Chapter 1. Λ-adic forms

§1 Some p-adic analysis

Let fix p an odd prime.

<u>(1.1) Description of $\mathbb{Z}_p^\times$</u> : Let $T := 1+p\mathbb{Z}_p \subseteq \mathbb{Z}_p^\times$. Then there is a split short exact sequence

$$0 \longrightarrow T \longrightarrow \mathbb{Z}_p^\times \longrightarrow (\mathbb{Z}/p\mathbb{Z})^\times \longrightarrow 0$$

$$\omega : (\mathbb{Z}/p\mathbb{Z})^\times \to \mathbb{Z}_p^\times$$

with the splitting homomorphism $\omega : (\mathbb{Z}/p\mathbb{Z})^\times \longrightarrow \mathbb{Z}_p^\times$ called the <u>Teichmuller character</u>.

- Therefore, for $a \in \mathbb{Z}_p^\times$, we can decompose it as

$$\begin{aligned} \mathbb{Z}_p^\times &\xrightarrow{\sim} T \times (\mathbb{Z}/p\mathbb{Z})^\times \\ a &\longmapsto (\langle a\rangle, \ a \bmod p) \end{aligned}$$

The Teichmuller character ω provide a lift of the "a mod p" part :

$$a \equiv \omega(a) \bmod p. \quad \text{—} (*)$$

We define $\langle a\rangle := \omega(a)^{-1}a$. Then it immediately follows from $(*)$ that $\langle a\rangle \equiv 1 \bmod p$, i.e. $\langle a\rangle \in T = 1+p\mathbb{Z}_p$.

<u>Intuition</u> : Consider for $a \in \mathbb{Z}_p^\times$ its p-adic development

$$a = a_0 + \sum_{n=1}^\infty a_n p^n, \qquad a_0 \in \{1,2,\cdots,p-1\} \simeq (\mathbb{Z}/p\mathbb{Z})^\times, \quad a_i \in \{0,\cdots,p-1\},\ i \geq 1$$

$$= a_0\left(1+\sum_{n=1}^\infty \frac{a_n}{a_0}p^n\right)$$

$$= a_0\underbrace{\left(1+p\underbrace{\sum_{n=1}^\infty \frac{a_n}{a_0}p^{n-1}}_{\in \mathbb{Z}_p}\right)}_{1+p\mathbb{Z}_p} \quad \langle a\rangle$$

$$\omega : (\mathbb{Z}/p\mathbb{Z})^\times \xrightarrow{\text{lift}} \mathbb{Z}_p^\times$$

<u>(1.2) Exponential and logarithms</u>

We formaly define

$$\exp(X) = \sum_{n=0}^\infty \frac{X^n}{n!} \in \mathbb{C}_p[\![X]\!], \qquad \log_p(X+1) = \sum_{n=1}^\infty \frac{(-1)^n}{n}X^n$$

<u>Fact 1</u> : $\exp(X)$ converges for $\{x \in \mathbb{C}_p \mid |x|_p < p^{-\frac{1}{p-1}}\}$ [Was, p49]

<u>Fact 2</u> : $\log_p(X)$ converges for $\{x \in \mathbb{C}_p \mid |x|_p < 1\}$. Moreover, there exists a

unique extension of $\log_p$ to $\mathbb{C}_p^\times$ s.t. $\log_p(p)=0$, $\log_p(xy)=\log_p(x)+\log_p(y)$.

[Was, p50]

Fact 3: $\log_p x = 0$ for $x\in\mathbb{C}_p^\times \iff x =$ rational power of p $\times$ a root of unity.

[Was, Prop. 5.6]

Fact 4: For $|x| < p^{-\frac{1}{p-1}}$, $\log_p(\exp(x)) = x$ and $\exp(\log_p(1+x)) = 1+x$.

Moreover, recall for $a\in\mathbb{Z}_p^\times$, $\langle a\rangle \equiv 1 \bmod p$. Hence $\log_p(\langle a\rangle) = \log_p(a)$. Thus for suitable $x\in\mathbb{C}_p$, we define

$$\langle a\rangle^x := \exp(x\log_p(a)) = \exp(x\log_p(\langle a\rangle)). \qquad (**)$$

Then one checks:

Fact 5: The above $(**)$ makes sense when $|x| < p^{-\frac{1}{p-1}+1}$ (Exercise)

Let u be a topological generator of Γ, we define

$$s:\ \Gamma = 1+p\mathbb{Z}_p \longrightarrow \mathbb{Z}_p$$
$$a \longmapsto \frac{\log_p(a)}{\log_p(u)} \qquad \left(\text{Fact 3} \Rightarrow \log_p(u)\neq 0\right)$$

Then by Fact 4 and Fact 5,

$$u^{s(a)} = \exp(s(a)\cdot\log_p(u)) = \exp\left(\frac{\log_p(a)}{\log_p(u)}\cdot\log_p(u)\right) = \exp(\log_p(a)) \overset{\text{Fact 4}}{=} a$$

for $a\in\Gamma$. This is an essential observation!

(1.3) Binomial coefficients

We formally define

$$\binom{X}{n} := \frac{1}{n!}X(X-1)\cdots(X-n+1)$$

as a polynomial of degree n in X. Then $x\longmapsto\binom{x}{n}$ is a continuous function on $\mathbb{Z}_p$.

Fact: $\binom{x}{n}\in\mathbb{Z}_p$ for any $x\in\mathbb{Z}_p$. (note: $\mathbb{N}\underset{\text{dense}}{\hookrightarrow}\mathbb{Z}_p$, $\binom{m}{n}\in\mathbb{N}$ for all $m\in\mathbb{N}$)

Mahler theorem: any continuous function $f:\mathbb{Z}_p\to\mathbb{C}_p$ can be written uniquely as

$$f(x) = \sum_{n=0}^{\infty} a_n\binom{x}{n}, \quad \text{with } a_n\to\infty \text{ for } n\to\infty.$$

We may turn back to it when discussing p-adic L-functions.

§2 Characters

We fix $u = 1+p$ as a topological generator for Γ. $(p,N)=1$ with $N \in \mathbb{Z}_{>0}$.

<u>Teichmuller character</u> : $\omega : (\mathbb{Z}/p\mathbb{Z})^\times \longrightarrow \mathbb{Z}_p^\times$, as a Dirichlet character.

<u>Tame character</u> : Let $\chi : (\mathbb{Z}/Np\mathbb{Z})^\times \longrightarrow \overline{\mathbb{Q}}_p$ be a Dirichlet character.

- $\mathbb{Z}[\chi] := \mathbb{Z}[\operatorname{im}\chi]$
- Assume $F | \mathbb{Q}_p$ such that $\mathbb{Z}[\chi] \subseteq F$.
- Let $\Lambda_F := \mathcal{O}_F[\![X]\!]$ be the <u>Iwasawa algebra</u> of F. We have canonical iso

$$\begin{aligned} \mathcal{O}_F[\![\Gamma]\!] \text{ as a completed grp algebra } &\simeq \mathcal{O}_F[\![X]\!] \\ u &\longmapsto 1+X \end{aligned}$$

(For proof, see [Was, theorem 7.1]).

Such a character is also called a <u>Dirichlet character of the first kind</u>.

<u>Wild character</u> : Let ε be a finite order character $\Gamma \longrightarrow \overline{\mathbb{Q}}_p^\times$.

- Recall there is a filtration

$$\Gamma = 1+p\mathbb{Z}_p \underbrace{\supseteq}_{\mathbb{Z}/p} \Gamma_2 := 1+p^2\mathbb{Z}_p \underbrace{\supseteq}_{\mathbb{Z}/p} \cdots \underbrace{\supseteq}_{\mathbb{Z}/p} \Gamma_n = 1+p^n\mathbb{Z}_p \underbrace{\supseteq}_{\mathbb{Z}/p} \cdots$$

- Since ε is of finite order, $\exists$ (a minimal) $r_\varepsilon \in \mathbb{Z}_{\geq 1}$ s.t.

$$\varepsilon|_{\Gamma_{r_\varepsilon}} = 1, \quad \varepsilon|_{\Gamma_{r_\varepsilon - 1}} \neq 1$$

  We call r_ε the <u>conductor</u> of ε. Therefore, to define such an ε, it is enough to fix a primitive p^r-th root of unity $\zeta \in \overline{\mathbb{Q}}_p^\times$ and set it as the image of u. In this way, we denote ε as ε_ζ.

- For such a character ε of conductor r, it induces

$$\varepsilon : \Gamma/\Gamma_r \simeq \mathbb{Z}/p^r\mathbb{Z} \longrightarrow \overline{\mathbb{Q}}_p^\times.$$

  Recall by "<u>Euler's theorem</u>"

$$(\mathbb{Z}/p^{r+1}\mathbb{Z})^\times \simeq (\mathbb{Z}/p\mathbb{Z})^\times \times \mathbb{Z}/p^r\mathbb{Z}.$$

  ε gives a Dirichlet character of level p^{r+1}, by assuming trivial map on the $(\mathbb{Z}/p\mathbb{Z})^\times$-component.

Such a character is often called a <u>Dirichlet character of the second kind</u>.

(<u>Slightly notation issume</u> : when focusing on ε as an Dirichlet character above, ε has <u>conductor</u> p^{r+1}. When we say ε_ψ is a Dirichlet character of conductor p^r, we are actually using ε of conductor $r-1$ and $\psi \in \mu_{p^{r-1}}^{\text{prim}}(\overline{\mathbb{Q}}_p)$.)

- <u>Fact</u> : Every Dirichlet character θ can be decomposed into $\theta = \chi \varepsilon_\psi \omega^{-k}$ for some tame character χ, wild character ε_ψ and some $k \geq 1$. ? 存疑.

§3 Λ-adic forms

Let $\mathcal{L}$ be a finite extension of $\mathrm{Frac}(\Lambda_F)$ and $\mathbb{I}$ be the integral closure of Λ_F in $\mathcal{L}$.

Note : We will see later that when interpolating CM forms, going beyond Λ_F is necessary.

Exercise : $\mathbb{I}$ is a finite flat Λ_F-algebra (why?)

Specialization on Λ_F : An O_F-algebra homomorphism $\varphi : \Lambda_F \longrightarrow \overline{\mathbb{Q}}_p$ is called a specialization.

- Given a group homomorphism $\alpha : \Gamma \longrightarrow \overline{\mathbb{Q}}_p^\times$,
  - It is determined by the image of $u = 1 + p \in \Gamma$.
  - It determines a specialization $\varphi : \Lambda_F \longrightarrow \overline{\mathbb{Q}}_p$ by setting
$$\varphi(f(X)) = f(\alpha(u) - 1) \quad \text{for } f \in \Lambda_F.$$
    (recall the canonical isomorphism $O_F[[\Gamma]] \xrightarrow{\sim} O_F[[X]]$, $u \longmapsto 1+X$, this is reasonable.)

- A specialization φ is called
  - arithmetic if φ is determined by a continuous group homomorphism $\alpha : \Gamma \longrightarrow \overline{\mathbb{Q}}_p^\times$ such that $\alpha(\gamma) = \varepsilon(\gamma)\gamma^k$ for a finite order character $\varepsilon : \Gamma \longrightarrow \overline{\mathbb{Q}}_p^\times$ and some $k \geq 1$. Denoted by $\varphi_{k,\varepsilon}$

    note : i.e. α is obtained by a wild character ε twisted by $\gamma \longmapsto \gamma^k$.
    hence it reflects two pieces of informations :
    - ε : nebentypus of specialized forms
    - k : weight of specialized forms (as main contribution).
  - classical if $\varphi = \varphi_{k,\varepsilon}$ for $k \geq 2$.

Remark : More explicity, $\varphi_{k,\varepsilon}(X) = \varepsilon(u)u^k - 1$.
So $\varphi_{k,\varepsilon}$ is actually the evaluation of X at $\varepsilon(u)u^k - 1$.
In particular for $\varphi_{k,\varepsilon_\zeta}$, it is actually the evaluation of X at $\zeta u^k - 1$.

Specialization on $\mathbb{I}$: Similarly, an O_F-algebra homomorphism $\varphi : \mathbb{I} \longrightarrow \overline{\mathbb{Q}}_p$ is a specialization.
It is called arithmetic / classical if $\varphi|_{\Lambda_F}$ is arithmetic / classical.

We can apply specializations to formal power series :

- $\mathbb{f} := \sum_{n=0}^{\infty} a(n,\mathbb{f})\, q^n \in \mathbb{I}[[q]]$ be a formal power series .
- Denote $\varphi(\mathbb{f}) := \mathbb{f}_\varphi := \sum_{n=0}^{\infty} \varphi(a(n,\mathbb{f}))\, q^n \in \overline{\mathbb{Q}}_p[[q]]$.

Then it's time to define :

<u>Defn</u> : A formal q-expansion $\mathbb{f} \in \mathbb{I}[[q]]$ is called an <u>$\mathbb{I}$-adic form</u> of <u>tame character</u> χ and level N if for <u>almost all</u> (i.e. except for finitely many) classical specialization $\Phi : \mathbb{I} \longrightarrow \overline{\mathbb{Q}}_p$ given by $\varphi_{k,\varepsilon}$ on Λ_F ,

$$\mathbb{f}_\Phi \in M_k(\Gamma_0(Np^{r_\varepsilon+1}),\ \varepsilon\chi\omega^{-k},\ \overline{\mathbb{Q}}_p)$$

Analogously one defines <u>$\mathbb{I}$-adic cusp form</u> for $\mathbb{f} \in \mathbb{I}[[q]]$ as lying in $S_k(\cdots)$ for almost all classical specialization Φ .

<u>Remark</u> :

(1) When $\mathcal{L} = F$ and $\mathbb{I} = \Lambda_F$, we get the definition of <u>Λ_F-adic forms</u> .

(2) Geometrically , we can think $\mathrm{Spec}\,\mathbb{I}$ as the space of specializations . The specializations above can be viewed as $\overline{\mathbb{Q}}_p$-points of $\mathrm{Spec}\,\mathbb{I}$.

§4 Example : Eisenstein families over $GL_2/\mathbb{Q}$.

As always, assume p is an odd prime, $(N,p)=1$.

Classical Eisenstein series :

- $k \geq 1$ integer
- $\psi : (\mathbb{Z}/Np^r\mathbb{Z})^\times \longrightarrow \mathbb{C}$ be a Dirichlet character, $\psi(-1)=(-1)^k$.

Then we define

$$E_{k,\psi} = \frac{L(1-k,\psi)}{2} + \sum_{n=1}^{\infty} \sigma_{k-1,\psi}(n) q^n , \quad \sigma_{k-1,\psi}(n) = \sum_{d|n} \underbrace{\psi(d)}_{\text{twist by } \psi} d^{k-1}$$

It is known that $E_{k,\psi} \in M_k(\Gamma_0(Np^r),\psi)$.

- <u>p-stablization</u> : $E_{k,\psi}^{(p)} := E_{k,\psi}(z) - \psi(p)p^{k-1}E_{k,\psi}(pz) \in M_k(\Gamma_0(Np^{r+1}),\psi)$.

  It has a q-expansion as

$$E_{k,\psi}^{(p)} = \frac{L^{(p)}(1-k,\psi)}{2} + \sum_{n=1}^{\infty} \sigma_{k-1,\psi}^{(p)}(n) q^n ,$$

where :
- $L^{(p)}(1-k,\psi) = \underbrace{\left(1-\psi(p)p^{-s}\right)}_{\text{Euler factor at } p} L(s,\psi)$
- $\sigma_{m,\psi}^{(p)}(n) = \sum_{\substack{d|n \\ p \nmid d}} \psi(d) d^m$.

<u>Remark</u> : Actually if $r>0$, then $E_{k,\psi}^{(p)}(z) = E_{k,\psi}(z)$.

- In particular we take the following nebentypus ψ :
  - χ : <u>even</u> Dirichlet character modulo Np for some N prime to p. $(\chi(-1)=1)$
  - ε_φ is a Dirichlet character of conductor p^r given by a primitive p^{r-1}-th root of unity $\varphi \in \overline{\mathbb{Q}}_p$ (for $r \geq 1$)
  - ω : Techimuller character.

Then for $k \geq 1$, we take $\psi = \chi\omega^{-k}\varepsilon_\varphi$. Then ψ is a Dirichlet character mod Np^r with $\psi(-1)=(-1)^k$

Theorem : Let $\mathbb{I} = \mathcal{O}[[X]]$ with $\mathcal{O} := \mathbb{Z}_p[\chi]$. Let $\Lambda := \mathbb{Z}_p[[X]]$.

Assume $\chi \neq 1$. Then there exists an $\mathbb{I}$-adic form

$$\mathbb{E}_\chi = \sum_{n=0}^{\infty} A_{n,\chi}(X) q^n \in \mathbb{I}[[q]]$$

which specializes to $E^{(p)}_{k,\psi}$ with $\psi = \chi\omega^{-k}\varepsilon_\varphi$ under the classical specializations

$\varphi : \mathbb{I} \to \overline{\mathbb{Q}}_p^\times$ with $\varphi|_\Lambda = \varphi_{k,\varepsilon_\varphi}$ for $k>1$ and ε_φ as above.

Motivating calculation : Suppose $d \in \mathbb{Z}_{>0}$ such that $d \equiv 1 \mod p$. Then by §1, we have seen $d = u^{s(d)}$ as $d \in \Gamma$. Hence setting

$$A_d(X) := \frac{1}{d}(1+X)^{s(d)}$$

one immediately see

$$A_d(u^k-1) = \frac{1}{d}(1+u^k-1)^{s(d)} = \frac{1}{d}(u^{s(d)})^k = d^{k-1}.$$

General calculation : For $d \in \mathbb{Z}_{>0}$ coprime to p, how can we swift d to something in Γ? Recall : $d = \omega(d)\langle d\rangle$ where $\langle d\rangle \in 1+p\mathbb{Z}_p$, we set

$$A_d(X) := \frac{1}{d}(1+X)^{s(\langle d\rangle)}.$$

Then

$$\begin{aligned} A_d(\varphi u^k-1) &= \frac{1}{d}(\varphi u^k)^{s(\langle d\rangle)} \\ &= \frac{1}{d}\cdot \varphi^{s(\langle d\rangle)}\cdot u^{ks(\langle d\rangle)} \\ &= \frac{\varepsilon_\varphi(\langle d\rangle)\cdot\langle d\rangle^k}{d} \\ &= \frac{\varepsilon_\varphi(\langle d\rangle)\,\omega(d)^{-k} d^k}{d} \\ &= \omega^{-k}(d)\,\varepsilon_\varphi(d)\, d^{k-1} \end{aligned}$$

Recall : $\langle d\rangle = u^{s(\langle d\rangle)}$

$\xRightarrow{\text{apply } \varepsilon_\varphi}$ $\varepsilon_\varphi(\langle d\rangle) = \varepsilon_\varphi\left(u^{s(\langle d\rangle)}\right) = \varepsilon_\varphi(u)^{s(\langle d\rangle)}$
$= \varphi^{s(\langle d\rangle)}$

Then one notes $\varepsilon_\varphi(\langle d\rangle) =: \varepsilon_\varphi(d)$. we see indeed

$$\varphi^{s(\langle d\rangle)} = \varepsilon_\varphi(d)$$

Finally for $n \geq 1$, we define

$$A_{n,\chi}(X) = \sum_{\substack{d|n \\ (d,p)=1}} \chi(d) A_d(X) \in \mathbb{Z}_p[\chi][[X]] = \mathbb{I}[[X]].$$

then

$$A_{n,\chi}(\varphi u^k-1) = \sum_{\substack{d|n \\ (d,p)=1}} \chi(d)\,\omega^{-k}(d)\,\varepsilon_\varphi(d)\, d^{k-1} = \sum_{\substack{d|n \\ (d,p)=1}} \psi(d)\, d^{k-1} = \sigma^{(p)}_{k-1,\psi}(n)$$

This interpolates the nonconstant term of $E^{(p)}_{k,\psi}$.

Constant terms : To interpolate the constant term, we need Kubota–Leopoldt p-adic L-functions. We omit the calculation here.

$\square$

§5 Example : p-adic families of cusp forms ("cuspidal families")

Keep notations as in §4.

Let $f \in S_1(\Gamma_0(Np), \psi')$.

Recall $E^{(p)}_{k-1,\psi} \in M_{k-1}(\Gamma_0(Np^r), \psi)$ for $\psi = \chi\omega^{1-k}\varepsilon_\zeta$, we form the product

$$f \cdot E^{(p)}_{k-1,\psi} \in S_k(\Gamma_0(Np^r), \chi'\omega^{-k}\varepsilon_\zeta)$$

where $\chi' := \psi'\chi\omega$. We now interpolate all such cusp forms :

- Assume $f \cdot \mathbb{E}_\chi = \sum_{n=0}^{\infty} B_n(X) q^n \in \mathbb{I}[[q]]$. We define

$$\mathbb{F} := \sum_{n=0}^{\infty} B_n(u^{-1}X + u^{-1} - 1) q^n \in \mathbb{I}[[q]].$$

One checks : the substitution $X \mapsto u^{-1}X + u^{-1} - 1$ is an automorphism of $\mathbb{I}[[q]]$, hence $\mathbb{F}$ is well-defined. Details see [Hida93, §7.1 Lemma 1].

- Then substituting $X = \zeta u^k - 1$, we get

$$\begin{aligned}
\mathbb{F}(\zeta u^k - 1) &= \sum_{n=0}^{\infty} B_n(u^{-1}(\zeta u^k - 1) + u^{-1} - 1) q^n \\
&= \sum_{n=0}^{\infty} B_n(\zeta u^{k-1} - 1) q^n \\
&= (f \cdot \mathbb{E}_\chi)(\zeta u^{k-1} - 1) \\
&= f \cdot \left(\mathbb{E}_\chi(\zeta u^{k-1} - 1)\right) \\
&= f \cdot E^{(p)}_{k-1,\psi}
\end{aligned}$$

So $\mathbb{F}$ is indeed the desired cuspidal family, a cuspidal $\mathbb{I}$-adic form.

Question : Is all cusp forms of this form ?

Remark : (See [Zhao Bin, Exercise 1.9]) This can be generalized :

Suppose $f \in M_\ell(\Gamma_0(p), \chi_0 ; \mathcal{O})$. We write

$$f \cdot \mathbb{E}_\chi(X) = \sum_{n=0}^{\infty} B_n(X) q^n \in \Lambda[[q]].$$

Yet a more appropriate "product" is the convolution

$$\mathbb{F}(X) := f * \mathbb{E}_\chi(X) := \sum_{n=0}^{\infty} B_n(u^{-\ell}X + u^{-\ell} - 1) q^n \in \Lambda[[q]]$$

(note "One checks" above). Then $\mathbb{F}$ is a Λ-adic form with character $\chi_0\chi$.

In particular, when f is a cusp form, $\mathbb{F}$ is a Λ-adic cusp form.

§6 CM forms and CM families

We first introduce CM forms ("modular forms with complex multiplication"), then interpolate them in Hida families.

<u>Setup</u>: • K imaginary quadratic field with discriminant $-D$ for some $D>0$. It has an associated Dirichlet character mod D:

$$\chi_{K/\mathbb{Q}}(n) = \left(\frac{-D}{n}\right) \quad \text{as the } \underline{\text{Jacobi symbol}}.$$

• Let $\Sigma_K = \{\sigma_1, \sigma_2 : K \hookrightarrow \mathbb{C}\}$ be the two embeddings into $\mathbb{C}$. Say σ_1 is the trivial embedding.

6.1 Some number theory reviewed [黎景辉, §1.7, §4.2–4.4]

- Let $\mathfrak{m}$ be an integral ideal of K. We regard it as a modulus with trivial infinite part $\mathfrak{m}_\infty = 1$.
- Define $I_K^{\mathfrak{m}}$:= group of fractional ideals of K coprime to $\mathfrak{m}$.

  $P_K^{\mathfrak{m}}$:= group of principal fractional ideals (α) such that $\underline{\alpha \equiv 1 \bmod^{\times} \mathfrak{m}}$.

  <u>Recall</u>: write $\mathfrak{m} = \prod_{i=1}^{g} \mathfrak{p}_i^{e_i}$, then here by saying $a \equiv b \bmod \mathfrak{m}$, we mean

$$v_{\mathfrak{p}_i}(a-b) \geq e_i, \quad \text{for any } i = 1, \dots, g.$$

  Then we can form the <u>ray class group</u> $Cl_K^{\mathfrak{m}} := I_K^{\mathfrak{m}} / P_K^{\mathfrak{m}}$.

- <u>Fact</u>: We have a short exact sequence

$$1 \longrightarrow Cl_{\mathbb{I}_K}^{\mathfrak{m}} \longrightarrow Cl_K \longrightarrow Cl_K^{\mathfrak{m}} \longrightarrow 0$$

  * Here $Cl_{\mathbb{I}_K}^{\mathfrak{m}} := \mathbb{I}_K^{\mathfrak{m}} K^\times / K^\times$ is the <u>idèle class group</u> of $\mathfrak{m}$. Recall $\mathbb{I}_K^{\mathfrak{m}} := \prod_{i=1}^{g} U_{\mathfrak{p}_i}^{(e_i)}$ where $U_{\mathfrak{p}_i}^{(e_i)} := 1 + \mathfrak{p}_i^{e_i}$.

  Then as a <u>corollary</u>: $Cl_K^{\mathfrak{m}}$ is a finite group. This will be used frequently later.

- Define a <u>Hecke Grössencharakter</u> ψ of <u>infinite type</u> (k_1, k_2) modulo $\mathfrak{m}$ as a group homomorphism $\psi : I_K^{\mathfrak{m}} \longrightarrow \mathbb{C}^\times$ (we don't require φ maps into S^1)

  such that for any $(\alpha) \in P_K^{\mathfrak{m}}$, $\psi((\alpha)) = \sigma_1(\alpha)^{k_1} \sigma_2(\alpha)^{k_2}$.

  <u>Historical review</u>: Weil called such a character of type (A_0).

  We extend ψ by zero on the entire group I_K.

- <u>Fact</u> : A Hecke Grössencharacter takes values in a number field
  Let $M_\psi | \mathbb{Q}$ be the number field generated by values of ψ. (<u>Exercise</u>)

- For a Grössencharacter ψ, there exists a unique divisor $\mathfrak{f}$ of $\mathfrak{m}$ such that
  (i) ψ is a restriction of a Grössencharacter $\tilde{\psi} : I_K^{\mathfrak{f}} \longrightarrow \mathbb{C}^\times$
  (ii) no proper divisor of $\mathfrak{f}$ has the property (i).

  Then $\mathfrak{f}$ is called the <u>conductor</u> of ψ, denote $\mathrm{cond}(\psi) := \mathfrak{f}$

  - If $\mathrm{cond}(\psi) = \mathfrak{m}$, then ψ is called <u>primitive</u>.

- <u>Example</u> : We shall frequently use the Grössencharacter of infinite type $(k-1, 0)$ for some $k \geq 2$. Then on principal ideals

$$\psi((\alpha)) = \alpha^{k-1} \qquad (\alpha) \in P_K^{\mathfrak{m}}.$$

  Moreover, we associate ψ of conductor $\mathfrak{f}$ with a Dirichlet character

$$\check{\psi} : \left(\mathbb{Z}/N(\mathfrak{f})\right)^\times \longrightarrow \mathbb{C}^\times \qquad \text{for } (n, N(\mathfrak{f})) = 1.$$
$$n \longmapsto \frac{\varphi((n))}{n^{k-1}}$$

6.2 Theta series and CM forms

Let $\psi : I_K^{\mathfrak{m}} \to \mathbb{C}^\times$ be a primitive Grössencharacter of infinite type $(k-1, 0)$, $k \geq 2$.

Then define the formal power series

$$\Theta(z, \psi) := \sum_{\mathfrak{a} \text{ integral ideal}} \psi(\mathfrak{a})\, q^{N(\mathfrak{a})} \in M_\psi[[q]],$$

where $N(\mathfrak{a})$ is the norm of the ideal $\mathfrak{a}$. This is called the <u>theta series</u> associated to ψ.

<u>Note</u>: We always assume that the sum is over all <u>nonzero</u> integral ideals of $\mathcal{O}_K$.

And recall $\psi(\mathfrak{b}) = 0$ if $\mathfrak{m} \mid \mathfrak{b}$.

<u>Theorem</u> (Shimura, 1971 Nagoya paper) So defined $\Theta(z, \psi)$ is a newform (hence an Hecke eigenform) in $S_k(\Gamma_0(M), \chi)$ where

- $M := D \cdot N(\mathfrak{m})$
- χ is a Dirichlet character of level M, given by $\chi = \chi_{K|\mathbb{Q}} \cdot \check{\psi}$, i.e.

$$\chi(n) = \left(\frac{-D}{n}\right) \cdot \frac{\psi((n))}{n^{k-1}} \quad \text{for } (n, M) = 1$$

<u>Defn</u>: Let $f \in S_k(\Gamma_1(Np^m))$ be an normalized cuspidal eigenform for some $(N,p)=1$, $m \geq 1$, with the field of definition $F | \mathbb{Q}_p$. We define the <u>p-slope</u> of f as the number

$$Sl_p(f) := v_p(a_p), \quad \text{normalized as } v_p(p) = 1$$

(<u>note</u>: $Sl_p(f) \in \mathbb{Q} \sqcup \{\infty\}$). We call f

- <u>p-ordinary</u> if $Sl_p(f) = 0$ (consistent with previous definition)
- has <u>finite slope</u> (<u>finite p-slope</u>) if $Sl_p(f) \in \mathbb{Q}$
- has <u>infinite slope</u> if $Sl_p(f) = \infty$, i.e. $a_p = 0$.

Similarly we talk about "strictly positive slope", etc..

We compute the p-slope of the theta series, always assume $p \nmid \mathfrak{m} = \mathrm{cond}(\psi)$.

① p is inert in K : there are no ideals of norm p in K. So $a_p = 0$, implying $Sl_p(f) = \infty$.

② p splits as $p\mathcal{O}_K = \mathfrak{p}\bar{\mathfrak{p}}$: then $a_p = \psi(\mathfrak{p}) + \psi(\bar{\mathfrak{p}})$.

- Recall $Cl_K^{\mathfrak{m}}$ is finite, $\exists\ n > 0$ s.t. $\mathfrak{p}^n = (\alpha)$, $\alpha \equiv 1 \mod \mathfrak{m}$.

  So $\psi((\alpha)) = \alpha^{k-1}$. Hence $\psi(\mathfrak{p})^n = \psi(\mathfrak{p}^n) = \alpha^{k-1}$.

- note $\alpha \in \mathfrak{p}$ but $\alpha \notin \bar{\mathfrak{p}}$ (otherwise $\alpha \in \bar{\mathfrak{p}}$, hence $\bar{\mathfrak{p}} \supseteq \mathfrak{p}^n$, implying $\mathfrak{p} = \mathfrak{p}$)

  So $\psi(\mathfrak{p}) \in \mathfrak{p}$ but $\psi(\mathfrak{p}) \notin \bar{\mathfrak{p}}$ (otherwise $\alpha^{k-1} \in \bar{\mathfrak{p}}$, showing $\alpha \in \bar{\mathfrak{p}}$)

  Similarly $\psi(\bar{\mathfrak{p}}) \in \bar{\mathfrak{p}}$ but not in $\mathfrak{p}$.

$\Longrightarrow a_p \notin \mathfrak{p}$ and $a_p \neq 0$. Hence $Sl_p(f) = 0$.

③ p ramifies as $p\mathcal{O}_K = \mathfrak{p}^2$: then $a_p = \psi(\mathfrak{p})$.

- Do the same thing as ② : $\psi(\mathfrak{p})^n = \alpha^{k-1} \underset{\text{go back}}{=} \mathfrak{p}^{n(k-1)}$

- Take v_p on both sides, we see

$$n\, v_p(\psi(\mathfrak{p})) = \tfrac{1}{2} n(k-1),$$

hence $Sl_p(f) = v_p(\psi(\mathfrak{p})) = \dfrac{k-1}{2}$. □

Hence when we consider (at least elementary) Hida theory, we will use the assumption that p split completely in K starting from the next section.

Definition A classical normalized eigenform $g \in S_k(\Gamma_1(Np^m))$ is called a CM form with CM given by an imaginary quadratic extension K if its Hecke eigenvalues for the operators T_ℓ ($\ell \nmid Np$) coincide with those of $\Theta(\mathfrak{z}, \psi)$ for some Grössencharacter φ of K of infinite type $(k-1, 0)$.

(notation issue : $p \nmid \mathrm{cond}(\psi)$, $(N, p) = 1$)

why defined as such : to guarantee the associated galois rep ρ_g, $\rho_{\Theta(\mathfrak{z},\psi)} : \mathrm{Gal}_{\mathbb{Q}} \longrightarrow GL_2(\bar{\mathbb{Q}}_p)$ are isomorphic. So it is the galois reps / Hecke eigensystems that really matters !

A result of Ribet : when $k \geq 2$, g is a newform, then g is of the form $\Theta(\mathfrak{z}, \psi)$.

(proved by examining carefully on the galois representation)

* Main reference : arXiv 1508.01598.

6.3 CM Λ-adic forms

<u>Goal</u> : Construct a Λ-adic form containing a fixed modular form of the form

$$g = \Theta(\mathfrak{z}, \psi), \quad \varphi : I_K^{\mathfrak{m}} \longrightarrow \mathbb{C}^\times .$$

- Suppose g has nebentypus $\vartheta = \chi_{K/\mathbb{Q}} \cdot \check{\psi}$, then we break it into $\vartheta = \chi \omega^{-k} \varepsilon_{\varphi_0}$ in the same way as <u>§4</u>. Here $\varphi_0 \in \mu_{p^{r_0}-1}^{\text{prim}}$.

<u>Setup</u> : • Suppose $p \nmid \mathfrak{m}$, p splits as $p\mathcal{O}_K = \mathfrak{p}\bar{\mathfrak{p}}$.
We pick a distinguished $\mathfrak{p}$ above p by fixing an embedding $\iota : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}_p}$.

- Auxillary Grössencharacter $\lambda : I_K^{\mathfrak{p}} \longrightarrow \mathbb{C}$ of conductor $\mathfrak{p}$ and infinite type $(1,0)$.
The field of definition is M_λ.

<u>Preparation</u> :

* $E :=$ completion of M_λ at a prime over p detemied by ι.
Then $E | \mathbb{Q}_p$ is a finite extension.
  - $\mathcal{O}_E$: ring of integer of E
  - Similar to <u>§1.1.1</u>, $\mathcal{O}_E^\times \xrightarrow{\sim} \mu_E \times W_E$ where
    * μ_E is finite
    * W_E is the free part as a finite free $\mathbb{Z}_p$-module.

  Then we can project any $x \in \mathcal{O}_E^\times$ to W_E, write $\langle x \rangle \in W_E$ for the image

* $W_E^\circ :=$ the subgroup of W_E topologically generated by $\{ \langle \lambda(\mathfrak{a}) \rangle \mid \mathfrak{a} \text{ integral ideals}, \mathfrak{p} \nmid \mathfrak{a} \}$

<u>Check</u> : We need to check well-definedness : for any integral ideal $\mathfrak{a}$ with $\mathfrak{p} \nmid \mathfrak{a}$, we have $\lambda(\mathfrak{a}) \in \mathcal{O}_E^\times$

<u>b/c</u> : • for such an integral ideal $\mathfrak{a}$, $\exists\, n$ s.t. $\mathfrak{a}^n = (\alpha)$ for some $\alpha \equiv 1 \bmod \mathfrak{p}$. in $\mathcal{O}_K$. Hence $(\lambda(\mathfrak{a}))^n = \lambda(\mathfrak{a}^n) = \lambda((\alpha)) = \alpha$. — $(*)$

• Then since $\alpha = 1 + \mathfrak{p}\mathcal{O}_K$, we see indeed $\lambda(\mathfrak{a}) \in \mathcal{O}_E^\times$ (<u>why</u>?) #

<u>Claim</u> : W_E° is isomorphic to $\mathbb{Z}_p$

<u>note</u> : we shall see : here we use the assumption that $p\mathcal{O}_K$ splits !

<u>b/c</u> : 1° <u>View $\Gamma = 1 + p\mathbb{Z}_p$ as a subgroup of W_E°</u> :

Since p splits in K, we can identify $\mathcal{O}_{K,\mathfrak{p}} \simeq \mathbb{Z}_p$, and further identify Γ as lying in $\mathcal{O}_{K_\mathfrak{p}}^\times \simeq \mathbb{Z}_p^\times$ in <u>§1.1.1</u>. Then we apply the map $\langle\lambda\rangle$ sending $\xi \in \mathcal{O}_{K,\mathfrak{p}}^{\times}$ to $\langle \lambda((\xi))\rangle \in W_E^{\circ}$. 如何对 $\xi \in \mathcal{O}_{K,\mathfrak{p}}^{\times}$ 作用 λ 呢?

$\Longrightarrow$ W_E° has at least rank 1 as $\mathbb{Z}_p$-modules .

? 2° Moreover write $m := \# Cl_K^{\mathfrak{p}}$. Then essentially follows from $(*)$ again

$$W_E^{\circ,(m)} := \{ x^m \mid x \in W_E^{\circ} \} \subseteq \Gamma$$

$\Longrightarrow$ W_E° is free of rank 1 over $\mathbb{Z}_p$, as desired . #

Get ready !

- Write $[W_E^{\circ} : \Gamma] = p^{\gamma}$ for some $\gamma \geq 0$, with a topological generator w of W_E° such that $w^{p^{\gamma}} = 1+p \in \Gamma$.
- Recall in §1.1.1 , we defined a powerful $s : \Gamma \longrightarrow \mathbb{Z}_p$, $a \longmapsto \dfrac{\log_p(a)}{\log_p(u)}$ satisfying $u^{s(a)} = a$ for any $a \in \Gamma$. We define a similar one here :

  For any integral ideal $\mathfrak{a}$ in $\mathcal{O}_K$ coprime to $\mathfrak{p}$, we define
  $$s(\mathfrak{a}) := \frac{\log_p(\langle \lambda(\mathfrak{a})\rangle)}{\log_p(w)} \in \mathbb{Z}_p$$
  the one checks : $w^{s(\mathfrak{a})} = \langle \lambda(\mathfrak{a}) \rangle$. (Exercise)
- Further extend E to $\widetilde{E} := E(\zeta_0)$, with ring of integer $\mathcal{O}$. Then our $\mathbb{I}$ is an extension of $\Lambda = \mathbb{Z}_p[[X]]$ given by
  $$\mathbb{I} := \mathcal{O}[[Y]] . \quad \text{with} \quad \zeta_0 (1+Y)^{p^{\gamma}} = 1+X \qquad \text{—} (**)$$
  Then recall the classical specializations are merely substituting :
  $$Y \rightsquigarrow \zeta w^{\ell} - 1 , \ \text{for} \ \ell \geq 2 , \ \zeta \in \mu_{p^r}^{\text{prim}} \ \text{for some} \ r \geq 1 .$$
  $$\xRightarrow[(**)]{\text{i.e.}} \quad X \rightsquigarrow \zeta_0 \zeta^{p^{\gamma}} (1+p)^{\ell} - 1 .$$

Interpolation : We finally define

$$\mathbb{F} = \sum_{(\mathfrak{a},\mathfrak{p})=1} \psi(\mathfrak{a}) \langle \lambda(\mathfrak{a}) \rangle^{-k} (1+Y)^{s(\mathfrak{a})} q^{N(\mathfrak{a})} \quad \in \mathbb{I}[[q]] = \mathcal{O}[[Y]][[q]]$$

- Note : One checks $\mathbb{F}$ does not depend on the auxillary Grössen λ.

  b/c : Any two Grössencharacters λ and λ' of infinite type $(1,0)$ of conductor $\mathfrak{p}$ differs by a finite order character. Therefore $\langle\lambda\rangle = \langle\lambda'\rangle$.

- Hope to check the interpolation property :

$$\varphi_{\ell,\varsigma} : Y \rightsquigarrow \varsigma\omega^{\ell}-1 \implies \mathbb{F}_{\varphi_{\ell,\varsigma}} \overset{\text{dream}}{=\!=} \sum_{\mathfrak{a}} \underbrace{\psi_{\ell,\varsigma}}(\mathfrak{a}) q^{N(\mathfrak{a})} = \Theta(\varsigma, \psi_{\ell,\varsigma})$$

for some Grössencharacter $\psi_{\ell,\varsigma}$

So we first define such a Grössencharacter :

- Set $\delta_\varsigma(\mathfrak{a}) = \varsigma^{s(\mathfrak{a})}$
- Let $\psi_{\ell,\varsigma}(\mathfrak{a}) = \psi(\mathfrak{a}) \langle \lambda(\mathfrak{a}) \rangle^{\ell-k} \delta_\varsigma(\mathfrak{a})$, for $\mathfrak{p} \nmid \mathfrak{a}$. $\mathfrak{a} \subseteq \mathcal{O}_K$ integral ideal.

  – Recall ψ is fixed at the very beginning of this section.

- Exercise : One checks that

  (i) δ_ς is a Grössencharacter $/K$ of infinite type $(t,0)$. What is t ??

  and $\check{\delta_\varsigma} = \varepsilon_{\varsigma p^r}$

  (ii) $\langle\lambda\rangle : \mathfrak{a} \longmapsto \langle \lambda(\mathfrak{a}) \rangle$ is a Grössencharacter $/K$ of infinite type $(1,0)$

  and $\langle\lambda\rangle^{\vee} = \omega^{-1}$

$\implies$ (iii) $\psi_{\ell,\varsigma}$ is a Grössencharacter over K of infinite type $(\ell-1,0)$.

It follows from (i) and (ii) that $\boxed{\check{\psi}_{\ell,\varsigma} = \check{\psi}\,\omega^{k-\ell}\,\varepsilon_{\varsigma p^r}}$

- Then one compute

$$Y \rightsquigarrow \zeta w^{\ell}-1 \implies (1+Y)^{s(\mathfrak{a})} \rightsquigarrow (\zeta w^{\ell})^{s(\mathfrak{a})} = \zeta^{s(\mathfrak{a})}(w^{s(\mathfrak{a})})^{\ell} = \delta_{\wp}(\mathfrak{a}) \langle \lambda(\mathfrak{a}) \rangle^{\ell}$$

Hence indeed

$$\mathbb{F}_{\varphi_{\ell,\zeta}} = \sum_{(\mathfrak{a},\wp)=1} \psi(\mathfrak{a}) \langle \lambda(\mathfrak{a}) \rangle^{-k} \delta_{\wp}(\mathfrak{a}) \langle \lambda(\mathfrak{a}) \rangle^{\ell} q^{N(\mathfrak{a})} = \sum_{(\mathfrak{a},\wp)=1} \psi_{\ell,\zeta}(\mathfrak{a}) q^{N(\mathfrak{a})}$$

recall we extend every Grössencharakter by zero to $\mathfrak{a}$ such that $\wp \mid \mathfrak{a}$, we see

$$\mathbb{F}_{\varphi_{\ell,\zeta}} = \sum_{\mathfrak{a}} \psi_{\ell,\zeta}(\mathfrak{a}) q^{N(\mathfrak{a})} = \Theta(z, \psi_{\ell,\zeta}),$$

with the resulting theta series of

$\theta = \chi\omega^{-k}\varepsilon_{\zeta_0}$

- nebentypus $\chi_{K/\mathbb{Q}} \cdot \check{\psi}\, \omega^{k-\ell} \varepsilon_{\zeta p^r} \underset{\theta = \chi_{K/\mathbb{Q}} \cdot \check{\psi}}{\overset{\text{recall } \Theta(z,\psi)}{=\!=}} \theta \cdot \omega^{k-\ell} \varepsilon_{\zeta p^r} = \chi \omega^{-\ell} \varepsilon_{\zeta_0 \zeta p^r}$.
- level $Np^{r'}$ with $p^{r'-1}$ the exact order of $\zeta_0 \zeta p^r$.

So finally we obtain a $\mathbb{I}$-adic family of theta series from a fixed one $\Theta(z, \psi)$.

- Moreover, $\mathbb{F}$ is <u>p-ordinary</u> since

$$a(p, \mathbb{F}_{\varphi_{\ell,\zeta}}) = \psi_{\ell,\zeta}(\bar{\wp}) + \underbrace{\psi_{\ell,\zeta}(\wp)}_{=0} = \psi_{\ell,\zeta}(\bar{\wp})$$

has the same <u>p-adic valuation as $\psi(\bar{\wp})$</u>, which is zero.

(recall in <u>§6.2 ②</u>, we have see $\psi(\bar{\wp}) \in \bar{\wp} \setminus \wp$. Hence $v_p(\psi(\bar{\wp})) = 0$)

凑不出一整个 $p = \wp\bar{\wp}$，故进！

i.e. Starting with an ordinary Θ, we get an ordinary family!

<u>Get back our original g</u> : Take the specialization $\varphi_{k,1}$, we see immediately:

- $\psi_{k,1} = \psi$,
- $\mathbb{F}_{k,1} = \Theta(z, \psi)$, as desired!

<u>Defn</u> An $\mathbb{I}$-adic form arising in this way is called an <u>$\mathbb{I}$-adic form with CM</u>.

§ 6.4 Further Phenomenons

<u>Attempt 1</u> : Take ψ = trivial character, we may recover the Eisenstein family.

$\Longrightarrow$ illustrating why control theorem fails when $k=1$.

<u>Attempt 2</u> : What will happen when p is inert / ramified in K ?

$\Longrightarrow$ We will obtain an " $\mathcal{O}[\![X,Y]\!]$-adic form ".

<u>Attempt 3</u> : Does there exist any Λ-adic form without CM ?

See <u>Hida's blue book, the end of Chapter 7</u>.

Chapter 2 Ordinary projectors

Setup: $K|\mathbb{Q}_p$ finite ext'n with
- $\mathcal{O}$: ring of integers
- $\mathbb{F}$: residue field
- ϖ : uniformizer

§1 Hida ordinary projector

(1.1) **Lemma**: Let A be a commutative $\mathcal{O}$-algebra which is free of finite rank over $\mathcal{O}$ endowed with p-adic topology. Then for any $x \in A$, the limit $\lim_{n\to\infty} x^{n!}$ exists in A and gives an idempotent of A.

Proof: By dévissage, we only treat the case $A = \mathcal{O}_L$, where L/K is a finite extension, with residue field k_L.

① $\boxed{a \in \mathcal{O}_L^\times}$: one counts for $r \geq 1$, $\#\left(\mathcal{O}_L/\mathfrak{m}_L^{r+1}\right)^\times = q^r(q-1)$, where $q = \# k_L$

Therefore, for $a \in \mathcal{O}_L^\times$, $a^{q^r(q-1)} \equiv 1 \mod \mathfrak{m}_L^{r+1}$. Therefore,

$$\lim_{n\to\infty} a^{n!} \overset{(*)}{=} \lim_{r\to\infty} a^{q^r(q-1)} = 1 \text{ in } \mathcal{O}_L \quad (\text{under } p\text{-adic topology})$$

key: See why $(*)$ holds: what will happen when replace "$\lim_{n\to\infty} a^{n!}$" by "$\lim_{n\to\infty} a^n$" ?

- Start with an example: $a = -1 \in \mathcal{O}_L^\times$. Then clearly,
  - $\{a^n\}$ 中有 -1 和 1 交替出现，因此 $\lim_{n\to\infty} a^n$ 不存在.
  - $\{a^{n!}\}$ 在 $n \geq 2$ 时 $a^{n!} = 1$，因此 $\lim_{n\to\infty} a^{n!}$ 是存在的.
- So where have we used "$n!$".

  For any $n \geq 1$, we define r_n as the largest natural number for which $p^{r_n - 1}(p-1) \mid n!$. Then

  $$|a^{n!} - 1| = p^{-r_n} \quad \text{in the } p\text{-adic norm}.$$

  Then $\{r_n\}$ is both (nonstrictly) monotonically increasing and unbounded. (when use n, this may not holds). This forces $|a^{n!} - 1| \to 0$. □

② $\boxed{a \in \mathfrak{m}_L}$: we have $\lim_{n\to\infty} a^{n!} = 0$ since $a \in \mathfrak{m}_L$.

- Note since $a \in \mathfrak{m}_L$, by definition we see $|a| < 1$. Hence

$$\lim_{n\to\infty} |a^{n!}| = \lim_{n\to\infty} |a|^{n!} = 0$$

<u>Here</u> : "$n!$" can be replaced by "n". It will cause no problem.

Hence the limit exists and is an idempotent in $\mathcal{O}_L$.

(See [Zhao Bin's note, <u>Lemma 1.14</u>] for the process of dévissage.)

<u>(1.2) Hida ordinary projector</u>

We apply (1.1) to the Hecke algebra $H_k(\Gamma_0(Np^{r+1}), \chi; \mathcal{O})$ with U_p-operator in it. Then $e := \lim_{n\to\infty} U_p^{n!}$ exists and is an idempotent in $H_k(\Gamma_0(Np^{r+1}), \chi; \mathcal{O})$. It is called the <u>Hida ordinary projector</u>.

Therefore we have a projection

$$H_k(\Gamma_0(Np^{r+1}), \chi, \mathcal{O}) \simeq \underbrace{e H_k(\Gamma_0(Np^{r+1}), \chi, \mathcal{O})}_{\substack{\| \\ H_k^{ord}(\Gamma_0(Np^{r+1}), \chi, \mathcal{O})}} \times (1-e) H_k(\Gamma_0(Np^{r+1}), \chi, \mathcal{O})$$

$H_k^{ord}(\Gamma_0(Np^{r+1}), \chi, \mathcal{O})$ <u>ordinary part</u> of the Hecke algebra as the largest direct summand (as an $\mathcal{O}$-algebra) of $H_k(\Gamma_0(Np^{r+1}), \chi, \mathcal{O})$ in which the image of U_p is a unit.

- As $e \in H_k(\Gamma_0(Np^{r+1}), \chi, \mathcal{O})$, it acts on modular forms. We hence form subspaces

$$M_k^{ord}(\Gamma_0(Np^{r+1}), \chi, \mathcal{O}) := M_k(\Gamma_0(Np^{r+1}), \chi, \mathcal{O}) \big| e .$$

- Similarly we define h_k^{ord}, m_k^{ord}, S_k^{ord}.

(1.3) Duality theorem

Recall that for any $\mathbb{Z}[\chi]$-algebra A as a subalgebra of $\mathbb{C}$ or $\overline{\mathbb{Q}_p}$, the pairing

$$\langle -,- \rangle : H_k(\Gamma_0(N), \chi; A) \times m_k(\Gamma_0(N), \chi; A) \longrightarrow A$$

(here N is <u>not</u> nec. coprime to p) given by $\langle h, f \rangle = a(1, f|h) \in A$ is perfect, where

$$m_k(\Gamma_0(N), \chi, A) := \{ f \in M_k(\Gamma_0(N), \chi, \mathrm{Frac}(A)) : a(n,f) \in A, \ \forall n \geq 1 \}$$

This also holds where replacing H_k by h_k and m_k by S_k.

Moreover, it still holds when restricting to the ordinary part of H_k and m_k (or h_k and S_k).

<u>b/c</u>: One notes $\underbrace{\langle T, f|e \rangle = a(1, f|eT)}_{\text{using } (f|e)|T = f|eT} = \langle eT, f \rangle$. $\square$

(1.4) Ordinary form

Let $f \in M_k(\Gamma_0(Np^{r+1}), \chi; \mathcal{O})$ be a Hecke eigenform, with $f|U_p = \lambda_p f$. Then one checks:

$$f|e = \begin{cases} f & \text{if } |\lambda_p| = 1 \\ 0 & \text{if } |\lambda_p| < 1. \end{cases}$$

<u>note</u>:
- The only possibility is that $|a_p| \leq 1$, since we know $a(p,f) = a(1,f) \cdot \lambda_p \in \mathcal{O}$ and $a(1,f) \in \mathcal{O}$.
- Then as $f|U_p = \lambda_p f$, we see $f|e = \left(\lim\limits_{n \to \infty} \lambda_p^{n!} \right) f$. Then recall the proof of (1.1) to see what happens.

When $f|e = f$, we call f a <u>p-ordinary eigenform</u>. By the above computation, we see:

- for eigenform f, f is p-ordinary $\Longleftrightarrow$ λ_p is a p-adic unit.
- for normalized eigenform, f is p-ordinary $\Longleftrightarrow$ $\underbrace{a(p,f) \text{ is a p-adic unit}}_{\text{recall: Talk 1 in [illegible]}}$.

§2 Cohomological preparations

(2.1) Definitions and properties

- Torsion-free condition : Let $\Gamma \subseteq SL_2(\mathbb{Z})$ be a congruence subgroup. Γ is called torsion-free if $\forall \gamma \in \Gamma$, $\forall n \geq 1$, $\gamma^n = 1$ implies $\gamma = 1$.
  - * Γ is torsion-free $\Longleftrightarrow$ $-1 \notin \Gamma$ and Γ has no elliptic points. ([李, 命题1.3.9])
  - * Example : $N \geq 3$, $\Gamma(N)$ / $N \geq 4$, $\Gamma_1(N)$ are torsion-free.

    i.e. for "sufficiently" deep congruence subgroups, the condition is satisfied.
  - * Non-example : $\Gamma_0(N)$ always has torsions since $-1 \in \Gamma_0(N)$ for any $N \geq 1$.

    So we need a way to get around this!
- Let $Y(N) \subseteq X(N)$ be the modular curve, with $S := X(N) \setminus Y(N)$ the finite set of cusps.
- Let $\Gamma_s := \{\gamma \in \Gamma : \gamma(s) = s\}$ be the stablizer of s in Γ.
- Let M be a left $R[\Gamma]$-module, where R is a commutative ring.

Then we can define the parabolic cohomology:

Defn : The parabolic cohomology group $H^1_p(\Gamma, M)$ is defined as

$$H^1_p(\Gamma, M) := \ker\Big(H^1(\Gamma, M) \xrightarrow{\text{res}} \bigoplus_{s \in S} H^1(\Gamma_s, M)\Big)$$

One can describe this in terms of cocycles and coboundries, see Zhao Bin's note.

Vanishing result : For torsion-free congruence subgroup $\Gamma \subseteq SL_2(\mathbb{Z})$ and any Γ-module M,
$H^2(\Gamma, M) = 0$

Proof : See [Hida93, Prop.6.1.1], it uses modular curves and its triangularization. Quite complicated! □

(2.2) Hecke action

Let $\Gamma = \Gamma_1(N)$. For $\alpha \in \Gamma$, set $\Gamma_\alpha := \Gamma \cap \alpha^{-1}\Gamma\alpha$, $\Gamma^\alpha := \Gamma \cap \alpha\Gamma\alpha^{-1}$.

Let $\Delta_1^n := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}) \;\middle|\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix} \bmod N,\ ad-bc=n \right\}$

Let $\Delta := \Delta_1^n = \bigcup_{n\geq 1} \Delta_1^n(N)$.

<u>Fact</u> : $[\Gamma : \Gamma_\alpha]$ and $[\Gamma : \Gamma^\alpha]$ are finite. [Somewhere in 李文威]

<u>Shimura involution</u> : for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z})_{\neq 0}$, define $\alpha^\iota := \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

<u>Hecke action</u> : For $\alpha \in \Delta$, define the action of α on $H^1(\Gamma, \alpha)$ by T_α :

$$\begin{tikzcd} H^1(\Gamma, V) \arrow[r, "\mathrm{Res}"] \arrow[ddr, "T_\alpha"'] & H^1(\Gamma^\alpha, V) \arrow[d, "\mathrm{conj}_\alpha"] \\ & H^1(\Gamma_\alpha, V) \arrow[d, "\mathrm{Cores}"] \\ & H^1(\Gamma, V) \end{tikzcd}$$

$$c \longmapsto \mathrm{conj}_\alpha(c) : \Gamma_\alpha \longrightarrow V,\quad g_\alpha \longmapsto \alpha^\iota \cdot c(\alpha g_\alpha \alpha^{-1})$$

Here we require that the Γ-action extends to a semigroup action by the semigroup $\{\alpha^\iota : \alpha \in \Delta\}$.

- <u>Check</u> : T_α also restricts to an operator on the parabolic subspace.
- <u>Explicit expression</u> : Suppose $\Gamma\alpha\Gamma = \bigsqcup_{i=1}^n \Gamma\delta_i$. Then for $c \in H^1(\Gamma, V)$, $g \in \Gamma$.

$$(T_\alpha(c))(g) = \sum_{i=1}^n \delta_i^\iota \cdot c(\delta_i g \delta_{\sigma_g(i)}^{-1}) \qquad \sigma_g(i) \text{ is the index s.t. } \delta_i g \delta_{\sigma_g(i)}^{-1} \in \Gamma.$$

Following this, one defines the action of α on higher cohomology groups :

$$(T_\alpha(c))(g_1, \dots, g_q) = \sum_{i=1}^n \delta_i^\iota \cdot c(\underbrace{\xi_i(g), \xi_{i(g_1)}(g_2), \dots, \xi_{i(g_1\cdots g_{q-1})}(g_q)}_{\text{we will not explain these terms}})$$

- For a positive integer n, $T_n := \sum \{T_\alpha : \alpha \text{ a set of rep' of } \Gamma\backslash\Delta^n/\Gamma\}$.

  <u>In particular</u>, if p is a prime number, $T_p = T_{\alpha_p}$, where $\alpha_p = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$.

<u>Remark</u> : For details, see [Greise, Stage 7].

(2.3) Eichler–Shimura isomorphism [Giese, Stage 6 & 7]

- $V_n(R) :=$ homogeneous polynomial of degree n with two indeterminates X and Y with coefficient R.

- $\mathrm{Mat}_2(\mathbb{Z})_{\neq 0} := \mathrm{Mat}_2(\mathbb{Z}) \cap GL_2(\mathbb{Q})$ acts on $V_n(R)$ via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} P(X,Y) := P\left((X,Y)\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = P(aX+cY, bX+dY).$$

<u>note</u> : When $n=0$, Γ acts trivially on $V_0(R) \simeq R$.

<u>Nebentypus version</u> : $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \longrightarrow R^\times$ be a Dirichlet character.

- Define R^χ to be the $R[\Gamma_0(N)]$-module R with action

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot r := \chi(d) r$$

for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$.

- Define $V_n^\chi(R) := V_n(R) \otimes_R R^\chi$ with $\Gamma_0(N)$ acts diagonally.

Then for fixed $z_0, z_1 \in \mathfrak{H}$, for $f \in M_k(\Gamma)$ with $k \geq 2$, $g, h \in SL_2(\mathbb{Z})$, define

$$I_f(gz_0, hz_0) := \int_{gz_0}^{hz_0} f(z)(Xz+Y)^{k-2}\, dz \in V_{k-2}(\mathbb{C}).$$

$$I_{\bar{f}}(gz_0, hz_0) := \int_{gz_0}^{hz_0} \overline{f(z)}\,(X\bar{z}+Y)^{k-2}\, d\bar{z} \in V_{k-2}(\mathbb{C}).$$

Then we have

<u>Eicher–Shimura isomorphism</u> : With notations above, the map

$$ES : M_k(\Gamma) \oplus \overline{S_k(\Gamma)} \longrightarrow H^1(\Gamma, V_{k-2}(\mathbb{C}))$$
$$(f, \bar{g}) \longmapsto \left(\gamma \mapsto I_f(z_0, \gamma z_0) + I_{\bar{g}}(z_1, \gamma z_1)\right)$$

is a well-defined isomorphism of $\mathbb{C}$-vector spaces, not depending on the choice of z_0, z_1.

Moreover, ES is Hecke-equivariant.

Furthermore, ES restricts to an isomorphism

$$ES : S_k(\Gamma) \oplus \overline{S_k(\Gamma)} \longrightarrow H^1_p(\Gamma, V_{k-2}(\mathbb{C})).$$

<u>Corollary</u> : For $\Gamma = \Gamma_1(N)$, the map
$$S_k(\Gamma, \mathbb{C}) \longrightarrow H^1_p(\Gamma, V_{k-2}(\mathbb{R}))$$
$$f \longmapsto \left(\gamma \mapsto \mathrm{Re}(I_f(z_0, \gamma z_0))\right)$$

is a Hecke-equivariant isomorphism of $\mathbb{C}$-vector spaces.

§3 Vertical control theorem

<u>Theorem 1</u> : Let $p \geq 3$ be a prime number and $(N,p)=1$. Then the integer $\mathrm{rank}_{\mathbb{Z}_p}(S_k^{ord}(\Gamma_1(Np^{r+1}), \mathbb{Z}_p))$ is bounded indepently of k if $k\geq 2$, $r \geq 0$.

- <u>Notation in the proof</u> : $\Gamma := \Gamma_1(Np^{r+1})$.
- <u>Integral structure</u> :
  - Define $L' := \mathrm{im}\left(H^1(\Gamma, V_{k-2}(\mathbb{Z})) \longrightarrow H^1(\Gamma, V_{k-2}(\mathbb{R}))\right)$
  - Define $L := L' \cap H^1_p(\Gamma, V_{k-2}(\mathbb{R})) \subseteq S_k(\Gamma, \mathbb{C})$ by ES isomorphism

    Then $\mathfrak{h}_k(\Gamma, \mathbb{Z})$ is (by definition) a commitatie $\mathbb{Z}$-subalgebra of $\mathrm{End}_{\mathbb{Z}}(L)$ which is finite free over $\mathbb{Z}$.
  - $L \otimes_{\mathbb{Z}} \mathbb{R} = H^1_p(\Gamma, V_{k-2}(\mathbb{R}))$.
  - $L_p := L \otimes_{\mathbb{Z}} \mathbb{Z}_p$.

    Then $\mathfrak{h}_k(\Gamma, \mathbb{Z}_p) = \mathfrak{h}_k(\Gamma, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}_p$ is a $\mathbb{Z}_p$-subalgebra of $\mathrm{End}_{\mathbb{Z}_p}(L_p)$.

    $\Longrightarrow$ we have $e := \lim_{n\to\infty} U_p^{n!}$ on L_p, hence we can consider

    * $L_p^{ord} := L_p|e$
    * $\mathfrak{h}_k^{ord}(\Gamma, \mathbb{Z}_p) = e\,\mathfrak{h}_k^{ord}(\Gamma, \mathbb{Z}_p) \subseteq \mathrm{End}_{\mathbb{Z}_p}(L_p^{ord})$.

- <u>Turn the problem into a cohomological problem</u>

  ① By duality, $\mathrm{rank}_{\mathbb{Z}_p}(S_k^{ord}(\Gamma, \mathbb{Z}_p)) = \mathrm{rank}_{\mathbb{Z}_p}(\mathfrak{h}_k^{ord}(\Gamma, \mathbb{Z}_p)) \leq \mathrm{rank}_{\mathbb{Z}_p}(\mathrm{End}_{\mathbb{Z}_p}(L_p^{ord}))$.

  (<u>note</u> : $\mathfrak{h}_k^{ord}(\Gamma, \mathbb{Z}_p)$ is a subalgebra of $\mathrm{End}_{\mathbb{Z}_p}(L_p^{ord})$)

  Now as $\mathrm{rank}_{\mathbb{Z}_p}(\mathrm{End}_{\mathbb{Z}_p}(L_p^{ord})) \overset{?}{=} \mathrm{rank}_{\mathbb{Z}_p}(L_p^{ord})^2$ (at least it holds once L_p^{ord} is free of finite rank over $\mathbb{Z}_p$?), we see it suffines to bound $\mathrm{rank}_{\mathbb{Z}_p}(L_p^{ord})$.

  ②
  - $L/pL \simeq L_p/pL_p$
  - $L/pL \hookrightarrow L'/pL'$
  - $H^1(\Gamma, V_k(\mathbb{Z})) \otimes \mathbb{Z}/p\mathbb{Z} \twoheadrightarrow L'/pL'$

  $\Longrightarrow \mathrm{rank}_{\mathbb{Z}_p}(L_p) \leq \mathrm{rank}_{\mathbb{F}_p}\left(H^1(\Gamma, V_k(\mathbb{Z})) \otimes_{\mathbb{Z}} \mathbb{F}_p\right)$

  $\overset{ord}{\Longrightarrow} \mathrm{rank}_{\mathbb{Z}_p}(L_p^{ord}) \leq \mathrm{rank}_{\mathbb{F}_p}(H^1(\Gamma, V_k(\mathbb{Z})) \otimes_{\mathbb{Z}} \mathbb{F}_p)$

③ Consider the SES of Γ-modules (write $n := k-2$)

$$0 \longrightarrow V_n(\mathbb{Z}) \xrightarrow{\times p} V_n(\mathbb{Z}) \longrightarrow V_n(\mathbb{F}_p) \longrightarrow 0$$

It derives $\cdots \longrightarrow H^1(\Gamma, V_n(\mathbb{Z})) \xrightarrow{\times p} H^1(\Gamma, V_n(\mathbb{Z})) \longrightarrow H^1(\Gamma, V_n(\mathbb{F}_p)) \longrightarrow \cdots$

This implies $H^1(\Gamma, V_n(\mathbb{Z})) \otimes_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z}$ can be embedded in $H^1(\Gamma, V_n(\mathbb{F}_p))$.

So combining ①, ②, ③, STS: $\dim_{\mathbb{F}_p} H^1_{ord}(\Gamma, V_n(\mathbb{F}_p))$ is bounded indep. of n.

- Construct an isomorphism $\Phi : H^1_{ord}(\Gamma, V_n(\mathbb{F}_p)) \xrightarrow{\Phi} H^1_{ord}(\Gamma, \mathbb{F}_p)$.

- Start with $\varphi : V_n(\mathbb{F}_p) \longrightarrow \mathbb{F}_p$. One checks
$$P(x,y) \longmapsto P(0,1).$$

① φ is a Γ-module homomorphism

b/c : Note Γ acts trivially on $\mathbb{F}_p$, NTS

$$\varphi(\gamma . P) = (\gamma P)(0,1) = P(0,1), \quad \forall \gamma \in \Gamma.$$

To show this, note:

* For all $\gamma \in \Gamma$, $\gamma \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \bmod p$

* As $\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} P(X,Y) = \sum_{i=0}^{n} a_i X^{n-i}(mX+Y)^i$
$$= \sum_{i=0}^{n} a_i X^{n-i}(mX+Y)^i + a_n (mX+Y)^n,$$

we see $\left(\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} P \right)(0,1) = a_n Y^n = P(0,1)$. #

② $\ker\varphi = \langle X^{n-i} Y^i \mid i = 0, \cdots, n-1 \rangle_{\mathbb{F}_p}$ follows from the above computation.

- Then it induces $\Phi : H^1(\Gamma, V_n(\mathbb{F}_p)) \longrightarrow H^1(\Gamma, \mathbb{F}_p)$, given by $c \longmapsto \varphi \circ c$.

It fits in the long exact sequence

(*) — $\cdots \longrightarrow H^1(\Gamma, \ker\varphi) \longrightarrow H^1(\Gamma, V_n(\mathbb{F}_p)) \xrightarrow{\Phi} H^1(\Gamma, \mathbb{F}_p) \xrightarrow{\delta} H^2(\Gamma, \ker\varphi) \longrightarrow \cdots$

- Now comes the Hecke action : $\alpha_p := \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$

  * α_p^ι leaves $\ker\varphi \subseteq V_n(\mathbb{F}_p)$ invariant. Hence $T_p \curvearrowright H^j(\Gamma, \ker\varphi)$, $\forall j \geq 0$.

  * Check : $\alpha_p^\iota \cdot (X^{n-i} Y^i) = (pX)^{n-i} Y^i = 0$ in char p, when $i = 0, \cdots, n-1$.

    So the action of α_p^ι kills $\ker\varphi$.

  * Compute : $\Gamma \alpha_p \Gamma = \bigsqcup_{i=0}^{p-1} \Gamma \alpha_p \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix}$. $\delta_i := \alpha_p \cdot \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix}$.

    By explicit action on cocycles, we see that T_p kills $H^j(\Gamma, \ker\varphi) = 0$ for $j > 0$.

- So note "taking ordinary parts" is exact $\cdot\times\cdot$

  Taking "ord" along $(*)$ gives $\Phi_{ord} : H^1_{ord}(\Gamma, V_n(\mathbb{F}_p)) \xrightarrow{\sim} H^1_{ord}(\Gamma, \mathbb{F}_p)$, as desired.

$\square$

<u>Theorem 1'</u> Let $p \geq 3$, $(N,p) = 1$. $\chi : (\mathbb{Z}/Np)^\times \to \mathcal{O}_F^\times$ where $F | \mathbb{Q}_p$ finite extension

Then

$$\mathrm{rank}_{\mathcal{O}_F}\left(M_k^{ord}(\Gamma_1(Np), \chi\omega^{-k}, \mathcal{O}_F) \right) \text{ and } \mathrm{rank}_{\mathcal{O}_F}\left(S_k^{ord}(\Gamma_1(Np), \chi\omega^{-k}, \mathcal{O}_F) \right)$$

are bounded independently of k if $k \geq 2$.

<u>pf</u> :

- <u>Cusp form case</u> : follows from <u>Thm 1</u> as $S_k(\Gamma_1(Np), \mathcal{O}_F) \simeq S_k(\Gamma_1(Np), \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathcal{O}_F$
- <u>Modular form case</u> : add the "Eisenstein part" $=$ # of cusps of $\Gamma_1(Np)$, which is independent of k.

$\square$

<u>Theorem 2</u> (Vertical control theorem) Let $p \geq 5$, $(N,p)=1$. $\chi : (\mathbb{Z}/Np)^\times \to O_F^\times$ where $F|\mathbb{Q}_p$ finite extension. Then for all $k \geq 2$,

$$\mathrm{rank}_{O_F}\left(M_k^{\mathrm{ord}}(\Gamma_0(Np), \chi\omega^{-k}, O_F)\right) = \mathrm{rank}_{O_F}\left(M_2^{\mathrm{ord}}(\Gamma_0(Np), \chi\omega^{-2}, O_F)\right)$$

and

$$\mathrm{rank}_{O_F}\left(S_k^{\mathrm{ord}}(\Gamma_0(Np), \chi\omega^{-k}, O_F)\right) = \mathrm{rank}_{O_F}\left(S_2^{\mathrm{ord}}(\Gamma_0(Np), \chi\omega^{-2}, O_F)\right).$$

- <u>Level shifting trick</u>:
  - <u>Problem</u>: $H^2(\Gamma, M) = 0$ holds for all torsion-free $\Gamma \leq SL_2(\mathbb{Z})$.
    Yet here $\Gamma_0(Np) \ni \begin{pmatrix} -1 & \\ & -1 \end{pmatrix}$, hence not torsion-free!
  - Let M be any O_F-module with $\Gamma_0(Np)$-action.
    We find a subgroup $\Gamma \leq \Gamma_0(Np)$ such that
    1° $\Gamma \trianglelefteq \Gamma_0(Np)$ is torsion-free, normal subgroup
    2° Γ is a finite index subgroup of $\Gamma_0(Np)$, with $p \nmid [\Gamma_0(Np) : \Gamma]$.

    then via restriction & corestriction map.

$$\begin{tikzcd} H^j(\Gamma_0(Np), M) \arrow[r, "\mathrm{res}"] \arrow[rr, bend right, "\times [\Gamma_0(Np):\Gamma]"'] & H^j(\Gamma, M) \arrow[r, "\mathrm{cores}"] & H^j(\Gamma_0(Np), M) \end{tikzcd}$$

[李二, 命题 6.8.5]

   - By $\boxed{2°}$, the composition is an isomorphism.
   - On the other hand, by $\boxed{1°}$, we know $H^2(\Gamma, M) = 0$. Hence the composition is zero when $j=2$.

   Therefore, $H^2(\Gamma_0(Np), M) = 0$, as desired.
  - Such a subgroup Γ of $\Gamma_0(Np)$ exists: Take $\Gamma = \Gamma_1(p) \cap \Gamma_0(Np)$.
    - When $p \geq 5$, $\Gamma_1(p)$ is torsion-free, hence so does Γ.
    - Γ is actually the kernel of the well-defined projection

$$\Gamma_0(Np) \twoheadrightarrow (\mathbb{Z}/p\mathbb{Z})^\times$$
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto d \bmod p.$$

   Hence $\Gamma \trianglelefteq \Gamma_0(Np)$ and $[\Gamma_0(Np) : \Gamma] = p-1$ coprime to p. $\square$

- <u>Setup</u> : – Fix a Dirichlet character ψ mod Np with value in $\mathcal{O}_F^\times$.
  – Let $\varpi \in \mathcal{O}_F$ be a uniformizer, $\mathbb{F}$ residue field ($\mathbb{F}|\mathbb{F}_p$ finite).

<u>Notation</u> : Denote $m_k := \mathrm{rank}_{\mathcal{O}_F}\left(M_k^{\mathrm{ord}}(\Gamma_0(Np), \chi\omega^{-k}, \mathcal{O}_F)\right)$

$s_k := \mathrm{rank}_{\mathcal{O}_F}\left(S_k^{\mathrm{ord}}(\Gamma_0(Np), \chi\omega^{-k}, \mathcal{O}_F)\right)$

- <u>Cohomological interpretation</u> : we show independently $m_k + s_k$ and $m_k - s_k$ are independent of $k \geq 2$. Hence so do m_k and s_k.

① By <u>Eichler–Shimura isomorphism</u>, <u>base change to $\mathcal{O}_F$</u>, (<u>handwaving</u>)

$$m_k + s_k = \mathrm{rank}_{\mathcal{O}_F}\left(H^1_{\mathrm{ord}}(\Gamma_0(Np), V_{k-2}^{\chi\omega^{-k}}(\mathcal{O}_F)\right).$$

The main task below is to show that RHS is independent of $k \geq 2$.

② <u>SLOGAN</u> : The difference $M_k \setminus S_k$ is essentially the Eisenstein series.

<u>Denote</u> :
$$\mathcal{E}_k^{\mathrm{ord}}(\Gamma_0(Np), \chi\omega^{-k}, \mathcal{O}_F) := \frac{M_k^{\mathrm{ord}}(\Gamma_0(Np), \chi\omega^{-k}, \mathcal{O}_F)}{S_k^{\mathrm{ord}}(\Gamma_0(Np), \chi\omega^{-k}, \mathcal{O}_F)},$$

called the space of <u>ordinary Eisenstein series</u>.

<u>Calculation by Hida</u> (1986 Inventiones §5) : $\mathrm{rank}_{\mathcal{O}_F}(\mathcal{E}_k^{\mathrm{ord}})$ is independent of $k \geq 2$.

- <u>We claim</u> :

$$(\star) \text{ — } \mathrm{rank}_{\mathcal{O}_F}\left(H^1_{\mathrm{ord}}(\Gamma_0(Np), V_n^\psi(\mathcal{O}_F)\right) = \dim_{\mathbb{F}}\left(H^1_{\mathrm{ord}}(\Gamma_0(Np), V_n^\psi(\mathbb{F}))\right), \quad n \geq 0$$

$\boxed{\text{Step 1}}$ For $n \geq 0$, consider the SES of $\mathcal{O}_F[\mathrm{Mat}_2(\mathbb{Z})_{\neq 0}]$–module

$$(\star\star) \text{ — } \quad 0 \to V_n^\psi(\mathcal{O}_F) \xrightarrow{\times\varpi} V_n^\psi(\mathcal{O}_F) \longrightarrow V_n^\psi(\mathbb{F}) \to 0$$

deriving long exact sequence

$$\cdots \to H^1(\Gamma_0(Np), V_n^\psi(\mathcal{O}_F)) \xrightarrow{\times\varpi} H^1(\Gamma_0(Np), V_n^\psi(\mathcal{O}_F)) \longrightarrow H^1(\Gamma_0(Np), V_n^\psi(\mathbb{F})) \xrightarrow{\delta} 0$$

Here we use $H^2(\Gamma_0(Np), V_n^\psi(\mathcal{O}_F)) = 0$ as seen at the very beginning. Hence

$$H^1(\Gamma_0(Np), V_n^\psi(\mathcal{O}_F)) \otimes_{\mathcal{O}_F} \mathbb{F} \xrightarrow{\sim} H^1(\Gamma_0(Np), V_n^\psi(\mathbb{F})).$$

Taking ordinary part, we see

$$H^1_{\mathrm{ord}}(\Gamma_0(Np), V_n^\psi(\mathcal{O}_F)) \otimes_{\mathcal{O}_F} \mathbb{F} \xrightarrow{\sim} H^1_{\mathrm{ord}}(\Gamma_0(Np), V_n^\psi(\mathbb{F})).$$

Hence to show $(\star)$, STS : $H^1_{\mathrm{ord}}(\Gamma_0(Np), V_n^\psi(\mathcal{O}_F))$ is a free $\mathcal{O}_F$–module.

Equivalently, show $H^1_{\mathrm{ord}}(\Gamma_0(Np), V_n^\psi(\mathcal{O}_F))$ is ϖ–torsion free.

$\boxed{\text{Step 2}}$ Derive (☆☆) again :

$$\cdots \to H^0(\Gamma_0(Np), V_n^{\psi}(\mathbb{F})) \xrightarrow{\delta} H^1(\Gamma_0(Np), V_n^{\psi}(\mathcal{O}_F))[\varpi] \to 0$$

Taking ordinary part ,

$$H^0_{ord}(\Gamma_0(Np), V_n^{\psi}(\mathbb{F})) \xrightarrow{\delta} H^1_{ord}(\Gamma_0(Np), V_n^{\psi}(\mathcal{O}_F))[\varpi] \to 0$$

So STS : $H^0_{ord}(\Gamma_0(Np), V_n^{\psi}(\mathbb{F})) = 0$.

- We do this by examing explicitly the T_p-action : for $i = 0, \cdots, n$,

$$X^{n-i}Y^i \mid T_p = \sum_{j=0}^{p-1} \begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix}^{\iota} \cdot X^{n-i}Y^i = \sum_{j=0}^{p-1} (pX)^{n-i}(-jX+Y)^i$$

① For $i = 0, \cdots, n-1$, $X^{n-i}Y^i \mid T_p = 0$, as we are in the char p world

② For $i = n$, $Y^n \mid T_p = \sum_{j=0}^{p-1} (Y-jX)^n$

$$= Y^n + (Y-X)^n + \cdots + (Y-(p-1)X)^n$$

$$= pY^n + (\text{lower terms in } Y)$$

has no Y^n involved in the char p world , we see $Y^n \mid T_p^2 = 0$.

Hence indeed $H^0_{ord}(\Gamma_0(Np), V_n^{\psi}(\mathbb{F})) = 0$.

- Now we have proved (☆) , we work on $\dim_{\mathbb{F}}(H^1_{ord}(\Gamma_0(Np), V_n^{\psi}(\mathbb{F})))$.

<u>Final claim</u> :

$$\dim_{\mathbb{F}}(H^1_{ord}(\Gamma_0(Np), V_n^{\psi}(\mathbb{F}))) = \dim_{\mathbb{F}}(H^1_{ord}(\Gamma_0(Np), V_0^{\psi\omega^n}(\mathbb{F})))$$

Granting the claim , we apply :

- $n = k-2$ for $k \geq 2$.
- $\psi = \chi\omega^{-k}$

we see $\dim_{\mathbb{F}}(H^1_{ord}(\Gamma_0(Np), V_{k-2}^{\chi\omega^{-k}}(\mathbb{F}))) = \dim_{\mathbb{F}}(H^1_{ord}(\Gamma_0(Np), V_0^{\chi\omega^{-2}}(\mathbb{F})))$

- To show the <u>final claim</u> , we do the same thing as in the proof of <u>theorem 1</u> :
  - Consider $\varphi : V_n^{\psi}(\mathbb{F}) \twoheadrightarrow V_0^{\psi\omega^n}(\mathbb{F})$, $P(X,Y) \mapsto P(0,1)$
    <u>Check</u>: φ is a morphism of $\mathbb{F}[\Gamma_0(Np)]$-module .
  - Deriving long exact sequence from

$$0 \to \ker\varphi \to V_n^{\psi}(\mathbb{F}) \twoheadrightarrow V_0^{\psi\omega^n}(\mathbb{F}) \to 0$$

and show $H^i_{ord}(\Gamma_0(Np), \ker\varphi) = 0$ for $i = 1, 2$. (Again use vanishing result)

□

§4 Ordinary Λ-adic forms

<u>Step 1</u> : Define an appropriate Hida projector $e \in \mathrm{End}_{\Lambda_F}(M(N,\chi,\Lambda_F))$.

<u>Step 2</u> : Define $M^{\mathrm{ord}}(N,\chi,\Lambda_F) := M(N,\chi,\Lambda_F)|e$: the space of ordinary Λ-adic forms.

<u>Step 3</u> : (Main theorem, Wiles) The Λ-modules $M^{\mathrm{ord}}(N,\chi,\Lambda_F)$ and $S^{\mathrm{ord}}(N,\chi,\Lambda_F)$ are Λ_F-free of finite rank.

For details, see Hida's blue book or Marannino's note Chapter 2.