J. SIAM Numer. Anal.
Ser. B, Vol. 2, No. 1
Printed in U.S.A., 1964

A FINITE DIFFERENCE ANALOG OF THE NEUMANN PROBLEM FOR POISSON'S EQUATION*

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1. Introduction. This paper is concerned with estimates for the order of convergence of certain discrete analogues of the Neumann problem for Poisson's equation. Compared with the Dirichlet problem this subject has received little attention in the literature. Three papers [12], [15], and [25] are of particular importance.

The paper by J. Geise [15] gives two finite difference analogues of the Neumann problem for Poisson's equation on a rectangle with error in approximating $\partial u/\partial n$ which are $O(h^2)$ and $O(h)$ respectively. The method of Fourier series is used to obtain estimates for the total error which are $O(h^2 \mid \log h \mid)$ and $O(h \mid \log h \mid)$. K. O. Friedrichs [12] formulates a finite difference analog for the Neumann problem for elliptic systems of second order and shows convergence in the mean. The method of approach used there is to apply a variational principle for the differential equation to piecewise linear functions. Volkov [25] derives estimates of the type $O(h^2 \log^2 h)$.

In this paper a finite difference analog is formulated with $O(h^2)$ local truncation error in such a manner that the matrix of the resulting linear system is of "positive type". A maximum principle is then applied to yield estimates for the order of convergence which are $O(h^2 \mid \log h \mid)$. An $O(h \mid \log h \mid)$ analog is also treated. In §3 examples are given which show that these estimates are sharp.

We note that by [9] the estimate (2.40) on $e(p)$ arising from (2.17) can be used to show that the various differences of $e(p)$ have the same order of convergence, i.e., $O(h^2 \mid \log h \mid)$ on any compact subset of R. Similar results hold for differences of $e(p)$ arising from (2.43).

2. The finite difference analog. Consider the Neumann problem for Poisson's equation:

$$
\begin{aligned}
-\Delta u &= f \qquad \text{in } R, \\
\frac{\partial u}{\partial n} &= g \qquad \text{on } C,
\end{aligned}
\tag{2.1}
$$

where R is a bounded connected region with sufficiently smooth boundary C and $\partial/\partial n$ denotes the normal derivative taken in the outward direction.

* Received by the editors November 15, 1963, and in revised form July 13, 1964.

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It follows immediately from Green's first identity that f and g cannot be chosen independently, but that they must satisfy the relation

$$\int_R f\,dv = \int_C g\,ds. \tag{2.2}$$

Furthermore it is clear that the solution of (2.1) is unique only up to an additive constant. This constant is normally determined by a normalization, e.g.,

$$\int_C u\,ds = 0. \tag{2.3}$$

Hence we shall consider the problem solved once any solution of (2.1) is obtained.

The authors [2], [3], [4], [7], [8] have previously posed finite difference analogues of the Dirichlet and mixed boundary value problems for elliptic operators. In each of these cases the matrix of the resulting system of linear equations was shown to be nonsingular. In fact, either the matrix of the original system or that of a "reduced" system was shown to have an inverse with nonnegative elements.

However, in the present situation, if one formulates a finite difference analog following the usual rules, then the matrix of the resulting system will be singular with rank one less than its order and a condition of the type (2.2) would need to be imposed on the right side of the linear system to insure consistency. Since the dimension of the null space of the matrix is one, we see that the solution would be unique only up to a vector in the null space, thus imitating the situation in (2.1).

We shall now define certain finite difference analogues of the differential operators involved in (2.1). First we place a square mesh of width h on the region R and call the intersections as well as the boundary crossings "mesh points". The set R_h will consist of those mesh points in R whose four nearest neighbors are also in R. The remaining mesh points in R will make up the set C_h^*. We note that each point of C_h^* has at least one adjacent point which is a boundary crossing. The boundary crossings themselves will be denoted by C_h. If we desire a uniform discretization error then previous experience (e.g., [3], [4], [7], [8]) indicates that the local truncation error, i.e., the error committed by substituting finite difference operators for differential operators, need not be uniform. In fact if we select the local truncation error on R_h and C_h to be $O(h^2)$ then experience indicates that we need only choose an $O(h)$ approximation at points of C_h^*. Hence, we define the following operators. At a point (x, y) of R_h let

$$\begin{aligned}\Delta_h V(x, y) \equiv h^{-2}\{&V(x + h, y) + V(x - h, y)\\ &+ V(x, y + h) + V(x, y - h) - 4V(x, y)\}.\end{aligned} \tag{2.4}$$

It is well known that if $u \in C^4(\bar{R})$, then

$$|\Delta u(p) - \Delta_h u(p)| \leqq M_4 h^2, \qquad p \in R_h, \tag{2.5}$$

where M_4 is a constant depending on the fourth derivatives of u. On C_h^* we use the operator of Shortley and Weller [19]. For example, if (x, y) is a point of C_h^* with $(x - \lambda h, y)$, $(x, y - \mu h) \in C_h$, where $0 < \lambda, \mu < 1$, then

$$\begin{aligned}\Delta_h V(x, y) \equiv 2h^{-2} \Big\{ & \frac{1}{\lambda(1 + \lambda)} V(x - \lambda h, y) + \frac{1}{1 + \lambda} V(x + h, y) \\ & + \frac{1}{\mu(1 + \mu)} V(x, y - \mu h) + \frac{1}{1 + \mu} V(x, y + h) \\ & - \left(\frac{1}{\lambda} + \frac{1}{\mu}\right) V(x, y) \Big\}.\end{aligned} \tag{2.6}$$

This is simply the five point divided difference analog of Δ and it is easily verified that

$$|\Delta_h u(p) - \Delta u(p)| \leqq M_3 h, \qquad p \in C_h^*, \tag{2.7}$$

where M_3 depends on the third derivatives of u.

The construction of an $O(h^2)$ approximation to the normal derivative at points of C_h is more complicated, particularly if we wish the resulting difference operator to be of "positive type" [11]. Many different authors have proposed analogues to $\partial u/\partial n$ among whom we might mention Batschelet [1], Friedrichs [12], Fox [13], Shaw [20], Uhlmann [22], and Viswanathan [24]. For a good account of this work the reader is referred to the recent book by L. Fox [13]. We shall use still another analog proposed by the authors [8]. In the above paper the authors prove that within a circle of radius βh about each point of C_h one can find three points of $R_h + C_h^*$ such that the resulting four point finite difference analog of $\partial u/\partial n$ has an $O(h^2)$ local truncation error. Furthermore the coefficients corresponding to the three points of $R_h + C_h^*$ will be nonpositive and that of the point of C_h will be positive, thus giving rise to an equation which is of positive type. As we shall see, this property is useful when showing that the inverse matrix of the finite difference problem exists and is nonnegative.

We shall consider an arbitrary point 0 on C at which C is smooth. Choose 0 to be the origin of a Cartesian coordinate system (x, y) such that the x-axis is tangent to C at 0. The positive y direction is taken along the inward normal. It can be shown, for any smooth enough function v, that

$$v_{xy} = -v_{ns} + K v_s \tag{2.8}$$

at the origin (cf. Synge and Schild [21] for the use of geodesic normal coordinates). The subscripts denote the indicated partial differentiation, n being the outward normal direction, s arc length, and K the curvature of

C. Thus we have

$$v_{xy} = -\frac{\partial}{\partial s}(v_n) + Kv_s \tag{2.9}$$

at 0. Also, of course,

$$v_{yy} = \Delta v - v_{xx}. \tag{2.10}$$

Now consider the Taylor expansion of v about 0; i.e.,

$$\begin{aligned} v(P) &= v(0) + xv_x(0) + yv_y(0) \\ &\quad + \tfrac{1}{2}\{x^2 v_{xx}(0) + 2xyv_{xy}(0) + y^2 v_{yy}(0)\} + O(x^3 + y^3). \end{aligned} \tag{2.11}$$

We note that $v_x(0) = v_s(0)$ and $v_y(0) = -v_n(0)$. Thus, using (2.9) and (2.10),

$$\begin{aligned} v(P) &= v(0) + x[1 + yK(0)]v_s(0) - yv_n(0) \\ &\quad + \frac{1}{2}[x^2 - y^2]v_{xx}(0) + \frac{y^2}{2}\Delta v(0) - xyv_{ns} + O(x^3 + y^3) \end{aligned} \tag{2.12}$$

Let $P_i = (x_i, y_i) \in R_h + C_h^*$, $i = 1, 2, 3$. We wish to determine three numbers A_i, $i = 1, 2, 3$, such that

$$\begin{aligned} \sum_{i=1}^{3} A_i\{v(p_i) - v(0)\} &= -v_n(0) \\ &\quad + \sum_{i=1}^{3} A_i\left\{\frac{y_i^2}{2}\Delta v(0) - x_i y_i v_{ns}(0)\right\} + O\left(\sum_{i=1}^{3} A_i[x_i^3 + y_i^3]\right). \end{aligned} \tag{2.13}$$

For (2.13) to hold for any v, A_i must satisfy

$$\begin{gathered} \sum_{i=1}^{3} A_i y_i = 1, \\ \sum_{i=1}^{3} A_i x_i[1 + y_i K(0)] = 0, \\ \sum_{i=1}^{3} A_i[x_i^2 - y_i^2] = 0. \end{gathered} \tag{2.14}$$

In [8] it is shown that the points P_i may be chosen so that $A_i \geqq 0$, if h is sufficiently small. Thus we define the boundary operator

$$\delta_n V(p) = \sum_{i=1}^{3} A_i\{V(p) - V(p_i)\}. \tag{2.15}$$

Furthermore as a consequence of (2.13) and the fact that $A_i = O(h^{-1})$ as is proved in [8], it can be shown that

$$\left| \delta_n u(p) - \left\{ g(p) + \sum_{i=1}^{3} \left[A_i \frac{y_i^2}{2} f(p) + x_i y_i \frac{\partial g(p)}{\partial s} \right] \right\} \right| \leqq k_0 h^2, \tag{2.16}$$

where k_0 is a constant independent of h. The construction of δ_n in general requires that the region have no acute corners. In the case of a rectilinear region however special considerations can be made as we shall see in §3.

Let $o \in R_h$ be a mesh point in the interior of R and define R_h' to be the set $R_h - o$. A finite difference analog of (2.1) which gives a consistent system of linear equations is

$$\begin{aligned}
&\text{(a)} \quad -\Delta_h U(p) = f(p), && p \in R_h' + C_h^*, \\
(2.17)\quad &\text{(b)} \quad \delta_n U(p) = g(p) + \sum_{i=1}^{3} A_i(p) \left[\frac{y_i^2}{2} f(p) + x_i y_i \frac{\partial g}{\partial s}(p) \right], && p \in C_h, \\
&\text{(c)} \quad U(o) = \text{a given value.}
\end{aligned}$$

It is seen that the matrix of the system (2.17) satisfies the following definition [7].

DEFINITION 2.1. An $N \times N$ matrix B with elements b_{ij} is said to be of *positive type* if the following conditions are satisfied.

(a) $b_{ij} \leqq 0, \quad i \neq j$,

(b) $\sum_k b_{jk} \geqq 0$ for all j, with $\sum_k b_{jk} > 0$ for $j \in J(B) \neq \emptyset$, $J(B) \subset \{1, 2, \cdots, N\}$,

(c) for $i \notin J(B)$ there exists a finite sequence of nonzero elements of the form $b_{ik_1}, b_{k_1k_2}, \cdots, b_{k_rj}$, where $j \in J(B)$. Such a sequence is called a *connection* in B from i to $J(B)$.

This definition is a modification of well known sufficient conditions for a matrix to be an M-matrix [23].

If B were the matrix of (2.17) then we see that $J(B) =$ (the index of o) and that every point of $R_h' + C_h^* + C_h$ is connected to $J(B)$.

We note that if (2.17c) were replaced by the condition

$$-\Delta_h U(o) = f(o), \tag{2.18}$$

then the resulting system would not be of positive type since $J(B)$ would be empty. In fact the matrix B would be singular with the vector $\eta(p)$,

$$\eta(p) = 1, \qquad p \in R_h + C_h^* + C_h, \tag{2.19}$$

spanning the null space, i.e.,

$$B\eta = 0. \tag{2.20}$$

If $\bar{\eta}$ lies in the null space of B^T we see that the consistency condition in this case is

$$\sum_{p \in R_h + C_h^*} \bar{\eta}(p) f(p) + \sum_{p \in C_h} \bar{\eta}(p) g(p) = 0, \tag{2.21}$$

which is the analog of (2.2). In general (2.21) will not be satisfied and, since $\bar{\eta}$ is difficult to find, the proper modification of the right side of the linear system is not easy to achieve except for special geometries. The formulation of the finite difference analog (2.17), which we shall now consider, provides a consistent system, without altering the right side, by eliminating one equation and replacing it by the normalization (2.17c). As was observed by J. H. Geise [15] in the case of the rectangle, the resulting error estimate for a slightly different analog is $O(h^2 \mid \log h \mid)$ which for all practical purposes is as good as $O(h^2)$.

The factor $\log h$ arises from the fundamental solution of Δ_h in two dimensions which is discussed in the Appendix. Consequently we would expect that the same process in higher dimensions would yield materially lower rates of convergence as indeed can be shown to be the case. The finite difference analog of Giese differs from ours in two respects. First, in Giese's formulation, the condition (2.18) is added to (2.17) and the resulting problem is turned into a consistent system by adding an $O(h^2)$ term at each boundary equation. Secondly, equations of the type (2.17a) are given on C_h which introduces another band of mesh points outside of R. Centered difference approximations to $\partial u/\partial n$ are also given at points of C_h which completes the formulation. Each pair of equations at a given boundary point could be combined to give a limiting case of (2.17b).

It is easy to prove (cf [6]) that positive-type matrices are nonsingular and have inverses with nonnegative elements. Hence, we define $N(p, q)$, $p, q \in R_h + C_h^* + C_h$, by

$$\begin{aligned} -\Delta_{h,p} N(p, q) &= h^{-2}\delta(p, q), \qquad p \in R_h' + C_h^*, \\ \delta_{n,p} N(p, q) &= h^{-1}\delta(p, q), \qquad p \in C_h, \\ N(o, q) &= \delta(o, q), \end{aligned} \tag{2.22}$$

where the Kronecker delta is defined by

$$\delta(p, q) = \begin{cases} 1 & \text{if } p = q, \\ 0 & \text{if } p \neq q. \end{cases} \tag{2.23}$$

It follows that

$$N(p, q) \geqq 0. \tag{2.24}$$

For any mesh function $V(p)$ it is clear that

$$\begin{aligned} V(p) = h^2 \sum_{q \in R_h' + C_h^*} & N(p, q)[-\Delta_h V(q)] \\ & + h \sum_{q \in C_h} N(p, q)[\delta_n V(q)] + N(p, o) V(o). \end{aligned} \tag{2.25}$$

Substituting the function $V(p) \equiv 1$ gives

$$N(p, o) = 1, \qquad p \in R_h + C_h^* + C_h . \tag{2.26}$$

Consequently we may rewrite (2.25) as

$$\begin{aligned} V(p) - V(o) = h^2 \sum_{q \in R_h' + C_h^*} N(p, q)\{-\Delta_h V(q)\} \\ + h \sum_{q \in C_h} N(p, q)[\delta_n V(q)]. \end{aligned} \tag{2.27}$$

To obtain the desired bound on the discretization error we shall first show that

$$h^2 \sum_{q \in R_h' + C_h^*} N(p, q) = O(|\log h|). \tag{2.28}$$

This is accomplished by considering the finite difference Green's function of a corresponding Dirichlet problem. We define $G(p, q)$ by

$$\begin{aligned} -\Delta_{h,p} G(p, q) &= \delta(p, q) h^{-2}, \qquad p \in R_h + C_h^*, \\ G(p, q) &= \delta(p, q), \qquad p \in C_h . \end{aligned} \tag{2.29}$$

It is well known [3] that $G(p, q)$ exists and is nonnegative. If $g(p, q)$ is the Green's function of the corresponding continuous problem then it is shown in the Appendix that for C sufficiently smooth,

$$|G(p, o) - g(p, o)| = O(h^2), \qquad p \in C_h , \tag{2.30}$$

and consequently

$$|\delta_{n,p} G(p, o) - \delta_{n,p} g(p, o)| = O(h), \qquad p \in C_h . \tag{2.31}$$

Moreover, since o is far from C, we see that on smooth portions of C,

$$\left| \frac{\partial}{\partial n_p} g(p, o) - \delta_{n,p} g(p, o) \right| = O(h^2). \tag{2.32}$$

If we assume for the moment that C has no corners then it follows from a lemma of Hopf [10, p. 327] and the fact that g has bounded derivatives in R that

$$\beta^{-1} \geqq -\frac{\partial g}{\partial n_p}(p, o) \geqq \beta > 0. \tag{2.33}$$

A consequence of the lemma is that the normal derivative of a function harmonic on a region with smooth boundary cannot vanish at points of the boundary where the function takes on either a maximum or a minimum.

If R_ϵ is the region R with a small circle about o deleted then we see that $-g(p, o)$ takes on a maximum (zero) at each point of C, from which we infer (2.33). If C has corners then the above argument may fail with the result that $-\partial g(p, o)/\partial n_p = 0$ at these corners. Since (2.30) and (2.32) may be violated as well, we shall assume, for the sake of simplicity, that C is a smooth curve and later treat the case where C has corners. It now follows from (2.31), (2.32), and (2.33) that for h chosen sufficiently small,

$$\delta^{-1} \geqq -\delta_{n,p} G(p, o) \geqq \delta > 0, \qquad p \in C_h . \tag{2.34}$$

We now substitute $V(p) = -G(p, o)$ into (2.27) and use (2.34) to obtain

$$\delta^{-1} h \sum_{q \in C_h} N(p, q) \geqq -G(p, o) + G(o, o) \geqq \delta h \sum_{q \in C_h} N(p, q). \tag{2.35}$$

Hence it follows from (2.35) and (4.8) that

$$h \sum_{q \in C_h} N(p, q) \leqq k_1 \mid \log h \mid . \tag{2.36}$$

Now if $r(p)$ is defined to be the distance from o to p and $V(p) = -r^2(p)$ in (2.27), we infer from (2.36) that

$$h^2 \sum_{q \in R_h' + C_h^*} N(p, q) \leqq k_2 \mid \log h \mid . \tag{2.37}$$

Further if we let $V(p)$ in (2.27) be given by

$$V(p) = \begin{cases} 1 & \text{if} \quad p \in R_h + C_h^*, \\ 0 & \text{if} \quad p \in C_h , \end{cases} \tag{2.38}$$

then (2.36) yields

$$h^2 \sum_{q \in C_h^*} N(p, q) \leqq k_3 h \mid \log h \mid . \tag{2.39}$$

Let $u(p)$, $U(p)$ be the solutions of (2.1) and (2.17) respectively, and let $e(p) \equiv u(p) - U(p)$ be the discretization error; then from (2.24) and (2.27) we see that

$$\begin{aligned} \mid e(p) \mid \leqq & \left\{\max_{q \in R_h'} \mid \Delta_h e(q) \mid\right\} h^2 \sum_{q \in R_h'} N(p, q) \\ & + \left\{\max_{q \in C_h^*} \mid \Delta_h e(q) \right\} h^2 \sum_{q \in C_h^*} N(p, q) \\ & + \left\{\max_{q \in C_h} \mid \delta_n e(q) \mid\right\} h \sum_{q \in C_h} N(p, q), \end{aligned} \tag{2.40}$$

where we assume that $u(o) = U(o)$. It now follows from (2.5), (2.7), (2.16), (2.36), (2.37), and (2.39) that

$$\mid e(p) \mid \leqq kh^2 \mid \log h \mid , \tag{2.41}$$

provided that $u \in C^4$ in $\bar{R}$.

In the case where C has a finite number of corners the preceding analysis requires some modification. We assume that for any given mesh, δ_n can be defined at every point of C_h so that the resulting matrix is of positive type.

In order to establish the inequality (2.36) we may no longer use simply $G(p, o)$ in (2.34) since that inequality will not in general be satisfied on all of C_h. To avoid this apparent difficulty let C_1 be a smooth arc of nonzero length, whose end points are not corners. Let R_1 be a region contained in R which has o as an interior point and whose boundary is smooth and contains C_1. We further insist that R_1 be so chosen that δ_n on C_1 involves only mesh points of R_1. The preceding analysis holds for $G_1(p, o)$. Hence there is a constant δ independent of h (for sufficiently small h) such that (2.34) holds for $G_1(p, o)$, i.e.,

$$-\delta_n G_1(p, o) \geqq \delta > 0 \tag{2.42}$$

for any p on the mesh boundary of R_1 and in particular for $p \in C_{1h}$. We note from (2.15) that

$$\sum_{i=1}^{3} A_i G_1(p_i, o) = \delta_n G(p, o) \geqq \delta > 0. \tag{2.43}$$

On the other hand it was shown in [5] that

$$G(p, o) \geqq G_1(p, o) \tag{2.44}$$

for points at which G and G_1 are commonly defined. It follows then that

$$-\delta_n G(p, o) \geqq \delta > 0 \tag{2.45}$$

for $p \in C_{1h}$.

Let ϕ be a smooth function satisfying

$$\begin{aligned} -\Delta\phi &\geqq 1 \quad \text{in} \quad R, \\ \frac{\partial\phi}{\partial n} &\geqq 1 \quad \text{on} \quad C - C_1, \\ \left|\frac{\partial\phi}{\partial n}\right| &< \delta_1 \quad \text{on} \quad C_1, \end{aligned} \tag{2.44}$$

where δ_1 is some constant. Thus we note that

$$\begin{aligned} -\Delta_h\left[-G(p, o) + \frac{\delta}{2\delta_1}\phi(p)\right] &\geqq \frac{\delta}{2\delta_1} + O(h) \geqq \lambda, \quad p \in R_h' + C_h^*, \\ \delta_n\left[-G(p, o) + \frac{\delta}{2\delta_1}\phi(p)\right] &\geqq \frac{\delta}{2} + O(h^2) \geqq \lambda, \qquad p \in C_{1h}, \\ \delta_n\left[-G(p, o) + \frac{\delta}{2\delta_1}\phi(p)\right] &\geqq \frac{\delta}{2\delta_1} + O(h^2) \geqq \lambda, \ p \in C_h - C_{1h}, \end{aligned} \tag{2.47}$$

where λ is a positive constant, if h is sufficiently small. Inequalities (2.36) and (2.37) follow immediately if we take

$$V(p) = -G(p, o) + \frac{\delta}{2\delta_1}\phi(p)$$

in (2.27).

We remark here than an $O(h)$ approximation to $\partial u/\partial n$ which is of positive type is easily obtained (cf. Kantorovich and Krylov [16]). Choosing only two interior points P_1 and P_2 in (2.11) we have

$$\begin{aligned} -\bar{\delta}_n V(0) &\equiv \sum_{i=1}^{2} b_i[V(p_i) - V(0)] \\ &= -V_n(0) + O\left[\sum_{i=1}^{2} b_i(x_i^2 + y_i^2)\right], \end{aligned} \tag{2.48}$$

provided

$$\begin{bmatrix} y_1 & y_2 \\ x_1 & x_2 \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{2.49}$$

Clearly $y_1, y_2 > 0$, $x_1 > 0$, $x_2 < 0$ will guarantee the existence of non-negative numbers b_1 and b_2. Thus, analogous to (2.16), we are led to

$$| \bar{\delta}_n u(p) - g(p)| \leqq \bar{k}_1 h. \tag{2.50}$$

The finite difference problem then becomes

$$\begin{aligned} -\Delta_h U(p) &= f(p), \qquad p \in R_h' + C_h^*, \\ \bar{\delta}_n U(p) &= g(p), \qquad p \in C_h, \\ U(0) &= u(0). \end{aligned} \tag{2.51}$$

Following the same method of proof it can be shown that for this analog,

$$| e | \leqq \bar{k}h \,| \log h \,|, \tag{2.52}$$

if $u \in C^3$ in $\bar{R}$.

In the next section examples will be given which show that the error estimates (2.41) and (2.46) are, in fact, sharp.

3. Sharpness of the estimates. We first give an example using (2.17) for which the order of convergence is no better than $h^2 \,| \log h \,|$. Let R be the rectangle with vertices (0, 0), (0, 1), (1, 1), (1, 0). In this case δ_n degenerates into the three point formula. At a typical point, say $(1, y)$,

$$\begin{aligned} &\delta_n V(1, y) \\ &\quad \equiv h^{-1}\{V(1, y) - \tfrac{1}{2}[V(1 - h, y + h) + V(1 - h, y - h)]\}. \end{aligned} \tag{3.1}$$

We see that

$$\delta_n u = \frac{\partial u}{\partial n} - \frac{h}{2}\Delta u + \frac{h^2}{6}F(u) + O(h^3), \tag{3.2}$$

where

$$F(u) = \begin{cases} u_{xxx} + 3u_{xyy}, & x = 1, \\ -u_{xxx} - 3u_{xyy}, & x = 0, \\ u_{yyy} + 3u_{xxy}, & y = 1, \\ -u_{yyy} - 3u_{xxy}, & y = 0. \end{cases} \tag{3.3}$$

Consider the function

$$u = x^2y^2 + (x-1)^2(y-1)^2, \tag{3.4}$$

and define the finite difference analog based on (2.17), i.e.,

$$\begin{aligned} -\Delta_h U(p) &= -\Delta u, \qquad p \in R_h' + C_h^*, \\ \delta_n U(p) &= \frac{\partial u}{\partial n} - \frac{h}{2}\Delta u, \qquad p \in C_h, \\ U(o) &= u(o). \end{aligned} \tag{3.5}$$

We note that for $e(p) \equiv u(p) - U(p)$,

$$\begin{aligned} \Delta_h e(p) &= \frac{h^2}{12}[u_{xxxx}(p) + u_{yyyy}(p)] = 0, \\ \delta_n e(p) &= \frac{h^2}{6}F(u) + O(h^3) = [2 + O(h)]h^2. \end{aligned} \tag{3.6}$$

Consequently we see that (2.27) becomes

$$e(p) = h\{\sum_{q \in C_h} N(p, q)[2 + O(h)]h^2\}. \tag{3.7}$$

By considerations similar to those which produced (2.36) we see that for p bounded away from o,

$$h\sum_{q \in C_h} N(p, q) \geqq k_4 \mid \log h \mid. \tag{3.8}$$

It now follows from (3.7) and (3.8) that

$$e(p) \geqq k_5 h^2 \mid \log h \mid. \tag{3.9}$$

In the same manner we can exhibit an example of convergence of the solution of (2.51) which is no better than $h \mid \log h \mid$.

For $\bar{\delta}_n$ we need only the two point operator, e.g.,

$$\bar{\delta}_n V(1, y) \equiv h^{-1}[V(1, y) - V(1, y - h)]. \tag{3.10}$$

Clearly,

$$\bar{\delta}_n u = \frac{\partial u}{\partial n} + \frac{h}{2} \bar{F}(u), \tag{3.11}$$

where

$$\bar{F}(u) = \begin{cases} u_{xx}, & x = 0, 1, \\ u_{yy}, & y = 0, 1. \end{cases} \tag{3.12}$$

Let u be the function

$$u = x^2 + y^2. \tag{3.13}$$

We see that $e(p)$ satisfies the identity

$$e(p) = h^2 \sum_{q \in C_h} N(p, q). \tag{3.14}$$

If p is bounded away from o, then (3.14) shows that

$$e(p) \geqq k_4 h \mid \log h \mid. \tag{3.15}$$

The overdetermined system (3.5) with R_h' replaced by R_h in this example is clearly incompatible. Indeed, if it were compatible then (3.6) shows that $\Delta_h e(o) = 0$. We note that since $\Delta_h G(o, o) = h^{-2}$, the analog of (2.35) for (2.51) yields for p_i, the four neighbors of o,

$$\sum_{i=1}^{4} h \sum_{q \in C_h} N(p_i, q) \geqq -\delta h^2 \Delta_h G(o, o) = \delta. \tag{3.16}$$

Hence, since $e(o) = 0$, we see that $\Delta_h e(o) = O(h^{-1})$, which clearly is a contradiction.

4. Appendix. In this section the estimate (2.30) is developed. Let $\bar{\Delta}_h$ be the five point operator (2.4) at each grid point in the (x, y)-plane. We know from the work of McCrea and Whipple (cf. [11, p. 317]) that there exists a fundamental solution $\Gamma(p, o)$ of

$$\bar{\Delta}_{h,p} \Gamma(p, o) = -h^2 \delta(p, o), \tag{4.1}$$

with the property that

$$\left| \Gamma(p, o) + \frac{1}{2\pi} \log r_{po} \right| = O(h^2) \tag{4.2}$$

for $\overline{po} \geqq \delta > 0$. Define $\Phi(p)$ and $\phi(p)$ by the relations

$$\begin{aligned} G(p, o) &= \Gamma(p, o) + \Phi(p), \\ g(p, o) &= -\frac{1}{2\pi} \log r_{po} + \phi(p). \end{aligned} \tag{4.3}$$

We note that $\Phi(p) - \phi(p)$ has the following properties:

$$\Phi(p) - \phi(p) = 0, \qquad p \in C_h,$$

$$\Delta_h[\Phi(p) - \phi(p)] = \begin{cases} O(h^2) & \text{if} \quad p \in R_h, \\ O(1) & \text{if} \quad p \in C_h^*, \end{cases} \tag{4.4}$$

where the last statement holds for regions with sufficiently smooth boundaries (cf. Laasonen [17]). As was pointed out in [3], $\Phi(p) - \phi(p)$ satisfies the finite difference Poisson formula

$$\Phi(p) - \phi(p) = h^2 \sum_{q \in R_h + C_h^*} G(p, q)[-\Delta_h(\Phi(q) - \phi(q))]. \tag{4.5}$$

It is further shown in that paper that

$$h^2 \sum_{q \in R_h} G(p, q) = O(1),$$
$$\sum_{q \in C_h^*} G(p, q) = O(1). \tag{4.6}$$

Hence, from (4.4), (4.5), and (4.6) we see that

$$|\Phi(p) - \phi(p)| = O(h^2). \tag{4.7}$$

The relation (2.30) now follows from (4.2), (4.3), and (4.7). We note the close connection between (2.28) and Theorem 23.8 of Forsythe and Wasow [11].

In [18] P. Laasonen shows that the finite difference Green's function for a rectangle behaves like $\log h$ at the singular point. It was shown in [5] that $G(p, q)$ is a monotone function of region and hence it is true that

$$|G(p, q)| \leqq \alpha |\log h|. \tag{4.8}$$

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