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Rayleigh-Ritz-Galerkin Methods for Dirichlet's Problem Using Subspaces Without Boundary Conditions*

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1. Introduction

Recently there has been much interest in Rayleigh-Ritz-Galerkin type methods for the approximation of solutions to elliptic boundary value problems. These methods consist in the construction of a certain finite-dimensional subspace of a Sobolev space containing the solution of the boundary value problem, and then projecting the solution onto the subspace in such a way that the projection can be computed from the data of the original problem. One then wants to study the error in approximation when the dimension of the finite-dimensional space is increased in a systematic manner. Most of these methods are based on classical type variational principles and present little difficulty when the boundary conditions are "natural", e.g., Neumann type problems (c.f. Friedrichs and Keller [14], Aubin [2] and Schultz [19]). For such problems the functions admissible in the associated variational problems are not required to satisfy any boundary conditions and hence domains in two or more dimensions of general shape may be treated.

The situation is different for Dirichlet's problem. Except in one dimension (c.f. [6], [11], [21]) and special situations in two dimensions (e.g. rectilinear domains, polygons, c.f. [6], [22]) little has been done until recently. Nitsche [17] has obtained very nice results for second order operators in convex domains satisfying homogeneous Dirichlet boundary conditions. He uses subspaces consisting of piece-wise linear functions on triangles (of largest side h) and shows that the error in the L_2 norm is of second order. Schultz [20] has constructed subspaces by multiplying functions in other spaces by a function which vanishes on the boundary of the domain. Babuška [4] has proposed and studied a method in which he perturbs the usual variational principle and thus is able to avoid satisfying boundary conditions.

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The basic difficulty has always been the construction of subspaces each element of which satisfies boundary conditions. The purpose of this paper is to present and give error estimates for simple methods for treating the Dirichlet problem by Rayleigh-Titz-Galerkin type methods which do not require that the functions in the subspace satisfy any boundary conditions whatsoever. In this paper we shall treat the Dirichlet problem for second order elliptic operators in a bounded domain in $\mathbb{R}^N$. In a subsequent paper we shall treat 2^m-th order elliptic equations with general boundary conditions. Error estimates are obtained for the approximation which depend only on the "goodness" of the chosen subspace and the regularity of the data. There are several difficulties in standard Rayleigh-Ritz-Galerkin methods for Dirichlet's problem which are not present in our method. The following special features of the methods of this paper should be noted:

1. The elements of the subspace are *not* required to satisfy any boundary conditions.
2. The approximate solutions are computed using only L_2 inner products involving data.
3. Non-selfadjoint problems are treated with equal ease.
4. Problems whose associated quadratic form is *not* positive definite present no difficulties.
5. The quadratic form associated with our method is always positive definite and the resulting finite-dimensional problem is symmetric.

As an example of the type of result obtained, we mention the important special case of the classical Dirichlet problem, $\Delta u = f$ in R (a bounded domain in $\mathbb{R}^N$) and $u = g$ on ∂R (the boundary of R). In Section 4 we present various schemes for approximating u. Choosing, for instance, cubic splines or cubic Hermite polynomials on a uniform mesh of width h in one of our schemes (c.f. equation (4.1) with $\gamma = \frac{3}{2}$), we obtain for the error e in the L_2 norm $\|\cdot\|_0$ on R,

$$\|e\|_0 \leqq C(h^2 \|f\|_0 + h^{1/2} |g|_0)$$

and

$$\|e\|_0 \leqq Ch^4(\|f\|_2 + |g|_{7/2}) \,.$$

Here $\|\cdot\|_s$ and $|\cdot|_s$, denote the Sobolev norms of order s on R and ∂R, respectively.

A general outline of the paper is as follows: In Section 2 some spaces of functions are defined and the boundary value problems which we shall consider are described. In Section 3 the subspaces to be used for the approximation are introduced and some new approximation theoretic results are proved. In Section 4 various approximation schemes are introduced and error estimates proved. Section 5 contains some examples which illustrate the results in specific cases.

2. The Dirichlet Problem. Preliminaries

Let R be a bounded domain in $\mathbb{R}^N$ with boundary ∂R. We shall assume (for convenience) that ∂R is of class C^∞ and consider in R the second order operator with C^∞ real coefficients

$$Au = A(x, D)u = -\sum_{i,j=1}^{N} a_{ij}(x)D_i D_j u + \sum_{i=1}^{N} a_i(x)D_i u + a(x)\,, \tag{2.1}$$

where $D_i = \partial/\partial x_i$.

It is assumed that A is uniformly elliptic; i.e., there exists a constant $C > 0$ independent of x such that

$$C^{-1}|\xi|^2 \leqq \left|\sum_{i,j=1}^{N} a_{ij}\xi_i\xi_j\right| \leqq C|\xi|^2$$

for all $x \in \bar{R}$ and all $\xi \in \mathbb{R}^N$.

We shall consider the boundary value problem

$$\begin{aligned} Au &= f \quad \text{in} \quad R\,,\\ u &= g \quad \text{on} \quad \partial R\,, \end{aligned} \tag{2.2}$$

where f and g are given.

All functions in this paper will be real valued and $C^\infty(\bar{R})$ will denote the set of infinitely differentiable functions on $\bar{R}$.

The conditions we shall place on our problem are as follows:

Condition A. (i) A is uniformly elliptic with coefficients in $C^\infty(\bar{R})$; (ii) the only solution of (2.2) in $C^\infty(\bar{R})$ with zero data is the zero solution.

Some function spaces. Let p be a non-negative integer. Then $H^p(R)$ will denote the Sobolev space of order p, i.e., the completion of $C^\infty(\bar{R})$ in the norm

$$\|\varphi\|_p = \left(\sum_{|\alpha|\leqq p} \|D^\alpha\varphi\|_0^2\right)^{1/2}, \tag{2.3}$$

where

$$\|\varphi\|_0 = \left(\int_R |\varphi|^2\,dx\right)^{1/2} \tag{2.4}$$

$H^p(R)$ is a Hilbert space with the inner product

$$(\varphi, \psi)_p = \sum_{|\alpha|\leqq p} (D^\alpha\varphi, D^\alpha\psi)_0\,, \tag{2.5}$$

where

$$(\varphi, \psi)_0 = \int_R \varphi\psi\,dx\,.$$

If p is any positive real number which is not an integer, then the Hilbert space $H^p(R)$ is defined by interpolation between successive integers. More generally, let $X \subset Y$ be two Hilbert spaces, X dense in Y with a continuous injection. One can then define new Hilbert spaces denoted by $[X, Y]_\theta$ for $0 \leqq \theta \leqq 1$ with $[X, Y]_0 = X$, $[X, Y]_1 = Y$ and $X \subset [X, Y]_\theta \subset Y$ for $0 \leqq \theta \leqq 1$. For details we refer the reader to [16], pp. 11–13.

If p is a non-negative real number which is not an integer, then $H^p(R) = [H^{i+1}(R), H^i(R)]_\theta$, where i is an integer, $i < p < i+1$ and $\theta = p - i$. For p real and negative, we define $H^p(R)$ as the completion of $C^\infty(\bar{R})$ with respect to the norm

$$\|u\|_p = \sup_{v \in C^\infty(\bar{R})} \frac{(u, v)_0}{\|v\|_{-p}} . \tag{2.6}$$

Remark 2.1. $H^p(R) = (H^{-p}(R))'$ is the dual space of $H^{-p}(R)$ and, if r and p are any two real numbers $r > p$, then $[H^r(R), H^p(R)]_\theta = H^{r\theta + p(1-\theta)}(R)$ for any $0 \leqq \theta \leqq 1$ with equivalent norms.

There are many equivalent ways of defining the spaces $H^s(\partial R)$, the Sobolev spaces of functions having L_2 derivatives of order s on ∂R. For simplicity we use the following:

Let $\Delta_{\partial R}$ denote the Laplace-Beltrami operator on ∂R. There exists a sequence $\lambda_j > 0$ of eigenvalues and corresponding eigenfunctions $w_j \in C^\infty(\partial R)$ satisfying

$$-\Delta_{\partial R} w_j = \lambda_j w_j \qquad \text{on} \qquad \partial R , \qquad\qquad j = 1, 2, \cdots ,$$

where the w_j are orthonormal in $H^0(\partial R) = L_2(\partial R)$ with respect to the inner product

$$\langle u, v\rangle_0 = \int_{\partial R} uv \, d\sigma . \tag{2.7}$$

Here σ is the measure on ∂R induced by the Lebesgue measure on $\mathbb{R}^{n-1}$.

For any $s \in \mathbb{R}$, we mean by $H^s(\partial R)$ the completion of $C^\infty(\partial R)$ with respect to the norm

$$|u|_s = \left(\sum_{j=1}^{\infty} \lambda_j^{2s} \langle u, w_j\rangle_0^2\right)^{1/2} . \tag{2.8}$$

$H^s(\partial R)$ is a Hilbert space with the inner product

$$\langle u, v\rangle_s = \sum_{j=1}^{\infty} \lambda_j^{2s} \langle u, w_j\rangle_0 \langle v, w_j\rangle_0 \tag{2.9}$$

and the two inner products (2.7) and (2.9) coincide when $s = 0$.

Remark 2.2. If p is any non-positive real number, then

$$(2.10) \qquad |u|_p = \sup_{v \in C^\infty(\partial R)} \frac{\langle u, v\rangle_0}{|v|_{-p}}.$$

Remark 2.3. If r and p are any two real numbers $r > p$, then

$$[H^r(\partial R), H^p(\partial R)]_\theta = H^{r\theta + p(1-\theta)}(\partial R)$$

for any $0 \leqq \theta \leqq 1$.

The boundary value problem. For any two real numbers l and s denote by $H^{(l,s)}$ the product space $H^{(l,s)} = H^l(R) \times H^s(\partial R)$ with the inner product

$$(2.11) \qquad (F, F_1)_{(l,s)} = (f, f_1)_l + \langle g, g_1\rangle_s$$

and norm

$$(2.12) \qquad \|F\|_{(l,s)} = (\|f\|_l^2 + |g|_s^2)^{1/2},$$

where $F = (f, g)$ and $F_1 = (f_1, g_1)$.

It will be essential for later purposes to consider a different inner product structure on the elements of $H^{(0,0)}$. Let $0 < h < \infty$ and $0 \leqq \gamma < \infty$ be given. Then by $H(h, \gamma)$ we mean the Hilbert space whose elements are those of $H^{(0,0)}$ but furnished with the inner product

$$(2.13) \qquad (F, F_1)_{(0,0)}^{(h,\gamma)} = (f, f_1)_0 + h^{-2\gamma}\langle g, g_1\rangle_0$$

and norm

$$(2.14) \qquad \|F\|_{(0,0)}^{(h,\gamma)} = (\|f\|_0^2 + h^{-2\gamma}|g|_0^2)^{1/2}.$$

We note that the norms (2.14) and (2.12) in the case $l = s = 0$ are equivalent and that they are equal when $\gamma = 0$.

We next give definitions of weak solutions of (2.2) under various assumptions on the regularity of the data. We shall use the following result which will be basic in what follows. In this theorem and throughout the paper we shall use C to denote a generic constant not necessarily the same in any two places.

THEOREM 2.1. (c.f. [18]) *Under Condition* A, *for any real number* p,

$$(2.15) \qquad \|u\|_p \leqq C(\|Au\|_{p-2} + |u|_{p-1/2})$$

for all $u \in C^\infty(\bar{R})$, *where* C *is independent of* u.

DEFINITION. Let $F = (f, g) \in H^{(p-2,p-1/2)}$ and $F_n = (f_n, g_n) \in C^\infty(\bar{R}) \times C^\infty(\partial R)$ converge to F in $H^{(p-2,p-1/2)}$, as $n \to \infty$. Let $u_n \in C^\infty(\bar{R})$ be the corresponding solution of (2.2) with data F_n (it is well known that such a solution exists and is unique). Then in view of (2.15) we define the weak solution of (2.2) to be the unique limit in $H^p(R)$ of the sequence $\{u_n\}$.

In the case $p \geqq 2$ we have

THEOREM 2.2. (c.f. [16]) *If Condition* A *holds and if* p *is a real number* $p \geqq 2$, *the norms* $\|u\|_p$ *and* $(\|Au\|^2_{p-2} + |u|^2_{p-1/2})^{1/2}$ *are equivalent norms. In other words, the mapping*

$$\mathscr{P}u = (Au, u)$$

is a homeomorphism of $H^p(R)$ *onto* $H^{(p-2,p-1/2)}$.

It will be convenient for later purposes to give an alternative form for (2.15) in the case when $p \leqq \frac{1}{2}$.

LEMMA 2.1. *Suppose that Condition* A *is satisfied and let* $h > 0$, $\gamma \geqq 0$ *and* $p \leqq \frac{1}{2}$ *be given. Then for all* $u \in H^p(R)$ *such that* $Au \in H^{(p-2)}(R)$ *and* $u \in H^{p-1/2}(\partial R)$ *(in the sense defined above)*,

$$\|u\|_p \leqq C \sup_{v \in C^\infty(\bar{R})} \frac{(Au, Av)_0 + h^{-2\gamma}\langle u, v\rangle_0}{(\|Av\|^2_{2-p} + h^{-4\gamma}|v|^2_{1/2-p})^{1/2}}, \tag{2.16}$$

where C *is a constant which is independent of* u, h *and* γ.

Proof: In view of (2.6) and (2.10), (2.15) may be rewritten in the form

$$\|u\|_p \leqq C\left\{\sup_{\psi \in C^\infty(\bar{R})} \frac{(Au, \psi)_0}{\|\psi\|_{2-p}} + \sup_{\varphi \in C^\infty(\partial R)} \frac{\langle u, \varphi\rangle_0}{|\varphi|_{1/2-p}}\right\}, \tag{2.17}$$

where we may restrict ourselves to those functions ψ and φ for which

$$\|\psi\|_{2-p} = |\varphi|_{1/2-p} = 1 .$$

Now let $\{\psi_n\}$ and $\{\varphi_n\}$ be maximizing sequences (subject to the above conditions) for the respective terms on the right-hand side of (2.17). For each n, let $v_n \in C^\infty(\bar{R})$ be the unique solution of $Av_n = \psi_n$ in R, $v_n = h^{2\gamma}\varphi_n$ on ∂R. Then,

$$\|u\|_p \leqq C\left\{\lim_{n\to\infty} \frac{(Au, Av_n)_0 + h^{-2\gamma}\langle u, v_n\rangle_0}{(\|Av_n\|^2_{2-p} + h^{-4\gamma}|v_n|^2_{1/2-p})^{1/2}}\right\}$$

from which the lemma easily follows.

3. Finite-Dimensional Subspaces of $H^k(R)$

Let h, $0 < h < 1$, be a parameter and Ω any fixed hypercube containing $\bar{R}$. For any two given non-negative integers k and r with $k < r$ let $S^h_{k,r}(\Omega)$ be any finite-dimensional subspace of $H^k(\Omega)$ with norm $\|\cdot\|^\Omega_k$ which satisfies the following condition:

(*) For any $u \in H^r(\Omega)$ there exists a $\bar{u} \in S^h_{k,r}(\Omega)$ and a constant C independent of h and u such that

$$\|u - \bar{u}\|^\Omega_k \leqq C\, h^{r-k}\, \|u\|^\Omega_r . \tag{3.1}$$

$H^k(\Omega)$ is the Sobolev space defined in Section 2, but relative to Ω.

This is obviously equivalent to the condition that

$$\inf_{\chi \in S^h_{k,r}} \|u - \chi\|^\Omega_k \leqq C\, h^{r-k}\, \|u\|^\Omega_r . \tag{3.2}$$

Remark 3.1. In the literature, the spaces $S^h_{k,r}(\Omega)$ usually refer to subspaces satisfying what might seem to be the stronger condition:

(**) For any $u \in H^j(\Omega)$ there exists a constant C independent of h and u such that

$$\inf_{\chi \in S^h_{k,r}} \|u - \chi\|^\Omega_l \leqq C\, h^{j-l}\, \|u\|^\Omega_j$$

for all non-negative integers j and l with $l \leqq k$ and $l \leqq j \leqq r$.

The property (*) will be sufficient for our needs in this paper. However, in another communication, we shall show (under reasonable conditions on Ω) that (*) implies that (**) holds in the more general case when l and j can be any real numbers (positive or negative) with $l \leqq k$, $l \leqq j \leqq r$, and where the constant C may depend on l and j.

The construction of such spaces has attracted much attention recently. For example S. Hilbert [15] constructs spaces of splines on uniform meshes in $\mathbb{R}^N$ and multi-dimensional Hermite functions both of which satisfy condition (*) for certain choices of k and r. In addition, Schultz [19] has studied many finite-dimensional subspaces on rectilinear domains in $\mathbb{R}^N$ (Ω could be a rectilinear domain) which satisfy (*). The papers of Aubin [3], Bramble and Zlámal [10] and DiGuglielmo [12] also contain examples of subspaces satisfying (*). Moreover, the works of Babuška [5] and Fix and Strang [13] are important in this regard.

The subspaces with which we shall be concerned will be defined as follows. Let $S^h_{k,r}$ be defined as the restriction to $\bar{R}$ of the functions in $S^h_{k,r}(\Omega)$. We then have

THEOREM 3.1. *Let* $u \in H^r(R)$, *and suppose that* $S^h_{k,r}(\Omega)$ *satisfies* (*). *Then there exists a constant* C *independent of* h *and* u *such that*

$$\inf_{\chi \in S^h_{k,r}} \|u - \chi\|_k \leqq C\, h^{r-k} \|u\|_r . \tag{3.3}$$

The proof follows immediately from (*) by an easy application of the Calderón extension theorem (c.f. [1]).

We shall now show that the spaces $S^h_{k,r}$ have certain approximation-theoretic properties relative to the "data spaces" of the differential operators considered here. For simplicity we shall only prove results which are directly related to the approximation schemes which will be presented in Section 4. Many variations of the following results can be proved which are useful in studying other types of approximation schemes for solutions of elliptic equations higher than second order with more general boundary conditions. We wish to remark also that for convenience we shall confine ourselves here to "L_2" theory; however, the Hilbert space structures on the spaces we consider are not essential to the proofs.

The remainder of this section will be devoted to proving the following approximation theoretic result.

THEOREM 3.2. *Let* $S^h_{k,r}(\Omega)$ *satisfy* (*) *with* $2 = k < r$. *Suppose that* $0 \leqq \gamma \leqq \frac{3}{2}$, $\sigma = (r - 2 + \gamma)/(r - \frac{1}{2})$ *and* $F = (f, g) \in H^{(\lambda, \lambda_0)}$, *where* $0 \leqq \lambda \leqq r - 2$ *and* $0 \leqq \lambda_0 \leqq r - \frac{1}{2}$. *Then there exists a constant* C *independent of* F *and* h *such that*

$$\inf_{\chi \in S^h_{k,r}} (\|f - A\chi\|_0^2 + h^{-2\gamma} |g - \chi|_0^2)^{1/2} \leqq C\,(h^{2\lambda} \|f\|_\lambda^2 + h^{2(\sigma\lambda_0 - \gamma)} |g|_{\lambda_0}^2)^{1/2} , \tag{3.4}$$

or equivalently

$$\inf_{\chi \in S^h_{k,r}} (\|f - A\chi\|_0 + h^{-\gamma} |g - \chi|_0) \leqq C\,(h^{\lambda} \|f\|_\lambda + h^{\sigma\lambda_0 - \gamma} |g|_{\lambda_0}) . \tag{3.5}$$

The proof of Theorem 3.2 is lengthy. The following preliminary remarks will be useful.

Remark 3.2. If X is a Banach space and N a closed subspace of X, then the quotient space X/N is a Banach space with norm

$$\|\{u\}\|_{X/N} = \inf_{v \in N} \|u - v\|_X ,$$

where $\{u\}$ denotes the equivalence class to which u belongs. The triangle inequality then states that

$$\inf_{v \in N} \|u_1 + u_2 - v\|_X \leqq \inf_{v_1 \in N} \|u_1 - v_1\|_X + \inf_{v_2 \in N} \|u_2 - v_2\|_X .$$

Remark 3.3. Suppose X and Y are Banach spaces, $X \subset Y$, and N is a closed subspace of both X and Y. Suppose C is a constant such that for all $u \in X$

$$\inf_{v \in N} \|u - v\|_Y \leqq C \|u\|_X .$$

Then

$$\inf_{v \in N} \|u - v\|_Y \leqq C \inf_{v \in N} \|u - v\|_X$$

for all $u \in X$ with the same constant C.

In order to prove Theorem 3.2 we shall need some lemmas.

LEMMA 3.1. *Let $S^h_{k,r}(\Omega)$ satisfy (*) with $2 = k < r$. Then for any $F = (f, g) \in H^{(r-2, r-1/2)}$ there exists a constant $C_1 \geqq 1$ independent of h and F such that*

$$\inf_{\chi \in S^h_{k,r}} (\|f - A\chi\|_0^2 + |g - \chi|_{3/2}^2)^{1/2} \leqq C_1 h^{r-2} (\|f\|_{r-2}^2 + |g|_{r-1/2}^2)^{1/2} . \tag{3.6}$$

The proof follows immediately from (3.3) and Theorem 2.2.

The following lemma is easily proved by a standard technique involving the "three lines theorem".

LEMMA 3.2. *Let Y, X_0 and X_1 be Hilbert spaces, X_1 being densely contained in X_0 with a continuous injection. Suppose that T is a bounded linear mapping from X_i into Y, $i = 0, 1$, and for some positive constants m_i, $i = 0, 1$,*

$$\|Tu\|_Y \leqq m_i \|u\|_{X_i} , \qquad i = 0, 1 . \tag{3.7}$$

Then T is a bounded linear mapping from $[X_1, X_0]_\theta$ into Y and

$$\|Tu\|_Y \leqq (m_0)^{1-\theta} (m_1)^\theta \|u\|_{[X_1, X_0]_\theta} , \qquad 0 \leqq \theta \leqq 1 . \tag{3.8}$$

We shall also need the following technical lemma.

LEMMA 3.3. *Let $S^h_{k,r}(\Omega)$ satisfy (*) with $2 = k < r$. Suppose that for some constants $\beta \geqq 0$ and $M_1 > 0$*

$$\inf_{\chi \in S^h_{k,r}} (\|f - A\chi\|_0^2 + h^{-3} |g - \chi|_0^2)^{1/2} \leqq 2(\|f\|_0^2 + M_1^2 h^{2(r-2-\beta)} |g|_{r-1/2}^2)^{1/2} \tag{3.9}$$

for all $F \in H^{(0, r-1/2)}$. Then

$$\inf_{\chi \in S^h_{k,r}} (\|f - A\chi\|_0^2 + h^{-3} |g - \chi|_0^2)^{1/2} \leqq 2(\|f\|_0^2 + M_2^2 h^{2(r-2-\alpha\beta)} |g|_{r-1/2}^2)^{1/2} , \tag{3.10}$$

where $M_2 = 2^{1+\alpha}C_1C_2M_1^\alpha$, $\alpha = \frac{3}{2}/(r - \frac{1}{2})$, C_1 *is as in* (3.6) *and* $C_2 \geqq 1$ *is a constant such that*

$$|g|_{[H^{r-1/2}(\partial R), H^0(\partial R)]_\alpha} \leqq C_2 \, |g|_{3/2}$$

for all $g \in H^{3/2}(\partial R)$.

Proof: Setting $(f, g) = (f, 0) + (0, g)$, we see in view of Remark 3.2 that

(3.11)

$$\inf_{\chi\in S^h_{k,r}} (\|f - A\chi\|_0^2 + h^{-3}|g - \chi|_0^2)^{1/2} \leqq \|f_0\| + \inf_{\psi\in S^h_{k,r}} (\|A\psi\|_0^2 + h^{-3}|g - \psi|_0^2)^{1/2} .$$

Now

$$\inf_{\psi\in S^h_{k,r}} (\|A\psi\|_0^2 + h^{-3}|g - \psi|_0^2)^{1/2} \leqq h^{-3/2}|g|_0 \tag{3.12}$$

and, in view of (3.9),

$$\inf_{\psi\in S^h_{k,r}} (\|A\psi\|_0^2 + h^{-3}|g - \psi|_0^2)^{1/2} \leqq 2M_1h^{r-2-\beta}|g|_{r-1/2} . \tag{3.13}$$

Let us interpolate the inequalities (3.12) and (3.13). In Lemma 3.2, take $Y = H(h, \frac{3}{2})/\mathscr{P}(S^h_{k,r})$, i.e., the quotient space of $H(h, \frac{3}{2})$ modulo the finite-dimensional subspace $\mathscr{P}(S^k_{k,r})$ (the image of $S^h_{k,r}$ under the mapping $\mathscr{P}\psi \to (A\psi, \psi)$), $X_0 = H^0(\partial R)$, $X_1 = H^{r-1/2}(\partial R)$, $m_0 = h^{-3/2}$, $m_1 = 2M_1h^{r-2-\beta}$ and $T = T_2 \circ T_1$. Here T_1 maps $H^0(\partial R)$ into H with $T_1 g = (0, g)$ and T_2 is the canonical map of $H(h, \frac{3}{2})$ onto $H(h, \frac{3}{2})/\mathscr{P}(S^h_{k,r})$. In this case, (3.8) implies that for all $g \in H^{(r-1/2)\theta}(\partial R)$, $0 \leqq \theta \leqq 1$,

$$\inf_{\psi\in S^h_{k,r}} (\|A\psi\|_0^2 + h^{-3}|g - \psi|_0^2)^{1/2} \leqq C_2(\theta)(2M_1)^\theta h^{-3/2+(r-1/2-\beta)\theta}|g|_{(r-1/2)\theta} , \tag{3.14}$$

where $C_2(\theta)$ is a constant which depends on θ with

$$\|g\|_{[H^{r-1/2}(\partial R), H^0(\partial R)]_\theta} \leqq C_2(\theta) \, |g|_{(r-1/2)\theta} .$$

Choosing $\theta = \frac{3}{2}/(r - \frac{1}{2}) = \alpha$, (3.14) becomes

$$\inf_{\psi\in S^h_{k,r}} (\|A\psi\|_0^2 + h^{-3}|g - \psi|_0^2)^{1/2} \leqq C_3h^{-\beta\alpha}|g|_{3/2} , \tag{3.15}$$

where $C_3 = \max(C_2(2M_1)^\alpha, 1)$ which is independent of g and h and where for simplicity we have set $C_2(\alpha) = C_2$. Now from (3.15) and (3.11) we have

$$\inf_{\chi\in S^h_{k,r}} (\|f - A\chi\|_0^2 + h^{-3}|g - \chi|_0^2)^{1/2} \leqq \|f\|_0 + C_3h^{-\beta\alpha}|g|_{3/2} ,$$

and in view of Remark 3.3 we see that, for any $\psi \in S^h_{k,r}$,

$$\inf_{\chi \in S^h_{k,r}} (\|f - A\chi\|_0^2 + h^{-3}|g - \chi|_0^2)^{1/2} \leqq \|f - A\psi\|_0 + C_3 h^{-\beta\alpha}|g - \psi|_{3/2}$$

and hence

$$\begin{aligned}\inf_{\chi \in S^h_{k,r}} &(\|f - A\chi\|_0^2 + h^{-3}|g - \chi|_0^2)^{1/2} \\ &\leqq \|f\|_0 + 2C_3 h^{-\beta\alpha} \inf_{\psi \in S^h_{k,r}} (\|A\psi\|_0^2 + |g - \psi|_{3/2}^2)^{1/2} \\ &\leqq \|f\|_0 + 2C_1C_3 h^{r-2-\beta\alpha}|g|_{r-1/2} \\ &\leqq 2(\|f\|_0^2 + M_2^2 h^{2(r-2-\beta\alpha)}|g|_{r-1/2}^2)^{1/2},\end{aligned}$$

where we have used Lemma 3.1. This completes the proof.

We shall now prove Theorem 3.2 in the case that $\gamma = \frac{3}{2}$, $\lambda = 0$ and $\lambda_0 = r - \frac{1}{2}$.

LEMMA 3.4. *Let $S^h_{k,r}(\Omega)$ satisfy (*) with $2 = k < r$. Suppose that $F = (f, g) \in H^{(0,r-1/2)}$. Then there exists a constant C_4 independent of h and F such that*

$$\inf_{\chi \in S^h_{k,r}} (\|f - A\chi\|_0^2 + h^{-3}|g - \psi|^2)^{1/2} \leqq C_4(\|f\|_0^2 + h^{2(r-2)}|g|_{r-1/2}^2)^{1/2}. \tag{3.16}$$

Proof: For any $\psi \in S^h_{k,r}$ we have

$$\inf_{\chi \in S^h_{k,r}} (\|f - A\chi\|_0^2 + h^{-3}|g - \chi|_0^2)^{1/2} \leqq \|f\|_0 + (\|A\psi\|_0^2 + h^{-3}|g - \psi|_0^2)^{1/2},$$

and therefore

$$\begin{aligned}\inf_{\chi \in S^h_{r,k}} &(\|f - A\chi\|_0^2 + h^{-3}|g - \chi|_0^2)^{1/2} \\ &\leqq \|f\|_0 + C_5 h^{-3/2} \inf_{\psi \in S^h_{k,r}} (\|A\psi\|_0^2 + |g - \psi|_{3/2}^2)^{1/2} \\ &\leqq \|f\|_0 + C_1C_5 h^{r-2-3/2}|g|_{r-1/2} \\ &\leqq 2(\|f\|_0^2 + C_1^2C_5^2 h^{2(r-2-3/2)}|g|_{r-1/2}^2)^{1/2},\end{aligned} \tag{3.17}$$

where we have used (3.6) and where $|g|_0 \leqq C_5|g|_{3/2}$ for all $g \in H^{3/2}(\partial R)$. The inequality (3.17) is a rather bad estimate. However, it may be used to yield (3.16) by iterating it with the help of Lemma 3.3. Namely, we start with $M_1 = C_1C_5$ and $\beta = \frac{3}{2}$ in Lemma 3.3 and then apply the lemma again to the result of the first application. Repeating this process s times, it is easily seen that

$$(3.18)\qquad \inf_{\chi \in S^h_{k,r}} (\|f - A\chi\|_0^2 + h^{-3}|g - \chi|_0^2)^{1/2} \leqq 2(\|f\|_0 + M_s^2 h^{2(r-2-3\alpha^s/2)} |g|^2_{r-1/2})^{1/2}$$

with $M_s \leqq (4C_1C_2)^{1/(1-\alpha)} C_5^{(\alpha)^s}$ and $\alpha = \frac{3}{2}/(r - \frac{1}{2}) < 1$, where we have used the fact that $C_1, C_2 \geqq 1$.

Taking the limit as s tends to infinity in (3.18), we obtain (3.16) which completes the proof.

Proof of Theorem 3.2: Suppose that $0 \leqq \gamma \leqq \frac{3}{2}$. Now,

$$(3.19)\qquad \begin{aligned} \inf_{\chi \in S^h_{k,r}} (\|f - A\chi\|_0^2 + h^{-2\gamma}|g - \chi|_0^2)^{1/2} &\leqq \inf_{\psi \in S^h_{k,r}} (\|A\psi\|_0^2 + h^{-2\gamma}|g - \psi|_0^2)^{1/2} \\ &\quad + \inf_{\varphi \in S^h_{k,r}} (\|f - A\varphi\|_0^2 + h^{-2\gamma}|\varphi|_0^2)^{1/2}. \end{aligned}$$

Let us first estimate the first term on the right-hand side of (3.19) in terms of $|g|_{\lambda_0}$ for any $0 \leqq \lambda_0 \leqq r - \frac{1}{2}$. We have trivially

$$(3.20)\qquad \inf_{\psi \in S^h_{k,r}} (\|A\psi\|_0^2 + h^{-2\gamma}|g - \psi|_0^2)^{1/2} \leqq h^{-\gamma}|g|_0$$

and, in view of (3.16),

$$(3.21)\qquad \begin{aligned} \inf_{\psi \in S^h_{k,r}} (\|A\psi\|_0^2 + h^{-2\gamma}|g - \psi|_0^2)^{1/2} &\leqq \inf_{\psi \in S^h_{k,r}} (\|A\psi\|_0^2 + h^{-3}|g - \psi|_0^2)^{1/2} \\ &\leqq C_4 h^{r-2}|g|_{r-1/2}. \end{aligned}$$

Interpolating the two inequalities (3.20) and (3.21) in the same fashion as in the proof of Lemma 3.3, we easily obtain

$$(3.22)\qquad \inf_{\psi \in S^h_{k,r}} (\|A\psi\|_0^2 + h^{-2\gamma}|g - \psi|_0^2)^{1/2} \leqq C_6 h^{-\gamma + (r-2+\gamma)\lambda_0/(r-1/2)}|g|_{\lambda_0},$$

where $0 \leqq \lambda_0 \leqq r - \frac{1}{2}$ and C_6 is a constant which is independent of g and h but may depend on λ_0.

We shall now estimate the second term in the right-hand side of (3.19) in terms of $\|f\|_\lambda$ for any $0 \leqq \lambda \leqq r - 2$. Let $\gamma = \frac{3}{2}$ and $\lambda_0 = \frac{3}{2}$ in (3.22). Then, in view of (3.11), Remark 3.2 and (3.6), we have

$$(3.23)\qquad \begin{aligned} &\inf_{\psi \in S^h_{k,r}} (\|f - A\psi\|_0^2 + h^{-3}|g - \psi|_0^2)^{1/2} \\ &\qquad \leqq C_7 \inf_{\psi \in S^h_{k,r}} (\|f - A\psi\|_0^2 + |g - \psi|^2_{3/2})^{1/2} \\ &\qquad \leqq C_8 h^{r-2}(\|f\|^2_{r-2} + |g|^2_{r-1/2})^{1/2} \end{aligned}$$

for all $F = (f, g) \in H^{(r-2, r-1/2)}$, where C_8 is a constant which is independent of F and h. Clearly,

$$\inf_{\psi \in S^h_{k,r}} (\|f - A\,\psi\|_0^2 + h^{-2\gamma}\,|\psi|_0^2)^{1/2} \leqq \|f\|_0 , \tag{3.24}$$

and from (3.23)

$$\begin{aligned} \inf_{\psi \in S^h_{k,r}} (\|f - A\,\psi\|_0^2 + h^{-2\gamma}\,|\psi|_0^2)^{1/2} &\leqq \inf_{\psi \in S^h_{k,r}} (\|f - A\,\psi\|_0^2 + h^{-3}\,|\psi|_0^2)^{1/2} \\ &\leqq C_8\, h^{r-2}\, \|f\|_{r-2} \end{aligned} \tag{3.25}$$

for all $f \in H^{r-2}(R)$.

Let us interpolate the two inequalities (3.24) and (3.25). In Lemma 3.2 we take $Y = H(h, \gamma)/\mathscr{P}(S^h_{k,r})$, $X_0 = H^0(R)$, $X_1 = H^{r-2}(R)$, $m_0 = 1$, $m_2 = C_8 h^{r-2}$, $T = T_2 \circ T_1$, where T_1 maps $H^0(R)$ into $H(h, \gamma)$ with $T_1 f = (f, 0)$ and T_2 is the canonical mapping of $H(h, \gamma)$ onto $H(h, \gamma)/\mathscr{P}(S^h_{k,r})$. We then obtain from (3.8)

$$\inf_{\psi \in S^h_{k,r}} (\|f - A\,\psi\|_0^2 + h^{-2\gamma}\,|\psi|_0^2)^{1/2} \leqq C_9\, h^{\lambda}\, \|f\|_{\lambda} \tag{3.26}$$

for all $f \in H^{\lambda}(R)$, where $0 \leqq \lambda \leqq r - 2$ and C_9 is a constant which is independent of f and g but may depend on λ. The inequality (3.4) now follows easily from (3.19), (3.22) and (3.26) which completes the proof of Theorem 3.2.

4. Methods of Rayleigh-Ritz-Galerkin Type for Dirichlet's Problem

In this section we propose schemes (including the usual least squares scheme) for finding approximate solutions of Dirichlet's problem using the subspaces $S^h_{k,r}$ defined in Section 3. We shall then prove some facts concerning the rates of convergence of these methods. Among the schemes considered here, one seems the most promising (the case $\gamma = \frac{3}{2}$ given below) in that it gives the best possible results relative to the properties of the subspaces $S^h_{k,r}$ used. A discussion of this scheme and also a comparison with the usual least squares method is given immediately after the main results of this section, i.e., Theorem 4.1 and Corollary 4.1. Further results for the case $\gamma = \frac{3}{2}$ are given in Theorems 4.2, 4.3 and Corollary 4.2.

Some approximation schemes. Let u be a solution of (2.2), where for the present we shall assume that $F = (f, g) \in H^{(0,0)}$. The approximate schemes we shall consider are as follows:

Let $S^h_{k,r}$ and γ be given with $2 = k < r$ and $0 \leqq \gamma \leqq \frac{3}{2}$. Find $w \in S^h_{k,r}$ such that

$$(4.1)\qquad \begin{aligned} &(f - A\,w, A\,\varphi)_0 + h^{-2\gamma}\langle g - w, \varphi\rangle_0 \\ &\qquad \equiv \int_R (f - A\,w)A\,\varphi\,dx + h^{-2\gamma}\int_{\partial R}(g - w)\varphi\,d\sigma = 0 \end{aligned}$$

for all $\varphi \in S^h_{k,r}$.

Since $\mathscr{P}(S^h_{k,r})$ (the image of $S^h_{k,r}$ under the mapping $\mathscr{P}\varphi = (A\varphi, \varphi)$) is a finite-dimensional subspace of $H(h, \gamma)$, by (ii) of Condition A (uniqueness), w exists and is unique. It is determined by solving a linear system of algebraic equations whose coefficients depend only on f, g, h and γ. An alternative way of stating (4.1) is the following: Among all $\chi \in S^h_{k,r}$ find the one which minimizes the functional

$$(4.2)\qquad \|f - A\,\chi\|_0^2 + h^{-2\gamma}|g - \chi|_0^2\,.$$

Remarks. 1. The trial functions are not required to satisfy the boundary conditions.

2. In all cases only L_2 norms over the domain and boundary are involved in the computation of the solution.

3. We have restricted ourselves here to schemes where $0 \leqq \gamma \leqq \frac{3}{2}$. One could of course also consider $\frac{3}{2} < \gamma < \infty$. Some remarks concerning this case will be given in this section immediately after the proof of Theorem 4.1.

Let $e = u - w$. The following theorems give error estimates for the error function e in the schemes discussed above.

THEOREM 4.1. *Suppose that Condition* A *is satisfied and* u *is the solution of* (2.2) *with* $F = (f, g) \in H^{(0,0)}$. *For given* $S^h_{k,r}$ *and* γ, *with* $2 = k < r$ *and* $0 \leqq \gamma \leqq \frac{3}{2}$, *let* w *be the solution of the approximate problem* (4.1).

Case i. *Suppose that* $r \geqq 4$ *and* l *satisfies* $4 - r \leqq l \leqq \frac{1}{2}$. *Then there exists a constant* C *independent of* h *and* F *such that*

$$(4.3)\qquad \|e\|_l \leqq C\,h^{\rho_1}(\|A\,e\|_0^2 + h^{-2\gamma}|e|_0^2)^{1/2} \leqq C\,h^{\rho_1}(\|f\|_0 + h^{-\gamma}|g|_0)$$

where

$$\rho_1 = \frac{r - 2 + \gamma}{r - \frac{1}{2}}\left(\tfrac{1}{2} - l\right) + \gamma\,.$$

Case ii. *Suppose that* $r = 3$ *and* $0 \leqq l \leqq \frac{1}{2}$. *Then there exists a constant* C *independent of* h *and* F *such that*

$$(4.4)\qquad \|e\|_l \leqq h^{\rho_2}(\|A\,e\|_0^2 + h^{-2\gamma}|e|_0^2)^{1/2} \leqq C\,h^{\rho_2}(\|f\|_0 + h^{-\gamma}|g|_0)\,,$$

where

$$\rho_2 = \min\left(1, \frac{2(1+\gamma)(\frac{1}{2}-l)}{5} + \gamma\right).$$

If u is smoother i.e., if the data are smoother we have

COROLLARY 4.1. *Suppose that the conditions of Theorem* 4.1 *are satisfied and in addition* $F = (f, g) \in H^{(\lambda,\lambda_0)}$, *where* $0 \leqq \lambda \leqq r-2$ *and* $0 \leqq \lambda_0 \leqq r - \frac{1}{2}$. *Let* $\sigma = (r-2+\gamma)/(r-\frac{1}{2})$.

Case i. *If* $r \geqq 4$ *and* $4 - r \leqq l \leqq \frac{1}{2}$, *then*

$$\|e\|_l \leqq Ch^{\rho_1}(h^\lambda \|f\|_\lambda + h^{(\sigma\lambda_0-\gamma)}|g|_{\lambda_0}). \tag{4.5}$$

Case ii. *If* $r = 3$, *and* $0 \leqq l \leqq \frac{1}{2}$, *then*

$$\|e\|_l \leqq Ch^{\rho_2}(h^\lambda \|f\|_\lambda + h^{(\sigma\lambda_0-\gamma)}|g|_{\lambda_0}. \tag{4.6}$$

In (4.5) *and* (4.6), C *is a constant which is independent of* h *and* F.

Proof of Corollary 4.1: Since

$$(\|Ae\|_0^2 + h^{-2\gamma}|e|_0^2)^{1/2} = \inf_{\chi\in S^h_{k,r}} (\|f - A\chi\|_0^2 + h^{-2\gamma}|g-\chi|_0^2)^{1/2},$$

both (4.5) and (4.6) follow immediately from (4.3) and (4.4), respectively, after applying Theorem 3.2.

Proof of Theorem 4.1: In view of Lemma 2.1,

$$\|e\|_l \leqq C \sup_{\psi\in C^\infty(\bar{R})} \frac{(Ae, A\psi)_0 + h^{-2\gamma}\langle e, \psi\rangle_0}{(\|A\psi\|_{2-l}^2 + h^{-4\gamma}|\psi|_{1/2-l}^2)^{1/2}}.$$

Since e satisfies (4.1) we know that, for any $\chi \in S^h_{k,r}$,

$$\|e\|_l \leqq C \sup_{\psi\in C^\infty(\bar{R})} \frac{(Ae, A\psi - A\chi)_0 + h^{-2\gamma}\langle e, \psi-\chi\rangle_0}{(\|A\psi\|_{2-l}^2 + h^{-4\gamma}|\psi|_{1/2-l}^2)^{1/2}},$$

and, using Schwarz's inequality,

$$\|e\|_l \leqq C(\|Ae\|_0^2 + h^{-2\gamma}|e|_0^2)^{1/2} \sup_{\psi\in C^\infty(\bar{R})} \frac{(\|A\psi - A\bar{\chi}\|_0^2 + h^{-2\gamma}|\psi-\bar{\chi}|_0^2)^{1/2}}{(\|A\psi\|_{2-l}^2 + h^{-4\gamma}|\psi|_{1/2-l}^2)^{1/2}}, \tag{4.7}$$

where $\bar{\chi} \in S^h_{k,r}$ may depend on ψ.

Case i. Suppose that $r \geqq 4$ and $4 - r \leqq l \leqq \frac{1}{2}$. Then for each $\psi \in C^\infty(\bar{R})$ we can choose $\bar{\chi} \in S^h_{k,r}$ so that

$$
\begin{aligned}
&(\|A\psi - A\bar{\chi}\|_0^2 + h^{-2\gamma}|\psi - \bar{\chi}|_0^2)^{1/2} \\
&\qquad = \inf_{\varphi \in S^h_{k,r}} (\|A\psi - A\varphi\|_0^2 + h^{-2\gamma}|\psi - \varphi|_0^2)^{1/2} \\
&\qquad \leqq C(h^{2(2-l)}\|A\psi\|_{2-l}^2 + h^{2(\sigma(1/2-l)-\gamma)}|\psi|_{1/2-l}^2)^{1/2} \\
&\qquad \leqq C\,h^{\rho_1}(\|A\psi\|_{2-l}^2 + h^{-4\gamma}|\psi|_{1/2-l}^2)^{1/2}\,;
\end{aligned}
\tag{4.8}
$$

we have used Theorem 3.2 in the case $\lambda = 2 - l$ and $\lambda_0 = \frac{1}{2} - l$. The inequality (4.3) then follows from (4.7) and (4.8).

Case ii. If $r = 3$ and $0 \leqq l \leqq \frac{1}{2}$, then we proceed as in Case i except that now we can only apply Theorem 3.2 in the case $\lambda = 1$ and $\lambda_0 = \frac{1}{2} - l$. Instead of (4.8) we obtain

$$
\begin{aligned}
(\|A\psi - A\bar{\chi}\|_0^2 + h^{-2\gamma}|\psi - \bar{\chi}|_0^2)^{1/2} &\leqq C(h^2\|A\psi\|_1^2 + h^{2(\sigma(1/2-l)-\gamma)}|\psi|_{1/2-l}^2)^{1/2} \\
&\leqq C\,h^{\rho_2}(\|A\psi\|_1^2 + h^{-4\gamma}|\psi|_{1/2-l}^2)^{1/2}\,.
\end{aligned}
\tag{4.9}
$$

Inequality (4.4) follows easily from (4.7) and (4.9), which completes the proof.

A discussion of Theorem 4.1 and Corollary 4.1. The results of Theorem 4.1 and Corollary 4.1 show that in the case $r \geqq 4$ the approximation scheme (4.1) with $\gamma = \frac{3}{2}$ gives the best possible results relative to the approximation-theoretic properties of the space $S^h_{k,r}$.

To illustrate this let us estimate the error in the $L_2(R) = H^0(R)$ norm. According to our assumptions concerning $S^h_{k,r}$, the best approximation $\bar{u}$ in $S^h_{k,r}$ to $u \in H^{2+\rho}(R)$ (with respect to the L_2 norm) satisfies in general only the inequality

$$
\|u - \bar{u}\|_0 \leqq Ch^{2+\rho}\,\|u\|_{2+\rho}\,. \tag{4.10}
$$

Suppose now that w is the solution of the approximate problem (4.1) for any $0 \leqq \gamma \leqq \frac{3}{2}$. In order to make a comparison with (4.10), let us assume that $\lambda_0 = \lambda + \frac{3}{2}$. Then $u \in H^{2+\lambda}(R)$ and the norm $\|f\|_\lambda + |g|_{\lambda+3/2}$ is equivalent to the norm $\|u\|_{2+\lambda}$. We then see from Corollary 4.1 that the error $u - w$ (where of course w depends on γ) may be estimated by

$$
\|u - w\|_0 \leqq Ch^{(r-2+\gamma)(\lambda+2)/(r-1/2)}\,\|u\|_{2+\lambda} \tag{4.11}
$$

for any $0 \leqq \lambda \leqq r - 2$. Therefore, for any given λ in this range the maximum rate of convergence occurs for $\gamma = \frac{3}{2}$ and we have

$$
\|u - w\|_0 \leqq Ch^{2+\lambda}\,\|u\|_{2+\lambda}\,, \tag{4.12}
$$

which essentially reproduces the property (4.10).

If $u \in H^r(R)$ (i.e., $F = (f, g) \in H^{(r-2, r-1/2)}$) we see from (4.12) that for $\gamma = \frac{3}{2}$ the maximum power of h is r, while for $\gamma = 0$ (the usual least squares approximation scheme) the maximum power is $(r-2)r/(r-\frac{1}{2})$. The difference between the exponents is at least $\frac{3}{2}$ for any r. This is most significant when r is chosen to be close to 4 which is desirable from the point of view of computing. A more general comparison of the schemes can be made. It is easy to see from (4.5) that for given data $F \in H^{(\lambda, \lambda_0)}$, where $0 \leqq \lambda \leqq r-2$ and $0 \leqq \lambda_0 \leqq r - \frac{1}{2}$, the scheme with $\gamma = \frac{3}{2}$ gives the highest rate of convergence. The same holds true if the error is estimated in the $H^l(R)$ norm for any $4 - r \leqq l \leqq \frac{1}{2}$.

For $r = 3$, it is more difficult to compare the results for various values of γ. However, for given smoothness of the data and given norm in which we wish to estimate the error, one can always find values of γ, $0 \leqq \gamma \leqq \frac{3}{2}$, for which the highest rate of convergence occurs. Our estimates do not indicate that the properties of w are as good as those of the best approximation in $S^h_{2,3}$. Examples will be given in the following section.

The case $\gamma > \frac{3}{2}$ could also have been treated by methods similar to those used in the case $0 \leqq \gamma \leqq \frac{3}{2}$. However, we always obtain for the scheme (4.1) with $\gamma > \frac{3}{2}$ lower rates of convergence than with $\gamma = \frac{3}{2}$.

Non-smooth data; $f \in H^\lambda(R)$, $-2 \leqq \lambda < 0$. Thus far our schemes only apply when $f \in H^\lambda(R)$, with $\lambda \geqq 0$. We shall now propose a method of dealing with less smooth functions f. For simplicity we shall restrict our attention to the case in which $r \geqq 4$ and $\gamma = \frac{3}{2}$.

Suppose $f \in H^\lambda(R)$ with $-2 \leqq \lambda < 0$ and $\operatorname{supp}(f)$ is a compact subset of R. We first smooth the data in the following way. Let

$$\varphi_h(x) = \begin{cases} 1/h^N, & -h/2 < x_i \leqq h/2, \quad i = 1, \cdots, N, \\ 0, & \text{elsewhere}, \end{cases} \tag{4.13}$$

and define $\Phi_h = \varphi_h * \varphi_h$, $*$ denoting the operation of convolution. Set $f_h = \Phi_h * f$.

The following lemma is easily proved using properties of Fourier transforms and convolutions (c.f. [8]).

LEMMA 4.1. *Let $f \in H^\lambda(R)$, where $-2 \leqq \lambda \leqq 0$, f_h is given as above and h is chosen small enough so that* $\operatorname{supp}(f_h) \subset \bar{R}$. *Then we have*

$$\|f_h\|_0 \leqq Ch^\lambda \|f\|_\lambda \tag{4.14}$$

and

$$\|f - f_h\|_{-2} \leqq Ch^{2+\lambda} \|f\|_\lambda, \tag{4.15}$$

where C is independent of h and f.

This lemma shows that $f \in H^{-2}(R)$ implies $f_h \in H^0(R)$. Now let $w_h \in S^h_{k,r}$ be the unique solution of

$$(f_h - Aw_h, A\varphi)_0 + h^{-3}\langle g - w_h, \varphi\rangle_0 = 0 \tag{4.16}$$

for all $\varphi \in S^h_{k,r}$. We shall consider w_h as an approximate solution of (2.2) for given $F = (f, g)$. The following theorem gives an estimate for the error $e = u - w_h$.

THEOREM 4.2. *Suppose that Condition* A *is satisfied and let* $S^h_{k,r}$ *be given with* $k = 2$ *and* $r \geqq 4$. *Let* u *be the solution of* (2.2) *with* $F = (f, g) \in H^{(\lambda,\lambda_0)}$, *where* $-2 \leqq \lambda < 0$ *and* $0 \leqq \lambda_0 \leqq r - \frac{1}{2}$. *Suppose further that* f *has compact support in* R. *Furthermore, let* w_h *be the solution of the approximate problem* (4.16), h *being chosen so that the support of* f_h *is contained in* $\bar{R}$. *Then*

$$\|u - w_h\|_0 \leqq C(h^{2+\lambda}\|f\|_\lambda + h^{1/2+\lambda_0}|g|_{\lambda_0}), \tag{4.17}$$

where C *is a constant independent of* h *and* F.

Proof: Let u_h be the solution of (2.2) with f replaced by f_h. Then

$$\|u - w_h\|_0 \leqq \|u - u_h\|_0 + \|u_h - w_h\|_0. \tag{4.18}$$

In view of (2.15) with $p = 0$, we have

$$\|u - u_h\|_0 \leqq C(\|f - f_h\|_{-2}).$$

Since the conditions of Lemma 4.1 are satisfied, we obtain

$$\|u - u_h\|_0 \leqq Ch^{2+\lambda}\|f\|_\lambda. \tag{4.19}$$

From (4.5) we have

$$\begin{aligned} \|u_h - w_h\|_0 &\leqq C(h^2\|f_h\|_0 + h^{1/2+\lambda_0}|g|_{\lambda_0}) \\ &\leqq C(h^{2+\lambda}\|f\|_\lambda + h^{1/2+\lambda_0}|g|_{\lambda_0}), \end{aligned} \tag{4.20}$$

where we have also used (4.14). The theorem follows using (4.19) and (4.20) in (4.18).

Remark. The particular choice of smoothing operator is somewhat arbitrary. The results of Theorem 4.3 will still hold provided we define f_h in such a way that Lemma 4.1 remains valid.

Interior estimates.

THEOREM 4.3. *Suppose that Condition* A *is satisfied and* u *is a solution of* (2.2). *For given* $S^h_{k,r}$ *with* $2 = k < r$, *let* w *be the solution of the approximate problem* (4.1) *in the case* $\gamma = \frac{3}{2}$, *where* $F = (f, g) \in H^{(\lambda,\lambda_0)}$ *with* $0 \leqq \lambda \leqq r - 2$ *and* $0 \leqq \lambda_0 \leqq r - \frac{1}{2}$. *Let* R_1 *be any compact subset of* R. *Then for each* θ, $0 \leqq \theta \leqq 1$, *there exists a constant* C *independent of* h *and* $F = (f, g)$ *(but in general depending on* R_1*) such that the following hold:*

Case 1. *If* $r \geqq 4$, *then*

$$\|e\|_{2\theta}^{R_1} \leqq Ch^{2(1-\theta)}(h^{\lambda} \|f\|_{\lambda} + h^{-3/2+\lambda_0} |g|_{\lambda_0}) . \tag{4.21}$$

Case 2. *If* $r = 3$, *then*

$$\|e\|_{1+\theta}^{R_1} \leqq Ch^{(1-\theta)}(h^{\lambda} \|f\|_{\lambda} + h^{-3/2+\lambda_0} |g|_{\lambda_0}) . \tag{4.22}$$

$\|\cdot\|_s^{R_1}$ *denotes the norm on* $H^s(R_1)$.

Proof: We shall use the well known interior estimate (c.f. [1])

$$\|e\|_2^{R_1} \leqq C(\|Ae\|_0 + \|e\|_0) , \tag{4.23}$$

where C is a constant which depends on R_1 but is independent of e.

Suppose now that $r \geqq 4$; it then follows from (4.3) that

$$\|e\|_0^{R_1} \leqq Ch^2(\|Ae\|_0 + h^{-3/2} |e|_0) , \tag{4.24}$$

and from (4.3) and (4.23)

$$\|e\|_2^{R_1} \leqq C(\|Ae\|_0 + h^{-3/2} |e|_0) . \tag{4.25}$$

Hence using a well known convexity inequality, we have

$$\|e\|_{2\theta}^{R_1} \leqq Ch^{2(1-\theta)}(\|Ae\|_0 + h^{-3/2} |e|_0) . \tag{4.26}$$

The inequality (4.21) follows from (4.26) and Theorem 3.2.

If $r = 3$, the inequality (4.25) remains valid and, by using well known interior estimates and the techniques in the proof of Theorem 4.1, it can be shown that

$$\|e\|_1^{R_1} \leqq Ch(\|Ae\|_0 + h^{-3/2} |e|_0) . \tag{4.27}$$

Inequalities (4.25) and (4.27) imply that

$$\|e\|_{1+\theta}^{R_1} \leqq Ch^{1-\theta}(\|Ae\|_0 + h^{-3/2} |e|_0) , \tag{4.28}$$

and (4.22) follows from Theorem 3.2 which completes the proof.

Using Theorem 4.3, it is easy to derive some pointwise estimates for the error $e = u - w$ on compact subsets of R.

COROLLARY 4.2. *Suppose that the conditions of Theorem* 4.3 *are satisfied and* $N = 2, 3$. *Then, if* ϵ *is any fixed positive real number, there exists a constant* C *independent of* h *and* F *(but depending on* R_1*) such that*

$$\sup_{x\in R_1} |e(x)| \leqq Ch^{2-N/2+\lambda-\epsilon}(\|f\|_\lambda + |g|_{\lambda+3/2}) \tag{4.29}$$

for all $0 \leqq \lambda \leqq r - 2$.

Proof: By Sobolev's lemma

$$\sup_{x\in \bar{R}_1} |e(x)| \leqq C \|e\|^{R_1}_{N/2+\epsilon}$$

for any fixed $\epsilon > 0$.

If $r \geqq 4$ we set $\theta = \frac{1}{2}(\frac{1}{2}N + \epsilon)$ which is less than 1 provided $N < 4$. Then from (4.21) we have

$$\|e\|^{R_1}_{N/2+} \leqq Ch^{2-N/2-\epsilon}(h^\lambda \|f\|_\lambda + h^{-3/2+\lambda_0} |g|_{\lambda_0})$$

and (4.29) now results by taking $\lambda_0 = \lambda + \frac{3}{2}$. The case $r = 3$ follows in a similar fashion.

5. Examples

In this section we shall illustrate the results of the main theorems of Section 4 by giving specific examples in special cases.

EXAMPLE 1. Consider Dirichlet's problem

$$\begin{aligned} \Delta u + qu &= f \quad \text{in} \quad R\,, \\ u &= g \quad \text{on} \quad \partial R\,, \end{aligned} \tag{5.1}$$

where $q = q(x)$ is any smooth function for which the solution of (5.1) is unique.

Case 1. $k = 2$, $r \geqq 4$, $\gamma = \frac{3}{2}$. Let w be the solution of the approximate problem (4.1) in the case $\gamma = \frac{3}{2}$. If we take $k = 2$, $r = 4$ and assume $S^h_{2,4}$ to be, say, cubic splines or cubic Hermite polynomials (c.f. [15]), then Corollary 4.1 yields

$$\|u - w\|_0 \leqq C(h^{2+\lambda} \|f\|_\lambda + h^{1/2+\lambda_0} |g|_{\lambda_0})$$

for any $0 \leqq \lambda \leqq 2$ and $0 \leqq \lambda_0 \leqq \frac{7}{2}$. The two extreme cases give us

$$\|u - w\|_0 \leqq C(h^2 \|f\|_0 + h^{1/2} |g|_0)$$

and

$$\|u - w\|_0 \leqq Ch^4(\|f\|_2 + |g|_{7/2})\,.$$

If the data are smoother and we take $k = 2$, $r = 6$ and assume $S^h_{2,6}$ to be, say, quintic splines, Corollary 4.1 yields

$$\|u - w\|_0 \leqq Ch^6(\|f\|_4 + |g|_{11/2}) .$$

In general if $f \in H^{r-2}(R)$ and $g \in H^{r-1/2}(\partial R)$, then we obtain, using any $S^h_{2,r}$,

$$\|u - w\|_0 \leqq Ch^r(\|f\|_{r-2} + |g|_{r-1/2}) .$$

Interior estimates in the $H^2(R_1)$ norm (or the $H^{2\theta}(R_1)$ norm for any $0 \leqq \theta \leqq 1$) and the maximum norm over any closed subdomain R_1 of R can be easily obtained from Theorem 4.3 and Corollary 4.2. We take w again to be the solution of the approximate problem (4.1) in the case that $\gamma = \frac{3}{2}$. If we let $k = 2$ and $r = 4$, then we have from (4.2) that

$$\|u - w\|_2^{R_1} \leqq Ch^2(\|f\|_2 + |g|_{7/2}) .$$

For $N = 2$, we have from (4.29)

$$\max_{x \in R_1} |u - w| \leqq Ch^{3-\epsilon}(\|f\|_2 + |g|_{7/2})$$

and, for $N = 3$,

$$\max_{x \in R_1} |u - w| \leqq Ch^{5/2-\epsilon}(\|f\|_2 + |g|_{7/2})$$

with any fixed $\epsilon > 0$.

Case 2. $k = 2$, $r = 3$. Suppose we take $k = 2$, $r = 3$ and assume $S^h_{2,3}$ to be, say, quadratic splines (c.f. [15]).

If $f \in H^0(R) = L_2(R)$ and $g \in H^0(\partial R) = L_2(\partial R)$, then Theorem 4.1 indicates that the maximum rate of convergence is realized when w is the solution of (4.1) with $\gamma = \frac{2}{3}$. We then obtain

$$\|u - w\|_0 \leqq C(h\|f\|_0 + h^{1/3}|g|_0) .$$

If the data are smoother, say $f \in H^1(R)$ and $g \in H^{5/2}(\partial R)$, and if w is the solution of (4.1) for any $\frac{2}{3} \leqq \gamma \leqq \frac{3}{2}$, then again taking $S^h_{2,3}$ we obtain from (4.6) that

$$\|u - w\|_0 \leqq Ch^2(\|f\|_1 + |g|_{5/2}) .$$

If we let w be the solution of (4.1) for any γ with $1 \leqq \gamma \leqq \frac{3}{2}$, we obtain

$$\|u - w\|_{1/2} \leqq Ch^2(\|f\|_1 + |g|_{5/2}) .$$

Interior estimates may be found for the case $k = 2$, $r = 3$ in the same fashion as the case $k = 2$, $r \geqq 4$.

It should be pointed out that q is not required to satisfy a sign condition and hence, for example, q could be a positive constant which lies between two of the fixed membrane eigenvalues. In that case the usual bilinear form associated with the problem with $g = 0$ would be

$$\sum_{j=1}^{N} \int_R \frac{\partial \phi}{\partial x_i} \frac{\partial v}{\partial x_i}\, dx - q \int_R \phi v \, dx$$

which is not positive definite. Hence even the existence of a solution in the usual Rayleigh-Ritz method for a given h is unclear. This difficulty is not present in our methods since the forms which are associated with (4.1) are always positive definite in view of uniqueness. Of course it should be noted that no change in the results of this example occurs when $\Delta u + qu$ is taken to be any second order uniformly elliptic operator with smooth coefficients.

Example 2. For simplicity let us assume that the origin is contained in R. Consider Dirichlet's problem

$$\begin{aligned} Au &= \delta(x) \quad && \text{in} \quad R, \\ u &= 0 && \text{on} \quad \partial R, \end{aligned}$$

where $\delta(x)$ is the "Dirac-delta function". If $N = 2$, then $\delta \in H^{-1-\epsilon}(R)$ and, if $N = 3$, then $\delta \in H^{-3/2-\epsilon}(R)$ for any $\epsilon > 0$. In this case $u(x) = G(x, 0)$, the Greens function for the operator A with singularity at the origin.

If we take $\delta_h(x)$ to be the smoothed data as defined in (4.13), then we simply have $\delta_h = \Phi_h * \delta = \Phi_h$. Let w_h be the solution of the approximate problem

$$(\Phi_h - Aw_h, \chi)_0 + h^{-3}\langle -w_h, \chi\rangle_0 = 0$$

for all $\chi \in S^h_{2,r}$.

If we take $k = 2$, $r = 4$ and assume $S^h_{2,4}$ to be, for instance, cubic splines or cubic Hermite polynomials, we obtain from (4.17) that if $N = 2$ and h is sufficiently small

$$\|u - w_h\|_0 \leqq Ch^{1-\epsilon} \|\delta\|_{-1-\epsilon}.$$

If $N = 3$, then

$$\|u - w_h\|_0 \leqq Ch^{1/2-\epsilon} \|\delta\|_{-3/2-\epsilon}$$

for any fixed $\epsilon > 0$.

The fact that the trial functions need not satisfy the boundary conditions in any one of these methods should be an important feature.

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