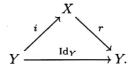
EXAM IN CATEGORY THEORY

Let ${\mathfrak C}$ be a category. A retract of an object $X\in {\mathfrak C}$ is a commutative diagram in ${\mathfrak C}$:



By abusing the terminology, we also say that Y is a retract of X.

Note that if we let $e = i \circ r$ then the morphism $e: X \to X$ is idempotent, i.e. $e^2 = e$. We say that \mathcal{C} is *idempotent complete* if every idempotent morphism in \mathcal{C} comes from a retract.

Problem 1. Show that the following conditions are equivalent for an idempotent morphism $e: X \to X$:

- (1) e comes from a retract of X.
- (2) The diagram $X \xrightarrow[\mathrm{Id}_X]{e} X$ admits an equalizer.
- (3) The diagram $X \xrightarrow[\mathrm{Id}_X]{e} X$ admits a coequalizer.

Problem 2. Show that if an idempotent morphism $e: X \to X$ comes from a retract Y of X then Y is unique up to unique isomorphism.

Problem 3. Show that for any idempotent complete category \mathcal{D} , Fun(\mathcal{C} , \mathcal{D}) is also idempotent complete.

We construct a category $\hat{\mathbb{C}}$ as follows. An object of $\hat{\mathbb{C}}$ is a pair (X,e) where $X \in \mathbb{C}$ and $e: X \to X$ is an idempotent morphism in \mathbb{C} . A morphism $(X,e) \to (Y,d)$ in $\hat{\mathbb{C}}$ is a morphism $f: X \to Y$ in \mathbb{C} such that $d \circ f = f = f \circ e$. The composition law of $\hat{\mathbb{C}}$ is induced from that of \mathbb{C} .

Problem 4. Show that the category $\hat{\mathbb{C}}$ is well-defined.

Problem 5. Show that $\hat{\mathbb{C}}$ is idempotent complete.

Problem 6. Show that the assignment $X \mapsto (X, \mathrm{Id}_X)$, $f \mapsto f$ defines a fully faithful functor $I : \mathcal{C} \to \hat{\mathcal{C}}$.

Let \mathcal{E} be an idempotent complete category. We say that a functor $I: \mathcal{C} \to \mathcal{E}$ exhibits \mathcal{E} as idempotent completion of \mathcal{C} if for any idempotent complete category \mathcal{D} , composition with I induces an equivalence $\operatorname{Fun}(\mathcal{E}, \mathcal{D}) \simeq \operatorname{Fun}(\mathcal{C}, \mathcal{D})$.

Problem 7. Show that the embedding $I: \mathcal{C} \to \hat{\mathcal{C}}$ exhibits $\hat{\mathcal{C}}$ as idempotent completion of \mathcal{C} .

Problem 8. Use Problem 7 to show that \mathcal{C} is idempotent complete if and only if $\mathcal{C} \simeq \hat{\mathcal{C}}$.



Let S be a locally small category which admits all small colimits. We say that an object $X \in \mathcal{S}$ is *completely compact* if the representable functor $\mathrm{Hom}_{\mathcal{S}}(X,-): \mathcal{S} \to \mathcal{S}et$ preserves all small colimits.

Problem 9. Show that if $X \in \mathcal{S}$ is completely compact then any retract of X is completely compact.

In what follows, we assume $\mathcal C$ is small and let $j:\mathcal C\to\mathcal P(\mathcal C)$ be the Yoneda embedding.

Problem 10. Show that $\mathcal{P}(\mathcal{C}) \simeq \mathcal{P}(\hat{\mathcal{C}})$. [Hint: Use Problem 7.]

Problem 11. Show that $j(X) \in \mathcal{P}(\mathcal{C})$ is completely compact for any $X \in \mathcal{C}$.

Problem 12. Show that if $S \in \mathcal{P}(\mathcal{C})$ is completely compact, then there exists $X \in \mathcal{C}$ such that S is a retract of j(X). [Hint: Use the fact that every object of $\mathcal{P}(\mathcal{C})$ is a small colimit of objects in $j(\mathcal{C})$.]

Problem 13. Let $\tilde{\mathbb{C}} \subset \mathcal{P}(\mathbb{C})$ be the full subcategory consisting of the completely compact objects. Show that the embedding $j: \mathbb{C} \to \tilde{\mathbb{C}}$ exhibits $\tilde{\mathbb{C}}$ as idempotent completion of \mathbb{C} . [Hint: Combine the above four problems.]