|_{x=x(k)} \underset{x}{grad}^T\{J(x)\}\Big|_{x=x(k)}$$

Where:

$$\text{grad}^T\{J(x)\} = Qx - b$$

calculation

Given any initial value:

$$x(0) = \begin{bmatrix} 10 \\ 10 \end{bmatrix}$$

$$x_{\text{extreme}} = x(1) = x(0) - Q^{-1}\,\text{grad}_x\{J(x)\}\big|_{x(0)} \approx \begin{bmatrix} 0.0869 \\ 0.3931 \end{bmatrix}$$

Code

```
%newton
clear all
x_0 =[10;10];Q = [12 5;5 4];b = [3;2];
x_extremum = x_0 - inv(Q)*grad(x_0,Q,b);
function g = grad(x,Q,b)
g = Q*x-b;
end
```

3.2. Using method of steepest descent (gradient descent method) for two different γ (corresponding to oscillation and aperiodic convergence) find the extrema with step-by-step calculation.

Use the same Q, b, and initial values as in 3.1

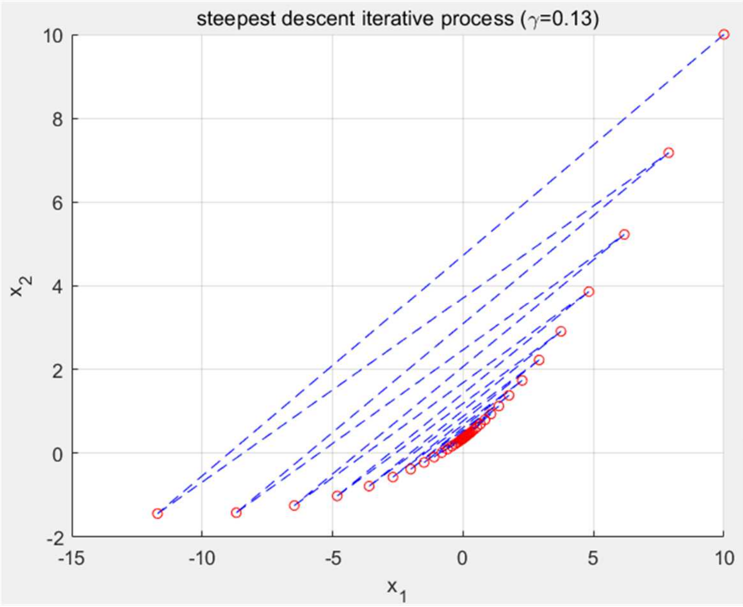
Iterate using the following formula:

$$x(k+1) = x(k) - \gamma\,\text{grad}_x^T\{J(x)\}\big|_{x=x(k)}$$

Where:

$$\text{grad}^T\{J(x)\} = Qx - b$$

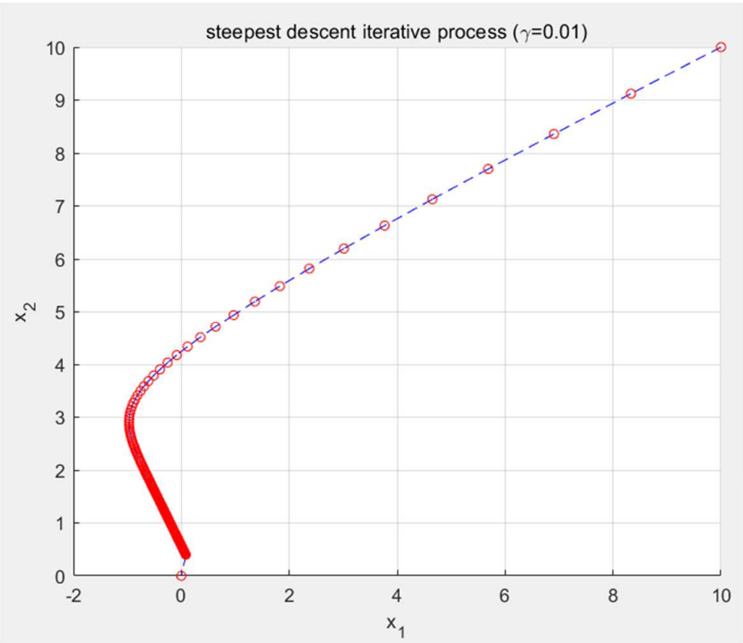
- $\gamma = 0.13$(oscillation)



x_convergence =

0.0870 0.3913

- $\gamma = 0.01$(aperiodic convergence)



x_convergence =

0.0870 0.3913

Code

```
%steepest
clear allQ = [12 5;5 4];b = [3;2];convergence_step = 10000;
X = zeros(2,10000);X(:,1) = [10;10];gamma = 0.13;
for i = 1:1:10000
    X(:,i+1) = X(:,i) - gamma*grad(X(:,i),Q,b);
    if i> 2
        if (X(1,i-1)==X(1,i))&&(X(2,i-
1)==X(2,i))&&(X(1,i+1)==X(1,i))&&(X(2,i+1)==X(2,i))
            convergence_step = i
            break;
        end
    end
end
X = X(:,1:convergence_step);
```

```
% draw process
xx = X(1,:);yy = X(2,:);x_convergence = [xx(convergence_step-
1) ,yy(convergence_step-1)]
figure
for i = 1: 1: length(xx)-1
    hold on

plot([xx(i),xx(i+1)],[yy(i),yy(i+1)],'Color','b','LineStyle','--
');
end
hold on
plot(xx,yy,'LineStyle','none','Marker','o','MarkerSize',5,'Color'
,'r');
hold on
grid on
function g = grad(x,Q,b)
g = Q*x-b;
end
```