# Exam in Category Theory

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**Problem 1.** Show that the following conditions are equivalent for an idempotent morphism $e : X \to X$:

(1) $e$ comes from a retract of $X$;

(2) The diagram $X \underset{\mathrm{id}_X}{\overset{e}{\rightrightarrows}} X$ admits an equalizer.

(3) The diagram $X \underset{\mathrm{id}_X}{\overset{e}{\rightrightarrows}} X$ admits a coequalizer.

*Proof.* For "(1) ⇒ (2)", since $e$ comes from a retract of $X$, there exists an object $Y \in \mathsf{C}$, morphisms $i : Y \to X$ and $r : X \to Y$, such that $r \circ i = \mathrm{id}_Y$ and $e = i \circ r$. We claim that $Y \xrightarrow{i} X \underset{\mathrm{id}_X}{\overset{e}{\rightrightarrows}} X$ is a equalizer diagram.

We first check that $e \circ i = \mathrm{id}_X \circ i$. In fact,

$$e \circ i = (i \circ r) \circ i = i \circ (r \circ i) = i \circ \mathrm{id}_Y = \mathrm{id}_X \circ i.$$

Then suppose $W \in \mathsf{C}$ and $h : W \to X$ such that $e \circ h = \mathrm{id}_X \circ h$.

$$\begin{tikzcd} & W \arrow[dl, dashed, "\exists! h'"'] \arrow[d, "h"] & \\ Y \arrow[r, "i"] & X \arrow[r, shift left, "e"] \arrow[r, shift right, "\mathrm{id}_X"'] & X \end{tikzcd} \tag{1}$$

If there exist a morphism $h' : W \to Y$ making the diagram commute, i.e. $i \circ h' = h$, then $i \circ h' = h = e \circ h$. Compositing $r$ on both sides, we obtain

$$r \circ i \circ h' = r \circ e \circ h.$$

Note that $r \circ i = \mathrm{id}_Y$, we have

$$h' = r \circ e \circ h.$$

This implies that such a morphism $h'$ exists and is unique.

For "(2) $\Rightarrow$ (3)", let

$$\ker(e, \mathrm{id}_X) \xrightarrow{i} X \underset{\mathrm{id}_X}{\overset{e}{\rightrightarrows}} X \tag{2}$$

be the equalizer diagram. Then $e \circ i = i$. Consider

$$\begin{tikzcd} & X \arrow[dl, dashed, "\exists! r"'] \arrow[d, "e"] & \\ \ker(e, \mathrm{id}_X) \arrow[r, "i"] & X \arrow[r, shift left, "e"] \arrow[r, shift right, "\mathrm{id}_X"'] & X \end{tikzcd} \tag{3}$$

Noting that $e \circ e = e \circ \mathrm{id}_X$, by the universal property of equalizers, there exists a unique morphism $r : X \to \ker(e, \mathrm{id}_X)$ such that $e = i \circ r$. We claim that

$$X \underset{\mathrm{id}_X}{\overset{e}{\rightrightarrows}} X \xrightarrow{r} \ker(e, \mathrm{id}_X) \tag{4}$$

is a coequalizer diagram. To show this, we first need to check that $r \circ e = r$. An observation is that both $[X \xrightarrow{r \circ e} \ker(e, \mathrm{id}_X)]$ and $r$ make the triangle in (3) commutes. In fact,

$$i \circ [X \xrightarrow{r \circ e} \ker(e, \mathrm{id}_X)] = (i \circ r) \circ e = e \circ e = e,$$

since $i \circ r = e$ and $e$ is an idempotent morphism. By the uniqueness part of the universal property of equalizers, we have $r \circ e = r$.

Then we check that (4) satisfies the universal property of coequalizers. Let $Z \in \mathsf{C}$ and $h : X \to Z$ such that $h \circ e = h \circ \mathrm{id}_X = h$, as shown in the following diagram.

$$\begin{tikzcd} X \arrow[r, shift left, "e"] \arrow[r, shift right, "\mathrm{id}_X"'] & X \arrow[r, "r"] \arrow[d, "h"'] & \ker(e, \mathrm{id}_X) \arrow[dl, dashed, "\exists! h'"] \\ & Z & \end{tikzcd} \tag{5}$$

If there exists a morphism $h' : \ker(e, \mathrm{id}_X) \to Z$ such that the triangle in (5) commutes, then $h' \circ r = h$. Composing with $i$, we have

$$h' \circ r \circ i = h \circ i. \tag{6}$$

Now we claim that $r \circ i = \mathrm{id}_{\ker(e,\mathrm{id}_X)}$. To show this, consider the commutative diagram

$$\begin{tikzcd}
 & \ker(e,\mathrm{id}_X) \arrow[dl, "\mathrm{id}_{\ker(e,\mathrm{id}_X)}"'] \arrow[d, "i"] & \\
\ker(e,\mathrm{id}_X) \arrow[r, "i"] & X \arrow[r, shift left, "e"] \arrow[r, shift right, "\mathrm{id}_X"'] & X
\end{tikzcd} \tag{7}$$

and we shall check that $r \circ i$ makes the triangle in (7) commute as well. In fact,

$$i \circ (r \circ i) = (i \circ r) \circ i = e \circ i = i$$

as desired. Hence, by the uniqueness part of the universal property of equalizers, we obtain that $r \circ i = \mathrm{id}_{\ker(e,\mathrm{id}_X)}$. Therefore, (6) can be written as

$$h' = h \circ i,$$

which implies both the existence and the uniqueness of $h'$ making the triangle in (5) commute. Hence, (4) is indeed a coequalizer diagram, as desired.

For "(3) $\Rightarrow$ (1)", let

$$\begin{tikzcd}
X \arrow[r, shift left, "e"] \arrow[r, shift right, "\mathrm{id}_X"'] & X \arrow[r, "r"] & \operatorname{coker}(e,\mathrm{id}_X)
\end{tikzcd} \tag{8}$$

be the coequalizer diagram. Then $r \circ e = r$. Consider

$$\begin{tikzcd}
X \arrow[r, shift left, "e"] \arrow[r, shift right, "\mathrm{id}_X"'] & X \arrow[r, "r"] \arrow[d, "e"] & \operatorname{coker}(e,\mathrm{id}_X) \arrow[dl, dashed, "\exists ! i"] \\
 & X &
\end{tikzcd} \tag{9}$$

Then since $e$ is idempotent, by the universal property of coequalizers, there exists a unique $i : \operatorname{coker}(e,\mathrm{id}_X) \to X$ such that $i \circ r = e$. Now we claim that the following diagram is a retract of $X$ and $e$ comes from this retract:

$$\begin{tikzcd}
 & X \arrow[dr, "r"] & \\
\operatorname{coker}(e,\mathrm{id}_X) \arrow[ur, "i"] \arrow[rr, "\mathrm{id}_{\operatorname{coker}(e,\mathrm{id}_X)}"'] & & \operatorname{coker}(e,\mathrm{id}_X)
\end{tikzcd} . \tag{10}$$

Since we already have $i \circ r = e$, it suffices to prove that the above diagram commutes, i.e. $r \circ i = \mathrm{id}_{\operatorname{coker}(e,\mathrm{id}_X)}$. If the claim is proved, then "(3) $\Rightarrow$ (1)" holds, and we are done.

To prove the claim, consider the diagram

$$\begin{tikzcd} X \arrow[r, shift left, "e"] \arrow[r, shift right, "\mathrm{id}_X"'] & X \arrow[r, "r"] \arrow[d, "r"] & \mathrm{coker}(e, \mathrm{id}_X) \arrow[dl, "\mathrm{id}_{\mathrm{coker}(e,\mathrm{id}_X)}"] \\ & \mathrm{coker}(e, \mathrm{id}_X) & \end{tikzcd} \tag{11}$$

We claim that $r \circ i$ also makes the triangle in (11) commutes. In fact,

$$(r \circ i) \circ r = r \circ (i \circ r) = r \circ e = r,$$

as desired. By the uniqueness part of the universal property of coequalizers, we obtain that $r \circ i = \mathrm{id}_{\mathrm{coker}(e,\mathrm{id}_X)}$. So the claim is proved.

□

**Problem 2.** Show that if an idempotent morphism $e : X \to X$ comes from a retract $Y$ of $X$ then $Y$ is unique up to unique isomorphism.

*Proof.* In the proof of Problem 1, we saw that $Y$ is the equalizer of the diagram $X \underset{\mathrm{id}_X}{\overset{e}{\rightrightarrows}} X$ , which is unique up to unique isomorphism. □

**Problem 3.** Show that for any idempotent complete category $\mathsf{D}$, $\mathrm{Fun}(\mathsf{C}, \mathsf{D})$ is also idempotent complete.

*Proof.* Let $\phi : F \to F$ be a natural transformation of a functor $F \in \mathrm{Fun}(\mathsf{C}, \mathsf{D})$ such that $\phi^2 = \phi$, we shall prove that $\phi$ comes from a retract of $F$. To do so, we shall first construct a retract of $F$ *objectwise* and then show the naturality.

Let $X$ be an object in $\mathsf{C}$. Then $\phi_X : F(X) \to F(X)$ is an idempotent morphism in $\mathsf{D}$. Since $D$ is idempotent complete, we have a retract of $F(X)$ as

$$\begin{tikzcd} & F(X) \arrow[dr, "\psi_X"] & \\ G_X \arrow[ur, "\iota_X"] \arrow[rr, "\mathrm{id}_{G_X}"'] & & G_X \end{tikzcd} \tag{12}$$

and $\phi_X = \iota_X \circ \psi_X$. Then we define a functor $G : \mathsf{C} \to \mathsf{D}$ as $G(X) := G_X$, and

$$G([X \xrightarrow{f} Y]) := [G_X \xrightarrow{\iota_X} F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{\psi_Y} G_Y] = \psi_Y \circ F(f) \circ \iota_X.$$

We now check that $G$ is indeed a functor:

- $G(\mathrm{id}_X) = \iota_X \circ F(\mathrm{id}_X) \circ \psi_X = \iota_X \circ \mathrm{id}_{F(X)} \circ \psi_X = \iota_X \circ \psi_X = \mathrm{id}_{G(X)}$.
- For $X \xrightarrow{f} Y \xrightarrow{g} Z$,

$$\begin{aligned} G(g) \circ G(f) &= (\psi_Z \circ F(g) \circ \iota_Y) \circ (\psi_Y \circ F(f) \circ \iota_X) \\ &= \psi_Z \circ F(g) \circ (\iota_Y \circ \psi_Y) \circ F(f) \circ \iota_X \\ &= \psi_Z \circ F(g) \circ \mathrm{id}_{G(Y)} \circ F(f) \circ \iota_X \\ &= \psi_Z \circ F(g) \circ F(f) \circ \iota_X \\ &= \psi_Z \circ F(g \circ f) \circ \iota_X \\ &= G(g \circ f). \end{aligned}$$

Thus, we can rewrite (12) as

$$\begin{tikzcd} & F(X) \arrow[dr, "\psi_X"] & \\ G(X) \arrow[ur, "\iota_X"] \arrow[rr, "\mathrm{id}_{G_X}"'] & & G(X) \end{tikzcd} \quad . \tag{13}$$

Now we show that $(\iota_X : G(X) \to F(X))$ and $(\phi_X : F(X) \to G(X))$ are natural transformations. For $X \xrightarrow{f} Y$, the square

$$\begin{tikzcd} G(X) \arrow[r, "\iota_X"] \arrow[d, "G(f)"'] & F(X) \arrow[d, "F(f)"] \\ G(Y) \arrow[r, "\iota_Y"'] & F(Y) \end{tikzcd} \tag{14}$$

commutes, since

$$\begin{aligned} \iota_Y \circ G(f) &= \iota_Y \circ (\psi_Y \circ F(f) \circ \iota_X) \\ &= (\iota_Y \circ \psi_Y) \circ F(f) \circ \iota_X \\ &= \mathrm{id}_{G(Y)} \circ F(f) \circ \iota_X \\ &= F(f) \circ \iota_X. \end{aligned}$$

Hence, $(\iota_X : G(X) \to F(X))$ is indeed a natural transformation. Similarly, the square

$$\begin{tikzcd} F(X) \arrow[r, "\psi_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] \\ F(Y) \arrow[r, "\psi_Y"'] & G(Y) \end{tikzcd} \tag{15}$$

commutes, since

$$\begin{aligned} \circ G(f) \circ \psi_X &= (\psi_Y \circ F(f) \circ \iota_X) \circ \psi_X \\ &= \psi_Y \circ F(f) \circ (\iota_X \circ \psi_X) \\ &= \psi_Y \circ F(f) \circ \mathrm{id}_{F(X)} \\ &= \psi_Y \circ F(f). \end{aligned}$$

Hence, $(\phi_X : F(X) \to G(X))$ is indeed a natural transformation. Therefore, we can rewrite (13) as

$$\begin{tikzcd} & F \arrow[dr, "\psi"] & \\ G \arrow[ur, "\iota"] \arrow[rr, "\mathrm{id}_G"'] & & G \end{tikzcd} . \tag{16}$$

and $\phi = \iota \circ \psi$ since $\phi_X = \iota_X \circ \psi_X$ for each object $X \in \mathsf{C}$. Therefore, $\phi$ indeed comes from a retract, as desired. □

**Problem 4.** Show that the category $\widehat{\mathsf{C}}$ is well-defined.

*Proof.* It suffices to define the identity morphism for each object $(X,e) \in \widehat{\mathsf{C}}$ and check the associative law and the identity law for morphisms.

For object $(X,e) \in \widehat{\mathsf{C}}$, we define the identity morphism as $\mathrm{id}_{(X,e)} := [X \xrightarrow{e} X]$. [1] We need to check that this is indeed a morphism in $\widehat{\mathsf{C}}$, i.e.

$$e \circ e = e = e \circ e,$$

[1]Note that it is not defined as $[X \xrightarrow{\mathrm{id}_X} X]$.

but this obviously holds since $e$ is an idempotent morphism. The composition law of $\widehat{\mathsf{C}}$ is induced from that of $\mathsf{C}$, so for associative law, it suffices to check that for

$$(X,e) \xrightarrow{f} (Y,d) \xrightarrow{g} (Z,c),$$

the composition $g \circ f$ is indeed a morphism in $\widehat{\mathsf{C}}$, i.e. check whether the following holds or not:

$$c \circ (g \circ f) = g \circ f = (g \circ f) \circ e.$$

Starting from the left hand side, we see that

$$c \circ (g \circ f) = (c \circ g) \circ f = (g \circ d) \circ f = g \circ (d \circ f) = g \circ (f \circ e) = (g \circ f) \circ e,$$

and

$$c \circ (g \circ f) = (c \circ g) \circ f = g \circ f$$

gives the middle one.

For the identity law, consider

$$(X,e) \xrightarrow{e} (X,e) \xrightarrow{f} (Y,d),$$

then $f \circ e = f$ since $f$ itself is a morphism in $\widehat{\mathsf{C}}$. Similar result holds for

$$(Y,d) \xrightarrow{f} (X,e) \xrightarrow{e} (X,e),$$

where $e \circ f = f$ for the same reason. Now we can conclude that the category $\widehat{\mathsf{C}}$ is well-defined. $\square$

**Problem 5.** Show that the category $\widehat{\mathsf{C}}$ is idempotent complete.

*Proof.* Let $(X,e) \in \widehat{\mathsf{C}}$ and $f : (X,e) \to (X,e)$ be an idempotent morphism. Now consider the diagram

$$\begin{tikzcd} & (X,e) \arrow[dr, "r:=f"] & \\ (X,f) \arrow[ur, "i:=f"] \arrow[rr, "\mathrm{id}_{(X,f)}"'] & & (X,f) \end{tikzcd} . \tag{17}$$

We claim that this diagram is commute and $f = i \circ r$. Before that, we have to check that $f : (X, f) \to (X, e)$ and $f : (X, e) \to (X, f)$ are indeed morphisms in $\widehat{\mathsf{C}}$. In other words, we have to check

$$e \circ f = f = f \circ f, \text{ and } f \circ f = f = f \circ e.$$

Yet the two follows from that $f$ is idempotent and $e \circ f = f = f \circ e$ since $f : (X, e) \to (X, e)$ is a morphism in $\widehat{\mathsf{C}}$.

The commutativity of the diagrams follows easily from that $f$ is an idempotent morphism, after noting that $\mathrm{id}_{(X,f)} = f$ (not $\mathrm{id}_X$). For $f = i \circ r$, this is again obvious since $f$ is idempotent. Therefore, the claim has been proved and hence $\widehat{\mathsf{C}}$ is idempotent complete. □

> **Problem 6.** Show that the assignment $X \mapsto (X, \mathrm{id}_X), f \mapsto f$ defines a fully faithful functor $I : \mathsf{C} \to \widehat{\mathsf{C}}$.

*Proof.* We first need to check that $I$ is indeed a functor. It suffices to check that $f : (X, \mathrm{id}_X) \to (Y, \mathrm{id}_Y)$ is indeed a morphism in $\widehat{\mathsf{C}}$, i.e.

$$\mathrm{id}_Y \circ f = f = f \circ \mathrm{id}_X,$$

yet this is merely the identity law in the category $\mathsf{C}$.The associative law and the identity law of $\widehat{\mathsf{C}}$ follow from those in the category $\mathsf{C}$.

For the fully-faithful-ness, we shall check that the map

$$\mathrm{Hom}_{\mathsf{C}}(X, Y) \to \mathrm{Hom}_{\widehat{\mathsf{C}}}((X, \mathrm{id}_X), (Y, \mathrm{id}_Y))$$

induced by the functor $I$ is a bijection. Yet this is clear from the definition of $I$ since $I$ does nothing but copying the morphism $f : X \to Y$ to $f : (X, \mathrm{id}_X) \to (Y, \mathrm{id}_Y)$. □

> **Problem 7.** Show that the embedding $I : \mathsf{C} \to \widehat{\mathsf{C}}$ exhibits $\widehat{\mathsf{C}}$ as idempotent completion of $\mathsf{C}$.

*Proof.* We shall prove that for any idempotent complete category $\mathsf{D}$, composition with $I$ induces an equivalence of category $\mathrm{Fun}(\widehat{\mathsf{C}},\mathsf{D}) \cong \mathrm{Fun}(\mathsf{C},\mathsf{D})$, i.e. the *restriction* functor

$$\begin{aligned} \phi : \mathrm{Fun}(\widehat{\mathsf{C}},\mathsf{D}) &\to \mathrm{Fun}(\mathsf{C},\mathsf{D}) \\ F &\mapsto F \circ I \end{aligned}$$

is an equivalence of categories.

Before actually proving, we make a remark. Since $\mathsf{D}$ is idempotent complete, for each idempotent morphism $X \xrightarrow{e} X$ in $\mathsf{D}$, we have a retract of $X$ as

$$\begin{tikzcd} & X \arrow[dr, "r"] & \\ Y \arrow[ur, "i"] \arrow[rr, "\mathrm{id}_Y"'] & & Y \end{tikzcd} . \tag{18}$$

For each such $e$, we *fix* such a retract and denote $X_e := Y$, $i_X := i$ and $r_X := r$. Note that retract of an idempotent morphism may not be unique, so here we might use the axiom of choice. [2] Moreover, for the identity morphism $\mathrm{id}_X : X \to X$, we *fix* the retract of $X$ as

$$\begin{tikzcd} & X \arrow[dr, "\mathrm{id}_X"] & \\ X \arrow[ur, "\mathrm{id}_X"] \arrow[rr, "\mathrm{id}_X"'] & & X \end{tikzcd} . \tag{19}$$

To prove that the restriction functor $\phi$ is an equivalence of categories, it suffices to show that $\phi$ is fully faithful and essentially surjective. We first show the latter one by constructing a *extension* functor as follows. Let $G \in \mathrm{Fun}(\mathsf{C},\mathsf{D})$, then we shall define a functor $\widehat{G} \in \mathrm{Fun}(\widehat{\mathsf{C}},\mathsf{D})$:

- For any $(X,e) \in \widehat{\mathsf{C}}$, since $e$ is idempotent, $G(e) : G(X) \to G(X)$ is an idempotent morphism in $\mathsf{D}$. We define

  $$\widehat{G}((X,e)) := G(X)_{G(e)}.$$

- For any morphism $(X,e) \xrightarrow{f} (Y,d)$, define $\widehat{G}(f)$ as the composition

  $$\widehat{G}((X,e)) = G(X)_{G(e)} \xrightarrow{i_{G(X)}} G(X) \xrightarrow{G(f)} G(Y) \xrightarrow{r_{G(Y)}} G(Y)_{G(d)} = \widehat{G}((Y,d)) = r_Y \circ G(f) \circ i_X,$$

  where we simplify the notations $r_Y := r_{G(Y)}$ and $i_X := i_{G(X)}$. [3]

[2] Unfortunately, this is not quite "canonical".

[3] We will make similar simplifications on other objects when discussing $r$ and $i$ in this proof.

We need to check that $\widehat{G}$ is indeed a functor. For the identity law, consider $\mathrm{id}_{(X,e)} : (X,e) \to (X,e)$ for any element $(X,e) \in \widehat{\mathsf{C}}$. Then

$$\widehat{G}(\mathrm{id}_{(X,e)}) = r_X \circ G(e) \circ i_X = r_X \circ (i_X \circ r_X) \circ i_X = (r_X \circ i_X) \circ (r_X \circ i_X) = \mathrm{id}_{\hat{G}((X,e))}.$$

For the associative law, consider $(X,e) \xrightarrow{f} (Y,d) \xrightarrow{g} (Z,c)$, then

$$\begin{aligned}
\widehat{G}(g) \circ \widehat{G}(f) &= (r_Z \circ G(g) \circ i_Y) \circ (r_Y \circ G(f) \circ i_X) \\
&= r_Z \circ G(g) \circ (i_Y \circ r_Y) \circ G(f) \circ i_X \\
&= r_Z \circ G(g) \circ (id\widehat{G}(Y)) \circ G(f) \circ i_X \\
&= r_Z \circ G(g) \circ G(f) \circ i_X \\
&= r_Z \circ G(g \circ f) \circ i_X \\
&= \widehat{G}(g \circ f).
\end{aligned}$$

Hence $\widehat{G}$ is indeed a functor. In this way, we can define an *extension* functor

$$\begin{aligned}
\psi : \mathrm{Fun}(\mathsf{C},\mathsf{D}) &\to \mathrm{Fun}(\widehat{\mathsf{C}},\mathsf{D}) \\
G &\mapsto \widehat{G}.
\end{aligned}$$

We claim that $\phi(\psi(G)) = G$, i.e. $\widehat{G} \circ I = G$. In fact, for any morphism $f : X \to Y$ in $\mathsf{C}$, $I(f) = [(X,\mathrm{id}_X) \xrightarrow{f} (Y,\mathrm{id}_Y)]$. Then with (19) in mind, we have

$$\begin{aligned}
\widehat{G}(I(f)) &= r_Y \circ G(f) \circ i_X \\
&= \mathrm{id}_{G(Y)} \circ G(f) \circ \mathrm{id}_{G(X)} \\
&= G(f).
\end{aligned}$$

Hence $\widehat{G} \circ I = G$ holds indeed. As another interpretation of this equality, $\phi$ is essentially surjective.

For the fully faithfulness[4], our goal is to show that the map induced from the restriction functor

$$\overline{\phi} : \mathrm{Hom}_{\mathrm{Fun}(\widehat{\mathsf{C}},\mathsf{D})}(F_1,F_2) \to \mathrm{Hom}_{\mathrm{Fun}(\mathsf{C},\mathsf{D})}(F_1 \circ I, F_2 \circ I)$$

[4]The proof is adapted from here: `https://ncatlab.org/toddtrimble/published/Karoubi+envelope`.

is bijective. Let $(\theta_X : F_1 \circ I(X) \to F_2 \circ I)$ be a natural transformation in $\mathrm{Hom}_{\mathsf{Fun}(\mathsf{C},\mathsf{D})}(F_1 \circ I, F_2 \circ I)$, then this can be rewritten as

$$\theta_X : F_1((X, \mathrm{id}_X)) \to F_2((X, \mathrm{id}_X)).$$

For any $(X,e) \in \widehat{\mathsf{C}}$, there exists a unique morphism [5] $F_1((X,e)) \xrightarrow{\xi_{(X,e)}} F_2((X,e))$, such that the following diagram commutes

$$\begin{tikzcd}
F_1((X,\mathrm{id}_X)) \arrow[r, "F_1(e)"] \arrow[d, "\theta_X"'] & F_1((X,e)) \arrow[r, "F_1(e)"] \arrow[d, "\xi_{(X,e)}"] & F_1((X,\mathrm{id}_X)) \arrow[d, "\theta_X"] \\
F_2((X,\mathrm{id}_X)) \arrow[r, "F_2(e)"'] & F_2((X,e)) \arrow[r, "F_2(e)"'] & F_2((X,\mathrm{id}_X))
\end{tikzcd} . \tag{20}$$

We now check that such defined $(\xi_{(X,e)} : F_1((X,e)) \to F_2((X,e)))$ is a natural transformation $F_1 \to F_2$. It suffices to check the naturality in $(X,e)$. This can be deduced from the naturality of $\theta$ and the diagram (20). In fact, for any $f : (X,e) \to (Y,d)$,

$$F_1(f) = F_1(d) \circ (F_1 \circ I)(f) \circ F_1(e),$$

and the same holds replacing $F_1$ by $F_2$ since $d \circ f = f = f \circ e$, where the latter follows from that $f$ is a morphism in $\widehat{\mathsf{C}}$. Note that the following diagram commutes:

$$\begin{tikzcd}
F_1((X,e)) \arrow[r, "F_1(e)"] \arrow[d, "\xi_{(X,e)}"'] \arrow[rrr, bend left, "F_1(f)"] & F_1((X,\mathrm{id}_X)) \arrow[r, "(F_1\circ I)(f)"] \arrow[d, "\theta_X"] & F_1((Y,\mathrm{id}_Y)) \arrow[r, "F_1(d)"] \arrow[d, "\theta_Y"] & F_1((Y,d)) \arrow[d, "\xi_{(Y,d)}"] \\
F_2((X,e)) \arrow[r, "F_2(e)"'] \arrow[rrr, bend right, "F_2(f)"'] & F_2((X,\mathrm{id}_X)) \arrow[r, "(F_2\circ I)(f)"'] & F_2((Y,\mathrm{id}_Y)) \arrow[r, "F_2(d)"'] & F_2((Y,d))
\end{tikzcd} , \tag{21}$$

where the commutativity of the left and the right square follows from (20), and the middle square commutes since $\theta$ is a natural transformation. In this way, we uniquely defined a natural transformation $(\xi_{(X,e)} : F_1((X,e)) \to F_2((X,e)))$, such that $\overline{\phi}(\xi) = \theta$. This finishes the proof of the fully faithfulness of the restriction $\phi$.

Therefore, $\phi$ is an equivalence of categories, as desired. □

[5] **A GIANT GAP** here that I'm not quite understand why "there exists a unique morphism".

**Problem 8.** Use Problem 7 to show that $\mathsf{C}$ is idempotent complete if and only if $\mathsf{C} \cong \widehat{\mathsf{C}}$.

*Proof.* If $\mathsf{C} \cong \widehat{\mathsf{C}}$, since $\widehat{\mathsf{C}}$ is idempotent complete, as shown in Problem 5, $\mathsf{C}$ is also idempotent complete. Conversely, suppose $\mathsf{C}$ is idempotent complete, we shall show that $\mathsf{C} \cong \widehat{\mathsf{C}}$. In Problem 6, we have shown that $I : \mathsf{C} \to \widehat{\mathsf{C}}$ is a fully faithful functor. So it remains to show that $I$ is essentially surjective. This is where we use Problem 7 [6].

Since $\mathsf{C}$ is idempotent complete, applying $\mathsf{D} := \mathsf{C}$ in Problem 7, we obtain

$$\mathrm{Fun}(\widehat{\mathsf{C}}, \mathsf{C}) \cong \mathrm{Fun}(\mathsf{C}, \mathsf{C}).$$

Consider the identity functor $\mathrm{id} : \mathsf{C} \to \mathsf{C}$, then by the above equivalence of category, we have its extension $\widehat{\mathrm{id}} : \widehat{\mathsf{C}} \to \mathsf{C}$. We claim that for any $(X, e) \in \widehat{\mathsf{C}}$,

$$(X, e) \cong I(\widehat{\mathrm{id}}((X, e))).$$

If the claim holds, then $I$ is indeed essentially surjective.

To show the claim, let $(X, e) \in \widehat{\mathsf{C}}$, then

$$\widehat{\mathrm{id}}((X, e)) = (\mathrm{id}_X)_{\mathrm{id}_e} = X_e,$$

where $X_e$ is a a priori fixed diagram

$$\begin{tikzcd} & X \arrow[dr, "r_X"] & \\ X_e \arrow[ur, "i_X"] \arrow[rr, "\mathrm{id}_X"'] & & X \end{tikzcd} \tag{22}$$

and

$$e = i_X \circ r_X. \tag{23}$$

Hence $I(\widehat{\mathrm{id}}((X, e))) = (X_e, \mathrm{id}_{X_e})$. We shall check that

$$i_X : (X_e, \mathrm{id}_{X_e}) \to (X, e) \text{ and } r_X : (X, e) \to (X_e, \mathrm{id}_{X_e})$$

gives the desired isomorphism $(X, e) \cong I(\widehat{\mathrm{id}}((X, e)))$.

[6] In fact, we are using the proof of Problem 7, especially the construction of the extension functor, not just the result.

Before that, we shall check that $i_X$ and $r_X$ are indeed morphisms in $\widehat{\mathsf{C}}$. For $i_X$, we need to check that

$$e \circ i_X = i_X = i_X \circ \mathrm{id}_{X_e},$$

yet the left hand side is

$$e \circ i_X = (i_X \circ r_X) \circ i_X = i_X \circ (r_X \circ i_X) = i_X \circ \mathrm{id}_{X_e},$$

as desired. For $r_X$, we need to check that

$$\mathrm{id}_X \circ r_X = r_X = r_X \circ e,$$

yet the right hand side is

$$r_X \circ e = r_X \circ (i_X \circ r_X) = (r_X \circ i_X) \circ r_X = \mathrm{id}_{X_e} \circ r_X,$$

as desired. Therefore, $i_X$ and $r_X$ are indeed morphisms in $\widehat{\mathsf{C}}$.

It remains to check that $i_X \circ r_X = \mathrm{id}_{(X,e)}$ and $r_X \circ i_X = \mathrm{id}_{(X_e,\mathrm{id}_{X_e})}$. Yet the first one is

$$[X \xrightarrow{r_X} X_e \xrightarrow{i_X} X] = [X \xrightarrow{e} X],$$

which is merely (23), and the second one is

$$[X_e \xrightarrow{i_X} X \xrightarrow{r_X} X_e] = [X_e \xrightarrow{\mathrm{id}_{X_e}} X_e],$$

which is merely the commutativity of the diagram (22). Therefore we have $(X,e) \cong I(\widehat{\mathrm{id}}((X,e)))$, as desired. $\square$

> **Problem 9.** Show that if $X \in \mathsf{S}$ is completely compact then any retract of $X$ is completely compact.

*Proof.* Let $X \in \mathsf{S}$ is completely compact and $Y$ is a retract of $X$, i.e. the following diagram commutes

$$\begin{tikzcd} & X \arrow[dr, "r"] & \\ Y \arrow[ur, "i"] \arrow[rr, "\mathrm{id}_Y"'] & & Y \end{tikzcd} . \tag{24}$$

We shall show that $Y$ is completely compact.

Applying the Hom-functor $\mathrm{Hom}_{\mathsf{S}}(-,Z) : \mathsf{S}^{\mathrm{opp}} \to \mathsf{Set}$ to the diagram (24), we obtain a commutative diagram

$$\begin{tikzcd}
& \mathrm{Hom}_{\mathsf{S}}(X,Z) \arrow[dl, "i_Z^*"'] & \\
\mathrm{Hom}_{\mathsf{S}}(Y,Z) & & \mathrm{Hom}_{\mathsf{S}}(Y,Z) \arrow[ul, "r_Z^*"'] \arrow[ll, "\mathrm{id}_{\mathrm{Hom}_{\mathsf{S}}(Y,Z)}"]
\end{tikzcd} . \tag{25}$$

We claim that this is natural in $Z$, i.e. $\mathrm{Hom}_{\mathsf{S}}(X,-)$ is a retract of $\mathrm{Hom}_{\mathsf{S}}(Y,-)$ via the above diagram in the funtor category $\mathrm{Fun}(\mathsf{S},\mathsf{Set})$. To prove this claim, it suffices to check that $(i_Z^* : \mathrm{Hom}_{\mathsf{S}}(X,Z) \to \mathrm{Hom}_{\mathsf{S}}(Y,Z))$ and $(r_Z^* : \mathrm{Hom}_{\mathsf{S}}(Y,Z) \to \mathrm{Hom}_{\mathsf{S}}(X,Z))$ are natural in $Z$.

For $i_Z^*$, consider $Z \xrightarrow{f} W$, the square

$$\begin{tikzcd}
\mathrm{Hom}_{\mathsf{S}}(X,Z) \arrow[r, "i_Z^*"] \arrow[d, "f_*"'] & \mathrm{Hom}_{\mathsf{S}}(Y,Z) \arrow[d, "f_*"] \\
\mathrm{Hom}_{\mathsf{S}}(X,W) \arrow[r, "i_W^*"'] & \mathrm{Hom}_{\mathsf{S}}(Y,W)
\end{tikzcd} \tag{26}$$

commutes, since for any $g \in \mathrm{Hom}_{\mathsf{S}}(X,Z)$, the two ways leading to $\mathrm{Hom}_{\mathsf{S}}(Y,W)$ both sends $g$ to $f \circ g \circ i$. Hence, $(i_Z^* : \mathrm{Hom}_{\mathsf{S}}(X,Z) \to \mathrm{Hom}_{\mathsf{S}}(Y,Z))$ is indeed a natural transformation.

Similarly, for $r_Z^*$, consider $Z \xrightarrow{f} W$, the square

$$\begin{tikzcd}
\mathrm{Hom}_{\mathsf{S}}(Y,Z) \arrow[r, "r_Z^*"] \arrow[d, "f_*"'] & \mathrm{Hom}_{\mathsf{S}}(X,Z) \arrow[d, "f_*"] \\
\mathrm{Hom}_{\mathsf{S}}(Y,W) \arrow[r, "r_W^*"'] & \mathrm{Hom}_{\mathsf{S}}(X,W)
\end{tikzcd} \tag{27}$$

commutes, since for any $g \in \mathrm{Hom}_{\mathsf{S}}(Y,Z)$, the two ways leading to $\mathrm{Hom}_{\mathsf{S}}(X,W)$ both sends $g$ to $f \circ g \circ r$. Hence, $(r_Z^* : \mathrm{Hom}_{\mathsf{S}}(Y,Z) \to \mathrm{Hom}_{\mathsf{S}}(X,Z))$ is indeed a natural transformation.

To sum up, we have proved that the following diagram gives a retract of $\mathrm{Hom}_{\mathsf{S}}(Y,-)$:

$$\begin{tikzcd}
& \mathrm{Hom}_{\mathsf{S}}(X,-) \arrow[dl, "i^*"'] & \\
\mathrm{Hom}_{\mathsf{S}}(Y,-) & & \mathrm{Hom}_{\mathsf{S}}(Y,-) \arrow[ul, "r^*"'] \arrow[ll, "\mathrm{id}_{\mathrm{Hom}_{\mathsf{S}}(Y,-)}"]
\end{tikzcd} . \tag{28}$$

As shown in Problem 1, $\mathrm{Hom}_{\mathsf{S}}(Y,-)$ is a coequalizer of the diagram

$$\mathrm{Hom}_{\mathsf{S}}(X,-) \underset{\mathrm{id}_{\mathrm{Hom}_{\mathsf{S}}(X,-)}}{\overset{r^*\circ i^*}{\rightrightarrows}} \mathrm{Hom}_{\mathsf{S}}(X,-) \ ,$$

which is in fact a colimit. Since $X$ is completely compact, the functor $\mathrm{Hom}_{\mathsf{S}}(X,-)$ preserves all small colimits. Hence a small colimit of such functors (in fact, here is merely a finite limit) still preserves colimits as any two colimits commute. Hence, $\mathrm{Hom}_{\mathsf{S}}(Y,-)$ preserves colimits, which means that $Y$ is also completely compact, as desired. $\square$

**Problem 10.** Show that $\mathscr{P}(\mathsf{C}) \cong \mathscr{P}(\widehat{\mathsf{C}})$.

*Proof.* In Problem 7, we proved that for any idempotent complete category $\mathsf{D}$, we have an equivalence of categories

$$\mathrm{Fun}(\widehat{\mathsf{C}},\mathsf{D}) \cong \mathrm{Fun}(\mathsf{C},\mathsf{D}).$$

*Suppose* the category of sets is idempotent complete, then put $\mathsf{C} := \mathsf{C}^{\mathrm{opp}}$, then

$$\mathrm{Fun}(\widehat{\mathsf{C}^{\mathrm{opp}}},\mathsf{Set}) \cong \mathrm{Fun}(\mathsf{C}^{\mathrm{opp}},\mathsf{Set}).$$

Note that $\widehat{\mathsf{C}^{\mathrm{opp}}} = \widehat{\mathsf{C}}^{\mathrm{opp}}$, we have

$$\mathrm{Fun}(\widehat{\mathsf{C}}^{\mathrm{opp}},\mathsf{Set}) \cong \mathrm{Fun}(\mathsf{C}^{\mathrm{opp}},\mathsf{Set}),$$

which is just $\mathscr{P}(\mathsf{C}) \cong \mathscr{P}(\widehat{\mathsf{C}})$, as desired.

So it remains to prove that the category of sets, $\mathsf{Set}$, is idempotent complete. Let $X$ be a set and $e : X \to X$ be an idempotent map. We claim that for any $y \in \mathrm{im}(e)$, $y$ is fixed by $e$. In fact, for $y \in \mathrm{im}(e)$, $y = e(x)$ for some $x \in X$. Then

$$e(y) = e(e(x)) = (e \circ e)(x) = e(x)$$

since $e$ is idempotent. In other words, $e|_{\mathrm{im}(e)} = \mathrm{id}_{\mathrm{im}(e)}$. This can be interpreted as the commutativity of the following diagram:

$$\begin{tikzcd} & X \arrow[dr, "r:=e"] & \\ \mathrm{im}(e) \arrow[ur, "i:=\mathrm{incl.}"] \arrow[rr, "\mathrm{id}_{\mathrm{im}(e)}"'] & & \mathrm{im}(e) \end{tikzcd} \ , \tag{29}$$

where incl. : $\mathrm{im}(e) \to X$ is the inclusion map. Obviously, $i \circ r = \mathrm{incl.} \circ e = e$. Hence, $\mathrm{im}(e)$ is a retract of $X$ and $e$ comes from this retract. Therefore, Set is indeed idempotent complete. $\square$

> **Problem 11.** Show that $j(X) \in \mathscr{P}(\mathsf{C})$ is completely compact for any $X \in \mathsf{C}$.

*Proof.* Let $\alpha : \mathsf{I} \to \mathscr{P}(\mathsf{C})$ be a diagram in $\mathscr{P}(\mathsf{C})$, which admits a colimit $\varinjlim_{i\in\mathsf{I}} \alpha(i)$ in $\mathscr{P}(\mathsf{C})$. Then we have

$$\begin{aligned}
\varinjlim_{i\in\mathsf{I}} \mathrm{Hom}_{\mathscr{P}(\mathsf{C})}(j(X), \alpha(i)) &\xrightarrow{\sim} \varinjlim_{i\in\mathsf{I}} \alpha(i)(X) \\
&\xrightarrow{\sim} (\varinjlim_{i\in\mathsf{I}} \alpha(i))(X) \\
&\xrightarrow{\sim} \mathrm{Hom}_{\mathscr{P}(\mathsf{C})}(j(X), \varinjlim_{i\in\mathsf{I}} \alpha(i)).
\end{aligned}$$

Here [7], the first and the third "$\xrightarrow{\sim}$" come from the Yoneda Lemma. The second one is by the concrete construction of colimits in the category $\mathscr{P}(\mathsf{C})$, where colimits of presheaves are computed objectwise. Therefore, $j(X) \in \mathscr{P}(\mathsf{C})$ is completely compact, as desired. $\square$

> **Problem 12.** Show that if $S \in \mathscr{P}(\mathsf{C})$ is completely compact, then there exists $X \in \mathsf{C}$ such that $S$ is a retract of $j(X)$.

*Proof.* As $S \in \mathscr{P}(\mathsf{C})$, $S$ can be written as a small colimit of objects in $j(\mathsf{C})$. In other words, there exists a functor $\alpha : \mathsf{I} \to \mathscr{P}(\mathsf{C})$ such that $\alpha(i) = j(X_i)$ for some $X_i \in \mathsf{C}$, for each $i \in \mathsf{I}$, and

$$S = \varinjlim_{i\in\mathsf{I}} \alpha(i) = \varinjlim_{i\in\mathsf{I}} j(X_i).$$

Let $\iota_i : \alpha(i) \to \varinjlim_{i\in\mathsf{I}} j(X_i)$ be the canonical morphism.

Then we have [8]

$$\mathrm{Hom}_{\mathscr{P}(\mathsf{C})}(S,S) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{P}(\mathsf{C})}(S, \varinjlim_{i\in\mathsf{I}} j(X_i)) \xleftarrow{\sim} \varinjlim_{i\in\mathsf{I}} \mathrm{Hom}_{\mathscr{P}(\mathsf{C})}(S, j(X_i)), \tag{30}$$

[7] Note that the colimits on the first line are in the category of sets, while on the second line, the colimit is taken in $\mathscr{P}(\mathsf{C})$.

[8] Here the direction of each arrow is the "canonical" direction, yet each one is an isomorphism.

where the second isomorphism holds since $S$ is completely compact.

Now we consider the image of the identity morphism $\mathrm{id}_S \in \mathrm{Hom}_{\mathscr{P}(\mathsf{C})}(S,S)$ via the isomorphism (30). By the concrete construction of small colimits in the category of sets, the image lies in one $\mathrm{Hom}_{\mathscr{P}(\mathsf{C})}(S, j(X_{i_0}))$ for some $i \in \mathsf{I}$. We denote it by $\psi : S \to j(X_{i_0})$.

We claim that $S$ is a retract of $j(X_{i_0})$ via the diagram

$$\begin{tikzcd} & j(X_{i_0}) \arrow[dr, "\iota_{i_0}"] & \\ S \arrow[ur, "\psi"] \arrow[rr, "\mathrm{id}_S"'] & & S \end{tikzcd} . \tag{31}$$

To prove the claim, it suffices to check that the above diagram commutes. This is obvious once we explicitly spell out the isomorphism (30), especially the second isomorphism. The second arrow come from the following construction. We have the canonical isomorphisms $\iota_i : \alpha(i) \to \varinjlim_{i\in\mathsf{I}} j(X_i)$, then applying the functor $\mathrm{Hom}_{\mathscr{P}(\mathsf{C})}(S,-) : \mathscr{P}(\mathsf{C}) \to \mathsf{Set}$, we obtain

$$(\iota_i)_* : \mathrm{Hom}_{\mathscr{P}(\mathsf{C})}(S, j(X_i)) \to \mathrm{Hom}_{\mathscr{P}(\mathsf{C})}(S, \varinjlim_{i\in\mathsf{I}} j(X_i)) \cong \mathrm{Hom}_{\mathscr{P}(\mathsf{C})}(S,S).$$

The second isomorphism in (30) is obtained by the universal property of colimit

$$\varinjlim_{i\in\mathsf{I}} \mathrm{Hom}_{\mathscr{P}(\mathsf{C})}(S, \alpha(i))$$

and the complete compactness of $S$. Now focusing on $(\iota_{i_0})_*$, we obtain that

$$(\iota_{i_0})_*(\psi) = \psi \circ \iota_{i_0} = \mathrm{id}_S.$$

This is exactly the commutativity of the diagram (31). Hence we have finished the proof. □

**Problem 13.** Let $\widetilde{\mathsf{C}} \subset \mathscr{P}(\mathsf{C})$ be the full subcategory consisting of the completely compact objects. Show that the embedding $j : \mathsf{C} \to \widetilde{\mathsf{C}}$ exhibits $\widetilde{\mathsf{C}}$ as idempotent completion of $\mathsf{C}$.

*Proof (Not finished).* Firstly, we shall provide an alternative description of $\widetilde{\mathsf{C}}$. We claim that $\widetilde{\mathsf{C}}$ is the full subcategory of $\mathscr{P}(\mathsf{C})$ consisting of retracts of objects in $j(\mathsf{C})$.

- Let $S \in \widetilde{\mathsf{C}}$ be a completely compact object in $\mathscr{P}(\mathsf{C})$, then by Problem 12, $S$ is a retract of $j(X)$ for some $X \in \mathsf{C}$.
- In Problem 11, we have shown that $j(X)$ is completely compact for any $X \in \mathsf{C}$. Then by Problem 9, any retract of $j(X)$ is completely compact.

By the above two points, we see that the claim indeed holds.

**(NOT FINISHED)** $\square$