Talk 08 — Local computations : p-adic places

Now we come to the most difficult part : choose Siegal sections at places above p .

Recall our assumption

(ord) $\quad p$ is unramified in F, every prime $v \mid p$ in F splits completely in K.

In [Wan2015], he assumed even stronger

(ord$^+$) $\quad p$ splits completely in K.

<u>Remark</u> : (ord$^+$) is used in [Wan15], (ord) is used in [Hsieh14][EHLS]. There are no many differences between the two assumptions.

In what follows, we use the assumption (ord) instead.

§8.1 Setup:

- By (ord), for a place v of F above p, we have

$$K_v \simeq K_w \times K_{\bar{w}} \xrightarrow{\sim} F_v \times F_v .$$

Throughout, we use the first projection $P_w : K_v \to F_v$ identifying K_w with F_v.

- By Talk 01, we see under the fixed projection P_w

$$GU(r,s)(F_v) \simeq GL_{r+s}(F_v) \times F_v^\times . \qquad \text{—— } (*) .$$

So at this most difficult places, the groups are actually simpler!

- We write $n = r+s$ or $n = r+s+1$, based on ♡ or ◇. Then the quasi-split unitary group we are using to construct Siegal sections is $GU(n,n)$.

Eisenstein datum @ p:

$$(\pi, \tau, \Sigma) \xrightarrow{@ v|p} (\pi_v, \tau_v) \xrightarrow[\text{to } GL_n\text{-case}]{\text{under proj. } P_w} (\pi_w, \tau_w) .$$

(2) Recall the Hecke character $\tau : \mathbb{A}_K^\times \to \mathbb{C}^\times$ as a part of the Eisenstein datum. We break it locally into

$$\tau = \otimes'_w \tau_w , \quad \tau_w : K_w^\times \to \mathbb{C}^\times \quad \text{local characters}. \quad w: \text{places of } K$$

Here for each $w|v|p$, we isolate τ_w and $\tau_{\bar{w}}$ and let

$$\tau_1 = \tau_w , \quad \tau_2 := \tau_{\bar{w}}^{-1} \quad \text{when } w \text{ (and } v) \text{ is clear}.$$

Write $\tau_v = (\tau_1, \tau_2)$. Then τ_v determines a character

$$\tau_{w,z} : B_n(K_w) \to \mathbb{C}^\times , \quad \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \mapsto \tau_1(\det D)\, \tau_2(\det A)\, |AD^{-1}|_w^{z}$$

Then given for each $w|v|p$, $f_{w,z} \in \mathrm{Ind}_{B_{2n}(K_w)}^{GL_{2n}(K_w)}(\tau_{w,z})$, we have

$$f_{v,z}(g_v) = f_{w,z}(g_w) \in I_{Q,v}(\tau_v, z) , \quad \text{under the first projection}.$$

as a Siegal section. Actually this can be left as an exercise? Note the difference between [EHLS] and [Wan15] on Siegal parabolic, we modified a little bit.

$\Longrightarrow$ This reduces the construction to GL_{2n}, which might be a bit easier.

(1) Recall the ultimate goal is to understand the automorphic representation (π, V) of $GU(r,s)(\mathbb{A}_F)$. For each $v|p$ with a distinguished $w|v$ (as the first projection), we consider π_v locally:

- Let $\chi_1, \cdots, \chi_{r+s}$ be characters of $K_w^\times$ of conductor $\varpi^{t_1}, \cdots, \varpi^{t_{r+s}}$. Then we form the character

$$\chi = (\chi_1, \cdots, \chi_{r+s}) : B_{r+s}(F_v) \to \mathbb{C}^\times \quad : \quad \begin{pmatrix} b_1 & & * \\ & \ddots & \\ & & b_{r+s} \end{pmatrix} \mapsto \chi_1(b_1) \cdots \chi_{r+s}(b_{r+s})$$

and the induced representation

$$\pi_w := \mathrm{Ind}_{B_{r+s}(F_v)}^{GL_{r+s}(F_v)}(\chi) := \mathrm{Ind}_{B_{r+s}(F_v)}^{GL_{r+s}(F_v)}(\chi_1, \cdots, \chi_{r+s}).$$

<u>Recall</u>: π_w has a model as

$$\pi_w = \left\{ f : GL_{r+s}(F_v) \to \mathbb{C} \text{ smooth} \;\middle|\; f(bx) = \chi(b)\delta(b)^{1/2} f(x) \right\}$$

with $GL_{r+s}(F_v)$-action as right translations. Here $\delta := \delta_B$ is the modulus function for the upper-trigular Borel subgroup.

- Then we require that under the first projection p_w, π_v is isomophic to π_w.

<u>Note</u>: In [EHLS], a general setup is considered: they generalize the Borel B_{r+s} to Levis wrt the partitions of r and s:

$$r = r_1 + \cdots + r_\ell, \quad s = s_1 + \cdots + s_m. \quad \text{—}(\star)$$

Then for characters $\chi_1, \cdots, \chi_\ell, \lambda_1, \cdots, \lambda_m$, they formed the induced representation to GL_n from the parobolic wrt $(\star)$

$$\pi_w := \mathrm{Ind}_{P_{r,s}}^{GL_n}\left(\mathrm{Ind}_{P_{r_1,\cdots,r_\ell}}^{GL_r}(\chi_1, \cdots, \chi_\ell) \otimes \mathrm{Ind}_{P_{s_1,\cdots,s_m}}^{GL_s}(\lambda_1, \cdots, \lambda_m) \right)$$

and moreover, they only need $\pi_w \twoheadrightarrow \pi_v$, not "isomorphism" as we are requiring here.

There are two types of simplifications:

(i) One is to assume that local reps are unramified

(ii) The other is to assume that certain characters are sufficiently ramified.

[CLW22] deals with general cases, while [Wan15, Wan20] focus on (ii). In this talk we mainly focus on (ii).

§ 8.2 Nearly ordinary sections

Note : In [Wan15, §4.0.1], the ongoing assumption is (ord$^+$). Here I'm trying to generalize to (ord), but unfortunately I've found no reference. (actually also no ref. on [Wan15]'s results)

Recall in §8.1, we required that under the first projection, $\pi_v = \mathrm{Ind}_{B_{r+s}(F_v)}^{GL_{r+s}(F_v)}(\chi_1, \cdots, \chi_n)$

Defn [Wan15, Defn 4.14] Let $n = r+s$, $\underline{k} = (c_{s+r}, \cdots, c_{s+1}, c_1, \cdots, c_s)$ be a weight (in the sense of [Wan15], see talk 02) We say $\chi := (\chi_1, \cdots, \chi_n)$ is nearly ordinary wrt $\underline{k}$ if

$$\{ v_p(\chi_1(p)), \cdots, v_p(\chi_{r+s}(p)) \} = \{ c_1 + s - 1 - \tilde{n}, c_2 + s - 2 - \tilde{n}, \cdots, c_s - \tilde{n} ;$$
$$c_{s+1} + r + s - 1 - \tilde{n}, \cdots, c_{s+r} + s - \tilde{n} \}, \quad \tilde{n} = \frac{n-1}{2}$$

We call $\{ v_p(\chi_i(p)) \}_{i=1,\cdots,r+s}$ the p-adic weights of χ.

Remark : In [Hsieh14], he call π_v regular if $\{ v_p(\chi_i(p)) \}$ are distinct. Then Wan's "nearly ordinary rep'n wrt. $\underline{k}$" is automatically regular. We write

$$k_1 > \cdots > k_{r+s} \quad \text{for the p-adic weight } \{ v_p(\chi_i(p)) \}.$$

- Later we always rearrange $\chi_1, \cdots, \chi_{r+s}$ such that $k_1 < \cdots < k_{r+s}$. Then as seen in previous section, [Wan15] required these characters to be "sufficiently ramified": we say χ (or π_v) is generic at $v | p$ if

$$t_1 > t_2 > \cdots > t_r > t_{r+1} > \cdots > t_{r+s}$$

Remark : In [Wan20], another "generic condition" is used (in $K/\mathbb{Q}$ case): $t_1 = \cdots = t_r = s_1 = s_2 =: t \geq 2$. But as the paragraph below [Wan20, Def. 6.30] goes, the argument goes through. In [CLW22], the ramification condition is completely removed.

Remark : In [Wan15, Defn 4.42], Wan defined genericness of the Eisenstein datum (π, τ). We shall come back to it later when talking about Klingen sections. Here we are only talking about π. Note : π is an autorep of $GU(r,s)$ for general r, s.

(1) <u>Atkin–Lehner operator</u>

It seems hard to find a reference. The original one seems to be

Atkin, Lehner "Hecke operators on $\Gamma_0(m)$", Math. Ann. jun. 1970.

and a general introduction may be found in [Hida04, pAFSV]. Wan's reference is [Hsieh11].

<u>Defn</u>: The <u>Atkin–Lehner ring</u> of $GL_n(F_v)$ is

$$\mathcal{A}_v := \mathcal{O}_{F_v}[t_1, t_2, \cdots, t_n, t_n^{-1}]$$

where t_i is defined as $\underline{t_i = [N(\mathcal{O}_{F_v})\alpha_i N(\mathcal{O}_{F_v})]}$, $\alpha_i := \begin{pmatrix} 1_{n-i} & \\ & \varpi_v 1_i \end{pmatrix}$

(it is a double coset operator — a kind of Hecke operator). More explicitly it acts on $\pi_v^{N(\mathcal{O}_{F_v})}$ by $v|t_i := \sum_{x \in N/\alpha_i^{-1}N\alpha_i} x\alpha_i^{-1} \cdot v$. Elements in $\mathcal{A}_v$ are called <u>Atkin–Lehner operators</u>. We normalize them <u>wrt the weight k</u> as

$$v\|t_i := \delta(\alpha_i)^{-\frac{1}{2}} p^{k_1 + \cdots + k_i} v|t_i .$$

(actually I cannot see how this is related to k. Maybe the p-adic weights are related to k since they are nearly ordinary wrt. k.).

<u>Remark</u>: In [Hsieh14], he defined the <u>U_v-operator</u> as the normalized operator

$$U_v \cdot v := v \| t_1 t_2 \cdots t_{n-1} .$$

Then <u>Hida's idempotent</u> e attached to U_v is defined as $e := \lim_{n\to\infty} (U_v)^{n!}$.

<u>Defn</u>: A vector $v \in \pi_v^{N(\mathcal{O}_{F_v})}$ is called <u>nearly ordinary</u> if it is an eigenvector for all "$\|t_i$" operators with eigenvalues that are p-adic unit.

<u>Question</u>: In [Hida04, pAFSV, p234], it seems that a <u>nearly p-ordinary vector</u> is an element in $e\pi_v^{N(\mathcal{O}_{F_v})}$ $(\neq 0)$. Same for [Hsieh11, Hsieh14] as well. Are the two definitions equivalent? It seems <u>that they are equivalent</u>.

(2) A big-cell section

Warning : When choosing Siegel section, we are in $\mathrm{Ind}_{B_{2n}}^{GL_{2n}}$. Here we are in π_v, as the induced representation $\mathrm{Ind}_{B_n}^{GL_n}$! Note the different setup ! To distinguish, we use "long" for the big-cell section as in [Wan15].

Let $w^{long} := \mathrm{antidiag}(1, \cdots, 1) := \begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix}$. Define f^{long} as the unique (up to scalar) in π_v that is supported in $B(F_v) w^{long} N(\mathcal{O}_{F_v})$ and is right-invariant under $N(\mathcal{O}_{F_v})$. (So $f^{long} \in \pi_v^{N(\mathcal{O}_{F_v})}$)

Proposition [Wan15, Lemma 4.16-4.17]

(i) f^{long} is an eigenvector for all t_i

(ii) Suppose χ is nearly ordinary wrt $\underline{k}$, then the eigenvalues of t_i on f^{long} are p-adic units. In other words, f^{long} is a nearly ordinary vector.

Proof : Exercise. See [Wan15, Lemma 4.16-4.17] for proofs.

Actually the nearly ordinary vector is unique up to scalar, see [Hida04, pAFSV].
In practice, a more convenient result is given as follows, generalizing [SU14, Prop.9.5]. Wan in [Wan15] break it into two statements:

<u>Lemma</u> [Wan15, Lemma 4.19] We rearrange χ as $\chi_{(1)}, \cdots, \chi_{(r+s)}$ such that

$$t_{(s+r)} > \cdots > t_{(s+1)} > t_{(1)} > \cdots > t_{(s)}$$

and define a <u>χ-level group</u> $K_\chi \subseteq GL_{r+s}(\mathcal{O}_{F_v})$. Let χ^{op} be a character of K_χ defined as

$$g = (g_{ij}) \in K_\chi \longmapsto \chi_{(r+s)}(g_{11})\chi_{(r+s-1)}(g_{22}) \cdots \chi_{(1)}(g_{r+s}) \in \mathbb{C}^\times .$$

Then f^{long} is the unique (up to scalar) vector in π such that the action of K_χ is given by multiplying χ^{op}. Here K_χ be the matrices such that

- below diagonal entries of the i-th column is contained in $\cancel{\varpi_v}^{t(r+s+1-i)}\mathcal{O}_v$, $1 \leq i \leq r$
($r+s+1-i$ goes from $r+s+1$ to $s+1$ downwards)
- left-to-diag entries of the j-th row is contained in $\cancel{\varpi_v}^{t(r+s+1-j)}\mathcal{O}_v$, $r+2 \leq j \leq r+s$
($r+s+1-j$ goes from $s-1$ to 1 downwards)

<u>Corollary</u> [Wan15, Coro.4.20]. Let $w_1 := \mathrm{diag}\left(1_r, \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}_s\right)$ and $\tilde{K}_\chi := w_1^{-1}K_\chi w_1$
Suppose χ is generic. Then the unique (up to scalar) nearly ordinary section wrt $\tilde{B} := w_1^{-1}Bw_1$ is

$$f^{ord}(g) = \mathbb{1}_{\tilde{K}_\chi}(g)\chi_1(g_{11})\chi_2(g_{22}) \cdots \chi(g_{r+s,r+s})$$

<u>Proof of lemma</u>: See [Wan15, Lemma 4.19].

§ 8.3 Siegal sections

§§ 8.3.1 A new basis

To simplify the future explanations, in this section we use new basis:

- Let $V_{r,s}^{new}$ be the skew-Hermitian space with the metric $\begin{pmatrix} S & & 1_b \\ & & \\ -1_b & & \end{pmatrix}$, i.e.

$$\begin{array}{c|c} y^i & \\ w^i & \\ x^i & \end{array}\begin{pmatrix} & & 1_b \\ & \zeta 1_a & \\ -1_b & & \end{pmatrix} \xrightarrow[\text{the basis}]{\text{reorder}} \begin{array}{cc} \begin{array}{c} w^i \\ y^i \\ x^i \end{array} \begin{pmatrix} \zeta 1_a & & \\ & & 1_b \\ & -1_b & \end{pmatrix} \begin{array}{c} a \\ b \\ b \end{array} \\ \begin{array}{ccc} w^i & y^i & x^i \\ a & b & b \end{array} & \end{array}, \quad P = \begin{array}{cc} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \begin{array}{c} b \\ a \\ b \end{array} \\ \begin{array}{ccc} a & b & b \end{array} & \end{array}$$

新基在老基下的矩阵.

$P^{-1} = P$

In $GU(r+1, s+1)$-case, P is further written as

$$P = \begin{array}{cc} \left(\begin{array}{c|cc|cc} & 1 & & & \\ & & 1 & & \\ \hline 1 & & & & \\ \hline & & & 1 & \\ & & & & 1 \end{array}\right) & \begin{array}{c} b \\ 1 \\ a \\ b \\ 1 \end{array} \\ \begin{array}{ccccc} a & b & 1 & b & 1 \end{array} & \end{array}$$

Then one checks that this P plays the role of w_{Borel} in [Wan15, p1996].

(This is only a guess from [Wan15] & [Wan20]: putting in the $GU(3,1)$-case where $a=2$ and $b=0$, $P = \begin{pmatrix} & 1 & \\ 1 & & \\ & & 1 \end{pmatrix}\begin{array}{c} 1 \\ a=2 \\ 1 \end{array} = \begin{pmatrix} & & 1 & \\ 1 & & & \\ & 1 & & \\ & & & 1 \end{pmatrix} \in GL_4$, this coincide with w'_{3} $_2$ in [Wan20, Defn 5.6])

- Under the new basis, $GU(r,s)_{new} := GU(V_{r,s}^{new})$ and the relation to the previous GU is by conjugation under P:

$$P: \quad g \in GU(r,s) \xrightarrow[\text{new basis}]{\text{under the}} g_{new} = P^{-1} g P \in GU(r,s)_{new}$$

$$= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} g_1 & g_2 & g_3 \\ g_4 & g_5 & g_6 \\ g_7 & g_8 & g_9 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

1, 2, 3 $\longmapsto$ 2, 1, 3
row — row
column — column
b a b — a b b

$$= \begin{pmatrix} g_4 & g_5 & g_6 \\ g_1 & g_2 & g_3 \\ g_7 & g_8 & g_9 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} g_5 & g_4 & g_6 \\ g_2 & g_1 & g_3 \\ g_8 & g_7 & g_9 \end{pmatrix}\begin{array}{c} a \\ b \\ b \end{array}$$

- Moreover, the setup of doubling method changes as :

$\boxed{\text{First step}}$ Under this new basis, we want to restate the embedding α as α_{new} :

$$
\begin{array}{rccccc}
\alpha_{new}: & GU(r+1,s+1)_{new} & \times & GU(r,s)_{new} & \longrightarrow & GU(r+s+1,r+s+1)_{new} \\
 & \underline{1},\underline{2,3},\underline{4,5} & & 1',2',3' & & 2,3,1,3',4,5,1',2' \\
 & P \Big\downarrow \text{go back} & & P \Big\downarrow \text{go back} & & P \Big\uparrow \text{go back: for Siegel case, the basis is } \underline{\text{not}} \text{ changed.} \\
 & \underset{b\ 1}{\underline{2,3}}, \underset{a}{\underline{1}}, \underset{b\ 1}{\underline{4,5}} & , & \underset{b}{\underline{2'}}, \underset{a}{\underline{1'}}, \underset{b}{\underline{3'}} & \xrightarrow{\alpha} & \underset{b\ 1}{\underline{2,3}}, \underset{a}{\underline{1}}, \underset{b}{\underline{3'}}, \underset{b\ 1}{\underline{4,5}}, \underset{a}{\underline{1'}}, \underset{b}{\underline{2'}} \quad ! \\
\alpha: & GU(r+1,s+1) & \times & GU(r,s) & \longrightarrow & GU(r+s+1,r+s+1)
\end{array}
$$

$\boxed{\text{Second step}}$ The matrix S used in β is changed accordingly as in [Wan15, p1997].

§§ 8.3.2 Godement–Jacquet section

(1) Schwarz function : $\Phi_w \in \mathrm{Sch}(M_{n\times 2n}(K_w))$

- In the following, we always break $M_{n\times 2n}(K_w)$ into blocks $\left(\boxed{n\times n}, \boxed{n\times n}\right)$
- In [EHLS], there is a base-free introduction on Φ, but it seems not necessary while bringin more troublesome notations.

(2) Godement–Jacquet section : Define the section $f^{\Phi_w} \in \mathrm{Ind}_{B_n(K_w)}^{GL_{2n}(K_w)} \tau_{w,z}$ as

$$f^{\Phi_w}(g) := \tau_2(\det g)\,|\det g|_w^{\frac{n}{2}-z} \int_{GL_n(K_w)} \Phi\big((0,X)\,g\big)\,\tau_1^{-1}\tau_2(\det X)\,|\det X|_w^{-2z+n}\, dX .$$

Here dX is the right Haar measure on $GL_n(K_w)$. Wan used $d^\times X$, which is actually corresponding to the base-free notation on Φ.

Now we compute the Fourier coefficients of such a Godement section : Let $\beta \in S_n(F_v)$,

$$W_\beta(1, f^{\Phi_w}, z) = \int_{M_n(K_w)} f^{\Phi_w}\left(w_n \begin{pmatrix} 1 & \sigma \\ 0 & 1 \end{pmatrix} \right) e_w(-\mathrm{Tr}\beta\sigma)\, d\sigma$$

$$= \int_{M_n(K_w)} \int_{GL_n(K_w)} \Phi\left((0,X)\, w_n \begin{pmatrix} 1 & \sigma \\ 0 & 1 \end{pmatrix} \right) \tau_1^{-1}\tau_2(\det X)\,|\det X|_w^{-2z+n}\, dX\, e_w(-\mathrm{Tr}\beta\sigma)\, d\sigma$$

$g := w_n\begin{pmatrix} 1 & \sigma \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\begin{pmatrix} 1 & \sigma \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -\sigma \end{pmatrix}$ is of determinant one.

$$= \int_{M_n(K_w)} \int_{GL_n(K_w)} \Phi\big((-X, -\sigma X) \big)\, \tau_1^{-1}\tau_2(\det X)\,|\det X|_w^{-2z+n}\, dX\, e_w(-\mathrm{Tr}\beta\sigma)\, d\sigma$$

$(0,X)\,g = (0,X)\begin{pmatrix} 0 & 1 \\ -1 & -\sigma \end{pmatrix} = (-X, -\sigma X)$

General Philosophy :

- Break Φ up : $\Phi(X,Y) = \Phi_1(X)\,\Phi_2(Y)$ for $\Phi_i \in \mathrm{Sch}(M_{n\times n}(K_w))$.

  Then the integral becomes

$$\int_{M_n(K_w)} \int_{GL_n(K_w)} \Phi_1(-X)\,\Phi_2(-\sigma X)\,\tau_1^{-1}\tau_2(\det X)\,|\det X|_w^{-2z+n}\, dX\, e_w(-\mathrm{Tr}\beta\sigma)\, d\sigma$$

- Then interchange the two integrals :

$$\int_{GL_n(K_w)} \Phi_1(-X) \int_{M_n(K_w)} \Phi_2(-\sigma X)\, e_w(-\mathrm{Tr}\beta\sigma)\, d\sigma\; \tau_1^{-1}\tau_2(\det X)\,|\det X|_w^{-2z+n}\, dX$$

• $\boxed{\Phi_2\text{-part}}$ We recognize the inner integral $-$ as the <u>Fourier inverse transform</u> of Φ_2:

$$- \underset{\sigma \mapsto -\sigma X^{-1}}{=\!=} \int_{M_n(K_w)} \Phi_2(\sigma)\, e_w(\mathrm{Tr}\,\beta\sigma X^{-1})\, d\sigma$$

$$= \int_{M_n(K_w)} \Phi_2(\sigma)\, e_w(\mathrm{Tr}\,\beta X^{-1}\sigma)\, d\sigma$$

<u>Recall</u>: The <u>Fourier transform</u> of a Schwarz function Φ on $M_n(K_w)$ is defined as

$$\mathcal{F}\Phi(x) = \int_{M_n(K_w)} \Phi(y)\, e_w(\mathrm{Tr}\, x^t y)\, dy$$

and the corresponding <u>inverse Fourier transform</u> of Φ is defined as

$$\mathcal{F}^{-1}\Phi(x) = \int_{M_n(K_w)} \Phi(y)\, e_w(-\mathrm{Tr}\, x^t y)\, dy$$

then $\mathcal{F}\mathcal{F}^{-1} = \mathcal{F}^{-1}\mathcal{F} = \mathrm{id}$.

Then $- = \mathcal{F}^{-1}\Phi_2(-X^{-t}\beta^t)$. Now the integral becomes

$$\int_{GL_n(K_w)} \Phi_1(-X)\cdot \mathcal{F}^{-1}\Phi_2(-X^{-t}\beta^t)\, \tau_1^{-1}\tau_2(\det X)\, |\det X|_w^{-2z+n}\, dX$$

So in practice, we choose Φ_2 as the Fourier transformation $\mathcal{F}\Phi_3$ of a Schwarz function Φ_3 on $M_{n\times n}(K_w)$. Then $\mathcal{F}^{-1}\Phi_2 = \mathcal{F}^{-1}\mathcal{F}\Phi_3 = \Phi_3$. Hence

$$W_\beta(1, f^{\Phi_w}, z) = \int_{GL_n(K_w)} \Phi_1(-X)\, \Phi_3(-X^{-t}\beta^t)\, \tau_1^{-1}\tau_2(\det X)\, |\det X|_w^{-2z+n}\, dX.$$

• $\boxed{\Phi_1\text{-part}}$ $\Phi_1(-X)$ with really <u>small and delicate</u> support T_w such that

① $T_w \subseteq GL_n(\mathcal{O}_{K_w}) \cap \ker(\tau_1^{-1}\tau_2)$: get rid of $\bullet$-part

② The choosen Φ_3 above has good left T_w-invariance property : take care of $\bullet$-part : $\Phi_3(-X^{-t}\beta^t) = \varphi_3(-X^{-t})\, \Phi_3(\beta^t)$ for some φ_3.

Then Φ_1 on T_w is defined to exterminate this <u>extra φ_3 part</u>.

Hence finally,

$$W_\beta(1, f^{\Phi_w}, z) \approx \underline{\mathrm{vol}(T_w)\, \Phi_3(\beta^t)}.$$

⚠ Hence with delicate choices of Φ_3 and Φ_1, the Fourier coefficient is easy to deal with!

(3) <u>Choice of Φ_3</u> :

- <u>Conductors and characters</u> : Write $\varpi := \varpi_K$ the uniformizer of O_{K_w} .
  - Let ϖ^{s_i} be the conductor of τ_i , $i=1,2$.
  - Let $\chi_1, \cdots, \chi_{r+s}$ be characters of $O_{K_w}^\times$ of conductor $\varpi^{t_1}, \cdots, \varpi^{t_{r+s}}$
  - Denote $\xi_i^\diamondsuit = \begin{cases} \chi_i\tau_1^{-1} & 1\le i\le r \\ \chi_i^{-1}\tau_2 & r+1\le i\le s \end{cases}$, $\xi_i^\heartsuit = \begin{cases} \chi_i\tau_1^{-1} & 1\le i\le r \\ 1 & i=r+1 \\ \chi_{i-1}^{-1}\tau_2 & r+2\le i\le r+s+1 \end{cases}$

    and denote $\xi_i' := \xi_k\tau_1\tau_2^{-1}$. (we can explicitly express ξ_i' as in [Wan15] , but it is not quite necessary) .

- <u>Choice of Φ_3</u> : Define $\mathfrak{X}_w^\heartsuit$ to be

$$\mathfrak{X}_w^\heartsuit = \left\{ x = \begin{array}{cc} \begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{pmatrix} & \begin{matrix} a \\ b \\ 1 \\ b \end{matrix} \\ \begin{matrix} a & \;\; b & \;\; 1 & \;\; b \end{matrix} & \end{array} \in M_{r+s+1}(O_{K_w}) \;\middle|\; \begin{array}{l} \begin{pmatrix} A_{11} & A_{14} \\ A_{21} & A_{24} \end{pmatrix} \text{ has } i\text{-th upper-left minors } A_i \text{ such that} \\ \det A_i \in O_{K_w}^\times ,\ i=1,\cdots,r . \\ A_{42} \text{ has } i\text{-th upper-left minors } B_i \text{ such that } \det B_i \in O_{K_w}^\times , \\ i=1,\cdots,s \end{array} \right\}$$

and $\mathfrak{X}_w^\diamondsuit$ by deleting the ● -rows & columns . Then we define

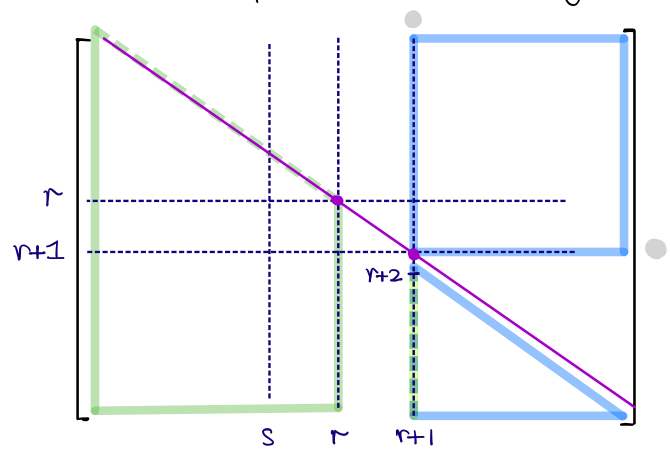
$$\Phi_3^\heartsuit(x) = \mathbb{1}_{\mathfrak{X}_w^\heartsuit}(x)\, \frac{\xi_1}{\xi_2}(\det A_1)\cdot \frac{\xi_2}{\xi_3}(\det A_2) \cdots \cdot \frac{\xi_{r-1}}{\xi_r}(\det A_{r-1})\cdot \xi_r(\det A_r) \cdot$$
$$\times \frac{\xi_{r+2}}{\xi_{r+3}}(\det B_1)\cdot \frac{\xi_{r+3}}{\xi_{r+4}}(\det B_2) \cdots \cdot \frac{\xi_{r+s}}{\xi_{r+s+1}}(\det B_{s-1})\cdot \xi_{r+s+1}(\det B_s)$$

and do similar modification to define $\Phi_3^\diamondsuit$. Later we mainly focus on $\heartsuit$-case without further notice .

<u>Note</u> : Though in [EHLS] , the T_w is defined not as delicate as [Wan] , it is delicate in another way : they considered $\begin{pmatrix} A_{11} & A_{14} \\ A_{21} & A_{24} \end{pmatrix}$, A_{42} as arbitary blocked matrices : $r = n_1 + \cdots + n_t$, $s = n_{t+1} + \cdots + n_{t'}$ and defined Φ_3 with minors under this partition . See [EHLS, P51] : this has more theoretical reason !

(4) <u>Choice of Φ_1</u> :

- Let $t := \max\{\underline{\underline{1}}, s_i, t_i\}$.
- Define $T_w^{\heartsuit}$ to be the subgroup of $GL_n(\mathcal{O}_{K_w})$ consisting of matrices



where the ●-filled, ▮, ●-filled entries are contained in $\varpi^t\mathcal{O}$, and the rest entries are in $\mathcal{O}$.

<u>Note</u>: In [Wan15], a more delicated version of T_w is designed:

- ● : the k-th column (for $1 \le k \le r$) is contained in $\varpi^k\mathcal{O}$.
- ● : the $(r+1)$-th column is contained in $\varpi^{s_i}\mathcal{O}$ for rows $i = r+2, \cdots, r+s+1$.
- ● : the j-th column (for $r+2 \le j \le r+s+1$) is contained in $\varpi^{t_{j-1}}$ for rows $i > j$ (i.e. below-diagonal entries) and $i \le r+s+1$.

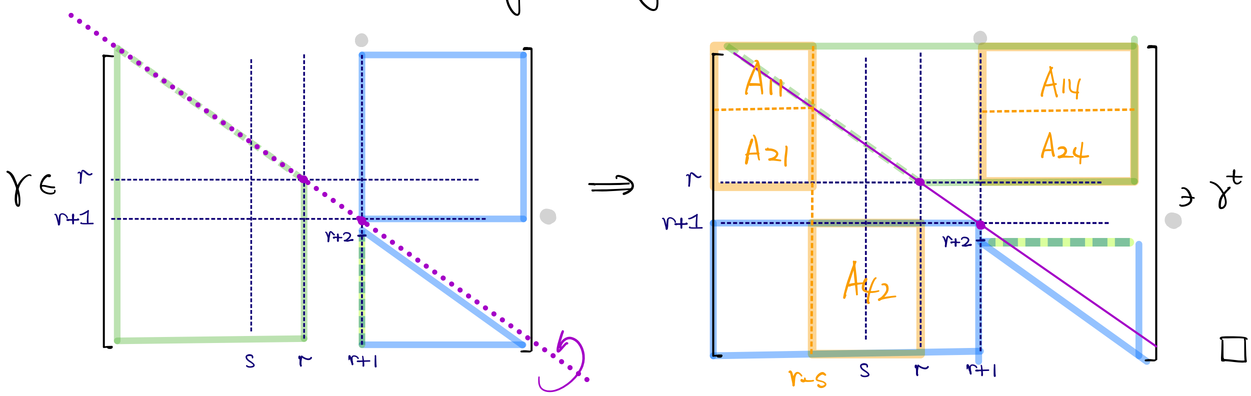
Our choice is based on [EHLS], since I guess choose t sufficiently large is enough for our purposes. (note our $t \ge 1$, allowing <u>unramified</u> local characters). For $T_w^{\heartsuit}$, simply deleting the extra column/rows ●.

Then we define $\Phi_1^{\bullet}(x) = \frac{1}{\mathrm{vol}(T_w)} \mathbb{1}_{T_w}(x) \cdot \prod_{k=1}^{n^{\bullet}} \varphi_{3k}'^{\bullet}(x_{kk})$.

Lemma [Wan15, Lemma 4.23] [EHLS, Lemma 4.2.3.(i)]

For $\gamma \in T$, $\Phi_3(\gamma^t \cdot X) = \prod_{k=1}^{n'} \varphi_{3k}(\gamma_{kk})\, \tau_1^{-1}\tau_2(\det\gamma)\, \Phi_3(X)$.

Proof: The proof is straightforward, by examing the blocks: γ^t can be thought as filpping the blocks along the diagonal:



$\square$

Therefore, we have the Fourier coefficient of f^{Φ_w} for $\Phi_w = \Phi_1 \otimes \mathcal{F}\Phi_3$:

Note: In [Wan15], he claimed the lemma without $\tau_1^{-1}\tau_2(\det\gamma)$. But then the result seems not fit the calculation on the Fourier coefficient.

Then by our choice of Φ_w, the Whittaker integral is now easy to compute :

<u>Proposition</u>. $W_\beta(1, f^{\Phi_w}, z) = \Phi_3(\beta^t)$.

<u>Proof</u> : Continue our previous computation :

$$W_\beta(1, f^{\Phi_w}, z) = \int_{GL_n(K_w)} \Phi_1(-X)\Phi_3(-X^{-t}\beta^t)\, \tau_1^{-1}\tau_2(\det X)\, \underbrace{|\det X|_w^{-2z+n}}_{|\det X|_w = 1}\, dX$$

$$= \int_{\Gamma_w} \mathrm{vol}(\Gamma_w)^{-1} \prod_{k=1}^{n} \underline{\xi_k'(-X_{kk})\,\xi_k(-(X^{-1})_{kk})}\, \Phi_3(\beta^t)\, \underline{\tau_1\tau_2^{-1}(\det(-X^{-1}))\,\tau_1^{-1}\tau_2(\det X)}\, dX$$

$$= \int_{\Gamma_w} \mathrm{vol}(\Gamma_w)^{-1} \underline{(\tau_1\tau_2^{-1}(-1))^n}\, \underline{(\tau_1\tau_2^{-1}(-1))^n}\, \Phi_3(\beta^t)\, dX$$

$$= \Phi_3(\beta^t).$$

(?) Actually I'm confused at how $\xi_k'(X_{kk})$, $\xi_k((X^{-1})_{kk})$ get cancelled out ? Even in the 2x2 matrix, $\begin{pmatrix} a & b \\ c & d\end{pmatrix}^{-1} = \det\cdot\begin{pmatrix} d & -b \\ -c & a\end{pmatrix}$. Then $d \neq a^{-1}$ actually.

Interprete in another way, write $\xi(y) = \prod_{k=1}^{n} \xi_k(y_{kk}) : \Gamma_w \to \mathbb{C}$,

then ξ may not be multiplicative !! This is maybe where we use the complicated assumption on Γ_w. $\square$

Sum up : So now we have constructed a Siegel section that is supported on the big cell.

- For p-adic L-function : it is enough, as in [EHLS].
- For Klingen Eisenstein family : it is not enough !

* Problem : A computation shows that the nearly ordinary Klingen section (in [CLW22] : semi-ord. Klingen section) is supported on the "small cell", which is hard to compute directly.

* Solution by Skinner–Urban : Use an intertwining operator $\mathcal{M}$ to compare the small-cell and big-cell section. with functional equations on

① Fourier coefficients : [Wan15, Lemma 4.2] ⟵ [SU14, Lemma 11.12]

(Fourier–Jacobi coefficient : See [Wan15, §4D3] for treatments following [SU14, Lemma 11.20])

② doubling integral : [Wan15, Prop. 4.40] ⟵ [SU14, Prop. 11.13]

Next our task is :

1° Introduce the good nearly ordinary Klingen section, as the final goal of pullback.

2° Compute the doubling integral of (modified) f^{Φ_ω}

3° Compute the doubling integral of (modified) $\mathcal{M} f^{\Phi_\omega}$

Along with 3°, we compute the Fourier coefficient of $\mathcal{M} f^{\Phi_\omega}$.

§ 8.4 Pullback section, before applying the intertwining operator

Note: A rough sketch is given in [Wan20, Lemma 6.32].

Defn [Wan15, Defn 4.32] Recall after we changed the basis, we have S_{new} for the second step of the doubling setup. Note S_{new} is not necessarily in $U(n,n)(F_v)$. Yet under the first projection,

$$\begin{array}{ccc} U(n,n)(F_v) & \xrightarrow{\sim} & GL_{2n}(F_v) \\ \Upsilon & \longmapsto & S_{new}^{-1} \end{array}$$

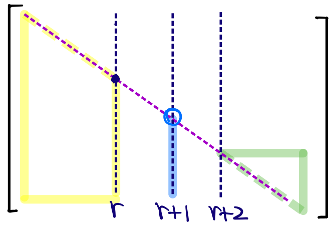
We define Υ as the element in $U(n,n)(F_v)$ corresponding to S_{new}^{-1}. Moreover, for any Siegal section $f_v \in I(\tau, z) \overset{\rho_w}{\simeq} \mathrm{Ind}_{B_{2n}(K_w)}^{GL_{2n}(K_w)}(\tau_w, z)$, we denote f_v^{Υ} as the right translation of f_v by Υ. (as a GL_{2n}-section, it is the right translation by S_{new}^{-1}).

Remark: Here we are using the notation of [Hsieh14, §5.6.3].

(1) Some "Klingen" level groups.

We define a (χ,τ)-level group $K_{\chi,\tau}$ as a subgroup of $GL_{r+s+2}(O_{F_v})$ so that

- for i-th column, $1\le i\le r$, $\varpi_v^{t_i} O_{F_v}$ contains the below-diagonal entries.
- for $(r+1)$-th column, $\varpi_v^{s'} O_{F_v}$ contains the below-diagonal entries.
- for $(r+1+j)$-th row, $1\le j\le s$, $\varpi_v^{t_{r+j}} O_{F_v}$ contains the right-to-diagonal entries

i.e. $K_{\chi,\tau} :=$ 

$\xrightarrow[\text{blocks}]{\text{break in}}$

$$\begin{array}{c c}
\left[\begin{array}{c:c:c:c:c}
a_1 & a_2 & a_3 & b_1 & b_2 \\ \hdashline
a_4 & a_5 & a_6 & b_3 & b_4 \\ \hdashline
a_7 & a_8 & a_9 & b_5 & b_6 \\ \hdashline
c_1 & c_2 & c_3 & d_1 & d_2 \\ \hdashline
c_4 & c_5 & c_6 & d_3 & d_4
\end{array}\right] & \begin{array}{c} a \\ b \\ 1 \\ b \\ 1 \end{array} \\
\begin{array}{ccccc} a & b & 1 & b & 1 \end{array} &
\end{array}$$

(Notation: The name of blocks follows from [Wan15, Lemma 4.33]).

We define a character ν of K as

$$\nu(k) := \tau_1(k_{r+1,r+1})\,\tau_2(k_{r+s+2,r+s+2}) \prod_{i=1}^{r} \chi_i(k_{ii}) \prod_{i=1}^{s} \chi_{r+i}(k_{r+i+1,r+i+1}).$$

(Notation: when discussing Klingen section, we often use letter ν as a shorthand for the combination (π,τ). For example, we will denote $K_{\chi,\tau}$ by K_ν.)

We write K_ν as a product $K = K'_\nu \cdot K''_\nu = K''_\nu \cdot K'_\nu$ as

the (i,j)-th entry of	c_1, $1\le i\le b$, $1\le j\le a$	c_2, $1\le i,j\le b$	remaing entries
K'_ν	$\varpi_v^{t_{r+i}+t_j}$	$\varpi_v^{t_{r+i}+t_{s+j}}$	—
K''_ν	$\varpi_v^{t_j}$	$\varpi_v^{t_{s+j}}$	· diagonal : 1 · off-diagonal : 0

Wan also brough in auxillary level groups of $GL_{r+s}(O_{F_v})$, but the strange labels are confusing at the first glance. We sort them out here:

$$\begin{array}{c c}
\left[\begin{array}{c:c:c:c:c}
a_1 & a_2 & a_3 & b_1 & b_2 \\ \hdashline
a_4 & a_5 & a_6 & b_3 & b_4 \\ \hdashline
a_7 & a_8 & a_9 & b_5 & b_6 \\ \hdashline
c_1 & c_2 & c_3 & d_1 & d_2 \\ \hdashline
c_4 & c_5 & c_6 & d_3 & d_4
\end{array}\right] & \begin{array}{c} a \\ b \\ 1 \\ b \\ 1 \end{array} \\
\begin{array}{ccccc} a & b & 1 & b & 1 \end{array} &
\end{array}
\xleftarrow[\substack{a_9=d_4=1 \\ \text{remaing } 0}]{\text{inclusion } \iota}
\begin{array}{c c}
\begin{bmatrix}
a_1 & a_2 & b_1 \\
a_4 & a_5 & b_3 \\
c_1 & c_2 & d_1
\end{bmatrix} & \begin{array}{c} a \\ b \\ b \end{array} \subseteq GL_{r+s}(O_{F_v}) \\
\begin{array}{ccc} a & b & b \end{array} &
\end{array}$$

$:= K_\nu^0 \subseteq GL_{r+s}(O_{F_v})$.

And accordingly we define the subgroups $K_\nu^{\circ,\prime} := (\iota^\circ)^{-1}(K_\nu^\prime)$ and $K_\nu^{\circ,\prime\prime} := (\iota^\circ)^{-1}(K_\nu^{\prime\prime})$.

Moreover we define the corresponding character $\nu^\circ(k) := \nu(\iota^\circ(k))$ for $k \in K_\nu^\circ$.

<u>Remark</u>: This is essentially what Wan was doing in [Wan15, Defn 4.34]. But he reordered the blocks as

$$\begin{array}{cc} \begin{matrix}a\\b\\b\end{matrix} & \begin{bmatrix}a_1&a_2&b_1\\a_4&a_5&b_3\\c_1&c_2&d_1\end{bmatrix} \\ & \begin{matrix}a&b&b\end{matrix}\end{array}
\xleftrightarrow[\text{in [Wan15, Defn 4.34]}]{\text{rename on blocks}}
\begin{array}{cc} \begin{matrix}a\\b\\b\end{matrix} & \begin{bmatrix}a_9&a_7&a_8\\a_3&a_1&a_2\\a_6&a_4&a_5\end{bmatrix} \\ & \begin{matrix}a&b&b\end{matrix}\end{array}
\xrightarrow[\left(1\leftrightarrow 2\ \substack{\text{rows}\\\text{colums}}\right)]{\text{rearrangement of Wan}}
\begin{array}{cc} \begin{bmatrix}a_1&a_3&a_2\\a_7&a_9&a_8\\a_4&a_6&a_5\end{bmatrix} & \begin{matrix}b\\a\\b\end{matrix} \\ \begin{matrix}b&a&b\end{matrix} & \end{array}$$

"$(-)^\iota$" ② the resulting rearrangement $(2\leftrightarrow 3$ column$)$

$$\begin{array}{cc} \begin{matrix}a\\b\\b\end{matrix} & \begin{bmatrix}a_1&b_1&a_2\\c_1&d_1&c_2\\a_4&b_3&a_5\end{bmatrix} \\ & \begin{matrix}a&b&b\end{matrix}\end{array}$$

↻

$\downarrow (-)^\iota$

① the resulting rearrangement: $\left(2\leftrightarrow 3\ \substack{\text{rows}\\\text{columns}}\right)$

$$\begin{array}{cc} \begin{matrix}a\\b\\b\end{matrix} & \begin{bmatrix}a_9&a_8&a_7\\a_6&a_5&a_4\\a_3&a_2&a_1\end{bmatrix} \\ & \begin{matrix}a&b&b\end{matrix}\end{array}$$

$\underbrace{\hphantom{xxxxxxxxxxxxxxxxxxxx}}_{\text{[Wan15, Defn4.34]}}$

Here the involution $(-)^\iota$ is defined in [Wan15, p2002]: $g \in GL_{r+s}(F_v)$,

$$g^\iota = \begin{pmatrix} & 1_a & \\ & & 1_b \\ 1_b & & \end{pmatrix} g \begin{pmatrix} 1_a & & 1_b \\ & & \\ & 1_b & \end{pmatrix}, \text{ as permuting rows \& columns of } g$$

Then Wan's $(K_\nu^\circ)^\iota$ in [Wan15, Lemma4.35] is actually swapping the 2^{nd} & 3^{rd} column and rows of our K_ν°. We also denote it as $(-)^\iota$.

<u>Note</u>: Actually I'm hoping:

1° $(\widetilde{K''})^\iota$ can give us exactly the reasonable K_ν° defined above.

2° $\widetilde{K''}$ is from a conjugation of K'' by w_1.

But both attempts failed. I'm not quite confident <u>on my strange $(-)^\iota$</u>.

(2) A guiding lemma for pullback sections

From now on in this subsection, we use w to denote

$$\begin{pmatrix} 1_a & & \\ & & 1_{b+1} \\ & -1_{b+1} & \end{pmatrix}_{a+2b+2} \quad \text{or} \quad \begin{pmatrix} 1_a & & \\ & & 1_b \\ & -1_b & \end{pmatrix}_{a+2b+1} \quad \text{according to the size.}$$

A guiding lemma [Wan15, Lemma 4.35] Let $\varphi \in \pi_v$. Assume ν is generic, and

(PB1) $F_\varphi(z, f^{\Phi_w, \chi}, g)$ as a function of g is supported in $P_w K_\nu$.

(PB2) $F_\varphi(z, f^{\Phi_w, \chi}, gk) = \nu(k) F_\varphi(z, f^{\Phi_w, \chi}, g)$, for $k \in K'$

(PB3) $F_\varphi(z, f^{\Phi_w, \chi}, w)$ is invariant under the action of $(K_\nu^{0,\prime\prime})^c$

Then $F_\varphi(z, f^{\Phi_w, \chi}, g)$ is the unique (up to scalar) section whose action by $k \in K_\nu$ is given by multiplying by $\nu(k)$.

Proof: It is actually a result of previous corollary (lemma), applied to the $GU(r+1, s+1)$-case, and note that $K_\nu = K_\nu' K_\nu'' = K_\nu'' K_\nu$. □

Moreover, combining (PB1) − (PB3), we see that it suffices to compute explicitly the pullback section $F_\varphi(z, f^{\Phi_w, \chi}, w)$. Before that, we sort out [Wan15] on justifying (PB1) − (PB3).

So to settle down the pullback section, we need to check (PB1) $\sim$ (PB3).

- (PB1) is justified in [Wan15, Lemma 4.36]: this is the MOST techical and complicated proof in the paper: through p2007–2012: (So we will not include a proof here).

  Lemma PB1: If $\iota_{\heartsuit}(g,1) \in \mathrm{supp}(f^{\Phi_w,\Upsilon})$, then $g \in P w K_v$.

  Therefore if $g \notin P w K_v$, then $\iota_B(g,1) = 0$ for all $g \in GU(r+1,s+1)(F_v)$. Then in the doubling integral,

$$f^{\Phi_w,\Upsilon}\left(\iota_B(g,g_1)\right) = f^{\Phi_w}\left(\underbrace{\iota_B(1,g_1)}_{\in \text{Siegal parabolic}} \cdot \iota_B(g,1)\right) = 0$$

  for any $g \in GU(r+1,s+1)(F_v)$.

- (PB2) is justified in [Wan15, Lemma 4.33]: Wan said that this follows directly from the action of K' on the GJ-section f^{Φ_w}.

- (PB3) involves a particular choice of φ:
  - * We emphasis again that π_v is nearly ordinary wrt $\underline{k}$, and here assume π_v is generic, to make the following φ well-defined:
  - * We define φ to be the unique (up to scalar) nearly ordinary section in π_v wrt the Borel $\widetilde{B}$.
  - * Let φ^w be right translation of φ by w
  - * Let $\varphi^{\Xi w}$ be the right translation of φ^w by Ξ, where

$$\Xi := \mathrm{diag}\Big(\underbrace{\varpi_v^{-t_{a+b+1}}, \cdots, \varpi^{-t_{a+2b}}}_{b}, \underbrace{\varpi^{t_1}, \cdots, \varpi^{t_a}}_{a}, \underbrace{\varpi^{t_{a+1}}, \cdots, \varpi^{t_{a+b}}}_{b}\Big)^{\iota} \cdot \begin{array}{c} \begin{pmatrix} & & -1_b \\ & 1_a & \\ -1_b & & \end{pmatrix}^{\iota} \\ \begin{array}{ccc} b & a & b \end{array} \end{array}$$

Notation: Here we use $\varphi^{w\Xi}$ to denote φ' in [Wan15, p2012], and our Ξ here is different from Wan's in loc.cit.

Lemma (PB3) [Wan15, proof of Lemma 4.37] $F_{\varphi^{\Xi}}(z, f^{\Phi_w,\Upsilon}, w)$ is invariant under the action of $(K_v^{0,\prime\prime})^{\iota}$.

This is actually the first sentence of [Wan15, Lemma 4.37].

(3) Concrete computations :

Now our tasks are to
- Verify (PB1) $\sim$ (PB3) .
- Compute $F_\varphi(z, f^{\Phi_w, \chi}, w)$

These are based on concrete computations of the pullback integral. We shall :

① Compute the Godement-Jacquet section explicitly : the GJ section is useful for computing the Whittaker integral, but the integral expression is not convinient for computing pullback integral. We will identify f^{Φ_w} with an "average" of big-cell sections .

② Verify (PB1) – (PB3) .

③ Along the way of ②, we carry on computing $F_\varphi(z, f^{\Phi_w, \chi}, w)$.

① Compute the Godement–Jacquet section explicitly

$\boxed{\text{Step 1}}$ Compute $\Phi_2 = \mathcal{F}\Phi_3$:

Lemma [Wan15, Lemma 4.28]

Suppose $\xi_i : \mathcal{O}_{K_w}^\times \to \mathbb{C}^\times$ for $i=1,\cdots,n$ are characters of conductor ϖ^{t_i}, and $t_1 > \cdots > t_n$. Let $\mathfrak{X}_n$ be the subset of $M_n(\mathcal{O}_{K_w})$ such that the i-th upper-left minor M_i has determinant in $\mathcal{O}_{K_w}^\times$. Define

$$\Phi_{\xi,n}(M) := \frac{\xi_1}{\xi_2}(\det M_1)\,\frac{\xi_2}{\xi_3}(\det M_2)\cdots\frac{\xi_{n-1}}{\xi_n}(\det M_{n-1})\,\xi_n(\det M_n)\,\mathbb{1}_{\mathfrak{X}_n}.$$

Then
$$\mathcal{F}\Phi_{\xi,n}(M) = \mathbb{1}_{\mathfrak{X}_n^{\mathcal{F}}}\, p^{-\sum_{i=1}^{r+s} i t_i} \prod_{i=1}^{n} \mathfrak{g}(\xi_i) \prod_{i=1}^{n} \overline{\xi_i}(x_i \varpi^{t_i})$$

where $\mathfrak{X}_{n,\xi}^{\mathcal{F}} := N(\mathcal{O}_{K_w})\begin{pmatrix}\varpi^{-t_1}\mathcal{O} & & & \\ & \varpi^{-t_2}\mathcal{O} & & \\ & & \ddots & \\ & & & \varpi^{-t_n}\mathcal{O}\end{pmatrix} N^{op}(\mathcal{O}_{K_w})$ and

write $x = \begin{pmatrix}1 & & * \\ & \ddots & \\ & & 1\end{pmatrix}\begin{pmatrix}x_1 & & \\ & \ddots & \\ & & x_n\end{pmatrix}\begin{pmatrix}1 & & \\ & \ddots & \\ * & & 1\end{pmatrix} \in \mathfrak{X}_n^{\mathcal{F}}$.

and $\mathfrak{g}(\xi)$ is the Gauss sum of the character ξ. We will define it in the proof.

Proof: The lemma is proved by induction on n.

First suppose $x \in$ big cell $N(K_w)T(K_w)N^{op}(K_w)$. then we write x as

$$x = \begin{pmatrix}1_{n-1} & u \\ & 1\end{pmatrix}\begin{pmatrix}z & \\ & w\end{pmatrix}\begin{pmatrix}1_{n-1} & \\ v & 1\end{pmatrix} \qquad \text{— } (*)$$

where $z \in GL_{n-1}(K_w)$. $w \in K_w^\times$, $u \in M_{n-1,1}(K_w)$, $v \in M_{1,n-1}(K_w)$. (Exercise. Actually $(*)$ is not that obvious, as in general $N(K_w)$ takes the form $\begin{pmatrix}\text{▨} & u \\ & 1\end{pmatrix}$ with ▨ uni-upper triangular. We need to turn it into a diagonal 1_{n-1}.)

Claim 1: If $\mathcal{F}\Phi_{\xi,n}(x) \neq 0$, then $v \in M_{1,n-1}(\mathcal{O}_{K_w})$ and $u \in M_{n-1,1}(\mathcal{O}_{K_w})$.

Claim 2: $\mathcal{F}\Phi_{\xi,n}$ is invariant under right multiplication by $N^{op}(\mathcal{O}_{K_w})$ and left multiplication by $N(\mathcal{O}_{K_w})$.

Some calculation: write $y = \begin{pmatrix} 1_{n-1} & \\ \ell & 1 \end{pmatrix} \begin{pmatrix} a & \\ & b \end{pmatrix} \begin{pmatrix} 1_{n-1} & m \\ & 1 \end{pmatrix} = \begin{pmatrix} a & am \\ \ell a & \ell a m + b \end{pmatrix}$

for $a \in \mathfrak{X}_{n-1}$, $b \in \mathcal{O}_{K_w}^\times$, $\ell \in M_{1,n-1}(\mathcal{O}_{K_w})$, $m \in M_{n-1,1}(\mathcal{O}_{K_w})$.

Actually it seems that we need an observation: every matrix in $\mathfrak{X}_n$ can be written in this form of y as above. [Wan15] used this observation without explicitly claiming!

Then

$$\mathcal{F}\Phi_\xi(x) = \int_{M_n(K_w)} \Phi_\xi(y)\, e_w(\mathrm{Tr}\, y x^t)\, dy$$

$$= \int_{\ell,a,b,m} \Phi_\xi\left(\begin{pmatrix} 1_{n-1} & \\ \ell & 1 \end{pmatrix} \begin{pmatrix} a & \\ & b \end{pmatrix} \begin{pmatrix} 1_{n-1} & m \\ & 1 \end{pmatrix} \right)$$

$$\times\, e_w\left(\mathrm{Tr} \begin{pmatrix} 1_{n-1} & \\ \ell & 1 \end{pmatrix} \begin{pmatrix} a & \\ & b \end{pmatrix} \begin{pmatrix} 1_{n-1} & m \\ & 1 \end{pmatrix} \begin{pmatrix} 1_{n-1} & v^t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \xi^t & \\ & w^t \end{pmatrix} \boxed{\begin{pmatrix} 1 & \\ u^t & 1 \end{pmatrix}} \right) dy$$

$$= \int_{\ell,a,b,m} \Phi_\xi\left(\begin{pmatrix} a & \\ & b \end{pmatrix} \right) e_w\left(\mathrm{Tr} \begin{pmatrix} 1 & \\ \ell+u^t & 1 \end{pmatrix} \begin{pmatrix} a & \\ & b \end{pmatrix} \begin{pmatrix} 1 & m+v^t \\ & 1 \end{pmatrix} \begin{pmatrix} \xi^t & \\ & w^t \end{pmatrix} \right) dy$$

$$= \int_{\ell,a,b,m} \Phi_\xi\left(\begin{pmatrix} a & \\ & b \end{pmatrix} \right) e_w\left(\mathrm{Tr} \begin{pmatrix} a & a(m+v^t) \\ (\ell+u^t)a & (\ell+u^t)a(m+v^t)+b \end{pmatrix} \begin{pmatrix} \xi^t & \\ & w^t \end{pmatrix} \right) dy$$

$$= \int_{\ell,a,b,m} \Phi_\xi\left(\begin{pmatrix} a & \\ & b \end{pmatrix} \right) e_w\left(\mathrm{Tr}\left(a\xi^t + \left((\ell+u^t)\, a(m+v^t) + b \right) w \right) \right) dy$$

w is scalar.

For claim 1: If $\mathcal{F}\Phi_\xi(x) \neq 0$ for x in the big cell, then:

- $w \in \varpi^{-t_n}\mathcal{O}_{K_w}^\times$. (Notice: focus on the term $a\xi^t w$, as $a \in \mathfrak{X}_{n-1}$. I'm guessing.)
- Suppose $v \notin M_{1,n-1}(\mathcal{O}_{K_w})$: then $m + v^t \notin M_{1,n-1}(\mathcal{O}_{K_w})$.
- Let us fix a, m, b and let ℓ vary in $M_{1,n-1}(\mathcal{O}_{K_w})$. Then we find the integral must be zero.

  (Notice: $a \in \mathfrak{X}_{n-1}$, $w \in \varpi^{-t_n}\mathcal{O}_{K_w}^\times$. Thus $a(m+v^t)w \notin M_{1,n-1}(\mathcal{O}_{K_w})$.)

Similarly, $u \notin M_{n-1,1}(\mathcal{O}_{K_w})$. So Claim 1 holds.

For claim 2: Left as an exercise. See [Hsieh14, Lemma 5.2.2].

Therefore WLOG, we assume $u = v = 0$, i.e. $x = \mathrm{diag}(z, w)$. We write $\Phi_{\varphi_{\xi,n-1}}$ as the restriction of $\Phi_{\varphi_{\xi,n}}$ to the upper-left $(n-1)\times(n-1)$ minor,

$$\mathcal{F}\Phi_{\varphi_{\xi,n}}(x) = \int_{M_n(K_w)} \Phi_{\varphi_{\xi,n}}(y)\, e_w(\mathrm{Tr}\, y \cdot x^t)\, dy$$

$$= \int_{l,a,b,m} \Phi_{\varphi_{\xi,n}}\left(\begin{pmatrix} a & \\ & b \end{pmatrix}\right) e_w\left(\mathrm{Tr}\left(az^t + (lam + b)w\right)\right) dl\, da\, db\, dm$$

Here I guess Wan first integrate $\int_{l,m} e_w(\mathrm{Tr}(lamw))\, dl\, dm$ and get 1 (or some constant: So may be the mysterious coefficient p^{-ntn} comes from the integral here?) So later he ignored this part and turn the integral into a product of "$\mathcal{F}\Phi_{\varphi_{\xi,n-1}}$" and a Gauss sum. See [EHLS, P53] for similar results.

$$= p^{-ntn} \int_{b \in O_{K_w}^\times} \xi_n(b)\, e_w(\mathrm{Tr}\, bw)\, db \cdot \int_{a \in \mathfrak{X}_{n-1}} \Phi_{\varphi_{\xi,n-1}}(a)\, e_w(\mathrm{Tr}(az^t))\, da$$

Note: In all future works of Wan, especially [Wan20], [CLW22], the factor p^{-ntn} appear. So we'd better include it.

Recall: in [Hsieh14, p56], the Gauss sum of a character ξ of conductor ϖ^t is defined as

$$\mathcal{G}(\xi) = \int_{x \in O_{K_w}^\times} \xi\left(\frac{x}{\varpi^t}\right) e_w\left(\mathrm{Tr}\frac{x}{\varpi^t}\right) dx$$

as we have seen above that $w \in \varpi^{-t_n} O_{K_w}^\times$, write $w = \varpi^{-t_n} w_0$. Then

$$\int_{b \in O_{K_w}^\times} \xi_n(b)\, e_w(\mathrm{Tr}\, bw)\, db = \int_{b \in O_{K_w}^\times} \xi_n(b)\, e_w\left(\mathrm{Tr}\frac{bw_0}{\varpi^t}\right) db$$

$$\underset{b \mapsto bw_0^{-1}}{=} \int_{b \in O_{K_w}^\times} \xi_n(bw_0^{-1})\, e_w\left(\mathrm{Tr}\frac{b}{\varpi^t}\right) db$$

$$= \xi_n^{-1}(\varpi^{t_n} w)\, \mathcal{G}(\xi_n).$$

Therefore for x in the big-cell, we get

- If $\mathcal{F}\Phi_{\varphi_{\xi,n}}(x) \neq 0$, then $x_n \in \varpi^{-t_n} O_{K_w}^\times$.
- Moreover, $\mathcal{F}\Phi_{\varphi_{\xi,n}}(x) = \xi_n^{-1}(\varpi^{t_n} x_n)\, \mathcal{G}(\xi_n) \cdot \mathcal{F}\Phi_{\varphi_{\xi,n-1}}(x)$.

Hence inductively, on the big cell,

$$\mathcal{F}\Phi_{\varphi_\xi, n} = \begin{cases} 0 & \text{if } x \in (\mathcal{X}_n^{\mathcal{F}})^c \cap \text{big cell} \\ p^{-\sum_{i=1}^{n} i t_i} \cdot \prod_{i=1}^{n} \xi_i^{-1}(\varpi^{t_i} x_i) \mathcal{G}(\xi_i) & \text{if } x \in \mathcal{X}_n^{\mathcal{F}} \cap \text{big cell}. \end{cases}$$

Then (as [Wan15] said) since $\mathcal{X}^{\mathcal{F}}_{\xi, n}$ is compact, $\mathcal{F}\Phi_{\varphi_\xi, n}$ itself must be supported in $\mathcal{X}_n^{\mathcal{F}}$, as desired. $\square$

<u>Remark</u>: Note that in the <u>Lemma</u>, we had requirement on the conductors. This is where the "generic assumption" is used.

With the above lemma, we see:

Lemma [Wan15, Lemma 4.29] Assume the generic condition in [Wan2015].

Let $\mathfrak{X}^{\mathcal{F}}_{n,\xi}$ be the support of $\Phi_2 = \mathcal{F}\Phi_3$. Then

(1) a complete set of representatives of $\mathfrak{X}^{\mathcal{F}}_{n,\xi}$ mod $M_n(\mathcal{O}_{K_w})$ is given by elements

$$x = x\left(\begin{pmatrix}A & B\\ C & D\end{pmatrix}, E\right) := \begin{array}{cc} \begin{array}{c}a\\ b\\ 1\\ b\end{array} & \left(\begin{array}{cc|c|c} A & & & B\\ C & & & D\\ \hline & & & \\ \hline & E & & \end{array}\right)\\ & \begin{array}{cccc} a & b & 1 & b\end{array}\end{array} \qquad \text{where:}$$

- $\begin{pmatrix}A & B\\ C & D\end{pmatrix}$ runs over $\begin{pmatrix}1 & & m_{ij}\\ & \ddots & \\ 0 & & 1\end{pmatrix}_{r\times r}\begin{pmatrix}\varpi^{-t_1}\mathcal{O}_w & & & \\ & \varpi^{-t_2}\mathcal{O}_w & & \\ & & \ddots & \\ & & & \varpi^{-t_r}\mathcal{O}_w\end{pmatrix}\begin{pmatrix}1 & & 0\\ & \ddots & \\ n_{ij} & & 1\end{pmatrix}_{r\times r}$

  with m_{ij} runs through $\underline{\mathcal{O}_w \bmod \varpi^{t_j}}$, n_{ij} runs over $\underline{\mathcal{O}_w \bmod \varpi^{t_i}}$ (here we mean m_{ij} runs through a complete set of representatives of $\mathcal{O}_w/\varpi^{t_i}$, and similarly for n_{ij})

- E runs over $\begin{pmatrix}1 & & k_{ij}\\ & \ddots & \\ 0 & & 1\end{pmatrix}_{s\times s}\begin{pmatrix}\varpi^{-t_{r+1}}\mathcal{O}_w & & & \\ & \varpi^{-t_{r+2}}\mathcal{O}_w & & \\ & & \ddots & \\ & & & \varpi^{-t_{r+s}}\mathcal{O}_w\end{pmatrix}\begin{pmatrix}1 & & 0\\ & \ddots & \\ l_{ij} & & 1\end{pmatrix}_{s\times s}$

  with k_{ij} runs through $\mathcal{O}_w \bmod \varpi^{t_{r+j}}$, n_{ij} runs over $\mathcal{O}_w \bmod \varpi^{t_{r+i}}$.

(2) Moreover,

$$\Phi_2(x) = \mathcal{F}\Phi_3(x) = \mathbb{1}_{\mathfrak{X}^{\mathcal{F}}_{n,\xi}}(x)\cdot p^{-\sum_{i=1}^{r} i t_i - \sum_{j=1}^{s} j t_{r+j}} \prod_{i=1}^{n} g(\xi_i)\, \xi_i^{-1}(x_i \varpi^{t_i})$$

where $x = x\left(\begin{pmatrix}A & B\\ C & D\end{pmatrix}, E\right)$ with diagonal representatives $x_1, \ldots, x_{r+s}$.

Remark : A fancy way to express explicity is to note that (almost correct I guess)

- For $i=1,\cdots,s$, $\varphi_{\xi_i}^{-1}(x_i\varpi^{t_i}) = \varphi_{\xi_i}^{-1}\left(\frac{\det A_i}{\det A_{i-1}}\varpi^{t_i}\right)$
- For $i=s+1,\cdots,r$, $\varphi_{\xi_i}^{-1}(x_i\varpi^{t_i}) = \varphi_{\xi_i}^{-1}\left(\frac{\det D_i}{\det D_{i-1}}\varpi^{t_{s+i}}\right)$
- For $i=r+1,\cdots,r+s$, $\varphi_{\xi_i}^{-1}(x_i\varpi^{t_i}) = \varphi_{\xi_i}^{-1}\left(\frac{\det E_{i-r}}{\det E_{i-r-1}}\varpi^{t_{i-r}}\right)$

  In this case, as in [Wan15], we tend to let $i=1,\cdots,s$ and write
$$\varphi_{\xi_{r+i}}^{-1}(x_{r+i}\varpi^{t_{r+i}}) = \varphi_{\xi_{r+i}}^{-1}\left(\frac{\det E_i}{\det E_{i-1}}\varpi^{t_i}\right).$$

Here A_i is the i-th upper-left minor of A, D_i is the $(s+i)$-th minor of $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ (not D!), E_i is the i-th upper-left minor of E. When $i=0$, we set $\det A_0 = 1$ and similar for D and E.

- This is more explicit, but makes the notation even more heavy! It has the advantage in $\boxed{\text{Step 3}}$ below.

$\boxed{\text{Step2}}$ Identify Godement–Jacquet section f^{Φ_w} with the big-cell section

<u>Defn</u>: Define $f_v^{\text{big-cell}}$ be the Siegal section supported on $Q(F_v)w_n\begin{pmatrix}1 & M_n(\mathcal{O}_{F_v})\\ & 1\end{pmatrix}$ such that $f_v^{\text{big-cell}}\left(w_n\begin{pmatrix}1 & X\\ 0 & 1\end{pmatrix}\right) = 1$ for all $X \in M_n(\mathcal{O}_{F_v})$.

Note that this is identically the same big-cell section in the ramified case in <u>Talk07</u>.

<u>Lemma</u> [Wan15, Lemma4.25] We identify $f_v^{\text{big-cell}}$ as a GL_{2n}-section under the first projection P_w. Then for $\beta \in M_n(K_w)$, its Fourier coefficient at β is

$$f_{w,\beta}^{\text{big-cell}}(1) = \mathbb{1}_{M_n(\mathcal{O}_{K_w})}(\beta)$$

The proof is the same (or even simpler) as the ramified case, though there we deal with unitary group, while here is GL_{2n}.

$\sum_{\cdots} \Phi_2\left(\begin{pmatrix}A & B\\ C & D\\ E & \end{pmatrix}\right)$ 要证明什么?

<u>Theorem</u>. We have

$$f^{\Phi_w}(z,g) = p^{-\sum_{i=1}^{r} i t_i - \sum_{j=1}^{s} j t_{r+j}} \prod_{i=1}^{n} g(\xi_i)$$

$$\times \sum_{A,B,C,D,E} \prod_{i=1}^{s} \xi_i^{-1}\left(\frac{\det A_i}{\det A_{i-1}} \varpi^{t_i}\right) \prod_{i=1}^{r-s} \xi_i^{-1}\left(\frac{\det D_i}{\det D_{i-1}} \varpi^{t_{s+i}}\right) \prod_{i=1}^{s} \xi_{r+i}^{-1}\left(\frac{\det E_i}{\det E_{i-1}} \varpi^{t_i}\right)$$

$$\times f_w^{\text{big-cell}}\left(z, g\left(\begin{array}{c|c} 1_n & \begin{pmatrix}A & B\\ C & D\\ & E\end{pmatrix} \\ \hline & 1_n \end{array}\right)\right)$$

<u>Proof</u>: Actually we are almost done.

- It suffices to check both sides coincide on $w_n N_n(K_w)$ since the big-cell $Q_n(K_w) w N_n(K_w)$ is dense in GL_{2n}.
- To see this, we just need to know that they have the same β-th Fourier coefficient for <u>all $\beta \in M_n(K_w)$</u>. (In [Wan15], he wrote $\beta \in S_n(K_w)$. Yet don't forget here we are in the GL_{2n}-case)

For the <u>GJ-section (LHS)</u>, we have seen $f_\beta^{\Phi_w}(1) = \Phi_3(\beta^t) = \mathcal{F}\Phi_2(\beta^t)$.

For the <u>big-cell section (RHS)</u>, we have seen $f_\beta^{\text{big-cell}}(1) = \mathbb{1}_{M_n(\mathcal{O}_{K_w})}(\beta)$.

So is the summation here works as the Fourier transform of Φ_2 ?? $\square$

$\boxed{\text{Step 3}}$ Rewrite the sum " $\sum_{A,B,C,D,E} f^{\text{big-cell}}$ " to fit in the context of pullback formula

We do the same thing as [Wan15, p2003–2004]:

<u>Recall</u>:

- $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ runs over $\begin{pmatrix} 1 & & m_{ij} \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_{r\times r} \begin{pmatrix} \varpi^{-t_1}\mathcal{O}_w & & & \\ & \varpi^{-t_2}\mathcal{O}_w & & \\ & & \ddots & \\ & & & \varpi^{-t_r}\mathcal{O}_w \end{pmatrix} \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ n_{ij} & & 1 \end{pmatrix}_{r\times r}$

- E runs over $\begin{pmatrix} 1 & & k_{ij} \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_{s\times s} \begin{pmatrix} \varpi^{-t_{r+1}}\mathcal{O}_w & & & \\ & \varpi^{-t_{r+2}}\mathcal{O}_w & & \\ & & \ddots & \\ & & & \varpi^{-t_{r+s}}\mathcal{O}_w \end{pmatrix} \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ l_{ij} & & 1 \end{pmatrix}_{s\times s}$

and we re-participate them as:

$$\mathcal{B}' := \text{set of } {\color{blue}\bullet}, \quad \mathcal{C}' := \text{set of } {\color{orange}\bullet}, \quad \mathcal{D}' := \text{set of } {\color{green}\bullet}, \quad \mathcal{E}' := \text{set of } {\color{violet}\bullet}$$

Then for a single $x = x\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}, E\right) \xlongequal{\text{rewrite as}} : x'(B', C', D', E')$

$$\prod_{i=1}^{s} \xi_i^{-1}\left(\frac{\det A_i}{\det A_{i-1}}\varpi^{t_i}\right) \prod_{i=1}^{r-s} \xi_i^{-1}\left(\frac{\det D_i}{\det D_{i-1}}\varpi^{t_{s+i}}\right) \prod_{i=1}^{s} \xi_{r+i}^{-1}\left(\frac{\det E_i}{\det E_{i-1}}\varpi^{t_i}\right)$$

$$= \prod_{i=1}^{r} \xi_i^{-1}(B'_{ii}) \prod_{i=1}^{s} \xi_{r+i}^{-1}(C'_{ii})$$

A more complicated part is the matrix in $f^{\text{big-cell}}$. Denote

- $y(B', C', D', E') := \alpha\left(\begin{pmatrix} B'_r & & & \\ & 1 & & \\ & & C'_s & \\ & & & 1 \end{pmatrix}_{r+s+2}, \begin{pmatrix} E' & \\ & D' \end{pmatrix}^{\iota}_{r+s} \right)$

- $$A' := \begin{array}{c|l} \left[\begin{array}{c:c:c:c} A'_{11} = \begin{pmatrix} \varpi^{-t_1} & & \\ & \ddots & \\ & & \varpi^{-t_{r-s}} \end{pmatrix} & & & \\ \hdashline & & & A'_{24} = \begin{pmatrix} \varpi^{-t_{r-s+1}} & & \\ & \ddots & \\ & & \varpi^{-t_r} \end{pmatrix} \\ \hdashline & & 0 & \\ \hdashline & A'_{42} = \begin{pmatrix} \varpi^{-t_{r+1}} & & \\ & \ddots & \\ & & \varpi^{-t_{r+s}} \end{pmatrix} & & \end{array}\right]_{(r+s+1)\times(r+s+1)} & \begin{array}{l} r-s \\ s \\ 1 \\ s \end{array} \end{array}$$

Lemma : The Godement–Jacquet section can be written as

$$f^{\Phi_w}(z,g) = p^{-\sum_{i=1}^{r} i t_i - \sum_{j=1}^{s} j t_{r+j}} \prod_{i=1}^{n} g(\varphi_{\xi_i})$$

$$\times \sum_{B',C',D,E'} \underbrace{\prod_{i=1}^{r} \varphi_{\xi_i}^{-1}\tau_1^{-1}(B'_{ii}) \prod_{i=1}^{s} \varphi_{\xi_{r+i}}^{-1}\tau_2^{-1}(C'_{ii})\, f^{\text{big-cell}}\left(z,\ g y(B',C',D',E')\cdot\begin{pmatrix}1 & A'\\ 0 & 1\end{pmatrix}\right)}_{} .$$

$$=: f^{\square}_{B',C',D,E'}(g) .$$ (notation in [Wan20]).

Proof : It seems the proof is just by matrix manipulation. But I failed to give details, after I tried in the following -pages. See [Wan15, p2004].

(actually I cannot checked this, the subtle thing is that the size of blocks does not match very well! Is it because Wan used a different basis as claimed in [Wan15, §4.D.2, p1996–1997]? No! Even in the new basis, I cannot see how the computation goes!) $\square$

③ Pullback section of $f^{\Phi_w, \Upsilon}$

To say anything on the pullback section $F_\varphi(z, f^{\Phi_w,\Upsilon}, g)$, for example, showing previous (PB1) — (PB3), we need to do explicit computations. The final result is in [Wan15, Lemma 4.38]. Combining with the guiding lemma, we get

Theorem [Wan15, Prop. 4.39] If φ and $\varphi^{\equiv w}$ is defined as in the previous section. Then

(1) $F_{\varphi^{\equiv w}}(z, f^{\Phi_w,\Upsilon}, g)$ is the unique section supported in PwK_v such that the right translation of K_v is given by the character ν.

(2) It value at $g = w$ is

$$F_{\varphi^{\equiv w}}(z, f^{\Phi_w,\Upsilon}, w) = \tau\left(\left(\varpi^{t_1+\cdots+t_r}, \varpi^{t_{r+1}+\cdots+t_{r+s}}\right)\right) \left|\varpi^{t_1+\cdots+t_{r+s}}\right|_v^{-z-\frac{r+s+1}{2}}$$
$$\cdot \operatorname{vol}(K_v^{0,\prime}) \cdot p^{-\sum_{i=1}^{r} i t_i - \sum_{j=1}^{s} j t_{r+j}} \prod_{i=1}^{n} g(\xi_i)\, \varphi^w$$

<u>Comment on the proof</u>:

- By previous computation on f^{Φ_w}, it suffices to compute $F_{\varphi^{w\equiv}}\left(z, f^{\square,\Upsilon}_{B',C',D',E'}\right)$. This is the starting point of the computation on [Wan15, p2012].
- The computation is a continuation of the proof of [Wan15, Lemma 4.36].

§ 8.5 Applying the intertwing operator

<u>Defn–Prop</u> [Wan15, p1978] For $f_v \in I_n(\tau)$, $z \in \mathbb{C}$. We formally define the <u>intertwing operator</u> as

$$M_v : I_n(\tau) \longrightarrow I_n(\tau^{-c}) \qquad \tau^{-c}(x) = \tau(\bar{x})^{-1}.$$

$$f_v \longmapsto M_v(z, f_v) := \int_{N_{Q_n}(F_v)} f_v\left(z, \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} r g\right) dr$$

For z in the compact subsets of $\{ Re(z) > \frac{n}{2} \}$, the integral converges absolutely and uniformly in g. Moreover, it has a meromorphic continuation on all of $\mathbb{C}$.

(0) Ramification condition

Recall previously we defined the "genericness" of χ (or say π_v). The same can be applied to $\pi \boxtimes \tau$. Here we consider the further assumptions as in [Wan15, p2015].

- π_∞ is a holomorphic discrete series associated to the scalar weight $\underline{k} := (0, \cdots, 0, k, \cdots, k)$.
- $\pi_v \simeq \mathrm{Ind}_{B_n(F_v)}^{GL_n(F_v)}(\chi_1, \cdots, \chi_{r+s})$ is nearly ordinary wrt $\underline{k}$: we reorder χ_i's s.t.

$$\begin{aligned} &v_p(\chi_1(p)) = s - \tilde{n}, \quad \cdots, \quad v_p(\chi_r(p)) = s + r - 1 - \tilde{n}, \\ &v_p(\chi_{r+1}(p)) = k + s - 1 - \tilde{n}, \quad \cdots, \quad v_p(\chi_{r+s}(p)) = k - \tilde{n}, \qquad \tilde{n} := \tfrac{1}{2}(n-1) \\ & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad n = r+s \end{aligned}$$

Then

$$v_p(\chi_1(p)) < \cdots < \underset{\substack{\| \\ r+s-1-\tilde{n}}}{v_p(\chi_r(p))} < \underset{\substack{\| \\ k-\tilde{n}}}{v_p(\chi_{r+s}(p))} < \cdots < \underset{\substack{\| \\ k+s-1-\tilde{n}}}{v_p(\chi_{r+1}(p))}$$

- $\tau_v = (\tau_1, \tau_2^{-1})$ with $v_p(\tau_1(p)) = v_p(\tau_2(p)) = \frac{k}{2}$.

$$\begin{aligned} &\cdot \ v_p(\tau_2(p) p^{-\mathfrak{z}_k}) = v_p(\tau_2(p)) - \mathfrak{z}_k = \tilde{n} + 1 \\ &\cdot \ v_p(\tau_1(p) p^{\mathfrak{z}_k}) = v_p(\tau_1(p)) + \mathfrak{z}_k = k - 1 - \tilde{n} \end{aligned}$$

$k - 1 - \tilde{n} > \tilde{n} + 1$

$k > 2\tilde{n} + 2$

$= n - 1 + 2 = n+1$

We require that

$$\underset{\substack{\| \\ r+s-1-\tilde{n}}}{v_p(\chi_1(p)) < \cdots < v_p(\chi_r(p))} < v_p(\tau_2(p) p^{-\mathfrak{z}_k})$$ — this holds trivially.

$$< v_p(\tau_1(p) p^{\mathfrak{z}_k})$$ $\Longleftrightarrow$ $k > n+1$

$$< \underset{\substack{\| \\ k-\tilde{n}}}{v_p(\chi_{r+s}(p))} < \cdots < \underset{\substack{\| \\ k+s-1-\tilde{n}}}{v_p(\chi_{r+1}(p))}$$ — these hold trivially.

where $\mathfrak{z}_k = \frac{1}{2}(k - r - s - 1) = \frac{k}{2} - \tilde{n} - 1$. It is easy to see that

$$I(\rho_v := \pi_v \boxtimes \tau_v, \mathfrak{z}_k) \simeq \mathrm{Ind}_{B_{r+s+2}(F_v)}^{GL_{r+s+2}(F_v)}(\chi_1, \cdots, \chi_{r+s}, \tau_2 \cdot |\cdot|^{\mathfrak{z}_k}, \tau_1 \cdot |\cdot|^{-\mathfrak{z}_k})$$

The setup of τ is to make $I(\rho_v, \mathfrak{z}_k)$ nearly ordinary wrt

$$(\underbrace{0, \cdots, 0}_{r+1}, \underbrace{k, \cdots, k}_{s+1}).$$

Remark: Actually in • above, we already need $k > r+s$. Bringing in τ, we further need $k > n+1$. This is a restriction on π_∞ when $GU(r,s)$ is fixed

So is this the source of picking good signature? This is actually strange!

Actually the generic condition is defined for $I(P_v, z_k)$ up to some reordering:

Defn [Wan15, Defn 4.42]. With the notations and assumptions above, say (π, τ) is generic if

$$t_1 > \cdots > t_r > s_2 > t_{r+1} > \cdots > t_{r+s} > s_1 .$$

Note that here we have $s_2 > s_1$, since we will apply the intertwing operator here.

(1) the pullback section

Theorem [SU14, Prop.11.13] [Wan15, Prop.4.40] Then there exists a meromorphic function $\gamma^{\bullet}(v,z)$ on $\mathbb{C}$ such that

(1) $F^{\diamondsuit}_{\varphi^{u}}(-z, M(z,f), g) = \gamma^{\diamondsuit}(v,z)\, F^{\diamondsuit}_{\varphi}(f; z, \eta g)$.

Moreover, if $\pi_v \simeq \mathrm{Ind}(\chi_1, \cdots, \chi_{n+s})$, then

$$\gamma^{\diamondsuit}(v,z) = \psi_v(-1)\, c_n(\tau', z)\, \mathfrak{g}(\tau_v')^n\, \varepsilon(\pi, \tau^c, z+\tfrac{1}{2}) \frac{L(\tilde{\pi}, \tau^c, \frac{1}{2}-z)}{L(\pi, \tau^c, z+\frac{1}{2})}.$$

(2) Write $\gamma^{\heartsuit}(v,z) = \gamma^{\diamondsuit}(v, z-\frac{1}{2})$, suppose (π_v, τ_v) comes from a global data (π, τ) as before, then

$$F^{\heartsuit}_{\varphi^{u}}(-z, M(z,f), g) = \gamma^{\heartsuit}(v,z)\, A(\rho, z, F_{\varphi}(f; z, -))(g).$$

Here:

- $\tau' = \tau|_{F_v}$: recall (ord), this is nothing but τ composed with the first projection
- $c_n(\tau', z) := \begin{cases} \tau'(\varpi_v^{nt})\, p^{2ntz - tn\frac{n+1}{2}} & t := \mathrm{ord}_v(\mathrm{cond}(\tau')) > 0 \\ p^{2nz - n\cdot\frac{n+1}{2}} & t = 0 \end{cases}$
- ε is the standard ε-factor, [Wan15, p206 (15)] gives an explicit expression.
- $A(\rho, z, f^{Klig}_{z,v})$ is the local Böchner integral for Klingen sections

$$A(f^{Klig}_{z})(g) = \int_{N_P(F_v)} f^{Klig}_{z,v}(wng)\, dn$$

recall
- we defined global Böchner integral in Talk04.
- The Klingen section $f^{Klig}_{z,v} : G \to V$ is actually vector-valued, so here the integral is actually a vector-valued integral.

- φ^{u} is the section after right translation of φ by $\eta = \begin{pmatrix} & & 1_b \\ & 1_a & \\ -1_b & & \end{pmatrix}$.

Recall : We changed basis ! So corresponding η should be $\eta_{new} = \begin{pmatrix} 1_a & & \\ & & 1_b \\ & -1_b & \end{pmatrix}$

That is our ongoing w.

A useful observation is that $\varphi^{uu} = \varphi$ since $\eta^2 = 1$.

Remark : Please see [Wan15, Remark 4.41] for comparing [SU14, Prop. 11.13] of different L-functions.

Now applying the intertwining operator to $f^{\Phi_w} \in I_n(\tau^c)$, we define

$$f^{\alpha}(z,g) := M(-z, f^{\Phi_w})(g).$$

Then our goal is:

<u>Goal</u>: The pullback section $F_{\varphi w\Xi}(z, f^{\alpha,\chi}, g)$ is a constant multiple of the nearly ordinary vector φ if (π,τ) comes from a global Eisenstein datum.

Actually the computation is straight forward as we have: [Wan15] said this!

- The explicit formula of $F_{\varphi w\Xi}(z, f^{\chi}, g)$.
- Previous theorem on the effect of M.

The difficulty is to compute the Böchner integral. In [Wan15, p2016], Wan claimed that "it is easy to compute". But I'm confused.

$$A(\rho, z, F_{\varphi w\Xi}(f; z, -))(1) = \int_{N_p(F_v)} F_{\varphi w\Xi}(f; z, wn)\, dn$$

$$\approx \underbrace{\text{"}\mathrm{Vol}(N_p(F_v))\text{" up to an integration on } K_v}_{\text{If } K_v \supseteq N_p(F_v)\text{, but I doubt on this}} \cdot \underline{\underline{\varphi^w}}$$

? doubt

by noting that previously we have seen $F_{\varphi w\Xi}(f, z, g)$ is supported on $P w K_v$ with prescribed right K_v-action by ν. Compare the results on [Wan15, p2016], I guess this part gives an extra factor $p^{\sum_{i=1}^{r+s} t_i}$?

⚠ The most suspicious part is how the Böchner integral turned φ^w into φ?

$$F_{(\varphi w\Xi)^{\nu}}(f, z, g) = \int_{Gl_{r+s}(F_v)} f(z, \iota_{\mathcal{D},\mathrm{new}}(g, g_1))\, \tau^{-1}(\det g_1)\, \pi(g_1)(\varphi^{w\Xi})^{\nu}\, dg_1$$

$$= \int_{Gl_{r+s}(F_v)} f(z, \iota_{\mathcal{D},\mathrm{new}}(g, g_1))\, \tau^{-1}(\det g_1)\, \pi(wg_1)\varphi^{w\Xi}\, dg_1$$

$wg_1 = g_1'$, $w = \bar{w}'$

$$\rightarrow\; = \int_{Gl_{r+s}(F_v)} f(z, \iota_{\mathcal{D},\mathrm{new}}(g, \underline{wg_1}))\, \tau^{-1}(\det wg_1)\, \pi(g_1)\varphi^{w\Xi}\, dg_1$$

$$= \tau^{-1}(\det w)\int_{Gl_{r+s}(F_v)} f(z, \underbrace{\iota_{\mathcal{D},\mathrm{new}}(g,w)}_{\text{take it out}}\, \iota_{\mathcal{D},\mathrm{new}}(g,g_1))\, \tau^{-1}(\det g_1)\cdots dg_1$$

dreaming

$$= \pi(w)\, \tau^{-1}(\det w)\, F_{\varphi w\Xi}(f, z, g)$$

$\rightsquigarrow$ So in general we need a result <u>linking F_{φ} and $F_{\varphi^{\nu}}$</u>!!

Proposition [Wan15, p2016] Under the genericness of (π, τ^{-c}) (note: not (π,τ))

$F_{\varphi\equiv w}(z, f^{\alpha,x}, g)$ is supported in $P(F_v)K_v$, with

(1) The right action of K_v is given by

$$g \longmapsto \chi_1(g_{11})\cdots\chi_r(g_{rr})\,\tau_2(g_{r+1,r+1})\,\chi_{r+1}(g_{r+2,r+2})\cdots\chi_{r+s+1}(g_{r+s+1,r+s+1})\,\tau_1(g_{r+s+2,r+s+2})$$

(2) The value at $g=1$ is (under my guess in the previous page).

$$F^{\heartsuit}_{\varphi\equiv w}(z, f^{\alpha,x}, 1) = \tau^{-c}\left((\varpi^{t_1+\cdots+t_r}, \varpi^{t_{r+1}+\cdots+t_{r+s}})\right)\left|\varpi^{t_1+\cdots+t_{r+s}}\right|_v^{z-\frac{r+s+1}{2}}$$
$$\times \operatorname{vol}(K_v^{0,\prime})\cdot p^{-\sum_{i=1}^{r}(i+1)t_i - \sum_{j=1}^{s}(j+1)t_{r+j}} \prod_{i=1}^{n} g(\varphi_{z_i}^{\alpha})\,\gamma^{\heartsuit}(v, -z)\cdot\varphi .$$

where $\varphi_{z_i}^{\alpha}$ is the define as previous φ_{z_i} but replace τ by τ^{-c}. Similar results hold for $F^{\diamondsuit}_{\varphi\equiv w}(\ldots)$.

Remark: It should be a direct computation as sketched (and guessed) on the previous page. In [Wan15, p2016], Wan continued to put in explicit expression of $\gamma^{\heartsuit}$ and ε-factor, but I think it is not necessary to do so!

(Actually, there are some typos on the second row of the formula for "$F_v^{0}(z,1)$" there. For example the lost $\psi_v(-1)$ and L-functions, and the extra ε-factor "$\varepsilon(\pi, \tau^c, z)$" there. Am I getting this wrong?)

The result of $F^{\diamondsuit}_{\varphi\equiv w}$ can be found in [Wan15, p2016].

Therefore, $F^{\heartsuit}_{\varphi\equiv w}(z, f^{\alpha,x}, -)$ is a nearly ordinary vector in I_v.

(2) the GJ-section

Recall we had rewritten the GJ-section as a summation of big-cell section.

Let $\mathrm{Cond}(\tau') = (\varpi_v^t)$ for $t \geq 1$. Then we define a new level group $K_Q(\varpi_v^t)$ as matrices in $Q(\mathcal{O}_{F_v})$ modulo ϖ_v^t, called the v-part level group.

Remark: confusing? It consists of matrices in $GL_{2n}(\mathcal{O}_{F_v})$ that is in $\left(\begin{array}{c|c} * & * \\ & * \end{array}\right)$ mod ϖ_v^t. Wan likes to write in this way. Note in [Wan15, Defn 4.43], he mentioned "$K_Q(p^t)$" without defined previously. I found the definition in the proof on [Wan15, p.2019] and [Wan20, Defn 5.6].

For every $k \in K_Q(\varpi^t)$, write $k = \begin{pmatrix} a_k & b_k \\ c_k & d_k \end{pmatrix}\begin{matrix} n \\ n \end{matrix}$.

Defn: [Wan15, Defn 4.43] We define f_t to be the section supported in $Q(F_v)K_Q(\varpi_v^t)$ with $f_t(k) = \tau'(\det d_k)$ on $K_Q(\varpi_v^t)$.

Proposition [Wan15, Lemma 4.44] $f^{\text{big-cell},\alpha} := M(-z, f^{\text{big-cell}}) = f_{t,z}$.

Proof: See [SU14, Lemma 11.10].

Then as a corollary, we have an explicit expression of $f^{\alpha} := M(-z, f^{\Phi_v})(g)$:

Corollary [Wan15, Corollary 4.45] With notations same as above.

$$f^{\alpha}(z,g) = p^{-\sum_{i=1}^{r} i t_i - \sum_{j=1}^{s} j t_{r+j}} \prod_{i=1}^{n} g(\varphi^{\alpha}_{\xi_i})$$

$$\times \sum_{A,B,C,D,E} \prod_{i=1}^{s} (\varphi^{\alpha}_{\xi_i})^{-1}\left(\frac{\det A_i}{\det A_{i-1}} \varpi^{t_i}\right) \prod_{i=1}^{r-s} (\varphi^{\alpha}_{\xi_i})^{-1}\left(\frac{\det D_i}{\det D_{i-1}} \varpi^{t_{s+i}}\right) \prod_{i=1}^{s} (\varphi^{\alpha}_{\xi_{r+i}})^{-1}\left(\frac{\det E_i}{\det E_{i-1}} \varpi^{t_i}\right)$$

$$\times f_t\left(z, g\left(\begin{array}{c|c} 1_n & \begin{pmatrix} A & B \\ C & D \\ & E \end{pmatrix} \\ \hline & 1_n \end{array}\right)\right)$$

Remark: In [Wan15, Coro. 4.45], "$\hat{f}_t$" had not been defined. It should be f_t! Moreover, ξ_i was not made into ξ_i^{α}. I "corrected" them up here.

(3) <u>the Fourier coefficients</u>

Recall we defined

$$\mathrm{Herm}_n^{\star}(\mathcal{O}_{F_v}) := \{ \beta \in \mathrm{Herm}_n(F_v) : \mathrm{Tr}\beta\sigma \in \mathcal{O}_{F_v} \text{ for any } \sigma \in \mathrm{Herm}_n(\mathcal{O}_{F_v}) \}$$

The Fourier coefficients after applying the intertwing operator M on f^{Φ_w} are computed as:

<u>Theorem</u> [SU14, Lemma 11.12] [Wan15, Lemma 4.46] Suppose $\det\beta \neq 0$. Then

$$W_\beta(1, f^\alpha, z) = (\tau^{-1})'(\det\beta)\,|\det\beta|_v^{2z}\, g(\tau')^n\, c_n((\tau^{-1})', -z)\, \Phi_3(\beta^t)\, \mathbb{1}_{\mathrm{Herm}_n^{\star}(\mathcal{O}_{F_v})}(\beta) .$$

Proof can be found in [SU14, Lemma 11.12]. I hope one day I will read it and copy a proof below.

(3) Applying the intertwing operator : the Fourier–Jacobi coefficients

Recall we haven't compute the FJ coefficients for f^{Φ_w}. So at the first glance, we should first compute it for f^{Φ_w} and then build up how the intertwing operator affects the FJ–coefficient.

BUT we can actually be more straight forward : we have computed the f^{α} as a "linear combination" of f_t. So we can directly start with f_t.

Unfortunately : currently we only have results for $m = s+1$.

Proposition [Wan15, Lemma 4.47] Let $m = s+1$, $\beta \in S_m(F_v) \cap GL_m(\mathcal{O}_{F_v})$. Let $x := \begin{array}{c} \begin{array}{cc} r & r \end{array} \\ \begin{pmatrix} 1 & 0 \\ D & 1 \end{pmatrix} \end{array}$

Then

(1) $FJ_\beta^{\heartsuit}(f_t ; z, v, x\eta^{-1}, 1) = 0$ if $D \notin \varpi_v^{t} M_r(\mathcal{O}_{F_v})$.

(2) If $D \in \varpi_v^{t} M_r(\mathcal{O}_{F_v})$, then

$$FJ_\beta^{\heartsuit}(f_t, z, v, x\eta^{-1}, 1) = c(\beta, \tau, z)\,\Phi_0(v).$$

Φ_0 是出现在 Θ-correspondence 之中的

而且我发现前面都没有算过

Fourier-Jacobi coefficient!!!