Talk 07 Local computations : ramified places

SLOGAN :
- The choices at ramified places are not quite important. In applications, we are going to change it according to the needs.
- The purpose for this section is only to convince the reader that such kinds of sections do exist !

- **Big cell section** : $f_v^{\text{big-cell}}$ be the Siegal section that
  - i) supported on the big cell $\underline{Q(F_v)\,w\,N_Q(\mathcal{O}_{F_v})} \subseteq Q(F_v)\,w\,N_Q(F_v)$.

    where $w = \begin{pmatrix} & 1_n \\ -1_n & \end{pmatrix}$.
  - ii) $f_v^{\text{big-cell}}\underline{\big(w N_Q(\mathcal{O}_{F_v})\big)} = 1$.

Note :
- In [Wan2015ANT], he only required $f_v^{\text{big-cell}}$ supported on $Q(F_v)\,w\,N_Q(F_v)$ and $f_v^{\text{big-cell}}(N_Q(\mathcal{O}_{F_v})) = 1$. His two conditions are not compatible and are different from [SU14, §11.4.3]. We make modifications according to [Hsieh14, (5.5)] and [SU14]. The modifications will lead to correct Fourier coefficients !
- In [Wan2020ANT], he corrected these in his low rank case !

The Siegal section f_v^{sieg} is defined to be a translation of $f_v^{\text{big-cell}}$ by γ_v :

$$f_v^{\text{sieg},\bullet}(g) := f_v^{\text{big-cell},\bullet}(g\gamma_v^{\bullet})$$

where

$$\gamma_v^{\heartsuit} := \left(\begin{array}{cccc|cccc} 1_b &&&&&&& x^{-1}1_b \\ & 1 &&&&&& \\ && 1_a &&&& (y\bar{y})^{-1}1_a & \\ &&& 1_b & \bar{x}^{-1}1_b &&& \\ \hline &&&& 1_b &&& \\ &&&&& 1 && \\ &&&&&& 1_a & \\ &&&&&&& 1_b \end{array}\right), \quad \gamma_v^{\diamondsuit} = \gamma_v^{\heartsuit} \text{ deleting } \bullet \text{ rows \& columns.}$$

γ_v^{UR} UR: upper-right

here $x, y \in K$ which are divisible by some high power of ϖ_v (the uniformizer of K_v). They are fixed when "varying p-adically".

$\boxed{\text{Fourier coefficients}}$

Recall the definition of the local Whittaker integral, we compute directly :

$$W_\beta(1; f_v^{\text{sieg},\bullet}, z) := \int_{\mathrm{Herm}_n(F_v)} f_v^{\text{big-cell}}\left(z, w_n \begin{pmatrix} 1_n & \sigma \\ & 1_n \end{pmatrix} \gamma_v\right) e_v(-\mathrm{Tr}\beta\sigma)\, d\sigma$$

$$= \int_{\mathrm{Herm}_n(F_v)} f_v^{\text{big-cell}}\left(z, w_n \begin{pmatrix} 1_n & \sigma+\gamma_v^{UR} \\ & 1_n \end{pmatrix}\right) e_v(-\mathrm{Tr}\beta\sigma)\, d\sigma$$

$$= \int_{\underline{\mathrm{Herm}_n(F_v)-\gamma_v^{UR}}} f_v^{\text{big-cell}}\left(z, w_n \begin{pmatrix} 1_n & \underline{\sigma'} \\ & 1_n \end{pmatrix}\right) e_v\left(-\mathrm{Tr}\beta(\sigma'-\gamma_v^{UR})\right) d\sigma'$$

big cell conditions

$$= e_v(\mathrm{Tr}(\beta\cdot\gamma_v^{UR})) \int_{\underline{\mathrm{Herm}_n(\mathcal{O}_{F_v})}} e_v(-\mathrm{Tr}\beta\sigma')\, d\sigma$$

To handle the latter integral, consider

$$\mathrm{Herm}_n^{\star}(\mathcal{O}_{F_v}) := \{\beta \in \mathrm{Herm}_n(F_v) : \mathrm{Tr}\beta\sigma \in \mathcal{O}_{F_v} \text{ for any } \sigma \in \mathrm{Herm}_n(\mathcal{O}_{F_v})\}$$

Then for

- $\underline{\beta \in \mathrm{Herm}_n^{\star}(\mathcal{O}_{F_v})}$: $-\mathrm{Tr}\beta\sigma' \in \mathcal{O}_{F_v}$, hence it has trivial fractional part.

  Hence the integrand $e_v(-\mathrm{Tr}\beta\sigma') = 1$ and hence

  the integral $= \mathrm{vol}(\mathrm{Herm}_n(\mathcal{O}_{F_v}), d\sigma)$.

- $\underline{\beta \notin \mathrm{Herm}_n^{\star}(\mathcal{O}_{F_v})}$: the integral is zero as we integrating the exponential along a complete circle. (I'm a little bit confused here.)

The remaing coefficient is direct : take $\heartsuit$-case as an example : note $\underline{\beta_{14} = \beta_{41}^*}$

$$\beta = \begin{pmatrix} \beta_{11} & \beta_{12} & \beta_{13} & \beta_{14} \\ & & \vdots & \\ \beta_{41} & \beta_{42} & \beta_{43} & \beta_{44} \end{pmatrix} \begin{matrix} b \\ 1 \\ a \\ b \end{matrix} ; \quad \beta\cdot\gamma_v^{UR} = \begin{pmatrix} \underline{\bar{x}^{-1}\beta_{14} = \bar{x}^{-1}\beta_{41}} & * & * & * \\ * & 0 & * & * \\ * & * & (y\bar{y})^{-1}\beta_{33} & * \\ * & * & * & x^{-1}\beta_{41} \end{pmatrix}$$

Hence

$$\mathrm{Tr}(\beta\cdot\gamma_v^{UR}) = \underbrace{\frac{\overline{\beta_{a+b+1,1}} + \cdots + \overline{\beta_{a+2b+1,b}}}{\bar{x}}} + \frac{\beta_{a+b+1,1} + \cdots + \beta_{a+2b+1,b}}{x} + \frac{\beta_{b+1,b+1} + \cdots + \beta_{a+b+1,a+b+1}}{y\bar{y}}$$

Both [SU14, Lemma 11.14] and [Wan15ANT, Lemma 8.12] lack this term

We summarize it into the following theorem.

Theorem [SU14, Lemma 11.14] [Wan15ANT, Lemma 4.12]

Let $\beta \in \mathrm{Herm}_{n^\circ}(F_v)$. Then:

① $$W_\beta(1; f_v^{\mathrm{sieg},\heartsuit}, z) = \mathbb{1}^{\star}_{\mathrm{Herm}_{\heartsuit}(\mathcal{O}_{F_v})}(\beta) \cdot \mathrm{vol} \cdot e_v\left(\mathrm{tr}^{K_v}_{F_v}\left(\frac{\mathrm{Tr}\,\overline{\beta_{41}}}{\bar{x}} + \frac{\mathrm{Tr}\,\beta_{41}}{x} + \frac{\mathrm{Tr}\,\beta_{33}}{y\bar{y}}\right)\right).$$

② $$W_\beta(1; f_v^{\mathrm{sieg},\diamondsuit}, z) = \mathbb{1}^{\star}_{\mathrm{Herm}_{n^\diamondsuit}(\mathcal{O}_{F_v})}(\beta) \cdot \mathrm{vol} \cdot e_v\left(\mathrm{tr}^{K_v}_{F_v}\left(\frac{\mathrm{Tr}\,\overline{\beta_{31}}}{\bar{x}} + \frac{\mathrm{Tr}\,\beta_{31}}{x} + \frac{\mathrm{Tr}\,\beta_{22}}{y\bar{y}}\right)\right).$$

Note:

(1) When we only use the big-cell section without translation, from the proof we see

$$W_\beta(1; f_v^{b\text{-}c,\heartsuit}, z) = \mathbb{1}^{\star}_{\mathrm{Herm}_{\heartsuit}(\mathcal{O}_{F_v})}(\beta) \cdot \text{volume term}.$$

$$W_\beta(1; f_v^{b\text{-}c,\diamondsuit}, z) = \mathbb{1}^{\star}_{\mathrm{Herm}_{n^\diamondsuit}(\mathcal{O}_{F_v})}(\beta) \cdot \text{volume term}.$$

This is quite neat. We see there is almost no need to interpolate at the ramified places.

(2) Using Talk 4, Property 3, we actually have $W_\beta\left(\begin{pmatrix} A & \\ & A^{-*}\end{pmatrix}; f_v^{\mathrm{sieg},\diamondsuit}, z\right)$ for all $A \in GL_n(K_v)$, as [Wan2020ANT, Lemma 6.21] goes:

$$W_\beta\left(\begin{pmatrix} A & \\ & A^{-*}\end{pmatrix}; f_v^{\mathrm{sieg},\diamondsuit}, z\right) = \chi(\det A)\, |\det A\bar{A}|^{-z+\frac{n}{2}}\, W_{A^*\beta A}(1; f_v^{\mathrm{sieg}}, z)$$

Pullback integrals

Natural question: Why we need a modification γ_v? In [SU14, §11.4.4], they explained that they are needed to be well-adapted to pullback formulas.

We write
$$F_\phi(f_v^{\mathrm{sieg}}, z, g; \chi) := \int_{U(r,s)(F_v)} \bar{\chi}(\det g_1)\, f_v^{\mathrm{sieg}}(z, \iota_\heartsuit(g, g_1))\, \phi(g_1)\, dg_1$$
$$= \int_{U(r,s)(F_v)} \bar{\chi}(\det g_1)\, f_v^{\text{big-cell}}(z, \iota_\heartsuit(g, g_1)\gamma_v)\, \phi(g_1)\, dg_1$$

The first difficulty is how to judge
$$\iota_\heartsuit(g, g_1)\gamma_v \in \operatorname{supp} f_v^{\text{big-cell}} \subseteq Q(F_v)\, w\, N_Q(\mathcal{O}_{F_v})$$

We can already see how difficult it could be. We seperate it into

$$\iota_\heartsuit(g,1)\gamma_v \in \operatorname{supp} f_v^{\text{big-cell}}$$
$$\Big\Downarrow \quad \text{[Wan15, Lemma 4.9], generalized from [SU14, (11.32)]}$$
$$g \in P w K_v^{(2)} \subseteq U(r+1, s+1)(F_v).$$

$$\iota_\heartsuit(1,g_1)\gamma_v \overset{?}{\in} \operatorname{supp} f_v^{\text{big-cell}}$$
$$\Big\Updownarrow \quad \text{[Wan15, Lemma 4.10] generalized from [SU14, (11.34)]}$$
$$g_1 \in \mathcal{Y}_v \subseteq U(r,s)(F_v)$$

Here:

- $$K_v^{(2)} = \left\{ \begin{array}{c|c} \begin{array}{ccccc|c} & & & & & \\ \begin{pmatrix} 1_b & & & & d \\ a & 1 & f & b & c \\ & & 1_a & & g \\ & & & 1_b & e \\ & & & & 1 \end{pmatrix} & \begin{matrix} b \\ 1 \\ a \\ b \\ 1 \end{matrix} \\ \begin{matrix} b & \quad 1 & \quad a & \quad b & \quad 1 \end{matrix} & \end{array} & \begin{array}{l} e = -a^* \\ b = d^* \in M(\mathcal{O}_{F_v}) \\ g = -Sf^* \\ c - fSf^* \in \mathcal{O}_{F_v} \\ a \in x\mathcal{O}_{F_v},\ e \in \bar{x}\mathcal{O}_{F_v} \\ f \in M(y\bar{y}\mathcal{O}_{F_v}) \end{array} \end{array} \right\} \subseteq U(r+1, s+1)(F_v)$$

  subgroup

  [Wan15] has this extra condition, I'm confused at it. ([Wan20] included this but [EW16] not) $\longrightarrow g \in 2Sy\bar{y}\mathcal{O}_{F_v}$

- $$\mathcal{Y}_v = \begin{array}{c} \begin{matrix} b & \qquad\qquad a & \qquad\qquad b \end{matrix} \\ \begin{pmatrix} 1 + x\mathcal{O}_{F_v} & x\bar{y}\mathcal{O}_{F_v} & x\bar{x}\mathcal{O}_{F_v} \\ \frac{1}{2}y\bar{y}S\mathcal{O}_{F_v} & 1 - y\bar{y}S(1 + y\bar{y}M_a(\mathcal{O}_{F_v})) & yyxS\mathcal{O}_{F_v} \\ \mathcal{O}_{F_v} & y\bar{y}\mathcal{O}_{F_v} & 1 + \bar{x}\mathcal{O}_{F_v} \end{pmatrix} \begin{matrix} b \\ a \\ b \end{matrix} \end{array} \subseteq U(r,s)(F_v).$$

- Fix once and for all $x, y \in K$ with sufficiently large N such that $\varpi_v^N | x, y$ with

$$\xi_v^N = \varpi_v^N \in \mathcal{S}_K \operatorname{cond}(\xi_v),\ \operatorname{cond}(\psi_v),\ \operatorname{cond}(\chi_v),\ \operatorname{cond}_{\tilde\pi}(\phi_v).$$

Since there are only finitely many ramified places, we can find a uniform N for all $v \in \Sigma_{\mathrm{ram}}$.

— Since for $v \in \Sigma_{\mathrm{ram}}$, these characters are ramified, we hope to use x and y to "eliminate" these ramification when computing the pullback sections.

<u>Theorem</u> [Wan15ANT, Lemma 4.9.4.10], [SU14, Prop. 11.16]

① The pullback section $F_\phi(f_v^{\mathrm{Sieg}}, \mathfrak{z}, g; \chi)$ is supported in $PwK_v^{(2)}$, and it is invariant under the right action of $K_v^{(2)}$ (explicit in [Wan20ANT])

$\Longrightarrow$ Being a Klingen section, the left translation of P is known. Moreover, it is right $K_v^{(2)}$-invariant. So the only essential value is at $w \in U(r+1, s+1)(F_v)$

(<u>recall</u>: In previous ∞ and unr. places, we have explicit expression on $F_\phi(f_v^{\mathrm{Sieg}}, g)$).

② Assume $\phi_v \in \pi_v$ to be invariant under the action of $\mathcal{Y}_v$ and modify it as

$$\phi_{v,x} := \pi\left(\operatorname{diag}(\bar{x}, 1, x^{-1})\eta^{-1}\right)\phi_v, \quad \eta = \begin{pmatrix} & & 1_b \\ & 1_a & \\ -1 & & \end{pmatrix}$$

Then:

- $F^{\heartsuit}_{\phi_{v,x}}(f_v^{\mathrm{Sieg}}, \mathfrak{z}, w) = \chi(y\bar{y}x)\, |(y\bar{y})^2 x\bar{x}|_v^{-\mathfrak{z} - \frac{a+2b+1}{2}} \cdot \operatorname{vol}(\mathcal{Y}_v) \cdot \phi_v$
- $F^{\diamondsuit}_{\phi_{v,x}}(f_v^{\mathrm{Sieg}}, \mathfrak{z}, w) = \chi(y\bar{y}x)\, |(y\bar{y})^2 x\bar{x}|_v^{-\mathfrak{z} - \frac{a+2b}{2}} \cdot \operatorname{vol}(\mathcal{Y}_v) \cdot \phi_v$

<u>Note</u>: Similar to the "new condition" at unramified primes, here at ramified places, we have a requirement on our chosen ϕ_v. Using "K-types", we need to "fix" an max. open compact K_v containing $\mathcal{Y}_v$ such that ϕ_v is invariant under K_v. (as $\mathcal{Y}_v$ may not be max. open compact.)

Here the above $F_{\phi_{v,x}}^{\bullet}$ is called a <u>good Klingen section</u> at $v \in \Sigma_{\mathrm{ram}}$.