p-ADIC L-FUNCTIONS FOR DIRICHLET CHARACTERS

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Contents

Introduction

Number theorists care about the connections between arithmetic objects and analytic objects. Iwasawa theory, roughly speaking, is the study of such connections in p-adic families. On the analytic side, this is often done by cert+ain constructions of p-adic L-function, which primarily can be regarded as a p-adic interpolation of critical values of classical L-functions.

In this short note, we shall introduce the construction of the p-adic L-function of Dirichlet L-functions for Dirichlet characters from various perspectives. Such a p-adic L-function is first constructed and studied by Kubota and Leopoldt in [KL64], so it is often called the **Kubota-Leopoldt p-adic L-function**.

Notations and Conventions. Throughout this note, we assume p be an *odd* prime to avoid some technical issues, and let $\mathbb{Z}_p$, $\mathbb{Q}_p$, $\mathbb{C}_p$ be the ring of p-adic integers, field of p-adic rational numbers and the completion of the algebraic closure of $\mathbb{Q}_p$, respectively. The field $\mathbb{C}_p$ plays the same role in the p-adic analysis as the complex plane $\mathbb{C}$ in the classical complex analysis.

We fix an algerbaic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$, and its completion $\mathbb{C}_p$. We fix an isomorphism $\iota_p : \mathbb{C} \to \mathbb{C}_p$. Let $|-|_p$ and $v_p(-)$ be the absolute value and the valuation on $\mathbb{Q}_p$ respectively, normalized as $|p|_p = p^{-1}$. We often write $|-|$ and v when there will be no confusion.

Structure of this note. In Section 1, we will introduce the setups of this note and make some preparations on both the p-adic analysis and the classical Dirichlet L-functions. In Section 2, we will construct the Kubota-Leopoldt p-adic L-function as a p-adic meromorphic function on a certain disk on the p-adic complex plane.

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Nowadays, to better fit in the study of Iwasawa theory, people tends to regard p-adic L-functions as a p-adic measure. In Section 3, we will first briefly recall the notion of distributions and measures, and then construct the p-adic L-functions of Dirichlet characters from this perspective. In Section 4, we reformulate the Kubota-Leopoldt p-adic L-function as a function over the weight space, and use the automorphic method, namely the p-adic Eisenstein family to construct them directly.

Acknowledgement. The materials in this note are largely borrowed from [Was97, Chapter 5,7,12], [Eis21] and Pak-Hin Lee's course material at `https://warwick.ac.uk/fac/sci/maths/people/staff/phlee/tcc/`.

1. Basic Setup and Preliminaries

1.1. **Dirichlet characters and their Dirichlet L-function.** First we recall that a **Dirichlet character modulo** N is a group homomorphism $\chi : (\mathbb{Z}/N)^\times \to \mathbb{C}^\times$. The number N is called its **degree**. It is called an **odd** character if $\chi(-1) = -1$, and **even** if $\chi(-1) = 1$. We write $\delta_\chi = 0$ if χ is even and $\delta_\chi = 1$ if χ is odd.

For any multiple M of N, we can construct a Dirichlet character modulo M by simply composing χ by the natural projection, i.e.

$$\chi : (\mathbb{Z}/M)^\times \twoheadrightarrow (\mathbb{Z}/N)^\times \xrightarrow{\chi} \mathbb{C}^\times.$$

Then for any Dirichlet character χ modulo N, we can find a smallest divisor $\mathfrak{f}_\chi$ of N such that χ is induced from a Dirichlet character modulo $\mathfrak{f}_\chi$. The number $\mathfrak{f}_\chi$ is called the **conductor** of χ. We say a Dirichlet character χ modulo N is **primitive** if it has conductor N.

One subtlety is the multiplication of two Dirichlet characters. For two characters χ_1 and χ_2 of the same degree N, then we can define their multiplication by simply putting $\chi_1\chi_2(a) := \chi_1(a)\chi_2(a)$ for $a \in (\mathbb{Z}/N)^\times$. If χ_1 and χ_2 are of different degrees N_1 and N_2, then we let N be the least common multiple of them and regard both χ_1 and χ_2 as of degree N, then we can multiply them as in the previous case. Note that then

- in general, $\chi_1\chi_2(a) \neq \chi_1(a)\chi_2(a)$, and
- even if χ_1 and χ_2 are both primitive, their product $\chi_1\chi_2$ may not be primitive.

Convention 1.1. From now on, we assume χ is a primitive Dirichlet character of conductor $\mathfrak{f} := p^n M$, where $n \geq 0$ and $M \geq 1$ are integers and $p \nmid M$.

We define the Dirichlet L-function of χ to be the series

$$L(s,\chi) = \sum_{n=0}^{\infty} \frac{\chi(n)}{n^s},$$

for $\mathrm{Re}(s) > 1$. Notice that for the trivial character $\chi = \mathbb{1}$, the L-series is nothing but the Riemann zeta function. It is well-known that

- The L-series $L(s,\chi)$ can be analytically continued to the entire complex plane, except for a simple pole at $s = 1$ when $\chi = \mathbb{1}$.

- Let $\mathbb{P}$ be the set of all finite prime numbers, then $L(s,\chi)$ has the Euler product expansion as
$$L(s,\chi)=\prod_{\ell\in\mathbb{P}}(1-\chi(\ell)\ell^{-s})^{-1}=\prod_{\ell\nmid p^n M}(1-\chi(\ell)\ell^{-s})^{-1}.$$

What we may care most are the special values of the Dirichlet L-function at "critical points". Before stating the results, we need to define the following.

- The **Bernoulli numbers** B_n are defined as those rational numbers such that
$$\frac{t}{e^t-1}=\sum_{n=0}^{\infty}\frac{B_n t^n}{n!}.$$
- For a Dirichlet character χ, the **generalized Bernoulli numbers** $B_{n,\chi}$ are defined by
$$\sum_{a=1}^{\mathfrak{f}}\frac{\chi(a)te^{at}}{e^{\mathfrak{f}t}-1}=\sum_{n=0}^{\infty}\frac{B_{n,\chi}t^n}{n!}.$$
- We define the **Bernoulli polynomial** $\mathbf{B}_n(X)$ as those polynomials such that
$$\frac{te^{Xt}}{e^t-1}=\sum_{n=0}^{\infty}\frac{\mathbf{B}_n(X)t^n}{n!}.$$

The following properties of them will be used frequently in the construction of Kubota-Leopoldt p-adic L-functions.

- It is easy to see that the defining relation for the $B_{n,\chi}$ is an even function of t when χ is even and odd when χ is odd. Therefore
$$B_{n,\chi}=0 \text{ if } n\not\equiv\delta_\chi \pmod 2. \tag{1.1}$$
- $\mathbf{B}_n(1-X)=(-1)^n\mathbf{B}_n(X)$. This is by a direct computation.
- Since the generating function of $\mathbf{B}_n(X)$ is the product of $\frac{t}{e^t-1}$ and e^{Xt}, it follows that
$$\mathbf{B}_n(X)=\sum_{i=0}^{n}\binom{n}{i}B_iX^{n-i}. \tag{1.2}$$
- Let F be any multiple of $\mathfrak{f}$, then
$$B_{n,\chi}=F^{n-1}\sum_{a=1}^{F}\chi(a)\mathbf{B}_n\left(\frac{a}{F}\right). \tag{1.3}$$
This is not quite obvious at first glance. Readers can find proof in [Was97, Proposition 4.1].
- For any positive integer $N\geq 1$, we have the "distribution relation" of Bernoulli polynomials as
$$\mathbf{B}_n(X)=N^{k-1}\sum_{a=0}^{N-1}\mathbf{B}_n\left(\frac{X+a}{N}\right). \tag{1.4}$$
Its proof can be found in [Lan90, p. 35]. The reason for its name will become clear in Section 3.

Then we record the special value of Dirichlet L-function at nonpositive integers.

Theorem 1.2. *For positive integers $n \geq 1$, $L(1-n,\chi) = -\frac{B_{n,\chi}}{n}$.*

Proof. The proof can be found in, for example, [Was97, Theorem 4.2]. $\square$

1.2. **Some p-adic analysis.** In this section, we develop the necessary tools in p-adic analysis, but without proofs.

Recall we are requiring that p is an odd prime. We have the following classical description for the units of $\mathbb{Z}_p$.

Proposition 1.3. *There is a decomposition $\kappa : \mathbb{Z}_p^\times \cong \Delta \times \Gamma$ induced by the split short exact sequence*

$$1 \to \Gamma \to \mathbb{Z}_p^\times \to \Delta \to 1,$$

where $\Gamma = 1 + p\mathbb{Z}_p$ and Δ is isomorphic to the maximal torsion subgroup $\boldsymbol{\mu}_{p-1}$ of $\mathbb{Z}_p^\times$, given by the group of $(p-1)$-th root of unity.

Under the isomorphism κ in 1.3, every $a \in \mathbb{Z}_p^\times$ can be written uniquely as $a = \omega(a)\langle a\rangle$ for some $\langle a\rangle \in \Gamma$ and $\omega(a) \in \boldsymbol{\mu}_{p-1}$. Then we have the fundamental property that $\omega(a) \equiv a \pmod{p}$. Therefore, by modding out $p\mathbb{Z}_p$, ω induces an isomorphism

$$\omega : (\mathbb{Z}/p)^\times = (\mathbb{Z}_p/p)^\times \to \boldsymbol{\mu}_{p-1}, \quad a \mod p \mapsto \omega(a).$$

By embedding $\boldsymbol{\mu}_{p-1}$ into $\mathbb{C}^\times$ via our fixed identification ι, we get a Dirichlet character $(\mathbb{Z}/p)^\times \to \mathbb{C}^\times$. We still denote it by ω and call it the **Teichmuller character**. One can show that it is an odd character.

1.2.1. *p-adic exponential and logarithm.* Set formally that

$$\exp(X) = \sum_{n=0}^{\infty} \frac{X^n}{n!}.$$

Then one can check that the series converges for $\{x \in \mathbb{C}_p : |x| < p^{-1/(p-1)}\}$. This is the **$p$-adic exponential**. The **p-adic logarithm** is defined again via a formal power series

$$\log_p(X+1) = \sum_{n=1}^{\infty} \frac{(-1)^n X^n}{n}.$$

Then it is easy to check that this series converges for $\{x \in \mathbb{C}_p : |x| < 1\}$. Many properties of exponential and logarithm comes formally from the properties of corresponding power series, so they still holds over $\mathbb{C}_p$, whenever the exponential and logarithm make sense.

Several properties are listed below.

- There exists a unique extension of $\log_p$ to all of $\mathbb{C}_p^\times$ such that $\log_p(p) = 0$ and $\log_p(xy) = \log_p(x) + \log_p(y)$ for all $x, y \in \mathbb{C}_p^\times$. This is [Was97, Proposition 5.4].
- If $|x| < p^{-1/(p-1)}$, then $\log_p \exp(x) = x$ and $\exp(\log_p(1+x)) = 1 + x$. Here one notices that the first identity is true whenever $\exp(x)$ converges, but the second one is not true for all x. This is [Was97, Proposition 5.7].

Now for $a \in \mathbb{Z}_p^\times$, recall we have set $\langle a \rangle = \omega(a)^{-1}a$. One can easily see that $\log_p(a) = \log_p(\langle a \rangle)$. Thus it makes sense to define for suitable $x \in \mathbb{C}_p$,

$$\langle a \rangle^x := \exp(x \log_p(a)) = \exp(x \log_p(\langle a \rangle)).$$

Then this converges if $|x| < p^{1-1/(p-1)}$.

1.2.2. *Binomial coefficients and Mahler's theorem.* We introduce the generalized binomial coefficient

$$\binom{X}{n} := \frac{X(X-1)\cdots(X-n+1)}{n!}.$$

Note that if $P(X) \in \mathbb{Q}_p(X)$ is any polynomial, then $x \mapsto P(x)$ is continous on $\mathbb{Z}_p$. Here as $\mathbb{N}$ is dense in $\mathbb{Z}_p$ and $\binom{m}{n} \in \mathbb{N}$ for all $\mathbb{N}$, we conclude that $\binom{x}{n} \in \mathbb{Z}_p$ for every $x \in \mathbb{Z}_p$.

A classical theorem of Mahler says that one can use binomial coefficients to interpolate continuous functions $f : \mathbb{Z}_p \to \mathbb{Q}_p$. More precisely, we have the following.

Theorem 1.4 (Mahler). *Any continuous function $f : \mathbb{Z}_p \to \mathbb{Q}_p$ can be written uniquely in the form*

$$f(X) = \sum_{n=0}^{\infty} a_n \binom{X}{n}, \quad a_n \to 0 \text{ as } n \to \infty.$$

The coefficients a_n for $n \geq 0$ in the above theorem are called **Mahler coefficients**. One can find its proof in [Hid93, Theorem 3.2.1].

Moreover, we can expect that if $a_n \to 0$ sufficiently rapidly, then $f(X)$ is analytic, in other words, $f(X)$ may be expanded in a power series. This is the following estimate that will be used in our construction.

Theorem 1.5. *Let $f : \mathbb{Z}_p \to \mathbb{Q}_p$ be a continuous function with Mahler coefficients a_n for $n \geq 0$. Suppose $r < p^{-1/(p-1)} < 1$ and for any $n \geq 0$, $|a_n| \leq Mr^n$ for some M. Then $f(X)$ may be expressed as a power series with radius of convergence at least $R = (rp^{1/(p-1)})^{-1} > 1$.*

Proof. It is proved in [Was97, Proposition 5.8]. □

1.3. **Von Staudt-Clausen congruences.** In the construction of Kubota-Leopoldt p-adic L-function, the following classical p-adic estimate on the Bernoulli numbers are used.

Theorem 1.6 (Von Staudt-Clausen). *Let n be even and positive. Then*

$$B_n + \sum_{p \in \mathbb{P},\, p-1 | n} \frac{1}{p} \in \mathbb{Z}.$$

Consequently, $pB_n \in \mathbb{Z}_p$ for all n and all p.

Proof. It is proved in [Was97, Theorem 5.10]. □

2. Kubota-Leopoldt p-adic L-function: as a p-adic meromorphic function

Recall we have fixed an identification $\iota_p : \mathbb{C} \to \mathbb{C}_p$, so Dirichlet characters can be viewed as taking value in $\mathbb{C}_p$.

The main theorem is the following.

Theorem 2.1. *Let χ be a Dirichlet character of conductor $\mathfrak{f}$. Then there exists a p-adic meromorphic (analytic if $\chi \neq \mathbb{1}$) function $\mathcal{L}_p(s, \chi)$ on $\{s \in \mathbb{C}_p : |s| < p^{1-1/(p-1)}\}$ such that*

$$\begin{aligned} \mathcal{L}_p(1-n, \chi) &= -(1 - \chi\omega^{-n}(p)p^{n-1})\frac{B_{n,\chi\omega^{-n}}}{n} \\ &= (1 - \chi\omega^{-n}(p)p^{n-1})L(1-n, \chi\omega^{-n}), n \geq 1. \end{aligned} \tag{2.5}$$

If $\chi = \mathbb{1}$, then $\mathcal{L}_p(s, \mathbb{1})$ is analytic except for a pole at $s = 1$ with residue $1 - 1/p$. In fact, let F be any multiple of p and f, then we have the formula

$$\mathcal{L}_p(s, \chi) = \frac{1}{F}\frac{1}{s-1}\sum_{a=1, p\nmid a}^{F} \chi(a) \langle a \rangle^{1-s} \sum_{j=0}^{\infty} \binom{1-s}{j} B_j \left(\frac{F}{a}\right)^j.$$

Here the second equality in (2.5) directly follows from the formula in Theorem 1.2. We see from (2.5) that $\mathcal{L}_p$ indeed interpolates the special value of Dirichlet L-functions, up to the removal of the Euler factor at p and a twist by Teichmuller character.

2.1. Hurwitz zeta function and its p-adic interpolation.

Before interpolating Dirichlet L-functions, we first start with a “patial” zeta function, namely the Hurwitz zeta function.

Definition 2.2. Let $s \in \mathbb{C}$ be a complex number with $\mathrm{Re}(s) > 1$, and for a real number $b \in (0, 1]$, we define the **Hurwitz zeta function** as

$$\zeta(s, b) = \sum_{n=0}^{\infty} \frac{1}{(b+n)^s}.$$

Let a, F be integers with $0 < a < F$, then we define

$$H(s, a, F) = \sum_{m \equiv a \bmod F, m>0} \frac{1}{m^s} = \sum_{m=0}^{\infty} \frac{1}{(a+nF)^s} = F^{-s}\zeta\left(s, \frac{a}{F}\right).$$

It can be viewed as a partial zeta function.

Similar to Theorem 1.2, we have the special value formula for Hurwitz zeta functions. It is proved in [Was97, Theorem 4.2].

Theorem 2.3. *For positive integers $n \geq 1$, $\zeta(1-n, b) = -\dfrac{\mathbf{B}_n(b)}{n}$ with $b \in (0, 1]$.*

Then we can p-adically interpolate them as follows.

Theorem 2.4. *Suppose that $1 \leq a \leq F$ and p is an odd prime such that $p \mid F$, $p \nmid a$. Then there exists a p-adic analytic function $\mathcal{H}_p(a, F, s)$ on $\{x \in \mathbb{C}_p : |x| < p^{1-1/(p-1)}\}$ except for a simple pole at $s = 1$ with residue $1/F$, such that*

$$\mathcal{H}_p(1-n, a, F) = \omega^{-n}(a)H(1-n, a, F).$$

Proof. Firstly, since $p \nmid a$, $a \in \mathbb{Z}_p^\times$ and $\langle a \rangle$ is well-defined in $1 + p\mathbb{Z}_p$. Now we set

$$\mathcal{H}_p(s,a,F) := \frac{1}{s-1}\frac{1}{F}\langle a \rangle^{1-s} \sum_{j=0}^{\infty} \binom{1-s}{j} B_j \left(\frac{F}{a}\right)^j .$$

We assume the convergence issue for the moment. Then

$$\begin{aligned}
\mathcal{H}_p(1-n,a,F) &= -\frac{1}{nF}\langle a \rangle^n \sum_{j=0}^{\infty}\binom{n}{j} B_j \left(\frac{F}{a}\right)^j \\
&= -\frac{1}{nF}\langle a \rangle^n \left(\frac{a}{F}\right)^{-n} \sum_{j=0}^{\infty}\binom{n}{j} B_j \left(\frac{a}{F}\right)^{n-j} \\
&= -\frac{F^{n-1}\omega^{-n}(a)}{n}\mathbf{B}_n\left(\frac{a}{F}\right) \\
&= \omega^{-n}(a) H(1-n,a,F).
\end{aligned}$$

Here in the third equality, we used (1.2) and that $a = \omega(a)\langle a \rangle$, and in the last equality, we applied the special value formula in Theorem 2.3. Moreover, at $s = 1$, we have residue

$$\lim_{s \to 1} \mathcal{H}_p(s,a,F) = \frac{1}{F}\langle a \rangle^0 \sum_{j=0}^{\infty}\binom{0}{j} B_j \left(\frac{F}{a}\right)^j = \frac{1}{F}.$$

It remains to prove the convergence.

- Firstly, $\langle a \rangle^{1-s}$ is analytic in $\{s \in \mathbb{C}_p : |1-s| < p^{1-1/(p-1)}\}$. Since $p^{1-1/(p-1)} > 1$, this is the same set as $D := \{s \in \mathbb{C}_p : |s| < p^{1-1/(p-1)}\}$.
- The more challenging part is the analyticity of $\sum \binom{1-s}{j} B_j (F/a)^j$. We have that $\left|B_j (F/a)^j\right| \leq |B_j|\,|F|^j$, and by Theorem 1.6, $pB_j \in \mathbb{Z}_p$. Thus, $|pB_j| \leq 1$. This implies that $|B_j| \leq 1/|p| = p$. Moreover, by hypothesis, $p \mid F$ and $p \nmid a$. Therefore, $\left|B_j (F/a)^j\right| \leq p \cdot p^{-j}$. Then we apply Theorem 1.5 with $M = p$ and $r = p^{-1} < p^{-1/(p-1)}$, we get $R = p^{1-1/(p-1)} > 1$.

Thus, $\mathcal{H}_p(s,a,F)$ is p-adic analytic on D except for a simple pole at $s = 1$ with residue $1/F$. □

2.2. **Construction of Kubota-Leopoldt p-adic L-function.** In this section we give the proof of Theorem 2.1, namely the construction of Kubota-Leopoldt p-adic L-function.

Fix an F such that $\mathfrak{f} \mid F$ and $p \mid F$. In particular we write $F = pt$ with $t \geq 1$.

2.2.1. *Proof: the analyticity and the residue at $s = 1$.* By the definition of $\mathcal{H}_p(s,a,F)$ given in the previous proof, we have that

$$\mathcal{L}_p(s,\chi) = \sum_{a=1,\, p\nmid a}^{F} \chi(a)\mathcal{H}_p(a,F,s).$$

This implies that $\mathcal{L}_p(s,\chi)$ is p-adic analytic on D except possibly for a simple pole at $s = 1$. Now by Theorem 2.4, we see that at $s = 1$, $\mathcal{L}_p(s,\chi)$ has residue $\sum_{a=1,\,p\nmid a}^{F} \chi(a)/F$. Then there are two cases: χ is trivial or not.

- Suppose $\chi = \mathbb{1}$. Then this sum is equal to $1 - 1/p$. Indeed, now $\mathfrak{f} = 1$ so $p \nmid \mathfrak{f}$. Then notice that a runs through exactly $(p-1)t$ values such that $p \nmid a$. Thus the sum is clearly equal to $(p-1)t \cdot \frac{1}{pt} = 1 - 1/p$.
- Suppose $\chi \nmid \mathbb{1}$. Then the same sum is equal to
$$\frac{1}{F}\sum_{a=1}^{F}\chi(a) - \frac{1}{F}\sum_{b=1}^{F/p}\chi(pb).$$
However the first sum is zero since $\sum_{a=1}^{\mathfrak{f}}\chi(a) = 0$ and F is a multiple of $\mathfrak{f}$. For the second sum, we see that if $p \mid \mathfrak{f}$, then $\chi(pb) = 0$ for all b. If otherwise $p \nmid \mathfrak{f}$, then $\mathfrak{f} \mid F/p$, so also the second sum is zero by the previous argument. Indeed, one notice that as b runs through 1 to F/p, pb runs through an entire period of χ as $\mathfrak{f} \mid F/p$.

To conclude, $\mathcal{L}_p(s,\chi)$ has no pole at $s = 1$ if χ is not trivial.

2.2.2. *Proof: the interpolation property at nonpositive integers.* Let $n \geq 1$, then

$$\begin{aligned}
\mathcal{L}_p(1-n,\chi) &= \sum_{a=1,\,p\nmid a}^{F}\chi(a)\mathcal{H}_p(1-n,a,F)\\
&= \sum_{a=1,\,p\nmid a}^{F}\chi(a)\omega^{-n}(a)H(1-n,a,F)\\
&= \sum_{a=1,\,p\nmid a}^{F}\chi(a)\omega^{-n}(a)\frac{-F^{n-1}\mathbf{B}_n(a/F)}{n}\\
&= -\frac{1}{n}F^{n-1}\sum_{a=1,\,p\nmid a}^{F}\chi\omega^{-n}(a)\mathbf{B}_n\left(\frac{a}{F}\right)\\
&= \underbrace{-\frac{1}{n}F^{n-1}\sum_{a=1}^{F}\chi\omega^{-n}(a)\mathbf{B}_n\left(\frac{a}{F}\right)}_{:=S_1} + \underbrace{\frac{1}{n}p^{n-1}\left(\frac{F}{p}\right)^{n-1}\sum_{b=1}^{F/p}\chi\omega^{-n}(pb)\mathbf{B}_n\left(\frac{b}{F/p}\right)}_{:=S_2}
\end{aligned}$$

Now by (1.3), we see the sum S_1 is equal to $-\frac{1}{n}B_{n,\chi\omega^{-n}}$. For the sum S_2, if $p \mid \mathfrak{f}_{\chi\omega^{-n}}$, then $\chi\omega^{-n}(pb) = 0$ and so we obtain directly the formula of the statement. Otherwise $\mathfrak{f}_{\chi\omega^{-n}}$ divides F/p. Again by (1.3) we have S_2 is equal to $\frac{1}{n}\chi\omega^{-n}(p)p^{n-1}B_{n,\chi\omega^{-n}}$. To conclude, we obtain
$$\mathcal{L}_p(1-n,\chi) = S_1 + S_2 = (1-\chi\omega^{n}(p)p^{n-1})\frac{-B_{n,\chi\omega^{-n}}}{n}.$$
This completes the proof. □

2.2.3. *Remark: Vanishing for odd characters.* It is frustrating to note that if χ is an odd character (hence $\chi \neq \mathbb{1}$), then n and the character $\chi\omega^{-n}$ have different parities. So by (1.1), we see that $B_{n,\chi\omega^{-n}}$ is identically zero for any $n \geq 1$. Then since $\mathcal{L}_p(s,\chi)$ is analytic, we see that $\mathcal{L}_p(s,\chi)$ is identically zero for all odd χ. We will give an explanation of this problem in Section 4.1.1.

3. Kubota-Leopoldt p-adic L-function: as a p-adic measure

3.1. **Generalities on p-adic measures.** We first introduce the notion of distributions, and a measure is nothing but a bounded distribution, which enables us to do integration against continuous functions.

3.1.1. *Distributions.* There are three different formulations of distributions.

Let I be a directed partially ordered set, i.e. for each $i, j \in I$, there is a $k \in I$ such that $k \geq i$ and $k \geq j$. Let $\{X_i : i \in I\}$ be a collection of finite sets, and if $i \geq j$, we assume that there is a surjective map $\pi_{ij} : X_i \to X_j$ such that $\pi_{ij} \circ \pi_{jk} = \pi_{ik}$ whenever $i \geq j \geq k$. So we can form the set $X := \varprojlim_n X_n$. Let W be an abelian group. Note that X is compact since each X_i is finite.

Suppose that for each $i \in I$ we have a function $\phi_i : X_i \to W$ such that if $i \geq j$,

$$\phi_j(x) = \sum_{\pi_{ij}(y)=x} \phi_j(y).$$

The collection of maps $\{\phi_i\}_{i \in I}$ is called a **W-valued distribution on X**. We denote $\mathrm{Dist}(X, W)$ as the set of W-valued distribution on X.

There is a second equivalent definition of distributions. Let ϕ be a finitely additive function on the collection of compact-open subsets of X, with value in W. By "finitely additive function", we mean a function ϕ on the collection of compact-open subsets of X, such that for any finite collection of *disjoint* compact-open subsets $\{U_k\}_{k=1}^n$,

$$\phi\left(\bigsqcup_{k=1}^n U_k\right) = \sum_{k=1}^n \phi(U_k).$$

We shall show that such a ϕ gives raise to a distribution. Since each π_{ij} is surjective, for each i there is a surjective map $\pi : X \to X_i$. If $a \in X_i$, then $\pi_i^{-1}(a)$ is a compact-open subset of X, and all compact-open subsets are obtained as finite unions of such $\pi_i^{-1}(a)$, as i and a vary. In fact, these sets form a basis for the topology of X. Suppose $b \in X_j$. For $i \geq j$,

$$\pi_j^{-1}(b) = \bigsqcup_{a \in X_i,\, \pi_{ij}(a)=b} \pi_i^{-1}(a), \qquad \text{a disjoint union.}$$

Therefore,

$$\phi(\pi_j^{-1}(b)) = \sum_{\pi_{ij}(a)=b} \phi(\pi_i^{-1}(a)),$$

so $b \mapsto \phi(\pi_j^{-1}(b))$ satisfies the distribution relation, giving a distribution $\{\phi_i\}_{i \in I}$ on X. Conversely, any distribution on X gives a finitely additive function on compact-open sets of X.

Finally we give a third formulation of distributions. Let K be a field extension of $\mathbb{Q}_p$ in $\mathbb{C}_p$. A function $f : X \to K$ is called a **step function** if for each $x \in X$, there is a neighborhood U of x such that f is constant on U. Since X is compact, this means that f is a finite linear combination of characteristic functions of disjoint compact-open sets. In fact, f is a finite linear combination of characteristic functions of sets of the form $\pi_i^{-1}(a)$.

We call these characteristic functions $\mathbb{1}_{i,a}$. Let $\mathrm{Step}(X, K)$ be the set of step functions on X.

Then given a finitely additive function on compact-opens with values in W, we may extend ϕ linearly to obtain a linear functional

$$\phi : \mathrm{Step}(X, K) \to W, \quad f \mapsto \phi(f) =: \int_X f \, \mathrm{d}\phi$$

More precisely, if $\{\phi_i\}_{i \in I}$ is the associated distribution, then $\phi(\mathbb{1}_{i,a}) = \phi_i(a)$. Conversely, a linear function on $\mathrm{Step}(X, K)$ may be restricted to characteristic functions to yield a finitely additive function on compact-opens.

Here are some examples.

Example 3.1 (Dirac distribution)**.** Let I be then positive integers with the usual ordering and let $X_i = \mathbb{Z}/p^i$, with π_{ij} the obvious map. Then $X \cong \mathbb{Z}_p$. Fix $a \in \mathbb{Z}_p$, let

$$\delta_{a,i}(y) = \begin{cases} 1, & \text{if } y \equiv a \bmod p^i, \\ 0, & \text{otherwise.} \end{cases}$$

Then $\{\delta_{a,i}\}$ forms a distribution, called the Dirac distribution. Moreover, $\{\delta_{a,i}\}$ gives a function on the compact open subsets of $\mathbb{Z}_p$. Let $U \subset \mathbb{Z}_p$ be a compact-open subset, then

$$\delta_a(U) = \begin{cases} 1, & \text{if } a \in U, \\ 0, & \text{if } a \notin U. \end{cases}$$

Indeed, $\pi_i^{-1}(y \bmod p^i) = y + p^i\mathbb{Z}_p$. If $f \in \mathrm{Step}(X)$, we have $\delta_a(f) = f(a)$. This is exactly how the classical delta function acts.

Example 3.2 (Haar measure)**.** There is a natural function on compact-opens of $\mathbb{Z}_p$, namely $\phi(U) = \mathrm{vol}(U)$, where $\mathrm{vol}(-)$ is the Haar measure normalized by $\mathrm{vol}(\mathbb{Z}_p) = 1$. We have $\phi(y + p^i\mathbb{Z}_p) = p^{-i}$, so the associated distribution satisfies $\phi_i(y \bmod p^i) = p^{-i}$. However, this Haar distribution will not be very useful to us when we develop p-adic integration. As $y + p^i\mathbb{Z}_p$ get smaller, that is, $i \to \infty$, their Haar measures becomes p-adically larger. This is not desirable since a small change in a function could produce a large change in its integral. We will exclude such distributions this later.

3.1.2. *Measures.* In this section, we let W be a Banach space over a field K, where K is a finite extension of $\mathbb{Q}_p$ in $\mathbb{C}_p$. For example, W can be $\mathbb{C}_p$, or the space of p-adic modular forms $M^\dagger$, as we shall introduce in Section 4.

We say a distribution $\phi \in \mathrm{Dist}(X, W)$ is a **measure**, if there exists a constant M such that $|\phi_i(a)| \leq M$ for all i and all $a \in X_i$. The set of measures is denoted by $\mathrm{Meas}(X, W)$.

Let $\mathcal{C}(X, K)$ be the K-Banach space of continuous K-valued functions on X with norm $||f|| := \sup_{x \in X} |f(x)|$. Then $\mathrm{Step}(X, K)$ is dense in $\mathcal{C}(X, K)$.

Proposition 3.3. *Let $\phi : \mathrm{Step}(X, K) \to W$ be a measure, then it extends uniquely to a continuous K-linear map $\phi : \mathcal{C}(X, K) \to W$.*

Proof. Since the step functions are dense, the map must be unique if it exists. Observe that if M is the constant used above and $\mathbb{1}_{i,a}$ is the characteristic function of the previous section,

$$\left|\int_X \mathbb{1}_{i,a}\,\mathrm{d}\phi\right| = |\phi_i(a)| \leq M.$$

Since the absolute value is non-archimedean,

$$\left|\int_X f\,\mathrm{d}\phi\right| \leq M\,||f||\,, \quad f \in \mathrm{Step}(X,K).$$

If $g \in \mathcal{C}(X,K)$ and $\{g_n\}$ is a Cauchy sequence in $\mathrm{Step}(X,K)$ converging to g, then

$$\left|\int_X g_n\,\mathrm{d}\phi - \int_X g_m\,\mathrm{d}\phi\right| \leq K\,||g_n - g_m|| \to 0,$$

as $m, n \to \infty$. Therefore, let

$$\int_X g\,\mathrm{d}\phi = \lim_{n\to\infty}\int_X g_n\,\mathrm{d}\phi. \tag{3.6}$$

This has desired properties, so the proof is complete. □

For example, let $X = \mathbb{Z}_p = \varprojlim_n \mathbb{Z}/p^n$ as usual, and $f \in \mathcal{C}(\mathbb{Z}_p, \mathbb{C}_p)$. Then we can integrate f by approximating f on $a + p^m\mathbb{Z}_p$ following (3.6), as

$$\int_{\mathbb{Z}_p} f\,\mathrm{d}\phi = \lim_{n\to\infty}\sum_{a=0}^{p^n-1} f(a)\phi(a + p^m\mathbb{Z}_p). \tag{3.7}$$

This can be regarded as a "Riemann sum" of the integration. In practice, X can also be several copies of $\mathbb{Z}_p$.

Remark. Let ϕ be a measure on X and U be a compact-open subset of X. Then for $f \in \mathcal{C}(U,K)$, we extend it to X by zero on $X \smallsetminus U$, denoted by $f^!$ then we usually write

$$\int_U f\,\mathrm{d}\phi := \int_X f^!\,\mathrm{d}\phi.$$

3.1.3. *Distributions and measures on profinite groups.* In this subsection, we consider the case when

- X is a profinite group. A typical example is $X = \mathbb{Z}_p$ as seen in Example 3.1 and 3.2, and
- let $K = W = \mathbb{C}_p$ and $\mathcal{O} \subseteq \mathbb{C}_p$ be a p-adic integer ring, then we consider the **$\mathcal{O}$-valued measures** on X as a linear functional on $\mathrm{Step}(X, \mathbb{C}_p)$ with values in $\mathcal{O}$. Note that $\mathcal{O}$ is bounded, such a linear functional is automatically bounded and hence a measure. So according to Proposition 3.3, it can be extended to $\mathcal{C}(X, \mathbb{C}_p)$.

We write $X = \varprojlim_j X/X_j$ with open subgroups X_j of X. Hence they are the finite finite index, normal subgroups of X. Then for any p-adic integer ring $\mathcal{O}$, we have an isomorphism of $\mathcal{O}$-algebras

$$\gamma : \mathrm{Meas}(X,\mathcal{O}) \to \Lambda_X := \mathcal{O}[\![X]\!], \quad \phi \mapsto \gamma_\phi := \left(\sum_{g\in X/X_j}\phi_i(g)e_g\right)_{j\geq 0}.$$

Here Λ_X is the **Iwasawa algebra** of X, namely the completed group algebra

$$\Lambda_X := \mathcal{O}[\![X]\!] := \varprojlim_j \mathcal{O}[X/X_j],$$

with $\mathcal{O}[X/X_j]$ the group algebra of the finite group X/X_j, i.e.

$$\mathcal{O}[X/X_j] = \left\{ \sum_{g \in X/X_j} a_g e_g : a_g \in \mathcal{O} \right\},$$

with formal symbols e_g as a basis. Moreover, $\mathrm{Meas}(X, \mathcal{O})$ has a structure of $\mathcal{O}$-algebra given by convolution of measures.

In practice, we care most about the case when X is d copies of $\mathbb{Z}_p$ for $d \geq 1$. Then Λ_X is isomorphic to the power series ring $\mathcal{O}[\![T_1, \dots, T_d]\!]$. In this case, we can regard measures $\mathrm{Meas}(X, \mathcal{O})$ as formal power series. This fits into the context of classical Iwasawa theory. In fact, there is a construction of Kubota-Leopoldt p-adic L-function by Iwasawa by giving a compactible sequence of elements $\Xi_n(\chi) \in \mathcal{O}[\mathbb{Z}_p/p^n\mathbb{Z}_p]$, called **Stickelberger elements**. One can turn to [Was97, Chapter 7] for his construction. The main technical preparation is Proposition 3.5 that we will quote later.

3.2. **Recognize the Kubota-Leopoldt p-adic L-function.** Recall in our setup, χ is a Dirichlet character of conductor $p^n M$ with $p \nmid M$ and $n \geq 0$. Let $\mathbb{Z}_{p,M} = \mathbb{Z}/M \times \mathbb{Z}_p$ and $\mathbb{Z}_{p,M}^\times = (\mathbb{Z}/M)^\times \times \mathbb{Z}_p^\times$, so χ is a character on $\mathbb{Z}_{p,M}$.

3.2.1. *Bernoulli distribution.* As a first attempt, for positive integers k, we define a family of distributions $\mathcal{B}_k$ on $\mathbb{Z}_{p,M}$ by assigning

$$\mathcal{B}_k(a + p^m M\mathbb{Z}_{p,M}) = -\frac{(p^m M)^{k-1}}{k} \mathbf{B}_k\left(\left\{\frac{a}{p^m M}\right\}\right),$$

where $\{\frac{a}{p^m M}\}$ is the fractional part of $\frac{a}{p^m M}$. We check that it is indeed a distribution on $\mathbb{Z}_{p,M}$, i.e. the finitely additivity holds:

$$\mathcal{B}_k(a + p^m M\mathbb{Z}_{p,M}) = \sum_{b=0}^{p-1} \mathcal{B}_k(a + p^m bM + p^{m+1} M\mathbb{Z}_{p,M}) \tag{3.8}$$

for any $a \in \mathbb{Z}/p^m M$ and $m \geq 0$. In fact, to verify this, we choose a distinguished integral representatives $a \in \{0, 1, \ldots, p^m M - 1\}$ for $\mathbb{Z}/p^m M$, then the right hand side expands as

$$\begin{aligned}
\text{Right hand side of (3.8)} &= -\frac{(p^{m+1}M)^{k-1}}{k} \sum_{b=0}^{p-1} \mathbf{B}_k\left(\left\{\frac{a+p^m bM}{p^{m+1}M}\right\}\right) \\
&= -\frac{(p^{m+1}M)^{k-1}}{k} \sum_{b=0}^{p-1} \mathbf{B}_k\left(\frac{\frac{a}{p^m M}+b}{p}\right) \\
&= -\frac{(p^{m+1}M)^{k-1}}{k} \frac{1}{p^{k-1}} \mathbf{B}_k\left(\frac{a}{p^m M}\right) \\
&= -\frac{(p^m M)^{k-1}}{k} \mathbf{B}_k\left(\left\{\frac{a}{p^m M}\right\}\right) \\
&= \text{Left hand side of (3.8)}
\end{aligned}$$

Here the second and the fourth equality follow from our choice of representatives of a, which does not matter. The third equality follows from (1.4). So (3.8) holds, and therefore $\mathcal{B}_k \in \mathrm{Dist}(\mathbb{Z}_{p,M}, \mathbb{Q}_p)$. This is called the **$k$-th Bernoulli distribution**.

However, $\mathcal{B}_k$ is not bounded, because $\mathbf{B}_k(\frac{a}{p^m M})$ is p-adically large. To get a measure, we need a modification.

3.2.2. *Kubota-Leopoldt p-adic L-function.* Let $c \in \mathbb{Z}$ such that $(c, pM) = 1$, then $c \in \mathbb{Z}_{p,M}^\times$ in a natural way, so let c^{-1} be its inverse in $\mathbb{Z}_{p,M}^\times$. We define

$$\mathcal{E}_{k,c}(a + p^m M\mathbb{Z}_{p,M}) = \mathcal{B}_k(a + p^m M\mathbb{Z}_{p,M}) - c^k \mathcal{B}_k(c^{-1}a + p^m M\mathbb{Z}_{p,M}).$$

Then $\mathcal{E}_{k,c}$ is clearly a distribution by that of the Bernoulli distribution. Moreover, $\mathcal{E}_{k,c}$ is bounded. In fact, for each $m \geq 0$, there is a unique integer $0 \leq a'_m < p^m M$ such that $c^{-1}a \equiv a'_m \pmod{p^m M}$. Then

$$\begin{aligned}
\mathcal{E}_{k,c}(a + p^m M\mathbb{Z}_{p,M}) &= -\frac{(p^m M)^{k-1}}{k}\left(\mathbf{B}_k\left(\frac{a}{p^m M}\right) - c^k \mathbf{B}_k\left(\frac{a'_m}{p^m M}\right)\right) \\
&= -\frac{1}{k}\sum_{i=0}^{k}\binom{k}{i} B_i (p^m M)^{i-1}(a^{k-i} - c^k (a'_m)^{k-i}).
\end{aligned}$$

Here we used (1.2). Then $i \neq 0$ terms are p-adically bounded as m gets large. For $i = 0$ term, that is $\frac{1}{p^m M}(a^k - c^k (a'_m)^k)$, we note that

$$a^k - c^k (a'_m)^k \equiv 0 \mod p^m M.$$

So this term is also p-adically bounded for m large. Therefore, $\mathcal{E}_{k,c}$ is indeed a measure.

Here to simplify our computation, from now on we only treat the case $k = 1$ and write $\mathcal{E}_c := \mathcal{E}_{1,c}$. Then as $\mathbf{B}_1(X) = X - \frac{1}{2}$, we see

$$\mathcal{E}_c(a + p^m M\mathbb{Z}_{p,M}) = \frac{a}{p^m M} - c\left\{\frac{c^{-1}a}{p^m M}\right\} + \frac{c-1}{2}.$$

The key is that $\mathcal{E}_c$ is nothing but our previously constructed Kubota-Leopoldt p-adic L-function, in the following sense.

Theorem 3.4. *Let χ be a Dirichlet character of conductor $\mathfrak{f} = p^n M$ with $p \nmid M$. Then for positive integers t,*

$$\int_{\mathbb{Z}_{p,M}^\times} \chi\omega^{-1}(a) \langle a \rangle^{t-1} \, \mathrm{d}\mathcal{E}_c = -(1 - \chi(c) \langle c \rangle^t)(1 - \chi\omega^{-t}(p)p^{t-1})L(1-t, \chi\omega^{-t}).$$

So in this sense, we can view the measure $\mathcal{E}_c$ *itself* as a p-adic interpolation of special values of the Dirichlet L-function for χ.

To prove Theorem 3.4, we need the following estimate.

Proposition 3.5. *Let $q_n := p^{n+1}M$ and m be a positive integer, then*

$$\lim_{n\to\infty} \frac{1}{q_n} \sum_{0<a<q_n, (a,q_0)=1} \chi\omega^{-m}(a)a^m = (1 - \chi\omega^{-m}(p)p^{m-1})B_{m,\chi\omega^{-m}}.$$

Proof. The proof can be found in [Was97, Lemma 7.11]. □

Proof of Theorem 3.4. To compute the integral on the left hand side, we approximate it by Riemann sums, as in (3.7),

$$\begin{aligned}\int_{\mathbb{Z}_{p,M}^\times} \chi\omega^{-1}(a) \langle a \rangle^{t-1} \, \mathrm{d}\mathcal{E}_c &= \lim_{m\to\infty} \sum_{b=0,\, b\nmid p}^{p^mM-1} \chi\omega^{-1}(b) \langle b \rangle^{t-1} \mathcal{E}_c(b + p^m M\mathbb{Z}_{p,M}) \\ &= \lim_{m\to\infty} \sum_{b=0,\, b\nmid p}^{p^mM-1} \chi\omega^{-t}(b)b^{t-1} \left(\frac{b}{p^mM} - c \left\{ \frac{c^{-1}b}{p^mM} \right\} + \frac{c-1}{2} \right)\end{aligned}$$

By Proposition 3.5, the term with $(c-1)/2$ tends to zero as $m \to \infty$, so we may ignore it. By the same proposition,

$$\lim_{m\to\infty} \frac{1}{p^mM} \sum_{b=0,\, b\nmid p}^{p^mM-1} \chi\omega^{-t}(b)b^t = (1 - \chi\omega^{-t}(p)p^{t-1})B_{t,\chi\omega^{-t}} =: U.$$

The remaining term is the hardest to evaluate. We first do some preparation work. Let

$$c^{-1}b = b_1 + p^m M b_2, \quad \text{with } 0 \le b_1 \le p^m M.$$

Note that

$$\chi\omega^{-t}(b) = \chi\omega^{-t}(c)\chi\omega^{-t}(b_1), \quad \text{if } n \ge m \tag{3.9}$$

and

$$b^t \equiv c^t(b_1 + p^m M b_2)^t \equiv c^t(b_1^t + tp^m M b_2 b_1^{t-1}) \bmod p^{2n}. \tag{3.10}$$

Therefore,

$$\begin{aligned}\sum_{b=0,\, b\nmid p}^{p^mM-1} \chi\omega^{-t}(b)b^t \equiv{} & \chi\omega^{-t}(c)c^t \sum_{b=0,\, b\nmid p}^{p^mM-1} \chi\omega^{-t}(b_1)b_1^t \\ & + tp^m M\chi\omega^{-t}(c)c^t \sum_{b=0,\, b\nmid p}^{p^mM-1} \chi\omega^{-t}(b_1)b_2 b_1^{t-1}.\end{aligned}$$

But b_1 runs through the same values as b, in a different order. Consequently, we obtain

(3.11)
$$\chi\omega^{-t}(c)c^t \sum_{b=0,\, b\nmid p}^{p^m M-1} \chi\omega^{-t}(b_1)b_2 b_1^{t-1} \equiv (1-\chi\omega^{-t}(c)c^t)\frac{1}{tp^m M}\sum_{b=0,\, b\nmid p}^{p^m M-1} \chi\omega^{-t}(b)b^t \bmod \frac{p^n}{k}.$$

The remaining term in the original sum involves $c\{c^{-1}b/p^m M\} = cb_1/p^m M$. We have modulo p^n,

$$-\frac{c}{p^m M}\sum_{b=0,\, b\nmid p}^{p^m M-1} \chi\omega^{-t}(b)b^{t-1}b_1 \equiv -\frac{c}{p^m M}\sum_{b=0,\, b\nmid p}^{p^m M-1} \chi\omega^{-t}(b)c^{t-1}(b_1^t + (t-1)p^m M b_2 b_1^{t-1})$$
$$\equiv -\frac{c^t}{p^m M}\sum_{b=0,\, b\nmid p}^{p^m M-1} \chi\omega^{-t}(b_1)b_1^t - (t-1)c^t\chi\omega^{-t}(c)\sum_{b=0,\, b\nmid p}^{p^m M-1} \chi\omega^{-t}(b_1)b_2 b_1^{t-1}.$$

Here the first $\equiv$ follows from (3.10) with t replaced by $t-1$ and the second $\equiv$ follows from (3.9). By Proposition 3.5, taking $m\to\infty$, the first term yields

$$-\chi\omega^{-t}(c)(1-\chi\omega^{-t}(p)p^{t-1})B_{t,\chi\omega^{-t}} =: V.$$

The second term is congruent modulo $(1/k)p^n$ to

$$-(t-1)(1-\chi\omega^{-t}(c)c^t)\frac{1}{tp^m M}\sum_{b=0,\, b\nmid p}^{p^m M-1} \chi\omega^{-t}(b)b^t$$
$$\to -\frac{t-1}{t}(1-\chi\omega^{-t}(c)c^t)(1-\chi\omega^{-t}(p)p^{t-1})B_{t,\chi\omega^{-t}} =: W$$

as $m\to\infty$, using (3.11). Addition of the relevant terms shows that the original sum approximating the integral becomes, as $n\to\infty$,

$$U+V+W = (1-\chi\omega^{-t}(c)c^t)(1-\chi\omega^{-t}(p)p^{t-1})\frac{B_{t,\chi\omega^{-t}}}{t},$$

which equals to $-(1-\chi\omega^{-t}(c)c^t)(1-\chi\omega^{-t}(p)p^{t-1})L(1-kt,\chi\omega^{-t})$. This completes the proof. □

3.2.3. *Remark: Comparing with the previous construction.* Comparing with Theorem 2.1, we see that for positive integers t,

$$\int_{\mathbb{Z}_{p,M}^\times} \chi\omega^{-1}(a)\langle a\rangle^{t-1}\, \mathrm{d}\mathcal{E}_c = -(1-\chi(c)\langle c\rangle^t)\mathcal{L}_p(1-t,\chi).$$

Put $t=s+1$, we rewrite it as

$$\int_{\mathbb{Z}_{p,M}^\times} \chi\omega^{-1}(a)\langle a\rangle^{s}\, \mathrm{d}\mathcal{E}_c = -(1-\chi(c)\langle c\rangle^t)\mathcal{L}_p(s,\chi).$$

It is shown in [Was97, Corollary 12.5] that the left hand side is actually analytic in $s\in\mathbb{Z}_p$. Since we have established the equality for all $s\geq 0$, the equality holds for all $s\in\mathbb{Z}_p$, because such an analytic function must have only finitely many zeros in $\{x\in\mathbb{C}_p : |x|<1\}$, by [Was97, Corollary 7.4].

3.3. **Application: Kummer congruence.** As an application of the Kubota-Leopoldt p-adic L-function, we prove the following.

Corollary 3.6 (Kummer congruence). *If $m \equiv n \bmod p^{b-1}(p-1)$ and $m \not\equiv 0 \bmod p-1$, then*
$$(1-p^{m-1})\frac{B_m}{m} \equiv (1-p^{n-1})\frac{B_n}{n}, \quad \bmod p^b.$$

Proof. We put $\chi = \omega^m$, $M = 1$ and $t = m$. Then
$$(1-c^m)(1-p^{m-1})\frac{B_m}{m} = \int_{\mathbb{Z}_p^\times} a^{m-1}\, \mathrm{d}\mathcal{E}_c.$$
Since $\mathcal{E}_c$ is $\mathbb{Z}_p$-valued, and $a^{m-1} \equiv a^{n-1} \bmod p^b$,
$$\int_{\mathbb{Z}_p^\times} a^{m-1}\, \mathrm{d}\mathcal{E}_c \equiv \int_{\mathbb{Z}_p^\times} a^{n-1}\, \mathrm{d}\mathcal{E}_c \bmod p^b.$$
Also, $1-c^m \equiv 1-c^n \bmod p^b$. Choose c such that $c^m \not\equiv 1 \bmod p$. The result now follows easily, since m and n are interchangable. □

This Kummer congruence relation has independent proofs not using the p-adic L-function, for example, see [IK14, Theorem 3.2]. Historically, this "higher congruences relation" inspired Swinnerton-Dyer and Serre to discover the "p-adic modular forms". We will introduce such notions in the next section.

4. Kubota-Leopoldt p-adic L-function: as a function on the weight space

Based on the calculations in the previous sections, we can regard the Kubota-Leopoldt p-adic L-function as a function on the p-adic weight space.

4.1. **Weight spaces.** Let $\mathcal{X}$ be the set of all continuous homomorphisms $\psi : \mathbb{Z}_p^\times \to \mathbb{C}_p^\times$. It is called the **$p$-adic weight space**. Recall that there is a canonical decomposition $\mathbb{Z}_p^\times \cong (\mathbb{Z}/p)^\times \times (1+p\mathbb{Z}_p)$, we can sort out two kinds of characters: a character $\phi \in \mathcal{X}$ is called a **wild character** if it is trivial on $(\mathbb{Z}/p)^\times$, and is called a **tame character** if it is trivial on $1+p\mathbb{Z}_p$. Then every character $\psi \in \mathcal{X}$ can be uniquely written as $\psi = \psi_{\mathrm{t}}\psi_{\mathrm{w}}$ with ψ_{t} tame and ψ_{w} wild.

Obviously there are only $p-1$ distinct tame characters. Recall the Teichmuller character
$$\omega : (\mathbb{Z}/p)^\times \to \mathbb{Z}_p^\times \hookrightarrow \mathbb{C}_p^\times$$
is a tame character of order $p-1$, we see $\{\omega^u : u = 0, \dots, p-2\}$ are all the possible tame characters.

Wild characters are also easy to characterise, since $1+p\mathbb{Z}_p$ is topologically generated by a single element γ, so the character is determined by the image of γ. More precisely, for $z \in B := \{z \in \mathbb{C}_p : |z-1|_p < 1\}$, we define a wild character
$$\psi_z : \mathbb{Z}_p^\times \to 1+p\mathbb{Z}_p \to \mathbb{C}_p^\times,$$

where the first map sends the x to $\langle x \rangle$ and the second map sends the topological generator to z. Since the continuity of a character ψ requires that $|\psi(\gamma)-1|_p < 1$, the set $\{\psi_z \in \mathcal{X} : z \in B\}$ is the set of all wild characters. Therefore, the following proposition holds.

Proposition 4.1. *The weight space $\mathcal{X}$ can be identified with a disjoint union of $p-1$ copies of the open unit disk $B = \{z \in \mathbb{C}_p : |z-1|_p < 1\}$.*

We have another characterizations of wild characters. Let $s \in \mathbb{Z}_p$ and consider the character $\langle - \rangle^s : 1 + p\mathbb{Z}_p \to \mathbb{C}_p^\times$. Then it can be regarded as the wild character γ^s. Conversely, for any $z \in B$, let $s = \frac{\log_p(z)}{\log_p(\gamma)}$, then $s \in \mathbb{Z}_p$ and $\psi_z = \langle - \rangle^s$. To conclude, every character $\psi \in \mathcal{X}$ can be uniquely written as $\omega^i \langle - \rangle^s$ for $i = 0, \dots, p-1$ and $s \in \mathbb{Z}_p$.

4.1.1. *Reformulate the Kubota-Leopoldt p-adic L-function: as a p-adic meromorphic function.* Recall we have constructed a p-adic L-function $\mathcal{L}_p(s,\chi)$ on $\{s \in \mathbb{C}_p : |s| < p^{1-1/(p-1)}\}$ interpolating the critical values of the Dirichlet L-function. Now we define

$$\mathcal{L}_{p,\chi} : \mathcal{X} \to \mathbb{C}_p^\times, \quad \omega^u \langle - \rangle^s \mapsto \mathcal{L}_p(-s, \chi\omega^u).$$

Geometrically, $\mathcal{L}_{p,\chi}$ has $p-1$ branches as u runs over $u = 0, \dots, p-2$, which corresponds to $p-1$ Kubota-Leopoldt p-adic L-functions $\mathcal{L}_p(s, \chi\omega^u)$. Each of these branches are defined over a p-adic open disk B, and its value on integral points of this disk are the critical values.

This formulation enables us to collect these $p-1$ branches together as a *sole* function over the weight space. This can explain the phenomenon mentioned in Section 2.2.3, namely the p-adic L-function $\mathcal{L}_p(\chi, s)$ vanishes identically for odd characters χ. Indeed, for odd characters, in the current formulation, such a vanishing result only means that $\mathcal{L}_{p,\chi}$ vanishes on the branch $u = 0$, and similarly also on the branches with even u. However, for odd u, $\chi\omega^u$ are even, hence $\mathcal{L}_p(-s, \omega^u)$ do not vanish. Actually the same story holds for even characters, that half of these branches vanishes while the other half does not. This is one of the advantage of this formulation.

4.1.2. *Reformulate the Kubota-Leopoldt p-adic L-function: as a p-adic measure.* Recall that we construncted a p-adic measure $\mathcal{E}_c$ interpolating the critial values of Dirichlet characters. Based on this, we define

$$\mathcal{L}_{p,\chi} : \psi \in \mathcal{X} \mapsto \frac{1}{1-\chi(c)\psi(c)\langle c \rangle} \int_{\mathbb{Z}_{p,M}^\times} \chi(a)\omega(a)^{-1}\psi(a)\, \mathrm{d}\mathcal{E}_c \in \mathbb{C}_p^\times.$$

In particular, by taking $\psi = \langle - \rangle^{t-1}$ for positive integers t, we get back the interpolation property of Theorem 3.4.

4.1.3. *Remark: Generalities on weight spaces.* What we have defined to be the weight space $\mathcal{X}$ is really the "set of $\mathbb{C}_p$ points" of a rigid analytic variety $\mathcal{X}$ which is represented by the following functor: for any affinoid algebra A over $\mathbb{Q}_p$, $\mathcal{X}(A)$ denotes the set $\mathrm{Hom}_{\mathrm{cts}}(\mathbb{Z}_p^\times, A^\times)$. See [Buz04, Section 2] for details.

4.2. **Construction via p-adic modular forms à la Serre.** Previously we have defined the p-adic L-function $\mathcal{L}_{p,\chi}$ as a function over the weight space, but in a quite indirect way. One may ask if we can construct it directly? The first such construction is given by Serre in 1970s using his definition of p-adic modular forms.

4.2.1. *p-adic modular forms à la Serre.* We will not introduce the classical theory of modular forms. Readers can turn to the brief introduction [Ser73b, Chapter VII], which is far more than enough for our purpose.

Starting with a formal q-expansion $f(q) = \sum_{n=0}^{\infty} a_n q^n \in \mathbb{Q}_p[\![q]\!]$, we define

$$v_p(f) := \inf_n v_p(a_n).$$

An upshot is that $f \equiv g \bmod p^m$ if and only if $v_p(f-g) \geq m$. So this valuation v_p on $\mathbb{Q}_p[\![q]\!]$ measures the higher congruence of f and g modulo p. Let $\{f_i\}_{i=1}^{\infty} \subseteq \mathbb{Q}_p[\![q]\!]$, we say the sequence converges to f and write $f_i \to f$ if $v_p(f_i - f) \to \infty$ as $i \to \infty$. We remark that $f \in \mathbb{Q}_p[\![q]\!]$ has bounded coefficient if and only if $f \in \mathbb{Z}_p[\![q]\!] \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, if and only if $v_p(f) > -\infty$. This includes all classical modular forms with Fourier coefficients in $\mathbb{Q}$.

The guiding theorem is the following.

Theorem 4.2. *Let m be a positive integer, and let $f_1, f_2 \in \mathbb{Q}[\![q]\!]$ be modular forms of weight k_1 and k_2 respectively, with $v_p(f_1 - f_2) \geq v_p(f_1) + m$. If $f_1 \neq 0$, then $k_1 \equiv k_2 \bmod (p-1)p^{m-1}$.*

Remark on the Proof. It is [Ser73c, Théorème 1]. The proof is beyond the scope of this note. Its basic ingredients are

- The sturcture of the space of mod p modular forms $\tilde{M}$. It uses the Kummer congruence in a crucial way. Moreover, the more advanced ingredients are
  - A filtration on $\tilde{M}$: introduced in both [SD73] and [Ser73a].
  - The geometry of $\tilde{M}$: presented in [Ser73a] only.

  These are omitted in the original article of Serre.
- Clausen–von Staudt theorem, i.e. Theorem 1.6.

So this theorem is far from obvious. □

Let $X_m := \mathbb{Z}/(p-1)p^{m-1}$ and let $X = \varprojlim_m X_m = \mathbb{Z}_p \times \mathbb{Z}/(p-1)$. Then elements in X can be identified with the weight space via

$$\epsilon : X \to \mathcal{X}, \quad k = (s,u) \mapsto \epsilon_k := \omega^u \langle - \rangle^s .$$

Via the projection $\pi : X \twoheadrightarrow \mathbb{Z}/(p-1)$, X can be regarded as a disjoint union of $p-1$ copies of $\mathbb{Z}_p$, i.e.

$$X = \bigsqcup_{u \in \mathbb{Z}/(p-1)} X_u, \quad X_u := \pi^{-1}(u) = \{(s,u) \in X : s \in \mathbb{Z}_p\}.$$

Since p is odd, we can project X to $X \twoheadrightarrow \mathbb{Z}/(p-1) \twoheadrightarrow \mathbb{Z}/2$. Then we say $k \in X$ is **even** if it lies in the fiber of $0 \in \mathbb{Z}/2$, otherwise we say k is **odd**. Equivalently, $k \in X$ is even if $k = (s,u) \in 2X$, that is $u \in 2\mathbb{Z}/(p-1)$.

Also, we note that the canonical projections $\mathbb{Z} \twoheadrightarrow X_m$ induce a natural inclusion $\mathbb{Z} \hookrightarrow X$ with dense image. For $k \in \mathbb{Z}$, it maps to the character $x \mapsto x^k$ for $k \in \mathbb{Z}$.

Remark (Motivation of the weight space). This definition of the weight space is motivated by higher congruences of modular forms. Recall Fermat's little theorem says that for any integer a prime to p and any integer k, $a^k \equiv a^{\phi(p^m)+k} \bmod p^m$. Apply this to the nonconstant coeffieients of Eisenstein series E_k for even integers $k > 2$, we may guess that $E_k \equiv E_{k+\phi(p^m)} \bmod p^m$ granting that the constant terms satisfy the same "higher congruences property". Indeed, this is precisely the Kummer congruence. Working with these congruences modulo p^m, it is then natural to pass to the projective system $\{X_m = \mathbb{Z}/\phi(p^m)\}_{m\geq 1}$.

Definition 4.3. A **p-adic modular form (à la Serre)** is a power series $f = \sum_{n=0}^{\infty} a_n q^n \in \mathbb{Q}_p[\![q]\!]$ such that there exists a sequence of modular forms $\{f_i\}_{i=1}^{\infty}$ of integral weights $\{k_i\}_{i=1}^{\infty}$ converging to f.

At first glance, the sequence $\{f_i\}$ might seem arbitrary. However, Theorem 4.2 imposes strong conditions on the behavior of $\{k_i\}$. We see that a nonzero modular form $f = \lim_i f_i$ has a well-defined **weight** $k = \lim_i k_i \in X$. In fact, by the convergence of the sequence $\{f_i\}$, we have $v_p(f_i) = v_p(f)$ for $i \gg 0$ and it is not ∞ since f is nonzero. For such a large i and each $m \geq 1$, we have for $j \geq i \gg 0$,

$$v_p(f_j - f_i) \geq v_p(f_i) + m = v_p(f) + m.$$

Hence by Theorem 4.2, $k_i \equiv k_j \bmod p^{m-1}(p-1)$ for $i, j \gg 0$. This means that k_i is stationary in i. So $k = \lim_i k_i$ exists in X. To see that the limit k depends only on f but not the choice of the sequence $\{f_i\}$, we suppose there is another such sequence $\{f_i'\}$, then consider $f_1, f_1', f_2, f_2', \ldots$.

We denote by $M^\dagger \subset \mathbb{Q}_p[\![q]\!]$ (in fact, contained in $\mathbb{Z}_p[\![q]\!] \otimes \mathbb{Q}_p$) the space of p-adic modular forms, and denote the space of p-adic modular forms of weight k ny $M_k^\dagger$. Then our previous arguments carry over to p-adic modular forms.

Proposition 4.4. *(1) Suppose $f \in M_k^\dagger$ and $f' \in M_{k'}^\dagger$ satisfies $f \neq 0$ and $v_p(f - f') \geq v_p(f) + m$ for some $m \geq 1$. Then k and k' have the same image in X_m.*

(2) If $f_i \in M_{k_i}^\dagger$ is a sequence of p-adic modular forms of weight $k_i \in X$ with $f_i \to f \in \mathbb{Q}_p[\![q]\!]$, then f is a p-adic modular form of weight $k = \lim_i k_i$.

Therefore, $M^\dagger$ is a p-adic Banach space, as a closed subspace of $\mathbb{Z}_p[\![q]\!] \otimes \mathbb{Q}_p$, equipped with a continuous map $M^\dagger \to X$.

The slogan of Serre's construction is that *the non-constant Fourier coefficients govern the constant term*. We will make it precise in the following two results.

Corollary 4.5. *Let $f = \sum_{n=0}^{\infty} a_n q^n \in \mathbb{Q}_p[\![q]\!]$ be a p-adic modular form of weight $k \in X$. Suppose the image of k in X_{m+1} is nonzero, then $v_p(a_0) + m \geq \inf_{n\geq 1} v_p(a_n)$.*

This is [Ser73c, Corollaire 1].

Proof. If $a_0 = 0$ then the proposition is immediate. Suppose now that $a_0 \neq 0$. Let $f' = a_0$, so f' is a modular form of weight $k' = 0$, and

$$v_p(f - f') = \inf_{n \geq 1} v_p(a_n).$$

Also, since the image of k in X_{m+1} is nonzero, $k \not\equiv k'$ in X_{m+1}. So by Theorem 4.2,

$$v_p(f - f') < v_p(f') + m + 1 = v_p(a_0) + m + 1.$$

The result then follows. □

The corollary gives a condition on the p-divisibility of a_0 in terms of a_n for $n \geq 1$. More concretely, in the setting of the corollary, we see if a_n are p-integral for all $n \geq 1$, then so is $p^m a_0$, and when $(p-1) \nmid k$, m can be taken to be 0. Even for classical modular forms, this is a new result. The following theorem, in [Ser73c, Corollaire 2], is even more remarkable.

Theorem 4.6. *Consider a sequence of p-adic modular forms $f^{(i)} = \sum_{n=0}^{\infty} a_n^{(i)} q^n$ of weights $k^{(i)}$ for $i = 1, 2, \ldots$, respectively. Suppose both of the following holds*

- $\lim_i a_n^{(i)} \to a_n \in \mathbb{Q}_p$ *uniformly for all* $n \geq 1$,
- $\lim_i k^{(i)} \to k \in X$ *with* $k \neq 0$.

Then $a_0^{(i)}$ converges p-adically to an element $a_0 \in \mathbb{Q}_p$, and $f = \sum_{n=0}^{\infty} a_n q^n$ is a p-adic modular form of weight k.

Proof. Since $\lim_i k^{(i)} \to k \neq 0$, there exists an integer m such that all $k^{(i)}$ has nonzero image in X_m. Indeed, we can pick m large enough such that $k \neq 0 \bmod p^m(p-1)$. Since $\lim_i a_n^{(i)} \to a_n \in \mathbb{Q}_p$ uniformly in n, there exists an integer t such that $v_p(a_i^{(n)}) \geq t$ for all $n \geq 1$ and $i \gg 0$. Then apply Corollary 4.5, for all $i \gg 0$,

$$v_p(a_0^{(i)}) + m \geq \inf_{n \geq 1} v_p(a_n^i) \geq t,$$

i.e. $v_p(a_0^{(i)}) \geq t - m$. Now since $p^{t-m}\mathbb{Z}_p$ is compact, we have that some subsequence of $\{a_0^{(n)}\}_{i \geq 0}$ has a limit a_0. Suppose this subsequence is labeled by $\{i_j\}_{j \geq 0}$. Then

$$f = \lim_j f^{(i_j)} = a_0 + a_1 q + a_2 q^2 + \cdots$$

is a p-adic modular form of weight k. Suppose we have another subsequence $\{a_0^{(i')}\}$ converging to a_0', then

$$f' = \lim_{j'} f^{(i_{j'})} = a_0' + a_1 q + a_2 q^2 + \cdots$$

is another p-adic modular form of weight k. Hence $f - f' = a_0 - a_0'$ is a p-adic modular form of weight $k \neq 0$. Then apply Corollary 4.5 to $f - f'$, we see that $v_p(a_0 - a_0') \geq \infty$, which forces $a_0 = a_0'$. Hence a_0 does not deppend on the choice of subsequences. Hence $\{a_0^{(i)}\}$ is a convergent sequence to some $a_0 \in \mathbb{Q}_p$. □

In the proof, we have seen that a non-zero constants cannot be p-adic modular form of weights $k \neq 0$.

We will apply this to the Eisenstein series, for which the condition on the nonconstant terms $a_n^{(i)}$ and $k^{(i)}$ are easy to check, but not so much for critical values on the constant term.

4.3. **p-adic Eisenstein series and the Kubota-Leopoldt p-adic L-function.** We first remark that in [Ser73c], Serre only considered the p-adic zeta functions instead of the more general p-adic L-functions for nontrivial Dirichlet characters χ. Here we try to use Serre's formulation to construct the more general Kubota-Leopoldt p-adic L-function.

Let $k \geq 4$ be an even integer, consider the level 1, weight k Eisenstein series, whose Fourier expansion is given by

$$E_{k,\chi}(z) = \frac{L(1-k,\chi)}{2} + \sum_{n\geq 1} \underbrace{\left(\sum_{d|n} \chi(d)d^{k-1}\right)}_{=:\sigma_{k-1,\chi}(n)} q^n.$$

Now for nonzero $k = (s,u) \in 2X$, we take a sequence $\{k_i\}_{i=1}^{\infty}$ such that $k_i \geq 4$ are even and

- $\lim_{i\to\infty} k_i = k \in X$. This can be done since $\mathbb{Z}$ is dense in X. Note that this implies that $\{k_i\}_{i=1}^{\infty} \subset X_u$.
- Replacing k_i by $k_i + p^{m_i}(p-1)$ for $m_i \gg 0$, we may assume additionally that $k_i \to \infty$ as a sequence of real numbers.

For integers k_i,

$$\sigma_{k_i-1,\chi}(n) = \sum_{d|n,\,(p,d)=1} \chi(d)d^{k_i-1} + \sum_{d|n,\,p|d} \chi(d)d^{k_i-1}.$$

Then

- For d prime to p, $d \in \mathbb{Z}_p^\times$. Since $k_i \to k$ in X, $d^{k_i} \to d^k$. Hence the terms in first sum converge to $\chi(d)d^{k-1}$ for each $d \mid n$.
- For $p \mid d$, $d^{k_i-1} \to 0$ in $\mathbb{Q}_p$ as $i \to \infty$ because $k_i \to \infty$ in $\mathbb{R}$.

Moreover, one checks that this convergence is uniform in n. We denote the limit as $\boldsymbol{\sigma}_{k-1,\chi}$.

We then consider the sequence of Eisenstein series $E_{k_i,\chi}$ for $i = 1, 2, \ldots$. By Theorem 4.6, we see that the sequence converges to a p-adic modular form

$$\mathbf{E}_{k,\chi} = (\mathbf{a}_0)_{k,\chi} + \sum_{n\geq 1} \boldsymbol{\sigma}_{k-1,\chi}(n)q^n \in \mathbb{Q}_p[\![q]\!]$$

of weight $k \in 2X$, called the **p-adic Eisenstein series** of weight k. Moreover, its constant term is given by

$$(\mathbf{a}_0)_{k,\chi} = \lim_{i\to\infty} \frac{L(1-k_i,\chi)}{2},$$

which is a p-adic limit of values of Dirichlet L-functions.

Therefore, $k \mapsto 2(\mathbf{a}_0)_{k,\chi}$ defines a function for even nonzero elements k of X. Moreover, it is continuous in k. In fact, suppose $k_i \to k$, all of which are even elements of $X \smallsetminus \{0\}$, then the nonconstant coefficients of $\mathbf{E}_{k_i,\chi}$ tend to those of $\mathbf{E}_{k,\chi}$ uniformly. Then apply Theorem 4.6 again, we see that the constant term $(\mathbf{a}_0)_{k_i,\chi} \to (\mathbf{a}_0)_{k,\chi}$ as $i \to \infty$.

Then the question is, what are the special values $2(\mathbf{a}_0)_{k,\chi}$ for even integers $k \geq 2$?

4.3.1. *Serre's approach to identify with the Kubota-Leopoldt p-adic L-function.* Serre answers this question by identifying $(\mathbf{a}_0)_{k,\chi}$ as the Kubota–Leopoldt p-adic L-function. We recall Serre's statement and proof here in [Ser73c, Théorème 4].

Theorem 4.7. *If $k = (s, u) \in 2X$ is nonzero and even, then* $(\mathbf{a}_0)_{k,\chi} = \dfrac{\mathcal{L}_{p,\chi}(s-1, u)}{2}$.

Proof. Recall we have defined $\mathcal{L}_{p,\chi} : k = (s, u) \mapsto \mathcal{L}_p(-s, \chi\omega^u)$ for the Kubota-Leopoldt p-adic L-function $\mathcal{L}_p(-, \chi\omega^u)$, and

$$\mathcal{L}_{p,\chi}(n-1, u) = (1 - \chi\omega^{u-n}(p)p^{n-1})L(1-n, \chi\omega^{u-n}), \quad n \geq 1. \tag{4.12}$$

If $k \in 2X$, then we can construct the p-adic Eisenstein series $\mathbf{E}_{k,\chi}$ with previous assumption that $k_i \to k \in X$ and $k_i \to \infty$ in the archimedian topology. Then

$$\begin{aligned}
\mathcal{L}_{p,\chi}(s-1, u) &= \lim_{i\to\infty} \mathcal{L}_{p,\chi}(k_i - 1, u) \\
&= \lim_{i\to\infty} (1 - \chi\omega^{u-k_i}(p)p^{k_i-1})L(1-k_i, \chi\omega^{u-k_i}) \\
&= \lim_{i\to\infty} L(1-k_i, \chi) \\
&= 2(\mathbf{a}_0)_{k,\chi}.
\end{aligned}$$

Here the first equality holds since $\mathcal{L}_{p,\chi}$ is continuous over the weight space $\mathcal{X}$, and note that $k_i \in X_u$ makes ω^{u-k_i} a trivial character. $\square$

Then as a result, for nonzero even integer $k \geq 2$,

$$2(\mathbf{a}_0)_{k,\chi} = \mathcal{L}_{p,\chi}(k-1, k) = (1 - \chi(p)p^{k-1})L(1-k, \chi) \tag{4.13}$$

4.3.2. *Remark: clarification on the appearance of Teichmuller characters.* One may wonder that the demanding interpolation formula should be something like

$$2(\mathbf{a}_0)_{k,\chi} \overset{?}{=} \mathcal{L}_{p,\chi}(k-1, k) = (1 - \chi\omega^{-k}(p)p^{k-1})L(1-k, \chi\omega^{-k}),$$

while the Teichmuller character disappeared in (4.13). The upshot is that the above "$\overset{?}{=}$" is the story on the disk X_0. Indeed, for nonzero even integers $k \geq 2$, consider the weight $k^\flat = (k, 0) \in X_0$, then Theorem 4.7 implies

$$2(\mathbf{a}_0)_{k^\flat,\chi} = \mathcal{L}_{p,\chi}(k-1, 0) \overset{(4.12)}{=\!=} (1 - \chi\omega^{-k}(p)p^{k-1})L(1-k, \chi\omega^{-k}), \quad n \geq 1,$$

as desired. But $k^\flat$ is *not* an integral weight! This is the place that causes many confusions.

Still, Serre's proof here requires *a priori* the knowledge of the Kubota-Leopoldt p-adic L-function. However, it seems more natural to turn the development around: an alternative calculation of $(\mathbf{a}_0)_{k,\chi}$ would imply (4.13).

4.3.3. *Another approach: using Hecke operators.* As the case of classical modular forms, we can also define Hecke operators for p-adic modular forms. For example, we can define the V_p-operator acting on $f = \sum_{n=0}^{\infty} a_n q^n \in \mathbb{Q}_p[\![q]\!]$ via

$$f|V_p := \sum_{n=0}^{\infty} a_n q^{np}.$$

It is shown in [Ser73c, Théorème 4] that V_p preserves the space of p-adic modular forms of weight $k \in X$. Note that this is different from the operators T_p and U_p, rather in the classical setting, the level-raising operator $(f|V_p)(z) := f(pz)$ is given by the same formula.

Taking the operator V_p for granted, we can relate for even integer $k \geq 2$,

$$\mathbf{E}_{k,\chi} = (\mathbf{a}_0)_{k,\chi} + \sum_{n=1}^{\infty} \boldsymbol{\sigma}_{k-1,\chi}(n) q^n$$

with

$$E_{k,\chi} = \frac{L(1-k,\chi)}{2} + \sum_{n=1}^{\infty} \sigma_{k-1,\chi}(n) q^n.$$

In fact, the following equality is straightforward.

Lemma 4.8. *If $k \geq 2$ is an even integer, then*

$$\mathbf{E}_{k,\chi} = E_{k,\chi}|(1-\chi(p)p^{k-1}V_p).$$

Then comparing the constant terms yields

$$(\mathbf{a}_0)_{k,\chi} = (1-\chi(p)p^{k-1})\frac{L(1-k,\chi)}{2}.$$

Recall that $2(\mathbf{a}_0)_{k,\chi}$ is a continuous function on the even elements of $X \smallsetminus \{0\}$ and interpolates the values $\mathcal{L}_p(k, k-1) = (1-\chi(p)p^{k-1})L(1-k,\chi)$ for even integers $k \geq 2$ at the integral weights $k \in X$, which form a dense subset of X. Therefore, $2(\mathbf{a}_0)_{k,\chi}$ must coincide with the Kubota-Leopoldt p-adic L-function $\mathcal{L}_p(s-1,u)$. We then obtain Theorem 4.7 again, without invoking the previous constructions of Kubota-Leopoldt p-adic L-functions.

To sum up, we have constructed a p-adic family of Eisenstein series, as a continuous map

$$\mathbf{E}_\chi : 2X \smallsetminus 0 \to M^\dagger.$$

with constant terms as the Kubota-Leopoldt p-adic L-function.

4.3.4. *Remark: the case $k = 0 \in X$.* In fact, we do not just ignore the case $k \in X_0$ in the weight space X. For simplicity, we consider the case $\chi = \mathbb{1}$. Consider the Eisenstein series E_k with even $k > 2$ and $k \equiv 0 \bmod p-1$,

$$E_k(z) = -\frac{B_k}{2k} + \sum_{n\geq 1}\left(\sum_{d|n} d^{k-1}\right) q^n.$$

By Theorem 1.6, we see that different from the $k \notin X_0$ case, the constant term has negative p-adic valuation. We modify it by simply dividing its constant term as

$$G_k(z) := 1 - \frac{2k}{B_k} \sum_{n \geq 1} \left(\sum_{d|n} d^{k-1} \right) q^n.$$

Then one sees immediately that $G_k(z) \equiv 1 \bmod p$. Moreover, $G_k(z)^{p^m} \equiv 1 \bmod p^m$ for each integer $m \geq 1$. Therefore, by letting $m \to \infty$, as $\lim_{m \to \infty} p^m(p-1) = 0 \in X$, we see that 1 is a p-adic modular form of weight $0 \in X$. It is trivial but maybe rather surprising.

Unfortunately, this (trivial) p-adic modular form does not fits into the family $\mathbf{E}_\chi$ that we have constructed previously, and since G_k has constant term 1, it is not helpful for us to construct the p-adic L-function. However, it is useful in its own rights, especially in the algebro-geometrical understanding of modular forms.

4.3.5. *Remark: Measure-theoretic perspective.* We can also view the above construction measure-theoretically. Consider what we have done

$$\mathbf{E}_\chi : 2\mathcal{X} \smallsetminus 0 \xrightarrow{\sim} 2X \smallsetminus 0 \to M^\dagger,$$

by first applying ϵ^{-1} and then associate a weight $k \in X$ with the p-adic Eisenstein series $\mathbf{E}_{k,\chi}$. This is called a **p-adic family of Eisenstein series**, or a **p-adic Eisenstein family** for short.

As the weight space $2\mathcal{X} \smallsetminus 0$ is a dense subset of $\mathcal{C}(2\mathbb{Z}_p^\times, \mathbb{C}_p)$, it is natural to ask if $\mathbf{E}_\chi$ can be extended to a measure in $\mathrm{Meas}(2\mathbb{Z}_p^\times, M^\dagger \otimes_{\mathbb{Q}_p} \mathbb{C}_p)$. This can be done by the **abstract Kummer congruences**.

Theorem 4.9 (Abstract Kummer congruences)**.** *Let W be a p-adic Banach space over a p-adic field K and G be a profinite group. Let $\mathcal{C}_0$ be a Zariski dense subset of $\mathcal{C}(G, K)$. Let $m : \chi \mapsto m_\chi$ be a function from $\mathcal{C}_0$ to W, and let $\lambda(m)$ denote the corresponding $W[1/p]$-valued measure. Then $\lambda(m)$ extends a p-adic measure in $\mathrm{Meas}(G, W)$ if and only if, for every integer n and for any finite sum $\sum_j \alpha_j \chi_j$ with $\alpha_j \in K[1/p]$ and $\chi_j \in \mathcal{C}_0$ such that $\sum \alpha_j \chi_j(t)$ for all $t \in p^n G$, we have*

$$\sum_j \alpha_j m_{\chi_j} \in p^n W.$$

This is an adaption of [Kat78, Proposition (4.0.6)], appears in [HLS06, Lemma (3.4.1)]. From this we can deduce that $\mathbf{E}_\chi \in \mathrm{Meas}(2\mathbb{Z}_p^\times, M^\dagger \otimes \mathbb{C}_p)$. So we often call it the **Eisenstein measure**.

4.4. **Summary and generalizations.** Reviewing what we have done so far in this section,

- we first constructed a *family* of p-adic modular forms $\mathbf{E}_\chi$, namely the **p-adic Eisenstein family**, such that as k runs through all even nonzero weights $k \in 2X$, the p-adic Eisenstein series $\mathbf{E}_{k,\chi}$ has constant terms interpolating the critical values of Dirichlet L-functions of χ.

- Then we took the constant term of the family $\mathbf{E}_\chi$ to obtain a continuous function over the weight space $2X$, as the Kubota-Leopoldt p-adic L-function.

We regard the operation of "taking constant terms" as a linear functional $\Phi : M^\dagger \otimes \mathbb{C}_p \to \mathbb{C}_p$, the two steps can be summarised as first constuct a family of p-adic modular forms, as a p-adic measure in $\mathrm{Meas}(\mathbb{Z}_p^\times, M^\dagger \otimes \mathbb{C}_p)$ and then applying the linear functional

$$\Phi_* : \mathrm{Meas}(2\mathbb{Z}_p^\times, M^\dagger \otimes \mathbb{C}_p) \to \mathrm{Meas}(2\mathbb{Z}_p^\times, \mathbb{C}_p)$$

to obtain a p-adic L-function. This method generalizes vastly to many other contexts, by

- replacing $M^\dagger \otimes \mathbb{C}_p$ with other more delicate spaces of p-adic modular forms, or
- replacing the linear functional Φ by other ones.

For example,

- In [Ser73c], Serre constructed the p-adic Dedekind zeta function $\zeta_{p,F}$ for totally real number fields F using some well-chosen p-adic Eisenstein family.
- Deligne and Ribet [DR80] constructed a p-adic L-function for finite order Hecke characters χ of a totally real number field F unramified away from p, that is, interpolating values of $L(s,\chi) \in \mathbb{Q}(\chi)$ for negative integers, where $\mathbb{Q}(\chi)$ is the field extension of $\mathbb{Q}$ obtained by adjoining all values of χ. They work in the space of Hilbert modular forms. Similar to Serre's approach, a crucial step is the construction of Eisenstein series with $L(1-k,\chi)$ as constant terms. This approach requires the higher congruence results in the Hilbert modular form setting. This need more geometry than the previous approach.
- It is then natural to move on to CM fields K, i.e. a imaginary quadratic extension of a totally real number field F. In [Kat78], Katz considered the case where $\chi : K^\times \backslash \mathbb{A}_K^\times \to \mathbb{C}$ is a Hecke character of type A_0 (à la Weil). In a recent work [EHLS20], the four authors constructed p-adic L-functions over general unitary groups $\mathrm{GU}(r,s)$ by constructing p-adic Siegal Eisenstein families and using the doubling method to compute their constant terms as critical values of certain L-functions. Taking the special case of $\mathrm{GU}(1,1)$, we can get back to Katz's construction of p-adic L-function for A_0-type Hecke characters over CM-fields.
- We can also replace the linear functional of "taking constant terms" by some other linear functional, such as pairing with another modular form. This is, for instance, the construction of the (three-variable) Rankin-Selberg p-adic L-functions. The construction is first given by Hida in [Hid85] and introduced with more details in [Hid93, Chapter 10].

Some details on these generalizations can be found in [Eis21, Section 4-5].

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