Assignment 5, Optimization methods (Maxim Kaledin)

Problem 3

Apply Sylvester's criterium to ensure that matrix is positive semidefinite. All principal minors should be non-negative, these are

$$\begin{aligned} -25y_1^2 &\geq 0, \\ (4y_1+2)(-25y_1^2) &\geq 0, \\ 3y_2(4y_1+2) &\geq 0, \\ 3y_2 &\geq 0, \\ 4y_1+2 &\geq 0. \end{aligned}$$

The first inequality gives $y_1 = 0$, and as result constraints are $D = \{y : y_1 = 0, y_2 \geq 0\}$. Obviously, all feasible solutions are optimal and optimal goal is 0.

Derive the dual problem (after we dismiss redundant constraints):

$$\max_{\lambda,\nu} \inf_{y\in D} \left[2y_1 + \nu y_1 - \lambda y_2\right], \quad \text{s.t. } \lambda \geq 0.$$

Since $y \in D$, $\inf_{y\in D} \left[2y_1 + \nu y_1 - \lambda y_2\right] = \inf_{y_2\in D} \left[-\lambda y_2\right]$ and

$$\inf_{y_2\in D} \left[-\lambda y_2\right] = \begin{cases} -\infty, & \lambda \geq 0, \\ 0, & \lambda = 0. \end{cases}$$

So $\lambda = 0$ is the solution to dual problem with optimal goal equals 0.

Problem 4

1

The problem can be reformulated as

$$\begin{aligned} &\min_{x,t} t \quad s.t. \\ &t \geq a_i^T x + b_i \quad \forall i \in \{1, .., m\}. \end{aligned}$$

Corresponding Lagrange dual problem is

$$\begin{aligned} &\max_{\lambda} \inf_{x,t} \left[t + \sum_{i=1}^{m} \lambda_i (a_i^T x + b_i - t)\right], \\ &s.t. \quad \lambda \geq 0, \end{aligned}$$

where $\lambda_i \geq 0$ are dual variables.

2

Let

$$\hat{f}_2(y) = \frac{1}{\alpha}\log\left(\sum_{i=1}^{m}\exp \alpha y_i\right),$$

and $y_i = a_i^T x + b_i$ be new constraints. First, formulate the primal problem:

$$\begin{aligned}
&\min_{x,y} \hat{f}_2(y) \\
&s.t. \quad y_i = a_i^T x + b_i \quad \forall i \in \{1,..,m\}\,.
\end{aligned}$$

The dual one is defined as

$$\max_{\lambda}\inf_{x,t}\left[t + \sum_{i=1}^{m}\lambda_i(a_i^T x + b_i - t)\right]$$

without constraints on λ because the constraints are equalities.

Problem 5

Define Lagrange function of a problem $L(x,\lambda) = \|Ax - b\|_2^2 + \sum_{i=1}^{m}\lambda_i(c_i.x - d_i)$. Since there are no inequality constraints, KKT conditions are exactly

$$\begin{aligned}
\frac{\partial L}{\partial x} &= 0 \\
\frac{\partial L}{\partial \lambda} &= 0,
\end{aligned}$$

where derivatives denote the gradients truncated to needed set of variables. These sets of constraints are exactly

$$\begin{aligned}
&\frac{\partial L}{\partial x_j} = 2\sum_{i=1}^{m} a_{ij}\sum_{k=1}^{n}(a_{ik}x_k - b_i) + \sum_{i=1}^{p}\lambda_i c_{ij}, \\
&Cx = d.
\end{aligned}$$

Let us write the first expression shorter:

$$\nabla_x L = 2A^T(Ax - b) + C^T\lambda.$$

Since the goal function is convex, solution to system

$$\begin{cases} 2A^T(Ax - b) + C^T\lambda = 0, \\ Cx = d \end{cases}$$

exists and is an optimal solution of primal problem. With pseudo-inverse (it exists since C has full row rank):

$$\begin{aligned}
&x = (C^T C)^{-1} C^T d, \\
&\lambda : \quad C^T\lambda = -2A^T(Ax - b).
\end{aligned}$$

Now derive the dual problem:

$$\max_{\lambda} \inf_{x} L(x, \lambda).$$

Lagrangian L remains convex for all λ, so the solution to inner problem is given by the system of normal equations

$$2A^T(Ax - b) + C^T\lambda = 0$$

and can be found based on any λ since matrix A^TA has full rank. We could write the solution explicitly:

$$x = (A^TA)^{-1}\left(-\frac{1}{2}C^T\lambda + A^Tb\right).$$