Talk 05 Local computations : archemedian places

There are two different choices, one by Xin Wan, the other by Skinner–Urban.
We will focus on Wan's version :

Runing hypothesis on π_∞ :

Suppose π_∞ is the holomorphic discrete series rep'n associated to the scalar weight $(0,\cdots,0,k,\cdots,k)$ as a rep of $GU(r,s)(F_\infty)$.

Intuition : $GL_2(\mathbb{R})$ basically has

- " holomorphic discrete series " $\longleftrightarrow$ " holomorphic automorphic forms "
- " principle series " $\longleftrightarrow$ " Maaß waveforms "

So here we are actually choosing with the first case .

Recall for $GU(n,n)$, the corresponding Hermitian symmetric domain $X^+_{n,n}$ is

$$X^+_{n,n} = \{ x \in M_n(\mathbb{C}) \mid i(x^* - x) > 0 \} .$$

Then we define a distinguished point in $X^+_{n,n}$ (we are in the constant signature case)

$$\mathring{i}^{\heartsuit} := \begin{array}{c} \begin{pmatrix} \frac{1}{2} i \cdot 1_s & & & \\ & i & & \\ & & \frac{1}{2} S & \\ & & & \frac{1}{2} i \cdot 1_s \end{pmatrix} \end{array} \begin{array}{l} s \\ 1 \\ r-s \\ s \end{array} \in M_{r+s+1}(\mathbb{C}) \quad \text{for } GU(r+s+1, r+s+1)$$

$$(n^{\heartsuit} = r+s+1)$$

$$\mathring{i}^{\diamondsuit} := \begin{pmatrix} \frac{1}{2} i \cdot 1_s & & \\ & \frac{1}{2} S & \\ & & \frac{1}{2} i \cdot 1_s \end{pmatrix} \begin{array}{l} s \\ r-s \\ s \end{array} \in M_{r+s}(\mathbb{C}) \quad \text{for } GU(r+s, r+s) .$$

$$(n^{\diamondsuit} = r+s)$$

Recall for $g = \begin{pmatrix} a & c \\ h & d \end{pmatrix} \in GU(n,n)$, we defined the second automorphy factor as

$$\mu : GU(n,n)(\mathbb{R}) \times X^+_{n,n} \longrightarrow GL_n(\mathbb{C})$$
$$(g, x) \longmapsto hx + d$$

$\boxed{\text{Siegal section at } \infty}$ For $v \in \{\infty\}$, the Siegal section we choose is

$$f_v^{sieg,\heartsuit}(g,z) := \mu(g,\mathring{i})^{-k} |\mu(g,\mathring{i})|^{k-2z-n-1}$$

$$f_v^{sieg,\diamondsuit}(g,z) := \mu(g,\mathring{i})^{-k} |\mu(g,\mathring{i})|^{k-2z-n}$$

(or write $n^\bullet$ for $\bullet = \{\heartsuit, \diamondsuit\}$, then $f_v^{sieg,\bullet}(g,z) = \mu(g,\mathring{i})^{-k} |\mu(g,\mathring{i})|^{k-2z-n^\bullet}$).

In Skinner-Urban, they used

$$f_v^{SU,\bullet}(g,z) = \mu(g, i1_{n^\bullet})^{-k} |\mu(g, i1_{n^\bullet})|^{k-2z-n^\bullet}$$

To use SU's computation, take $\heartsuit$-case below, we relate Wan's $\mathring{i}$ with SU's $i1_n$ by

$g_0^{\heartsuit} = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3, \lambda_4 ; \mu_1, \mu_2, \mu_3, \mu_4)$ s.t.

$$g_0^{\heartsuit} \cdot i1_n = g_0 \cdot \begin{pmatrix} i1_{n^\heartsuit} \\ 1 \end{pmatrix} = \begin{pmatrix} \mathring{i}^{\heartsuit} \\ 1 \end{pmatrix}$$

which gives

$$\begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \lambda_3 & \\ & & & \lambda_4 \end{pmatrix} \begin{pmatrix} \mu_1^{-1} & & & \\ & \mu_2^{-1} & & \\ & & \mu_3^{-1} & \\ & & & \mu_4^{-1} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} 1_s & & & \\ & 1 & & \\ & & \frac{S}{2i} & \\ & & & \frac{1}{2} 1_s \end{pmatrix} . \quad \text{——} \ (*)$$

Recall by our assumption on S, $i^{-1}S = \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_{r-s} \end{pmatrix} > 0$, so there exists $a_1', \cdots, a_{r-s}'$ such that

$$\begin{pmatrix} a_1' & & \\ & \ddots & \\ & & a_{r-s}' \end{pmatrix}^2 = \frac{S}{2i}, \quad a_1', \cdots, a_{r-s}' \in \mathbb{R}_{>0}$$

$$a_1' \cdots a_{r-s}' = \sqrt{a_1 \cdots a_{r-s}/2}$$

Then we can take

- $\lambda_1 = \frac{1}{\sqrt{2}}$, $\mu_1 = \sqrt{2}$
- $\lambda_2 = 1$, $\mu_2 = 1$
- $\lambda_3 = \begin{pmatrix} a_1' & & \\ & \ddots & \\ & & a_{r-s}' \end{pmatrix}$, $\mu_3 = \lambda_3^{-1}$
- $\lambda_4 = \frac{1}{\sqrt{2}}$, $\mu_4 = \sqrt{2}$

to make (*) holds. Then by cocycle condition, i.e. $\mu(gg_0, i1_n) = \mu(g, i)\mu(g_0^{\heartsuit}, i1_n)$, and

$$\mu(g_0^{\heartsuit}, i1_n) = \det\begin{pmatrix} \mu_1^{-1} & & \\ & \ddots & \\ & & \mu_4^{-1} \end{pmatrix} = \left(\frac{1}{\sqrt{2}}\right)^{2s} \cdot a_1' \cdots a_{r-s}' = \frac{1}{2^s}\sqrt{\frac{a_1 \cdots a_{r-s}}{2}}$$

make it real to get rid of "$1-1$"

Then

$$f_v^{\mathrm{Siig}, \heartsuit}(g, z) = \mu(g, i)^{-k} \, |\mu(g, i)|^{k-2z-n-1}$$

$$= \mu(gg_0, i1_n)^{-k} \, |\mu(gg_0, i1_n)|^{k-2z-n-1} \cdot \mu(g_0, i1_n)^{k} \, |\mu(g_0, i1_n)|^{k-2z-n-1}$$

$$= f_v^{SU, \heartsuit}(gg_0, z) \cdot \mu(g_0, i1_n)^{2k-2z-n-1} \leftarrow \text{a "constant" related to } z$$

Let's name the extra factor as

$$S_v(n, z) := \mu(g_0, i1_n)^{2k-2z-n-1}$$

<u>Note</u>: In the $\diamondsuit$-case, suffices to delete the second (block) row and the rests are the same.

$\boxed{\text{Fourier coefficient}}$

We first state the computation for $f_v^{SU,\bullet}$. The following result is stated for both ♡ and ◊ :

<u>Theorem</u> : [SU14, Lemma 11.4] Suppose $\beta \in S_n(\mathbb{R})$. Then

① The local Whittaker integral

$$z \longmapsto W_\beta(h; f_v^{SU}, z) := \int_{\mathrm{Herm}_n(F_v)} f_v^{SU}\left(z, w_n \begin{pmatrix} 1_n & \sigma \\ & 1_n \end{pmatrix} h\right) e_v(-\mathrm{Tr}\beta\sigma)\, d\sigma$$

has a meromorphic continuation to all of $\mathbb{C}$.

② If $k \geq n$, $W_\beta(h; f_v^{SU}, z)$ is holomorphic at $z_k = \frac{k-n}{2}$

③ For $y \in GL_n(\mathbb{C})$,

$$W_\beta\left(\mathrm{diag}(y, \bar{y}^{-*}); f_v^{SU}, z_k\right) = \begin{cases} 0 & \det\beta \leq 0 \\ C_v(n,k) \cdot e_v(i\,\mathrm{Tr}(\beta y y^*))(\det\beta)^{k-n}(\det\bar{y})^k & \det\beta > 0 \end{cases}$$

Here

$$C_v(n,k) = \frac{(-2)^{-n}(2\pi i)^{nk}\left(\frac{2}{\pi}\right)^{\frac{n(n-1)}{2}}}{\prod_{j=0}^{n-1}(k-j-1)!}$$ is a constant related to k and n. (not related to β, y, z)

As a corollary, by the previous section,

$$W_\beta\left(\mathrm{diag}(y, y^{-*}); f_v^{sieg}, z_k\right) = W_\beta\left(\mathrm{diag}(y^{SU}, y^{SU,-*}); f_v^{SU}, z_k\right) \cdot \delta_v(n, z).$$

where $y^{SU} := y \cdot \underline{\begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \lambda_3 & \\ & & & \lambda_4 \end{pmatrix}}$. Here we see the importance to have λ_i to be real.

This is the computation originally by Shimura.

Proof: Since $G_n(\mathbb{R}) = Q_n(\mathbb{R}) \cdot K_{n,\infty}$ by Iwasawa decomposition, it suffices to prove ①, ② for $h = \mathrm{diag}(y, y^{-*})$. Then direct computation shows:

$$x := w_n \begin{pmatrix} 1_n & \sigma \\ & 1_n \end{pmatrix} \begin{pmatrix} y & \\ & y^{-*} \end{pmatrix} = \begin{pmatrix} 0 & y^{-*} \\ -y & -\sigma y^{-*} \end{pmatrix},$$

then $\mu_n(x, i1_n) = \det(-yi - \sigma y^{-*}) = \underline{(-1)^n \det y^{-*}}\ \underline{\det(iyy^* + \sigma)}$.

Hence

$$W_\beta(h; f_v^{SU}, z) := \underline{(-1)^{-nk} (\det y^*)^k \left|\det y^*\right|^{-(k-2z-n)}} \int_{\mathrm{Herm}_n(\mathbb{R})} \underline{\det(iyy^* + \sigma)^{-k} \left|\det(iyy^* + \sigma)\right|^{k-2z-n}} \times e_{\mathbb{R}}(-\mathrm{Tr}\beta\sigma)\, d\sigma$$

- $k - n = 2z_k$
- get rid of $|\cdot|$

$$= (-1)^{-nk} (\det \bar{y})^k \left|\det \bar{y}\right|^{2(z - z_k)} \int_{\mathrm{Herm}_n(\mathbb{R})} e_{\mathbb{R}}(-\mathrm{Tr}\beta\sigma) \det(iyy^* + \sigma)^{-k + z_k - z} \det(-iyy^* + \sigma)^{z_k - z} \cdot d\sigma$$

$$= 2^{\frac{n(n-1)}{2}} (-1)^{-nk} (\det \bar{y})^k \left|\det \bar{y}\right|^{2(z - z_k)} \cdot \xi_{Shi}(yy^*, \beta, z - z_k + k, z - z_k)$$

Here $\xi_{Shi}(-, -, s, s')$ is the function in [Shi97, (18.11.4)]. In Lemma 18.12 loc cit:

- ξ_{Shi} is convergent at least for $\mathrm{Re}(s) > n$, and can be meromorphically continued as a function of s and s' on $\mathbb{C} \times \mathbb{C}$.
- "The poles can be controled by finitely many Γ-functions"

$\Longrightarrow$ $W_\beta(h; f_v^{SU}, z)$ is holomorphic at $z = z_k$ if $k \geq n$ and is zero at $z = z_k$ if $\beta \leq 0$.

For ③, by previous computation,

$$W_\beta\left(\mathrm{diag}(y, \bar{y}^{-*}); f_v^{SU}, z_k\right) = 2^{\frac{n(n-1)}{2}} (-1)^{-nk} (\det \bar{y})^k \cdot \xi_{Shi}(yy^*, \beta, k, 0)$$

By another computation of [Shimura 1982, (1.23)], we obtain the result. □

$\boxed{\text{Pullback integrals}}$

Confused : Go back to the definition.

$$F_\phi(f^{\mathrm{Sieg},\heartsuit}, z, g, \chi) = \int_{U(r,s)(\mathbb{A}_F)} \bar{\chi}(\det g_1 h)\, f^{\mathrm{Sieg},\heartsuit}(z, \iota_\heartsuit(g, g_1 h))\, \phi(g_1 h)\, dg_1$$

$$= S_v(n,z) \int_{U(r,s)(\mathbb{A}_F)} \bar{\chi}(\det g_1 h)\, f^{SU,\heartsuit}(z, \iota_\heartsuit(g, g_1 h) g_0)\, \phi(g_1 h)\, dg_1$$

Here the adjusting g_0 does <u>NOT</u> lie in the image of

$$\iota_\heartsuit : GU(r+1, s+1) \underset{\mathbb{G}_m}{\times} -GU(r,s) \longrightarrow GU(r+s+1, r+s+1) \simeq GU(V_\heartsuit, \eta_{r,s}^\heartsuit).$$

$$(g_1, g_2) \longmapsto M_\heartsuit^{-1} \begin{pmatrix} g_1 & \\ & g_2 \end{pmatrix} M_\heartsuit$$

by very explicit calculations. Actually diagonal matrices in $\iota_\heartsuit$ must have the form

$$\mathrm{diag}(\underset{s}{a}, \underset{1}{g}, \underset{r-s}{m}, \underset{s}{s}, \underset{1}{s}\,;\, \underset{s}{y}, \underset{r-s}{m}, \underset{s}{a}) \longleftarrow\!\!\bullet\ \mathrm{diag}(a, g, m, s, y\,;\, a, m, s)$$

So how can we use the pullback integral computed in SU to Wan's case ?

- In [Wan15ANT], he directly gave a formula without mentioning SU, but refering to [Shimura97]. <u>I choose to believe in him</u>.
- In [Wan20ANT], he ignored this issue completely : he doesn't care the possible changes (<u>Lemma 6.7</u>) and the coefficients.

<u>Theorem</u> [Wan15ANT, Lemma 4.3] The pullback integrals $F^\heartsuit$ and $F^\diamondsuit$ converges absolutely for $\mathrm{Re}(z)$ sufficiently large, and for such z,

- $F_\phi^\heartsuit(f_v^{\mathrm{Sieg},\heartsuit}, z, g) = c_v^\heartsuit(k,z)\, F_v^{\mathrm{Klig}}(k, z\,; g) \in V$
- $F_\phi^\diamondsuit(f_v^{\mathrm{Sieg},\diamondsuit}, z, g) = c_v^\diamondsuit(k,z)\, \underline{\pi(g)\varphi} \quad \in V$

here. $c_v^\diamondsuit(k,z) = 2^\nu |\det S|_v^b \cdot \begin{cases} \pi^{(a+b)b}\, \Gamma_b(z + \frac{n+k}{2} - a - b)\, \Gamma_b(z + \frac{n-k}{2})^{-1} & b > 0, \\ 1 & b = 0 \end{cases}$

- $\Gamma_m(s) := \pi^{\frac{m(m-1)}{2}} \prod_{k=0}^{m-1} \Gamma(s-k)$, $\quad \nu := (a+2b)db$, $\quad d := [F:\mathbb{Q}]$.
- $c_v^\heartsuit(k,z) = c_v^\diamondsuit(k, z+\frac{1}{2})$.

Now let me explain the distinguished F_v^{Kling} :

As in the theorem above, the following discussion is only for $\heartsuit$.

- By assumption on π_∞, there exists a unique (up to scalar) vector $v \in \pi$ such that
$$g \cdot v = \det \mu_{GU(r,s)}(g, i)^{-k} \cdot v, \quad \forall g \in GU(r,s)(F_v).$$

For a GL_2-case, one may see the original [SU, §9.2.1]. But I haven't found any reference dealing with the general case!

- Then by Frobenius reciprocity, there exists a unique (up to scalar) vector F^{Kling} in the Klingen section space $I_P(\pi, \chi)$ such that
$$g \cdot F^{Kling}(k, z, -) = \det \mu_{GU(r+1,s+1)}(g, i)^{-k} \cdot F^{Kling}(k, z, -), \quad \forall g \in GU(r+1, s+1)(F_v).$$

- Normalization : Fix v and multiply F^{Kling} by a constant so that $F^{Kling}(1) = v$

Remark : In the previous theorem, the constants "$c_v^\bullet(k,z)$" is cumbersome. In [Wan2020ANT], Wan further choice v so that after the above normalization, the constant $c_v^\bullet$ is absorbed in F^{Kling}, so that
$$F_\phi(f_v^{sieg,\bullet}, z, g) = F^{Kling}(k, z, g).$$

This section is called the "good Klingen section" at $v \in \infty$.

Archemedian Böchner integral :

Recall the Böchner integral when computing the constant terms :

$$M_v(f^{kling}_{z,v})(g) = \int_{N_P(F_v)} f^{kling}_{z}(wng)\, dn , \qquad w := \begin{pmatrix} & & 1_{b+1} \\ & 1_a & \\ -1_{b+1} & & \end{pmatrix}$$

It actually acts as an intertwing operator :

- For π, we define $(\pi^\vee, V)$ be the $(\mathfrak{gu}(\mathbb{R}), K'_\infty)$-module given by
$$\pi^\vee(x) := \pi(\mathrm{Ad}(\eta)x) := \pi(\eta^{-1}x\eta) , \qquad \eta := \begin{pmatrix} & & 1_b \\ & 1_a & \\ -1_b & & \end{pmatrix}$$
for $x \in \mathfrak{gu}(\mathbb{R})$ or K'_∞.

- Again we formulate the Klingen section space
$$I^\vee_P(\pi, \chi, z) := I_P(\pi^\vee \otimes \chi\circ\det , \bar{\chi}^c, z)$$
Then the Böchner integral is actually a linear operator
$$M_v(-) \in \mathrm{Hom}_{\mathbb{C}}\left(I_P(\pi,\chi,z) , I^\vee_P(\pi,\chi,z) \right) .$$ (One may check directly).

With suitable adjustments, the key is that
$$M_v(F^{kling}) = c(\pi, z)\cdot F^{kling,\vee}$$

Theorem : [SU14, Lemma 9.3] [Wan15ANT, Lemma 5.10].

$$c(\pi,z) = \pi^{a+2b+1} \prod_{i=0}^{b-1}\left(z+\frac{k}{2}-\frac{1}{2}-i-a\right)^{-1}\left(z-\frac{k}{2}+\frac{1}{2}-i\right)^{-1} \prod_{i=0}^{a-1}\left(-1+i-2z+2b\right)^{-1}$$

$$\times \frac{\Gamma(2z+a)\, 2^{-1-2z+2b}}{\Gamma\left(\frac{a+1}{2}+z+\frac{k}{2}\right)\Gamma\left(\frac{a+1}{2}+z-\frac{k}{2}\right)} \det\left(\frac{iS}{2}\right)^{-2}$$

Here when a or $b=0$, the product is regarded as 1 respectively.

Note : In [SU14], $r=s=1$, hence $b=1$, $a=0$. In this case, we get the same Gamma part of [SU14] and a slight difference on the power of 2 and i, once we use the functional equation $\Gamma(x+1) = x\Gamma(x)$.

Corollary [SU14, Lemma 9.7] [Wan15AN], Corollary 5.11]

When $k > \frac{3}{2}a+2b$, or $k \geq 2b$ and $a=0$, we have $c(\pi, z_k)=0$ for $z_k = \frac{k-(a+2b+1)}{2}$.

(actually I think Wan's bound is not enough. k should be larger. See my modification :))

Proof: Directly use the explicit expression above.

- Recall as a complex function, $\Gamma(z)$ has no zeros and has simple poles at negative integers and zero.
- Let $z = z_k = \frac{k-(a+2b+1)}{2}$, then the decomominator of $c(\pi, z)$ is

$$\prod_{i=0}^{b-1}\left(k-\left(\tfrac{3}{2}a+b+i+1\right)\right)\left(-\tfrac{a}{2}-b-i\right)\prod_{i=0}^{a-1}\left(a+4b+i-k\right)\cdot \det\left(\tfrac{iS}{2}\right)^2\Gamma(k-b)\,\Gamma(-b)$$

As $b \geq 0$, $\Gamma(-b)$ contributes a pole already. Therefore, we need to guarantee that the other factors are nonzero:

- $k-\left(\frac{3}{2}a+b+i+1\right) \neq 0$, $\forall\, i=0,\cdots,b-1$
  A sufficient condition is that $\boxed{k > \frac{3}{2}a+2b}$.
  | When $b \neq 0$ this bound appears
- $a+4b+i-k \neq 0$, $\forall\, i=0,\cdots,a-1$.
  A sufficient condition is that $\boxed{k > 2a+4b-1}$
  | When $a \neq 0$ this bound appears

and moreover no poles appear in the numerator:

$$\text{numerator} = \pi^{a+2b+1}\,\Gamma(k-(2b+1))\,2^{a+2b-k}$$

So we require $k-(2b+1) > 0$, i.e. $\boxed{k > 2b+1}$.
| this bound always appears.

So as I calculate above, the bound for k should be:

- $\underline{a=0}$ $(\Rightarrow b \neq 0)$: $k > 2b+1$.
- $\underline{b=0}$ $(\Rightarrow a \neq 0)$: $k > \max\{1, 2a-1\} = \begin{cases} 2a-1 & a>1 \quad (\text{otherwise}) \\ 1 & a=1 \quad (u(1,0)\text{-case}) \end{cases}$
- $\underline{a \neq 0 \text{ and } b \neq 0}$: $k > \max\left\{\frac{3}{2}a+2b,\ 2a+4b-1,\ \cancel{2b+1}\right\}$.

$$= \max\left\{\tfrac{3}{2}a+2b,\ 2a+4b-1\right\}$$

Compute: $\delta = 2a+4b-1-\left(\frac{3}{2}a+2b\right)$

$$= \tfrac{1}{2}a+2b-1 \geq 0 \iff \tfrac{1}{2}a+2b \geq 1, \text{ always holds!}$$

$a \neq 0\ \&\ b \neq 0$

So in this case, $k > 2a+4b-1$. In general, Wan only touched "$\frac{3}{2}a+2b$"

$\square$