Talk 04 Eisenstein series

§4.1 Setups

- Same set up as in previous talks
- Let $r \geq s \geq 0$. consider the general unitary group $G = GU(r,s)$ over F.
  We fix a choice of $K = \prod_{v|\infty} K_v \times \prod_{v\nmid\infty} K_v$ a maximal (open) compact subgroup of $G(\mathbb{A}_F)$ inexplicity.

Datum for constructing Klingen Eisenstein series:

- (π, V) be an irreducible cuspidal tempered automorphic rep'n of $GU(r,s)(\mathbb{A}_F)$.

Recall: • (π, V) is an admissible $(\mathfrak{gu}(\mathbb{R}), K_\infty)_{v|\infty} \times GU(r,s)(\mathbb{A}_{F,f})$-module, which lies in the space of cuspidal automorphic forms over $GU(r,s)(\mathbb{A}_F)$.

By Flath's theorem, it admits a decomposition of local representations

$$(\pi, V) \simeq \Big(\bigotimes_{v|\infty} \pi_v, \bigotimes_{v|\infty} V_v\Big) \otimes \Big(\bigotimes_{v\nmid\infty}{}' \pi_v, \bigotimes_{v\nmid\infty}{}' V_v\Big), \quad \text{—}(*)$$

where • for $v|\infty$, (π_v, V_v) is an irreducible admissible $(\mathfrak{gu}(r,s)_{\mathbb{R}}, K_v)$-module

• for $v\nmid\infty$, (π_v, V_v) is an irreducible admissible $G(F_v)$-module,

and for all but finitely many $v\nmid\infty$, one has $V_v^{K_v} \neq 0$. By Satake isomorphism, $\dim_{\mathbb{C}} V_v^{K_v} = 1$. Such a rep'n is called unramified rep'n. We take and fix a nonzero vector in $V_v^{K_v}$, called the spherical vector at v, and the restricted tensor product in $(*)$ is taken wrt these sphreical vectors.

Note: Write $G = GU(r,s)$ temporially. The complex conjugation $\bar{\pi} \subseteq \mathcal{A}\big(G(F)\backslash G(\mathbb{A})\big)$ is the contragredient of π. We require for the tensor product decomposition above that for factorizable $\phi_1, \phi_2 \in \pi$ with $\phi_1 = \otimes \phi_{1,v}$, $\bar{\phi}_2 = \otimes \bar{\phi}_{2,v}$,

$$\langle \phi_1, \bar{\phi}_2 \rangle = \prod_v \langle \phi_{1,v}, \bar{\phi}_{2,v} \rangle$$

where

- LHS is the bi-$\mathbb{C}$-linear Petersson inner product on forms over G, wrt our fixed Haar measure on $G(\mathbb{A})$.

- RHS is the natural pairing between π_v and $\tilde{\pi}_v$.

- Let $\psi: \mathbb{A}_K^\times \longrightarrow \mathbb{C}^\times$ be a Hecke character such that $\psi' := \psi|_{\mathbb{A}_F^\times}$ is a central character of π.
- Let $\tau: \mathbb{A}_K^\times \longrightarrow \mathbb{C}^\times$ be a Hecke character.

Then we divide the places of F into parts

finite places: unramified places, ramified places $v \nmid p$, places above p

Σ_∞		Σ_{ram}	Σ_p
archimedian places	unramified places	ramified places $v \nmid p$	places above p

good places : infinitely many (unramified places)

bad primes $=: \Sigma$ (ramified places $v \nmid p$ and places above p)

here "unramified" means places $v \nmid \infty$ such that π_v, ψ_v, τ_v are all unramified.

Denote the set of ramified places as Σ_{ram} and places above p as Σ_p.

<u>Definition</u> ([Wan15ANT, Defn 3.2]) We call the triple $\mathcal{D} := (\pi, \tau, \Sigma)$ an <u>Eisenstein datum</u>.

<u>Notation</u>: For conviences, we often write $a = r - s$, $b = s$. then $r = a + b$, $s = b$ and $r + s = a + 2b$

§4.2 Klingen Eisenstein series

Recall in <u>Talk2</u>, we defined a Klingen parabolic subgroup P of $GU(r+1,s+1)$ as

$$P := \left\{ \begin{array}{c} \begin{pmatrix} x & * & * & * & * \\ & A & E & * & B \\ & F & M & * & G \\ & & & \lambda\bar{x}^{-*} & \\ & C & H & * & D \end{pmatrix} \\ \begin{array}{ccccc} 1 & s & r-s & 1 & s \end{array} \end{array} \in GU(r+1,s+1) \;\middle|\; \begin{array}{l} h = \begin{pmatrix} A & E & B \\ F & M & G \\ C & H & D \end{pmatrix} \in GU(r,s), \\ \lambda = \lambda_{GU(r,s)}(h) \\ x \in \mathrm{Res}_F^K \mathbb{G}_m \end{array} \right\}$$

and its Levi decomposition $P = M \ltimes N$, where

$$M := \left\{ m(h,x) := \begin{pmatrix} x & & & & \\ & A & E & & B \\ & F & M & & G \\ & & & \lambda\bar{x}^{-*} & \\ & C & H & & D \end{pmatrix} \in GU(r+1,s+1) \;\middle|\; \begin{array}{l} h = \begin{pmatrix} A & E & B \\ F & M & G \\ C & H & D \end{pmatrix} \in GU(r,s) \\ \lambda = \lambda_{GU(r,s)}(h) \\ x \in \mathrm{Res}_F^K \mathbb{G}_m \end{array} \right\}$$

is the Levi part with N the unipotent radical.

Let δ_P be the modulus character of P and $\zeta \in \mathbb{C}$ a complex number.

<u>Remark</u>: One can compute explicitly compute δ_P, but not quite useful.

- It will be used in proving pullback formulas, from [Bump, Lie groups, §1], we can compute

$$\delta_P(p = m(h,x)\cdot n) = |x\bar{x}|^{r+s+1}$$

(actually I'm guessing from the proof of the pullback formula.)

Definition : The space of Klingen sections is the parabolic induction space

$$I_P(z) := \mathrm{Ind}_{P(\mathbb{A})}^{GU(r+1,s+1)(\mathbb{A})}\left(\delta_P^{\frac{1}{2}+z}\cdot(\pi\boxtimes\tau)\boxtimes\mathbb{1}_N\right)$$

where $\pi\boxtimes\tau$ is a rep'n of $M(\mathbb{A})$ by $(\pi\boxtimes\tau)(m(g,x)) := \tau(x)\cdot\pi(g)$.

More explicity, it is the set of smooth functions $f^b : GU(r+1,s+1)(\mathbb{A}) \longrightarrow V$ such that

(1) $f^b(m(h,x)ng) = \delta_P(m(h,x)n)^{\frac{1}{2}+z}\cdot\tau(x)\,\pi(h)f_z^b(g)$

(2) f^b is right K-finite.

As $V\subseteq \mathcal{A}_{cusp}$, we have further modifications of the space I_P : define for each $f^b\in I_P(z)$, a $\mathbb{C}$-valued function over $g\in GU(r+1,s+1)(\mathbb{A})$ as

$$f(g) := f^b(g)(1) \qquad 1\in GU(r,s), \quad \forall g\in GU(r+1,s+1)(\mathbb{A}).$$

Then $f(nmg) = f^b(g)(m)$ for $m\in M$. $(*)$

Definition Let $f\in I_P(z)$ be a Klingen section. Define the Klingen Eisenstein series

$$E^{Kling}(g;f,z) := \sum_{\gamma\in P(F)\backslash GU(r+1,s+1)(F)} f(\gamma g)$$

Fact : $E^{Kling}(g;f,z)$ converges absolutely and uniformly for (z,g) in compact subsets of $\left\{z\in\mathbb{C} \;\middle|\; \mathrm{Re}(z) > \frac{r+s+1}{2}\right\}\times GU(r+1,s+1)(\mathbb{A}_F)$, and can be meromorphically continued to $\mathbb{C}$.

Remark : Sometimes we write f_z for $f\in I_P(z)$ to explicit show what our z is, and to distinguish the later defined Siegal sections, we write f_z^{Klig} as well.

§4.3 Siegal Eisenstein series

Let $m>0$. In this section we consider $GU(m,m)$, the Hermitian unitary group.

Recall in this case: the metric becomes $\begin{pmatrix} & 1_m \\ -1_m & \end{pmatrix}$, so according to this, we break $2m\times 2m$ matrices into such blocks $\begin{pmatrix} m\times m & m\times m \\ m\times m & m\times m \end{pmatrix}$.

- Siegal parabolic subgroup:

$$Q := \left\{ \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \in \mathrm{Res}^K_F GL_{2m} \;\middle|\; \begin{array}{l} A=\lambda D^{-*}, \ \lambda(g)\in \mathbb{G}_m \\ A^{-1}B \in \mathrm{Res}^K_F \mathrm{Herm}_m \end{array} \right\} \subseteq GU(m,m)$$

$$= \left\{ \begin{pmatrix} \lambda D^{-*} & \\ & D \end{pmatrix}\begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix} \in \mathrm{Res}^K_F GL_{2m} \;\middle|\; \begin{array}{l} X\in \mathrm{Res}^K_F \mathrm{Herm}_m \\ D\in \mathrm{Res}^K_F GL_{2m} \end{array} \right\}.$$

is a parabolic subgroup of $GU(m,m)$, with Levi component

$$M_Q := \left\{ \begin{pmatrix} \lambda D^{-*} & \\ & D \end{pmatrix} \in Q \right\} \xrightarrow{\sim} \mathrm{Res}^K_F GL_{2m} \times \mathbb{G}_m$$

$$m(D,\lambda) \longleftarrow\!\shortmid \ (D,\ \lambda)$$

and unipotent part

$$N_Q := \left\{ \begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix} \in Q \right\} \xrightarrow{\sim} \mathrm{Res}^K_F \mathrm{Herm}_m$$

$$n(X) \longleftarrow\!\shortmid \ X$$

- Siegal sections are defined locally
  - Let v be any place of F.
  - Let $\tau: \mathbb{A}_K^\times \longrightarrow \mathbb{C}^\times$ be a Hecke character with decomposition $\tau_v: K_v^\times \longrightarrow \mathbb{C}^\times$.

Define

$$I_{Q,v}(\tau_v,\zeta) := \left\{ f_v: G(F_v)\longrightarrow \mathbb{C} \;\middle|\; \begin{array}{l} \bullet\ f_v \text{ is smooth, } K_v\text{-finite} \\ \bullet\ f_v(n(X)\, m(D,\lambda) g) \\ \qquad = \tau_v(\det D)\,\left|\det AD^{-1}\right|_v^{\zeta+\frac{m}{2}} f(g) \end{array} \right\}$$

Suppose τ is unramified at $v\nmid\infty$, we fix a spherical Siegal section at v to be $f_v^\circ: G(F_v)\longrightarrow \mathbb{C}$ such that $f_v^\circ(GU(m,m)(\mathcal{O}_{F,v})) = 1$.

Then we formulate $I_Q := \bigotimes_{v|\infty} I_{Q,v} \otimes \bigotimes'_{v\nmid\infty} I_{Q,v}$, where the restricted tensor product are taken wrt the sphrecial vectors f_v^0 in $I_{Q,v}$.

<u>Definition</u>. Let $f \in I_Q(\chi, z)$ be a Siegal section. Define the Siegal Eisenstein series as
$$E^{Sieg}(g; f, z, \tau) := \sum_{\gamma \in Q(F) \backslash GU(m,m)(F)} f(\gamma g).$$

<u>Fact</u>: $E^{Sieg}(g, f, z, \tau)$ converges absolutely and uniformly for (z, g) in compact subsets of $\{ z \in \mathbb{C} \mid \mathrm{Re}(z) > \frac{m}{2} \} \times GU(m,m)(\mathbb{A})$.

<u>The philosophy of the holomorphic parameter</u> ($z \in \mathbb{C}$).

Recall in the elementary courses on modular forms, the Eisenstein series E_k does In practice here, we are primarily interested in $z = z_k := \frac{k-n}{2}$. But it is often more convienent to put a single Eisenstein series into a <u>holomorphic family</u>. This enables us to build up:

- meromorphic continuation
- functional equations
- intertwing operators, ...

which can be used to investigate the original Eisenstein series. This is how [Wan15 ANT] goes (eg. § 3.1.2. 3.2.1):

<u>Klingen</u>: $I(\rho) \left(= I_P(\pi, \tau, 0) \text{ in our notation} \right) \xrightarrow{\text{bringing in } z} I(\rho, z)$

$$f \longmapsto f_z : g = mk \longmapsto \delta(m)^{\frac{n}{2}+z} (\pi \boxtimes \tau)(m) f(k).$$

<u>Siegal</u>: $I(\chi) \left(= I_Q(\chi, 0) \text{ in our notation} \right) \xrightarrow{\text{bringing in } z} I(\tau, z)$

$$f \longmapsto f_z : g = qk \longmapsto \chi(\det D_q) \left| \det A_q D_q^{-1} \right|_v^{z+\frac{n}{2}} f(k)$$

Here $g = mk$ or $g = qk$ are Iwasawa decompositions, where the compact groups are corresponding level groups.

§4.4 Doubling method

1. <u>Setup</u> : We have two setups for the doubling method :

- Doubling method for <u>Klingen Eisenstein series</u>. (Wan2015ANT)
- Doubling method for <u>p-adic L-functions</u>. (EHLS)

- "<u>Negative space</u>" : Recall $V_K = Y_K \oplus W \oplus X_K$ with metric $\theta_{r,s} = \begin{pmatrix} & & 1_s \\ & S & \\ -1_s & & \end{pmatrix}$.

We define $-V := (-V, -\theta)$ as the same K-vector space but with the metric $-\theta_{r,s}$.

Let $-GU(r,s) := GU(-V)$ be the general unitary group of this space.

$\boxed{\heartsuit}$ Start with

- $(V_{r+1,s+1}, \theta_{r+1,s+1})$: ordered K-basis $y^1, \cdots, y^s, y^{s+1}, w^1, \cdots, w^{r-s}, x^1, \cdots, x^s, x^{s+1}$
- $(V_{r,s}, \theta_{r,s})$: ordered K-basis $\tilde{y}^1, \cdots, \tilde{y}^s, \tilde{w}^1, \cdots, \tilde{w}^{r-s}, \tilde{x}^1, \cdots, \tilde{x}^s$.

we form the doubled space

$$V_\heartsuit := (V_{r+1,s+1}, \theta_{r+1,s+1}) \oplus -(V_{r,s}, \theta_{r,s})$$
$$:= (V_{r+1,s+1} \oplus V_{r,s},\ \theta_{r+1,s+1} \oplus -\theta_{r,s})$$

with ordered K-basis $\{y^i, w^i, x^i, \tilde{y}^i, \tilde{w}^i, \tilde{x}^i\}$ and the metric $\eta^{\oplus}_{r,s} = \left(\begin{array}{c|c} \theta_{r+1,s+1} & \\ \hline & -\theta_{r,s} \end{array}\right)$.

clearly it is of signature $(r+s+1, r+s+1)$.

<u>Recall</u> we denote $a_i = \theta_{r+1,s+1}(w^i, w^i) = \theta_{r,s}(\tilde{w}^i, \tilde{w}^i) \in K$.

- <u>Change of basis</u> : Denote $y^1_\heartsuit, \cdots, y^{r+s+1}_\heartsuit, x^1_\heartsuit, \cdots, x^{r+s+1}_\heartsuit$ be our new basis of $V_\heartsuit$, designed as

• $y^i_\heartsuit := y^i - \tilde{y}^i$	• $x^i_\heartsuit = \frac{1}{2}(\tilde{x}^i + x^i)$	$i = 1, \cdots, s$
• $y^{s+1}_\heartsuit = y^{s+1}$	• $x^{s+1}_\heartsuit = x^{s+1}$	
• $y^{s+1+i}_\heartsuit = w^i - \tilde{w}^i$	• $x^{s+1+i}_\heartsuit = -\frac{1}{2}a_i(w^i + \tilde{w}^i)$	$i = 1, \cdots, r-s$
• $y^{r+1+i}_\heartsuit = -\tilde{x}^i + x^i$	• $x^{r+1+i}_\heartsuit = -\frac{1}{2}(y^i + \tilde{y}^i)$	$i = 1, \cdots, s$

Then under this basis, the metric on $V_\heartsuit$ becomes

$$\eta^{\heartsuit}_{r,s} = \left(\begin{array}{cccc|cccc} & & & & 1 & & & \\ & & & & & 1 & & \\ & & & & & & -s\bar{s} & \\ & & & & & & & 1 \\ \hline -1 & & & & & & & \\ & -1 & & & & & & \\ & & s\bar{s} & & & & & \\ & & & -1 & & & & \end{array}\right)$$

Now up to some insignificant $a_i \cdot \bar{a}_i$, we get the standard metric on $V_\heartsuit$.

- We can take some time to write out the transition matrix of the two basis, denoted by $M_\heartsuit$. Then we accordingly have the embedding of unitary groups

$$\iota_\heartsuit : GU(r+1, s+1) \times_{\mathbb{G}_m} -GU(r,s) \longrightarrow GU(r+s+1, r+s+1) \simeq GU(V_\heartsuit, \eta^\heartsuit_{r,s}).$$

$$(g_1, g_2) \longmapsto M_\heartsuit^{-1} \begin{pmatrix} g_1 & \\ & g_2 \end{pmatrix} M_\heartsuit$$

$\boxed{\diamondsuit}$ Start with

- $(V_{r,s}, \theta_{r,s})$: ordered K-basis $y^1, \dots, y^s, w^1, \dots, w^{r-s}, x^1, \dots, x^s$
- $(V_{r,s}, \theta_{r,s})$: ordered K-basis $\tilde{y}^1, \dots, \tilde{y}^s, \tilde{w}^1, \dots, \tilde{w}^{r-s}, \tilde{x}^1, \dots, \tilde{x}^s$.

Then we form the doubled space

$$V_\diamondsuit = (V_{r,s}, \theta_{r,s}) \oplus -(V_{r,s}, \theta_{r,s})$$

with ordered K-basis $\{y^i, w^i, x^i, \tilde{y}^i, \tilde{w}^i, \tilde{x}^i\}$ and the metric $\eta^{\oplus}_{r,s} = \left(\begin{array}{c|c} \theta_{r,s} & \\ \hline & -\theta_{r,s} \end{array}\right)$

clearly it is of signature $(r+s, r+s)$.

- <u>Change of basis</u> : Denote $y^1_\diamondsuit, \dots, y^{r+s+1}_\diamondsuit, x^1_\diamondsuit, \dots, x^{r+s+1}_\diamondsuit$ be our new basis of $V_\diamondsuit$, designed as
  - $y^i_\diamondsuit := y^i - \tilde{y}^i$ • $x^i_\diamondsuit = \frac{1}{2}(\tilde{x}^i + x^i)$ $i = 1, \dots, s$
  - $y^{s+i}_\diamondsuit = w^i - \tilde{w}^i$ • $x^{s+i}_\diamondsuit = -\frac{1}{2} a_i (w^i + \tilde{w}^i)$ $i = 1, \dots, r-s$
  - $y^{r+i}_\diamondsuit = -\tilde{x}^i + x^i$ • $x^{r+i}_\diamondsuit = -\frac{1}{2}(y^i + \tilde{y}^i)$ $i = 1, \dots, s$

Then under this basis, the metric on $V_\diamond$ becomes

$$\eta^\diamond_{r,s} = \left(\begin{array}{ccc|ccc} & & & 1 & & \\ & & & & -s\bar{s} & \\ & & & & & 1 \\ \hline -1 & & & & & \\ & s\bar{s} & & & & \\ & & -1 & & & \end{array}\right)$$

Now up to some insignificant $a_i \cdot \bar{a_i}$, we get the standard metric on $V_\diamond$.

- We can take some time to write out the <u>transition matrix</u> of the two basis, denoted by $M_\diamond$. Then we accordingly have the embedding of unitary groups

$$\begin{aligned} \iota_\diamond : \quad GU(r,s) \underset{\mathbb{G}_m}{\times} -GU(r,s) &\longrightarrow GU(r+s,r+s) \simeq GU(V_\diamond, \eta^\diamond_{r,s}). \\ (g_1, g_2) &\longmapsto M_\diamond^{-1} \begin{pmatrix} g_1 & \\ & g_2 \end{pmatrix} M_\diamond . \end{aligned}$$

<u>Remark</u>: Here by "transition matrix", we mean columns as coordinates of <u>new</u> basis under the original basis.

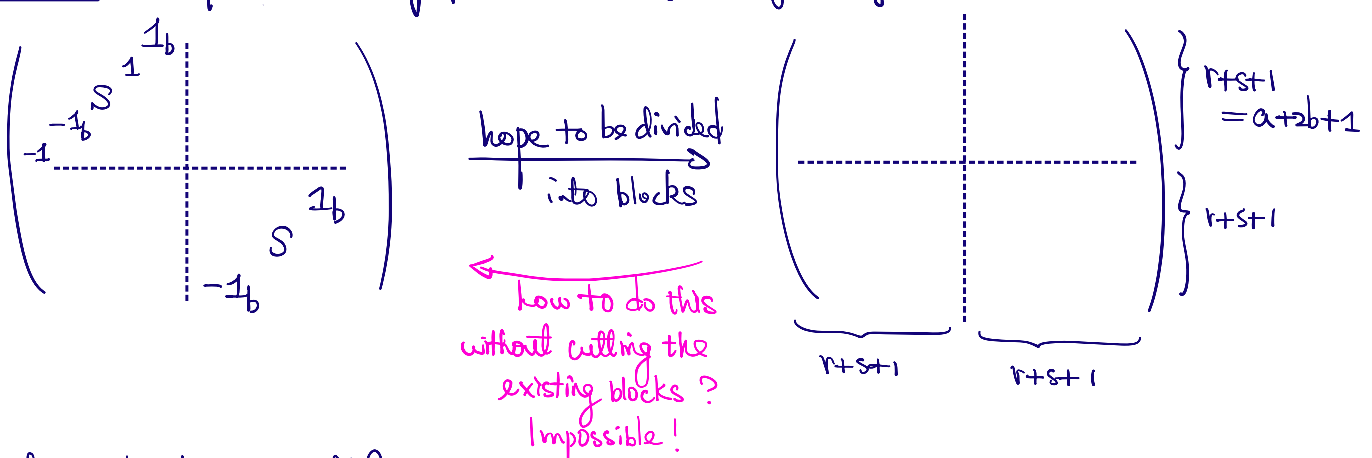
Remark: Above is a summary of the doubling setup in [Wan15]. It's worth to mention that Wan considered the doubling in two steps. We only use $\heartsuit$-case as an example.

$\boxed{\text{First step}}$ $\alpha: \mathrm{GU}(r+1,s+1) \times_{\mathbb{G}_m} -\mathrm{GU}(r,s) \to \mathrm{GU}(r+s+1, r+s+1)'$. This is defined by

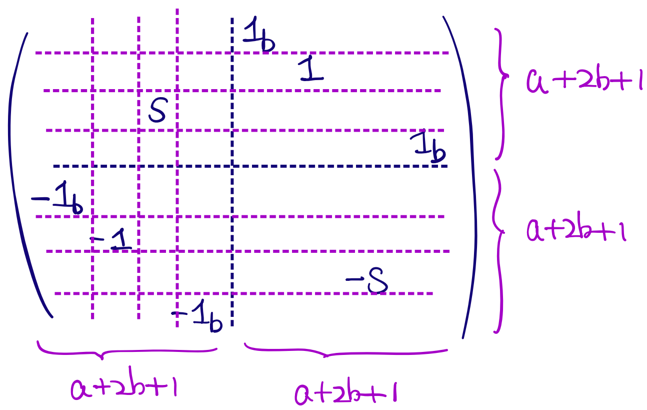
- "1,2,3,4,5-th row and column" of G_1 embed to the "1,2,3,5,6"-th row and col.
- "1,2,3-rd row and colums" of $-\mathrm{GU}(r,s)$ to "8,7,4-th row and column"

Note: There was a typo in the upper matrix on p1962: (8,4)-entry should be "-1_b" and (4,8)-entry should be "1_b".

Why bother? If we directly put g_1 and g_2 diagonally.



But if we do the permutation as in [Wan15, p1962]:



Remark: In [Hsieh14] and [CLW22], they are using $\mathrm{GU}(2,1)$ and $\mathrm{GU}(3,1)$, i.e. $s=0$ case. In this case,

$$\overset{1}{b}+\overset{2}{1}+\overset{3}{a}+\overset{5}{b}+\overset{6}{1} \xrightarrow{\text{becomes}} \overset{2}{1}+\overset{3}{a}+\overset{6}{1}$$

$$\overset{8}{b}+\overset{7}{a}+\overset{4}{b} \xrightarrow{\text{becomes}} \overset{7}{a}$$

, therefore α is still the canonical diagonal embedding $(g_1, g_2) \mapsto \mathrm{diag}(g_1, g_2)$. So α here does not contradict to these references!

<u>Explicit permutation</u> : In this step, the original direct sum of standard basis are permuted as

$$\{ y^1, \ldots, y^s \,;\, y^{s+1} \,;\, w^1, \ldots, w^{r-s} \,;\, \tilde{x}^1, \ldots, \tilde{x}^s \,;\, x^1, \ldots, x^s, x^{s+1} \,;\, \tilde{w}^1, \ldots, \tilde{w}^{r-s} \,;\, \tilde{y}^1, \ldots, \tilde{y}^s \}.$$

$\boxed{\text{Second step}}$ He gave $\beta : GL(r+s+1, r+s+1)' \longrightarrow GL(r+s+1, r+s+1)$ by $g \longmapsto S^{-1} g S$ by an explicitly defined S on [Wan15, p1963].

Putting these together, I obtained the basis above. It is the same for the doubling for p-adic L-function (and the same typos, with typo on P1963, 9-th row "6.5.4" should be "6.5.3"), and I will not explain later.

<u>Remark</u> : Wan's S in β is

$$S = \begin{pmatrix} 1_b & & & & & & & -\frac{1}{2}\cdot 1_b \\ & 1 & & & & & & \\ & & 1_a & & & & -\frac{\varsigma}{2} & \\ & & & -1_b & \frac{1}{2}\cdot 1_b & & & \\ & & & 1_b & \frac{1}{2}\cdot 1_b & & & \\ & & & & & 1 & & \\ & & -1_a & & & & -\frac{\varsigma}{2} & \\ -1_b & & & & & & & -\frac{1}{2}\cdot 1_b \end{pmatrix} \xrightarrow[\text{case}]{b=0} S = \begin{array}{cc} \begin{pmatrix} 1 & & & \\ & 1_a & & -\frac{1}{2}\varsigma \\ & & 1 & \\ & -1_a & & -\frac{1}{2}\varsigma \end{pmatrix} & \begin{matrix} 1 \\ a \\ 1 \\ a \end{matrix} \end{array}$$

This is consistent with [Hsieh14, (5.16)] and [CLW22, (5.5.1)]. So finally there is no problem on comparing different literatures.

- So in this sense, α can be regarded as a <u>more correct "diagonal embedding"</u> while β is the genuine doubling adjustment !! This perspective is extremely useful when computing the pullback sections at bad places !

2. Pullback formula

- Let χ be a unitary Hecke character $\chi : \mathbb{A}_K^\times \longrightarrow \mathbb{C}^\times$.

  Let ϕ be a unitary tempered cuspidal eigenform on $GU(r,s)$, which is a pure tensor.

  (We can think that ϕ generates an automorphic representation π_ϕ, which is used in the Klingen setup.)

$\heartsuit$ For any fixed $g \in GU(r+1,s+1)(\mathbb{A}_F)$, we define the pullback integral (or say doubling integral) formally

$$F_\phi(f,z,g;\tau) := \int_{U(r,s)(\mathbb{A}_F)} \bar{\tau}(\det g_1 h)\, f(z, \iota_\heartsuit(g, g_1 h))\, \phi(g_1 h)\, dg_1$$

where $f \in I_{Q_{r+s+1}}(z,\tau)$ is a Siegel section, $h \in GU(r,s)(\mathbb{A}_F)$ such that $\lambda_{GU(r+1,s+1)}(g) = \lambda_{GU(r,s)}(g_1 h)$. The integral is independent of h.

Theorem (Pullback formula) (Due to: Piatetski-Shapiro, Rallis, Shimura)

(i) If $f \in I_{Q_{r+s+1}}(z,\tau)$, then $F_\phi(f,z,g,\tau)$ converges absolutely and uniformly for (z,g) in a compact subset of $\{z \in \mathbb{C} \mid \mathrm{Re}(z) > r+s+\frac{1}{2}\} \times GU(r,s)(\mathbb{A}_F)$.

(ii) Moreover, $F_\phi(f,z,g,\tau) \in I_{P_{r+1,s+1}}(\pi,\tau)$ is a Klingen section in g.

(iii) Moreover,

$$\int_{[U(r,s)]} E^{\mathrm{sieg}}(f,z,\iota_\heartsuit(g,g_1 h))\,\bar{\tau}(\det g_1 h)\,\phi(g_1 h)\,dg_1 = E^{\mathrm{klig}}(F_\phi(f,z,-,\tau),z,g)$$

Here $[U(r,s)] := U(r,s)(F) \backslash U(r,s)(\mathbb{A}_F)$.

Proof (Sketch):

(i)
- Write F_ϕ as a product of local integrals.
- Prove the absolute convergence of local integrals.
- Global absolute convergence follows from local ones, computation at unramified places and temperedness assumption.

(ii) We only prove the $s=0$ case as a baby version : now

$$P(\mathbb{A}_F) = \left\{ \begin{pmatrix} x & * & * \\ & M & * \\ & & \bar{x}^{-*} \end{pmatrix} \in U(r+1,1) \;\middle|\; \begin{array}{l} M \in U(r,0)(\mathbb{A}_F) \\ x \in \mathrm{Res}^K_F \mathbb{G}_m(\mathbb{A}_F) = \mathbb{A}_K^\times \end{array} \right\}$$

and the pullback embedding

$$(\star) \quad \iota_\heartsuit\left(p := \begin{pmatrix} x & * & * \\ & M & * \\ & & \bar{x}^{-*} \end{pmatrix}, g_1 \right) = \left(\begin{array}{cc|cc} x & * & * & * \\ & S^{-1} g_1 S & * & * \\ \hline 0 & & \bar{x}^{-*} & * \\ & & & g_1 \end{array} \right) \in Q_{r+1}(\mathbb{A}_F)$$

So one computes

$$F_\phi(pg) = \int_{U(r,0)(\mathbb{A})} f(\iota_\heartsuit(pg, g_1)) \bar{\tau}(\det g_1) \phi(g_1) dg_1$$

$$\overset{g_1 \mapsto Mg_1}{=} \int_{U(r,0)(\mathbb{A})} f(\iota_\heartsuit(pg, Mg_1)) \bar{\tau}(\det Mg_1) \phi(Mg_1) \underline{dg_1} \quad \text{left invariant}$$

$$= \int_{U(r,0)(\mathbb{A})} f\left(\iota_\heartsuit\left(\begin{pmatrix} x & * & * \\ & M & * \\ & & \bar{x}^{-*} \end{pmatrix}, M\right) \cdot \iota_\heartsuit(g, g_1)\right) \bar{\tau}(\det Mg_1) \phi(Mg_1) dg_1$$

$$\overset{(\star)}{=} \int_{U(r,0)(\mathbb{A})} f\left(\left(\begin{array}{cc|cc} x & * & * & * \\ & S^{-1}MS & * & * \\ \hline 0 & & \bar{x}^{-*} & * \\ & & & M \end{array} \right) \cdot \iota_\heartsuit(g, g_1)\right) \bar{\tau}(\det Mg_1) \phi(Mg_1) dg_1$$

$$\overset{f \text{ is Siegal}}{=} \int_{U(r,0)(\mathbb{A})} \underbrace{\boxed{\text{induced Siegal terms}} \cdot \underline{\bar{\tau}(\det M)}}_{\text{independent of } g_1} \cdot f(\iota_\heartsuit(g, g_1)) \bar{\tau}(\det g_1) \phi(Mg_1) dg_1$$

Here by the definition of Siegal sections, we have

$$\boxed{\text{induced Siegal terms}} = \tau\left(\det\begin{pmatrix} \bar{x}^{-*} & * \\ 0 & M \end{pmatrix}\right) \left| \det\begin{pmatrix} x & * \\ & S^{-1}MS \end{pmatrix} \cdot \det\begin{pmatrix} \bar{x}^{-*} & \\ & M \end{pmatrix}^{-1} \right|_v^{s+\frac{r+1}{2}}$$

$$= \tau(\bar{x}^{-1} \cdot \det M) \left| x \cdot \det M \cdot \bar{x} (\det M)^{-1} \right|_v^{s+\frac{r+1}{2}}$$

$$= \underline{\overline{\tau(\bar{x})}}\, \underline{\tau(\det M)} \, |x\bar{x}|_v^{s+\frac{r+1}{2}}$$

$$= \underline{\tau(x)}\, \underline{\tau(\det M)} \, \delta_P(p)^{\frac{1}{2}+s}$$

<u>note</u> : here we used τ is a unitary character. so $\bar{\tau} = \tau^{-1}$.

Put them together,

$$F_\phi(pg) = \delta_P(p)^{\frac{1}{2}+z}\,\tau(x) \int_{U(r,0)(\mathbb{A}_F)} f(\iota_\heartsuit(g,g_1))\,\overline{\tau}(\det g_1)\,\phi(Mg_1)\,dg_1$$

$$= \delta_P(p)^{\frac{1}{2}+z} \cdot (\pi \boxtimes \tau)(p)\, F_\phi(g)$$

recall the comparision between f and f^b, especially $(*)$ above!

This gives $F_\phi \in I_P(\pi, \tau)$.

(iii) Here we essentially use our "baby" version condition:

Shimura 1997, §2 : We have the coset decomposition:

$$Q(F)\backslash U(r+1,r+1)(F) = \iota_\heartsuit\left(P(F)\backslash U(r+1,1)(F),\ U(r,0)\right).$$

Then

$$E^{\mathrm{Sieg}}(f, z, \iota_\heartsuit(g,g_1)) = \sum_{\gamma \in Q(F)\backslash U(r+1,r+1)(F)} f_z(\gamma\,\iota_\heartsuit(g,g_1))$$

$$= \sum_{\gamma_1 \in P(F)\backslash U(r+1,1)(F)} \ \sum_{\gamma_2 \in U(r,0)(F)} f_z(\iota_\heartsuit(\gamma_1 g, \gamma_2 g_1))$$

Then we unfold the integral

$$\int_{U(r,0)(F)\backslash U(r,0)(\mathbb{A})} E^{\mathrm{Sieg}}(f, z, \iota_\heartsuit(g,g_1))\,\phi(g_1)\,\overline{\tau}(\det g_1)\,dg_1$$

$$= \int_{\cancel{U(r,0)(F)}\backslash U(r,0)(\mathbb{A})} \ \sum_{\gamma_1 \in P(F)\backslash U(r+1,1)(F)} \ \sum_{\cancel{\gamma_2 \in U(r,0)(F)}} f_z(\iota_\heartsuit(\gamma_1 g, \gamma_2 g_1))\,\phi(g_1)\,\overline{\tau}(\det g_1)\,dg_1$$

$$= \int_{U(r,0)(\mathbb{A})} \ \sum_{\gamma_1 \in P(F)\backslash U(r+1,1)(F)} f_z(\iota_\heartsuit(\gamma_1 g, g_1))\,\phi(g_1)\,\overline{\tau}(\det g_1)\,dg_1$$

$$= \sum_{\gamma_1 \in P(F)\backslash U(r+1,1)(F)} \int_{U(r,0)(\mathbb{A})} f_z(\iota_\heartsuit(\gamma_1 g, g_1))\,\phi(g_1)\,\overline{\tau}(\det g_1)\,dg_1$$

we suppose: z is in a proper range so there are no convergence issue.

$$= \sum_{\gamma_1 \in P(F)\backslash U(r+1,1)(F)} F_\phi(f, z, g, \tau)$$

$$\overset{(ii)}{=} E^{\mathrm{Klig}}(F_\phi, z), \text{ as desired!}$$

Here we can see the essential input is Shimura's coset decomposition. In general $s>0$ case, similar decomposition is described in loc.cit as well, yet the problem is: there will be other cosets! We have to show that the integral on these cosets vanishes ("neligible cosets"). For the argument, one may go back to PS-R's original papers for details. $\square$

Remark:

① From the proof we see "τ is unitary" is necessary. In general for a Hecke character $\tau : K^\times \backslash \mathbb{A}_K^\times \to \mathbb{C}^\times$ of infinite type $k = \sum_{\sigma \in I_K} k_\sigma \cdot \sigma$, we can associate τ with a unitary Hecke character τ_0 such that

?

$$\tau = \tau_0 \cdot |-|_{\mathbb{A}_K}^{-\frac{|k|}{2}}, \quad |k| = \sum_{\sigma \in I_K} k_\sigma .$$

(guessed from [CLW22].)

② As the formula shows, $E^{\mathrm{Kling}} = \int_{[U(r,s)]} \boxed{E^{\mathrm{Sieg}}(f, z, \iota_\phi(g, g_1 h))\,\overline{\tau}(\det g_1 h)}\,\phi(g_1 h)\, dg_1$

- For any fixed $g \in U(r+1, s+1)$, we consider the twisted Siegel Eisenstein series
$$E^{\mathrm{Sieg}}(f, z, \iota_\phi(g, -)) \cdot \overline{\chi} \circ \det = E^{\mathrm{Sieg},*}(f, z, \iota_\phi(g, -))$$

(note: Here the definition has nothing to do with the cuspidal autorep π. It is merely on the lower group $GU(r,s)$.)

- For a fixed cuspidal automorphic form $\varphi \in \mathcal{A}_0\left(U(r,s)(F) \backslash U(r,s)(\mathbb{A}_F)\right)$, we define the linear functional

$$\mathcal{G}_\phi : \mathcal{A}\left(G(F) \backslash G(\mathbb{A}_F)\right) \longrightarrow \mathcal{A}\left(U(r+1,s+1)(F) \backslash U(r+1,s+1)(\mathbb{A}_F)\right)$$
$$F \longmapsto \left(g \mapsto \int_{[U(r,s)]} F(\iota_\phi(g, g_1 h))\, \phi(g_1 h)\, dg_1 \right)$$

where $G =$ the doubled group $GU(r+s+1, r+s+1)$, called the Garrett's map.

Then the pullback formula can be rewritten as
$$E^{\mathrm{Kling}}(F_\phi(f, z, -, \tau), g) = \mathcal{G}_\phi\left(E^{\mathrm{Sieg},*}(f, z, --)\right).$$

3. Relation with L-functions

To relate the above with L-functions, we need to

- Decompose global pullback integrals into local ones.
- Compute the pullback integrals for unramified places. (outside a finitely many of places.)

archimedian places	unramified places	ramified places $v \nmid p$	places above p

finite places: unramified places, ramified places $v \nmid p$, places above p

good places : infinitely many (unramified places)

bad primes $=: \Sigma$ (ramified places $v \nmid p$, places above p)

Theorem ([Wan15ANT, Lemma 4.5], [SU14, Thm 11.9])

For $v \notin \Sigma \cup \{\infty\}$, suppose $\phi \in \pi$ is a new vector.

Then if $\mathrm{Re}(z) > \frac{a+b}{2}$, the pullback integral converges and

- $$F^{\heartsuit}_{\phi}(f_v^{\circ}, z, g) = \frac{L(\tilde{\pi}, \xi, z+1)}{\prod_{i=0}^{a+2b-1} L(2z+a+2b+1-i, \tau' \eta^i_{K/F})} F^{\circ}(z, g) \quad \in V$$
- $$F^{\diamondsuit}_{\phi}(f_v^{\circ}, z, g) = \frac{L(\tilde{\pi}, \xi, z+\frac{1}{2})}{\prod_{i=0}^{a+2b-1} L(2z+a+2b-i, \tau' \eta^i_{K/F})} \pi(g)\varphi \quad \in V$$

where F° is the sphrecical Klingen section, taken value $\phi \in V$ at $1 \in GU(r+1, s+1)(\mathbb{A}_F)$

Here:

- $\tilde{\pi}$: the contragredient rep'n of π.
- $\xi = \psi/\tau$, recall ψ is the central character of π.
- $\eta_{K/F}$: the quadratic character of $F^\times \backslash \mathbb{A}_F^\times$ associated to K/F. (recall CFT)
- τ' : the restriction of the Hecke character τ on K to F.

Note : $L(\tilde{\pi}, \xi, z+1) := L(BC(\tilde{\pi}) \otimes \xi, z+1)$, where $BC(\tilde{\pi})$ is the base change of $\tilde{\pi}$ over $G(\mathbb{A}_F)$ to $G(\mathbb{A}_K)$, to be "compatible" with ξ. (This was made explicit in [SU14, Introduction, P8].

§ 4.5 Fourier–Jacobi coefficients of Eisenstein series

Setup: Fix an additive character

$$e_{\mathbb{A}_\mathbb{Q}} = \bigotimes_v e_v : \mathbb{Q}^\times \backslash \mathbb{A}_\mathbb{Q}^\times \longrightarrow \mathbb{C}^\times$$

such that for each v,

$$e_v : \mathbb{Q}_v^\times \longrightarrow \mathbb{C}^\times$$
$$x \longmapsto \begin{cases} e^{2\pi i \{x\}_v} & v \neq \infty \\ e^{-2\pi i x} & v = \infty \end{cases}, \quad \{x\}_v : \text{fractional part of } x.$$

and define $e_\mathbb{A} : F^\times \backslash \mathbb{A}_F^\times \longrightarrow \mathbb{C}^\times$ by $e_\mathbb{A}(x) := e_{\mathbb{A}_\mathbb{Q}}(\mathrm{Tr}^{\mathbb{A}_F}_{\mathbb{A}_\mathbb{Q}} x)$, for $x \in \mathbb{A}_F^\times$.

note: the convention here differs from [CLW22] by a minus sign:

$$e_{v,\mathrm{Wan}}(x) = e_{v,\mathrm{CLW}}(-x).$$

it makes almost no difference.

Definition: Let $\phi : GU(r,s)(\mathbb{A}_F) \longrightarrow \mathbb{C}^\times$ be any automorphic form. Let $0 \le t \le s$. For $\beta \in \mathrm{Herm}_t(F)$, we define the β-th Fourier–Jacobi coefficient of ϕ at $g \in GU(r,s)(\mathbb{A}_F)$ as

$$\phi_\beta(g) := \int_{\mathrm{Herm}_t(F) \backslash \mathrm{Herm}_t(\mathbb{A}_F)} \phi\left(\left(\begin{array}{c|c|c} 1_s & & \begin{matrix} \sigma & 0 \\ 0 & 0 \end{matrix} \\ \hline & 1_{r-s} & \\ \hline & & 1_s \end{array} \right) \cdot g \right) e_\mathbb{A}(-\mathrm{Tr}\,\beta\sigma)\, d\sigma$$

Special case: $r = s = n$, $t = n$. Then for $\phi : GU(n,n)(\mathbb{A}_F) \longrightarrow \mathbb{C}^\times$, we have

$$\phi_\beta(g) = \int_{\mathrm{Herm}_n(\mathbb{A}_F) \backslash \mathrm{Herm}_n(\mathbb{A}_F)} \phi\left(\left(\begin{array}{c|c} 1_s & \sigma \\ \hline 0 & 1_s \end{array} \right) g \right) e_\mathbb{A}(-\mathrm{Tr}\,\beta\sigma)\, d\sigma$$

called the β-th Fourier coeffient of ϕ at g.

To construct Siegal Eisenstein family, we interpolate its <u>Fourier-Jacobi coefficients</u>.

<u>Definition</u>: Let $1 \leq t \leq s$, $f_v(z,\tau) \in I_{Q_n}(\tau)$.

For $\beta \in \mathrm{Herm}_t(F)$, $g_v \in U(n-t, n-t)(F_v)$, $x_v \in GL_t(K_v)$, and $y_v \in M_{t\times(n-t)}(K_v)$,

We define the <u>Fourier-Jacobi integral</u> at place v as

to fit in expansion of Siegal Eis.

$$FJ_{\beta,v}\left(y_v, g_v, x_v, h_v; f_v(z,\tau)\right) := \int_{\mathrm{Herm}_t(F_v)} f_v\left(w_n \begin{pmatrix} 1_n & \begin{matrix} \sigma & y_v \\ y_v^* & 0 \end{matrix} \\ & 1_n \end{pmatrix} \cdot j\left(\mathrm{diag}(x_v, x_v^{-*}), g_v\right) h_v \right) e_{F_v}(-\mathrm{Tr}\,\beta\sigma)\, d\sigma$$

if replaced by unitary groups at both places

where
$$j: U(n-t, n-t)(F_v) \times GL_t(K_v) \longrightarrow GL_n(F_v)$$
$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}, \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} \longmapsto \begin{pmatrix} A & & B & \\ & D' & & C' \\ C & & D & \\ & B' & & A' \end{pmatrix}$$

and $w_n := \begin{pmatrix} & 1_n \\ -1_n & \end{pmatrix}$.

For $h \in GU(n,n)(\mathbb{A}_F)$, $y \in M_{t\times(n-t)}(\mathbb{A}_K)$, $f(z,\tau) := \bigotimes_v f_v(z,\tau)$, let

$$FJ_\beta\left(y, g, x, h, f(z,\tau)\right) := \bigotimes_v FJ_{\beta,v}\left(y_v, g_v, x_v, h_v; f_v(z,\tau)\right)$$

<u>Note</u>: Here we always assume $\beta \in \mathrm{Herm}_t(F)$, not decomposed into local pieces.

Proposition: Let $f \in I_{Q_n}(\mathfrak{z}, \chi)$ be a Siegal section, with corresp. Siegal Eisenstein series $E^{\mathrm{sieg}}(f_{\mathfrak{z}})$. Then:

① [SU14. Lemma 11.2] Further suppose that for some place v_0 of F,

$$\operatorname{supp} f_{v_0} \subseteq \underbrace{Q_n(F_{v_0}) w_n Q_n(F_{v_0})}\ \text{"big cell"}$$

Let $\beta \in \mathrm{Herm}_n(F)$, then the β-th Fourier coefficient of $E^{\mathrm{sieg}}(f_{\mathfrak{z}})$ is given by

$$E^{\mathrm{sieg}}_{\beta}(f_{\mathfrak{z}}, h) = \prod_v \underbrace{\int_{\mathrm{Herm}_n(F_v)} f_v\left(\mathfrak{z}, w_n \begin{pmatrix} 1_n & \sigma \\ & 1_n \end{pmatrix} h_v\right) e_v(-\mathrm{Tr}\beta\sigma)\, d\sigma}_{\text{"local Whittaker integral"}}$$

if $\mathrm{Re}(\mathfrak{z}) > \frac{n}{2}$, when the RHS integrals converge absolutely.

② [SU14, Lemma 11.3] [Wan15ANT, Prop. 3.7] Let $\beta \in \mathrm{Herm}_t(F)_{>0}$ for $t > 0$. Then FJ coefficient of $E^{\mathrm{sieg}}(f_{\mathfrak{z}})$ is given by

$$E^{\mathrm{sieg}}_{\beta}(f_{\mathfrak{z}}, h) = \sum_{\gamma \in Q_{n-t}(F) \backslash GL_{n-t}(F)} \ \sum_{y \in M_{n\times(n-t)}(K)} FJ_{\beta}(y, \gamma, 1_{n-t}, h; f_{\mathfrak{z}}).$$

for sufficiently large $\mathrm{Re}(\mathfrak{z})$ with the integrals FJ_β on RHS converge absolutely.

Proof: For ①, see Shimura's computation [Shi97, (18.10.3)] based on Lemma 18.8 loc.cit. For ②, it is an art of unfolding everything, see [Wan15, Prop. 3.7] or the original [Ikeda 1994, §2–3].

So we see that in future computations, we only need to compute the <u>local Whittaker integrals</u> and <u>local Fourier–Jacobi integrals</u>.

In the following computations, we need some properties of the FJ-integral.

Property 1: $FJ_{\beta,v}\left(X, \boxed{\begin{pmatrix} A & BA^{-*} \\ 0 & A^{-*}\end{pmatrix}} g, Y, h; f_{\mathfrak{z}}\right)$ for $A \in GL_{n-t}(K_v)$, $B \in GL_{n-m}(F_v)$, $X \in M_{t\times(n-t)}(K)$

$$= \chi_v^c(\det A)^{-1} \left|\det A\bar{A}\right|_v^{\mathfrak{z}+\frac{n}{2}} e_v\left(-\mathrm{Tr}(X^*\beta X B)\right) FJ_{\beta,v}(XA, g, Y, h; f_{\mathfrak{z}})$$

Proof: Honestly write the integrand

$$f_v\left(w_n \left(\begin{array}{c:c} 1_n & \begin{matrix}\sigma & y_v\\ y_v^* & 0\end{matrix} \\ \hdashline & 1_n\end{array}\right) \cdot \jmath\left(\mathrm{diag}(Y, \bar{Y}^{-*}), \boxed{\begin{pmatrix} A & BA^{-*} \\ 0 & A^{-*}\end{pmatrix}} g_v\right) \underline{h_v}\right) e_{F_v}(-\mathrm{Tr}\,\beta\sigma)$$

$$= f_v\left(w_n \left(\begin{array}{c:c} 1_n & \begin{matrix}\sigma & y_v\\ y_v^* & 0\end{matrix} \\ \hdashline & 1_n\end{array}\right) \cdot \jmath\left(1, \boxed{\begin{pmatrix} A & BA^{-*} \\ 0 & A^{-*}\end{pmatrix}}\right) \jmath\left(\mathrm{diag}(Y, \bar{Y}^{-*}), g_v\right) \underline{h_v}\right) e_{F_v}(-\mathrm{Tr}\,\beta\sigma)$$

‖ identity by direct verification

‖ <u>purpose</u>: move ● to the front by one matrix

$$= f_v\left(w_n \left(\begin{array}{cc:cc} 1 & XBA^{-*} & & \\ & A^{-*} & & \\ \hdashline & & 1 & \\ & BA^{-*} & \underline{-BX^*} & A\end{array}\right) \left(\begin{array}{c:c} 1_n & \begin{matrix}\sigma - XX^* & XA \\ A^*X^* & \end{matrix} \\ \hdashline & 1_n\end{array}\right) \jmath\left(\mathrm{diag}(Y, \bar{Y}^{-*}), g_v\right) \underline{h_v}\right) e_{F_v}(-\mathrm{Tr}\,\beta\sigma)$$

Wan missed this

‖ again move one step forward

$$= f_v\left(\left(\begin{array}{c:c} \underbrace{\begin{matrix}1 & \\ -BX^* & A\end{matrix}}_{\mathbb{A}} & \begin{matrix} & \\ & -BA^{-*}\end{matrix} \\ \hdashline & \underbrace{\begin{matrix}1 & XBA^{-*} \\ & A^{-*}\end{matrix}}_{\mathbb{D}}\end{array}\right) w_n \left(\begin{array}{c:c} 1_n & \begin{matrix}\sigma - XBX^* & XA \\ A^*X^* & \end{matrix} \\ \hdashline & 1_n\end{array}\right) \jmath\left(\mathrm{diag}(Y, \bar{Y}^{-*}), g_v\right) \underline{h_v}\right) e_{F_v}(-\mathrm{Tr}\,\beta\sigma)$$

One checks: $\mathbb{A}\mathbb{D}^* = 1$, hence the leftmost matrix lies in the Siegel parabolic. Hence take it out:

$$\cdots = \text{the constants } \bullet \cdot f_v\left(w_n \left(\begin{array}{c:c} 1_n & \begin{matrix}\sigma - XBX^* & XA \\ A^*X^* & \end{matrix} \\ \hdashline & 1_n\end{array}\right) \jmath\left(\mathrm{diag}(Y, \bar{Y}^{-*}), g_v\right) h_v\right) e_{F_v}(-\mathrm{Tr}\,\beta\sigma).$$

Note that $XX^* \in \mathrm{Her}_t(F)$, we change the integral variable $\sigma \longleftrightarrow \sigma - XBX^*$, we get

$$\cdots = \bullet \cdot f_v\left(w_n \left(\begin{array}{c:c} 1_n & \begin{matrix}\sigma' & XA \\ A^*X^* & \end{matrix} \\ \hdashline & 1_n\end{array}\right) \jmath\left(\mathrm{diag}(Y, \bar{Y}^{-*}), g_v\right) h_v\right) e_{F_v}(-\mathrm{Tr}\,\beta\sigma')\, e_{F_v}(-\mathrm{Tr}\,\beta(XBX^*))$$

$$= \bullet \cdot FJ_{\beta,v}(XA, g, Y, h_v; f_{\mathfrak{z}}) \cdot e_v\left(-\mathrm{Tr}(X^*\beta XB)\right) \qquad \square$$

<u>Property 2</u> : $FJ_{\beta,\upsilon}(y,g,x,h;f_z)$

$$= \chi_\upsilon(\det x)\,|\det x\bar{x}|_\upsilon^{-(z+\frac{n}{2}-t)} \cdot FJ_{x^*\beta x}(x^{-1}y, g, 1, h)$$

<u>Proof</u> : This is proved in the same way as above. But there is a tricky point, so let me prove this :

- Again the key is <u>shifting x forward</u> :

$$w_n\left(\begin{array}{c|c} 1_n & \begin{matrix}\sigma & y\\ y^* & \end{matrix} \\ \hline & 1_n \end{array}\right) j(\mathrm{diag}(x, x^{-*}), 1)\, j(1,g)\, h$$

$$= w_n \begin{pmatrix} x & & & \\ & 1 & & \\ & & x^{-*} & \\ & & & 1 \end{pmatrix}\left(\begin{array}{c|c} 1_n & \begin{matrix}\sigma^+ & x^{-1}y\\ y^*x^{-*} & \end{matrix} \\ \hline & 1_n \end{array}\right) j(1,g)\,h \qquad \sigma^+ := x^{-1}\sigma x^{-*}$$

$$= \begin{pmatrix} x^{-*} & & & \\ & 1 & & \\ & & x & \\ & & & 1 \end{pmatrix}\left(\begin{array}{c|c} 1_n & \begin{matrix}\sigma^+ & x^{-1}y\\ y^*x^{-*} & \end{matrix} \\ \hline & 1_n \end{array}\right) j(1,g)\,h$$

Then the leftmost matrix lies in Siegal parabolic, so gives the constant

$$\chi_\upsilon(\det x)\,|\det x\bar{x}|_\upsilon^{-(z+\frac{n}{2})}$$

and the rest gives

$$\int_{\mathrm{Herm}_t(F_\upsilon)} f_\upsilon\left(w_n\left(\begin{array}{c|c} 1_n & \begin{matrix}\sigma^+ & x^{-1}y\\ y^*x^{-*} & \end{matrix} \\ \hline & 1_n \end{array}\right) j(1,g)\,h\right) e_{F_\upsilon}(-\mathrm{Tr}\,\beta\sigma)\,d\sigma$$

- Now its time to change the variable. Here $\sigma \longleftrightarrow \sigma^+$, we need to note :

$$d(x^{-1}\sigma x^{-*}) = |\det xx^*|^{-t}\,d\sigma \qquad [\text{Shimura97}, (18.9.5)]$$

this is different from the previous property.

- Note that

$$e_{F_\upsilon}(-\mathrm{Tr}\,\beta\sigma) = e_{F_\upsilon}\left(-\mathrm{Tr}\,(x^*\beta x)(x^{-1}\sigma x^{-*})\right)$$

So our FJ_β becomes $FJ_{x^*\beta x}$. $\square$

Property 3: For $y \in GL_n(K_v)$, the local Whittaker integral satisfies

$$W_\beta(\mathrm{diag}(y, y^{-*}); f_v, z)$$

$$:= \int_{\mathrm{Herm}_n(F_v)} f_v\left(z, w_n \begin{pmatrix} 1_n & \sigma \\ & 1_n \end{pmatrix} \begin{pmatrix} y & \\ & y^{-*} \end{pmatrix}\right) e_v(-\mathrm{Tr}\beta\sigma)\, d\sigma$$

$$= \int_{\mathrm{Herm}_n(F_v)} f_v\left(z, \begin{pmatrix} y^{-*} & \\ & y \end{pmatrix} w_n \begin{pmatrix} 1 & y^{-1}\sigma y^{-*} \\ 0 & 1 \end{pmatrix}\right) e_v(-\mathrm{Tr}\beta\sigma)\, d\sigma$$

$$= \chi(\det y) \left|\det y\bar{y}\right|^{-z-\frac{n}{2}} \int_{\mathrm{Herm}_n(F_v)} f_v\left(z, w_n \begin{pmatrix} 1 & y^{-1}\sigma y^{-*} \\ 0 & 1 \end{pmatrix}\right) e_v\left(-\mathrm{Tr}(y^{*}\beta y)(y^{-1}\sigma y^{-*})\right) d\sigma$$

$$= \chi(\det y) \left|\det y\bar{y}\right|^{-z+\frac{n}{2}} \int_{\mathrm{Herm}_n(F_v)} f_v\left(z, w_n \begin{pmatrix} 1 & \sigma' \\ 0 & 1 \end{pmatrix}\right) e_v\left(-\mathrm{Tr}(y^{*}\beta y)\right) d\sigma'$$

$$= \chi(\det y) \left|\det y\bar{y}\right|^{-z+\frac{n}{2}} W_{y^{*}\beta y}(1; f_v, z).$$

So when computing Fourier coefficients, sometimes we only compute $W_\beta(1; f_v, z)$ for conviences.

§4.6 Constant terms.

Definition: Let R be *any* parabolic subgroup of $GU(r+1,s+1)$, ϕ be an automorphic form on $GU(r+1,s+1)$. Define the *constant term* of ϕ along R as

$$\phi_R(g) = \int_{N_R(F)\backslash N_R(\mathbb{A}_F)} \phi(ng)\,dn .$$

For the constant terms of Klingen Eisenstein series $E^{Kling}(f_z^{Kling})$, we have:

Theorem. [Wan15AN]. Lemma 3.4] [MgW1994, II.1.7] Suppose $Re(z) > \frac{a+b+1}{2}$, then

$$E^{Kling}(f_z^{Kling})_P(g) = f_z^{Kling}(g) + A(f_z^{Kling})(g),$$

where $A(f_z^{Kling})$ is the *Bröchner integral* defined as

$$A(f_z^{Kling})(g) = \int_{N_P(\mathbb{A}_F)} f_z^{Kling}(wng)\,dn, \qquad w := \begin{pmatrix} & & 1_{b+1} \\ & 1_a & \\ -1_{b+1} & & \end{pmatrix}.$$

Proof: The proof is "elementary": use Bruhat decomposition and unfolding things. Proof is also availible in Shahidi "Eisenstein series and Automorphic L-functions". §6.2.

Note: The Bröchner integral can be decomposed into local ones:

$$M(f_z^{Kling})(g) = \prod_v M_v(f_{z,v}^{Kling})(g_v)$$

In practice, we take advantange of archmedian sections to kill this Böchner integral part.