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MEAN VALUE THEOREMS FOR POLYHARMONIC FUNCTIONS

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I. Introduction. It is well known that polyharmonic functions satisfy various mean value theorems which may be considered as generalizations of the Gauss mean value theorem for harmonic functions. For example Nicolesco [**3**] gave an expression in terms of certain iterated means and showed that a converse was also true. Cheng [**1**] established a converse for a different mean value expression. Other work on mean value theorems for polyharmonic functions has been carried out by Pizetti [**5**], Picone [**4**], and others (see e.g., [**2**]).

In this paper we derive two rather simple mean value theorems for polyharmonic functions of order p in N dimensions. The expressions involve means over p distinct spheres and seem to be a very natural generalization of the Gauss "peripheral" and "solid" theorems for harmonic functions. A strong converse is given in each case.

We shall consider an N dimensional region R. A function ϕ is called polyharmonic of order p in R if $\phi \in C^{2p}$ and $\Delta^p\phi = 0$ in R where Δ denotes the Laplace operator. For an arbitrary point O of R let S_ρ be the interior of the sphere of radius ρ and center at O. The variable r will be used as the radial variable with respect to O, and the quantity ω_N will denote the surface area of the N dimensional unit sphere.

II. Derivation of the mean value expressions. We start with the following result due to Pizetti [**5**]. Let O be an arbitrary point of R and suppose that $\Delta^p\phi = 0$ in $S_{\rho_p} \subset R$. Then for any $\rho_j \leqq \rho_p$

$$\phi(0) + \sum_{i=2}^{p} \rho_j^{2(i-1)} A_i = \frac{1}{\omega_N} \int_{r=\rho_j} \phi \, d\Omega, \tag{2.1}$$

where the A_i's are independent of ρ_j.

Let the $(p \times p)$ matrix P_{ij} be defined as

$$P_{ij} = \rho_j^{2(i-1)} \tag{2.2}$$

for the p given numbers $0 < \rho_1 < \cdots < \rho_p$, and let P^{ij} be its inverse. Then from (2.1) it follows that

$$\omega_N \phi(0) = \frac{\sum_{j=1}^{p} P^{j1} \int_{r=\rho_j} \phi \, d\Omega}{\sum_{j=1}^{p} P^{j1}} . \tag{2.3}$$

It is easy to see that

$$\sum_{j=1}^{p} P^{j1} = 1. \tag{2.4}$$

Thus

$$\omega_N \phi(0) = \sum_{j=1}^{p} P^{j1} \int_{r=\rho_j} \phi \, d\Omega. \tag{2.5}$$

Because of the form of P_{ij}, and Cramer's rule, we obtain the result in Table A.

$$\omega_N \phi(0) = \frac{\begin{vmatrix} \int_{\rho_1} \phi \, d\Omega & \int_{\rho_2} \phi \, d\Omega & \cdots & \int_{\rho_p} \phi \, d\Omega \\ \rho_1^2 & \rho_2^2 & & \rho_p^2 \\ \rho_1^4 & \vdots & & \vdots \\ \vdots & & & \\ \rho_1^{2(p-1)} & \cdots & & \rho_p^{2(p-1)} \end{vmatrix}}{\begin{vmatrix} 1 & 1 & \cdots & 1 \\ \rho_1^2 & \rho_2^2 & & \rho_p^2 \\ \vdots & \vdots & & \vdots \\ \rho_1^{2(p-1)} & \cdots & & \rho_p^{2(p-1)} \end{vmatrix}} \equiv \frac{D_1}{D_2} \tag{2.6}$$

TABLE A

We can also obtain an expression involving solid means since the identity

$$\phi(0) + \sum_{i=2}^{p} \rho_j^{2(i-1)} B_i = \frac{N}{\rho_j^N \omega_N} \int_{r \leqq \rho_j} \phi \, dV \tag{2.7}$$

can be shown to hold for any function ϕ satisfying $\Delta^p \phi = 0$ in S_{ρ_p}, the B_i's being independent of ρ_j.

In exactly the same way we obtain the result in Table B.

$$\frac{\omega_N}{N} \phi(0) = \frac{\begin{vmatrix} \frac{1}{\rho_1^N} \int_{r \leqq \rho_1} \phi \, dV & \cdots & & \frac{1}{\rho_p^N} \int_{r \leqq \rho_p} \phi \, dV \\ \rho_1^2 & & & \rho_p^2 \\ \vdots & & & \vdots \\ \rho_1^{2(p-1)} & \cdots & & \rho_p^{2(p-1)} \end{vmatrix}}{\begin{vmatrix} 1 & 1 & \cdots & 1 \\ \rho_1^2 & \rho_2^2 & & \rho_p^2 \\ \vdots & \vdots & & \vdots \\ \rho_1^{2(p-1)} & \cdots & & \rho_p^{2(p-1)} \end{vmatrix}} \tag{2.8}$$

TABLE B

III. Converses. Let

$$\Gamma = \begin{cases} \eta K r^{2p+2-N}, & N \text{ odd or } N > 2p+2 \\ \eta K r^{2p+2-N} \log r, & N \text{ even and } N \leqq 2p+2, \end{cases} \tag{3.1}$$

where K is a constant and η is a nonnegative infinitely differentiable function of r which is 1 in $S_{r_1/2}$ and zero outside S_{r_1}. We assume that $S_{r_1} \subset R$.

For each N and p there is a constant K such that

$$(3.2) \qquad v(0) = -\int_R \Gamma\Delta^p v\, dV + \int_{R-S_{r_1/2}} v\Delta^p \Gamma\, dV,$$

for every sufficiently smooth v in R. With (3.2) it is easy to prove the following.

THEOREM 1. *Let ϕ be a function integrable over all spheres in R and let ϕ satisfy* (2.6) *almost everywhere, for all $0<\rho_1< \cdots <\rho_p$, with ρ_p sufficiently small. Then ϕ is equal, almost everywhere, to a function $\bar{\phi}$ which is polyharmonic of order p.*

Proof. Let $\rho_2, \cdots, \rho_p$ be fixed and keep $0<r<r_1<\rho_2$. Then, from (2.6),

$$(3.3) \qquad \omega_N\phi(0) = \int_{r_1/2<r<r_1} D_2\Delta^p\Gamma r^{N-1}\, dr = \int_{r_1/2<r<r_1} D_1\Delta^p\Gamma r^{N-1}\, dr.$$

Expanding D_1 and D_2 by means of their first columns, we observe that every term except the first in each case vanishes, because

$$(3.4) \qquad \omega_N \int_{r_1/2<r<r_1} \Delta^p\Gamma r^{2i} r^{N-1}\, dr = \int_{r_1/2<r<r_1} r^{2i}\Delta^p\Gamma\, dV,$$

$i=1, \cdots, p-1$ (note that Γ depends only on r). Since $\Delta^p r^{2i}=0$ and $r^{2i}=0$ for $r=0$, $i=1, \cdots, p-1$, we conclude from (3.2) and (3.4), by setting $v=r^{2i}$, that

$$(3.5) \qquad \int_{r_1/2<r<r_1} \Delta^p\Gamma r^{2i} r^{N-1}\, dr = 0, \qquad i = 1, \cdots, p-1.$$

Hence (3.3) reduces to

$$(3.6) \qquad \phi(0) = \int_{r_1/2<r<r_1} \phi\Delta^p\Gamma\, dV, \text{ almost everywhere.}$$

Since $\Delta^p\Gamma$ is infinitely differentiable for $r_{1/2}<r<r_1$ and a relation such as (3.6) holds almost everywhere for all sufficiently small r, it follows by standard arguments that there is an infinitely differentiable function $\bar{\phi}$ such that $\phi=\bar{\phi}$ almost everywhere in R. Hence (3.6) holds for $\bar{\phi}$. But from (3.2) we have

$$(3.7) \qquad \int_{r<r_1} \Gamma\Delta^p\bar{\phi}\, dV = \int_R \Gamma\Delta^p\bar{\phi}\, dV = 0.$$

Now Γ is of one sign and r is arbitrarily small, so that $\Delta^p\bar{\phi}(0)$ must be zero. Since O is an arbitrary point it follows that $\Delta^p\bar{\phi}=0$ in R and the theorem is proved.

THEOREM 2. *Let ϕ be a locally integrable function in R and satisfy* (2.8) *almost everywhere for all $0<\rho_1< \cdots <\rho_p$ with ρ_p sufficiently small. Then ϕ is equal almost everywhere to a function $\bar{\phi}$ which is polyharmonic of order p.*

Proof. Multiplying numerator and denominator by ρ_1^N of (2.8), and differentiating with respect to ρ_1, we obtain the result in Table C.

$$
(3.8)\qquad \begin{vmatrix} 1 & \cdots & 1 \\ \frac{(N+2)}{N}\rho_1^2 & \cdots & \rho_p^2 \\ \vdots & & \vdots \\ \frac{[N+2(p-1)]}{N}\rho_1^{2(p-1)} & \cdots & \end{vmatrix} \omega_N\phi(0) = \begin{vmatrix} \int_{r=\rho_1}\phi\, d\Omega & \frac{1}{\rho_2^N}\int_{r\le\rho_2}\phi\, dV \cdots & \frac{1}{\rho_p^N}\int_{r\le\rho_p}\phi\, dV \\ (N+2)\rho^2 \cdots & & \rho_p^2 \\ \vdots & & \vdots \\ (N+2(p-1))\ \rho_1^{2(p-1)} \cdots & & \end{vmatrix}
$$

TABLE C

Note that (2.8) implies that ϕ is bounded almost everywhere and hence there is a bounded function $\bar{\phi}$ for which (2.8) is satisfied and $\rho=\bar{\phi}$ almost everywhere. Also note that if ρ_1 is small enough then the determinant on the left is not zero, since for $\rho_1=0$ it is a Vandermonde determinant which is different from zero if $0<\rho_2<\cdots<\rho_p$.

Just as before we now obtain (3.6) from (3.8).

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References

1. M. Cheng, On a theorem of Nicolesco and generalized Laplace operators, Proc. Amer. Math. Soc., 2 (1951) 77–86.

2. Ghermanesco, Sur les moyennes successives des fonctions, Bull. Soc. Math. France, 52 (1934) 245–264.

3. M. Nicolesco, Les fonctions polyharmoniques, Actualités Sci. Ind., 4 (1936).

4. M. Picone, Nuovi indirizzi di ricerca teoria e nel calcolo soluzioni di talune equazioni lineari alle derivate parziali della Fisica-Matematica, Ann. Scuola Norm. Sup. Pisa, 4 (1935) 213–288.

5. P. Pizetti, Sul significato geometrico del secondo parametro differenziale di una funzione sopra una superficie qualunque, Rend. Lincei, 18 (1909) 309–316.