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ERROR ESTIMATES FOR DIFFERENCE METHODS IN FORCED VIBRATION PROBLEMS*

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1. Introduction. In some recent papers [1], [2], [3], [4], [5], Bramble and Hubbard have studied the error in finite difference approximations to solutions of various boundary value problems for second order elliptic partial differential equations. In addition to examining various standard difference operators they analyzed some operators which failed to possess standard properties such as diagonal dominance (cf. [7]).

This paper is concerned with obtaining error estimates for some difference methods for another type of problem which leads, in general, to difference approximations that are not diagonally dominant. The class of problems considered is sometimes referred to as "forced vibration problems."

Specifically we shall consider the problem

$$\begin{aligned} \Delta u + ku &= F \quad \text{in} \quad R, \\ u &= f \quad \text{on} \quad C, \end{aligned} \tag{1.1}$$

where R is an N-dimensional bounded region with sufficiently smooth boundary C, and Δ denotes the Laplace operator. The function k is only required to be bounded and such that the solution u is unique. If, in addition, the function k were restricted to be nonpositive, then quite standard techniques apply in the study of errors in related difference problems. Our concern will be with the more general situation and hence some techniques, other than those most often used for such problems, will be employed. This method was suggested by the work of Payne and the author [6] in which a priori inequalities relating to forced vibration problems were derived.

For the sake of clarity we shall confine our attention to two dimensions. The N-dimensional case is discussed however in §7 as is the extension to more general operators.

2. Notation. As mentioned we shall discuss in detail the case $N = 2$. Thus C is a smooth, simple closed curve in the (x, y)-plane bounding the region R. The plane will be covered with a square grid with mesh width h. The mesh points are the intersection of the grid lines. The set R_h is defined to consist of those mesh points in R whose four nearest neighbors also

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belong to R. Those mesh points in R which do not belong to R_h will make up the set C_h^*. The points of intersection of the grid with C form the set C_h. Finally $\bar{R}_h = R_h \cup C_h^* \cup C_h$.

The following difference operators will be needed. Let $V(x, y)$ be any function defined on $\bar{R}_h$. Then

$$\begin{aligned}\Delta_h V(x, y) &= h^{-2}\{V(x + h, y) + V(x - h, y) \\ &\quad + V(x, y + h) + V(x, y - h) - 4V(x, y)\}, \quad (x, y) \in R_h. \end{aligned} \tag{2.1}$$

This is the usual five point approximation to Δ. If $v \in C^4$ in $\bar{R}$ then

$$| \Delta v(x, y) - \Delta_h V(x, y)| \leqq \frac{h^2}{6} M_4, \qquad (x, y) \in R_h, \tag{2.2}$$

where we have used the notation

$$M_j = \sup_{P \in R}\left\{ \left| \frac{\partial^j v(p)}{\partial x^i \partial y^{j-i}} \right|, i = 0, \cdots, j\right\}. \tag{2.3}$$

At points of C_h^*, Δ_h is defined to be the five point divided difference approximation to Δ. For example, if $(x, y) \in C_h^*$ and if $(x - \alpha h, y)$ and $(x, y - \beta h) \in C_h$ while $(x + h, y)$ and $(x, y + h) \in R_h$, then Δ_h would take the form

$$\begin{aligned}\Delta_h^{(1)} V(x, y) &= 2h^{-2}\left\{\left(\frac{1}{\alpha + 1}\right) V(x + h, y) + \frac{1}{\alpha(\alpha + 1)} V(x - \alpha h, y)\right. \\ &\quad + \left(\frac{1}{\beta + 1}\right) V(x, y + h) + \frac{1}{\beta(\beta + 1)} V(x, y - \beta h) \\ &\quad \left. - \left(\frac{1}{\alpha} + \frac{1}{\beta}\right) V(x, y)\right\}. \end{aligned} \tag{2.4}$$

Because of the definition of C_h^*, $0 < \alpha \leqq 1$ and $0 < \beta \leqq 1$. At each point of C_h^* the appropriate analog of (2.4) is assumed. In this case

$$| \Delta v(x, y) - \Delta_h^{(1)} v(x, y)| \leqq \frac{2h}{3} M_3, \qquad (x, y) \in C_h^*, \tag{2.5}$$

for $v \in C^3$.

We shall consider also a slightly different operator at points of C_h^*. Let

$$\begin{aligned}\Delta_h^{(2)} V(x, y) &= h^{-2}\left\{V(x + h, y) + \frac{1}{\alpha} V(x - \alpha h, y)\right. \\ &\quad \left. + V(x, y + h) + \frac{1}{\beta} V(x, y - \beta h) - \left(\frac{\alpha + 1}{\alpha} + \frac{\beta + 1}{\beta}\right) V(x, y)\right\}. \end{aligned} \tag{2.6}$$

It is easy to see that

$$| \Delta_h^{(2)} v(x, y) | \leqq 2M_2, \qquad (x, y) \in C_h^*,$$

if $v \in C^2$. As is easily seen, the operators (2.1) and (2.6) give rise to a difference analog of (1.1) whose associated matrix is symmetric. For some purposes this seems to be an advantage.

3. Lemmas and preliminaries. In this section we will present some lemmas and results which will be needed in establishing the results in the subsequent sections. The difference analogs of (1.1) which we shall study are

$$\begin{aligned} \Delta_h U(P) + kU(P) &= F(P), &\quad P \in R_h \cup C_h^*, \\ U(P) &= f(P), &\quad P \in C_h. \end{aligned} \tag{3.1}$$

In (3.1) the operator Δ_k on C_h^* is taken to be *either* everywhere of the type $\Delta_h^{(1)}$ or everywhere of the type $\Delta_h^{(2)}$.

The first lemma is the well-known maximum principle.

LEMMA 1. *Let $V(P)$ and $q(P)$ be defined on $\bar{R}_h$ and satisfy*

$$\Delta_h V(P) - q(P)V(P) \geqq 0, \qquad P \in R_h \cup C_h^*, \tag{3.2}$$

with $q \geqq 0$. Then either $V(P) \leqq 0$, $P \in \bar{R}_h$, or

$$\max_{P \in \bar{R}_h} V(P) \leqq \max_{P \in C_h} V(P). \tag{3.3}$$

For a detailed proof, cf. [3].

LEMMA 2. *The solution of* (3.1), *with $k \equiv -q \leqq 0$, exists and is unique for any given F and f.*

Proof. Uniqueness follows immediately from the maximum principle. But for linear systems uniqueness implies existence.

Thus, with Lemma 2, ($q \equiv 0$) we can introduce the discrete analog of the Green's function. Let $G(P, Q)$ satisfy

$$\begin{aligned} \Delta_{h,P} G(P, Q) &= -h^{-2}\delta(P, Q), &\quad P \in R_h \cup C_h^*, \\ G(P, Q) &= \delta(P, Q), &\quad P \in C_h, \end{aligned} \tag{3.4}$$

for $Q \in \bar{R}_h$. The symbol $\Delta_{h,P}$ means that the operator Δ_h operates on variable P and $\delta(P, Q)$ is the Kronecker delta

$$\delta(P, Q) = \begin{cases} 1 & \text{if} \quad P = Q, \\ 0 & \text{if} \quad P \neq Q. \end{cases}$$

By applying Lemma 1 to $-G(P, Q)$ for fixed but arbitrary Q we obtain

$$\max_{P \in \bar{R}_h} -G(P, Q) \leqq \max_{P \in C_h} -G(P, Q) = 0.$$

This yields immediately the following.

LEMMA 3.

$$G(P, Q) \geqq 0, \qquad P, Q \in \bar{R}_h . \tag{3.5}$$

The next lemma gives the useful representation formula for an arbitrary mesh function V in terms of G.

LEMMA 4. *For any* $V(P)$ *defined on* $\bar{R}_h$,

$$V(P) = h^2 \sum_{Q \in R_h \cup C_h^*} G(P, Q)[-\Delta_h V(Q)] + \sum_{Q \in C_h} G(P, Q) V(Q). \tag{3.6}$$

Proof. Let the right-hand side of (3.6) be denoted by $W(P)$. Using (3.4) we see that

$$\Delta_h W(P) = \Delta_h V(P), \qquad P \in R_h \cup C_h^*,$$

$$W(P) = V(P), \qquad P \in C_h .$$

It follows from Lemma 2 that $V(P) \equiv W(P)$, $P \in \bar{R}_h$.

If we take $V(P) \equiv 1$, $P \in \bar{R}_h$, we obtain:

LEMMA 5.

$$\sum_{Q \in C_h} G(P, Q) = 1, \qquad P \in \bar{R}_h . \tag{3.7}$$

The next lemma was given by Bramble and Hubbard in [1]. It is this result which is crucial in showing that less accurate approximations to Δ may be made on C_h^* than on R_h without affecting the overall accuracy.

LEMMA 6.

$$\sum_{Q \in C_h^*} G(P, Q) \leqq 1, \qquad P \in R_h . \tag{3.8}$$

Proof. In (3.6) take $V(P) = 1$ for $P \in R_h \cup C_h^*$, and $V(P) = 0$ for $P \in C_h$. Examination of $\Delta_h^{(1)}$ and $\Delta_h^{(2)}$ shows that

$$-\Delta_h V(P) \geqq h^{-2}, \qquad P \in C_h^*. \tag{3.9}$$

Clearly $\Delta_h V(P) = 0$, $P \in R_h$. Using (3.5) we obtain (3.8).

In obtaining results, even in the case $k \equiv 0$, an additional inequality is required.

LEMMA 7.

$$h^2 \sum_{Q \in R_h \cup C_h^*} G(P, Q) \leqq C, \qquad P \in \bar{R}_h , \tag{3.10}$$

where C is a constant independent of h.

Proof. Let $r(P)$ be the distance from an arbitrary but fixed point. Then $\Delta_h r^2 = \Delta r^2 = 4$. Let $V(P) = -r(P)^2$ in (3.6). The result then follows from Lemmas 3 and 5 and the fact that R is bounded.

We shall use the notation C for a generic constant, independent of h

(and the particular difference problem), but not necessarily always the same.

A simple consequence of Lemmas 4, 5, 6 and 7 is that

$$|V|_{\bar{R}_h} \leqq C|\Delta_h V|_{R_h} + h^2 |\Delta_h V|_{C_h^*} + |V|_{C_h} \tag{3.11}$$

for any V, where we have used the notation

$$|W|_S = \sup_{P \in S} |W(P)| \tag{3.12}$$

for a function W defined on a set S. This is the result given in [1].

In order to treat the case with general k we shall need two additional lemmas giving information about G.

LEMMA 8.

$$h^2 \sum_{Q \in R_h \cup C_h^*} G^2(P, Q) \leqq C, \qquad P \in \bar{R}_h. \tag{3.13}$$

This lemma is proved in [2] for the operator $\Delta_h^{(1)}$ on C_h^*. The argument for $\Delta_h^{(2)}$ on C_h^* follows in precisely the same manner.

Finally we need a bit more information "near the boundary."

LEMMA 9.

$$h^2 \sum_{Q \in R_h \cup C_h^*} G(P, Q) \leqq Ch, \qquad P \in C_h^*. \tag{3.14}$$

Proof. We have assumed that the boundary of R is sufficiently smooth. In this instance we need the function ϕ defined by

$$\begin{aligned} \Delta\phi &= -1 \quad \text{in} \quad R, \\ \phi &= 0 \quad \text{on} \quad C, \end{aligned} \tag{3.15}$$

to exist and possess uniformly bounded third derivatives. Now the mesh function

$$\phi_h(P) = h^2 \sum_{Q \in R_h \cup C_h^*} G(P, Q) \tag{3.16}$$

satisfies

$$\begin{aligned} \Delta_h\phi(P) &= -1, \qquad & P \in R_h \cup C_h^*, \\ \phi(P) &= 0, \qquad & P \in C_h. \end{aligned} \tag{3.17}$$

Taking $V(P) = \phi(P) - \phi_h(P)$ in (3.11), we find that

$$|\phi(P) - \phi_h(P)| \leqq Ch, \qquad P \in \bar{R}_h. \tag{3.18}$$

Hence

$$|\phi_h(P)| \leqq |\phi(P)| + Ch. \tag{3.19}$$

But since $\phi = 0$ on the boundary it follows that

$$(3.20) \qquad | \phi_h(P) | \leqq Ch, \qquad P \in C_h^*,$$

which is the same as (3.14).

4. Associated estimates for the discrete L_2 norm. Let V be any mesh function defined in $\bar{R}_h$. We define

$$(4.1) \qquad \| V \|^2 = h^2 \sum_{Q \in R_h} V(Q)^2.$$

In this section we shall prove the inequality

$$(4.2) \qquad \| V \| \leqq C(\| \Delta_h V + kV \| + | V |_{C_h^*}),$$

where k is a bounded function,

$$(4.3) \qquad | k |_{\bar{R}} \leqq \bar{k},$$

with bound $\bar{k}$.

The uniqueness condition for (1.1) we take in the following form. Consider the eigenvalue problem

$$(4.4) \qquad \begin{aligned} \Delta W + (k - \bar{k})W + \lambda W &= 0 \quad \text{in} \quad R, \\ W &= 0 \quad \text{on} \quad C. \end{aligned}$$

Thus we assume that $\lambda = \bar{k}$ is not an eigenvalue of (4.4). It is well known that the eigenvalues form an increasing sequence $\lambda_1, \lambda_2, \cdots$ of positive real numbers. Analogously we have the discrete problem

$$(4.5) \qquad \begin{aligned} \Delta_h U + (k - \bar{k})U + \mu U &= 0 \quad \text{in} \quad R_h, \\ U &= 0 \quad \text{on} \quad C_h^*. \end{aligned}$$

This is just a symmetric positive definite matrix eigenvalue problem and thus the real positive eigenvalues μ_i with eigenvectors U_i satisfy

$$(4.6) \qquad \begin{aligned} \Delta_h U_i + (k - \bar{k})U_i + \mu_i U_i &= 0 \quad \text{in} \quad R_h, \\ U_i &= 0 \quad \text{on} \quad C_h^*, \end{aligned}$$

and the eigenvectors span the m-dimensional space of vectors indexed by the m points of R_h. Furthermore it is also known (cf. [10]) that if $i = \alpha$, where α is fixed, then

$$(4.7) \qquad \mu_\alpha \to \lambda_\alpha \quad \text{as} \quad h \to 0.$$

Clearly we have, then, for any V,

$$(4.8) \qquad \| V \|^2 = \sum_{i=1}^{m} (h^2 \sum_{P \in R_h} V(P) U_i(P))^2,$$

provided the U_i have been chosen so that

$$h^2 \sum_{P \in R_h} U_i(P) U_j(P) = \delta_{ij}, \tag{4.9}$$

where δ_{ij} is the Kronecker delta.

Now consider for any fixed i,

$$\mu_i h^2 \sum_{P \in R_h} V U_i = -h^2 \sum_{P \in R_h} V[\Delta_h U_i + (k - \bar{k}) U_i], \tag{4.10}$$

where the argument P in the mesh functions will be omitted for simplicity. By virtue of Lemma 2 we may introduce the mesh function H as the solution of

$$\begin{aligned} \Delta_h H + (k - \bar{k}) H &= 0 \text{ in } R_h, \\ H &= V \quad \text{on} \quad C_h^*. \end{aligned} \tag{4.11}$$

Applying (4.10) to both V and H and combining we have

$$\begin{aligned} &\mu_i h^2 \sum_{P \in R_h} V U_i \\ &\quad = -h^2 \sum_{P \in R_h} (V - H)[\Delta_h U_i + (k - \bar{k}) U_i] + \mu_i h^2 \sum_{P \in R_h} H U_i \cdot \end{aligned} \tag{4.12}$$

But since $V - H$ and U_i both vanish on C_h^* we have

$$\begin{aligned} &h^2 \sum_{P \in R_h} (H - V)[\Delta_h U_i + (k - \bar{k}) U_i] \\ &\qquad = -h^2 \sum_{P \in R_h} U_i[\Delta_h V + (k - \bar{k}) V]. \end{aligned} \tag{4.13}$$

Combining (4.12) and (4.13) yields

$$(\mu_i - \bar{k}) h^2 \sum_{P \in R_h} V U_i = -h^2 \sum_{P \in R_h} [\Delta_h V + k V] U_i + \mu_i h^2 \sum_{P \in R_h} H U_i. \tag{4.14}$$

Now since $\lambda \neq \bar{k}$ and $\mu_i \to \lambda_i$ as $h \to 0$, we may choose h_0 sufficiently small that $\mu_i - \bar{k} \neq 0$ for $h < h_0$. Hence

$$\begin{aligned} \sum_{i=1}^{m} \Big(h^2 \sum_{P \in R_h} V U_i\Big)^2 &= \sum_{i=1}^{m} \left[\left(\frac{-1}{\mu_i - \bar{k}}\right) h^2 \sum_{P \in R_h} [\Delta_h V + k V] U_i \right. \\ &\qquad \left. + \left(\frac{\mu_i}{\mu_i - \bar{k}}\right) h^2 \sum_{P \in R_h} H U_i \right]^2 \\ &\leqq 2 \sum_{i=1}^{m} \left(\frac{1}{\bar{k} - \mu_i}\right)^2 \Big(h^2 \sum_{P \in R_h} [\Delta_h V + k V] U_i\Big)^2 \\ &\qquad + 2 \sum_{i=1}^{m} \left(\frac{\mu_i}{\bar{k} - \mu_i}\right)^2 \Big(h^2 \sum_{P \in R_h} H U_i\Big)^2 \\ &\leqq 2C^2(\| \Delta_h V + k V \|^2 + \| H \|^2), \end{aligned} \tag{4.15}$$

where $C = \max(1/\rho, 1 + \bar{k}/\rho)$, $\rho = \min_i |\mu_i - \bar{k}|$. But by Lemma 1 applied to R_h,

$$|H|_{R_h} \leqq |H|_{C_h^*} = |V|_{C_h^*}. \tag{4.16}$$

Combining (4.8), (4.15), and (4.16) there is another constant C such that

$$\|V\| \leqq C(\|\Delta_h V + kV\| + |V|_{C_h^*}). \tag{4.17}$$

5. Estimates for the maximum norm. We start with the representation (3.6) of Lemma 4, adding and subtracting the appropriate terms

$$\begin{aligned} V(P) = &-h^2 \sum_{Q \in R_h \cup C_h^*} G(P, Q)\,[\Delta_h V(Q) + kV(Q)] \\ &+ h^2 \sum_{Q \in R_h \cup C_h^*} G(P, Q) k V(Q) + \sum_{Q \in C_h} G(P, Q) V(Q). \end{aligned} \tag{5.1}$$

Breaking up the sums, using Schwarz's inequality, and the bound for k we have

$$\begin{aligned} |V(P)| \leqq &\Big(h^2 \sum_{Q \in R_h} G^2(P, Q)\Big)^{1/2} \{\|\Delta_h V + kV\| + \bar{k}\|V\|\} \\ &+ \Big(h^2 \sum_{Q \in C_h^*} G(P, Q)\Big)\{|\Delta_h V + kV|_{C_h^*} + \bar{k}|V|_{C_h^*}\} \\ &+ \Big(\sum_{Q \in C_h} G(P, Q)\Big)|V|_{C_h}, \end{aligned} \tag{5.2}$$

where we have used the fact that $G(P, Q) \geqq 0$ (Lemma 3). By virtue of Lemmas 6, 7, and 8 we obtain

$$\begin{aligned} |V|_{\bar{R}_h} \leqq C\{&|\Delta_h V + kV|_{R_h} + \|V\| \\ &+ h^2|\Delta_h V + kV|_{C_h^*} + h^2|V|_{C_h^*} + |V|_{C_h}\}. \end{aligned} \tag{5.3}$$

Now (5.3) together with (4.17) yields

$$|V|_{\bar{R}_h} \leqq \bar{C}\{|\Delta_h V + kV|_{R_h} + h^2|\Delta_h V + kV|_{C_h^*} + |V|_{C_h} + |V|_{C_h^*}\}, \tag{5.4}$$

for some $\bar{C}$. To obtain a bound in terms of only "data" we must still bound $|V|_{C_h^*}$. This may be accomplished by considering again the representation (5.1) with P restricted to lie on C_h^*. Then Lemma 9 yields

$$\begin{aligned} &|V|_{C_h^*} \\ &\leqq \bar{\bar{C}}\{h|\Delta_h V + kV|_{R_h} + h^2|\Delta_h V + kV|_{C_h^*} + |V|_{C_h} + h|V|_{\bar{R}_h}\}, \end{aligned} \tag{5.5}$$

with $\bar{\bar{C}}$ an appropriate constant. Combining (5.4) and (5.5) we obtain

$$|V|_{\bar{R}_h} \leqq C\{|\Delta_h V + kV|_{R_h} + h^2|\Delta_h V + kV|_{C_h^*} + |V|_{C_h}\}, \tag{5.6}$$

provided h is taken strictly less than $\min(h_0, h_1)$, where $h_1 \leqq 1/\bar{C}\bar{\bar{C}}$.

6. Error estimates. Let $|k| \leqq \bar{k}$ and suppose $\bar{k}$ is not an eigenvalue of (4.4). Denote by u the unique solution of (1.1) and suppose that u has continuous fourth partial derivatives in the closure of R. Further define U to be the solution of (3.1) for small h. If $e(P) = u(P) - U(P)$, $P \in \bar{R}_h$, then there is an $\bar{h}$ such that if $h \leqq \bar{h}$,

$$|e|_{\bar{R}_h} \leqq Ch^2. \tag{6.1}$$

This is easily seen by taking $V = e$ in (5.6) and $\bar{h}$ such that (5.6) holds for $h \leqq \bar{h}$. The bound on the first term on the right-hand side is obtained from (2.2):

$$\begin{aligned} |\Delta_h e + ke|_{R_h} &= |(\Delta_h u + ku) - (\Delta_h U + kU)|_{R_h} \\ &= |\Delta_h u + ku - F|_{R_h} = |\Delta_h u - \Delta u|_{R_h} \leqq \frac{h^2}{6} M_4 . \end{aligned} \tag{6.2}$$

A similar consideration for the second term yields

$$|\Delta_h e + ke|_{C_h^*} \leqq \tfrac{2}{3} h M_3 \tag{6.3}$$

in case we are using $\Delta_h^{(1)}$, and

$$|\Delta e + ke|_{C_h^*} \leqq 4M_2 \tag{6.4}$$

if we use $\Delta_h^{(1)}$. In either case the contribution of this term to the error is no worse than $O(h^2)$. The last term on the right vanishes.

Bounds on difference quotients of e can be obtained by exactly those considerations of [5], making use of (5.6). We only state the result here. If P and P^1 are neighboring points of $\bar{R}_h$ let $\delta e(P) = [e(P) - e(P^1)]/h$. Then if $\Delta_h = \Delta_h^{(1)}$ on C_h^*,

$$|\delta e|_{\bar{R}_h} \leqq Ch^2. \tag{6.5}$$

Thus the difference quotients are uniformly $O(h^2)$ if the "better" approximation $\Delta_h^{(1)}$ is used on C_h^*.

7. Extension of the preceding results to higher dimensions, more general second order elliptic operators, and higher order elliptic equations.

(a) N-dimensional case.

If $N \leqq 3$ then all of the preceding considerations are valid, defining, of course, the mesh and operators analogously. For more than three dimensions the only lemma of §3 which is not a straightforward generalization is Lemma 8. However, it is tedious but not difficult to show that

$$\begin{aligned} h^N \sum_{Q \in R_h \cup C_h^*} & r_{PQ}^{\alpha} G^2(P, Q) \\ & + h^{N-2} G(P, P) \leqq C, \qquad P \in \bar{R}_h, \quad N - 2 < \alpha. \end{aligned} \tag{7.1}$$

The failure of Lemma 8 to hold corresponds to the fact that the Green's function is not square integrable for $N > 3$.

Now in §4 all steps are valid for any N. Since we may no longer use (5.2) to derive (5.4) we must modify the proof. We look now at the analog of (5.4):

$$V(P) = h^N \sum_{Q \in R_h} G(P, Q) k V(Q) + T(P), \tag{7.2}$$

where $T(P)$ stands for terms which can be dealt with as before. Now

$$\begin{aligned} | V(P) | \leqq \bar{k} h^N \sum_{\substack{Q \in R_h \\ Q \neq P}} G(P, Q) | V(Q) | \\ + \bar{k} h^N G(P, P) | V(P) | + | T(P) |. \end{aligned} \tag{7.3}$$

Actually we can obtain

$$\begin{aligned} | T |_{\bar{R}_h} &\leqq C \{ | \Delta_h V + kV |_{R_h} + h^2 | \Delta_h V + kV |_{C_h^*} + | V |_{C_h^*} + | V | C_h \} \\ &\equiv T. \end{aligned} \tag{7.4}$$

Thus by virtue of (7.1), with h small enough that $\bar{k} h^N G(P, P) \leqq \bar{k} C h^2 < 1$' we obtain

$$| V(P) | \leqq C \{ h^N \sum_{\substack{Q \in R_h \\ Q \neq P}} G(P, Q) | V(Q) | + T \}. \tag{7.5}$$

Again using (7.1) we obtain

$$V(P)^2 \leqq C \left\{ h^N \sum_{\substack{Q \in R_h \\ Q \neq P}} \frac{V(Q)^2}{r_{PQ}^{\alpha}} + T^2 \right\} \tag{7.6}$$

for $N - 2 < \alpha < N$.

We now extend the definition of $V(P)$ to be piecewise constant on hypercubes of side h centered at P, and zero for $P \notin R_h$.

Now since $r_{PQ}^{-\alpha}$ is a subharmonic function of Q for $P \neq Q$ we have that

$$r_{PQ}^{-\alpha} \leqq \frac{1}{\omega_N h^N} \int_{S_h(Q)} r_{PR}^{-\alpha} \, dv_R, \tag{7.7}$$

where $S_h(Q)$ is the sphere of radius $h/2$ centered at Q and ω_N is the surface area of the N-dimensional unit sphere. Thus since $V(Q)$ is constant on $S_h(Q)$,

$$\begin{aligned} V^2(P) &\leqq C \left\{ h^N \sum_{\substack{Q \in R_h \\ Q \neq P}} \int_{S_h(Q)} \frac{V^2(R)}{r_{PR}^{\alpha}} \, dv_R + T^2 \right\} \\ &\leqq C \left\{ \int_{E_N} \frac{V^2(R)}{r_{PR}^{\alpha}} \, dv_R + T^2 \right\}, \end{aligned} \tag{7.8}$$

for $P \in R_h$, where E_N is N-dimensional euclidean space. It is easy to see that this inequality can be extended to hold for all points P of E_N. Now it is well known (see [8]) that for $\alpha < N$ such an inequality implies

$$V^2(P) \leqq C\left\{\int_{E_N} V^2(Q)\, dv_Q + T^2\right\}. \tag{7.9}$$

(This can be seen by iterating inequality (7.8) once and using the identity of Riesz and Hölder's inequality.) But since V is piecewise constant on cubes and vanishes outside R, (7.9) may be written as

$$| V(P) | \leqq C\{\| V \| + T\}, \qquad P \in R_h, \tag{7.10}$$

or

$$| V |_{R_h} \leqq C\{\| V \| + | \Delta_h V + kV |_{R_h} + h^2 | \Delta_h V + kV |_{C_h^*} + | V |_{C_h}\}. \tag{7.11}$$

This inequality combined with the other appropriate inequalities yields (5.6) for any N.

(b) The previous results carry over to second order self-adjoint operators with variable coefficients provided the difference analog chosen is symmetric in R_h and has a positive discrete Green's function and in addition Lemma 8 is obtainable. For example, for $N = 2, 3$, if no mixed partial derivatives are present a symmetric difference analog with positive Green's function is easily derivable. The question of obtaining Lemma 8 for such difference operators should be studied.

(c) In his paper [9] Thomée studies a general class of difference methods for the Dirichlet problem for elliptic equations with constant coefficients of order $2m$ which contain only the highest derivatives. If L is one of the operators studied by Thomée then the methods used in §4 can easily be used to obtain convergence estimates for problems corresponding to $L + k$, provided k is such that a condition analogous to (3.1) is satisfied and estimates for the eigenvalues in the discrete eigenvalue problem are known. In fact if $| k |$ is less than the first eigenvalue in the continuous problem then the result is complete since Thomée has shown convergence for the first discrete eigenvalue.

REFERENCES

[1] J. H. Bramble and B. E. Hubbard, *On the formulation of finite difference analogues of the Dirichlet problem for Poisson's equation*, Numer. Math., 4 (1962), pp. 313–327.

[2] ———, *A priori bounds on the discretization error in the numerical solution of the Dirichlet problem*, Contributions to Differential Equations, 2 (1963), pp. 229–252.

[3] ———, *On a finite difference analogue of an elliptic boundary problem which is*

neither diagonally dominant nor of non-negative type, J. Math. and Phys., 43 (1964), pp. 117–132.

[4] ———, *New monotone type approximations for elliptic problems*, Math. Comp., 18 (1964), pp. 349–367.

[5] ———, *Approximations of derivatives by finite difference methods in elliptic boundary value problems*, to appear.

[6] J. H. Bramble and L. E. Payne, *Upper and lower bounds in equations of forced vibration type*, Arch. Rational Mech. Anal., 14 (1963), pp. 153–170.

[7] G. Forsythe and W. Wasow, *Finite Difference Methods for Partial Differential Equations*, John Wiley, New York, 1960.

[8] M. Riesz, *L'intégrale de Riemann-Liouville et le probléme de Cauchy*, Acta Math., 81 (1949), pp. 1–223.

[9] V. Thomée, *Elliptic difference operators and Dirichlet's probléme*, Contributions to Differential Equations, 3 (1964), pp. 301–324.

[10] H. F. Weinberger, *Lower bounds for higher eigenvalues by finite difference methods*, Pacific J. Math., 8 (1958), pp. 339–368.