Talk 1 Motivations and Setups

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§1 Ribet's theorem

- Everything starts from Ramanujan's observation that

$$\Delta \equiv E_{12} \mod p = 691 .$$

and Serre used it to study the cyclotomic field $\mathbb{Q}(\zeta_{691})$.

Theorem 1 (Kummer, Herbrand, Ribet) We take a sketchy formulation

Let p be an odd prime. Then

$p \mid$ the numerator of some B_m with $m \in [2, p-3]$ even number iff $\underbrace{H^1_{ur}(\mathbb{Q}, \mathbb{F}_p(\omega^{m-1}))}_{\text{the group of everywhere unramified cohomology classes.}} \neq 0$

Here $\omega: \mathbb{F}_p^\times \to \mathbb{C}^\times$ is the Teichmüller character: regard it as a "mod-p" version of the cyclotomic character $\epsilon_{cyc}: \mathbb{Z}_p^\times \to \mathbb{C}^\times$.

Proof ("Not even wrong"): * Kummer (& Herbrand): "$\Leftarrow$"

* Ribet: "$\Rightarrow$" Consider the weight m, level $SL_2(\mathbb{Z})$ Eisenstein series

$$E_m(q) := \frac{\zeta(1-m)}{2} + \sum_{n \geq 1} \sigma_{m-1}(n) q^n , \quad \sigma_{m-1}(n) = \sum_{d \mid n} d^{m-1} .$$

recall: $\zeta(1-m) = (-1)^{m-1} \frac{B_m}{m}$. $(1-m < 0)$

Then:

$p \mid B_m \Rightarrow p \mid$ ~~constant~~ the constant term of E_m

$\Rightarrow E_m \equiv f$ for some eigencuspform $f \in S_2(SL_2(\mathbb{Z}), \mathcal{O})$

⚠ Warning: f is not simply the function $\sum_{n \geq 1} \sigma_{m-1}(n) q^n$.

- The latter is not necessarily a modular form.
- We have to guarantee f being an eigenform to have a Galois rep'n attached.

translation

$\Longrightarrow \rho_f^{ss} \equiv \rho_{E_m}^{ss} \mod p$ as Galois reps.

- ρ_f is an irreducible Galois rep'n (since f is a cuspform)
- $\rho_{E_m} = \varepsilon_{cyc}^{m-1} \oplus \mathbb{1}$ is reducible.

$$\rightsquigarrow \boxed{\rho_f \equiv \begin{pmatrix} \omega^{m-1} & * \\ & 1 \end{pmatrix} \mod p}$$

Then the entry "$*$" gives the desired cohomology class.

Let's do a computation: let $\sigma, \tau \in \mathrm{Gal}_{\mathbb{Q}}$,

$$\bar\rho_f(\sigma)\bar\rho_f(\tau) = \bar\rho_f(\sigma\tau) \Longrightarrow \begin{pmatrix} \omega^{m-1}(\sigma) & *(\sigma) \\ & 1 \end{pmatrix}\begin{pmatrix} \omega^{m-1}(\tau) & *(\tau) \\ & 1 \end{pmatrix} = \begin{pmatrix} \omega^{m-1}(\sigma\tau) & *(\sigma\tau) \\ & 1 \end{pmatrix}$$

$$\Longrightarrow \begin{pmatrix} \omega^{m-1}(\sigma\tau) & \omega^{m-1}(\sigma)*(\tau) + *(\sigma) \\ & 1 \end{pmatrix} = \begin{pmatrix} \omega^{m-1}(\sigma\tau) & *(\sigma\tau) \\ & 1 \end{pmatrix}$$

$$\Longrightarrow \underbrace{*(\sigma\tau) = *(\sigma) + \omega^{m-1}(\sigma)*(\tau)}_{\text{this is a cocycle condition for Galois coh } H^1.}$$

A more detailed study gives further "local properties" of $*$. $\square$

This technique was used by Mazur-Wiles, Wiles in proving Iwasawa main conjecture of class groups.

Theorem 2 (Mazur-Wiles, Wiles) IMC for class groups is "true".

"Proof": $E_m \rightsquigarrow$ a p-adic family of Eisenstein series $\mathbb{E} \in \mathbb{Z}_p[[T]][[q]]$.
s.t. its constant term is given by Kubota-Leopoldt p-adic L-function $\zeta_p \in \mathbb{Z}_p[[T]] =: \Lambda$

Let $\mathfrak{p}$ be a height one prime of $\mathbb{Z}_p[[T]]$, then

$\mathfrak{p} \mid \zeta_p \Rightarrow \mathfrak{p} \mid$ constant term of $\mathbb{E}$

$\Rightarrow \mathbb{E} \equiv \mathbb{f} \mod \mathfrak{p}$ for some "p-adic family of eigencuspforms" $\mathbb{f}$

$$\Longrightarrow \quad \rho_f \equiv \begin{pmatrix} \chi & * \\ & 1 \end{pmatrix} \quad \text{for characters } \chi, * : (\Lambda/p)^\times \to \mathbb{C}^\times$$

Then $*$ belongs to $H^1(\mathbb{Q}, (\Lambda/p)(\chi))$, together with certain local properties by more detailed study. More carefully, we see

$(\xi_p) \mid \mathrm{char}_\Lambda \, Cl_\infty$. Note: Here we only captured the "prime divisors" of ξ_p, not paying attention to their "powers", and $\mathrm{length}_{\Lambda_p} M_p$ accordingly.

This is the starting point of the machinary of Eisenstein congruences: proving the lower bound of Iwasawa-theoretic Selmer groups.

- Generalization: What if we want to study other arithmetic objects M?
  eg: elliptic curves, or higher dimensional Galois representations ...

Step 0: "Galois $\Rightarrow$ Automorphic": To study the "automorphic rep" π attached to the arithmetic object: $L(\pi, s) = L(M, s)$.

eg: • Elliptic curves $E/\mathbb{Q} \underset{\text{BCDT}}{\overset{\text{Wiles}}{\rightsquigarrow}} f \in S_2(\Gamma_1(N))$ s.t.

$$L(E, s) = L(f, s)$$

• Higher weight modular forms $f \in S_k(\Gamma_1(N))$ $\rightsquigarrow$ A_f AV of GL_2-type attached to f

• Other examples growing out of these:

  – $E_1 \times E_2$: product of two elliptic curves
  $\rightsquigarrow f_1 \times f_2$: Rankin-Selberg product

  – $\mathrm{ad}^\circ(E)$: adjoint rep'n of (the Gal rep of) E
  $\rightsquigarrow \mathrm{ad}^\circ(f)$: Adjoint rep of f.

Granting such "modularity": we focus on the automorphic side: Iwasawa theory of automorphic rep'ns π / G

Step 1 : Construct a (p-adic family of) Eisenstein series $E/\mathcal{G}$, here $\mathcal{G}$ is a larger group than G, such that the constant term of E sees the L-value of π.

$\rightsquigarrow$ This is the motivation for studying automorphic reps over larger groups other than GL_2 (where classical modular forms live)

- For a single Eisenstein series, this is morally "Langlands-Shahidi" method.
- The task is to construct "p-adic families of Eisenstein series".

Step 2 : Establish the congruence of $\mathbb{E}/\mathcal{G}$ with eigencuspform $\mathbb{f}/\mathcal{G}$.

There are two problems that have been ignored in the proof of Thm 1 and Thm 2 :

(a) The existence of the cuspidal family $\mathbb{f}$: in the GL_2-case, we understand the space of modular forms (char 0 & char p) so well that this can be done by hand.

$\rightsquigarrow$ What about the story over general groups ? This is far from easy.

- For any "cusp" $[g]$, there is a "Siegel operator" $\Phi_{[g]}$ s.t. $\Phi_{[g]}$ is "taking the q-expansion" :

$$\Phi_{[g]}(\mathbb{E}) = \underset{\text{the } p\text{-adic L-function}}{\mathbb{L}} \cdot \mathbb{F}_{[g]}, \qquad \mathbb{F}_{[g]} : \text{the modular form left after factoring out } \mathbb{L}$$

Then (1) The surjectivity of $\oplus \Phi_{[g]}$ implies $\exists\, \mathbb{F}$ s.t. $\Phi_{[g]}\mathbb{F} = \mathbb{F}_{[g]}$ for any "cusp" $[g]$. So $\Phi_{[g]}(\mathbb{E}) = \mathbb{L}\,\Phi_{[g]}(\mathbb{F}) = \Phi_{[g]}(\mathbb{L}\mathbb{F})$.

Then (2) The kernel of $\oplus \Phi_{[g]}$ are "cuspforms" :

$$\mathbb{E} - \mathbb{L}\mathbb{F} \in \ker \Phi_{[g]} = \{\text{cusp forms}\} \rightsquigarrow \mathbb{E} - \mathbb{L}\mathbb{F} = \widetilde{\mathbb{F}}.$$

In this sense, when "modulo p", p divides $\mathbb{L}$, hence

$$\mathbb{E} \equiv \widetilde{\mathbb{F}} \mod p .$$

The above (1) & (2) are described into the following exact seq:

$$0 \to \{\text{cusp forms} / G\} \to \{\text{Modular forms} / G\} \xrightarrow{\oplus \Phi_{[g]}} \bigoplus_{[g]} \{\text{"}q\text{-expan"}\}$$

this is the style of the "fundamental exact sequence for Eisenstein congruences".

(b) The p-primitivity of Eisenstein families. But after all, we have to guarantee "$\mathbb{E} \not\equiv 0 \mod \mathfrak{p}$" for those "$\mathfrak{p} \mid \mathbb{L}$".

⚠ This innocent-looking statement is the most technical part of the machinary.

- In the GL_2-case: this is trivial:

$$a_p(E_m) = 1 + p^{m-1}, \quad \text{hence } a_p(E_m) \equiv 1 \mod p .$$

⇝ In general cases: we shall construct and compute a "clever" functional "a_p" on the space of automorphic forms.

Method: Take "a_p" to be a "period integral" $P_\bullet$:

$$\boxed{\mathbb{E} \mod \mathfrak{p} \text{ nonvanishing}} \Longleftarrow \boxed{P_\bullet(\mathbb{E}) \mod p \text{ nonvanishing}}$$

$\Updownarrow$ the philosophy of period integral

$$\boxed{L(\mathbb{E}, \bullet, s) \mod p \text{ nonvanishing}}$$

$\wr\wr$

$L(\pi, \bullet, s)$ and other terms

- Period integrals: $H \leq G$ a "spherical subgroup" of G, then for automorphic form f over G, we formally compute

$$\int_{[H]} f(\iota(h))\, dh \sim \text{"special value of L-functions"}$$

This is only a "philosophical" connection, and we have to prove it in each individual case. (See later talks).

Step3 : "Automorphi $\Rightarrow$ Galois" : Translate the congruence $\mathbb{E} \equiv \widetilde{\mathbb{F}}$ mod $\mathfrak{p}$ into congruences of Galois reps : "Galois arguments" / "Lattice Construction".

Slogan : $\rho_{\mathbb{E}}$ is "less" irreducible than $\rho_{\widetilde{\mathbb{F}}}$. Their "mod p" congruences hence provide nontrivial "$*$" as in Thm 1 and Thm 2.

$\rightsquigarrow$ In general case, this machinery of "lattice construction" is generalized and axiomized by [Bellaiche – Chenevier, 2009 Asterisque].

In this series of talks, we shall focus on the case of Iwasawa theory over unitary groups.

§3 Unitary groups

The first thing is : we should be sufficiently familiar with unitary groups.

Setup 1: Let K/F be a CM field :

- F be a totally real number field of degree d over $\mathbb{Q}$.
- K be a totally imaginary quadratic extension of F

then K (or K/F) is called a CM-field. Let $c \in \mathrm{Gal}(K/F)$ be the nontrivial ele called the complex conjugation.

Setup 2 : Let $K = F$ be a totally real number field, $c = \mathrm{id}_F \in \mathrm{Gal}(K/F)$.

Let
- $\epsilon \in \{\pm 1\}$ be a fixed number.
- V be an d-dimensional vector space over K.

Definition : An ϵ-Hermitian form on V is a map $\phi : V \times V \to K$ s.t.

(1) $\phi(v,w)^c = \epsilon \phi(w,v)$, $\forall v,w \in V$.

(2) $\phi(v_1+v_2, w) = \phi(v_1,w) + \phi(v_2,w)$

$\phi(av, bw) = ab^c \phi(v,w)$

It is called nondegenerate if $v \mapsto \phi(v,-)$ is an isomorphism from V to V^*.

We call (V,ϕ) is

- isotropic, if $\phi(x,x) = 0$ for some $x \in V \setminus \{0\}$.
- anisotropic, if ——— only for $x = 0$.

For a subspace U of V, define

$$R^\phi(U) = \{ x \in U : \phi(U,x) = 0 \} \qquad \text{isotropic radical}.$$

Call U totally ϕ-isotropic if $U = R^\phi(U)$.

Check : (V,ϕ) nondegenerate iff $R^\phi(V) = \{0\}$.

Example :

- $K = \mathbb{Q}(\sqrt{-d}) \,/\, F = \mathbb{Q}$: imaginary quadratic fields
- $K = \mathbb{Q}(\zeta_m) \,/\, F = \mathbb{Q}(\zeta_m + \zeta_m^{-1})$ maximal totally real subfield.
- Let E be an elliptic curve over a totally real field F with complex multiplication $\mathcal{O}_M$, where $M/\mathbb{Q}$ is an imaginary quadratic field. Now let $K = FM$, then K/F is a CM-field extension.

We can extend ϕ to any S-module $V \otimes_F S$ for any F-algebra S.

- Once we fix an K-basis $\{v_1, \cdots, v_d\}$ of V, we can identify:

$$\phi_S / V \underset{F}{\otimes} S \longleftrightarrow d \times d \text{ matrix } A_S \text{ of entries in } S \text{, s.t. } \underbrace{A_S^{ct} =: A_S^{\star} = \epsilon A_S}_{\epsilon\text{-Hermitian matrix}}$$

$$V \otimes_F S \longleftrightarrow (K \otimes_F S)^{\oplus d} \text{, regard elements as row vectors.}$$

$$\text{s.t. } \boxed{\phi_S(v_i, v_j) = e_i^{\star} A_S e_j} \quad \text{for } v_i \longleftrightarrow e_i \in (K \underset{F}{\otimes} S)^{\oplus d}.$$

Definition : The <u>general unitary group</u> attached to (V, ϕ) is the algebraic group

$$GU(V, \phi)(S) := \left\{ g \in GL_{K \underset{F}{\otimes} S}\left(V \underset{F}{\otimes} S\right) : \phi_S(gv, gw) = \nu(g)\, \phi_S(v, w),\ \nu(g) \in S^{\times} \right\}$$

for F-algebras S. The homomorphism $\nu : GU(V, \phi) \to \mathbb{G}_m$ is called the <u>silimitude factor</u>. We define:

$$U(V, \phi)(S) := \ker(\nu)(S) = \left\{ g \in GL_{K \underset{F}{\otimes} S}\left(V \underset{F}{\otimes} S\right) : \phi_S(gv, gw) = \phi_S(v, w) \right\}$$

called the <u>unitary group</u> attached to (V, ϕ).

- By fixing a K-basis $\{v_1, \cdots, v_d\}$ of V, these are simply matrix groups

$$\text{"}\phi_S(gv, gw) = \phi_S(v, w)\text{"} \iff \text{"} g \in GL_n(K \underset{F}{\otimes} S) \text{ s.t. } g^{\star} A_S\, g = A_S \text{"}$$

(<u>Skew-Hermitian</u> : this condition is the <u>same</u>, the difference is at A_S !)

Example :

1) In Setup 1, we get "unitary groups" of Hermitian spaces and skew-Hermitian spaces.

We remark the relation of skew-Hermitian spaces and Hermitian spaces :

Fix an totally imaginary element $i \in K$ $\left(\text{i.e. } \bar{i} = -i \right)$, then

$$\phi \text{ is an Hermitian form } \Longleftrightarrow i\phi \text{ is a skew-Hermitian form.}$$

So (V, ϕ) and $(V, -i\phi)$ are by nomeans "isomorphic" as "inner product spaces". But, as groups, one checks immediately that

$$\boxed{U(V,\phi) \simeq U(V, -i\phi)}.$$

So we are "free" to change the setup between ϕ and $i\phi$ (Hermitian and skew-Hermitian.)

Canonical basis in skew-Hermitian settings :

Let $s \geq 0$ such that $s \leq$ "Witt index" of (V,ϕ) := maximal totally ϕ-isotropic subspace, $\dim \leq \lfloor \frac{d}{2} \rfloor$.

Then there is a decomposition $V = Y \oplus W \oplus X$, with

$$Y = \bigoplus_{i=1}^{s} K y^i, \quad X = \bigoplus_{i=1}^{s} K x^i, \quad W = \bigoplus_{j=1}^{r-s} K w^j \qquad r + s = d$$

s.t. $\phi := J_{r,s} := \begin{bmatrix} & & 1_s \\ & S & \\ -1_s & & \end{bmatrix}$ under the canonical basis $\{ y^i, w^j, x^i \}$.

Here $S = \mathrm{diag}[a_1, \cdots, a_{r-s}]$ s.t. $\bar{a_i} = -a_i$ (totally imaginary elements).

Then $(V, J_{r,s})$ is a skew-Hermitian space.

$\rightsquigarrow$ $\infty_F : F \hookrightarrow \mathbb{R}$ infinite place, then $\left\{ \sqrt{-1}\, \infty(a_i) \right\}_{i=1}^{r-s}$ determines the signature of $(V, J_{r,s})$ at the place ∞_F as we shall see in the following example.

2) In Setup 2, we have two cases:

- $\varepsilon = -1$: We get "general symplectic group" $GSp(V, \phi) := GU(V, \phi)$. One checks that in this case, $d = 2d_0$ is even. A canonical metric is $\begin{bmatrix} & \mathbb{1}_{d_0} \\ -\mathbb{1}_{d_0} & \end{bmatrix} =: A$. One checks: $A^* = A^t = -A$.

- $\varepsilon = 1$: We get "general orthogonal group" $GO(V, \phi) := GU(V, \phi)$. Let $a, b \geq 0$ be integers (for certainty, let $a \geq b \geq 0$), a canonical metric is $\begin{bmatrix} \mathbb{1}_a & \\ & -\mathbb{1}_b \end{bmatrix}$ One checks: $A^* = A^t = A$.

These are useful backgrounds on understanding the setup of [Lemma – Ochiai].

Reference: Shimura « Euler products and Eisenstein series » CBMS № 93.
Chapter I & II.

§3.1 Local unitary groups

① $S=\mathbb{R}$. Here $\mathbb{R}$ is an F-algebra by a real place $\infty_i : F \hookrightarrow \mathbb{R}$. Then

$$\left[\begin{array}{l} V \otimes_{F,\infty_i} \mathbb{R} \simeq \mathbb{C}^{\oplus n} \\ \phi_{\mathbb{R},i} \longleftrightarrow \text{Hermitian matrix } A_i \\ \qquad\qquad A_i^{*} = A_i \end{array}\right] \text{ upon a choice of an } F\text{-basis of } V.$$

$\underset{\sim\sim\sim}{\text{a change of}}$ basis $\quad I_{a_i,b_i} = \begin{bmatrix} \mathbb{1}_{a_i} & \\ & -\mathbb{1}_{b_i} \end{bmatrix}$, or when $a_i = b_i$, $\eta_{a_i} := \begin{bmatrix} & -\mathbb{1}_{a_i} \\ \mathbb{1}_{b_i} & \end{bmatrix}$.

The tuple $\{(a_i,b_i) : i=1,\cdots,d\}$ is called the <u>signature</u> of $U(V,\phi)$.

- We denote $U(a_i,b_i)$ for the unitary group $U(V,\phi)_{\mathbb{R},\infty_i}$.
- If the signature is "<u>pure</u>" in the sense that $a_i = a_j$ for any $1 \le i,j \le d$, we denote $U(a,b) := U(V,\phi)$ if cause no confusion.
- $U(a_i,b_i)$ is compact iff $b_i = 0$ $\rightsquigarrow$ called "definite case".

② S = a K-algebra with induced F-algebra structure, <u>then</u>

$$K \otimes_F S \xrightarrow{\sim} S \oplus S \quad \text{by} \quad k \otimes s \mapsto (ks, \bar{k}s) \quad \text{— } (*)$$

Then the induced complex conjugation on $K \otimes_F S$ is simply swapping two factors of S on the right-hand-side.

<u>Exercise</u>: This essentially dues to: $K \otimes_F K \xrightarrow{\sim} K \oplus K$ by $k_1 \otimes k_2 \mapsto (k_1k_2, \bar{k}_1k_2)$

Then

$$U(V,\phi)(S) = \{(g_1,g_2) \in GL_n(S) \times GL_n(S) : \underbrace{(g_2^t, g_1^t)\, A_S\, (g_1,g_2) = A_S}_{\text{coordinate-wise}}\}$$

[upon choosing a basis with respect to $(*)$:
- $K \otimes_F S \xrightarrow{can} S \oplus S$
- $V \otimes_F S \xrightarrow{\sim} (K \otimes_F S)^{\oplus n} \xrightarrow{can} S^{\oplus n} \oplus S^{\oplus n}$
- A_S : $\underline{n \times n}$ matrix with entry in S.]

$$= \{(g_1,g_2) \in GL_n(S) \times GL_n(S) : \underbrace{g_2^t A_S g_1 = A_S, \; g_1^t A_S g_2 = A_S}_{\text{equivalent, and } g_2 = A_S^{-1} g_1^{-t} A_S}\}$$

$$\simeq GL_n(S)$$

In particular $U(V,\phi)(\mathbb{C}) \simeq GL_n(\mathbb{C})$, and similar for all $\mathbb{C}$-algs

$\implies U(V,\phi)_{\mathbb{C}} \simeq GL_{n,\mathbb{C}}$, or in fancy terms:

"unitary groups are always <u>forms</u> of GL_n".

③ $S =$ local field F_v st. v splits in K: $v\mathcal{O}_K = w\bar{w}$. Similar to ②,

$$\begin{array}{ccccc} K \otimes_F F_v & \underset{\text{can}}{\simeq} & K_w \oplus K_{\bar{w}} & \xrightarrow[\text{can}]{\sim} & F_v \oplus F_v \\ a & & \longmapsto & & (\rho_w(a), \rho_{\bar{w}}(a)) \end{array},$$

and complex conjugation acts by swapping the two factors. Then

$$U(V,\phi)(F_v) = \{(g_w, g_{\bar{w}}) \in GL_n(F_v) \times GL_n(F_v) \;:\; g_1^t A_{F_v} g_2 = A_{F_v}\} \xrightarrow{\rho_w} GL_n(F_v)$$

i.e. $U(V,\phi)(F_v) \simeq GL_n(F_v)$.

<u>Note</u>: This relies highly on the "split" assumption on v. The inert and ramified cases are not that easy.

§3.2 Special cases & low rank cases

We use Witt's decomposition and the Witt basis in this section

Special cases :

① "Hermitian case" : $r = s$. In this case :

- $V = Y \oplus X$, with W degenerates into 0.
- Under the standard basis, the skew-Hermitian form becomes $J = \begin{pmatrix} & -1_r \\ 1_r & \end{pmatrix}$.

Then

$$\begin{aligned} GU(r,r)(R) &= \left\{ g \in GL_{2r}(K \otimes_F R) \;\middle|\; gJg^* = \lambda(g) J,\ \lambda(g) \in R^\times \right\} \\ &= \left\{ \underbrace{g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}}_{\text{block-matrix by } r/r;\ r/r} \in GL_{2r}(K \otimes_F R) \;\middle|\; \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} & -1_r \\ 1_r & \end{pmatrix} \begin{pmatrix} A^* & C^* \\ B^* & D^* \end{pmatrix} = \lambda(g) \begin{pmatrix} & -1_r \\ 1_r & \end{pmatrix} \right\} \\ &= \left\{ g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GL_{2r}(K \otimes_F R) \;\middle|\; \begin{array}{l} AB^*,\ CD^* \in \mathrm{Herm}_r(K \otimes_F R) \\ \underline{\lambda(g) = AD^* - BC^* \in R} \ \text{ hence } \overline{\lambda(g)} = \lambda(g) \end{array} \right\} \end{aligned}$$

(see the difference between λ and $\det$)

② "Definite case" : $s = 0$. Then

- $V = W$ with the skew-Hermitian paring θ
- Under the standard basis, θ can be written as $S = \mathrm{diag}[a_1, \cdots, a_r]$.

Then $GU(r,0)(R) = \left\{ g \in GL_r(K \otimes_F R) \;\middle|\; gSg^* = \lambda(g) S,\ \lambda(g) \in R^\times \right\}$.

Why unitary group matters? See three low-rank cases (usually: $r+s\leq 2$)

① $GU(1,0) \simeq \mathrm{Res}_F^K \mathbb{G}_m$

- Recall our computation ② on definite case above:

$$GU(1,0)(R) = \{ g \in (K\otimes_F R)^\times \mid gg^* = \lambda(g) \in R^\times \}$$

Yet for any $g \in (K\otimes_F R)^\times$, $(gg^*)^* = gg^*$. hence $gg^* \in R^\times$ automatically. Hence

$$GU(1,0)(R) = (K\otimes_F R)^\times \quad \text{for any } F\text{-algebra } R.$$

Therefore $GU(1,0) \simeq \mathrm{Res}_F^K \mathbb{G}_m$ (the Weil restriction)

P.S. Weil restriction: Let E/F be a finite extension of number fields and G be an algebraic group over E. We define the Weil restriction of G to F to be

$$(\mathrm{Res}_F^E G)(R) := G(R\otimes_F E) \quad \text{for any } F\text{-algebra } R.$$

- Moreover, $U(1,0)(R) = \{ g \in (K\otimes_F R)^\times \mid gg^* = 1\}$, as a "unit circle".

Now let $\chi : GU(1,0)(\mathbb{A}_F) \to \mathbb{C}^\times$ be an automorphic form over $GU(1,0)$:

- $GU(1,0)(\mathbb{A}_F) = (K \otimes_F \mathbb{A}_F)^\times = \mathbb{A}_K^\times$.
- χ is left-invariant by $GU(1,0)(F) = (K\otimes_F F)^\times = K^\times$

$\rightsquigarrow$ χ is nothing but a Hecke character over K.

⚠ Warning: Over unitary group $U(1,0)$,

$$\left[\chi : \mathbb{A}_K^\times \to \mathbb{C}^\times\right] \xrightarrow{\text{instead}} \chi : (\mathbb{A}_K^\times)^{(1)} \to \mathbb{C}^\times \text{ left invariant by } \underbrace{K^\times}_{\text{product formula}}$$

So χ is only defined on "norm-one" adèles. This is a little bit subtle.

But anyway, given a Hecke character χ over K, we can regard it as an automorphic form of $U(1,0)$ over F.

② $GU(1,1) \simeq (Gl_2 \times Res_F^K \mathbb{G}_m)/\mathbb{G}_m$. Here $\mathbb{G}_m$ is identified with $\{(a^{-1}\mathbb{1}_2, a) : a \in \mathbb{G}_m\}$

Proof : We work on its F-points. Then we first have a map

$$m: \quad Gl_2(F) \times Res_F^K \mathbb{G}_m(F) = K^\times \longrightarrow GU(1,1)(F)$$
$$(g, a) \longmapsto ag$$

Then one checks directly that $ag \in GU(1,1)(F)$ by the "Hermitian" case computation above (Since $r=1$ and entries in g lie in F, all conditions there on g are satisfied)

Moreover, for any pair (g,a) such that $ag = \mathbb{1}_2 \in GU(1,1)(F)$, hence

$$(g,a) = (a^{-1}\mathbb{1}_2, a) \in \text{the copy of } \mathbb{G}_m \text{ in } Gl_2(F) \times K^\times$$

(note : a priori $g \in Gl_2(F)$, hence $ag = 1$ implies $a \in F^\times$. Hence $(g,a) \in \mathbb{G}_m$ indeed !)

Conversely for each $(a^{-1}\mathbb{1}_2, a)$, clearly $a \cdot a^{-1}\mathbb{1}_2 = \mathbb{1}_2 \in GU(1,1)(F)$. Hence $\ker(m) = \{(a^{-1}\mathbb{1}_2, a) : a \in F^\times\}$.

The key is to show m is surjective. So for any $g \in GU(1,1)(F)$, we need to find $g' \in Gl_2(F)$, $a \in K^\times$ st. $g = ag'$:

- Step1 : Show the condition on g is equivalent to $\bar{g} = \frac{\lambda(g)}{\det(g)} g$.

In fact we compute in general :

$$gJg^* = \lambda(g)J \iff g^* = \lambda(g)(gJ)^{-1}J = \lambda(g)J^{-1}g^{-1}J = \frac{\lambda(g)}{\det(g)} J^{-1}g^{\iota}J$$
$$\iff \bar{g} = \frac{\lambda(g)}{\det(g)} \underbrace{(J^{-1}g^{\iota}J)^t}_{=:\, g^!} \qquad g^{\iota} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Then when $r = s = 1$.

$$(g^!)^t = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

Hence $g^! = g$, as desired.

- Step2 : Then we see $\mathrm{Norm}_F^K \left(\frac{\lambda(g)}{\det(g)} \right) = 1$.

Indeed :
$$\mathrm{Norm}_F^K \left(\frac{\lambda}{\det}(g) \right) = \frac{\lambda(g)}{\det(g)} \cdot \frac{\lambda(\bar{g})}{\det(\bar{g})} \overset{\boxed{\text{Step1}}}{=} \bar{g}g^{-1} \cdot g \cdot \bar{g}^{-1} = 1$$

Here we note that $\lambda(g) \in F^\times$, hence $\overline{\lambda(g)} = \lambda(g)$, and

$$\bar{g}J\bar{g}^* = \lambda(\bar{g})J \xrightarrow{\text{conjugation}} gJg^* = \overline{\lambda(\bar{g})}\,\bar{J} = \lambda(\bar{g})\bar{J}$$

$\Longrightarrow_{J=\begin{pmatrix} & 1\\ -1 & \end{pmatrix}}$ $gJg^* = \lambda(\bar{g})J$. hence $\lambda(g) = \lambda(\bar{g})$.

- $\underline{\text{Step 3}}$: Then by $\underline{\text{Hilbert Satz 90}}$, $\exists\, a(g) \in K^\times$ s.t. $\dfrac{\lambda(g)}{\det(g)} = \dfrac{a(g)}{\bar{a}(g)}$.

This implies $\underbrace{\bar{g} = \frac{a(g)}{\bar{a}(g)} g}$, i.e. $a(g)g = \bar{a}(g)\bar{g}$, which means that actually $a(g)\cdot g \in GL_2(F)$. Hence under m, step 1

$$\left(a(g)\cdot g,\ \frac{1}{a(g)}\right) \longmapsto g, \quad \text{as desired !} \qquad \square$$

③ $GU(2,0) \simeq (\mathbb{D}^\times \times \mathrm{Res}^K_F \mathbb{G}_m)/\mathbb{G}_m$. Here $\mathbb{G}_m$ is identified with $\{(a1_2, \bar{a}^{-1}) : a \in \mathbb{G}_m\}$

- Recall in the previous calculation on the definite case,

$$GU(2,0)(F) = \{ g \in GL_2(K) : gSg^* = \lambda(g)S,\ \lambda(g) \in F^\times \},\quad S = \begin{bmatrix} a_1 & \\ & a_2 \end{bmatrix}$$
skew-Hermitian.

One computes: $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$:

$$gSg^* = \lambda(g)S \iff gS = \lambda(g)S\bar{g}^{-t} = \frac{\lambda(g)}{\det(g)} S\cdot(\bar{g}^*)^{\iota} = \frac{\lambda(g)}{\det(g)} S\cdot \begin{bmatrix} \bar{d} & -\bar{c} \\ -\bar{b} & \bar{a} \end{bmatrix}$$

$$\iff \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a_1 & \\ & a_2 \end{bmatrix} = \frac{\lambda(g)}{\det(g)}\begin{bmatrix} a_1 & \\ & a_2 \end{bmatrix}\begin{bmatrix} \bar{d} & -\bar{c} \\ -\bar{b} & \bar{a} \end{bmatrix}$$

$$\iff \begin{bmatrix} a_1 a & a_2 b \\ a_1 c & a_2 d \end{bmatrix} = \frac{\lambda}{\det}(g)\begin{bmatrix} a_1\bar{d} & -a_1\bar{c} \\ -a_2\bar{b} & a_2\bar{a} \end{bmatrix}$$

$$\iff d = \frac{\lambda}{\det}(g)\,\bar{a},\quad b = -\frac{\lambda}{\det}(g)\frac{a_1}{a_2}\bar{c}.$$

$$\rightsquigarrow GU(2,0)(F) = \left\{ \begin{bmatrix} a & -\frac{\lambda}{\det}(g)\frac{a_1}{a_2}\bar{c} \\ c & \frac{\lambda}{\det}(g)\,\bar{a} \end{bmatrix} : a, c \in K,\ \lambda(g) \in F^\times \right\}$$

For simplicity, we take a special $S = \delta\begin{bmatrix} s & \\ & 1 \end{bmatrix}$, here:

- δ is a fixed totally imaginary element in K. ($\bar{\delta} = -\delta$ in particular)
- $\mathrm{Nm}(\delta) = \delta\bar{\delta}$ is a p-adic unit.
- s is a $\square$-free positive integer.

(Indeed, S is skew-Hermitian: $S^* = \bar{\delta}\begin{bmatrix} s & \\ & 1 \end{bmatrix} = -\delta\begin{bmatrix} s & \\ & 1 \end{bmatrix}$)

Then
$$GU(2,0)(F) = \left\{ \begin{bmatrix} a & -\frac{\lambda}{\det}(g)\, s\bar{c} \\ c & \frac{\lambda}{\det}(g)\,\bar{a} \end{bmatrix} : a, c \in K,\ \lambda(g) \in F^\times \right\}$$

- We define the <u>quaternion algebra</u> D as the matrix group:

$$(\star)\quad D := M_2(K) = \{ g \in M_2(K) : gSg^* = \det(g) S \} \quad \text{previous calculation, put } \lambda = \det$$
$$= \left\{ \begin{bmatrix} a & -s\bar{c} \\ c & \bar{a} \end{bmatrix} : a, c \in K \right\}.$$

Comparing these computations and the proof of "GU(1,1)"-case: we guess:

$$m: \quad D^\times \times \mathrm{Res}^K_F \mathbb{G}_m(F) = K^\times \longrightarrow GU(2,0)(F)$$
$$(g, a) \longmapsto ag$$

induces the desired isomorphism. This is indeed true and we invite the to provide a proof.

- We associate D with an algebraic group $\mathbb{D}$ over F:

$$\mathbb{D}(R) := (D \otimes_F R)^\times, \quad \text{for any } F\text{-algebra } R.$$

Then essentially we obtain the conclusion.

<u>More on quaternion algebras</u>: Our simplification on "S" is to make here easier to formulate.

- An algebra D over F is called a <u>quaternion algebra</u> if $\exists\, i, j \in D$ s.t. $1, i, j, ij$ is an F-basis of D and $i^2 = a$, $j^2 = b$, $ji = -ij$ for some $a, b \in F^\times$. Denote $D = \left(\frac{a,b}{F}\right)$.
- Let $K = F(\sqrt{a})$, then D can be described as a matrix group:

$$D \longrightarrow M_2(K) \quad \text{injection, iso on its image.}$$

$$i, j \longmapsto \begin{bmatrix} \sqrt{a} & \\ & -\sqrt{a} \end{bmatrix}, \begin{bmatrix} & b \\ 1 & \end{bmatrix}.$$

$$t + xi + yj + zij \longmapsto \begin{bmatrix} t + x\sqrt{a} & b(y + z\sqrt{a}) \\ y - z\sqrt{a} & t - x\sqrt{a} \end{bmatrix}.$$

Then it is direct to see that our D defined $(\star)$ is simply the quaternion algebra $\left(\frac{\Delta, -s}{F}\right)$, where <u>$\Delta \in F$ is such that $K = F(\sqrt{\Delta})$</u>.
(Since K is totally imaginary, $\forall\, \sigma: F \hookrightarrow \mathbb{R}$, $\sigma(\Delta) < 0$)

- Local properties :

(a) The formation of quaternion algebras respects base change : $\left(\frac{a,b}{F}\right) \otimes_F E = \left(\frac{a,b}{E}\right)$

(b) Classification over $\mathbb{R}$: $B = \left(\frac{a,b}{\mathbb{R}}\right) \simeq M_2(\mathbb{R})$ or $\underbrace{\mathbb{H}}$ Hamiltonian algebra

* $B \cong \mathbb{H}$ iff $a<0$ and $b<0$. [Voight, ex 2.4.4]

Let B be a quaternion algebra over F. Then B is said to be

- ramified at $\sigma : F \hookrightarrow \mathbb{R}$ if $B \otimes_{F,\sigma} \mathbb{R} \simeq \mathbb{H}$.
- split at ———————— $\simeq M_2(\mathbb{R})$.

Then B is called totally definite if B is ramified at all infinite places.

$\rightsquigarrow$ It follows immediately that our D is a totally definite quaternion algebra.

(c) Classification over finite places : $B = \left(\frac{a,b}{F_v}\right) \simeq \underset{\text{split}}{M_2(\mathbb{Q}_p)}$ or $\underset{\text{ramified}}{\text{division algebra}}$

computing Hilbert symbols.

Let $\mathrm{Ram}(B)$ = set of ramified places. Then :

* $\mathrm{Ram}(B)$ is a finite set of even cardinality.
* $\mathrm{Ram}(B)$ determines B uniquely up to isomorphism. ($\overset{1:1}{\longleftrightarrow} D \in \mathbb{Z}_{>0}$ squarefree)

$\rightsquigarrow$ · If $[F:\mathbb{Q}]$ = odd, there are oddly many ramified finite places. ($F=\mathbb{Q}$)

· If $[F:\mathbb{Q}]$ = even, —— evenly ———————— .

⚠ So $GU(2,0)$ does not cover all possible (coherent) quaternion algebras !

(There are $\mathbb{B}$ quaternion algebra over $\mathbb{A}_F$ s.t. $\mathrm{Ram}(\mathbb{B})$ is odd, then $\mathbb{B}$ is not from any quaternion algebra B over F (i.e. $\mathbb{B} \neq B \otimes_F \mathbb{A}_F$). In this case, $\mathbb{B}$ is called an incoherent quaternion algebra.)

④ Mixed signature case : How to "solve" the problem in ⚠ ?

$\rightsquigarrow$ We consider the unitary group $U(V,\phi)$ of mixed signature

$$\mathcal{P} := \{ \underbrace{(2,0)}_{\substack{\text{ramified} \\ \text{quaternion alg.}}}, \underbrace{(1,1)}, \cdots, \underbrace{(1,1)} \} \; / \; F.$$

split quaternion alg "GL_2" (rank 1)

This can be used to study the arithmetic of elliptic curves over F (not nec. $\mathbb{Q}$).

§ 3.3 Hermitian symmetric domain

Ref : Kai-Wen Lan « An example-based introduction to Shimura varieties »

Recall $\mathrm{Res}^F_{\mathbb{Q}}\, GU(\mathbb{R}) = \prod_{\sigma\in\Sigma} GU(r_\sigma, s_\sigma)(\mathbb{R})$. So we now deal with one individual piece.

Σ : set of real places of F

We only consider $U(r,s)(\mathbb{R})$ as a start :

- Unbounded realization

$$X_{r,s} := \left\{ \tau = \begin{pmatrix} x \\ y \end{pmatrix} \in \overbrace{M_s(\mathbb{C}) \times M_{r-s,s}(\mathbb{C})}^{M_{r\times s}(\mathbb{C}) \,=} \;\middle|\; -i \begin{pmatrix} \bar{x} \\ \bar{y} \\ 1 \end{pmatrix}^t J_{r,s} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = -i(x^* - x + y^* S y) < 0 \right\}$$

Then $g = \begin{pmatrix} a & b & c \\ g & e & f \\ h & \ell & d \end{pmatrix} \in U(r,s)(\mathbb{R})$ acts on $X_{r,s}$ by

$$g\cdot\tau = \begin{pmatrix} a & b & c \\ g & e & f \\ h & \ell & d \end{pmatrix}\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} ax+by+c \\ gx+ey+f \\ hx+\ell y+d \end{pmatrix} \sim \begin{pmatrix} ax+by+c \\ gx+ey+f \\ 1 \end{pmatrix} \cdot (hx+\ell y+d)^{-1} \in X_{r,s}$$

One checks :

- The action is well-defined : $g\cdot\tau \in X_{r,s}$.
- In the case $\boxed{U(1,1)}$:

$$X_{1,1} = \left\{ \tau = x \in \mathbb{C} : -i(x^* - x) < 0 \right\}$$

$x = a+bi$

$x^* = a-bi$

$-i(x^*-x) = -2b < 0 \Rightarrow b > 0$

$\rightsquigarrow X_{1,1} = \mathbb{H}$ upper half plane.

We can regard $X_{r,s}$ as a generalized upper half plane :

- Let $\mathrm{Herm}_s(\mathbb{C}) := \{ g \in M_s(\mathbb{C}) \mid g^* = g \}$.
- We have a canonical isomorphism

$$\begin{aligned} M_s(\mathbb{C}) &\xrightarrow{\sim} \mathrm{Herm}_s(\mathbb{C}) \otimes_{\mathbb{R}} \mathbb{C} \\ X &\longmapsto \mathrm{Re}(X) + i\,\mathrm{Im}(X) \end{aligned}$$

where $\mathrm{Re}(X) = \frac{1}{2}(X+X^*)$ and $\mathrm{Im}(X) = \frac{1}{2i}(X+X^*)$, called Hermitian real / imaginary parts.

Note : This is not the "entry-wise" real / imaginary part : for $X = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in M_2(\mathbb{C})$,

$$\mathrm{Re}(X) = \frac{1}{2}\left(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} + \begin{pmatrix} \bar{\alpha} & \bar{\gamma} \\ \bar{\beta} & \bar{\delta} \end{pmatrix} \right) = \begin{pmatrix} \mathrm{Re}\,\alpha & \frac{1}{2}(\beta+\bar{\gamma}) \\ \frac{1}{2}(\gamma+\bar{\beta}) & \mathrm{Re}\,\delta \end{pmatrix}$$

and similar for the imaginary part.

Then we can rewrite the condition

$$-i(x^* - x + y^* S y) < 0 \iff \mathrm{Im}(x) > \frac{1}{2} y^*(-iS) y .$$

- The action of $U(r,s)(\mathbb{R})$ on $X_{r,s}$ is transitive:

$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

$$\wr$$

$$\begin{pmatrix} x' \\ 0 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & * & * \\ & 1 & -y \\ & & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} x' \\ 0 \\ 1 \end{pmatrix}$$

$\Longrightarrow$ x' satisfies $\mathrm{Im}(x') > 0$. Here there exists some $A \in GL_s(\mathbb{C})$ s.t. $\mathrm{Im}(x') = A^* A$.

$$\wr$$

$$\begin{pmatrix} i\,\mathrm{Im}(x') \\ 0 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & & -\mathrm{Re}(x') \\ & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} x' \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} i\,\mathrm{Im}(x') \\ 0 \\ 1 \end{pmatrix}$$

$$\wr$$

$$\begin{pmatrix} i 1_s \\ 0 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} A^{-*} & & \\ & 1 & \\ & & A \end{pmatrix}\begin{pmatrix} i\,\mathrm{Im}(x') \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} i A^{-*}\mathrm{Im}(x') \\ 0 \\ A \end{pmatrix} \sim \begin{pmatrix} i A^{-*}\mathrm{Im}(x') A^{-1} \\ 0 \\ 1 \end{pmatrix}$$

Check: The above "transition matrices" indeed lies in the group $U(r,s)(\mathbb{R})$.

- The stablizer of $\begin{pmatrix} i1_s \\ 0 \end{pmatrix}$ under this action: we use the bounded realization:
  - Let $1_{r,s} = \begin{pmatrix} 1_r & \\ & -1_s \end{pmatrix}$ as an Hermitian matrix. Define the (bounded realization) unitary group

    $$GU(r,s)(\mathbb{R})' := \{ g \in GL_{r+s}(\mathbb{C}) \mid g 1_{r,s} g^* = \lambda(g) 1_{r,s},\ \lambda(g) \in \mathbb{R}^\times \}$$

    and $U(r,s)(\mathbb{R})' := \ker(\lambda)$.
  - The two Hermitian matrices $1_{r,s}$ and $J_{r,s}$ has the following relation:
    * As $i^{-1}S > 0$, there exists some Hermitian matrix $T \in GL_n(\mathbb{C})$ s.t. $i^{-1}S = T^*T$.
    * One checks for $\mathcal{C} := \begin{pmatrix} \frac{1}{\sqrt{2}} & & \frac{1}{\sqrt{2}} \\ & T^{-1} & \\ \frac{i}{\sqrt{2}} & & -\frac{i}{\sqrt{2}} \end{pmatrix}$, $\quad -i\,\mathcal{C}^* J_{r,s}\, \mathcal{C} = \begin{pmatrix} 1_s & & \\ & 1_{r-s} & \\ & & -1_s \end{pmatrix}$

    Then we have an isomorphism

    $$\Phi : U(r,s)(\mathbb{R}) \longrightarrow U(r,s)(\mathbb{R})' \qquad g \longmapsto \mathcal{C}^{-1} g\, \mathcal{C}$$

    The matrix / operation is called the Cayley transform.

- Consider the "bounded region":

$$\mathcal{D}_{r,s} := \left\{ u \in M_{r,s}(\mathbb{C}) : \begin{pmatrix} \bar{u} \\ 1 \end{pmatrix}^t 1_{r,s} \begin{pmatrix} u \\ 1 \end{pmatrix} = u^* u - 1_s < 0 \right\}$$

and $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in U(r,s)(\mathbb{R})$ acts on $\mathcal{D}_{r,s}$ by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} u \\ 1 \end{pmatrix} = \begin{pmatrix} Au+B \\ Cu+D \end{pmatrix} \sim \begin{pmatrix} (Au+B)(Cu+D)^{-1} \\ 1 \end{pmatrix}$$

Then there is a Φ-equivariant isomorphism

$$\Phi^b : X_{r,s} \xrightarrow{\sim} \mathcal{D}_{r,s} : \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \longmapsto e^{-1}\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \sim \begin{pmatrix} u \\ 1 \end{pmatrix},$$

mapping the distinguished point $\begin{pmatrix} i1_s \\ 0 \end{pmatrix} \in X_{r,s}$ to $\begin{pmatrix} 0 \\ 1 \end{pmatrix} \in \mathcal{D}_{r,s}$.

- So we can compute the stablizer of $\begin{pmatrix} 0 \\ 1 \end{pmatrix} \in \mathcal{D}_{r,s}$ now:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} B \\ D \end{pmatrix} \sim \begin{pmatrix} BD^{-1} \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \iff B = 0.$$

Then <u>automatically</u> $C = 0$ (<u>in fact</u>, by assumption

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} 1_r & \\ & -1_s \end{pmatrix}\begin{pmatrix} A^* & C^* \\ B^* & D^* \end{pmatrix} = \begin{pmatrix} A & -B \\ C & -D \end{pmatrix}\begin{pmatrix} A^* & C^* \\ B^* & D^* \end{pmatrix} = \begin{pmatrix} AA^*-BB^* & AC^*-BD^* \\ CA^*-DB^* & CC^*-DD^* \end{pmatrix} = \begin{pmatrix} 1_r & \\ & -1_s \end{pmatrix}$$

implies $\begin{cases} AA^* - BB^* = 1_r \\ AC^* - BD^* = 0 \\ CC^* - DD^* = -1_s \end{cases}$. Then once $B = 0$: $\begin{cases} AA^* = 1_r \\ AC^* = 0 \\ CC^* - DD^* = -1_s \end{cases}$,

forcing $C = 0$, and $AA^* = 1_r$, $DD^* = 1_s$.)

Moreover, by above computation, $A \in U(r,0)(\mathbb{R})$, $D \in U(s,0)(\mathbb{R})$.

Hence $\mathrm{Stab}_{U(r,s)(\mathbb{R})'}\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) \cong U(r,0)(\mathbb{R}) \times U(s,0)(\mathbb{R})$.

Therefore, via Φ^b, $X_{r,s} \simeq \mathcal{D}_{r,s} = U(r,s)(\mathbb{R}) / U(r,0)(\mathbb{R}) \times U(s,0)(\mathbb{R})$.

- <u>Note on the similitude</u>: Throughout, if we consider the similitude,
  - $X_{r,s}$ should be replaced by $X_{r,s}^{\pm} :=$

    $$\left\{ \tau = \begin{pmatrix} x \\ y \end{pmatrix} \in M_s(\mathbb{C}) \times M_{r-s,s}(\mathbb{C}) \;\middle|\; -i(x^* - x + y^* S y) \text{ is } \begin{matrix}\text{either positive definite or} \\ \text{negative definite.}\end{matrix} \right\}$$

    The connected component containing $\begin{pmatrix} i1_s \\ 0 \end{pmatrix}$ is still $X_{r,s}$.
  - In the bounded realization, the stablizer of $\begin{pmatrix} 0 \\ 1 \end{pmatrix} \in \mathcal{D}_{r,s}$ should be

    $$GU(r,0)(\mathbb{R}) \underset{\mathbb{R}^\times}{\times} GU(s,0)(\mathbb{R}) := \{ (A,D) \in GU(r,0)(\mathbb{R}) \times GU(s,0)(\mathbb{R}) : \lambda(A) = \lambda(D) \}$$

    This can be seen from the calculation above.

- Back to the original unitary group

  $$\mathrm{Res}_{\mathbb{R}}^{F_\infty} GU(r,s)(\mathbb{R}) = \overset{\text{Wan prefers this notation}}{\overbrace{GU(r,s)(F_\infty)}} = \prod_{\sigma \in \Sigma} GU(r,s)(\mathbb{R}),$$

  we should consider the Hermitian symmetric domain as (by abuse of notation)

  $$X_{r,s} := \prod_{\sigma \in \Sigma} X_{r,s}$$

  $$\overset{\text{identify}}{=\!=} \left\{ \tau = \begin{pmatrix} x \\ y \end{pmatrix} \in M_s(\mathbb{C}^\Sigma) \times M_{r-s,s}(\mathbb{C}^\Sigma) \;\middle|\; -i \begin{pmatrix} \bar{x} \\ \bar{y} \\ 1 \end{pmatrix}^t J_{r,s} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = -i(x^* - x + y^* S y) < 0 \right\}$$

  Here keep in mind that $\mathbb{C}^\Sigma = \underbrace{\mathbb{C} \times \cdots \times \mathbb{C}}_{\#\Sigma \text{ times}}$ and we write the vectors as <u>row</u> vectors. explicitly $x = (x_\sigma)_{\sigma \in \Sigma}$ for $x \in \mathbb{C}^\Sigma$, with usual coordinate-wise addition and scalar multiplication.

§3.4 Shimura varieties of unitary groups

- Let $K \subseteq G(\mathbb{A}_{F,f})$ be a <u>neat</u> compact open subgroup, called the <u>level group</u>. (See [Hsieh, §1.10] : neat $\Leftarrow$ torsion-free).

Put

$$\boxed{M_G(K) := G(F)^+ \backslash X^+ \times G(\mathbb{A}_{F,f}) / K}$$

where $G(F)^+ = \{ g \in G(F) \mid \lambda(g) > 0 \}$. Then $M_G(X^+, u)$ is a complex manifold.

* <u>Parametrizing Hodge structures</u> : Define a "distinguished" Hodge structure

$$h_0 : \mathbb{S}(\mathbb{R}) = \mathbb{C}^\times \longrightarrow \mathrm{Res}^F_{\mathbb{Q}} GU(r,s)(\mathbb{R}) = \prod_{\sigma \in \Sigma} GU(r,s)(\mathbb{R})$$

$$x+yi \longmapsto \left(\begin{pmatrix} x & 0 & -y \\ 0 & x+yi & 0 \\ y & 0 & x \end{pmatrix} \right)_{\sigma \in \Sigma}$$

For the bounded version, apply <u>Cayley transformation</u> :

$$h_0' : \mathbb{S}(\mathbb{R}) = \mathbb{C}^\times \longrightarrow \mathrm{Res}^F_{\mathbb{Q}} GU(r,s)(\mathbb{R}) = \prod_{\sigma \in \Sigma} GU(r,s)'(\mathbb{R})$$

$$x+yi \longmapsto \left(\begin{pmatrix} x+yi & \\ & x-yi \end{pmatrix} \right)_{\sigma \in \Sigma}$$

Then via h_0', one checks (more easily) that $Z_{\mathrm{Res}\cdots}(h_0) = \prod_{\sigma \in \Sigma} \left(GU(r,0) \times_{\mathbb{G}_m} GU(s,0) \right)(\mathbb{R})$

Therefore, $X^{\pm}_{r,s} = \mathrm{Res}^F_{\mathbb{Q}} GU(r,s)(\mathbb{R}) \cdot h_0$.

* <u>Upshot</u> : $\left(\mathrm{Res}^F_{\mathbb{Q}} GU(r,s),\ X^{\pm}_{r,s} \right)$ is a Shimura datum (satisfying (SV1)—(SV3) in [Milne])

<u>Approach</u> : This is actually of PEL-type. One imitates the proof in [Milne] there.

① <u>Descent to reflex fields</u> :

Here comes the magic word "<u>Shimura variety</u>": the complex manifold $M_G(K)$ is "<u>algebraic</u>": it is the $\mathbb{C}$-point of a variety $Sh_G(K)$ over a <u>number field E</u> :

$$Sh_G(K)(\mathbb{C}) = M_G(K) \qquad \text{reflex field of } (G,K)$$

The E-variety $Sh_G(K)$ is called the <u>Shimura variety</u> of G of <u>level</u> K.

<u>Method</u> : • Realize $M_G(K)$ as a moduli space of abelian varieties $/\mathbb{C}$ with extra "PEL-datum"

• This moduli interpretation can be descend to the reflex field E.

But in the following up talks, we will treat the explicit construction of $Sh_G(K)$ as a black box.

② <u>Descend to integral models</u> :

<u>Problem</u> : We hope to make $Sh_G(K)$ "integral", not just "algebraic" to consider the arithmetic properties :

- $O_E \subseteq E$ be the ring of integers of E, the reflex field.
- Does there exists a good <u>integral model</u> of $Sh_K(G,X)$ over $O_{E,(\mathfrak{p})}$ for some prime ideal $\mathfrak{p} \subseteq O_E$? i.e.

$$\begin{tikzcd} Sh_G(K) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathscr{S}_G^{(p)}(K) \arrow[d] \\ \mathrm{Spec}\, E \arrow[r] & \mathrm{Spec}\, O_{E,(\mathfrak{p})} := \mathrm{Spec}(O_E \otimes \mathbb{Z}_{(p)}). \end{tikzcd}$$

integral model

<u>Solution</u> : We have extra assumption on $K = K_p K^p$, $K_p \subseteq G(\mathbb{Q}_p)$, $K^p \subseteq G(\mathbb{A}^{(p)})$.

• K_p should be <u>hyperspecial</u> : "$\exists$ model $\mathcal{G}/\mathbb{Z}_p$ of $G/\mathbb{Q}_p$" such that $K_p = \mathcal{G}(\mathbb{Z}_p)$.

In this case, the integral model $\mathscr{S}_G^{(p)}(K_pK^p)$ exists.

<u>Remark</u> : For general Shimura datum (G,X), the existence of integral model is still a conjecture (of Langlands, Milne, Moonen). It is solved at least in our case of PEL-type Shimura varieties. (?)

<u>Sum up</u> : So far we have discussed $Sh_G(K)$ from the $\mathbb{C}$-level to E-level, then to $O_{E,(\mathfrak{p})}$. This is an analogue of $Y(K) = Y(N), Y_0(N)$ or $Y_1(N)$ depending on the level group. $\rightsquigarrow$ Where is $X(K)$, the <u>compactified modular curve</u> ?

§4 Compactifications

§4.1 Minimal compactification

<u>Philosophy</u>: On the complex analytic level, we add some "<u>rational boundary component</u>" to $X_{r,s}$, i.e. $X^*_{r,s} = X_{r,s} \cup \left(\bigcup_{1 \le t \le s} G(\mathbb{Q}) \cdot X_{(t)} \right)$

<u>Recall</u>: When discussing the Hermitian symmetric domain, we use

$$\mathrm{Res}^{E}_{\mathbb{Q}} GU(\mathbb{R}) = \prod_{\sigma \in \Sigma} GU(r,s)(\mathbb{R}),$$

Then we only focus on one component $GU(r,s)(\mathbb{R})$, which is precisely the unitary group defined in <u>Lan's "example note"</u>.

Throughout, let $1 \le t \le s$. Let $V_K = Y_K \oplus W \oplus X_K$, decomposed into

$$V_K = Y_K^{(t)} \oplus Y_{K,(t)} \oplus W \oplus X_K^{(t)} \oplus X_{K,(t)}, \quad V_{K,(t)} := Y_{K,(t)} \oplus W \oplus X_{K,(t)}$$

where $Y_K^{(t)} := Ky^1 \oplus \cdots \oplus Ky^t$, $Y_{K,(t)} = Ky^{t+1} \oplus \cdots \oplus Ky^s$ and similarly for X.

Then the induced metric on $V_K^{(t)}$ is nothing but $J_{r-t,s-t}$ as

$$J_{r,s} = \begin{pmatrix} & & 1_s \\ & S & \\ -1_s & & \end{pmatrix} = \begin{array}{c} \begin{pmatrix} & & & 1_t & \\ & & & & 1_{s-t} \\ & & S & & \\ -1_t & & & & \\ & -1_{s-t} & & & \end{pmatrix} \\ y^1,\dots,y^t \qquad w^i \qquad x^1,\dots,x^t \end{array} \begin{array}{l} y^1,\dots,y^t \\ \\ w^i \\ x^1,\dots,x^t \\ \\ \end{array}$$

Then $G^{(t)} := GU(V_{K,(t)}) \simeq GU(r-t, s-t)$.

Define a subgroup $P^{(t)}$ of $GU(r,s)$ of the form

$$P^{(t)} = \left\{ g = \begin{array}{c} \begin{pmatrix} X & * & * & * & * \\ & A & E & * & B \\ & F & M & * & G \\ & & & \lambda X^* & \\ & C & H & * & D \end{pmatrix} \\ \underbrace{t \quad s-t}_{s} \quad \underbrace{r-s}_{r-s} \quad \underbrace{t \quad s-t}_{s} \end{array} \begin{array}{l} t \\ s-t \\ r-s \\ t \\ s-t \\ \\ \end{array} \in GU(r,s) \;\middle|\; \begin{array}{l} \ell_1(g) := X \in \mathrm{Res}^K_F GL_t \\ \ell_2(g) := \begin{pmatrix} A & E & B \\ F & M & G \\ C & H & D \end{pmatrix} \in G^{(t)}, \\ \lambda = \nu(\ell_2(g)) \end{array} \right\}$$

<u>One checks</u>: $P^{(t)}$ is the stablizer of the totally isotropic K-subspace $Y_K^{(t)}$.

Then by the <u>classification of parabolics</u> (cf. [Shi97, §2]), $P^{(t)}$ is a maximal parabolic subgroup of G. When $t=1$, $P^{(1)}$ is called the <u>Klingen parabolic subgroup</u> of G.

More on $P^{(t)}$

(1) Levi decomposition : $P^{(t)} = L^{(t)} \ltimes N^{(t)}$, where

- $$L^{(t)} = \left\{ g = \begin{pmatrix} X & & & & \\ & A & E & & B \\ & F & M & & G \\ & & & \lambda X^{*} & \\ & C & H & & D \end{pmatrix} : \begin{array}{l} \ell_1(g) := X \in \mathrm{Res}^K_F GL_t \\ \ell_2(g) := \begin{pmatrix} A & E & B \\ F & M & G \\ C & H & D \end{pmatrix} \in G^{(t)} \\ \lambda := \lambda(\ell_2(g)) \end{array} \right\} \simeq \mathrm{Res}^K_F GL_t \times G^{(t)}$$

- $$N^{(t)} = \left\{ \begin{pmatrix} 1 & * & * & * & * \\ & 1 & & * & \\ & & 1 & * & \\ & & & 1 & \\ & & & * & 1 \end{pmatrix} \in GU(r,s) \right\}$$ is the unipotent radical.

(2) Moreover, $N^{(t)}$ has a further extension as

$$1 \longrightarrow W^{(t)} \longrightarrow N^{(t)} \longrightarrow V^{(t)} \longrightarrow 1$$

- $$W^{(t)} := \left\{ \begin{pmatrix} 1 & & & Y & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix} : Y \in \mathrm{Res}^K_F \mathrm{Herm}_t \right\} \simeq \mathrm{Res}^K_F \mathrm{Herm}_t$$

  One checks : $W^{(t)}$ is the centre of $N^{(t)}$.

- $V^{(t)}(R) \simeq (K \otimes_F R)^{(r-t)+(s-t)}$

Now we restricts to the unitary case. Consider for $1 \leq t \leq s$

$$X^{+}_{(t)} := \begin{pmatrix} \infty_r & \\ & X_{r-t,s-t} \end{pmatrix} = \left\{ \begin{array}{c} \\ \end{array} \begin{array}{c|c} t & s-t \\ \hline \begin{pmatrix} 1_t & \\ & x \\ & y \\ 0_t & \\ & 1_{s-t} \end{pmatrix} & \end{array} \begin{array}{l} t \\ s-t \\ r-s \\ t \\ s-t \end{array} \;\; \underbrace{}_{s} \;\; s+r \; : \; \begin{pmatrix} x \\ y \end{pmatrix} \in X^{+}_{r-t,s-t} \right\} \simeq X^{+}_{r-t,s-t}$$

So $U(r,s)(\mathbb{R})$ can act on this thing

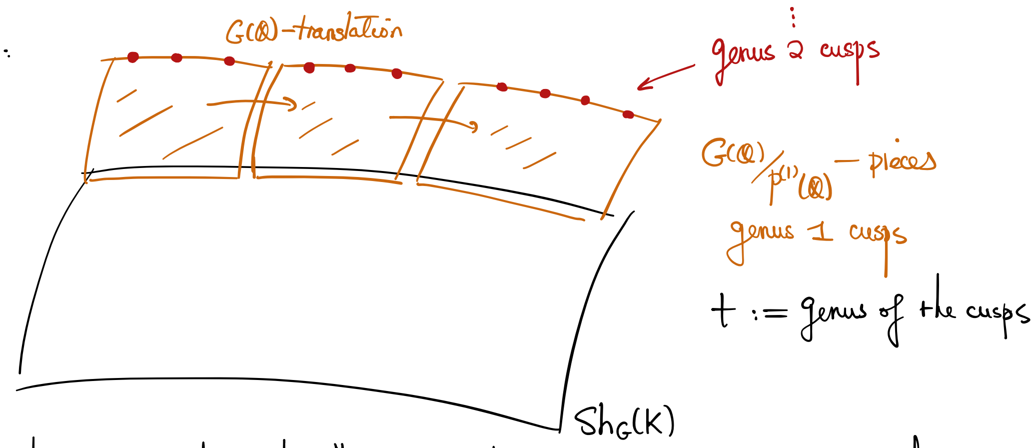
As $U(r,s)(\mathbb{R})$ acts on $X_{(t)}$ by left multiplication (under projective coordinate), one checks $X^{(t)}$ is stablized by $P^{(t)}(\mathbb{Q})$. Then on the complex analytic level,

$$X^{*}_{r,s} = X_{r,s} \cup \Big(\bigcup_{1 \leq t \leq s} G(\mathbb{Q}) \cdot X^{+}_{(t)} \Big)$$

here G is the unitary group without similitude.

as the minimal compactification of $X^{+}_{r,s}$.

An illustration :



$\rightsquigarrow$ Upshot : There is a "hierachey" among the rational boundary components, hence when it comes to modular forms, there are also "hierachies" :

$$\underbrace{\{\text{cusp forms}\}}_{\substack{\text{vanishing at all}\\ \text{boundaries}}} \subseteq \underbrace{\{\text{genus 1 modular form}\}}_{\substack{\text{vanishing on boundaries}\\ \text{of genus} \geq 2}} \subseteq \quad \cdots$$

$\nwarrow$ where our Klingen Eisenstein series lives.

I'm making this remark to understand why we only consider $P^{(1)}$-Eisensries but not higher.

Exercise : Can we recover the construction $Y(N) \hookrightarrow X(N)$ in the case of the Shimura variety of $U(1,1)$?

Problem : • The $\mathbb{C}$-story is already so complicated. Then what about the algebrai story of compactification ?

• In the baby case $Y(N) \hookrightarrow X(N)$, $\{\text{cusps}\}$ has large codimension. But here we are gluing a large dimension piece to $X^+_{r,s}$
$\rightsquigarrow$ does not look like a compactification at all.

• We are lazy on the topology of X^*_{rs}.

This compactification is called Baily-Borel-Satake compactification.

§4.2 Toroidal compactification

- Cusp labels : Let $1 \le t \le s$, define

means $\mathrm{Res}^{K}_{F} GL_t$ in the previous section

$$\mathcal{C}_t(K) = \Big(\big(\boxed{GL(Y_{K,(t)})} \ltimes G^{(t)}\big)(\mathbb{A}_{F,f}) \Big) N^{(t)}(\mathbb{A}_{F,f}) \Big\backslash G(\mathbb{A}_{F,f}) \Big/ K$$

The set is finite, called the set of cusp labels for $S_G(K)$

Exercise : For each $[g] \in \mathcal{C}_t(K)$, we can choose $g = pk^\circ$ for $p \in P^{(t)}(\mathbb{A}_{F,f}^{(pN_0)})$ and $k^\circ \in K^\circ$ with similitude $\lambda(g) \in \widehat{\mathcal{O}_F}$.

One writes $\mathcal{C}(K) := \bigsqcup_{1 \le t \le s} \mathcal{C}_t(K)$.

- $\boxed{\text{Black box}}$ To a datum of "smooth rational cone decomposition" $\{\xi_{[g]}\}_{[g] \in \mathcal{C}(K)}$ of F_+, we attach a toroidal compactification $\overline{S}^{(p)}_G(K)/\mathcal{O}$ of $S^{(p)}_G(K)/\mathcal{O}$.
  - $\overline{S}^{(p)}_G(K)/\mathcal{O}$ is a proper smooth scheme over $\mathcal{O}$ containing $S^{(p)}_G(K)/\mathcal{O}$
  - The complement $\overline{S}^{(p)}_G \setminus S^{(p)}_G$ is a relative Cartier divisor with normal crossings.
  - Let $\underline{\mathcal{A}} = (\mathcal{A}, \lambda, \iota, \overline{\eta}^{(p)})$ be the universal quadruple over $S^{(p)}_G(K)$. Then $\underline{\mathcal{A}}$ has an extension $\underline{\mathcal{G}} = (\mathcal{G}, \lambda, \iota, \overline{\eta}^{(p)})$ where
    - * $\mathcal{G}$ is a semiabelian scheme over $\overline{S}^{(p)}_G(K)$, $\mathcal{G}|_{S^{(p)}_G} = \mathcal{A}$.
    - * $\lambda : \mathcal{G} \longrightarrow \mathcal{G}^\vee$ a "$\mathbb{Z}_{(p)}$-polarization"
    - * $\overline{\eta}^{(p)}$ is the level structure in $\underline{\mathcal{A}}$.

  Moreover, $\underline{\omega_{\mathcal{G}}} := e^* \Omega_{\mathcal{G}/\overline{S}^{(p)}_G}$, the sheaf of invariant differential, is a locally free coherent $\mathcal{O}_{\overline{S}^{(p)}_G}$-module.

- The minimal compactification $S^{*,(p)}_G(K)/\mathcal{O}$ is defined as

$$S^{*,(p)}_G(K) := \mathrm{Proj} \bigoplus_{k=0}^{\infty} \Gamma(\overline{S}^{(p)}_G(K), \det \underline{\omega}^{k})$$

Then there is a commutative diagram of $\mathcal{O}$-schemes

$$\begin{tikzcd}
 & \overline{S}^{(p)}_G(K) \arrow[d, "\pi"] \\
S^{(p)}_G(K) \arrow[ur, "\overline{i}"] \arrow[r, "i"'] & S^{*,(p)}_G(K)
\end{tikzcd}$$

$i, \overline{i}$: open immersion

π : blow-down map.

Theorem (Lan Kai-wen)

(1) $\pi_*(\det \underline{\omega})$ is an ample line bundle on S^*, and S^* is a normal projective scheme of finite type over $\mathcal{O}$.

(2) $\pi_* \mathcal{O}_{\overline{S}} = \mathcal{O}_{S^*}$, hence π has geometrically connected fibres.

(3) $S^*(\mathbb{C})$ is the classical Satake–Baily–Borel compactifications.

(4) $\exists$ a natural stratification of $\partial S^* := S^* \setminus S$ indexed by $\underline{C}(K)$:

$$\partial S^* = \bigsqcup_{1 \leq t \leq s} \bigsqcup_{[g]_t \in \underline{C}_t(K)} S^{(p)}_{G^{(t)}}(K^g_{(t)}), \quad K^g_{(t)} = G^{(t)}(\mathbb{A}_{F,f}) \cap gKg^{-1}.$$

So the rational boundary components are indeed Shimura varieties of smaller unitary groups.