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Bounds for a Class of Linear Functionals with Applications to Hermite Interpolation*

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Abstract. A general estimation theorem is given for a class of linear functionals on Sobolev spaces. The functionals considered are those which annihilate certain classes of polynomials. An interpolation scheme of Hermite type is defined in N-dimensions and the accuracy in approximation is bounded by means of the above mentioned theorem. In one and two dimensions our schemes reduce to the usual ones, however our estimates in two dimensions are new in that they involve only the pure partial derivatives.

1. Introduction

In a recent paper [4] the authors gave estimates for a certain class of linear functionals on Sobolev spaces. These functionals have the property that they annihilate the set of polynomials P_{k-1} of degree $k-1$. The bounds were given in terms of the L_p norms of all k-th order partial derivatives. These results were applied to obtain estimates for the difference between discrete and continuous Fourier transforms as well as to study convergence properties for a class of spline interpolants in E^N.

In this paper the approximation theorems of [4] are generalized in that further results are obtained for linear functionals which annihilate certain classes of polynomials, P_K, intermediate to P_{k-1} and P_k. The estimates only involve L_p norms of those partial derivatives of order k which also annihilate all polynomials in P_K.

These results are applied to the estimation of errors in Hermite interpolation. For one and two dimensions Birkhoff, Schultz and Varga [3] obtained bounds for errors in Hermite interpolation. In the two dimensional case their bounds always involved all partial derivatives of a certain order. It was conjectured by Birkhoff that these bounds could be replaced by similar ones involving only the pure partial derivatives. This result is obtained as an application of our estimates.

For $N \geqq 3$ and any integer m, a class of Hermite interpolants is defined which is $2m$-th order accurate. These functions involve derivatives (of a given function) of order at most $2m-1$ for any N. By applying our estimates, bounds are obtained which involve only a (small) proper subset of the partial derivatives of order $2m$.

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2. Estimation of Linear Functionals

We will consider complex valued functions on a domain R in N dimensional Euclidean space, E^N. It will always be assumed that R is a bounded domain with $\operatorname{diam}(R)=\varrho$ which satisfies a strong cone condition; that is, there exists a finite collection of open subsets $\{\mathcal{O}_i\}$ which covers ∂R and cones $\{C_i\}$ with vertices at the origin such that for any $x\in\mathcal{O}_i\cap R$, $x+C_i$ is contained in R, where ∂R is the boundary of R.

The notations α, β, γ and τ will all be used for multi-indices, with $|\alpha|=\sum_{j=1}^{N}\alpha_j$ and $D^\alpha=D_1^{\alpha_1}\dots D_N^{\alpha_N}$ where $D_j=\partial/i\,\partial x_j$, and $\beta!=\beta_1!\dots\beta_N!$.

As usual we define $L_p(R)$ as the set of all functions u such that $\int_R |u(x)|^p dx$ exists and is finite. The norm on L_p is given by $\|u\|_{p,R}=\left(\int_R |u(x)|^p dx\right)^{1/p}$. We will also use the notation (u,v) for $\int_R u(x)\,\overline{v(x)}\,dx$ where $u\in L_p(R)$, $v\in L_{p'}(R)$ and $\frac{1}{p}+\frac{1}{p'}=1$. By $H_p^m(R)$ we mean the set of all functions in $L_p(R)$ whose distributional derivatives of order less than or equal to m (a non-negative integer) are in $L_p(R)$. Henceforth it will be assumed that p lies in the interval $[1,\infty\rangle$. The norm on $H_p^m(R)$ is $\|u\|_{m,p,R}=\sum_{|\alpha|\le m}\|D^\alpha u\|_{p,R}$. We will also use the semi-norm $|u|_{k,p,R}=\sum_{|\alpha|=k}\|D^\alpha u\|_{p,R}$. As usual $C^j(R)$ will denote the space of functions with continuous derivatives up to and including order j in R.

We shall also need the following results due to Rellich, and Aronszajn and Smith. Rellich's lemma states that bounded subsets of $H_p^m(R)$ are conditionally compact as subsets of $H_p^{m-1}(R)$. Thus, if $\{u_n\}$ is a bounded sequence of elements of $H_p^m(R)$, there is a subsequence $\{u_{n_j}\}$ such that $\{u_{n_j}\}$ converges in $H_p^{m-1}(R)$. This is proved in Agmon [1, p. 30] for the special case $p=2$. His proof is easily generalized to arbitrary p greater than or equal to one.

Aronszajn and Smith have shown that for any $u\in H_p^m(R)$ there is a constant C independent of u such that

$$\|u\|_{m,p,R}\le C\left(\sum_{j=1}^{N}\|D_j^m u\|_{p,R}+\|u\|_{p,R}\right).$$

The result is essentially proved by Smith [6].

Let K be any subset of the set of multi-indices γ of length k, (i.e., $|\gamma|=k$) which contains the indices of the form $\gamma_\ell=k$, $\gamma_j=0$ for $j\ne\ell$, $\ell=1,\dots,N$. The set of polynomials q such that $D^\tau q=0$ for all $\tau\in K$ will be denoted by P_K. (We always assume all functions are restricted to the domain R.) Now form the quotient space $Q=H_p^k(R)/P_K$. It is well known that under the norm

$$\|[u]\|_Q=\inf_{v\in P_K}\|u+v\|_{k,p,R},$$

where $[u]$ is the equivalence class containing u, Q is a Banach space.

We will now prove the main theorem of this section.

Theorem 1. Let $u\in H_p^k(R)$ and assume that $\operatorname{diam}(R)=1$. Then there is a constant C independent of u such that

$$\sum_{\tau\in K}\|D^\tau u\|_{p,R}\le\|[u]\|_Q\le C\sum_{\tau\in K}\|D^\tau u\|_{p,R}.$$

Proof. Now since any polynomial q in P_K is such that $D^\tau q = 0$ for all $\tau \in K$, we have

$$\sum_{\tau\in K} \|D^\tau u\|_{p,R} = \sum_{\tau\in K} \|D^\tau (u + q)\|_{p,R} \leqq \|u+q\|_{k,p,R} \quad \text{for all} \quad q \in P_K.$$

Taking the infimium over P_K we obtain $\sum_{\tau\in K} \|D^\tau u\|_{p,R} \leqq \|[u]\|_Q$, which is the first inequality.

Define a sequence of sets of multi-indices by $K(0) \equiv K$,

$$K(j) = K \cup \{\gamma(1)\} \cup \dots \cup \{\gamma(j)\} \quad \text{for} \quad j = 1, \dots, s$$

where $\{\gamma(1), \dots, \gamma(s)\}$ is the set of multi-indices of length k, arranged in any order, which are not in K. We now define two sets of polynomials. Let

$$P_{K(j)} = \{q \mid D^\tau q = 0 \text{ for } \tau \in K(j)\} \quad \text{and} \quad P_{\gamma(j)} = \{r \mid r = D^{\gamma(j)} q \text{ for } q \in P_{K(j-1)}\}.$$

We need the following lemma, whose proof will be deferred until after the theorem is proved.

Lemma 1. Let $u \in H_p^k(R)$. Then there exists a polynomial $q \in P_K$ such that

1) $\left(r, D^{\gamma(j)}(u+q)\right) = 0$ for all $r \in P_{\gamma(j)}$ for $j = 1, \dots, s$

and

2) $\left(D^\gamma(u+q), 1\right) = 0$ for all γ such that $|\gamma| \leqq k-1$.

We will now finish the proof of the theorem assuming that u satisfies conditions 1) and 2) of Lemma 1. The lemma states that there is an element of the class $[u]$ which satisfies conditions 1) and 2). So we want to show that there is a constant C independent of u such that $\|[u]\|_Q \leqq C \sum_{\tau\in K} \|D^\tau u\|_{p,R}$. Note that both sides of the inequality are invariant under translation by any element of P_K, so it is sufficient to prove the statement for the case when u satisfies condition 1) and 2) of Lemma 1.

Assume there is no such constant; then it follows that there is a sequence $\{u_n\}$ such that $\|u_n\|_{p,k,R} = 1$ and u_n satisfies 1) and 2) for each n and $\|D^\tau u_n\|_{p,R} \to 0$ as $n \to \infty$ for all $\tau \in K$. Since $\{u_n\}$ is bounded in $H_p^k(R)$, by the Rellich lemma, there is a subsequence, which we will also denote by $\{u_n\}$, such that $\{u_n\}$ converges in $H_p^{k-1}(R)$ and hence in $L_p(R)$. Now by the result of Aronszajn and Smith since $\|D^\tau u_n\|_{p,R} \to 0$, we have that $\{u_n\}$ converges in $H_p^k(R)$. We will denote by $\tilde{u}$ the limit of $\{u_n\}$ in $H_p^k(R)$. Also $\tilde{u}$ will satisfy conditions 1) and 2) because of the result of Aronszajn and Smith.

We shall first show that $D^{\gamma(1)}\tilde{u} = 0$. For each $\tau \in K(0) \equiv K$, let β_τ be the unique multi-index of minimum length such that $\beta_\tau + \gamma(1) = \beta_\tau^* + \tau$ with β_τ^* a multi-index. Now for any function $\varphi \in C_0^\infty(R)$, we have $(D^{\beta_\tau}\varphi, D^{\gamma(1)}\tilde{u}) = (D^{\beta_\tau^*}\varphi, D^\tau \tilde{u}) = 0$. Since $K(0)$ contains all derivatives of length k in each coordinate direction, it follows that $D^{\gamma(1)}\tilde{u}$ is almost everywhere equal to a polynomial. Further, $D^{\gamma(1)}\tilde{u} = \sum_\alpha a_{\gamma(1)\alpha} x^\alpha$, a.e., where $\alpha \in \{\alpha \mid$ for each $\tau \in K(0)$, $\exists j$ with $\alpha_j \leqq \beta_{\tau j} - 1\}$. Now for each such j, $\beta_{\tau j} > 0$ and hence, because β_τ is minimal,

$\beta^*_{\tau_j}=0$ so that $\beta_{\tau_j}=\tau_j-\gamma(1)_j$. Thus we have $D^{\gamma(1)}\tilde{u}=\sum_\alpha a_{\gamma(1)\alpha}x^\alpha$, a.e., where

$$\alpha\in\{\alpha|\text{ for each }\tau\in K(0),\ \exists j\text{ such that }\alpha_j\leqq\tau_j-\gamma(1)_j-1\}.$$

Hence, $D^{\gamma(1)}\tilde{u}=D^{\gamma(1)}\left(\sum_\alpha b_{\gamma(1)\alpha}x^\alpha\right)$, a.e., for $\alpha\in\{\alpha|$ for each $\tau\in K(0)$, $\exists i$ such that $\alpha_i\leqq\tau_i-1\}$. Clearly this implies $D^\tau\left(\sum_\alpha b_{\gamma(1)\alpha}x^\alpha\right)=0$ for all $\tau\in K(0)$. So we have $D^{\gamma(1)}\tilde{u}=D^{\gamma(1)}q$ a.e. for some $q\in P_{K(0)}$, hence $D^{\gamma(1)}\tilde{u}$ is equal almost everywhere to an element of $P_{\gamma(1)}$. But by condition 1), this means $D^{\gamma(1)}\tilde{u}=0$, a.e.

Now consider $D^{\gamma(2)}\tilde{u}$, using $K(1)$ in place of $K(0)$ and $P_{\gamma(2)}$ instead of $P_{\gamma(1)}$ in the argument of the preceeding paragraph, we obtain $D^{\gamma(2)}\tilde{u}=0$, a.e. For each $\gamma(j)$ using $K(j-1)$ and the above argument we can obtain finally $D^{\gamma(j)}\tilde{u}=0$, a.e. for $j=1,\ldots,s$. But we already know $D^\tau\tilde{u}=0$, a.e., for all $\tau\in K$. Thus, $D^\gamma\tilde{u}=0$, a.e., for all multi-indices of length k.

Next, let β be an arbitrary multi-index of length $k-1$. For any multi-index α of length one and any $\varphi\in C_0^\infty(R)$ we have $(D^\alpha\varphi, D^\beta\tilde{u})=(\varphi, D^{\alpha+\beta}\tilde{u})=0$. Hence, $D^\beta\tilde{u}$ is equal to a constant, a.e. But by condition 2) this means $D^\beta\tilde{u}=,0$, a.e., for each multi-index β of length $k-1$. Similarly, we find that $D^\gamma\tilde{u}=0$, a.e., for all multi-indices of length less than or equal to k. Thus $\|\tilde{u}\|_{k,p,R}=0$. But since each u_n has norm 1 in $H_p^k(R)$, this is a contradiction. Hence, there is a constant C such that $\|u\|_{k,p,R}\leqq C\sum_{\tau\in K}\|D^\tau u\|_{p,R}$. Thus, $\|[u]\|_Q\leqq C\sum_{\tau\in K}\|D^\tau u\|_{p,R}$ where C is independent of u, which proves the theorem. Now we will prove Lemma 1.

For $u\in H_p^k(R)$, consider $(r_j, D^{\gamma(1)}u)=c_j$ where $\{r_j\}$ is an orthonormal basis for $P_{\gamma(1)}$ (with respect to $(u,v)=\int u\bar{v}$). Let q_0 be a polynomial in $P_{K(0)}$ such that $D^{\gamma(1)}q_0=\sum_j\bar{c}_j r_j$. Then $\left(r_j, D^{\gamma(1)}(u-q_0)\right)=(r_j, D^{\gamma(1)}u)-c_j=0$ for all j. Hence, $\left(r, D^{\gamma(1)}(u-q_0)\right)=0$ for all $r\in P_{\gamma(1)}$. In the same manner we can choose $q_1\in P_{K(1)}$ such that $\left(r, D^{\gamma(2)}(u-q_0-q_1)\right)=0$ for all $r\in P_{\gamma(2)}$. But by the definition of $P_{K(1)}$ we have $D^{\gamma(1)}q_1=0$, so that $\left(r, D^{\gamma(1)}(u-q_0-q_1)\right)=0$ for all $r\in P_{\gamma(1)}$. Proceeding in this way, we can find $\tilde{q}\in P_K$ such that

$$\left(r, D^{\gamma(j)}(u-\tilde{q})\right)=0\quad\text{for all}\quad r\in P_{\gamma(j)}\quad\text{for}\quad j=1,\ldots,s.$$

Now for each multi-index β of length $k-1$, let $\beta!\,c_\beta=\left(D^\beta(u-\tilde{q}),1\right)$. Then $\left(D^\beta[(u-\tilde{q})-\sum_{|\alpha|=k-1}c_\alpha x^\alpha],1\right)=0$, for all β with $|\beta|=k-1$. Clearly for any index γ of length k, $D^\gamma\left(\sum_{|\alpha|=k-1}c_\alpha x^\alpha\right)=0$. Proceeding inductively we find a polynomial $q\in P_K$ such that conditions 1) and 2) of the lemma are satisfied.

We will now apply our theorem to linear functionals on Sobolev spaces.

Theorem 2. Let F be a linear functional on $H_p^k(R)$ satisfying

a) $|F(u)|\leqq C\sum_{j=0}^k\varrho^{j-N/p}|u|_{j,p}$ where C is independent of ϱ and u

and

b) $F(q)=0$ for $q\in P_K$.

Then there is a constant C_1 independent of ϱ and u such that

$$|F(u)|\leqq C_1\varrho^{k-N/p}\sum_{\tau\in K}\|D^\tau u\|_{p,R}.$$

Proof. We will first prove the theorem for the case when $\varrho = \operatorname{diam}(K) = 1$. In this case the right hand side of a) becomes $\|u\|_{k,p,R}$, so we have $|F(u)| = |F(u+q)| \leqq C\|u+q\|_{k,p,R}$ for any $q \in P_K$. By taking the infinium over the set P_K we have $|F(u)| \leqq C\|[u]\|_Q$ where $Q = H_p^k(R)/P_K$. But by Theorem 1, $\|[u]\|_Q \leqq C_2 \sum_{\tau \in K} \|D^\tau u\|_{p,R}$, which proves Theorem 2 for the case $\varrho = 1$.

Now for any arbitrary R, we choose the coordinate system such that one of the extreme points of R is the origin. (By extreme point we mean a point x on ∂R, such that there is a point $y \in \partial R$ and $|x-y| = \varrho$.)

So R is contained in a sphere of radius ϱ with center at the origin. Now for $|x| \leqq 1$ let $u(\varrho x) = v(x)$. This gives a correspondence between functions in $H_p^k(R)$ and $H_p^k(\tilde{R})$ where $\tilde{R}$ has diameter 1. Now $\int_{\tilde{R}} |D^\alpha v|^p dx = \varrho^{|\alpha|p - N} \int_{\tilde{R}} |D^\alpha u|^p d\tau$. So a) becomes $|F(u)| \leqq C\|v\|_{k,p}$ where C is independent of ϱ and u. Define a functional $\bar{F}$ on $H_p^k(\tilde{R})$ by $\bar{F}(v) = F(u)$, then by b) $F(p) = 0$ for $p \in P_K$ restricted to $\tilde{R}$. Hence by the first part of the proof $|\bar{F}(v)| \leqq C_1 \sum_{\tau \in K} \|D^\tau v\|_{p,\tilde{R}}$ where C is independent of ϱ and v. But $\|D^\tau v\|_{p,\tilde{R}} = \varrho^{k-N/p} \|D^\tau u\|_{p,R}$, $\bar{F}(v) = F(u)$, and since we have the same constant for any ϱ or u, the theorem is proved.

By applying Sobolev's lemma we obtain the following corollary.

Corollary. Let F be a linear functional on $C^j(R)$ such that $0 \leqq j < k$

a') $|F(u)| \leqq C \sum_{l=0}^{j} \varrho^l |u|_{l,R}$ where C is independent of ϱ and u

and

$$|u|_{l,R} = \sup_{x \in R} \sum_{|\alpha| = l} |D^\alpha u(x)|$$

and

b) $F(q) = 0$ for all $q \in P_K$.

Then for $p > N/(k-j)$ there is a constant C_1 independent of ϱ and u such that

$$|F(u)| \leqq C_1 \varrho^{k-N/p} \sum_{\tau \in K} \|D^\tau u\|_{p,R}.$$

Proof. By Sobolev's lemma (Morrey [5, p. 78]) for $p > N/(k-j)$ condition a') of the Corollary implies condition a) of Theorem 2. Hence, using Theorem 2, the corollary follows.

3. Hermite Interpolation in N-Dimensions

We will now apply our theorems to the problem of Hermite interpolation in N dimensions. Consider a hypercube I in E^N such that $\operatorname{diam}(I) = 1$ and I has the origin as a vertex. So I is of the form $I = \{x \in E^N \mid 0 \leqq x_j \leqq 1/\sqrt{N}$ for $j = 1, \ldots, N\}$. If at each vertex of the cube we assign m^N values, which will correspond to the values of the derivatives of order j in each variable where $0 \leqq j \leqq m-1$, then we can solve the resulting system of equations and obtain the coefficients of a unique polynomial of degree at most $2m-1$ in each variable. This polynomial's derivatives at the vertices will match the given values for any derivative of order j in each variable for $0 \leqq j \leqq m-1$. The proof of this result is an easy generalization of the proof for the case $N=2$ due to Ahlin [2].

Now we define the m-th Hermite interpolate of a function u, denoted by u_m, to be the polynomial of degree at most $2m-1$ in each variable such that for all γ with $0 \leqq \gamma_j \leqq m-1$ for $j=1, \ldots, N$

$$D^\gamma u_m(x) = \begin{cases} D^\gamma u(x) & \text{for} \quad |\gamma| < 2m \\ 0 & \text{for} \quad |\gamma| \geqq 2m \end{cases}$$

where x is any vertex of I. Now for $N \geqq 3$ if $u \in C^{2m-1}(I)$ then there is a unique u_m. If $N=2$ then $u \in C^{2m-2}(I)$ is sufficient. We note that if $N=1$ or 2, our definition coincides with the usual definitions for Hermite interpolates. The generalization to arbitrary hyper-rectangles is obvious.

Now for any point $y \in I$, we define a linear functional by $F(u; y) = u_m(y) - u(y)$.

By Sobolev's lemma, the functional is defined and bounded for $u \in H_p^{2m}(I)$ where $p > N$ (if $N=2$, $p > 1$ is sufficient). Now since the m-th Hermite interpolate is unique, it is easy to see that if q is a polynomial such that the total degree of q is at most $2m-1$ or if $D^\gamma q = 0$ for all indices γ such that $|\gamma| \geqq 2m$ and $0 \leqq \gamma_j \leqq m-1$ for $j=1, \ldots, N$, and q is of degree at most $2m-1$ in each variable, then $q_m = q$. We denote by P_k the set of polynomials q such that the Hermite interpolate of q is q. Thus $P_k = \{q \mid q_m = q\}$ and as in Section 2, $K = \{\tau \mid |\tau| = 2m,\ D^\tau q = 0$ for all $q \in P_k\}$. Hence $F(q; y) = 0$ for all $q \in P_K$ and all $y \in I$.

Since $|F(u; y)| \leqq C |u|_{2m-1, I}$ where C is independent of y and u, by the corollary of Theorem 2 we have

$$|F(u; y)| \leqq C \sum_{\tau \in K} \|D^\tau u\|_{p, I_h}$$

where C is independent of y and u, and $p > N$.

If we consider a cube I_h of diameter h, by a change of variables we easily obtain for $u \in H_p^{2m}(I_h)$

$$|F(u; y)| \leqq C h^{2m - N/p} \sum_{\tau \in K} \|D^\tau u\|_{p, I_h} \quad \text{for all} \quad y \in I_h,$$

where $p > N$ and K is defined as above.

Consider a domain R consisting of the union of a finite number of sets $R_\ell = \prod_{j=1}^{N} [a_j^{(\ell)}, b_j^{(\ell)}]$ for $\ell = 1, \ldots, s$. So $R = \bigcup_{\ell=1}^{s} R_\ell$, furthermore we can choose R_ℓ such that $R_i \cap R_j$ is empty or part of the boundary of R_i for $1 \leqq i, j \leqq s$. Now let $\mathscr{C}$ be a collection of partitions of R. Then any $\pi \in \mathscr{C}$ induces a partition on R_ℓ denoted by $\pi^{(\ell)}$. In turn each partition $\pi^{(\ell)}$ induces a partition $\varrho_j^{(\ell)}$ of $[a_j^{(\ell)}, b_j^{(\ell)}]$. If the points of the partition $\varrho_j^{(\ell)}$ are denoted by $X_{i,j}^{(\ell)}$, then define

$$R_{\ell i} = \prod_{j=1}^{N} [X_{i,j}^{(\ell)}, X_{i+1,j}^{(\ell)}].$$

Now $R_\ell = \bigcup_i R_{\ell,i}$ for all ℓ. We define

$$\underline{\varrho}_j^{(\ell)} = \inf_i |X_{i,j}^{(\ell)} - X_{i+1,j}^{(\ell)}| \quad \text{and} \quad \bar{\varrho}_j^{(\ell)} = \sup_i |X_{i,j}^{(\ell)} - X_{i+1,j}^{(\ell)}|.$$

Following [3, p. 244] we say that $\mathscr{C}$ is a regular collection of partitions of R if and only if there exists three positive constants σ, θ, and η such that for all $\pi\in\mathscr{C}$, $\underline{\varrho}_j^{(\ell)}\geqq\sigma\bar{\varrho}_j^{(\ell)}$ and $\theta\leqq\bar{\varrho}_j^{(\ell)}/\bar{\varrho}_k^{(\ell)}\leqq\eta$ for $\ell=1,\ldots,s$ and $j, k=1,\ldots,N$. Finally we define $\nu=\max\limits_{j,\ell}(\bar{\varrho}_j^{(\ell)})$ and note that $\operatorname{diam}(R_{\ell,i})\leqq\sqrt{N}\nu$ for all ℓ and i, and denote by u_m the function which coincides with the Hermite interpolate of u in each $R_{\ell i}$.

Theorem 3. Let $u\in H_p^{2m}(R)$ where $R=\bigcup\limits_{\ell=1}^{s}R_\ell$ and each R_ℓ is as above, and let $\mathscr{C}$ be a regular collection of partitions of R. Then for any $\pi\in\mathscr{C}$ and $p>N$ (for $N=2$, $p>1$) there exists a constant C independent of π and u such that

$$\|u_m-u\|_{j,p,R}\leqq C\,\nu^{2m-j}\sum_{\tau\in K}\|D^\tau u\|_{p,R}\quad\text{for}\quad 0\leqq j\leqq m. \tag{3.1}$$

Proof. Since $\mathscr{C}$ is regular, there is a constant C independent of u and $R_{\ell,i}$ such that for any $y\in R_{\ell,i}$ $|F(u;y)|\leqq C\nu^{2m-N/p}\sum\limits_{\tau\in K}\|D^\tau u\|_{p,R_{\ell,i}}$. Raising F to the p-th power and integrating over $R_{\ell,i}$ and summing over i and ℓ we obtain $\|u_m-u\|_{p,R}\leqq C\nu^{2m}\sum\limits_{\tau\in K}\|D^\tau u\|_{p,R}$ for any $\pi\in\mathscr{C}$ where C is independent of ν. It is easy to see by a similar argument that

$$\|D^j(u_m-u)\|_{p,R}\leqq C\,\nu^{2m-j}\sum_{\tau\in K}\|D^\tau u\|_{p,R}\quad\text{for}\quad 0\leqq j\leqq m.$$

Thus, the theorem is proved.

Now consider the case $N=2$. If $u\in H_p^{2m}$ then for $p>1$, $u\in C^{2m-2}$ by Sobolev's lemma, so we can form u_m. The set of indices K such that any $\tau\in K$ annihilates $q\in P_k$ is the set of indices of length $2m$ which annihilate all polynomials of degree at most $2m-1$ in each variable. Hence, K consists only of $(0, 2m)$ and $(2m, 0)$ and we can bound $\|u_m-u\|_{p,R}$ by $C\nu^{2m}\sum\limits_{j=1}^{2}\|D_j^{2m}u\|_{p,R}$.

Now the final remark we wish to make is that the results of Section 2 always have a non-trivial application to Hermite interpolation in N dimensions for $N>1$. We will show that for $N\geqq 3$ there are always derivatives of order $2m$ which are not included in the error bound (3.1). Now the only derivatives which are needed in (3.1) are those which annihilate all elements of P_k. Let α be any index of the form $\alpha_\ell=2m-1$, $\alpha_j=1$ for $\ell\neq j$ (with j and ℓ fixed) and $\alpha_i=0$, $i\neq\ell, j$. So $|\alpha|=2m$ and it is easy to see that the interpolate of x^α is x^α. Hence, $x^\alpha\in P_k$. So, D^α will not enter into the error bound. In fact, the number of derivatives which can be deleted from the error bound is usually quite large, as the following example illustrates.

Let $m=N=3$, take $u\in H_p^6(R)$ with $p>3$. Then we obtain

$$\|u_3-u\|_{p,R}\leqq C\,\nu^6\sum_{\tau\in K}\|D^\tau u\|_{p,R}.$$

The set K consists only of the indices $(0, 0, 6)$, $(0, 6, 0)$, $(6, 0, 0)$ and $(2, 2, 2)$. This is true since any other index α of length 6 has at least one component $\geqq 3$ and therefore $x^\alpha\in P_k$. Hence for the bound we only need 4 derivatives of the 28 different derivatives of order 6.

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