Talk 2 Main Conjectures

§0 Preparations on algebraic stuffs : Let A be a commutative ring.

0.1 Objects level

- Let $P_{fg}(A) :=$ the category of finitely generated projective module.

  For each $P \in P_{fg}(A)$, we can define

$$\mathrm{Det}_A(P) \text{``}=\text{''} \Lambda_A^{rk_A(P)} P \quad (\text{at least when } P \text{ is finite free as } A\text{-module})$$

More precise definition : See [Stacks project, 0FJ9] : $P \in P_{fg}(A)$, then $\exists$

$$A = A_0 \times \cdots \times A_t,\quad P = P_0 \times \cdots \times P_t \quad \text{s.t. } P_i \text{ is finite loc. free } A_i\text{-module.}$$

Then in this situation, we define

$$\mathrm{Det}_A(P) = \Lambda^0_{A_0} P_0 \times \Lambda^1_{A_1} P_1 \times \cdots \times \Lambda^t_{A_t} P_t \quad \text{as an R-module.}$$

This is a finite locally free A-module of rank one. (since each $\Lambda^i_{A_i} P_i$ is locally free A_i-module of rank one.)

We list its properties :

(i) $\mathrm{Det}_A(0) = A$: this recovers the coefficient ring A !

(ii) Let $0 \to P' \to P \to P'' \to 0$ in $P_{fg}(A)$, then

$$\mathrm{Det}_A(P) \xrightarrow{\sim} \mathrm{Det}_A(P') \underset{A}{\otimes} \mathrm{Det}_A(P'') \quad (\text{``}\otimes_A\text{'' usually written as ``}\cdot\text{''})$$

(iii) <u>Base change property</u> : $\mathrm{Det}_A(P) \otimes_A B \simeq \mathrm{Det}_B(P \underset{A}{\otimes} B)$

- The target category can be seen as the "category of finite locally free A-mod of rank one", which is a "<u>Picard category</u>" in the sense that :
  - the isomorphism classes of projective A-modules of rank one form an <u>abelian group</u>, called the <u>Picard group</u> of A, under

$$[P] \cdot [Q] := [P \otimes_A Q] \quad \text{in } \mathrm{Pic}(A).$$

- The neutral element is $[A]$ and the inverse of $[P]$ is $[P]^{-1} := [P^*]$.
  (We omit the brackets everywhere.)

- We record another property: [Kato. Italy P72] Let Λ = regular noetherian ring, M := finitely generated torsion Λ-module. Then

$$\mathrm{Det}_\Lambda^{-1}(M) \simeq \mathrm{char}_\Lambda(M) \quad \text{as fractional ideals in } Q(\Lambda).$$

0.2 Category of chain complexes

- Let Perf_A := category of complexes of A-modules, qis to a bounded complexes of modules in $P_{fg}(A)$

For $C^\bullet \in \mathrm{Perf}_A$, fix a quasi-isomorphism $\tilde{C}^\bullet \to C^\bullet$, we define

$$\mathrm{Det}_A(C^\bullet) := \prod_{i \in \mathbb{Z}} \mathrm{Det}_A^{(-1)^i}(\tilde{C}^i)$$

$\leadsto$ Det_A can be regarded as a functor from the "sub" derived category of A-mods consisting of complexes in Perf_A.

Corresponding to the properties (i) to (iii), we have:

$(i)^D$: If $C^\bullet$ is acyclic, then $\mathrm{Det}_A(C^\bullet) = \mathrm{Det}_A(0) = A$.

$(ii)^D$: Let $P' \to P \to P''$ be a distinguished triangle (exclude some of them?)
then there is a canonical isomorphism

$$\mathrm{Det}_A P \underset{\longleftarrow}{\xrightarrow{\sim}} \mathrm{Det}_A P' \otimes \mathrm{Det}_A P''.$$

<u>In pratice</u>: Let $f: P' \to P$ be a morphism of complexes, then
$P' \xrightarrow{f} P \longrightarrow \mathrm{cone}(f) \in$ distinguished triangle. We apply Det_A:

$$\mathrm{Det}_A P \underset{\longleftarrow}{\xrightarrow{\sim}} \mathrm{Det}_A P' \otimes \mathrm{Det}_A(\mathrm{Cone}(f))$$

(!) Note: this isomorphism carries the information of f, not just an abstract isomorphism!

We often rewrite it as

$$\mathrm{Det}_A(f) : \mathrm{Det}_A P \cdot \mathrm{Det}_A^{-1} P' \underset{\circledast}{\overset{\sim}{\rightleftarrows}} \mathrm{Det}_A(\mathrm{Cone}(f)) .$$

Take the inverse of this map ($\circledast$), we see

$$\mathrm{triv}_{A,f} : \mathrm{Det}_A^{-1} P \underset{A}{\otimes} \mathrm{Det}_A P' \xrightarrow{\sim} \mathrm{Det}_A^{-1}(\mathrm{Cone}(f))$$

this map is often called the <u>trivialization</u> of f.

$(\mathrm{iii})^D$: <u>Base change property</u>: similar as before.

(iv): If $C^\bullet \in \mathrm{Perf}_A$ s.t. $H^j(C^\bullet) \in P_{fg}(A)$ for all $j \in \mathbb{Z}$, then one has

$$\boxed{\mathrm{Det}_A(C^\bullet) \simeq \prod_{j \in \mathbb{Z}} \mathrm{Det}_A^{(-1)^j}(H^j(C^\bullet))}$$

This can be regarded as an algebrai object analogue of "Euler characteristic" of the complex $C^\bullet$.

<u>A quick review</u>:

Recall in <u>Talk 1</u>: we fixed an eigencusp form f with coefficient field L, with ramified set $\Sigma \supseteq \{p\}$ and Galois rep'n

$$\rho : \mathrm{Gal}_{\mathbb{Q},\Sigma} \to GL_2(E)$$

for some $E/\mathbb{Q}_p$, with integers $\mathcal{O}$, uniformizer ϖ and residue field $\mathbb{F}$. We formed the deformation space $\mathfrak{X}_\Sigma(\bar\rho)$ when $\bar\rho$ satisfies extra properties ($\circledast$).

§1 Kato's Iwasawa main conjecture and a reformulation

Motivationally, we introduce Kato's IMC using his zeta elements.

Advantage: This formulation does not require any ordinality condition at p.

- Iwasawa cohomology: Let $T \subseteq V$ be a $G_{\mathbb{Q}}$-stable lattice of a Galois rep V of $\mathrm{Gal}_{\mathbb{Q}}$, of coefficient $\mathcal{O}_E$. We define
  - $T_{\mathrm{Iw}} := T \otimes_{\mathcal{O}} \mathcal{O}[\![\mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})]\!]$
  - $H^i_{\mathrm{Iw}}(T) := H^i_{\text{ét}}(\mathbb{Z}[\tfrac{1}{p}], T_{\mathrm{Iw}}) \cong \varprojlim_n H^i_{\text{ét}}(\mathbb{Z}[\tfrac{1}{p}], T \otimes_{\mathcal{O}} \mathcal{O}[G_n])$

    $= \varprojlim_n H^i_{\text{ét}}(\mathbb{Z}[\zeta_{p^n}, \tfrac{1}{p}], T)$

  then this coincide with [Kato, Astérisque]'s $\mathbb{H}^i(T)$

- Kato's zeta element: Let $f \in S_k(\Gamma_0(N), \epsilon)$ be an eigencuspform, then Kato constructed $z_f^{\mathrm{Kato}} \in H^1_{\mathrm{Iw}}(T_f)$ that "related to the critical L-values of $L(f,s)$".
- Kato's Iwasawa main conjecture (in [Kato, Astérisque, Conjecture 12.10]):

$$\boxed{\mathrm{char}_\Lambda H^2_{\mathrm{Iw}}(T_f) \overset{?}{=} \mathrm{char}_\Lambda H^1_{\mathrm{Iw}}(T_f)/\Lambda_{\mathrm{Iw}} \cdot z_f^{\mathrm{Kato}}}$$

algebraic side

analytic side: z_f^{Kato} is an algebraic avatar of L-values. (so name "zeta element")

Let $S \supseteq \{p\}$ be any finite set of primes, actually what Kato really constructed in his Astérisque article is a "zeta morphism":

$$\mathcal{Z}^{\mathrm{Kato}}_{f,\mathrm{Iw},S} : T_{f,\mathrm{Iw}}(-1)^+ \longrightarrow H^1_{\text{ét}}(\mathbb{Z}[\tfrac{1}{S}], T_{f,\mathrm{Iw}})$$

called the "S-imprimitive zeta morphism". Taking $S = \{p\}$, z_f^{Kato} is actually the image of a generator of $T_{f,\mathrm{Iw}}(-1)^+$ under $\mathcal{Z}^{\mathrm{Kato}}_{f,\mathrm{Iw},\{p\}}$.

Note : In [Xin Wan's ICTS 2022 Survey], Kato's IMC is expressed in the following way :

(a) algebraic side : We define the strict Selmer group

$$\mathrm{Sel}^{str}_{\mathbb{Q}_n}(f) := \ker\Big(\underbrace{H^1\big(\mathbb{Q}_n^{\Sigma}, V_f(-\tfrac{k-1}{2})/T_f(-\tfrac{k-1}{2})\big)}_{\circledast} \to \prod_{v\nmid p} \frac{H^1(\mathbb{Q}_{n,v}, \circledast)}{H^1_f(\mathbb{Q}_{n,v}, \circledast)} \times H^1(\mathbb{Q}_{n,p}, \circledast)\Big)$$

and $\mathrm{Sel}_{\mathbb{Q}_\infty}(f) := \varinjlim_n \mathrm{Sel}^{str}_{\mathbb{Q}_n}(f)$, $X^{str}_{\mathbb{Q}_\infty}(f) := \mathrm{Sel}^{str}_{\mathbb{Q}_\infty}(f)^{\vee} \leftarrow$ Pontryagin dual

(b) analytic side :

$$H^1_{Iw}\big(\mathbb{Q}_\infty^{\Sigma}, T_f(-\tfrac{k-2}{2})\big) := \varprojlim_n H^1\big(\mathbb{Q}_n^{\Sigma}, T_f(-\tfrac{k-2}{2})\big)$$ a torsion-free rk one module over Λ

The Kato defined a zeta element

$$z_{Kato} \in H^1_{Iw}\big(\mathbb{Q}_\infty^{\Sigma}, T_f(-\tfrac{k-2}{2})\big) \otimes \mathbb{Q}_p$$

that is "closely related" to the special value of L-functions (an explicit reciprocity law)

Then Kato's Iwasawa main conjecture is as follows :

KIMC :

(A) $X^{str}_{\mathbb{Q}_\infty}(f)$ is a torsion Λ-module.

(B) If $\mathrm{im}(\rho_f) \supseteq SL_2(\mathbb{Z}_p)$, then $z_{Kato} \in H^1_{Iw}\big(\mathbb{Q}_\infty^{\Sigma}, T_f(-\tfrac{k-2}{2})\big)$

(C) $$\mathrm{char}_\Lambda X^{str}_{\mathbb{Q}_\infty}(f) = \mathrm{char}_\Lambda\left(\frac{H^1_{Iw}\big(\mathbb{Q}_\infty^{\Sigma}, T_f(-\tfrac{k-2}{2})\big)}{\Lambda \cdot z_{Kato}}\right)$$

Problem : How to relate this version with the one in [Kato's Astérisque article] as above ?

Let $\Sigma \supseteq \{p\}$ be the set of primes considered in Talk 01. In this seminar we can only touch the "Σ-imprimitive IMCs", though the reason is not easy to explain. We write $z^{Kato}_{f,Iw}$ for $z^{Kato}_{f,Iw,\Sigma}$ from now on.

Problem: How to formulate a "universal zeta morphism" on the entire universal deformation space?

(I think this is one of the three most nontrivial part of [Fouquet–Wan]:

- zeta morphism over deformation rings (by Nakamura or Colmez–Wang)
- Crystalline IMC (Wan)
- the formulation of fundamental lines (Fouquet).)

This is the result of K. Nakamura:

Theorem (Nakamura 2022 Invent.) Under assumption $(\star)^+$, there exists a "zeta morphism"

$$z_\Sigma^{univ} : T_\Sigma^{univ}(-1)^+ \longrightarrow H^1_{ét}\left(\mathbb{Z}[\tfrac{1}{\Sigma}], T_\Sigma^{univ}\right)$$

s.t. for any modular points λ_g, the following diagram commutes:

$$\begin{tikzcd} T_\Sigma^{univ}(-1)^+ \arrow[r, "z_\Sigma"] \arrow[d] & H^1_{ét}\left(\mathbb{Z}[\tfrac{1}{\Sigma}], T_\Sigma^{univ}\right) \arrow[d] \\ T_{f,I_w}(-1)^+ \arrow[r, "z^{Kato}_{f,I_w}"'] & H^1_{ét}\left(\mathbb{Z}[\tfrac{1}{\Sigma}], T_{f,I_w}\right) \end{tikzcd}$$

With z_Σ^{univ}, we have zeta morphisms at any specialization $\lambda : \mathbb{T}^\Sigma_{\mathfrak{m}_{\bar{\rho}}} \to A$,

$$z_{\Sigma,\lambda} : T_\lambda(-1)^+ \longrightarrow H^1_{ét}\left(\mathbb{Z}[\tfrac{1}{\Sigma}], T_\lambda\right).$$

Next, we compute the trivialization of $z_{\Sigma,\lambda}$ to see what could possibly happen. The computation relies on some further "technical assumptions" on the specialization of λ:

(A) $R\Gamma_{ét}(\mathbb{Z}[\tfrac{1}{\Sigma}], T_\lambda)$ is concentrated at degree 1 & 2. Then

$$\cdots \longrightarrow 0 \xrightarrow{d^0_{ét}} R\Gamma^1_{ét}(\cdot\cdot) \xrightarrow{d^1_{ét}} R\Gamma^2_{ét}(\cdots) \xrightarrow{d^2_{ét}} 0 \to \cdots$$

$$H^1_{ét}(\mathbb{Z}[\tfrac{1}{\Sigma}], T_\lambda) := \frac{\ker d^1_{ét}}{\operatorname{im} d^0_{ét}} \qquad H^2_{ét}(\mathbb{Z}[\tfrac{1}{\Sigma}], T_\lambda) = \frac{\ker d^2_{ét}}{\operatorname{im} d^1_{ét}}$$

So we upgrade

$$\begin{tikzcd}
T_\lambda(-1)^+ \arrow[d, "z_\lambda"'] & \arrow[r, Rightarrow, "\text{upgrade}"] & {} & \overbrace{\underbrace{T_\lambda(-1)^+}_{\text{concentrated at degree } 0}[-1]}^{\text{concentrated at degree } 1} \arrow[d, "z_\lambda"] \\
H^1_{ét}(\mathbb{Z}[\tfrac{1}{\Sigma}], T_\lambda) & {} & \arrow[l, Rightarrow, "\text{taking coh.}"] {} & R\Gamma_{ét}(\mathbb{Z}[\tfrac{1}{\Sigma}], T_\lambda)
\end{tikzcd}$$

Then we compute the mapping cone of z_λ, then some magic comes out:

- Recall the definition of mapping cones: Let $\mathcal{A}$ be an additive category and $C(\mathcal{A})$ be the category of complexes. Let $f \in \mathrm{Hom}_{C(\mathcal{A})}(X, Y)$, we define

$$\mathrm{Cone}(f) := \left[\mathrm{Cone}(f)^n := X^{n+1} \oplus Y^n, \quad d^n_{\mathrm{Cone}(f)} := \begin{pmatrix} -d_X^{n+1} & 0 \\ f^{n+1} & d_Y^n \end{pmatrix} \right]$$

$\mathrm{Cone}(z_\lambda)$:

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & \boxed{\deg 0} & \xrightarrow[\begin{pmatrix} 0 & 0 \\ z_\Sigma & d^0_{ét} \end{pmatrix}]{d^0} & \boxed{\deg 1} & \xrightarrow[\begin{pmatrix} 0 & 0 \\ 0 & d^1_{ét} \end{pmatrix}]{d^1} & \boxed{\deg 2} & \xrightarrow[\begin{pmatrix} 0 & 0 \\ 0 & d^2_{ét} \end{pmatrix}]{d^2} & \boxed{\deg 3} \\
& T_\lambda(-1)^+ & & 0 & & 0 & & 0 \\
& \oplus & & \oplus & & \oplus & & \oplus \\
& R\Gamma^0_{ét}(\mathbb{Z}[\tfrac{1}{\Sigma}], T_\lambda) & & R\Gamma^1_{ét}(\mathbb{Z}[\tfrac{1}{\Sigma}], T_\lambda) & & R\Gamma^2_{ét}(\mathbb{Z}[\tfrac{1}{\Sigma}], T_\lambda) & & R\Gamma^3_{ét}(\mathbb{Z}[\tfrac{1}{\Sigma}], T_\lambda) \\
& \overset{(A)}{\parallel} & & & & & & \overset{(A)}{\parallel} \\
& 0 & & & & & & 0
\end{array}$$

Then

$$H^1(\mathrm{Cone}(f)) = \frac{\ker d^1}{\operatorname{im} d^0} = \frac{\ker d^1_{ét}}{\operatorname{im} z_\Sigma + \operatorname{im} d^0_{ét}} \simeq \frac{H^1_{ét}(\mathbb{Z}[\frac{1}{\Sigma}], T_\lambda)}{\operatorname{im}(z_\Sigma)}$$

$$\left[\begin{array}{l} \begin{pmatrix} 0 & 0 \\ 0 & d^1_{ét} \end{pmatrix} \begin{pmatrix} \gamma \\ \alpha \end{pmatrix} = \begin{pmatrix} 0 \\ d^1_{ét}(\alpha) \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ z_\Sigma & d^0_{ét} \end{pmatrix} \begin{pmatrix} \gamma \\ \alpha \end{pmatrix} = \begin{pmatrix} 0 \\ z_\Sigma(\gamma) + d^0_{ét}(\alpha) \end{pmatrix} \end{array} \right]$$

$$H^2(\mathrm{Cone}(f)) = \frac{\ker d^2}{\operatorname{im} d^1} = \frac{\ker d^2_{ét}}{\operatorname{im} d^1_{ét}} = H^2_{ét}(\mathbb{Z}[\tfrac{1}{\Sigma}], T_\lambda).$$

$$H^0(\mathrm{Cone}(f)) = \frac{\ker d^0}{\operatorname{im} d^{-1}} \overset{(A)}{=} \ker z_\Sigma \quad \text{—— another annoying term.}$$

$$H^{i \geq 3 \text{ or } i < 0}(\mathrm{Cone}(f)) = 0$$

(B) The cohomology of $\mathrm{cone}(f_1)$ is concentrated at degree $[1,2]$.

Then under <u>(A) and (B)</u>, we take $\mathrm{Det}(\mathrm{cone}(f_1))$:

$$\mathrm{Det}_A(\mathrm{Cone}(f_1)) = \mathrm{Det}_A^{-1} H^1(\mathrm{Cone}(f_1)) \cdot \mathrm{Det}_A H^2(\mathrm{Cone}(f_1))$$
$$= \mathrm{Det}_A^{-1}\left(\frac{H^1_{et}(\mathbb{Z}[\frac{1}{\Sigma}], T_\lambda)}{\mathrm{im}\, z_\Sigma}\right) \cdot \mathrm{Det}_A H^2_{et}(\mathbb{Z}[\tfrac{1}{\Sigma}], T_\lambda).$$

If we have $\boxed{\mathrm{Cone}(f) \text{ is acyclic}}$, then by (i)[D]

$$\mathrm{Det}_A\left(\frac{H^1_{et}(\mathbb{Z}[\frac{1}{\Sigma}], T_\lambda)}{\mathrm{im}\, z_{\Sigma,\lambda}}\right) = \mathrm{Det}_A H^2_{et}(\mathbb{Z}[\tfrac{1}{\Sigma}], T_\lambda).$$

this can be regarded as the <u>main conjecture</u> at the specialization $\lambda : R_\Sigma(\bar{\rho}) \to A$.

For example,

- For an Iwasawa-theoretic modular point $\lambda_{g,I_w} : R_\Sigma(\bar{\rho}) \to \Lambda_{I_w}$,

$$\begin{array}{ccc}
\mathrm{Det}_{\Lambda_{I_w}}\left(\dfrac{H^1_{et}(\mathbb{Z}[\frac{1}{\Sigma}], T_{g,I_w})}{\mathrm{im}\, z^{Kato}_{g,I_w}}\right) & = & \mathrm{Det}_{\Lambda_{I_w}} H^2_{et}(\mathbb{Z}[\tfrac{1}{\Sigma}], T_{g,I_w}) \\
\| & & \| \\
\mathrm{char}^{-1}_{\Lambda_{I_w}}\left(\dfrac{H^1_{et}(\mathbb{Z}[\frac{1}{\Sigma}], T_{g,I_w})}{\mathrm{im}\, z^{Kato}_{g,I_w}}\right) & = & \mathrm{char}^{-1}_{\Lambda_{I_w}} H^2_{et}(\mathbb{Z}[\tfrac{1}{\Sigma}], T_{g,I_w})
\end{array}$$

  The lower line is a reformulation of <u>Kato's Iwasawa main conjecture</u>.

- For a modular point $\lambda_g : R_\Sigma(\bar{\rho}) \to \mathcal{O}$, this is the <u>Tamagawa number conjecture</u>.

- For the universal point $\lambda_{id} : R_\Sigma(\bar{\rho}) \to R_\Sigma(\bar{\rho})$, this is the <u>universal Iwasawa main conjecture</u>.

Recall the definition of the trivialization map

$$\mathrm{triv}_{z_{\Sigma,\lambda}} : \mathrm{Det}_A^{-1} R\Gamma_{et}(\mathbb{Z}[\tfrac{1}{\Sigma}], T_\lambda) \otimes \mathrm{Det}_A\, T_\lambda(-1)^+[-1] \to \underset{\underset{\mathrm{Det}_A^{-1} T_\lambda(-1)^+}{\|}}{\mathrm{Det}_A^{-1}(\mathrm{cone} f_1)}$$

then the "<u>main conjecture</u>" claims that $\mathrm{triv}_{z_{\Sigma,\lambda}}$ gives an isomorphism

$$\mathrm{triv}_{z_{\Sigma,\lambda}} : \mathrm{Det}_A^{-1} R\Gamma_{\acute{e}t}(\mathbb{Z}[\tfrac{1}{\Sigma}], T_\lambda) \otimes_A \mathrm{Det}_A^{-1} T_\lambda(-1)^+ \xrightarrow{\sim} A .$$

The left hand side is called the <u>Σ-imprimitive fundamental line</u> of the specialization λ, denoted by $\Delta_A(T_\lambda)$. So the slogan of main conjectures is:

$$\boxed{\text{The zeta morphism trivializes the fundamental line.}} \quad ☆$$

<u>Note</u>: This computation is highly motivational but not accurate at many places. For example, we put extra (A) and (B), and actually we ignored the issue that "Det" is applicable: it can only be defined for <u>perfect complexes</u>.

$\rightsquigarrow$ We formulate the main conjectures on the "<u>Iwasawa suitable points</u>". (in the sense of Fouquet in his Besançon article).

<u>Definition</u>: Let $\lambda : R_\Sigma(\bar\rho) \to A$ be a specialization with A a reduced ring, and write $T_\lambda := T_\Sigma^{univ}(\bar\rho) \otimes_{R_\Sigma(\bar\rho),\lambda} A$. We say λ is <u>Iwasawa suitable</u> if the complex

$$\mathrm{Cone}(z_{\Sigma,\lambda}) := \mathrm{Cone}\left(T_\lambda(-1)^+[-1] \longrightarrow R\Gamma_{\acute{e}t}(\mathbb{Z}[\tfrac{1}{\Sigma}], T_\lambda)\right)$$

"induced from" $z_{\Sigma,\lambda} : T_\lambda(-1)^+[-1] \to H^1_{\acute{e}t}(\mathbb{Z}[\tfrac{1}{\Sigma}], T_\lambda)$ is acyclic after extension of scalars to the total ring of quotient $Q(A)$ of A.

Let λ be a Iwasawa-suitable point, then the <u>main conjecture</u> is the existence of the commutative diagram:

$$\begin{tikzcd}
\Delta_\Sigma(T_\lambda) \otimes_A Q(A) \arrow[r, "\sim", "\mathrm{triv}_{z_{\Sigma,A}}"'] & Q(A) \\
\Delta_\Sigma(T_\lambda) \arrow[u] \arrow[r, dashed, "\sim", "\mathrm{triv}_{z_{\Sigma,A}}"'] & A \arrow[u, hook]
\end{tikzcd}$$

Finally we introduce a weaker version of the universal IMC :

- Recall $\Lambda = \mathcal{O}[\![X_1, X_2, X_3]\!]$ and $R_\Sigma^{univ}(\bar{\rho})$ is Cohen–Macaulay of relative dim zero over Λ, hence a free module of rank d.
- T_Σ^{univ} is then a $\Lambda[Gal_{\mathbb{Q},\Sigma}]$-module which is free of rank $2d$ as a Λ-mod.

Define the fundamental line

$$\Delta_\Lambda := \mathrm{Det}_\Lambda^{-1} R\Gamma_{ét}(\mathbb{Z}[\tfrac{1}{\Sigma}], T_\Sigma^{univ}) \otimes_\Lambda \mathrm{Det}_\Lambda^{-1} T_\Sigma(-1)^+$$

then <u>due to some reasons</u> (to be explained in <u>Talk 3</u>), we trivialize z_Σ^{univ} as

$$z_{\Sigma,\Lambda} : \Delta_\Lambda \otimes \mathrm{Frac}(\Lambda) \xrightarrow{\sim} \mathrm{Frac}(\Lambda).$$

<u>Conjecture</u> (Universal Iwasawa main conjecture, in families) The universal zeta morphism z_Σ^{univ} induces an isomorphism

$$z_\Sigma : \Delta_\Lambda \xrightarrow{\sim} \Lambda.$$

This is actually the correct "Universal Iwasawa main conjecture" in <u>Theorem A</u> of <u>Talk 1</u>.

<u>Remark</u> : In the classical setup of IMC, there are the so called "lower bounds" and "upper bounds" of Selmer group. In this language of fundamental lines, there are also two sides of "divisibilities". They are hidden in the proof of <u>Theorem A</u>. (Have not fully understood this.)

$$\mathrm{Det}_{\Lambda_{I_w}}\left(\frac{H^1_{ét}(\mathbb{Z}[\frac{1}{\Sigma}], T_{g,I_w})}{\mathrm{im}\, z_{g,I_w}^{Kato}}\right) \underset{\supseteq}{\subseteq} \mathrm{Det}_{\Lambda_{I_w}} H^2_{ét}(\mathbb{Z}[\tfrac{1}{\Sigma}], T_{g,I_w})$$

$$\|$$

$$\mathrm{char}^{-1}_{\Lambda_{I_w}}\left(\frac{H^1_{ét}(\mathbb{Z}[\frac{1}{\Sigma}], T_{g,I_w})}{\mathrm{im}\, z_{g,I_w}^{Kato}}\right) \underset{\subseteq}{\supseteq} \mathrm{char}^{-1}_{\Lambda_{I_w}} H^2_{ét}(\mathbb{Z}[\tfrac{1}{\Sigma}], T_{g,I_w})$$

$$\Updownarrow$$

upper bound of Selmer grps

Lower bounds: ([Skinner-Urban, Wan])

$$\boxed{\mathrm{char}_{\Lambda_{I_w}}\left(\frac{H^1_{ét}(\mathbb{Z}[\frac{1}{\Sigma}], T_{g,I_w})}{\mathrm{im}\, z^{\mathrm{Kato}}_{g,I_w}}\right) \,\Big|\, \mathrm{char}_{\Lambda_{I_w}} H^2_{ét}(\mathbb{Z}[\tfrac{1}{\Sigma}], T_{g,I_w})} \iff \boxed{z^{\mathrm{Kato}}_{g,I_w}(\Delta_{g,I_w}) \subseteq \Lambda_{I_w}}$$

Upper bounds: ("Kato")

$$\boxed{\mathrm{char}_{\Lambda_{I_w}} H^2_{ét}(\mathbb{Z}[\tfrac{1}{\Sigma}], T_{g,I_w}) \,\Big|\, \mathrm{char}_{\Lambda_{I_w}}\left(\frac{H^1_{ét}(\mathbb{Z}[\frac{1}{\Sigma}], T_{g,I_w})}{\mathrm{im}\, z^{\mathrm{Kato}}_{g,I_w}}\right)} \iff \boxed{z^{\mathrm{Kato}}_{g,I_w}(\Delta^{-1}_{g,I_w}) \subseteq \Lambda_{I_w}}$$

§2-4 Very short quick record on the notations

- Let $f \in S_k(\Gamma_0(N), \epsilon)$ as always, with coeff. field $L \hookrightarrow E/\mathbb{Q}_p$. $\rightsquigarrow$ $\boxed{\text{motive } M}$.
- Let $\chi \in \widehat{G_n} := \mathrm{Gal}(\mathbb{Q}_n/\mathbb{Q})^\wedge$ be a character of $G_n := \mathrm{Gal}(\mathbb{Q}_n/\mathbb{Q}) \simeq \mathbb{Z}/p^n$.

  We denote $L_\chi :=$ ext'n of $\mathbb{Q}$ gen. by L and image of χ.

  Let $\mathfrak{p} | p$ a prime of $\mathcal{O}_{L_\chi}$, and $E_\chi :=$ corresp. ext'n of E.

  $\rightsquigarrow$ $\boxed{M_\chi := M \otimes_{\mathbb{Q}} h^0(\mathrm{Spec}\,\mathbb{Q}_n)_\chi}$: the χ-isotypic part of "M", over $\mathbb{Q}_n$.
- Let $1 \leq r \leq k-1$, $\boxed{M_\chi(r)}$:= r-times Tate twist of M_χ.

Then for all but finitely many (n, χ), $M_\chi(r)$ is "<u>strictly critical</u>" à la Kato.

§2 Preparations on motives

Let $X \longrightarrow \operatorname{Spec}\mathbb{Q}$ be a smooth projective variety, $i, j \in \mathbb{Z}$.

$$\boxed{M := h^i(X)(j)}$$

motive 原相

étale realization, ℓ prime

$M_\ell := H^i_{ét}(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)(j)$: a Galois rep'n of $\mathrm{Gal}_{\mathbb{Q}}$.

- $$P_p(T) := \begin{cases} \det\left(1 - \mathrm{Fr}_p^{-1} \cdot T \mid M_\ell^{I_\ell}\right) & p \neq \ell \\ \det\left(1 - \phi T \mid D_{cris}(M_\ell)\right) & p = \ell \end{cases} \in \mathbb{Q}_\ell[T]. \quad \left(\mathrm{Fr}_p \in \mathrm{Gal}_{\mathbb{Q}}\right)$$

  When X has good reduction at p, $P_p(T) \in \mathbb{Q}[T]$, indep of ℓ. Expected to be correct in general.

- $L(M, s) := \prod_p P_p(p^{-s})^{-1}$, for $\mathrm{Re}(s)$ suff. large, with conjectural merom. continuation at $s = 0$

$$L(M, s) = L^*(M)\, s^{r(M)} + \cdots$$

<u>Main theme</u> is to understand $L^*(M) \in \mathbb{R}^\times$, $r(M) \in \mathbb{Z}$.

Betti realization 现相

$M_B := H^i(X(\mathbb{C}), \mathbb{Q}(j))$: complex conjugation & Hodge structure

- The Hodge structure on M_B is of weight $i - 2j$
- Complex conjugation $\iota : X(\mathbb{C}) \to X(\mathbb{C})$ and $\mathbb{Q}(j) = (2\pi i)^j \mathbb{Q} \subseteq \mathbb{C}$ via $(-1)^j$. Then the complex conj on M_B is the composition.

  Denote $M_B^+ :=$ fixed part by complex conjugation.

de Rham realization

$M_{dR} := H^i_{dR}(X/\mathbb{Q})$ with a Hodge filtration

$$\mathrm{Fil}^n M_{dR} := \mathrm{im}\left(H^i\left(\Omega^{\geq n+j}_{X/\mathbb{Q}}\right) \longrightarrow H^i\left(\Omega^\bullet_{X/\mathbb{Q}}\right) \right), \quad n \in \mathbb{Z},$$

where $\Omega^\bullet_{X/\mathbb{Q}}$ is the de Rham complex of $X/\mathbb{Q}$. The <u>tangent space</u> of M is defined as $t(M) := M_{dR} / \mathrm{Fil}^0 M_{dR}$.

and together with comparison isomorphisms.

§3 Fundamental lines

3.1 Modular motives

- Let $f(z) = \sum_{n=1}^{\infty} a_n(f) q^n \in S_k(U)$ be an eigencuspform with coeff $L \overset{\iota}{\hookrightarrow} \mathbb{C}$ and assume $L \hookrightarrow E$. Let $M :=$ pure motive attached to f.
- Let $\chi \in \widehat{G_n} := \mathrm{Gal}(\mathbb{Q}_n/\mathbb{Q})^\wedge$ be a character of G_n.

  We denote $L_\chi :=$ ext'n of $\mathbb{Q}$ gen. by L and image of χ.

  Let $\mathfrak{p} | p$ a prime of $\mathcal{O}_{L_\chi}$, and $E_\chi :=$ corresp. ext'n of E.

  Then let $h^0(\mathrm{Spec}\,\mathbb{Q}_n) :=$ pure motive over $\mathbb{Q}$, with coefficient in L_χ.

  let $h^0(\mathrm{Spec}\,\mathbb{Q}_n)_\chi :=$ direct summand of $h^0(\mathrm{Spec}\,\mathbb{Q}_n)$ that G_n acts through χ.

$\rightsquigarrow$ $M_\chi := M \underset{\mathbb{Q}}{\times} h^0(\mathrm{Spec}\,\mathbb{Q}_n)_\chi$, with realization of M_χ is the corresp. realization of M with scalar extended from L to L_χ together with an action of G_n thru χ.

(a) <u>algebraic side</u> : $S \supseteq \{p\}$, and $1 \leq r \leq k-1$, $n \in \mathbb{N}$, $\chi \in \widehat{G_n}$.

- <u>complex period map</u> : Let $\mathrm{per}_\mathbb{C}$ be the following composition

$$\underbrace{\mathrm{Fil}^0 V_{\chi,dR}(r) \otimes_{L_\chi} \mathbb{C} \hookrightarrow V_{\chi,dR}(r) \underset{L_\chi}{\otimes} \mathbb{C} \underset{\substack{\text{Betti} \\ \text{to} \\ \text{deRham}}}{\overset{\sim}{\rightleftarrows}} \left(V_{\chi,\mathbb{C}}(r) \underset{L_\chi}{\otimes} \mathbb{C}\right)^+ \xleftarrow{\sim} V_{\chi,\mathbb{C}}(r-1)^+ \underset{L_\chi}{\otimes} \mathbb{C}}_{\mathrm{per}_\mathbb{C}}$$

- <u>p-adic period map</u> : Let per_p^{-1} be the map of E_χ-spaces

$$\begin{array}{c} H^1_{et}\left(\mathbb{Z}[\tfrac{1}{S}], V_\chi(r)\right) \longrightarrow H^1\left(\mathrm{Gal}_{\mathbb{Q}_p(\zeta_{p^n})}, V_\chi(r)\right) \xrightarrow[\text{map}]{\text{dual exp.}} \mathbb{D}^0_{dR}(V_\chi(r)) \\ \downarrow \\ \mathrm{Fil}^0 V_{\chi,dR}(r) \otimes_{L_\chi} E_\chi \end{array}$$

<u>notation</u> : p-adic realization per_p^{-1} of $M_\chi(r)$.

Defn: We say $M_\chi(r)$ is <u>strictly critical</u> if $\mathrm{per}_{\mathbb{C}}$ and per_p^{-1} are bijective and $H^i_{\text{ét}}(\mathbb{Z}[\frac{1}{S}], V_\chi(r)) = 0$ for all $i \neq 1$. conjectured to be automatic when $\mathrm{per}_{\mathbb{C}}$ and per_p^{-1} are bijective.

Theorem:

(1) $\mathrm{per}_{\mathbb{C}}$ is an isomorphism

(2) For all but finitely many $n \in \mathbb{N}$ and $\chi \in \widehat{G_n}$, $H^1_{\text{ét}}(\mathbb{Z}[\frac{1}{S}], V_\chi(r))$ is of dimension one and per_p^{-1} is an isomorphism.

Hence for such n and χ, $M_\chi(r)$ is strictly critical.

- <u>Fundamental line</u>: Apply Det to strictly critical period maps:

$$\begin{array}{ccc}
\mathrm{Det}_{E_\chi} H^1_{\text{ét}}(\mathbb{Z}[\tfrac{1}{S}], V_\chi(r)) \otimes_{E_\chi} \mathrm{Det}^{-1}_{E_\chi} V_\chi(r-1)^+ & & \left(\mathrm{Det}_{\mathbb{C}} \mathrm{Fil}^0 V_{\chi,dR}(r) \otimes_{L_\chi} \mathbb{C}\right) \otimes_{\mathbb{C}} \left(\mathrm{Det}^{-1}_{\mathbb{C}} V_{\chi,\mathbb{C}}(r-1)^+ \otimes_{L_\chi} \mathbb{C}\right) \\
\simeq \big\downarrow \mathrm{per}_p^{-1} \quad \big\uparrow \mathrm{per}_p & \& & \simeq \big\downarrow \mathrm{per}_{\mathbb{C}} \\
\mathrm{Det}_{E_\chi}\left(\mathrm{Fil}^0 V_{\chi,dR}(r) \otimes_{L_\chi} E_\chi\right) \otimes_{E_\chi} \mathrm{Det}^{-1}_{E_\chi} V_\chi(r-1)^+ & & \mathbb{C}
\end{array}$$

Defn: The <u>motivic fundamental line</u> $(\Delta_{\text{mot}}(M_\chi(r)), \mathrm{per}_{\mathbb{C}}, \mathrm{per}_p)$ of strictly critical motive $M_\chi(r)$ is the one-dimensional L_χ-space

$$\Delta_{\text{mot}}(M_\chi(r)) := \mathrm{Det}_{L_\chi} \mathrm{Fil}^0 V_{\chi,dR}(r) \otimes_{L_\chi} \mathrm{Det}^{-1}_{L_\chi} V_{\chi,c}(r-1)^+$$

together with two isomorphisms

$$\mathrm{per}_p : \Delta_{\text{mot}}(M_\chi(r)) \otimes_{L_\chi} E_\chi \xrightarrow{\sim} \mathrm{Det}_{E_\chi} H^1_{\text{ét}}(\mathbb{Z}[\tfrac{1}{S}], V_\chi(r)) \otimes_{E_\chi} \mathrm{Det}^{-1}_{E_\chi} V_\chi(r-1)^+$$

$$\mathrm{per}_{\mathbb{C}} : \Delta_{\text{mot}}(M_\chi(r)) \otimes_{L_\chi} \mathbb{C} \xrightarrow{\sim} \mathbb{C}$$

(b) <u>Analytic side</u>: We define the <u>S-partial L-function</u> $L_S(M,\chi,s)$ to be the holomorphic complex function satisfying

$$L_S(M,\chi,s) := \prod_{\ell \notin S} \left(1 - a_\ell \chi(\mathrm{Fr}_\ell) \ell^{-s} + \epsilon(\ell)\chi(\mathrm{Fr}_\ell)\ell^{k-1-2s}\right)^{-1}$$

for all $s \in \mathbb{C}$ with $\mathrm{Re}(s) \gg 0$.

<u>Question</u>: In this framework, how to link the algebraic side with analytic side?

3.2 Zeta morphisms ⚠ record analytic information algebraically!

By definition, $V_\chi(r-1)^+$ and $H^1_{ét}(\mathbb{Z}[\frac{1}{S}], V_\chi(r))$ are isomorphic as E_χ-vector space. Let

$$Z : V_\chi(r-1)^+ \xrightarrow{\sim} H^1_{ét}(\mathbb{Z}[\tfrac{1}{S}], V_\chi(r))$$

be an isomorphism between them. Apply Det_{E_χ}:

$$\mathrm{Det}_{E_\chi}(Z) : \mathrm{Det}_{E_\chi} H^1_{ét}(\mathbb{Z}[\tfrac{1}{S}], V_\chi(r)) \otimes_{E_\chi} \mathrm{Det}^{-1}_{E_\chi} V_\chi(r-1)^+ \xrightarrow{\sim} E_\chi .$$

Let $\Delta_{L,Z}(M_\chi)$ be the inverse image of L_χ through this isomorphism.

Go to Iwasawa level, suppose there is a nonzero morphism

$$Z_{Iw} : T_{f,Iw}(-1)^+ \longrightarrow H^1_{ét}(\mathbb{Z}[\tfrac{1}{S}], T_{f,Iw})$$

where $T_{f,Iw} := T_f \otimes_{\mathcal{O}} \Lambda_{Iw}$ for T_f a stable $G_{\mathbb{Q}}$-lattice inside V. Then for all n and $\chi \in \widehat{G_n}$ in the strictly critical range, Z_{Iw} induces a nonzero morphism

$$Z_\chi : V_\chi(r-1)^+ \longrightarrow H^1_{ét}(\mathbb{Z}[\tfrac{1}{S}], V_\chi(r))$$

that is an isomorphism.

Definition: A morphism Z is called

(1) motivic, if $\Delta_{L,Z}(M_\chi) = \mathrm{per}_p(\Delta_{mot}(M_\chi))$.

(2) the S-partial zeta morphism of M_χ if

$$\mathrm{per}_{\mathbb{C}}\Big(\underbrace{\mathrm{per}_p^{-1}\big(\mathrm{Det}_{E_\chi}(Z)^{-1}(1_{E_\chi})\big)}_{\in \Delta_{mot}(M_\chi(r)) \text{ since } Z \text{ is motivic}} \otimes 1_{\mathbb{C}}\Big) = L_S(M^*(1), \chi^{-1}, -r) \in \mathbb{C} .$$

A morphism Z_{Iw} is the S-partial zeta morphism of $T_{f,Iw}$ if Z_χ is the S-partial zeta morphism for V_χ for all χ s.t. M_χ is strictly critical.

Theorem of Kato: Let f corresponds to a classical point of $\mathcal{X}_{\Sigma}(\bar{\rho})$, then there exists a partial zeta morphism

$$z(f)_{S,I_w} : T_{f,I_w}(-1)^{+} \longrightarrow H^1_{\text{ét}}\left(\mathbb{Z}[\tfrac{1}{S}], T_{f,I_w}\right).$$

Question: how to interpolate zeta morphisms $z(f)_{S,I_w}$ as f varies on the universal deformation space? This is a hard problem!

§4 Main conjectures

4.1 Tamagawa number conjecture

The zeta morphism

$$Z(f)_{\chi,r} = Z : V_\chi(r-1)^+ \xrightarrow{\sim} H^1_{ét}(\mathbb{Z}[\tfrac{1}{S}], V_\chi(r))$$

induces a trivialization map by applying Det_{E_χ}:

$$\mathrm{triv}_{Z(f)_{\chi,r}} := \mathrm{Det}^{-1}_{E_\chi} \underset{R\Gamma_{ét}}{H^1_{ét}}(\mathbb{Z}[\tfrac{1}{S}], V_\chi(r)) \otimes_{E_\chi} \mathrm{Det}^{-1}_{E_\chi} V_\chi(r-1)^+ \xrightarrow{\sim} E_\chi .$$

(recall: $\mathrm{Det}_{E_\chi} R\Gamma_{ét}(\mathbb{Z}[\tfrac{1}{S}], V_\chi(r)) = \prod_{i\in\mathbb{Z}} \mathrm{Det}^{(-1)^i}_{E_\chi} H^i_{ét}(\mathbb{Z}[\tfrac{1}{S}], V_\chi(r))$

$= \mathrm{Det}^{-1}_{E_\chi} H^1_{ét}(\mathbb{Z}[\tfrac{1}{S}], V_\chi(r))$, by strict criticality)

We define the p-adic fundamental line as a one-dim'l E_χ-vector space

$$\Delta_{S,p}(M_\chi(r)) := \mathrm{Det}^{-1}_{E_\chi} R\Gamma_{ét}(\mathbb{Z}[\tfrac{1}{S}], V_\chi(r)) \otimes_{E_\chi} \mathrm{Det}^{-1}_{E_\chi} V_\chi(r-1)^+$$

together with a free rank-one $\mathcal{O}_\chi$-module

$$\Delta_{S,p}(T_{f,\chi}(r)) := \mathrm{Det}^{-1}_{\mathcal{O}_\chi} R\Gamma_{ét}(\mathbb{Z}[\tfrac{1}{S}], T_\chi(r)) \otimes \mathrm{Det}^{-1}_{\mathcal{O}_\chi} T_\chi(r-1)^+$$

obtained by choosing a Gal$_\mathbb{Q}$-stable $\mathcal{O}_\chi$-lattice T_χ inside V_χ. It does not depend on the choice of T_χ.

Tamagawa number conjecture (TNC) The S-partial zeta morphism of $M_\chi(r)$ exists and $\mathrm{triv}_{Z(f)_{\chi,r}} : \Delta_{S,p}(M_\chi(r)) \xrightarrow{\sim} E_\chi$ induces an isomorphism

$$\mathrm{triv}_{Z(f)_{\chi,r}} : \Delta_{S,p}(T_\chi(r)) \xrightarrow{\sim} \mathcal{O}_\chi .$$

In [Fouquet-Wan], this is always interpreted as the commutativity of the diagram

$$\begin{tikzcd}
\Delta_{S,p}(M_\chi(r)) \arrow[r, "\mathrm{triv}_{Z(f)_{\chi,r}}", "\sim"'] & E_\chi \\
\Delta_{S,p}(T_\chi(r)) \arrow[u, hook] \arrow[r, "\sim"] & \mathcal{O}_\chi \arrow[u, hook]
\end{tikzcd}$$

4.2 Iwasawa main conjecture

Similar to before, Kato provides us with the Iwasawa theoretic zeta morphism

$$z(f)_{Iw} : T_{f,Iw}(-1)^+ \longrightarrow H^1_{ét}(\mathbb{Z}[\tfrac{1}{S}], T_{f,Iw})$$

Applying Det_Λ to both sides, we obtain the trivialization map

$$\mathrm{triv}_{z(f)_{Iw}} : \mathrm{Det}_\Lambda^{-1} \underset{R\Gamma_{ét}}{H^1_{ét}}(\mathbb{Z}[\tfrac{1}{S}], T_{f,Iw}) \otimes_\Lambda \mathrm{Det}_\Lambda^{-1} T(f)_{Iw}(-1)^+ \hookrightarrow \mathrm{Frac}(\Lambda)$$

Note: Unlike $z(f)_{\chi,r}$, here the Iwasawa-theoretic zeta morphism is NOT a priori an isomorphism. So $\mathrm{triv}_{z(f)_{Iw}}$ is not necessarily an isomorphism of Λ-module. Instead, we only know the inclusion above. (Descent to finite level n and together with a choice of $\chi \in \widehat{G_n}$ and a choice of twist r, we indeed have isom "after inverting p", i.e. for $V_\chi(r-1)^+ \xrightarrow{\sim} H^1_{ét}(\mathbb{Z}[\tfrac{1}{S}], V_\chi(r))$.)

We define the Iwasawaic fundamental line as a free Λ-module of rk 1:

$$\Delta(T(f)_{Iw}) := \mathrm{Det}_\Lambda^{-1} R\Gamma_{ét}(\mathbb{Z}[\tfrac{1}{S}], T_{f,Iw}) \otimes_\Lambda \mathrm{Det}_\Lambda^{-1} T(f)_{Iw}(-1)^+$$

Then $\mathrm{triv}_{z(f)_{Iw}}$ gives a morphism

$$\mathrm{triv}_{z(f)_{Iw}} : \Delta(T(f)_{Iw}) \otimes_\Lambda \mathrm{Frac}(\Lambda) \xrightarrow{\sim} \mathrm{Frac}(\Lambda).$$ (why?)

Iwasawa main conjecture (IMC): The S-partial zeta morphism $z(f)_{Iw}$ induces an isomorphism

$$\mathrm{triv}_{z(f)_{Iw}} : \Delta(T(f)_{Iw}) \xrightarrow{\sim} \Lambda.$$

In other words, the diagram

$$\begin{tikzcd} \Delta(T(f)_{Iw}) \otimes_\Lambda \mathrm{Frac}(\Lambda) \arrow[r, "\mathrm{triv}_{z(f)_{Iw}}"] & \mathrm{Frac}(\Lambda) \\ \Delta(T(f)_{Iw}) \arrow[u, hook] \arrow[r, "\sim"] & \Lambda \arrow[u, hook] \end{tikzcd}$$

commutes.

Remark:

$$(\mathrm{IMC}) \xrightarrow[(\star)]{f \text{ is ordinary @ } p} (\mathcal{L}_p^{cyc}(f))\Lambda = \underbrace{\mathrm{char}_\Lambda \widetilde{H}^2_f(G_{\mathbb{Q},\Sigma}, T(f)_{I_w})}$$

$$(\mathrm{IMC}) \xrightarrow[a_p(f)=0]{} \text{Kobayashi's } \pm\text{-main conjectures.}$$

Greenberg's Selmer group à la Nekovář.

Why ($\star$)? There are two cases: for $\pi(f)_p := GL_2(\mathbb{Q}_p)$-rep

(1) $\pi(f)_p$ is in principal series: treated by [Kato, Astérique 295].

(2) $\pi(f)_p$ is Steinberg: treated in [Colmez, Astérique 294].