§1 Category Theory Review

(1.0) We shall ignore "set theory" issue.

(1.1) Defn

1) category, subcategory, full subcategory

2) monomorphism, epimorphism

(1.2) Defn

1) functor (covariant / contravariant functors)

2) A functor $F: \mathcal{C}' \longrightarrow \mathcal{C}$ is

- <u>essentially surjective</u>, if any object in $\mathcal{C}$ is isomorphic to FX for some X in $\mathcal{C}'$

- <u>faithful</u> (resp. <u>full</u>) if for any $X, Y \in Ob(\mathcal{C}')$, the map

$$Hom_{\mathcal{C}'}(X, Y) \longrightarrow Hom_{\mathcal{C}}(FX, FY)$$

is injective (resp. surjective)

3) <u>natural transformation</u>
$$\begin{tikzcd} \mathcal{C}' \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\theta"] & \mathcal{C} \end{tikzcd}$$

4) <u>functor category</u>

<u>Remark</u>: Fully faithful functor preserves and reflect isomorphism !

In other words, let $F: \mathcal{C} \longrightarrow \mathcal{D}$ be a fully faithful functor. $X, Y \in Ob(\mathcal{C})$,

then: 1° $f: X \longrightarrow Y$ is an isomorphism iff $F(f): FX \longrightarrow FY$ is an isomorphism

2° For any isomorphism $g: FX \to FY$ in $\mathcal{D}$, $\exists!$ isomorphism $f: X \to Y$

s.t. $Ff = g$

$\divideontimes$ 3° X, Y are isomorphic iff FX and FY are isomorphi.

This can be worked out as an exercise or see MSE 3566069.

(1.3) Defn

1) equivalence of category: $\mathcal{C}_1 \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{C}_2$ s.t. $\exists$ isomorphism of functors $\theta: FG \xrightarrow{\sim} \mathrm{id}_{\mathcal{C}_2}$, $\psi: GF \xrightarrow{\sim} \mathrm{id}_{\mathcal{C}_1}$. Then we say $\mathcal{C}_1$ and $\mathcal{C}_2$ are equivalent, and G is the quasiinverse of F.

2) Moreover, if $FG = id_{\mathcal{C}_2}$, $GF = id_{\mathcal{C}_1}$, then we say $\mathcal{C}_1$ and $\mathcal{C}_2$ are <u>isomorphic</u>, and G is the inverse of F.

(1.4) Theorem – Reading

Let $F: \mathcal{C}_1 \longrightarrow \mathcal{C}_2$ be a functor. Then TFAE:

1. F is an equivalence of categories
2. F is fully faithful and essentially surjective.

<u>Proof</u>: cf. 李文威《代数学方法》第一卷定理2.2.13.

(1.5) Defn

$X \in ob(\mathcal{C})$ is called an <u>initial object</u>, if for all object Y, the set $Hom_{\mathcal{C}}(X, Y)$ is a singleton. It is called a final object, if for all object Y, $Hom_{\mathcal{C}}(Y, X)$ is a singleton. If X is both an initial object and a final object, then it is called a <u>zero object</u>.

(1.6) Example – Exc. Fill in the form

Cat.	initial obj.	final object	zero object	morphism
Set	ϕ	pt	–	map
Top	ϕ	pt	–	continous map
Top_*	(pt, pt)	(pt, pt)	(pt, pt)	..
Grp	$\{1\}$	$\{1\}$	$\{1\}$	group homomorphism
Ab	$\{0\}$	$\{0\}$	$\{0\}$	abel group homomorphism
Ring	$\mathbb{Z}$	$\{0\}$	–	ring homomorphism
CRing	$\mathbb{Z}$	$\{0\}$	–	..
Poset	minimal element	maximal element	–	"$\leq$" (isomorphism: "$=$")

(1.7) Defn

Let $\mathcal{C}$ be a category. Define $\mathcal{C}^\wedge := \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Set})$ called <u>presheaf category</u> over $\mathcal{C}$. We have a canonical functor

$$\begin{aligned} h_{\mathcal{C}} : \mathcal{C} &\longrightarrow \mathcal{C}^\wedge = \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Set}) \\ S &\longmapsto \mathrm{Hom}(-, S) = h_S \end{aligned}$$

and an evaluation functor

(?)

$$\begin{aligned} ev^\wedge : \mathcal{C}^{op} \times \mathcal{C}^\wedge &\longrightarrow \mathrm{Set} \\ (S, A) &\longmapsto A(S) \end{aligned}$$

(1.8) Yoneda's lemma

Let $S \in \mathrm{Ob}(\mathcal{C})$, $A \in \mathcal{C}^\wedge$, then the map

$$\begin{aligned} \theta_S : \mathrm{Hom}_{\mathcal{C}^\wedge}(h_{\mathcal{C}}(S), A) &\longrightarrow A(S) = ev^\wedge(S, A) \\ \left[\mathrm{Hom}_{\mathcal{C}}(-, S) \xrightarrow{\phi} A \right] &\longmapsto \phi_S(\mathrm{id}_S) \end{aligned}$$

Recall that ϕ is a natural transformation

$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}}(S, S) &\xrightarrow{\phi_S} A(S) \\ \mathrm{id}_S &\longmapsto \phi_S(\mathrm{id}_S) \end{aligned}$$

is bijective. It gives an isomorphism of functors $\mathrm{Hom}_{\mathcal{C}^\wedge}(h_{\mathcal{C}}(-), -) \xrightarrow{\sim} ev^\wedge$.

<u>Proof</u>: cf. 李文威《代数学方法》第一卷 定理2.5.1.

(1.9) Corollary-Defn.

The functor $h_{\mathcal{C}}$ is fully faithful. $h_{\mathcal{C}}$ is called the <u>Yoneda's embedding</u>.

<u>Proof</u>: Apply (1.8) with $A = h_{\mathcal{C}}(T)$ for any $T \in \mathrm{Ob}(\mathcal{C})$, we get

$$\mathrm{Hom}_{\mathcal{C}^\wedge}(h_{\mathcal{C}}(S), h_{\mathcal{C}}(T)) \xrightarrow[\text{bijective}]{} h_{\mathcal{C}}(T)(S) = \mathrm{Hom}_{\mathcal{C}}(S, T).$$

Hence $h_{\mathcal{C}}$ is fully faithful. $\square$

<u>Remark</u>: Recall in functional analysis, for any normed vector space V, its dual space V^* is a Banach space, so we can consider V^{**}, called the <u>double dual space</u>. Then we have a canonical map

$$T: V \longrightarrow V^{**}$$
$$x \longmapsto [X : f \longmapsto f(x)]$$

<u>Fact</u>: V is isometric to a subspace of V^{**} via T. If T is surjective, then V is called <u>reflexible space</u>.

Here we can regard $\mathcal{C}$ as the space V and $\mathcal{C}^\wedge$ as the space V^{**}, then the Yoneda embedding can be regarded as the natural map T.

(1.10) Defn

We say a functor $A: \mathcal{C}^{op} \longrightarrow \mathrm{Set}$ is <u>representable</u>, if $\exists X \in \mathrm{Ob}(\mathcal{C})$ and an isomorphism $\phi: h_{\mathcal{C}}(X) \xrightarrow{\sim} A$. We say A is <u>represented</u> by (X, ϕ).

<u>Remark</u>: We can regard a representable functor as functors in $\mathcal{C}^\wedge$ falling in the image of $h_{\mathcal{C}}$. It may be natural to ask when $h_{\mathcal{C}}$ is essentially surjective. But Yoneda embedding is never essentially surjective: consider the empty set set functor

$$\begin{aligned} F_\phi : \ & \mathcal{C}^{op} \longrightarrow Set \\ & A \longmapsto \phi, \quad \forall A \in ob(\mathcal{C}^{op}) \end{aligned}$$

Yet suppose F_ϕ is represented by $X \in ob(\mathcal{C}^{op})$, then $F_\phi(X) = Hom_{\mathcal{C}}(X, X) \ni id_X$

So $F_\phi(X)$ is not empty!

Fortunately, we have some description on its essential image. See 李文威《代数学方法》第2卷 定理1.7.3.

(1.11) Defn

An adjoint pair is a tuple (F, G, φ), where $\mathcal{C}_1 \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{C}_2$ is a pair of functors and φ is an isomorphism of functors

$$\varphi : Hom_{\mathcal{C}_2}(F(-), -) \xrightarrow{\sim} Hom_{\mathcal{C}_1}(-, G(-)).$$

We call F a left adjoint of G and G a right adjoint of F.

Remark: In some sense, we can regard F as a "left inverse" of G and G a "right inverse" of F. We shall make this clearer in (1.12)

(1.12) Defn

Let (F, G, φ) be an adjoint pair. We define $\eta = (\eta_X)_{X \in ob(\mathcal{C}_1)} : id_{\mathcal{C}_1} \longrightarrow GF$ as follows

$$\begin{aligned} Hom_{\mathcal{C}_2}(FX, FX) & \xrightarrow{\varphi_{X,FX}} Hom_{\mathcal{C}_1}(X, GF(X)) \\ id_{FX} & \longmapsto [\eta_X : X \longrightarrow GF(X)] \end{aligned}$$

Here η is called a unit of (F, G, φ). Similarly, we can define the counit $\varepsilon := (\varepsilon_Y)_{Y \in ob(\mathcal{C}_2)} : FG \longrightarrow id_{\mathcal{C}_2}$.

Remark: Let's uncover the definition: for any fixed $X \in ob(\mathcal{C}_1)$,

$$X \xrightarrow{id_X} X \overset{F(\cdot)}{\rightsquigarrow} FX \xrightarrow{id_{FX}} FX$$

adjointness $\quad X \xrightarrow[\eta_X]{} G(FX) = GFX$.

Can you write ε_Y starting with $Y \xrightarrow{id_Y} Y$ in $\mathcal{C}_2$? (Exc.)

Exercise : Verify that η_X is indeed a natural transformation .

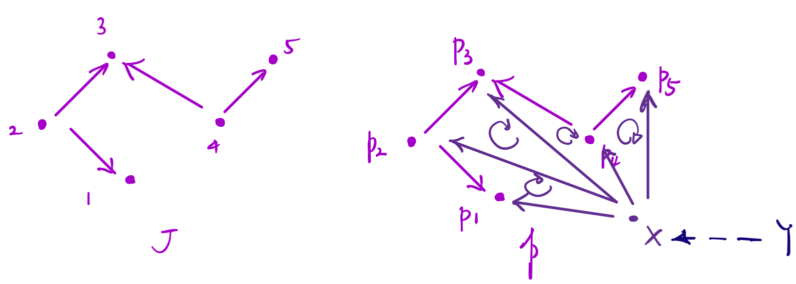
(1.13) Defn

Let $\mathcal{C}$ be a category. A diagram in $\mathcal{C}$ is a functor $p : J \to \mathcal{C}$. Here J is called an index category .

A limit of p is an object $X \in Ob(\mathcal{C})$ and a family of morphisms

$$\{ f_\alpha : X \longrightarrow p(\alpha) \}_{\alpha \in J}$$

such that any morphism $t : \alpha \to \beta$ in J, we have $f_\beta = p(t) \circ f_\alpha$, satisfying the universal property : for any $Y \in Ob(\mathcal{C})$ and such a family $\{g_\alpha\}$, $\exists !\, h : Y \to X$ s.t. $g_\alpha = f_\alpha \circ h$. Denote $X := \lim\limits_{\alpha \in J} p(\alpha) = \varprojlim\limits_{\alpha \in J} p(\alpha)$.



The limit of $p^{op} : J^{op} \longrightarrow \mathcal{C}^{op}$ is called the colimit of p .

Remark : Here " $\varprojlim$ " " $\varinjlim$ " looks similar to " final objects " and " initial objects " in " some category " . This can be made more precise if we bring in the notion of " comma categories " .

(1.14) Examples

1) Let Λ be a set regarded as a discrete category, then

$$\lim_{\alpha\in\Lambda} p(\alpha) =: \prod_{\alpha\in\Lambda} p(\alpha) \quad \underline{\text{product}} \;;$$

$$\operatorname{colim}_{\alpha\in\Lambda} p(\alpha) =: \coprod_{\alpha\in\Lambda} p(\alpha) \quad \underline{\text{coproduct}} \;.$$

2) Let Λ be a <u>filtered (cofiltered) poset</u>, i.e. if any finite subset of Λ has a upper (lower) bound.

$$\lim_{\alpha\in\Lambda} p(\alpha) \;:\; \text{inverse limit}$$

$$\operatorname{colim}_{\alpha\in\Lambda} p(\alpha) \;:\; \text{direct limit}$$

3) Let $\Lambda := \{x, y, z\}$ be a poset with $x \le z$, $y \le z$

$$\begin{tikzcd} & \bullet\, y \arrow[d] \\ \bullet\, x \arrow[r] & \bullet\, z \end{tikzcd}$$

(identity ignored)

$\lim_{\Lambda} p$: pullback $\qquad \operatorname{colim}_{\Lambda^{op}} p$: pushforward

$$\begin{tikzcd} L \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & \lim_{\Lambda} p \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & Y \arrow[d] \\ & X \arrow[r] & Z \end{tikzcd} \qquad \begin{tikzcd} Z \arrow[r] \arrow[d] \arrow[dr, phantom, "\boxplus"] & X \arrow[d] \arrow[ddr, bend left] & \\ Y \arrow[r] \arrow[drr, bend right] & \operatorname{colim}_{\Lambda^{op}} p \arrow[dr, dashed] & \\ & & M \end{tikzcd}$$

denoted as $p(x) \times_{p(z)} p(y)$ $\qquad$ denoted as $p(x) \coprod_{p(z)} p(y)$

4) Let $J := \{x \rightrightarrows y\}$.

- $\lim_{J} p$: equalizer, denoted as $Eq\left(p(x) \overset{f}{\underset{g}{\rightrightarrows}} p(y)\right) = \ker(f, g)$

$$\begin{tikzcd} & L \arrow[dl, dashed, "\exists!"'] \arrow[d] & \\ \lim_{J} p \arrow[r] & p(x) \arrow[r, shift left] \arrow[r, shift right] & p(y) \end{tikzcd}$$

- $\operatorname{colim}_{J} p$: coequalizer, denoted as $Coeq\left(p(x) \rightrightarrows p(y)\right) = \operatorname{coker}(f, g)$

$$\begin{tikzcd} p(x) \arrow[r, shift left] \arrow[r, shift right] & p(y) \arrow[r] \arrow[d] & \operatorname{colim}_J p \arrow[dl, dashed, "\exists!"] \\ & L & \end{tikzcd}$$

5) Example - Exe.

Let X be a smooth manifold. In the category Man, we have an coequalizer diagram

$$\coprod_{i,j} U_i \cap U_j \underset{\iota_j}{\overset{\iota_i}{\rightrightarrows}} \coprod_i U_i \longrightarrow \bigcup_i U_i .$$

where $\{U_i\}$ is a family of open subsets of X. Meanwhile, there is a equalizer diagram

$$C^\infty(\bigcup_i U_i) \longrightarrow \prod_i C^\infty(U_i) \underset{res_j}{\overset{res_i}{\rightrightarrows}} \prod_{i,j} C^\infty(U_i \cap U_j) .$$

(1.15) Defn

A category J is called <u>finite</u>, if J has only finitely many objects and its morphisms are generated by finitely many of them. A limit with index category a finite one is called a <u>finite limit</u>.

(1.16) Theorem

Let $\mathcal{C}$ be a category. TFAE:

1) $\mathcal{C}$ has all (small) limits (Then we say $\mathcal{C}$ is <u>complete</u>)
2) $\mathcal{C}$ has all finite limits and inverse limit
3) $\mathcal{C}$ has all product and pullback
4) $\mathcal{C}$ has all product and equalizer

Moreover, if $\mathcal{C}$ is complete, $F : \mathcal{C} \longrightarrow \mathcal{D}$ is a functor, then TFAE :

1) F preserves all (small) limits
2) F preserves all finite limits and inverse limit
3) F preserves all product and pullback
4) F preserves all product and equalizer.

<u>Proof</u> : cf. 李文威《代数学方法》第一卷 定理 2.8.3，推论 2.8.4 .

(1.17) Theorem

Let $\mathcal{C}$ be a category. TFAE :

1) $\mathcal{C}$ has all finite limits
2) $\mathcal{C}$ has final object and pullback
3) $\mathcal{C}$ has finite product and equalizer

Suppose $\mathcal{C}$ has all finite limit and $F : \mathcal{C} \longrightarrow \mathcal{D}$ is a functor. TFAE :

1) F preserves all finite limits
2) F preserves final object and pullback
3) F preserves finite product and equalizer

<u>Proof</u> : <u>(1) $\Longrightarrow$ (2)</u> Clear. (? final object is the limit over the empty diagram)

<u>(2) $\Longrightarrow$ (3)</u> finite product :

$$\begin{tikzcd} X \times Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d] \\ Y \arrow[r] & \{*\} \end{tikzcd}$$ final object

Check : $X \times Y := X \times_{\{*\}} Y$ is a product of X and Y.

equalizer: more generally, suppose

$$\begin{tikzcd} Z \arrow[r, "f'"] \arrow[d, "g'"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\ Y \arrow[r, "g"'] & W \end{tikzcd}$$

then

$$\begin{tikzcd} & & X \arrow[dr, "f"] & \\ Z \arrow[r, "f' \times g'"] & X \times Y \arrow[ur, bend left, "pr_X"] \arrow[dr, bend right, "pr_Y"'] \arrow[rr, shift left] \arrow[rr, shift right] & & W \\ & & Y \arrow[ur, "g"'] & \end{tikzcd}$$

is an equalizer.

Now set $X = Y$ and $f, g : X \to W$ any two morphisms. Then we are done.

(3) $\Rightarrow$ (1): Let β be a finite diagram. Then how to describe "$\lim \beta$"?
$(\beta : J \longrightarrow \mathcal{C})$

$$\prod_{i \in ob(J)} \beta(i) \overset{\Phi}{\underset{\Psi}{\rightrightarrows}} \prod_{\substack{\sigma : s(\sigma) \to t(\sigma) \\ \sigma \in Mor(J)}} \beta(t(\sigma)) \qquad (*)$$

where Φ: diagonal morphism

Ψ: target morphism:

$$\left\{ \beta(i) \xrightarrow{\Psi} \prod_{\substack{\sigma : \beta(i) \to t(\sigma) \\ \sigma \in Mor(J)}} \beta(t(\sigma)) \right\}_{i \in ob(J)} \xRightarrow{\text{induces}} \Psi.$$

then the equalizer of $(*)$ is $\lim \beta$. $\square$

(1.18) Prop.

Let $\mathcal{C}$ be a category and $X \in \mathcal{C}$. Then any representable functor

$$h_X := \mathrm{Hom}_{\mathcal{C}}(-, X) : \mathcal{C}^{op} \longrightarrow \mathrm{Set}$$

(resp. $$k_X := \mathrm{Hom}_{\mathcal{C}}(X, -) : \mathcal{C} \longrightarrow \mathrm{Set}$$)

preserves limits in $\mathcal{C}^{op}$ (resp. $\mathcal{C}$). In other words,

$$\mathrm{Hom}_{\mathcal{C}}(\operatorname{colim}_i \alpha, -) \xrightarrow{\sim} \lim_i \mathrm{Hom}_{\mathcal{C}}(\alpha(i), -)$$

$$\mathrm{Hom}_{\mathcal{C}}(-, \lim \beta) \xrightarrow{\sim} \lim_i \mathrm{Hom}_{\mathcal{C}}(-, \beta(i))$$

<u>Proof</u> : cf. 李文威《代数学方法》第一卷 命题 2.8.11.

(1.19) Theorem

Let (F, G, φ) be an adjoint pair, where $\mathcal{C}_1 \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{C}_2$, then F preserves colim and G preserves limit.

<u>Proof</u> : We write

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{C}_1}(X, G(\lim_\alpha Y_\alpha)) &\underset{\varphi}{\xleftarrow{\sim}} \mathrm{Hom}_{\mathcal{C}_2}(FX, \lim_\alpha Y_\alpha) \\
&\xrightarrow[(1.18)]{\sim} \lim_\alpha \mathrm{Hom}_{\mathcal{C}_2}(FX, Y_\alpha) \\
&\xrightarrow[\varphi]{\sim} \lim_\alpha \mathrm{Hom}_{\mathcal{C}_1}(X, G(Y_\alpha)) \\
&\xrightarrow[(1.18)]{\sim} \mathrm{Hom}_{\mathcal{C}_1}(X, \lim_\alpha G(Y_\alpha))
\end{aligned}$$

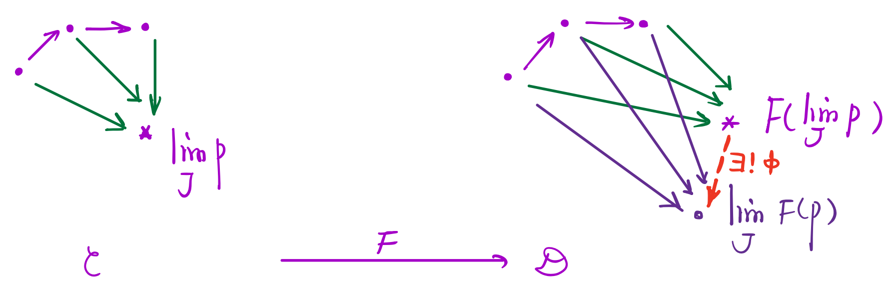
Then by Yoneda lemma (1.9), we see $G(\lim_\alpha Y_\alpha) = \lim_\alpha G(Y_\alpha)$. (?) $\square$

(We can regard the isomorphism above as

$$h_{G(\lim_\alpha Y_\alpha)} \xrightarrow{\sim} h_{\lim_\alpha G(Y_\alpha)} .$$

Since the Yoneda embedding is fully faithful, we obtain the result .)

<u>Remark</u>: What does F preserves "lim" really mean ?



We say F preserves lim if the canonical ϕ is an isomorphism. So in the proof of (1.19), we actually need to show that the isomorphism brought by Yoneda's lemma is the canonical one here. Wen-wei Li bring in the notion of "lim" in his book for a clearer discussion.

(1.20) Theorem

Limits commute with limits. In other words, for any diagram $p : J_1 \times J_2 \to \mathcal{C}$,

$$\lim_{\alpha\in J_1} \lim_{\beta\in J_2} p(\alpha,\beta), \quad \lim_{\beta\in J_2} \lim_{\alpha\in J_1} p(\alpha,\beta)$$

exist and they are both the limit $\lim_{\alpha,\beta} p(\alpha,\beta)$. So they are canonically isomorphic.

<u>Proof</u>: By Yoneda's lemma (1.9) & (1.18), Exc.
or cf. 李文威 引理 2.7.10.

(1.21) Theorem - Exc.

In Set, cofiltered colimit commutes with finite limits. In other words, for any cofiltered set Λ, finite category J and a diagram $p : \Lambda \times J \to \mathrm{Set}$,

$$\operatorname{colim}_{\alpha\in\Lambda} \lim_{\beta\in J} p(\alpha,\beta) \simeq \lim_{\beta\in J} \operatorname{colim}_{\alpha\in\Lambda} p(\alpha,\beta)$$

<u>Proof</u> (1.17), Exc.

(1.22) Defn

A category $\mathcal{A}$ is called an additive category, if

1) $\mathcal{A}$ has a zero object, finite product and coproduct.

2) By 1), for $X, Y \in ob(\mathcal{A})$,

(pullback)
$$\begin{tikzcd} X \sqcup Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & Y \arrow[d, "0 \times id"] \\ X \arrow[r, "id \times 0"'] & X \times Y \end{tikzcd}$$
(push-forward)

recall:
$$\begin{tikzcd} 0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\boxplus"] & X \arrow[d] \\ Y \arrow[r] & X \times Y \end{tikzcd}$$

Then $X \sqcup Y \xrightarrow{\sim} X \times Y$ is an isomorphism.

We write $X \sqcup Y \simeq X \times Y$ as $X \oplus Y$, called the <u>direct sum</u> of X and Y.

3) By 1) 2), for any two object $X, Y \in ob(\mathcal{A})$, $\mathrm{Hom}_{\mathcal{A}}(X, Y)$ has a structure of an additive monoid: $f+g$ is defined as the composition

$$X \xrightarrow{\Delta} X \times X \xrightarrow{f \times g} Y \times Y \xrightarrow{\sim} Y \sqcup Y \longrightarrow Y \quad (?)$$

(as shown in the following diagram on the right)

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\boxplus"] & X \arrow[d] \arrow[drr, bend left, "f"] & & \\ X \arrow[r] \arrow[drr, bend right, "g"'] & X \times X \arrow[dr, dashed, "\exists ! \, f \times g"] & & \\ & & Y \times Y & Y \arrow[l] \\ & & Y \arrow[u] & 0 \arrow[l] \arrow[u] \arrow[ul, phantom, "\boxplus"] \end{tikzcd}$$

and the zero element is the zero morphism. Then we require $\mathrm{Hom}_{\mathcal{A}}(X, Y)$ is an abelian group.

Then $\mathcal{A}$ is called an <u>additive category</u>.

Let $\mathcal{A}, \mathcal{B}$ be two additive categories and $F: \mathcal{A} \longrightarrow \mathcal{B}$ a functor. F is called <u>additive</u>, if F preserves all finite products (or equivalently all direct sum)

(1.23) Remark – Exe.

1) $\forall\, X, Y, Z \in ob(\mathcal{A})$, the composition

$$Hom_{\mathcal{A}}(X,Y) \times Hom_{\mathcal{A}}(Y,Z) \longrightarrow Hom_{\mathcal{A}}(X,Z)$$

is a bilinear map of abelian groups. Therefore $h_X := Hom_{\mathcal{A}}(-,X)$ is a functor $\mathcal{A}^{op} \longrightarrow Ab$.

2) Let $F: \mathcal{A} \longrightarrow \mathcal{B}$ be an additive functor, then for any $X, Y \in \mathcal{A}$,

$$F: Hom_{\mathcal{A}}(X,Y) \longrightarrow Hom_{\mathcal{B}}(FX, FY)$$

is an abelian group homomorphism.

(1.24) Defn

Let $\mathcal{A}$ be a category with 0. Let $f: X \longrightarrow Y$ be a morphism.

- <u>kerf</u> is defined as the pullback and <u>cokerf</u> is defined as the pushforward

$$\begin{tikzcd} \ker f \arrow[r, "\alpha"] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\ 0 \arrow[r] & Y \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r] \arrow[d, "f"'] \arrow[dr, phantom, "\boxplus"] & 0 \arrow[d] \\ Y \arrow[r, "\beta"'] & \operatorname{coker} f \end{tikzcd}$$

- <u>imf</u> is defined as the pullback and <u>coimf</u> is defined as the pushforward

$$\begin{tikzcd} \ker f \arrow[r, "\alpha"] \arrow[d] \arrow[dr, phantom, "\boxplus"] & X \arrow[d, "p"] \\ 0 \arrow[r] & \operatorname{coim} f \end{tikzcd} \qquad \begin{tikzcd} \operatorname{im} f \arrow[r, "\gamma"] \arrow[d] \arrow[dr, phantom, "\square"] & Y \arrow[d, "\beta"] \\ 0 \arrow[r] & \operatorname{coker} f \end{tikzcd}$$

" $\operatorname{coim} f = \operatorname{coker}(\ker f)$ " " $\operatorname{im} f = \ker(\operatorname{coker} f)$ "

- Put these together,

$$\begin{tikzcd}
\ker f \arrow[r, "\alpha"] \arrow[d] \arrow[dr, phantom, "\boxplus"] & X \arrow[d, "p"] \arrow[ddrr, bend left, "f"] \arrow[dddr, bend left] & & \\
0 \arrow[r] \arrow[ddrr, bend right] \arrow[drrr, bend right, "0"] & \operatorname{coim} f \arrow[dr, dashed, "\exists!\,\bar f"'] \arrow[rr, dotted, "\text{univ. of } \operatorname{im} f : \psi"] \arrow[ddr, dashed] & & \\
& & \operatorname{im} f \arrow[r, hook, "\iota"] \arrow[d] \arrow[dr, phantom, "\square"] & Y \arrow[d, "\beta"] \\
& & 0 \arrow[r] & \operatorname{coker} f
\end{tikzcd}$$

(f) defn of ker f

Step1

Step2

Step3

we obtain a canonical map $\bar f : \operatorname{coim} f \to \operatorname{im} f$, and $f : X \to Y$ can be decomposed as

$$X \overset{p}{\twoheadrightarrow} \operatorname{coim} f \xrightarrow{\bar f} \operatorname{im} f \overset{\iota}{\hookrightarrow} Y \quad . \ (\text{coimage-image factorization})$$

(1.25) Defn

An additive category $\mathcal{A}$ is said to be an <u>abelian category</u>, if $\mathcal{A}$ has a finite limit and finite colimit, and for any morphism f in $\mathcal{C}$, the canonical map $\operatorname{coim} f \xrightarrow{\bar f} \operatorname{im} f$ is an isomorphism. (strict morphism)

(1.26) Defn

1) A <u>chain complex</u> in $\mathcal{A}$ is a sequence of objects and morphisms

$$C_\bullet : \quad \cdots \longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots$$

Such that $\partial_n \circ \partial_{n+1} = 0, \ \forall n \in \mathbb{Z}$.

2) The <u>q-th homology of $C_\bullet$</u> is defined as

$$H_q(C_\bullet) := \operatorname{coker}(\operatorname{im} \partial_{n+1} \to \ker \partial_n)$$

where the map "$\to$" is defined as

$$
\begin{tikzcd}
\ker\partial_{n+1} \arrow[r, "\alpha"] \arrow[d] \arrow[dr, phantom, "\boxplus"] & C_{n+2} \arrow[d, "p"] \arrow[ddrr, bend left, "\partial_n \circ \partial_{n+1} = 0"] \arrow[ddr, bend left, "\partial_n \circ \partial_{n+1} = 0"'] & & \\
0 \arrow[r] \arrow[drrr, bend right] \arrow[ddrr, bend right] & \operatorname{coim}\partial_{n+1} \arrow[dr, dashed, "\exists !"'] \arrow[drr, dashed, "\psi"] \arrow[ddr, dashed] & & \\
& & \ker\partial_n \arrow[r, "\alpha"] \arrow[d] \arrow[dr, phantom, "\square"] & C_n \arrow[d, "\partial_n"] \\
& & 0 \arrow[r] & C_{n-1}
\end{tikzcd}
$$

(recall we are in an abelian category $\mathcal{A}$)

3) If $H_q(C_\bullet) = 0$, then we say $C_\bullet$ is <u>exact</u> at C_n. This is equivalent to that $\operatorname{im}\partial_{n+1} = \ker\partial_n$. An everywhere exact chain complex is called an <u>exact sequence</u>. Such a chain complex is also said to be <u>acyclic</u>.

(1.27) Defn

Let $\mathcal{C}$ be a category. $F: \mathcal{C} \to \mathcal{D}$ be a functor.

1) $F: \mathcal{C} \longrightarrow \mathcal{D}$ is called <u>left exact</u>, if F preserves finite limit

2) . . <u>right exact</u>, ... colimit

3) . . <u>exact</u>, if F is both left and right exact.

(1.28) Remark

1) By (1.18), $\mathrm{Hom}_{\mathcal{C}}(X, -): \mathcal{C} \longrightarrow \mathrm{Set}$ and $\mathrm{Hom}_{\mathcal{C}}(-, X): \mathcal{C}^{op} \to \mathrm{Set}$ are left exact (in $\mathcal{C}$ and $\mathcal{C}^{op}$ accordingly)

2) By (1.19), any left (resp. right) adjoint functor $F: \mathcal{C} \to \mathcal{D}$ is right (resp. left) exact.

(1.29) Prop.

Let $\mathcal{A}, \mathcal{B}$ be two abelian categories. Then $F: \mathcal{A} \to \mathcal{B}$ is left exact, iff

1) F is additive

2) F maps exact sequences $0 \to A \to B \to C$ in $\mathcal{A}$ to exact sequences

$$0 \longrightarrow FA \longrightarrow FB \longrightarrow FC.$$

<u>Proof</u>: 1) $\Leftrightarrow$ F preserves all finite product ((1.22), by defn of additive functor)

2) $\overset{(?1)}{\Leftrightarrow}$ F preserves kernel $\overset{(?2)}{\Leftrightarrow}$ F preserves equalizers

So 1) + 2) $\Leftrightarrow$ F preserves all finite product and equalizer

$\underset{(1.17)}{\Leftrightarrow}$ F preserves all finite limits. $\square$

(?1) cf. 李文威《代数学方法》第二卷 命题 2.8.1.

(?2): consider

$$T \xrightarrow{h} X \overset{f}{\underset{g}{\rightrightarrows}} Y$$

Then $fh = gh$ is equivalent to $(f-g)h = 0$. Therefore

$$\ker(f,g) = \ker(f-g, 0) = \ker(f-g) \text{ as we can check}$$

So we can "interchange" kernel with equalizers in additive categories.

(1.30) Defn

Let $\mathcal{A}$ be an abelian category. We say $P \in \mathrm{Ob}(\mathcal{A})$ is projective if

$$\mathrm{Hom}_{\mathcal{A}}(P, -) : \mathcal{A} \longrightarrow \mathrm{Ab}$$

is exact. We say $I \in \mathrm{Ob}(\mathcal{A})$ is injective if $\mathrm{Hom}_{\mathcal{A}}(-, I) : \mathcal{A}^{op} \longrightarrow \mathrm{Ab}$ is exact.

(1.31) Motivations

Question? How to describe a circle on the plane?

1. Definition: All point P on the plane that has distance r to a fixed point O.

2. Equation: $x^2 + y^2 = 1$

3. Algebra:

Let's consider the ring $\mathbb{C}[x,y]/(x^2+y^2-1) = R$. Now what are the maximal ideals of R?

By correspondence theorem, (? cf. 李文威 第一卷 命题 5.3.4)

$$\{\text{maximal ideals of } R\} \longleftrightarrow \left\{\begin{matrix}\text{maximal ideal of } \mathbb{C}[x,y] \text{ containg} \\ (x^2+y^2-1)\end{matrix}\right\}$$

$$\mathfrak{m} \longmapsto \mathfrak{m}/(x^2+y^2-1).$$

By Hilbert nullstellensatz, maximal ideals of $\mathbb{C}[x,y]$ are of the form $(x-a, y-b)$ for $(a,b) \in \mathbb{C}^2$. Then

$$(x-a, y-b) \supseteq (x^2+y^2-1) \Leftrightarrow a^2+b^2-1=0$$

$x^2+y^2-1 = \alpha(x-a)+\beta(y-b)$. $\therefore$ (a,b) is a zero.

Hence we can use the $\underbrace{\text{maximal ideals of}}_{\text{MaxSpec}}$ $\mathbb{C}[x,y]/(x^2+y^2-1)$ to describle the circle.

But also, we shall ask the following questions:

1. If we replace $\mathbb{C}$ with arbitary field, e.g. $\mathbb{Q}$, then nullstellensatz fails.

2. Can we recover (in some sense) the topology of the circle on MaxSpec?

3. Can we expect some functorality of MaxSpec? No!

   Why functorality? $\varphi: X_1 \longrightarrow X_2$ (be affine varieties), then we have corresponding $\varphi^\flat: \mathbb{C}[x,y]/(p_1(x,y)) \longleftarrow \mathbb{C}[x,y]/(p_2(x,y))$

   (cf. Hartshorne Prop. I.3.5). Now can $\varphi^\flat$ induce canonical map on MaxSpec? Not necessarily!

Recall in algebra, if $\varphi: R_1 \longrightarrow R_2$ be any ring homomorphism, then φ induces

$$\varphi^\#: \operatorname{Spec} R_2 \longrightarrow \operatorname{Spec} R_1$$
$$I_2 \longmapsto \varphi^{-1}(I_2)$$

But for MaxSpec, we do not have similar results:

eg: $\mathfrak{p} \in \operatorname{Spec} R$, consider $R_{\mathfrak{p}}$ the localization of R at $\mathfrak{p}$, then $R_{\mathfrak{p}}$ has a unique maximal ideal $\mathfrak{p}R_{\mathfrak{p}}$. However, via $\varphi: R \to R_{\mathfrak{p}}$, $\varphi^{-1}(\mathfrak{p}R_{\mathfrak{p}}) = \mathfrak{p}$ is not nece. maximal !

$\Longrightarrow$ Grothendieck : "Spec".

(1.32) Motivation

Question : How to describle a (topological / differential / complex) manifold ?

Answer 1 : Describle its continous / smooth / holomorphic functions

Attempt 1 : $C^{\infty}(M)$: ~~ring of~~ continous / smooth / holomorphic functions.

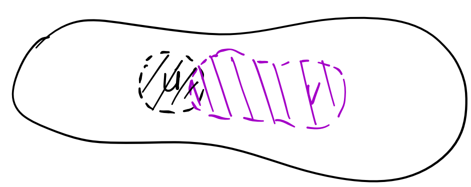
$\Downarrow$ But : If M is a compact Riemann surface, then M has no nonstant holomorphic function

cf. ① MSE 412/552

② Hartshorne, Theorem I.3.4.(a)

Attempt 2 : Local properties : for any $U \underset{\text{open}}{\subseteq} M$, describle continuous / smooth / holomorphic functions and how they are glued together.

M



Answer 2 : Linearlize the manifold and do calculus locally

—— tangent space & bundles on M

All of these can be answered in sheaf theory.

cf. Torsten Wedhorn "Manifolds, sheaves and cohomology".