Talk 06 Local computations : unramified places

The local Siegal section at unramified places are merely spherical sections f_v° defined in §4.3 (both in ♡ and in ◊ case) .

$\boxed{\text{Fourier expansion}}$ Recall for $\beta \in \mathrm{Herm}_n(F)$, we have the local Whittaker integral

$$W_\beta(h; f_v^\circ, z) := \int_{\mathrm{Herm}_n(F_v)} f_v^\circ\left(z, w_n \begin{pmatrix} 1_n & \sigma \\ & 1_n \end{pmatrix} h\right) e_v(-\mathrm{Tr}\beta\sigma)\, d\sigma$$

Theorem : [Wan15ANT, Lemma 4.6] [SU14, Lemma 11.7, 11.8]

Let $\beta \in \mathrm{Herm}_n(F_v)$ and $r = \mathrm{rank}\,\beta$. Then for $y \in GL_n(K_v)$,

$$W_\beta\left(\mathrm{diag}(y, \bar{y}^*); f_v^\circ, z\right) = \chi(\det y) \left|\det y\bar{y}\right|_v^{-z+\frac{n}{2}} D_v^{\frac{-n(n-1)}{4}} h_{v, y^*\beta y}\left(\bar{\chi}'(\varpi) q_v^{-2z-n}\right)$$

$$\times \frac{\prod_{i=r}^{n-1} L(2z+i-n+1, \bar{\chi}'\eta^i_{K/F})}{\prod_{i=0}^{n-1} L(2z+n-i, \bar{\chi}'\eta^i_{K/F})} .$$

Here :
- χ here are the local piece χ_v at the place v.
- D_v is the absolute discriminant of K_v/F_v, ϖ the uniformizer of F_v.
- $h_{v, y^*\beta y} \in \mathbb{Z}[X]$ is a monic polynomial depending on v and $y^*\beta y$, but not on χ. Moreover, $h_{v,y^*\beta y} = 1$ when β is v-primitive : $\beta \in \mathrm{Herm}_n(\mathcal{O}_{F,v})$ with $\det\beta \in \mathcal{O}_{F,v}^\times$ (i.e. β has full rank $r = n$)

Proof : The original proof is in [Shimura97, Prop. 18.14, Prop. 19.2] .

If I have enough time, I will try to include a full proof of this fact .

$\boxed{\text{Pullback integrals}}$ We have stated in §4.4.3 the following theorem.

<u>Theorem</u> ([Wan15 ANT, Lemma 4.5], [SU14, Thm 11.9])

For $v \notin \Sigma \cup \{\infty\}$, suppose $\phi \in \pi$ is a new vector.

Then if $\mathrm{Re}(z) > \frac{a+b}{2}$, the pullback integral converges and

- $$F_\phi^{\heartsuit}(f_v^\circ, z, g) = \frac{L(\tilde{\pi}, \xi, z+1)}{\prod_{i=0}^{a+2b-1} L(2z+a+2b+1-i, \bar{\chi}' \eta_{K/F}^i)} F^\circ(z, g) \quad \in V$$

- $$F_\phi^{\diamondsuit}(f_v^\circ, z, g) = \frac{L(\tilde{\pi}, \xi, z+\frac{1}{2})}{\prod_{i=0}^{a+2b-1} L(2z+a+2b-i, \bar{\chi}' \eta_{K/F}^i)} \pi(g)\varphi \quad \in V$$

where F° is the <u>sphrecical Klingen section</u>, taken value $\phi \in V$ at $1 \in GU(r+1, s+1)(\mathbb{A}_F)$

Here:

- $\tilde{\pi}$: the contragredient rep'n of π.
- $\xi = \psi/\chi$, recall ψ is the central character of π.
- $\eta_{K/F}$: the quadratic character of $F^\times \backslash \mathbb{A}_F^\times$ associated to K/F. (<u>recall CFT</u>)
- χ' : the restriction of the Hecke character χ on K to F.

<u>Note</u> : $L(\tilde{\pi}, \xi, z+1) := L(BC(\tilde{\pi}) \otimes \xi, z+1)$, where $BC(\tilde{\pi})$ is the <u>base change</u> of $\tilde{\pi}$ over $G(\mathbb{A}_F)$ to $G(\mathbb{A}_K)$, to be "compatible" with ξ. (This was made explicit in [SU14, Introduction, p8].

<u>Remark</u> : Here to do pullback, we require $\phi \in \pi$ is a new vector. Later in the arguments, <u>I often forget / ignore this</u>. This leads to many confusions of mine.

→ So next, let's review the <u>old-new story</u> of modular forms in the setup of GL_2. The general story should be similar I guess.

(cf. Hsieh, "Hida families and p-adic triple products L-functions)

Old-new story for $GL_2/\mathbb{Q}$:

- Let ℓ be a prime.
- (π, V) be an irreducible admissible infinite dim'l rep'n of $GL_2(\mathbb{Q}_\ell)$.

For $n \geq 0$, let

$$\mathcal{U}_1(\ell^n) = GL_2(\mathbb{Z}_\ell) \cap \begin{pmatrix} \mathbb{Z}_\ell & \mathbb{Z}_\ell \\ \ell^n \mathbb{Z}_\ell & 1+\ell^n\mathbb{Z}_\ell \end{pmatrix}.$$

Let $c(\pi)$ be the <u>exponent of the conductor</u> of π: smallest integer such that $V^{\mathcal{U}_1(\ell^{c(\pi)})}$ is nonzero. Then the <u>new-space</u> is defined as

$$V^{new} := \left\{ \phi \in V : \pi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) v = v, \text{ for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{U}_1(\ell^{c(\pi)}) \right\}$$

Then we have:

<u>Multiplicity one theorem</u>: $\dim_{\mathbb{C}} V^{new} = 1$. So we call V^{new} the <u>new line</u> of π.

The story for $GL_n/\mathbb{Q}$ is defined similarly, see [Goldfeld–Hundley, Vol II, §13.8] for proofs:

- $c(\pi)$ is well-defined, i.e. $c(\pi) < \infty$: <u>Lemma 13.8.2</u> in loc.cit.
- Multiplicity one: <u>Theorem 13.8.7</u> in loc.cit.

By the tensor product decomposition, we can talk about "<u>global new vectors</u>" or with some "<u>new condition</u>".

<u>Note</u>: Recall for local admissible rep π (at nonarchemedian places),

$$\pi = \bigcup_{\substack{K \text{ max. open} \\ \text{compact of } GL_2(\mathbb{Q}_\ell)}} \pi^K$$

So here by saying "new line", we are actually choosing a particular "<u>K-type</u>". (i.e. how K acts for a particular K?)