Talk 02 Unitary Shimura varieties

2023-5-15

Reference :

- [Hsieh14] Eisenstein congruence on unitary groups and Iwasawa main conjectures for CM fields.

§1 Quick degressions on PEL data

Remember : PEL = Polarization + Endormorphism + Level structure .

Defn : We say $\mathcal{D} = (B, *, V, \psi, h)$ is a PEL-data (or Shimura data of PEL type)

- a finite semisimple $\mathbb{Q}$-algebra B with a positive involution $*$
- a symplectic $(B,*)$-module V, that is, a B-module V with a skew-symmetric nondegenerate $\mathbb{Q}$-bilinear form $\psi : V \times V \longrightarrow \mathbb{Q}$ such that
$$\psi(b^* u, v) = \psi(u, bv), \quad \forall\, u, v \in V,\ b \in B .$$
Let $\mathcal{C} := \mathrm{End}_B(V)$.

Recall : For $\alpha \in \mathrm{End}_B(V)$, define α^* to be the endomorphism such that
$$\psi(\alpha^* v, w) = \psi(v, \alpha w), \quad \forall\, v, w \in V .$$
Then this "$*$" extends by identity to $\mathrm{End}_B(V)_{\mathbb{R}}$. This is called the adjoint involution.

- an $\mathbb{R}$-algebra map $h : \mathbb{C} \longrightarrow \mathcal{C}_{\mathbb{R}}$ such that
  - $h(\bar{z}) = h(z)^*$, where "$*$" on the RHS is the adjoint involution on $\mathrm{End}_B(V)_{\mathbb{R}}$
  - $(u, v) \longmapsto \psi(u, h(i)v)$ is positive-definite and symmetric.

We say $\mathcal{D}$ is a simple PEL data if B is a simple $\mathbb{Q}$-algebra.

Moreover, we associate $\mathcal{D}$ with :

- $F :=$ the centre of B
- $F_0 := \{ b \in F : b^* = b \}$, i.e. the subalgebra of $*$-invariants in F.

and two $\mathbb{Q}$-groups : for $\mathbb{Q}$-algebra R.

- $G(R) := \{ g \in GL_{B \otimes_{\mathbb{Q}} R}(V \otimes_{\mathbb{Q}} R) \mid \langle gu, gv \rangle = \lambda(g) \langle u, v \rangle ,\ u, v \in V,\ \lambda(g) \in R^\times \}$
- $G_1(R) := \{ g \in GL_{B \otimes_{\mathbb{Q}} R}(V \otimes_{\mathbb{Q}} R) \mid \langle gu, gv \rangle = \lambda(g) \langle u, v \rangle ,\ u, v \in V,\ \lambda(g) \in (F_0 \otimes_{\mathbb{Q}} R)^\times \}$

Then clearly, G is a subgroup of G_1

Exercise : We can reinterprete G and G_1 as
$$G_1(\mathbb{Q}) = \{ x \in \mathcal{C} \mid x^* x \in F_0^\times \} \supseteq G(\mathbb{Q}) = \{ x \in \mathcal{C} \mid x^* x \in \mathbb{Q}^\times \} .$$

Reflex field of $\mathcal{D}$:

- We can associate a Hecke cocharacter $\mu_h : \mathbb{C}^\times \longrightarrow G(\mathbb{C})$ that induces a decomposition $V_{\mathbb{C}} \simeq V^{-1,0} \oplus V^{0,-1}$ such that
    - $\mu_h(z)$ acts on $V_{\mathbb{C}}$ by z
    - $\mu_h(z)$ acts on $V^{0,-1}$ by identity.

  This is done in two steps :

  1° Associate h with a (by abuse of notation) homomorphism $h : \mathbb{S} \longrightarrow G_{\mathbb{R}}$ s.t. (V, h) has Hodge type $\{(-1,0), (0,-1)\}$, with $2\pi i \psi$ a polarization of (V, h). This is done by Deligne and Zink, see [Milne, Prop. 8.13, 8.14].

  2° For such a HS $h : \mathbb{S} \longrightarrow G_{\mathbb{R}}$, we build

$$\begin{aligned} \mu_h : \mathbb{C}^\times &\longrightarrow \mathbb{C}^\times \times \mathbb{C}^\times \xrightarrow{h_{\mathbb{C}}} G(\mathbb{C}) \\ z &\longmapsto (z, 1) \longmapsto h_{\mathbb{C}}(z, 1). \end{aligned}$$

  See [Milne, p111].

- Then the reflex field $E := E(\mathcal{D})$ is defined as the subfield of $\overline{\mathbb{Q}}$ generated by $\mathrm{tr}_{\mathbb{Q}}(b \,|\, V^{(0,-1)})$ for $b \in B$.

Classification theorem : We have three families of PEL data according to $\mathcal{C}_{\mathbb{R}}$:

(A) $\mathcal{C}_{\mathbb{R}} \simeq$ a product of copies of $M_n(\mathbb{C})$ for some n. (unitary type)

(C) $\mathcal{C}_{\mathbb{R}} \simeq$ . . . . $M_{2n}(\mathbb{R})$ for some n. (symplectic type)

(D) $\mathcal{C}_{\mathbb{R}} \simeq$ . . . . . . $M_n(\mathbb{H})$ for some n. (orthogonal type)

Fact (Milne, p81) Assume B is free as an F-module. Then :

- (A) $\Rightarrow [F : F_0] = 2$
- (C) or (D) $\Rightarrow F = F_0$.

New data for integral models

Recall: In general, given a Shimura datum (G, X) (of Hodge type), we have a canonical model $Sh_K(G,X)$ over a reflex field $E(G,X) =: E$

Let $O_E \subseteq E$ be the ring of integers of E.

Question: Does there exists a good integral model of $Sh_K(G,X)$ over $O_{E,\wp}$ for some prime ideal $\wp \subseteq O_E$?

- O_B : a $\mathbb{Z}_{(p)}$-order in B such that
  - O_B is stable under the involution $*$ on B.
  - $O_B \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_p$ becomes a maximal order in B.
- Λ : a $\mathbb{Z}_p$-lattice in $V_{\mathbb{Q}_p} := V \otimes_{\mathbb{Q}} \mathbb{Q}_p$ such that
  - Λ is stable under O_B (recall $\Lambda \subseteq V_{\mathbb{Q}_p}$ with $V_{\mathbb{Q}_p}$ a B-module acting on V)
  - Λ is self-dual wrt the pairing ψ.

Throughout, we always assume

(unr_p) B is unramified at p, i.e. $B_{\mathbb{Q}_p} := B \otimes_{\mathbb{Q}} \mathbb{Q}_p$ is isomorphic to a product of matrix algebras over an unramified extension of $\mathbb{Q}_p$.

Hyperspecial subgroup at p:

Defn: Let $K_p^\circ := \{ g \in G(\mathbb{Q}_p) \mid g\Lambda \subseteq \Lambda \}$, i.e. the subgroup of $G(\mathbb{Q}_p)$ stablizing Λ.

Fact: $G_{\mathbb{Q}_p}$ has a smooth reductive model $\mathcal{G}$ over $\mathbb{Z}_p$, with K_p° a hyperspecial subgroup.

As a result, $G_{\mathbb{Q}_p}$ is unramified.

§2 Unitary PEL data

We construct our <u>unitary PEL data</u> as :

- Start with a CM field K/F, $[F:\mathbb{Q}] = d$
- Take $B = K$, with the involution as the complex conjugation in $\mathrm{Gal}(K/F)$

  Then " $F = K$ " and " $F_0 = F$ " .

- Let V be a finite dimensional K-vector space with a skew-Hermitian form $\theta : V \times V \longrightarrow K$, of signature (r,s). Define

$$\psi_{r,s} : V \times V \longrightarrow \mathbb{Q} , \quad \psi_{r,s}(u,v) := \mathrm{Tr}^K_{\mathbb{Q}}(\theta(u,v))$$

  Then one <u>checks</u> that $(V, \psi_{r,s})$ is a symplectic $(B,*)$-module.

- $h : \mathbb{C} \longrightarrow \mathcal{E}_{\mathbb{R}}$ is provided in [Milne, prop. 8.12] :

  * To give such an h is equivalent to give $J \in \mathcal{E}_{\mathbb{R}}$ (as $J = h(i)$) such that

  $(*)$ — $J^2 = -1$, $\psi_{r,s}(Ju, Jv) = \psi_{r,s}(u,v)$, $\psi_{r,s}(u, Jv) > 0$ if $v \neq 0$.

  * Name a basis (e_j) of V such that

$$\big(\theta(e_j, e_k)\big)_{j,k} = \mathrm{diag}(\underbrace{i, \cdots, i}_{r}, \underbrace{-i, \cdots, -i}_{s})$$

  Then one <u>checks</u> that J such that $J(e_j) = -\theta(e_j,e_j)e_j$ satisfies the condition $(*)$.

In this way we have a PEL data $\mathcal{D} = (K, *, V, \psi_{r,s}, h)$ of type A, called a <u>unitary PEL data</u>.

Unitary groups: Then for such a unitary data $\mathcal{D}$, we have the $\mathbb{Q}$-groups

$$G(R) = \{ g \in GL_{K\otimes_{\mathbb{Q}} R}(V \otimes_{\mathbb{Q}} R) \mid \langle gu, gv \rangle = \lambda(g) \langle u,v \rangle, \ u,v \in V, \ \lambda(g) \in R^\times \}$$

$$G_1(R) = \{ g \in GL_{K\otimes_{\mathbb{Q}} R}(V \otimes_{\mathbb{Q}} R) \mid \langle gu, gv \rangle = \lambda(g) \langle u,v \rangle, \ u,v \in V, \ \lambda(g) \in (F \otimes_{\mathbb{Q}} R)^\times \}$$

So if we take θ as in Talk 01 given by $J_{r,s}$ under the standard basis $\{\underline{x}, \underline{w}, \underline{y}\}$,

we see $\boxed{\mathrm{Res}^F_{\mathbb{Q}} GU = G_1}$

· Indeed, for $\mathbb{Q}$-algebra R,

$$\begin{aligned} &\mathrm{Res}^F_{\mathbb{Q}} GU(R) \\ &= GU(F \otimes_{\mathbb{Q}} R) \\ &= \{ g \in GL_{K\otimes_F (F\otimes_{\mathbb{Q}} R)} \mid \langle gu, gv \rangle = \lambda(g) \langle u,v \rangle, \ u,v \in V, \ \lambda(g) \in (F \otimes_{\mathbb{Q}} R)^\times \} \\ &= \{ g \in GL_{K\otimes_{\mathbb{Q}} R} \mid \langle gu, gv \rangle = \lambda(g) \langle u,v \rangle, \ u,v \in V, \ \lambda(g) \in (F \otimes_{\mathbb{Q}} R)^\times \} = G_1 \end{aligned}$$

But:
- in general, nice Shimura varieties of PEL types arise from G, not G_1!
  ("nice" ≥ "nice moduli description")
- But G is not convenient for local computations since we can't treat the primes of F each independently.

How to solve the problem?

Remark 2.1 in [Wan15ANT]:

- For analytic constructions, we write down forms on G_1 and then restricts to G (implicitly).
- For algebraic constructions: we only do pullbacks for unitary groups (without similitude).

Anyway, a good news: The Hermitian symmetric domain are the same.

So implicitly: Later when discussing Shimura varieties, we use G and when writing $GU(r,s)$, we also mean the $\mathbb{Q}$-group G above.

Additional data for integral model

- O_K is taken simply as the ring of integers of K.
- Let ∂_K be the absolute difference of $K/\mathbb{Q}$. Define :
  - $X^\vee := \partial_K^{-1} x^1 \oplus \cdots \oplus \partial_K^{-1} x^s \simeq (\partial_K^{-1})^s$
  - $Y := O_K y^1 \oplus \cdots \oplus O_K y^s \simeq O_K^s$
  - Choose an O_K-lattice L in W such that $\psi_{r,s}(L,L) \subseteq \mathbb{Z}$ and

$$L_p := L \otimes_{\mathbb{Z}} \mathbb{Z}_p = \sum_{i=1}^{r-s} (O_K \otimes_{\mathbb{Z}} \mathbb{Z}_p) w^i \simeq (O_K \otimes_{\mathbb{Z}} \mathbb{Z}_p)^{r-s}$$

  Then we can define the O_K-lattice M in V by

$$M := Y \oplus L \oplus X^\vee$$

  Check : (1) $\psi_{r,s}(M,M) \subseteq \mathbb{Z}$

  (2) $M_p := M \otimes_{\mathbb{Z}} \mathbb{Z}_p$ is self-dual wrt $\psi_{r,s}$. (as p is unramified in K)

  Proof : Actually I think it is useful to recall ∂_K and its property.

- Remark . The assumption (unr_p) is satisfied as p is unramified in K.

Hyperspecial subgroups :

- Please distinguish : M/O_K , $M_p := M \otimes_{\mathbb{Z}} \mathbb{Z}_p \,/\, \mathbb{Z}_p$, $M_v := M \otimes_{O_F} O_{F,v}$ for $v \in I_F^\circ$
- For $G = GU(r,s)/_F$ (in this section), we put

$$K_v^\circ := \{ g \in G(F_v) \mid gM_v = M_v \} \quad , \quad K^\circ := \prod_{v \in I_F^\circ} K_v^\circ$$

Through out the discussion, we fix an open compact subgroup $K \subseteq G(\mathbb{A}_{F,f})$ such that $K_p = \prod_{v|p} K_v^\circ$. (People say " K is hyperspecial at p " . Often, as in Wan's paper, people often assume $K_v = K_v^\circ$, even for $v \nmid p$; yet they write " $K_v = G(\mathbb{Z}_p)$ " which may cause confusion since G is over F (or sometimes $\mathbb{Q}$) .)

$\Rightarrow$ $K_v^\circ \simeq GL_{r+s}(O_v) \times O_v^\times$, write $g_v \in K_v^\circ$ as $g_v = (g_v^{(1)}, \lambda(g_v))$

- Some useful subgroups of K_p° : write $K_p^\circ = \prod_{v|p} K_v^\circ$ and $g_p = (g_p^{(1)}, \lambda(g_p))$.

Put $I_1(p^n) = \{ g_p \in K_p^\circ \mid g_p \in N_{r+s}(O_{F,p}) \times \{1\} \mod p^n \}$

and define for fixed K ,

$$\begin{aligned} K^n &= \{ g \in K \mid g_p \equiv \begin{pmatrix} 1_r & * \\ & 1_s \end{pmatrix} \times \{1\} \mod p^n \} \\ &\cap| \\ K_1^n &= K^{(p)} I_1(p^n) \\ &\cap| \\ K_0^n &= \{ g \in K \mid g_p^{(1)} \in B_{r+s}(O_{F,p}) \mod p^n , \text{ no requirement on similitude} \} \end{aligned}$$

- Additional requirement on $K^{(p)}$: Let $N_0 \geq 3$ be a prime-to-p integer .

Defn : (1) A compact open subgroup of $G(\mathbb{A}_{F,f}^p)$ is called a level subgroup .

(2) A principal level subgroup of level N_0 is the compact open subgroup

$$K^p(N_0) = \{ g \in G(\mathbb{A}_{F,f}^p) \mid (g-1) M^{(p)} \subseteq N_0 M^{(p)} \}$$

where $M^{(p)} = \prod_{v \nmid p} M_v$.

Note in Hsieh : He considered $K(N_0) = \{ g \in G(\mathbb{A}_{F,f}) : (g-1) M \subseteq N_0 \cdot M \}$, including primes at p .

Then K is neat if $K \subseteq K(N_0)$ and $\lambda(K) \cap O_{F,+}^\times \subseteq (K \cap O_F^\times)^2$

§3 Moduli interpretation

Setup:

- $\mathcal{K}$ is a finite extension of $K^{nc}(e^{2\pi i/N_0})$ which is unramified at p.
  Let $\mathfrak{p}$ be a prime ideal of $\mathcal{K}$ above p corresponding to $\iota_p : K \to \mathbb{C}_p$
- Let $\mathcal{O} := \mathcal{O}_{\mathcal{K},(\mathfrak{p})}$ be the localization and $\mathcal{O}_{\mathfrak{p}}$ the $\mathfrak{p}$-adic completion of $\mathcal{O}$.
  Note: Here $\mathcal{O}$ is large enough to contain $\mathcal{O}_E \otimes_{\mathbb{Z}} \mathbb{Z}_{(\square)}$.
- Let $\square$ = a finite set of primes not dividing N_0. Then we can define
$$\mathbb{Z}_{(\square)} = \left\{ \tfrac{a}{b} \in \mathbb{Q} \;\middle|\; b\mathbb{Z} + q\mathbb{Z} = \mathbb{Z},\ \forall q \in \square \right\}$$
  and for any ring R, $R_{(\square)} := R \otimes_{\mathbb{Z}} \mathbb{Z}_{(\square)}$.
  Example:
  - $\square = \phi$: $\mathbb{Z}_{(\square)} = \mathbb{Q}$
  - $\square = \{p\}$: write $\mathbb{Z}_{(\square)} = \mathbb{Z}_{(p)}$
- Let $Sch/_T$ be the category of locally noetherian connected T-scheme, equipped with a geometric point $\bar{s}$, i.e. $\bar{s} \in \mathrm{Hom}(\mathrm{Spec}\,\Omega, S)$ where Ω is an seperablely closed field, usually denoted as $k(s)$. Here $T = \mathcal{O}$ or $\mathcal{K}$.

Isogeny category: Define the category $AV^{\square}_{/S}$, as

- objects: A, an abelian scheme over S
- morphisms: $\mathrm{Hom}_{AV^{\square}_{/S}}(A_1, A_2) := \mathrm{Hom}_{AV_{/S}}(A_1, A_2) \otimes_{\mathbb{Z}} \mathbb{Z}_{(\square)}$, called $\mathbb{Z}_{(\square)}$-isogenies.

Then accordingly we can define a $\mathbb{Z}_{(\square)}$-polarization $\lambda : A \to A^{\vee}$ as a polarization of A that is also a $\mathbb{Z}_{(\square)}$-isogeny. (Sometimes called a prime-to-$\square$ polarization)

Remark: When $\square = \phi$, we have the notion of "$\mathbb{Q}$-isogeny". It is a "standard fact that $\mathrm{Hom}_{AV^{\phi}_{/S}}(A_1, A_2) = \mathrm{Isog}(A_1, A_2)$ in the usual sense.

Then for any $\square$, we see
$$\mathrm{Hom}_{AV^{\square}_{/S}}(A_1, A_2) \overset{(*)}{\hookrightarrow} \mathrm{Hom}_{AV^{\phi}_{/S}}(A_1, A_2) \xrightarrow{\sim} \mathrm{Isog}(A_1, A_2).$$
So Indeed elements on the LHS can be regarded as genuine "isogenies".

Actually I doubt if $(*)$ is injective. When $S = \mathrm{Spec}\,k$, [李克正, P277 最下方] gives the $\mathbb{Z}$-flatness of $\mathrm{Hom}_k(A_1, A_2)$. But what about general abelian schemes?

Defn : We say (A, λ, ι) is a $\mathbb{Z}_{(\square)}$-polarized abelian scheme with O_K-action over S

if (1) $A \in AV^{\square}_{/S}$ of dimension $(r+s)d$.

(2) λ is a $\mathbb{Z}_{(\square)}$-polarization.

(3) $\iota : O_K \hookrightarrow \mathrm{End}^{\square}(A)$ an injective ring homomorphism, satisfying :

(R) Rossati condition : for any $b \in O_K$, $\iota(b)^{+} = \iota(\bar{b})$ where "$+$" is the Rossati involution induced by λ on $\mathrm{End}^{\square}(A)$.

(K) Kottwitz determinant condition : explain later in more detail.

Kottwitz determinant condition : How do we think of it ?

- Recall for such an abelian scheme A/S, we have an short exact sequence induced by the Hodge to de-Rham spectral sequence :

$$(\ast) \quad 0 \longrightarrow \omega_{A/S} \longrightarrow \mathcal{H}_1^{dR}(A/S) \longrightarrow \mathrm{Lie}\, A^{\vee}_{/S} \longrightarrow 0 \quad \text{as an } O_S\text{-module.}$$

- Then (K) actually implies that this SES can be decomposed into direct summands "at each places in Σ" :

$$\bigoplus_{i=1}^{d} \text{ of } \quad 0 \longrightarrow \begin{matrix} \omega_{A/S,\sigma_i} \\ \oplus \\ \omega_{A/S,\sigma_i^c} \end{matrix} \longrightarrow \begin{matrix} \mathcal{H}_1^{dR}(A/S)_{\sigma_i} \\ \oplus \\ \mathcal{H}_1^{dR}(A/S)_{\sigma_i^c} \end{matrix} \longrightarrow \begin{matrix} \mathrm{Lie}\, A^{\vee}_{/S,\sigma_i} \\ \oplus \\ \mathrm{Lie}\, A^{\vee}_{/S,\sigma_i^c} \end{matrix} \longrightarrow 0 \qquad \bigoplus_{i=1}^{d} \begin{matrix} O_S \\ \oplus \\ O_S \end{matrix}$$

Here the upper lines "—" are locally free O_S-modules of rank r, and the lower lines "—" are locally free O_S-modules of rank s.

Notations : For $\sigma \in \Sigma$, we often write $\omega_\sigma^{+} := \omega_{A/S,\sigma}$ and $\omega_\sigma^{-} := \omega_{A/S,\sigma^c}$, which are locally free O_S-modules of rank r, s.

cf. [SU14, §5.3.1]

This will enable us to define automorphic bundles and geometric modular forms.

Remark : I only know the proof of $(\ast)$ in the case of elliptic curves. Where can I find a proof in the setup of abelian schemes?

Kottwitz determinant condition :

For each $b \in O_K$, $\iota(b) \in \mathrm{End}^{\square}(A)$ acts on A, hence O_K acts on $\underline{\mathrm{Lie}}_{A/S}$. The condition (K) characterizes this action :

- Recall : $\underline{\mathrm{Lie}}_{A/S}$ is a locally free O_S-module of rank $(r+s)d$.

  Then for each affine open $U \subseteq S$, $\Gamma(U, \underline{\mathrm{Lie}}_{A/S})$ is a finite projective $\Gamma(U, O_S)$-module.
- So we can define the determinant

$$\det_U\left(X - \iota(b)\Big|_{\Gamma(U, \underline{\mathrm{Lie}}_{A/S})}\right) \in \Gamma(U, O_S)[X] \text{ of degree } (r+s)d.$$

  Moreover these sections are compatible. Hence we glue them together

$$\det_S\left(X - \iota(b)\Big|_{\Gamma(S, \underline{\mathrm{Lie}}_{A/S})}\right) \in \Gamma(S, O_S)[X] \text{ of degree } (r+s)d.$$

  often written as $\det\left(X - \iota(b)\big|_{\underline{\mathrm{Lie}}_{A/S}}\right) \in O_S[X]$. Then ι satisfies (K) if

$$\det\left(X - \iota(b)\Big|_{\underline{\mathrm{Lie}}_{A/S}}\right) = \prod_{\sigma \in \Sigma} (X - (\sigma^c)(b))^r (X - \sigma(b))^s \in O_S[X], \ \forall b \in O_K$$

- Application : The sheaf of invariant differential $\underline{\omega}_{A/S}$ has the factorization

$$\underline{\omega}_{A/S} = \underline{\omega}^+_{A/S} \oplus \underline{\omega}^-_{A/S}$$

  where $\underline{\omega}^{\pm}_{A/S}$ is locally free of rank dr and ds.

* Hidden algebraic version :

Motivation : O_K acts on $\underline{\mathrm{Lie}}_{A/S}$ the same way it acts on $V^{(0,-1)}$.

Starting point : Fix a generating family $\alpha_1, \cdots, \alpha_t$ of O_K as a $\mathbb{Z}^{(\square)}$-module.

- Let R be an algebra over O.
- Let M be a finite projective R-module, with an O_K-action by R-linear endomorphisms.

Defn : Consider the action of $O_K[X_1, \cdots, X_t]$ on $M \otimes_R R[X_1, \cdots, X_t]$. Then define Det_M as the determinant of the element $X_1\alpha_1 + \cdots + X_t\alpha_t$ for this action.

Tate modules :

Notation : We write $\mathbb{A}_{\bullet}$ without specify the field when we mean things over $\mathbb{Q}$.

- Recall $\hat{\mathbb{Z}}^{(\square)} = \prod_{\ell \notin \square} \mathbb{Z}_\ell$ and $\mathbb{A}_f^{\square} = \hat{\mathbb{Z}}^{(\square)} \otimes_{\mathbb{Z}} \mathbb{Q}$
- Let $V^{(\square)} := V \otimes_{\mathbb{Q}} \mathbb{A}_f^{\square}$. (distinguish the rational Tate module later !)

Let (A, λ, ι) as before.

Let $S \in Sch/_{\mathcal{O}}$ with a geometric point s.

Defn (1) $T(A_s) := \varprojlim_N A_s[N]$, $T^{(\square)}(A_s) := \varprojlim_{N \text{ away from } \square} A_s[N] = T(A_s) \otimes_{\mathbb{Z}} \mathbb{Z}_{(\square)}$

(2) $V^{(\square)}(A_s) = T^{(\square)}(A_s) \otimes_{\hat{\mathbb{Z}}^{(\square)}} \mathbb{A}_f^{\square}$.

Defn (1) Let $\eta^{(\square)} : V^{(\square)} \longrightarrow V^{(\square)}(A_s)$ be an $\mathcal{O}_K$-module morphism such that

$$\begin{tikzcd} V^{(\square)} \times V^{(\square)} \arrow[rr, "\psi_{r,s}"] \arrow[d, "\eta^{(\square)} \times \eta^{(\square)}"'] & & \mathbb{A}_f^{\square}(1) \arrow[d, dashed, "\exists\, u(\eta) \text{ for some } u(\eta) \in (\mathbb{A}_f^{(\square)})^{\times}"] \\ V^{(\square)}(A_s) \times V^{(\square)}(A_s) \arrow[rr, "e^{\lambda}"'] & & \mathbb{A}_f^{\square}(1) \end{tikzcd}$$

Commutes, i.e. $\eta^{(\square)}$ respects the bilinear forms on both sides up to a scalar and compatible with the $\mathcal{O}_K$-action on both sides, called a "<u>level morphism</u>".

(2) Let $g \in G(\mathbb{A}_f^{\square})$, then one <u>checks</u> $\eta \circ [g]$ is a level morphism.

(<u>note</u> : $[g]$ is the left multiplication map by g on $V^{(\square)}$)

(3) Let $U \subseteq K$ and $U^{(\square)} \subseteq G(\mathbb{A}_f^{\square})$. A <u>level structure</u> of <u>level</u> $U^{(\square)}$ on (A, λ, ι) $\bar{\eta}^{(\square)}$ is a $U^{(\square)}$-orbit of level morphisms $\eta^{(\square)} : V^{(\square)} \longrightarrow V^{(\square)}(A_s)$ such that the orbit is fixed under the action of $\pi_1(s, S)$.

<u>Remark</u> : The last condition ensures that a level structure is independent of the choice of s.

Moduli problem : Let $\mathcal{E}^{(\square)}_{U,T}$ be the following "category fibered in groupoid over $Sch/_T$".

- Objects over S : quadruples $\underline{A} := (A, \lambda, \iota, \overline{\eta}^{(\square)})$ where
  - (A, λ, ι) is a $\mathbb{Z}_{(\square)}$-polarized abelian scheme with O_K-action over S
  - $\overline{\eta}^{(\square)}$ a level structure of level $U^{(\square)}$.
- Morphism over S : $f : \underline{A} \longrightarrow \underline{A}'$ over S are given by a $\mathbb{Z}^{(\square)}$-isogeny $f : A \to A'$

s.t.
  - compatible with polarization : $\lambda = r(f^\vee \circ \lambda' \circ f)$ for some $r \in \mathbb{Z}^{(\square),\times}$
  - compatible with O_K-action : for any $b \in O_K$,

$$\begin{tikzcd} A \arrow[r, "\iota(b)"] \arrow[d, "f"'] & A \arrow[d, "f"] \\ A' \arrow[r, "\iota'(b)"'] & A' \end{tikzcd}$$

in $\mathrm{Hom}^\square$ respectively

Sometimes we write $f \in \mathrm{Hom}^\square_{O_K}(A, A')$ for such an f, as in [Hsieh14].

  - compatible with level structure : $\overline{\eta}'^{(\square)} = V^{(\square)}(f) \circ \overline{\eta}^{(\square)}$

So then actually we form

$$\begin{aligned} \mathcal{E}^{(\square)}_{U,T} : \quad Sch/_T &\longrightarrow Sets \\ S &\longmapsto \{ S\text{-quadruples } (A, \lambda, \iota, \overline{\eta}^{(\square)}) \} / \sim \text{"isogeny"} \end{aligned}$$

and discuss its representability.

Theorem 1 (Deligne–Shimura) When $\square = \phi$, $T = \mathcal{K}$, K is neat.

(1) $\mathcal{E}_{U,\mathcal{K}}$ is represented by a quasi-projective scheme $S_G(U)$ over $\mathcal{K}$.

We call it the Shimura variety attached to G of level U.

(2) Let $Sh(G,X)_U$ be the canonical model over $\mathcal{K}$. Then

$$S_G(U) = \coprod_{\ker^1(\mathbb{Q},G)} Sh(G,X)_U$$

where $\ker^1(\mathbb{Q},G)$ are locally trivial elements of $H^1(\mathbb{Q},G)$. When $r+s=n$ or $F=\mathbb{Q}$, $\ker^1(\mathbb{Q},G) = \{1\}$. (" failure of Hasse principle")

Theorem 2 (Kottwitz) When $\square = \{p\}$, $T = \mathcal{O}$,

(1) $\mathcal{E}_U^{(p)}$ is a smooth Deligne–Mumford stack.

(2) When K is neat, $\mathcal{E}_U^{(p)}$ is represented by a quasi-projective smooth scheme $S_G^{(p)}(U)$ over $\mathcal{O}$.

Proof: See [Rozensztajn, §4] for representability and §5 in loc.cit. for smoothness.

Theorem 3 (Generic fiber of the integral model) Recall: $\mathcal{O} \hookrightarrow \mathcal{K}$ with $\mathcal{K} = \mathrm{Frac}(\mathcal{O})$

For each $U \subseteq K$, recall K (and hence U) is hyperspecial at p and neat, we have

$$S_G^{(p)}(U) \times_{\mathrm{Spec}\,\mathcal{O}} \mathrm{Spec}\,\mathcal{K} \xrightarrow{\sim} S_G(U)$$

Proof: Essentially by the moduli interpretation. See [Rozensztajn, §7.2].

Remark: We briefly introduce the moduli for G_1 following [Hida04, §7.1.3]: $\mathcal{E}^{(p)}_{U,T,1}$ as

- objects over S: The same, yet modify:
  - level morphism $\eta^{(p)}$ as

$$\begin{tikzcd}
V^{(\square)} \times V^{(\square)} \arrow[r, "\psi_{r,s}"] \arrow[d, "\eta^{(\square)} \times \eta^{(\square)}"'] & \mathbb{A}_{F,f}^{\square}(1) \arrow[d, dotted, "\exists\, u(p) \text{ for some } u(p) \in (\mathbb{A}_{F,f}^{(\square)})^{\times}"] \\
V^{(\square)}(\mathbb{A}_S) \times V^{(\square)}(\mathbb{A}_S) \arrow[r, "e^{\lambda}"'] & \mathbb{A}_{F,f}^{\square}(1)
\end{tikzcd}$$

- morphisms over S: $f: A \to A'$ over S are given by a $\mathbb{Z}^{(p)}$-isogeny $f: A \to A'$
  - compatible with polarization: $\lambda = r(f^{\vee} \circ \lambda' \circ f)$ for some $r \in \mathcal{O}_{F,(\square),+}$
  - compatible with $\mathcal{O}_K$-action:
  - compatible with level structure:

  } same as before.

Then again we can formulate $S_{G_1}^{(p)}(U)$ and $S_{G_1}(U)$ as its generic fibre.

Now the good point is that Hasse principle holds for G_1, hence

$$S_{G_1}(U)(\mathbb{C}) \simeq Sh(G_1, X)_U = \underbrace{G(F)^+ \backslash X^+_{r,s} \times G(\mathbb{A}_{F,f}) / U}_{\text{in the sense of Talk 1}}$$

(cf. [Hida04, §7.1.5, p.319]). Hsieh used this.

§4 Compactifications

§4.1 Minimal compactification

<u>Philosophy</u> : On the complex analytic level, we add some "<u>rational boundary component</u>" to $X_{r,s}$, i.e. $X^*_{r,s} = X_{r,s} \cup \left(\bigcup_{1 \le t \le s} G(\mathbb{Q}) \cdot X_{(t)} \right)$

<u>Recall</u> : When discussing the Hermitian symmetric domain, we use

$$\mathrm{Res}_{\mathbb{Q}}^{F} GU(\mathbb{R}) = \prod_{\sigma \in \Sigma} GU(r_\sigma, s_\sigma)(\mathbb{R}),$$

Then we only focus on one component $GU(r,s)(\mathbb{R})$, which is precisely the unitary group defined in <u>Lan's "example note"</u>.

Throughout, let $1 \le t \le s$. Let $V_K = Y_K \oplus W \oplus X_K$, decomposed into

$$V_K = Y_K^{(t)} \oplus Y_{K,(t)} \oplus W \oplus X_K^{(t)} \oplus X_{K,(t)}, \quad V_{K,(t)} := Y_{K,(t)} \oplus W \oplus X_{K,(t)}$$

where $Y_K^{(t)} := Ky^1 \oplus \cdots \oplus Ky^t$, $Y_{K,(t)} = Ky^{t+1} \oplus \cdots \oplus Ky^s$ and similarly for X.

Then the induced metric on $V_K^{(t)}$ is nothing but $J_{r-t,s-t}$ as

$$J_{r,s} = \begin{pmatrix} & & 1_s \\ & S & \\ -1_s & & \end{pmatrix} = \begin{array}{c} \begin{pmatrix} & & & 1_t & \\ & & & & 1_{s-t} \\ & & S & & \\ -1_t & & & & \\ & -1_{s-t} & & & \end{pmatrix} \begin{array}{l} y^1,\dots,y^t \\ \\ w^i \\ x^1,\dots,x^t \\ \\ \end{array} \\ \begin{array}{ccc} y^1,\dots,y^t & w^i & x^1,\dots,x^t \end{array} \end{array}$$

Then $G^{(t)} := GU(V_{K,(t)}) \simeq GU(r-t, s-t)$.

Define a subgroup $P^{(t)}$ of $GU(r,s)$ of the form

$$P^{(t)} = \left\{ g = \begin{array}{c} \begin{pmatrix} X & * & * & * & * \\ & A & E & * & B \\ & F & M & * & G \\ & & & \lambda X^* & \\ & C & H & * & D \end{pmatrix} \begin{array}{l} t \\ s-t \\ r-s \\ t \\ s-t \end{array} \\ \begin{array}{ccc} \underbrace{t \quad s-t}_{s} & \underbrace{r-s}_{r-s} & \underbrace{t \quad s-t}_{s} \end{array} \end{array} \in GU(r,s) \;\middle|\; \begin{array}{l} \ell_1(g) := X \in \mathrm{Res}_F^K GL_t \\ \ell_2(g) := \begin{pmatrix} A & E & B \\ F & M & G \\ C & H & D \end{pmatrix} \in G^{(t)}, \\ \lambda = \nu(\ell_2(g)) \end{array} \right\}$$

One <u>checks</u> : $P^{(t)}$ is the stabilizer of the totally isotropic K-subspace $Y_K^{(t)}$.

Then by the <u>classification of parabolics</u> (cf. [Shi97, §2]), $P^{(t)}$ is a maximal parabolic subgroup of G. When $t=1$, $P^{(1)}$ is called the <u>Klingen parabolic subgroup</u> of G.

More on $P^{(t)}$

(1) Levi decomposition : $P^{(t)} = L^{(t)} \ltimes N^{(t)}$, where

- $L^{(t)} = \left\{ g = \begin{pmatrix} X & & & & \\ & A & E & & B \\ & F & M & & G \\ & & & \lambda X^* & \\ & C & H & & D \end{pmatrix} : \begin{array}{l} \ell_1(g) := X \in \mathrm{Res}_F^K GL_t \\ \ell_2(g) := \begin{pmatrix} A & E & B \\ F & M & G \\ C & H & D \end{pmatrix} \in G^{(t)} \\ \lambda := \lambda(\ell_2(g)) \end{array} \right\} \simeq \mathrm{Res}_F^K GL_t \times G^{(t)}$

- $N^{(t)} = \left\{ \begin{pmatrix} 1 & * & * & * & * \\ & 1 & & * & \\ & & 1 & * & \\ & & & 1 & \\ & & & * & 1 \end{pmatrix} \in GU(r,s) \right\}$ is the unipotent radical.

(2) Moreover, $U^{(t)}$ has a further extension as

$$1 \longrightarrow W^{(t)} \longrightarrow N^{(t)} \longrightarrow V^{(t)} \longrightarrow 1$$

- $W^{(t)} := \left\{ \begin{pmatrix} 1 & & & Y & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix} : Y \in \mathrm{Res}_F^K \mathrm{Herm}_t \right\} \simeq \mathrm{Res}_F^K \mathrm{Herm}_t$

  One checks : $W^{(t)}$ is the centre of $N^{(t)}$.

- $V^{(t)}(R) \simeq (K \otimes_F R)^{(r-t)+(s-t)}$

Now we restricts to the unitary case. Consider for $1 \leq t \leq s$

$$X^+_{(t)} := \begin{pmatrix} \infty_r & \\ & X_{r-t,s-t} \end{pmatrix} = \left\{ \begin{array}{c|c} & \begin{array}{cc} t & s-t \end{array} \\ \begin{array}{c} t \\ s-t \\ r-s \\ t \\ s-t \end{array} & \begin{pmatrix} 1_t & \\ & x \\ & y \\ 0_t & \\ & 1_{s-t} \end{pmatrix} \end{array} \; \Bigg]\, s+r \; : \; \begin{pmatrix} x \\ y \end{pmatrix} \in X^+_{r-t,s-t} \right\} \simeq X^+_{r-t,s-t}$$

with the two columns t, $s-t$ together of size $\underbrace{t \quad s-t}_{s}$.

As $U(r,s)(\mathbb{R})$ acts on $X_{(t)}$ by left multiplication (under projective coordinate), one checks $X^{(t)}$ is stablized by $P^{(t)}(\mathbb{Q})$. Then on the complex analytic level,

$$X^*_{r,s} = X_{r,s} \cup \Big(\bigcup_{1 \leq t \leq s} G(\mathbb{Q}) \cdot X^+_{(t)} \Big)$$

here G is the unitary group without silimitude.

as the minimal compactification of $X^+_{r,s}$.

§4.2 Toroidal compactification

- <u>Cusp labels</u> : Let $1 \le t \le s$, define

  means $\mathrm{Res}^{K}_{F} GL_t$ in the previous section

$$\underline{C}_t(K) = \left(\left(\boxed{GL(Y_{K,(t)})} \times G^{(t)} \right)(\mathbb{A}_{F,f}) \right) N^{(t)}(\mathbb{A}_{F,f}) \backslash G(\mathbb{A}_{F,f}) / K$$

  The set is finite, called <u>the set of cusp labels</u> for $S_G(K)$

  <u>Exercise</u> : For each $[g] \in \underline{C}_t(K)$, we can choose $g = pk^\circ$ for $p \in P^{(t)}(\mathbb{A}_{F,f}^{(pN_0)})$ and $k^\circ \in K^\circ$ with similitude $\lambda(g) \in \widehat{O_F}$.

  One writes $\underline{C}(K) := \bigsqcup_{1 \le t \le s} \underline{C}_t(K)$.

- $\boxed{\text{Black box}}$ To a datum of "smooth rational cone decomposition" $\{\Sigma_{[g]}\}_{[g] \in \underline{C}(K)}$ of F_+, we attach a <u>toroidal compactification</u> $\overline{S}^{(p)}_G(K)/\mathcal{O}$ of $S^{(p)}_G(K)/\mathcal{O}$.
  - $\overline{S}^{(p)}_G(K)/\mathcal{O}$ is a proper smooth scheme over $\mathcal{O}$ containing $S^{(p)}_G(K)/\mathcal{O}$
  - The complement $\overline{S}^{(p)}_G \setminus S^{(p)}_G$ is a relative Cartier divisor with normal crossings.
  - Let $\underline{\mathcal{A}} = (\mathcal{A}, \lambda, \iota, \overline{\eta}^{(p)})$ be the universal quadruple over $S^{(p)}_G(K)$. Then $\underline{\mathcal{A}}$ has an extension $\underline{\mathcal{G}} = (\mathcal{G}, \lambda, \iota, \overline{\eta}^{(p)})$ where
    - $\mathcal{G}$ is a semiabelian scheme over $\overline{S}^{(p)}_G(K)$. $\mathcal{G}|_{S^{(p)}_G} = \mathcal{A}$.
    - $\lambda : \mathcal{G} \longrightarrow \mathcal{G}^\vee$ a "$\mathbb{Z}_{(p)}$-polarization"
    - $\overline{\eta}^{(p)}$ is the level structure in $\underline{\mathcal{A}}$.

  Moreover, $\underline{\omega_{\mathcal{G}}} := e^*\Omega_{\mathcal{G}/\overline{S}^{(p)}_G}$, the <u>sheaf of invariant differential</u>, is a locally free coherent $\mathcal{O}_{\overline{S}^{(p)}_G}$-module.

- The <u>minimal compactification</u> $S^{*(p)}_G(K)/\mathcal{O}$ is defined as

$$S^{*,(p)}_G(K) := \mathrm{Proj} \bigoplus_{k=0}^{\infty} \Gamma(\overline{S}^{(p)}_G(K), \det \underline{\omega}^k)$$

Then there is a commutative diagram of $\mathcal{O}$-schemes

$$\begin{tikzcd} & \overline{S}^{(p)}_G(K) \arrow[d, "\pi"] \\ S^{(p)}_G(K) \arrow[ur, "\overline{i}"] \arrow[r, "i"'] & S^{*(p)}_G(K) \end{tikzcd}$$

$i, \overline{i}$: open immersion

π : blow-down map.

Theorem (Lan Kai-wen)

(1) $\pi_*(\det \underline{\omega})$ is an ample line bundle on S^*, and S^* is a normal projective scheme of finite type over $\mathcal{O}$.

(2) $\pi_* \mathcal{O}_{\overline{S}} = \mathcal{O}_{S^*}$, hence π has geometrically connected fibres.

(3) $S^*(\mathbb{C})$ is the classical Satake–Baily–Borel compactifications.

(4) $\exists$ a natural stratification of $\partial S^* := S^* \setminus S$ indexed by $\underline{C}(K)$:

$$\partial S^* = \bigsqcup_{1 \leq t \leq s} \bigsqcup_{[g]_t \in \underline{C}_t(K)} S^{(p)}_{G^{(t)}}(K^g_{(t)}), \quad K^g_{(t)} = G^{(t)}(\mathbb{A}_{F,f}) \cap gKg^{-1}.$$

§5 Igusa schemes

Philosophy: To go from Shimura varieties to Igusa schemes, we add level structures j at p.

§5.1 Polarizations

- Recall $M = Y \oplus L \oplus X^\vee$ is a $\mathcal{O}_K$-lattice in V_K, $M_p := M \otimes_{\mathbb{Z}} \mathbb{Z}_p$.

Defn: An ordered polarization of M_p is a pair of $\mathbb{Z}_p$-sublattices $\mathrm{Pol}_p := \{N^{-1}, N^0\}$ of M_p, such that

(1) N^0, N^{-1} are maximal isotropic submodules in M_p

(2) N^0 and N^{-1} are dual to each other w.r.t. $\psi_{r,s}$

(3) for each $v = w\bar{w}$ for $v|p$, $w \in \Sigma_p$,

$$\mathrm{rank}\, N_w^{-1} = \mathrm{rank}\, N_{w^c}^{0} = r ,$$
$$\mathrm{rank}\, N_{w^c}^{-1} = \mathrm{rank}\, N_w^{0} = s .$$

Example: The standard polarization $\mathrm{Pol}_p^{std} = \{N^{std,-1}, N^{std,0}\}$ of M_p is given on each $v|p$ by

$$\begin{cases} N_v^{std,-1} := Y_w \oplus L_w \oplus Y_{w^c} \\ N_v^{std,0} := X_{w^c} \oplus L_w \oplus X_w \end{cases}$$

for $v = w\bar{w}$, $w \in \Sigma_p$.

§5.2 Moduli problems

Let $Pol_p = \{N^{-1}, N^0\}$ be a polarization of M_p.

<u>Defn</u>: Let $\mathcal{J}^{(p)}_{K,n,Pol_p,\mathcal{O}}$ be the following "category fibered in groupoid over $Sch/_{\mathcal{O}}$",

- <u>Objects over S</u> : S-quintuples $(\underline{A}, j)_S$ of <u>level</u> K^n, where
  - $\underline{A} := (A, \lambda, \iota, \bar{\eta}^{(p)}) \in \mathcal{C}^{(p)}_{K,\mathcal{O}}$,
  - $j : \mu_{p^n} \otimes_{\mathbb{Z}} N^0 \hookrightarrow A[p^n]$ a monomorphism as $\mathcal{O}_K$-group schemes over S.

    Called the <u>level-p^n structure</u> of A.
- <u>Morphism over S</u> : $f : (\underline{A}, j) \longrightarrow (\underline{A}', j')$ over S as
  - $f : \underline{A} \longrightarrow \underline{A}'$ in $\mathcal{C}^{(p)}_{K,\mathcal{O}}$
  - $f \circ j = j'$.

<u>Theorem</u> (Hida ?) $\mathcal{J}^{(p)}_{K,n,Pol_p}$ is relative representable by a scheme $I_G(K^n, Pol_p)$ over the Kottwitz model $S^{(p)}_G(K)/\mathcal{O}$.

- Let $\underline{\mathcal{A}}$ be the universal quadraple of $S^{(p)}_G(K)/\mathcal{O}$. Then one checks

$$I_G(K^n, Pol_p^{std}) = \underline{\mathrm{Inj}}_{\mathcal{O}_K}(\mu_{p^n} \otimes_{\mathbb{Z}} M^0, \mathcal{A}).$$

where $Pol_p^{std} = \{M^{-1}, M^0\}$ defined previously. Later on, denote LHS $=: I^0_G(K^n)$.

- Moreover, given two polarizations

$$Pol_{p,1} = \{M^{-1}, M^0\} \quad \text{and} \quad Pol_{p,2} = \{N^{-1}, N^0\}.$$

Then choose $\gamma \in K^0_p$ such that $N^{-1} = \gamma \cdot M^{-1}$, $N^0 = \gamma \cdot M^0$ (<u>recall</u> : K^0_p is by definition the subgroup of $G(\mathbb{F}_0)$ preserving M_p.)

Then $j \longmapsto \gamma j$ is an isomorphism from level p^n-structure of $Pol_{p,1}$ to that of $Pol_{p,2}$.

Therefore $[(\underline{A}, j)] \mapsto [(\underline{A}, \gamma j)]$ induces an isomorphism of groupoids

$$\mathcal{J}^{(p)}_{K,n,Pol_{p,1},\mathcal{O}} \xrightarrow{\sim} \mathcal{J}^{(p)}_{K,n,Pol_{p,2},\mathcal{O}}.$$

Proposition : Let L be an field extension of $K(e^{2\pi i/p^n})$. Then there is an non-canonical isomorphism over $\mathrm{Spec}\, L$:

$$I_G^{\circ}(K^n)/_L \xrightarrow{\sim} S_G(K^n)/_L$$

Proof : Since $e^{2\pi i/p^n}$ is included in L, we choose an isomorphism $\varphi_{p^n} : \mathbb{Z}/n\mathbb{Z} \longrightarrow \mu_{p^n}$, then a level p^n-structure becomes

$$j' : \mathbb{Z}/n \otimes_{\mathbb{Z}} N^{std,0} \xrightarrow[\sim]{\varphi_{p^n}^{-1}} \mu_{p^n} \otimes_{\mathbb{Z}} N^{std,0} \xhookrightarrow{j} A[p^n] \quad — ①$$

Then by taking Cartier dual of j (not j')

$$j^{\vee} : A[p^n] \longrightarrow\!\!\!\!\rightarrow M_p/N^{std,0} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \simeq N^{std,-1} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \quad — ②$$

comparing ① and ②, we see $M \otimes \mathbb{Z}/p^n\mathbb{Z} \simeq A[p^n]$. Then taking inverse limit, we get $M \otimes \mathbb{Z}_p \simeq T_p(A)$. Together with $\overline{\eta}^{(p)}$, we get the result. $\square$

§5.3 Igusa schemes over $\bar{S}^{(p)}_G(K)$

Inspired by §5.2, we define the Igusa scheme $Ig_G(K^n)$ over $\bar{S}^{(p)}_G(K)$ to be the scheme representing the functor

$$Ig_G(K^n) := \underline{\mathrm{Inj}}_{O_K}\left(\mu_{p^n} \otimes_{\mathbb{Z}} M^\circ, \mathcal{G}\right).$$

Defn: (0) Let $\mathbb{H} := GL_{O_K}(M^\circ) = GL_{O_F}(M^\circ_{\Sigma_p}) \times GL_{O_F}(M^\circ_{\Sigma_p^c})$. Accordingly we have $\mathbb{N}, \mathbb{B}$.

(Recall $H := \mathrm{Res}^{O_F}_{\mathbb{Z}} GL_r \times \mathrm{Res}^{O_F}_{\mathbb{Z}} GL_s$ before. One checks $\mathbb{H} \simeq H(\mathbb{Z}_p)$).

Then $\mathbb{H}$ acts on $Ig_G(K^n)$ by

$$h \cdot j(m^\circ) = j(h \cdot m^\circ), \quad \text{for } h \in \mathbb{H},\ m^\circ \in M^\circ \otimes_{\mathbb{Z}} \mu_{p^n}.$$

$$\left.\begin{array}{ll} (1) & Ig_G(K_1^n) := Ig_G(K^n)/\mathbb{N} \\ (2) & Ig_G(K_0^n) := Ig_G(K^n)/\mathbb{B} \end{array}\right\}$$

here by "moding" $\mathbb{N}$ and $\mathbb{B}$, we mean only consider the $\mathbb{N}$-orbits or $\mathbb{B}$-orbits of level p^n-structures under the action of (0).

- Varying n, we have an Igusa tower.

$\pi_{m,n}$: $M^\circ \otimes_{\mathbb{Z}} \mu_{p^n} \hookrightarrow M^\circ \otimes_{\mathbb{Z}} \mu_{p^m}$

π_n: forgetful map $(\underline{A}, j) \longmapsto \underline{A}$

étale $\Downarrow$ $Ig_G(K^n)$ is smooth over $\mathrm{Spec}\, \mathcal{O}$

$$\begin{tikzcd}
Ig_G(K^m) \arrow[r, two heads] \arrow[d] \arrow[dd, bend right=60, "\pi_{m,n}"'] & Ig_G(K_1^m) \arrow[r, two heads] \arrow[d] & Ig_G(K_0^m) \arrow[d] \\
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] \\
Ig_G(K^n) \arrow[r, two heads] \arrow[d] \arrow[ddd, bend right=60, "\pi_n"'] & Ig_G(K_1^n) \arrow[r, two heads] \arrow[d] & Ig_G(K_0^n) \arrow[d] \\
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] \\
Ig_G(K) \arrow[r, two heads] \arrow[d] & Ig_G(K_1) \arrow[r, two heads] \arrow[dl] & Ig_G(K_0) \arrow[dll] \\
\bar{S}^{(p)}_G(K) & \boxed{\mathbb{N}\text{-orbits}} & \boxed{\mathbb{B}\text{-orbits}}
\end{tikzcd}$$

§6 Geometric modular forms

§6.1 Recap on torsors

"Definition": Let G be a flat group scheme over S. A G-torsor (for the Zariski top) is a scheme $\pi: \mathcal{E} \longrightarrow S$ such that

(1) G acts on $\mathcal{E}$, given by $\Phi: G \times_S \mathcal{E} \longrightarrow \mathcal{E}$, satisfying "obvious axioms".

(2) $\mathcal{E}$ is locally trivial in a sense that $\exists$ a Zariski open covering $\{U_i\}_{i\in I}$ of S and an isomorphism

$$\phi_i : \mathcal{E} \times_S U \xrightarrow{\sim} G \times_S U \quad \text{for each } i \in I$$

such that

$$\begin{tikzcd} G \times_S (\mathcal{E} \times_S U) \arrow[r, "\Phi"] \arrow[d, "\phi_i"', "\simeq"] & \mathcal{E} \times_S U \arrow[d, "\phi_i", "\simeq"'] \\ G \times_S G \times_S U \arrow[r, "m"] & G \times_S U \end{tikzcd}$$

commutes.

Fact. $\{$ isomorphism classes of G-torsors on $S \} \longleftrightarrow \check{H}^1(S, G)$ (Čech cohomology)

Example 1: $\{$ vector bundles of rank n over $S \}/\sim$ = locally free $\mathcal{O}_X$-module of rank n.

$$\updownarrow$$

$\{ GL_n$-torsors over $S \} \longleftrightarrow \check{H}^1(S, GL_n(\mathcal{O}_S))$

$U \longmapsto GL_n(\Gamma(U, \mathcal{O}_S))$

Example 2: Let $\rho: G \longrightarrow GL(V)$ be an algebraic representation over S (i.e. a rep'n as an algebraic group for algebraic groups G). Then there is a natural functor preserving "tensor" and "dual": fix any G-torsor $\mathcal{E}$ on S,

$$\{\text{Alg. rep of } G\} \longrightarrow \{\text{Vect. bundles on } S\}$$

$$V \longmapsto \mathcal{E} \overset{G}{\times} V := \mathcal{E} \times V / \text{diagonal } G\text{-action}$$

Here the "quotient" can be seen locally as $(G \times U \times V)/G \simeq \text{"}X\text{"} \times U$, but there is a global twist.

Another way: $\check{H}^1(X, G) \xrightarrow{\rho} \check{H}^1(X, GL(V))$, $[\mathcal{E}] \longmapsto [\mathcal{E} \overset{G}{\times} V]$.

§6.2 Geometric modular forms

Let $\underline{\omega} := e^*\Omega_{\mathcal{G}/\overline{S}_G^{(\varphi)}(K)}$. Recall the "determinant condition" (and the notations there).

<u>Defn</u>: $\mathcal{E}^+ := \bigoplus_{\sigma\in\Sigma} \underline{\mathrm{Isom}}(\mathcal{O}_S^r, \underline{\omega}_\sigma^+)$ and $\mathcal{E}^- := \bigoplus_{\sigma\in\Sigma} \underline{\mathrm{Isom}}(\mathcal{O}_S^s, \underline{\omega}_\sigma^-)$

Then $\mathcal{E} := \mathcal{E}^+ \oplus \mathcal{E}^-$ is an <u>H-torsor</u> over $\overline{S}_G^{(\varphi)}(K)$ with structure map $\pi : \mathcal{E} \longrightarrow \overline{S}_G^{(\varphi)}(K)$

<u>R-points of $\mathcal{E}$</u>: Let R be an $\mathcal{O}$-algebra, then an R-point of $\mathcal{E}$ is a pair (x,ω) where:

- $x \in \overline{S}_G^{(\varphi)}(R)$, it also has a moduli interpretation.
- $\omega = (\omega^+, \omega^-) : \bigoplus_{\sigma\in\Sigma}(\mathcal{O}_K\otimes_{\mathbb{Z}} R^r) \oplus \bigoplus_{\sigma\in\Sigma}(\mathcal{O}_K\otimes_{\mathbb{Z}} R^s) \to \underline{\omega}^+_{/x} \oplus \underline{\omega}^-_{/x}$ is an $\mathcal{O}_K \otimes_{\mathbb{Z}} R$-module isomorphism.

and its H-torsor structure is obtained by

$$h\cdot(x,\omega) := (x, h\cdot\omega), \quad h\cdot(\omega^+,\omega^-)(v^+,v^-) := (\omega^+(h^+v^+), \omega^-(h^-v^-)).$$

<u>Assume in the following talks</u>:

- $\underline{k}$ is a <u>parallel weight</u>, i.e. $k = (a_1, \cdots, a_r, b_1, \cdots, b_s) \in \mathbb{Z}[\Sigma]^{r+s}$ satisfies $k_\sigma = k_\tau$ for any $\sigma, \tau \in \Sigma$.
- $\underline{k}$ is a <u>dominant weight</u> (in the sense of Hsieh, not Wan.)

<u>Defn</u>: $\underline{\omega_k} := \mathcal{E} \times^H L_{\underline{k}}$ as the <u>automorphic bundle</u> over $\overline{S}_G^{(\varphi)}(K)$ of <u>weight</u> $\underline{k}$.

<u>Remark</u>: Recall the <u>meeting with Prof. Wan on 5/18/2023</u>,

- $\underline{k}$ is called a <u>scalar weight</u> if $\underline{\omega_k}$ here is a line bundle. For general unitary group case here, it is equivalent to say that $a_i = a$ and $b_j = b$ for any $1\le i\le r$, $1\le j\le s$.

In fact, in [Wan. 2015 ANT], the weight is even assumed as parallel and $(\underbrace{0,\cdots,0}_{r}, \underbrace{k,\cdots,k}_{s})$ at archemedian places.

- Then a section f of $\underline{\omega_k}$ is a morphism $f : \mathcal{E} \longrightarrow L_{\underline{k}}$ such that

$$f(x, h\omega) = \rho_{\underline{k}}(h) f(x,\omega), \quad \text{for any } h\in H,\ x\in \overline{S}_G^{(\varphi)}(K).$$

Defn 3 (Geometric version)

Let R be a flat $\mathbb{Z}_{(p)}$-algebra. Define the space of <u>geometric modular forms</u> over R of <u>weight</u> $\underline{k}$ and level $K_\bullet^n$ (for $\bullet = 0, 1, \phi$) by

$$M_{\underline{k}}(K_\bullet^n, R) := H^0(\mathcal{I}_G(K_\bullet^n)_{/R}, \underline{\omega_k})$$

note: by abuse of notation, the automorphic bundle $\underline{\omega_k}$ over $\overline{S}_G^{(p)}(K)$ can be pulled pack and pushed-forward on the Igusa towers. We still use $\underline{\omega_k}$ to denote it.

Koechner's principle: When $\operatorname{char} R = 0$, $rs > 1$ (or $F \neq \mathbb{Q}$), then

$$M_{\underline{k}}(K_\bullet^n, R) = H^0(\overline{S}_G^{(p)}(K_\bullet^n)_{/R}, \underline{\omega_k}).$$

Remark:

- When $rs = 1$, the only option is $GU(1,1)$, which is essentially GL_2. Moreover, $F = \mathbb{Q}$, means just $GL_2/\mathbb{Q}$. So even when Kochner's principle fails here, we can say alright.
- When $s = 0$, i.e. G is the definite unitary group, and $F = \mathbb{Q}$. This is the key case when Kochner's principle fails.

§7 p-adic modular forms

- <u>Hasse invariant</u> : $\overline{H}_{p-1} \in H^0(S_G^{(\varphi)}(K)/\overline{\mathbb{F}}_p\,,\ \det(\underline{\omega})^{p-1})$.
  - Since $\pi_*(\det\omega)$ is ample, we can lift a power of the Hasse invariant $\overline{H}_{p-1}^{\,t}$ to a global section $E \in H^0(S_G^{(\varphi)}(K),\ \det(\omega)^{t(p-1)})$.
  - Moreover, we can further extend E to $E \in H^0(\overline{S}_G^{(\varphi)}(K),\ \det(\omega)^{t(p-1)})$

  This helps us to characterize the <u>ordinary locus</u> $T_{0,m} := \overline{S}_G^{(\varphi)}(K)[\frac{1}{E}]/\mathcal{O}_m$

  Moreover, by condition (ord), $T_{0,m}$ is open and dense in $\overline{S}_G^{(\varphi)}(K)/\mathcal{O}_m$.

- For $n > 1$, set $T_{n,m} := I_G(K^n)/\mathcal{O}_m$

  Then we can define the <u>Igusa tower</u> $T_{\infty,m} := \varprojlim_n T_{n,m}$, which is a Galois cover of $T_{0,m}$ of Galois group $\mathbb{H} = GL_{\mathcal{O}_K}(M^0)$.

$$
\begin{tikzcd}
T_{\infty,m} \arrow[dddddd, bend left=40, "\text{Galois cover of Galois group } \mathbb{H}"] & \\
T_{n',m} \arrow[d] \arrow[dd, bend right=40, swap, "\substack{n' \geq n \geq 0 \\ \text{finite étale}}"] & \\
\vdots \arrow[d] & \\
T_{n,m} \arrow[d] & \\
\vdots \arrow[d] & \\
\ \arrow[d] & \\
T_{0,m} \arrow[r] \arrow[d, swap, "\substack{\text{open} \\ \text{dense}}"] \arrow[dr, phantom, "\lrcorner", very near start] & \overline{S}_G^{(\varphi)}(K)[\frac{1}{E}] \arrow[d] \\
\overline{S}_G^{(\varphi)}(K)/\mathcal{O}_m \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \overline{S}_G^{(\varphi)}(K) \arrow[d] \\
\operatorname{Spec}\mathcal{O}_m \arrow[r] & \operatorname{Spec}\mathcal{O}
\end{tikzcd}
$$

Defn 4 Let R be a p-adic $\mathcal{O}_p$-algebra. Let $R_m = R/p^m R$ for $m \in \mathbb{Z}_+$. Define

- $V_{n,m} := H^0(T_{n,m}, \mathcal{O}_{T_{n,m}})$ ← 真正构造 p-adic modular form
- $V_{\underline{k}}(K^n, R_m) = H^0(T_{n,m}/R_m, \underline{\omega}_{\underline{k}})$
- $V_{\underline{k}}(K_1^n, R_m) = H^0(T_{n,m}/R_m, \underline{\omega}_{\underline{k}})^N$

} 过渡手段: "false modular form" à la Deligne

Then let $V_{\infty,m} := \varinjlim_n V_{n,m}$ and $V_{\infty,\infty} = \varprojlim_m V_{\infty,m}$. Then we call the space $V_p(G,K) := V_{\infty,\infty}^N$ the <u>space of p-adic modular forms</u>.

– Moreover, let $\Lambda_{\mathbb{T}} := \mathcal{O}_p[[\mathbb{T}]]$ where $\mathbb{T} = T(\mathbb{Z}_p) \subseteq H$.

Recall in [Wan2015ANT], the weight space is defined to be $\Lambda_{r,s} := \mathcal{O}_L[[T(1+p\mathbb{Z}_p)]]$.

Here $\mathbb{T} = T(\mathbb{Z}_p)$. Yet one <u>notes</u> that via exponential and logarithm map,

$$(\mathbb{Z}_p, +) \xrightarrow{\sim} (p\mathbb{Z}_p, +) \underset{\log}{\overset{\exp}{\rightleftarrows}} (1+p\mathbb{Z}_p, \times)$$

So in Hsieh, $\Lambda_{\mathbb{T}}$ is exactly the <u>weight space</u> $\Lambda_{r,s}$ of Wan.

<u>Note</u>: $\Lambda_{\mathbb{T}}$ is isomorphic to a $\mathcal{O}_p$-power series ring of $(r+s)$-variables.

Then explicitly, the $\mathbb{T}$-action on $V_{\infty,m}^N$ is defined as

$$[t]\cdot f(\underline{A}, j) := f(\underline{A}, tj), \quad t \in \mathbb{T},\ f \in V_{\infty,m}^N.$$

In this way, $V_{\infty,m}^N$ is a discrete $\Lambda_{\mathbb{T}}$-module.

<u>Hodge–Tate map</u>