§1 Basic AG mod p : a preparation

Fix a prime number p.

Let S be a $\mathbb{F}_p$-scheme : $\forall U \underset{\text{open}}{\subseteq} S$, $\Gamma(U, \mathcal{O}_S)$ is a ring of characteristic p.

Defn : The absolute Frobenius of S is the morphisms of $\mathbb{F}_p$-schemes $\mathrm{Frob}_S : S \to S$ which is
- identity on the underlying topological spaces
- $\mathrm{Frob}_S^\# : \mathcal{O}_S \to \mathcal{O}_S$ is given by $a \mapsto a^p$ on $\Gamma(U, \mathcal{O}_S)$.

(it is a morphism of rings b/c $\Gamma(U, \mathcal{O}_S)$ is of char p.)

Remark :

(1) When $S = \mathrm{Spec}\, R$ is affine, Frob_S is merely $a \mapsto a^p$ on R.

(2) Let X be an S-scheme, then the diagram commutes

$$\begin{tikzcd} X \arrow[r, "\mathrm{Frob}_X"] \arrow[d] & X \arrow[d] \\ S \arrow[r, "\mathrm{Frob}_S"'] & S \end{tikzcd} \quad — (*)$$

Problem : Frob_X is not an S-morphism.

So we need a relative version of Frob_X.

Defn
- Define the pullback $X^{(p)}$ as

$$\begin{tikzcd} X^{(p)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & X \arrow[d, "\pi"] \\ S \arrow[r, "\mathrm{Frob}_S"'] & S \end{tikzcd}$$

- Then recall the diagram $(*)$, we have a unique $F_{X/S} : X \to X^{(p)}$ making the diagram commute :

$$\begin{tikzcd} X \arrow[dr, dashed, "F_{X/S}"] \arrow[drr, bend left, "\mathrm{Frob}_X"] \arrow[ddr, bend right, "\pi"'] & & \\ & X^{(p)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & X \arrow[d, "\pi"] \\ & S \arrow[r, "\mathrm{Frob}_S"'] & S \end{tikzcd}$$

called the relative Frobenius of X over S.

- The relative Frobenius can be iterated as follows :

$$\cdots \Longleftarrow \begin{tikzcd} X^{(p)} \arrow[dr, "F_{X^{(p)}/S}"] \arrow[drr, bend left, "\mathrm{Frob}_{X^{(p)}}"] \arrow[ddr, bend right, "\pi_p"'] & & \\ & X^{(p^2)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & X^{(p)} \arrow[d, "\pi_p"] \\ & S \arrow[r, "\mathrm{Frob}_S"'] & S \end{tikzcd} \Longleftarrow \begin{tikzcd} X \arrow[dr, "F_{X/S}"] \arrow[drr, bend left, "\mathrm{Frob}_X"] & & \\ & X^{(p)} \arrow[r] \arrow[d, "\pi_p"'] \arrow[dr, phantom, "\ulcorner", very near start] & X \arrow[d, "\pi"] \\ & S \arrow[r, "\mathrm{Frob}_S"'] & S \end{tikzcd}$$

Then $F^n_{X/S} := F_{X^{(\varphi^{n-1})}/S} \circ \cdots \circ F_{X^{(\varphi)}/S} \circ F_{X/S} : X \longrightarrow X^{(\varphi^n)}$

<u>Remark</u> :

(1) Composing the diagrams above, we see the following square is Cartesian

$$\begin{tikzcd} X^{(\varphi^n)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & X \arrow[d, "\pi"] \\ S \arrow[r, "\mathrm{Frob}^n_S"'] & S \end{tikzcd}$$

iterated Frobenius

but $F^n_{X/S}$ is <u>not</u> defined <u>directly</u> using the universal property of this square !

(2) Assume $X = \operatorname{Spec} B$ and $S = \operatorname{Spec} R$. Then we have two descriptions of $F_{X/S}$:

$\boxed{\text{Des 1}}$ Naturally B is an R-algebra, written as $B = \dfrac{R[X_i : i \in I]}{(f_j : j \in J)}$.

- For $f \in R[X_i : i \in I]$, $f = \sum_{\nu \in \mathbb{N}^{(I)}} a_\nu X^\nu$, define

$$f^{(\varphi)} := \sum_{\nu \in \mathbb{N}^{(I)}} a_\nu^p X^\nu \quad (\text{raising the coefficient, not the variables})$$

- Recall $X^{(\varphi)} = \operatorname{Spec}(B^{(\varphi)})$, where $B^{(\varphi)} = B \otimes_{R, \mathrm{Frob}_R} R$, as an R-algebra.
  (that is : $a_\nu X^\nu \otimes 1 = X^\nu \otimes a_\nu^p$ in $B^{(\varphi)}$)

  Then one <u>checks</u> :

  * $B^{(\varphi)} = \dfrac{R[X_i : i \in I]}{(f_j^{(\varphi)} : j \in J)}$ as in the following diagram :

$$\begin{tikzcd} B^{(\varphi)} = B \otimes_{R, \mathrm{Frob}_R} R \arrow[dr, phantom, "+"] & B \arrow[l, "\sigma_{B/R}"'] \\ R \arrow[u] & R \arrow[l, "\mathrm{Frob}_R"] \arrow[u] \end{tikzcd}$$

  * The map $\sigma_{B/R} : B \longrightarrow B^{(\varphi)}$ is the map $f \longmapsto f^{(\varphi)}$.

- Then

$$\begin{tikzcd} B & & B \arrow[ll, bend right, "\mathrm{Frob}_B"'] \arrow[dl, "\sigma_{B/R}"] \\ & B^{(\varphi)} \arrow[ul, "F_{B/R}"] & \end{tikzcd} \quad \overset{\text{翻译}}{\Longleftrightarrow} \quad \begin{tikzcd} X \arrow[dr, dashed, "F_{X/S}"'] \arrow[drr, bend left, "\mathrm{Frob}_X"] & & \\ & X^{(\varphi)} \arrow[r, "\sigma_{X/S}"'] & X \end{tikzcd}$$

as

$$\begin{tikzcd}[column sep=large] B \arrow[r, "\sigma_{B/R}"] & B^{(\varphi)} \arrow[r, "F_{B/R}"] & B \\ \sum a_\nu X^\nu \arrow[r, mapsto, "\substack{\text{raising} \\ \text{coefficients}}"'] \arrow[rr, bend right, "\mathrm{Frob}_B \text{ absolute Frobenius}"'] & \sum a_\nu^p X^\nu \arrow[r, mapsto, "\text{relatie Frobenius}", "\substack{\text{raising} \\ \text{variables}}"'] & \sum a_\nu^{(p)} (X^p)^\nu \end{tikzcd}$$

- As a byproduct, from above we see $\sigma_{B/R} \circ F_{B/R} = \mathrm{Frob}_{B^{(p)}}$.
  This implies $F_{X/S} \circ \sigma_{X/S} = \mathrm{Frob}_{X^{(p)}}$.

$\boxed{\text{Des 2}}$ Define

$$S^p(B) \underset{\text{Symmetrization } S}{\overset{\text{inclusion as sym. tensors}}{\rightleftarrows}} T^p(B) = \overbrace{B \otimes_R \cdots \otimes_R B}^{p \text{ times}}$$

$$\sum_{\sigma \in G_p} b_{\sigma(1)} \otimes \cdots \otimes b_{\sigma(p)} \longleftarrow\!\shortmid\; b_1 \otimes \cdots \otimes b_p$$

as a morphism of R-modules.

Check 1: $J := S(T^p(B))$ is an ideal of $S^p(B)$

(indeed, STS $\forall t \in T^p(B)$, $s \in S^p(B)$, $s \cdot S(t) = S(st)$)

Check 2: The well-defined map

$$\varphi_{B/R} : B^{(p)} = B \otimes_{R, \mathrm{Frob}_R} R \longrightarrow S^p(B)/J$$
$$b \otimes a \longmapsto a \cdot (\underbrace{b \otimes \cdots \otimes b}_{p \text{ times}}) \mod J$$

is a morphism of R-algebras.

Proposition: If B is flat over R, then $\varphi_{B/R}$ is an isomorphism of R-algebras.

Pf (Sketch): By devissage, STS B is a free R-module with a basis $(e_i)_{i \in I}$.
Then $(e_{i_1} \otimes \cdots \otimes e_{i_p} \mid (i_1, \cdots, i_p) \in I^p)$ is a basis of $T^p(B)$.
For each $(i_1, \cdots, i_p) \in I^p$, take its stablizer $H \subseteq G_p$ and define

$$s_{i_1, \cdots, i_p} := \sum_{\sigma \in H \backslash G_p} e_{i_{\sigma(1)}} \otimes \cdots \otimes e_{i_{\sigma(p)}}$$

Then such vectors spans $S^p(B)$. Observe:

- $S(e_i \otimes \cdots \otimes e_i) = p!\, s_{i, \cdots, i} = 0$, $\forall i \in I$
- $S(e_{i_1} \otimes \cdots \otimes e_{i_p}) = u s_{i_1, \cdots, i_p}$ for some $u \in R^\times$ if $(i_1, \cdots, i_p) \neq (i_1, \cdots, i_1)$.

Therefore $(e_i \otimes \cdots \otimes e_i \mid i \in I)$ forms a basis of $S^p(B)/J$.
This implies $\varphi_{B/R}$ is an isomorphism. $\square$

Defn Let E and E' be two elliptic curves over a scheme S.

A homomorphism $f : E \longrightarrow E'$ of group schemes $/S$ is called an <u>isogeny</u> if it is surjective, finite and locally free.

- In this case, $\ker f$ is a finite locally free group scheme $/S$ of locally constant rank. If this rank is constant equal to d, we say f has <u>degree</u> d.
- For an isogeny $f : E \to E'$ of degree d, we say an isogeny $g : E' \to E$ of degree d is <u>dual</u> to f if $g \circ f = [d]$.

<u>Example</u> :

- The relative Frobenius $F : E \longrightarrow E^{(p)}$ is an isogeny of degree p.
  - a <u>group homomorphism</u> : F is functorial, compatible with products and base extensions.
  - an <u>isogeny</u> : follows from the expression locally on affines.
- There exists a dual isogeny of F, called the <u>Verschiebung morphism</u> of E over S denoted by $V_{E/S} : E^{(p)} \longrightarrow E$

(<u>Remark</u> : $V_{E/S}$ can be defined for any finite commutative group scheme over $\mathbb{F}_p$. See Pink's note for details)

Describe $V_{E/S}$ locally: $\quad E = \mathrm{Spec}\, B, \quad S = \mathrm{Spec}\, R$.

- By $\boxed{\text{Des 2}}$, we have seen $E^{(p)} = \mathrm{Spec}(B^{(p)}) \xrightarrow{\sim} \mathrm{Spec}(S^p(B)/J)$ via $\varphi_{B/R}$. Moreover one checks that $F_{B/R}$ factors through $\varphi_{B/R}$:

$$\begin{tikzcd} B^{(p)} \arrow[r, "\varphi_{B/R}", "\sim"'] \arrow[rr, bend left, "F_{B/R}"] & S^p(B)/J \arrow[r, "\widetilde{F}_{B/R}"] & B \end{tikzcd}$$

$$b_1 \otimes \cdots \otimes b_p \bmod J \longmapsto b_1 \cdots b_p$$

- Since E is commutative, the comultiplication $m : B \longrightarrow T^p(B)$ factors through $S^p(B)$:

$$\begin{tikzcd} B \arrow[r, "\widetilde{m}"] \arrow[rr, bend right, "m"'] & S^p(B) \arrow[r, hook] & T^p(B) \end{tikzcd}$$

Put them together:

$$\begin{tikzcd} B \arrow[r, "\widetilde{m}"] \arrow[d, dashed, "\exists!\, V_{B/R}"'] & S^p(B) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & T^p(B) \arrow[d, "\Delta \text{ diagonal morphism}"] \\ B^{(p)} \arrow[r, "\varphi_{B/R}", "\sim"'] \arrow[rr, bend right, "F_{B/R}"'] & S^p(B)/J \arrow[r, "\widetilde{F}_{B/R}"] & B \end{tikzcd}$$

Then $V_{B/R} : B \longrightarrow B^{(p)}$ is the unique morphism making the diagram commute.

- Flipping them into the language of schemes: $V_{E/S} \circ F_{E/S} = [p]$.
- Dually, we claim: $F_{E/S} \circ V_{E/S} = [p]$:
  - One checks $V_{E/S}$ in general is compatible with base change. Hence $(V_{E/S})^{(p)} = V_{E^{(p)}/S}$.
  - By functoriality of $F_{E/S}$, the diagram commutes:

$$\begin{tikzcd} E^{(p)} \arrow[r, "F_{E^{(p)}/S}"] \arrow[d, "V_{E/S}"'] \arrow[dr, bend left, "(*)"] & E^{(p^2)} \arrow[d, "(V_{E/S})^{(p)} = V_{E^{(p)}/S}"] \\ E \arrow[r, "F_{E/S}"'] & E^{(p)} \end{tikzcd}$$

Then as we have seen $(*) = [p]$, we obtain $F_{E/S} \circ V_{E/S} = [p]$.

$\square$

§2 Hasse invariant

Let R be an $\mathbb{F}_p$-algebra.

Let $(E/R, \omega)$ be a pair where

- E is an elliptic curve over R.
- ω is a basis of $H^0(\operatorname{Spec} R, \underline{\omega}_{E/R}) = H^0(E, \Omega^1_{E/R})$.

We also take a dual basis $\eta \in H^1(E, \mathcal{O}_E)$ via Serre duality.

Then check: if we replace ω by $\lambda\omega$ for some $\lambda \in R^\times$, then the dual basis becomes $\lambda^{-1}\eta$.

Let $\mathrm{Frob}_E : E \to E$ be the absolute Frobenius. Then we obtain an induced $\mathbb{F}_p$-linear morphism

$$\mathrm{Frob}_E^* : H^1(E, \mathcal{O}_E) \longrightarrow H^1(E, \mathcal{O}_E)$$
$$\eta \longmapsto A(E/R, \omega)\eta, \quad A(E/R, \omega) \in R.$$

Prop: The assignment $(E/R, \omega) \longmapsto A(E/R, \omega) \in R$ is a Katz modular form over $\mathbb{F}_p$ of weight $p-1$ and level 1.

Proof: One checks

$$\mathrm{Frob}_E^*(\lambda^{-1}\eta) = \lambda^{-p}\,\mathrm{Frob}_E^*(\eta) = \lambda^{-p} \cdot A(E/R, \omega)\eta = \lambda^{1-p} A(E/R, \omega)(\lambda^{-1}\eta).$$

Recall $\mathrm{Frob}_E^*(\lambda^{-1}\eta) = A(E/R, \lambda\omega)(\lambda^{-1}\eta)$ by definition, we see

$$A(E/R, \lambda\omega) = \lambda^{1-p} A(E/R, \omega).$$

We left to the reader for that A respects base change.

Note: $N = 1$ is the "trivial" level structure, we simply omit them. □

Defn: We call A the Hasse invariant. (or $A(E/R, \omega)$ the Hasse invariant of $(E/R, \omega)$)

Properties:

(1) The q-expansion of the Hasse invariant is equal to 1 in $\mathbb{F}_p((q))$.

Background: Ancient congruences: for $\boxed{\text{fixed } p \geq 5}$ and any integer $k > 0$

- Clausen–von Staudt congruence: $p-1 \mid 2k \implies pB_{2k} \in \mathbb{Z}_{(p)}$, $pB_{2k} \equiv -1 \bmod p$.

  In particular, $v_p(B_{2k}) = -1$.

- Kummer congruence: $p-1 \nmid 2k \implies \frac{B_{2k}}{2k} \in \mathbb{Z}_{(p)}$ and its residue class mod p depends only on $2k$ mod $p-1$.

Consider the Eisenstein series

$$E_{2k}(z) = 1 - 2\frac{2k}{B_{2k}} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n$$

(another normalization, distinct from Talk 01)

Then in particular:

- $k = \frac{p-1}{2}$, by CS congruence, $v_p\left(2\frac{p-1}{B_{p-1}}\right) = 1$.

  Hence $E_{p-1} \equiv 1 \mod p$.

  $\Longrightarrow$ Upshot: Hasse invariant tells us that "1" is indeed a mod p modular form!

- treat $k = \frac{p+1}{2}$ case via Kummer congruences: see [Gispert, Cor.1.36].

$\Longrightarrow$ We can lift Hasse invariant to coefficient $\mathbb{Q} \cap \mathbb{Z}_p$ when $p \geq 5$:

indeed, we have seen above $E_{p-1} \equiv 1 \mod p$.

(2) Let k be a field of characteristic p, then

$$A(E/k, \omega) = 0 \iff E/k \text{ is a supersingular elliptic curve}.$$

See the sketch of proof in [Hida2004, §3.2.6] or [Hida2011, Prop.2.9.1] for details.

Upshot: When working on p-adic modular forms, we need to "remove supersing. points", this can be done elegantly via Hasse invariant!

Question: In Zhao's lecture, he require R is a field, algebraically closed of char p

But in [Hida11], there is no such requirement?

Exercise: Do sheet 5 in Liang Xiao's 2021 spring seminar.

Another interpretation of Hasse invariant :

Let E/S be an elliptic curve over a scheme S.

- Recall in §1, we see the relative Frobenius $F_{E/S}$ factors as

$$E \xrightarrow{F_{E/S}} E^{(p)} \xrightarrow{V_{E/S}} E \qquad V_{E/S} : \text{Verschiebung morphism.}$$

- Then $V_{E/S}$ induces a morphism

$$H^1_{dR}(E/S) \xrightarrow{V^*_{E/S}} H^1_{dR}(E^{(p)}/S) \xrightarrow[\sim\ (*)]{\text{Exercise}} H^1_{dR}(E/S) \otimes_{\mathcal{O}_S, \mathrm{Frob}_S} \mathcal{O}_S$$

— note : the induced morphism is contravariant. We can regard it as the "pullback" of differential forms along $V_{E/S}$.

- Recall the Hodge-to-Rham exact sequence

$$0 \longrightarrow \pi_* \Omega^1_{E/S} = \omega_{E/S} \longrightarrow H^1_{dR}(E/S) \longrightarrow \mathrm{Lie}(E/S) \longrightarrow 0$$

we see $\omega_{E/S}$ embeds into $H^1_{dR}(E/S)$ as a submodule.

Claim 1: The image of $V^*_{E/S}$ is precisely $\omega_{E^{(p)}/S} \simeq \omega_{E/S} \otimes_{\mathcal{O}_S, \mathrm{Frob}_S} \mathcal{O}_S$.
(the isomorphism is induced from $(*)$, please check it !)

Claim 2 : $\omega_{E^{(p)}/S} \simeq \omega_{E/S}^{\otimes p}$

Then actually $V^*_{E/S}$ restricts to a map $\omega_{E/S} \longrightarrow \omega_{E/S}^{\otimes p}$.

- Apply all these to the universal elliptic curve $\mathbb{E}/Y_1(N)$.

$$A := V^*_{\mathbb{E}/Y_1(N)} \in \mathrm{Hom}_{\mathcal{O}_{Y_1(N)}}\left(\omega_{\mathbb{E}/Y_1(N)}, \omega^{\otimes p}_{\mathbb{E}/Y_1(N)}\right) \cong H^0\left(Y_1(N), \omega^\vee_{\mathbb{E}/Y_1(N)} \otimes \omega^{\otimes p}_{\mathbb{E}/Y_1(N)}\right)$$
$$\cong H^0\left(Y_1(N), \omega^{\otimes (p-1)}_{\mathbb{E}/Y_1(N)}\right)$$

This is called the Hasse invariant as a weight $p-1$ mod p modular form.

Exercise : Quoted from Xiao Liang's 2021 Summer course Exercise 6.2 and note of Lecture 6 (not mentioned in the lecture video)

§3 Ordinary loci of modular schemes

<u>Setup</u>:
- Fix prime $p \geq 5$.
- Let W be the ring of integers of a finite extension of $\mathbb{Q}_p$. $\pi \in W$ be a uniformizer.

  <u>note</u>: for any N coprime to p, N is invertible in W!
- Let $\Gamma = \Gamma(N)$ or $\Gamma_1(N)$, with $N \gg 0$ making the corresponding moduli problem representable. ($N \geq 3$ for $\Gamma(N)$, $N \geq 4$ for $\Gamma_1(N)$)
  - When $\Gamma = \Gamma(N)$, we use the <u>arithmetic level structure</u> and assume $\zeta_N \in W$.
- Let $Y_{\Gamma/W}$ (resp. $X_{\Gamma/W}$) be the base change of modular curve Y_Γ (resp. X_Γ) to W.
- Let $\pi_{univ} : \mathbb{E} \longrightarrow Y_{\Gamma/W}$ be the universal elliptic curve with extended generalized elliptic curve $\tilde{\pi}_{univ} : \widetilde{\mathbb{E}} \longrightarrow X_\Gamma/W$. (by abuse of notation, denote also by $\widetilde{\mathbb{E}}$)

$$\begin{tikzcd}
\widetilde{\mathbb{E}} \arrow[d, "\tilde{\pi}_{univ}"'] & & \mathbb{E} \arrow[d, "\pi_{univ}"'] \arrow[r] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{E} \arrow[d] \\
X_{\Gamma/W} \arrow[drr] & & Y_{\Gamma/W} \arrow[ll, hook'] \arrow[d] \arrow[r] & Y_\Gamma \arrow[d] \\
& & \operatorname{Spec} W \arrow[r] & \operatorname{Spec} \mathbb{Z}[\frac{1}{N}]
\end{tikzcd}$$

Define ordinary locus

- Let $\mathcal{E} \in M_{p-1}(\Gamma, W)$ as a lift of the Hasse invariant A.

  (As $p \geq 5$, such a lift always exists: $\mathcal{E} = E_{p-1}$ works)

  We regard $\mathcal{E}$ as a global section $H^0(Y_\Gamma/W, \underline{\omega}_{\mathbb{E}}^{\otimes(p-1)})$ or $H^0(X_\Gamma/W, \underline{\omega}_{\mathbb{E}}^{\otimes p-1})$.

- Let $Y_\Gamma^{\mathrm{ord}}/W$ (resp. $X_\Gamma^{\mathrm{ord}}/W$) be the open subscheme of Y_Γ/W (resp. X_Γ/W) where the global section $\mathcal{E}$ is invertible. This is called the <u>ordinary locus</u> of Y_Γ/W (resp. X_Γ/W)

  <u>In AG, what does this mean?</u>

<u>Thickening the ordinary locus</u>: for $m \geq 1$, let $W_m := W/\pi^m W$

- Define the base change $S_m^\circ := Y_\Gamma^{\mathrm{ord}} \otimes_W W_m$, $S_m := X_\Gamma^{\mathrm{ord}} \otimes_W W_m$ as the <u>order m (infinitesemal) thickening</u> of Y_Γ^{ord} or X_Γ^{ord}

  <u>note</u>: $S_0^\circ = Y_\Gamma^{\mathrm{ord}}$, $S_0 = X_\Gamma^{\mathrm{ord}}$

  $S_1^\circ = Y_\Gamma^{\mathrm{ord}} \otimes_W \mathbb{F}$, $S_1 = X_\Gamma^{\mathrm{ord}} \otimes_W \mathbb{F}$, ...

- <u>Proposition</u>: S_m° and S_m are affine smooth curves over W_m with geometrically connected fibers.

  <u>Idea</u>: • $\omega_{\mathbb{E}/X_\Gamma}$ is ample $\xRightarrow{\text{[SP 01PV]}}$ X_Γ^{ord} is affine $\Rightarrow$ S_m is affine.
  [Hida04, §3.2.7]
  Similar for S_m°.

  • What about the claim on fibers?

  We denote $S_m = \mathrm{Spec}(V_{m,0})$ for a W_m-flat algebra $V_{m,0}$.

- In this way, we obtain a tower

$$\begin{tikzcd}
\cdots & S_2 \arrow[l] \arrow[d] \arrow[rrr, bend left] & S_1 \arrow[l, "\varphi_1"'] \arrow[d] \arrow[r] & X_\Gamma^{\mathrm{ord}} \arrow[d] \arrow[r, hook] & X_\Gamma \\
\cdots & \mathrm{Spec}\, W_2 \arrow[l] \arrow[rr, bend left] & \mathrm{Spec}\, W_1 \arrow[l] \arrow[r] & \mathrm{Spec}\, W & \\
\cdots \arrow[r] & W/\pi^3 W \arrow[r, "\mathrm{mod}\,\pi"] & W/\pi^2 W \arrow[r, "\mathrm{mod}\,\pi"] & W/\pi W & W \arrow[l]
\end{tikzcd}$$

$\underline{\omega}_{\mathbb{E}}$ on X_Γ^{ord}, $\underline{\omega}_{\mathbb{E}}$ on X_Γ

(similarly for S_m°). Let $\underline{\omega}_m$ be the pullback of $\underline{\omega}_{\mathbb{E}}$ to S_m. Then it gives an inverse system of global sections $\{H^0(S_m, \underline{\omega}_m^{\otimes k}) \longrightarrow H^0(S_{m-1}, \underline{\omega}_{m-1}^{\otimes k})\}_{m \geq 1}$

P.S. How the green arrow is constructed?

$$\begin{tikzcd}
S_1 \arrow[d] \arrow[dr, "\exists ! \varphi_1"] \arrow[drr, bend left] & & \\
\mathrm{Spec}\, W_1 \arrow[dr, "\text{``mod } \pi \text{''}"'] & S_2 \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & X_T^{ord} \arrow[d] \\
& \mathrm{Spec}\, W_2 \arrow[r] & \mathrm{Spec}\, W
\end{tikzcd}$$

- Define $H^0(S_\infty, \underline{\omega}^{\otimes k}) := \varprojlim_m H^0(S_m, \underline{\omega}_m^{\otimes k})$, as the space of false modular forms of weight k. (à la Deligne)

Note: Here S_∞ is not just a "symbol", it is actually the formal scheme of $S = X_T^{ord}/W$ completed at the special fiber $S_1 = X_T^{ord} \otimes_W W/\varpi W$.

(see [GME, example 1.13.3]).

Igusa tower :

- For $n \geq 1$, we have the connected-étale exact sequence of finite flat group schemes
$$0 \longrightarrow \mathbb{E}^{\circ}[p^n] \longrightarrow \mathbb{E}[p^n] \longrightarrow \mathbb{E}^{\text{ét}}[p^n] \longrightarrow 0 \qquad \text{over } S_m^{\circ}$$
where
  - $\mathbb{E}^{\circ}[p^n]$ is the connected component of the identity section of $\mathbb{E}[p^n]$?
  or say the kernel of $[p^n] : \widehat{\mathbb{E}} \longrightarrow \widehat{\mathbb{E}}$ as formal schemes.
  - $\mathbb{E}^{\text{ét}}[p^n]$ is the maximal étale quotient of $\mathbb{E}[p^n]$, as the Cartier dual of $\mathbb{E}^{\circ}[p^n]$.
  - On étale site $(S_m^{\circ})_{\text{ét}}$, $\mathbb{E}^{\circ}[p^n] \simeq \mu_{p^n}$, $\mathbb{E}^{\text{ét}}[p^n] \simeq \underline{(\mathbb{Z}/p^n)}_{S_m^{\circ}}$

  [GME, §1.12.2] for this exact sequence.

- By a theorem of Katz, the SES extends to S_m as well.

  Key : Then $\mathbb{E}^{\text{ét}}[p^n]$ extends to a finite flat group scheme over S_m. It is isomorphic to $\mathbb{Z}/p^n\mathbb{Z}$ after a finite étale extension of S_m.

- This motivates us to consider the functor
$$T_{m,n} := \underline{\mathrm{Isom}}_{S_m}(\mathbb{Z}/p^n, \mathbb{E}^{\text{ét}}[p^n]) : \mathrm{Sch}/S_m \longrightarrow \mathrm{Set}$$
$$T \longmapsto \{\phi : \underline{(\mathbb{Z}/p^n\mathbb{Z})}_T \xrightarrow{\sim} \mathbb{E}^{\text{ét}}[p^n]_T\}/\simeq$$

Proposition : $T_{m,n}$ is representable by an affine scheme $T_{m,n} = \mathrm{Spec}(V_{m,n})$ over S_m.

We will prove this proposition next section axiomatically.

- <u>Tower structure</u> :

  ① For fixed m, $\{T_{m,n}\}_{n\geq 1}$ gives a tower of representable functors

$$\begin{tikzcd}
T_{m,n+1}(T) = \{ \arrow[d, "\bmod p"'] & \phi : \arrow[d, "\bmod p"'] & (\mathbb{Z}/p^{n+1}\mathbb{Z})_T \xrightarrow{\sim} \mathbb{E}^{\acute{e}t}[p^{n+1}]/T \}/\simeq \arrow[d, "\bmod p"] \\
T_{m,n}(T) \supseteq \{ & \phi \bmod p : & (\mathbb{Z}/p^{n}\mathbb{Z})_T \xrightarrow{\simeq} \mathbb{E}^{\acute{e}t}[p^{n}]/T \}/\simeq
\end{tikzcd}$$

Hence we have a tower of representing objects

$$\cdots\cdots \longrightarrow T_{m,2} \xrightarrow{\gamma_{m,2}} T_{m,1} \xrightarrow{\gamma_{m,1}} S_m$$

and their global sections

$$\cdots\cdots \supseteq V_{m,2} \supseteq V_{m,1} \supseteq V_{m,0}$$

Then we define

- $V_{m,\infty} := \varinjlim_n V_{m,n}$, $\forall m\geq 1$
- $T_{m,\infty} := \operatorname{Spec} V_{m,\infty}$, $\forall m \geq 1$.

② Horizontally, these representing object is compatible with base change, in the sense that we have :

$$\begin{tikzcd}
\cdots & T_{2,\infty} \arrow[l] \arrow[d] & T_{1,\infty} = \operatorname{Spec}(V_{1,\infty} := \varinjlim_n V_{1,n}) \arrow[l] \arrow[d] & & \\
& \vdots \arrow[d] & \vdots \arrow[d] & & \\
\cdots & T_{2,2} \arrow[l] \arrow[d] & T_{1,2} = Ig(p^2) \arrow[l] \arrow[d] & & \\
\cdots & T_{2,1} \arrow[l] \arrow[d] & T_{1,1} = Ig(p) \arrow[l] \arrow[d] & & \\
\cdots\cdots & S_2 \arrow[l] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & S_1 \arrow[l] \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & X_T^{ord} \arrow[r, hook] \arrow[d] & X_T \\
\cdots\cdots & \operatorname{Spec} W_2 \arrow[l] & \operatorname{Spec} W_1 \arrow[l] \arrow[r] & \operatorname{Spec} W &
\end{tikzcd}$$

<u>Igusa tower</u>

Then horizontally we define $V := \varinjlim_m V_{m,\infty}$ as the space of <u>p-adic modular forms</u> .

<u>Goal</u>: We shall see that

(1) There are canonical inclusions

$$\{\text{true modular forms}\} \subseteq \{\text{false modular forms}\} \subseteq \{p\text{-adic modular forms}\}$$
$$\| \qquad\qquad\qquad\qquad \| \qquad\qquad\qquad\qquad \|$$
$$H^0(X_T/W, \underline{\omega}^{\otimes k}) \qquad H^0(S_\infty, \underline{\omega}^{\otimes k}) = \varprojlim_m H^0(S_m, \underline{\omega}_m^{\otimes k}) \qquad V$$

(2) Moreover, the images of these inclusions are dense under p-adic topology.

To prove these, we prefer an axiomatic setup as follows.

§4 False modular forms à la Deligne

- W mixed characteristic completed DVR with residue field $k | \mathbb{F}_p$.

~~Let $k | \mathbb{F}_p$ be a finite field~~

~~Let $W := W(k)$ be its Witt ring and to $\varpi \in W$ is a uniformizer~~

For $m \geq 1$, let $W_m := W/\varpi^m W$ be the quotient

<u>eg</u>: $k = \mathbb{F}_p$, $W = \mathbb{Z}_p$ with $W_m \simeq \mathbb{Z}/p^m\mathbb{Z}$. Or W is the ring of integer of some $F | \mathbb{Q}_p$.

<u>Axiomatically, we are given</u>:

① For $m \geq 1$, an affine flat scheme S_m / W_m with horizontal Cartesian squares

$$\begin{tikzcd}
\cdots & S_{m+1} \arrow[l] \arrow[d] & S_m \arrow[l, "\varphi_m"'] \arrow[d] & \cdots \arrow[l] & S_2 \arrow[l, "\varphi_2"'] \arrow[d] & S_1 \arrow[l, "\varphi_1"'] \arrow[d] \\
\cdots & \operatorname{Spec} W_{m+1} \arrow[l] & \operatorname{Spec} W_m \arrow[l] & \cdots \arrow[l] & \operatorname{Spec} W_2 \arrow[l] & \operatorname{Spec} W_1 \arrow[l]
\end{tikzcd}$$

- <u>Assume later on</u>: S is a flat W-scheme and all these S_i are produced by $S_i := S \times_{\operatorname{Spec} W} \operatorname{Spec} W_i$.

<u>eg</u>: $S = X_T^{\mathrm{ord}}/W$, $S_i = S_m$ previously.

② For $m \geq 1$, a <u>rank 1 p-adic étale sheaf</u> $\mathcal{P}$ on S_m. <u>By definition</u>, $\mathcal{P}$ is an inverse system of étale sheaves $\mathcal{P}_n / S_m$ <u>associated</u> to a projective system of finite étale schemes G_n / S_m such that

1° <u>compatibility among different m</u>: $G_n/S_{m+1} \times_{S_{m+1}} S_m = G_n/S_m$

2° $G_n \times_{S_m} S_m' \simeq \underline{(\mathbb{Z}/p^n\mathbb{Z})}_{S_m'}$ over a finite faithfully flat étale extension S_m'/S_m

(So <u>recall</u> for any étale map $U \longrightarrow S_m$, $\mathcal{P}_n(U) = \operatorname{Hom}_{S_m}(U, G_n)$ is an étale sheaf) eg: $G_n/S_m = \mathbb{E}^{\acute{e}t}[p^n]$ where $\mathbb{E}$ is the universal elliptic curve over S_m as seen in the previous section.

ref: [GME, p241].

③ For $m \geq 1$, an étale sheaf $\omega_m = P_m \otimes_{\mathbb{Z}/p^m\mathbb{Z}} \mathcal{O}_{S_m}$ (See [SP, Tag 03EK]), such that

1° ω_m is an invertible coherent sheaf on S_m

2° compatibility among different m: $\varphi_m^*(\omega_{m+1}) = \omega_m$

eg: In the case of modular forms, this coincide with the invertible sheaves ω_m we already had. But this is nontrivial (see [GME, Coro. 3.2.9])

Axiomatic Igusa tower:

For $m, n \geq 1$, we define a functor

$$T_{m,n}: \quad (Sch/S_m)^{op} \longrightarrow Set$$

$$\begin{array}{c} X \\ \downarrow \gamma \\ S_m \end{array} \longmapsto \left\{ \text{isomorphism of étale sheaves } \psi_n : \underline{\mathbb{Z}/p^n\mathbb{Z}}_X \to \gamma^* P_n \right\}$$

Note: $T_{m,n}$ is a contravariant functor: for $f: X \longrightarrow Y \in Sch/S_m$,

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\gamma_X"'] & & Y \arrow[dl, "\gamma_Y"] \\ & S_m & \end{tikzcd}$$

$$T_{m,n}^*(f)\left(\psi_{n,Y} : \underline{\mathbb{Z}/p^n\mathbb{Z}}_Y \xrightarrow{\sim} \gamma_Y^* P_n\right)$$

$$= \left(\begin{array}{ccc} f^*\psi_{n,Y} : f^* \underline{\mathbb{Z}/p^n\mathbb{Z}}_Y & \xrightarrow{\sim} & f^*\gamma_Y^* P_n \\ \| & & \| \\ \underline{\mathbb{Z}/p^n\mathbb{Z}}_X & & \gamma_X^* P_n \end{array} \right) \in T_{m,n}\left(\begin{array}{c} X \\ \downarrow \gamma_X \\ S_m \end{array}\right)$$

Moreover, the group $(\mathbb{Z}/p^n\mathbb{Z})^\times$ acts on $T_{m,n}$ via $\alpha.\psi_n = \alpha^{-1}\psi_n$ for $\alpha \in (\mathbb{Z}/p^n\mathbb{Z})^\times$.

Note on notations: In [GME], $\alpha.\psi_n = \alpha\psi_n$ and hence frequently we see "$V[-k]$" to adjust the notation there.

Prop. A ① $T_{m,n}$ is represented by a finite étale S_m-scheme $T_{m,n} = \mathrm{Spec}(V_{m,n})$.

② $(\mathbb{Z}/p^n\mathbb{Z})^\times$ acts freely on $T_{m,n}$ with quotient S_m

Proof: A standard descent trick: Assume S_m is connected

(1) $\mathcal{P}_n = \underline{\mathbb{Z}/p^n\mathbb{Z}}_{S_m}$ is the constant sheaf. Then for each $\gamma: X \to S_m$,

$$T_{m,n}(X) = \{ \psi_n : \underline{\mathbb{Z}/p^n\mathbb{Z}}_X \xrightarrow{\sim} \gamma^* \underline{\mathbb{Z}/p^n\mathbb{Z}}_{S_m} = \underline{\mathbb{Z}/p^n\mathbb{Z}}_X \text{ as an isomorphism of étale sheaves over } X \}$$

$$\simeq \mathrm{Aut}(\underline{\mathbb{Z}/p^n\mathbb{Z}})(X) \text{ the automorphism functor over } S_m$$

$$\underset{(*)_X}{\simeq} \mathrm{Hom}_{Sch/S_m}(X, \coprod_{\alpha \in (\mathbb{Z}/p^n\mathbb{Z})^\times} S_m).$$

and the isomorphism $(*)_X$ is natural in X. Hence $T_{m,n}$ is represented by the scheme $\coprod_{\alpha \in (\mathbb{Z}/p^n\mathbb{Z})^\times} S_m$, which is affine, finite, étale over S_m.

(2) For a general $\mathcal{P}_n$, by definition we have an étale galois covering S_m'/S_m with galois group G such that $\mathcal{P}_n' := \mathcal{P}_n \otimes_{S_m} S_m' = \underline{\mathbb{Z}/p^n\mathbb{Z}}_{S_m'}$.

- Since $\mathcal{P}_n$ is defined over S_m, σ induces an automorphism of $\mathcal{P}_n'$, hence inducing an automorphism of the functor $T_{m,n}'/S_m'$ by composing σ.

Hence by Yoneda's Lemma, $T_{m,n}'/S_m'$ has an action of G. Then via Galois descent (cf. [GME, Example 1.11.1]), we descend $T_{m,n}'$ to an étale finite scheme S_m representing the functor. $\square$

Remark: We can also regard $T_{m,n}/S_m$ as a Hilbert scheme, see [李克正, p160, 定理2.iv)].

Remark : Moreover, these functors forms a projective system $\{T_{m,n}\}_{n\geq 1}$: for any $\gamma : X \to S_m$, we consider

$$
\begin{tikzcd}
T_{m,n+1} \arrow[d, "\theta_{n+1}"'] & T_{m,n+1}(X) = \{ \psi_{n+1} : \underline{\mathbb{Z}/p^{n+1}\mathbb{Z}}_X \xrightarrow{\sim} \gamma^* \mathcal{P}_{n+1} \} \arrow[d, "\theta_{n+1,X}"'] \\
T_{m,n} & T_{m,n}(X) = \{ \psi_n : \underline{\mathbb{Z}/p^n\mathbb{Z}}_X \xrightarrow{\sim} \gamma^* \mathcal{P}_n \}
\end{tikzcd}
\qquad
\begin{tikzcd}
\vdots \arrow[d] \\
\mathcal{P}_{n+1} \arrow[d, "S_{n+1}"] \\
\mathcal{P}_n \arrow[d] \\
\vdots
\end{tikzcd}
$$

here $\theta_{n+1,X}$ is defined as

$\theta_{n+1,X}(\psi_{n+1})$ = the isomorphism :

$$
\begin{tikzcd}
\underline{\mathbb{Z}/p^{n+1}\mathbb{Z}}_X \arrow[r, "\psi_{n+1}", "\sim"'] \arrow[d, "\mathrm{mod}\, p"'] & \gamma^* \mathcal{P}_{n+1} \arrow[d, "\gamma^* S_{n+1}"] \\
\underline{\mathbb{Z}/p^n\mathbb{Z}}_X \arrow[r, "\theta_{n+1}(\psi_{n+1})", "\sim"'] & \gamma^* \mathcal{P}_n
\end{tikzcd}
$$

One checks $(\theta_{n+1,X})_{X/S_m}$ is natural in X, hence it indeed gives a projective system $\{T_{m,n}\}_{n\geq 1}$. Therefore, taking global sections, we have a direct system

$$V_{m,1} \longrightarrow V_{m,2} \longrightarrow \cdots \longrightarrow V_{m,n} \longrightarrow V_{m,n+1} \longrightarrow \cdots$$

and $V_{m,\infty} := \varinjlim_n V_{m,n}$, $T_{m,\infty} := \mathrm{Spec}(V_{m,\infty})$.

Moreover, $T_{m,\infty}$ represents the functor

$$
\begin{aligned}
T_{m,\infty} : \mathrm{Sch}/S_m &\longrightarrow \mathrm{Set} \\
(X \xrightarrow{\gamma} S_m) &\longmapsto \left\{ \begin{array}{c} \text{isomorphisms of rank one } p\text{-adic étale sheaves} \\ \psi : \underline{\mathbb{Z}_p}_X \xrightarrow{\sim} \gamma^* \mathcal{P} \end{array} \right\}
\end{aligned}
$$

and the group $\mathbb{Z}_p^\times$ acts freely on $T_{m,\infty}$ via $\alpha.\psi := \alpha^{-1}\psi$ with quotient S_m.

- Then horizontally, we build up $V_{\infty,\infty} := \varprojlim_m V_{m,\infty} = \varprojlim_m \varinjlim_n V_{m,n}$, as the (axiomatic) space of p-adic modular forms.

Note : Actually we have the formal completion $T_{\infty,\infty} := \varprojlim_n T_{m,n}$.

<u>Define the canonical embedding $\beta(\infty)$</u>

Then we define two graded rings (<u>推测</u>：加"′"的记号暗示在 false modular form 空间中)

- $R'_m = \bigoplus_{k\geq 0} H^0(S_m, \underline{\omega}_m^{\otimes k})$
- $R'_\infty = \bigoplus_{k\geq 0} \varprojlim_m H^0(S_m, \underline{\omega}_m^{\otimes k}) = \bigoplus_{k\geq 0} H^0(S_\infty, \underline{\omega}^{\otimes k})$.

<u>Lemma H1</u>: For every $m \geq 1$, $\exists$ natural isomorphism $R'_\infty / \varpi^m R'_\infty \simeq R'_m$.

<u>Pf sketch</u>: For all $k \geq 0$, we have the SES

$$0 \to \underline{\omega}^{\otimes k} \xrightarrow{\times \varpi^m} \underline{\omega}^{\otimes k} \longrightarrow \underline{\omega}_m^{\otimes k} \to 0 .$$

Then it gives rise to the SES

(H1) — $0 \to H^0(S, \underline{\omega}^{\otimes k}) \xrightarrow{\times \varpi^m} H^0(S, \underline{\omega}^{\otimes k}) \longrightarrow H^0(S_m, \underline{\omega}_m^{\otimes k}) \to 0$

Since S_m is by assumption an affine scheme. (Serre's affineness criterion, see [GME, §1.10.2 (5)]). In other words,

$$H^0(S, \underline{\omega}^{\otimes k}) / \varpi^m H^0(S, \underline{\omega}^{\otimes k}) \xrightarrow{\sim} H^0(S_m, \underline{\omega}_m^{\otimes k}) .$$

Hence taking direct sum over all $k \geq 0$, we see $R'_\infty / \varpi^m R'_\infty \simeq R'_m$. $\square$

<u>Lemma H2</u>: The space of p-adic modular forms V is p-adically complete, and V is a flat W-algebra.

(In particular, for every $m \geq 1$, $V_{m,\infty} = V/\varpi^m V$, and V is ϖ-torsion free.)

<u>Proof sketch</u>: Simply recall $T_{m,n} = \mathrm{Spec}(V_{m,n})$ is finite étale over S_m.

Hence $V_{m,n}$ is a flat W_m-algebra.

- Hence taking direct limit, $V_{m,\infty} := \varinjlim_n V_{m,n}$ is a flat W_m-algebra
  (<u>Note</u>: $\varinjlim$ is right exact and commutes with "$\otimes$").
- Then as $V = \varprojlim_m V_{m,\infty}$, we see that V is ϖ-adically complete and V is a flat W-algebra.

(a <u>little bit commutative algebra</u>?) $\square$

<u>Goal</u> : Define a canonical homomorphism $\beta(\infty) : R'_\infty \longrightarrow V := \varprojlim_m V_{m,\infty}$
and show it is an embedding.

- <u>Extend ω_m to Igusa towers</u> :

$$\underline{\omega}_{m,n} := P_m \otimes_{\mathbb{Z}/p^m\mathbb{Z}} \mathcal{O}_{T_{m,n}} \quad \text{as an invertible sheaf over } T_{m,n}.$$

- Start with the universal element of the representable functor $\mathcal{T}_{m,n}$:

$$\begin{tikzcd} T_{m,n} \arrow[d, "\gamma_{m,n}"'] \\ S_m \end{tikzcd} \qquad \begin{array}{ccc} \mathrm{Hom}_{Sch/S_m}(T_{m,n}, T_{m,n}) & \xrightarrow{\sim} & \mathcal{T}_{m,n}(T_{m,n}) \\ \mathrm{id} & \longmapsto & \psi_{m,n,can} : \underline{\mathbb{Z}/p^n\mathbb{Z}}_{T_{m,n}} \xrightarrow{\sim} \gamma_{m,n}^* P_n \end{array}$$

Then we tensor $\psi_{m,n,can}$ with $\mathcal{O}_{T_{m,n}}$, we obtain

$$\begin{tikzcd} \psi_{m,n,can}\otimes \mathrm{id} : \underline{\mathbb{Z}/p^n\mathbb{Z}}_{T_{m,n}} \otimes_{T_{m,n}} \mathcal{O}_{T_{m,n}} \arrow[r, "\sim"] \arrow[d, phantom, "\wr"] & \gamma_{m,n}^* P_n \otimes \mathcal{O}_{T_{m,n}} \arrow[d, phantom, "\wr"] \\ \underline{\mathbb{Z}/p^n\mathbb{Z}}_{T_{m,n}} & \underline{\omega}_{m,n} \end{tikzcd}$$

So here $\underline{\omega}_{m,n}$ is a trivial invertible sheaf over $T_{m,n}$.

- So we take global section, $1 \in \mathbb{Z}/p^n\mathbb{Z}$ is sent to a global section of $\underline{\omega}_{m,n}$, denoted by $\omega_{can}(m,n) \in H^0(T_{m,n}, \underline{\omega}_{m,n})$.

- In particular, when $m=n$, $\omega_{can}(m) := \omega_{can}(m,m)$. Then we can define

$$\begin{array}{rccccc} \beta(m,k) : & H^0(S_m, \underline{\omega}_m^{\otimes k}) & \longrightarrow & V_{m,m} & \hookrightarrow & V_{m,\infty}. \\ & f_k & \longmapsto & \dfrac{f_k}{\omega_{can}(m)^{\otimes k}} & & \end{array}$$

(here the latter f_k is the section in $H^0(T_{m,m}, \underline{\omega}_m^{\otimes k})$)

$\Longrightarrow$ ① Take $\bigoplus_{k\geq 0}$ directly, we get $\beta(m)$:

$$\begin{array}{rccccc} \beta(m) : & R'_m = \bigoplus_{k\geq 0} H^0(S_m, \underline{\omega}_m^{\otimes k}) & \longrightarrow & V_{m,m} & \hookrightarrow & V_{m,\infty} \\ & \sum_{k\geq 0} f_k & \longmapsto & \sum_{k\geq 0} \dfrac{f_i}{\omega_{can}(m)^{\otimes k}} & & \end{array}$$

② Note for each fixed k, $\beta(m,k) \equiv \beta(n,k) \mod p^m$, for all $n > m$. Then we can take inverse limit

$$\beta(\infty,k) : \varprojlim_m H^0(S_m, \underline{\omega}_m^{\otimes k}) \longrightarrow \varprojlim_m V_{m,\infty} = V$$

then take direct sum :

$$\beta(\infty) : R'_\infty = \bigoplus_{k\geq 0} \varprojlim_m H^0(S_m, \underline{\omega}_m^{\otimes k}) \longrightarrow V.$$

Proposition : $\beta(m)$ is not injective for $m \geq 1$, and $\beta(\infty)$ is injective.

* Reinterprete elements in $V_{m,m}$: By definition of $V_{m,m}$, $f_m \in V_{m,m}$ is the assignment

$$(X \xrightarrow{\gamma} S_m ,\ \psi_m : \underline{\mathbb{Z}/p^m\mathbb{Z}}_X \xrightarrow{\sim} \gamma^* \mathcal{P}_m) \overset{f_m}{\rightsquigarrow} f(\gamma, \psi_m) \in \Gamma(X, \mathcal{O}_X)$$

which are compatible with base change.

Recall :

$$\mathcal{T}_{m,m}(X) = \mathrm{Hom}_{Sch/S_m}(X, \mathrm{Spec}\, V_{m,m}) \overset{(*)}{=} \mathrm{Hom}_{Alg/W_m}(V_{m,m}, \Gamma(X, \mathcal{O}_X))$$

$$(\gamma, \psi_m) \longmapsto (\gamma, \psi_m) : V_{m,m} \longrightarrow \Gamma(X, \mathcal{O}_X),\quad f_{m,m} \longmapsto f_{m,m}(\gamma, \psi_m).$$

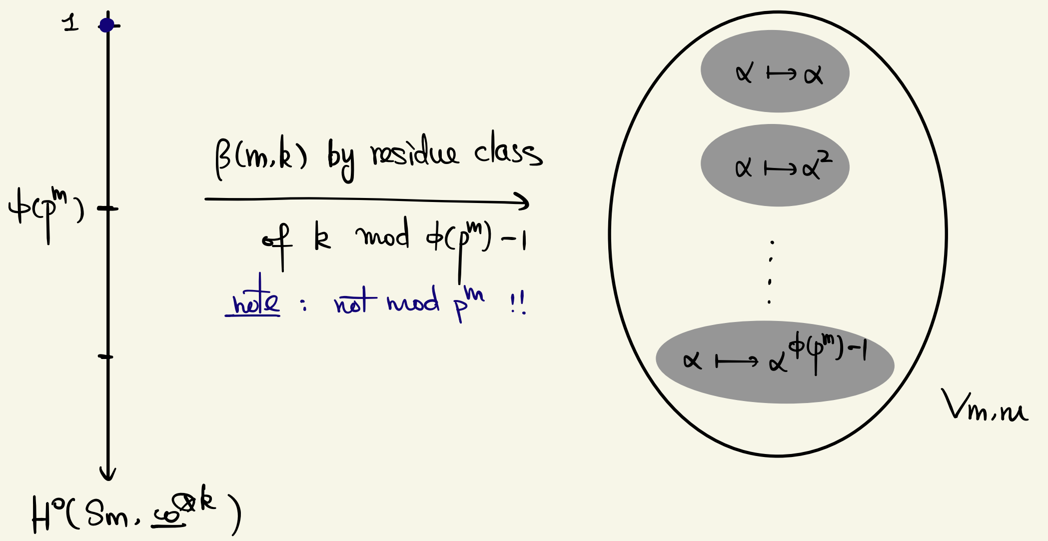
This is just abstract nonsense.

Recall : Here $(*)$ follows from [Har73, ex II.2.4].

* Reinterprete the $(\mathbb{Z}/p^m\mathbb{Z})^\times$-action : in the above setup, for $\alpha_m \in (\mathbb{Z}/p^m\mathbb{Z})^\times$,

$$\alpha_m . f_m \left(X \xrightarrow{\gamma} S_m ,\ \psi_m : \underline{\mathbb{Z}/p^m}_X \xrightarrow{\sim} \gamma^* \mathcal{P}_m \right) = f_m(\gamma, \alpha_m^{-1} \psi_m) .$$

* Reinterpretation of $\beta(m,k)$: then $\beta(m,k)$ identifies $H^0(S_m, \underline{\omega}_m^{\otimes k})$ with the subspace of "assignment" in $V_{m,m}$ on which $(\mathbb{Z}/p^m\mathbb{Z})^\times$ acts via the character $\alpha \longmapsto \alpha^k$. Hence for fixed m,



So ① When taking "$\oplus$" over m, we see $\beta(m)$ is not injective since $H^0(S_m, \underline{\omega}_m^{\otimes k})$ and $H^0(S_m, \underline{\omega}_m^{\otimes k+\phi(p^m)})$ has the same image in $V_{m,m}$ for each k.

② When passing to $\beta(\infty)$, we have similar interpretations as above, yet key difference is that :

- $\beta(\infty)$ identifies $\varprojlim_m H^0(S_m, \underline{\omega}_m^{\otimes k})$ with functions in V on which $\mathbb{Z}_p^\times$ acts by $\alpha \mapsto \alpha^k$ for all $k \geq 0$.

use Lemma H2.

- Note "$\mathbb{Z}_p^\times$ has no torsion", things in ① will not happen ! $\square$

Density theorem :

Since V and R'_∞ are flat W-algebra, we have the commutative diagram of inclusions :

$$\begin{tikzcd} R'_\infty \arrow[r, hook, "\beta(\infty)"] \arrow[d, hook] & V \arrow[d, hook] \\ R'_\infty[\frac{1}{p}] \arrow[r, hook, "\beta(\infty)"] & V[\frac{1}{p}] \end{tikzcd}$$

Then we form $D' := \beta(\infty)(R'_\infty[\frac{1}{p}]) \cap V \subseteq V$.

<u>Density theorem A</u> : The natural inclusion $\iota : D' \longrightarrow V$ induces an isomorphism $\iota_m : D'/\varpi^m D' \xrightarrow{\sim} V/\varpi^m V$ for every $m \geq 1$. In other words, V is the ϖ-adic completion of D'.

<u>Proof</u> :

(1) <u>Injectivity of ι_m</u> : We tensor the SES $0 \to D' \to V \to V/D' \to 0$ with W_m over W, we have

$$\cdots \to \mathrm{Tor}_1^W(V/D', W_m) \to D' \otimes_W W_m \to V \otimes_W W_m \to V/D' \otimes_W W_m \to 0$$

Then note that V/D' is W-flat, we see

$$\begin{tikzcd} 0 \arrow[r] & D' \otimes_W W_m \arrow[r] \arrow[d, phantom, "\wr"] & V \otimes_W W_m \arrow[d, phantom, "\wr"] \\ & D'/\varpi^m D' \arrow[r, "\iota_m"'] & V/\varpi^m V \end{tikzcd}$$

that ι_m is injective indeed.

(2) <u>Surjectivity of ι_m</u> :

- By Nakayama, STS $\iota_1 : D'/\varpi D' \to V/\varpi V \simeq V_{1,\infty}$ is surjective.

  <u>Note</u> : Here we use <u>Lemma H2</u> for $V_{1,\infty} \simeq V/\varpi V$.

- <u>Descent to finite $\beta(m)$-level</u> : Let $f \in V_{1,n}$.

  ⁎ We choose $m > n$ large enough so that $\frac{\varpi^m}{(p^n-1)!} \in W$.

  We choose a lifting $F \in \varprojlim_{\ell} V_{\ell,n}$ of f. Then it suffices to show :

  $$\varpi^{m-1} \cdot F \in \beta(\infty)(R'_\infty) + \varpi^m V$$

  as it implies $F \in D' + \varpi V$

  ⁎ We project F to F_m under $\varprojlim_{\ell} V_{\ell,n} \longrightarrow V_{m,n} \hookrightarrow V_{m,m}$. Then it suffices to show $\varpi^{m-1} F_m \in \beta(m)(R'_m)$.

- We turn to show: $\varpi^{m-1} V_{m,n} \subseteq \beta(m)(R_m') \subseteq V_{m,m}$.

  * <u>Case when P_m is constant</u>:

    - $T_{m,m} = \coprod_{\alpha \in (\mathbb{Z}/p^m\mathbb{Z})^\times} S_m$, $\quad V_{m,m} = \prod_{\alpha \in (\mathbb{Z}/p^m\mathbb{Z})^\times} V_{m,0}$, $\quad V_{m,0} := \Gamma(S_m, \mathcal{O}_{S_m})$.

    - The invertible sheaf ω_m is trivial on S_m. Hence
    $$R_m' = \bigoplus_{k \geq 0} H^0(S_m, \omega_m^{\otimes k}) \simeq V_{m,0}[X] \quad \text{polynomial ring}.$$

    - Recall the proof of noninjectivity of $\beta(m)$, we have the reinterpretation

      - $\beta(m): R_m' \longrightarrow V_{m,m}$ is regarding the polynomials as functions over $(\mathbb{Z}/p^m\mathbb{Z})^\times$ with values in $V_{m,0}$.

        (<u>note</u>: $V_{m,0}$ is a $\mathbb{Z}/p^m\mathbb{Z}$-algebra, hence $p^m V_{m,0} = 0$. This indeed makes sense)

      - The inclusion $V_{m,n} \hookrightarrow V_{m,m}$ becomes the natural inclusion
      $$\mathrm{Map}((\mathbb{Z}/p^n\mathbb{Z})^\times, V_{m,0}) \longrightarrow \mathrm{Map}((\mathbb{Z}/p^m\mathbb{Z})^\times, V_{m,0}).$$

      Then we aim to show
      $$\varpi^{m-1}\mathrm{Map}((\mathbb{Z}/p^n\mathbb{Z})^\times, V_{m,0}) = \mathrm{Map}((\mathbb{Z}/p^n\mathbb{Z})^\times, \varpi^{m-1}V_{m,0})$$
      lies in the image of $\beta(m)$: that are consist of polynomial functions!

    - <u>Fact</u>: The $\mathbb{F}$-vector space $\mathrm{Map}((\mathbb{Z}/p^n\mathbb{Z})^\times, \mathbb{F})$ admits a Mähler basis of binomial functions $\left\{\binom{x}{i} \,\middle|\, 0 \leq i \leq p^n-1\right\}$.

      <u>Ref</u>: See [GME, Lemma 3.2.3].

      Now as
      $$\mathrm{Map}((\mathbb{Z}/p^n\mathbb{Z})^\times, \varpi^{m-1}V_{m,0}) \simeq \mathrm{Map}((\mathbb{Z}/p^n\mathbb{Z})^\times, \mathbb{F}) \otimes_{\mathbb{F}} \varpi^{m-1}V_{m,0},$$
      any element of LHS can be written as
      $$\sum_{i=0}^{p^n-1} a_i \binom{x}{i}, \qquad a_i \in \varpi^{m-1}V_{m,0}.$$

    - Then it follows from our choice of m that $\varpi^{m-1}\binom{x}{i} \in W[x]$ for all $i = 0, \cdots, p^n-1$, as desired!

  * For general P_n, we use faithfully flat descent to finish the work. □

More axiomatic assumptions

④ Let M be a proper smooth scheme over W, whose fibers are geometrically connected curves. Set $M_m := M \otimes_W W_m$ for $m \geq 1$.

<u>eg</u> : $M = X_\Gamma / W$ is the compact modular curve over W.

<u>note</u> : M_m are infinitesimal thickening of M_1. hence topologically they are homeomorphic this makes ⑤ below make sense.

⑤ Let Ω be a finite set of closed points of M_1 such that $S_m = M_m \setminus \Omega$ is affine.

<u>eg</u> : Ω is the supersingular locus on M_1, and $S_m = M_m \setminus \Omega$ is indeed affine.

⑥ We have an invertible sheaf $\underline{\omega}$ on M, and the restriction of $\underline{\omega}$ on S_m is precisely the étale sheaf $\underline{\omega}_m \simeq \mathbb{P}_m \otimes_{\mathbb{Z}/p^m\mathbb{Z}} \mathcal{O}_{S_m}$.

<u>note</u> : It is nontrivial to see that in our previous setup,

- $\underline{\omega} = \pi_* \Omega^1_{\mathbb{E}/X_\Gamma, W}$
- $\mathbb{P}_m = \mathbb{E}[p^m]/W$

fits the isomorphism here. We shall see this later.

<u>Note</u> : $\underline{\omega}_1^{\otimes(p-1)}$ is trivial on S_1. In fact $\mathbb{P}_1^{\otimes(p-1)} \simeq \underline{\mathbb{Z}/p\mathbb{Z}}_{S_1}$

$$\underline{\omega}_1^{\otimes(p-1)} \simeq (\mathbb{P}_1 \otimes_{\mathbb{Z}/p\mathbb{Z}} \mathcal{O}_{S_1})^{\otimes(p-1)} \simeq \mathbb{P}_1^{\otimes(p-1)} \otimes_{\mathbb{Z}/p\mathbb{Z}} \mathcal{O}_{S_1} \simeq \underline{\mathbb{Z}/p\mathbb{Z}}_{S_1} \otimes_{\mathbb{Z}/p\mathbb{Z}} \mathcal{O}_{S_1} \simeq \underline{\mathbb{Z}/p\mathbb{Z}}_{S_1}$$

Then taking global section, we see

$$H^0(S_1, \underline{\omega}_1^{\otimes p-1}) \xrightarrow{\sim} \mathbb{Z}/p\mathbb{Z}$$

Let $A \in H^0(S_1, \underline{\omega}_1^{\otimes p-1})$ be the section corresponding to $1 \in \mathbb{Z}/p\mathbb{Z}$.

⑦ Suppose A extends to a (necessarily unique) section $A \in H^0(M_1, \omega^{\otimes p-1})$, which vanishes at each point of Ω.

<u>eg</u>. A is the Hasse invariant in GL_2-case.

Then we define the third graded ring

- $R'_m = \bigoplus_{k \geq 0} H^0(S_m, \underline{\omega}_m^{\otimes k})$
- $R'_\infty = \bigoplus_{k \geq 0} \varprojlim_m H^0(S_m, \underline{\omega}_m^{\otimes k}) = \bigoplus_{k \geq 0} H^0(S_\infty, \underline{\omega}^{\otimes k})$. <u>fake modular form</u>
- $R_\infty = \bigoplus_{k \geq 0} H^0(M, \underline{\omega}^{\otimes k})$ <u>true modular form</u>

Lemma : $R_\infty \subseteq R'_\infty \subseteq V$.

Proof sketch: We have seen $R'_\infty \subseteq V$ canonically via the injection $\beta(\infty)$.

1° By the properness of M, $H^1(M, \underline{\omega}^{\otimes k})$ is a W-module of finite type [GME, §1.10.2.(8)] .

Note : In [GME], Hida prefers the derived direct image functor since when $f: X \to S$ with S is a singleton, $R^i f_* \mathcal{F} \simeq H^i(X, \mathcal{F})$. This shall not cause any confusion !

2° Recall the proof of Lemma H1 , we have the SES for each m :

$$0 \to H^0(M, \underline{\omega}^{\otimes k}) \otimes_W W_m \longrightarrow H^0(M_m, \underline{\omega}_m^{\otimes k}) \longrightarrow H^1(M, \underline{\omega}^{\otimes k})[p^m] \to 0 ,$$

Taking inverse limit wrt m , by 1° we see

$$0 \to \varprojlim_m \left(H^0(M, \underline{\omega}^{\otimes k}) \otimes_W W_m \right) \to \varprojlim_m H^0(M_m, \underline{\omega}_m^{\otimes k}) \to \varprojlim_m H^1(M_m, \underline{\omega}^{\otimes k})[p^m] \overset{②}{=} 0$$

$$\wr\| \text{ finiteness result in 1° ①}$$

$$H^0(M, \underline{\omega}^{\otimes k}) \otimes_W \varprojlim_m W_m$$

$$\wr\|$$

$$H^0(M, \underline{\omega}^{\otimes k}) .$$

Hence $H^0(M, \underline{\omega}^{\otimes k}) \simeq \varprojlim_m H^0(M_m, \underline{\omega}_m^{\otimes k})$.

Explaination :

① In general, $\varprojlim$ not commute with $\otimes$, but here we have :

- $H^0(M, \underline{\omega}^{\otimes k})$ is a finitely generated W-module by 1° .

? • W is a noetherian ring , $\{W_m\}$ is a surjective system of flat W-modules

② I have no idea .

Note cananonically $H^0(M_m, \underline{\omega}_m^{\otimes k})$ embeds into $H^0(S_m, \underline{\omega}_m^{\otimes k})$, we see

$$H^0(M, \underline{\omega}^{\otimes k}) \hookrightarrow \varprojlim_m H^0(S_m, \underline{\omega}_m^{\otimes k}) = H^0(S_\infty, \underline{\omega}^{\otimes k})$$

and hence taking direct sum over k, we see $R_\infty \subseteq R'_\infty$. $\square$

Therefore , we have the overall diagram

$$\begin{tikzcd} R_\infty \arrow[r, hook] \arrow[d] & R'_\infty \arrow[r, hook, "\beta(\infty)"] \arrow[d] & V \arrow[d] \\ R_\infty[\frac{1}{p}] \arrow[r, hook] \arrow[rr, bend right, "{:= \beta}"'] & R'_\infty[\frac{1}{p}] \arrow[r, hook, "\beta(\infty)"] & V[\frac{1}{p}] \end{tikzcd}$$

$$\begin{array}{l} V \\ \iota \uparrow \\ D' := \beta(\infty) R'_\infty[\frac{1}{p}] \cap V \\ j \uparrow \\ D = \beta\left(R_\infty[\frac{1}{p}]\right) \cap V \hookrightarrow V . \end{array}$$

Density theorem B . The canonical inclusion $j: D \hookrightarrow D'$ induces isomorphisms

$$j_m : D/\varpi^m D \xrightarrow{\sim} D'/\varpi^m D' \quad \text{for all } m \geq 1 .$$

- Combining with the <u>Density theorem A</u>, we see for all $m \geq 1$,

$$D/\varpi^m D \xrightarrow[\sim]{j_m} D'/\varpi^m D' \xrightarrow[\sim]{\iota_m} V/\varpi^m V .$$

Hence V is the ϖ-adic completion of D.

Therefore, D is ϖ-adically dense in V !

<u>Pf of Density B</u> .

(1) <u>Injectivity of j_m</u> : by the same argument in <u>Density A</u>, $\iota_m \circ j_m$ is injective. Since ι_m is an isomorphism, j_m is injective.

(2) <u>Surjectivity of j_m</u> :

- Again by NAK, STS $j_1 : D/\varpi D \xrightarrow{\sim} D'/\varpi D'$ is surjective.
- <u>Build a bridge between S_m and M_m</u> :
  - * The invertible sheaf $\underline{\omega}_1$ on M_1 has positive degree since $A \in H^0(M_1, \underline{\omega}_1^{\otimes(p-1)})$ is a nonzero section which has zeros on M_1.

    $\Longrightarrow$ $\exists$ an integer $\nu >> 0$ s.t. $\underline{\omega}_1^{\otimes \nu(p-1)}$ has degree $> 2g-2$, where $g =$ genus of M_1
  - * Hence we can lift $A^\nu \in H^0(M_1, \omega^{\otimes \nu(p-1)})$ to a section $\mathcal{E} \in H^0(M, \omega^{\otimes \nu(p-1)})$

    <u>Note</u> : The obstruction of this lifting lies in $H^1(M_1, \underline{\omega}_1^{\otimes \nu(p-1)})$, which is zero by our assumption on ν. In fact, by the proof of <u>Lemma H1</u>,

$$\cdots \longrightarrow H^0(M, \omega^{\otimes \nu(p-1)}) \longrightarrow H^0(M_1, \underline{\omega}_1^{\otimes \nu(p-1)}) \longrightarrow \underbrace{H^1(M_1, \underline{\omega}_1^{\otimes \nu(p-1)})}_{\text{obstruction here !}} \to \cdots$$

$$? \longmapsto A^\nu$$

  - * Then we view $\mathcal{E}$ as an element of $1 + \varpi V$, as we recall A is defined as the section corresponding to $1 \in \mathbb{Z}/p\mathbb{Z}$, hence the image of A in $V_{1,1}$ is the function "1".
  - * The open subscheme is defined by the open subset where $\mathcal{E} \in H^0(M, \underline{\omega}^{\otimes \nu(p-1)})$ is invertible.

this interpretation will be used later on frequently

$$\Longrightarrow \quad H^0(S_m, \underline{\omega}_m^{\otimes k}) = \varinjlim_n \frac{H^0(M_m, \underline{\omega}^{\otimes k + n\nu(p-1)})}{\mathcal{E}^n} \quad — \quad \circledast$$

<u>Interpretation</u> : For any $f \in H^0(S_m, \underline{\omega}_m^{\otimes k})$, $\exists$ an integer $N >> 0$ s.t. $\mathcal{E}^N f \in H^0(M_m, \underline{\omega}^{\otimes k + N\nu(p-1)})$!

- <u>Now we can show the surjectivity</u> :

  * Given $f \in R'_\infty \cap \varpi^m V$, or equivalently $\frac{1}{\varpi^m} f \in R'_\infty[\frac{1}{p}] \cap V = D'$.

    We want to find $g \in D$ s.t. $f \equiv g \mod \varpi^{m+1} V$.

    $\Longrightarrow \frac{1}{\varpi^m} f \equiv \frac{1}{\varpi^m} g \in D$. mod ϖV, showing the surjectivity of J_1.

  * WMA $f = f_i \in \varprojlim_m H^0(S_m, \underline{\omega}_m^{\otimes i})$ for some $i \geq 0$.

    Then $\circledast_{m+1}$ implies $\exists$ integer $N \gg 0$, $g \in H^0(M_m, \underline{\omega}^{\otimes i + Np^m \upsilon(p-1)})$ s.t.

    $$f_i \equiv \frac{g}{\varepsilon^{Np^m}} \quad \underbrace{\text{mod } \varpi^{m+1} R'_\infty}$$

    <u>note</u>: f_i is in the inverse limit $\varprojlim_m H^0(S_m, \cdots)$.
    So by "mod ϖ^{m+1}", we mean focusing on the "m+1" coordinate of f_i.

    Then as $\varepsilon \in 1 + \varpi V$, we have $\frac{1}{\varepsilon^{Np^m}} \in 1 + \varpi^{m+1} V$. It follows that $f_i \equiv g \mod \varpi^{m+1} V$, as desired ! $\square$

§5 Interpolating Hecke algebra

- $V = \varprojlim_m V/\varpi^m V = \varprojlim_m \frac{1}{\varpi^m}V/V$
- Define $\mathcal{V} := \varinjlim_m \frac{1}{\varpi^m}V/V = \bigcup_{m\geq 1} \mathcal{V}[\varpi^m]$.

  Note : • The direct limit is taken under the "inclusions" $\frac{1}{\varpi^m}V/V \to \frac{1}{\varpi^{m+1}}V/V$, So we can image $\mathcal{V}$ as taking the "compatible union" of all $\frac{1}{\varpi^m}V/V$.

  • Imagine $V = \mathbb{Z}_p$ and $\varpi = p$. Then $\mathcal{V}$ looks like $\mathbb{Q}_p/\mathbb{Z}_p$. In this sense we can regard $\mathcal{V}$ as some kind of "dual" of V.

  Then : • $\mathcal{V}$ is a ϖ-divisible W-module by construction.

  • V has a $\mathbb{Z}_p^\times$-action, inducing a $\mathbb{Z}_p^\times$-action on $\frac{1}{\varpi^m}V/V$ and hence $\mathcal{V}$.

- Note $\mathcal{V}$ can be regarded as a discrete W-module. We define

$$\mathcal{V}^* := \mathrm{Hom}_W(\mathcal{V}, K/W) = \varprojlim_m \mathrm{Hom}_W\left(\mathcal{V}[\varpi^m], \tfrac{1}{\varpi^m}W/W\right)$$

  as the Pontryagin dual of $\mathcal{V}^*$ with the profinite topology. With the indued $\mathbb{Z}_p^\times$-action, $\mathcal{V}^*$ becomes a compact $W[\![\mathbb{Z}_p^\times]\!]$-module.

- We seperate $\mathcal{V}$ into isotypic parts of $\mathbb{Z}_p^\times$-action : $\iota : \mathbb{Z}_p^\times \hookrightarrow W^\times$ be the inclusion
  - M : W-module, with a W-linear action of $\mathbb{Z}_p^\times$.
  - For $k\geq 1$, define $M[\iota^k]$ for the submodule of M on which $\mathbb{Z}_p^\times$ acts by ι^k.

  Then
$$\mathcal{V}[\iota^k] = \varinjlim_m V_{m,\infty}[\iota^k] \overset{(*)}{=} \varinjlim_m V_{m,m}[\iota^k],$$
  here note $W_{m,\infty}$ is a W_m-module to get $(*)$ (i.e. check ι^k-isotypic part comes from $V_{m,m}$ in $V_{m,\infty} = \varinjlim_n V_{m,n}$.)

- Recall in Proposition §4, we have seen the image of $\beta(m) : H^0(S_m, \underline{\omega}_m^{\otimes k}) \to V_{m,m}$ is $V_{m,m}[\iota^k]$. i.e. $\mathcal{V}[\iota^k]$ essentially comes from false modular forms.

Hida ordinary projector : We assume there exists a W-linear idempotent map $e : V \to V$ compatible with $\mathbb{Z}_p^\times$-action. We make two assumptions :

(C) We have $e(Af) = A\cdot e(f)$ for all $f\in H^0(S_1, \underline{\omega}_1^{\otimes k})$ for all k.
("C" stands for compatible with Hasse invariant.)

(F) The dimension $\dim_K eH^0(M/_K, \underline{\omega}^{\otimes k})$ is bounded independent of k.

("F" stands for "finite". Let D be such a bound independent of k).

Sometimes we replace (F) with a stronger condition

(F^+) The dimension $\dim_K eH^0(M_{/K}, \underline{\omega}^{\otimes k})$ depends only on $k \bmod (p-1)$ for $k \geq k_0$ with some given integer k_0.

<u>Proposition</u> : Under assumption (C) and (F), $\dim_{\mathbb{F}} eH^0(S_1, \underline{\omega}_1^{\otimes k})$ is finite and bounded independent of $k \geq 1$.

<u>Proof</u> : Let $\{\bar{f}_1, \cdots, \bar{f}_\ell\}$ be a set of $\mathbb{F}$-linearly independent sections in $eH^0(S_1, \underline{\omega}_1^{\otimes k})$.

- <u>Lift 1</u> : Note that $H^0(S_1, \underline{\omega}_1^{\otimes k}) \simeq H^0(S_\infty, \underline{\omega}^{\otimes k}) / \varpi H^0(S_\infty, \underline{\omega}^{\otimes k})$, we lift $\bar{f}_i$ to a section $f_i \in H^0(S_\infty, \underline{\omega}^{\otimes k})$. Then $\{f_1, \cdots, f_\ell\}$ are K-linearly independent.
- <u>Lift 2</u> : For simplicity, we assume $A \in H^0(M_1, \underline{\omega}_1^{\otimes(p-1)})$ can be lifted to a section $\mathcal{E} \in H^0(M, \underline{\omega}^{\otimes(p-1)})$.
  - i.e. Take $\nu = 1$ in the proof of <u>Density B</u>.
  - In the Gl_2-case, this holds for $p \geq 5$.

Since A is nowhere vanishing on S_1, we have an injective map

$$\times A^s : H^0(S_1, \underline{\omega}_1^{\otimes k}) \hookrightarrow H^0(S_1, \underline{\omega}_1^{\otimes s(p-1)+k}). \quad \forall s \geq 1.$$
$$\bar{f}_i \longmapsto A^s \bar{f}_i$$

Hence $\{A^s \bar{f}_i\}_{i=1}^{\ell}$ are $\mathbb{F}$-linearly independent. One computes

$$e \cdot A^s \bar{f}_i \underset{\text{condition (C)}}{=\!=} A^s e \bar{f}_i = A^s \cdot \bar{f}_i \quad \text{since } f_i \in eH^0(S_1, \underline{\omega}^{\otimes k})$$

By <u>Lift 1</u>, we then see

$$\{e \cdot (\mathcal{E}^s f_i)\}_{i=1}^{\ell} \text{ are } K\text{-linearly independent in } H^0(S_\infty, \underline{\omega}^{\otimes s(p-1)+k})$$

for any $s \geq 1$. Recall the proof of <u>Density B</u>, we see

$$\mathcal{E}^s \cdot f_i \in H^0(M_{/K}, \underline{\omega}^{\otimes s(p-1)+k})$$

for sufficiently large s (depend on i). Hence we can find $s \gg 0$ s.t.

$$\{e(\mathcal{E}^s \cdot f_i)\}_{i=1}^{\ell} \text{ are } K\text{-linearly independent in } eH^0(M_{/K}, \underline{\omega}^{\otimes s(p-1)+k}).$$

Now by assumption (F), we see $\dim_K eH^0(M_{/K}, \underline{\omega}^{\otimes s(p-1)+k}) \leq D$ for a bound D independent of s and k. Hence $\ell \leq D$. $\square$

<u>Next goal</u> : Translate the above <u>proposition</u> to the space of p-adic modular forms V.

- $e : V \to V$ induces idempotent $e : V/\varpi^m V \to V/\varpi^m V$ for every $m \geq 1$, hence inducing idempotents on $\mathcal{V}$ and $\mathcal{V}^*$. Denote

$$\mathcal{V}_{\mathrm{ord}} := e\mathcal{V}, \quad \mathcal{V}^*_{\mathrm{ord}} := e\mathcal{V}^*.$$

- <u>Obs</u>: Taking ν^k-isotypic part and ν^k-coisotypic part, we see $\forall\ k \geq 1$
  - $\mathcal{V}^*_{\mathrm{ord}} \otimes_{W[\![\mathbb{Z}_p^\times]\!], \nu^k} W$ is the Pontryagin dual of $\mathcal{V}_{\mathrm{ord}}[\nu^k]$.
  - $\left(\mathcal{V}^*_{\mathrm{ord}} \otimes_{W[\![\mathbb{Z}_p^\times]\!], \nu^k} W\right) \otimes_W W/\varpi W$ is the Pontryagin dual of $\mathcal{V}_{\mathrm{ord}}[\nu^k][\varpi]$.

  Then as we recalled previously,

$$\beta(1) : H^0(S_1, \underline{\omega}_1^{\otimes k}) \longrightarrow V_{1,1}[\nu^k] \hookrightarrow \mathcal{V}[\nu^k].$$

  Moreover, taking ordinary part, it is known that $e\beta(1)$ gives an isomorphism

$$eH^0(S_1, \underline{\omega}_1^{\otimes k}) \xrightarrow{\sim} \mathcal{V}_{\mathrm{ord}}[\nu^k].$$

  Combining with the previous <u>Obs</u>, we get

<u>Coro</u> : $\mathcal{V}^*_{\mathrm{ord}} \otimes_{W[\![\mathbb{Z}_p^\times]\!], \nu^k} \mathbb{F}$ is finite dim'l over $\mathbb{F}$. $\square$

- Now recall $\mathbb{Z}_p^\times \simeq \Delta \times (1+p\mathbb{Z}_p)$, we further decompose $\mathcal{V}^*_{\mathrm{ord}}$ into Δ-isotypic parts as

$$\mathcal{V}^*_{\mathrm{ord}} = \bigoplus_{\chi \in \Delta^\vee} \mathcal{V}^*_{\mathrm{ord}}[\chi], \quad \{\chi : \Delta \to W^\times \text{ characters}\} =: \Delta^\vee$$

  and each $\mathcal{V}^*_{\mathrm{ord}}[\chi]$ can be regarded as a $\Lambda := W[\![1+p\mathbb{Z}_p]\!]$-module.

- Then invoke the above <u>corollary</u>, we see

$$\mathcal{V}^*_{\mathrm{ord}}[\chi] \otimes_{\Lambda, \nu^k} \mathbb{F} \simeq \mathcal{V}^*_{\mathrm{ord}}[\chi] / (1+p, \varpi)\mathcal{V}^*_{\mathrm{ord}}[\chi] \text{ is finite dim'l over } \mathbb{F}.$$

  Note that $\mathcal{V}^*_{\mathrm{ord}}[\chi]$ is a compact Λ-module. by <u>topological Nakayama lemma</u>, we see that for each $\chi \in \Delta^\vee$,
  - $\mathcal{V}^*_{\mathrm{ord}}[\chi]$ is a finitely generated Λ-module that can be generated by d_χ elements where $d_\chi := \dim_{\mathbb{F}}\left(\mathcal{V}^*_{\mathrm{ord}}[\chi] \otimes_\Lambda \mathbb{F}\right)$.

  $\Longrightarrow$ $\exists$ surjective Λ-linear homomorphism

$$\varphi_\chi : \Lambda^{d_\chi} \twoheadrightarrow \mathcal{V}^*_{\mathrm{ord}}[\chi].$$

- We claim: $\mathcal{V}^*_{\mathrm{ord}}[\chi]$ is a free Λ-module of rank d_χ.

To show this, we prove φ_χ is an isomorphism for each $\chi \in \Delta^\vee$.

* Recall from the beginning, for every $k \geq 1$ and corresponding $\chi = \kappa_k|_\Delta$,

(here χ can be regarded as the "classical weights")

$\mathcal{V}^*_{\mathrm{ord}}[\chi] \otimes_{\Lambda, \nu^k} W$ is the pontryagin dual of $\mathcal{V}_{\mathrm{ord}}[\nu^k]$

and is W-torsion free, we see it is a free Λ-module of rank d_χ.

$\Rightarrow$ $\varphi_\chi \otimes_{\nu^k} W$ is an isomorphism for a Zariski dense subset of classical weights

we see φ_χ is indeed an isomorphism.

- Next we claim: $eH^0(S_1, \underline{\omega}_1^{\otimes k+s(p-1)}) = eH^0(M_1, \underline{\omega}_1^{\otimes k+s(p-1)})$ for $s \gg 0$.

* Note: $eH^0(M_1, \underline{\omega}_1^{\otimes k+s(p-1)}) \hookrightarrow eH^0(S_1, \underline{\omega}_1^{\otimes k+s(p-1)})$ always holds.

By Proposition, LHS is finite dim'l over $\mathbb{F}$, so it suffices to compare the dimensions of two sides.

* Imitating the trick of Prop. as multiplying a large power of $\mathcal{E}^s$, we are done.

And similarly for any "mod ~~∞~~ ^{m}"-case. Putting them together, we obtain

$$\begin{array}{c} eH^0(S_\infty, \underline{\omega}^{\otimes k} \otimes K/W) = eH^0(M, \underline{\omega}^{\otimes k} \otimes K/W) \quad \text{for } k \gg 0 \\ \| \\ \mathcal{V}_{\mathrm{ord}}[\nu^k] \end{array}$$

Then taking pontryagin dual, we see

$$\mathcal{V}^*_{\mathrm{ord}} \otimes_{W[[\mathbb{Z}_p^\times]], \nu^k} W \simeq \mathrm{Hom}_W(eH^0(M, \underline{\omega}^{\otimes k}), W) \quad \text{for } k \gg 0$$

Therefore we obtain the theorem:

Theorem: Suppose the existence of the idempotent e and the assumption (C) and (F), then:

(1) $\mathcal{V}^*_{\mathrm{ord}}$ is a finitely generated projective $W[[\mathbb{Z}_p^\times]]$-module.

(2) $\mathcal{V}^*_{\mathrm{ord}} \otimes_{W[[\mathbb{Z}_p^\times]], \nu^k} W \simeq \mathrm{Hom}_W(eH^0(M, \underline{\omega}^{\otimes k}), W)$ for $k \gg 0$ — $(**)$

(3) For every character $\chi \in \Delta^\vee$, $\mathcal{V}^*_{\mathrm{ord}}[\chi]$ is a free Λ-module of rank d_χ

Under the stronger (F^+), $(**)$ holds for all $k \geq k_0$.

Next tasks :

1. The above theorem is just interpolating the " Hecke algebra" $\mathbb{T}^*_{\mathrm{ord}}$. But recall our ultimate goal is to interpolate the space of modular forms .
2. How to construct Hida projectors on the space of p-adic modular forms .
3. In the axiomatic process , we donnot know if $\mathcal{P}_n$ really fits in the previous framework .

§ 6 More on Igusa towers

- Previous approach to Igusa towers : convienient to establish density theorems

$\rightarrow$ Now : reformulate as a moduli of elliptic curves (back to modular curve case)

Setup :
- Fix a prime $p \geq 5$.
- Fix a positive integer N that is coprime to p .
- Let Γ be a congruence subgroup $\Gamma(N)$ or $\Gamma_1(N)$.
- Let $\mathbb{F}/\mathbb{F}_p$ be a finite extension with $W = W(\mathbb{F})$ Witt vectors and $K = \mathrm{Frac}(W)$. $W_m := W/\varpi^m W$.
- When $\Gamma = \Gamma(N)$, we require W contains a primitive N-th root of unity .

- $p\text{-}\mathrm{Alg}/W_m$: the category of p-adically complete W_m-algebras .

Here and later on , $m \geq 1$, with $m = \infty$ included , when $W_\infty := W$.

- Consider the functor for integer $n \geq 0$, congruence subgroup $\Gamma = \Gamma_0(N), \Gamma_1(N), \Gamma(N)$.

$$\mathcal{E}^{ord}_{\Gamma,m,n} : p\text{-Alg}/W_m \longrightarrow \mathcal{S}et$$
$$R \longmapsto \{(E/R, \phi_\Gamma, \phi_{p^n})\}/\simeq$$

where :
- E/R : an elliptic curve over $\mathrm{Spec} R$.
- ϕ_Γ : Γ-level structure
- ϕ_{p^n} : inclusion of group schemes $\mu_{p^n} \hookrightarrow E[p^n]$ (identifying the locally free group scheme μ_{p^n} with a subgroup scheme of $E[p^n]$ over $\mathrm{Spec} R$.)

- Note : When $n' > n$, we can define a projection of functors $\pi^m_{n',n} : \mathcal{E}^{ord}_{\Gamma,m,n'} \longrightarrow \mathcal{E}^{ord}_{\Gamma,m,n}$ by $(E/R, \phi_\Gamma, \phi_{p^n} : \mu_{p^{n'}} \hookrightarrow E[p^{n'}]) \mapsto (E/R, \phi_\Gamma, \phi_{p^{n'}}|_{\mu_{p^n}})$.

Preliminary observations

1° When $n=0$, $m=\infty$, $\mathcal{E}^{ord}_{\Gamma,\infty,0}$ is given by Y_Γ/W.

Yet from now on we take the ordinary locus Y^{ord}_Γ/W since it is the image of all vertical morphisms $\pi_n := \pi^\infty_{n,0}$'s

2° The scheme $Y^{ord}_{\Gamma,\infty,n}$ and the morphism π_n extends to the cusps of X^{ord}_Γ, so get a finite étale morphism $\pi_n : X^{ord}_{\Gamma,\infty,n} \longrightarrow X^{ord}_\Gamma$

For simplicity, we denote
- $S/W := X^{ord}_\Gamma/W$, $S_m = S \times_W W_m$ as before.
- $X^{ord}_{\Gamma,\infty,n}$ are affine for $n \geq 1$. We denote $X^{ord}_{\Gamma,\infty,n} \times_W W_m = \mathrm{Spec}(V_{\Gamma,m,n})$ for all $m \geq 1$.

Then the finite étale morphisms $\pi^\infty_{n+1,n}$ induces injections $V_{\Gamma,m,n} \longrightarrow V_{\Gamma,m,n+1}$ for all m.

Appropriate $\mathcal{D}_n$: Let $\mathbb{E} \longrightarrow X_\Gamma$ be the generalized universal elliptic curve. Then we have a connected-étale exact sequence

$$0 \longrightarrow \mathbb{E}^\circ[p^n] \longrightarrow \mathbb{E}[p^n] \longrightarrow \mathbb{E}^{\acute{e}t}[p^n] \longrightarrow 0, \quad \forall n \geq 1, \quad \text{over } S := X^{ord}_\Gamma/W$$

Let $\mathcal{D}_n = \mathbb{E}^{\acute{e}t}[p^n]$ over S, $\mathcal{C}_n := \mathbb{E}^\circ[p^n]$ over S.

– Use this $\mathcal{D}_n$, we define the previous $T_{m,n}$ and $V_{m,n}$.

Theorem. The scheme $X^{ord}_{\Gamma,m,n} = \mathrm{Spec}(V_{\Gamma,m,n})$ represents the functor $T_{m,n}$, $m \geq 1$.

Proof sketch. Let $\gamma : A \to S_m$ be an S_m-scheme, then recall the definition of $T_{m,n}$,

$$\mathrm{Spec}(V_{m,n})(A) = \{ \psi_n : \gamma^* P_n \xrightarrow{\sim} \underline{\mathbb{Z}/p^n\mathbb{Z}}_A \}$$

- Then over A we have $E_A[p^n] \twoheadrightarrow \gamma^* P_n \xrightarrow{\psi_n} \underline{\mathbb{Z}/p^n\mathbb{Z}}_A$, by the con.-étale exact sequence.
- Taking Cartier dual, we get $(\mu_{p^n})_A \xrightarrow{\sim} (\gamma^* P_n)^D \hookrightarrow E_A[p^n]$ as $E_A[p^n]^D \simeq E_A[p^n]$ by Weil pairing, and the Cartier dual of the constant group scheme $\underline{\mathbb{Z}/p^n\mathbb{Z}}_A$ is $(\mu_{p^n})_A$.

In this way for any $R \in p\text{-Alg}/W_m$, using $\gamma : \mathrm{Spec}R \to S_m$, we have a natural map

$$\mathrm{Spec}(V_{m,n})(\mathrm{Spec}R) \xrightarrow{\theta_R} \mathcal{E}^{ord}_{\Gamma,m,n}(R)$$

Since taking Cartier dual is "reflexive", turn the above process around, we get the natural inverse

$$\mathrm{Spec}(V_{m,n})(\mathrm{Spec}R) \underset{\xi_R}{\overset{\theta_R}{\rightleftarrows}} \mathcal{E}^{ord}_{\Gamma,m,n}(R)$$

as desired! $\square$

Remark: So essentially the description is merely the two perspectives of the connected-étale exact sequence, linked via the Cartier dual.

Ref: [GME, Theorem 3.2.8].

Let's consider a byproduct, solving Task ③ at the end of §5.

Corollary: Let $\pi: \mathbb{E} \to S_m$ be the generalized elliptic curve over S_m with zero section $0: S_m \to \mathbb{E}$. Then there are canonical isomorphisms

$$\pi_* \Omega^1_{\mathbb{E}/S_m} =: \underline{\omega}_{\mathbb{E}/S_m} \simeq P_m \otimes_{\mathbb{Z}/p^m\mathbb{Z}} \mathcal{O}_{S_m}.$$

Proof sketch:

(1) Establish $\pi_* \Omega^1_{\mathbb{E}/S_m} \simeq 0^* \Omega^1_{\mathbb{E}/S_m}$

- Away from cusps, this is a basic property of elliptic curves.
- It is subtle near cusps. (See [GME, proof of Coro. 3.2.9])

(2) Recall $\mathcal{E}_m := \mathbb{E}^\circ[p^m] \subseteq \mathbb{E}[p^m]$, and after an étale base change S_m'/S_m, $\mathcal{E}_{m/S_m'} \simeq \mu_{p^m}$.

Fact: Let A be a $\mathbb{Z}/p^m\mathbb{Z}$-algebra. $\mu_{p^m/A} := \mathrm{Spec}\left(A[T]/(1+T)^{p^m}-1\right)$.
Then $\Omega^1_{\mu_{p^m}/A}$ is an invertible sheaf over μ_{p^m}/A. (Exercise)

Hence descent to S_m, we see $\Omega^1_{\mathcal{E}_m/S_m}$ is an invertible sheaf over S_m.

Recall we have the fundamental exact sequence

$$\Omega^1_{\mathbb{E}/S_m} \otimes_{\mathcal{O}_{\mathbb{E}}} \mathcal{O}_{\mathcal{E}_m} \twoheadrightarrow \Omega^1_{\mathcal{E}_m/S_m} \quad \text{over } \mathcal{E}_m$$

Then pullback along the zero section $0: S_m \to \mathbb{E}^\circ[p^m] = \mathcal{E}_m$, we obtain

$$0^* \Omega^1_{\mathbb{E}/S_m} \twoheadrightarrow 0^* \Omega^1_{\mathcal{E}_m/S_m}.$$

Since both sides are invertible sheaves over S_m by Fact, this is an isomorphism.

(3) Recall the connected-étale sequence stuff, $P_m = \mathrm{Hom}_{\mathrm{GrpSch}}(\mathcal{E}_m, \mathbb{G}_m)$.
Writing $\mathbb{G}_m = \mathrm{Spec}\, \mathbb{Z}[t, \frac{1}{t}]$, we associate

$$\varphi \in P_m \longmapsto \varphi^* \frac{dt}{t} \in \Omega^1_{\mathcal{E}_m/S_m}. \qquad (*)$$

It induces a surjective map $P_m \otimes_{\mathbb{Z}/p^m\mathbb{Z}} \mathcal{O}_{S_m} \twoheadrightarrow 0^* \Omega_{\mathcal{E}_m/S_m}$ by pulling back $(*)$ along the zero section. It is an isomorphism since both sides are invertible sheaves over S_m.

Combining (1) — (3), we are done! $\square$

Now we work toward Task ②, by the above reinterpretation of the Igusa tower:

- Write $V_{T/W} = V$ as the p-adic modular form on T.
- For each $A \in p\text{-Alg}/W$, put

$$V_{T/A} := \varprojlim_n V_{T/W} \otimes_W A/p^n A \quad : \quad \text{p-adic modular form on } T \text{ with coeff. in } A.$$

Then similar to modular forms à la Katz, we reinterprete $f \in V_{T/A}$ as a rule assigning

$$(E/R,\ \phi_T,\ \phi_{p^\infty}: \mu_{p^\infty} \hookrightarrow E) \longmapsto f(E/R, \phi_T, \phi_{p^\infty}) \in R$$

for any p-adic A-algebra R, such that

① The assignment only depends on isomorphism classes.

② Compatibility with base change: if $\rho: R \to R'$ is a continuous A-algebra homomorphism, then

$$f\big((E/R, \phi_N, \phi_{p^\infty}) \times_{R,\rho} R'\big) = \rho\big(f(E/R, \phi_N, \phi_{p^\infty})\big).$$

③ q-expansion: for all level T-structure of "type N" (i.e. $T = T(N), T_1(N), T_0(N)$) on Tate curve $\mathrm{Tate}(q^N)$,

$$f(\mathrm{Tate}(q^N)/R,\ \phi_N,\ \zeta\phi_{p^\infty}^{can}) \in A[\![q^{\frac{1}{N}}]\!], \quad \forall \zeta \in \mathbb{Z}_p^\times.$$

Recall: We have canonical injections $\mu_n \hookrightarrow \mathrm{Tate}(q^N)$.

Replacing ③ by ③' that $f(\mathrm{Tate}(q^N)) \in q^{\frac{1}{N}} A[\![q^{\frac{1}{N}}]\!]$ for any $\zeta \in \mathbb{Z}_p^\times$, we define the subspace of p-adic cusp forms on T.

Then we restate the density theorem.

Density theorem: For any W-algebra A, define $M(T, A) := \bigoplus_{k \geq 0} H^0(M_{T/A}, \underline{\omega}^{\otimes k})$

Then $M(T, K) \cap V_{T/W} \subseteq V_{T/W}[\frac{1}{p}]$ is dense in $V_{T/W}$.

Note: Here we are actually cheating!

- In the definition of $\mathcal{E}^{ord}_{T,m,n}$, we only allowed finite n at $\phi_{p^n}: \mu_{p^n} \hookrightarrow E[p^n]$.
  Yet here we write $\phi_{p^\infty}: \mu_{p^\infty} \hookrightarrow E$.
- So here by ϕ_{p^∞}, we actually means a bunch of compatible $\{\phi_{p^n}: \mu_{p^n} \hookrightarrow E[p^n]\}_{n \geq 1}$
  or regard

$$\mu_{p^\infty} = \bigcup_{n \geq 1} \mu_{p^n}, \quad E[p^\infty] = \bigcup_{n \geq 1} E[p^n] \quad \text{as p-divisible groups}$$

  and ϕ_{p^∞} as morphisms of p-divisible groups.

$\Longrightarrow$ So in the following description of β, we always work on finite height $n \geq 1$.

Moreover, with the geometric interpretation, the inclusion

$$\beta : M(\Gamma, A) = \bigoplus_{k\geq 0} H^0(M_\Gamma/A, \underline{\omega}^{\otimes k}) \longrightarrow V_{\Gamma/A}$$

is quite explicit! We now describe it:

<u>Aim</u> : Let $f \in H^0(M_\Gamma/A, \underline{\omega}^{\otimes k})$, then $\beta(f) \in V_{\Gamma/A}$ sends

$$\left(E/A, \phi_\Gamma, \phi_{p^\infty} : \mu_{p^\infty} \hookrightarrow E \right) \rightsquigarrow \beta(f)(E/A, \phi_\Gamma, \phi_{p^\infty}) \overset{\text{should be}}{=\!=} f(E/A, \omega, \phi_\Gamma)$$

for a canonically defined ω from the datum $(E/A, \phi_\Gamma, \phi_{p^\infty})$.

<u>Method</u> : Similar to the previous proof of the <u>Corollary</u>.

- <u>Finite height</u>: For any $n \geq 1$, we have inclusion $i_n : \mu_{p^n} \hookrightarrow \mathbb{G}_m$. Then the canonical differential $\frac{dt}{t}$ over $\mathbb{G}_m$ pulls back to $i_n^*(\frac{dt}{t})$.

($\overrightarrow{\text{putting them together}}$ $\omega_{can} = i_\infty^*(\frac{dt}{t})$ gives an invariant differential ω_{can} on μ_{p^∞}. regarding μ_{p^∞} as a <u>p-divisible group</u>?)

- <u>Go to EC</u>: For $m \geq 1$, over $A_m := A/p^m A$, $\phi_{p^\infty} : \mu_{p^\infty} \hookrightarrow E$ induces isomorphism

$$\phi_{p^m} : \mu_{p^m} \xrightarrow{\sim} \mathcal{E}_m \hookrightarrow E[p^m]/A_m \qquad \text{over } A_m$$

Then review the proof of the <u>Corollary</u>, we have

$$0^*\Omega^1_{\mu_{p^m}/A_m} \xrightarrow{\sim} 0^*\Omega^1_{E/A_m} \simeq \underline{\omega}_{E/A_m}$$

Now for $\phi_m^*(\frac{dt}{t}) \in \Omega^1_{\mu_{p^m}/A_m}$, we pull it back along zero section $0 : \operatorname{Spec} A_m \to E$ and send it to $\underline{\omega}_{E/A_m}$, we get an invariant differential in $\underline{\omega}_{E/A_m}$.

- Putting them together, we get an invariant differential in $\underline{\omega}_{E/A}$, as desired. It is denoted by $\phi_{p^\infty,*}\,\omega_{can}$

$\Longrightarrow$ In this way, we have the slogan:

<u>SLOGAN</u> : All modular forms are p-adic modular forms!

<u>Furthermore</u> : One checks if $f \in H^0(M/A, \underline{\omega}^{\otimes k})$, then we have $\beta(f) \in V_{\Gamma/A}[\nu^k]$.

* <u>Interlude</u> :

We shall establish the q-expansion principle and compare $V_{T/W}$ with p-adic modular forms à la Serre. They both relies on the deep result :

<u>Irreducibility of the Igusa tower</u> : Suppose that p is a prime outside N. Then the scheme $T^{\mathrm{ord}}_{\Gamma(N),1,n}/\mathbb{F}_p[\mu_N]$ is geometrically irreducible over the finite field $\mathbb{F}_p[\mu_N]$ and is smooth if $N \geq 3$ or $p^n \geq 4$. See [GME, Prop. 2.9.6] or [Katz-Mazur].

<u>Katz's philosophy of q-expansions</u> : If f is a global section of a coherent sheaf over an irreducible smooth curve M over W_m, which is zero at a formal completion at a closed point of M, then $f = 0$. See [Katz73] for q-expansion principle.

$\Longrightarrow$ $\boxed{q\text{-expansion principle}}$: Let $f \in V_{T/A}$. If $f(\mathrm{Tate}(q^N), \phi_T, \phi_{p^\infty}^{\mathrm{can}})$ vanishes for one level T-structure ϕ_T for Tate curve $\mathrm{Tate}(q^N)$, then f vanishes identically.

Moreover, as a result, we have :

<u>Prop.</u> The p-adic topology on $V_{T/W}$ coincides with the norm topology induced by q-expansions à la Serre : for $f \in V_{T/W}$, $\|f\| := \sup_{r \geq 0} |a(r,f)|_p$ with

- $|\cdot|_p$: p-adic valuation on W
- $\sum_{r \geq 0} a(r,f) q^r$ is the q-expansion of f of Tate curve $(\mathrm{Tate}(q^N), \phi_T, \phi_{p^\infty}^{\mathrm{can}})$ for any fixed level T-structure ϕ_T on $\mathrm{Tate}(q^N)$.

<u>Proof idea</u> : Note that if $f \in p^n W[[q^{\frac{1}{N}}]]$, then $f \bmod p^n$ vanishes in V_{T/W_m}, hence by q-expansion "philosophy", $f \in p^n V_{T/W}$. $\square$

§7 Hecke operators

Recall we had explicit description

$$\beta : M(\Gamma, A) = \bigoplus_{k \geq 0} H^0(M_\Gamma/A, \underline{\omega}^{\otimes k}) \longrightarrow V_{\Gamma/A}.$$

To describe Hida projector, we now define Hecke operators on both sides, making β a Hecke-equivariant map.

§7.1 Hecke operator on true modular forms

<u>Setup</u>:
- R is a ring with prime ℓ invertible in R.
- $\Gamma = \Gamma_1(N)$ or $\Gamma(N)$ with N sufficiently large to make the moduli representable.
- Require $(\ell, N) = 1$.

Let E/R be an elliptic curve. Then $\exists\ R \to R'$ finite étale, such that after base change to R',

- $E[\ell]_{R'} \simeq \underline{(\mathbb{Z}/\ell\mathbb{Z})^2}_{R'}$
- $E[\ell]_{R'}$ has $\ell+1$ finite flat subgroup scheme H of rank ℓ over R'.

For such a subgroup scheme H, we can define two isogenies

$$\pi : E_{R'} \longrightarrow E_{R'}/H, \qquad \pi^t : E_{R'}/H \longrightarrow E_{R'}.$$

Then for a triple $(E/R, \omega, \phi_\Gamma)$, we can define another triple

- $E'/R' := E_{R'}/H$ as an elliptic curve over R'
- <u>Two versions of differentials</u>: Let $\omega_{R'}$ be the pullback of ω over E/R to E/R'
  - $\omega_H := (\pi^t)^* \omega_{R'}$ (Katz's definition)
  - The ω'_H such that $\omega_{R'} := \pi^* \omega'_H$ (Hida's definition)
- <u>Level structure</u> given by

$$\begin{tikzcd} & \underline{(\mathbb{Z}/N\mathbb{Z}_{R'})}^{\star} & \\ E_{R}[N] \arrow[ur, "\phi_\Gamma"] \arrow[rr, "\sim", "\pi"'] & & E'[N] \arrow[ul, "\phi_{\Gamma,H}"'] \end{tikzcd}$$

($\star = 1$ or 2 depend on $\Gamma = \Gamma(N)$ or $\Gamma_1(N)$)

Now for $f \in H^0(M_\Gamma, \underline{\omega}^{\otimes k})$,

$$(f|T(\ell))(E_{/R}, \omega, \phi_\Gamma) = \ell^{-1} \sum_{H \subseteq E[\ell]_{/R'}} f(E_{R'}/H, \omega'_H, \phi_{\Gamma,H}) \quad \text{(Hida's defn)}$$

$$= \ell^{k-1} \sum_{H \subseteq E[\ell]/R'} f(E_{R'}/H, \omega_H, \phi_{\Gamma,H}) \quad \text{(Katz's defn)}$$

Check: By definition, $(f|T(\ell))(E_{/R}, \omega, \phi_\Gamma) \in R'$, yet we can show by q-expansion principle that

- $(f|T(\ell))(E_{/R}, \omega, \phi_\Gamma) \in R$
- It does not depend on the choice of the étale cover $R \to R'$.

Generally:

① When ℓ is not invertible in R, but ℓ is not a zero divisor:

Step 1: Define $T(\ell)$ on modular forms of level Γ and weight k over $R[\frac{1}{\ell}]$.

Step 2: Check $T(\ell)$ leaves the space of those over R stable.

② When $\ell | N$, $\Gamma = \Gamma_1(N)$. (we can "only" consider the level $\Gamma_1(N)$-structure)

we do a little modification: for tuple $(E_{/R}, \omega, \phi_\Gamma)$.

- Let $C_N :=$ the image of $\phi_\Gamma : \underline{\mathbb{Z}/N\mathbb{Z}} \longrightarrow E$
- Define

$$(f|U(\ell))(E_{/R}, \omega, \phi_\Gamma) = \ell^{-1} \sum_{\substack{H \subseteq E[\ell]_{R'} \\ H \cap C_N = (0)}} f(E_{/R'}, \omega'_H, \phi_{\Gamma,H}).$$

§ 7.2 Hecke operator for p-adic modular forms

Setup :
- $p \geq 5$
- A : a p-adic W-algebra

We now define the $\boxed{U(p)\text{-operators}}$:

- For triples $(E/R, \phi_T, \phi_{p^\infty} : \mu_{p^\infty} \to E)$, we have a connected-étale exact sequence
$$0 \to \mu_p \xrightarrow{\phi_p} E[p] \xrightarrow{\psi_p} E^{\text{ét}}[p] \to 0 \quad / R$$
- Again for some finite étale extension $R \to R'$, $\exists$ exactly p finite flat subgroup scheme H of $E[p]$ such that
$$\psi_p : H \xrightarrow{\sim} E^{\text{ét}}[p] \quad \text{is an isomorphism (as restriction of } \psi_p)$$
(Hence such an H provides a splitting of the connected-étale exact sequence)
- The tame level structure $\phi_{\Gamma,H}$ is defined similarly.
- Define
$$\phi_{p^\infty,H} : \mu_{p^\infty,R'} \xrightarrow{\phi_{p^\infty,R'}} E_{R'} \overset{\pi}{\twoheadrightarrow} E_{R'}/H$$

Then we define for $f \in V_{\Gamma/A}$, we define
$$(f|U(p))(E/R, \phi_T, \phi_{p^\infty}) = p^{-1} \sum_{H \subseteq E[p]/R'} f(E_{R'}/H, \phi_{\Gamma,H}, \phi_{p^\infty,H})$$
and make similar "<u>Check</u>" here.

Remark : The map
$$\beta : M(\Gamma, A) = \bigoplus_{k \geq 0} H^0(M_\Gamma/A, \underline{\omega}^{\otimes k}) \to V_{\Gamma/A}.$$
is <u>NOT</u> Hecke-equivariant :
- On LHS, we are actually using $T(p)$-operator in general as $p \nmid N$
- On RHS, we are using $U(p)$-operator.

To modify this, we consider
$$\beta : M(\Gamma \cap \Gamma_1(p), A) = \bigoplus_{k \geq 0} H^0(M_{\Gamma \cap \Gamma_1(p)}/A, \underline{\omega}^{\otimes k}) \to V_{\Gamma/A}.$$
is $U(p)$-equivariant !

* We can also define Hecke operators
- $T(\ell)$, $\ell \nmid Np$ on $V_{\Gamma/A}$
- $U(\ell)$, $\ell \mid Np$ on $V_{\Gamma/A}$

We omit the definition here.

Proposition :

(1) The Hecke operators $T(\ell)$, $\ell \nmid Np$ and $U(\ell)$, $\ell \mid Np$ on V_T/W give continuous endomorphisms on V_T/W

(2) The limit $e := \lim_{n\to\infty} U(p)^{n!}$ exists and gives an idempotent in V_T/W.

Proof sketch :

(1) An explicit computation on q-expansions.

- See Zhao Bin's lectures or [GME, Prop. 3.2.12].
- See Katz's computation in [Katz73], yet Katz only considered full level structure.

(2) Warning : Previous algebraic argument fails since V_T/W is very large !

Invoke : Density theorem ! Recall

$$D = \left(\bigoplus_{k\geq 0} H^0(M_{T/K}, \underline{\omega}^{\otimes k}) \right) \cap V_{T/W} \text{ is dense in } V_T/W$$

- Recall $e = \lim_{n\to\infty} U(p)^{n!}$ is well-defined on $H^0(M_{T/K}, \underline{\omega}^{\otimes k})$, hence on their direct sum $\bigoplus_{k\geq 0} H^0(M_{T/K}, \underline{\omega}^{\otimes k})$.
- Hence e is well-defined over each

$$D/\pi^m D \xrightarrow[\sim]{\iota_m \circ J_m} V_T/\pi^m V_T \quad , m \geq 1 .$$

Then taking inverse limit, we have well-defined e over V_T/W. $\square$

§8 Families of p-adic modular forms

Assume p is an odd prime.

- We can also define weights of p-adic modular forms:
  - * Diamond operators: for $z \in \mathbb{Z}_p^\times$, we define the diamond operator $\langle z \rangle$ acting on $f \in V_{T/A}$:

$$(f|\langle z\rangle)(E/R, \phi_N, \phi_{p^\infty} : \mu_{p^\infty} \hookrightarrow E) = f(E/R, \phi_N, z^{-1}\phi_{p^\infty})$$

Ref: See [GME, p263 top line], there for general $a := (a_p, a_N) \in \mathbb{Z}_p^\times \times (\mathbb{Z}/N\mathbb{Z})^\times$,

$$(f|\langle a\rangle)(E/R, \phi_N, \phi_{p^\infty} : \mu_{p^\infty} \hookrightarrow E) = f(E/R, a_N\phi_N, a_p^{-1}\phi_{p^\infty}).$$

  - * A p-adic modular form $f \in V_{T/W}$ is called of weight $s \in \mathbb{Z}_p$ on T if

$$f|\langle z\rangle = z^s f, \quad \forall\, z \in 1+p\mathbb{Z}_p$$

Recall: $z^s \in 1+p\mathbb{Z}_p$ is the p-adic power given by the p-adically convergent series

$$z^s = \sum_{n\geq 0} \binom{s}{n} (z-1)^n.$$

Rmk: So here we have a feeling: we only care about the "profinite action" of $\mathbb{Z}_p^\times$.

Setup: $\Lambda := W[\![1+p\mathbb{Z}_p]\!] \xrightarrow{\sim} W[\![T]\!]$ power series ring

$$[u = 1+p] \longmapsto 1+T$$

Def: (1) A formal power series

$$\Phi(X; q) = \sum_{n\geq 0} a(n,\Phi)(X)\, q^n \in \Lambda[\![q]\!]$$

is called a Λ-adic form if for any $s \in \mathbb{Z}_p$, $\Phi(u^s-1; q) \in W[\![q]\!]$ is a p-adic modular form on T of weight s.

$\Longrightarrow \{\Phi(u^s-1; q)\}_{s\in\mathbb{Z}_p}$ is called a family of p-adic modular forms.

(2) A Λ-adic modular form Φ is called arithmetic if for all sufficiently large positive integers k, $\Phi(u^k-1; q)$ is a true modular form.

(3) A Λ-adic form Φ is called a cusp form if $\Phi(u^s-1; q)$ is a p-adic cusp form for infinitely many $s \in \mathbb{Z}_p$. (equivalently ?, for all $s \in \mathbb{Z}_p$).

Reinterpretation of Λ-adic forms as measures :

① $\Lambda = W[[X]]$ can be identified with $\mathrm{Meas}(\mathbb{Z}_p; W) \subseteq \mathrm{Hom}_W(C(\mathbb{Z}_p, W), W)$

- A W-linear homomorphism $\varphi : C(\mathbb{Z}_p, W) \longrightarrow W$ is called a p-adic measure (or measure) if $\exists$ a constant $B > 0$, s.t.

$$|\varphi(f)|_p \leq B, \quad \text{for any } f \in C(\mathbb{Z}_p, W)$$

Remark : Often we equip $\mathrm{Meas}(\mathbb{Z}_p, W)$ with a norm

$$\|\varphi\|_p = \sup_{\|f\|_p = 1} |\varphi(f)|_p = \sup_{f \neq 0} \frac{|\varphi(f)|_p}{\|f\|_p}, \qquad \|\text{-}\|_p : \text{norm on } C(\mathbb{Z}_p, W).$$

Then $\mathrm{Meas}(\mathbb{Z}_p, W)$ becomes a p-adic Banach W-module under this norm.

Remark : We often write $\varphi(f) =: \int_{\mathbb{Z}_p} f \, d\varphi := \int_{\mathbb{Z}_p} f(x) \, d\varphi(x)$

- Given a measure $\varphi \in \mathrm{Meas}(\mathbb{Z}_p, W)$, we define a formal power series in the following way :
  - By Mähler's theorem, write $f \in C(\mathbb{Z}_p, W)$ uniquely into

$$f(x) = \sum_{n=0}^{\infty} a_n(f) \binom{x}{n}, \qquad a_n(f) \in W.$$

  with $\lim_{n \to \infty} a_n(f) = 0$.

  - Since the partial sum $\left\{ f_m := \sum_{n=0}^{m} a_n(f) \binom{x}{n} \right\}_{m \geq 0}$ converges to f uniformly, for $\varphi \in \mathrm{Meas}(\mathbb{Z}_p, W)$,

$$\begin{aligned}
\int_{\mathbb{Z}_p} f(x) \, d\varphi(x) &= \lim_{m \to \infty} \int_{\mathbb{Z}_p} f_m(x) \, d\varphi(x) \\
&= \lim_{m \to \infty} \sum_{n=0}^{m} a_n(f) \int_{\mathbb{Z}_p} \binom{x}{n} d\varphi(x) \\
&= \sum_{n=0}^{\infty} a_n(f) \int_{\mathbb{Z}_p} \binom{x}{n} d\varphi(x)
\end{aligned}$$

  Thus φ is determined by the sequence $\left\{ \int_{\mathbb{Z}_p} \binom{x}{n} d\varphi(x) : n \geq 0 \right\} \subseteq W$.

  The sequence is bounded as

$$\left| \int_{\mathbb{Z}_p} \binom{x}{n} d\varphi(x) \right|_p \leq \|\varphi\|_p \left\| \binom{x}{n} \right\|_p \leq \|\varphi\|_p$$

  - Then we define a formal power series in T :

$$\boxed{\Phi(T) = \sum_{n \geq 0} \int_{\mathbb{Z}_p} \binom{x}{n} d\varphi(x) \, T^n = \int_{\mathbb{Z}_p} (1+T)^x \, d\varphi(x) \in W[[T]].} \qquad (*)$$

Reverse the above process, we see indeed $\Lambda \simeq \mathrm{Meas}(\mathbb{Z}_p, W)$.

Ref : [Hida blue, §3.3].

Sum up :

- Meas$(\mathbb{Z}_p, W) \rightsquigarrow \Lambda = W[[T]]$: Compute the sequence $\{\int_{\mathbb{Z}_p} \binom{x}{n} d\varphi(x) : n \geq 0\}$ and put them into the "generating power series"

$$\Phi(T) := \sum_{n\geq 0} \left(\int_{\mathbb{Z}_p} \binom{x}{n} d\varphi(x) \right) T^n \in W[[T]] .$$

By $(*)$, more directly $\Phi(T) := \int_{\mathbb{Z}_p} (1+T)^x d\varphi(x) \in W[[T]]$.

- $\Lambda = W[[T]] \rightsquigarrow$ Meas$(\mathbb{Z}_p, W)$: Directly read off the sequence $\{\int_{\mathbb{Z}_p} \binom{x}{n} d\varphi(x)\}_{n\geq 0}$ from the coefficients of $\Phi(T) \in W[[T]]$. Then by Mähler's theorem, this recovers the measure φ.

② Moreover, note the homeomorphism $\theta: \begin{array}{ccc} \mathbb{Z}_p & \longrightarrow & 1+p\mathbb{Z}_p \\ x & \longmapsto & u^x \end{array}$, Λ is identified with Meas$(1+p\mathbb{Z}_p, W)$. This is better for us to consider specializations $T = u^s - 1$:

$$\Phi(u^s-1) = \int_{\mathbb{Z}_p} (1+(u^s-1))^x d\varphi(x) = \int_{\mathbb{Z}_p} (u^x)^s d\varphi(x) \overset{\theta}{=} \int_{1+p\mathbb{Z}_p} x^s d\varphi^\theta(x) , \quad s \in \mathbb{Z}_p$$

Here $\varphi^\theta \in$ Meas$(1+p\mathbb{Z}_p, W)$ is the modified measure over $1+p\mathbb{Z}_p$ via θ .

③ By q-expansion "at a cusp", $V_{T/W} \hookrightarrow W[[q]]$.

- Over $V_{T/W}$ and $W[[q]]$, we have a p-adic norm $\left| \sum_{n\geq 0} a(n,f) q^n \right| = \sup_n |a(n,f)|_p$.

Then we can consider as in ① and ②, replacing W by $V_{T/W}$ and $W[[q]]$.

$\overset{1^\circ}{\Longrightarrow}$ Meas$(1+p\mathbb{Z}_p, W[[q]]) \simeq W[[q]][[X]] = \Lambda[[q]]$: for $\Phi \in \Lambda[[q]]$,

$$\Phi(u^s-1) = \sum_{n\geq 0} a(n,\Phi)(u^s-1) q^n =: \int_{1+p\mathbb{Z}_p} x^s d\varphi(x) \in W[[q]]$$

$\overset{2^\circ}{\Longrightarrow}$ $\boxed{\text{Meas}(1+p\mathbb{Z}_p, V_{T/W}) \simeq \text{the space of } \Lambda\text{-adic forms}} \subseteq W[[q]]$

Action on $V_{T/W}$: let $T: V_{T/W} \longrightarrow V_{T/W}$ be a W-linear bounded map. Then for any Λ-adic form $\Phi \in$ Meas$(1+p\mathbb{Z}_p, V_{T/W})$, $T \circ \Phi \in$ Meas$(1+p\mathbb{Z}_p, V_{T/W})$ corresponds to a Λ-adic form.

eg : T can be a $\mathbb{Z}_p^\times$-action or Hecke operator.

Defn : Let A be a p-adic W-algebra

(1) $f \in V_{\Gamma/A}$ is **p-ordinary** if $f \in eV_{\Gamma/A}$.

(2) A Λ-adic form Φ is **p-ordinary** if $\Phi(u^s-1, q) \in eV_{\Gamma/A}$ is p-ordinary for any $s \in \mathbb{Z}_p$.

Write $M^{ord}(\Gamma, \Lambda, A)$ (resp. $S^{ord}(\Gamma, \Lambda, A)$) be the space of p-ordinary Λ-adic form (resp. cusp form) with level Γ and coefficient A.

- Let $H^{ord}(\Gamma, \Lambda, A)$ (resp $h^{ord}(\Gamma, \Lambda, A)$) be the Λ-subalgebra of $\mathrm{End}_\Lambda(M^{ord}(\Gamma, \Lambda, A))$ (resp. $\mathrm{End}_\Lambda(S^{ord}(\Gamma, \Lambda, A))$)

Vertical control theorem

① $H^{ord}(\Gamma, \Lambda, W) \simeq V_{ord}^*$, $H^{ord}(\Gamma, \Lambda) \otimes_{\Lambda, \nu_k} W \simeq H_k^{ord}(\Gamma \cap \Gamma_1(p), W)$, $k \geq 2$.

② $M^{ord}(\Gamma, \Lambda, W) \simeq \mathrm{Hom}_\Lambda(V_{ord}^*, \Lambda)$.

③ $M^{ord}(\Gamma, \Lambda, W)$ is free of finite rank over Λ.

④ For $k \geq 2$, $\Phi \longmapsto \Phi(u^k-1, q)$ induces an isomorphism

$$M^{ord}(\Gamma, \Lambda) \otimes_{\Lambda, \nu_k} W \xrightarrow{\sim} M_k^{ord}(\Gamma \cap \Gamma_1(p), W)$$

Therefore, every p-ordinary Λ-adic form is arithmetic.

Similar results holds for cusp forms.

Proof sketch : Put together previous results.

- For any $n \geq 0$, $a(n) : V_{\Gamma/W} \to W$ mapping p-adic modular forms f to its q^n-coefficient via a q-expansion.

$\Longrightarrow$ Mod ϖ^m and passing to "$\varinjlim_m$", we have, after taking ordinary part

$$a(n) : V_{ord} \longrightarrow K/W$$

Hence $a(n) \in V_{ord}^*$.

- Then consider $a : \mathrm{Hom}_\Lambda(V_{ord}^*, \Lambda) \longrightarrow \Lambda[[q]]$

$$\phi \longmapsto \sum_{n \geq 0} \phi(a(n)) q^n =: \Phi(X, q).$$

- Now note $\mathrm{Hom}_\Lambda(V_{ord}^*, \Lambda) \otimes_{\Lambda, \nu_k} W \overset{(*)}{\simeq} \mathrm{Hom}_W(V_{ord}^* \otimes_{\Lambda, \nu_k} W, W)$

$$\simeq \mathrm{Hom}_W(\mathrm{Hom}_W(eH^0(M_{\Gamma/W}, \omega^{\otimes k}), W), W)$$

$$\simeq eH^0(M_{\Gamma/W}, \omega^{\otimes k})$$

$$\simeq \mathrm{Hom}_W(\mathrm{Hom}_W(M_k^{ord}(\Gamma \cap \Gamma_1(p), W), W), W)$$

$$\simeq M_k^{\mathrm{ord}}(T \cap T_1(\varphi), W)$$

Here (☆) holds since

$\mathcal{V}_{\mathrm{ord}}^*$ is free of finite rank over Λ. —— ㉑

Moreover via a, if we embed $\mathrm{Hom}_\Lambda(\mathcal{V}_{\mathrm{ord}}^*, \Lambda)$ into $\Lambda[[q]]$, tracking that the isomorphism brings $\varphi \in \mathrm{Hom}_\Lambda(\mathcal{V}_{\mathrm{ord}}^*, \Lambda)$ to $\Phi(u^k-1, q) \in eH^0(M_{\Gamma/W}, \underline{\omega}^{\otimes k})$

$\Longrightarrow$ Therefore a induces

$$\mathrm{Hom}_\Lambda(\mathcal{V}_{\mathrm{ord}}^*, \Lambda) \simeq M^{\mathrm{ord}}(T, \Lambda, W) \text{ — (!)}, \text{ implies } ②.$$

- Note ㉑ again, we obtain ③. ④
- Take Λ-dual on both sides of (!), we see
  - $\mathrm{Hom}_\Lambda(\mathrm{Hom}_\Lambda(\mathcal{V}_{\mathrm{ord}}^*, \Lambda)) \simeq \mathcal{V}_{\mathrm{ord}}^*$ by ㉑.
  - $\mathrm{Hom}_\Lambda(M^{\mathrm{ord}}(T, \Lambda, W)) \simeq H_{\mathrm{ord}}(T, \Lambda, W)$: this is the Λ-adic version of the perfect pairing of M_{ord} and H_{ord} : [GME, Lemma 3.2.14].

    Remark. Actually the Hecke character $T(n)$ corresponds to the $a(n)$ above under this perfect pairing.

Therefore we get ①.

- All these arguments work for cusp forms. $\square$

§ 9 Galois representations attached to Λ-adic forms

Setup :
- $\Gamma = \Gamma_0(Np)$
- Dirichlet character $\chi : (\mathbb{Z}/Np\mathbb{Z})^\times \to \overline{\mathbb{Q}}^\times$ as nebentypus.
- $\mathbb{K} := \mathrm{Frac}(\Lambda)$.

For a Λ-algebra A, we define $H^{ord}(\Gamma,\chi,A) := H^{ord}(\Gamma,\chi,\Lambda) \otimes_\Lambda A$.

Integral structure of Λ-adic forms :

- **Theorem** : $H^{ord}(\Gamma,\chi,\Lambda)$ is reduced. $H^{ord}(\Gamma,\chi,\mathbb{K})$ is semisimple.

  Corollary :
  - $\exists$ a finite extension $\mathbb{L}|\mathbb{K}$ s.t. $M^{ord}(\Gamma,\chi,\mathbb{L})$ has a basis consisting of common Hecke eigenforms of all Hecke operators.
  - If we normalize the basis element so that the coefficient of q is 1, then the elements in the basis belongs to $m^{ord}(\Gamma,\chi,\mathbb{I})$, where $\mathbb{I}$ is the integral closure of Λ in $\mathbb{L}$.

  * So this is why we often go to $\mathbb{I}$-adic form instead of only Λ-adic forms.

- **Duality** : Then we have isomorphism

$$\theta : \mathrm{Hom}_{\mathbb{L}}(H^{ord}(\Gamma,\chi,\mathbb{L}),\mathbb{L}) \xrightarrow{\sim} m^{ord}(\Gamma,\chi,\mathbb{L})$$

$$\varphi \longmapsto \mathbb{I}\text{-adic form } \sum_{n=0}^{\infty} \varphi(T(n))\, q^n$$

Galois representation : Let $F \in S^{ord}(\Gamma,\chi,\mathbb{I})$ be an $\mathbb{I}$-adic normalized cuspidal Hecke eigenform, corresponds to the eigensystem

$$\lambda : h^{ord}(\Gamma,\chi,\Lambda) \longrightarrow \mathbb{I} \quad , \quad \mathbb{L} = \mathrm{Frac}(\mathbb{I})$$

Then $\exists!$ Galois representation $\rho_F : \mathrm{Gal}_{\mathbb{Q}} \longrightarrow \mathrm{Gl}_2(\mathbb{L})$ s.t.

① ρ_F is continuous and irreducible

② ρ_F is unramified outside Np.

③ For $q \nmid Np$. $\det(1-\pi(\mathrm{Frob}_q)T) = 1-\lambda(T(q))T + \chi(q)\,\kappa(\langle q\rangle)\, q^{-1} T^2$,

where
- $\langle q\rangle = \omega(q)^{-1} q \in 1+p\mathbb{Z}_p$
- $\kappa : 1+p\mathbb{Z}_p \longrightarrow \Lambda^\times$ sending u to $1+X$.

Proof approach :
- Hida : use the p-divisible group of the Jacobian variety of modular curves. Hard to generalize!
- Wiles : pseudo-representations. Many generalizations!

Remark :

(1) This galois representation is quite different from the familiar ones (as continuous group homomorphisms $\rho : Gal_{\mathbb{Q}} \longrightarrow GL_2(\bar{\mathbb{Q}}_p)$). Here by "continuous", we mean $\exists$ an $\mathbb{I}$-submodule $L \subseteq \mathbb{L}^2$ such that

- L is finitely generated over $\mathbb{I}$, s.t. $L \otimes_{\mathbb{I}} \mathbb{L} \simeq \mathbb{L}^2$
  (note : L is not necessarily free $\mathbb{I}$-module)
- L is stable under ρ_F.
- The induced $\rho_F^{\#} : Gal_{\mathbb{Q}} \longrightarrow End_{\mathbb{I}}(L)$ is continuous with L is equipped with the m-adic topology, where m is the maximal ideal of $\mathbb{I}$.

* Note : It is "impossible" to equip a topology on $\mathbb{L}$ making π continuous at the same time ρ_F is useful.

(2) Interpolation property :

- Note : Krull dimension of $\mathbb{I}$ is 2.
- Let $\wp$ be a height one prime of $\mathbb{I}$. Then we have

Facts : $L_{\wp} := L \otimes_{\mathbb{I}} \mathbb{I}_{\wp}$ is a free $\mathbb{I}_{\wp}$-module of rank 2.

$$\Longrightarrow L/\wp L \simeq L_{\wp}/\wp L_{\wp} \text{ is a free } \mathbb{I}/\wp\text{-module of rank } 2.$$

Then we can reduce ρ_F further to

$$\rho_{F,\wp} : Gal_{\bar{\mathbb{Q}}} \longrightarrow GL_2(\mathbb{I}) \longrightarrow GL_2(\mathbb{I}/\wp)$$

- As we saw in Special Talk – Hida theory, every specialization $\varphi : X \longmapsto u^s - 1$ corresponds to prime ideals $\wp := \ker\varphi$ of height one in $\mathbb{I}$.

$\Longrightarrow$ Proposition : For such a height one prime $\wp$ in $\mathbb{I}$, $\rho_{F,\wp}$ is precisely the Galois representation attached to $F(u^s-1)$ correspoding to the specialization of $\wp$.