Talk4 Beilinson–Kato zeta elements

In this talk, we introduce the construction of Beilinson–Kato zeta elements and their p-adic interpolation.

Original article : Kato, Astérisque

Survey article : · Chan-Ho Kim, "A users' guide to Beilinson-Kato's zeta elements".
· Tadashi Ochiai, Volume 2, §6.4.

§1. Beilinson–Kato zeta element

§1.1. Motivic construction

(1) Theta function : certain special functions on elliptic curves.

Proposition 4.1. *Let E be an elliptic curve over a scheme S. Let c be an integer with* $(c,6)=1$.

(1) *There exists a unique ${}_c\theta_E \in \mathcal{O}(E\setminus E[c])^\times$ such that*
   (a) $\mathrm{Div}({}_c\theta_E) = c^2\cdot(0) - E[c]$ *on* E
   (b) $\mathrm{N}_a({}_c\theta_E) = {}_c\theta_E$ *for an integer a with $(a,c)=1$ where N_a is the norm map $\mathrm{N}_a : \mathcal{O}(E\setminus E[ac])^\times \to \mathcal{O}(E\setminus E[c])^\times$ associated to the pull-back homomorphism by the multiplication by a.*

(2) *Let d be another integer with $(d,6)=1$. Then*
$$({}_d\theta_E)^{c^2}\cdot(c^*({}_d\theta_E))^{-1} = ({}_c\theta_E)^{d^2}\cdot(d^*({}_c\theta_E))^{-1}$$
in $\mathcal{O}(E\setminus E[ac])^\times$ where
$$c^* : \mathcal{O}(E\setminus E[d])^\times \to \mathcal{O}(E\setminus E[cd])^\times,$$
$$d^* : \mathcal{O}(E\setminus E[c])^\times \to \mathcal{O}(E\setminus E[cd])^\times$$
are the pull-back homomorphisms by the multiplication by c and d, respectively.

(3) *Over $\mathbb{C}$, ${}_c\theta_E$ can be written more explicitly. Let $\mathfrak{h}$ be the upper-half plane. Let $\tau\in\mathfrak{h}$, $z\in\mathbb{C}\setminus c^{-1}(\mathbb{Z}\tau+\mathbb{Z})$. Then ${}_c\theta(\tau,z)$ is the value at z of ${}_c\theta_{\mathbb{C}/(\mathbb{Z}\tau+\mathbb{Z})}$ over $\mathbb{C}$. Therefore, we have the explicit formula*
$${}_c\theta(\tau,z) = q^{\frac{1}{12}\cdot(c^2-1)}\cdot(-t)^{\frac{1}{2}\cdot(c-c^2)}\cdot\gamma_q(t)^{c^2}\cdot\gamma_q(t^c)^{-1}$$
where $q=e^{2\pi i\tau}$, $t=e^{2\pi i z}$, and
$$\gamma_q(t) = \prod_{n\geq 0}(1-q^n\cdot t)\cdot\prod_{n\geq 1}(1-q^n\cdot t^{-1}).$$

(4) *If $h : E\to E'$ be an isogeny of elliptic curve of degree prime to c, then we have the corresponding norm map $h_* : {}_c\theta_E \to {}_c\theta_{E'}$.*

This is proved by Kato in [Astérisque, §1].

Before next step, we recall a bit K-theory: Let X be a regular separate Noetherian scheme. Then Quillen's K-group $K_i(X)$ are defined for $i \geq 0$ with cup product

$$\smile \; : \; K_i(X) \times K_j(X) \longrightarrow K_{i+j}(X)$$

① $\exists$ a canonical homomorphism $\mathcal{O}(X)^\times \longrightarrow K_1(X)$

② $\exists$ the universal Steinberg symbol map via cup product

$$\mathcal{O}(X)^\times \times \mathcal{O}(X)^\times \longrightarrow K_2(X) ; \quad (u,v) \longmapsto \{u,v\} .$$

and $K_i(-)$ has the functorality properties.

(2) Siegal units from theta functions

- Modular curve $Y(M,N)/\mathbb{Q}$:

$$\begin{array}{c} T \\ \uparrow \\ \mathbb{Q}\text{-scheme} \end{array} \longmapsto \left\{ (E, e_1, e_2) \;\middle|\; \begin{array}{l} \cdot\ E \to T \text{ an elliptic curve} \\ \cdot\ e_1, e_2 \in E \text{ of order } M, N \text{ respectively, s.t.} \\ \qquad \langle e_1 \rangle \cap \langle e_2 \rangle = 0 \end{array} \right\} .$$

Then we have an universal elliptic curve

$$\left(\mathbb{E}/Y(M,N)_{\mathbb{Q}} ,\ e_1, e_2 \right) \text{ being an elliptic curve over } Y(M,N)_{\mathbb{Q}} .$$

- Let $(a,b) \in \mathbb{Z}^2$, define $(a,b) := ae_1 + be_2 \in \mathbb{E}_{\text{tors}}$.

When the order of (a,b) is prime to c, (*) we define

$$\begin{array}{ccccc} K_1(Y(M,N)_{\mathbb{Q}}) & \xleftarrow{\text{pullback map}} & K_1(\mathbb{E} \setminus \mathbb{E}[c]) & \longleftarrow & \mathcal{O}(\mathbb{E} \setminus \mathbb{E}[c])^\times \\ \cup\!\!\!\!\!\;\; & & & & \\ {}_c g_{a,b} & \longleftarrow\!\!\!\!\longleftarrow & & \longmapsto\!\!\!\!\!\! & {}_c\theta_{\mathbb{E}} \end{array}$$

here the pullback map is induced by the section

$$\iota_{(a,b)} = ae_1 + be_2 : Y(M,N)_{\mathbb{Q}} \longrightarrow \mathbb{E} \setminus \mathbb{E}[c] \hookrightarrow \mathbb{E} , \text{ at the point } (a,b) .$$

Elements "${}_c g_{a,b}$" as Siegal units.

(3) <u>Motivic zeta element</u>

Let c, d be integers satisfying $(c, 6M) = (d, 6N) = 1$. Let $M, N \geq 2$ s.t. $M+N \geq 5$. Then define

$$ {}_{c,d}z^{mot}_{M,N} := \{ c g_{1,0}, \ d g_{0,1} \} \in K_2(Y(M,N)_{\mathbb{Q}}) $$

The element called the <u>(motivic) zeta element of Beilinson–Kato</u>.

- For $M | M'$ and $N | N'$, we have a natural norm map
$$ K_2(Y(M',N')_{\mathbb{Q}}) \longrightarrow K_2(Y(M,N)_{\mathbb{Q}}) $$
then how the behavior of $\{ {}_{c,d}z^{mot}_{M,N} \}_{M,N}$ for varying M and N ?

<u>Theorem</u> ([Astérique, Proposition 2.3 & Proposition 2.4])

(a) $M | M'$, $N | N'$ and $\underline{\mathrm{div}(M) = \mathrm{div}(M')}$ (set of prime divisors), $\mathrm{div}(N) = \mathrm{div}(N')$

Then ${}_{c,d}z^{mot}_{M',N'}$ is mapped to ${}_{c,d}z^{mot}_{M,N}$ under the norm map.

(b) For q a prime number such that $(q, M) = 1$.

(i) If $(q, N) = 1$, ${}_{c,d}z^{mot}_{qM,qN}$ is mapped to
$$ \left(1 - T^*(q) \begin{pmatrix} 1/q & 0 \\ 0 & 1 \end{pmatrix}^* + \begin{pmatrix} 1/q & 0 \\ 0 & 1/q \end{pmatrix}^* q \right) {}_{c,d}z^{mot}_{M,N} . $$

(ii) If $q | N$, ${}_{c,d}z^{mot}_{qM,qN}$ is mapped to $\left(1 - T^*(q) \begin{pmatrix} 1/q & 0 \\ 0 & 1 \end{pmatrix}^* \right) {}_{c,d}z^{mot}_{M,N}$.

<u>Note</u>: The K-group is a motivic object which has various realizations :

${}_{c,d}z^{mot}_{M,N}$ $\longrightarrow$ <u>étale realization</u> : Kato's zeta element $\xrightarrow{\text{Coleman map}}$ p-adic L-functions

${}_{c,d}z^{mot}_{M,N}$ $\longrightarrow$ <u>de Rham realization</u> : Rankin-Selberg method $\xrightarrow{\text{Hida}}$ p-adic L-functions

${}_{c,d}z^{mot}_{M,N}$ $\longrightarrow$ <u>Betti realization</u> : modular symbols $\xrightarrow{\text{Mazur-Tate-Teitelbaum}}$ p-adic L-functions

§1.2 étale realization

$\boxed{\text{Step 1}}$ Chern class map :

- $K_1(Y(M,N)_{\mathbb{Q}}) = \mathcal{O}(Y(M,N)_{\mathbb{Q}})^{\times}$ by the definition of $K_1(-)$.
- By Kummer theory, we have discription of $H^1_{\text{ét}}$:

$$H^1_{\text{ét}}(Y(M,N)_{\mathbb{Q}}, \mathbb{Z}_p(1)) \simeq \varprojlim_n \frac{\mathcal{O}(Y(M,N))^{\times}}{\mathcal{O}(Y(M,N))^{\times p^n}}$$

Then the first Chern class map

$$c_1 : K_1(Y(M,N)_{\mathbb{Q}}) \longrightarrow H^1_{\text{ét}}(Y(M,N)_{\mathbb{Q}}, \mathbb{Z}_p(1))$$

is the embedding into the p-adic completion. Then the 2nd Chern class map

$$c_2 : K_2(Y(M,N)_{\mathbb{Q}}) \longrightarrow H^2_{\text{ét}}(Y(M,N)_{\mathbb{Q}}, \mathbb{Z}_p(2))$$

maps $\{x,y\}$ to $c_1(x) \smile c_1(y)$.

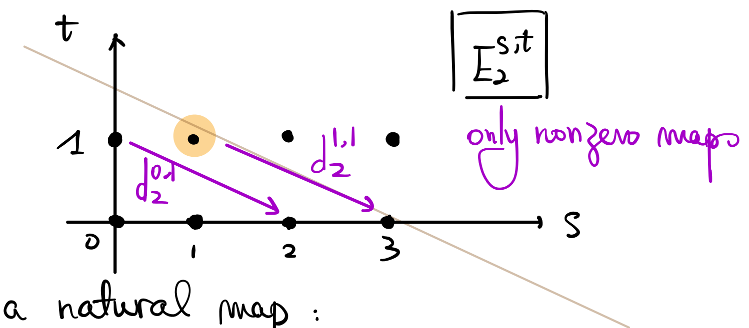
- We denote $c,d\,\mathfrak{Z}^{\text{ét}}_{M,N}(2,2)$ the image of $c,d\,\mathfrak{Z}^{\text{mot}}_{M,N} \in K_2(Y(M,N)_{\mathbb{Q}})$ by the Chern class map c_2.

$\boxed{\text{Step 2}}$ Hochschild-Serre spectral sequence

not the "relative étale".

$$E_2^{s,t} := H^s(\mathbb{Q}, H^t_{\text{ét}}(Y(M,N)_{\overline{\mathbb{Q}}}, \mathcal{M})) \Rightarrow E^{s+t} := H^{s+t}_{\text{ét}}(Y(M,N)_{\mathbb{Q}}, \mathcal{M})$$

- Since $Y(M,N)_{\overline{\mathbb{Q}}}$ is affine over alg closed fld, $E_2^{s,\geq 2} = 0$
- For p an odd prime, the p-coh. dim of $\mathrm{Gal}_{\mathbb{Q}}$ equals two, hence $E_2^{\geq 3,t} = 0$.



Hence we have a natural map :

$$\text{edge map} : H^2_{\text{ét}}(Y(M,N)_{\mathbb{Q}}, \mathbb{Z}_p(2)) \longrightarrow H^1(\mathbb{Q}, H^1_{\text{ét}}(Y(M,N)_{\overline{\mathbb{Q}}}, \mathbb{Z}_p(2)))$$

$$c,d\,\mathfrak{Z}^{\text{ét}}_{M,N} \longmapsto \text{by abuse of notation} : c,d\,\mathfrak{Z}^{\text{ét}}_{M,N}.$$

$\boxed{\text{Step 3}}$ Twists

- [Ochiai, Vol 2, Defn A.1.3] Let A be a ring, we define :

$$L_k(A) := \bigoplus_{i=0}^{k-2} A X^{k-2-i} Y^i \subseteq A[X,Y], \quad \text{homog. polys of degree } k-2.$$

as a free A-module of rank $k-1$. It carries an action of $M_2(\mathbb{Z}) \cap GL_2(\mathbb{Q})$:

$$g \cdot p(X,Y) := p\left((X,Y) \begin{pmatrix} a & c \\ b & d \end{pmatrix} \right), \quad g \in M_2(\mathbb{Z}) \cap GL_2(\mathbb{Q})$$

- We define a local system

$$\mathcal{L}_k(A) \text{ on } Y_1(M)_{\mathbb{C}} = \Gamma_1(M) \backslash \mathbb{H}, \quad \mathcal{L}_k(A) := \Gamma_1(M) \backslash (\mathbb{H} \times L_k(A)).$$

and we treat as a black-box of a local system in the étale world.

$$\begin{tikzcd}
\varprojlim_n H^1(\mathbb{Q}, H^1_{ét}(Y(Mp^n, Np^n)_{\overline{\mathbb{Q}}}, \mathbb{Z}_p(2))) \arrow[d, "(?)\ \otimes e_1^{k-2} \otimes (\varprojlim \zeta_{p^n})^{\otimes(k-j-2)}"'] & ({}_{c,d}z^{ét}_{Mp^n, Np^n}(2,2))_n \arrow[dd, mapsto, "\mathrm{Tw}_{j,k}"] \\
\varprojlim_n H^1(\mathbb{Q}, H^1_{ét}(Y(Mp^n, Np^n)_{\overline{\mathbb{Q}}}, \mathcal{L}_k(\mathcal{O})(k-j))) \arrow[d, "\text{Corestriction}"] & \\
H^1(\mathbb{Q}, H^1_{ét}(Y(M,N)_{\overline{\mathbb{Q}}}, \mathcal{L}_k(\mathcal{O})(k-j))) & {}_{c,d}z^{ét}_{M,N}(j,k)
\end{tikzcd}$$

for $k \geq 2$. ("k": weight of modular form, "j": Soulé twist).

$\boxed{\text{Step 4}}$ Let A be an auxiliary integer, $\alpha \in GL_2(\mathbb{Z}/(A))$, $r \geq 1$. Assume $n_0, n_1 \leq n$.

$$\begin{tikzcd}
Y(Arp^n, Arp^n N)_{\mathbb{Q}} \arrow[d, "\alpha_*"] & H^1(\mathbb{Q}, H^1_{ét}(Y(Arp^n, Arp^n N)_{\overline{\mathbb{Q}}}, \mathcal{L}_k(\mathcal{O})(k-j))) \arrow[dd, "Q^{n,\alpha,r}_{n_0,n_1}"] & {}_{c,d}z^{ét}_{Arp^n, Arp^n N}(j,k) \arrow[dd, mapsto] \\
Y(Arp^n, Arp^n N)_{\mathbb{Q}} \arrow[d, "\text{quotient map}"', "Y(N', NN')_{\mathbb{Q}} \twoheadrightarrow Y(1,N)_{\mathbb{Q}(\mu_{N'})}"] & & \\
Y_1(Np^{n_1})_{\mathbb{Q}(\mu_{rp^{n_0}})} & H^1(\mathbb{Q}(\mu_{rp^{n_0}}), H^1_{ét}(Y_1(Np^{n_1})_{\overline{\mathbb{Q}}}, \mathcal{L}_k(\mathcal{O})(k-j))) & {}_{c,d}z^{ét,\alpha,rp^{n_0}}_{Np^{n_1}}(j,k)
\end{tikzcd}$$

(why A is missing?)

Recall the construction of Galois rep'ns attached to an eigencuspform $f \in S_k(\Gamma_1(M))$ normalized, and $1 \leq j \leq k-1$, there is a map of $\mathrm{Gal}_{\mathbb{Q}}$-modules:

$$H^1_{\acute{e}t}(Y_1(Np^{n_1})_{\overline{\mathbb{Q}}}, \mathcal{L}_k(0)(k-j)) \longrightarrow V_f^*(1-j)$$

by taking "$[f]$-component" with respect to the action of the Hecke algebra. Therefore we obtain

$$H^1(\mathbb{Q}(\mu_{rp^{n_0}}), H^1_{\acute{e}t}(Y_1(Np^{n_1})_{\overline{\mathbb{Q}}}, \mathcal{L}_k(0)(k-j))) \longrightarrow H^1(\mathbb{Q}(\mu_{rp^{n_0}}), V_f^*(1-j))$$

$${}_{c,d}\zeta^{\acute{e}t,\alpha,rp^{n_0}}_{Np^{n_1}}(j,k) \longmapsto {}_{c,d}\zeta^{\acute{e}t,\alpha,rp^{n_0}}_{Np^{n_1}}(j)[f]$$

This is not far from the Beilinson–Kato's zeta element: $\exists$ a set

$$S := \{ a^{c,d,\alpha} \in \mathbb{Q}_f : \text{almost all } a^{c,d,\alpha} \text{ are nonzero} \}_{c,d,\alpha}$$

which is indexed by $c,d,\alpha \in GL_2(\mathbb{Z}/(A))$ s.t.

$$\zeta^{\acute{e}t,rp^n}(j,\{b_T^{\pm}\}) = \sum_{c,d,\alpha} \underline{a^{c,d,\alpha}}\, {}_{c,d}\zeta^{\acute{e}t,\alpha,rp^n}_{M}(j)[f] \in H^1(\mathbb{Q}(\mu_{rp^n}), V_f^*(1-j)).$$

related to complex period

an appropriate linear combination.

interpolates $L^{alg}_{(r)}(f,\phi,j)$ (!) for any Dirichlet character ϕ of conductor p^n.

(★) This sounds magic! It dates back to another realization of ${}_{c,d}\zeta^{mot}_{M,N}$, the de Rham realization, where L^{alg}-value appears in Rankin–Selberg methods.

(★) Doubt: In [Ochiai, Proposition 6.64], $\zeta^{\acute{e}t,rp^n}(j,\{b_T^{\pm}\}) \in H^1(\mathbb{Q}(\mu_{rp^n}), V_f^*(1-j))$, but in the goal [Theorem 6.62],

$$\left\{ \mathfrak{z}_{n,r}(j,\{b_T^{\pm}\}) \in H^1\left(\mathbb{Q}(\mu_{rp^n})_{\underbrace{\Sigma_{\mathbb{Q}(\mu_{rp^n})}}}/\mathbb{Q}(\mu_{rp^n}), \frac{1}{\varpi^a} T^*(1)\right) \right\}_{(n,r)\in \mathbb{Z}_{\geq 1}\times R} \quad ?$$

(footnote "37" on page 139?)

- finite set of primes of $\mathbb{Q}(\mu_{rp^n})$ which consists of infinite primes, primes above (p) and ramified primes of T.

§1.3 de Rham realization

- As before, $\mathbb{E}/Y(M,N)_{\mathbb{Q}}$ be the universal elliptic curve. Denote

$$\omega := p_* \Omega^1_{\mathbb{E}/Y(M,N)}, \quad p : \mathbb{E} \to Y(M,N).$$

Let $H^1_{dR}(Y(M,N)_{\mathbb{Q}}, \omega^{\otimes k}) :=$ first de Rham coh. of the line bundle.

It has a de Rham filtration $\mathrm{Fil}^i H^1_{dR}(Y(M,N)_{\mathbb{Q}}, \omega^{\otimes k})$.

- For any integer s, the twist $H^1_{dR}(Y(M,N)_{\mathbb{Q}}, \omega^{\otimes k})(s)$:
  - space : remains the same.
  - filtration : $\mathrm{Fil}^i H^1_{dR}(s) := \mathrm{Fil}^{i+s} H^1_{dR}$.

and $\forall\, 1 \leq j \leq k-1$, $\mathrm{Fil}^0 H^1_{dR}(Y(M,N)_{\mathbb{Q}}, \omega^{\otimes k})(k-j) = \mathrm{Fil}^{k-j} H^1_{dR} = M_k(\Gamma(M,N), \mathbb{Q})$.

- Eisenstein series : $t \in \mathbb{Z}_{>0}$, $\alpha, \beta \in \mathbb{Q}/\mathbb{Z}$,

$$E^{(t)}_{\alpha,\beta}(z) := (-1)^t (t-1)! \left(2\pi\sqrt{-1}\right)^{-t} \sum_{(m,n)\in\mathbb{Z}^2} \frac{1}{(\tilde{\alpha} z + \tilde{\beta} + mz + n)^t}$$

and for $c \nmid 6$, coprime to denominators of α and β, define

$$ {}_cE^{(t)}_{\alpha,\beta}(z) := c^2 E^{(t)}_{\alpha,\beta}(z) - c^t E^{(t)}_{c\alpha, c\beta}(z)$$

$\boxed{\text{Step1}}$ (Step3 in étale realization) Define

$${}_{c,d}z^{dR}_{M,N}(j,k) := (-1)^j (k-2)!^{-1} M^{j-k} N^{-j}\, {}_cE^{(k-j)}_{1/M,0}\, {}_dE^{(j)}_{0,1/N} \in \mathrm{Fil}^0 H^1_{dR}(\cdots, \omega^{\otimes k})(k-j)$$

(!)

$\boxed{\text{Step2}}$ (Step4 in étale realization)

$$\begin{tikzcd}
Y(Arp^n, Arp^nN)_{\mathbb{Q}} \arrow[d, "\alpha_*"] & H^1_{dR}(Y(Arp^n, Arp^nN)_{\mathbb{Q}}, \omega^{\otimes k})(k-j) \arrow[dd, "Q^{n,\alpha,r}_{n_0,n_1}"] & {}_{c,d}z^{dR}_{Arp^n, Arp^nN}(j,k) \arrow[dd, mapsto] \\
Y(Arp^n, Arp^nN)_{\mathbb{Q}} \arrow[d, "\text{quotient map}"', "Y(N',NN')_{\mathbb{Q}} \twoheadrightarrow Y(1,N)_{\mathbb{Q}(\mu_{N'})}"] & & \\
Y_1(Np^{n_1})_{\mathbb{Q}(\mu_{rp^{n_0}})} & H^1_{dR}(Y_1(Np^{n_1})_{\mathbb{Q}(\mu_{rp^{n_0}})}, \omega^{\otimes k})(k-j) & {}_{c,d}z^{dR,\alpha,rp^{n_0}}_{Np^{n_1}}(j,k)
\end{tikzcd}$$

Theorem A: (Rankin-Selberg method) Let $f \in S_k(\Gamma_1(M))$ be p-ordinary and p-stablized normalized eigencuspform of $k \geq 2$, and $1 \leq j \leq k-1$. Then $\exists$ a complex period $\Omega_\infty(\alpha)^+$ (resp. $\Omega_\infty(\alpha)^-$) depending only on $\alpha \in GL_2(\mathbb{Z}/(A))$ s.t. $\forall$ Dirich. char. ϕ of conductor p-th power

$$\sum_{a \in (\mathbb{Z}/rp^{n_0}\mathbb{Z})^\times} \phi^{-1}(\sigma_a) \cdot \sigma_a \left({}_{c,d}\mathfrak{Z}_{Np^{n_1}}^{dR,\alpha,rp^{n_0}}(j,k) \right) [f^\rho] = (c^2 - c^{k-j})(d^2 - d^j) \frac{L^{(r)}(f, \phi^{-1}, j)}{(2\pi\sqrt{-1})^j \Omega_\infty(\alpha)^{\mathrm{sgn}(j,\phi)}} f^\rho$$

where:

- $\mathrm{sgn}(j,\phi) \in \{\pm\}$ is the signature of $(-1)^j \phi(-1)$
- f^ρ := "complex conjugate" of f, with $[f^\rho]$ a projection to a certain Hecke eigenspace.
- $\sigma_a \in \mathrm{Gal}(\mathbb{Q}(\mu_{rp^{n_0}})/\mathbb{Q})$ s.t. $\sigma_a(\zeta_{rp^{n_0}}) = \zeta_{rp^{n_0}}^a$

  $\circlearrowleft$ functorially

  $$H^1_{dR}(Y_1(Np^{n_1})_{\mathbb{Q}(\mu_{rp^{n_0}})}, \omega^{\otimes k})(k-j) \ni {}_{c,d}\mathfrak{Z}_{Np^{n_1}}^{dR,\alpha,rp^{n_0}}(j,k)$$

- $L^{(r)}$:= removing Euler factors at primes dividing r.

This is "essentially" the Rankin-Selberg method ([Kato, Astérisque, Thm 6.6(1)], see the thread of proof in [Ochiai, p138].). Corresponding to the étale realization, an appropriate linear combination on c, d, α gives an element $\mathfrak{Z}^{dR,rp^n}(j,k,\{\Omega_\infty(f)^\pm\})$ parallel to $\mathfrak{Z}^{\text{ét},rp^n}(j,\{b_T^\pm\})$.

Now we relate z^{dR} with $z^{ét}$ to obtain the explicit reciprocity law :

- Define a map

$$\overline{\mathrm{loc}_p} : H^1(\mathbb{Q}, V) \xrightarrow{\mathrm{loc}_p} H^1(\mathbb{Q}_p, V) \twoheadrightarrow H^1_{/f}(\mathbb{Q}_p, V) := \frac{H^1(\mathbb{Q}_p, V)}{H^1_f(\mathbb{Q}_p, V)}$$

- Dual exponential map : This is a standard way of relating étale and de Rham
  - Fundamental exact sequence :

$$0 \to \mathbb{Q}_p \xrightarrow{\alpha} B_{cris} \oplus B_{dR}^+ \overset{\text{hard}}{\underset{\beta}{\twoheadrightarrow}} B_{cris} \oplus B_{dR} \to 0$$

$$x \mapsto (x, x)$$

$$(x, y) \longmapsto ((1-\varphi_{cris})x, x-y)$$

We apply $V \otimes_{\mathbb{Q}_p} -$ and take long exact seq of Galois coh :

$$0 \to H^0(\mathbb{Q}_p, V) \xrightarrow{\alpha} D_{cris}(V) \oplus D_{dR}^+(V) \xrightarrow{\beta} D_{cris}(V) \oplus D_{dR}(V)$$

$$\xrightarrow{\delta} \underbrace{H^1(\mathbb{Q}_p, V) \xrightarrow{\alpha_1} H^1(\mathbb{Q}_p, V \otimes B_{cris}) \oplus H^1(\mathbb{Q}_p, V \otimes B_{dR}^+)} \to \cdots$$

one checks : $\ker \alpha_1 = H^1_f(\mathbb{Q}_p, V)$
$:= \ker(H^1(\mathbb{Q}_p, V) \to H^1(\mathbb{Q}_p, V \otimes B_{cris}))$

(not that easy, see [Bellaïche's Hawaii note, proof of prop. 2.8])

So the exact sequence becomes

$$0 \to V^{Gal_{\mathbb{Q}_p}} \to D_{cris}(V) \oplus D_{dR}^+(V) \to D_{cris}(V) \oplus D_{dR}(V) \to H^1_f(\mathbb{Q}_p, V) \to 0$$

- We define the Bloch-Kato exponential map as the map

$$\exp : D_{dR}(V) \to H^1_f(\mathbb{Q}_p, V)$$

- We have two degenerate pairs for either sides :

$$H^1(\mathbb{Q}_p, V) \times H^1(\mathbb{Q}_p, V^*(1)) \to H^2(\mathbb{Q}_p, \mathbb{Q}_p(1)) \simeq \mathbb{Q}_p$$

$$D_{dR}(V) \times D_{dR}(V^*(1)) \to D_{dR}(\mathbb{Q}_p(1)) \simeq \mathbb{Q}_p$$

So taking the "Kummer dual" of exp, we get :
← "$\mathbb{Q}_p$-dual" with respect to the pairings above ?

$$\exp^* : H^1(\mathbb{Q}_p, V^*(1)) \longrightarrow D_{dR}(V^*(1))$$

This is the famous Bloch-Kato dual exponential map.

Further $\exp^*$ induces :

$$\exp^* : H^1_f(\mathbb{Q}_p, H^1_{et}(Y(M,N)_{\overline{\mathbb{Q}}_p}, \mathcal{L}_k(0))(k-j)) \longrightarrow \underbrace{\mathrm{Fil}^0 D_{dR}(H^1_{et}(Y(M,N)_{\overline{\mathbb{Q}}_p}, \mathcal{L}_k(0))(k-j))}$$

Theorem B (The most important technical part of [Astérisque] :)

Under the above setting, we have

$$\boxed{\exp^*\left(\overline{\mathrm{loc}}_p\left({}_{c,d}z^{et}_{M,N}(j,k)\right)\right) = \underbrace{{}_{c,d}z^{dR}_{M,N}(j,k)}_{\in \ \mathrm{Fil}^0 H^1_{dR}(Y(M,N)_{\mathbb{Q}}, \omega^{\otimes k})(k-j)}}$$

Comparison map hidden in Theorem B

This fundamental link between $z^{et}_{M,N}$ and $z^{dR}_{M,N}$ gives the desired relation of $z^{et}_{M,N}$ with special L-values.

Next question : How to apply such a system z^{et} to Iwasawa theory ? Note that our $\boxed{z^{et}}$ and $\boxed{\text{special L-values}}$ are loosely connected but not well-viewed in cyclotomic towers !

$\rightsquigarrow$ interpolating "Dual exponential map" as "Coleman map".

This is explained in [Ochiai, Vol.2, § 6.5].

$\rightsquigarrow$ Watch Lei's video course at MSRI.

§2 Preparation : p-adic local langlands

2.1 A big picture

$$\left\{\begin{array}{c}\text{Abs irred, adm, unitary}\\ \text{Banach reps of } GL_2(\mathbb{Q}_p)\end{array}\right\} \overset{?}{\dashrightarrow} \left\{\begin{array}{c}\text{two-dim'l abs irred}\\ p\text{-adic rep of } Gal_{\mathbb{Q}_p}\end{array}\right\}$$

$$\begin{tikzcd} ? \arrow[r] \arrow[d, squiggly] & \rho^{univ} \arrow[d, squiggly] \\ \overline{\pi} \arrow[r, mapsto, "\text{mod } p \text{ local Langlands}"] & \overline{\rho} \end{tikzcd}$$

$$\left\{\text{adm. sm. } \overline{\mathbb{F}}_p\text{-rep'n of } GL_2(\mathbb{Q}_p)\right\} \xrightarrow[\text{MF}]{} \left\{\text{2-dim'l cont. } \overline{\mathbb{F}}_p\text{-reps of } Gal_{\mathbb{Q}_p}\right\}$$

This is the deformation-theoretic description of p-adic local Langlands.

Note : We will not spend too much time introducing mod p local Langlands. But add few points here :

- To have the mod p local Langlands ([LGC, Theorem 3.3.2]), we put further assumptions (☆): $\overline{\rho} \nsim \chi \otimes \begin{pmatrix} 1 & * \\ 0 & \overline{\varepsilon}_{cyc} \end{pmatrix}$ for some char $\chi : Gal_{\mathbb{Q}_p} \to \mathbb{F}^\times$.
- The functor "MF" is called the magic functor of Colmez : for $A \in \mathrm{Comp}(\mathcal{O})$: i.e. complete noetherian local $\mathcal{O}$-algebra with res fld an ext'n of $\mathbb{F}$.

$$\left\{\begin{array}{c}\text{orthonormalizable adm } \pi :\\ A \text{ reps of } GL_2(\mathbb{Q}_p) \text{ st.}\\ \pi/\mathfrak{m}\pi \text{ has finite length}\end{array}\right\} \xrightarrow{\text{MF}} \left\{\begin{array}{c}Gal_{\mathbb{Q}_p}\text{-reps on}\\ \text{finite free } A\text{-modules}\end{array}\right\}$$

The construction of MF is by :

- first reduced to Artinian objects in $\mathrm{Comp}(\mathcal{O})$
- factoring thru Fontaine's functor on (φ, Γ)-modules.

See [LGC, §3.2] for details.

2.2 p-adic local Langlands

We define two deformation functors, on both sides :

$$\begin{tikzcd} & \left\{ \begin{array}{l} \cdot \text{ free rank 2 } A\text{-modules } V \overset{\text{cts}}{\circlearrowleft} \mathrm{Gal}_{\mathbb{Q}_p} \\ \cdot \ A/\mathfrak{m}\text{-linear } \mathrm{Gal}_{\mathbb{Q}_p}\text{-equiv. isomorphism} \\ \qquad V/\mathfrak{m}V \xrightarrow{\sim} A/\mathfrak{m}A \otimes_{\mathbb{F}} \bar{\rho} \end{array} \right\} \\ A \in \mathrm{Comp}(\mathcal{O}) \arrow[ur, bend left, "\mathrm{Def}(\bar{\rho})"] \arrow[dr, bend right, "\mathrm{Def}(\bar{\pi})"'] & \\ & \left\{ \begin{array}{l} \cdot \text{ ONable adm rep } \pi \text{ of } GL_2(\mathbb{Q}_p) \text{ over } A \\ \cdot \ A/\mathfrak{m}\text{-linear } GL_2(\mathbb{Q}_p)\text{-equiv. isom} \\ \qquad \pi/\mathfrak{m}\pi \xrightarrow{\sim} A/\mathfrak{m}A \otimes_{\mathbb{F}} \bar{\pi} \end{array} \right\} \arrow[uu, "MF"'] \end{tikzcd}$$

And define subfunctors :

- $\mathrm{Def}^{cris}(\bar{\rho}) :=$ full subgroupoid of $\mathrm{Def}(\bar{\rho})$ obtained as the Zariski closure in $\mathrm{Def}(\bar{\rho})$ of the set of cristalline points in the generic fibre of $\mathrm{Def}(\bar{\rho})$.

- $\mathrm{Def}^{cris}(\bar{\pi}) := \underset{\text{(fixed central char.)}}{\mathrm{Def}^{*}(\bar{\pi})} \times_{\mathrm{Def}(\bar{\rho}), \text{"}MF\text{"}} \mathrm{Def}^{cris}(\bar{\rho})$.

<u>Theorem</u> (Kisin) Under the hypothesis ($\star$), $MF : \mathrm{Def}^{*}(\bar{\pi}) \longrightarrow \mathrm{Def}(\bar{\rho})$ is fully faithful and induces an equivalence

$$MF : \mathrm{Def}^{cris}(\bar{\pi}) \xrightarrow{\ \sim\ } \mathrm{Def}^{cris}(\bar{\rho})$$

Now given $V_E :=$ 2-dim'l E-vector space with continuous $\mathrm{Gal}_{\mathbb{Q}_p}$-action lifting $\bar{\rho}$.

- Suppose $V_E \in \mathrm{Def}^{cris}(\bar{\rho})$. We actually fix a $\mathcal{O}$-lattice of V_E, and abuse notation.
- Then by <u>Kisin's theorem</u>, $\exists!$ π of $GL_2(\mathbb{Q}_p)$ over $\mathcal{O}$ lifting $\bar{\pi}$ in $\mathrm{Def}^{cris}(\bar{\pi})$ s.t. $MF(\pi) = V$. We define $\boxed{B(V_E) = E \otimes_{\mathcal{O}} \pi}$. This association $V_E \mapsto B(V_E)$ is the p-<u>adic local Langlands correspondence</u>.

2.3 Paškūnas's theory

Recall the picture:

$$\begin{tikzcd}
\left\{\begin{array}{c}\text{Abs irred, adm, unitary} \\ \text{Banach reps of } GL_2(\mathbb{Q}_p)\end{array}\right\} \arrow[r, dashed, "?"] & \left\{\begin{array}{c}\text{two-dim'l abs irred} \\ p\text{-adic rep of } \mathrm{Gal}_{\mathbb{Q}_p}\end{array}\right\} \\
(?) \circlearrowright \; ? \arrow[r] \arrow[d, squiggly] & \rho^{univ} \circlearrowleft R^{univ}_{\bar\rho} \arrow[d, squiggly] \\
\bar\pi \arrow[r, mapsto, "\text{mod } p \text{ local Langlands}"] & \bar\rho \\
\{\text{adm. sm. } \bar{\mathbb{F}}_p\text{-rep'n of } GL_2(\mathbb{Q}_p)\} \arrow[r, "MF"'] & \{2\text{-dim'l cont. } \bar{\mathbb{F}}_p\text{-reps of } \mathrm{Gal}_{\mathbb{Q}_p}\}
\end{tikzcd}$$

One has $? = \pi^{univ}$, the universal object of $\mathrm{Def}^*(\bar\pi)$ with $MF(\pi^{univ}) = \rho^{univ}$.

Alternatively, we have a more direct description of "$?$": as the Paškūnas's module, denoted by $\tilde{P}$. It is the "projective envelope" of $\bar\pi$ (actually its socle $\bar\pi^{soc}$) in a certain category $\mathcal{C}(\mathcal{O})$. It has the following two properties:

① There exists a $\mathrm{Gal}_{\mathbb{Q}_p}$-equivariant $R_{\bar\rho}$-linear topological isomorphism
$$\tilde{P} \xrightarrow{\sim} \pi^{univ} = B(\rho^{univ}) \text{ in the notation of §2.2.}$$
(This is [Nakamura, Prop. B.26].)

② For any compact $R_{\bar\rho}$-module M, the $R_{\bar\rho}$-linear map
$$M \longrightarrow \mathrm{Hom}_{\mathcal{C}(\mathcal{O})}(\tilde{P}, \tilde{P}\hat{\otimes}_{R_{\bar\rho}} M) : \quad m \mapsto [v \mapsto v\hat{\otimes} m]$$
is a topological $R_{\bar\rho}$-linear isomorphism.

(• This is [Nakamura, Prop. B.27]

• This seems not easy to understand. It makes lots of sense to first prove a more basic property: $\boxed{R_{\bar\rho} \xrightarrow{\sim} \mathrm{End}_{\mathcal{C}(\mathcal{O})}(\tilde{P})}$, and the above isomorphism essentially follows from the projectivity of $\tilde{P}$.).

2.4 Completed cohomology

Together with classical local langlands for $l \neq p$, we have local informations at every places.

Ask : What is the correct global object ?

Answer : The completed cohomology, and the local-global compatibility.

2.4.1 Definition

- Let $K_f :=$ open compact subgroup of $GL_2(\mathbb{A}_f)$. Let $Y(K_f)$ be the modular curve over $\mathbb{Q}$:

$$Y(K_f)(\mathbb{C}) = GL_2(\mathbb{Q}) \backslash (\mathbb{C} \setminus \mathbb{R}) \times GL_2(\mathbb{A}_f) / K_f$$

We write $H^1(K_f)_A := H^1_{ét}(Y(K_f)_{\overline{\mathbb{Q}}}, A)$ for $A = E, \mathcal{O}, \mathcal{O}/\varpi^s$ for $s \geq 1$.

- We write $K_f = K_p K^p$ for $K^p \subseteq GL_2(\mathbb{A}_f^p)$ some fixed compact open (i.e. tame level group), we write :

$$H^1(K^p)_A := \varinjlim_{K_p} H^1(K_p K^p)_A \qquad \text{for } K_p \leqslant GL_2(\mathbb{Q}_p) \text{ compact open.}$$

There are natual commuting action of $G := GL_2(\mathbb{Q}_p)$ and $Gal_{\mathbb{Q}}$ on $H^1(K^p)_A$.

(※)
- On complex points : $g \in GL_2(\mathbb{A}_f)$. the multiplication by g on the right induces $Y(gK_f g^{-1})(\mathbb{C}) \xrightarrow{\sim} Y(K_f)(\mathbb{C})$, $(\tau, [h]) \mapsto (\tau, [hg])$

  and this induces A-linear isomorphism

$$H^1_{sing}(K_f)_A \xrightarrow{\sim} H^1_{sing}(gK_f g^{-1})_A$$

  By the singular-étale comparision, regard them as étale cohs. When K^p is fixed and $g \in GL_2(\mathbb{Q}_p)$. this gives an action of g on $H^1(K^p)_A$.

- Clearly étale coh has $Gal_{\mathbb{Q}}$-action.

- One checks $H^1(K^p)_{\mathcal{O}}/\varpi^s H^1(K^p)_{\mathcal{O}} \xrightarrow{\sim} H^1(K^p)_{\mathcal{O}/\varpi^s\mathcal{O}}$. We take

$$\widehat{H}^1(K^p)_{\mathcal{O}} := \varprojlim_s H^1(K^p)_{\mathcal{O}}/\varpi^s H^1(K^p)_{\mathcal{O}} = \varprojlim_s H^1(K^p)_{\mathcal{O}/\varpi^s\mathcal{O}}.$$

this is called the <u>completed cohomology</u> with <u>tame level</u> K^p, and we write

$$\widehat{H}^1(K^p)_E := \widehat{H}^1(K^p)_{\mathcal{O}} \otimes_{\mathcal{O}} E.$$

  - $\widehat{H}^1(K^p)_{\mathcal{O}}$ and $\widehat{H}^1(K^p)_E$ carry continuous $G \times G_{\mathbb{Q}}$-action from $H^1(K^p)_{\mathcal{O}}$.
  - The G-action makes $\widehat{H}^1(K^p)_{\mathcal{O}}$ a ϖ-adically adm rep of G over $\mathcal{O}$, and makes $\widehat{H}^1(K^p)_E$ a unitary admissible Banach space rep'n of G over E, with $\widehat{H}^1(K^p)_{\mathcal{O}}$ as the unit ball.

- For $A = \mathcal{O}$ or E, we define

$$\widehat{H}^1_A := \varinjlim_{K^p} \widehat{H}^1(K^p)_A \quad \text{with } \mathcal{O}\text{-linear inductive topology}.$$

Called the <u>completed cohomology</u> of coefficient in A. Following $(\ast)$, it has a smooth action of $\mathrm{Gal}_{\mathbb{Q}} \times \mathrm{GL}_2(\mathbb{A}_f)$.

- <u>Note</u>: Another version is to keep track of bad primes:
  - let Σ_0 = finite set of primes excluding p, $\Sigma := \Sigma_0 \cup \{p\}$.
  - Write $G_{\Sigma_0} = \prod_{\ell \in \Sigma_0} \mathrm{GL}_2(\mathbb{Q}_\ell)$ and $K_0^{\Sigma_0} := \prod_{\ell \in \Sigma} \mathrm{GL}_2(\mathbb{Z}_\ell)$, and $K_{\Sigma_0} \subseteq G_{\Sigma_0}$ open compact.
  - Define (for $A = E, \mathcal{O}$): $\widehat{H}^1_{A,\Sigma} := \varinjlim_{K_{\Sigma_0}} \widehat{H}^1(K_{\Sigma_0} K_0^{\Sigma})_A$, being the compact cohomology.

2.4.2 Hecke action

- $\mathbb{T}(K^pK_p) :=$ $\mathcal{O}$-alg of $G_{\mathbb{Q}} \times G$-equivariant end. of $H^1(K_pK^p)_E$ generated by Hecke operators S_ℓ and T_ℓ for $\ell \neq p$, and unramified in K^p.

Under inclusion $K_p' \leq K_p$, we have $\mathbb{T}(K_p'K^p) \twoheadrightarrow \mathbb{T}(K_pK^p)$, so it defines

$$\mathbb{T}(K^p) := \varprojlim_{K^p} \mathbb{T}(K_pK^p). \quad \overset{\text{faithfuly}}{\circlearrowleft} \quad \widehat{H}^1(K^p)_{\mathcal{O}} \text{ and } \widehat{H}^1(K^p)_E.$$

How to understand this Hecke algebra?

- * $\mathbb{T}(K^p)$ is a reduced, commutative and complete wrt its local topology.
- * $\mathbb{T}(K^p) =$ a finite product of complete local $\mathcal{O}$-algebras, corresponding to finitely many Galois conjugacy classes of eigensystems of modular forms of level K^p over $\mathbb{F}$. $\rightsquigarrow$ pick out one!

- Let $\bar{\rho} : \mathrm{Gal}_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{F})$ so that $\bar{\rho}$ is unramified outside Σ. Assuming $\bar{\rho}$ is odd, we know by Khare-Wintenberger that $\bar{\rho}$ is modular. We say K_{Σ_0} is an allowable level if $\exists$ maximal ideal $\mathfrak{m} \subseteq \mathbb{T}(K_{\Sigma_0})$ of residue field $\mathbb{F}$ s.t.

$$\begin{aligned} T_\ell \bmod \mathfrak{m} &= \mathrm{trace}(\bar{\rho}(\mathrm{Frob}_\ell)) \\ \ell S_\ell \bmod \mathfrak{m} &= \det(\bar{\rho}(\mathrm{Frob}_\ell)) \end{aligned}, \quad \ell \notin \Sigma.$$

(or: iff $\exists$ a new form f of tame level K_{Σ_0} s.t. ρ_f lifts $\bar{\rho}$. Then $\mathfrak{m}$ is attached to f.)

$\Longrightarrow$ Define $\underbrace{\mathbb{T}(K_{\Sigma_0})_{\bar{\rho}} := \mathbb{T}(K_{\Sigma_0})_{\mathfrak{m}}}_{\text{actually a direct factor of } \mathbb{T}(K_{\Sigma_0})}$, localization at $\mathfrak{m}$.

- We define $\mathbb{T}_{\bar{\rho},\Sigma} := \varprojlim_{K_{\Sigma_0}} \mathbb{T}(K_{\Sigma_0})_{\bar{\rho}}$ as K_{Σ_0} runs over all allowable levels.

Moreover, define $\widehat{H}^1(K_{\Sigma_0})_{A,\bar{\rho}} := \mathbb{T}(K_{\Sigma_0})_{\bar{\rho}} \otimes_{\mathbb{T}(K_{\Sigma_0})} \widehat{H}^1(K_{\Sigma_0})_A$ and

$$\boxed{\widehat{H}^1_{A,\bar{\rho},\Sigma} := \varprojlim_{K_{\Sigma_0}} \widehat{H}^1(K_{\Sigma_0})_{A,\bar{\rho}}}$$ as K_{Σ_0} runs over allowable levels.

This object $\widehat{H}^1_{A,\bar\rho,\Sigma}$ is the central object of interest.

- $\widehat{H}^1_{A,\bar\rho,\Sigma}$ carries $G_{\mathbb{Q}} \times G \times G_\Sigma$-action, being an invariant summand of $\widehat{H}^1_{A,\Sigma}$.
- $\widehat{H}^1_{A,\bar\rho,\Sigma}$ is also a $\mathbb{T}_{\bar\rho,\Sigma}$-module, s.t. $G_{\mathbb{Q}} \times G \times G_\Sigma$-action is $\mathbb{T}_{\bar\rho,\Sigma}$-linear.

The "<u>(refined) local-global compatibility</u>" provides a "factorization" of $\widehat{H}^1_{A,\bar\rho,\Sigma}$ with respect to these actions.

2.4.3 Local-global compatibility.

- $\rho^{univ}_\Sigma :=$ the universal deformation of $\bar\rho$ over $\mathbb{T}_{\bar\rho,\Sigma}$.
- $\rho^{univ}_\Sigma|_{G_{\mathbb{Q}_p}} \in \mathrm{Def}^{cris}(\bar\rho)$ (one checks this (?)), so we can define π^{univ}_Σ as the p-adic local Langlands of $\rho^{univ}_\Sigma|_{G_{\mathbb{Q}_p}}$. (i.e. $MF(\pi^{univ}_\Sigma) = \rho^{univ}_\Sigma|_{G_{\mathbb{Q}_p}}$)
- $\bigotimes_\ell \rho^{univ}_\Sigma|_{G_{\mathbb{Q}_\ell}}$ for $\ell \in \Sigma_0$ has classical local Langlands $\pi_{\Sigma_0}(\rho^{univ}_\Sigma)$.

Then <u>(refined) LGC</u> states follows:

<u>Theorem</u> [Emerton, LGC, Conj. 6.1.6 and Thm 6.4.16] Suppose $\mathrm{End}(\bar\rho|_{G_{\mathbb{Q}_p}}) = 1$, then there is a $\mathrm{Gal}_{\mathbb{Q}} \times G \times G_{\Sigma_0}$-equivariant, $\mathbb{T}_{\bar\rho,\Sigma}$-linear isomorphism

$$\boxed{\widehat{H}^1_{O,\bar\rho,\Sigma} \xleftarrow{\sim} \underbrace{\rho^{univ}_\Sigma}_{\mathrm{Gal}_{\mathbb{Q}}} \otimes_{\mathbb{T}_{\bar\rho,\Sigma}} \underbrace{\pi^{univ}_\Sigma}_{G = GL_2(\mathbb{Q}_p)} \widehat{\otimes}_{\mathbb{T}_{\bar\rho,\Sigma}} \underbrace{\pi_{\Sigma_0}(\rho^{univ}_\Sigma)}_{G_{\Sigma_0}}}$$

§3 Interpolation of Kato's zeta element : Nakamura's method

Let N be fixed integer prime to p. Set $\Sigma_0 = \{ \ell | N \}$, $\Sigma = \Sigma_0 \cup \{p\}$.

In §1, we constructed motivic zeta elements

$$ {}_{c,d}z^{mot}_{Np^r} := {}_{c,d}z^{mot}_{Np^r, Np^r} \in K_2(Y(Np^r)_{\mathbb{Q}}) \qquad \left(\underline{\text{note}} : \quad \Gamma(Np^r, Np^r) = \Gamma(Np^r) \right) $$

and they are compatible for various N, r via the norm map. So we have a universal K_2-class

$$ {}_{c,d}z^{mot}_{Np^\infty} := \left({}_{c,d}z^{mot}_{Np^r} \right)_{r \geq 1} \in \varprojlim_{r\geq 1} K_2(Y(Np^r)_{\mathbb{Q}}) =: K_2(Y(Np^\infty)_{\mathbb{Q}}) . $$

We focus on the étale realization :

$\boxed{\text{Step 1}}$ Realize zeta element (morphism) in the completed cohomology

· By Step 1 & 2 of étale realization, we obtain

$$ {}_{c,d}z^{ét}_{Np^\infty} \xrightarrow[\text{2. Hochschild-Serre}]{\text{1. Chern class map}} \in H^1_{ét}\left(\mathbb{Z}[\tfrac{1}{Np}], \varprojlim_m \underbrace{\varprojlim_r H^1_{ét}\left(Y(Np^r)_{\overline{\mathbb{Q}}}, \mathbb{Z}/p^m\mathbb{Z} \right)} \right) $$

(actually we need to be careful on Tate twists. Here to illustrate the idea, we ignore this issue. The correct term should be : $\left(\varprojlim_r H^1_{ét}(Y(Np^r)_{\overline{\mathbb{Q}}}, \mathbb{Z}_p(2)) \right)(1)$)

and we localize it at the maximal ideal $\mathfrak{m}_{\bar\rho}$ of the Hecke algebra $\mathbb{T}^{\Sigma}(N)$, determined by $\bar\rho$, and obtain

$$ {}_{c,d}z^{ét}_{Np^\infty, \mathfrak{m}_{\bar\rho}} \in H^1_{ét}\left(\mathbb{Z}[\tfrac{1}{Np}], \underbrace{\varprojlim_m \varprojlim_r H^1_{ét}\left(Y(Np^r)_{\overline{\mathbb{Q}}}, \mathbb{Z}/p^m\mathbb{Z} \right)_{\mathfrak{m}_{\bar\rho}}}_{= \widetilde{H}^1_{\mathbb{Z}_p}(\Gamma(N))_{\mathfrak{m}_{\bar\rho}}} \right) . $$

Take inverse limit on N with prime factors included in Σ (i.e. in the allowable levels), it yields

$$ \boxed{ {}_{c,d}z^{ét}_{p^\infty, \mathfrak{m}_{\bar\rho}} \in H^1_{ét}\left(\mathbb{Z}[\tfrac{1}{\Sigma}], \widetilde{H}^1_{\mathbb{Z}_p, \mathfrak{m}_{\bar\rho}, \Sigma} \right) } $$

Following [Kato, Astérisque Theorem 12.4], we view it as a morphism

$$ {}_{c,d}\delta^{\text{ét}}_{p^\infty, m_{\bar\rho}} : \tilde{H}^1_{\mathbb{Z}_p, m_{\bar\rho}, \Sigma}(-1)^+ \longrightarrow H^1_{\text{ét}}\left(\mathbb{Z}[\tfrac{1}{\Sigma}], \tilde{H}^1_{\mathbb{Z}_p, m_{\bar\rho}, \Sigma}\right)$$

Remark: Careful reader may have noticed that our "$\tilde{H}^1$" is not exactly the completed cohomology "$\hat{H}^1$" above: at level groups at p, we are taking "$\varprojlim$" instead of "$\varinjlim$". This is named as "completed homology" (or Borel–Moore homology) in [Nakamura]. This is a dual version of [Emerton, LGC], and Nakamura carefully dealt with it in Section 2, so we now ignore the issue.

$\boxed{\text{Step 2}}$ Invoke local-global compatibility:

By LGC, we reinterprete ${}_{c,d}\delta^{\text{ét}}_{p^\infty, m_{\bar\rho}}$ as the morphism (we use the dual version)

$$\begin{tikzcd}
& \rho^u_\Sigma(-1)^+ \otimes B\left(\rho^{u,*}_\Sigma(1)\big|_{G_{\mathbb{Q}_p}}\right) \hat{\otimes}\, \pi_{\Sigma_0}\left(\rho^{u,*}_\Sigma(1)\big|_{Gal_{\mathbb{Q}_\Sigma}}\right) \arrow[d, "{}_{c,d}\delta^{\text{ét}}_{p^\infty, m_{\bar\rho}}"] \\
\text{Goal:} & H^1_{\text{ét}}\left(\mathbb{Z}[\tfrac{1}{\Sigma}], \rho^u_\Sigma \otimes B\left(\rho^{u,*}_\Sigma(1)\big|_{G_{\mathbb{Q}_p}}\right) \hat{\otimes}\, \pi_{\Sigma_0}\left(\rho^{u,*}_\Sigma(1)\big|_{Gal_{\mathbb{Q}_\Sigma}}\right)\right)
\end{tikzcd}$$

So next we need to factor out the remaining parts "•" and "•".

$\boxed{\text{Step 3}}$ Factoring "•": use Emerton-Helm's theory of classical local Langlands in families: There exists the so-called "Bernstein-Zelevinsky derivative functor" Φ_ℓ for each $\ell \in \Sigma_0$ such that $\Phi_\ell\left(\rho^{u,*}_\Sigma(1)\big|_{Gal_{\mathbb{Q}_\ell}}\right)$ is a free $\mathbb{T}_{\bar\rho,\Sigma}$-module of rank one. Applying $\Phi_{\Sigma_0} := \prod_{\ell\in\Sigma_0} \Phi_\ell$ to ${}_{c,d}\delta^{\text{ét}}_{p^\infty, m_{\bar\rho}}$, we thus obtain

$$\rho^u_\Sigma(-1)^+ \otimes B\left(\rho^{u,*}_\Sigma(1)\big|_{G_{\mathbb{Q}_p}}\right) \xrightarrow{\Phi_{\Sigma_0}\left({}_{c,d}\delta^{\text{ét}}_{p^\infty, m_{\bar\rho}}\right)} H^1_{\text{ét}}\left(\mathbb{Z}[\tfrac{1}{\Sigma}], \rho^u_\Sigma \otimes B\left(\rho^{u,*}_\Sigma(1)\big|_{G_{\mathbb{Q}_p}}\right)\right)$$

$\boxed{\text{Step 4}}$ Factoring " ● " : use Paškūna's theory :

We apply $\mathrm{Hom}_{\mathcal{E}(\mathcal{O})}(\widetilde{P}, -)$ to the above map :

- On LHS , the property ① and ② in §2.3 gives precisely $\rho_\Sigma^u(-1)^+$
- On RHS , ——— ● ——— " ——— $H^1_{et}(\mathbb{Z}[\frac{1}{\Sigma}], \rho_\Sigma^u)$

(Here we secretly commute $\mathrm{Hom}_{\mathcal{E}(\mathcal{O})}(\widetilde{P}, -)$ with $H^1_{et}(\mathbb{Z}[\frac{1}{\Sigma}], \rho_\Sigma^u \otimes -)$. This is "doable" as explained by [Nakamura, §B.4] . Another way of "cheating" is as in [Fouquet, Bourbaki] : we secretly write

$$\rho_\Sigma^u(-1)^+ \otimes B(\rho_\Sigma^{u,*}(1)|_{G_{\mathbb{Q}_p}}) \xrightarrow{\cdot\cdot} H^1_{et}(\mathbb{Z}[\tfrac{1}{\Sigma}], \rho_\Sigma^u) \otimes B(\rho_\Sigma^{u,*}(1)|_{G_{\mathbb{Q}_p}}) .$$

Actually Paškūna's theory provides " $\mathrm{End}(\widetilde{P}) \overset{①}{=} \mathrm{End}(\text{"}B(\rho_\Sigma^{u,*}(1)|_{G_{\mathbb{Q}_p}})\text{"}) \cong \mathbb{T}_{m_{\bar\rho},\Sigma}$.

So we can cancel the " ● " part.)

In this way , we obtain the map

$$\text{"} z^{univ} \text{"} : \rho_\Sigma^u(-1)^+ \longrightarrow H^1_{et}(\mathbb{Z}[\tfrac{1}{\Sigma}], \rho_\Sigma^u) \qquad \text{as desired .}$$

One missing point : Step 2−4 complicate things so it may be difficult to verify the interpolation property . For this purpose , we need to describe

$$\lambda_f : R_\Sigma(\bar\rho) \longrightarrow \mathcal{O}$$

using the objects appearing in our construction , in particular Paškūna's module $\widetilde{P}$.
This is done in [Nakamura , §4] (while [Nak, §3] introduces the above construction)

- Arrangements of [Nakayama] :
  - §1, §2 : Introduction & Preparation on completed cohomology.
  - §3 Construction of " z^{univ} "
  - §4 Interpolation property.
  - §5 Applications to IMC.

on Kato's construction : App. A .
on more p-adic LLC : App. B .

§4 Interpolation of Kato's zeta element : Colmez–Wang's method

Keep notations above, we introduce Colmez–Wang's method, through Betti's realization of Kato's zeta element. Our introduction follows from Wan's talk at 2022 ICTS conference.

This is very vague :

- Consider the "modular symbol" $\{0,\infty\} \in H_1(K^p K_p)$ (as homology of modular curve)

  Realizing in the completed cohomology, it provides a functional

$$f_{\{0,\infty\}} : \rho^u_\Sigma \otimes B(\rho^{u,v}_\Sigma|_{\mathrm{Gal}_{\mathbb{Q}_p}}) \otimes \pi_{\Sigma_0}(\rho^u_\Sigma) \underset{\text{LGC}}{\simeq} \widehat{H}^1_{0,\bar\rho,\Sigma} \longrightarrow \mathcal{O}.$$

  To illustrate the idea, we ignore the "π_{Σ_0}" part above, and $f_{\{0,\infty\}}$ gives :

$$f_{\{0,\infty\}} : \rho^u_\Sigma \longrightarrow \left(B(\rho^{u,v}_\Sigma|_{\mathrm{Gal}_{\mathbb{Q}_p}})\right)^\vee$$

- Theorem of Colmez–Wang (also partial result of Yiwen Zhou 周意闻) :

  The image of $f_{\{0,\infty\}}$ is invariant under the action of $\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \in GL_2(\mathbb{Q}_p)$.

  Then we invoke the theory of p-adic local langlands (by Colmez) : $\exists$ a can. isomorphism

$$\left(B(\rho^{u,v}_\Sigma|_{\mathrm{Gal}_{\mathbb{Q}_p}})\right)^{\vee,\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}} \xrightarrow{\sim} \underbrace{H^1_{Iw}(\mathbb{Q}_{p,\infty}, \rho^u)}_{\text{local Iwasawa cohomology}}.$$

  So altogether : we obtain

$$\underset{\text{"modular symbol"}}{z_{MS}} : \rho^u_\Sigma \longrightarrow H^1_{Iw}(\mathbb{Q}_{p,\infty}, \rho^u)$$

  called the "zeta morphism of modular symbols". Problem : $H^1_{Iw}(\mathbb{Q}_{p,\infty}, \rho^u)$ is only a local Iwasawa cohomology !

- Another crucial point of Colmez–Wang : (rather convoluted !)

  1° z_{MS} comes from global Iwasawa coh. $H^1_{Iw}(\mathbb{Q}^\Sigma_\infty, \rho)$

  2° At classical points, z_{MS} and z_{Kato} are equal up to p-adic units.