Talk 2 Klingen Eisenstein series

In this talk, the task is to make sense of the following diagram:

$$\begin{tikzcd}
\mathcal{D} = (\sigma, \chi, \Sigma) \arrow[r, squiggly] & \boxed{\text{Klingen Eisenstein series } E^{\mathrm{Kling}}_{\sigma,\chi,s}} \arrow[dl, "\text{constant term (See Talk 3)}"'] \arrow[dr, "\text{Galois representation}"] & \\
L(\sigma, \chi, s+\tfrac{1}{2}) & & \text{“}\begin{bmatrix} \rho_\chi & & \\ & \rho_\sigma & \\ & & \rho_{\chi^{-c}} \end{bmatrix}\text{”}
\end{tikzcd}$$

Setup:
- K/F CM field.
- $m \geq n \geq 0$ be two integers.
- Consider the unitary group $G := U(m,n)$, $G^{\heartsuit} := U(m+1, n+1)$.

  i.e. they are all unitary groups of pure signature. It is still confusing to me how to define everything for mixed signature case.

  $\rightsquigarrow$ eg: The definition of Klingen parabolic subgroup of $G^{\heartsuit}$: mixed sgn?

§1 Klingen Eisenstein series

Recall the Klingen parabolic subgroup $P := P^{(1)} \subseteq U(m+1, n+1)$ defined in Talk 1:

$$P := \left\{ g = \begin{array}{c} \begin{bmatrix} a & & b & c & * \\ * & \bar{x}^{*} & * & * & * \\ d & & e & f & * \\ h & & \ell & k & * \\ & & & & x \end{bmatrix} \begin{matrix} n \\ 1 \\ m-n \\ n \\ 1 \end{matrix} \\ \begin{matrix} n & 1 & m-n & n & 1 \end{matrix} \end{array} \in G^{\heartsuit} \;\middle|\; \begin{matrix} \begin{bmatrix} a & b & c \\ d & e & f \\ h & \ell & k \end{bmatrix} \in G \\ x \in \mathrm{Res}^K_F \mathbb{G}_m \end{matrix} \right\} \subseteq G^{\heartsuit}.$$

with Levi decomposition $P = MN$, where

$$M = \left\{ g = \begin{array}{c} \begin{bmatrix} a & & b & c & \\ & \bar{x}^{*} & & & \\ d & & e & f & \\ h & & \ell & k & \\ & & & & x \end{bmatrix} \begin{matrix} n \\ 1 \\ m-n \\ n \\ 1 \end{matrix} \\ \begin{matrix} n & 1 & m-n & n & 1 \end{matrix} \end{array} \in P \right\} \xleftarrow[m]{\sim} G^{\heartsuit} \times \mathrm{Res}^K_F \mathbb{G}_m$$

$$g = m(g_0, x) \qquad \left(g_0 = \begin{bmatrix} a & b & c \\ d & e & f \\ h & \ell & k \end{bmatrix},\ x \in \mathrm{Res}^K_F \mathbb{G}_m \right)$$

Here comes the main object we want to study :

- Let σ be a cuspidal automorphic representation of G, which is holomorphic discrete series at archimedean places. ($\approx$ comes from holomorphic cuspforms)
  - Weight : for any $\sigma \in \Sigma$. the weight $\underline{k}_\sigma$ is a tuple
$$\underline{k}_\sigma = (t^+_{1,\sigma} \geq \cdots \geq t^+_{m,\sigma} ; t^-_{1,\sigma} \geq \cdots \geq t^-_{n,\sigma})$$
such that $t^+_{1,\sigma} \geq \cdots \geq t^+_{m,\sigma} ; t^-_{1,\sigma} \geq \cdots \geq t^-_{n,\sigma}$ are integers. Such weights classify the (L-packet of) discrete series representation of $G(\mathbb{R},\sigma)$.
$\rightsquigarrow$ such a discrete series rep is denoted by $\pi_{\sigma,\underline{k}_\sigma}$
  - The holomorphic discrete series corresponds to weights s.t. $t^+_{m,\sigma} - t^-_1 \geq d$.

    Example : Let $f \in S_k(\Gamma_1(N))$ be a classical holomorphic modular form, then in $U(1,1)$-perspective, $wt(f) = (t^+_1 = 0 ; t^-_1 = -k)$. So "holomorphic" $\Leftrightarrow$ $k \geq 2$. quite reasonable.
  - Scalar weight (In [Wan 2015 ANT] and so on) : $\underline{k}_\sigma = (0, \cdots, 0 ; -k, \cdots, -k)$ for a positive integer $k \geq d$. to be a holomorphic discrete series.

- Let $\chi : \mathbb{A}_K^\times \to \mathbb{C}^\times$ be a Hecke character over K, of infinite type $(\ell_1, \ell_2) \in \mathbb{Z}^2$, i.e. a continuous group homomorphism s.t.
$$\chi(\alpha \cdot x \cdot z_\infty) = \chi(x)\, z_\infty^{-\ell_1}\, \bar{z}_\infty^{-\ell_2} \quad \forall \alpha \in K^\times, z_\infty \in K_\infty^\times$$
such that $\ell_1 + \ell_2$ has the same parity with d.

There are equivalent definitions :

(a) χ regard as an ideal class character, $\chi((\alpha)) = \alpha^{\ell_1} \bar{\alpha}^{\ell_2}$ for $\alpha \equiv 1 \bmod f_\chi$.

(b) χ is an idèle character s.t.
$$\chi(z) = \left(\frac{z}{\bar{z}}\right)^{\kappa} (z\bar{z})^{\kappa'}, \quad \kappa, \kappa' \in \tfrac{1}{2}\mathbb{Z} \quad \left(\Longrightarrow \begin{array}{l} \kappa = \frac{1}{2}(\ell_2 - \ell_1) \\ \kappa' = -\frac{1}{2}(\ell_1 + \ell_2) \end{array} \right)$$

The character χ is central if $\kappa' = 0$.

<u>Definition</u> ([Wan15AN], Defn 3.2) The triple (σ, χ, Σ) is called an <u>Eisenstein datum</u>, where Σ is a finite set of primes of F containing all the infinite places, primes dividing p and places where σ or χ is ramified. (In a word, Σ is the set of "bad" primes).

Now we are ready to define Klingen Eisenstein series.

<u>Definition</u>: Let $s \in \mathbb{C}$ be a complex number.

(1) Regard $\sigma \boxtimes \chi$ as an automorphic rep of the Klingen Levi subgroup
$$M \xleftarrow[\sim]{m} U(m,n) \times \mathrm{Res}^K_F \mathbb{G}_m$$
and inflate it to an autorep of $P = MN$. We define
$$I^{\mathrm{Kling}}(\sigma, \chi, s) := \mathrm{Ind}_{P(\mathbb{A})}^{G^{\heartsuit}(\mathbb{A})}\left(\sigma \boxtimes \chi \cdot \delta_P^{1/2}\right)$$
$$= \left\{ \underset{\substack{\text{smooth} \\ K\text{-finite}}}{f^{\mathrm{Kling}}} : G^{\heartsuit}(\mathbb{A}) \to V_\sigma : \begin{array}{l} f(m(g_0,x)ng) = \sigma(g_0)\chi(x)\,\delta_P^{\frac{1}{2}+s}(m(g_0,x))\,f(g), \\ \forall g_0 \in U(m,n)(\mathbb{A}),\ x \in \mathbb{A}_K^\times,\ g \in G^{\heartsuit}(\mathbb{A}) \end{array} \right\}$$
Here δ_P is the modulus character of P, actually $\delta_P(m(g_0,x)) = |x\bar{x}|^{-(d+1)}$.

Elements of $I^{\mathrm{Kling}}(\sigma, \chi, s)$ are called <u>Klingen Eisenstein sections</u>.

(2) Now that $V_\sigma \hookrightarrow \mathcal{A}_{\mathrm{cusp}}(U(m,n))$, elements in V_σ are cuspforms, so that we have for $g \in U(m,n)(\mathbb{A})$
$$\mathrm{ev}_g : I^{\mathrm{Kling}}(\sigma, \chi, s) \to \mathrm{Fun}(G^{\heartsuit}(\mathbb{A}), \mathbb{C})$$
$$f^{\mathrm{Kling}} \mapsto f^{\mathrm{Kling}}(-)(g).$$

We finally define
$$E^{\mathrm{Kling}}(f^{\mathrm{Kling}}_{\sigma,\chi,s})(g) := \sum_{\gamma \in P(\mathbb{Q}) \backslash G^{\heartsuit}(\mathbb{Q})} \underbrace{\boxed{f^{\mathrm{Kling}}_{\sigma,s,\chi}}}_{}(\gamma g)(1) \quad \leftarrow P(\mathbb{Q})\text{-invariant}.$$
This is the <u>Klingen Eisenstein series</u> over $G^{\heartsuit}(\mathbb{A})$, defined by the Klingen Eisenstein section $f^{\mathrm{Kling}}_{\sigma,s,\chi}$.

Exercise: Let's seen an extreme case: $U(m,n) = U(0,0)$. Then the Eisenstein datum deduce to a single Hecke character χ over K.

- The Klingen parabolic subgroup: $P = \left\{ \begin{bmatrix} x^{-\star} & * \\ & x \end{bmatrix} \right\}$, $M = \left\{ \begin{bmatrix} x^{-\star} & \\ & x \end{bmatrix} \right\}$.

  and the modulus character is $|x\bar{x}|^{-1}$.

  (In the case of general unitary group $GU(1,1)$, we get the upper triangular Borel and the diagonal torus.)
- The Klingen Eisenstein sections and Eisenstein series are defined accordingly, coincide with the classical definition:

$$f^{\mathrm{Kling}} : U(1,1)(\mathbb{A}) \longrightarrow \mathbb{C}, \quad f\left(\begin{bmatrix} x^{-\star} & n \\ & x \end{bmatrix} g \right) = \chi(x) |x\bar{x}|^{-(\frac{1}{2}+s)} f(g)$$

  for any $x \in \mathbb{A}_K^\times$, $n \in \mathbb{A}$, and the Eisenstein series is defined from f^{Kling}.

Remark: Note that our notation is consistent with [SU2006 ICM report], but not in [SU14, Wan15 AN], Wan20, CLW, etc] on the complex variable s:

- We are writting: $\quad \sigma(g_0)\chi(x)\,\delta_P^{\frac{1}{2}+s}(m(g_0,x)) f(g)$,
- They are using: $\quad \delta$ a character of P such that $\delta^{d+1} = \delta_P$, i.e.

  $\delta(m(g_0,x)) = |x\bar{x}|^{-1}$, and the induced representation with factor

$$\delta^{\frac{d+1}{2}+z} = \delta_P^{\frac{1}{d+1}\left(\frac{d+1}{2}+z\right)} = \delta_P^{\frac{1}{2}+\frac{z}{d+1}}$$

  So their $z = (d+1)s$. This is really a subtlety but it is claimed that this saves notation.

The critical point $s_0 := \dfrac{k'+1}{2(d+1)}$.

The contribution of Langlands: the black box of Eisenstein series:

(v) If $\mathrm{Re}(s)$ is sufficiently large, $E^{\mathrm{Kling}}_{\sigma,\chi,s}$ (as a function in $\underset{\mathbb{C}\times G^{\heartsuit}(\mathbb{A})}{(s,g)}$)

converges absolutely and uniformly for s, g in compact sets.

$\rightsquigarrow$ $E^{\mathrm{Kling}}_{\sigma,\chi,s}$ is holomorphic in s, defines an autoform on $G^{\heartsuit}$.

[When π is tempered and χ is unitary, $\mathrm{Re}(s) > \frac{1}{2}$ suffices].

(v) There is a meromorphic continuation of $E^{\mathrm{Kling}}_{\sigma,\chi,s}$ to all $s \in \mathbb{C}$

(?) What about the holomorphy of $E^{\mathrm{Kling}}_{\sigma,\chi,s}$ in the variable $g \in G^{\heartsuit}(\mathbb{A})$.

— Surely not always the case: weight 2 classical Eisenstein series.

Slogan of Skinner: The order of nonholom $\rightsquigarrow$ nonvanishing of central L.

$\rightsquigarrow$ To grant it, the strategy is to

(1) guarantee $\rho := \sigma \boxtimes \chi$ is a holomorphic discrete series at each infinite places, as autorep over $M(\mathbb{A})$

(2) Choose a canonical Klingen Eisenstein section $f^{\mathrm{Kling}}_{\sigma,\chi,s,\infty} \in I^{\mathrm{Kling}}(\rho_\infty \cdot \delta_P^{1/2+s})$.

<u>Condition on (1)</u>:

Let $\chi : \mathbb{A}_K^\times \to \mathbb{C}^\times$ be a Hecke character over K, of <u>infinite type</u> $(l_1, l_2) \in \mathbb{Z}^2$.

We assume $l_1 + l_2 \equiv d \mod 2$.

- <u>Fact</u>: The weight of ρ is

$$(t_1^+, t_2^+, \cdots, t_m^+, \kappa + \tfrac{d}{2} + 1 ; \kappa - (\tfrac{d}{2}+1), t_1^-, \cdots, t_n^-)$$

Therefore, for ρ to be a holomorphic discrete series, the condition is

$$t_m^+ \geqslant \underbrace{\kappa + \tfrac{d}{2} + 1 \geqslant \kappa - (\tfrac{d}{2}+1)}_{\text{the central gap is } d+2.} \geqslant t_1^- \qquad — (\divideontimes)$$

If we put κ in the central, $(\divideontimes)$ is equivalent to

$$t_1^- + (\tfrac{d}{2}+1) \leqslant \kappa \leqslant t_m^+ - (\tfrac{d}{2}+1) \qquad — (\divideontimes)'$$

<u>Condition on (2)</u>: We shall focus on the archimedean stuff in a seperate talk.

<u>Example of U(1,1) case</u>: As before, $f \in S_k(\Gamma_1(N))$, corresponding to $(t_1^+ = 0 ; t_1^- = -k)$

Then $(\divideontimes)'$ becomes: $\boxed{-k+2 \leqslant \kappa \leqslant -2}$, the central gap is $\boxed{k \geqslant 4}$

$\rightsquigarrow$ weight$(\theta_\chi) = 2|\kappa|+1 \in [5, 2k-3]$. <u>weird</u>.

<u>Example of U(2,0) case</u>: The scalar weight becomes $(t_1^+ = t_2^+ = 0)$. Then $(\divideontimes)'$ becomes $\boxed{\kappa \leqslant -2}$. The central gap condition is empty.

$\rightsquigarrow$ weight$(\theta_\chi) = 2|\kappa|+1 \geqslant 5$. <u>weired</u>.

<u>Example in [Wan2015AN]</u>:

- σ is in the scalar case $(t_1^+ = \cdots = t_m^+ = 0 ; t_1^- = \cdots = t_n^- = -k)$, with $k \geqslant d$ an even integer. (holomorphic condition)
- χ is Hecke character of <u>weight</u> $(-\frac{k}{2}, 0)$ in OUR notation

⚠ In [Wan15AN], it goes like "<u>χ has infinite type $(-\frac{k}{2}, \frac{k}{2})$</u>". But in loc.cit, the infinite type is <u>not</u> defined. In our setup, I guess it should be <u>$(\frac{k}{2}, -\frac{k}{2})$</u> (I put extra "$-$" to be consistent with

[BDP13] and [Lei-Loeffler-Zerbes]. Then compute

$$\kappa = \frac{1}{2}\left(-\frac{k}{2} - \frac{k}{2}\right) = -\frac{k}{2}$$

$$\kappa' = -\frac{1}{2}\left(\frac{k}{2} - \frac{k}{2}\right) = 0$$

So χ is a Hecke character of weight $(-\frac{k}{2}, 0)$. It is a <u>central</u> Hecke character.

$\rightsquigarrow$ Then $(\divideontimes)$ gives

$$\overset{\ominus}{-k+\left(\frac{d}{2}+1\right)} \leqslant \overbrace{-\frac{k}{2}}^{\text{“}\kappa\text{”}} \leqslant \overset{\oplus}{0-\left(\frac{d}{2}+1\right)}$$

$\Longrightarrow$ $k \geqslant d+2$, and the central gap condition is the same.

<u>Remark</u>: On comparing [Skinner–Urban] and [Wan 2020 ANT]:

- For simplicity, we assume $K/F = K/\mathbb{Q}$ imaginary quadratic field. Let:
  - $f \in S_k(\Gamma_0(N), \epsilon_f)$ a normalized newform
  - $\chi : \mathbb{A}_K^\times \to \mathbb{C}^\times$ a Hecke character of K of infinite type (l_1, l_2), i.e. a continuous group homomorphism s.t.

$$\chi(\alpha \cdot x \cdot z_\infty) = \chi(x)\, z_\infty^{-l_1}\, \bar{z}_\infty^{-l_2} \quad \forall \alpha \in K^\times,\ z_\infty \in K_\infty^\times$$

Then we can attach a theta function $\theta_\chi \in S_{l+1}(\Gamma_1(N'))$, with $l = |l_1 - l_2|$.

$\rightsquigarrow$ The object we want to study: <u>the Rankin–Selberg product</u> $\pi_f \times \pi_\chi$, where π_f and π_χ are unitary automorphic reps of $GL_2(\mathbb{A}_\mathbb{Q})$ associated to f and θ_χ accordingly:

$$L(f, \chi, s) = L\left(\pi_f \times \pi_\chi,\ s - \frac{k-1+l_1+l_2}{2}\right)$$

We say χ is <u>critical</u> for f if "$s=1$" is a <u>critical value in the sense of Deligne</u>:

- We complete $L(f,\chi,s)$ by

$$L_\infty(f,\chi,s) = \Gamma_\mathbb{C}(s - l_0)\,\Gamma_\mathbb{C}(s - \min(k-1, l) - l_0), \quad l_0 = \min(l_1, l_2).$$

$$\Gamma_\mathbb{C}(s) = 2(2\pi)^{-s}\,\Gamma(s).$$

and define $\Lambda(f,\chi,s) := L_\infty(f,\chi,s)\, L(f,\chi,s)$. Then

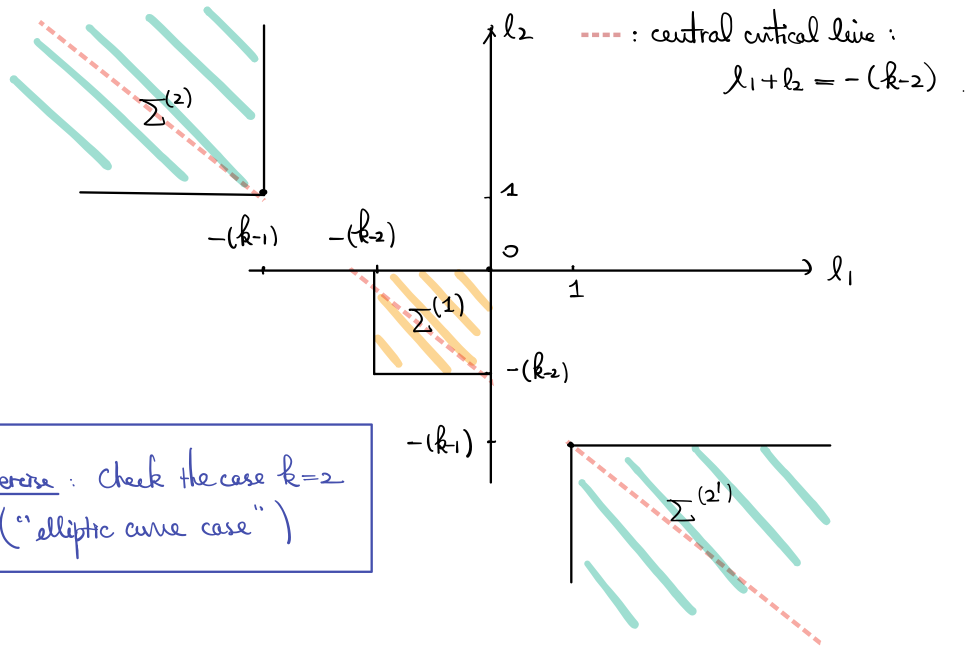
$$\Lambda(f,\chi,s) = \epsilon(f,\chi,s)\,\Lambda(f_\rho, \bar{\chi}, k + l_1 + l_2 - s)$$

certain "involution" of f.

- Then "$s=n$" is a <u>critical value</u> in the sense of Deligne if neither $L_\infty(\text{LHS})$ nor $L_\infty(\text{RHS})$ has a pole at $s=n$.

$\rightsquigarrow$ Fixing f and let χ vary, it is an exercise to determine the region of critical characters:

$$\Sigma_{\text{crit}} = \Sigma^{(1)} \sqcup \Sigma^{(2)} \sqcup \Sigma^{(2')}, \text{ as drawn below:}$$



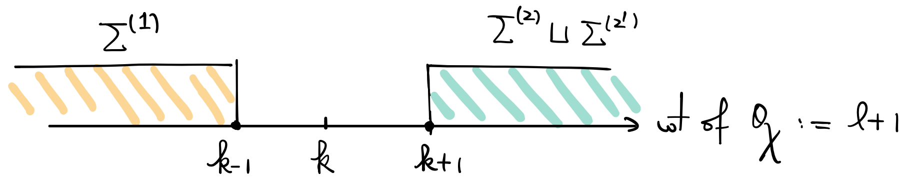
Exercise : Check the case $k=2$ ("elliptic curve case")

$\Sigma^{(1)} := \{(\ell_1,\ell_2) : -(k-2) \leq \ell_1, \ell_2 \leq 0\} \rightsquigarrow \ell \leq k-2$, i.e. $wt(g_\chi) \leq k-1$.

$\Sigma^{(2)} := \{(\ell_1,\ell_2) : \ell_1 \leq -(k-1),\ \ell_2 \geq 1\} \rightsquigarrow \ell \geq k$, i.e. $wt(g_\chi) \geq k+1$.

$\Sigma^{(2')} := \{(\ell_1,\ell_2) : \ell_1 \geq 1,\ \ell_2 \leq -(k-1)\}$

i.e.



Then :
- [Skinner–Urban] dealt with $\Sigma^{(1)}$: $wt(f) > wt(g_\chi)$.
- [Wan 2020 AN] dealt with $\Sigma^{(2)} \sqcup \Sigma^{(2')}$: $wt(f) < wt(g_\chi)$

It turns out that the two types of regions shall be dealt with independently :
- $\mathbb{E}^{Kling}$ / $U(2,2)$ has constant term interpolating in $\Sigma^{(1)}$
- $\mathbb{E}^{Kling}$ / $U(3,1)$ ———————— in $\Sigma^{(2)} \cup \Sigma^{(2')}$

Reference :

- Bertolini–Darmon–Prasanna : « Generalized Heegner cycles & p-adic Rankin L-series » 2013 Duke . §4 .
- Lei–Loeffler–Zerbes : « Euler systems for modular forms over imaginary quadratic fields » 2015 Compositio , §6 , note Remark 6.1.1 in loc. cit .

Question : why the Rankin-Selberg product $\pi_f \times \pi_\chi$?

- Iwasawa thy with p-adic L-function is well-behaved under certain p-ordinary condition (more precisely the Panchishkin condition à la Greenberg)
  - In the elliptic curve / modular curve case : p-ordinary condition
  - In the Rankin-Selberg case : $\pi_f \times \pi_g$, say $wt(g) > wt(f)$. Then it is requiring g being p-ordinary, with no condition put on f.
- Here the CM form θ_χ is automatically p-ordinary (this depends on that p splits in K)

$\rightsquigarrow$ So when $wt(\theta_\chi) > wt(f)$, we can use Iwasawa main conjecture to touch nonordinary modular forms f.

Remark : • In [Wan 2020] though, it is required that f is ordinary, (applied to rk one BSD and p-converse of GZK)

but in [CLW 2022], this condition is partically removed.

• Of course, it is a long story from

$$\boxed{\text{Greenberg IMC over } \mho(2,0)} \Longrightarrow \boxed{\text{Kobayashi–Pollack's } \pm\text{-IMC}}$$

recent work of [Burungale–Skinner–Tian–Wan].

$$\boxed{\text{Greenberg IMC over } \mho(2,0)} \Longrightarrow \boxed{\text{Crystalline IMC}}$$

[Fouquet–Wan, chapter 4]

§2 Constant Term of Klingen Eisenstein series

<u>Defn</u>: Given a smooth function $\varphi : G(F)\backslash G(\mathbb{A}) \to \mathbb{C}$. We define the <u>constant term</u> along a parabolic $P \subseteq G$ is the following function

$$\varphi_P : g \in G(\mathbb{A}) \longmapsto \int_{N(F)\backslash N(\mathbb{A})} \varphi(ng)\, dn \in \mathbb{C}$$

<u>§2.1</u> GL_2-case: Let's do the simplest GL_2-case as a start. $(F = \mathbb{Q})$.

Recall the Eisenstein series for GL_2 is

$$E_{f,s}(g) := \sum_{\gamma \in B(\mathbb{Q})\backslash GL_2(\mathbb{Q})} f_s(\gamma g).$$

Then we compute its constant term at the parabolic subgroup B.

<u>Fact</u>: <u>Bruhat decomposition</u>:

$$GL_2(\mathbb{Q}) = B(\mathbb{Q}) \sqcup B(\mathbb{Q})\, w\, N(\mathbb{Q}), \quad w := \begin{bmatrix} & 1 \\ -1 & \end{bmatrix}.$$

$$\Longrightarrow B(\mathbb{Q})\backslash GL_2(\mathbb{Q}) = \{1\} \sqcup \left\{ \begin{bmatrix} & 1 \\ -1 & \end{bmatrix} N(\mathbb{Q}) \right\}.$$

$\boxed{\text{Step 1}}$

$$E_{B,f,s}(g) = \int_{N(\mathbb{Q})\backslash N(\mathbb{A})} \sum_{\gamma \in B_2(\mathbb{Q})\backslash GL_2(\mathbb{Q})} f_s(\gamma n g)\, dn$$

$$= \underline{\int_{N(\mathbb{Q})\backslash N(\mathbb{A})} f_s(ng)\, dn} + \int_{N(\mathbb{Q})\backslash N(\mathbb{A})} \sum_{m \in N(\mathbb{Q})} f_s\left(\begin{bmatrix} & 1 \\ -1 & \end{bmatrix} m n g \right) dn$$

$$= \underline{f_s(g)} + \int_{N(\mathbb{A})} f_s\left(\begin{bmatrix} & 1 \\ -1 & \end{bmatrix} n g \right) dn$$

<u>Recall</u>: $f_s \in I_s(\chi, \psi)$: $f(ng) = f(g)$.

So if we insist that the $\mathrm{vol}(N(\mathbb{Q})\backslash N(\mathbb{A})) = 1$, then indeed the integral is $f_s(g)$.

$\boxed{\text{Step2}}$ Break into local product:

- $f \in I(\chi,\psi) = \underbrace{\otimes'_v I(\chi_v,\psi_v)}_{(*)}$.

To make sense of $(*)$: we need to specify what we mean by "$\otimes'$"

- For places v s.t. χ_v and ψ_v are unramified (this is for a.e. v),

$$I(\chi_v,\psi_v)^{GL_2(\mathbb{Z}_\ell)} = \mathbb{C}\cdot f_v^{sph}, \qquad f_v^{sph}(GL_2(\mathbb{Z}_\ell)) = 1.$$

(Since Iwasawa decomposition $GL_2(\mathbb{Q}_\ell) = B(\mathbb{Q}_\ell)GL_2(\mathbb{Z}_\ell)$, this defines an elt $f_v^{sph} \in I(\chi_v,\psi_v)$ definitely.)

So we define the restricted tensor product wrt these f_v^{sph}.

- For $f \in I(\chi,\psi)$, as we require it to be smooth (here to make sense of it, we mean $GL_2(\mathbb{Z}^S)$-smooth), it is trivial on $\prod_{\ell\notin S} GL_2(\mathbb{Z}_\ell) =: GL_2(\mathbb{Z}^S)$.

$\rightsquigarrow \quad f = f^{S,sph} f_S, \quad S =$ "bad primes" $\supseteq$ ramified, ∞-places

Then:

$$E_{B,f,s}(g) = \underline{f_s(g)} + \underbrace{\int_{N(\mathbb{Q}_S)} f_S\left(\begin{bmatrix} & 1 \\ -1 & \end{bmatrix} ng\right) dn}_{\substack{\text{since we haven't specified the}\\ \text{bad sections } f_S \text{, we cannot}\\ \text{say further words on this part}}} \cdot \underbrace{\prod_{\ell\notin S} \int_{N(\mathbb{Q}_\ell)} f_\ell^{sph}\left(\begin{bmatrix} & 1 \\ -1 & \end{bmatrix} ng\right) dn}_{\text{But we can compute this part.}}$$

$\boxed{\text{Step3}}$ Compute the good intertwining term.

$$M(f_s)(g) := \int_{N(\mathbb{Q}_\ell)} f_{\ell,s}^{sph}\left(\begin{bmatrix} & 1 \\ -1 & \end{bmatrix} ng\right) dn.$$

Fact: $GL_2(\mathbb{Q}_\ell) = B(\mathbb{Q}_\ell)\, T(\mathbb{Q}_\ell)\, GL_2(\mathbb{Z}_\ell)$. Iwasawa decomposition

$$g = \begin{bmatrix}1 & b\\ & 1\end{bmatrix}\begin{bmatrix}a & \\ & d\end{bmatrix} k$$

Then: $$M(f_s)(g) = \int_{\mathbb{Q}_\ell} f^{sph}_{\ell,s}\left(\begin{bmatrix} & 1\\ -1 & \end{bmatrix}\underbrace{\begin{bmatrix}1 & m\\ & 1\end{bmatrix}\begin{bmatrix}1 & b\\ & 1\end{bmatrix}\begin{bmatrix}a & \\ & d\end{bmatrix} k}_{\text{hope}}\right) dm$$

change of variable $\to$ $$= \int_{\mathbb{Q}_\ell} f^{sph}_{\ell,s}\left(\begin{bmatrix} & 1\\ -1 & \end{bmatrix}\begin{bmatrix}1 & m\\ & 1\end{bmatrix}\begin{bmatrix}a & \\ & d\end{bmatrix}\right) dm$$

$$= \int_{\mathbb{Q}_\ell} f^{sph}_{\ell,s}\left(\begin{bmatrix} & 1\\ -1 & \end{bmatrix}\begin{bmatrix}a & \\ & d\end{bmatrix}\begin{bmatrix}1 & \frac{d}{a}m\\ & 1\end{bmatrix}\right) dm$$

change of variable $\to$ $$= \int_{\mathbb{Q}_\ell} f^{sph}_{\ell,s}\left(\underbrace{\begin{bmatrix}d & \\ & a\end{bmatrix}}_{\text{drop out}}\begin{bmatrix} & 1\\ -1 & \end{bmatrix}\begin{bmatrix}1 & m\\ & 1\end{bmatrix}\right) \left|\frac{a}{d}\right| dm$$

$$= \int_{\mathbb{Q}_\ell} \underbrace{\chi_\ell(d)\,\psi_\ell(a)\left|\frac{d}{a}\right|^{s+\frac{1}{2}}\left|\frac{a}{d}\right|}_{=\ \psi_\ell(a)\,\chi_\ell(d)\left|\frac{a}{d}\right|^{-s+\frac{1}{2}}} f^{sph}_{\ell,s}\left(\begin{bmatrix} & 1\\ -1 & \end{bmatrix}\begin{bmatrix}1 & m\\ & 1\end{bmatrix}\right) dm$$

$$\Longrightarrow\ M(f_s) \in I_{-s}(\psi_\ell, \chi_\ell)^{GL_2(\mathbb{Z}_\ell)} \ \underset{\text{unramifiedness}}{=}\ \underbrace{c_\ell(s)}_{\in\, \mathbb{C}} f^{sph}_{\ell,-s}$$

Putting $g=1$ on both sides:

$$c_\ell(s) = M(f^{sph}_{\ell,s})(1) = \int_{\mathbb{Q}_\ell} f^{sph}_{\ell,s}\left(\begin{bmatrix} & 1\\ -1 & \end{bmatrix}\begin{bmatrix}1 & m\\ & 1\end{bmatrix}\right) dm .$$

$\boxed{\text{Step 4}}$ Computing the coefficient $c_\ell(s)$: $\mathbb{Q}_\ell = \mathbb{Z}_\ell \sqcup \bigsqcup_{r=1}^{\infty} \ell^{-r}\mathbb{Z}_\ell^\times$

$$c_\ell(s) = \underbrace{\int_{\mathbb{Z}_\ell} f^{sph}_{\ell,s}\left(\begin{bmatrix} & 1\\ -1 & \end{bmatrix}\begin{bmatrix}1 & m\\ & 1\end{bmatrix}\right) dm}_{=\ 1} + \sum_{r=1}^{\infty}\int_{\mathbb{Z}_\ell^\times} f^{sph}_{\ell,s}\left(\begin{bmatrix} & 1\\ -1 & \end{bmatrix}\begin{bmatrix}1 & \ell^{-r}u\\ & 1\end{bmatrix}\right)\ell^r\, du$$

$$\begin{bmatrix} & 1\\ -1 & \end{bmatrix}\begin{bmatrix}1 & \ell^{-r}u\\ & 1\end{bmatrix} = \begin{bmatrix}+\ell^r/u & 1\\ & +\ell^r u\end{bmatrix}\underbrace{\begin{bmatrix}0 & +1\\ -1 & +\ell^r/u\end{bmatrix}\begin{bmatrix} & 1\\ -1 & \end{bmatrix}}_{\in\, GL_2(\mathbb{Z}_\ell)}$$

so $$\underbrace{f^{sph}_{\ell,s}(\dots)\,\ell^r}_{\text{inside the integral}} = \frac{\chi_\ell}{\psi_\ell}(\ell^r)\,|\ell^{2r}|^{s+\frac{1}{2}}\,\ell^r = \frac{\chi_\ell}{\psi_\ell}(\ell^r)\,\ell^{-2rs}$$

So:

$$c_\ell(s) = 1 + \mathrm{vol}(\mathcal{I}_\ell^X) \sum_{r=1}^{\infty} \frac{\chi_\ell}{\psi_\ell}(\ell^r)\,\ell^{-2rs} = 1 + \frac{\frac{\chi_\ell}{\psi_\ell}(\ell)\,\ell^{-2s}\left(1-\frac{1}{\ell}\right)}{1 - \frac{\chi_\ell}{\psi_\ell}(\ell)\,\ell^{-2s}} = \frac{L_\ell\left(\frac{\chi}{\psi}, 2s\right)}{L_\ell\left(\frac{\chi}{\psi}, 2s+1\right)}$$

Therefore:

$$E_{B,f_s}(g) = \underbrace{f_s(g)}_{\text{original section}} + \frac{L^S(\chi/\psi, 2s)}{L^S(\chi/\psi, 2s+1)} \underbrace{M(f_{S,s})}_{\substack{\text{intertwing operator}\\ \text{at bad places}}} \widetilde{f}_{-s}^{S,0}$$

Remark:

$$\boxed{\begin{array}{l} 1^\circ\ E(f_s, g) \text{ and hence } E_B(f_s, g) \text{ is merom. in } s. \\ 2^\circ\ \text{Local intertwing operator } M(-) \text{ is mero. in } s \\ 3^\circ\ \text{We choose } f_s \text{ to be "holomorphic section"} \end{array}} \Longrightarrow \frac{L^S(\chi/\psi, 2s)}{L^S(\chi/\psi, 2s+1)} \text{ is merom. in } s.$$

i.e. we can "propagate" the meromorphy of $L^S(\chi/\psi, 2s)$ to the left by 1 step by step.

This is the original idea of Langlands (& Langlands–Shahidi method)

$\boxed{\text{Step 5}}$ The choice of bad sections

We point out that in practice on the choice of archimedean places, we choose $f_{s,\infty} \in DS_k$, i.e. discrete series of weight k, for $s = \frac{k-1}{2}$

$$\xrightarrow[\text{Blackbox}]{} M(f_{s,\infty}) = 0.$$

Therefore, as long as we don't obtain any pole from $\frac{L^S(\chi/\psi, k-1)}{L^S(\chi/\psi, k)}$, we obtain $\boxed{E_B(f_{s_k}) = f_{s_k}}$.

<u>Example</u>: If $\chi = \psi$, $k=2$, then the L-quotient has a pole at $s_k = \frac{k-1}{2}$. In this case,

- $E(f_{s_k})$ is the weight 2 Eisenstein series, which is not holom.
- The representation $I_s(\chi,\psi)$ is <u>reducible</u>:

$$0 \to \mathrm{St} \otimes \chi \to I_s(\chi,\chi) \to \chi \to 1 .$$

<u>Skinner said</u>: on the Galois side, any deformation is locally steinberg, so it has monodromy at a prime ℓ, so not necessarily lie in the Bloch-Kato f-Selmer group (but somewhere else with different condition.)

§2.2 The case of Klingen Eisenstein series

__Fact__: For a standard F-parabolic of G (i.e. $R \supseteq B$). If $R \neq P$, then $E^{Kling}_{\sigma,\chi,s,R} = 0$.

For the constant term along the Klingen parabolic P:

- __Bruhat decomposition__:

$$G = P \sqcup P\omega N, \qquad \omega = \begin{array}{c} \begin{bmatrix} 1_n & & & \\ & & & 1 \\ & & 1_m & \\ & -1 & & \end{bmatrix} \\ \begin{array}{cccc} n & 1 & m & 1 \end{array} \end{array}$$
longest Weyl elts.

$\rightsquigarrow$ Immitate the previous calculation in Gl_2-case:

$$E_P(g) = f^{Kling}_{\sigma,\chi,s}(g) + \prod_{v:\text{ place of } F} \underbrace{M(f^{Kling}_{\sigma,\chi,s})(g)}_{\in I(\sigma,\chi^{-c},-s)}$$

$$= f^{Kling}_{\sigma,\chi,s}(g) + \prod_{v\in S} M(f^{Kling}_{\sigma,\chi,s})(g) \cdot \prod_{v\notin S} c_v(s) f^{Kling,sph}_{\sigma,\chi^{-c},-s}$$

__Theorem__ (Gindikin–Karpelevich formula)

$$c_v(s) = \frac{L_v(\pi_v,\chi_v^{-1},(d+1)s)}{L_v(\pi_v,\chi_v^{-1},(d+1)s+1)} \; \frac{L_v(\chi_v', 2(d+1)s)}{L_v(\chi_v', 2(d+1)s+1)}$$

here $\chi_v' = \chi_v^{-1}|_{F_v} \cdot \eta_{K_v/F_v}$, and the Euler factor

$$L_v(\pi_v,\chi_v^{-1},s) := \underbrace{L_v}_{\text{Godement-Jacquet L-function}}(\underbrace{BC(\pi_v)}_{\text{automorphic rep of } GL_d} \otimes \chi_v^{-1}, s)$$

__P.S.__ Surely there will be references to this proof but I haven't found a precise published reference.

Then one checks: (I'm still confused at the choice of $s_0 := \frac{k-d-1}{2}$)

① In our holomorphic range, the GK term $c_\ell(s)$ has no pole.

② Under explicit choice at the archimedean place: $M_\infty(f^{\mathrm{Kling},\infty}_{\sigma,\chi,s}) \equiv 0$

So: $\boxed{E_p(g) = f^{\mathrm{Kling}}_{\sigma,\chi,s}(g)}$.

QUESTION: Thank you for your paitience! But still: where is "L-function" as promised?

- Though the GK-term with L-factor disappears in the final result, it can give us some insight:

"integrals"

$\boxed{\text{L-factors at good primes show up from intertwining operator}}$

$\Longrightarrow$ In fact, the Klingen Eisenstein series we used is constructed via pulling back from Siegel Eisenstein series over larger unitary groups.

Then:

$$\underbrace{\mathrm{Pb}_\varphi\left(f^{\mathrm{Sieg},S,\mathrm{sph}}_{\chi,s}\right)}_{\text{to Talk 3}} =: f^{\mathrm{Kling},S}_{\sigma,\chi,s}(g) = \underbrace{c^S_{LR}(s)\, f^{\mathrm{Kling},S,\mathrm{sph}}_{\sigma,\chi,s}}_{\text{the L-value term}}, \quad \sigma := \langle\varphi\rangle.$$

We will give a detailed explanation of this part in Talk 3.

Another reason of pulling back:

$$\boxed{\begin{matrix}\text{Fourier coeff}\\ \text{of}\\ \text{Siegel ES}\end{matrix}} \xRightarrow[\text{"q-expansion principle"}]{} \boxed{\begin{matrix}\text{Siegel Eisenstein}\\ \text{family}\end{matrix}} \xrightarrow[\text{(p-adically interp.)}]{\text{pullback}} \boxed{\begin{matrix}\text{Klingen Eisenstein}\\ \text{family}\end{matrix}}$$

(p-adically interpolatable)

§3 Galois reps of Klingen Eisenstein series

The original reference should be [SU14, §9.5]. Assume σ has trivial central char.

<u>Goal</u>: Let Π be the automorphic representation of $\underline{G^{\heartsuit}}$ (written as G from now on) generated by the Klingen Eisenstein series $E^{\mathrm{Klig}}_{\sigma,\chi,s_0}$. Then

Then we shall show that there exists a Galois rep R <u>associated</u> to Π:

This means the <u>standard partial L-functions</u> are the same.

① <u>Black box on the unramified reps</u>

Let $\Pi = \otimes'_v \Pi_v$, and Π_v is unramified for $v \notin S$.

- From now on we pick $\underline{v \notin S}$. We require v splits in K to simplify our computation.

  $\rightsquigarrow$ $G(F_v) \simeq GL_d(F_v)$ depending on a choice of $w|v$ in K, and
  $B(F_v) \simeq$ upper triangular matrices.
  $T(F_v) \simeq$ diagonal torus

  We will write them as G, B, T accordingly to simplify the notations.

- <u>Langlands parametrization</u>: $\Pi_v \hookrightarrow \mathrm{Ind}_B^G \psi_v$ for some unramified character $\psi_v : T \to \mathbb{C}^\times$. Then we can associate a tuple of <u>Satake parameters</u>:

$$\left\{ \psi_i := \psi_v\left(\begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & \varpi_v & & \\ & & & & 1 & \\ & & & & & \ddots \\ & & & & & & 1 \end{bmatrix}\right) \right\}_{i=1}^{d+2}$$

(ϖ_v at the i-th entry)

In this case, the <u>standard local L-factor</u> is determined by these Satake parameters:

$$L_v(BC(\Pi_v), s, \mathrm{std}) = \prod_{i=1}^{d+2} (1 - q_v^{-s}\psi_i)^{-1}(1 - q_v^{-s}\psi_i^{-1})^{-1} \quad \text{—} (*)$$

② <u>Understand the relation between Π and "$\sigma \boxtimes \chi$"</u>

- For $v \in S$, σ is unramified at v

$$\sigma \xrightarrow[\text{Langlands}]{} \sigma \hookrightarrow \mathrm{Ind}_{B_H}^{H}(\psi_{H,v})$$

$$\rightsquigarrow \quad \Pi \hookrightarrow \mathrm{Ind}_P^G\left(\mathrm{Ind}_{B_H}^{H}\psi_{H,v} \otimes \chi_v\delta_P^{s_0}\right) = \mathrm{Ind}_B^G\left(\overbrace{\psi_{H,v} \otimes \chi_v\delta_P^{s_0}}^{\psi_v}\right)$$

Here let's recall (<u>note</u>: here we use "v splits in K" to simplify)

- $P = \left\{ \begin{bmatrix} \bar{t}^{-1} & & \\ & h & \\ & & t \end{bmatrix} : \begin{array}{l} t \in K_v^\times \\ h \in H \end{array} \right\} \Longrightarrow \begin{array}{l} B \cap P \simeq B_H \times K_v^\times \\ T \cap M \simeq T_H \times K_v^\times \end{array}$

- Here by <u>splitness</u>, $K_v^\times = F_v^\times \times F_v^\times$ and $\chi_v = (\chi_1, \chi_2)$ with χ_i being characters of $F_v^\times$. Moreover,

$$K_v^\times = F_v^\times \times F_v^\times \hookrightarrow P$$
$$(a, b) \mapsto \mathrm{diag}[b^{-1}, \mathbb{1}_d, a].$$

Therefore:

$$\psi_v\left(\begin{bmatrix} t = b^{-1} & & \\ & \mathbb{1}_d & \\ & & t' = a \end{bmatrix}\right) = \chi_1(t')\chi_2^{-1}(t)\psi_{H,v}(t_H)\underbrace{\left|\frac{t'}{t}\right|^{(d+1)s_0}}$$

here we used the explicit description of the modulus character δ_P.

i.e. $\psi_v = [\chi_2^{-1}|\cdot|^{-(d+1)s_0}, \psi_{H,v}, \chi_1|\cdot|^{(d+1)s_0}]$.

Therefore:

$L_v(BC(\Pi_v), s, \mathrm{std})$ — (※)

$$= L_v(BC(\sigma_v), s, \mathrm{std})\left(1 - \chi_2^{-1}(\varpi_v)q_v^{(d+1)s_0 - s}\right)^{-1}\left(1 - \chi_1(\varpi_v)q_v^{-(d+1)s_0 - s}\right)^{-1}$$
$$\left(1 - \chi_2(\varpi_v)q_v^{-(d+1)s_0 - s}\right)^{-1}\left(1 - \chi_1(\varpi_v)^{-1}q_v^{(d+1)s_0 - s}\right)^{-1}$$

$$= L_v(BC(\sigma_v), s, \mathrm{std})\, L_v(\chi_v^c, s + (d+1)s_0)\, L_v(\chi_v^{-1}, s - (d+1)s_0)$$

Remark:

1° Here χ_v^c is simply swaping χ_1 and χ_2, so actually there is little difference between χ_v and χ_v^c. We choose to write χ_v^c to be consistent with the inert computation.

2° This is not really a precise proof. To determine the Satake parameters, one should compute the Satake isomorphism explicitly. This is exactly the method of [SU14, §9.5].

③ Relate to galois reps

From (※※), we see it suffices to attach Galrep to each factors.

1° GL$_1$-factors: We have class field theory. $\chi \rightsquigarrow \rho_\chi^{-1}$

Moreover, suppose χ has infinite type (ℓ_1, ℓ_2) following our convention, being a character over K. Then

$$\boxed{\rho_\chi \rho_\chi^c = \epsilon_{cyc}^{-2k'}}$$

I'm still confused at the conventions.

2° $BC(\sigma_v)$-factor: For cuspidal automorphic reps σ over H, we have the black box to associate it to Galreps R_σ:

Theorem (Skinner 2012 ANT, Theorem B, etc.) $\exists$ a continuous semisimple representation $R_\sigma : \mathrm{Gal}_K \to GL_d(\overline{\mathbb{Q}}_p)$ s.t.

(1) $R_\sigma^c \simeq R_\sigma^\vee \otimes \epsilon_{cyc}^{1-d}$

(2) R_σ is unramified at all finite places away from $S_\pi^{bad} \cup \{p\}$ and "the local L-factors there match":

$$L_w(R_\sigma, s) = L_v\left(BC(\sigma)_w^\vee \otimes |\cdot|^{\frac{1-d}{2}}, s\right). \quad \text{— } (※)$$

(3) For $w \mid p$, $R_\sigma|_{G_{K_w}}$ is potentially semisimple.
When $p \notin S_\Pi^{bad}$, it is cristalline.

Imitating $(\cdot\!*\!\cdot)$, we can see that there is a Gal rep $R_\Pi : G_K \to GL_{d+2}(\overline{\mathbb{Q}}_p)$ s.t.

$$\boxed{L^S(R_\Pi, s) = L^S\left(BC(\Pi)^\vee, s - \frac{1-(d+2)}{2}\right) = L^S\left(BC(\Pi)^\vee, s + \frac{d+1}{2}\right)}$$

Indeed, by $(\cdot\!*\!*\!\cdot)$, the latter value is:

$$L_v\left(BC(\Pi_v)^\vee, s + \frac{d+1}{2}\right)$$

$$= \underbrace{L_v\left(BC(\sigma_v)^\vee, s + \frac{d+1}{2}\right)} \underbrace{L_v\left(\chi_v^{-c}, s - (d+1)s_0 + \frac{d+1}{2}\right)} \underbrace{L_v\left(\chi_v, s + (d+1)s_0 + \frac{d+1}{2}\right)}$$

$$L_v\left(BC(\sigma_v), s \,\Big\|\, s + d\right) \quad L_v\left(\chi_v^c, s + (d+1)s_0\right) \quad L_v\left(\chi_v^{-1}, s - (d+1)s_0\right)$$

compare the effect of taking dual

$$L_v\left(BC(\sigma_v)^\vee, s - \frac{1-d}{2} + 1\right)$$

$$= L_v\left(\rho_\sigma(1) \oplus \rho_\chi^{-c}\left(\frac{d+1}{2} - (d+1)s_0\right) \oplus \rho_\chi\left(\frac{d+1}{2} + (d+1)s_0\right)\right)$$

i.e. $R_\Pi = \rho_\sigma(1) \oplus \rho_\chi^{-c}\left(\frac{d+1}{2} - (d+1)s_0\right) \oplus \rho_\chi\left(\frac{d+1}{2} + (d+1)s_0\right)$

If we only care about the critical value $s_0 = \frac{\kappa' + \frac{1}{2}}{d+1}$, then

$$\boxed{R_\Pi = \rho_\sigma(1) \oplus \rho_\chi^{-c}\left(\frac{d}{2} - \kappa'\right) \oplus \rho_\chi\left(\frac{d}{2} + \kappa' + 1\right)}.$$

Here the "half-integers" on the Tate twist are actually integers bc $\kappa' \equiv d \bmod 2$

$$R_\Pi = \begin{bmatrix} \rho_\sigma(1) & & \\ & \rho_\chi^{-c}\left(\frac{d}{2} - \kappa'\right) & \\ & & \rho_\chi\left(\frac{d}{2} + \kappa' + 1\right) \end{bmatrix}$$

④ <u>A slight adjustment</u> : Following [SU06] :

$$R_\pi \otimes \rho_\chi^c\left(k'-\frac{d}{2}\right) = \rho_\sigma(1) \otimes \rho_\chi^c\left(k'-\frac{d}{2}\right) \oplus \mathbb{1} \oplus \rho_\chi \rho_\chi^c(2k'+1)$$

$$= \underbrace{\rho_\sigma(1) \otimes \rho_\chi^c\left(k'-\frac{d}{2}\right)}_{\underset{V}{\parallel}} \oplus \mathbb{1} \oplus \varepsilon_{cyc}.$$

Then one checks : $\boxed{V^c = V^\vee(1)}$ ← I'm still a little bit confused here. though our definition of V matches the one on Skinner's 2009 lecture note.

Therefore it makes sense to study the Bloch–Kato conjecture for the conjugate self-dual Galois rep V.

<u>Further question</u> : Given a conjugate self-dual Gal rep V of a CM field K, does it necessarily come from an <u>Eisenstein datum</u> (σ, χ, s_0) ? If so, the machinery of <u>Eisenstein congruence of unitary groups</u> may work for such V.

§4 Previous work

		$G \hookrightarrow \mathcal{G}$	"p-primitivity"
Mazur–Wiles, Wiles	χ : Hecke character $/\mathbb{Q}$, totally real fields	$GL_1 \hookrightarrow GL_2$ $\mathbb{E}$	explicit q-expansion of the Eisenstein series $\mathbb{E}$.
Hsieh 2014 [JAMS]	χ : Hecke character $/K$	$U(1,0) \times \mathbb{G}_m \hookrightarrow U(2,1)$ $\chi \times \mathbb{1}$ $\quad$ $\mathbb{E}^{Kling}$	Fourier–Jacobi coefficient of $\mathbb{E}^{Kling}$
Skinner–Urban 2014 ⋮ Wan 2015 [Σ] to totally real number fields	$\pi_f \times \chi$, $wt(f) > wt(\chi)$ π_f : autorep attached to $f \in S_2^{ord}(\Gamma_1(N))$ χ : Hecke character $/K$	$U(1,1) \times \mathbb{G}_m \hookrightarrow U(2,2)$ $\pi_f \times \chi$ $\quad$ $\mathbb{E}^{Kling}$ $U(m,n) \times \mathbb{G}_m \hookrightarrow U(m+1,n+1)$ $\mathbb{E}^{Kling}$	Fourier coefficient of $\mathbb{E}^{Kling}$ $U(2,2)$ is quasi-split, hence this is doable.
Wan 2020 [ANT] Castella–Liu–Wan	$\pi_f \times \chi$, $wt(f) < wt(\chi)$ $f \in S_k(\Gamma_1(N))$	$U(2,0) \times \mathbb{G}_m \hookrightarrow U(3,1)$ $\pi_f \times \chi$ $\quad$ $\mathbb{E}^{Kling}$	Fourier–Jacobi coefficient of $\mathbb{E}^{Kling}$
Urban 2006 (preprint)	$Ad(E)$ adjoint rep of E E : elliptic curve $/\mathbb{Q}$	$Sp_2 \times \mathbb{G}_m \hookrightarrow Sp_4$ $\mathbb{E}^{Kling}$ $Sp(2n) \times \mathbb{G}_m \hookrightarrow Sp(2n+2)$	Fourier coefficient of $\mathbb{E}^{Kling}$