Talk 9 Global Construction : Siegal Eisenstein family

Recall through Talk 5 – Talk 8, we have choosen Siegal sections at each places. This gives a Siegal Eisenstein series $E_{\mathcal{D}}^{\mathrm{Sieg}}$ and computed the Whittaker integrals

§ 9.1 Siegal Eisenstein series and its normalization

Recall the local sections we used to pullback

- Archemedian places : $f_v^{\mathrm{Sieg}}(g,z) := \mu(g,i)^{-k} |\mu(g,i)|^{k-2z-n}$
- unramified places : spherical sections f_v°
- ramified places : $f_v^{\mathrm{Sieg}} = f_v^{\text{big-cell}, \gamma_v}$ (right translation of $f_v^{\text{big-cell}}$ by γ_v).
- p-adic places : $f_v^{\alpha, \chi}$

Then we can define a global Siegal section as

$$f_{\mathcal{D}}^{\mathrm{Sieg}} := \bigotimes_{v|\infty} f_v^{\mathrm{Sieg}} \bigotimes_{v\notin\Sigma} f_v^{\circ} \bigotimes_{v\in\Sigma_{\mathrm{ram}}} f_v^{\mathrm{Sieg}} \otimes_{v|p} f_v^{\alpha,\chi}.$$

To construct the Siegal Eisenstein family, we need to normalize f_v^{Sieg} to

- make the Whittaker integrals interpolatable : add normalization factor $B_{\mathcal{D}}$.
- fit the pullback integrals : modification at Σ_{ram} and " $\otimes \overline{\tau}(\det(-))$ "

(1) Recall the Whittaker integrals $\quad z := z_k$

- archmedian places :

$$W_\beta\left(\mathrm{diag}(y, \bar{y}^{*}) ; f_v^{\mathrm{Sieg}}, z_k\right) = \begin{cases} 0 & \det\beta \leq 0 \\ C_v(n,k)\,\delta_v(n,z)\, \underbrace{e_v(i\,\mathrm{Tr}(\beta y y^{*}))(\det\beta)^{k-n}(\det\bar{y})^{k}}_{\text{here } y \text{ should be } y^{SU}\text{, but we can a priori adjust it !}} & \det\beta > 0 \end{cases}$$

- unramified places :

$$W_\beta\left(\mathrm{diag}(y, \bar{y}^{*}) ; f_v^{\circ}, z\right) = \tau_v(\det y)\, |\det y\bar{y}|_v^{-z+\frac{n}{2}}\, D_v^{\frac{-n(n-1)}{4}}\, h_{v, y^{*}\beta y}\left(\overline{\tau}'(\varpi)\, q_v^{-2z-n}\right)$$

$$\times \frac{\prod_{i=r}^{n-1} L(2z+i-n+1, \overline{\chi}'\eta_{K/F}^{i})}{\prod_{i=0}^{n-1} L(2z+n-i, \overline{\chi}'\eta_{K/F}^{i})}$$

- <u>ramified places</u> :

$$W_\beta\left(\begin{pmatrix} A & \\ & A^{-*}\end{pmatrix}; f_v^{\mathrm{sieg}}, z\right) = \tau_v(\det A)\,|\det A\bar{A}|^{-z+\frac{n}{2}}\,\mathbb{1}_{\mathrm{Herm}_8^{\star}(\mathcal{O}_{F_v})}(\beta^A)$$

$$\times\ \mathrm{vol}\cdot e_v\left(\mathrm{tr}^{K_v}_{F_v}\left(\frac{\mathrm{Tr}\,\beta_{41}^{\bar{A}}}{\bar{x}} + \frac{\mathrm{Tr}\,\beta_{41}^{A}}{x} + \frac{\mathrm{Tr}\,\beta_{33}^{A}}{y\bar{y}}\right)\right)$$

where $\beta^A := A^*\beta A$. This is the $\heartsuit$-case and $\diamondsuit$-case is similar.

- <u>p-adic places</u> :

$$W_\beta\left(\begin{pmatrix} A & \\ & A^{-*}\end{pmatrix}, f^{\alpha}, z\right) = (\tau^{-1})'(\det\beta)\,|\det\beta|_v^{2z}\,\mathfrak{g}(\tau')^n\,c_n((\tau^{-1})', -z)\,\Phi_3((\beta^A)^t)\,\mathbb{1}_{\mathrm{Herm}_n^{\star}(\mathcal{O}_{F_v})}(\beta^A)$$

$$\times\ \tau_v(\det A)\,|\det A\bar{A}|^{-z+\frac{n}{2}}$$

So we define the normalization factor $B_{\mathcal{D}}$ for the Eisenstein datum $\mathcal{D} = \{\pi, \tau, \Sigma\}$ as

$$B_{\mathcal{D}} = \underbrace{\prod_{v|\infty}\left(c_v(n,k)\,\delta_v(n,z)\right)^{-1}}_{\infty\text{-places}} \cdot \underbrace{\prod_{i=0}^{n-1} L^{\Sigma}(2z+n-i, \bar{\chi}'\eta^i_{K/F})}_{\text{unramified places}} \cdot \underbrace{\mathfrak{g}(\tau')^n\,c_n((\tau^{-1})', -z)}_{\text{p-adic places}}$$

and define $E_{\mathcal{D}}^{\mathrm{sieg}}$ as the Eisenstein series associated to $B_{\mathcal{D}}\cdot f_{\mathcal{D}}^{\mathrm{sieg}}$. Then by previous results,

- $E^{\mathrm{sieg}}_{\mathcal{D},\beta}\left(\begin{pmatrix} A & \\ & A^{-*}\end{pmatrix}\right)$, as a function of β, is supported in

$$\mathfrak{S}_{n,A} := \left\{ \beta\in \mathrm{Herm}_n(\mathcal{O}_{F_v})_{>0} \;\middle|\; \begin{array}{l} A^*\beta A \in \mathrm{Herm}_n^{\star}(\mathcal{O}_{F_v}) \text{ for } v\in\Sigma \\ A^*\beta A \in \mathrm{Supp}\,\Phi_3 \text{ for } v\in\Sigma_p \end{array} \right\}$$

- And for $\beta\in\mathfrak{S}_{n,A}$, we have

$$E^{\mathrm{sieg}}_{\mathcal{D},\beta}\left(\begin{pmatrix} A & \\ & A^{-*}\end{pmatrix}\right) = \text{a } \underline{\text{constant independent of } \tau} \text{ (may depend on } \beta)$$

$$\times\ \tau(\det A)\prod_{v\notin\Sigma} h_{v, A^*\beta A}(\bar{\tau}'(\varpi)\,q_v^{-2z-n})$$

$$\times\ (\tau^{-1})'_p(\det\beta)\,\Phi_{3,p}(A^*\beta A).$$

So recall the definition of Φ_3, it suffices to "<u>interpolate the Hecke characters in families</u>"! A more convenient perspective is from measures.

§9.2 p-adic interpolations

Let L be a finite extension of $\mathbb{Q}_p$ that is fixed throughout.

- Let $H = \prod_{v|p} GL_r \times GL_s$ be a group scheme $/ \mathbb{Z}_p$.

  Recall $H(\mathbb{Z}_p)$ is the Galois group of the Igusa tower over the ordinary locus of the toroidal compactified Shimura variety.

- Let $T / \mathbb{Z}_p$ be the diagonal torus of H.

- Define the <u>weight algebra</u> as the completed group algebra $\Lambda_{r,s} := \mathcal{O}_L [\![T(1+p\mathbb{Z}_p)]\!]$.

  As $T(1+p\mathbb{Z}_p)$ acts on the Igusa scheme, the space of p-adic modular forms on $GU(r,s)$ has a structure of $\Lambda_{r,s}$-algebra.

<u>Characters</u>:

- Let $\underline{k} = (c_{s+1}, \cdots, c_{s+r} ; c_1, \cdots, c_s)$ be a weight (in the sense of Wan). Then we can associate it with a character of $T(1+p\mathbb{Z}_p)$ as

$$[\underline{k}] \cdot \operatorname{diag}(t_1, \cdots, t_r, t_{r+1}, \cdots, t_{r+s}) = t_1^{c_{s+1}} \cdots t_r^{c_{s+r}} t_{r+1}^{-c_1} \cdots t_{r+s}^{-c_s}$$

- Fix a finite order character χ_0 on $T(\mathbb{F}_p) \xleftarrow{\sim} T(\mathbb{Z}_p/p\mathbb{Z}_p) \twoheadleftarrow T(1+p\mathbb{Z}_p)$ as the torsion part of $T(1+p\mathbb{Z}_p)$, throughout.

Then we define a <u>$\overline{\mathbb{Q}}_p$-point ϕ of $\operatorname{Spec}\Lambda$</u> (i.e. $\phi : \Lambda_{r,s} \to \overline{\mathbb{Q}}_p$) to be <u>arithmetic</u> if $\exists$ $\underline{k} = (0, \cdots, 0, k, \cdots, k)$ s.t. ϕ is given by a character $\chi_0 \chi_\phi [\underline{k}]$ for some χ_ϕ of order and conductor power of p, and <u>$k > r+s+1$</u>. Here k is called the <u>weight</u> of ϕ, denoted by k_ϕ.

<u>Note</u>: In [Wan15, p2022], $k \geq 2(a+b+1)$, this is wierd? I guess from <u>Talk 8</u> that this should be $k > r+s+1$?

<u>Remark</u>: Here we can regard χ_ϕ as a character of $T(\mathbb{Z}_p)$ that is trivial on the torsion part $T(\mathbb{F}_p)$, while χ_0 is the effect on the torsion.

The family of Eisenstein datum :

There are various ways to define a family of Eisenstein datum. The differences are not essential. We take [Wan20, Defn 7.3] as a starting point, at the same time look at [Wan15, Defn 5.3].

<u>Definition</u> : A family of Eisenstein datum is a tuple $\mathbb{D} = (L, \mathbb{I}, \mathbf{f}, \tau_0, \chi_0)$:

- $L \mid \mathbb{Q}_p$ is a finite extension, used to define $\Lambda_{r,s}$.
- $\mathbb{I}$ is a normal domain over $\Lambda_{r,s}$ which is also a finite $\Lambda_{r,s}$-module.
- $\mathbf{f}$ is an $\mathbb{I}$-adic Hida family of cuspidal ordinary eigenforms on $\mathcal{U}(r,s)$.
- τ_0 is a finite order character of $\mathbb{A}_K^\times / K^\times$ whose conductors at primes above p divide (p).
- χ_0 as above.

Then we define the <u>Iwasawa algebra</u> $\Lambda_{\mathbb{D}} := \mathbb{I} \underset{\mathcal{O}_L}{\otimes} \Lambda_K$, $\Lambda_K := \mathcal{O}_L[\![T_K]\!]$.

For the character τ_0, we <u>deform it into p-adic family</u> as $\boldsymbol{\tau} := \tau_0 \cdot \Psi_K$, where

multiplication

ε_K : totological character

$$\begin{tikzcd}
\mathrm{Gal}_K \arrow[r, two heads] \arrow[d] \arrow[rrr, bend left, "\varepsilon_K"] & T_K \arrow[r, hook] & \Lambda_K \arrow[r] & \Lambda_{\mathbb{D}} \arrow[dr, "\phi"] & \\
\mathrm{Gal}_K^{ab} \arrow[urrr, dashed] & & & & \overline{\mathbb{Q}}_p \\
K^\times \backslash \mathbb{A}_K^\times \arrow[u, "\mathrm{rec}_K"] \arrow[uurrr, bend right, "\Psi_K"'] & & & &
\end{tikzcd}$$

abelian

Let ϕ be a $\overline{\mathbb{Q}}_p$-point of $\mathrm{Spec}\,\Lambda_{\mathbb{D}}$. Then $\tau_\phi := \phi \circ \boldsymbol{\tau}$ is the specialization of $\boldsymbol{\tau}$ at ϕ. Denote $\mathbb{D}_\phi := (\pi_{\mathbf{f}_\phi}, \tau_\phi, \Sigma_\phi)$ as the specialization of $\mathbb{D}$ at ϕ. It is an Eisenstein datum potentially.

Defn: Let ϕ be a $\overline{\mathbb{Q}}_p$-point of $\mathrm{Spec}\,\Lambda_{\mathbb{D}}$.

(1) it is called arithmetic if $\phi_{\mathbb{I}}$ is arithmetic (i.e. its image in $\Lambda_{r,s}$ is arithmetic) of weight k_ϕ and

$$\phi(\gamma^+) = (1+p)^{\frac{k_\phi}{2}}\zeta_+ , \quad \phi(\gamma^-) = (1+p)^{\frac{k_\phi}{2}}\zeta_- .$$

(2) moreover if the specialization f_ϕ is classical and generates an irreducible cuspidal autorep π_{f_ϕ} of $U(r,s)$, and $(\pi_{f_\phi}, \tau_\phi)$ is generic, we call ϕ a generic point.

Denote $\mathfrak{X}^{gen}$:= the set of generic arithmetic points. It is a dense subset of $\mathrm{Spec}\,\Lambda_{\mathbb{D}}$. (I cannot see why? Does this require p to be sufficiently large?).

Theorem: There exists a p-adic measure $E^{Sieg}_{\mathbb{D}} \in \mathrm{Meas}(T(1+p\mathbb{Z}_p)\times T_K, V_{U(r+s+1)})$ such that for every $\phi \in \mathfrak{X}^{gen}$, $E^{Sieg}_{\mathbb{D}}(\phi) = E^{Sieg}_{\mathbb{D}_\phi}$.

Proof: Recall we computed $E^{Sieg}_{\mathbb{D}_\phi,\beta}\left(\begin{pmatrix} A & \\ & A^{-*}\end{pmatrix}\right)$ as the β-th coefficient of the q-expansion of $E^{Sieg}_{\mathbb{D}_\phi}$ at the cusps indexed by $\mathrm{diag}(A, A^{-*})$, as

$$E^{Sieg}_{\mathbb{D}_\phi,\beta}\left(\begin{pmatrix} A & \\ & A^{-*}\end{pmatrix}\right) = \text{a constant independent of } \tau \text{ (may depend on } \beta)$$

$$\times\ \tau_\phi'(\det A) \prod_{v\notin\Sigma} h_{v,A^*\beta A}\left(\tau_\phi'(\varpi)\, q_v^{-2z-n}\right)$$

$$\times\ (\tau_\phi^{-1})_p'(\det\beta)\ \Phi^{(\phi)}_{z,p}(A^*\beta A) , \qquad \beta \in \mathfrak{S}_{n,A} .$$

Then the ●-parts are interpolated by the δ-measures in $\mathrm{Meas}(T(1+p\mathbb{Z}_p)\times T_K, \mathcal{O}_L)$. The convolution of these δ-measures gives an element in $\mathrm{Meas}(T(1+p\mathbb{Z}_p)\times T_K, \mathcal{O}_L)$ interpolating $E^{Sieg}_{\mathbb{D}_\phi,\beta}$. Then we see that there exists a p-adic measure

$$E^{Sieg}_{\mathbb{D}} \in \mathrm{Meas}\Big(T(1+p\mathbb{Z}_p)\times T_K, \bigoplus_{\text{cusps}} \mathcal{O}_L[[\mathfrak{S}_{n,A}]]\Big) .$$

By q-expansion principle and Kummer congruences, $E^{Sieg}_{\mathbb{D}} \in \mathrm{Meas}(\mathbb{T}\times T_K, V_{U(r+s+1)})$.

□