Talk 01

Algebro-geometric introduction to modular forms

2022.11.14
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A. Modular forms à là Katz.

§1 Elliptic curves over schemes

(1.1) Defn

An elliptic curve over a scheme S is a proper smooth morphism of schemes $\pi : E \longrightarrow S$ whose geometric fibers are connected curves of genus 1, together with a section $e : S \to E$.

- geometric fibre : $\forall s \in S$, $E_{\bar{s}}$ given by the following fiber product

$$\begin{tikzcd} E_{\bar{s}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & E \arrow[d, "\pi"] \\ \mathrm{Spec}\,\overline{k(s)} \arrow[r] & S \end{tikzcd}$$

is called the geometric fiber of E at s.

- curve : we take the convention in [Hartshorne, Chapter 4] : an integral scheme of dimension 1, proper over an algebraically closed field k, all of whose local rings are regular.

- Remark : The geometric fibres are elliptic curves in the usual sense.

(1.2) Some facts

(1) Let $\Omega_{E/S}$ be the sheaf of differentials of E over S. Then the pushforward $\omega_{E/S} := \pi_*(\Omega_{E/S})$ is an invertible sheaf over S.
 - In particular, this enables us to find a basis ω for $\omega_{E/S}$ locally over S.

(2) Any elliptic curves E/S admits a unique structure of abelian group scheme for which e is the identity section.

(1.3) Level structure

Let $N \in \mathbb{N}$. Define $\Gamma(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod N \right\}$.

(1) Let $[N] : E \longrightarrow E$ be the multiplication by N map with kernel $E[N]$. Then $E[N]$ is a finite flat abelian group scheme of order N^2 over S.

(2) Suppose S is a $\mathbb{Z}[\frac{1}{N}]$-scheme, (Equivalently : N is invertible in $\Gamma(S, \mathcal{O}_S)$) then $E[N]$ is étale over S, and vice versa.

(3) In the case of (2), there exists a finite étale covering S' of S such that $E_{S'}[N]$ is isomorphic to $(\mathbb{Z}/N\mathbb{Z})_{S'} \times (\mathbb{Z}/N\mathbb{Z})_{S'}$. (Here : $(\mathbb{Z}/N\mathbb{Z})_{S'}$ is the constant cyclic group scheme of order N over S').

(1.4) Defn

A Level $\Gamma(N)$-structure on E/S is a homomorphism

$$\alpha_N : E[N] \longrightarrow (\mathbb{Z}/N\mathbb{Z})_S \times (\mathbb{Z}/N\mathbb{Z})_S$$

of group schemes over S.

(1.5) Modular forms à la Katz

(1) A tuple (E_R, ω, α_N) is called a <u>test object</u>, where

- E_R is an elliptic curve over R, where R is a ring (i.e. over $\operatorname{Spec} R$)
- ω is a basis of $\Gamma(\operatorname{Spec} R, \underline{\omega}_{E/R})$.
- α_N is a level $\Gamma(N)$-structure of E_R.

We can talk about the <u>base change</u> of test objects: given a ring map $g: R \to R'$, we form:

$$\begin{tikzcd} E_{R'} \arrow[r, "g'"] \arrow[d, "\pi'"'] & E \arrow[d, "\pi"] \\ \operatorname{Spec} R' \arrow[r, "g"'] & \operatorname{Spec} R \end{tikzcd}$$

- $E_{R'} := \operatorname{Spec} R' \times_{\operatorname{Spec} R} E_R$
- $\omega_{R'}$ is a basis of $\Gamma(\operatorname{Spec} R', g^* \underline{\omega}_{E/R})$

here: we need to check $g^* \underline{\omega}_{E/R} = \underline{\omega}_{E_{R'}/R'}$. As $\underline{\Omega_{E_{R'}/R'} \simeq (g')^* \Omega_{E/R}}$, * we apply <u>smooth base change</u> to the diagram above to see
$$g^* \underline{\omega}_{E/R} = g^*(\pi_* \Omega_{E/R}) \cong \pi_* \left((g')^* \underline{\Omega_{E/R}}\right) = \underline{\omega}_{E_{R'}/R'}.$$
this is actually nontrivial.

- $\alpha_{R',N}$ is the corresponding level $\Gamma(N)$-structure of $E_{R'}/R$.

(2) A <u>modular form</u> of <u>level</u> $\Gamma(N)$, <u>weight</u> k is a rule f assigning each test object $(E/R, \omega, \alpha_N)$ to an element $f(E/R, \omega, \alpha_N) \in R$, such that:

(i) depends only on the R-isomorphism classes of the test objects.

(ii) f "commute with base change":
$$f(E/R', \omega_{R'}, \alpha_{R',N}) = g\left(f(E, \omega, \alpha_N)\right).$$

(iii) $f(E, \lambda\omega, \alpha_N) = \lambda^{-k} f(E, \omega, \alpha_N)$.

<u>Remark</u>: (1) Here (iii) implies that $f(E, \omega, \alpha_N)\, \omega^{\otimes k} \in \Gamma(\operatorname{Spec} R, \underline{\omega}_{E/R})$ <u>does not depend on the choice of</u> ω: $f(E, \lambda\omega, \alpha_N)(\lambda\omega)^{\otimes k} = \lambda^{-k} f(E, \omega, \alpha_N) \lambda^k \omega^{\otimes k}$
$= f(E, \omega, \alpha_N)\, \omega^{\otimes k}$.

So actually we can omit "ω" in the test object and even upgrade to the "scheme" level: assign each $(E/S, \alpha_N)$ to some element in $H^0(S, \underline{\omega}_{E/S}^{\otimes k})$ such that

(i) $f(E/S, \alpha_N)$ depends only on the S-isomorphism class of the test object.

(ii) $f(E_{S'}, g^*(\alpha_N)) = g^*(f(E/S, \alpha_N))$ for every morphism $g: S' \to S$.

(2) We also consider the relative sense: base rings R (or base schemes S) are defined over a fixed ring R_0, and only base change under R_0-morphisms.

The space is denoted by $F(R_\bullet; \Gamma(N), k)$.

<u>Goal</u> : Require some condition at ∞-points : we do this via the Tate curve.

§ 1.2 Tate curves

(1.6) Construction

Let $\tau \in \mathbb{H}$. Then we have:

- complex tori $\mathbb{C}/\Lambda(\tau) := \mathbb{C}/\mathbb{Z}\tau \oplus \mathbb{Z}$.
- ellipti curve E_τ: has affine equation
$$\tilde{y}^2 = 4\tilde{x}^3 - \frac{(2\pi i)^4}{12} E_4(\tau)\,\tilde{x} - \frac{(2\pi i)^6}{216} E_6(\tau)$$

where · $E_4(\tau)$, $E_6(\tau)$ are complex Eisenstein series with q-expansions in $\mathbb{C}[[q_\tau]]$, $q_\tau = e^{2\pi i\tau}$.

They are related via Weierstrass $\wp$-functions.

not the same normalization as Talk 1

Step 1: Replace E_4 and E_6 by their q-expansions
$$\tilde{y}^2 = 4\tilde{x}^3 - \frac{(2\pi i)^4}{12}\left(1 + 240\sum_{n=1}^{\infty} \sigma_3(n) q_\tau^n\right)\tilde{x} - \frac{(2\pi i)^6}{216}\left(1 - 504\sum_{n=1}^{\infty}\sigma_5(n) q^n\right)$$

and do
$$\frac{1}{(2\pi i)^2}\tilde{x} = x + \frac{1}{12}, \quad \frac{1}{(2\pi i)^3}\tilde{y} = 2y + x.$$

it becomes (Exercise)
$$E_\tau: \ y^2 + xy = x^3 + a_4(q_\tau)\,x + a_6(q_\tau)$$

where:
- $a_4(q_\tau) = \sum_{n=1}^{\infty} -5\sigma_3(n) q_\tau^n \in \mathbb{Z}[[q_\tau]]$
- $a_6(q_\tau) = \sum_{n=1}^{\infty} \frac{-5\sigma_3(n) - 7\sigma_5(n)}{12} q_\tau^n \in \mathbb{Z}[[q_\tau]]$.

here for $a_6(q_\tau) \in \mathbb{Z}[[q_\tau]]$, one checks $d^3 \equiv d^5 \bmod 12$ for every $d \in \mathbb{Z}$.

Step 2: We compute

- $\Delta_{E_\tau} = \Delta(\tau)$, the discriminant $q_\tau \prod_{n=1}^{\infty}(1-q_\tau^n)^{24}$

  Upshot: it is invertible in $\mathbb{Z}((q_\tau))$ but not in $\mathbb{Z}[[q]]$
  So to make E_τ an elliptic curve when removing "τ", we need it to be over $\mathbb{Z}((q_\tau))$ to guarantee smoothness.

- j-invariant $j(q_\tau) = \cdots \in \frac{1}{q_\tau}\mathbb{Z}[[q_\tau]]$.
- canonical differential $\omega_{can} = \frac{dx}{2y+x}$

Step 3: We ignore "τ" everywhere above, regard E_τ as an ellipti curve over $\mathbb{Z}((q))$. it is called the Tate curve over $\mathbb{Z}((q))$, denoted by $\mathrm{Tate}(q)$.
we have computed Δ, j and ω_{can} above.

Another description of Tate curve (use this to generalize to the rigid-analytic case)

(1) We have a commutative diagram

$$\begin{tikzcd}
z \arrow[r, mapsto] & t := \exp(2\pi i z) \\
\mathbb{C} \arrow[u, phantom, "\rotatebox{90}{$\in$}"] \arrow[r, two heads, "\text{“q-expansion”}"] \arrow[d] & \mathbb{C}^\times \arrow[u, phantom, "\rotatebox{90}{$\in$}"] \arrow[d] \\
\mathbb{C}/\Lambda_\tau \arrow[r, "\sim"] & \mathbb{C}^\times/q_\tau^{\mathbb{Z}} \\
2\pi i dz & \frac{dt}{t} \arrow[l, mapsto]
\end{tikzcd}
\qquad q_\tau = e^{2\pi i \tau},\ 0<|q|<1$$

(2) By Weierstrass theory, we have embedding by Weierstrass $\wp$-function

$$\mathbb{C}/\Lambda_\tau \xrightarrow{\sim} E_{q_\tau} \qquad [\text{李文威 定义-命题8.2.1}]$$

$\Longrightarrow$ Composing the above two isomorphisms above, we obtain

$$\begin{aligned}
\Phi_\tau :\ \mathbb{C}^\times/q_\tau^{\mathbb{Z}} &\xrightarrow{\quad\sim\quad} E_\tau \\
t\cdot q_\tau^{\mathbb{Z}} &\longmapsto \begin{cases} [x(t,q_\tau) : y(t,q_\tau) : 1], & t \notin q_\tau^{\mathbb{Z}}, \\ [0:1:0], & t \in q_\tau^{\mathbb{Z}}. \end{cases}
\end{aligned}$$

here x, y has explicit expression, see [Gispert, pp.34].

Now we omit "τ" everywhere, we see $\mathrm{Tate}(q)(\mathbb{C}) \simeq \mathbb{C}^\times/q^{\mathbb{Z}}$.

The standard level $T(N)$-structure on $\mathrm{Tate}(q)$

- To describe $E_\tau[N]$, we first describe $\mathbb{C}^\times/q_\tau^{\mathbb{Z}}[N]$, inspired by the Φ_τ above:

$$\alpha_N : \mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z} \xrightarrow{\sim} \mathbb{Z}/N\mathbb{Z} \times \mu_N \xrightarrow{\sim} \mathbb{C}^\times/q_\tau^{\mathbb{Z}}[N]$$
$$(i,j) \longmapsto (i, \zeta_N^j) \longmapsto q_\tau^{a}\zeta_N^{j}$$

<u>Upshot</u>: Via Φ_τ, coordinates of $E_\tau[N]$ lie in $\mathbb{Z}[q_\tau^{1/N}] \otimes_{\mathbb{Z}} \mathbb{Z}[\zeta_N]$.
So to equip $\mathrm{Tate}(q)$ with a standard level structure, this inspires us to add ζ_N and $\frac{1}{N}$.

- Again forget τ above, as in Step 3, we see for Tate curve $\mathrm{Tate}(q)$ over $\mathbb{Z}((q))$, we need to base change to $\mathbb{Z}((q^{1/N})) \otimes_{\mathbb{Z}} \mathbb{Z}[\zeta_N, \frac{1}{N}]$ at least to equip it with a level $T(N)$-structure.

$$\begin{tikzcd}
\mathrm{Tate}(q) \arrow[d] & \mathrm{Tate}(q) \arrow[l] \arrow[d] \arrow[dl, phantom, "\lrcorner"] \\
\operatorname{Spec}\mathbb{Z}((q)) & \operatorname{Spec}\left(\mathbb{Z}((q^{1/N})) \otimes_{\mathbb{Z}} \mathbb{Z}[\zeta_N, \frac{1}{N}]\right) \arrow[l]
\end{tikzcd}$$
(abuse of notation)

the level $T(N)$-structure is called α_{std}, <u>standard level $T(N)$-structure</u>.

(1.7) Relation with q-expansion

Let $\mathcal{R} := \{ f \in \mathbb{C}((q)) : f$ is holomorphic on $\mathbb{D}_0 \}$ where $\mathbb{D}_0$ is the punctured unit disc $\mathbb{D}_0 := \{ q \in \mathbb{C} : 0 < |q| < 1 \}$. Then $\mathrm{Tate}(q)$ is also defined over $\mathcal{R}$.

- Recall: Tate curve $\mathrm{Tate}(q)$ only $/ \mathbb{Z}((q))$.
- If we hope its coefficients to be really holo, STS: $q \in \mathbb{D}_0$

By definition of Tate curve, we have the Cartesian diagram

$$(*) \qquad \begin{tikzcd} E_\tau \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathrm{Tate}(q) \arrow[d] \\ \mathrm{Spec}\,\mathbb{C} \arrow[r] & \mathrm{Spec}\,\mathcal{R} \end{tikzcd},$$

here $\tau : \mathcal{R} \to \mathbb{C}$ is given by $q \mapsto q_\tau$ by abuse of notation, called the <u>specialization map</u> to $\tau \in \mathbb{C}$. (故名思义：τ 把 Tate curve 中的未定元 q 变为确切复数 $q_\tau = e^{2\pi i \tau / N}$. 由此可见 $\mathrm{Tate}(q)$ 以这种方式"编码"所有椭圆曲线).

Let $f \in \mathcal{M}_k(\Gamma(N))$, then to interprete it as a f^{Katz}, we first assign:

- $f^{\mathrm{Katz}}((E_\tau, \omega, \alpha_N)) \in \mathbb{C}$ being $f(\tau)$, as $f : \mathbb{H} \to \mathbb{C}$.

Then since f^{Katz} respects base change, using the diagram $(*)$, we see

$$\tau\Big(\underbrace{f^{\mathrm{Katz}}\big((\mathrm{Tate}(q), \omega_{\mathrm{can}}, \alpha_{\mathrm{std}}) \big)}_{=: \hat{f}(q) \in \mathcal{R}} \Big) = f^{\mathrm{Katz}}(E_\tau, \omega, \alpha_N) = f(\tau)$$

hence $f(\tau) = \hat{f}(q_\tau)$. This is exactly the Fourier expansion of f at ∞.

(1.8) Definition

We denote $O_N := \mathbb{Z}[\zeta_N, \frac{1}{N}]$. Let $R_\cdot$ be an O_N-algebra. Let $f \in F(R_\cdot, T(N), k)$.

<u>Note</u>: As seen before, the Tate curve $Tate(q)$ is defined over $\mathbb{Z}((q))$.
Yet to discuss its level structure, we have seen that we need to add $\frac{1}{N}$ and ζ_N.
$\leadsto$ so to talk about q-expansion, viewing Tate curve as a test object, we need to work over O_N.

(1) For every level $T(N)$-structure α_N on $Tate(q)$ over $\mathbb{Z}((q^{1/N})) \otimes_{\mathbb{Z}} R_\cdot$, the <u>$q$-expansion of f at α_N</u> is the Laurent series

$$\widehat{f_{\alpha_N}}(q) := f\left(Tate(q)/(\mathbb{Z}((q^{1/N})) \otimes_{\mathbb{Z}} R_\cdot), \omega_{can}, \alpha_N \right) \in \mathbb{Z}((q^{1/N})) \otimes_{\mathbb{Z}} R_\cdot$$

<u>Remark</u>: Here require $R_\cdot$ to be an O_N-algebra to make it contains $\frac{1}{N}$ and primitive root of unity ζ_N.

(2) Let R_0 be a ring in which N is not nilpotent. (imitating $\mathbb{Z}$). We say $f \in F(R_0; T(N), k)$ is <u>holomorphic at ∞</u> if its base change to $F(R_0[\zeta_N, \frac{1}{N}]; T(N), k)$ has all its q-expansions in $\mathbb{Z}[[q^{1/N}]] \otimes_{\mathbb{Z}} R_0[\zeta_N, \frac{1}{N}]$.
It is a <u>cusp form</u> if the q-expansions lie in $q^{1/N}\mathbb{Z}[[q^{1/N}]] \otimes_{\mathbb{Z}} R_0[\zeta_N, \frac{1}{N}]$.
The spaces are denoted by $M(R_0, T(N), k)$ and $S(R_0, T(N), k)$ respectively.

Remark : (From TCC course on p-adic modular forms , Lecture 6)

- Story over $\mathbb{C}$: by construction, we see every elliptic curve is parametrized by a specialization of the Tate's curve. (正是Tate curve妙处)

- Story over K , where K is a p-adic field : we have p-adic valuations :
  - The power series defining $\mathrm{Tate}(q)$ are convergent for $0<|q|<1$.
    Then $\mathrm{Tate}(q)$ is an elliptic curve with $|j|>1$. We can identify
    $$\overline{K}^{\times}/q^{\mathbb{Z}} \xrightarrow{\sim} \mathrm{Tate}(q)(\overline{K})$$
  - In particular, any elliptic curve E/K with $|j(E)|\leq 1$ cannot be parametrized by Tate's curve :

    Theorem of Tate :

    — Given an elliptic curve E/K with $|j(E)|>1$, there exists a unique $q\in K^{\times}$ with $|q|<1$ such that $E\simeq \mathrm{Tate}(q)$ over $\overline{K}$.

    — This isomorphism descends to K iff E has split multiplicative reduction.

  $\Longrightarrow$ This led Tate to develop the theory of rigid analytic geometry.

Goal : Find a universal " test object " so that it suffices to assign on this universal one.

§3 Modular curves

(1.9) Defn

Let R_0 be a ring. We define the category Ell/R_0 as

(i) objects : elliptic curves E/S where S is an R_0-scheme

(ii) morphisms from E'/S' to E/S is a Cartesian square

$$\begin{tikzcd} E' \arrow[r] \arrow[d, "\pi'"'] \arrow[dr, phantom, "\ulcorner", very near start] & E \arrow[d, "\pi"] \\ S' \arrow[r] & S \end{tikzcd}$$

of R_0-morphisms.

(1.10) Defn

(1) A <u>moduli problem for elliptic curves over R_0</u> is a functor

$$\mathcal{P} : (Ell/R_0)^{op} \to Set .$$

For $E/S \in Ell/R_0$, $\mathcal{P}(E/S)$ is called the set of <u>level $\mathcal{P}$-structures</u>.

(2) We say $\mathcal{P}$ is

(i) <u>representable</u> if $\exists\ \mathbb{E}/\mathcal{M}(\mathcal{P}) \in Ell/R_0$ s.t. $\mathcal{P}(-) \xrightarrow{\sim} \mathrm{Hom}_{Ell/R_0}(-, \mathbb{E}/\mathcal{M}(\mathcal{P}))$.

(ii) <u>relatively representable</u> if for every $E/S \in Ell/R_0$, the associated functor

$$\mathcal{P}_{E/S} : (Sch/S)^{op} \to Set$$
$$T \longmapsto \mathcal{P}(E_T/T)$$

where $E_T = E \times_S T$, is represented by an S-scheme $\mathcal{P}_{E/S}$.

(3) When $\mathcal{P}$ is representable, we have

$$\mathcal{P}(\mathbb{E}/\mathcal{M}(\mathcal{P})) \xrightarrow{\sim} \mathrm{Hom}_{Ell/R_0}(\mathbb{E}/\mathcal{M}(\mathcal{P}), \mathbb{E}/\mathcal{M}(\mathcal{P}))$$
$$\alpha_{univ} \longmapsto id$$

Then $(\mathbb{E}/\mathcal{M}(\mathcal{P}), \alpha_{univ})$ is called a <u>universal pair</u>.

(1.11) Lemma

If $\mathcal{P}$ is represented by $\mathbb{E}/\mathcal{M}(\mathcal{P})$, then the associated functor

$$\widetilde{\mathcal{P}} : (\mathrm{Sch}/R_0)^{op} \longrightarrow \mathrm{Set}$$
$$S \longmapsto \{ (E/S, \alpha) : E/S \in \mathrm{Ell}/R_0 ,\ \alpha \in \mathcal{P}(E/S) \} / \sim$$

here "$\sim$" is S-isomorphisms of such pairs, is represented by the R_0-scheme $\mathcal{M}(\mathcal{P})$.

<u>Proof</u>: We are to show:

$$\widetilde{\mathcal{P}}(S) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Sch}/R_0}(S, \mathcal{M}(\mathcal{P})) , \quad \text{naturally in } S .$$

Given an R_0-morphism $f : S \longrightarrow \mathcal{M}(\mathcal{P})$, we form the fiber product

$$\begin{array}{ccc} \mathbb{E}_S := \mathbb{E} \times_{\mathcal{M}(\mathcal{P})} S & \longrightarrow & \mathbb{E} \\ \big\downarrow \ \lrcorner & & \big\downarrow \\ S & \longrightarrow & \mathcal{M}(\mathcal{P}) \end{array} \qquad \text{—} (*)$$

and obtain $\mathbb{E}_S/S \in \mathrm{Ell}/R_0$, determined up to S-isomorphisms. Moreover, as $\mathbb{E}/\mathcal{M}(\mathcal{P})$ represents $\mathcal{P}$, i.e.

$$\mathcal{P}(\mathbb{E}_S/S) \xrightarrow{1:1} \mathrm{Hom}_{\mathrm{Ell}/R_0}(\mathbb{E}_S/S, \mathbb{E}/\mathcal{M}(\mathcal{P}))$$

the diagram $(*)$ gives a unique $\alpha \in \mathcal{P}(\mathbb{E}_S/S)$. $\square$

(1.12) Definition

Let $N \in \mathbb{N}$. Let R_0 be a ring in which N is not nilpotent. Define the moduli problem

$$T(N)_{R_0} : \quad E/S \longmapsto \{ \alpha_N : \text{level } T(N)\text{-structure on } E/S \}.$$

We write $T(N) = T(N)_{\mathbb{Z}}$.

(1.13) Theorem

(1) Let $N \geq 3$, the moduli problem $T(N)_{\mathbb{Z}[\frac{1}{N}]}$ is representable by some $\mathbb{E}/Y(N)$ in $Ell/\mathbb{Z}[\frac{1}{N}]$.

(2) The $\mathbb{Z}[\frac{1}{N}]$-scheme $Y(N)$ is a smooth affine curve over $\mathbb{Z}[\frac{1}{N}]$.

(3) If R_0 is a $\mathbb{Z}[\frac{1}{N}]$-algebra, then $T(N)_{R_0}$ is representable by $\mathbb{E}_{R_0}/Y_1(N)_{R_0}$, the base change of $\mathbb{E}/Y(N)$ to R_0.

Here $Y(N)$ is called the <u>modular curve</u> for $T(N)$ and $\mathbb{E}/Y(N)$ is a <u>universal elliptic curve</u>, with α_{univ} the <u>universal level $T(N)$-structure</u>.

For proof, see Katz & Mazur's book or see [Gispert] for references!

(1.14) Geometric modular forms

Upshot: ① modular forms (à la Katz) commute with base change.

② every elliptic curve with a level structure can be obtained by pulling-back from $(\mathbb{E}/Y(N), \alpha_{univ})$, as we're working over $\mathbb{Z}[\frac{1}{N}]$.

Let K be a $\mathbb{Z}[\frac{1}{N}]$-module. Let $k \in \mathbb{Z}$. A modular form for $\Gamma(N)$ of weight k and coefficient in K is an element in

$$F(K; \Gamma(N), k) := H^0(Y(N), \underline{\omega}_{\mathbb{E}/Y(N)}^{\otimes k} \otimes_{\mathbb{Z}[\frac{1}{N}]} K)$$

Remark: hence by ① and ②, for $f \in F(K, \Gamma(N), k)$, $(E/S, \alpha_N) \in \mathcal{E}ll/R_0$, $\exists$ a Cartesian diagram

$$\begin{tikzcd} E \arrow[r, "\tilde{g}"] \arrow[d, "\pi_E"'] \arrow[dr, phantom, "\ulcorner"] & \mathbb{E} \arrow[d, "\pi_{\mathbb{E}}"] \\ S \arrow[r, "g"'] & Y(N) \end{tikzcd} \qquad \text{--- } (*)$$

s.t. $(E/S, \alpha_N) = g^*(\mathbb{E}/Y(N), \alpha_{univ})$ and g is unique with this property. Then we define

$$f(E/S, \alpha_N) = g^*(f) \in H^0(S, \underline{\omega}_{E/S}^{\otimes k} \otimes_{\mathbb{Z}[\frac{1}{N}]} K)$$

here g^* is the induced map (?)

$$g^*: H^0(Y(N), \underline{\omega}_{\mathbb{E}/Y(N)}^{\otimes k} \otimes_{\mathbb{Z}[\frac{1}{N}]} K) \longrightarrow H^0(S, \underline{\omega}_{E/S}^{\otimes k} \otimes_{\mathbb{Z}[\frac{1}{N}]} K)$$
$$f \longmapsto g^*(f)$$

? GAP: how to obtain this map g^*?

Guess: (flat) base change theorem applied to the diagram $(*)$:

$$g^*\pi_{\mathbb{E},*}\Omega_{\mathbb{E}/Y(N)} \simeq \pi_{E,*}\,\underbrace{\tilde{g}^*\Omega_{\mathbb{E}/Y(N)} \simeq \pi_{E,*}\Omega_{E/S}}_{\text{property of sheaf of differential}}$$

i.e. $g^*\underline{\omega}_{\mathbb{E}/Y(N)} \simeq \underline{\omega}_{E/S}$.

Then g^* is basically obtained by comparing the global sections of pullbacks. □

This coincide with (1.5). Hence we can again define q-expansion of f.

(1.15) j-invariant morphism

We define a $\mathbb{Z}[\frac{1}{N}]$-morphism $j : Y(N) \longrightarrow \underbrace{\mathbb{A}^1_{\mathbb{Z}[\frac{1}{N}]} = \mathrm{Spec}\left(\mathbb{Z}[\tfrac{1}{N}][j]\right)}_{\text{affine } j\text{-line}}$; $\leftarrow$ indeterminate j

Let R be a $\mathbb{Z}[\frac{1}{N}]$-algebra,

- $Y(N)(R) = \mathrm{Hom}_{\mathbb{Z}[\frac{1}{N}]}(\mathrm{Spec} R, Y(N)) = \{(E/R, \alpha_N) : E/R \in \mathcal{Ell}/\mathbb{Z}[\tfrac{1}{N}]\}/\sim$.

  So choosing a point $P \in Y(N)(R)$ is equivalent to choose a pair $(E/R, \alpha_N)$.

- $\mathbb{A}^1_{\mathbb{Z}[\frac{1}{N}]}(R) = \mathrm{Hom}_{\mathbb{Z}[\frac{1}{N}]}(\mathrm{Spec} R, \mathrm{Spec}\,\mathbb{Z}[\tfrac{1}{N}][j]) = \mathrm{Hom}_{\mathbb{Z}[\frac{1}{N}]}(\mathbb{Z}[\tfrac{1}{N}][j], R) \simeq R$

- So we define

$$j(R) : Y(N)(R) \longrightarrow \mathbb{A}^1_{\mathbb{Z}[\frac{1}{N}]}(R)$$
$$(E/R, \alpha_N) \longmapsto \underbrace{j(E/R)}_{j\text{-invariant of } E/R} \in R$$

  assembling into $j : Y(N) \to \mathbb{A}^1_{\mathbb{Z}[\frac{1}{N}]}$.

Fact : j is finite and flat.

We further canonically embed $\mathbb{A}^1_{\mathbb{Z}[\frac{1}{N}]}$ into the "projective j-line" $\mathbb{P}^1_{\mathbb{Z}[\frac{1}{N}]} = \mathrm{Proj}(\mathbb{Z}[\tfrac{1}{N}][j])$.

So we get

$$\begin{tikzcd}
 & Y(N) \arrow[d, "j"] \arrow[ddl, bend right, "j'"'] \\
 & \mathbb{A}^1_{\mathbb{Z}[\frac{1}{N}]} \arrow[d, hook] \\
X(N) \arrow[r, "\tilde{j}"'] & \mathbb{P}^1_{\mathbb{Z}[\frac{1}{N}]}
\end{tikzcd}$$

The compactified modular curve $X(N)$ over $\mathbb{Z}[\frac{1}{N}]$ is defined as the normalization of $\mathbb{P}^1_{\mathbb{Z}[\frac{1}{N}]}$ in $Y(N)$. [Stacks project, 035H]

- normalization $X(N) = \underline{\mathrm{Spec}}_{\mathbb{P}^1_{\mathbb{Z}[\frac{1}{N}]}}(\mathcal{O}')$, where $\mathcal{O}'$ is the "integral closure" of $\mathcal{O}_{\mathbb{P}^1_{\mathbb{Z}[\frac{1}{N}]}}$ in $j_* \mathcal{O}_{Y(N)}$.

(1.16) Theorem

(1) The modular scheme $X(N)$ is a proper smooth curve over $\mathbb{Z}[\frac{1}{N}]$.

(2) $C(N) := X(N) \setminus Y(N)$ is a closed subscheme of $X(N)$. is finite étale over $\mathbb{Z}[\frac{1}{N}]$. called the scheme of cusps.

(3) There is a bijection

$$C(N)_{\mathbb{Q}_N} \xleftrightarrow{1:1} \text{level } \Gamma(N)\text{-structures on } \mathrm{Tate}(q) \text{ over } \mathbb{Z}((q^{1/N})) \otimes_{\mathbb{Z}} \mathcal{O}_N .$$

(1.17) Theorem

There is an invertible sheaf $\underline{\omega}$ on $X(N)$ whose restriction to $Y(N)$ is $\underline{\omega}_{\mathbb{E}/Y(N)}$.

Remark: This is done by extending the universal elliptic curve $\mathbb{E}/Y(N)$ to a generalized elliptic curve $\widetilde{\mathbb{E}}/X(N)$. Then $\underline{\omega} := \tilde{\pi}_* \Omega^1_{\widetilde{\mathbb{E}}/X(N)}$

$$\begin{tikzcd} \mathbb{E} \arrow[r, dashed] \arrow[d, "\pi"'] & \widetilde{\mathbb{E}} \arrow[d, "\tilde{\pi}"] \\ Y(N) \arrow[r, hook] & X(N) \end{tikzcd}$$

See Zhao Bin's note for references. (Or Gispert's note)

Inspired by (1.16.3) and (1.17), we define

(1.18) Defn

Let K be a $\mathbb{Z}[\frac{1}{N}]$-module, Let $k \in \mathbb{Z}$. A modular form for $T(N)$ of weight k with coefficient in K and holomorphic at ∞ is an element of

$$H^0(X(N), \underline{\omega}^{\otimes k} \underset{\mathbb{Z}[\frac{1}{N}]}{\otimes} K) =: M(K, T(N), k).$$

At each cusp C, by (1.16), there is a level $T(N)$-structure α_N of $\mathrm{Tate}(q)$. Then the q-expansion f at the cusp C is defined as

$$f(\mathrm{Tate}(q)/\mathbb{Z}((q^{1/N})) \underset{\mathbb{Z}}{\otimes} \mathbb{Z}[\tfrac{1}{N}, \zeta_N], \omega_{can}, \alpha_N) \in \mathbb{Z}[[q^{1/N}]] \underset{\mathbb{Z}}{\otimes} \mathbb{Z}[\tfrac{1}{N}, \zeta_N] \underset{\mathbb{Z}[\frac{1}{N}]}{\otimes} K.$$

In this way, f is a cusp form at C if the above expansion lies in $q^{1/N}\mathbb{Z}[[q^{1/N}]] \underset{\mathbb{Z}}{\otimes} \mathcal{O}_N \underset{\mathbb{Z}[\frac{1}{N}]}{\otimes} K$.

(1.19) Corollary

Let $f: R \to R'$ be an $\mathcal{O}_N$-algebra homomomorphism such that R' is a flat R-algebra via f. Then

$$M(R, T(N), k) \underset{R}{\otimes} R' \xrightarrow{\sim} M(R')$$

and similar for cusp forms.

Proof: Now this is purely an algebro-geometric problem: establish a "base change theorem". $\square$

<u>Remark on cusps</u> : Here it has always been difficult to analysing cusps. In our contexts, cusps on $X(N)$ gives a divisor D_{cusp} on $X(N)$. Then the space of cusp forms is actually

$$S(K, T(N), k) = H^0(X(N), K \underset{\mathbb{Z}[\frac{1}{N}]}{\otimes} \underline{\omega}^{\otimes k}(-D_{cusp})).$$

Here intuitively, $D := D_{cusp}$

$$H^0(X(N), K \underset{\mathbb{Z}[\frac{1}{N}]}{\otimes} \underline{\omega}^{\otimes k}(-D_{cusp})) = \{ f \in K(X(N)) \setminus \{0\} : \mathrm{div}(f) - D \geq 0 \} \cup \{0\}$$
$$= \{ f \in K(X(N)) \setminus \{0\} : f|_D = 0 \} \cup \{0\}.$$

Since D_{cusp} is an effective divisor, $H^0(X(N), K \underset{\mathbb{Z}[\frac{1}{N}]}{\otimes} \underline{\omega}^{\otimes k}(-D_{cusps})) \subseteq M(K, T(N), k)$.
Actually, $K \underset{\mathbb{Z}[\frac{1}{N}]}{\otimes} \underline{\omega}^{\otimes k}(-D_{cusp})$ is a sheaf of ideals of $K \underset{\mathbb{Z}[\frac{1}{N}]}{\otimes} \underline{\omega}^{\otimes k}$.

This has already quite "simple" since $X(N)$ is a curve. Things are more complicated for general moduli. A more correct <u>cusp divisor</u> D_{cusp} is the "<u>log cusps</u>", as local sections near cusps are of the form $\mathcal{O}_{X(N)} \cdot \frac{dq}{q} = \mathcal{O}_{X(N)} \cdot d(\log q)$.

<u>See</u> : Huber, Müller-Stach "Periods and Nori motives", § 3.1.6 (originated by Deligne, 1973).

§ 4 The q-expansion principle

To state this, we need to understand more on modular curves.

(1.20) Weil pairing

- Observe $\mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z}$ has a canonical $\mathbb{Z}/N\mathbb{Z}$ basis $((1,0),(0,1))$, so given a level $\Gamma(N)$-structure α_N on E, we have locally a canonical basis (P,Q) of $E(N)$.
- We have a Weil pairing
$$e_N : E[N] \times_S E[N] \longrightarrow \mu_{N,S}$$
and $e_N(P,Q)$ is a primitie N-th root of unity, called the <u>deteminant</u> of α_N.

(1.21) A submoduli problem

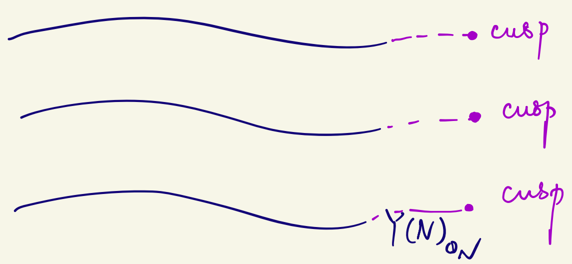
Fix a primitie N-th root of unity ζ_N, we consider
$$\begin{aligned} \Gamma(N)_{\zeta_N} : (\mathrm{Ell}/\mathcal{O}_N)^{op} &\longrightarrow \mathrm{Set} \\ E/S &\longmapsto \{ \alpha_N \in \Gamma(N)(E/S) : \det(\alpha_N) = \zeta_N \} \end{aligned}$$
Then it is represented by $\mathbb{E}_{\zeta_N}/Y(N)_{\zeta_N}$, where:

- $Y(N)_{\zeta_N}$ is a closed subscheme of $Y(N)_{\mathcal{O}_N}$.
- $\mathbb{E}_{\zeta_N}$ is the pullback of $\mathbb{E}$ via the closed emmersion $Y(N)_{\zeta_N} \hookrightarrow Y(N)_{\mathcal{O}_N}$.

We anologuely construct closed subschemes $X(N)_{\zeta_N}$ of $X(N)_{\mathcal{O}_N}$.

(1.22) Theorem

1) The curve $Y(N)_{\zeta_N}$ (resp. $X(N)_{\zeta_N}$) is an affine (resp. proper) smooth geometrically connected curve over $\mathcal{O}_N$.
2) $Y(N)$ (resp. $X(N)$) is a disjoint union of $\phi(N)$ geometrically connected components, one for each primitie N-th root of unity.
3) $C(N)_{\mathcal{O}_N}$ is a disjoint union of points.



(1.23) The q-expansion principal (I) : Uniqueness

Let $N, k \in \mathbb{Z}$ with $N \geq 3$. Let ζ_N be a pmitie N-th root of unity. Let K be a $\mathbb{Z}[\frac{1}{N}]$-module. Let $f \in M(K, T(N), k)$.

If each connected component of $X(N)_{\mathcal{O}_N}$ has at least one cusp at which the corresponding q-expansions of f vanishes identically, then $f = 0$.

<u>Proof</u> : [Katz, § 1.6].

(1.24) The q-expansion principal (II) : Cartesianality

Let $N, k \in \mathbb{Z}$ with $N \geq 3$. Let ζ_N be a pmitie N-th root of unity Let K be a $\mathbb{Z}[\frac{1}{N}]$-module with a submodule L. Let $f \in M(K, T(N), k)$.

If each connected component of $X(N)_{\mathcal{O}_N}$ has at least one cusp at which the corresponding q-expansions of f has all coefficients lies in $L \otimes_{\mathbb{Z}[\frac{1}{N}]} \mathcal{O}_N$, then $f \in M(L, T(N), k)$.

<u>Proof</u> : Since $\underline{\omega}$ is a locally free $\mathcal{O}_{X(N)}$-module (hence flat), and $X(N)$ is flat over $\mathbb{Z}[\frac{1}{N}]$, we have a short exact sequence of $\mathcal{O}_{X(N)}$-modules

$$0 \longrightarrow \underline{\omega}^{\otimes k} \otimes_{\mathbb{Z}[\frac{1}{N}]} L \longrightarrow \underline{\omega}^{\otimes k} \otimes_{\mathbb{Z}[\frac{1}{N}]} K \longrightarrow \underline{\omega}^{\otimes k} \otimes_{\mathbb{Z}[\frac{1}{N}]} K/L \longrightarrow 0$$

Then taking $H^0(X(N), -)$, we get

$$0 \longrightarrow M(L, T(N), k) \xrightarrow{\alpha} M(K, T(N), k) \xrightarrow{\beta} M(K/L, T(N), k).$$

Then $\beta(f) \in M(K/L, T(N), k)$ satisfies the hypothesis of (1.23), hence $\beta(f) = 0$, therefore $f \in \operatorname{im} \alpha$, i.e. $f \in M(L, T(N), k)$. $\square$

<u>Remark</u> : We naively think in $SL_2(\mathbb{Z})$-case. (<u>note</u> : $SL_2(\mathbb{Z}) = T(1)$, not meeting the conditions above), then ∞ is the unique cusp, and we have the "q-expansion map"

$$\phi_{K,\infty} : M(K, SL_2(\mathbb{Z}), k) \longrightarrow K[[q]]$$

which is "functorial" in A : let $L \subseteq K$, then the diagram comutes:

Here α is the map in the proof of (1.24)

$$\begin{array}{ccc} M(K, SL_2(\mathbb{Z}), k) & \xrightarrow{\phi_{K,\infty}} & K[[q]] \\ \alpha \uparrow & & \uparrow \text{incl.} \\ M(L, SL_2(\mathbb{Z}), k) & \xrightarrow[\phi_{L,\infty}]{} & L[[q]] \end{array} \quad — (*)_{L,K}$$

then the above two theorems are saying :

- $(1.23) \Longrightarrow \phi_{K,\infty}$ is injective
- $(1.24) \Longrightarrow (*)_{L,K}$ is Cartesian, i.e.

$$f'(K[[q]]) = \phi_{L,\infty}(M(L, SL_2(\mathbb{Z}), K)).$$

<u>Remark on level structures</u> : [李文威. pp.298 中段叙述]

In this section we only took care of level structures $\Gamma(N)$. Actually such disscussions can be carried out for other level structures, <u>eg</u>: $\Gamma_1(N)$, $\Gamma_0(N)$, ...

$$\Gamma(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod N \right\}$$

principal congruence subgroups

$$\cap\!|$$

$$\Gamma_1(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \bmod N \right\}$$

$$\cap\!|$$

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & * \\ & * \end{pmatrix} \bmod N \right\}.$$

- $\Gamma(1) = SL_2(\mathbb{Z})$.
- $\Gamma_1(N) \trianglelefteq \Gamma_0(N)$ with $\Gamma_0(N)/\Gamma_1(N) \simeq (\mathbb{Z}/N\mathbb{Z})^\times$.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \Gamma_1(N) \longmapsto d \bmod N$$

The corresponding moduli often has easier interpretation.

<u>Example 1</u>: For $\Gamma_1(N)$, the level structure in the moduli problem is a homomorphism

$$\beta_N : E[N]_S \longrightarrow (\mathbb{Z}/N)_S$$

for an elliptic curve E over S.

<u>Example 2</u> : For $\Gamma_0(N)$, see [Katz & Mazur, (3.4)].

We have parallel constructions as in this section.

B. Geometric Tools

§5 Algebraic de Rham cohomology

(1.25) Defn

Let $\pi : X \longrightarrow S$ be a morphism of schemes.

(1) The relative de Rham complex is defined as

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{d^0} \Omega^1_{X/S} \xrightarrow{d^1} \Omega^2_{X/S} \xrightarrow{d^2} \Omega^3_{X/S} \to \cdots \quad \in C^+(\mathcal{O}_X\text{-mod}).$$

A special case : X is smooth of relative dimension n over S. Then

- $\Omega^1_{X/S}$ is locally free of rank n
- $\Omega^i_{X/S}$ vanishes for $i > n$.
- The differential is uniquely characterised by :
  - (i) $d^0 = d_{X/S}$: the universal derivation $d_{X/S} : \mathcal{O}_S \to \Omega^1_{X/S}$.
  - (ii) $d^{p+1} d^p = 0$ for all $p \geq 0$
  - (iii) $d(\omega \wedge \omega') = d\omega \wedge \omega' + (-1)^p \omega \wedge d\omega'$ for all local sections ω of $\Omega^p_{X/S}$ and ω of $\Omega^{p'}_{X/S}$.

Indeed if $t_1, \dots, t_n$ are local parameters at $x \in X$. Then local sections near $\Omega^p_{X/S}$ can be expressed as

$$\omega = \sum_{1 \leq i_1 < \cdots < i_p \leq n} f_{i_1 \cdots i_p}\, dt_{i_1} \wedge \cdots \wedge dt_{i_p}$$

then

$$d^p \omega = \sum_{1 \leq i_1 < \cdots < i_p \leq n} df_{i_1 \cdots i_p} \wedge dt_{i_1} \wedge \cdots \wedge dt_{i_p}.$$

We only use this special case.

(2) Let $\mathbb{R}\pi_* : D^+(\mathcal{O}_X\text{-mod}) \longrightarrow D^+(\mathcal{O}_S\text{-mod})$ be the derived pushforward. Then the $\mathcal{O}_S$-module

$$H^n_{dR}(X/S) := \mathbb{R}^n \pi_* (\Omega^\bullet_{X/S}) \quad \in (\mathcal{O}_S\text{-mod})$$

is defined as the relative de Rham cohomology of X over S.

Remark :

(1) This is defined as "hypercohomology" of the de Rham complex $\Omega^\bullet_{X/S}$, instead of directly taking cohomology of $\Omega^\bullet_{X/S}$. (in char 0, this sequence is freq. exact)

(2) In some references ([SP]) we consider $\mathbb{R}\Gamma : D^+(\mathcal{O}_X\text{-mod}) \longrightarrow D^+(\Gamma(X, \mathcal{O}_X)\text{-mod})$ and define

$$H^n_{dR}(X/S) := \mathbb{R}^n\Gamma(\Omega^\bullet_{X/S}) \in (\Gamma(X, \mathcal{O}_X)\text{-mod}).$$

This is called the algebraic de Rham cohomology.

(3) Without language of derived cats? See [考=, §3.12].

(1.26) The spectral sequences

Algebraic background : 李文威《代数学方法 2》例 5.6.5.

Let $\mathcal{D}^{\bullet,\bullet}$ be a Cartan–Eilenberg resolution of $\Omega^\bullet_{X/S}$, i.e. we have a double complex

$$\mathcal{D}^{\bullet,\bullet}: \quad \begin{tikzcd} & \uparrow & \uparrow & \uparrow & \\ 0 \arrow[r] & \mathcal{D}^{1,0} \arrow[r] \arrow[u] & \mathcal{D}^{1,1} \arrow[r] \arrow[u] & \mathcal{D}^{2,1} \arrow[r] \arrow[u] & \cdots \\ 0 \arrow[r] & \mathcal{D}^{0,0} \arrow[r] \arrow[u] & \mathcal{D}^{1,0} \arrow[r] \arrow[u] & \mathcal{D}^{2,0} \arrow[r] \arrow[u] & \cdots \\ 0 \arrow[r] & \mathcal{O}_X \arrow[r] \arrow[u, "\varepsilon^0"] & \Omega^1_{X/S} \arrow[r] \arrow[u, "\varepsilon^1"] & \Omega^2_{X/S} \arrow[r] \arrow[u, "\varepsilon^2"] & \cdots \end{tikzcd} \qquad \begin{tikzcd} \vdots \\ H^p(I^{\bullet,1}, {}^{\partial}d) \arrow[u] \\ H^p(I^{\bullet,0}, {}^{\partial}d) \arrow[u] \\ H^p(X) \arrow[u, "\text{chain map}"] \end{tikzcd}$$

Recall : Cartan–Eilenberg resolution

$\mathcal{A}$ abcat with enough injecties. $\qquad \mathcal{A} := \mathcal{O}_X$-modules

$X \in C^+(\mathcal{A})$ $\qquad X := \Omega^\bullet_{X/S}$ the relatie de Rham complex

Then $\exists$ a double complex I and $\varepsilon: X \longrightarrow (I^{\bullet,0}, {}^{\partial}d^{\bullet,0}) \in C^+(\mathcal{A})$, such that

(i) (ii) bounded conditions

(iii) vertical resolution : $\forall a \in \mathbb{Z}$, $0 \to X^a \xrightarrow{\varepsilon^a} I^{a,0} \xrightarrow{{}^{\Delta}d^{a,0}} I^{a,1} \to \cdots$

is an injectie resolution of X^a.

(vi) cohomological resolution : $H^p := H^p(X)$ has an injectie resolution

$$0 \to H^p(X) \longrightarrow H^p(I^{\bullet,0}, {}^{\partial}d) \longrightarrow H^p(I^{\bullet,1}, {}^{\partial}d) \to 0$$

Then we form the associated total complex

$$\mathcal{C}^\bullet := \mathrm{tot}(\mathcal{D}^{\bullet,\bullet}) \quad : \quad \mathcal{C}^k = \bigoplus_{a+b=k} \mathcal{D}^{a,b}$$

as an injectie resolution of $\Omega^\bullet_{X/S}$ (here we mean $\Omega^\bullet_{X/S} \to \mathcal{C}^\bullet$ is a quasi-isomorphism).

Then :

- $\mathcal{H}^n_{dR}(X/S) = \mathbb{R}^n \pi_*(\mathcal{C}^\bullet) = H^n(\pi_* \mathcal{C}^\bullet)$.

associated with two decreasing filtrations

- ${}_I\mathrm{Fil}^i \mathcal{C}^\bullet$: ${}_I\mathrm{Fil}^i \mathcal{C}^k = \bigoplus_{a \geq i} \mathcal{D}^{a,k-a} \subseteq \mathcal{C}^k$
- ${}_{II}\mathrm{Fil}^i \mathcal{C}^\bullet$: ${}_{II}\mathrm{Fil}^i \mathcal{C}^k = \bigoplus_{b \geq j} \mathcal{D}^{k-b,b} \subseteq \mathcal{C}^k$.

This induces two filtrations on $\mathcal{H}^n_{dR}(X/S)$:

(i) Hodge filtration.

$$ {}_{\mathrm{I}}\mathrm{Fil}^i \mathcal{H}^n_{dR}(X/S) = \mathrm{im}\left(\mathbb{R}^n\pi_*({}_{\mathrm{I}}\mathrm{Fil}^i \mathcal{C}^\bullet) \to \mathbb{R}^n\pi_*(\mathcal{C}^\bullet) = \mathcal{H}^n_{dR}(X/S) \right) $$

Usually we write it as (see explanations in [SP, tag 0FM7])

$$ {}_{\mathrm{I}}\mathrm{Fil}^i \mathcal{H}^n_{dR}(X/S) = \mathrm{im}\left(\mathbb{R}^n\pi_*(\tau_{\geq i}\Omega^\bullet_{X/S}) \to \mathbb{R}^n\pi_*(\Omega^\bullet_{X/S}) = \mathcal{H}^n_{dR}(X/S) \right). $$

(ii) The conjugate filtration

$$ {}_{\mathrm{II}}\mathrm{Fil}^i \mathcal{H}^n_{dR}(X/S) = \mathrm{im}\left(\mathbb{R}^n\pi_*({}_{\mathrm{II}}\mathrm{Fil}^i \mathcal{C}^\bullet) \to \mathbb{R}^n\pi_*(\mathcal{C}^\bullet) = \mathcal{H}^n_{dR}(X/S) \right) $$

Moreover, such filtered complexes gives spectral sequences:

(i) The Hodge to de Rham spectral sequence

$$ {}_{\mathrm{I}}E_0^{a,b} = \pi_*(\mathcal{D}^{a,b}) \xrightarrow[\text{page}]{\text{turning}} {}_{\mathrm{I}}E_1^{a,b} = \mathbb{R}^b\pi_*\Omega^a_{X/S} $$ (by vertical resolution property)

$$ \begin{array}{cc} \vdots & \vdots \\ \uparrow & \uparrow \\ \pi_*(\mathcal{D}^{0,1}) & \pi_*(\mathcal{D}^{1,1}) \\ \uparrow & \uparrow \\ \pi_*(\mathcal{D}^{0,0}) & \pi_*(\mathcal{D}^{1,0}) \\ \uparrow & \uparrow \\ \pi_*\mathcal{O}_X \longrightarrow & \pi_*\Omega^1_{X/S} \longrightarrow \cdots \end{array} \qquad \begin{array}{l} \mathbb{R}^2\pi_*\mathcal{O}_X \longrightarrow \mathbb{R}^2\pi_*\Omega^1_{X/S} \longrightarrow \mathbb{R}^2\pi_*\Omega^2_{X/S} \longrightarrow \cdots \\ \mathbb{R}^1\pi_*\mathcal{O}_X \longrightarrow \mathbb{R}^1\pi_*\Omega^1_{X/S} \longrightarrow \mathbb{R}^1\pi_*\Omega^2_{X/S} \longrightarrow \cdots \\ \mathbb{R}^0\pi_*\mathcal{O}_X \longrightarrow \mathbb{R}^0\pi_*\Omega^1_{X/S} \longrightarrow \mathbb{R}^0\pi_*\Omega^2_{X/S} \longrightarrow \cdots \end{array} $$

giving $$ {}_{\mathrm{I}}E_1^{a,b} = \mathbb{R}^b\pi_*\Omega^a_{X/S} \Rightarrow \mathcal{H}^{a+b}_{dR}(X/S). $$

(Continue to turn to ${}_{\mathrm{I}}E_2$, we have ${}_{\mathrm{I}}E_2^{a,b} = H^a(\mathbb{R}^b\pi_*\Omega^\bullet_{X/S})$, but not interesting.)

(ii) The conjugate spectral sequence

$$ {}_{\mathrm{II}}E_0^{a,b} = \pi_*(\mathcal{D}^{b,a}) \xrightarrow[\text{page}]{\text{turning}} {}_{\mathrm{II}}E_1^{a,b} = H^b(\pi_*\mathcal{D}^{\bullet,a}) \overset{\text{如何交换}}{=} \pi_* H^b(\mathcal{D}^{\bullet,a}) $$

回到CE解消的具体构造

[李二, 笔记3.11.11]

$$ \begin{array}{ccccc} \pi_*\Omega^2_{X/S} & \longrightarrow & \pi_*\mathcal{D}^{2,0} & & \pi_*\mathcal{D}^{2,1} \\ \uparrow & & \uparrow & & \uparrow \\ \pi_*\Omega^1_{X/S} & \longrightarrow & \pi_*\mathcal{D}^{1,0} & \longrightarrow & \pi_*\mathcal{D}^{1,1} \longrightarrow \cdots \\ \uparrow & & \uparrow & & \uparrow \\ \pi_*\mathcal{O}_X & \longrightarrow & \pi_*\mathcal{D}^{0,0} & \longrightarrow & \pi_*\mathcal{D}^{0,1} \longrightarrow \cdots \\ \uparrow & & & & \\ 0 & & & & \end{array} \qquad \begin{array}{l} \pi_*H^1(\mathcal{D}^{\bullet,0}) \longrightarrow \pi_*H^1(\mathcal{D}^{\bullet,1}) \longrightarrow \pi_*H^1(\mathcal{D}^{\bullet,2}) \\ \pi_*H^0(\mathcal{D}^{\bullet,0}) \longrightarrow \pi_*H^0(\mathcal{D}^{\bullet,1}) \longrightarrow \pi_*H^0(\mathcal{D}^{\bullet,2}) \end{array} $$

turning page $$ {}_{\mathrm{II}}E_2^{a,b} = \mathbb{R}^a\pi_* H^b(\Omega^\bullet_{X/S}) $$ by cohomological resolution property.

giving $$ {}_{\mathrm{II}}E_2^{a,b} = \mathbb{R}^a\pi_* H^b(\Omega^\bullet_{X/S}) \Rightarrow \mathcal{H}^{a+b}_{dR}(X/S). $$

(1.27) Theorem

The Hodge-to-de Rham spectral sequence

$$ {}_{I}E_1^{a,b} = \mathbb{R}^b\pi_*(\Omega^a_{X/S}) \Rightarrow \mathcal{H}^{a+b}_{dR}(X/S) $$

degenerates at ${}_{I}E_1$ in the following cases:

(1) (Deligne) when S is a $\mathbb{Q}$-scheme (i.e. in the world of characteristic zero) and $\pi : X \to S$ is a proper smooth morphism of schemes.

(2) (Deligne-Illusie) When S is a $\mathbb{F}_p$-scheme and $\pi : X \to S$ satisfies some further properties besides (1).

Here conditions in (1) & (2) are satisfied by elliptic curves $\pi : E \to S$ over any base scheme S.

(3) (Oda) When A be an abelian variety over a field k (without any restriction on k).

Proof :

(1) char 0 case : Deligne, "Théorème de Lefschetz et critères de dégénérescence de suites spectrales", Théorème (5.5).

(2) char p case : Deligne, Illusie, "Relèvements modulo p^2 et décomposition du complexe de de Rham". Corollaires 4.1.5.

(3) See [van der Geer] Corollary 7.29 for a proof. $\square$

(1.28) Hodge to de Rham exact sequence

There is an exact sequence of $\mathcal{O}_S$-modules

$$0 \longrightarrow \pi_* \Omega^1_{E/S} \longrightarrow \mathcal{H}^1_{dR}(E/S) \longrightarrow R^1\pi_*(\mathcal{O}_E) \longrightarrow 0 .$$

Proof: As E/S is an elliptic curve, $\Omega^a_{E/S} \neq 0$ for only $a=0$ or 1.
So the Hodge-to-de Rham spectral sequence has only two nonzero columns $a=0$ and 1, implying (cf. 李 $\underline{=}$ §第 5.I.i)

$$0 \longrightarrow R^0\pi_*(\Omega^1_{E/S}) \longrightarrow \mathcal{H}^1_{dR}(E/S) \longrightarrow R^1\pi_*(\mathcal{O}_E) \longrightarrow 0,$$

as desired. $\square$

Remark:

(1) The conjugation spectral sequence plays a more important role in the world of characteristic p. See [Gispert §3.2] for a summary.

(2) We have another description of $R^1\pi_*(\mathcal{O}_E)$ as Lie algebras? Yes!

- For an abelian variety A/S, recall one way to define $A^\vee$ is that $A^\vee = \mathrm{Pic}^\circ_{A/S}$.
- Fact: There is a canonical isomorphism

$$\underline{\mathrm{Lie}}(\mathrm{Pic}_{A/S}) \xrightarrow{\sim} R^1\pi_*\mathcal{O}_A$$

see [Bosch–Lütkebohmert–Raynard, "Néron models"], §8.4 theorem 1.

As $\underline{\mathrm{Lie}}$ does not detect the difference b/w the identity component and the whole scheme, we see

$$\underline{\mathrm{Lie}}(A^\vee/S) \xrightarrow{\sim} R^1\pi_*\mathcal{O}_A .$$

In particular for elliptic curves E/S, $\underline{E^\vee/S \simeq E/S}$ (存疑).

$\Longrightarrow$ we can rewrite the Hodge-to-de Rham exact sequence as

$$0 \longrightarrow \pi_*\Omega^1_{E/S} = \omega_{E/S} \longrightarrow \mathcal{H}^1_{dR}(E/S) \longrightarrow \underline{\mathrm{Lie}}(E/S) \longrightarrow 0$$

(3) Via Serre–Grothendieck duality, we can actually see $R^1\pi_*\mathcal{O}_E \simeq \omega^{-1}_{E/S}$.
(this should hold only for elliptic curves)

§6 Gauss–Manin connection

- In this section, we assume $f: S \to T$ is a smooth morphism.
- Goal: Equip $\mathcal{H}^n_{dR}(X/S)$ with a structure of "connection".
- Largely follows: [Katz & Oda] "On the differentiation of de Rham coh. classes wrt para".

(1.29) Definition

Let $\mathcal{E}$ be a quasi-coherent $\mathcal{O}_S$-module.

(1) A connection of $\mathcal{E}$ is a homomorphism of abelian sheaves (not as $\mathcal{O}_S$-modules)

$$\nabla : \mathcal{E} \longrightarrow \mathcal{E} \otimes_{\mathcal{O}_S} \Omega_{S/T}$$

such that for every open subset U of S and $f \in \Gamma(U, \mathcal{O}_S)$, $e \in \Gamma(U, \mathcal{E})$,

$$\nabla(f \cdot e) = f \cdot \nabla(e) + e \otimes d_{S/T}(f)$$

where $d_{S/T} : \mathcal{O}_S \longrightarrow \Omega_{S/T}$ is the universal derivation.

(2) A connection can be extended to a homomorphism of abelian sheaves

$$\nabla^i : \Omega^i_{S/T} \otimes_{\mathcal{O}_S} \mathcal{E} \longrightarrow \Omega^{i+1}_{S/T} \otimes_{\mathcal{O}_S} \mathcal{E}$$

given by $\nabla^i(\omega \otimes e) = d^i\omega \otimes e + (-1)^i \omega \wedge \nabla(e)$, where $\omega \wedge \nabla(e)$ denotes the image of $\omega \otimes \nabla(e)$ under

$$\begin{array}{ccccc} \Omega^i_{S/T} & \otimes_{\mathcal{O}_S} & (\Omega^1_{S/T} \otimes_{\mathcal{O}_S} \mathcal{E}) & \longrightarrow & \Omega^{i+1}_{S/T} \otimes_{\mathcal{O}_S} \mathcal{E} \\ \omega & \otimes & (\tau \otimes e) & \longmapsto & (\omega \wedge \tau) \otimes e \\ \omega & \otimes & \nabla(e) & \longmapsto & \omega \wedge \nabla(e) \end{array}$$

(3) The curvature of a connection ∇ is the $\mathcal{O}_S$-linear map

$$K := \nabla^1 \circ \nabla : \mathcal{E} \longrightarrow \Omega^2_{S/T} \otimes_{\mathcal{O}_S} \mathcal{E}$$

Exercise: Check even though ∇ and ∇^1 is merely a morphism of abelian sheaves, the curvature K is $\mathcal{O}_S$-linear.

(4) The connection ∇ is called integrable / flat if $K = 0$. Such a connection gives a complex

$$0 \longrightarrow \mathcal{E} \xrightarrow{\nabla} \Omega^1_{S/T} \otimes_{\mathcal{O}_S} \mathcal{E} \xrightarrow{\nabla^1} \Omega^2_{S/T} \otimes_{\mathcal{O}_S} \mathcal{E} \xrightarrow{\nabla^2} \cdots$$

Exercise: Check $\underbrace{\nabla^{i+1} \circ \nabla^i}_{\text{走两步}} (\omega \otimes e) = \omega \wedge \underline{\underline{K(e)}}$ 走两步

This will be denoted by $(\Omega^\bullet_{S/T} \otimes \mathcal{E}, \nabla^\bullet)$.

(1.30) Induced connection on $\mathrm{Sym}^k \mathcal{E}$

For each $k \in \mathbb{N}$, a connection $\nabla : \mathcal{E} \longrightarrow \Omega^1_{S/T} \otimes_{\mathcal{O}_S} \mathcal{E}$ induces

$$\nabla^{\otimes k} : \mathcal{E}^{\otimes k} \longrightarrow \Omega^1_{S/T} \otimes_{\mathcal{O}_S} \mathcal{E}^{\otimes k}$$

by the <u>Leibniz rule</u> on sections: for any open subset $U \subseteq S$ and $e_1, \cdots, e_k \in \Gamma(U, \mathcal{E})$,

$$\nabla^{\otimes k}(e_1 \otimes \cdots \otimes e_k) = \nabla(e_1) \otimes e_2 \otimes \cdots \otimes e_k + \cdots + e_1 \otimes \cdots \otimes e_k \otimes \nabla(e_k).$$

This in turn induces a connection on $\mathrm{Sym}^k \mathcal{E}$.

<u>Proof</u>: Left as an exercise to check that $\nabla^{\otimes k}$ is well-defined and indeed is a connection

$\square$

(1.31) Koszul filtration and the associated spectral sequence

Let $\pi: X \to S$ be a smooth T-morphism. The smoothness gives the following short exact sequence of $\mathcal{O}_X$-modules

$$\underbrace{0 \longrightarrow \pi^*\Omega^1_{S/T}}_{\text{smoothness hypothesis.}} \longrightarrow \Omega^1_{X/T} \longrightarrow \Omega^1_{X/S} \longrightarrow 0 \qquad \text{— } (*)$$

- Koszul filtration on the de Rham complex

$$\mathrm{Fil}^i\Omega^\bullet_{X/T} = \mathrm{im}\left(\pi^*(\Omega^i_{S/T}) \otimes_{\mathcal{O}_X} \Omega^\bullet_{X/T}[-i] \longrightarrow \Omega^\bullet_{X/T}\right)$$

Note: Here π^* commutes with taking wedge product. ([Hartshorne, ex II.5.16])

Hence from $(*)$, we see indeed

$$\pi^*(\Omega^i_{S/T}) = (\pi^*\Omega^1_{S/T})^{\wedge i} \overset{(*)}{\subseteq} \Omega^i_{X/T}$$

and $$\left(\pi^*(\Omega^i_{S/T}) \otimes \mathcal{O}_X \Omega^\bullet_{X/T}[-i]\right)_j \subseteq \Omega^i_{X/T} \otimes \Omega^{j-i}_{X/T} \xrightarrow[(1.29.2)]{\wedge} \Omega^j_{X/T}.$$

Don't forget: For each i, $\mathrm{Fil}^i\Omega^\bullet_{X/T}$ is a chain complex, we can apply $\mathbb{R}\pi_*$ to it.

- Applying $\mathbb{R}\pi_*: D^+(\mathcal{O}_X\text{-mod}) \longrightarrow D^+(\mathcal{O}_S\text{-mod})$, we have an induced filtration

$$\mathrm{Fil}^i\,\mathbb{R}\pi_*(\Omega^\bullet_{X/T}) := \mathbb{R}\pi_*(\mathrm{Fil}^i\Omega^\bullet_{X/T}) \subseteq \mathbb{R}\pi_*(\Omega^\bullet_{X/T}).$$

This gives a spectral sequence (see 李二, §5.5, or 李二, 练习 5.7).

$$E_1^{a,b} := \mathbb{R}^{a+b}\pi_*\left(\pi^*(\Omega^a_{S/T}) \otimes_{\mathcal{O}_X} \Omega^\bullet_{X/S}[-a]\right) \Longrightarrow \mathcal{H}^{a+b}_{dR}(X/T).$$

fixed, 故两处的 a 可以抵消

- We simplify LHS:

$$\begin{aligned} E_1^{a,b} &= \mathbb{R}^b\pi_*(\pi^*(\Omega^a_{S/T}) \otimes_{\mathcal{O}_X} \Omega^\bullet_{X/S}) \\ &\simeq \Omega^a_{S/T} \otimes_{\mathcal{O}_S} \mathbb{R}^b\pi_*\Omega^\bullet_{X/S} \\ &= \Omega^a_{S/T} \otimes_{\mathcal{O}_S} \mathcal{H}^b_{dR}(X/S). \end{aligned}$$

(projection formula, See [Hartshorne, ex III.8.3], note $\Omega^a_{S/T}$ is a locally free $\mathcal{O}_S$-module.)

i.e. $$E_1^{a,b} = \Omega^a_{S/T} \otimes_{\mathcal{O}_S} \mathcal{H}^b_{dR}(X/S) \Longrightarrow \mathcal{H}^{a+b}_{dR}(X/T).$$

- Fix any $b \geq 0$,

$$E_1^{\bullet,b} = \Omega^\bullet_{S/T} \otimes_{\mathcal{O}_S} \mathcal{H}^b_{dR}(X/S):$$

$$\begin{array}{c} 0 \to \mathcal{O}_S \otimes_{\mathcal{O}_S} \mathcal{H}^b_{dR}(X/S) \xrightarrow{d_1^{0,b}} \Omega^1_{S/T} \otimes_{\mathcal{O}_S} \mathcal{H}^b_{dR}(X/S) \xrightarrow{d_1^{1,b}} \Omega^2_{S/T} \otimes_{\mathcal{O}_S} \mathcal{H}^b_{dR}(X/S) \to \cdots \\ \wr\!\wr \\ \mathcal{H}^b_{dR}(X/S) \end{array}$$

(1.32) Proposition

The spectral sequence in (1.30) has a multiplicative structure called <u>cup product</u>

$$\cup : E_r^{a,b} \times E_r^{a',b'} \longrightarrow E_r^{a+a',\, b+b'}$$

satisfying the following conditions sectionwise

(i) $e \cup e' = (-1)^{(a+b)(a'+b')} e' \cup e$

(ii) $d_r(e \cup e') = d_r(e) \cup e' + (-1)^{a+b} e \cup d_r(e')$ (we omit unnecessary indices)

<u>Proof</u> : See Katz & Oda's article for references.

<u>Better reference</u> : See [李二, §5.7], especially <u>命题 5.7.6</u> : spectral sequence induced by filtration has multiplicative structure.

(1.33) Proposition–Definition

The morphism $\nabla_{GM,n} := d_1^{0,n}$ defines an integrable connection on $\mathcal{H}^n_{dR}(X/S)$ for any $n \geq 0$.

Proof : When $n=0$, the complex

$$E_1^{\bullet,0} : \quad 0 \to \mathcal{H}^0_{dR}(X/S) \xrightarrow{d_1^{0,0}} \Omega^1_{S/T} \otimes_{\mathcal{O}_S} \mathcal{H}^0_{dR}(X/S) \xrightarrow{d_1^{1,0}} \Omega^2_{S/T} \otimes_{\mathcal{O}_S} \mathcal{H}^0_{dR}(X/S) \to \cdots$$

and the differential is given by $d_1^{i,0} = d^i \otimes 1$. So we can regard $\Omega^\bullet_{S/T}$ as a subcomplex of $E_1^{\bullet,0}$. Thus for open subset $U \subseteq S$

$$\omega \in \Gamma(U, \Omega^i_{S/T}) \subseteq E_1^{i,0}, \quad e \in \Gamma(U, \mathcal{H}^n_{dR}(X/S)),$$

we have

$$d_1^{i,n}(\omega \cup e) = d^i\omega \cup e + (-1)^i \omega \cup d_1^{0,n}(e)$$

Therefore $d_1^{0,n}$ is a connection (in fact : when $i=0$, $\omega \in \Gamma(U, \mathcal{O}_S)$,

$$d_1^{0,n}(\omega \cup e) = d_{S/T}(\omega) \cup e + \omega \cup d_1^{0,n}(e)$$

Here the cup product is actually

- $$\begin{array}{cccccc} \cup : & \mathcal{O}_S & \times & \mathcal{H}^n_{dR}(X/S) & \longrightarrow & \mathcal{H}^n_{dR}(X/S) \\ & \parallel\!\wr & & \parallel & & \parallel \\ & E_1^{0,0} & \times & E_1^{0,n} & \xrightarrow{\cup} & E_1^{0,n} \\ & (\omega & , & e) & \longmapsto & \omega \cdot e \end{array}$$
- $$\begin{array}{cccccc} \cup : & \Omega^1_{S/T} & \times & \mathcal{H}^n_{dR}(X/S) & \longrightarrow & \Omega^1_{S/T} \otimes \mathcal{H}^n_{dR}(X/S) \\ & \parallel\!\wr & & \parallel & & \parallel \\ & E_1^{1,0} & \times & E_1^{0,n} & \xrightarrow{\cup} & E_1^{1,n} \\ & (d_{S/T}(\omega) & , & e) & \longmapsto & d_{S/T}(\omega) \otimes e \end{array}$$
- $$\begin{array}{cccccc} \cup : & \mathcal{O}_S & \times & \mathcal{H}^n_{dR}(X/S) & \longrightarrow & \mathcal{H}^n_{dR}(X/S) \\ & \parallel\!\wr & & \parallel & & \parallel \\ & E_1^{0,0} & \times & E_1^{0,n} & \xrightarrow{\cup} & E_1^{0,n} \\ & (\omega & , & d_1^{0,n}(e)) & \longmapsto & \omega \cdot d_1^{0,n}(e) \end{array}$$

hence $d_1^{0,n}$ is a connection.) A more detailed study gives that $d_1^{\bullet,n}$ is given by (1.29.2). $E_1^{\bullet,b}$ being a complex in turns shows that $d_1^{0,n}$ is integrable. $\square$

(1.34) Defn

The connection $\nabla_{GM,n} : \mathcal{H}^n_{dR}(X/S) \to \Omega^1_{S/T} \otimes \mathcal{H}^n_{dR}(X/S)$ is called the <u>(n-th) Gauss–Manin connection</u>.

<u>Remark</u> : See <u>Liang Xiao's 2021 spring Exercise sheet 3</u>, in <u>Problem 3.4</u> :

<u>Fact</u> : Let X be a smooth variety over a field k of char zero. Let $\mathcal{E}$ be a coherent sheaf on X with an integral connection ∇, then $\mathcal{E}$ is necessarily a locally-free $\mathcal{O}_X$-module.

This is a theorem of Katz, where the condition " integral " can be removed, see MO 81338 for further discussion. $\square$

(1.35) Case study

Let K be a field, $T = \operatorname{Spec} K$, S/T is a smooth curve over K. Then $\Omega^i_{S/K} = 0$ for $i \geq 2$.

Let $\pi : X = E \longrightarrow S$ be a smooth curve of relatie dim. two.

Then we use (1.31)'s (*),

$$0 \to \pi^*(\Omega^1_{S/K}) \to \Omega^1_{E/K} \to \Omega^1_{E/S} \to 0$$

Inspired by this, we break the de Rham complex $\Omega^\bullet_{E/K}$ into

$$\begin{array}{lccccccc}
 & \mathcal{O}_E \longrightarrow & \Omega^1_{E/K} & \longrightarrow & \Omega^2_{E/K} & \longrightarrow & \Omega^3_{E/K} & \to 0 \to \cdots \\
\boxed{\text{sub}} & 0 \to & \pi^*(\Omega^1_{S/K}) & \longrightarrow & \pi^*(\Omega^1_{S/K}) \otimes_{\mathcal{O}_E} \Omega^1_{E/S} & \to & \pi^*(\Omega^1_{S/K}) \otimes_{\mathcal{O}_E} \Omega^2_{E/S} & \to 0 \to \cdots \\
 & & | & & | & & & \\
\boxed{\text{quot}} & \mathcal{O}_E \longrightarrow & \Omega^1_{E/S} & \longrightarrow & \Omega^2_{E/S} & \to & 0 & \to \cdots
\end{array}$$

hence we have a short exact sequence

$$0 \to \pi^*(\Omega^1_{S/K}) \otimes_{\mathcal{O}_E} \Omega^\bullet_{E/S}[-1] \to \Omega^\bullet_{E/K} \to \Omega^\bullet_{E/S} \to 0 \quad . \qquad \text{---} (\ast\ast)$$

Recall: In general the Koszul filtration is defined as

$$\operatorname{Fil}^i \Omega^\bullet_{X/T} = \operatorname{im}\left(\pi^*(\Omega^i_{S/T}) \otimes_{\mathcal{O}_X} \Omega^\bullet_{X/T}[-i] \to \Omega^\bullet_{X/T} \right), \quad \text{for } i \geq 0.$$

From (**) above, we see how Koszul filtration arises as $i = 1$.

Then applying $R\pi_*$ to the exact sequence (**), we obtain

$$R^n\pi_*(\Omega^\bullet_{E/S}) = H^n_{dR}(E/S)$$

$$\xrightarrow{\ \delta^n\ } R^{n+1}\pi_*\left(\pi^*(\Omega^1_{E/K}) \otimes_{\mathcal{O}_E} \Omega^\bullet_{E/S}[-1] \right)$$

$$\simeq H^n_{dR}(E/S) \otimes_{\mathcal{O}_E} \Omega^1_{E/K} \quad \text{(same computation in (1.31))}.$$

Then here δ^n is the n-th Gauss–Manin connection $\nabla_{GM,n}$.

Remark: $\pi : E \to S$ is often an elliptic curve over S, which is of relatie dimension one. We use the $\operatorname{rel.dim}_S(E) = 2$ to have more "nonzero" terms to get better illustration.

Remark : What is the (**) in general rel. dim n case ? (cf. Yi Qizheng's course)

In the same $\boxed{\text{Sub}}$, $\boxed{\text{quot.}}$ argument, we write

$$0 \to \pi^*\Omega^1_{S/k} \otimes_{\mathcal{O}_E} \Omega^\bullet_{E/S}[-1] \longrightarrow \Omega^\bullet_{E/k} \big/ \pi^*\Omega^2_{S/k} \wedge \Omega^\bullet_{E/k}[-2] \longrightarrow \Omega^\bullet_{E/S} \longrightarrow 0$$

as the only obvious "generalization" is the mid-term. Again derive using $\mathbb{R}\Gamma$, we get the connecting homomorphism

$$\nabla_{GM,n} : \mathcal{H}^n_{dR}(E/S) \longrightarrow \mathcal{H}^n_{dR}(E/S) \otimes_{\mathcal{O}_E} \Omega^1_{E/k}$$

as desired. $\square$

(1.36) Kodaira–Spencer isomorphism : construction of the map

Let's now focus on the elliptic curve case of (1.35).

- Recall in (1.28), by Hodge-to-de Rham spectral sequence, we have a SES :

$$0 \to R^0\pi_*(\Omega^1_{E/S}) \to \mathcal{H}^1_{dR}(E/S) \to R^1\pi_*(\mathcal{O}_E) \to 0,$$

  - $R^0\pi_*(\Omega^1_{E/S}) = \pi_*(\Omega^1_{E/S}) = \underline{\omega}_{E/S}$
  - By <u>Serre–Grothendieck duality</u>, $R^1\pi_*(\mathcal{O}_E) \xrightarrow{\sim} \underline{\omega}^{-1}_{E/S}$.

  Hence we have

$$0 \to \underline{\omega}_{E/S} \to \mathcal{H}^1_{dR}(E/S) \xrightarrow{p} \underline{\omega}^{-1}_{E/S} \to 0. \qquad \text{— } (\star)$$

- Then we have

$$\begin{tikzcd}
\underline{\omega}_{E/S} \arrow[d, hook] \arrow[ddr, bend left, "\kappa"] & \\
\mathcal{H}^1_{dR}(E/S) \arrow[r, "\nabla_{GM1}"] & \mathcal{H}^1_{dR}(E/S) \otimes_{\mathcal{O}_S} \Omega^1_{S/K} \arrow[d, "p\otimes \mathrm{id} =: pr"] \\
 & \underline{\omega}^{-1}_{E/S} \otimes_{\mathcal{O}_S} \Omega^1_{S/K}
\end{tikzcd}$$

  We tensor κ with $\underline{\omega}_{E/S}$, obtaining

$$KS : \underline{\omega}^{\otimes 2}_{E/S} \to \Omega^1_{S/K}$$

  This is the <u>Kodaira–Spencer map</u>.

- <u>Note</u> : In the previous construction, we see ∇_{GM1} is <u>NOT</u> an $\mathcal{O}_S$-linear morphism, but only a morphism of abelian sheaves. Despite of this, κ is still an <u>$\mathcal{O}_S$-linear</u> map. (<u>Exercise</u>)

  <u>Indeed</u>, Let $pr : \mathcal{H}^1_{dR}(E/S)\otimes_{\mathcal{O}_S}\Omega^1_{S/K} \to \underline{\omega}^{-1}_{E/S} \otimes_{\mathcal{O}_S} \Omega^1_{S/K}$. Then for $x \in \underline{\omega}_{E/S}$, and $a \in \mathcal{O}_S$,

$$pr\cdot\nabla_{GM}(ax) = pr\left(x\otimes d_{S/K}(a) + a\cdot\nabla_{GM}(x)\right)$$
$$= pr\left(x\otimes d_{S/K}(a)\right) + pr\left(a\cdot\nabla_{GM}(x)\right)$$

  - $pr(x\otimes d_{S/K}(a)) = p(x)\otimes d_{S/K}(a) = 0$, here $p(x)=0$ since $x \in \underline{\omega}_{E/S}$. (from $(\star)$ 走两步)
  - $pr(a\cdot\nabla_{GM}(x)) = a\cdot pr(\nabla_{GM}(x))$.

  hence $pr\cdot\nabla_{GM}(ax) = a\cdot pr(\nabla_{GM}(x))$.

(1.37) Theorem (Kodaira–Spencer isomorphism)

Let $N \geq 3$ and $\mathrm{char}(k) \nmid N$ (or $\mathrm{char}\, k = 0$). Let $\mathbb{E}_k / Y_1(N)_k$ be the universal elliptic curve for $Y_1(N)$ over k. Then

$$KS : \underline{\omega}^{\otimes 2}_{\mathbb{E}_k / Y_1(N)_k} \longrightarrow \Omega^1_{Y_1(N)/k}$$

is an isomorphism of $\mathcal{O}_{Y_1(N)}$-modules.

<u>Proof</u> : See Diamond & Taylor's article.

<u>Remark</u> : This has a complex-analytical version, see 李文成《模形式初步》§9.1.

<u>Remark on cusps</u> Again above we only deal with the "noncompactified" $Y_1(N)$. Actually for $X_1(N)$, we also have a Kodaira–Spencer isomorphism

$$KS : \underline{\omega}^{\otimes 2}_{\mathbb{E}_k / X_1(N)_k} \xrightarrow{\sim} \Omega^1_{X_1(N)/k}(-D_{\text{cusps}})$$

with a twist by the divisor of cusps.

(1.38) Griffith transversality

Motivation: We have seen on $\mathcal{H}^n_{dR}(X/S)$, we have two structures:

- Hodge filtration: (see (1.26))

$$\mathrm{Fil}^i\mathcal{H}^n_{dR}(X/S) = \mathrm{im}\left(\mathbb{R}^n\pi_*(\tau_{\geq i}\Omega^\bullet_{X/S}) \longrightarrow \mathbb{R}^n\pi_*(\Omega^\bullet_{X/S}) = \mathcal{H}^n_{dR}(X/S)\right).$$

- Gauss-Mainn connection:

$$\nabla_{GM,n} : \mathcal{H}^n_{dR}(X/S) \longrightarrow \Omega^1_{S/T} \otimes_{\mathcal{O}_S} \mathcal{H}^n_{dR}(X/S).$$

So a natural question is: are the two structures compatible? More precisely: does $\nabla_{GM,n}$ preserves filtration $\mathrm{Fil}^i\mathcal{H}^n_{dR}(X/S)$:

$$\begin{array}{cccc} \nabla_{GM,n} : & \mathcal{H}^n_{dR}(X/S) & \longrightarrow & \Omega^1_{S/T}\otimes_{\mathcal{O}_S}\mathcal{H}^n_{dR}(X/S) \\ & \cup\!| & & \cup\!| \\ & \mathrm{Fil}^i\mathcal{H}^n_{dR} & \overset{?}{\dashrightarrow} & \Omega^1_{S/T}\otimes_{\mathcal{O}_S}\mathrm{Fil}^i\mathcal{H}^n_{dR} \end{array}$$

Unfortunately, $\underline{\underline{\text{No}}}$!

Griffith transversality: $\nabla_{GM,n}(\mathrm{Fil}^i\mathcal{H}^n_{dR}) \subseteq \Omega^1_{S/T}\otimes_{\mathcal{O}_S}\mathrm{Fil}^{i-1}\mathcal{H}^n_{dR}$.

Proof on a special case: We only consider in the same setup in (1.35):

- k field, $T = \mathrm{Spec}\,k$
- S/T is a smooth curve over k.
- $\pi : E \longrightarrow S$ be a smooth curve of relatie dimension 2.

Then again we break the de Rham complex $\Omega^\bullet_{E/k}$ into

$$\begin{array}{lcccccccc} & \mathcal{O}_E & \longrightarrow & \Omega^1_{E/k} & \longrightarrow & \Omega^2_{E/k} & \longrightarrow & \Omega^3_{E/k} \to 0 \to \cdots & \\ \boxed{\text{sub}} & & 0 \to & \pi^*(\Omega^1_{E/k}) & \longrightarrow & \pi^*(\Omega^1_{E/k})\otimes_{\mathcal{O}_E}\Omega^1_{E/S} & \longrightarrow & \pi^*(\Omega^1_{E/k})\otimes_{\mathcal{O}_E}\Omega^2_{E/S} & \to 0 \to \cdots \\ & & & | & & | & & & \\ \boxed{\text{quot}} & \mathcal{O}_E & \longrightarrow & \Omega^1_{E/S} & \longrightarrow & \Omega^2_{E/S} & \longrightarrow & 0 \to \cdots & \end{array}$$

Now focusing on $\mathrm{Fil}^1\mathcal{H}^n_{dR}$, we cut off the shaded part above, obtain

$$0 \longrightarrow \pi^*(\Omega^1_{E/k})\otimes_{\mathcal{O}_E}(\tau_{\geq 1}\Omega^\bullet_{E/S})[-1] \longrightarrow \tau_{\geq 2}\Omega^\bullet_{E/k} \longrightarrow \tau_{\geq 2}\Omega^\bullet_{E/S} \longrightarrow 0$$

Then taking $\mathbb{R}\pi_*$:

$$\mathbb{R}^n\pi_*(\tau_{\geq 2}\Omega^\bullet_{E/S}) = \mathrm{Fil}^2\mathcal{H}^n_{dR}(E/S)$$

$$\xrightarrow{\nabla_{GM,n}} \mathbb{R}^{n+1}\pi_*\left(\pi^*(\Omega^1_{E/k})\otimes_{\mathcal{O}_E}(\tau_{\geq 1}\Omega^\bullet_{E/S})[-1]\right)$$

$$= \mathbb{R}^n\pi_*\left(\pi^*(\Omega^1_{E/k})\otimes_{\mathcal{O}_E}\tau_{\geq 1}\Omega^\bullet_{E/S}\right)$$

$$\cong \Omega^1_{E/k}\otimes_{\mathcal{O}_E}\mathbb{R}^n\pi_*(\tau_{\geq 1}\Omega^\bullet_{E/S}) = \Omega^1_{E/k}\otimes_{\mathcal{O}_E}\mathrm{Fil}^1\mathcal{H}^n_{dR}(E/S).$$

$\square$

(1.39) Example 1: Boring example of elliptic curves

Continuing (1.38), but require $\pi : E \to S$ is an elliptic curve (hence of relative dimension one) over S. Then

for Fil^1

$$\Omega^\bullet_{E/S} : \quad 0 \longrightarrow \mathcal{O}_E \longrightarrow \Omega^1_{E/S} \longrightarrow 0$$

Hence the only interesting case is $\mathcal{H}^1_{dR}(E/S)$, a locally free $\mathcal{O}_S$-module of rank 2, ?

with a Hodge filtration as

$$\begin{array}{ccccc} \mathcal{H}^1_{dR}(E/S) & \supseteq & \underline{\omega}_{E/S} & \supseteq & 0 \\ \| & & \| & & \| \\ Fil^0\mathcal{H}^1_{dR}(E/S) & & Fil^1\mathcal{H}^1_{dR}(E/S) & & Fil^2\mathcal{H}^1_{dR}(E/S) \end{array}$$

$\underline{\omega}^{-1}_{E/S}$

Then the Griffith transversality is

$$\nabla_{GM,1} : \quad \begin{array}{ccccc} \mathcal{H}^1_{dR}(E/S) & \supseteq & \underline{\omega}_{E/S} & \supseteq & 0 \\ \downarrow{\scriptstyle \nabla_{GM,1}} & & & & \\ \Omega^1_{S/K} \otimes_{\mathcal{O}_S} \mathcal{H}^1_{dR}(E/S) & \supseteq & \Omega^1_{S/K} \otimes \underline{\omega}_{E/S} & \supseteq & 0 \end{array}$$

$\nabla_{GM,1}$ $\quad$ $\nabla_{GM,1}$

which is trivial.

(1.40) Example 2 : for $k \geq 2$, consider $\mathrm{Sym}^{k-2} H^1_{dR}(E/S)$.

Since $H^1_{dR}(E/S)$ is locally free of rank 2, $\mathrm{Sym}^{k-2} H^1_{dR}$ is locally free of rank $k-1$.

- The <u>hodge filtration</u> on $H^1_{dR}(E/S)$ induces a filtration on $\mathrm{Sym}^{k-2} H^1_{dR}(E/S)$:

As an intuition, we consider the vector space case :

- Let V be a two-dim'l k-vector space with basis $\{e_1, e_2\}$.
  Then $\mathrm{Sym}^n V$ is of dimension $n+1$, with basis $\{ \underbrace{e_1 \cdots e_1}_{\ell \text{ times}} \underbrace{e_2 \cdots e_2}_{(n-\ell) \text{ times}} : 0 \leq \ell \leq n \}$.
- Then given a filtration on V as

$$V \supseteq \underbrace{k \cdot e_1}_{gr^1 \simeq k \cdot e_2} \supseteq 0$$

here we write $v_1 \cdot v_2 \in \mathrm{Sym}^n V$ as $v_1 \otimes v_2$

we have an induced filtration on $\mathrm{Sym}^n V$ as

$$\begin{aligned}
\mathrm{Fil}^{n+1} \mathrm{Sym}^n V = \{0\} \subseteq \mathrm{Fil}^n \mathrm{Sym}^n V &= k \cdot e_1^{\otimes n} \\
\subseteq \mathrm{Fil}^{n-1} \mathrm{Sym}^n V &= k \cdot e_1^{\otimes n} \oplus k \cdot e_1^{\otimes(n-1)} \otimes e_2 \\
\subseteq \mathrm{Fil}^{n-2} \mathrm{Sym}^n V &= k \cdot e_1^{\otimes n} \oplus k \cdot e_1^{\otimes(n-1)} \otimes e_2 \oplus k \cdot e_1^{\otimes(n-2)} \otimes e_2^{\otimes 2} \\
\subseteq & \cdots \\
\subseteq \mathrm{Fil}^1 \mathrm{Sym}^n V &= k \cdot e_1^{\otimes n} \oplus \cdots \oplus k \cdot e_1 \otimes e_2^{\otimes(n-1)} \\
\subseteq \mathrm{Fil}^0 \mathrm{Sym}^n V &= k \cdot e_1^{\otimes n} \oplus \cdots \oplus k \cdot e_1 \otimes e_2^{\otimes(n-1)} \oplus k \cdot e_2^{\otimes n} \quad \mathrm{Sym}^n V.
\end{aligned}$$

"sub"
"quotient"

Here the induced Hodge filtration from $H^n_{dR}(E/S)$, which fits into (1.36) as

$$0 \to \omega_{E/S} \to H^1_{dR}(E/S) \to \omega_{E/S}^{-1} \to 0$$

gives a filtration on $\mathrm{Sym}^{k-2} H^n_{dR}(E/S)$ in a similar way :

$$\underset{\mathrm{Fil}^0}{\overset{}{\mathrm{Sym}^{k-2} H^1_{dR}(E/S)}} \supseteq \underbrace{\mathrm{Fil}^1 \supseteq}_{\omega^{\otimes -(k-2)}} \mathrm{Fil}^2 \supseteq \cdots \supseteq \underbrace{\mathrm{Fil}^{k-4} \supseteq}_{\omega^{\otimes k-4} \otimes (\omega^{-1})^{\otimes 2} \,=\, \omega^{\otimes k-6}} \overbrace{\mathrm{Fil}^{k-3} \supseteq}^{\omega^{\otimes k-3} \otimes \omega^{-1} = \omega^{k-4}} \underbrace{\mathrm{Fil}^{k-2} \supseteq}_{\omega^{\otimes k-2}} \mathrm{Fil}^{k-1} = 0$$

- The <u>Gauss-Manin connection</u> $\nabla_{GM,1}$ on $H^1_{dR}(E/S)$ induces a connection on $\mathrm{Sym}^{k-2} H^1_{dR}(E/S)$ by Leibniz rule, in (1.30).

• The two structures on $\mathrm{Sym}^{k-2} H^1_{dR}(E/S)$ also satisfy Griffith transversality. Now as we know better on the graded piece, we consider the induced

$$
\begin{array}{ccc}
\overline{\nabla}_{GM}^{(i)} : gr^i H^1_{dR}(E/S) & \longrightarrow & gr^{i-1}\left(H^1_{dR}(E/S)\otimes_{\mathcal{O}_S}\Omega^1_{S/T}\right) \\
\| & \text{map into by } \nabla_{GM} & \| \\
\dfrac{\mathrm{Fil}^i H^1_{dR}(E/S)}{\mathrm{Fil}^{i+1} H^1_{dR}(E/S)} & & \dfrac{\mathrm{Fil}^{i-1} H^1_{dR}(E/S)\otimes_{\mathcal{O}_S}\Omega^1_{S/T}}{\mathrm{Fil}^{i} H^1_{dR}(E/S)\otimes_{\mathcal{O}_S}\Omega^1_{S/T}} \\
 & \text{maps into by } \nabla_{GM} &
\end{array}
$$

Then the Griffith transversality in the $\overline{\nabla}_{GM}^{(i)}$ version gives

$$
\begin{tikzcd}[column sep=small]
\mathrm{Sym}^{k-2} H^1_{dR}(E/S) \arrow[d, "\nabla_{GM}"'] & \underline{\omega}^{\otimes k-2} \arrow[dr, bend left, "\overline{\nabla}_{GM}^{(k-2)}"] & - & \underline{\omega}^{\otimes k-4} \arrow[dr, bend left, "\overline{\nabla}_{GM}^{(k-3)}"] & - & \underline{\omega}^{\otimes k-6} \arrow[dr, bend left, "\overline{\nabla}_{GM}^{(k-4)}"] & - & \cdots & - & \underline{\omega}^{\otimes -(k-2)} \\
\mathrm{Sym}^{k-2} H^1_{dR}(E/S)\otimes\Omega^1_{S/K} & \underline{\omega}^{\otimes k-2}\otimes\Omega^1_{S/K} & - & \underline{\omega}^{\otimes k-4}\otimes\Omega^1_{S/K} & - & \underline{\omega}^{\otimes k-6}\otimes\Omega^1_{S/K} & - & \cdots & - & \underline{\omega}^{\otimes 2-k}\otimes\Omega^1_{S/K}
\end{tikzcd}
$$

$\boxed{gr}$ $\quad k-2 \quad k-3 \quad k-4 \quad \cdots \quad 0$

• In particular, when $E/S = \mathbb{E}/Y(N)$, we can further apply Kodaira-Spencer isomorphism to the lower grading as

$$
\begin{tikzcd}[column sep=small]
\mathrm{Sym}^{k-2} H^1_{dR}(E/S) \arrow[d, "\nabla_{GM}"'] & \underline{\omega}^{\otimes k-2} \arrow[dr, bend left, "\overline{\nabla}_{GM}^{(k-2)}"] \arrow[ddr, bend left, red, "\Phi^{(k-2)}"] & - & \underline{\omega}^{\otimes k-4} \arrow[dr, bend left, "\overline{\nabla}_{GM}^{(k-3)}"] \arrow[ddr, bend left, red, "\Phi^{(k-3)}"] & - & \underline{\omega}^{\otimes k-6} \arrow[dr, bend left, "\overline{\nabla}_{GM}^{(k-4)}"] & - & \cdots\ \underline{\omega}^{\otimes 4-k} \arrow[dr, bend left, "\overline{\nabla}_{GM}^{(1)}"] \arrow[ddr, bend left, red, "\Phi^{(1)}"] & - & \underline{\omega}^{\otimes -(k-2)} = \underline{\omega}^{\otimes 2-k} \arrow[dllllllll, bend left, "\theta"] \\
\mathrm{Sym}^{k-2} H^1_{dR}(E/S)\otimes\Omega^1_{S/K} & \underline{\omega}^{\otimes k-2}\otimes\Omega^1_{S/K} \arrow[d, "\simeq"', "KS^{-1}"] & - & \underline{\omega}^{\otimes k-4}\otimes\Omega^1_{S/K} \arrow[d, "\simeq"', "KS^{-1}"] & - & \underline{\omega}^{\otimes k-6}\otimes\Omega^1_{S/K} \arrow[d, "\simeq"', "KS^{-1}"] & - & \cdots & - & \underline{\omega}^{\otimes 2-k}\otimes\Omega^1_{S/K} \arrow[d, "\simeq"', "KS^{-1}"] \\
 & \underline{\omega}^{\otimes k} & & \underline{\omega}^{\otimes k-2} & & \underline{\omega}^{\otimes k-4} & & & & \underline{\omega}^{\otimes 4-k}
\end{tikzcd}
$$

$\boxed{gr}$ $\quad k-2 \quad k-3 \quad k-4 \quad \cdots \quad 0$ (sub)

- <u>Fact</u> : The morphisms $\Phi^{(i)}$ are isomorphisms.

<u>Remark</u> : - Note that we are not saying that $\Phi^{(i)}$ are identity maps, as it's not clear (for me) how to construct KS^{-1} explicitly.

- In Xiao's course, he said this is related to the rep'n of SL_2.

- <u>Corollary</u> : We have a quasi-isomorphism of complexes

$$\mathrm{Sym}^{k-2}H^1_{dR}\otimes\Omega^\bullet_{Y(N)/K} = \left[0 \to \mathrm{Sym}^{k-2}H^1_{dR}(\mathbb{E}/Y_1(N)) \xrightarrow{\nabla_{GM}} \mathrm{Sym}^{k-2}H^1_{dR}(\mathbb{E}/Y_1(N))\otimes_{\mathcal{O}_{Y(N)}}\Omega^1_{Y_1(N)/K} \to \cdots\right]$$

$$\uparrow \text{qis}$$

$$\omega^\bullet := \left[0 \to \underline{\omega}^{\otimes 2-k} \xrightarrow[\text{some differential operator}]{\mathcal{D}} \omega^{\otimes k-2}\otimes\Omega^1_{Y(N)/K} \simeq \omega^{\otimes k} \to 0 \to \cdots\right]$$

- Then we take sheaf cohomology $R\Gamma : D^+(Sh_{Y(N)}) \to D^+(Ab)$ (<u>note</u> : not $R\pi_*$), under the above quasi-isomorphism, we have the spectral sequence associated to this hypercohomology :

$$E_1^{a,b} = H^b(Y_1(N), \omega^a) \Longrightarrow H^{a+b}(\mathrm{Sym}^{k-2}H^1_{dR}(\mathbb{E}/Y_1(N))\otimes_{\mathcal{O}_{Y_1(N)}}\Omega^\bullet_{Y(N)/K})$$

- Since $Y_1(N)$ is a smooth curve, the spectral sequence has only two nonzero columns at $a\in\{0,1\}$, giving

$$0 \to H^0(Y_1(N), \underline{\omega}^{\otimes k}) \to H^1(\mathrm{Sym}^{k-2}H^1_{dR}(\mathbb{E}/Y(N))\otimes_{\mathcal{O}_{Y_1(N)}}\Omega^\bullet_{Y(N)/K}) \to H^1(Y_1(N), \underline{\omega}^{\otimes(2-k)}) \to 0$$

$$\wr\downarrow \text{ Serre duality}$$

$$H^0(Y_1(N), \underline{\omega}^{\otimes k-2}\otimes_{\mathcal{O}_{Y(N)}}\Omega^1_{Y_1(N)/K})^\vee$$

<u>Here</u> : - $H^0(Y_1(N), \underline{\omega}^{\otimes k}) \overset{\text{"cusps"}}{\approx} S_k(\Gamma_1(N), K)$

- $H^0(Y_1(N), \underline{\omega}^{\otimes k-2}\otimes_{\mathcal{O}_{Y(N)}}\Omega^1_{Y(N)/K})^\vee \overset{KS}{=\!=} H^0(Y(N), \underline{\omega}^{\otimes k})^\vee \underset{\text{"cusps"}}{\approx} S_k(\Gamma_1(N), K)^\vee$.

This provides us basic intuition of the Eichler–Shimura isomorphisms.

§7 Eichler–Shimura isomorphism and Eichler–Shimura relation

<u>Goal</u> : Precisely formulate the last short exact sequence in (1.40).
So the key object there is $H^1(\mathrm{Sym}^{k-2} H^1_{dR}(\mathbb{E}/Y_1(N)) \otimes_{\mathcal{O}_{Y(N)}} \Omega^{\bullet}_{Y_1(N)/K})$.

<u>Method</u> : By comparision theorem

(i) Passing to <u>Betti cohomology</u> $/\mathbb{C}$: acquire <u>Hecke action</u>

(ii) Passing to <u>étale cohomology</u> $/\mathbb{Q}_\ell$: acquire <u>Galois action</u>.

<u>Convention</u> : Assume $K \subseteq \mathbb{C}$ is a field in this section.

(1.41) Betti cohomology : things over $\mathbb{C}$.

Let

$$\begin{tikzcd}
X_{\mathbb{C}} \arrow[r] \arrow[d, "\pi_{\mathbb{C}}"'] \arrow[dr, phantom, "\ulcorner"] & X \arrow[d, "\pi"] \\
S_{\mathbb{C}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & S \arrow[d] \\
\mathrm{Spec}\,\mathbb{C} \arrow[r] & \mathrm{Spec}\,K
\end{tikzcd}
\quad \xrightarrow[\text{use analytic topology}]{\text{GAGA}} \quad
\begin{tikzcd}
X^{an}_{\mathbb{C}} \arrow[d, "\pi^{an}_{\mathbb{C}} =: \pi^{an}"] \\
S^{an}_{\mathbb{C}}
\end{tikzcd}$$

- <u>Betti cohomology</u> : $H^n_B(X^{an}_{\mathbb{C}}/S^{an}_{\mathbb{C}}, A) := \mathbb{R}^n \pi^{an}_* \underline{A}_{X^{an}_{\mathbb{C}}}$

  here A usually takes noetherian comutatie rings like
  - $A = \mathbb{Q}, \mathbb{C}, \cdots$ fields
  - $A = \mathbb{Z}, \mathbb{Z}_\ell$ (with global dimension 1).

- <u>Betti – de Rham comparison</u>

  * As $H^n_B(X^{an}_{\mathbb{C}}/S^{an}_{\mathbb{C}}, \mathbb{C}) = \mathbb{R}^n \pi^{an}_* \underline{\mathbb{C}}_{X^{an}_{\mathbb{C}}}$ is locally constant (or say "<u>local system</u>", yet $H^n_{dR}(X^{an}_{\mathbb{C}}/S^{an}_{\mathbb{C}}) := \mathbb{R}^n \pi^{an}_*(\Omega^{\bullet}_{X^{an}_{\mathbb{C}}/S^{an}_{\mathbb{C}}})$ may not, we cannot expect that the two are equals. In some sense we may write

  $$\mathbb{R}^n \pi^{an}_* \underline{\mathbb{C}}_{X^{an}_{\mathbb{C}}} \xleftrightarrow{\ \sim\ } \mathbb{R}\pi^{an}_*(\Omega_{X^{an}_{\mathbb{C}}/S^{an}_{\mathbb{C}}})^{\nabla_{GM} = 0}$$

  This can be regarded as the Riemann–Hilbert correspondence.

  * A more precise statement as follows :

  $$H^n_B(X^{an}_{\mathbb{C}}/S^{an}_{\mathbb{C}}, \underline{\mathbb{C}}) \otimes_{\underline{\mathbb{C}}_{S^{an}_{\mathbb{C}}}} \mathcal{O}_{S^{an}_{\mathbb{C}}} \xrightarrow{\sim} H^n_{dR}(X^{an}_{\mathbb{C}}/S^{an}_{\mathbb{C}})$$

respecting Gauss–Manin connection

$$\underbrace{1}_{\substack{\text{trivial}\\ \text{connection}}} \otimes \underbrace{\nabla_{S^{an}_{\mathbb{C}}}}_{\substack{\text{connection from}\\ \text{complex geometry}}} \longleftrightarrow \nabla_{GM}.$$

- The <u>analytic resolution</u>: The following complex (given as (1.29.4)) is exact in the analytic topology as

$$0 \longrightarrow R^n \pi_* \underline{\mathbb{C}}_{X_{\mathbb{C}}^{an}} \longrightarrow \mathcal{H}_{dR}^n(X_{\mathbb{C}}^{an}/S_{\mathbb{C}}^{an}) \xrightarrow{\nabla_{GM}} \mathcal{H}_{dR}^n(X_{\mathbb{C}}^{an}/S_{\mathbb{C}}^{an}) \otimes \Omega^1_{S_{\mathbb{C}}^{an}} \xrightarrow{\nabla_{GM}^2} \cdots .$$

Moreover, this is an $\mathbb{R}\Gamma$-acyclic resolution of $R^n \pi_*^{an} \underline{\mathbb{C}}_{X_{\mathbb{C}}^{an}}$.

(1.42) Back to the case of (universal) elliptic curve

Let's consider the setup in (1.40). Define

$$\mathcal{L}_k := \mathrm{Sym}^{k-2}(R^1 \pi_*^{an} \underline{\mathbb{C}}_{\mathbb{E}_{\mathbb{C}}^{an}}), \quad \text{a locally constant sheaf on } Y(N)_{\mathbb{C}}^{an}.$$

Then the analytic resolution in (1.41) gives a resolution of $\mathcal{L}_k$:

$$0 \longrightarrow \mathcal{L}_k \longrightarrow \mathrm{Sym}^{k-2}\mathcal{H}_{dR}^1(\mathbb{E}_{\mathbb{C}}^{an}/Y_1(N)_{\mathbb{C}}^{an}) \xrightarrow{\nabla_{GM}} \mathrm{Sym}^{k-2}\mathcal{H}_{dR}^1(\mathbb{E}_{\mathbb{C}}^{an}/Y_1(N)_{\mathbb{C}}^{an}) \otimes \Omega_{Y(N)_{\mathbb{C}}^{an}} \longrightarrow 0$$

So

$$\begin{aligned} H^1(Y_1(N)_{\mathbb{C}}^{an}, \mathcal{L}_k) &\simeq H^1(Y_1(N)_{\mathbb{C}}^{an}, \mathrm{Sym}^{k-2}\mathcal{H}_{dR}^1(\mathbb{E}_{\mathbb{C}}^{an}/Y_1(N)_{\mathbb{C}}^{an}) \otimes \Omega_{Y(N)_{\mathbb{C}}^{an}}) \\ &\simeq H^1(Y_1(N), \mathrm{Sym}^{k-2}\mathcal{H}_{dR}^1(\mathbb{E}/Y_1(N)) \otimes \Omega_{Y(N)}). \end{aligned}$$

Now back to (1.40), the following result is convincing:

(1.43) Eichler–Shimura isomorphism (complex analytic)

There is a natural isomorphism

$$ES: \quad H^1(X_1(N)^{an}_{\mathbb{C}}, j_*\mathcal{L}_k) \xrightarrow{\sim} S_k(\Gamma_1(N)) \oplus \overline{S_k(\Gamma_1(N))} .$$

where $j : Y_1(N)^{an}_{\mathbb{C}} \longrightarrow X_1(N)^{an}_{\mathbb{C}}$ is the open immersion.

<u>Remark</u> : We write $\mathcal{L}_{k,A} = \mathrm{Sym}^{k-2} R^1\pi^{an}_* \underline{A}_{\mathbb{E}^{an}_{\mathbb{C}}}$ for any commutatie noetherian ring of finite global dimension, then

$$\begin{aligned} \mathbb{W}_A &:= H^1(X_1(N)^{an}_{\mathbb{C}}, j_*\mathcal{L}_{k,A}) \\ &\simeq \mathrm{im}\left(H^1_c(Y_1(N)^{an}_{\mathbb{C}}, \mathcal{L}_{k,A}) \longrightarrow H^1(Y_1(N)^{an}_{\mathbb{C}}, \mathcal{L}_{k,A})\right) \\ &=: \widetilde{H}^1(Y_1(N), \mathcal{L}_{k,A}) . \end{aligned}$$

here $H^1_c(\cdots)$ is the cohomology with compact support. In this notation, the theorem goes as

$$\mathbb{W}_{\mathbb{C}} \xrightarrow{\sim} S_k(\Gamma_1(N)) \oplus \overline{S_k(\Gamma_1(N))} .$$

This coincides with the so-called "<u>parabolic cohomology</u>"

<u>Notation</u> : Here the bar on $S_k(\Gamma_1(N))$ is the complex analogue of "$\vee$" in Serre's duality :

- <u>One perspectiue</u> : $\overline{S_k(\Gamma_1(N))} := \{ \overline{f} : f \in S_k(\Gamma_1(N)) \}$
- <u>Another perspectie</u> : $\overline{S_k(\Gamma_1(N))} \xleftarrow{\sim} S_k(\Gamma_1(N)) \otimes_{\mathbb{C}} \mathbb{C}$

  $z \cdot \overline{f} \longleftarrow\!\shortmid \quad f \otimes z , \quad z \in \mathbb{C}$

  here the second coordinate $\mathbb{C}$ has a $\mathbb{C}$-algebra structure given by complex conjugation $z \mapsto \overline{z}$.

(1.44) Hecke action

Let $T \in \mathrm{End}_{\mathbb{C}}(S_k(\Gamma_1(N)))$ be any operator on the space of cusp forms. Then we have a corresponding $\overline{T} \in \mathrm{End}_{\mathbb{C}}(\overline{S_k(\Gamma_1(N))})$ mapping $\overline{f}$ to $\overline{Tf}$. Then through the isomorphism ES in (1.43), $T \oplus \overline{T}$ acts on $\mathbb{W}_{\mathbb{C}}$.

<u>Example 1</u> : T_p be the Hecke operator (given by $\Gamma_1(N)\begin{pmatrix}1 & \\ & p\end{pmatrix}\Gamma_1(N)$), then T_p acts on $\mathbb{W}_{\mathbb{C}}$. Denote the action as $T[p] : \mathbb{W}_{\mathbb{C}} \longrightarrow \mathbb{W}_{\mathbb{C}}$.

<u>Example 2</u> : Let $d \in (\mathbb{Z}/N)^{\times}$. Then it gives an action on $\mathbb{W}_{\mathbb{C}}$ as

$$I_d^* : \mathbb{W}_{\mathbb{C}} \longrightarrow \mathbb{W}_{\mathbb{C}} .$$

Example 3 : Let W_N be the Fricke involution defined by

$$W_N : M_k(\Gamma_1(N)) \longrightarrow M_k(\Gamma_1(N))$$
$$f \longmapsto \left[i^k f\big|_k \begin{pmatrix} & -1 \\ N & \end{pmatrix} : \underset{\mathbb{H}}{\overset{\tau}{\in}} \longrightarrow i^k N^{-\frac{k}{2}} \tau^{-k} f\left(\frac{-1}{N\tau}\right) \right]$$

satisfying $W_N(S_k(\Gamma_1(N))) \subseteq S_k(\Gamma_1(N))$. It defines an action

$$i^{k+2} N^{-\frac{k}{2}} w_{\zeta}^{*} : W_{\mathbb{C}} \longrightarrow W_{\mathbb{C}} .$$

Unusual ? This involution appears when discussing functional equations of L-functions

$$\Lambda_N(s, f) = \Lambda_N(k-s, W_N f)$$

In this way, the Eichler-isomorphism is automatically Hecke equivariant (and " diamond equivariant " " Frickle equivariant ")

Remark : Here actually we are cheating : there's nothing surprising on the Hecke equivariance by definition. It is important to explicitly describle these actions on $W_{\mathbb{C}}$ without ES isomorphism, i.e., we ask :

Question : What is the moduli interpretations of these actions ?

Reference : 李文威《模形式初学》命题 :

(a) 10.3.4 Description of $T[p]$-operator : via cohomological correspondence.

(b) 10.3.5 Diamond operator : $d \in (\mathbb{Z}/N\mathbb{Z})^\times$

$$\begin{tikzcd}
\mathbb{E}_{univ} \arrow[r, dashed, "Id", "提升"'] \arrow[d] & \mathbb{E}_{univ} \arrow[d] \\
Y_1(N) \arrow[r, "Id"'] & Y_1(N)
\end{tikzcd}
\qquad \text{fiberwise} \qquad
\begin{tikzcd}
\mathbb{E}_{univ}|_{(E,P)} \arrow[d, equal] & \mathbb{E}_{univ}|_{(E,P)} \arrow[d, equal] \\
E \arrow[r, "id"] & E
\end{tikzcd}$$

$$(E, P) \longmapsto (E, dP)$$

(E,P) :
- E : $EC/\mathbb{C}$
- $P \in E[N]$ N-torsion pt

Then the action of $\langle d \rangle \oplus \overline{\langle d \rangle}$ on $W_{\mathbb{C}}$ equals Id^*.

(c) 10.3.6 Frickel operator $\zeta = e^{-\frac{2\pi i}{N}}$

$$\begin{tikzcd}
\mathbb{E}_{univ,\mathbb{C}} \arrow[r, dashed, "w_\zeta", "提升"'] \arrow[d] & \mathbb{E}_{univ,\mathbb{C}} \arrow[d] \\
Y_1(N)_{\mathbb{C}} \arrow[r, "w_\zeta"] & Y_1(N)_{\mathbb{C}}
\end{tikzcd}
\qquad \text{fiberwise} \qquad
\begin{tikzcd}
\mathbb{E}_{univ}|_{(E,P)} \arrow[d, equal] & \mathbb{E}_{univ}|_{(E/\langle p \rangle, P')} \arrow[d, equal] \\
E \arrow[r, "quotient"'] & E/\langle p \rangle
\end{tikzcd}$$

$$(E, P) \longmapsto (E/\langle p \rangle, P' \bmod \langle p \rangle)$$

Then the action of $W_N \oplus \overline{W_N}$ on $W_{\mathbb{C}}$ equals $i^{k+2} N^{-\frac{k}{2}} w_{\zeta}^{*}$

(1.45) Eichler–Shimura isomorphism (étale version)

Let $j : Y_1(N)_{\mathbb{Q}_\ell} \longrightarrow X_1(N)_{\mathbb{Q}_\ell}$ be the immersion. We consider

$$\mathcal{L}_k^{ét} = \mathrm{Sym}^{k-2}(R^1\pi_* \mathbb{Q}_\ell) \text{ over } Y_1(N)_{\mathbb{Q}_\ell}, \quad j_*\mathcal{L}_k^{ét} \text{ over } X_1(N)_{\mathbb{Q}_\ell}.$$

Upon choosing an isomorphism $\overline{\mathbb{Q}}_\ell \xleftarrow{\sim} \mathbb{C}$, the LHS of (1.43) can be written as the étale cohomology

$$H^1(X_1(N)^{an}_{\mathbb{C}}, j_*\mathcal{L}_k) \xrightarrow[\text{comparison thm}]{\sim} H^1_{ét}(X_1(N)_{\overline{\mathbb{Q}}_\ell}, j_*\mathcal{L}_k^{ét}).$$

Now similar to the remark in (1.43), we reinterprete RHS

- Let $a : Y_1(N) \longrightarrow \mathrm{Spec}\,\mathbb{Z}[\frac{1}{N\ell}]$.
- We define a $\mathbb{Q}_\ell$-module over $\mathrm{Spec}\,\mathbb{Z}[\frac{1}{N\ell}]$ as

$$\mathcal{W}_\ell := \mathrm{im}\left(R^1 a_!(\mathcal{L}_k^{ét}) \longrightarrow R^1 a_*(\mathcal{L}_k^{ét})\right)$$

This is a "$\mathbb{Q}_\ell$-local system in étale cohomology", commuting with any base change.

<u>In particular</u>,

(i) for any $p \nmid N\ell$, fix algebraic closure $\overline{\mathbb{F}}_p | \mathbb{F}_p$, the geometric fibre of $\mathcal{W}_\ell$ at p is isomorphic to

$$\mathcal{W}_{\ell,p} := \mathrm{im}\left(H^1_{ét,c}(Y_1(N)_{\overline{\mathbb{F}}_p}, \mathcal{L}^{ét}_{k,\overline{\mathbb{F}}_p}) \longrightarrow H^1_{ét}(Y_1(N)_{\overline{\mathbb{F}}_p}, \mathcal{L}^{ét}_{k,\overline{\mathbb{F}}_p})\right)$$

<u>Galois action</u>: $H^1_{ét,c}(Y_1(N)_{\overline{\mathbb{F}}_p}, \mathcal{L}^{ét}_{k,\overline{\mathbb{F}}_p})$ has a $\mathrm{Gal}_{\mathbb{F}_p}$-action, hence so does $\mathcal{W}_{\ell,p}$. This action pulls back as an action of $\mathrm{Gal}_{\mathbb{Q}_\ell}$ (as $\mathrm{Gal}_{\mathbb{Q}_\ell} \twoheadrightarrow \mathrm{Gal}_{\mathbb{F}_p}$).

(ii) Let η be the generic point of $\mathrm{Spec}\,\mathbb{Z}[\frac{1}{N\ell}]$, with residue field $\overline{\mathbb{Q}}$. The geometric fiber of $\mathcal{W}_\ell$ at η is isomorphic to

$$\mathcal{W}_{\ell,\mathbb{Q}} := \mathrm{im}\left(H^1_{ét,c}(Y_1(N)_{\overline{\mathbb{Q}}}, \mathcal{L}^{ét}_{k,\overline{\mathbb{Q}}}) \longrightarrow H^1_{ét}(Y_1(N)_{\overline{\mathbb{F}}_p}, \mathcal{L}^{ét}_{k,\overline{\mathbb{Q}}})\right) \simeq W_{\mathbb{Q}_\ell}$$

<u>Galois action</u>: $H^1_{ét,c}(Y_1(N)_{\overline{\mathbb{Q}}}, \mathcal{L}^{ét}_{k,\overline{\mathbb{Q}}})$ naturally has a $\mathrm{Gal}_{\mathbb{Q}}$-action, hence so does $\mathcal{W}_{\ell,\mathbb{Q}}$.

- Since $\mathcal{W}_\ell$ is a local system over $\mathrm{Spec}\,\mathbb{Z}[\frac{1}{N\ell}]$, we have isomorphisms of $\mathbb{Q}_\ell$-vector spaces

$$\underbrace{\mathcal{W}_{\ell,p} \xrightarrow{\sim} \mathcal{W}_{\ell,\mathbb{Q}}}_{\text{étale coh. version}} \xrightarrow{\sim} W_{\mathbb{Q}_\ell}, \quad \forall p \nmid N\ell.$$

Moreover, this isomorphism is compatible with a priori fixed $\mathrm{Gal}_{\mathbb{Q}_p} \hookrightarrow \mathrm{Gal}_{\mathbb{Q}}$.

- <u>Hecke action</u> : We have ℓ-adic version of $T[p]$, I_d^*, w_ζ^*, giving automorphisms on $\mathcal{W}_{\ell,p}$ and $\mathcal{W}_{\ell,\mathbb{Q}_\ell}$, compatible with the above isomorphism.

  <u>Galois action</u> :

  (i) <u>Frobenius</u> : the geometric frobenius $F \in \mathrm{Gal}_{\mathbb{Q}_p}$ acts on $\mathcal{W}_{\ell,p}$ for any $p \nmid N\ell$

  (ii) <u>Verschiebung</u> :

  - We have <u>Poincàre duality</u> on étale cohomology as a nondegenerate bilinear form $\langle -,- \rangle_\ell : \mathcal{W}_{\ell,p} \times \mathcal{W}_{\ell,p} \longrightarrow \underline{\mathbb{Q}_\ell(-k+1)}$ ← Tate twist, s.t.

$$\langle x,y \rangle_\ell = (-1)^{k-1} \langle y,x \rangle_\ell .$$

  - The transpose of F under $\langle -,- \rangle_\ell$ is defined as $V \in \mathrm{End}_{\mathbb{Q}_\ell}(\mathcal{W}_{\ell,p})$.
    (i.e. $\langle Fx,y \rangle_\ell = \langle x, Vy \rangle_\ell$)

(1.46) Eichler–Shimura relation

We have both <u>Hecke action</u> and <u>Galois action</u> on $\mathcal{W}_{\ell,p}$. For $p \nmid N\ell$, we have equalities in $\mathrm{End}_{\mathbb{Q}_\ell}(\mathcal{W}_{\ell,p})$:

(i) $T[p] = F + I_p^* V$

(ii) $FV = p^{k+1} \cdot \mathrm{id} = VF$

(iii) $(w_\zeta^*)^{-1} V w_\zeta^* = I_p^* V$, here ζ is a N-th primitive root of unity.

<u>Now, we are so close to the Galois rep'n of modular forms !</u>

§8 Galois representation associated to modular forms

(1.47) Preparation on Hecke algebras

- Let $\mathbb{T}_{\mathbb{Z}}$ be the subring of $\mathrm{End}_{\mathbb{C}}(S_k(\Gamma_1(N)))$ generated by T_p and $\langle d\rangle$ as p runs through primes and d runs through $(\mathbb{Z}/d)^\times$. For any commutative ring A,

$$\mathbb{T}_A := \mathbb{T}_{\mathbb{Z}} \otimes_{\mathbb{Z}} A \quad \text{is an } A\text{-algebra}.$$

<u>Example 1</u> : $A=\mathbb{Z}$, then $\mathbb{T}_{\mathbb{Z}}$ is a free $\mathbb{Z}$-module of finite rank.

cf: 李文威《模形式初步》引理 10.5.2.

<u>Idea of proof</u> : Let $\mathbb{W}'_{\mathbb{Z}}$ be the image of $\mathbb{W}_{\mathbb{Z}}$ in $\mathbb{W}_{\mathbb{C}}$.

- $\mathbb{T}_{\mathbb{Z}}$ preserves $\mathbb{W}'_{\mathbb{Z}}$ by the explicit construction of actions
- $\mathbb{W}_{\mathbb{Z}}$ finitely generated $\mathbb{Z}$-module $\Rightarrow$ so does $\mathbb{W}'_{\mathbb{Z}}$.
  $\mathbb{W}_{\mathbb{C}}$ is torsion free
  $\xRightarrow[\text{so}]{}$ $\mathbb{W}'_{\mathbb{Z}}$ is a free $\mathbb{Z}$-module of finite rank r
- Therefore via $\mathbb{T}_{\mathbb{Z}} \hookrightarrow \mathrm{End}_{\mathbb{Z}}(\mathbb{W}'_{\mathbb{Z}}) \simeq \mathbb{Z}^{r^2}$ is free of finite rank.

<u>Example 2</u> : $A=\mathbb{C}$. Consider the bilinear form

$$\begin{aligned}\mathbb{T}_{\mathbb{C}} \times S_k(\Gamma_1(N)) &\longrightarrow \mathbb{C}\\ (T, f) &\longmapsto a_1(Tf) =: \psi_f(T)\end{aligned}$$

(i) This bilinear form is nondegenerate, inducing isomorphisms of $\mathbb{T}_{\mathbb{C}}$-modules

- $\mathbb{T}_{\mathbb{C}} \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{C}}(S_k(\Gamma_1(N)), \mathbb{C}) =: S_k(\Gamma_1(N))^*$
- $S_k(\Gamma_1(N)) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{C}}(\mathbb{T}_{\mathbb{C}}, \mathbb{C}) =: \mathbb{T}_{\mathbb{C}}^*$.

(ii) $f\in S_k(\Gamma_1(N))$ is a normalized eigenform iff ψ_f is a ring homomorphism.
This gives a bijection

$$\begin{array}{ccc}\{f\in S_k(\Gamma_1(N)) : \text{normalized Hecke eigenform}\} & \ni & f\\ \big\downarrow 1:1 & & \big\downarrow\\ \{\text{ring homomorphism } \mathbb{T}_{\mathbb{Z}}\to\mathbb{C}\} & \ni & (\phi_f : \mathbb{T}_{\mathbb{Z}}\to\mathbb{T}_{\mathbb{C}}\xrightarrow{\psi_f}\mathbb{C}).\end{array}$$

cf: 李文威《模形式初步》命题 10.5.5.

<u>Coro</u> : Let $f\in S_k(\Gamma_1(N))$ be a normalized Hecke eigenform. Then

- each $a_n(f)$ is an algebraic integer
- Let $K_f :=$ the subfield of $\mathbb{C}$ generated by $\{a_n(f) : n\geq 1\}$.
  Then K_f is a finite extension of $\mathbb{Q}$.

cf: 李文威《模形式初步》推论 10.5.6. In fact, K_f is gen. by $\mathrm{im}(\phi_f)$.

Example 3 $A = \mathbb{Q}_\ell$. Write $\mathbb{T}_\ell = \mathbb{T}_{\mathbb{Q}_\ell}$.

- By Example 1, $\mathbb{T}_\ell$ is a finite dimensional $\mathbb{Q}_\ell$-vector space.
- For $p \nmid N\ell$, $\mathbb{T}_\ell$ acts on the $\mathbb{Q}_\ell$-vector spaces $\mathcal{W}_{\ell,\mathbb{Q}} \xrightarrow{\sim} \mathcal{W}_{\ell,p}$

  Choose a primitive N-th root of unity $\zeta \in \overline{\mathbb{Q}}$, identifying it with its image in $\overline{\mathbb{F}_p}$.

  We define a bilinear form on $\mathcal{W}_{\ell,p}$ by

$$[x,y]_\ell := \langle x, w_\zeta^* y \rangle_\ell$$

recall (1.45)

- Fact: (i) $\forall\, T \in \mathbb{T}_\ell$, T is self-adjoint on $\mathcal{W}_{\ell,p}$ wrt the bilinear form $[-,-]_\ell$.

  (ii) There exists isomorphisms of $\mathbb{T}_\ell$-modules

$$\mathcal{W}_{\ell,\mathbb{Q}} \xrightarrow{\sim} \mathbb{T}_\ell^{\oplus 2}, \quad \mathbb{T}_\ell \simeq \mathbb{T}_\ell^* := \mathrm{Hom}_{\mathbb{Q}_\ell}(\mathbb{T}_\ell, \mathbb{Q}_\ell)$$

Remark: The isomorphism $\mathbb{T}_\ell \xrightarrow{\sim} \mathbb{T}_\ell^*$ is equivalent to say $\mathbb{T}_\ell$ is a Gorenstein ring. See Tilouine's article in Fermat's volume for more. For a sketch of proof, see 李文威, 定理10.6.1.

(1.48) Construction of the contragredient $\check{\rho}_{f,\ell}$

Recall in (1.45), $Gal_{\mathbb{Q}}$ acts on $W_{\ell,\mathbb{Q}}$. This action is also $\mathbb{T}_\ell$-linear (we have only constructed Hecke action on the complex version $W_{\mathbb{C}}$. But here please accept this fact via comparison.) Write it as

$$\check{\rho}_\ell : Gal_{\mathbb{Q}} \longrightarrow GL_{\mathbb{T}_\ell}(W_{\ell,\mathbb{Q}})$$

- Let $f \in S_k(\Gamma_1(N))$ be a normalized Hecke eigenform. By (1.47), it corresponds to

$$\phi_f : \mathbb{T}_{\mathbb{Z}} \longrightarrow K_f \subseteq \mathbb{C}, \qquad \begin{array}{l} T_p \longmapsto T_p(f) \\ \langle d \rangle \longmapsto \langle d \rangle (f). \end{array}$$

  - Tensoring with $\mathbb{Q}$, $\phi_f : \mathbb{T}_{\mathbb{Q}} \twoheadrightarrow K_f$
  - Then tensoring with $- \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$, upon choosing a distinguished valuation λ lying over ℓ,

$$\phi_{f,\lambda} : \mathbb{T}_\ell \twoheadrightarrow K_f \otimes_{\mathbb{Q}} \mathbb{Q}_\ell = \prod_{\lambda' | \ell} K_{f,\lambda'} \twoheadrightarrow K_{f,\lambda}.$$

- Define $V^*_{f,\lambda} := W_{\ell,\mathbb{Q}} \otimes_{\mathbb{T}_\ell} K_{f,\lambda}$, here $K_{f\lambda}$ is regarded as a $\mathbb{T}_\ell$-algebra via $\phi_{f,\lambda}$
- Then $\check{\rho}_\ell$ together with the natural projection $W_{\ell,\mathbb{Q}} \to V^*_{f,\lambda}$ ($w \mapsto w \otimes 1$), gives a group homomorphism

$$\check{\rho}_{f,\lambda} : G_{\mathbb{Q}} \to GL_{K_{f,\lambda}}(V^*_{f,\lambda}) \xrightarrow[\approx]{\text{taking a basis}} GL_2(K_{f,\lambda}).$$

(1.49) Construction of $\rho_{f,\lambda}$

Define $\rho_{f,\lambda}$ as the contragredient rep'n of $\check{\rho}_{f,\lambda}$ on $V_{f,\lambda} := Hom_{K_{f,\lambda}}(V^*_{f,\lambda}, K_{f\lambda})$.

More precisely,

$$\begin{array}{rcl} \rho_{f,\lambda} : Gal_{\mathbb{Q}} & \longrightarrow & GL_{K_{\lambda,f}}(V_{f,\lambda}) \\ g & \longmapsto & [\underset{\substack{\cap \\ V_{f,\lambda}}}{\xi} \longmapsto \xi \circ \check{\rho}_{f,\lambda}(g)^{-1}] \end{array}$$

$$\begin{tikzcd} V^*_{f,\lambda} \arrow[r, "\check{\rho}_{f,\lambda}(g)"] \arrow[d, "\xi"'] & V^*_{f,\lambda} \arrow[dl, bend left, "\xi \circ \check{\rho}_{f,\lambda}(g)^{-1}"] \\ K_{f\lambda} & \end{tikzcd}$$

This is the Galois representation associated to the (normalized) eigenform f.

- Recall : Let $f \in S_k(\Gamma_1(N))$ be a normalized Hecke eigenform. Then it has a character

$$\chi_f : (\mathbb{Z}/N)^\times \to \mathbb{C}^\times \quad \text{s.t. } \langle d \rangle f = \chi_f(d) f$$

Upon choosing $\mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}}_\ell$,

$$\chi_f : (\mathbb{Z}/N)^\times \longrightarrow K_f^\times \hookrightarrow K_{f,\lambda}^\times$$

By Kronecker-Weber, χ_f gives

$$\chi_f : Gal_{\mathbb{Q}} \twoheadrightarrow Gal_{\mathbb{Q}}^{ab} \twoheadrightarrow Gal(\mathbb{Q}(\mu_N) | \mathbb{Q}) \simeq (\mathbb{Z}/N\mathbb{Z})^\times \longrightarrow K_{f,\lambda}^\times$$

Via this construction :
- $\chi_f(Fr_p) = \chi_f(p)$, for $p \nmid N\ell$
- χ_f(complex conjugation) $= -1$.

(1.50) Theorem (Deligne-Shimura ; Deligne-Serre ; K. Ribet)

The Galois rep'n $\rho_{f,\lambda}$ satisfies :

(i) $\rho_{f,\lambda}$ is unramified outside $N\ell$

(ii) For any prime $p \nmid N\ell$, $\rho_{f,\lambda}(\mathrm{Fr}_p)$ has characteristic polynomial, called <u>Hecke polynomial</u>

$$X^2 - a_p(f) X + \chi_f(p) p^{k-1} \in K_f[X] \subseteq K_{f,\lambda}[X].$$

(iii) $\det(\rho_{f,\lambda}) = \chi_f \varepsilon_\ell^{k-1}$, here ε_ℓ is the ℓ-adic cyclotomic character.

(iv) $\det(\text{complex conjugation}) = -1$, i.e. the rep'n is <u>odd</u>.

(v) This rep'n is absolutely irreducible.

<u>Proof sketch</u> :

(i) Back to the $\mathrm{Gal}_\mathbb{Q}$-action on $\mathcal{W}_{\ell,\bar{\mathbb{Q}}}$. since $\mathcal{W}_\ell$ is an étale local system over $\mathrm{Spec}\,\mathbb{Z}[\frac{1}{N\ell}]$, the geometric fiber $\mathcal{W}_{\ell,\bar{\mathbb{Q}}}$ at the generic point η is unramified outside $N\ell$.

(ii) <u>Step 1</u> : Compute the Frobenius action $F = \check{\rho}_f(\mathrm{Frob}_p^{-1}) \in \mathrm{End}_{\mathbb{T}_\ell}(\mathcal{W}_{\ell,\bar{\mathbb{Q}}})$;

via $\mathcal{W}_{\ell,\bar{\mathbb{Q}}} \simeq \mathcal{W}_{\ell,\bar{p}}$, we work on $\mathcal{W}_{\ell,\bar{p}}$. Then Eichler-Shimura relation (1.46) gives

$$(X-F)(X-I_p^* V) \boxed{=} X^2 - (I_p^* V + F) X + I_p^* V F$$

recall $\mathbb{T}_\ell$ is a commutative algebra $\longrightarrow$ $\boxed{=}\, X^2 - T[p] X + I_p^* p^{k+1}$ (i) & (ii)

Regard both sides as matrices in $M_2(\mathbb{T}_\ell[X])$ by fixing any $\mathbb{T}_\ell$-basis of $\mathcal{W}_{\ell,\bar{\mathbb{Q}}}$

(RHS is a constant matrix (RHS) $\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$). Taking determinant :

$$\det(X-F)\det(X-I_p^* V) = (X^2 - T[p] X + I_p^* p^{k+1})^2 \in \mathbb{T}_\ell[X]$$

<u>Observation</u> : Let A be a commutative ring with 2 not being a zero divisor. Then for any monic $g \in A[X]$, $\exists$ at most one monic $f \in A[X]$ s.t. $f^2 = g$.

By this observation, it suffices to show $\det(X - I_p^* V) = \det(X-F)$.

To show this, note

$$\begin{aligned}
[Fx, y]_\ell = \langle Fx, w_\zeta^* y\rangle_\ell &= \langle x, V w_\zeta^* y\rangle_\ell \\
&= [x, (w_\zeta^*)^{-1} V w_\zeta^* y]_\ell \\
&= [x, I_p^* V y]_\ell \quad \text{by ES relation (iii)}
\end{aligned}$$

Hence $I_p^* V$ is the transpose of F wrt $[-,-]_\ell$. By linear algebra and (1.47), we see $I_p^* V$ and F has the same characteristic polynomial!

<u>Step 2</u> : Then via $-\otimes_{\mathbb{T}_\ell} K_{f,\lambda}$, $\check{\rho}_{f,\lambda}(\mathrm{Frob}_p^{-1})$ has characteristic polynomial

$$X^2 - a_p(f)X + \chi_f(p)p^{k-1} \in K_{f,\lambda}[X].$$

Hence the contragredient $\rho_{f,\lambda}(\mathrm{Frob}_p)$ has the same characteristic polynomial.

(iii) We see from above

$$\det \rho_{f,\lambda}(\mathrm{Frob}_p) = \chi_f(p)p^{k-1} = \chi_f(\mathrm{Frob}_p)\,\varepsilon_\ell^{k-1}(\mathrm{Frob}_p) = \chi_f\,\varepsilon_\ell^{k-1}(\mathrm{Frob}_p).$$

By the standard <u>Chebotarev + BNS trick</u>, we see this holds for general $\sigma \in G_{\mathbb{Q}}$.

(iv) Note $f|_k \begin{pmatrix} -1 & \\ & -1 \end{pmatrix} = (-1)^k f$. Hence

$$\chi_f(\text{complex conjugation}) = \chi_f(-1) = (-1)^k$$

Put into $\det \rho_{f,\lambda}$ in (iii),

$$\det \rho_{f,\lambda}(\text{complex conjugation}) = \chi_f(-1)\,\varepsilon_\ell^{k-1}(-1) = (-1)^k \cdot (-1)^{k-1} = -1.$$

(v) Deep result by Deligne–Serre, K. Ribet. $\square$

<u>Remark</u>: We can also attach Eisenstein series with 2-dim Galois reps, they are reducible. See GTM 228, Thm 9.6.6, it can be written as $\begin{pmatrix} 1 & \\ & \varepsilon_\ell^{k-1} \end{pmatrix}$.

(1.51) Compatible systems

Question : Why we always keep "λ" in our notion ?

Answer : Let λ run !

Let $K|\mathbb{Q}$ be a number field. Let S be a finite set of rational primes.

Let $\rho := \{\rho_\lambda : \mathrm{Gal}_{\mathbb{Q}} \to GL_n(\overline{K_\lambda}) \ : \ \lambda \in M_K^\circ\}$ be a family of semisimple galois reps, where $\overline{K_\lambda}$ is an algeorai closure of K_λ. Write ℓ as the characteristic of residue fields of K_λ.

We say ρ is a compatible system over K unramfied outside S if for any prime number p,

(i) For $p \notin S \cup \{\ell\}$, ρ_λ is unraimfied at p. The characteristic polynomial of $\rho_\lambda(\mathrm{Frob}_p)$ lies in $K[X]$, independent of λ.

(ii) If $p = \ell$, then $\rho_\lambda|_{G_{\mathbb{Q}_p}}$ is de Rham. When $\ell \notin S$, $\rho_\lambda|_{G_{\mathbb{Q}_p}}$ is even crystalline.

(iii) The Hodge-Tate number of ρ_λ is independent of λ.

(1.52) Theorem

Let $S = \{$prime divisors of $N\}$. Then the family $\rho_f = \{\rho_{f,\lambda}\}$ constructed in (1.49) is a compatible system over K_f unramfied outside S.

Proof : We only sketched (i). The remaing conditions very deep results in p-adic Hodge theory. These verifies that ρ_f indeed "comes from geometry". $\square$

(1.53) A broader picture

$\left\{ \text{cuspidal cohomological automorphic rep'n of } GL_n \right\}$ - - - - - - - - $\left\{ \begin{array}{l} \text{Compatible systems} \\ \rho = (\rho_\lambda : \mathrm{Gal}_{\mathbb{Q}} \to GL_n(\overline{K_\lambda})) \end{array} \right\}$

? ?

motive
(a geometric object)

Ask : Which compatible systems arise from cuspidal cohomological autoreps ?
These problems are called modularity problems.

§ 9 Modularity

(1.53) Modularity

Let E be an elliptic curve over $\mathbb{Q}$. We say E is <u>modular</u> if $\exists$ a new form $f \in S_2(\Gamma_1(N_E))$ and a finite place $\lambda \mid \ell$ of K_f s.t. for any fixed $K_{f,\lambda} \hookrightarrow \overline{\mathbb{Q}}_\ell$,

$$\rho_{E,\ell} \underset{\mathbb{Q}_\ell}{\otimes} \overline{\mathbb{Q}}_\ell \xrightarrow{\sim} \rho_{f,\lambda} \underset{K_{f,\lambda}}{\otimes} \overline{\mathbb{Q}}_\ell \, .$$

(1.54) Modularity theorem

Every elliptic curve over $\mathbb{Q}$ is modular.

Fermat's last theorem

$\Uparrow$

<u>Proof</u>: Taylor–Wiles : when E is semistable ($\Leftrightarrow N_E$ is square-free).

Breuil–Conrad–Diamond–Taylor : all cases. $\square$

<u>Idea</u>:

$$\{ f \in S_k(\Gamma_1(N)) : \text{new form} \} \underset{\text{Wiles' work}}{\overset{\text{Deligne–Serre}}{\rightleftarrows}} \{ \rho_{f,\lambda} : \mathrm{Gal}_{\mathbb{Q}} \to \mathrm{GL}_2(K_{f,\lambda}) \}$$

$$\begin{array}{ccc} f & \overset{\text{modularity lifting}}{\dashrightarrow} & \rho \text{ from elliptic curve} \\ \Uparrow \text{ guess} & & \Downarrow \text{ reduction mod } \ell \\ \bar{f} \text{ modular form mod } \ell & \underset{\text{Langlands–Tunnell thm}}{\longleftrightarrow} & \bar{\rho} \text{ a residue galois rep.} \end{array}$$

$\mathbb{T}$: universal deformation ring of $\bar{f}$ (actually the "Hecke algebra")

R : universal deformation ring of $\bar{\rho}$ (parametrizing all possible lifts)

<u>Taylor–Wiles</u> : $R = \mathbb{T}$ theorem $\square$

<u>Remark</u> : How does this imply FLT : for prime $\ell \geq 5$, $x^\ell + y^\ell = z^\ell$ has no nonzero integer solution :

<u>Step1</u> : Suppose (a,b,c) is a solution, construct an elliptic curve

$$E : y^2 = x(x-a^\ell)(x+b^\ell) \quad \text{over } \mathbb{Q}.$$

s.t. (1) E has semistable reduction with $\Delta = 16(abc)^{2\ell}$ and conductor N_E = product of prime factors of abc

(2) $\bar{\rho}_{E,\ell} : \mathrm{Gal}_{\mathbb{Q}} \longrightarrow \mathrm{Aut}(E[\ell]) = \mathrm{GL}_2(\mathbb{F}_\ell)$, unramified at each odd prime $p \mid abc$.

<u>Step2</u> : By modularity, $\exists$ a new (normalized) Hecke eigenform $f \in S_2(\Gamma_0(N_E))$ with isomorphic galois rep'n over $\overline{\mathbb{Q}}_\ell$.

By <u>Ribet's level lowering</u>, $\exists f^\# \in S_2(\Gamma_0(2))$. But $X_0(2) = \mathbb{P}^1$ and

$H^0(\mathbb{P}^1, \Omega_{\mathbb{P}^1}) = 0$. Hence there is no such cusp forms.