Talk 1 An Overview

§0 Breakthrough on elliptic curves

Let p be an odd prime.

Theorem A: Let $f \in S_k(\Gamma_0(N))$ be a normalized eigenform such that $\bar{\rho}_f$ satisfies condition ($\star$). Then the Selmer group

$$\mathrm{Sel}_{\mathbb{Q}}(f) := \ker\left(H^1(\mathbb{Q}, V/T) \to \prod_{\ell \text{ prime}} \frac{H^1(\mathbb{Q}_\ell, V/T)}{\mathrm{im}\left(H^1_f(\mathbb{Q}_\ell, V/T)\right)}\right)$$

$\left(V = \rho_f\left(\frac{k}{2}\right),\ T \subseteq V \text{ a } \mathrm{Gal}_{\mathbb{Q}}\text{-stable lattice}\right)$ is a finite group if and only if $L(f, \frac{k}{2}) = 0$. (eg. $k=2$ in the elliptic curve case)

Theorem B: Let $A/\mathbb{Q}$ be an abelian variety of GL_2-type of conductor N ass. to $f \in S_2(\Gamma_0(N))$. Assume $L(A,1) \neq 0$ and $A[p]$ satisfies ($\star$), then

$$\frac{L(A,1)}{\Omega_f} \sim_p \left|Ш(A/\mathbb{Q})[p^\infty]\right| \cdot \prod_{\ell | N} \mathrm{Tam}_\ell(A/\mathbb{Q}).$$

That is, the p-part of the BSD formula for A.

Remark: ($\star$) is that $\bar{\rho}: \mathrm{Gal}_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{F})$ is absolutely irred and odd, s.t.

(1) $\bar{\rho}|_{G_{\mathbb{Q}_p}} \neq \bar{\chi} \oplus \bar{\chi}_{cyc}\bar{\chi}$, for any $\bar{\chi}: \mathrm{Gal}_{\mathbb{Q}_p} \to \mathbb{F}^\times$.

(2) $\exists\, \ell \neq p$, $\ell \| N$, s.t. $\bar{\rho}|_{\mathrm{Gal}_{\mathbb{Q}_\ell}}$ is a ramified ext'n

so $\mu = \mu^{-1}$, being of order two

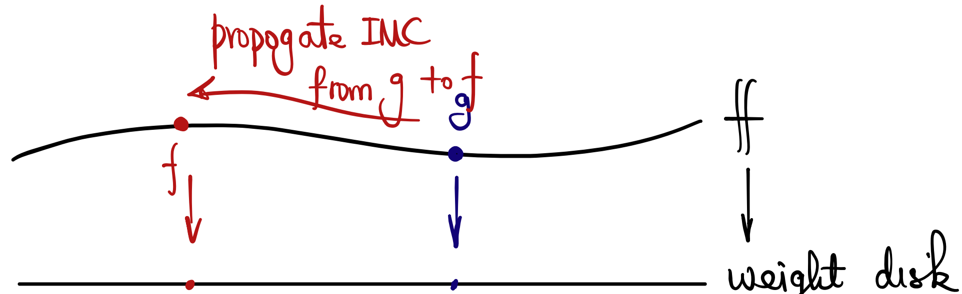
$$0 \to \mu \chi_{cyc}^{1-\frac{k}{2}} \to \bar{\rho}|_{\mathrm{Gal}_{\mathbb{Q}_\ell}} \to \mu \chi_{cyc}^{-\frac{k}{2}} \to 0,$$

with $\mu: \mathrm{Gal}_{\mathbb{Q}_\ell} \to \{\pm 1\}$ the nontrivial unramified quadratic character.

So we see that there are very few assumption on the behavior of f at p and the conditions are on residue reps.

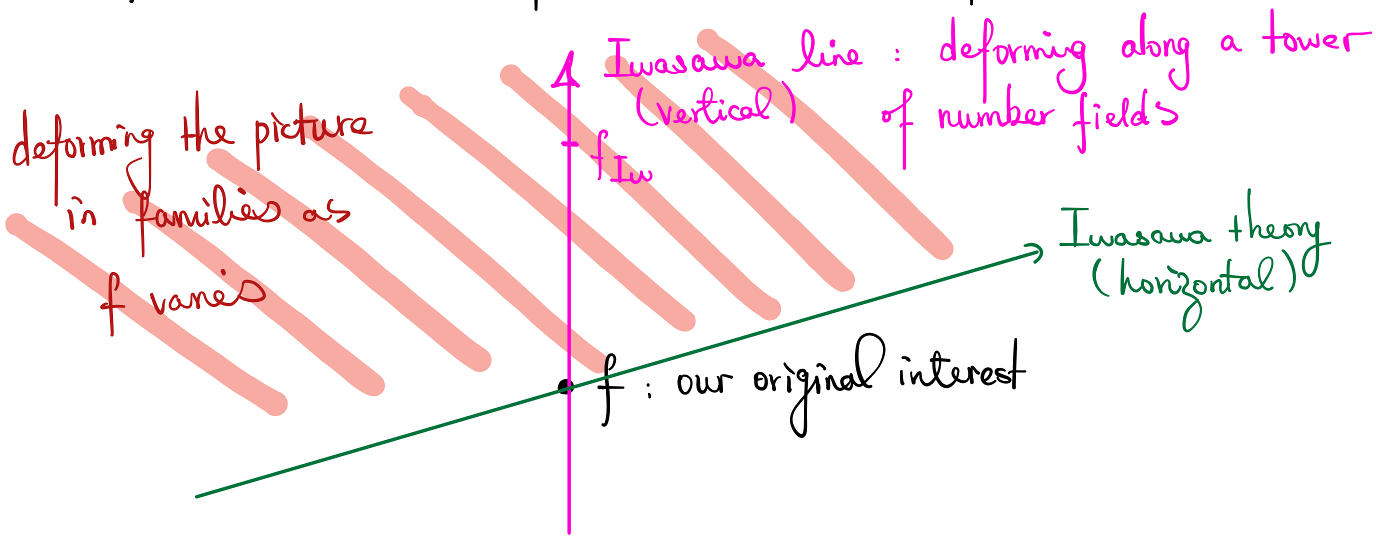
§1 Main method : Iwasawa main conjecture in families

- Greenberg's question : Let $f \in S_k(\Gamma_1(N), \epsilon)$ be a cuspidal eigenform. Let $\mathbb{f}$ be a Hida family (of p-ordinary forms) s.t. f is a specialization of $\mathbb{f}$ ($\therefore$ f itself is an ordinary form). Suppose $\exists$ a specialization g of $\mathbb{f}$ that is classical s.t. IMC holds for g, then can we deduce the IMC for f ?



- First answer by Emerton–Pollack–Weston : this holds if $\bar{\rho}_f$ satisfies some technical assumptions and the μ-invariant of g is zero.
- In Fouquet–Wan, they systematically deal with such problems :
  - For the "most" general family of modular forms.
  - Getting rid of the "$\mu = 0$" hypothesis. (much harder than IMCs)

To get a feel of what's going on, we first define what we mean by " universal families ". We hope to understand the picture :



1.1 Universal families

- Setups :

- $\Sigma \supseteq \{\ell \mid Np\}$.
- $G_{\mathbb{Q},\Sigma} = \mathrm{Gal}(\mathbb{Q}^{\Sigma\cup\infty}/\mathbb{Q})$: maximal ext'n of $\mathbb{Q}$ unramified ~~@~~ Σ and ∞.
- $E/\mathbb{Q}_p$ be a sufficiently large finite ext'n, integers $\mathcal{O}$ and residue field $\mathbb{F}$
- Let $\bar{\rho} := \bar{\rho}_f$ for $f \in S_k(\Gamma_1(N), \epsilon)$ with hypothesis ($\star$).

We define a deformation functor

$$D_{\bar{\rho}} : \left\{\begin{matrix} \text{finite local } \mathcal{O}\text{-algebras} \\ \text{w/ residue field } \mathbb{F} \end{matrix}\right\} =: \mathcal{C}_{\mathcal{O}} \to \mathrm{Sets}$$

$$A \mapsto \left\{ ((T, \rho, A), \iota) \;\middle|\; \begin{array}{l} \cdot\ T : \text{a free } A\text{-module of finite rank.} \\ \cdot\ \rho : G_{\mathbb{Q},\Sigma} \to \mathrm{Aut}_A(T) \text{ continuous group hom.} \\ \cdot\ \iota : \rho \otimes_A \mathbb{F} \simeq \bar{\rho} \quad \text{iso as } \mathbb{F}[G_{\mathbb{Q},\Sigma}]\text{-mod.} \end{array} \right\}$$

Theorem : The deformation functor is representable by a complete noetherian local $\mathcal{O}$-algebra $R_\Sigma(\bar{\rho})$ with residue field $\mathbb{F}$ (Denote this category by $CNL_{\mathcal{O}}$).

In other words, $\exists$ a ring $R_\Sigma(\bar{\rho})$ such that there is a natural isom for $A \in \mathcal{C}_{\mathcal{O}}$:

$$D_{\bar{\rho}}(A) = \mathrm{Hom}_{CNL_{\mathcal{O}}}(R_\Sigma(\bar{\rho}), A) = \mathrm{Hom}_{\mathcal{O}\text{-Sch}}(\mathrm{Spec}\, A, \mathrm{Spec}\, R_\Sigma(\bar{\rho})).$$

- Denote $(T_\Sigma^{univ}, \rho_\Sigma^{univ}, R_\Sigma(\bar{\rho}))$ the universal $G_{\mathbb{Q},\Sigma}$-representation.
- Denote $\mathfrak{X}_\Sigma(\bar{\rho}) := \mathrm{Spec}\, R_\Sigma(\bar{\rho})[\frac{1}{p}]$ (removing "the fibers at p"), called the universal deformation space.

Question: What does this space $\mathcal{X}_\Sigma(\bar\rho)$ look like?

(1) Let $A \in \mathcal{C}_\mathcal{O}$, then $x \in \mathcal{X}_\Sigma(\bar\rho)(A)$ iff $\exists$ a Galois representation

$$\rho_x : G_{\mathbb{Q},\Sigma} \longrightarrow GL_2(A)$$

s.t. $\rho_x \equiv \bar\rho \mod m_A$. Hence

$$(\ast)\left[\begin{array}{l} tr(\rho_x(Frob_\ell)) \mod m_A = tr(\bar\rho(Frob_\ell)) \quad \text{for } \ell \notin \Sigma. \\ det(\rho_x(Frob_\ell)) \mod m_A = det(\bar\rho(Frob_\ell)) \end{array}\right.$$

We say an $\mathcal{O}$-valued point x is modular if $\exists$ an eigencuspform g_x s.t. $\rho_{g_x} \simeq \rho_\Sigma \otimes_{R_\Sigma(\bar\rho), x} \mathcal{O}$. Therefore, if x and y are both modular points, then by $(\ast)$, $a_\ell(g_x) \equiv a_\ell(g_y) \mod \varpi$ for all $\ell \notin \Sigma$, i.e. g_x and g_y are congruent.

(recall: for eigencuspform g, charpoly $\rho_g(Frob_\ell) = X^2 - a_\ell(g)X + \epsilon_g(\ell)\ell^{k+1}$.)

In particular, since ρ_f itself is a modular point of $\mathcal{X}_\Sigma(\bar\rho)$, we see any eigencuspform attached to a modular point of $\mathcal{X}_\Sigma(\bar\rho)$ is congruent to f. (and vice-versa if we extend the scalars).

(2) Let $\mathbb{Q}_\infty/\mathbb{Q}$ be the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$.

Let $\Lambda := \mathcal{O}[\![Gal(\mathbb{Q}_\infty/\mathbb{Q})]\!]$ the Iwasawa algebra.

Let $T_f \subseteq V_f$ be a $Gal_\mathbb{Q}$-stable $\mathcal{O}$-lattice.

We define $T_{f,Iw} := T_f \otimes_\mathcal{O} \Lambda$ with tensor product action.

Note: $T_{f,Iw}$ is unramified @ Σ and $T_{f,Iw} \otimes_\Lambda \mathbb{F} \simeq \bar\rho_f$. Hence $T_{f,Iw}$ is a Λ-valued points of $\mathcal{X}_\Sigma(\bar\rho)$.

(3) Following the notations of (2), let $\mathfrak{p}$ be a height one prime of Λ, prime to (p). Then $\Lambda/\mathfrak{p}$ is a DVR, flat over $\mathbb{Z}_p$. One checks that $T_{f,\mathrm{Iw}} \otimes_{\Lambda} \Lambda/\mathfrak{p}$ is a Galois rep'n deforming $\bar{\rho}_f$, i.e. it is a $\Lambda/\mathfrak{p}$-point of $\mathcal{X}_\Sigma(\bar{\rho})$.

<u>Upshot</u>: So we see that in a <u>single space</u> "universal deformation space" $\mathcal{X}_\Sigma(\bar{\rho})$, we can "all objects" that we are interested in!

- all eigencuspforms congruent to f.
- their deformations along the Iwasawa-theoretic line
- and the informations on finite levels.

and a <u>uniformizing Galois representation</u> $(T_\Sigma, R_\Sigma(\bar{\rho}), \rho_\Sigma)$.

<u>What we need</u>: a formulation of Iwasawa main conjecture that can be defined over $\mathcal{X}_\Sigma(\bar{\rho})$, and respect various kinds of base changes.

— "fundamental lines" brought by Fontaine, Perrin-Riou, Kato, ...

Then one main theorem is:

<u>Theorem A</u>: There exists a Zariski-dense open subset $\mathcal{X}_\Sigma^{sm}(\bar{\rho}) \subseteq \mathcal{X}_\Sigma(\bar{\rho})$ s.t. TFAE:

(i) The "universal Iwasawa main conjecture" is true.

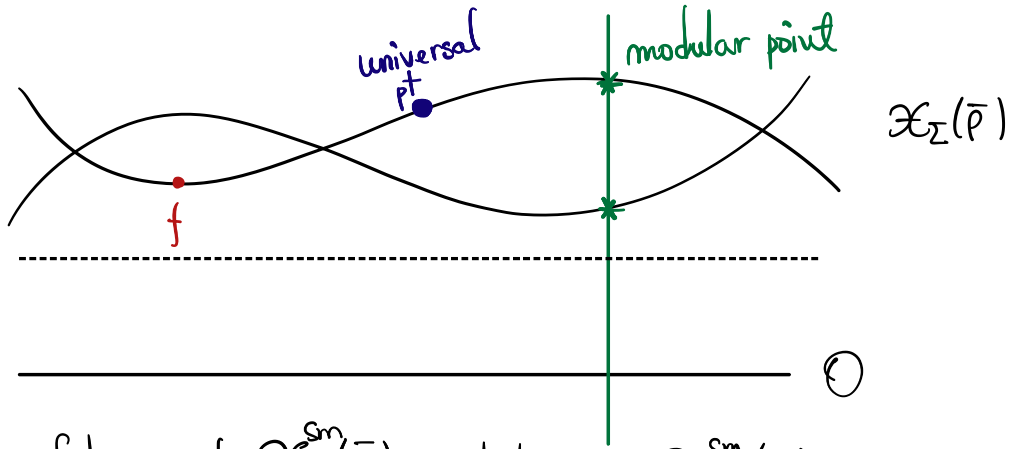
(ii) IMC is true for all modular points in $\mathcal{X}_\Sigma(\bar{\rho})$.

(iii) —— // —— in $\mathcal{X}_\Sigma^{sm}(\bar{\rho})$

(iv) IMC is true for the points in a single fiber of a modular point of $\mathcal{X}_\Sigma^{sm}(\bar{\rho})[\frac{1}{p}]$.

Kato's IMC (IMC wo p-adic L-function)

The nontrivial implications are (iv) $\Rightarrow$ (i) and (i) $\Rightarrow$ (ii).



- A "fact" on fibers of $\mathcal{X}_\Sigma^{sm}(\bar\rho)$: let $x \in \mathcal{X}_\Sigma^{sm}(\bar\rho)$ is a modular point.
  - Suppose x is ordinary, then all points in the fiber are ordinary.
  - Suppose x is cris and short, then —— " —— cris and short.

  (all such points lies in $\mathcal{X}_\Sigma^{sm}(\bar\rho)$ by requirement)

1.2 A strategy of proof: Start with a eigencuspform f.

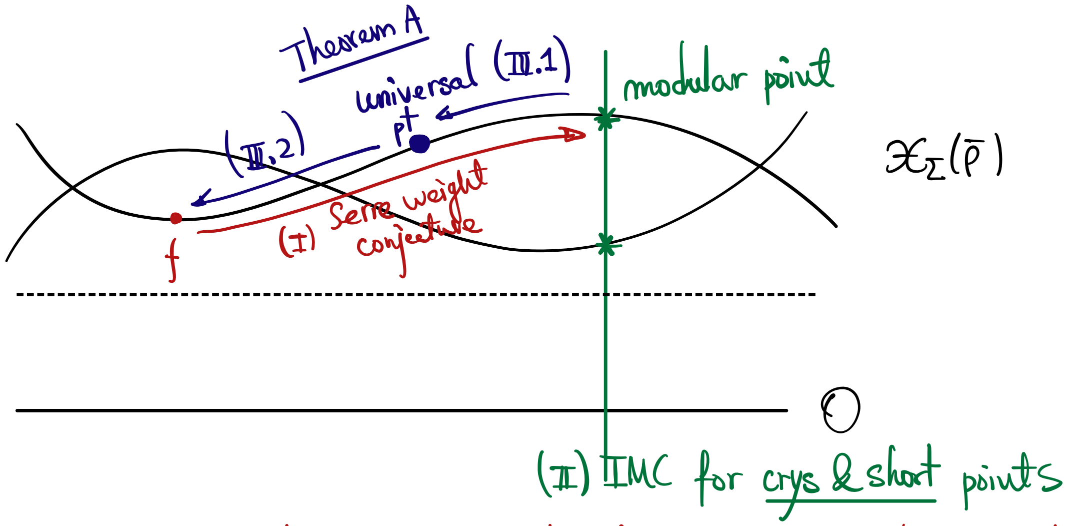
(I) Find a form g congruent to f such that
- if $\bar\rho_f|_{G_{\mathbb{Q}_p}}$ is irreducible, g is crystalline and short.
- if $\bar\rho_f|_{G_{\mathbb{Q}_p}}$ is reducible, g is ordinary.

This is a result of Serre weight conjecture (& some results of Edixhoven.)

(II) Prove IMCs for such g.
- Ordinary case: "well-known"
- Crystalline and short case: new result in [Fouquet-Wan]

  [Wan" Iwasawa theory for non-ordinary forms"]

(III) Apply Theorem A to get IMC for all modular points in $\mathcal{X}_\Sigma(\bar\rho)$.



In spirit, • it says that the IMC is not only true at each individual modular points of $\mathcal{X}_\Sigma(\rho)$, but also that these IMCs vary continuously on that space.

• and note: {ordinary} ∪ {crystalline & short} has positive codim, yet {modular} is dense.

In the following talks, I will mainly focus on Step (III), that is, the proof of Theorem A. So here we briefly talk about the proof on (II).

Model: in the supersingular case, we have

$\boxed{\text{an appropriate IMC à la Greenberg}} \Longrightarrow \boxed{\text{Kobayashi's } \pm\text{-main conjectures}}$

certain integral theory at p in "$\pm$"-settings.

Here the picture is similar:

$\boxed{\text{an appropriate IMC à la Greenberg}} \Longrightarrow \boxed{\text{Crystalline type Iwasawa main conj}}$

• "Triangulline Iwasawa theory" by Pottharst: CIMC $[\frac{1}{p}]$

• Then we treat the powers at p.

(see Wan's "Iwasawa 2017" conference article for details. That is "Theorem 6.1" in loc.cit.)

Still missing before Wan's work. See [Wan, arXiv 1607.07729v7] for a sketch of the Greenberg's IMC.

P.S. On crystalline & short :

- A modular point g is called crystalline if $\rho_g|_{\mathrm{Gal}_{\mathbb{Q}_p}}$ is a crystalline rep of $\mathrm{Gal}_{\mathbb{Q}_p}$, i.e. $D_{\mathrm{cris}}(V_g) := (\mathbb{B}_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} V_g)^{\mathrm{Gal}_{\mathbb{Q}_p}}$ is a dim 2 $\mathbb{Q}_p$-vector space.
- A modular point g is called crystalline and short, if it is crystalline and $2 \leq k :=$ weight of $g \leq p$.

  ($\Longleftrightarrow$ $\rho_g|_{\mathrm{Gal}_{\mathbb{Q}_p}}$ is in the image by the Fontaine-Laffille functor of a rank two Fontaine-Laffille module with nontrivial graded piece of the filtration in degree 0 & $k-1$.)

Often people say "modular / classical up a twist" if the point $x \in \mathfrak{X}_\Sigma(\bar{\rho})$ there exists a eigencuspform g and a character of Λ, s.t. $\rho_x \simeq \rho_g \otimes \chi$, and the corresponding defn of properties on such points is defined on g.

These descriptions are actually independent of the "$R = \mathbb{T}$" theorems.

1.3 Structure of [Fouquet – Wan]

- Chapter 2 : Some geometry of $\mathcal{X}_\Sigma(\bar\rho)$ and $\mathcal{X}_\Sigma^{Sm}(\bar\rho)$. — Backgrounds
- Chapter 3 : Formulation of Universal IMC : this is far from trivial !
  - A p-adic interpolation of Kato's zeta element $z(f)_{Kato}$ on $\mathcal{X}_\Sigma(\bar\rho)$
    (this is done by [Nakamura, 2023 Inventiones])
  - To get "primitive zeta elements", need to pass to each irreducible components of $\mathcal{X}_\Sigma(\bar\rho)$. (this is done by Fouquet–Wan, in §3.2)

  Such constructions need the huge machine of p-adic local Langlands !

  (Chapter 6 provides a missing part in p-adic local Langlands needed in Fouquet-Wan's construction.)
- Chapter 7 : Prove the appropriate Greenberg's main conjecture. (GMC)

  Chapter 4 : Show how GMC implies the crystalline IMC.

  - [Wan 2020] assumes the form f varies in a Hida family, hence itself being ordinary.
  - [Castella – Liu – Wan] removes "ordinary" condition of f, but only deals with the weight two case.

  In [Fouquet – Wan], Wan made a step forward, still using EC on $U(3,1)$.
- Chapter 5 (with statements in § 3.3–3.4) : Prove Thm A.

We warn that this talk is a very rough review, and missing "tons of details in [Fouquet - Wan].

1.4 Numerical example

- $E: y^2 = x^3 + x - 10 \rightsquigarrow f \in S_2(\Gamma_0(52))$ with $a_5(f) = 2$, hence 5-ordinary.
  - Check: $E[p]$ satisfies $(\star)$
  - Greenberg-Vatsal: f has zero μ-invariant and satisfies IMC.

Then:

(1) Let $K = \mathbb{Q}(x^5 + 294x^4 - \cdots) \supseteq \mathcal{O}_K$, with $5\mathcal{O}_K = \mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_3\mathfrak{p}_4$.

For exactly $\mathfrak{p}_1, \mathfrak{p}_2$ two primes, we find eigencuspforms f_1, f_2 congruent mod $\mathfrak{p}_1, \mathfrak{p}_2$ to f respectively.

- f_1 is known to be ordinary, so f_1 valids IMC by [EPW].
- f_2 has finite slope but not ordinary, yet f_2 satisfies IMC by Thm A.

(2) There is also $E' : y^2 = x^3 + 625x - 6250 \rightsquigarrow g \in S_2(\Gamma_0(1300))$.

- Check: E' is additive at $p = 5$. $(\Leftrightarrow \pi_{g,5}$ is sc as a rep of $GL_2(\mathbb{Q}_5))$
- Theorem A implies to E', shows the IMC is true for E' at $p = 5$.
  This is a brand new result!

§2 More on the universal deformation space

We assume Assumption 2.1 of [Fouquet–Wan], ensuring $D_{\bar\rho}$ is representable.

2.1 Hecke algebras and $R = \mathbb{T}$ theorem

- Let $U \subseteq GL_2(\mathbb{A}^\infty)$ a compact open subgroup, and $Y(U)$ the affine modular curve of level U:

$$Y(U)(\mathbb{C}) = GL_2(\mathbb{Q}) \backslash (\mathbb{C} \setminus \mathbb{R}) \times GL_2(\mathbb{A}^\infty) / U$$

We write $H^i_{ét}(U, -) := H^i_{ét}(Y(U) \times_{\mathbb{Q}} \overline{\mathbb{Q}}, -)$.

- Let Σ = finite set of finite primes containing $\underbrace{\{\ell : U_\ell \text{ is not a max. cpt}\} \cup p}_{=: \Sigma(U)}$

Let $\mathbb{T}^\Sigma(U) :=$ $\mathcal{O}$-subalg of $\mathrm{End}(H^1_{ét}(U, \mathcal{O}))$ gen. by $T(\ell)$ and $S(\ell)$ for $\ell \notin \Sigma$.

We fix Σ as above and a residue rep

$$\bar\rho : G_{\mathbb{Q},\Sigma} \longrightarrow GL_2(\mathbb{F})$$

that is abs. irred and modular ($\Rightarrow$ odd) Galois rep unr. @ Σ. Then $U \subseteq GL_2(\mathbb{A}^\infty)$ is allowable for $\bar\rho$ if $\exists\,(!)$ maximal ideal $m_{\bar\rho} \in \mathrm{Spec}\,\mathbb{T}^\Sigma(U)$ s.t.

$$\left[\begin{array}{l} T(\ell) \bmod m_{\bar\rho} \equiv \mathrm{tr}\,\bar\rho(\mathrm{Fr}_\ell) \\ S(\ell) \bmod m_{\bar\rho} \equiv \det\bar\rho(\mathrm{Fr}_\ell) \end{array}\right. \in \mathbb{F}, \quad \forall\, \ell \notin \Sigma.$$

For $U' \subseteq U$ allowable, $m_{\bar\rho}, m'_{\bar\rho}$ of $\mathbb{T}^\Sigma(U), \mathbb{T}^{\Sigma'}(U')$. Then:

$$\begin{array}{ccc} Y(U') & & \mathbb{T}^{\Sigma'}(U')_{m'_{\bar\rho}} \\ \downarrow \text{covering} & \overset{\text{induces}}{\rightsquigarrow} & \downarrow \text{surjection} \\ Y(U) & & \mathbb{T}^{\Sigma}(U)_{m_{\bar\rho}} \end{array}$$

, iso if U is sufficiently small.

We fix such a choice of small U as above, write $\mathbb{T}^\Sigma_{m_{\bar\rho}}$ for $\mathbb{T}^\Sigma(U)_{m_{\bar\rho}}$.

<u>Theorem of Carayol</u> : $\exists$ a $\mathbb{T}^{\Sigma}_{\mathfrak{m}_{\bar\rho}}$-module T_Σ, free of rank two, with a Galois representation

$$\rho_\Sigma : G_{\mathbb{Q},\Sigma} \longrightarrow \mathrm{Aut}_{\mathbb{T}^{\Sigma}_{\mathfrak{m}_{\bar\rho}}}(T_\Sigma) \simeq GL_2(\mathbb{T}^{\Sigma}_{\mathfrak{m}_{\bar\rho}})$$

uniquely characterized by $\mathrm{tr}\,\rho_\Sigma(\mathrm{Fr}_\ell) = T(\ell)$ for $\ell \notin \Sigma$.

$\xRightarrow{\text{reinterpretation}}$ $(T_\Sigma, \rho_\Sigma, \mathbb{T}^{\Sigma}_{\mathfrak{m}_{\bar\rho}}) \in D_{\bar\rho}$, i.e. $\exists$ a map $R_\Sigma(\bar\rho) \longrightarrow \mathbb{T}^{\Sigma}_{\mathfrak{m}_{\bar\rho}}$.

This map is surjective : its image contains the image of $\mathrm{tr}(\rho^{univ}_\Sigma(\mathrm{Fr}_\ell))$, which is $\mathrm{tr}\rho_\Sigma(\mathrm{Fr}_\ell) = T_\ell$ for all $\ell \notin \Sigma$.

<u>Geometry of $R_\Sigma(\bar\rho)$</u> :

- $R_\Sigma(\bar\rho)$ is a flat $\mathcal{O}$-algebra, a <u>reduced</u> complete intersection ring of Krull dimension 4.
- There exists an open subset $\mathfrak{X}^{sm} \subseteq \mathfrak{X}_\Sigma(\bar\rho)$ dense under Zariski and adic topology, containing modular pts on each irred component of $\mathfrak{X}_\Sigma(\bar\rho)$, s.t. the map $\mathfrak{X}_\Sigma(\bar\rho) \longrightarrow \mathrm{Spec}\,\mathcal{O}$ is formally smooth at $x \in \mathfrak{X}^{sm}$.

Then in particular, $R_\Sigma(\bar\rho) \twoheadrightarrow \mathbb{T}^{\Sigma}_{\mathfrak{m}_{\bar\rho}}$ is an isomorphism. This is the so-called "$R=\mathbb{T}$"-theorem. Hidden in the proof of above facts, we have :

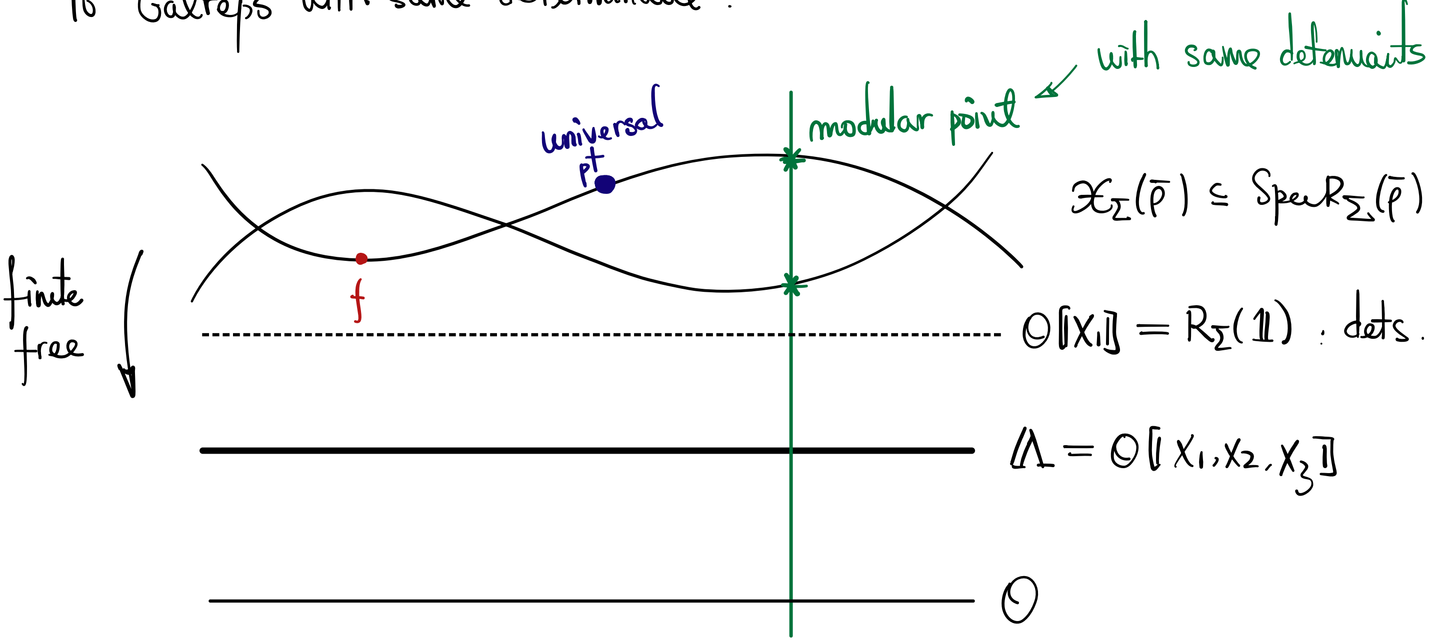
- By <u>a result of Edixhoven</u>, $\exists\ x \in \mathfrak{X}_\Sigma(\bar\rho)$ attached to a eigencuspform f of weight k and $\chi = \det\rho_f$ ([FW, Lemma 2.2] deals with this). Then

$$R_\Sigma(\bar\rho) = R^{\chi}_\Sigma(\bar\rho) \hat\otimes R_\Sigma(\mathbb{1}) \quad — (\ast)$$

- $R^{\chi}_\Sigma(\bar\rho)$ is the universal def space of fixed determinant χ.
- $R_\Sigma(\mathbb{1})$ is the deformation of trivial residue rep, $\simeq \mathcal{O}[\![X,Y]\!]$.

- Let $\Lambda := \mathcal{O}[\![X_1, X_2, X_3]\!]$. Then there is a length 3 regular sequence (x_1, x_2, x_3) inside $R_\Sigma(\bar\rho)$ s.t. the assignment $X_i \mapsto x_i$ endows $R_\Sigma(\bar\rho)$ a structure of Λ-algebra, s.t.
  - $R_\Sigma(\bar\rho)$ is finite and free as Λ-module
  - The variable X_1 corresponds to $R_\Sigma(\mathbb{1})$ in (*)
  - Modular points of $\mathbb{T}^\Sigma_{m_{\bar\rho}}$ which induces the same morphisms after restriction to Λ through the above specified $\Lambda \hookrightarrow \mathbb{T}^\Sigma_{m_{\bar\rho}}$ are either all crystalline & short up to a twist (resp. ordinary or up to a twist) or non of them are.

  So all points in the fiber of $\operatorname{Spec} \mathbb{T}^\Sigma_{\bar\rho}$ above a point in Λ corresp. to Galreps with same determinant.



This is actually the correct picture.

<u>Remark</u>: If we have enough time, we would give a proof of this "<u>geometric fact</u>" following [Fouquet, 2024 Besançon article]. The advantage of working over Λ can be seen through the proof of <u>Theorem A</u>.

this relies on certain modularity result. So we are not getting "$R = \mathbb{T}$"

from nothing.

<u>Remark</u>: How to understand "$R = \mathbb{T}$" as a modularity result?

- Let $A \in \mathcal{C}_O$, then:
  - An A-point x of R is a Galois rep $\rho_x : \mathrm{Gal}_{\mathbb{Q},\Sigma} \to GL_2(A)$ s.t. $\rho_x \equiv \bar{\rho} \mod m_A$
  - An A-point x of $\mathbb{T}_{m_{\bar{\rho}}}$ is an O-algebra homomorphism $\mathbb{T}_{m_{\bar{\rho}}} \xrightarrow{f_x} A$.
    Such a homomorphism can be regarded as a modular form f with coefficients in A and eigensystem f_x $\left(\text{or} : \mathbb{T} \to \mathbb{T}_{m_{\bar{\rho}}} \xrightarrow{f_x} A \right)$.
    It factor thru $\mathbb{T}_{m_{\bar{\rho}}}$, means "$f_x$" is congruent with f.

So "$R = \mathbb{T}$" is saying that

$$\left\{ \text{Gal reps } \rho : \mathrm{Gal}_{\mathbb{Q},\Sigma} \to GL_2(A) \right\} \xleftrightarrow{1:1} \left\{ \text{eigenforms over } A \right\}$$

This is a "modularity result".

2.2 Proof of the geometric facts on $\mathcal{X}_\Sigma(\bar\rho)$

We turn to [Fouquet. Besançon note] for the detailed proof. It is needed to understand certain subtleties!

(A) Extract the determinant part

- Let k be the Serre weight of $\bar\rho$. (standard notion in rep theory of $GL_2(\mathbb{F}_p)$) then by Edixhoven, $\exists\ x \in \mathcal{X}_\Sigma(\bar\rho)$ attached to a eigencuspform f of weight k.
  Denote $\chi := \det \rho_f : G_\mathbb{Q} \to \mathcal{O}^\times$.
- Let $R^\chi_\Sigma(\bar\rho) :=$ deformation ring of $\bar\rho$ parametrizing deformations ρ of $\bar\rho$ in $CNL_\mathcal{O}$ s.t. unramified @ Σ and $\det\rho = \chi$.
- Let $\rho \in R_\Sigma(\bar\rho)$ be an $\mathcal{O}$-point. Then $(\det\rho)^{-1}\chi$ has value in $\mathcal{O}^\times$ and
  $$(\det\rho)^{-1}\chi \equiv (\det\bar\rho)^{-1}(\det\bar\rho_f)^{-1} \equiv 1 \mod \varpi$$
  So $(\det\rho)^{-1}\chi \in 1+\varpi\mathcal{O}$. Hence $(\det\rho)^{-1}\chi : G_\mathbb{Q} \to 1+\varpi\mathcal{O}$ corresponds to a point in the deformation space $R_\Sigma(\mathbb{1})$ of trivial characters.
- Note: as p is odd, the multiplicative group $1+\varpi\mathcal{O}$ admit canonical square roots. (need to check this p-adic analytical fact.)

$\rightsquigarrow$ Let $\psi_\rho :=$ canonical square root of $(\det\rho)^{-1}\chi$. Then

- $\rho\otimes\psi_\rho \equiv \bar\rho \mod \varpi$
- $\det(\rho\otimes\psi_\rho) = \chi$

so $\rho\otimes\psi_\rho$ is a point of $R^\chi_\Sigma(\bar\rho)$.

So this gives an isomorphism $R_\Sigma(\bar\rho) \xrightarrow{\sim} \boxed{R^\chi_\Sigma(\bar\rho)} \otimes \boxed{R_\Sigma(\mathbb{1})}$

isomorphic to a power-series ring in one variable, over a complete intersection $\mathcal{O}$-algebra of relative dimension zero

"$\simeq \mathcal{O}[[X_1]]$"

- Next goal: $R^\chi_\Sigma(\bar\rho)$ is a flat $\mathcal{O}$-algebra, which is a complete intersection ring of relative dimension 2.

(ⓥ ⓥ $\Longrightarrow$ $R_\Sigma(\bar\rho)$ is a flat $\mathcal{O}$-algebra, which is a CI ring of rel. dim $= 3$)

(B) Extract the ordinary / crystalline part

(B.1) When $\bar\rho|_{G_{\mathbb{Q}_p}}$ is reducible : we may assume by a twist of character

$$(\bar\rho|_{G_{\mathbb{Q}_p}})^{ss} = \bar\chi_1 \oplus \bar\chi_2, \quad \bar\chi_1(I_p) = \{1\} \quad (\text{i.e. } \bar\chi_1 \text{ is unramified})$$

Let $R_\Sigma^{\mathrm{ord},\chi}(\bar\rho) :=$ quotient of $R_\Sigma^\chi(\bar\rho)$ parametrizing points $\rho \in R_\Sigma^\chi(\bar\rho)$ s.t. $\exists$ a SES of nonzero $G_{\mathbb{Q}_p}$-reps

$$0 \to \chi_1 \to \rho|_{G_{\mathbb{Q}_p}} \to \chi_2 \to 0, \quad \chi_1 \text{ unramified}.$$

Big input (Wiles, Taylor–Wiles) : $R_\Sigma^{\mathrm{ord},\chi} \simeq$ "$\mathbb{T}_\Sigma^{\mathrm{ord},\Sigma}$" a suitable Hecke algebra, hence flat of relative dimension zero over $\mathcal{O}$.

(B.2) When $\bar\rho|_{G_{\mathbb{Q}_p}}$ is irreducible, by "Serre modularity" (see [FW, Lemma 2.2]), we assume f is crystalline and short at p.

Let $R_\Sigma^{\mathrm{cris},\chi}(\bar\rho) :=$ quotient of $R_\Sigma^\chi(\bar\rho)$ parametrizing cryshort points.

Big input (Diamond–Flach–Guo) $R_\Sigma^{\mathrm{cris},\chi} \simeq$ "$\mathbb{T}_\Sigma^{\mathrm{cris},\chi}$" a suitable Hecke algebra, flat of relative dimension zero over $\mathcal{O}$

The result is then unified : $R_\Sigma^{*,\chi}$ is flat of rel. dim. zero over $\mathcal{O}$.

$\rightsquigarrow$ Goal : Describe the kernel of $R_\Sigma^\chi(\bar\rho) \to R_\Sigma^{*,\chi}(\bar\rho)$.

(C) Local deformation rings and then to global

(C.1) When $\bar\rho|_{G_{\mathbb{Q}_p}}$ is reducible : we use two big inputs :

1° Skinner–Wiles, 2001 : $R^{\square,\mathrm{ord}}(\bar\rho|_{G_{\mathbb{Q}_p}})$ is a regular ring of relative dim. 3.

2° Böckle, 2000 : $R^{\square,\mathrm{ord}}(\bar\rho|_{G_{\mathbb{Q}_p}})$ is a quotient of $R^\square(\bar\rho|_{G_{\mathbb{Q}_p}})$ by a length 2 regular sequence.

$\Longrightarrow$ $R^{\square,\mathrm{ord}}(\bar\rho|_{G_{\mathbb{Q}_p}})$ is a regular ring of rel. dim 5 over $\mathcal{O}$, and $\ker(R_p^\square \to R_p^{\square,\mathrm{ord}})$ is gen. by a subset of a syst. of para. of cardinal 2.

$\underset{R^{\square} \rightsquigarrow R}{\xRightarrow{\text{should hold:}}}$ $\ker(R_p^{\chi} \to R_p^{\mathrm{ord},\chi})$ is gen. by a regular seq. of length 2.

(C.2) When $\bar\rho|_{G_{\mathbb{Q}_p}}$ is irreducible: Diamond–Flach–Guo showed: the kernel of $R_p^{\chi} \to R_p^{\mathrm{cris},\chi}$ is generated by a regular sequence of length 2.

So in both cases, we have:

$$\begin{tikzcd} \boxed{R^{\chi}(\bar\rho|_{G_{\mathbb{Q}_p}}) \arrow[r] \arrow[d] \arrow[dd, bend right] & R^{*,\chi}(\bar\rho|_{G_{\mathbb{Q}_p}})} \arrow[d] \\ \boxed{R_{\Sigma}^{\chi}(\bar\rho) \arrow[r] & R_{\Sigma}^{*,\chi}(\bar\rho)} \\ {} \end{tikzcd}$$

$\longleftarrow$ induced by restriction from $\mathrm{Gal}_{\mathbb{Q},\Sigma}$ to $\mathrm{Gal}_{\mathbb{Q}_p}$.

where this kernel is generated by a regular sequence of length 2.

$\Rightarrow$ $R_{\Sigma}^{*,\chi}(\bar\rho)$ is a quotient of $R_{\Sigma}^{\chi}(\bar\rho)$ by an ideal gen. by at most two elts.

$\Rightarrow$ $R_{\Sigma}^{*,\chi}(\bar\rho)$ —— $R_{\Sigma}(\bar\rho)$ —— (x_1, x_2, x_3) by at most 3 elts.

In particular, $R_{\Sigma}(\bar\rho)$ is of Krull dimension at most 4. (recall characterization of $R_{\Sigma}^{*,\chi}(\bar\rho)$ in (B): $R_{\Sigma}^{*,\chi}$ is flat of rel. dim zero over $\mathcal{O}$.)

Another input: [Böckle 2001] $R_{\Sigma}(\bar\rho)$ is of Krull dimension at least 4

$\Rightarrow$ $R_{\Sigma}(\bar\rho)$ is of Krull dimension 4.

(D) Complete intersection:

- Also follows from [Böckle 2001]: the zero dim ring $R_{\Sigma}^{*,\chi}/\varpi$ admits a presentation

$$R_{\Sigma}^{*,\chi}(\bar\rho)/(\varpi) \xrightarrow{\sim} \mathcal{O}[\![X_1, \cdots, X_n]\!]/(\varpi, x_1, x_2, x_3, y_4, \cdots, y_n).$$

- Since $\dim(\mathrm{LHS}) = 0$, $(\varpi, x_1, x_2, x_3, y_4, \cdots, y_n)$ is a regular sequence in $\mathcal{O}[\![X_1, \cdots, X_n]\!]$.

$\xRightarrow{\text{in particular}}$ $R_{\Sigma}^{*,\chi}(\bar\rho)/(\varpi)$, $R_{\Sigma}^{*,\chi}(\bar\rho)$, $R_{\Sigma}(\bar\rho)$ are complete intersection of dim 0, 1, 4.

and (x_1, x_2, x_3) is a regular seq in $R_{\Sigma}(\bar\rho)$.

(E) Definition of $\mathfrak{X}^{sm}$

- Let (x_2, x_3) be the sub-regular sequence. Let $\Lambda^\chi := \mathcal{O}[[X_2, X_3]]$, consider

$$\begin{tikzcd} \Lambda^\chi \arrow[r, "X_i \mapsto x_i"] \arrow[d, "\pi"] & R^\chi_\Sigma(\bar\rho) \arrow[d, "\pi'"] \\ \mathcal{O} \arrow[r] & R^{*,\chi}_\Sigma(\bar\rho) \end{tikzcd}$$

π : quotient mod (X_2, X_3)
π' : —— (x_2, x_3)

- Let $\mathfrak{p} \in \operatorname{Spec} R^\chi_\Sigma(\bar\rho)$ be a point above $\pi : \operatorname{Spec}\mathcal{O} \to \operatorname{Spec}\Lambda^\chi$.
Since $R^\chi_\Sigma(\bar\rho)_\mathfrak{p}$ is flat over $R^\chi_\Sigma(\bar\rho)$, (x_2, x_3) remains a reg. seq. of $R^\chi_\Sigma(\bar\rho)_\mathfrak{p}$.
By construction of $R^\chi_\Sigma(\bar\rho)$ (see (B), (C) : characterize $R^{*,\chi}_\Sigma$ and the kernel) :

$$R^\chi_\Sigma(\bar\rho)_\mathfrak{p}/(x_2,x_3) \otimes_\mathcal{O} E = \text{localization of a minimal prime of "}\underbrace{\text{Hecke alg}}_{\text{reduced}}\text{"}$$

hence a separable ext'n of E and

$$\boxed{\operatorname{Spec} R^\chi_\Sigma(\bar\rho)_\mathfrak{p}/(x_2,x_3) \longrightarrow \operatorname{Spec} E \text{ is étale.}}$$

- Therefore, $\operatorname{Spec} R^\chi_\Sigma(\bar\rho) \longrightarrow \operatorname{Spec}\Lambda^\chi$ is unramified at $\mathfrak{p}$. Being flat + finite by (B)(C), it is étale at $\mathfrak{p}$.

$\rightsquigarrow$ Let $\mathfrak{X}^{sm,\chi} :=$ étale points of $\operatorname{Spec} R^\chi_\Sigma(\bar\rho) \longrightarrow \operatorname{Spec}\Lambda^\chi$, then $\mathfrak{p} \in \mathfrak{X}^{sm,\chi}$!

- Let $U \subseteq \operatorname{Spec}\Lambda^\chi$ be the complement of $\operatorname{supp}(\Omega^1_{R^\chi_\Sigma(\bar\rho)/\Lambda^\chi})$
Then U is nonempty, formally smooth over $\operatorname{Spec}\mathcal{O}$, and $\mathfrak{X}^{sm,\chi}$ is formally smooth over $\operatorname{Spec}\mathcal{O}$.

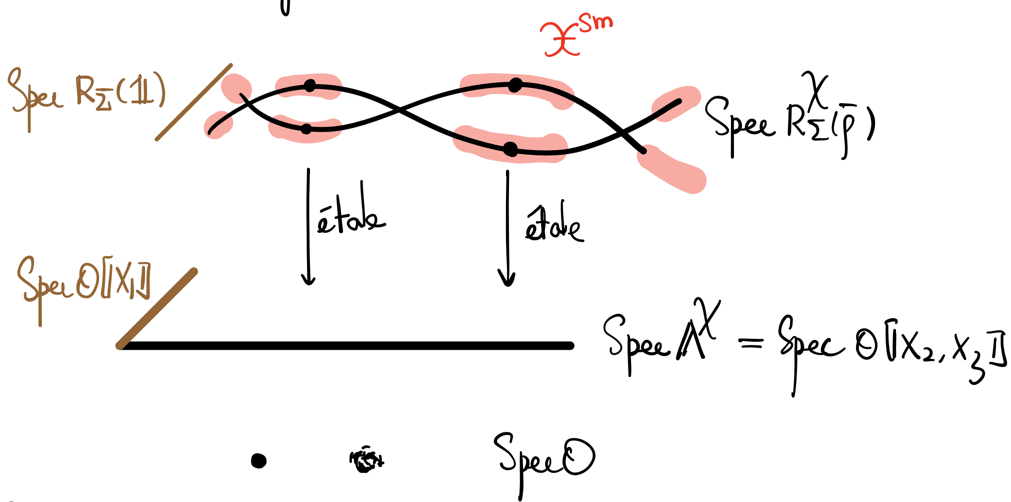
- U is open, nonempty, hence Zariski-dense in $\operatorname{Spec}\Lambda^\chi$.
$\rightsquigarrow$ $\operatorname{Spec}\Lambda^\chi \setminus U$ is at least of codim 1.
$\rightsquigarrow$ $\operatorname{Spec} R^\chi_\Sigma(\bar\rho) \setminus \mathfrak{X}^{sm}$ is also at least codim 1 ($\because$ $\operatorname{Spec} R^\chi_\Sigma(\bar\rho) \to \operatorname{Spec}\Lambda^\chi$ is finite)
$\rightsquigarrow$ $\mathfrak{X}^{sm,\chi}$ is Zariski dense in each irred component. ($\because$ $R^\chi_\Sigma(\bar\rho)$ is CM.)

- Let $\{\mathfrak{p}_1, \cdots, \mathfrak{p}_d\}$ be the finite set of minimal primes of $R^\chi_\Sigma(\bar\rho)$ and $\mathfrak{X}_i := \operatorname{Spec} R^\chi_\Sigma(\bar\rho)/\mathfrak{p}_i$

  $\rightsquigarrow$ $\mathfrak{X}^{sm,\chi} \cap \mathfrak{X}_i$ are nonempty and disjoint, and by étaleness, its generic degree equals its degree at p.

  Hence each $\mathfrak{X}_i$ contains a point p_i above π^* and p_i are pairwise distinct.

  Moreover, relation with Hecke algebras implies each irred component of $R^\chi_\Sigma(\bar\rho)$ contains points attached to eigencuspforms.



(F) Reducedness of $R_\Sigma(\bar\rho)$

- As $R^\chi_\Sigma(\bar\rho)$ is Cohen-Macauley, $\{\mathfrak{p}_1, \cdots, \mathfrak{p}_d\}$ is also the set of associated primes.
- By properties of $\mathfrak{X}^{sm}$ (?) : $\forall\, i = 1, \cdots, d$, $\exists\, a_i \in R^\chi_\Sigma(\bar\rho)$ s.t. $a_j \equiv 0 \mod \mathfrak{p}_i$ iff $i \neq j$, and s.t. $\operatorname{Spec} R^\chi_\Sigma(\bar\rho)_{a_i}$ is smooth over E.

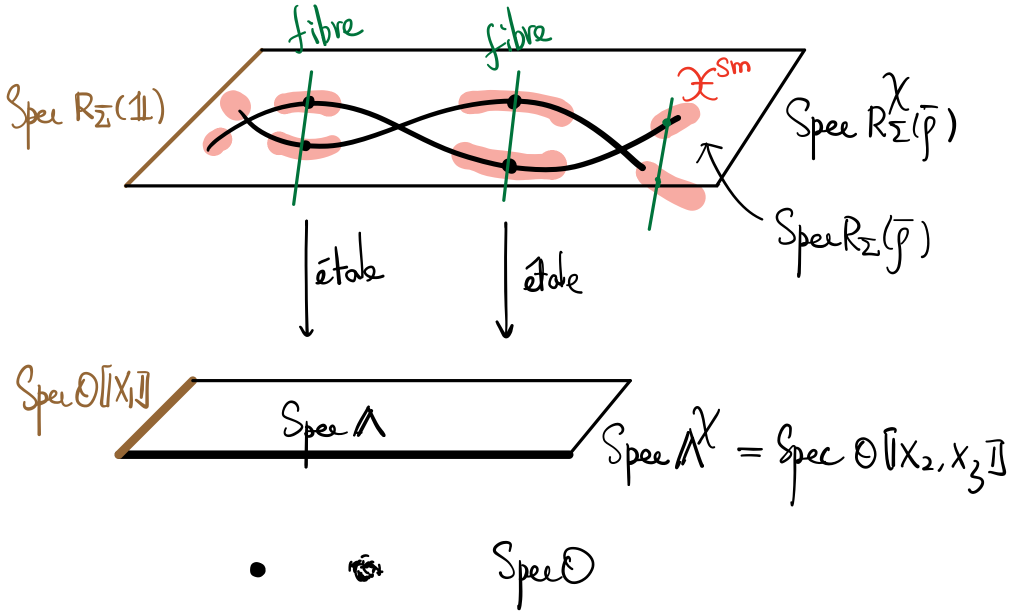
  $\rightsquigarrow$ $a = a_1 + \cdots + a_d \notin \mathfrak{p}_i$, $\forall\, i = 1, \cdots, d$.

  hence $a \notin$ any associated prime of $R^\chi_\Sigma(\bar\rho)$.

  hence a is not a zero divisor and $\exists$ embedding $R^\chi_\Sigma(\bar\rho) \hookrightarrow R^\chi_\Sigma(\bar\rho)_a$.
- As $\operatorname{Spec} R^\chi_\Sigma(\bar\rho)_a = \coprod_{i=1}^{d} \operatorname{Spec} R^\chi_\Sigma(\bar\rho)_{a_i}$, w/ $\operatorname{Spec} R^\chi_\Sigma(\bar\rho)_{a_i}$ is smooth

  we conclude : $R^\chi_\Sigma(\bar\rho)$ is reduced.

Then the same must be true for $R_\Sigma(\bar\rho)$. $\square$



Now we are clear to say "fibre": if $x \in \mathfrak{X}^{sm}(\bar\rho)$ is a modular point, we call the fibre of x the finite set of points x_i s.t. x_i are above the same point of Spec Λ, (not Spec $\mathcal{O}$ of course) denote by S_x.

we can think of Λ as the "weight algebra" of the universal deformation ring.

(G) Points on a fibre: fibres of modular points $x \in \mathfrak{X}_\Sigma^{sm}(\bar\rho)$:

- Recall (C): the kernel of $R_\Sigma(\bar\rho)/(x_1) = R_\Sigma^\chi(\bar\rho) \twoheadrightarrow R_\Sigma^{*,\chi}(\bar\rho)$ contains a regular sequence (x_2, x_3) which is the image of the regular seq. generating the kernel of $R_p^\chi \to R_p^{*,\chi}$.
- Then to a crystalline point ρ_f is thus attached a Λ-structure on $R_\Sigma(\bar\rho)$ defined by $X_i \mapsto x_i$ (note: x_2, x_3 depends on the choice of ρ_f).

Hence given such a Λ-structure on $R_\Sigma(\bar\rho)$, if two specializations $\psi_i : R_\Sigma(\bar\rho) \to S$ with values in a DVR coincide after restriction to Λ, then they have same value on x_2 and x_3. (this is the meaning of "in the same fibre"!)
So they both factor thru $R_\Sigma^{*,\chi}(\bar\rho)$ attached to this choice of Λ-structure or neither of them does. $\square$