EXAM IN CATEGORY THEORY

Let $\mathcal{C}$ be a category. A *retract* of an object $X \in \mathcal{C}$ is a commutative diagram in $\mathcal{C}$:

$$\begin{tikzcd} & X \arrow[dr, "r"] & \\ Y \arrow[ur, "i"] \arrow[rr, "\mathrm{Id}_Y"] & & Y. \end{tikzcd}$$

By abusing the terminology, we also say that Y is a retract of X.

Note that if we let $e = i \circ r$ then the morphism $e : X \to X$ is idempotent, i.e. $e^2 = e$. We say that $\mathcal{C}$ is *idempotent complete* if every idempotent morphism in $\mathcal{C}$ comes from a retract.

Problem 1. Show that the following conditions are equivalent for an idempotent morphism $e : X \to X$:

(1) e comes from a retract of X.

(2) The diagram $X \underset{\mathrm{Id}_X}{\overset{e}{\rightrightarrows}} X$ admits an equalizer.

(3) The diagram $X \underset{\mathrm{Id}_X}{\overset{e}{\rightrightarrows}} X$ admits a coequalizer.

Problem 2. Show that if an idempotent morphism $e : X \to X$ comes from a retract Y of X then Y is unique up to unique isomorphism.

Problem 3. Show that for any idempotent complete category $\mathcal{D}$, $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ is also idempotent complete.

We construct a category $\hat{\mathcal{C}}$ as follows. An object of $\hat{\mathcal{C}}$ is a pair (X, e) where $X \in \mathcal{C}$ and $e : X \to X$ is an idempotent morphism in $\mathcal{C}$. A morphism $(X, e) \to (Y, d)$ in $\hat{\mathcal{C}}$ is a morphism $f : X \to Y$ in $\mathcal{C}$ such that $d \circ f = f = f \circ e$. The composition law of $\hat{\mathcal{C}}$ is induced from that of $\mathcal{C}$.

Problem 4. Show that the category $\hat{\mathcal{C}}$ is well-defined.

Problem 5. Show that $\hat{\mathcal{C}}$ is idempotent complete.

Problem 6. Show that the assignment $X \mapsto (X, \mathrm{Id}_X)$, $f \mapsto f$ defines a fully faithful functor $I : \mathcal{C} \to \hat{\mathcal{C}}$.

Let $\mathcal{E}$ be an idempotent complete category. We say that a functor $I : \mathcal{C} \to \mathcal{E}$ exhibits $\mathcal{E}$ as *idempotent completion* of $\mathcal{C}$ if for any idempotent complete category $\mathcal{D}$, composition with I induces an equivalence $\mathrm{Fun}(\mathcal{E}, \mathcal{D}) \simeq \mathrm{Fun}(\mathcal{C}, \mathcal{D})$.

Problem 7. Show that the embedding $I : \mathcal{C} \to \hat{\mathcal{C}}$ exhibits $\hat{\mathcal{C}}$ as idempotent completion of $\mathcal{C}$.

Problem 8. Use Problem 7 to show that $\mathcal{C}$ is idempotent complete if and only if $\mathcal{C} \simeq \hat{\mathcal{C}}$.

Let $\mathcal{S}$ be a locally small category which admits all small colimits. We say that an object $X \in \mathcal{S}$ is *completely compact* if the representable functor $\mathrm{Hom}_{\mathcal{S}}(X, -) : \mathcal{S} \to Set$ preserves all small colimits.

Problem 9. Show that if $X \in \mathcal{S}$ is completely compact then any retract of X is completely compact.

In what follows, we assume $\mathcal{C}$ is small and let $j : \mathcal{C} \to \mathcal{P}(\mathcal{C})$ be the Yoneda embedding.

Problem 10. Show that $\mathcal{P}(\mathcal{C}) \simeq \mathcal{P}(\hat{\mathcal{C}})$. [Hint: Use Problem 7.]

Problem 11. Show that $j(X) \in \mathcal{P}(\mathcal{C})$ is completely compact for any $X \in \mathcal{C}$.

Problem 12. Show that if $S \in \mathcal{P}(\mathcal{C})$ is completely compact, then there exists $X \in \mathcal{C}$ such that S is a retract of $j(X)$. [Hint: Use the fact that every object of $\mathcal{P}(\mathcal{C})$ is a small colimit of objects in $j(\mathcal{C})$.]

Problem 13. Let $\tilde{\mathcal{C}} \subset \mathcal{P}(\mathcal{C})$ be the full subcategory consisting of the completely compact objects. Show that the embedding $j : \mathcal{C} \to \tilde{\mathcal{C}}$ exhibits $\tilde{\mathcal{C}}$ as idempotent completion of $\mathcal{C}$. [Hint: Combine the above four problems.]