A survey of elliptic curves for proof systems

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Abstract. Elliptic curves have become key ingredients for instantiating zero-knowledge proofs and more generally proof systems. Recently, there have been many tailored constructions of these curves that aim at efficiently implementing different kinds of proof systems. In this survey we provide the reader with a comprehensive view on existing work and revisit the contributions in terms of efficiency and security. We present an overview at three stages of the process: curves to instantiate a SNARK, curves to instantiate a recursive SNARK, and also curves to express an elliptic-curve related statement. We provide new constructions of curves for SNARKs and generalize the state-of-the-art constructions for recursive SNARKs. We also exhaustively document the existing work and open-source implementations.

1 Introduction

A proof system is an interactive protocol where one party (called the prover) tries to convince another party (called the verifier) that a given statement is true. In a zero-knowledge proof, we require further that the proof does not reveal anything beyond the truth of the statement. A proof is non-interactive if no communication is required between the prover and the verifier. In the class of non-interactive proofs, a particularly interesting concept for proving the computational integrity is the Succinct Non-interactive ARgument of Knowledge (SNARK). It provides a computationally sound proof cheap to verify and small compared to the size of the statement or the witness. SNARK systems can be further equipped with a zero-knowledge property that enables the proof to be verified without revealing anything about the intermediate steps (the witness generation).

Proof systems were introduced in [GMR89] and extensively studied both in theoretical and applied settings [Kil92,Mic94,GW11,BCCT12]. Recent constructions focus on a panoply of settings that range from cryptographic assumptions, asymptotic efficiency, concrete performance of implementations to numerous applications.

The mathematical security of many schemes relies on variants of the discrete logarithm problem (DLP): given a cyclic group $\mathbb{G}$ of prime order r written additively, a generator G, and an element $P \in \mathbb{G}$, compute $x \in \{0, 1, \ldots, r - 1\}$ such that $[x]G = P$. For example, the DLP-based zero-knowledge proofs (e.g. [BCC+16], Bulletproofs [BBB+18], Hyrax [WTs+18]) require a group $\mathbb{G}$ where the discrete logarithm problem is hard. They can be instantiated with any cryptographically secure elliptic curve, where the problem is then referred to as ECDLP. An efficient implementation uses the Ristretto group [Ham15,dV21] over `ed25519` [BDL+12] (e.g. Bulletproofs' Dalek library [dVYA22]).

Alternatively, a bilinear pairing is required in certain schemes. A cryptographic pairing is a bilinear non-degenerate map from two groups $\mathbb{G}_1$ and $\mathbb{G}_2$ to a target group $\mathbb{G}_T$, where the three groups have the same prime order r. The pairing is denoted $e\colon \mathbb{G}_1 \times \mathbb{G}_2 \to \mathbb{G}_T$, where $\mathbb{G}_1$ and $\mathbb{G}_2$ are distinct subgroups of the same order of an elliptic curve, and $\mathbb{G}_T$ is a multiplicative subgroup of some finite field. Pairing-based SNARKs cannot be instantiated with generic-purpose elliptic curves, but instead require tailored constructions of elliptic curves. More precisely, they need pairing-friendly elliptic curves with additional properties, designed on purpose to provide an efficient implementation. In this paper, we propose a view on the elliptic curves designed for different proof systems, revisit some constructions and propose some new ones.

Contributions. We present an overview of elliptic curves tailored for SNARKs, at three stages of the process: curves to instantiate a SNARK, curves to instantiate a recursive SNARK, and also curves to express an elliptic-curve related statement. We investigated the literature and collected the milestones in the constructions of pairing-friendly elliptic curves and 2-chains for SNARKs. We also cover different kinds of cycles of curves and summarize the impossibility results and open questions.

In the case of recursive proofs, the promising (pairing-friendly) curves with room for improvements are the 2-chains of curves. We generalize the constructions from [EG22] and include the BN curves in the general framework of 2-chains, in addition to BLS12 and BLS24 curves. We improve the final exponentiation for outer BW6 curves in the three cases (BLS12, BLS24 and BN).

We propose a systematic way to obtain seeds for inner curves such that we ensure a larger power of 2 in $r - 1$. For example, this more than doubles the 2-adicity of previously known SNARK-friendly BLS24. Although the results were already satisfying for BLS12, it was a drawback for using BLS24. As an application, we propose a new inner BLS24 more suitable for KZG-based SNARKs, that can be used to prove deeper arithmetic circuits ($2^{60} \mid r - 1$ instead of $2^{22} \mid r - 1$ at the 128-bit security level).

There are various library developments for SNARKs. We list (in Table 15) all known open-source library developments (to the best of our knowledge), together with the main projects using them, excluding forks and stand-alone implementations.

Organization. Section 2 provides a historical overview on the state of the art and section 2 lays out briefly mathematical preliminaries on bilinear pairings and zk-SNARKs. The core of the paper are sections 4, 5 and 6. First, in 4, we look into suitable pairing-friendly elliptic curves to some popular SNARKs, provide existing and new constructions and compare them in terms of efficiency and security. Next, in 5, we investigate tailored constructions for recursive SNARKs: different combinations of pairing-friendly and plain cycles, and 2-chains of pairing-friendly curves. Finally, in 6, we look into how to efficiently prove elliptic curves cryptography in a SNARK. After concluding, we catalog in appendix A a list of different independent implementations of discussed curves, cycles and 2-chains.

2 State of the art

Building on ideas from the pairing-based doubly-homomorphic encryption scheme [BGN05], Groth, Ostrovsky and Sahai [GOS06,Gro06,GS08] introduced the pairing-based non-interactive zero-knowledge proofs, yielding the first linear-size proofs based on standard assumptions. Groth [Gro10] combined these techniques with ideas from interactive zero-knowledge proofs to give the first constant-size proofs which are based on constructing a set of polynomial equations and using pairings to efficiently verify these equations. This work relies on two new introduced pairing-based cryptographic assumptions, namely the q-computational power Diffie-Hellman (q-CPDH) and the q-power knowledge of Exponent (q-PKE).

Following this direction of work, Gennaro et al. [GGPR13] proposed an insightful construction of polynomial equations that resulted in many interesting implementations [PHGR13,BFR+13,BCG+13,BCTV14b,KPP+14] leading to the most succinct and widely implemented pairing-based SNARK [Gro16]. The first implementation, Pinocchio [PHGR13] used a pairing-friendly elliptic curve in the Barreto–Naehrig family [BN06] (BN) targeting a 128-bit security level, but the source code was proprietary. They used the BN curve defined over a 256-bit field suggested in [NNS10] (seed $x = 1868033^3$). Next, as part of Pantry [BFR+13], authors re-implemented Pinocchio under a BSD-style license using a 254-bit BN curve from [BGM+10] (seed $x = -(2^{62} + 2^{55} + 1)$, first introduced in [NAS+08]). This new BN implementation partially builds on techniques from the previous BN paper [NNS10] Pinocchio used.

Later in [BCG+13], the authors observed that constructing a pairing-friendly curve with a subgroup order r where $r - 1$ is divisible by 2^L a large power of 2, results in an efficient proof generation via suitable Fast Fourier Transforms (FFTs) in $\mathbb{F}_r$. To speedup the arithmetic, they proposed to use the elliptic curve in Edwards form, by looking for a group order multiple of 4. To match these two constraints: 2^L divides $r - 1$ and the curve has order $4 \cdot r$, they designed a Galbraith-McKee-Valença curve [GMV07] of embedding degree 6 (GMV6) defined over $\mathbb{F}_p$ where p is a prime of 183 bits, and of order $4r$ where r is of 181 bits such that $2^{31} \mid r - 1$. This curve was targeting a 80-bit security level in 2013 and was implemented in `libff` [BSCT+a].

The same year, [BCTV14b] improved previous SNARK works and developed their implementations with two different curves: the same GMV6 curve for the estimated 80-bit security level ($2^{31} \mid r - 1$) and a new BN curve for the estimated 128-bit security level ($2^{28} \mid r - 1$). Note that, as of today, this BN curve is the one precompiled in the Ethereum blockchain. Last, Trinocchio [KPP+14] provided an implementation of a privacy-preserving version of Pinocchio using the BN curve from [BCTV14b]. As the number of applications increases, other implementations were released especially in the blockchain space. In particular, Zcash cryptocurrency first implemented Zerocash protocol [BCG+14] which uses the SNARK of [BCTV14b] and its BN curve, before switching to the SNARK of [Gro16] and a new curve.

The previous paragraphs told an overview over ten years of active and fruitful research on SNARK from the prelude [BGN05] in 2005 to 2016 with [Gro16]. Meanwhile, the research in discrete logarithm computation in finite fields related to pairings also saw a tremendous decade, with a prequel in 2012 with a discrete logarithm record-breaking computation in $GF(3^{6 \cdot 97})$ in [FNK12]. This announcement had a quite important impact over the community at that time [GM16, §9.3.8 p.30], probably because the broken curve was introduced with the BLS short signatures [BLS01]. The targeted finite field $GF(3^{6 \cdot 97})$ was considered safe for 80-bit security implementations in 2012. In 2014, two algorithms on fast computation of DL in small characteristic finite fields were published [BGJT14,GKZ14] (quasi-polynomial-time algorithm, zig-zag descent). In 2019, Granger et al. announced the largest record computation in a field $GF(2^{30750})$ [GKL+21], and finally [KW22] published a proved complexity. Nowadays, small-characteristic finite fields should be definitely avoided, as computing DL is not hard anymore especially if the extension degree is composite, which renders supersingular elliptic curves in small characteristic insecure.

Cryptanalysts also considered large and medium characteristic finite fields of the form $GF(p^k)$ that arise with pairings. In 2016, Kim and Barbulescu [KB16] presented a variant of the Number Field Sieve (NFS) algorithm which reduced the security level of the BN curves previously at 128-bit security to around 110. Menezes, Sarkar and Singh [MSS16] were the first to analyze thoroughly the consequences of the new NFS variants on the security of pairing-friendly curves. Their conclusions recommend the Barreto–Lynn–Scott curves of embedding degree 12 [BLS04] or BN curves over 384-bit prime fields instead of BN curves over 256-bit fields. Based on this work, the Zcash team derived the BLS12-381 curve defined over a 381-bit field with a 255-bit prime subgroup order r such that $2^{32} \mid r - 1$ [Bow17]. This curve is used today in several projects (e.g. Zcash, Ethereum 2.0, Skale, Algorand, Dfinity, Chia), implemented in different libraries and considered for IETF standards.

Besides their efficiency, SNARKs' succinctness makes them good candidates for recursive proof composition. Such proofs could themselves verify the correctness of other proofs. This would allow a single proof to inductively attest to the correctness of many former proofs. However, once a first proof is generated, it is highly impractical to use the same elliptic curve to generate a second proof

verifying the first one. A practical approach requires two different curves $E(\mathbb{F}_p)$ and $E'(\mathbb{F}_q)$ that are closely tied together (cf. Fig 1).

$$\#E'(\mathbb{F}_q) = p \qquad \boxed{E'(\mathbb{F}_q)} \qquad \boxed{E(\mathbb{F}_p)} \qquad \#E(\mathbb{F}_p) = q$$
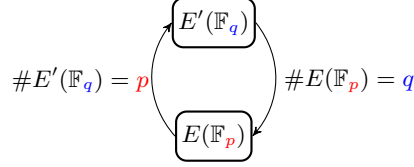
Fig. 1. A cycle of elliptic curves.

Therefore, we need to find new pairing-friendly curves constructions for this problem. Indeed, given a curve equation over a finite field, computing its order can be done efficiently with the SEA algorithm, while given p and r, computing the parameters of the curve equation involves computing a class polynomial (e.g. with Sutherland software [Sut11,ES10]), and it is infeasible if the curve discriminant D is too large (say more than 20 decimal digits). Ben-Sasson et al. [BCTV14a] presented the first practical setting of a recursive proof composition with a cycle of two MNT pairing-friendly elliptic curves [MNT01]. Proofs generated from one curve can feasibly reason about proofs generated from the other curve. To achieve this, one curve's order is the other curve's base field order and vice-versa (i.e. $\#E'(\mathbb{F}_q) = p$ and $\#E(\mathbb{F}_p) = q$). The two curves are necessarily of prime order, hence cannot admit an Edwards form. Moreover, they have low embedding degrees (4 and 6) resulting in large base fields to achieve a standard security level. The authors proposed a pair of MNT curves with parameters of 298 bits which they estimated to meet the 80-bit security level in 2014.

Around approximately the same time, Costello et al. [CFH+15] built on this idea to obtain a bounded recursive proof composition using a 2-chain of two elliptic curves (cf. Fig 2), i.e. a BN curve (with seed $x = -(2^{62} + 2^{55} + 1)$ from [NAS+08]) defined over a 254-bit field $\mathbb{F}_p$ and a BW6-509 curve constructed using the Brezing-Weng method (BW) in a way that it has a prime subgroup order equal to p the field characteristic of BN's $\mathbb{F}_p$. Note that both MNT4 and MNT6 are $\mathbb{F}_r^*$-smooth ($2^{17} \mid r - 1$) while Geppetto's BN and BW6 are not. Later on, [BCTV14a] authors found a new MNT4/6 pair with parameters of 753 bits which they estimated at the 128-bit security level. They shared the parameters of this latter pair with the Coda protocol [BMRS20] developers (now Mina) who used it to build a recursive SNARK-based light blockchain. They updated the pre-print with this pair of parameters only recently (2020).

Few years after [BCTV14a], motivated by a new proof composition application (decentralized private computation), ZEXE [BCG+20] proposed a new chain of curves. It is similar to Geppetto's one although the citation was missing. The first curve is a BLS12-377 defined over a 377-bit field with a subgroup r of 253 bits ($2^{47} \mid r - 1$) and, using the more rudimentary Cocks-Pinch method (CP), the second curve is a CP6-782 of embedding degree 6, defined over a 782-bit field

$$\#E'(\mathbb{F}_q) = h \cdot p$$
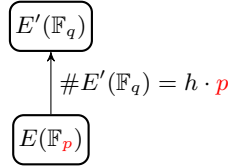
Fig. 2. A 2-chain of elliptic curves.

and having a group order divisible by BLS12-377's base field size p. Based on this inner BLS12-377, El Housni and Guillevic [EG20] proposed and alternative outer curve to CP6-782 that is much faster for a recursive SNARK [Gro16] generation and verification. They used a variant of the Brezing-Weng method to construct a BW6-761 curve of embedding degree 6 defined over a 761-bit field and enjoying properties for efficient implementation. Then the same authors [EG22] generalized the BW6-761 curve construction, the pairing formulas and the group operations to any BW6 curve defined on top of any BLS12 curve, forming a family of 2-chain pairing-friendly curves.

Up to this point, all the pairing-based SNARKs use the three pairing groups $\mathbb{G}_1, \mathbb{G}_2$ and $\mathbb{G}_T$ whether for proving, verifying or setting the SNARK up. Therefore, all previously mentioned curves were constructed in order to optimize operations in the three different groups. Note that the most efficient pairing-based SNARKs have a trapdoored pre-processing setup specific to the statement to prove (e.g. [Gro16]). Recently, a new kind of SNARKs was introduced, where the pre-processing phase is not specific to a given statement but is rather universal in that sense. Groth et al. [GKM+18] proposed a universal SNARK with a single setup to prove all statements of a given bounded size. However, Sonic [MBKM19] is considered to be the first practical universal SNARK. This work inspired many researchers and practitioners who then came up with new and elegant universal constructions (e.g. AuroraLight [Gab19], PLONK [GWC19], Marlin [CHM+20]).

A key building block of these universal constructions is the use of **polynomial commitment** (PC) schemes. While there are different PC schemes with tradeoffs, the KZG scheme [KZG10] remains the most efficient. It uses pairings and therefore its implementation requires a suitable pairing-friendly elliptic curve. Contrary to the previous pairing-friendly curves used in the SNARK context, KZG-based universal SNARKs need a curve optimized only for $\mathbb{G}_1$ arithmetic and pairings. This observation inspired El Housni and Guillevic [EG22], who investigated the use of Barreto–Lynn–Scott curves of embedding degree 24 (BLS24) to instantiate KZG-based universal SNARKs. They proposed a BLS24-315 curve defined over a 315-bit field with a subgroup order r of 253 bits such that $2^{22} \mid r - 1$. Moreover, similarly to the BLS12-BW6 chains, they characterized all 2-chains that can be formed with a BLS24 as an inner curve and BW6 as an outer curve. They shortlisted few BW6 curves (e.g. BW6-633 and BW6-672) and looked into Cocks-Pinch curves of higher embedding degrees (CP8 and CP12) for a more conservative

security. Their work was implemented in the `gnark` ecosystem [BPH⁺22] using PLONK.

Actually, the universality of these constructions comes from the nature of the polynomial commitment scheme. By swapping the KZG scheme for other PC schemes [BCC⁺16,VP19,BFS20], one gets new transparent (no setup) SNARKs [KPV19,BFS20] and new transparent recursive SNARKs as first proposed by Bowe, Grigg and Hopwood in [BGH19] (Halo) and then formalized and generalized in [BCMS20,BDFG21]. These recursive constructions build on the discrete logarithm based PC from [BCC⁺16] and Bulletproofs [BBB⁺18] and one might want to instantiate them with a non-pairing-friendly elliptic curve like `ed25519`. However, having an efficient recursion requires a **cycle** of elliptic curves as in [BCTV14a], hence the curves are of prime order, which means Edwards and Montgomery forms are not possible. Although this time, the curves do not need to be pairing-friendly. To this end, Halo's authors derived an efficient cycle for SNARK implementation, namely the Tweedledum-Tweedledee cycle. Later, Hopwood proposed the more efficient Pasta cycle [Hop20]. Note that finding such cycles is much easier than finding pairing-friendly cycles. It was investigated previously in a different context by Stange and Silverman [SS11]. Finally, as a future work, Halo suggests using the same SNARK techniques with a hybrid cycle where one curve is pairing-friendly and the other is not. Thereafter, Hopwood proposed a hybrid cycle, Pluto-Eris [Hop21]. Such a cycle can be constructed from any prime-order pairing-friendly curve (e.g. BN [BN06], Freeman [Fre10], MNT [MNT01]).

SNARKs enable verifying non-deterministic polynomial time (NP) computations with substantially lower complexity than those required for classical NP verification. In many applications (e.g. privacy-preserving cryptocurrencies, zk-rollups, verifiable computation outsourcing), the NP-computation revolves around proving the knowledge of a hash pre-image or the verification of a cryptographic signature. In CØCØ [KZM⁺15], Kosba et al. proposed a library of cryptographic primitives that can be efficiently proved in a SNARK. In particular, the authors looked into proving an Elliptic Curve Diffie-Hellman (ECDH) key exchange. They constructed a new elliptic curve to efficiently implement the operation required in key exchanges, i.e. the scalar multiplication (cf. Fig. 3).

This curve defined over the scalar field $\mathbb{F}_r$ of the BN curve, with the seed `0x44e992b44a6909f1` from [BCTV14a], was given in Montgomery form for further optimizing the arithmetic inside a SNARK. The paper also mentioned converting this associated curve to Edwards form in order to efficiently prove EdDSA signatures. Following this work, the Zcash team introduced the JubJub curve [ZCa21] which is a similar curve in twisted Edwards form associated to BLS12-381, alongside further algebraic optimizations. This curve allowed Zcash to efficiently implement a collision-resistant variant of an EC-based Pedersen hash inside a SNARK. Practitioners who proposed tailored elliptic curves for SNARKs (BN254, BLS12-377, BLS24-315, BW6-761, BW6-633) each time also proposed a twisted Edwards associated curve defined over the scalar field. Finally, Masson et al. [MSZ21] performed an exhaustive search of associated curves over

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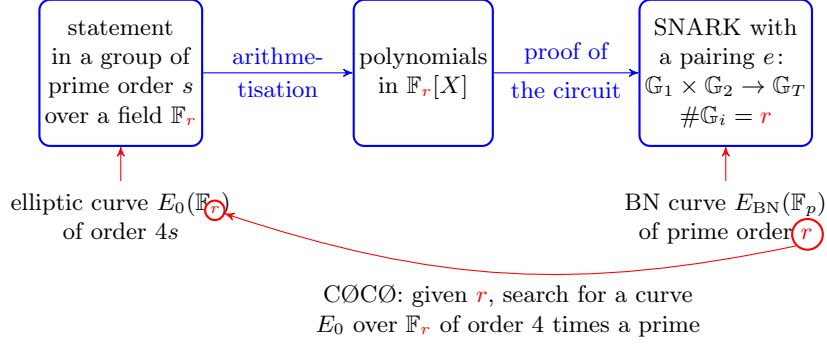
Fig. 3. Kosba et al. construction [KZM[+]15]

BLS12-381 with small Complex Multiplication discriminant in order to speed up the curve arithmetic using a fast endomorphism. They found an isolated curve with discriminant $D = -8$, called Bandersnatch. A similar associated curve is unlikely to be found for other SNARK curves of interest.

We stress that the problem solved in the CØCØ construction is partway similar to the one solved in the Geppetto construction. Remember that in a proof composition, one requires writing efficiently a pairing computation as a SNARK statement and to this end, researchers came up with new elliptic curves that efficiently encode the pairing in the SNARK (cf. Fig. 4). In CØCØ, the SNARK curve is fixed ($E_{BN}(\mathbb{F}_p)$ of order r) and the authors look for a curve ($E_0(\mathbb{F}_r)$) for the statement (ECDH). In Geppetto, the statement is "the verification of a previously generated proof" and its curve is fixed ($E_{BN}(\mathbb{F}_p)$ of order r) as it was already used to generate that previous proof. The authors look for a new SNARK curve (BW6($\mathbb{F}_s$) of order $h \cdot p$) to prove "the proof composition" statement.
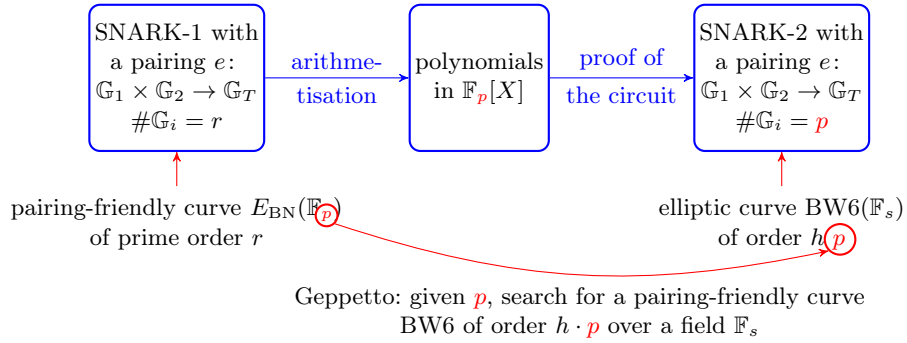
Fig. 4. Geppetto construction [CFH[+]15]

3 Mathematical preliminaries

3.1 Background on pairings

We briefly recall elementary definitions of pairings and present the computation of two pairings used in practice, the Tate and ate pairings. All elliptic curves discussed below are *ordinary* (i.e. non-supersingular).

Let E be an elliptic curve defined over a field $\mathbb{F}_p$, where p is a prime power. Let π_p be the Frobenius endomorphism: $(x, y) \mapsto (x^p, y^p)$. Its minimal polynomial is $X^2 - tX + p$ where t is called the *trace*. Let r be a prime divisor of the curve order $\#E(\mathbb{F}_p) = p + 1 - t$. The r-torsion subgroup of E is denoted $E[r] := \{P \in E(\overline{\mathbb{F}_p}), [r]P = \mathcal{O}\}$ and has two subgroups of order r (eigenspaces of π_p in $E[r]$) that are useful for pairing applications. We define the two groups $\mathbb{G}_1 = E[r] \cap \ker(\pi_p - [1])$ with a generator denoted by G_1, and $\mathbb{G}_2 = E[r] \cap \ker(\pi_p - [p])$ with a generator G_2. The group $\mathbb{G}_2$ is defined over $\mathbb{F}_{p^k}$, where the embedding degree k is the smallest integer $k \in \mathbb{N}^*$ such that $r \mid p^k - 1$.

We recall the Tate and ate pairing definitions, based on the same two steps: evaluating a function $f_{s,Q}$ at a point P, the Miller loop step, and then raising it to the power $(p^k - 1)/r$, the final exponentiation step. The function $f_{s,Q}$ has divisor $\operatorname{div}(f_{s,Q}) = s(Q) - ([s]Q) - (s - 1)(\mathcal{O})$ and satisfies, for integers i and j,

$$f_{i+j,Q} = f_{i,Q} f_{j,Q} \frac{\ell_{[i]Q,[j]Q}}{v_{[i+j]Q}},$$

where $\ell_{[i]Q,[j]Q}$ and $v_{[i+j]Q}$ are the two lines needed to compute $[i + j]Q$ from $[i]Q$ and $[j]Q$ (ℓ intersecting the two points and v the vertical). We compute $f_{s,Q}(P)$ with the Miller loop presented in Algorithm 1.

Algorithm 1: MillerLoop(s, P, Q)

Output: $m = f_{s,Q}(P)$ for $s = \sum_{i=0}^{t} s_i 2^i$

1   $m \leftarrow 1$; $S \leftarrow Q$;
2   **for** b *from* $t - 1$ *to* 0 **do**
3     $\ell \leftarrow \ell_{S,S}(P)$; $S \leftarrow [2]S$;   // DOUBLELINE
4     $v \leftarrow v_{[2]S}(P)$;   // VERTICALLINE
5     $m \leftarrow m^2 \cdot \ell/v$;   // UPDATE1
6   **if** $s_b = 1$ **then**
7       $\ell \leftarrow \ell_{S,Q}(P)$; $S \leftarrow S + Q$;   // ADDLINE
8       $v \leftarrow v_{S+Q}(P)$;   // VERTICALLINE
9       $m \leftarrow m \cdot \ell/v$;   // UPDATE2
10 **return** m;

The Tate and ate pairings are defined by

$$\operatorname{Tate}(P, Q) := f_{r,P}(Q)^{(p^k - 1)/r}$$

$$\operatorname{ate}(P, Q) := f_{t-1,Q}(P)^{(p^k - 1)/r}$$

where $P \in \mathbb{G}_1$ and $Q \in \mathbb{G}_2$. It is also important to recall some results with respect to the complex multiplication (CM) discriminant $-D$. When $D = 3$ (resp. $D = 4$), the curve has CM by $\mathbb{Q}(\sqrt{-3})$ (resp. $\mathbb{Q}(\sqrt{-1})$) so that twists of degrees 3 and 6 exist (resp. 4). When E has d-th order twists for some $d \mid k$, then $\mathbb{G}_2$ is isomorphic to $E'[r](\mathbb{F}_{p^{k/d}})$ for some twist E'. Otherwise, in the general case, E admits a single twist (up to isomorphism) and it is of degree 2.

3.2 Some pairing-friendly constructions

We recall some methods and families from the literature related to pairing-friendly constructions that will be of interest in the following sections. A detailed study of these constructions is available in the taxonomy paper by Freeman, Scott and Teske [FST10]. We focus on the Cocks–Pinch [FST10, Sec. 4.1] (CP) and Brezing–Weng [FST10, Sec. 6.1] (BW) methods, and Barreto–Lynn–Scott [FST10, Sec. 6.6] of embedding degrees 12 and 24 (BLS12 and BLS24), Barreto–Naehrig [BN06] (BN), Miyaji–Nakabayashi–Takano [FST10, Sec. 5.1] of embedding degrees 4 and 6 (MNT4 and MNT6), Galbraith–McKee-Valença [FST10, Sec. 5.2] of embedding degree 6 and cofactor 4 families and Kachisa–Schaefer–Scott [KSS08] of embedding degrees 16 and 18 (KSS16 and KSS18) families.

From a high level perspective, there are two ways to obtain pairing-friendly curves. Generic algorithms take as inputs k and r, and output (if it exists) an elliptic curve defined over a field $\mathbb{F}_p$ and of embedding degree k w.r.t. a subgroup of prime order r over $\mathbb{F}_p$. If r is an integer, this is the Cocks–Pinch method, if r is a polynomial, this is the Brezing–Weng method. The alternative is to consider precomputed tables of polynomial families $(k, r(x), D, t(x), p(x))$ as in [FST10]. To rank the families of the same k, the rho-value is defined as the ratio of the sizes of p and r, resp. the ratio of the degrees of the polynomials $p(x)$ and $r(x)$:

$$\rho = \frac{\log p}{\log r}, \quad \text{resp.} \quad \rho = \frac{\deg p(x)}{\deg r(x)} \tag{1}$$

and because $r \mid p + 1 - t$, then $\rho \geq 1$.

Cocks–Pinch is the most flexible method and can be used to construct a curve $E(\mathbb{F}_p)$ with arbitrary embedding degrees and a subgroup order r such that the ratio $\rho = \log_2 p / \log_2 r$ is approximately 2. It works by fixing r and the CM discriminant D and then computing the trace t and the prime p such that the CM equation $4p = t^2 + Dy^2$ (for some $y \in \mathbb{Z}$) is satisfied (cf. Alg. 2).

Brezing and Weng [FST10, Sec. 6.1], and independently, Barreto, Lynn and Scott [FST10, Sec. 6.1] generalized the Cocks–Pinch method by parameterizing t, r and p as polynomials. This led to curves with a ratio $\rho < 2$. Below, we sketch the idea of the algorithm in its generality for both BLS and BW constructions (cf. Alg. 3). Particular choices of polynomials for $k = 12$ and $k = 24$ yield two families of curves with good security/performance tradeoffs, denoted respectively BLS12 and BLS24. The parameters are given in Table 1.

MNT curves, however, have a fixed embedding degree $k \in \{3, 4, 6\}$ and a variable discriminant D. They are also parameterized by polynomials and form

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Algorithm 2: Cocks–Pinch method

Input: A positive integer k and a positive square-free integer D

Output: $E/\mathbb{F}_p$ with an order-r subgroup and embedding degree k

1   Choose a prime r such that k divides $r - 1$ and $-D$ is a square modulo r;

2   Compute $t = 1 + x^{(r-1)/k}$ for x a generator of $(\mathbb{Z}/r\mathbb{Z})^{\times}$;

3   Compute $y = (t - 2)/\sqrt{-D} \bmod r$;

4   Lift t and y in $\mathbb{Z}$;

5   Compute $p = (t^2 + Dy^2)/4$ in $\mathbb{Q}$;

6   **if** *p is a prime integer* **then**

7 | Use CM method ($D < 10^{12}$) to construct $E/\mathbb{F}_p$ with order-r subgroup;

8   **else**

9 | Go back to 1;

10   **return** $E/\mathbb{F}_p$ *with an order-r subgroup and embedding degree k*

Algorithm 3: Idea of BLS and BW methods

Input: A positive integer k and a positive square-free integer D

Output: $E/\mathbb{F}_{p(x)}$ with an order-$r(x)$ subgroup and embedding degree k

1   Choose an irreducible polynomial $r(x) \in \mathbb{Z}[x]$ with positive leading coefficient such that $\sqrt{-D}$ and the primitive k-th root of unity ζ_k are in $K = \mathbb{Q}[x]/(r(x))$;

2   Choose $t(x) \in \mathbb{Q}[x]$ be a polynomial representing $\zeta_k + 1$ in K;

3   Set $y(x) \in \mathbb{Q}[x]$ be a polynomial mapping to $(\zeta_k - 1)/\sqrt{-D}$ in K;

4   Compute $p(x) = (t^2(x) + Dy^2(x))/4$ in $\mathbb{Q}[x]$;

5   **while** *p(x) is not irreducible* **do**

6 | Go back to 1;

7   **return** $E/\mathbb{F}_{p(x)}$ *with an order-r(x) subgroup and embedding degree k*

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a sparse family (cf. Table 1) because one is required to solve a generalized Pell equation. Karabina and Teske [KT08] showed that there is a prime-order elliptic curve $E/\mathbb{F}_p$ from the MNT6 family if and only if there is a prime-order elliptic curve $E'/\mathbb{F}_q$ from the MNT4 family such that $\#E(\mathbb{F}_p) = q$ and $\#E'(\mathbb{F}_q) = p$.

GMV curves extend MNT curves with cofactors $2 \le h \le 5$. This is achieved by following the MNT strategy and substituting $h \cdot r$ for $\#E(\mathbb{F}_p)$ followed by an explicit analysis of cases $h = 2, 3, 4$ and 5. Polynomial parameters for GMV with $k = 6$ and $h = 4$ are given in Table 1.

BN curves form a family of prime-order pairing-friendly elliptic curves with $k = 12$ and $D = 3$ (cf. Table 1). The construction is based on a result from [GMV07] and a lucky try in which the right-hand side of the CM equation happens to be a constant times a perfect square polynomial. However, it was suggested in [FST10, Example 6.8] that the BN construction can be viewed as a complete family on its own.

Another strategy to build pairing-friendly constructions is to pick random small elements and take their minimal polynomials as the subgroup order polynomial $r(x)$. For well chosen embedding degrees $k = 16$ and $k = 18$, this yields the KSS16 and KSS18 families with $\rho = 5/4$ and $\rho = 4/3$ respectively (cf. Table 1).

Family	k	D	ρ	$r(x)$	$p(x)$	$t(x)$
BN	12	-3	1	$36x^4 + 36x^3 + 18x^2 + 6x + 1$	$36x^4 + 36x^3 + 24x^2 + 6x + 1$	$6x^2 + 1$
BLS12	12	-3	$3/2$	$x^4 - x^2 + 1$	$(x^6 - 2x^5 + 2x^3 + x + 1)/3 + x$	$x + 1$
BLS24	24	-3	$5/4$	$x^8 - x^4 + 1$	$(x^{10} - 2x^9 + x^8 - x^6 + 2x^5 - x^4 + x^2 + x + 1)/3$	$x + 1$
MNT4	4	D	1	$x^2 + 1$ or $x^2 + 2x + 2$	$x^2 + x + 1$	$-x$ or $x + 1$
MNT6	6	D	1	$4x^2 \pm 2x + 1$	$4x^2 + 1$	$1 \pm 2x$
GMV6 ($h = 4$)	6	D	1	$4x^2 + 2x + 1 = \Phi_6(t-1)$ $28x^2 + 10x + 1 = \Phi_6(t-1)/7$ $28x^2 + 18x + 3 = \Phi_6(t-1)/7$ $\mathbf{52x^2 + 14x + 1 = \Phi_6(t-1)/13}$ $52x^2 + 38x + 7 = \Phi_6(t-1)/13$	$16x^2 + 10x + 5$ $112x^2 + 54x + 7$ $112x^2 + 86x + 17$ $\mathbf{208x^2 + 30x + 1}$ $208x^2 + 126x + 19$	$2x + 2$ $14x + 4$ $14x + 6$ $\mathbf{-26x - 2}$ $-26x - 8$
KSS16	16	-1	$5/4$	$(x^9 + 48x^4 + 625)/61550$	$(x^{10} + 2x^9 + 5x^8 + 48x^6 + 152x^5 + 240x^4 + 625x^2 + 2398x + 3125)/980$	$(2x^5 + 41x + 35)/35$
KSS18	18	-3	$4/3$	$(x^6 + 37x^3 + 343)/343$	$(x^8 + 5x^7 + 7x^6 + 37x^5 + 188x^4 + 259x^3 + 343x^2 + 1763x + 2401)/21$	$(x^4 + 16x + 7)/7$

Table 1. Polynomial parameters of BN, BLS12, MNT4, MNT6, GMV6, KSS16 and KSS18 families.

3.3 zk-SNARKs

In this survey, we mainly focus on preprocessing zkSNARKs for NP languages for which we give a basic overview. Given a public NP program F, public inputs a and b and a private input w, such that the program F satisfies the relation

$F(a, w) := b$, a zk-SNARK consists in proving this relation succinctly without revealing the private input w. Given a security parameter λ, it consists of the `Setup`, `Prove` and `Verify` algorithms:

$$(\sigma_p, \sigma_v) \leftarrow \texttt{Setup}(F, 1^\lambda)$$
$$\pi \leftarrow \texttt{Prove}(a, b, w, \sigma_p)$$
$$0/1 \leftarrow \texttt{Verify}(a, b, \pi, \sigma_v)$$

where σ_p is the proving key which encodes the program F for the prover, σ_v the verification key that encodes F for the verifier and π the proof. If the `Setup` algorithm is trapdoored an additional secret input τ is required $(\sigma_p, \sigma_v) \leftarrow \texttt{Setup}(F, \tau, 1^\lambda)$.

4   SNARKs with pairing-friendly elliptic curves

Pairing-based SNARKs can be instantiated with any secure pairing-friendly elliptic curve. For efficiency, we require a fast arithmetic in $\mathbb{F}_r$ (where r is the curve prime subgroup order), in $\mathbb{G}_1$ and in $\mathbb{G}_2$, and a fast pairing computation. For security, we are interested in the 128-bit security level in $\mathbb{G}_1$, $\mathbb{G}_2$ and $\mathbb{G}_T$.

4.1   Efficiency

The `Setup` and `Prove` algorithms involve solving multiple large instances of tasks about polynomial arithmetic in $\mathbb{F}_r[X]$ and multi-scalar multiplication (MSM) over $\mathbb{G}_1$ and $\mathbb{G}_2$. The `Verify` algorithm involves computing a product of pairings and an evaluation in $\mathbb{G}_T$. Fast arithmetic in $\mathbb{F}_r[X]$, when manipulating large-degree polynomials, is best implemented using the Fast Fourier Transform (FFT) [Pol71] and MSMs of large sizes are best implemented using a variant of Pippenger's algorithm [BDLO12, Section 4]. For example, Table 2 reports the numbers of MSMs and FFTs required in the `Setup`, `Prove` and `Verify` algorithms in the [Gro16] SNARK and the KZG-based PLONK universal SNARK [GWC19]. The report follows the `gnark` open-source implementation [BPH+22] but we stress that these numbers might differ slightly from an implementation to another.

An efficient implementation of the FFT over $\mathbb{F}_r$ requires $r - 1$ to split into small primes or ideally to be divisible by a large power of 2. This narrows the search of pairing-friendly elliptic curves to the ones with a subgroup order r such that $2^L \mid r - 1$ for a large integer $L \geq 1$. This was suggested first in [BCG+13] and yielded the GMV6-183 curve with $2^{31} \mid r - 1$.

Recently, a new algorithm was proposed by Ben-Sasson et al. [BCG+13] that implements a variant of the FFT over non-smooth arbitrary fields $\mathbb{F}_{p'}$ (ECFFT). However, this algorithm suffers from two major drawbacks: finding an elliptic curve over $\mathbb{F}_{p'}$ with a large 2^L-torsion and efficiently implementing the algorithm in the canonical basis (ENTER and EXIT algorithms as in the paper notations). Not to mention that, asymptotically, it should be much slower than the basic

Table 2. Cost of `Setup`, `Prove` and `Verify` algorithms for [Gro16] and PLONK. $m =$ number of wires, $n =$ number of multiplications gates, $a =$ number of additions gates and $\ell =$ number of public inputs. $\mathtt{M}_\mathbb{G} =$ multiplication in $\mathbb{G}$ and P=pairing.

	Setup	Prove	Verify
[Gro16]	$3n$ $\mathtt{M}_{\mathbb{G}_1}$ m $\mathtt{M}_{\mathbb{G}_2}$	$(3n + m - \ell)$ $\mathtt{M}_{\mathbb{G}_1}$ n $\mathtt{M}_{\mathbb{G}_2}$ 7 FFT	3 P ℓ $\mathtt{M}_{\mathbb{G}_1}$
PLONK (KZG)	$d_{\geq n+a}$ $\mathtt{M}_{\mathbb{G}_1}$ 1 $\mathtt{M}_{\mathbb{G}_2}$ 8 FFT	$9(n + a)$ $\mathtt{M}_{\mathbb{G}_1}$ 8 FFT	2 P 18 $\mathtt{M}_{\mathbb{G}_1}$

FFT over smooth fields. Thus, designing a SNARK curve with smooth $r - 1$ remains the best approach.

This narrows the search to an elliptic curve with the following requirements:

 (i) valid parameters: integers p, r, t ($p(u), r(u) \in \mathbb{N}$, $t(u) \in \mathbb{Z}$), prime p.
 (ii) a subgroup order r such as $2^L \mid r - 1$ for a large integer $L \geq 1$,
(iii) a fast pairing,
(iv) a fast arithmetic in $\mathbb{G}_1$ and
 (v) a fast arithmetic in $\mathbb{G}_2$.

For KZG-based universal SNARKs, the last requirement can be dropped.

Elliptic curves proposed in the literature to suit SNARKs belong to one of the families discussed in the preliminaries in section 3. Except for the Cocks-Pinch curve CP6-782, all the curves have polynomial parameters. Condition (ii) becomes a congruence condition on the seed x of the polynomial $r(x)$ (cf. Table 3).

Pairing computations take place mainly in $\mathbb{F}_{p^k}$ and it is important to construct efficiently the $\mathbb{F}_{p^k}$ tower. Pairing-friendly tower extensions are built using a sequence of quadratic and cubic sub-extensions. The ideal way is to start with an optimal quadratic extension $\mathbb{F}_{p^2} = \mathbb{F}_p[u]/(u^2 + 1)$ that arises when $p \equiv 3 \bmod 4$. For the discussed families, satisfying condition (iii) boils down then to satisfying a congruence equation in the seed x alongside the previous congruence condition (cf. Table 3). Note that even seeds allow an additional speedup in the pairing computation (final exponentiation) for some families (e.g. BLS12 and BLS24 [GF16]). Finally, as is the custom in pairing-based cryptography, the Miller loop scalar (e.g. $u = t - 1$ for BLS curves) should have a small Hamming weight. For the curve families we are interested in, the optimal Miller loop scalar is a low-degree polynomial in the seed x, hence we are additionally looking for a sparse seed x in (signed) binary representation.

Condition (iv) is mainly related to the bitsize of p as the point coordinates are in $\mathbb{F}_p$. The bitsize of p varies for each family and is constrained by the security. One requires r of about 256 bits, and each family has a fixed parameter $\rho = \log_2 p / \log_2 r$, $1 \leq \rho \leq 2$, hence p is always larger than r.

Except for KZG-based universal SNARKs, one requires also a fast arithmetic in $\mathbb{G}_2$. As discussed in the preliminaries, the curves from Table 1 have a twist of

degree $d = 2$, 4 or 6 and thus $\mathbb{G}_2$ is isomorphic to $E'[r](\mathbb{F}_{p^{k/d}})$ for an appropriate d-twist E'. Condition (v) is immediately related to the choice of k and d (cf. Tab 3).

Congruence conditions on the seed x to achieve (ii). The subgroup order r is given by an irreducible polynomial $r(x)$ in $\mathbb{Q}[x]$. Computing the congruence conditions on x_0 such that $2^L \mid r(x_0) - 1$ given L is equivalent to finding the roots x_0 of $r(x) - 1$ modulo 2^L. Because 2 is a prime, we define in Alg. 4 a Hensel-like root-lifting technique inspired by [GS21, §A], with auxiliary functions in Algs. 5 and 6.

For BN curves, one has $r(x) - 1 = 6x(6x^3 + 6x^2 + 3x + 1)$. With $2^{L-1} \mid x$, then immediately $2^L \mid r - 1$. The other option is $2^{L-1} \mid 6x^3 + 6x^2 + 3x + 1$ and we run Alg. 4 (we were not able to derive a generic formula). Note that $\gcd(p(x)-1, r(x)-1) = 6x$. For BLS12 and BLS24 curves, $\gcd(p(x)-1, r(x)-1) = x - 1$. For BLS12, $r(x) - 1 = x^2(x^2 - 1)$ and for BLS24, $r(x) - 1 = x^4(x^4 - 1)$ and the results of Table 3 follow. For KSS16, we did not obtain a generic formula and we derived the condition for $L = 64$ in Table 3 with Alg. 4. Note that for curves with $D = 1$, $p \equiv 1 \bmod 4$ is required otherwise the curve would be supersingular. For KSS18 curves, Alg. 4 outputs another congruence but p is always even in that case so we discarded it. For BN and KSS16 we picked $L = 64$ and obtained the conditions $x_0 \equiv u_0 \bmod c \cdot 2^{L-1}$ but for a smaller $L_0 < L$, the reduction $u_0 \bmod c \cdot 2^{L_0 - 1}$ gives the answer. Algorithms 4, 5 and 6 are implemented in SageMath at `https://gitlab.inria.fr/tnfs-alpha/alpha`, in the file `sage/tnfs/gen/generate_curve_utils.py`.

4.2 Security

We look at the security of all the pairing-friendly families of the proposed curves both from a generic point of view (TNFS attack) and a SNARK-specific point of view (Cheon's attack).

TNFS attack. In 2015, [BGK15] revisited the Tower-NFS algorithm (TNFS) to compute discrete logarithms in $\mathbb{F}_{p^k}$. Then in 2016, Kim with Barbulescu combined it with other variants of NFS and exploited the extension fields to improve the TNFS algorithm. This resulted in an expected asymptotic complexity $L_Q(\alpha, c) = \exp((c + o(1))(\log Q)^\alpha (\log \log Q)^{1-\alpha})$ to be $L_{p^k}(1/3, (48/9)^{1/3} \approx 1.747)$ instead of $L_{p^k}(1/3, (96/9)^{1/3} \approx 2.201)$ with the NFS-HD algorithm of [BGGM15]. More important, the complexity of TNFS is lower than the general NFS algorithm in prime fields: $L_{p^k}(1/3, (64/9)^{1/3} \approx 1.923)$. The keysizes should be enlarged, and several papers deal with security estimates of TNFS [MSS16,BD19,GS21,Gui20,DGP20]. However these papers *estimate* the security level, but they do not scale with respect to a record computation. This is not yet possible as the first record computation with the TNFS algorithm was published after them, in [DGP21]: it

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[1] For GMV6 and KSS18, one needs to construct respectively the $\mathbb{F}_{p^6}$ and $\mathbb{F}_{p^{18}}$ towers starting from $\mathbb{F}_{p^3}$. In this case, one looks for the smaller cubic non-residue in $\mathbb{F}_p$.

Family, (i)	(ii)	(iii) $p \equiv 3$	(v) $\mathbb{G}_2$
$r,p \in \mathbb{N},\ t \in \mathbb{Z}$	$r \equiv 1 \bmod 2^L$	mod 4	coord. in
BN	$x \equiv 2570880382155688433 \bmod 2^{63} \Rightarrow 2^{64} \mid r-1$	✓	$\mathbb{F}_{p^2}$
any x	$x \equiv 0 \bmod 2^{L-1} \Rightarrow 2^L \mid r-1,\ 2^L \mid p-1$	✗	
BLS12	$x \equiv 1 \bmod 3 \cdot 2^{L-1} \Rightarrow 2^L \mid r-1,\ 2^{L-1} \mid p-1$	✗	
$x \equiv 1$	$x \equiv 2^{L-1} - 1 \bmod 3 \cdot 2^{L-1} \Rightarrow 2^L \mid r-1,\ 6 \mid p-1$	✓	$\mathbb{F}_{p^2}$
mod 3	$x \equiv 2^{L/2} \bmod 3 \cdot 2^{L/2} \Rightarrow 2^L \mid r-1,\ 6 \mid p-1$	✓	
BLS24	$x \equiv 1 \bmod 3 \cdot 2^{L-2} \Rightarrow 2^L \mid r-1,\ 2^{L-2} \mid p-1$	✗	
$x \equiv 1$	$x \equiv 2^{L-1} - 1 \bmod 3 \cdot 2^{L-2} \Rightarrow 2^L \mid r-1,\ 6 \mid p-1$	✓	$\mathbb{F}_{p^4}$
mod 3	$x \equiv 2^{L/4} \bmod 3 \cdot 2^{L/4} \Rightarrow 2^L \mid r-1,\ 6 \mid p-1$	✓	
MNT4, $t = x+1$	$x \equiv 0 \bmod 2^{L/2} \Rightarrow 2^L \mid r-1,\ 2^{L/2} \mid p-1$	✗	$\mathbb{F}_{p^2}$
MNT6	$x \equiv 0 \bmod 2^{L-1} \Rightarrow 2^L \mid r-1,\ 2^{2L} \mid p-1$	✗	$\mathbb{F}_{p^3}$
GMV6($h=4$) any x	$x \equiv 0 \bmod 2^{L-1} \Rightarrow 2^L \mid r-1,\ 2^{L-1} \mid p-1$	NA[1]	$\mathbb{F}_{p^3}$
KSS16 $(x \equiv \pm 25 \bmod 70)$	$\pm 14398186520986421885,\ \pm 37456616613123361405$ $\bmod 35 \cdot 2^{62} \Rightarrow 2^{64} \mid r-1,\ p \equiv 1 \bmod 4$	✗	$\mathbb{F}_{p^4}$
KSS18 $(x \equiv 14 \bmod 42)$	$x = 14 \cdot 2^{L/3} \bmod 42 \cdot 2^{L/3} \Rightarrow 2^L \mid r-1,\ 12 \mid p-1$	NA[1]	$\mathbb{F}_{p^3}$

Table 3. Conditions (ii), (iii), (iv), and (v) for Table 1 families. For BN curves with $p \equiv 3 \bmod 4$ and KSS16 curves, it was not possible to obtain a general rule. The residue of $x \bmod 2^L$ is computed by Alg. 4 with input $L = 64$ but any L can be given. For KSS18 curves, the other residues x do not give a prime p. Condition (iii) is not possible.

Algorithm 4: Congruence condition

Input: polynomial $s(x) \in \mathbb{Q}[x]$, modulus m and congruence conditions $\{a_i\}_{i \geq 0}$ such that $x_0 \equiv a_i \bmod m \Rightarrow s(x_0) \in \mathbb{Z}$, prime integer ℓ, integer $L > 0$

Output: Residues u_j, integers L_j and moduli m_j s.t. for all $x_j \equiv u_j \bmod m_j$, $s(x_j) \in \mathbb{Z}$ and $s(x_j) \equiv 0 \bmod \ell^{L_j}$, $L_j \geq L$

1 **for** $a_i \in \{a_i\}_{i \geq 0}$**:**
2     $s_i(x) \leftarrow s(m \cdot x + a_i) \in \mathbb{Z}[x]$ (this ensure that $s_i(x)$ has integer coefficients)
3     $v_i \leftarrow \mathrm{valuation}_\ell(\mathrm{content}(s_i(x)))$
4     $s_i(x) \leftarrow s_i(x)/\ell^{v_i}$
5   **for** $r_j \in \mathbb{Z}/\ell\mathbb{Z}$ *a simple root of $s_i(x)$ modulo ℓ*:
6         $r_j \leftarrow \mathrm{lift}_\mathbb{N}(r_j)$
7         $(u_j, m_j, L_j) \leftarrow \texttt{lift\_simple\_root}(s_i, r_j, a_i + r_j \cdot m, m \cdot \ell, 1 + v_i, \ell, L)$
8         $S \leftarrow S \cup \{(u_j, m_j, L_j)\}$
9   **for** $r_j \in \mathbb{Z}/\ell\mathbb{Z}$ *a multiple root of $s_i(x)$ modulo ℓ*:
10         $r_j \leftarrow \mathrm{lift}_\mathbb{N}(r_j)$
11         $S_j \leftarrow \texttt{lift\_multiple\_root}(s_i, r_j, a_i + r_j \cdot m, m \cdot \ell, 1 + v_i, \ell, L)$
12         $S \leftarrow S \cup S_j$
13 **return** S

Algorithm 5: Hensel lifting of simple roots

Input: polynomial $s_i(x) \in \mathbb{Z}[x]$, modulus m_i, residue $u_i \bmod m_i$, root $r_i \in \mathbb{N}$
      s.t. $s_i(r_i) \equiv 0 \bmod \ell$, prime integer ℓ, integers L_i and bound $L > 0$

Output: Residue u_j, integer L_j and modulus m_j s.t. for all $x_j \equiv u_j \bmod m_j$,
      $s_i(x_j) \equiv 0 \bmod \ell^{L_j - L_i}$

1   **def** *lift_simple_root*$(s_i, r_i, u_i, m_i, L_i, \ell, L)$:
2   **while** $L_i < L$**:**
3 $s_i(x) \leftarrow s_i(\ell \cdot x + r_i)/\ell$
4 $L_i \leftarrow L_i + 1$
5 $r_i \leftarrow \text{root}_{\mathbb{Z}/\ell\mathbb{Z}}(s_i(x))$ #(by Hensel, a simple root lifts to one root only)
6 $u_i \leftarrow u_i + \text{lift}_{\mathbb{N}}(r_i) \cdot m_i$
7 $m_i \leftarrow \ell \cdot m_i$
8   **return** (u_i, m_i, L_i)

Algorithm 6: Hensel lifting of multiple roots

Input: polynomial $s_i(x) \in \mathbb{Q}[x]$, modulus m_i, residue $u_i \bmod m_i$, multiple root
      $r_i \bmod \ell$, prime integer ℓ, integer L_i and bound $L > 0$

Output: Residues u_j, integers L_j and moduli m_j s.t. for all $x_j \equiv u_j \bmod m_j$,
      $s_i(x_j) \equiv 0 \bmod \ell^{L_j - L_i}$

1   **def** *lift_multiple_root*$(s_i, r_i, u_i, m_i, L_i, \ell, L)$:
2 $S \leftarrow [(s_i, r_i, u_i, m_i, L_i)]$ a linked list
3 $R \leftarrow \{\}$
4   **while** S *is not empty***:**
5 $(s_i, r_i, u_i, m_i, L_i) \leftarrow \text{pop}(S)$
6 $s_i(x) \leftarrow s_i(\ell \cdot x + r_i)/\ell$
7 $v_i \leftarrow \text{valuation}_\ell(\text{content}(s_i(x)))$
8 $s_i(x) \leftarrow s_i(x)/\ell^{v_i}$
9 $L_i \leftarrow L_i + 1 + v_i$
10   **if** $L_i \geq L$**:**
11 $R \leftarrow R \cup \{(u_i, m_i, L_i)\}$
12   **else:**
13   **for** $r_i \in \mathbb{Z}/\ell\mathbb{Z}$ *a simple root of* $s_i(x)$ *modulo* ℓ**:**
14 $r_i \leftarrow \text{lift}_{\mathbb{N}}(r_i)$
15 $(u_j, m_j, L_j) \leftarrow \text{lift\_simple\_root}(s_i, r_i, u_i + r_i \cdot m_i, m_i \cdot \ell, L_i, \ell, L)$
16 $R \leftarrow R \cup \{(u_j, m_j, L_j)\}$
17   **for** $r_i \in \mathbb{Z}/\ell\mathbb{Z}$ *a multiple root of* $s_i(x)$ *modulo* ℓ**:**
18 $r_i \leftarrow \text{lift}_{\mathbb{N}}(r_i)$
19 $S \leftarrow \text{append}(S, (s_i, r_i, u_i + r_i \cdot m_i, m_i \cdot \ell, L_i))$
20   **return** R

---

runs the TNFS in a field $\mathbb{F}_{p^6}$ but new security estimates are not extrapolated. Below we reproduce the keysizes from [GS21,Gui21].

For pairing-friendly curves with parameters given by polynomials evaluated at some seed, the *Special*-TNFS algorithm applies. It exploits the special form of the prime p, resulting in an asymptotic complexity of $L_{p^k}(1/3, (32/9)^{1/3} \approx 1.526)$ in the most favorable case. Is means that compared to prime fields $\mathbb{F}_q$, the total size $k \log p$ should be twice larger to ensure the same level of security: $k \log p = 2 \log q$. For MNT and GMV curves, p is given by a quadratic polynomial and the special variant of TNFS is not better than the generic methods (the degree of $p(x)$ should be at least 3 to make a difference). By the way, a change of variables $v = -26x - 3$ gives $t(v) = v + 1$, $r(v) = (v^2 - v + 1)/13$ and $p(v) = 4r(v) + t(v) - 1 = (4v^2 + 9v + 4)/13$ with smaller coefficients. We can reasonably assume that the sizes required for GMV6 curves are the same as for MNT6 curves. Such sizes were given in [Gui21, § MNT Curves]. We run the SageMath code of [GS21] and obtain a security estimate of 71 bits in GF(p^6) for the GMV6 curve (Table 4). Security levels for BLS12-377, CP6-782 and BW6-761 were provided in [EG20, § C] (Table 5).

Table 4. Security level estimates of MNT curves from [Gui21], and of the GMV curve obtained with the software from [GS21].

curve	k	D	ref	r bits	p bits	p^k bits	DL cost in $\mathbb{F}_{p^k}$
MNT4-298	4	614144978799019	[BCTV14a]	298	298	1192	2^{77}
MNT6-298	6	614144978799019	[BCTV14a]	298	298	1788	2^{87}
MNT4-753	4	241873351932854907	[BCTV14a]	753	753	3012	2^{113}
MNT6-753	6	241873351932854907	[BCTV14a]	753	753	4517	2^{137}
MNT4-992	4	95718723	[Gui21]	992	992	3966	2^{126}
MNT6-992	6	95718723	[Gui21]	992	992	5948	2^{156}
GMV6-183	6	21048712401	[BCG$^+$13]	181	183	1093	2^{71}

Cheon's attack. Cheon [Che10] showed that given $G, [\tau]G$ and $[\tau^{\mathrm{T}}]G$, with G a point in a subgroup $\mathbb{G}$ of order r with $\mathrm{T}|r - 1$, it is possible to recover τ in $2(\lceil \sqrt{(r-1)/\mathrm{T}} \rceil + \lceil \sqrt{\mathrm{T}} \rceil) \times (\mathtt{Exp}_{\mathbb{G}}(r) + \log r \times \mathtt{Comp}_{\mathbb{G}})$ where $\mathtt{Exp}_{\mathbb{G}}(r)$ stands for the cost of one exponentiation in $\mathbb{G}$ by a positive integer less than r and $\mathtt{Comp}_{\mathbb{G}}$ for the cost to determine if two elements are equal in $\mathbb{G}$. According to [Che10, Theorem 2], if $\mathrm{T} \leq r^{1/3}$, then the complexity of the attack is about $O(\sqrt{r/\mathrm{T}})$ exponentiation by using $O(\sqrt{r/\mathrm{T}})$ storage. Sakemi et al. reported an implementation on a 160-bit elliptic curve in [SHI$^+$12].

In SNARKs such as [Gro16] and KZG-based schemes, the `Setup` keys include elements $[\tau^i]_{i=0}^{\mathrm{T}} G \in \mathbb{G}$ where $\mathrm{T} \in \mathbb{N}^*$ is at least the size of the arithmetic circuit related to the statement to prove, and τ is the secret trapdoor. The property $\mathrm{T} \mid r - 1$ also holds since we need $r - 1$ to be highly *2-adic* (condition (ii)). So,

Table 5. Security level estimates of BLS curves, CP6-782, BW6 and CP8, CP12 curves from [EG20,EG22], with seeds $u_{377} = $ `0x8508c00000000001`, $u_{379} = $ `0x9b04000000000001`, $u_{315} = -$`0xbfcfffff`, $u_{317} = $ `0xe19c0001`.

curve	k	D	ref	r bits	p bits	p^k bits	DL cost in $\mathbb{F}_{p^k}$
BLS12-377, u_{377}	12	3	[BCG$^+$20]	253	377	4521	2^{126}
BLS12-379, u_{379}	12	3	[EG22, Tab. 9]	254	379	4537	2^{126}
BLS24-315, u_{315}	24	3	[EG22, Tab. 10]	253	315	7543	2^{160}
BLS24-317, u_{317}	24	3	[EG22, Tab. 10]	255	317	7599	2^{160}
outer curve with a BLS12 curve							
CP6-782, u_{377}	6	339	[BCG$^+$20]	377	782	4691	2^{138}
BW6-761, u_{377}	6	3	[EG20]	377	761	4566	2^{126}
BW6-764, u_{379}	6	3	[EG22, Tab. 11]	379	764	4584	2^{126}
outer curves with the BLS24-315 curve							
BW6-633, u_{315}	6	3	[EG22, Tab. 11]	315	633	3798	2^{124}
BW6-672, u_{315}	6	3	[EG22, Tab. 11]	315	672	4032	2^{128}
CP8-632, u_{315}	8	4	[EG22, Tab. 7]	315	632	5056	2^{140}
CP12-630, u_{315}	12	3	[EG22, Tab. 7]	315	630	7560	2^{166}

Table 6. Security level estimates of BN curves and outer curves with the software shipped with [GS21].

curve	k	D	ref	r bits	p bits	p^k bits	DL cost in $\mathbb{F}_{p^k}$
BN-256 $u = 1868033^3$ Pinocchio	12	3	[PHGR13]	256	256	3063	2^{103}
BN-254 $u = 2^{62} - 2^{54} + 2^{44}$ Pantry	12	3	[BFR$^+$13]	256	256	3038	2^{102}
BN-254 $u = -(2^{62} + 2^{55} + 1)$ Geppetto	12	3	[CFH$^+$15]	254	254	3038	2^{102}
BN-254 $u = $ `0x44e992b44a6909f1` Ethereum	12	3	[BCTV14b]	254	254	3044	2^{103}
outer curve with the Geppetto BN curve							
BW6-509	6	3		254	509	3051	2^{102}

given these auxiliary inputs, an attacker can recover the secret using Cheon's algorithm in time $O(\sqrt{r/\mathrm{T}})$, hence breaking the SNARK soundness. We stress that this attack vector is not inherent to the curve design but to the SNARK design. Given a curve where $r \equiv 1 \bmod 2^L$ used in a SNARK requiring a setup of size $2^{L'}$ where $L' < L$, Cheon's attack runs in $O(\sqrt{r/L'})$ and not $O(\sqrt{r/L})$. To the best of our knowledge, the Filecoin circuit ($L'_{\mathbb{G}_1} = 28, L'_{\mathbb{G}_2} = 27$) is the biggest application circuit of public interest.

While taking into account this attack at the curve design level might limit the attack vector, this prevents a nice speed up in the pairing computation. As observed in [EG22], having a large L s.t. $r(x) \equiv 1 \bmod 2^L$ is often entangled to having a large number of consecutive zeroes in the seed x. This allows mixing efficiently the Karabina [Kar13] and Granger-Scott [GS10] cyclotomic squaring algorithms, hence speeding up significantly the final exponentiation. That is said, Cheon's attack must be taken into account when implementing a SNARK circuit with a given elliptic curve.

4.3 Examples

In Table 7, we recall the literature curves presented in the state-of-the-art section 2 and summarize their SNARK-friendliness properties and security levels.

Curve	x	L	$r = \#\mathbb{G}_1$ (bits)	p, $\mathbb{G}_1$ (bits)	$p^{k/d}$, $\mathbb{G}_2$ (bits)	$p \equiv 3$ mod 4	security (bits) $\mathbb{G}_1$	$\mathbb{F}^*_{p^k}$
BN-256 [PHGR13]	1868033^3 $\mathrm{HW}_{2\text{-NAF}}(6x+2) = 19$	5	256	256	512	✓	128	103
BN-254 [BFR$^+$13]	$2^{62} - 2^{54} + 2^{44}$ $\mathrm{HW}_{2\text{-NAF}}(6x+2) = 7$	45	254	254	508	×	127	102
GMV6-183 [BCG$^+$13]	0x8eed757d90615e40000000 $\mathrm{HW}(-26x-2) = 16$	31	181	183	549	NA^2	90	71
BN-254 [BCTV14b]	0x44e992b44a6909f1 $\mathrm{HW}_{2\text{-NAF}}(6x+2) = 22$	28	254	254	508	✓	127	103
BLS12-381 [Bow17]	-0xd201000000010000 $\mathrm{HW}(x) = 6$	32	255	381	762	✓	127	126

Table 7. Properties of SNARK curves from the literature.

We see that only the BLS12-381 satisfies conditions (ii), (iii), (iv), (v) and has almost a 128-bit security level. KSS16 and KSS18 families were not investigated in the SNARK literature. BLS24 family was considered in [EG22] in the context of 2-chains, hence the proposed BLS24-315 does not satisfy the condition (iii) by definition (cf. 5.2). In table 8, we propose new BN and BLS24 that satisfy all the requirements. We also propose the first KSS16 and KSS18 SNARK curves and compare them to the existing curves. We omit to revisit the GMV6 family as a 128-bit secure curve would be defined over a large field (around 704 bits).

Curve	x	L	$r = \#\mathbb{G}_1$ (bits)	p, $\mathbb{G}_1$ (bits)	$p^{k/d}$, $\mathbb{G}_2$ (bits)	$p \equiv 3$ mod 4	security (bits) $\mathbb{G}_1$	$\mathbb{F}_{p^k}^*$
BN383	0x49e69d16fdc80216226909f1 $\mathrm{HW}_{2\text{-NAF}}(6x+2) = 30$	44	383	383	766	✓	191	123
BLS24-317	0xd9018000 $\mathrm{HW}_{2\text{-NAF}}(x) = 6$	60	255	317	1268	✓	127	160
KSS16-329	0x38fab7583 $\mathrm{HW}(x) = 12$	19	255	329	1316	✓	127	140
KSS18-345	0xc0c44000000 $\mathrm{HW}(x) = 6$	78	254	345	690	NA[3]	127	150

Table 8. New SNARK curves from the BN, BLS24, KSS16 and KSS18 families.

KSS16 and KSS18. The KSS family was not investigated previously in the SNARK context. KSS16 and KSS18 defined respectively over fields of size 328-bit and 348-bit offer 128 bits of security. We suggest the KSS16-329 and KSS18-345 that fulfill all conditions except $p \equiv 3 \mod 4$ (condition (iv) does not apply to KSS18 and is not possible for KSS16).

BLS24. The BLS family with embedding degree $k = 24$ was previously investigated in [EG22] in the recursive SNARK context. However, the authors only considered lifting of simple roots and proposed the BLS24-315 curve with $2^{22} \mid r - 1$. Here we consider multiple roots and propose the BLS24-317 curve with $2^{60} \mid r - 1$ that fulfills all SNARK-friendliness conditions. This curve is particularly suitable for KZG-based SNARKs compared to previous known curves. As a reference example, compared to the widely used BLS12-381, it takes 14% less time to generate a PLONK proof of a circuit of 40000 constraints (implemented in `gnark` [BPH+22]). The setup generation is also 23% faster but the verification is 30% slower, although this can be likely amortized with a batch verification.

5 Recursive SNARKs with elliptic curves

5.1 Cycles of pairing-friendly elliptic curves

A *cycle* of elliptic curves is a list of curves defined over finite fields in which the number of points in one curve is equal to the size of the field of definition of the next curve, in a cyclic manner.

Definition 1. *An m-cycle of elliptic curves is a list of m distinct elliptic curves*

$$E_1/\mathbb{F}_{p_1}, \ldots, E_m/\mathbb{F}_{p_m},$$

for primes $p_1, \ldots, p_m$, such that the numbers of points on these curves satisfy

$$\#E_1(\mathbb{F}_{p_1}) = p_2, \ldots, \#E_i(\mathbb{F}_{p_i}) = p_{i+1}, \ldots, \#E_m(\mathbb{F}_{p_m}) = p_1.$$

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[2] For GMV6-183, 3 is the smallest cubic non-residue on $\mathbb{F}_p$.
[3] For KSS18-345, 2 is the smallest cubic non-residue on $\mathbb{F}_p$.

This notion was initially introduced under different names, for example *amicable pairs* (or equivalently *dual elliptic primes* [Mih07]) for 2-cycles of ordinary curves, and *aliquot cycles* for the general case [SS11]. A *pairing-friendly m-cycle* is an *m*-cycle composed of ordinary pairing-friendly curves.

The state of the art. Chiesa, Chu and Weidner studied the existence of pairing-friendly cycles, beyond the previously known cycles of curves with embedding degrees 4 and 6, and arrived at a number of negative results which indicate that having a small embedding degree is a *strong* restriction on constructing cycles [CCW19]:

- There are no 2-cycles of elliptic curves with embedding degrees $(5, 10), (8, 8)$ or $(12, 12)$, which means that there are no *optimal* (in terms of parameter sizes) pairing-friendly 2-cycles at the 128-bit security level.
- There are no pairing-friendly cycles with more than 2 curves with the same CM discriminant $D > 3$, which implies that elliptic curves from families of varying discriminants must be used to construct cycles.
- There are no cycles of prime-order pairing-friendly curves only from the Freeman and Barreto-Naehrig families; or cycles of composite-order elliptic curves. This motivates the search for new constructions of prime-order pairing-friendly curves.

On the positive side, the authors characterize all cycles of MNT curves as consisting of curves with alternating embedding degrees of 4 or 6, and construct a new 4-cycle of MNT curves which consists of the union of 2-cycles. Table 9 collects the possibilities to generate MNT cycles, for which a specific cycle can be generated by substituting the possible choices of parameter x and checking that all polynomials $p(x)$ and $r(x)$ evaluate to prime numbers. Concrete parameters used in practice can be found in Table 4.

Open problems. The above restrictions and current constructions of cycles suggest that pairing-friendly 2-cycles will always have the inconvenience of including two elliptic curves at different security levels, which means that at least one curve would have sub-optimal performance, but there are several *open problems* for which a satisfying solution would potentially increase the available options:

- Are there cycles of elliptic curves with the same embedding degree, and possibly the same discriminant?
- Are there pairing-friendly cycles of embedding degrees greater than 6?
- Are there pairing-friendly cycles combining MNT, Freeman and Barreto-Naehrig curves?

We already know that *optimal* 2-cycles for recursive SNARKs cannot be found, but it is possible that better alternatives with higher embedding degrees and smaller parameter sizes than the inefficient MNT (6,4)-cycles exist. However,

(6,4,6,4) 4-cycle			
(6,4) 2-cycle		(6,4) 2-cycle	
E_1	E_2	E_3	E_4
k	6	4	6	4
$p(x)$	$4x^2 + 1$	$4x^2 + 2x + 1$	$4x^2 + 1$	$4x^2 - 2x + 1$
$r(x)$	$4x^2 + 2x + 1$	$4x^2 + 1$	$4x^2 - 2x + 1$	$4x^2 + 1$
$t(x)$	$-2x + 1$	$2x + 1$	$2x + 1$	$-2x + 1$

Table 9. Parameterized (6,4) 2-cycles and (6,4,6,4) 4-cycles of MNT curves, where 4-cycles are constructed as the union of the 2-cycles.

reconciling the conditions for pairing-friendly cycles (prime order, varying families and discriminants) with the requirements for SNARK-friendliness to construct efficient recursive SNARKs, i.e. conditions (i)–(v), further restricts the choice of parameters and increase the challenge. Since no methods are known to find prime-order pairing-friendly curves with embedding degree larger than 12, novel ideas are severely needed. In particular, finding 2-cycles of pairing-friendly curves with embedding degrees $(16, 16)$ or higher would already be a significant contribution to improving performance of constructions at current 128-bit and higher security levels.

Because of these difficulties in finding parameters, current efforts have focused on findings *chains* of pairing-friendly elliptic curves, *hybrid cycles* containing one pairing-friendly curve, or amicable pairs that allow alternative recursive proofs without pairings.

5.2 Chains of pairing-friendly elliptic curves

We have seen in Section 5.1 that constructing a *cycle* of two pairing-friendly curves is a very difficult task, and results in a very slow pairing computation, and large parameters. Relaxing the conditions on the curves, one can look for a chain of pairing-friendly elliptic curves, as in Definition 3 (following [EG22]).

Definition 2. *An m-chain of elliptic curves is a list of distinct curves*

$$E_1/\mathbb{F}_{p_1}, \ldots, E_m/\mathbb{F}_{p_m}$$

where $p_1, \ldots, p_m$ are large primes and

$$p_1 = r_2 \mid \#E_2(\mathbb{F}_{p_2}), \ldots, \ p_{i-1} = r_i \mid \#E_i(\mathbb{F}_{p_i}), \ldots, \ p_{m-1} = r_m \mid \#E_m(\mathbb{F}_{p_m}) \ . \tag{2}$$

Definition 3. *An m-chain of SNARK-friendly elliptic curves is an m-chain where each of the $\{E_i/\mathbb{F}_{p_i}\}_{1 \leq i \leq m}$ curves*

- *is pairing-friendly;*

− has a highly 2-adic subgroup, i.e. $r_i − 1 \equiv 0 \mod 2^L$ for a large $L \geq 1$.

In a 2-chain, the first curve is denoted the **inner curve**, while the second curve whose order is the characteristic of the inner curve, is denoted the **outer curve**.

Previous work on chains of pairing-friendly curves. Approximately at the same time as [BCTV14a] released the first cycle of MNT curves, Costello et al. [CFH+15] presented a 2-chain of pairing-friendly curves to obtain a bounded recursive proof composition (of depth 2). Their inner curve is the popular BN curve from [NAS+08] with seed $x = −(2^{62}+2^{55}+1)$ and their outer curve is a new Brezing–Weng curve of embedding degree 6 and 509 bits (just under the machine-word-aligned size of 512 bits). However, the curves do not satisfy Condition (ii) that is, no large power of 2 divides neither $r − 1$ nor $p − 1$. It is surprising that they did not use another widespread BN curve with seed $x = 2^{62} − 2^{54} + 2^{44}$ for example (Beuchat et al. [BGM+10], in the TEPLA library), which does satisfy Condition (ii): $2^{45} \mid r − 1$ and $2^{45} \mid p − 1$. Nevertheless, the Geppetto construction is the first example of a 2-chain of pairing-friendly curves (with parameters having a polynomial form) that we found in the literature, to the best of our knowledge.

Probably because Geppetto appeared *before* the new TNFS algorithm and the re-evaluation of all key sizes of pairing-friendly curves, the MNT cycle of 298 bits was a milestone construction, while the Geppetto 2-chain was not remembered in the subsequent construction named ZEXE [BCG+20]. In light of the need of larger keysizes, Bowe et al. released a 2-chain of pairing-friendly curves based on the BLS12 family. The inner curve is a BLS12-377 curve with $2^{46} \mid r − 1$ and $2^{46} \mid p − 1$. However the outer curve was a Cocks–Pinch curve that did not benefit from all possible speed-ups of pairing-friendly curves.

In [EG20], El Housni and Guillevic introduced a 2-chain of curves made of the previous BLS12-377 and a new BW6-761 curve, a Brezing-Weng curve of embedding degree 6 defined over a 761-bit prime field, which they demonstrated to be orders of magnitude faster than CP6-782. Indeed many optimizations are now available, from a twist of degree 6 to an optimal ate pairing, and 761 bits fit in one less machine word of 64 bits compared to 782 bits. The most recent work to date (in 2022) is [EG22]. The authors investigate many different possibilities of 2-chains with BLS12 and BLS24 inner curves, and Brezing–Weng outer curves of embedding degree 6, but also Cocks–Pinch curves of the larger embedding degrees 8 and 12.

Inner curves from polynomial families. The inner curves are either BN or BLS curves (BLS12 and BLS24). For efficiency reasons, the condition (ii) of $2^L \mid r − 1$ for a large power L applies to the two curves of the 2-chain: the inner curve and the outer curve. That is, the inner curve should have $2^L \mid r − 1$ and $2^L \mid p − 1$ at the same time. Table 3 shows that BN curves with $x \equiv 0 \mod 2^{L−1}$, BLS12 and BLS24 curves with $x \equiv 1 \mod 3 \cdot 2^L$, satisfy the 2-adicity condition

on p and r. We will then focus on these three families of curves: BN, BLS12 and BLS24.

Outer curves with the Brezing–Weng and Cocks-Pinch methods. The papers [EG20,EG22] consider 2-chains from a BLS12 and a BLS24 curve. The paper [EG20] introduces the curve BW6-761, an outer curve with BLS12-377. The paper [EG22] describes a general framework for all 2-chains made with a BW6 curve from a BLS12 curve, resp. all 2-chains of a BW6 curve with a BLS24 curve. The families of 2-chains are parameterized by the lifting cofactors h_t, h_y of Alg. 3, lines 2, 3. Non-zero lifting cofactors produce curves with a rho-value at least 2, but otherwise the polynomial defining $p_{\mathrm{bw}}(x)$ is not irreducible. Non-zero lifting cofactors appear in [FK19,GMT20] for pairing-friendly curves.

We reproduce the tables of parameters [EG22, Tables 3, 4] in Tables 10 and 11. Table 10 gives the parameters of the BW6-BLS12 curves in terms of the seed x, and the two lifting cofactors h_t, h_y. Table 11 gives the parameters of the BW6-BLS24 curves with the same parameters x, h_t, h_y.

Optimal ate pairing and optimal twisted ate pairing on outer curves. As can be seen in Tables 10 and 11, the Miller loop of the optimal ate pairing on the BW6 outer curves is not as short as for the well-known BN:

$$f_{6u+2,Q}(P)\ell_{[6u+2]Q,\pi(Q)}(P)\ell_{[6u+2]Q+\pi(Q),\pi^2(-Q)}(P)$$

or BLS12, BLS24 curves ($f_{u,Q}(P)$). It comprises two parts, $f_{a_0,Q}(P)f_{a_1,\pi(Q)}(P)$ for the optimal ate pairing, and $f_{a_0+a_1\lambda,P}(Q)$ for the optimal twisted ate pairing, with $\lambda = t_{\mathrm{bw}} - 1$ the eigenvalue of the endomorphism $(x,y) \mapsto (\omega x, y)$ on the appropriate subgroup, where $\omega^2 + \omega + 1 = 0$. Two optimization strategies are developed in [EG20,EG22]. First, the Miller loop can we shared thanks to the formula

$$f_{uv,Q}(P) = f_{u,Q}^v(P) \cdot f_{v,[u]Q}(P) \tag{3}$$

and the exponentiation $f_{u,Q}^v(P)$ is implicitly performed by initializing the second Miller function $f_{v,[u]Q}(P)$ at $f = f_{u,Q}(P)$ instead of $f = 1$. This is [EG20, Alg. 5]. The second optimization applies to the ate and twisted ate pairings and consists in computing a multi-scalar Miller function $f_{a_0+a_1\lambda,P}(Q)$ with shared doublings and squares, this is [EG22, Alg. 2] reproduced in Alg .7.

Our optimized final exponentiation on outer curves. The final exponentiation raises the result f of the Miller loop to the power $(p^k - 1)/r$ so that the final result is well-defined. It is usually divided into an easy part made of Frobenius powers, and a hard part, as follows:

$$\frac{p^k - 1}{r} = \underbrace{\frac{p^k - 1}{\Phi_k(p)}}_{\text{easy part}} \cdot \underbrace{\frac{\Phi_k(p)}{r}}_{\text{hard part}} \ . \tag{4}$$

Table 10. Parameters of a BW6 outer curve with a BLS12 inner curve, with $x \equiv 1 \bmod 3$.

parameter	value	property
r_{bw}	$p_{\text{bls}} = (x-1)^2/3(x^4 - x^2 + 1) + x$	generates prime
$\bar{\zeta_6}$	$-x^5 + 3x^4 - 3x^3 + x - 1$	
ζ_6	$x^5 - 3x^4 + 3x^3 - x + 2$	
$1/\sqrt{-3}$	$-(2x^5 - 6x^4 + 6x^3 - 2x + 3)/3$	
$t_{\text{bw},0}$	$-x^5 + 3x^4 - 3x^3 + x$	$6 \mid t_{\text{bw},0}$
$t_{\text{bw},3}$	$x^5 - 3x^4 + 3x^3 - x + 3$	$3 \mid t_{\text{bw},3}, 2 \nmid t_{\text{bw},3}$
$y_{\text{bw},0}$	$(x^5 - 3x^4 + 3x^3 - x)/3 = -t_{\text{bw},0}/3$	$2 \mid y_{\text{bw},0}$
$y_{\text{bw},3}$	$(x^5 - 3x^4 + 3x^3 - x + 3)/3 = t_{\text{bw},3}/3$	$2 \nmid y_{\text{bw},3}$
$p_{\text{bw},0}$	$((t_{\text{bw},0} + h_t r_{\text{bw}})^2 + 3(y_{\text{bw},0} + h_y r_{\text{bw}})^2)/4$	generates prime
$p_{\text{bw},3}$	$((t_{\text{bw},3} + h_t r_{\text{bw}})^2 + 3(y_{\text{bw},3} + h_y r_{\text{bw}})^2)/4$	generates prime
$\Phi_6(t_{\text{bw},i} - 1)$	$3r_{\text{bw}}(x^4 - 4x^3 + 7x^2 - 6x + 3)$	
$c_{\text{bw},0}$	$(h_t^2 + 3h_y^2)/4r_{\text{bw}} + (h_t - h_y)/2t_{\text{bw},0} + x^4 - 4x^3 + 7x^2 - 6x + 3 - h_t$	
$c_{\text{bw},3}$	$(h_t^2 + 3h_y^2)/4r_{\text{bw}} + (h_t + h_y)/2t_{\text{bw},3} + x^4 - 4x^3 + 7x^2 - 6x + 3 - h_t$	
$c'_{\text{bw},0}$ ($\mathbb{G}_2$)	$c_{\text{bw},0} + (h_t + 3h_y)/2$	
$c'_{\text{bw},3}$ ($\mathbb{G}_2$)	$c_{\text{bw},3} + (h_t - 3h_y)/2$	
Optimal ate Miller loop $f_{a_0, Q}(P) \cdot f_{a_1, Q}^p(P)$, Optimal twisted ate $f_{a_0 + a_1 \lambda, P}(Q)$		
$\lambda = t_{\text{bw},0} - 1$	$a_0 = x^3 - x^2 - x,\ a_1 = x + 1 \quad a'_0 = -(x+1),\ a'_1 = x^3 - x^2 + 1$	
$\lambda = t_{\text{bw},3} - 1$	$a_0 = x + 1,\ a_1 = x^3 - x^2 - x \quad a'_0 = x^3 - x^2 + 1,\ a'_1 = -(x+1)$	
Final exponentiation		
$e_{\text{bw},0}$	$(x+1)\Phi_6(p_{\text{bw},0})/r_{\text{bw}} =$ $((x+1)p_{\text{bw},0} - x^3 + x^2 - 1)(c_{\text{bw},0} + h_t) + 3(p_{\text{bw},0} - x^2 + 2x - 2)$	
$e'_{\text{bw},0}$	$(x^3 - x^2 - x)\Phi_6(p_{\text{bw},0})/r_{\text{bw}} =$ $((x^3 - x^2 - x)p_{\text{bw},0} + x + 1)(c_{\text{bw},0} + h_t) - 3(1 + (x^2 - 2x + 1)p_{\text{bw},0})$	
$e_{\text{bw},3}$	$(x+1)\Phi_6(p_{\text{bw},3})/r_{\text{bw}} =$ $(x^3 - x^2 - x + (x+1)p_{\text{bw},3})(c_{\text{bw},3} + h_t) + 3(x^2 - 2x + 1 + p_{\text{bw},3})$	
$e'_{\text{bw},3}$	$(x^3 - x^2 + 1)\Phi_6(p_{\text{bw},3})/r_{\text{bw}} =$ $((x^3 - x^2 + 1)p_{\text{bw},3} - x - 1)(c_{\text{bw},3} + h_t) - 3(1 - (x^2 - 2x + 2)p_{\text{bw},3})$	

Table 11. Parameters of a BW6 outer curve with a BLS24 inner curve, with $x \equiv$ 1 mod 3.

r_{bw}	$p_{\mathrm{bls}} = (x-1)^2/3(x^8 - x^4 + 1) + x$ $(x^{10} - 2x^9 + x^8 - x^6 + 2x^5 - x^4 + x^2 + x + 1)/3$	prime
$\overline{\zeta_6}$	$-x^9 + 3x^8 - 4x^7 + 4x^6 - 3x^5 + 2x^3 - 2x^2 + x - 1$	
ζ_6	$x^9 - 3x^8 + 4x^7 - 4x^6 + 3x^5 - 2x^3 + 2x^2 - x + 2$	
$1/\sqrt{-3}$	$(2x^9 - 6x^8 + 8x^7 - 8x^6 + 6x^5 - 4x^3 + 4x^2 - 2x + 3)/3$	
$t_{\mathrm{bw},0}$	$-x^9 + 3x^8 - 4x^7 + 4x^6 - 3x^5 + 2x^3 - 2x^2 + x$	$6 \mid t_{\mathrm{bw},0}$
$t_{\mathrm{bw},3}$	$x^9 - 3x^8 + 4x^7 - 4x^6 + 3x^5 - 2x^3 + 2x^2 - x + 3$	$3 \mid t_{\mathrm{bw},3}, 2 \nmid t_{\mathrm{bw},3}$
$y_{\mathrm{bw},0}$	$(x^9 - 3x^8 + 4x^7 - 4x^6 + 3x^5 - 2x^3 + 2x^2 - x)/3$	
$y_{\mathrm{bw},0}$	$-t_{\mathrm{bw},0}/3$	$2 \mid y_{\mathrm{bw},0}$
$y_{\mathrm{bw},3}$	$(x^9 - 3x^8 + 4x^7 - 4x^6 + 3x^5 - 2x^3 + 2x^2 - x + 3)/3$	
$y_{\mathrm{bw},3}$	$t_{\mathrm{bw},3}/3$	$2 \nmid y_{\mathrm{bw},3}$
$p_{\mathrm{bw},0}$	$((t_{\mathrm{bw},0} + h_t r_{\mathrm{bw}})^2 + 3(y_{\mathrm{bw},0} + h_y r_{\mathrm{bw}})^2)/4$	prime
$p_{\mathrm{bw},3}$	$((t_{\mathrm{bw},3} + h_t r_{\mathrm{bw}})^2 + 3(y_{\mathrm{bw},3} + h_y r_{\mathrm{bw}})^2)/4$	prime
$\Phi_6(t_{\mathrm{bw},i} - 1)$	$(x^8 - 4x^7 + 8x^6 - 12x^5 + 15x^4 - 14x^3 + 10x^2 - 6x + 3) \cdot 3 \cdot r_{\mathrm{bw}}$	
$c_{\mathrm{bw},0}$	$(h_t^2 + 3h_y^2)/4r_{\mathrm{bw}} + (h_t - h_y)/2t_{\mathrm{bw},0} + \Phi_6(t_{\mathrm{bw},0} - 1)/(3r_{\mathrm{bw}}) - h_t$	
$c_{\mathrm{bw},3}$	$(h_t^2 + 3h_y^2)/4r_{\mathrm{bw}} + (h_t + h_y)/2t_{\mathrm{bw},3} + \Phi_6(t_{\mathrm{bw},3} - 1)/(3r_{\mathrm{bw}}) - h_t$	
$c'_{\mathrm{bw},0}$ ($\mathbb{G}_2$)	$c_{\mathrm{bw},0} + (h_t + 3h_y)/2$	
$c'_{\mathrm{bw},3}$ ($\mathbb{G}_2$)	$c_{\mathrm{bw},3} + (h_t - 3h_y)/2$	
Optimal ate Miller loop $f_{a_0,Q}(P) \cdot f_{a_1,Q}^p(P)$, Optimal twisted ate $f_{a_0 + a_1\lambda, P}(Q)$		
$\lambda = t_{\mathrm{bw},0} - 1$	$a_0 = -(x+1),\ a_1 = x^5 - x^4 + 1 \quad a_0' = x^5 - x^4 - x,\ a_1' = x + 1$	
$\lambda = t_{\mathrm{bw},3} - 1$	$a_0 = x + 1,\ a_1 = x^5 - x^4 - x \quad a_0' = x^5 - x^4 + 1,\ a_1' = -(x+1)$	
Final exponentiation		
$e_{\mathrm{bw},0}$	$(x+1)\Phi_6(p_{\mathrm{bw},0})/r_{\mathrm{bw}} =$ $(-x^5 + x^4 - 1 + (x+1)p_{\mathrm{bw},0})(c_{\mathrm{bw},0} + h_t) + 3(x^4 + 2(-x^3 + x^2 - x + 1) - p_{\mathrm{bw},0})$	
$e'_{\mathrm{bw},0}$	$(x^5 - x^4 - x)\Phi_6(p_{\mathrm{bw},0})/r_{\mathrm{bw}} =$ $((x+1) + (x^5 - x^4 - x)p_{\mathrm{bw},0})(c_{\mathrm{bw},0} + h_t) - 3(1 + (x^4 - 2x^3 + 2x^2 - 2x + 1)p_{\mathrm{bw},0})$	
$e_{\mathrm{bw},3}$	$(x+1)\Phi_6(p_{\mathrm{bw},3})/r_{\mathrm{bw}} =$ $(x(x^4 - x^3 - 1) + (x+1)p_{\mathrm{bw},3})(c_{\mathrm{bw},3} + h_t) + 3(x^4 + 2(-x^3 + x^2 - x) + 1 + p_{\mathrm{bw},3})$	
$e'_{\mathrm{bw},3}$	$(x^5 - x^4 + 1)\Phi_6(p_{\mathrm{bw},3})/r_{\mathrm{bw}} =$ $((-x-1) + (x^5 - x^4 + 1)p_{\mathrm{bw},3})(c_{\mathrm{bw},3} + h_t) - 3(1 - (x^4 - 2x^3 + 2x^2 - 2x + 2)p_{\mathrm{bw},3})$	

Table 12. Parameters of a BW6 outer curve with a BN inner curve, any integer x.

r_{bw}	$p_{\mathrm{bn}} = 36x^4 + 36x^3 + 24x^2 + 6x + 1$	prime
$\overline{\zeta_6}$	$-18x^3 - 18x^2 - 9x - 1$	
ζ_6	$18x^3 + 18x^2 + 9x + 2$	
$1/\sqrt{-3}$	$12x^3 + 12x^2 + 6x + 1$	
$t_{\mathrm{bw},0}$	$-18x^3 - 18x^2 - 9x$	$9 \mid t_{\mathrm{bw},0}$
$t_{\mathrm{bw},3}$	$18x^3 + 18x^2 + 9x + 3$	$3 \mid t_{\mathrm{bw},3}$
$y_{\mathrm{bw},0}$	$-t_{\mathrm{bw},0}/3 = 6x^3 + 6x^2 + 3x$	$3 \mid y_{\mathrm{bw},0}$
$y_{\mathrm{bw},3}$	$t_{\mathrm{bw},3}/3 = 6x^3 + 6x^2 + 3x + 1$	
$p_{\mathrm{bw},0}$	$((t_{\mathrm{bw},0} + h_t r_{\mathrm{bw}})^2 + 3(y_{\mathrm{bw},0} + h_y r_{\mathrm{bw}})^2)/4$	prime
$p_{\mathrm{bw},3}$	$((t_{\mathrm{bw},3} + h_t r_{\mathrm{bw}})^2 + 3(y_{\mathrm{bw},3} + h_y r_{\mathrm{bw}})^2)/4$	prime
$\Phi_6(t_{\mathrm{bw},i} - 1)$	$(9x^2 + 9x + 3) \cdot r_{\mathrm{bw}}$	
$c_{\mathrm{bw},0}$	$(h_t^2 + 3h_y^2)/4r_{\mathrm{bw}} + (h_t - h_y)/2t_{\mathrm{bw},0} + \Phi_6(t_{\mathrm{bw},0} - 1)/(3r_{\mathrm{bw}}) - h_t$	
$c_{\mathrm{bw},3}$	$(h_t^2 + 3h_y^2)/4r_{\mathrm{bw}} + (h_t + h_y)/2t_{\mathrm{bw},3} + \Phi_6(t_{\mathrm{bw},3} - 1)/(3r_{\mathrm{bw}}) - h_t$	
$c'_{\mathrm{bw},0}$ $(\mathbb{G}_2)$	$c_{\mathrm{bw},0} + (h_t + 3h_y)/2$	
$c'_{\mathrm{bw},3}$ $(\mathbb{G}_2)$	$c_{\mathrm{bw},3} + (h_t - 3h_y)/2$	
Optimal ate Miller loop $f_{a_0,Q}(P) \cdot f_{a_1,Q}^p(P)$, Optimal twisted ate $f_{a_0+a_1\lambda,P}(Q)$		
$\lambda = t_{\mathrm{bw},0} - 1$	$a_0 = 2x,\ a_1 = 6x^2 + 2x + 1 \quad a'_0 = 6x^2 + 4x + 1,\ a'_1 = -2x$	
$\lambda = t_{\mathrm{bw},3} - 1$	$a_0 = 6x^2 + 2x + 1,\ a_1 = 2x \quad a'_0 = -2x,\ a'_1 = 6x^2 + 4x + 1$	
	Final exponentiation	
$e_{\mathrm{bw},0}$	$2x\Phi_6(p_{\mathrm{bw},0})/r_{\mathrm{bw}} =$ $(6x^2 + 2x + 1 + 2xp_{\mathrm{bw},0})(c_{\mathrm{bw},0} + h_t) - (3x + 1 + p_{\mathrm{bw},0})$	
$e'_{\mathrm{bw},0}$	$(6x^2 + 4x + 1)\Phi_6(p_{\mathrm{bw},0})/r_{\mathrm{bw}} =$ $(-2x + (6x^2 + 4x + 1)p_{\mathrm{bw},0})(c_{\mathrm{bw},0} + h_t) + 1 - (3x + 2)p_{\mathrm{bw},0}$	
$e_{\mathrm{bw},3}$	$2x\Phi_6(p_{\mathrm{bw},3})/r_{\mathrm{bw}} =$ $(-(6x^2 + 4x + 1) + 2xp_{\mathrm{bw},3})(c_{\mathrm{bw},3} + h_t) - (3x + 2) + p_{\mathrm{bw},3}$	
$e'_{\mathrm{bw},3}$	$(6x^2 + 2x + 1)\Phi_6(p_{\mathrm{bw},3})/r_{\mathrm{bw}} =$ $(2x + (6x^2 + 2x + 1)p_{\mathrm{bw},3})(c_{\mathrm{bw},3} + h_t) + 1 + (3x + 1)p_{\mathrm{bw},3}$	

Algorithm 7: [EG22, Alg. 2] Miller loop for optimal pairing with endomorphism ϕ on $\mathbb{G}_1$ (Tate), resp. $\mathbb{G}_2$ (ate) of eigenvalue λ and degree 2.

Input: $P \in \mathbb{G}_i$, $Q \in \mathbb{G}_{1-i}$, end. ϕ on $\mathbb{G}_i$ of eigenvalue λ, scalars a_0, a_1
        s. t. $a_0 + a_1\lambda = 0 \bmod r$

Output: $f_{a_0+a_1\lambda,P}(Q)$

1   $P_0 \leftarrow P$; $P_1 \leftarrow \phi(P)$
2   **if** $a_0 < 0$: $a_0 \leftarrow -a_0$; $P_0 \leftarrow -P_0$
3   **if** $a_1 < 0$: $a_1 \leftarrow -a_1$; $P_1 \leftarrow -P_1$
4   $P_{1+\lambda} \leftarrow P_0 + P_1$; $\ell_{1,\lambda} \leftarrow \ell_{P_0,P_1}(Q)$
5   $l_0 \leftarrow \mathrm{bits}(a_0)$; $l_1 \leftarrow \mathrm{bits}(a_1)$
6   **if** $\#l_0 = \#l_1$: $S \leftarrow P_{1+\lambda}$; $f \leftarrow \ell_{1,\lambda}$; $n \leftarrow \#l_0$
7   **elif** $\#l_0 < \#l_1$: $S \leftarrow P_1$; $f \leftarrow 1$; $n \leftarrow \#l_1$; pad l_0 with 0 s.t. $\#l_0 = n$
8   **else:** $S \leftarrow P_0$; $f \leftarrow 1$; $n \leftarrow \#l_0$; pad l_1 with 0 s.t. $\#l_1 = n$
9   **for** $i = n - 2$ *downto* 0:
10     $f \leftarrow f^2$; $\ell_t \leftarrow \ell_{S,S}(Q)$; $S \leftarrow [2]S$
11   **if** $l_0[i] = 0$ *and* $l_1[i] = 0$: $f \leftarrow f \cdot \ell_t$ // $m_{\text{full-sparse}}$
12   **elif** $l_0[i] = 1$ *and* $l_1[i] = 1$:
13       $S \leftarrow S + P_{1+\lambda}$; $\ell \leftarrow \ell_{S,P_{1+\lambda}}(Q)$
14       $f \leftarrow (f \cdot \ell_t) \cdot (\ell \cdot \ell_{1,\lambda})$ // $m_k + m_{\text{full-sparse}} + m_{\text{sparse-sparse}}$
15   **elif** $l_0[i] = 1$:
16       $S \leftarrow S + P_0$; $\ell \leftarrow \ell_{S,P_0}(Q)$
17       $f \leftarrow f \cdot (\ell_t \cdot \ell)$ // $m_k + m_{\text{sparse-sparse}}$
18   **else:** ($l_1[i] = 1$)
19       $S \leftarrow S + P_1$; $\ell \leftarrow \ell_{S,P_1}(Q)$
20       $f \leftarrow f \cdot (\ell_t \cdot \ell)$ // $m_k + m_{\text{sparse-sparse}}$
21   **return** f

We improve the hard part inspired by the work of Hayashida, Hayasaka and Teruya [HHT20], and the work of Cai, Hu and Zhao [CHZ21]. The key-ingredient is to factor as many terms as possible. The cost summary is in Tab. 13. The SageMath implementation is at https://gitlab.inria.fr/zk-curves/snark-2-chains.

BN-BW6 curves. A lattice reduction (with Magma on polynomials) gives the formulas in Tab. 12, where $d = \Phi_6(t_i - 1)/(3r) = 3x(x + 1) + 1$. We highlight with an underbrace the similar parts that can be shared. The sequential steps are $t_0 = -3x(2d + 1)$ then $r = 2(-xt_0 + d) - 1$ in Alg. 8, and $t_3 = 3(x(2d + 1) + 1)$,

Curve	Cost in terms of operations in $\mathbb{F}_{p^6}$
BN-BW6, t_0, Alg. 8	$6\mathbf{e}^x + \mathbf{e}^{h_2} + \mathbf{e}^{h_1} + 6\mathbf{s} + 14\mathbf{m} + \mathbf{f} + 3\mathbf{cj}$
BN-BW6, t_0, alternative	$6\mathbf{e}^x + \mathbf{e}^{h_2} + \mathbf{e}^{h_1} + 6\mathbf{s} + 14\mathbf{m} + \mathbf{f} + 4\mathbf{cj}$
BN-BW6, t_3, alternative	$6\mathbf{e}^x + \mathbf{e}^{h_2} + \mathbf{e}^{h_1} + 6\mathbf{s} + 15\mathbf{m} + \mathbf{f} + 2\mathbf{cj}$
BN-BW6, t_3, Alg. 9	$6\mathbf{e}^x + \mathbf{e}^{h_2} + \mathbf{e}^{h_1} + 6\mathbf{s} + 15\mathbf{m} + \mathbf{f}$
BLS12-BW6, t_0, Alg. 10	$5\mathbf{e}^{x-1} + \mathbf{e}^{(x-1)/3} + 3\mathbf{e}^{x+1} + \mathbf{e}^{h_2} + \mathbf{e}^{h_1} + 14\mathbf{m} + 2\mathbf{s} + \mathbf{f} + 4\mathbf{cj}$
BLS12-BW6, t_0, alt.	$5\mathbf{e}^{x-1} + \mathbf{e}^{(x-1)/3} + 3\mathbf{e}^{x+1} + \mathbf{e}^{h_2} + \mathbf{e}^{h_1} + 15\mathbf{m} + 2\mathbf{s} + \mathbf{f} + 4\mathbf{cj}$
BLS12-BW6, t_3, Alg. 11	$5\mathbf{e}^{x-1} + \mathbf{e}^{(x-1)/3} + 3\mathbf{e}^{x+1} + \mathbf{e}^{h_2} + \mathbf{e}^{h_1} + 14\mathbf{m} + 2\mathbf{s} + \mathbf{f} + 3\mathbf{cj}$
BLS12-BW6, t_3, alt.	$5\mathbf{e}^{x-1} + \mathbf{e}^{(x-1)/3} + 3\mathbf{e}^{x+1} + \mathbf{e}^{h_2} + \mathbf{e}^{h_1} + 15\mathbf{m} + 2\mathbf{s} + \mathbf{f} + 4\mathbf{cj}$
BLS24-BW6, t_0, Alg. 12	$5\mathbf{e}^{x-1} + \mathbf{e}^{(x-1)/3} + 3\mathbf{e}^{x^2+1} + 3\mathbf{e}^{x+1} + \mathbf{e}^{h_2} + \mathbf{e}^{h_1} + 14\mathbf{m} + 2\mathbf{s} + \mathbf{f} + 4\mathbf{cj}$
BLS24-BW6, t_0, alt.	$5\mathbf{e}^{x-1} + \mathbf{e}^{(x-1)/3} + 3\mathbf{e}^{x^2+1} + 3\mathbf{e}^{x+1} + \mathbf{e}^{h_2} + \mathbf{e}^{h_1} + 15\mathbf{m} + 2\mathbf{s} + \mathbf{f} + 4\mathbf{cj}$
BLS24-BW6, t_3, Alg. 13	$5\mathbf{e}^{x-1} + \mathbf{e}^{(x-1)/3} + 3\mathbf{e}^{x^2+1} + 3\mathbf{e}^{x+1} + \mathbf{e}^{h_2} + \mathbf{e}^{h_1} + 15\mathbf{m} + 2\mathbf{s} + \mathbf{f} + 2\mathbf{cj}$
BLS24-BW6, t_3, alt.	$5\mathbf{e}^{x-1} + \mathbf{e}^{(x-1)/3} + 3\mathbf{e}^{x^2+1} + 3\mathbf{e}^{x+1} + \mathbf{e}^{h_2} + \mathbf{e}^{h_1} + 16\mathbf{m} + 2\mathbf{s} + \mathbf{f} + 3\mathbf{cj}$

Table 13. Cost of hard part of final exponentiation. $\mathbf{e}^y$ stands for exponentiation to y, $\mathbf{m}$ is multiplication, $\mathbf{s}$ is squaring, $\mathbf{f}$ is Frobenius (x^p), $\mathbf{cj}$ is conjugation, all in $\mathbb{F}_{p^6}$. For BLS24-BW6 curves, $m^{x^2+1} = (m^x)^x \cdot m$. If $x^2 + 1$ has a lower Hamming weight than $2\,\mathrm{Hw}(x) + 1$, it is faster to do the former.

then $r = 2x(t_3 + 3x) + 1$ in Alg. 9.

$$(2x)\frac{\Phi_6(p_0)}{r} = \overbrace{\left(1 + 2x\underbrace{(3x+1+p_0)}_{a_0}\right)}^{b_0}\left(\tfrac{h_t^2+3h_y^2}{4}r + \tfrac{h_t-h_y}{2}t_0 + d\right) - \underbrace{(3x+1+p_0)}_{a_0}$$

$$(6x^2 + 4x + 1)\frac{\Phi_6(p_0)}{r} = \overbrace{\left(p_0 - 2x\underbrace{(1-(3x+2)p_0)}_{a_0'}\right)}^{b_0'}\left(\tfrac{h_t^2+3h_y^2}{4}r + \tfrac{h_t-h_y}{2}t_0 + d\right) + \underbrace{1 - (3x+2)p_0}_{a_0'}$$

$$(2x)\frac{\Phi_6(p_3)}{r} = \overbrace{\left(2x\underbrace{(p_3-(3x+2))}_{a_3}-1\right)}^{b_3}\left(\tfrac{h_t^2+3h_y^2}{4}r + \tfrac{h_t+h_y}{2}t_3 + d\right) + \underbrace{p_3 - (3x+2)}_{a_3}$$

$$(6x^2 + 2x + 1)\frac{\Phi_6(p_3)}{r} = \overbrace{\left(2x\underbrace{(1+(3x+1)p_3)}_{a_3'}+p_3\right)}^{b_3'}\left(\tfrac{h_t^2+3h_y^2}{4}r + \tfrac{h_t+h_y}{2}t_3 + d\right) + \underbrace{1 + (3x+1)p_3}_{a_3'}$$

<div style="display: flex;">

Algorithm 8: Hard part of final exp., BN-BW6, $t_{\text{bn},0}$
Input: x, m, $h_1 = (h_t - h_y)/2$, $h_2 = (h_t^2 + 3h_y^2)/4$
Output: $m^{(2x)\Phi_6(p_0)/r}$

1 $A \leftarrow m^x$
2 $B \leftarrow A^2 \cdot A \cdot m \cdot m^p$ $\triangleright m^{a_0}$
3 $A \leftarrow \overline{B}$
4 $B \leftarrow (B^2)^x \cdot m$ $\triangleright m^{b_0}$
5 $C \leftarrow (B^x \cdot B)^x$
6 $C \leftarrow C^2 \cdot C \cdot B$ $\triangleright B^d$
7 $D \leftarrow (C^2 \cdot B)^x$
8 $D \leftarrow D^2 \cdot D$ $\triangleright B^{-t_0}$
9 $F \leftarrow (D^x \cdot C)^2 \cdot \overline{F}$ $\triangleright B^r$
10 $F \leftarrow F^{h_2} \cdot \overline{D}^{h_1} \cdot C$
11 **return** $F \cdot A$

Algorithm 9: Hard part of final exp., BN-BW6, $t_{\text{bn},3}$
Input: x, m, $h_1 = (h_t + h_y)/2$, $h_2 = (h_t^2 + 3h_y^2)/4$
Output: $m^{(6x^2+2x+1)\Phi_6(p_3)/r}$

1 $B \leftarrow m^p$
2 $A \leftarrow B^x$
3 $A \leftarrow (A^2 \cdot A \cdot B) \cdot m$ $\triangleright m^{a'_3}$
4 $B \leftarrow (A^2)^x \cdot B$ $\triangleright m^{b'_3}$
5 $C \leftarrow B^x$
6 $D \leftarrow C^2 \cdot C$
7 $C \leftarrow D^x \cdot D \cdot B$ $\triangleright B^d$
8 $E \leftarrow (C^2 \cdot B)^x \cdot B$
9 $E \leftarrow E^2 \cdot E$ $\triangleright B^{t_3}$
10 $F \leftarrow ((E \cdot D)^x)^2 \cdot B$ $\triangleright B^r$
11 $F \leftarrow F^{h_2} \cdot E^{h_1} \cdot C$
12 **return** $F \cdot A$

</div>

BLS12-BW6 curves. Parameters are in Tab. 10, where $d = \Phi_6(t_i - 1)/(3r) = (x^4 - 4x^3 + 7x^2 - 6x + 3)$. We highlight with an underbrace the same exponent a_i that can be shared. The steps in Alg. 10 correspond to the exponents

$$
\begin{aligned}
a &= (x-1)/3 \\
b &= a(x-1) & &= (x-1)^2/3 \\
c &= b((x-1)^2 + 1) & &= (d-1)/3 = (x-1)^2/3(x^2 - 2x + 2) \\
e &= -(u+1)c + b - a & &= t_0/3 = (x-1)^2/3(-x^3 + x^2 - 1) - (x-1)/3 \\
f &= -(u+1)(e+b) + a + 1 & &= r = (x-1)^2/3(x^4 - x^2 + 1) + x
\end{aligned}
$$

The steps in Alg. 11 correspond to this sequence deduced from the former, with $t_3/3 = -t_0 + 1$.

$$
\begin{aligned}
a &= (x-1)/3 \\
b &= a(x-1) & &= (x-1)^2/3 \\
c &= b((x-1)^2 + 1) & &= (d-1)/3 \\
b' &= -b \\
e &= b' + 1 \\
f &= c(x+1) + e & &= (x-1)^2/3(x^3 - x^2 + 1) + 1 \\
g &= f + a & &= t_3/3 \\
h &= (f + b')(x+1) - e & &= r
\end{aligned}
$$

$$(x+1)\tfrac{\Phi_6(p_0)}{r} = \overbrace{\big((x+1)\underbrace{(p-(x-1)^2-1)}_{a_0}+1\big)}^{b_0}\Big(\tfrac{h_t^2+3h_y^2}{4}r+3\big(\tfrac{h_t-h_y}{2}\tfrac{t_0}{3}+\tfrac{d-1}{3}\big)+1\Big)-3\underbrace{(p-(x-1)^2-1)}_{a_0}$$

$$(x^3-x^2-x)\tfrac{\Phi_6(p_0)}{r} = \overbrace{\big((x+1)\underbrace{((x-1)^2p+1)}_{a_0'}-p\big)}^{b_0'}\Big(\tfrac{h_t^2+3h_y^2}{4}r+3\big(\tfrac{h_t-h_y}{2}\tfrac{t_0}{3}+\tfrac{d-1}{3}\big)+1\Big)-3\underbrace{((x-1)^2p+1)}_{a_0'}$$

$$(x+1)\tfrac{\Phi_6(p_3)}{r} = \overbrace{\big((x+1)\underbrace{((x-1)^2+p)}_{a_3}-1\big)}^{b_3}\Big(\tfrac{h_t^2+3h_y^2}{4}r+3\big(\tfrac{h_t+h_y}{2}\tfrac{t_3}{3}+\tfrac{d-1}{3}\big)+1\Big)+3\underbrace{((x-1)^2+p)}_{a_3}$$

$$(x^3-x^2+1)\tfrac{\Phi_6(p_3)}{r} = \overbrace{\big((x+1)\underbrace{(((x-1)^2+1)p-1)}_{a_3'}-p\big)}^{b_3'}\Big(\tfrac{h_t^2+3h_y^2}{4}r+3\big(\tfrac{h_t+h_y}{2}\tfrac{t_3}{3}+\tfrac{d-1}{3}\big)+1\Big)+3\underbrace{(((x-1)^2+1)p-1)}_{a_3'}$$

Algorithm 10: Hard part of final exp., BLS12-BW6, t_0

Input: x, m, $h_1 = (h_t - h_y)/2$, $h_2 = (h_t^2 + 3h_y^2)/4$

Output: $m^{(x^3-x^2-x)\Phi_6(p_0)/r}$

1   $Q \leftarrow m^p$

2   $A \leftarrow (Q^{x-1})^{x-1} \cdot m$ $\triangleright m^{a_0'}$

3   $B \leftarrow \overline{A^{x+1} \cdot \overline{Q}}$ $\triangleright m^{b_0'}$

4   $A \leftarrow \overline{A^2 \cdot A}$ $\triangleright m^{-3a_0'}$

5   $C \leftarrow B^{(x-1)/3}$

6   $D \leftarrow C^{x-1}$

7   $E \leftarrow (D^{x-1})^{x-1} \cdot D$ $\triangleright B^{(d-1)/3}$

8   $F \leftarrow \overline{E^{x+1} \cdot C} \cdot D$ $\triangleright B^{t_0/3}$

9   $G \leftarrow \overline{(F \cdot D)^{x+1}} \cdot C \cdot B$ $\triangleright B^r$

10   $H \leftarrow F^{h_1} \cdot E$

11   $H \leftarrow H^2 \cdot H \cdot B \cdot G^{h_2}$

12   **return** $A \cdot H$

Algorithm 11: Hard part of final exp., BLS12-BW6, t_3

Input: x, m, $h_1 = (h_t + h_y)/2$, $h_2 = (h_t^2 + 3h_y^2)/4$

Output: $m^{(x+1)\Phi_6(p_3)/r}$

1   $A \leftarrow m^p \cdot (A^{x-1})^{x-1}$ $\triangleright m^{a_3}$

2   $B \leftarrow \overline{m} \cdot A^{x+1}$ $\triangleright m^{b_3}$

3   $A \leftarrow A^2 \cdot A$ $\triangleright m^{3a_3}$

4   $C \leftarrow B^{(x-1)/3}$

5   $D \leftarrow C^{x-1}$

6   $E \leftarrow (D^{x-1})^{x-1} \cdot D$ $\triangleright B^{(d-1)/3}$

7   $D \leftarrow \overline{D}$

8   $F \leftarrow D \cdot B$

9   $G \leftarrow E^{x+1} \cdot F$

10   $H \leftarrow G \cdot C$ $\triangleright B^{t_3/3}$

11   $I \leftarrow (G \cdot D)^{x+1} \cdot \overline{F}$ $\triangleright B^r$

12   $J \leftarrow H^{h_1} \cdot E$

13   $K \leftarrow J^2 \cdot J \cdot B \cdot I^{h_2}$

14   **return** $A \cdot K$

BLS24-BW6 curves. The exponents of the hard part of the final exponentiation for BW6-BLS24 curves are the following.

$$c_0 = (x-1)^2(x^2+1)$$

$$(x^5 - x^4 - x)\frac{\Phi_6(p_0)}{r} = \Big((x+1)\overbrace{\underbrace{(1+c_0p)}_{a_0}}^{b_0}-p\Big)\Big(\tfrac{h_t^2+3h_y^2}{4}r + 3\big(\tfrac{h_t-h_y}{2}\tfrac{t_0}{3} + \tfrac{d-1}{3}\big)+1\Big) - 3\underbrace{(1+c_0p)}_{a_0}$$

$$c_0' = (x-1)^2(x^2+1)+1 = c_0 + 1$$

$$(x+1)\frac{\Phi_6(p_0)}{r} = \Big((x+1)\overbrace{\underbrace{(p-c_0')}_{a_0'}+1}^{b_0'}\Big)\Big(\tfrac{h_t^2+3h_y^2}{4}r + 3\big(\tfrac{h_t-h_y}{2}\tfrac{t_0}{3} + \tfrac{d-1}{3}\big)+1\Big) - 3\underbrace{(p-c_0')}_{a_0}$$

$$c_3 = (x-1)^2(x^2+1)$$

$$(x+1)\frac{\Phi_6(p_3)}{r} = \Big((x+1)\overbrace{\underbrace{(c_3+p)}_{a_3}}^{b_3}-1\Big)\Big(\tfrac{h_t^2+3h_y^2}{4}r + 3\big(\tfrac{h_t+h_y}{2}\tfrac{t_3}{3} + \tfrac{d-1}{3}\big)+1\Big) + 3\underbrace{(c_3+p)}_{a_3}$$

$$c_3' = (x-1)^2(x^2+1)+1 = c_3 + 1$$

$$(x^5 - x^4 + 1)\frac{\Phi_6(p_3)}{r} = \Big((x+1)\overbrace{\underbrace{(c_3'p-1)}_{a_3'}}^{b_3'}-p\Big)\Big(\tfrac{h_t^2+3h_y^2}{4}r + 3\big(\tfrac{h_t+h_y}{2}\tfrac{t_3}{3} + \tfrac{d-1}{3}\big)+1\Big) + 3\underbrace{(c_3'p-1)}_{a_3'}$$

The parameters are given in Tab. 11, and we set $d = \Phi_6(t_i - 1)/(3r) = x^8 - 4x^7 + 8x^6 - 12x^5 + 15x^4 - 14x^3 + 10x^2 - 6x + 3$. Part of the exponent is

$$\tfrac{h_t^2+3h_y^2}{4}r + \tfrac{h_t+h_y}{2}t_3 + d$$

$$\tfrac{h_t^2+3h_y^2}{4}r + \tfrac{h_t-h_y}{2}t_0 + d$$

We compute the exponents r, $t_0/3$, $t_3/3$ and $(d-1)/3$ as follows and obtain Alg. 12 and Alg. 13.

$a = (x-1)/3$	$a = (x-1)/3$
$b = a(x-1)(x^2+1)$	$b = a(x-1)(x^2+1)$
$c = b((x-1)^2(x^2+1)+1) \quad = (d-1)/3$	$c = b((x-1)^2(x^2+1)+1) = (d-1)/3$
	$e_3 = (x+1)c - b + a$
$f_0 = -(x+1)c + b - a \qquad = t_0/3$	$f_3 = e_3 + 1 \qquad\qquad\qquad = t_3/3$
$g = -(x+1)(f_0+b) + a + 1 = r$	$g = (x+1)(e_3-b) + a + 1 = r$

Algorithm 12: Hard part of final exp., BLS24-BW6, t_0

Input: x, m, $h_1 = (h_t - h_y)/2$, $h_2 = (h_t^2 + 3h_y^2)/4$

Output: $m^{(x^5 - x^4 - x)\Phi_6(p_0)/r}$

1   $m_p \leftarrow m^p$
2   $A \leftarrow (m_p^{u-1})^{u-1}$
3   $A \leftarrow A^{u^2+1}$ $\triangleright$or $(A^u)^u \cdot A$
4   $A \leftarrow m \cdot A$
5   $B \leftarrow A^{u+1} \cdot \overline{m_p}$ $\triangleright m^{b_0}$
6   $A \leftarrow \overline{A^2 \cdot A}$ $\triangleright m^{-3a_0}$
7   $C \leftarrow B^{(u-1)/3}$
8   $D \leftarrow C^{u-1}$
9   $D \leftarrow D^{u^2+1}$ $\triangleright$or $(D^u)^u \cdot D$
10   $E \leftarrow (D^{u-1})^{u-1}$
11   $E \leftarrow E^{u^2+1}$ $\triangleright$or $(E^u)^u \cdot E$
12   $E \leftarrow E \cdot D$ $\triangleright B^{(d-1)/3}$
13   $F \leftarrow \overline{E^{u+1} \cdot C} \cdot D$ $\triangleright B^{t_0/3}$
14   $G \leftarrow \overline{F \cdot D}$
15   $H \leftarrow G^{u+1} \cdot C \cdot B$ $\triangleright B^r$
16   $I \leftarrow F^{d_1} \cdot E$
17   $I \leftarrow I^2 \cdot I \cdot B \cdot H^{d_2}$
18   **return** $A \cdot I$

Algorithm 13: Hard part of final exp., BLS24-BW6, t_3

Input: x, m, $h_1 = (h_t + h_y)/2$, $h_2 = (h_t^2 - 3h_y^2)/4$

Output: $m^{(x+1)\Phi_6(p_3)/r}$

1   $A \leftarrow (m^{u-1})^{u-1}$
2   $A \leftarrow A^{u^2+1}$ $\triangleright$or $(A^u)^u \cdot A$
3   $A \leftarrow A \cdot m^p$
4   $B \leftarrow A^{u+1} \cdot \overline{m}$ $\triangleright m^{b_3}$
5   $A \leftarrow A^2 \cdot A$ $\triangleright m^{3a_3}$
6   $C \leftarrow B^{(u-1)/3}$
7   $D \leftarrow C^{u-1}$
8   $D \leftarrow D^{u^2+1}$ $\triangleright$or $(D^u)^u \cdot D$
9   $E \leftarrow (D^{u-1})^{u-1}$
10   $E \leftarrow E^{u^2+1}$ $\triangleright$or $(E^u)^u \cdot E$
11   $E \leftarrow E \cdot D$ $\triangleright B^{(d-1)/3}$
12   $D \leftarrow \overline{D}$
13   $F \leftarrow E^{u+1} \cdot D \cdot C$
14   $G \leftarrow F \cdot B$ $\triangleright B^{t_3/3}$
15   $H \leftarrow (F \cdot D)^{u+1} \cdot C \cdot B$ $\triangleright B^r$
16   $I \leftarrow G^{d_1} \cdot E$
17   $I \leftarrow I^2 \cdot I \cdot B \cdot H^{d_2}$
18   **return** $A \cdot I$

The limitation of the order of the twist with the Brezing–Weng method, and the Cocks–Pinch construction. The authors of [EG22] also considered the Cocks–Pinch construction for the outer curves. While this construction allows less efficient pairings, it provides curves with more conservative security when the embedding degree of the inner curve is large (e.g. BLS24). BW and CP constructions follow the same recipe, but CP deals with integers, while BW deals with polynomials [FST10, §4.1,§6]. They start from the subgroup order $r_{\mathrm{bw}}(x) = p_{\mathrm{bls}}(x)$, $r_{\mathrm{cp}}(u) = p_{\mathrm{bls}}(u)$, and look for k-th roots of unity $\zeta_k \bmod p_{\mathrm{bls}}$ to set the trace value $t = \zeta_k + 1$. For CP, the existence of ζ_k requires $p_{\mathrm{bls}}(u) \equiv 1 \bmod k$: for $k = 6, 12, 8$ resp., this means $u \equiv 1 \bmod 3$, $u \equiv 1, 10 \bmod 12$, and $u \equiv 1, 10 \bmod 24$ resp. For BW, the number field defined by $p_{\mathrm{bls}}(x)$ only contains $\zeta_k(x)$ for $k \mid 6$, limiting the BW construction to $k = 6$ at most.

5.3   Cycles of plain elliptic curves

Given the difficulties in finding optimal cycles and chains of pairing-friendly curves, a recent alternative is employing inner-product argument proofs together with amicable pairs (2-cycles) of *plain* elliptic curves, i.e., curves not equipped

with efficient bilinear maps, where just the hardness of the discrete logarithm holds. The advantages of this approach are higher performance, since parameters can scale just like traditional Elliptic Curve Cryptography, and conservativeness against advances in discrete logarithm computation over finite fields, observing that advances in solving elliptic curve discrete logarithms would impact pairing-friendly curves anyway. On the other hand, these advantages come at mild increase in proof sizes and verification times.

Silverman and Stange [SS11] showed that there exist elliptic curves with arbitrary long cycles, and conjectured that for any elliptic curve $E/\mathbb{Q}$, the number $\mathcal{Q}_E(X)$ of prime pairs (p, q) with $p < q$ and $p \leq X$ such that reducing E modulo p and q gives an amicable pair is such that:

1. $\mathcal{Q}_E(X) = \Theta\left(\frac{\sqrt{X}}{(\log X)^2}\right)$ as $X \to \infty$ if E does not have complex multiplication;
2. $\mathcal{Q}_E(X) \approx A_E \frac{X}{(\log X)^2}$ for a constant $A_E > 0$, otherwise.

In other words, there is a surprising difference between the CM and non-CM cases, and amicable pairs are asymptotically common. The reason for why CM curves have so many amicable pairs is attributed to the fact that if $E/\mathbb{Q}$ has CM and $\#E_p(\mathbb{F}_p) = q$ then there are generally only two possible values for $\#E_q(\mathbb{F}_q)$: p and $2q + 2 - p$, while non-CM curves have $\#E_q(\mathbb{F}_q)$ ranging throughout the interval given by the Hasse condition $p - 2\sqrt{p} + 1 \leq r \leq p + 2\sqrt{p} + 1$. Further research in [Jon13,Par19] has refined and proven these estimates on average.

Regarding security, one notes that the extreme case of an 1-cycle, i.e. $\#E(\mathbb{F}_p) = p$ leads to anomalous curves for which the discrete logarithm can be computed in linear time [Sma99]. For longer cycles, a set of requirements for choosing parameters can be derived from the SafeCurves [BL] criteria for ECC security:

1. Hardness of the ECDLP against Pollard's rho method on the curve and twist.
2. Large embedding degree and CM discriminant D.
3. Rigidity for generating parameters in a reproducible manner.
4. Availability of Montgomery ladder and complete addition laws.
5. Efficient bijective maps (encodings) between uniform random strings and a large set of rational points on the curve. Such an encoding can also be used for hashing efficiently to the curve.

A conservative choice of parameters attempts to finds curves that maximize the set of satisfied security requirements.

`sec(p|q)k256k1` **curves** . Probably the earliest 2-cycle of elliptic curves with application in zero-knowledge proofs was based on the elliptic curve adopted for signatures in Bitcoin. The initial motivation was being able to build zero-knowledge proofs over already deployed cryptosystems by using the closely related

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curve obtained with just swapping base field and order [Poe18]:

$$E_p/\mathbb{F}_p : y^2 = x^3 + 7 \text{ of order } q, \text{ called } \texttt{secp256k1}$$
$$E_q/\mathbb{F}_q : y^2 = x^3 + 7 \text{ of order } p, \text{ called } \texttt{secq256k1, with}$$
$$p = 2^{256} - 2^{32} - 2^9 - 2^8 - 2^7 - 2^6 - 2^4 - 1 \text{ and}$$
$$q = 2^{256} - 432420386565659656852420866394968145599.$$

These curves satisfy SafeCurves criteria 1 and 3, but have CM discriminant $D = -3$ for the efficient endomorphism [GLV01]; and prime order which does not allow the original Montgomery ladder or Elligator encoding [BHKL13]. However, an optimized version of the Elligator Squared [Tib14] encoding can be used [Wui21].

Tweedle curves. One of the first 2-cycles of elliptic curves suitable for recursive proofs was the Tweedledum-Tweedledee curves in Halo [BGH19]:

$$E_p/\mathbb{F}_p : y^2 = x^3 + 5 \text{ of order } q, \text{ called Tweedledum};$$
$$E_q/\mathbb{F}_q : y^2 = x^3 + 5 \text{ of order } p, \text{ called Tweedledee, with}$$
$$p = 2^{254} + 4707489545178046908921067385359695873 \text{ and}$$
$$q = 2^{254} + 4707489544292117082687961190295928334.$$

These curves satisfy some of the SafeCurves criteria at the 126-bit security, but have CM discriminant $D = 3$ as a consequence of how they were constructed, and do not benefit from the Montgomery ladder as prime-order curves. On the other hand, they have high 2-adicity, are equipped with fast endomorphisms for efficient scalar multiplication, and somewhat efficient encoding/hashing algorithms.

Pasta curves. Like the Tweedle curves, Pallas and Vesta form a 2-cycle of elliptic curves for recursive proofs in Halo 2 and beyond [Hop20]:

$$E_p/\mathbb{F}_p : y^2 = x^3 + 5 \text{ of order } q, \text{ called Pallas};$$
$$E_q/\mathbb{F}_q : y^2 = x^3 + 5 \text{ of order } p, \text{ called Vesta, with}$$
$$p = 2^{254} + 45560315531419706090280762371685220353 \text{ and}$$
$$q = 2^{254} + 45560315531506369815346746415080538113.$$

These curves provide several of the attractive properties of the Tweedle curves, while relaxing twist security to 100 bits for Pallas, and otherwise improving over them on a number of ways. Both curves have:

– Low-degree isogenies (3 instead of 23) from curves with a nonzero j-invariant, which accelerates hashing.
– The same 2-adicity of 32 (instead of 33 and 34) that accelerates square root extraction and simplifies point compression.
– Sparse moduli to accelerate base field arithmetic in the Montgomery representation.

– A reproducible generation method based on a search utility that is publicly available and should facilitate searching for similar 2-cycles in the future, satisfying rigidity concerns.

The generation method specializes the CM method for $D = 3$ to produce curves of form $E(\mathbb{F}) : y^2 = x^3 + b$ with $b \neq 0$. It looks for values (V, T) within a certain range such that $p = (3V^2 + T^2)/4$ is a prime with the desired length and $p \equiv 1 \pmod 6$ to equip the curve with a fast endomorphism. By requiring $(V - 1)/2$ and $(T - 1)/2$ to be multiples of 2^L, $p - 1$ be a multiple of 2^L and $q - 1$ will be a multiple of 2^L for $q \in \{p + 1 - T, p + 1 + (T - 3V)/2\}$ among the six possible orders given by the CM method. Algorithm 14 summarizes the generation strategy.

Algorithm 14: FindAmicablePair(ℓ, L)
Input: designed length ℓ for prime p, 2-adicity L, k-bit security
Output: 2-cycle of curves E_p, E_q.

1 **while** *true*:
2   Find (V, T) such that:
3   • $4p = 3V^2 + T^2$, $2^L \mid (V - 1)/2$, $2^L \mid (T - 1)/2$;
4   • p is prime, $|p| = \ell \wedge p \equiv 1 \pmod 6$;
5   **for** $q \in \{p + 1 - T, p + 1 + (T - 3V)/2\}$:
6   **if** $q \notin \{p, p + 1, p - 1\} \wedge q \equiv 1 \pmod 6 \wedge 2^L|(q - 1) \wedge q$ *is prime*:
7   Find b_p incrementally such that $E_p(\mathbb{F}_p) : y^2 = x^3 + b_p$ has order q;
8   Find b_q incrementally such that $E_q(\mathbb{F}_q) : y^2 = x^3 + b_q$ has order p;
9   **if** *ECDLPSecurity(E_p)* $\geq k \wedge$ *ECDLPSecurity(E_q)* $\geq k$:
10   **return** (E_p, E_q);

Generalizing the method for constructing 2-cycles described above gives the immediate corollary that every curve generated with the CM method using a small discriminant forms a cycle with another CM curve generated with the prescribed order given by the characteristic of the base field.

5.4 Hybrid cycles of elliptic curves

A middle-ground alternative to cycle of pairing-friendly curves and cycles of plain elliptic curves is *half-pairing cycles*, where only one of the curves in a 2-cycle is pairing-friendly. This can be potentially useful to combine recursive proofs with protocols based on cryptographic pairings.

Using the results from previous sections, it is possible to generate half-pairing cycles where the pairing-friendly curve is of prime order by just using the CM method. Hence this strategy applies to generating 2-cycles where the pairing-friendly curve is either from the BN, Freeman or MNT families. One such example is the Pluto-Eris curve.

Pluto-Eris curves. Pluto and Eris form a 2-cycle of elliptic curves [Hop21]:

$E_p/\mathbb{F}_p : y^2 = x^3 + 57$ is a BN curve of order q, called Pluto;

$E_q/\mathbb{F}_q : y^2 = x^3 + 57$ is a plain curve of order p, called Eris, with

$\quad p = 36x^4 + 36x^3 + 24x^2 + 6x + 1$ and

$\quad q = 36x^4 + 36x^3 + 18x^2 + 6x + 1$, for $x = -(2^{110} + 2^{60} + 2^{39} + 2^{35} - 2^{31})$.

The field size of 446 bits for Pluto is chosen to satisfy 128 bits of security for pairings [Gui20], although is naturally leaves a larger security margin for Eris. Regarding performance, both curves have the high 2-adicity of 32, are again equipped with fast endomorphisms for efficient scalar multiplication due to the choice of small CM discriminant and efficient encoding/hashing algorithms. The choice of seed for the BN curve gives a particularly fast pairing computation due to its low Hamming weight.

Another such example is the hybrid cycle based on a BN382 curve that was briefly used by the Mina protocol in early 2020 [Mec20].

BN382-Plain curves. BN382 forms, with a related plain curve, a 2-cycle of elliptic curves [Mec20]:

$E_p/\mathbb{F}_p : y^2 = x^3 + 14$ is a BN curve of order q;

$E_q/\mathbb{F}_q : y^2 = x^3 + 7$ is a plain curve of order p, with

$\quad p = 36x^4 + 36x^3 + 24x^2 + 6x + 1$ and

$\quad q = 36x^4 + 36x^3 + 18x^2 + 6x + 1$, for $x = 2^{94} + 2^{81} + 2^{74} + 2^{66}$.

The fields sizes of 382 bits are chosen to satisfy slightly less than 128 bits of security for pairings, while allowing an efficient 6-limb arithmetic on 64-bit architectures. Similarly to Pluto-Eris, both curves have the high 2-adicity of 64, are equipped with fast endomorphisms for efficient scalar multiplication due to the choice of small CM discriminant and efficient encoding/hashing algorithms. The choice of seed for the BN curve gives a particularly fast pairing computation due to its low Hamming weight.

6 Elliptic curves inside SNARKs

While SNARKs allow proving general-purpose computations, in many applications these computations revolve around proving some cryptographic operations such as hashings, encryptions, key exchanges or signatures. For example, in Zerocash [BCG+14], the authors consider proving the SHA256 hash computation to build a privacy-preserving cryptocurrency protocol. Zcash (ZEC) is a cryptocurrency that first implemented the Zerocash protocol with many improvements over the years. Among these, they replaced the SHA256 hash by a variant of the Pedersen hash based on elliptic curves. Gyges [JKS16] and Hawk [KMS+16]

consider proving the RSA encryption for privacy-preserving blockchain smart contracts. Cinderella [DFKP16] considered proving the validity and compliance of X.509 certificates by proving the RSA signature verification. Later, CØCØ's authors observed that the RSA scheme is not a well suited computation to be proven in a SNARK and considered replacing it by a hybrid encryption. They first proved an ECDH key exchange and then a lightweight symmetric encryption (Speck [BSS+13] and Chaskey-LTS [MMV+14]). More recently, zk-rollups were proposed to solve the scalability problem of the Ethereum blockchain. A zk-rollup operator validates a batch of transactions by verifying a proof of the correct verifications of many signatures (e.g. ConsenSys rollup and Polygon Hermez rollup verify EdDSA signatures on a twisted Edwards curve associated to BN254).

All those computations are elliptic curve based cryptographic primitives. They require an elliptic curve E_0 to express the computation, independently of the SNARK elliptic curve E to prove the computation. CØCØ's authors were the first to propose to construct a customized elliptic curve to efficiently express this kind of computations. Proving systems use different models to translate a generic-purpose computation into an equivalent arithmetized statement The Rank-1 Constraint System (R1CS) [BCC+16] is one of the most used arithmetization models. A key property of this model is that multiplications, squarings and inversions have the same cost while additions and subtractions are free. Therefore, an implementor should express the statement they want to prove with the least multiplicative complexity and, counter-intuitively, they should bear in mind that inverses are not costly. The arithmetization step takes place in $\mathbb{F}_r$ where r is the prime subgroup order of the SNARK curve, i.e. $r \mid \#E$. CØCØ constructs a custom curve E_0 over precisely $\mathbb{F}_r$ to avoid emulating non-native field operations (cf. Fig. 3). Following the guidelines described in constructing `curve25519`, the authors proposed $E_0(\mathbb{F}_r)$ in Montgomery form $y^2 = x^3 + 126932x^2 + x$ with an order 8×251-bit prime and with a twist order 4×252-bit. Implementing affine-coordinates scalar multiplication using this curve is very efficient in the R1CS model. Later, Zcash engineers proposed the use of a twisted Edwards curve instead that is more efficient than the CØCØ Montgomery curve. It was built on top of the BLS12-381 curve and named Jubjub. In fact, conditional statements are costly in R1CS and one resort to a *double-and-always-add*-like algorithm (as in a side-channel resistant implementation) instead of the classical *double-and-add* algorithm to express a scalar multiplication. Instinctively, it would seem that windowed multiplications would be expensive in this context but it turns out that constant windowed methods can be achieved efficiently in R1CS using a twisted Edwards curve. The affine group law on these curves is *complete* meaning that one can use a lookup table to select the precomputed points (x_i, y_i) or the zero point $(x_0, y_0) = (0, 1)$ to add in the algorithm. This is done through polynomials which vanish at the inputs that are not being selected. As an example, we show how to perform a 2-bit windowed scalar multiplication in R1CS in Algorithms 16 and 15.

In [MSZ21] Masson et al. performed an exhaustive search of a twisted Edwards curve defined over the BLS12-381 with small Complex Multiplication discriminant

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Algorithm 15: Lookup2: 2-bit lookup table in R1CS
Input: bits (b_0, b_1), and constants (c_0, c_1, c_2, c_3)

$$\textbf{Output: } r = \begin{cases} c_0, & \text{if } b_0 = 0, b_1 = 0 \\ c_1, & \text{if } b_0 = 0, b_1 = 1 \\ c_2, & \text{if } b_0 = 1, b_1 = 0 \\ c_3, & \text{if } b_0 = 1, b_1 = 1 \end{cases}$$

1 $t_1, t_2 \leftarrow$ temporary variables;
2 $(c_3 - c_2 - c_1 + c_0) \times b_1 = t_1 - c_1 + c_0$;
3 $t_1 \times b_0 = t_2$;
4 $(c_2 - c_0) \times b_1 = r - t_2 - c_0$;
5 return r;

Algorithm 16: 2-bit windowed scalar multiplication in R1CS
Input: a scalar s and a point $P \in E_0$
Output: $R = [s]P \in E_0$

1 $(b_0, \cdots, b_{n-1}) \leftarrow$ the binary decomposition of s (b_0 being the least significant bit and assuming n is even) ;　　　　　// THIS IS ALSO A STATEMENT TO PROVE
2 $R, T \in E_0$; $A \leftarrow [2]P$; $B \leftarrow A + P$;
3 $R_x \leftarrow \text{Lookup2}(b_{n-1}, b_{n-2}, 0, P_x, A_x, B_x)$;
4 $R_y \leftarrow \text{Lookup2}(b_{n-1}, b_{n-2}, 1, P_y, A_y, B_y)$;
5 for b_i *from* b_{n-3} *downto* b_0, $i \leftarrow i - 2$**:**
6 $R \leftarrow [4]R$;
7 $T_x \leftarrow \text{Lookup2}(b_i, b_{i-1}, 0, P_x, A_x, B_x)$;
8 $T_y \leftarrow \text{Lookup2}(b_i, b_{i-1}, 1, P_y, A_y, B_y)$;
9 $R \leftarrow R + T$;
10 return R;

---

in order to speed up the scalar multiplication using a fast endomorphism. They found an isolated curve with the discriminant $-D = -8$, called Bandersnatch. This curve has $a \neq -1$ and is not a square, hence the formulae are incomplete. However, all infinity points are of even order and checking that the points are of the right odd prime order is enough to rule out exceptions [HWCD08, Th. 1]. All SNARK curves can have associated twisted Edwards curves but it is unlikely to find such a curve with a small CM discriminant. In table 14, we gather the parameters of the proposed twisted Edwards associated to some SNARK curves discussed in this paper. Note that the multiplication by the curve coefficients is free in the R1CS model as they are constants with respect to the statement.

7 Conclusion

The application of elliptic curves in modern zero-knowledge proof systems has energized research efforts well beyond traditional elliptic curve cryptography. The initiative already developed practical solutions to several challenging problems: how to instantiate recursive proof systems with 2-cycles of MNT pairing-friendly

	associated twisted Edwards curve $ax^2 + y^2 = 1 + dx^2y^2$			
	a=-1?	curve order	twist cofactor	fast endomorphism?
BN254 (Ethereum)	✓	$8 \times$ (251-bit prime)	4	✗
BLS12-381	✓ (Jubjub)	$8 \times$ (252-bit prime)	4	✗
	✗($a = -5$) (Bandersnatch)	$4 \times$ (253-bit prime)	$2^7 \cdot 3^3$	✓
BLS24-317	✓	$8 \times$ (250-bit prime)	1	✗
BLS12-377	✓	$4 \times$ (251-bit prime)	8	✗
BW6-761	✓	$8 \times$ (374-bit prime)	4	✗
BLS24-315	✓	$8 \times$ (250-bit prime)	4	✗
BW6-633	✓	$8 \times$ (312-bit prime)	4	✗
MNT4-753	✓	$8 \times$ (750-bit prime)	4	✗
MNT4-298	✓	$4 \times$ (296-bit prime)	8	✗

Table 14. Associated twisted Edwards curves to some SNARK curves.

curves, how to use 2-chains to trade-off generality (as in the number of composition layers) for efficiency, and how to use 2-cycles of plain curves to further trade-off proof performance for deployability (transparent setup).

In this survey, we presented an overview of elliptic curves generated for SNARKs belonging to one of the three classes above. We summarize the mathematical background and revisit the efficiency and security requirements for (pairing-friendly) elliptic curves within SNARKs. We then collect results and examples from the literature of 2-cycles and 2-chains of elliptic curves, while introducing novel improvements for several families in the latter case. In particular, we generalize the 2-chain framework to include BN curves, improve final exponentiation for BW6 curves, and propose a way to obtain higher 2-adicity for BLS24 curves.

Still, a number of questions linger: Are there more efficient cycles of pairing-friendly curves? How to generate them? What are the optimal choices for 2-chains at higher security levels? Are there more efficient constructions of 2-chains with smaller outer curves? We hope that our comprehensive overview attracts new interest towards this topic, and that further developments translate into practical impact through one of the many frameworks for theoretical research or practical deployment in this space.

A Implementations

We report in Table 15 some libraries that implement different SNARK curves, 2-chains and 2-cycles. We only cite implementations that are used in zero-knowledge proofs based projects and we omit to cite forks that improve independently over the original work. The libraries are implemented in different languages and some use more assembly acceleration than others. Besides the different algorithmic and

software optimizations used across, it should also be noted that some libraries target constant-time implementations for some or all the operations.

Note. Libraries in Table 15 provide the classical implementation of elliptic curves. Few of these libraries provide also implementations of curves as SNARK computations. That is, the arithmetic of fields and groups of elliptic curves as statements to be proved in a SNARK using another elliptic curve (e.g. Alg. 16 for twisted Edwards). For example arkworks [aC22], gnark [BPH+22], libsnark [BSCT+b] and zcash [BS] provide such implementations within different proving systems.

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[4] gnark-crypto and arkworks implement for each SNARK curve an associated twisted Edwards curve, including Jubjub and Bandersnatch.

Library	curves implemented	programming language	license	zero-knowledge projects
arkworks-rs [aC22]	BN254 BLS12-381 BLS12-377/BW6-761 chain MNT753 cycle MNT298 cycle Pasta cycle (and all associated curves[4])	Rust	MIT Apache-2.0	arkworks-rs Celo Aleo
Barretenberg [Wil]	BN254	C++		Aztec rollup
blst [Sup]	BLS12-381	C	Apache-2.0	Filecoin
constantine [AR]	BN254 BLS12-381 BLS12-377 Jubjub, Bandersnatch Curve25519	Nim	MIT Apache-2.0	Status-Ethereum (ongoing adoption)
Dalek [dVYA22]	Ed25519 Curve25519	Rust	BSD-3-Clause	Dalek-bulletproofs Spartan
Geth (Cloudflare)	BN254	Go	LGPL-3.0	Geth Erigon
gnark-crypto[4] [BPH+]	BN254 BLS12-381 BLS24-317 BLS12-377/BW6-761 chain BLS24-315/BW6-633 chain (and all associated curves[4])	Go	Apache-2.0	gnark ConsenSys Rollup Baseline protocol geth (Fuzzing)
Kilic [Kil]	BN254 BLS12-381 BLS12-377/BW6-761 chain	Go	Apache-2.0	Geth Celo
libff [BSCT+a]	BN254 BLS12-381 GMV6-183 MNT298 cycle	C++	MIT	Libsnark Loopring
mcl [Shi]	BN254 BLS12-381	C++	BSD-3-Clause	DFINITY
RELIC [AGM+]	BN254 BLS12-377 BLS12-381 BLS24-315 Tweedle/Pasta cycles	C	Apache-2.0 LGPL-2.1	Chia Network
wasmcurves [Bay]	BN254 BLS12-381	JavaScript, WASM	GPL-3	Circom/snarkjs Polygon Hermez rollup
zcash [BS,Zca]	BN254 BLS12-381 Jubjub Pasta cycle	Rust	MIT Apache-2.0	Zcash Dusk Algorand OpenEthereum zkSync rollup

Table 15. Some implementations of SNARK curves.

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