Homework for Valuations

1. Prove Krasner's theorem: suppose K is a complete non-Archimedean discrete valuation field whose algebraic closure is $\bar{K}$. Let $\alpha, \beta \in \bar{K}$ be such that

$$|\beta - \alpha| < |\beta - \beta'|$$

for each Galois conjugate β' of β different from β. Show that $\beta \in K(\alpha)$.

2. Prove Hensel's Lemma. Let K be a finite extension of $\mathbb{Q}_p$ and $\mathcal{O}_K$ its valuation ring. Let k be the residue field of $\mathcal{O}_K$. Let $f(x) \in \mathcal{O}_K[x]$ be a polynomial whose image $\bar{f} \in k[x]$ is nonzero. Suppose

$$\bar{f}(x) = \bar{g}(x)\bar{h}(x)$$

with $\bar{g}(x)$ and $\bar{h}(x)$ relatively prime in $k[x]$. Then $f(x)$ admits a factorization as $g(x)h(x)$ in $\mathcal{O}_K[x]$ such that the reductions of $g(x)$ and $h(x)$ in $k[x]$ are $\bar{g}(x)$ and $\bar{h}(x)$ respectively, and such that the degrees of $g(x)$ and $\bar{g}(x)$ are equal. Moreover such factorization is unique up to a unit in $\mathcal{O}_K$.

3. Let $\varphi_1, \cdots, \varphi_n$ be non-Archimedean valuations of a field F which are pairwise inequivalent , and let $a_1, \cdots, a_n \in F^\times$. Show that there exists an $a \in F^\times$, such that $\varphi_i(a) = \varphi_i(a_i)$ for each i.

4. Let E/F and K/F be finite extensions of complete discrete valuation fields, whose residue fields are finite.

(1) If E/F is completely ramified, K/F is unramified, show that EK/K is completely ramified.

(2) If K/F and E/F are completely ramified, can we conclude that KE/F is completely ramified?

(3) If $F \subseteq K \subseteq E$ with K/F and E/K being completely ramified, then show that E/F is completely ramified.

5. Compute the ramification index and residue degrees of $\mathbb{Q}_2(\sqrt{3}, \sqrt{7})$ and $\mathbb{Q}_2(\sqrt{3}, \sqrt{2})$ over $\mathbb{Q}_2$.

6. Suppose F is a complete discrete valuation field, and $\alpha \in F$ with $v_F(\alpha) \geq 1$. Suppose the residue field of F is finite with characteristic p. If n is an integer prime to p, then $F(\sqrt[n]{\alpha})$ is an unramified extension of F.

7. Suppose K is a finite extension of $\mathbb{Q}_p$ which is completely ramified. Suppose π is a prime element in K. Show that the valuation ring of K is $\mathbb{Z}_p[\pi]$.

8. Let K be an unramified extension of $\mathbb{Q}_p$ of degree n, with valuation ring O. Show that there exists a $p^n - 1$-th primitive root of unity α, and that $O = \mathbb{Z}_p[\alpha]$.