Talk 01. Prerequisites on category theory. Motivations

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(1.1) Definition

1. category,

定义 2.1.1 一个范畴 $\mathcal{C}$ 系指以下资料:

1. 集合 $\mathrm{Ob}(\mathcal{C})$, 其元素称作 $\mathcal{C}$ 的**对象**.
2. 集合 $\mathrm{Mor}(\mathcal{C})$, 其元素称作 $\mathcal{C}$ 的**态射**, 配上一对映射 $\mathrm{Mor}(\mathcal{C}) \underset{t}{\overset{s}{\rightrightarrows}} \mathrm{Ob}(\mathcal{C})$, 其中 s 和 t 分别给出态射的**来源**和**目标**. 对于 $X, Y \in \mathrm{Ob}(\mathcal{C})$, 一般习惯记 $\mathrm{Hom}_{\mathcal{C}}(X,Y) := s^{-1}(X) \cap t^{-1}(Y)$ 或简记为 $\mathrm{Hom}(X,Y)$, 称为 Hom-集, 其元素称为从 X 到 Y 的态射.

   source

   target

3. 对每个对象 X 给定元素 $\mathrm{id}_X \in \mathrm{Hom}_{\mathcal{C}}(X,X)$, 称为 X 到自身的**恒等态射**.
4. 对于任意 $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$, 给定态射间的**合成映射**

$$\circ : \mathrm{Hom}_{\mathcal{C}}(Y,Z) \times \mathrm{Hom}_{\mathcal{C}}(X,Y) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(X,Z)$$
$$(f,g) \longmapsto f \circ g,$$

$$\begin{tikzcd} & B \arrow[dr] & \\ A \arrow[ur] \arrow[rr, dashed] & & C \end{tikzcd}$$

不致混淆时常将 $f \circ g$ 简记为 fg. 它满足

✓ (i) 结合律: 对于任意态射 $h, g, f \in \mathrm{Mor}(\mathcal{C})$, 若合成 $f(gh)$ 和 $(fg)h$ 都有定义, 则

$$f(gh) = (fg)h.$$

故两边可以同写为 $f \circ g \circ h$ 或 fgh;

✓ (ii) 对于任意态射 $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$, 有

$$f \circ \mathrm{id}_X = f = \mathrm{id}_Y \circ f.$$

(1.1) Definition

1. category,
2. subcategory,
3. full subcategory,

定义 2.1.2 称 $\mathcal{C}'$ 是 $\mathcal{C}$ 的**子范畴**, 如果

(i) $\mathrm{Ob}(\mathcal{C}') \subset \mathrm{Ob}(\mathcal{C})$;

(ii) $\mathrm{Mor}(\mathcal{C}') \subset \mathrm{Mor}(\mathcal{C})$, 并保持恒等态射;

(iii) 来源/目标映射 $\mathrm{Mor}(\mathcal{C}') \overset{s}{\underset{t}{\rightrightarrows}} \mathrm{Ob}(\mathcal{C}')$ 是由 $\mathcal{C}$ 限制而来的, 而且

(iv) $\mathcal{C}'$ 中态射的合成也是由 $\mathcal{C}$ 限制而来的.

简言之, 对任意 $\mathcal{C}'$ 中对象 X, Y, 有包含关系 $\mathrm{Hom}_{\mathcal{C}'}(X,Y) \subset \mathrm{Hom}_{\mathcal{C}}(X,Y)$, 它与态射的合成兼容. 如果 $\mathrm{Hom}_{\mathcal{C}'}(X,Y) = \mathrm{Hom}_{\mathcal{C}}(X,Y)$ 则称 $\mathcal{C}'$ 是**全子范畴**.

eg: Grp $\supseteq$ Ab

(1.1) Definition

1. category,
2. subcategory,
3. full subcategory,
4. monomorphism, epimorphism.

定义 2.1.7 设 X, Y 为范畴 $\mathcal{C}$ 中的对象, $f: X \to Y$ 为态射.

- ◇ 称 f 为**单态射**, 如果对任何对象 Z 和任一对态射 $g, h: Z \to X$ 有 $fg = fh \iff g = h$ (左消去律);
- ◇ 称 f 为**满态射**, 如果对任何对象 Z 和任一对态射 $g, h: Y \to Z$ 有 $gf = hf \iff g = h$ (右消去律).

(1.2) Definition

1. functor,
2. essentially surjective, : 对象
3. faithful / full, : 态射

$\begin{cases} \text{faithful}: & F \text{ is injective} \\ \text{full}: & F \text{ is surjective} \end{cases}$ + fully faithful

定义 2.2.1 (函子) 设 $\mathcal{C}', \mathcal{C}$ 为范畴. 一个函子 $F : \mathcal{C}' \to \mathcal{C}$ 意谓以下资料:

(i) 对象间的映射 $F : \mathrm{Ob}(\mathcal{C}') \to \mathrm{Ob}(\mathcal{C})$.

$\forall Y \in ob(\mathcal{C}),\ \exists X \in ob(\mathcal{C}'),\ s.t.\ Y \cong FX$

(ii) 态射间的映射 $F : \mathrm{Mor}(\mathcal{C}') \to \mathrm{Mor}(\mathcal{C})$, 使得

- ✓ F 与来源和目标映射相交换 (即 $sF = Fs$, $tF = Ft$), 等价的说法是对每个 $X, Y \in \mathrm{Ob}(\mathcal{C}')$ 皆有映射 $F : \mathrm{Hom}_{\mathcal{C}'}(X, Y) \to \mathrm{Hom}_{\mathcal{C}}(FX, FY)$;
- ✓ $F(g \circ f) = F(g) \circ F(f)$, $F(\mathrm{id}_X) = \mathrm{id}_{FX}$.

$$\begin{tikzcd} X \arrow[d, "f"'] \arrow[r, squiggly, "F"] & FX \arrow[d, "Ff"] \\ Y & FY \end{tikzcd}$$

- 对于 $F : \mathcal{C}_1 \to \mathcal{C}_2$, $G : \mathcal{C}_2 \to \mathcal{C}_3$, 合成函子 $G \circ F : \mathcal{C}_1 \to \mathcal{C}_3$ 的定义是显然的: 取合成映射

$$\begin{tikzcd} \mathcal{C}_1 \arrow[r, "F"] \arrow[rr, bend right, dashed, "G \circ F"'] & \mathcal{C}_2 \arrow[r, "G"] & \mathcal{C}_3 \end{tikzcd}$$

✓ $$\mathrm{Ob}(\mathcal{C}_1) \xrightarrow{F} \mathrm{Ob}(\mathcal{C}_2) \xrightarrow{G} \mathrm{Ob}(\mathcal{C}_3),$$

✓ $$\mathrm{Mor}(\mathcal{C}_1) \xrightarrow{F} \mathrm{Mor}(\mathcal{C}_2) \xrightarrow{G} \mathrm{Mor}(\mathcal{C}_3).$$

1. functor,
2. essentially surjective,
3. faithful / full,
4. natural transformation,
5. functor category. $\mathrm{Fun}(\mathcal{C}', \mathcal{C})$

$$\begin{tikzcd} \mathcal{C}' \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\theta"] & \mathcal{C} \end{tikzcd}$$

$$\begin{tikzcd} X \arrow[d, "f"'] & FX \arrow[r, "\theta_X"] \arrow[d, "Ff"'] & GX \arrow[d, "Gf"] \\ Y & FY \arrow[r, "\theta_Y"'] & GY \end{tikzcd}$$

$\mathcal{C}'$

$\theta = \{\theta_X\}_{X \in \mathrm{Ob}(\mathcal{C}')}$

定义 2.2.5 (自然变换, 或函子间的态射) 函子 $F, G : \mathcal{C}' \to \mathcal{C}$ 之间的自然变换 θ 是一族态射

$$\theta_X \in \mathrm{Hom}_{\mathcal{C}}(FX, GX), \quad X \in \mathrm{Ob}(\mathcal{C}'),$$

使得下图对所有 $\mathcal{C}'$ 中的态射 $f : X \to Y$ 交换

$$\begin{tikzcd} FX \arrow[r, "\theta_X"] \arrow[d, "Ff"'] & GX \arrow[d, "Gf"] \\ FY \arrow[r, "\theta_Y"'] & GY. \end{tikzcd} \tag{2.1}$$

$$\begin{tikzcd} X \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\theta\Downarrow"] & Y \end{tikzcd}$$

上述自然变换写作 $\theta : F \to G$, 或图解为

$$\begin{tikzcd} \mathcal{C}' \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\theta"] & \mathcal{C}. \end{tikzcd}$$

◇ 考虑 $\mathcal{C}'$ 到 $\mathcal{C}$ 的三个函子间的态射 $\theta : F \to G$, $\psi : G \to H$. 纵合成 $\psi \circ \theta$ 的定义是 $\{\psi_X \circ \theta_X : X \in \mathrm{Ob}(\mathcal{C})\}$, 图解作

$$\begin{tikzcd} \mathcal{C}' \arrow[r, bend left=60, "F"] \arrow[r, "G" description] \arrow[r, bend right=60, "H"'] & \mathcal{C} \end{tikzcd} \quad \text{合成为} \quad \begin{tikzcd} \mathcal{C}' \arrow[r, bend left, "F"] \arrow[r, bend right, "H"'] \arrow[r, phantom, "\Downarrow\psi\circ\theta"] & \mathcal{C}. \end{tikzcd}$$

(1.3) Definition

1. equivalence of categories,
2. isomorphism of categories.

$$\begin{tikzcd} \mathcal{C}_1 \arrow[r, bend left, "F"] & \mathcal{C}_2 \arrow[l, bend left, "G"] \end{tikzcd}$$

✓ ① equivalence : $\exists$ isomorphism of functors

.. $\theta : FG \xrightarrow{\sim} id_{\mathcal{C}_2}$

.. $\psi : GF \xrightarrow{\sim} id_{\mathcal{C}_1}$

say $\mathcal{C}_1$ & $\mathcal{C}_2$: are equivalent

G : quasi-inverse of F

F : ———— G

✓ ② isomorphism : $FG = id_{\mathcal{C}_2}$

$GF = id_{\mathcal{C}_1}$

say $\mathcal{C}_1$ & $\mathcal{C}_2$ are isomorphic

定理

Let $F : \mathcal{C} \to \mathcal{D}$ be a functor, then the following statements are equivalent:

(a) *F is an equivalence of categories,*

(b) *F is fully faithful and essentially surjective.*

- $F : \mathcal{C} \longrightarrow \mathcal{D}$
- prove: F: fully / faithful
- ? Pick subcat of $\mathcal{D}$, say $\mathcal{D}'$, s.t. $F : \mathcal{C} \longrightarrow \mathcal{D}'$

  and F: essentially surj.

$(Var) \xrightarrow{t} (Sch)$

$\cup$

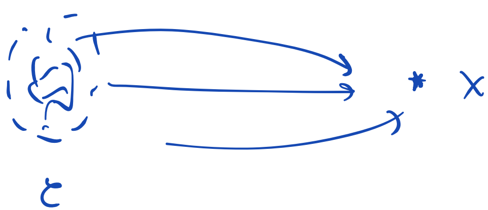
subcat of schemes

(1.5) Definition

- initial object,
- final object,
- zero object.

* initial object: $X \in ob(\mathcal{C})$, if $\forall\, Y \in ob(\mathcal{C})$, $Hom(X,Y)$ is a singleton

* final object: $X \in ob(\mathcal{C})$, if $\forall\, Y \in ob(\mathcal{C})$, $Hom(Y,X)$ is a sington



(1.6) Example-Exc

Category	Initial objects	Final object	Zero object	Morphism
Set				
Top				
Top_*	(X,x) $(X,x) \xrightarrow{f} (Y,y)$	$f: X \to Y$, $f(x) = y$	$\pi_1 : Top_* \longrightarrow Grp$, $(X,x) \longmapsto \pi_1(X,x)$	
Grp				
Ab				
Ring				
CRing				
Poset	minimal element	maximal obj.	—	"$\leq$"

Λ : poset

obj: Λ 中的元素

mor: $x \leq y$: $x \longrightarrow y$

$x \leq y, y \leq z \Longrightarrow x \leq z$: $x \longrightarrow y \longrightarrow z$

$x \leq x$: $x \circlearrowleft id_x$

(1.7) Definition

- Presheaf category

functional analysis : V normed vector space

V^* Banach space

V^{**} dual dual space

- $T : V \longrightarrow V^{**}$

$x \longmapsto [X : f \mapsto f(x)]$

- Fact : V is isometric to a subspace of V^{**}

Let $\mathcal{C}$ category. Defn : $\mathcal{C}^\wedge := \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Set})$: presheaf cat. of $\mathcal{C}$

$$h_{\mathcal{C}} : \mathcal{C} \longrightarrow \mathcal{C}^\wedge = \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Set})$$

$$S \longmapsto \mathrm{Hom}_{\mathcal{C}}(-, S) = h_{\mathcal{C},S} = h_S$$

② $h_{\mathcal{C}}$: fully faithful ? essentially surjective ?

(1.8) Yoneda Lemma

定理

Yoneda's lemma

Let $S \in ob(\mathcal{C})$, $A \in \mathcal{C}^\wedge$, then the map

$A \in \mathcal{C}^\wedge := Fun(\mathcal{C}^{op}, Set)$

$A : \mathcal{C}^{op} \to Set$

$S \mapsto A(S)$

$$\theta_S : Hom_{\mathcal{C}^\wedge}(h_{\mathcal{C}}(S), A) \longrightarrow A(S)$$

$$[\, Hom_{\mathcal{C}}(-, S) \xrightarrow{\phi} A \,] \longmapsto \phi_S(id_S)$$

$$Hom_{\mathcal{C}}(-, S) = h_{\mathcal{C}}(S)$$

$\phi : Hom_{\mathcal{C}}(-, S) \xrightarrow{\phi} A$

$\phi_S : Hom_{\mathcal{C}}(S, S) \xrightarrow{\phi_S} A(S)$

$id_S \xmapsto{\phi_S} \phi_S(id_S)$

is bijective. It gives an isomorphism of functors

$$\rightsquigarrow \quad Hom_{\mathcal{C}^\wedge}(h_{\mathcal{C}}(-), -) \xrightarrow{\sim} ev^\wedge : \mathcal{C}^{op} \times \mathcal{C}^\wedge \longrightarrow Set$$

$$(S, A) \longmapsto A(S)$$

(1.9) Corollary-Definition

定理

The functor $h_{\mathcal{C}}$ is fully faithful. It is called the Yoneda embedding *of* $\mathcal{C}$.

~~Proof of (1.9)~~ : $A = h_{\mathcal{C}}(T)$, $\forall T \in ob(\mathcal{C})$

$$\theta_S : \mathrm{Hom}_{\mathcal{C}^\wedge}(h_{\mathcal{C}}(S), h_{\mathcal{C}}(T)) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(S, T)$$

$$[\underbrace{\mathrm{Hom}_{\mathcal{C}}(-,S)}_{= h_{\mathcal{C}}(S)} \xrightarrow{\phi} \mathrm{Hom}_{\mathcal{C}}(-,T)] \longmapsto \phi_S(id_S)$$

$\Longrightarrow$ $h_{\mathcal{C}}$ is fully faithful.

$h_{\mathcal{C}} : \mathcal{C} \longrightarrow \mathcal{C}^\wedge$

Fact : fully faithful functors preserve and reflect isomorphisms ! $\square$

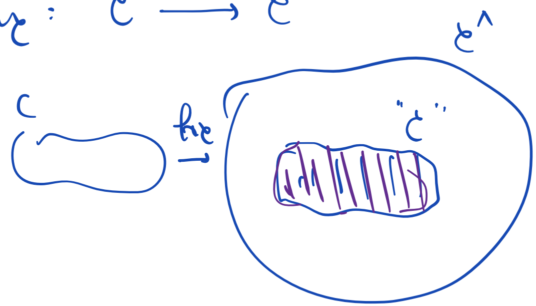
$F : \mathcal{C} \to \mathcal{D}$ fully faithful functor

$X, Y \in ob(\mathcal{C})$,

$$X \underset{f}{\simeq} Y \iff FX \overset{Ff}{\simeq} FY$$

- Representable functor

$h_{\mathcal{C}} : \mathcal{C} \longrightarrow \mathcal{C}^{\wedge}$



Defn: $A : \mathcal{C}^{op} \longrightarrow \mathrm{Set}$ is representable, if $\exists\, X \in \mathcal{C}$, s.t.

$$\phi : h_{\mathcal{C},X} \longrightarrow A$$

ϕ: $A(-) \overset{\phi}{\xleftarrow{\simeq}} \mathrm{Hom}_{\mathcal{C}}(-, X)$ is represented by (X, ϕ)

→ preserve 保

反 reflect ←

$\mathcal{C}: \quad X \to Y \in ob(\mathcal{C})$

$\downarrow$

$\mathcal{C}^{\wedge}: \quad h_{\mathcal{C}}(X) \simeq h_{\mathcal{C}}(Y)$

反例: $V \xrightarrow{\sim} V^{**}$

? V 自反的 ?

- Representable functor

1° Example: $\mathcal{C}$ category.

- $F: \mathcal{C}^{op} \longrightarrow Set$

$X \longmapsto \phi \qquad \forall X \in \mathcal{C}^{op}$

Fact: If F is rep. by (X, ϕ), then

$F(X) \simeq \underbrace{Hom_{\mathcal{C}}(X, X)}_{h_{\mathcal{C},X}(X)} \overset{\text{defn of cat.}}{\ni} id_X$ so $F(X)$ is nonempty

Contradiction!

2° Density of the essen. im. of $h_{\mathcal{C}}$: $\mathcal{C}^{\wedge}$ Liu. Vol.2. Thm 1.7.3

(1.11) Definition

- Adjoints

Defn: Adjoint pair : (F, G, φ), where

$$\begin{tikzcd} \mathcal{C}_1 \arrow[r, bend left, "F"] & \mathcal{C}_2 \arrow[l, bend left, "G"] \end{tikzcd}$$

and a natural isomorphism of functors

$$\varphi : \mathrm{Hom}_{\mathcal{C}_2}(F(-), -) \longrightarrow \mathrm{Hom}_{\mathcal{C}_1}(-, G(-))$$

- F is a left adjoint of G
- G - - right ——— F

- Units and counits

"$\phi_S(id_S)$"

For any fixed $X \in ob(\mathcal{C}_1)$,

$$\begin{array}{ccc} X & \xrightarrow{id_X} & X \\ & \Big\downarrow F(-) & \\ FX & \xrightarrow{id_{FX}} & FX \\ & \Big\downarrow \text{adjoint} & \\ X & \xrightarrow[\eta_X]{} & GFX \end{array} \qquad \forall X \in ob(\mathcal{C}_1)$$

$$\varphi : Hom_{\mathcal{C}_2}(F(-), -) \longrightarrow Hom_{\mathcal{C}_1}(-, G(-))$$

$-_1 : X$

$-_2 : FX$

$$Hom_{\mathcal{C}_2}(FX, FX) \xrightarrow[\sim]{1:1} Hom_{\mathcal{C}_1}(X, GFX)$$

$$id_{FX} \xmapsto[\varphi_{X,FX}]{} X \xrightarrow{\eta_X} GFX$$

Proof: η is natural transformati

$\eta = (\eta_X)_{X \in ob(\mathcal{C}_1)}$

- Units and counits

For $\forall Y \in ob(\mathcal{C}_2)$:

φ : $Hom_{\mathcal{C}_2}(F(-_1), -_2) \xrightarrow{\sim} Hom_{\mathcal{C}_1}(-_1, G(-_2))$

$-_1 : GY,$

$-_2 : Y$

$Hom_{\mathcal{C}_2}(FGY, Y) \xrightarrow{\sim} Hom_{\mathcal{C}_1}(GY, GY)$

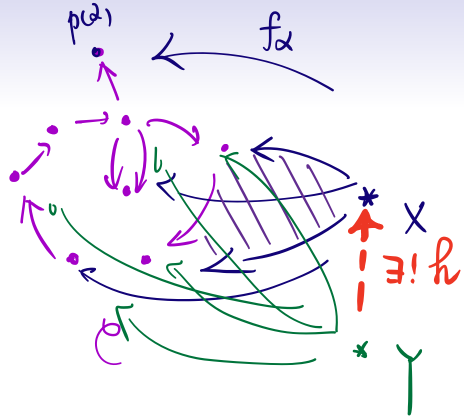
$[FGY \xrightarrow[\varepsilon_Y]{} Y] \xleftrightarrow{1:1} id_{GY}$

$$\begin{array}{c} Y \xrightarrow{id_Y} Y \\ \Downarrow G(-) \checkmark \\ GY \xrightarrow{id_{GY}} GY \\ \Downarrow \text{adjoint} \checkmark \\ FGY \xrightarrow[\varepsilon_Y]{} Y \quad ** \end{array}$$

$\forall Y \in ob(\mathcal{C}_2)$

$\therefore \varepsilon_Y$: natural tran.

(1.13) Definition



- Diagram,
- Limits, colimits.

∘ diagram: $\mathcal{J}$, $\mathcal{C}$. $p : \underbrace{\mathcal{J}}_{\text{index category 指标范畴}} \to \mathcal{C}$

∘ limit: a limit of p is an object $X \in ob(\mathcal{C})$, & a family of morphisms $\{f_\alpha : X \longrightarrow p(\alpha)\}_{\alpha \in \mathcal{J}}$

s.t. $\forall\, t : \alpha \to \beta$ in $\mathcal{J}$, s.t. $f_\beta = p(t) \circ f_\alpha$, satisfying the universal property: $\forall\, Y \in ob(\mathcal{C})$, $\{g_\alpha : Y \to p(\alpha)\}$ s.t. above conditions. then $\exists!\, h : Y \to X$ s.t. $g_\alpha = f_\alpha \circ h$

∘ colimit: limit of $p^{op} : \mathcal{J}^{op} \to \mathcal{C}^{op}$

(1.14) Examples

1) product & coproduct

Λ : set as a discrete category $\rightsquigarrow$ $\lim\limits_{\alpha\in\Lambda} p(\alpha) =: \prod\limits_{\alpha\in\Lambda} p(\alpha)$ <u>product</u>.

$p : \Lambda \to \mathcal{C}$ $\qquad \operatorname{colim}\limits_{\alpha\in\Lambda} p(\alpha) =: \coprod\limits_{\alpha\in\Lambda} p(\alpha)$ <u>coproduct</u>.

2) filtered / cofiltered limit / colimit

Λ : poset <u>filter / cofiltered</u>

$p : \Lambda \to \mathcal{C}$

filtered : $\{x,y\} \subseteq \Lambda$, $\exists z$ s.t. $x, y \le z$
x, y

cofiltered : $\exists z$, s.t. $x, y \ge z$

$\rightsquigarrow$ $\lim\limits_{\alpha\in\Lambda} p(\alpha) =: \varprojlim\limits_{\alpha\in\Lambda} p(\alpha)$ <u>inverse limit</u>

$\operatorname{colim}\limits_{\alpha\in\Lambda} p(\alpha) =: \varinjlim\limits_{\alpha\in\Lambda} p(\alpha)$ <u>direct limit</u>

3) Pullback, Pushforward

$$\begin{tikzcd} y \arrow[d, dash] \arrow[r, phantom, "\xrightarrow{p}"] & Y \arrow[d, dash] \\ {} & {} \end{tikzcd}$$

(1.14) Examples

$\Lambda := \{x, y, z\}, \quad x \leq z, \; y \leq z$

$p : \Lambda \to \mathcal{C}$

$x \bullet \longrightarrow \bullet\, z \qquad X \longrightarrow Z$

$\Lambda \longrightarrow \mathcal{C}$

$$\lim_{\Lambda} p = \text{pullback} = X \times_Z Y$$

$$\operatorname{colim}_{\Lambda^{op}} p = \text{pushforward} = X \sqcup_Z Y$$

$$\begin{tikzcd}
L \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& \lim_{\Lambda} p \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & Y \arrow[d] \\
& X \arrow[r] & Z
\end{tikzcd}
\qquad
\begin{tikzcd}
Z \arrow[r] \arrow[d] \arrow[dr, phantom, "\boxplus"] & X \arrow[d] \arrow[ddr, bend left] & \\
Y \arrow[r] \arrow[drr, bend right] & \operatorname{colim}_{\Lambda^{op}} p \arrow[dr, dashed, "\exists!"] & \\
& & M
\end{tikzcd}$$

4) Equalizer, coequalizer

$\Lambda := \{x \rightrightarrows y\}$

$$x \rightrightarrows y \xrightarrow{p} X \underset{g}{\overset{f}{\rightrightarrows}} Y$$

(1.14) Examples

- $\lim\limits_{\Lambda} p$ = equalizer $= \ker(f,g) = Eq\left(X \overset{f}{\underset{g}{\rightrightarrows}} Y\right)$
- $\operatorname{colim}\limits_{\Lambda^{op}} p$ = coequalizer $= \operatorname{coker}(f,g) = Coeq.\left(X \overset{f}{\underset{g}{\rightrightarrows}} Y\right)$

$$
\begin{tikzcd}
 & L \arrow[dl, dashed, "\exists!"'] \arrow[d, "\phi"] \arrow[dr, bend left] \arrow[dr, bend right] & \\
\ker(f,g) \arrow[r] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y
\end{tikzcd}
$$

equalizer

coequalizer

$$
\begin{tikzcd}
X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[dr, bend right] & Y \arrow[r] \arrow[d] & \operatorname{coker}(f,g) \arrow[dl, dashed, "\exists!"] \\
 & L &
\end{tikzcd}
$$

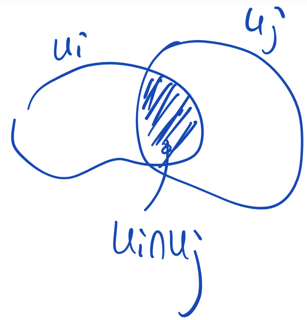
5) Example on equ / coeq.

X smooth maifold. Let Man be the cat. of sm. man.

$\{U_i\}$ a family of open subsets of X, then

$$\coprod_{i,j} (U_i \cap U_j) \rightrightarrows \coprod_i U_i \to \bigcup_i U_i$$

- finite category



$$\underbrace{C^\infty\Big(\bigcup_i U_i\Big) \longrightarrow}_{\text{equalizer}} \prod_i C^\infty(U_i) \overset{res_i}{\underset{res_j}{\rightrightarrows}} \prod_{i,j} C^\infty(U_i \cap U_j)$$

Λ : finite category

$p : \Lambda \to \mathcal{C}$: finite limit

(1.16) Theorem

极限的转化

定理

Let $\mathcal{C}$ be a category, then the following are equivalent:

1. *$\mathcal{C}$ has all ~~small~~ limits,* (complete category)
2. *$\mathcal{C}$ has all finite limits and inverse limits,*
3. *$\mathcal{C}$ has all products and pullback,*
4. *$\mathcal{C}$ has all products and equalizers.*

Moreover, if $\mathcal{C}$ is complete, $F : \mathcal{C} \to \mathcal{D}$ is a functor, then the following are equivalent:

1. *F preserves all small limits,*
2. *F preserves all finite limits and inverse limits,*
3. *F preserves all products and pullback,*
4. *F preserves all products and equalizers.*

(1.17) Theorem

定理

Let $\mathcal{C}$ be a category, then the following are equivalent:

1. *$\mathcal{C}$ has all finite limits,*
2. *$\mathcal{C}$ has final object and pullback,*
3. *$\mathcal{C}$ has finite products and equalizers.*

Moreover, if $\mathcal{C}$ is such a category, $F : \mathcal{C} \to \mathcal{D}$ is a functor, then the following are equivalent:

1. *F preserves all finite limits,*
2. *F preserves final objects and pullback,*
3. *F preserves all finite products and equalizers.*

Cont.

定理

Let $\mathcal{C}$ be a category, then the following are equivalent:

1. *$\mathcal{C}$ has all finite limits,*
2. *$\mathcal{C}$ has final object and pullback,*
3. *$\mathcal{C}$ has finite products and equalizers.*

定理

✓ *Representable functors preserve limits.*

Let $\mathcal{C}$ be a cat. $X \in \mathcal{C}$.

$$h_X := \mathrm{Hom}_{\mathcal{C}}(-, X) : \mathcal{C}^{op} \to \mathrm{Set}$$

$$\left(\text{resp.}\quad k_X := \mathrm{Hom}_{\mathcal{C}}(X, -) : \mathcal{C} \to \mathrm{Set}\right)$$

preserve limits in $\mathcal{C}^{op}$ (resp. $\mathcal{C}$), i.e.

- $\mathrm{Hom}_{\mathcal{C}}(\operatorname{colim}_i \alpha, -) \xrightarrow{\sim} \lim_i \mathrm{Hom}_{\mathcal{C}}(\alpha(i), -)$
- $\mathrm{Hom}_{\mathcal{C}}(-, \lim_i \beta) \xrightarrow{\sim} \lim_i \mathrm{Hom}_{\mathcal{C}}(-, \beta(i))$

定理

Let (F, G, ϕ) *be an adjoint pair, then* F *preserves* colim *and* G *preserves* lim.

Proof: $\forall X \in \mathcal{C}_1$

$$\mathrm{Hom}_{\mathcal{C}_1}(X, G(\lim_\alpha Y_\alpha)) \xleftarrow[\varphi]{\sim} \mathrm{Hom}_{\mathcal{C}_2}(FX, \lim_\alpha Y_\alpha)$$

$$\xrightarrow{(1.18)} \lim_\alpha \mathrm{Hom}_{\mathcal{C}_2}(FX, Y_\alpha)$$

$$\xrightarrow{\varphi} \lim_\alpha \mathrm{Hom}_{\mathcal{C}_1}(X, GY_\alpha)$$

$$\xrightarrow{(1.18)} \mathrm{Hom}_{\mathcal{C}_1}(X, \lim_\alpha GY_\alpha)$$

$$\rightsquigarrow h_{\mathcal{C}_1, G(\lim_\alpha Y_\alpha)} \simeq h_{\mathcal{C}_1, \lim_\alpha GY_\alpha}$$

Yoneda lemma

$$G(\lim_\alpha Y_\alpha) \xrightarrow{\sim} \lim_\alpha G(Y_\alpha)$$

Yoneda embedding is fully faithful

&

(1.20) Theorem

L Lie algebra $\mathcal{U}(L)$

$\mathcal{U}(L \oplus L) \xrightarrow{\sim} \mathcal{U}(L) \otimes_{\mathbb{C}} \mathcal{U}(L)$

定理

Limits commute with limits.

a finite subset of $\mathbb{Z}$ poset "$\leq$" $\rightsquigarrow$ morphism

lim : minimal element

colim : maximal element

$$\Lambda_{\alpha,\beta} = \{ n_{\alpha_1,\beta_1}, \dots \}$$

(1.21) Theorem

定理

In Set*, cofiltered colimit commutes with finite limits.*

direct limit

(1.22) Definition

- additive category,
- additive functor.

additive category : $\mathcal{A}$:

1) $\mathcal{A}$ has a zero object, finite product, finite coproduct

2) By 1), $\forall\, X, Y \in \mathrm{ob}(\mathcal{A})$

$$
\begin{tikzcd}
0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\boxplus"] & X \arrow[d, "\mathrm{id} \times 0"] \\
Y \arrow[r, "0 \times \mathrm{id}"'] & X \times Y
\end{tikzcd}
\quad \Longrightarrow \quad
\begin{tikzcd}
X \amalg Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d, "\mathrm{id} \times 0"] \\
Y \arrow[r, "0 \times \mathrm{id}"'] & X \times Y
\end{tikzcd}
$$

$X \amalg Y \xrightarrow{\sim} X \times Y$ is an isomorphism : <u>direct sum</u>

(1.22) Definition

$\mathcal{A}, \mathcal{B}$ additive cat. $F: \mathcal{A} \to \mathcal{B}$ functor

- additive category,
- additive functor.

if F preserves all finite products

3) By 1) 2), for any two objects $X, Y \in ob(\mathcal{A})$, $Hom_{\mathcal{A}}(X, Y)$ is an additive monoid : $f, g \in Hom_{\mathcal{A}}(X, Y)$

$$\underbrace{X \xrightarrow{\Delta} X \times X \xrightarrow{f \times g} Y \times Y \xrightarrow[2)]{\sim} Y \sqcup Y \longrightarrow Y}_{:= f+g}$$

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\boxplus"] & X \arrow[d] \arrow[ddrr, "f"] & & \\
X \arrow[r] \arrow[ddrr, "g"'] & X \times X \arrow[dr, dashed, "\exists ! f \times g"] & & \\
& & Y \times Y & Y \arrow[l] \\
& & Y \arrow[u] & 0 \arrow[l] \arrow[u] \arrow[ul, phantom, "\boxplus"]
\end{tikzcd}$$

Zero : zero morphism   is an abelian group ! $\square$

(1.23) Remark

- For any $X, Y, Z \in Obj(\mathcal{A})$, the composition

$$\ast \qquad \mathrm{Hom}_{\mathcal{A}}(X, Y) \times \mathrm{Hom}_{\mathcal{A}}(Y, Z) \to \mathrm{Hom}_{\mathcal{A}}(X, Z)$$

  is a bilinear map of abelian groups.

- Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor, then for any $X, Y \in \cancel{\mathcal{A}}$ ob($\mathcal{A}$),

$$F : \mathrm{Hom}_{\mathcal{A}}(X, Y) \to \mathrm{Hom}_{\mathcal{B}}(FX, FY)$$

  is an abelian group homomorphism.

(1.24) Definition

Let $\mathcal{A}$ be ~~a category with 0~~ additive category. Let $f : X \to Y$ be a morphism.

- $\ker f$, $\operatorname{coker} f$, $\operatorname{im} f$, $\operatorname{coim} f$.

$$\begin{tikzcd}
\ker f \arrow[r, dashed, "\alpha"] \arrow[d, dashed] \arrow[dr, phantom, "\sqcap"] & X \arrow[d, "f"] \\
0 \arrow[r] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r] \arrow[d, "f"'] \arrow[dr, phantom, "\boxplus"] & 0 \arrow[d, dashed] \\
Y \arrow[r, dashed, "\beta"'] & \operatorname{coker} f
\end{tikzcd}$$

"base change to 0"

$\operatorname{coim} f = \operatorname{coker}(\ker f)$

$\operatorname{im} f = \ker(\operatorname{coker} f)$

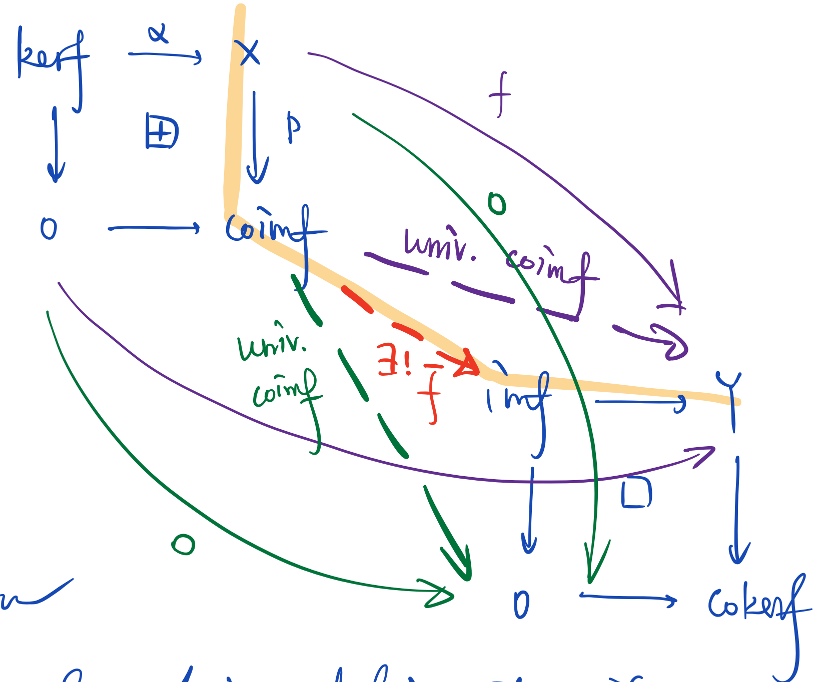
$$\begin{tikzcd}
\ker f \arrow[r, "\alpha"] \arrow[d] \arrow[dr, phantom, "\boxplus"] & X \arrow[d] & \operatorname{im} f \arrow[r] \arrow[d] \arrow[dr, phantom, "\sqcap"] & Y \arrow[d, "\beta"] \\
0 \arrow[r] & \operatorname{coim} f \arrow[ur, dashed, "?"' , "\bar{f}"] & 0 \arrow[r] & \operatorname{coker} f
\end{tikzcd}$$

- Abelian category.

$X \xrightarrow{p} \mathrm{coim} f \xrightarrow{\exists! \bar{f}} \mathrm{im} f \hookrightarrow Y$

f 的满单分解

- $\bar{f}$ 是同构：严格态射 strict morphism



Def: $\mathcal{A}$ additive cat. Say $\mathcal{A}$ is abeli cat, if

1° $\mathcal{A}$ has finite limit & finite colimit

2° all morphisms of $\mathcal{A}$ are strict.

(1.26) Definition

- Chain complex, homology,
- exactness.

(1.27) Definition

Let $\mathcal{C}$ be a category. $F : \mathcal{C} \to \mathcal{D}$ be a functor.

- Left and right exactness.

left : preserve all finite limit

right : ———————— colimit

(1.28) Remark

- Representable functors are left exact. in $\mathcal{C}^{op}$ & $\mathcal{C}$ resp.
- Any left (resp. right) adjoint functors are right (resp. left) exact.

(1.29) Proposition

定理

Let $\mathcal{A}, \mathcal{B}$ be two abelian categories. Then $F : \mathcal{A} \to \mathcal{B}$ is left exact if and only if

- *F is additive,*
- *F maps exact sequences $0 \to A \to B \to C$ in $\mathcal{A}$ to $0 \to FA \to FB \to FC$ in $\mathcal{B}$.*

1) F additive $\Longleftrightarrow$ F preserve all finite product

2) $\underset{\Delta}{\Longleftarrow}$ F persevere kernel $\underset{\Delta}{\Longleftarrow}$ F pres. equalser

1) + 2) $\Longleftrightarrow$ F presere all finite product & equalzie

$\underset{(1.17)}{\Longleftarrow}$ F presere all finite limit

$\square$

(1.30) Definition

Let $\mathcal{A}$ be an abelian category.

- Projective objects, $*$ $P \in Ob(\mathcal{A})$ projective if $Hom_{\mathcal{A}}(P, -) : \mathcal{A} \to Ab$ is exact, then P : projective object
- Injective objects.

$*$ $I \in Ob(\mathcal{A})$ injective if $Hom_{\mathcal{A}}(-, I) : \mathcal{A}^{op} \to Ab$ is exact.

(1.31) Motivation (I)

Question: How to describe a circle on the plane $\mathbb{C}^2$.

1. Defn :

2. Equation : $x^2+y^2=1$

3. Algebra : $R = \mathbb{C}[X,Y]/(X^2+Y^2-1)$

What are the maximal ideals of R ?

i $\{\text{maximal ideals of } R\} \xleftrightarrow{1:1} \{\text{max. ideals of } \mathbb{C}[X,Y] \text{ cont. } (X^2+Y^2-1)\}$

$\mathfrak{m} \longmapsto \mathfrak{m}/(X^2+Y^2-1)$

ii Hilbert nullstellensatz : $\mathbb{C}$ alg. closed

$\mathbb{C}[X,Y]$'s maximal ideal are of the form

$(X-a,\ Y-b) \subseteq \mathbb{C}[X,Y]$

(1.31) Motivation (I)

Question: How to describe a circle on the plane $\mathbb{C}^2$.

Check: $(x-a, y-b) \supseteq (x^2+y^2-1) \iff a^2+b^2-1=0$

✓ Circle $\overset{!!}{=}$ MaxSpec(R)

Question: 1° $\mathbb{C} \rightsquigarrow \mathbb{R}, \underline{\underline{\mathbb{Q}}}$ ~~null stellensatz~~ (eg: why?)

2° Can we recover topology of $\mathbb{C}$? (in some sense)

3° functoriality of MaxSpec? [H1]

$$\varphi: X_1 \to X_2$$
$$\mathcal{O} \to \alpha$$
$$\Downarrow$$
$$\varphi: R_1 \to R_2$$

Answer: Grothendieck: Spec

(1.32) Motivation (II)

Question: How to study a (topological/differential/complex) manifold?

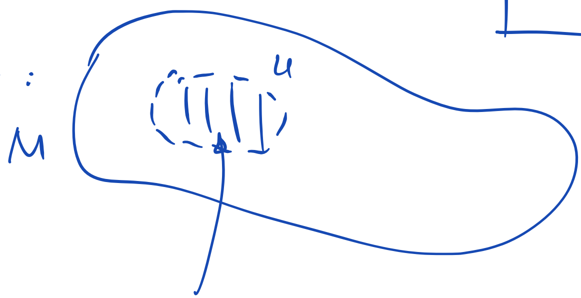
Answer 1 : Functions !

Attempt 1 : $\mathbb{C}$: no nonconstant bounded holomorphic funct. on $\mathbb{C}$

(Liouville)

$\Downarrow$

local properties :



i $C^\infty(U)$

ii Glue them together !

$\varphi: R_1 \longrightarrow R_2$ ring hom.

$\varphi: \operatorname{Spec} R_2 \longrightarrow \operatorname{Spec} R_1$

$I_2 \mapsto \varphi^{-1}(I_2)$

eg: R $\mathfrak{p} \in \operatorname{Spec} R$

$R_\mathfrak{p}$: $\mathfrak{p}R_\mathfrak{p}$: max.

$R \longrightarrow R_\mathfrak{p}$

$\varphi^\#: \operatorname{MaxSpec} R_2$

$? \Downarrow$

$\operatorname{MaxSpec} R_1$

Answer 2 : Linearlise !

~~tangent~~ sp.
tangent bundle

⟶ sheaf theory !

cf. Torsten Wedhorn
" Mans. sheaves & coho. "