The Gan-Gross-Prasad period of Klingen Eisenstein families over unitary groups

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Journées Arithmétiques 2025, at the University of Luxembourg

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July 1st, 2025

Contents

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1. The **class number formula**: The class number of a number field F, and its Dedekind ζ-functions $\zeta_F(s)$.
2. The **Birch-Swinnerton-Dyer (B-SD) conjecture**: The $\mathbb{Z}$-rank of $E(F)$ of an elliptic curves E over F, and its Hasse-Weil L-function $L(E/F, s)$. Also we have the refined B-SD formula predicted.

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Our focus: $\mathbb{Z}_p$-extensions

Let F be a number field. Consider a tower of number fields

$$F \subset F_1 \subset F_2 \subset \cdots \subset F_n \subset \cdots \subset F_\infty$$

such that $\mathrm{Gal}(F_\infty/F) \simeq (\mathbb{Z}_p, +)$. The extension F_∞/F is called a $\mathbb{Z}_p$-**extension** of F, and its intermediate fields are F_n for $n \geq 1$ with $\mathrm{Gal}(F_n/F) \simeq \mathbb{Z}/p^n$.

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One checks that $\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q}(\mu_p)$ is a $\mathbb{Z}_p$-extension, with the n-th layer being $\mathbb{Q}(\mu_{p^{n+1}})$.

Motivation: Iwasawa theory

Iwasawa's original problem

Let F_∞/F be a $\mathbb{Z}_p$-**extension** of F with intermediate fields

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The upshot: Cl_∞ is a **finitely generated torsion** Λ-module.

Iwasawa theory of class groups

Here comes some commutative algebras of Λ-modules:

Finitely generated torsion Λ-modules are "rigid":

Let M be a finitely generated torsion Λ-module. Then there is a ("unique") map

$$M \to \bigoplus_{i=1}^{r} \Lambda/p^{m_i} \oplus \bigoplus_{j=1}^{t} \Lambda/f_j(T)^{n_j}$$

whose kernel and cokernel are of finite cardinality. Here $f_j(T) \in \mathbb{Z}_p[T]$ are "distinguished polynomials".

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- λ-**invariant**: $\lambda(M) := \sum_{j=1}^{t} n_j \deg(f_j(T))$,
- **characteristic ideal**: $\mathrm{char}(M) := (p^{\mu(M)} \prod_{j=1}^{t} f_j(T)^{n_j})$, a principal ideal of Λ.

Iwasawa's class number formula, Iwasawa main conjecture

1. (Structural theorem, Iwasawa) We have the following **Iwasawa's class number formula**:
$$\#\mathrm{Cl}(F_n)[p^\infty] = p^{\mu p^n + \lambda n + \nu}, \quad \text{for } n \gg 0,$$
where $\mu = \mu(\mathrm{Cl}_\infty)$, $\lambda = \lambda(\mathrm{Cl}_\infty)$, and ν is an integer (possibly negative).

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2. ("Analytical" theorem, Mazur-Wiles) We have the following **Iwasawa main conjecture**, formulated in a very imprecise way:

$$\mathrm{char}(\mathrm{Cl}_\infty) = (\mathcal{L}_p), \text{ as principal ideals of } \Lambda.$$

Here $\mathcal{L}_p \in \Lambda$ is the *p-adic L-function of Kubota-Leopoldt*, interpolating special values of Dirichlet L-functions.

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So we can describe the μ-invariant and the λ-invariant of Cl_∞ by these of $\Lambda/(\mathcal{L}_p)$, which is a problem of analytic nature.

The machinery of Eisenstein congruences

Today, we focus on Iwasawa's main conjecture, of the following form

$$\mathrm{char}(\mathrm{Cl}_\infty) = (\mathcal{L}_p),$$

and especially the following divisibility:

Lower bound for Cl_∞: Show $\mathcal{L}_p \mid \mathrm{char}(\mathrm{Cl}_\infty)$

Use **Eisenstein congruences** to construct sufficiently many ideal classes.

Baby example: Serre's construction with Ramanujan's congruence

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Step 1 : Consider the weight 12, level $\mathrm{SL}_2(\mathbb{Z})$ Eisenstein series

$$E_{12}(q = \exp(2\pi \mathrm{i} z)) = -\frac{B_{12}}{24} + \sum_{n \geq 1} \left(\sum_{d \mid n} d^{11} \right) q^n, \quad -\frac{B_{12}}{24} = \frac{691}{156} \cdot \frac{1}{420}.$$

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Step 2 : Therefore, p divides the constant term of E_{12}. Therefore it is plausible to construct a cuspform $f \in S_k(\mathrm{SL}_2(\mathbb{Z}))$ such that $E_{12} \equiv f \pmod{p}$.

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Step 2 : Therefore, p divides the constant term of E_{12}. Therefore it is plausible to construct a cuspform $f \in S_k(\mathrm{SL}_2(\mathbb{Z}))$ such that $E_{12} \equiv f \pmod p$. Actually it is proved by Ramanujan that

$$\Delta \equiv E_{12} \mod p = 691,$$

where

$$\Delta(q = \exp(2\pi \mathrm{i} z)) := q \prod_{i=1}^{\infty} (1 - q^n)^{24} = \sum_{n=1}^{\infty} \tau(n) q^n \in S_{12}(\mathrm{SL}_2(\mathbb{Z})), \quad \mathrm{im}(z) > 0.$$

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Step 3 : Translate this to a modulo-p congruence of Galois representations, we have

$$\rho_\Delta \equiv \rho_{E_{12}} \equiv \begin{bmatrix} \omega^{11} & * \\ 0 & 1 \end{bmatrix} \pmod{691}.$$

The Galois cohomology class $*$, which turns out to be nontrivial, will provide a desired ideal class.

Iwasawa theory of unitary groups

Starting with B. Mazur, it was observed that the Iwasawa theory of class groups can be vastly generalized to study the arithmetic of elliptic curves, modular forms, and even general Galois representations.

Question: Iwasawa theory of modular forms

Suppose we want to study the Iwasawa theory of a cuspidal eigenform $f_0 \in S_k(\Gamma_1(N), \epsilon)$. How does the machinery of Eisenstein congruences proceed?

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Step 1 : Regard the L-function $L(f_0, s)$ as an automorphic L-function over the reductive group GL_2, the task is to construct

$$\text{an Eisenstein series } E_{??} \text{ over a reductive group } G^{??},$$

whose constant term should involve $L(f_0, s)$.

Iwasawa theory of unitary groups

Upshot: The **Klingen Eisenstein series** $E_{f_0}^{\mathrm{Kling}}$ over unitary groups $\mathrm{U}(2,2)$ or $\mathrm{U}(3,1)$ encodes $L(f_0, s)$.

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Unitary groups

Let K be an auxiliary imaginary quadratic field such that p splits in K, one defines

$$\mathrm{U}(m,n) := \left\{ g \in \mathrm{GL}_{m+n}(K) : \overline{g}^{\mathrm{t}} \begin{bmatrix} \mathbf{1}_m & 0 \\ 0 & -\mathbf{1}_n \end{bmatrix} g = \begin{bmatrix} \mathbf{1}_m & 0 \\ 0 & -\mathbf{1}_n \end{bmatrix} \right\}.$$

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This is called the **unitary group** of signature (m,n). This is an algebraic group over $\mathbb{Q}$. For example,

$$\mathrm{U}(1,1) \simeq \mathrm{GL}_2, \quad \mathrm{U}(2,0) \simeq \text{a nonsplit quaternion algebra.}$$

Step 1 Let's put ourself in the most general case: Let φ_0 be a cuspform over the unitary group $\mathrm{U}(m,n)$, we can define a Klingen Eisenstein series

$$E_{\varphi_0}^{\mathrm{Kling}} : \mathrm{U}(m+1,n+1)(\mathbb{A}_{\mathbb{Q}}) \to \mathbb{C}$$

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The key difficulty: "*Step 2*" of Eisenstein congruences

How to show that the congruence relation "$E_{\varphi_0}^{\mathrm{Kling}} \equiv f \pmod{p}$" is **nontrivial**? In other words, how to show that $E_{\varphi_0}^{\mathrm{Kling}} \not\equiv 0 \pmod{p}$? (after appropriate normalization to make it algebraic.)

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Example on lower rank cases

- Recall the GL_2-case, the p-th Fourier coefficient of E_{12} is $1+p^{11}$, which is nonzero modulo p. Hence $E_{12} \not\equiv 0 \pmod p$.

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- Recall the GL_2-case, the p-th Fourier coefficient of E_{12} is $1 + p^{11}$, which is nonzero modulo p. Hence $E_{12} \not\equiv 0 \pmod p$.
- For Klingen Eisenstein series $E_{f_0}^{\mathrm{Kling}}$ over $\mathrm{U}(2,2)$ or $\mathrm{U}(3,1)$, certain Fourier coefficients of $E_{f_0}^{\mathrm{Kling}}$ can be computed. See for example, [Skinner-Urban 2014][Wan 2020][Castella-Liu-Wan 2022].

Problem: these methods are hard to generalize to higher ranks.

Our work on the GGP period integral of Klingen Eisenstein series

For $E_{\varphi_0}^{\mathrm{Kling}}$ over $\mathrm{U}(m+1, n+1)$, we compute its **Gan-Gross-Prasad period integral**, defined as

$$\mathscr{P}_{\varphi}(E_{\varphi_0}^{\mathrm{Kling}}) := \int_{\mathrm{U}(m+1,n)(\mathbb{Q})\backslash \mathrm{U}(m+1,n)(\mathbb{A}_{\mathbb{Q}})} E_{\varphi_0}^{\mathrm{Kling}}(\iota(g))\varphi(g)\,\mathrm{d}g,$$

where φ is a cuspform over the smaller group $\mathrm{U}(m+1, n)$, and ι is a canonical embedding $\iota : \mathrm{U}(m+1, n) \hookrightarrow \mathrm{U}(m+1, n+1)$.

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Philosophy of period integrals

- To show $E_{\varphi_0}^{\mathrm{Kling}}$ is modulo-p nonvanishing, it suffices to **choose an approprite** φ such that $\mathscr{P}_{\varphi}(E_{\varphi_0}^{\mathrm{Kling}})$ is nonvanishing modulo p.

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- Such period integrals are closely related to **special values of L-functions**, and we have more tools on these L-functions.

Theorem (X., arXiv:2410.13132)

Under a list of technical hypotheses, including that π_φ and π_{φ_0} are p-ordinary, we have

$$\frac{\mathscr{P}_\varphi(E_{\varphi_0}^{\mathrm{Kling}})^2}{|\varphi|\,|\varphi_0|} \approx \mathcal{L}^\Sigma\left(\frac{1}{2}, \pi_\varphi \times \pi_{\varphi_0}\right)\mathcal{L}^\Sigma(s_0, \pi_\varphi)\,\mathcal{L}^\Sigma(s_0, \pi_\varphi^\vee),$$

where "$\approx$" means "up to explicit factors at some bad places $v \in \Sigma$", and $\mathcal{L}^\Sigma$ denotes appropriately normalized L-functions with local Euler factors at bad places $v \in \Sigma$ removed.

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- For applications to Iwasawa theory, we also need a p-adic variation of the above theorem: **deforming φ and φ_0 in Hida families**.

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- For applications to Iwasawa theory, we also need a p-adic variation of the above theorem: **deforming φ and φ_0 in Hida families**.
- **A byproduct**: A p-adic L-function for the Rankin-Selberg product of Hida families over $\mathrm{U}(m, n) \times \mathrm{U}(m+1, n)$.

These results are available in our preprint [arXiv:2410.13132].

Ongoing (unclear) works

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Unclear "Philosophy": reducing the rank of unitary groups

Choose φ (over $\mathrm{U}(m+1, n)$) to be the **theta lifting** of an appropriately chosen $\varphi^\sharp$ over a lower rank unitary group $\mathrm{U}(m, n)$ (or even smaller unitary groups).

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- When is a theta lifting $\theta(\varphi^\sharp)$ "algebraic", nonzero modulo p?
- Suppose $\varphi^\sharp$ is p-ordinary, when will the theta lifting $\theta(\varphi^\sharp)$ be p-ordinary? Can we deform them into Hida families accordingly?

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The general theory, however, remains to be fully explored.

Thank you!

Slides will be available on my webpage:
`https://xuruichen98.github.io/`

Ruichen XU, `xuruichen21@mails.ucas.ac.cn`