Talk 3 Control Theorems

§1 Iwasawa suitable specializations

* Note that there are various formulation of Iwasawa suitable specializations (eg: [Fouquet-Wan]. [Fouquet, 2024 Besançon]. [Wan 2024 ICTS]), we take the one in [Fouquet's Besançon article].

Recall by Nakamura, we have a universal zeta morphism

$$z_{\Sigma}^{univ}: T_{\Sigma}^{univ}(-1)^{+} \longrightarrow H^1_{et}\left(\mathbb{Z}[\tfrac{1}{\Sigma}], T_{\Sigma}^{univ}\right)$$

that interpolates Kato's zeta element.

Define : Let $\lambda: R_{\Sigma}(\bar{\rho}) \longrightarrow A$ be a specialization to a reduced ring A, then it is called Iwasawa suitable if $\mathrm{Cone}(z_{\Sigma,\lambda}) \overset{\mathbb{L}}{\underset{A}{\otimes}} Q(A)$ is acyclic.

(A) Tamagawa number conjecture : We consider $M_{\chi}(r)$ is strictly critical then automatically for the specialization $\lambda_{\chi,r}: R_{\Sigma}(\bar{\rho}) \longrightarrow \mathcal{O}_{\chi}$, the zeta morphism automatically gives an isomorphism between E_{χ}-vector spaces

$$z_{\chi,r} := z_{\lambda_{\chi,r}}: V_{\chi}(r-1)^{+} \xrightarrow{\sim} H^1_{et}\left(\mathbb{Z}[\tfrac{1}{\Sigma}], V_{\chi}(r)\right)$$

Hence $\mathrm{Cone}(z_{\chi,r}) \otimes_{\mathcal{O}_{\chi}} E_{\chi}$ is acyclic. (recall : $\mathrm{Cone}(f)$ is acyclic $\Longleftrightarrow$ f is a q.is.)

(B) Iwasawa main conjecture : Date back to the computation of $\mathrm{Cone}(f)$, it suffices to check : (naively, they are torsion, so $- \underset{\Lambda_{Iw}}{\otimes} \mathrm{Frac}(\Lambda_{Iw})$ kills them)

$$H^1_{et}\left(\mathbb{Z}[\tfrac{1}{\Sigma}], T_{f,Iw}\right) / \mathrm{im}\, z^{Kato}_{f,Iw} \quad \text{and} \quad H^2_{et}\left(\mathbb{Z}[\tfrac{1}{\Sigma}], T_{f,Iw}\right)$$

are torsion Λ_{Iw}-modules. The torsionness follows from [Kato, Astérisque, Theorem 12.4].

(1) Note : The cohomology of $R\Gamma_{et}(\mathbb{Z}[\tfrac{1}{\Sigma}], T_{f,Iw})$ is concentrated in deg $[1,2]$.

(2) To apply [Astérique], one may need to note the difference that

here we're using Σ-imprimitive H^1_{et} but in loc.cit, Kato uses primitive ones. This difference should be taken care of!

(3) In [Astérisque], it is proved that H^2_{et} is a torsion Λ_{I_w}-module. For H^1_{et}, if $p \neq 2$ and $\overline{T}_f$ is irred, then H^1_{et} is a free Λ_{I_w}-mod of rank one. Since $\mathrm{im}\, z^{Kato}_{f,I_w}$ is non-trivial, the quotient is then a torsion Λ_{I_w}-mod.

(C) Universal Iwasawa main conjecture: Similar to (B), it suffices to show

$$H^1_{et}\left(\mathbb{Z}[\tfrac{1}{\Sigma}], T^{univ}_{\Sigma}\right) / \mathrm{im}\, z^{univ}_{\Sigma} \quad \text{and} \quad H^2_{et}\left(\mathbb{Z}[\tfrac{1}{\Sigma}], T^{univ}_{\Sigma}\right)$$

are torsion $R_\Sigma(\overline{\rho})$-modules. The proof is by reducing to (B): the canonical isomorphism

$$\boxed{R\Gamma_{et}\left(\mathbb{Z}[\tfrac{1}{\Sigma}], T_\Sigma\right) \overset{\mathbb{L}}{\underset{R_\Sigma(\overline{\rho}),\,x}{\otimes}} \Lambda_{I_w} \xrightarrow[\simeq]{can} \underbrace{R\Gamma_{et}\left(\mathbb{Z}[\tfrac{1}{\Sigma}], T_{f,I_w}\right)}_{\text{case of (B)}}} \qquad (*)$$

By the result of (B) and "Nakayama's lemma", we see that:

- $R\Gamma_{et}\left(\mathbb{Z}[\tfrac{1}{\Sigma}], T_\Sigma\right)$ is concentrated in degree $[1,2]$.
- $H^2_{et}\left(\mathbb{Z}[\tfrac{1}{\Sigma}], T_\Sigma\right)$ is torsion
- $H^1_{et}\left(\mathbb{Z}[\tfrac{1}{\Sigma}], T_\Sigma\right)$ is generated by at most one element.

Hence to see $H^1_{et}/\mathrm{im}\, z_\Sigma$ is torsion, it suffices to a modular point λ_{f,I_w}, where the result follows.

Question : How many Iwasawa suitable specializations are there?

Theorem ✫ ([FW, Prop. 2.6 & 3.16])

The set of specializations $\lambda : R_\Sigma(\bar{\rho}) \to \mathcal{O}$ such that

- one of the points on the fibre of λ is not Iwasawa suitable, or
- $\operatorname{Spec} R_\Sigma(\bar{\rho})$ is not étale over Λ at λ

is of large codimension. (understand as : there are very few bad specializations.)

We shall see from §2.2 that étaleness is crucial, but our original point of interest may not be étale. This does not matter because of Theorem ✫ : we can approximate the original point by "good specializations" though itself maybe bad.

§2 Control theorem

We start with a warm up :

§2.0 Warm up

We prove the following :

Theorem : IMC for f implies TNC for all integers $1 \leq r \leq k-1$ and $\chi \in \widehat{G_n}$ such that $M_\chi(r)$ is strictly critical. (In [Fouquet24, Prop. 3.9], this condition is missed)

Proof : This boils down (chasing diagram) to the commutativity of the diagram

$$
\begin{tikzcd}
\Delta(T_{f,Iw}) \arrow[rr, "Z_{f,Iw}", "\sim"'] \arrow[d, "- \otimes_{\Lambda_{Iw,\chi}} \mathcal{O}_\chi"'] & & \Lambda_{Iw} \arrow[d, "\chi"] & \boxed{\text{IMC}} \\
\Delta(T_{f,\chi,r}) \arrow[rr, "Z_{f,\chi}(r)", "?\cong"'] & & \mathcal{O}_\chi & \boxed{\text{TNC}}
\end{tikzcd}
$$

with the left-hand-side vertical arrow being the isomorphism

$$\Delta(T_{f,Iw}) \otimes_{\Lambda_{Iw,\chi}} \mathcal{O}_\chi \xrightarrow[\text{can}]{\sim} \Delta(T_{f,\chi,r}).$$

induced from the canonical isomorphism of complexes

$$R\Gamma_{et}(\mathbb{Z}[\tfrac{1}{\Sigma}], T_{f,Iw}) \overset{\mathbb{L}}{\otimes}_{\Lambda_{Iw,\chi}} \mathcal{O}_\chi \xrightarrow{\sim} R\Gamma_{et}(\mathbb{Z}[\tfrac{1}{\Sigma}], T_{f,\chi}(r))$$

This latter " $\xrightarrow{\sim}$ " comes from the universal coefficient theorem of étale cohomology. (To apply UCT here, we crucially use that $T_{f,Iw}$ is a perfect complex of étale sheaves of Λ_{Iw}-modules on $\operatorname{Spec} \mathbb{Z}[\frac{1}{\Sigma}]$.) $\square$

This plays no role in the proof of Thm A actually, but it shows a prototype of the technique. This can be regarded as a "control theorem", and we want to formulate and generalize it to the universal deformation spaces.

§2.1 Fouquet-Wan control theorem

<u>Definition</u>: Let $\lambda : R_\Sigma(\bar{\rho}) \longrightarrow A$ and $\psi : R_\Sigma(\bar{\rho}) \longrightarrow B$ be two specializations.
We say <u>λ contains ψ</u> (write $\lambda \geq \psi$) if $\exists$ a morphism of local rings $\psi : A \to B$ s.t. the diagram

$$\begin{tikzcd} R_\Sigma(\bar{\rho}) \arrow[r, "\psi"] \arrow[d, "\lambda"'] & B \\ A \arrow[ur, "\psi"'] & \end{tikzcd}$$

(sometimes we denote $\psi : A \to B$ and do not assign a notation for $R_\Sigma(\bar{\rho}) \to B$.)

commutes.

The theorem in §2.0 is a typical example: $\lambda_{f,Iw}$ contains $\lambda_{f,\chi,r}$, then the validity of IMC implies TNC. Other examples are:

- universal specialization $id : R_\Sigma(\bar{\rho}) \to R_\Sigma(\bar{\rho})$ contains any specialization.
- Iwasawa-theoretic specialization $\lambda_{f,Iw} : R_\Sigma(\bar{\rho}) \to \Lambda_{Iw}$ contains the modular point $\lambda_f : R_\Sigma(\bar{\rho}) \to \mathcal{O}$.

For such containment, we have the following control theorem:

<u>Theorem</u>: Let $\lambda \geq \psi$ be two Iwasawa suitable specializations.
Let
- $(x,y) \in A^2$ be such that the image of $\mathrm{triv}_{z_\lambda}$ is equal to $\frac{x}{y}A$
- $(x',y') \in B^2$ ——— triv_{z_ψ} ——— $\frac{x}{y}B$

Then the canonical isomorphism

$$\Delta_\Sigma(T_\lambda) \otimes_{A,\psi} B \xrightarrow{\sim} \Delta_\Sigma(T_\psi)$$

fits into the commutative diagram:

$$\begin{tikzcd} \Delta_\Sigma(T_\lambda) \arrow[r, "\mathrm{triv}_{z_\lambda}"] \arrow[d, "- \otimes_{A,\psi} B"'] & \frac{x}{y}A \arrow[d, "\psi"] \\ \Delta_\Sigma(T_\psi) \arrow[r, "\mathrm{triv}_{z_\psi}"] & \frac{x'}{y'}B \end{tikzcd}$$

In particular, the morphism ψ extends to $x/_y A \to x'/_y B$.

Proof: Let $\mathfrak{p} := \ker(A \xrightarrow{\psi} B)$. $\mathrm{Frac}(A) = \mathrm{Frac}(A_\mathfrak{p}) = A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p} =: k(\mathfrak{p})$.

(i) The complex $R\Gamma_{\acute{e}t}(\mathbb{Z}[\frac{1}{\Sigma}], T_\lambda \otimes_A A_\mathfrak{p})$ is perfect & commutes with $- \overset{\mathbb{L}}{\otimes}_A k(\mathfrak{p})$.

(ii) The complex

$$\mathrm{Cone}\left((T_\lambda \otimes_A k(\mathfrak{p}))(-1)^+ \xrightarrow{z_\lambda} R\Gamma_{\acute{e}t}(\mathbb{Z}[\tfrac{1}{\Sigma}], T_\lambda \overset{\mathbb{L}}{\otimes}_A k(\mathfrak{p})) \right)$$

is acyclic since λ is Iwasawa suitable. By (i), and Nakayama's lemma,

$$\mathrm{Cone}\left((T_\lambda \otimes_A A_\mathfrak{p})(-1)^+ \xrightarrow{z_\lambda} R\Gamma_{\acute{e}t}(\mathbb{Z}[\tfrac{1}{\Sigma}], T_\lambda \overset{\mathbb{L}}{\otimes}_A A_\mathfrak{p}) \right)$$

is acyclic.

Then by the functorality of Det, (i) & (ii) give the commutative diagram

(?)
$$\begin{tikzcd}
\Delta_A(T_\lambda) \arrow[r, hook] \arrow[dd, "\psi"] & \Delta_A(T_\lambda \otimes_A A_\mathfrak{p}) \arrow[r, "\cong"', "\text{acyclic property}"] \arrow[d, "- \otimes_{A_\mathfrak{p}} k(\mathfrak{p})"'] & A_\mathfrak{p} \arrow[d, "- \otimes_{A_\mathfrak{p}} k(\mathfrak{p})"] \\
 & \Delta_A(T_A \overset{\mathbb{L}}{\otimes}_A k(\mathfrak{p})) \arrow[r] \arrow[d, "\psi"] & k(\mathfrak{p}) \arrow[d, "\psi"] \\
\Delta_B(T_\psi) \arrow[r, hook] & \Delta_{Q(B)}(T_\psi \otimes Q(B)) \arrow[r, "\cong"] & Q(B)
\end{tikzcd}$$

(ii)

ψ is Iwasawa suitable

functorality of Det.

In particular, if $\mathrm{triv}_{z_\lambda}(\Delta_A(T_\lambda))$ is generated by $\frac{x}{y}$, then y may be chosen such that $y \notin \mathfrak{p}$, so $\psi(x)/\psi(y)$ generates $\mathrm{triv}_{z_\psi}(\Delta_B(T_\psi))$. $\square$

Remark: The commutative diagram above is not exactly the one in the proof of [Fouquet. Besançon, Prop. 3.15], we adapt the diagram following the proof of [F-W, Prop. 3.22].

Application : The universal IMC implies the IMC at modular points.

Proof : We take :

$$
\begin{tikzcd}
R_\Sigma(\bar\rho) \arrow[r, "Z^{Kato}_{f,I_w}"] \arrow[d, "\lambda := id"'] & \Lambda_{I_w} \\
R_\Sigma(\bar\rho) \arrow[ur, "\psi"'] &
\end{tikzcd}
\Longrightarrow
\begin{tikzcd}
\Delta_\Sigma(T_\Sigma^{univ}) \arrow[rr, "triv_{Z_\Sigma^{univ}}", "\sim"'] \arrow[d, "- \otimes_\psi \Lambda_{I_w}"'] \arrow[drr, phantom, "\circlearrowright"] & & R_\Sigma^{univ}(\bar\rho) \arrow[d, "\psi"] \\
\Delta_\Sigma(T_{f,I_w}) \arrow[rr, "triv_{Z_\psi}", "\sim"'] & & \frac{x'}{y'}\Lambda_{I_w} = \Lambda_{I_w}
\end{tikzcd}
\quad (x = y = 1)
$$

The bottom red box is precisely the IMC of f. $\square$

However, this is <u>NOT</u> the UIMC we mean in <u>Theorem A</u>. The UIMC is the "universal IMC in families" involving Λ as the "weight algebra".

§2.2. Proof of Theorem A

Now we prove Theorem A. Recall:

> **Theorem A**: There exists a Zariski-dense open subset $\mathcal{X}^{sm}_\Sigma(\bar\rho) \subseteq \mathcal{X}_\Sigma(\bar\rho)$
> s.t. TFAE:
> (i) The "universal IMC for families" is true.
> (ii) IMC is true for all modular points in $\mathcal{X}_\Sigma(\bar\rho)$.
> (iii) —— " —— in $\mathcal{X}^{sm}_\Sigma(\bar\rho)$
> (iv) IMC is true for the points in a single fiber of a modular point of $\mathcal{X}^{sm}_\Sigma(\bar\rho)[\frac{1}{p}]$.

We divide it into three steps:

(1) z_Σ sends Δ_Λ^{-1} inside Λ. ("upper bound of UIMC in families").

(2) Prove: (iv) $\Rightarrow$ (i), so we only need to show the lower bound.

(3) Prove: (i) $\Rightarrow$ (ii).

$\boxed{\text{Step 1}}$ Assume by contradiction that this is not the case. Let $\lambda : R_\Sigma(\bar\rho) \to \Lambda_{Iw}$ [Prop. 5.5] be an Iwasawa suitable specialization. By control theorem:

$$\begin{tikzcd} \Delta_\Lambda^{-1} \arrow[r, "\mathrm{triv}_{z_\Sigma}"] \arrow[d, "-\otimes_\Lambda \Lambda_{Iw}"'] & \frac{x}{y}\Lambda \arrow[d] \\ \Delta_\Lambda(T_\lambda)^{-1} \arrow[r, "\mathrm{triv}_{z_\lambda}"'] & \frac{x}{y'}\Lambda_{Iw} \end{tikzcd} \qquad , \qquad \frac{x}{y} \notin \Lambda$$

- Λ is a UFD, so $\exists\, \mathfrak{p} \in \mathrm{Spec}\,\Lambda$ of height one s.t. $y \in \mathfrak{p}$, $x \notin \mathfrak{p}$.
  Since Λ is regular, we can take $\mathfrak{p} = (y_0)$ be principal.
- Further adjust λ in two steps:

  $\triangle$ s.t. $\lambda(y_0) \in \mathfrak{m}^n_{\Lambda_{Iw}}$ but $\lambda(x) \notin \mathfrak{m}^n_{\Lambda_{Iw}}$. —— ①

s.t. $\operatorname{Spec} \mathbb{T}^{\Sigma}_{\mathfrak{m}_{\bar{\rho}}}$ is étale over $\operatorname{Spec} \Lambda$ at λ, — ② and

every point in the fiber of λ are Iwasawa suitable ③

This second adjustment is by <u>Theorem</u>$^{☆}$.

- By étaleness ②, (? 为何 étale 起这样的作用，尚不很明确), we have

$$\boxed{T_\lambda \text{ is a lattice inside } \bigoplus_{r=1}^{d} V_{\lambda,r}}$$ along the fibre of λ.

of $\mathrm{Gal}_{\mathbb{Q},\Sigma}$-reps $V_{\lambda,r}$ which are free modules of rank 2 over $\mathrm{Frac}(\Lambda_{I_w})$.

$\rightsquigarrow$ a SES: $\quad 0 \to T_\lambda \to \bigoplus_{r=1}^{d} T_{\lambda,r} \to C \to 0$

torsion Λ_{I_w}-module

Then apply Det-functor:

$$\bigotimes_{r=1}^{d} \Delta_{\Lambda_{I_w}}(T_{\lambda,r}) \xrightarrow{\sim} \Delta_{\Lambda_{I_w}}(T_\lambda) \otimes_{\Lambda_{I_w}} \underbrace{\Delta_{\Lambda_{I_w}}(C)}_{=\text{ unit object } \simeq \Lambda_{I_w}}$$

Hence by our hypothesis:

$$\mathrm{triv}_{z_\lambda}\left(\bigotimes_{r=1}^{d} \Delta^{-1}_{\Lambda_{I_w}}(T_{\lambda,r})\right) \not\subseteq \Lambda_{I_w}. \quad (\text{by } ①)$$

So $\exists\, r_0$ s.t. $z_\lambda\left(\Delta^{-1}_{\Lambda_{I_w}}(T_{\lambda,r_0})\right) \not\subseteq \Lambda_{I_w}$.

- But ③ says λ_{r_0} is Iwasawa suitable. this <u>contradicts to [FW, Prop. 3.29]</u>.

Actually one understands this via:

$$\boxed{\begin{array}{c}\text{the image } \mathrm{triv}_{z_\lambda}\left(\Delta^{-1}_{\Lambda_{I_w}}(T_{\lambda,r_0})\right) \\ \text{is contained inside } \Lambda_{I_w}\end{array}} \; \text{“}\Longleftrightarrow\text{”} \; \boxed{\begin{array}{l}\text{One divisibility} \\ \mathrm{char}_{\Lambda_{I_w}} H^2_{\text{ét}}\left(\mathbb{Z}[\tfrac{1}{\Sigma}], T_{\lambda,I_w}\right) \,\Big|\, \mathrm{char}_{\Lambda_{I_w}} H^1_{\text{ét}} / \mathrm{im}\left(z^{\text{Kato}}_{\lambda,I_w}\right)\end{array}}$$

and the RHS divisibility is precisely the one proved by Kato! $\square$

$\boxed{\text{Step 2}}$ By assumption, $\exists$ a fibre $S_x := \{x_i : 1 \leq i \leq d\}$ of a point $x \in \underline{\mathcal{X}^{sm}_\Sigma(\bar\rho)}$

[Prop. 5.7 & Thm 5.2] s.t. IMC is true at all classical points x_i. By <u>étaleness</u> (missing in [Fouquet Besançon article, that should be essential!]), <u>control theorem</u> and $\boxed{\text{Step 1}}$,

$\exists\ \alpha \in \Lambda$ s.t. diagram commutes

$$\begin{tikzcd}
\Delta_\Lambda^{-1} \arrow[r, "\mathrm{triv}_{Z_\Sigma}"] \arrow[d, "-\otimes_{\Lambda, x} \Lambda_{I_w}"'] & \alpha\Lambda \arrow[d, "\lambda"] \\
\bigotimes_{\lambda_i \in S_x} \Delta_{\Lambda_{I_w}}(T_{\lambda_i})^{-1} \arrow[r, "\sim", "\bigotimes_{\lambda_i \in S_x} \mathrm{triv}_{Z_{\lambda_i}}"'] & \Lambda_{I_w}
\end{tikzcd}$$

Use the equalities of IMC on the fibre to obtain the "lower bound" of triv_{Z_Σ}

This implies α is a unit in Λ hence $\mathrm{triv}_{Z_\Sigma} : \Delta_\Lambda \longrightarrow \Lambda$ is an isomorphism, as desired!

$\boxed{\text{Step 3}}$ Let λ_f be a modular point, then we know KIMC holds for Kato's

[Prop. 5.8, Thm 5.1] inclusion, i.e. there is an inclusion:

$$\mathrm{triv}_{Z_{f,I_w}}\left(\Delta_{\Lambda_{I_w}}(T_{f,I_w})^{-1}\right) \subseteq \Lambda_{I_w}$$

so $\exists\ \gamma \in \Lambda_{I_w}$ s.t. $\mathrm{triv}_{Z_{f,I_w}} : \Delta_{\Lambda_{I_w}}(T_{f,I_w})^{-1} \xrightarrow{\sim} \frac{1}{\gamma}\Lambda_{I_w}$.

- By <u>control theorem</u>, there is a commutative diagram

$$\begin{tikzcd}
\Delta_\Sigma \arrow[r, "Z_\Sigma^{univ}"] \arrow[d, "-\otimes_{\lambda_{f,I_w}} \Lambda_{I_w}"'] & \frac{\alpha}{\beta} R_\Sigma(\bar\rho) \arrow[d, "\lambda_{f,I_w}"] \\
\Delta_{\Lambda_{I_w}}(T_{f,I_w})^{-1} \arrow[r, "\mathrm{triv}_{Z_{f,I_w}}"'] & \frac{1}{\gamma}\Lambda_{I_w}
\end{tikzcd}$$

in which we can choose β s.t. $\lambda_{f,I_w}(\beta) \neq 0$.

- Let $Z :=$ set of specializations sending β to zero. Then by <u>Theorem*</u>, $\exists$ a specialization $\lambda : \mathbb{T}^{\Sigma}_{m_{\bar\rho}} \to \mathcal{O}$ not in Z above $x \in \operatorname{Spec}\Lambda$, s.t. $\operatorname{Spec}\mathbb{T}^{\Sigma}_{m_{\bar\rho}}$ is étale over x on Λ and all specializations above x is IS.

$$\Longrightarrow \quad \begin{tikzcd}[column sep=large] \boxed{\Delta_\Sigma(T_\Sigma) \xrightarrow{z_{\Sigma,\Lambda}} \Lambda} \arrow[d, "-\otimes_{\Lambda,\chi_{I_w}} \Lambda_{I_w}"'] \arrow[dr, phantom, "\circlearrowright"] & \\ \bigotimes_{\psi \in S_x} \Delta_{\Lambda_{I_w}}(T_{\psi,I_w}) \arrow[r, "\otimes z_{\Sigma,\psi_{I_w}}"'] & \left(\prod_{\psi\in S_x} \frac{1}{\gamma_\psi}\right)\Lambda_{I_w} \end{tikzcd}$$

$$\begin{tikzcd}[column sep=large] \Delta_\Sigma(T_\Sigma) \arrow[r, "z_{\Sigma,\Lambda}"] \arrow[d, "-\otimes_{\Lambda,\chi_{I_w}} \Lambda_{I_w}"'] \arrow[dr, phantom, "\circlearrowright"] & \Lambda \arrow[d, "\chi"] \\ \bigotimes_{\psi \in S_x} \Delta_{\Lambda_{I_w}}(T_{\psi,I_w}) \arrow[r, "\otimes z_{\Sigma,\psi_{I_w}}"'] & \left(\prod_{\psi\in S_x} \frac{1}{\gamma_\psi}\right)\Lambda_{I_w} \end{tikzcd}$$

← universal IMC for families.

by control theorem

Hence γ_ψ is a unit, and the diagram becomes

$$\begin{tikzcd}[column sep=large] \Delta_\Sigma(T_\Sigma) \arrow[r, "z_{\Sigma,\Lambda}"] \arrow[d, "-\otimes_{\Lambda,\chi_{I_w}} \Lambda_{I_w}"'] \arrow[dr, phantom, "\circlearrowright"] & \Lambda \arrow[d, "\chi"] \\ \bigotimes_{\psi \in S_x} \Delta_{\Lambda_{I_w}}(T_{\psi,I_w}) \arrow[r, "\otimes z_{\Sigma,\psi_{I_w}}"'] & \Lambda_{I_w} \end{tikzcd}$$

- So this holds particular for λ and because $\lambda \notin Z$, we have a commutative diagram

$$\begin{tikzcd}[column sep=large] \Delta_\Sigma \arrow[r, "z_\Sigma^{univ}"] \arrow[d, "-\otimes_{\lambda,I_w} \Lambda_{I_w}"'] & \frac{\alpha}{\beta} R_\Sigma(\bar\rho) \arrow[d, "\lambda_{I_w}"] \\ \Delta_{\Lambda_{I_w}}(T_{\lambda,I_w})^{-1} \arrow[r, "\mathrm{triv}_{z_{\lambda,I_w}}"'] & \Lambda_{I_w} \end{tikzcd} \qquad \Longrightarrow \lambda_{I_w}\left(\frac{\alpha}{\beta}\right) \text{ is a unit.}$$

As we can choose $\lambda \in \mathfrak{X}^{sm}_\Sigma(\bar\rho)$ arbitarily p-adically close to $\lambda(f)$, this entails $\chi\left(\frac{1}{\gamma}\right)$ is a unit, hence γ is a unit. This gives the IMC for f.

$\square$