Some Inequalities for Vector Functions with Applications in Elasticity

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1. Introduction

In a recent paper [*1*], the authors derived certain *a priori* inequalities, which they employed in obtaining bounds in the first boundary value problem for the equations of elasticity. The derivation of the necessary *a priori* inequalities for treating the second boundary value problem (surface tractions prescribed on the boundary of the elastic medium) is much more difficult because of the fact that the corresponding variational problems have a number of zero eigenvalues.

In a later paper the authors [*2*] derived a lower bound for the first non-zero eigenvalue in the free membrane problem (the reciprocal of the Poincaré constant), as well as a lower bound for the first non-zero Steklov eigenvalue. *A priori* inequalities suitable for obtaining the bounds in the Neumann problem were also given.

In this paper we derive lower bounds for the first non-zero eigenvalues in the elasticity problems analogous to those mentioned above. That is, we obtain bounds for the first non-zero eigenvalue of a vibrating elastic medium with a traction free boundary, and for the first non-zero Steklov-type eigenvalue. This leads then to *a priori* inequalities which may be employed in obtaining pointwise bounds in the second boundary value problem in elasticity.

The inequalities mentioned above are not only of interest in themselves, but they are also useful in other connections. For instance, they have already been used [*3*] in establishing that the second boundary value problem in elasticity has a unique solution for a range of values of $\sigma > \frac{1}{2}$. Inequalities of the form (3.8) and (3.15) have previously been used to prove an existence theorem for the second boundary value problem in elasticity [*10*]. In [*10*], the establishment of the needed inequalities depended on the existence of Korn's constant (see also [*6*], [*9*], [*12*]). However, an explicit lower bound for Korn's constant is, in general, difficult to obtain. Our derivation of these inequalities is not dependent upon the existence of Korn's constant, and our method allows explicit computation of the constants for a large class of regions.

2. Definitions and preliminary inequalities

Let R be a simply connected bounded region with boundary C in three dimensions. We introduce the notation

$$L_i(u) = u_{i,jj} + \alpha u_{j,ji}, \qquad i = 1, 2, 3, \tag{2.1}$$

the operator being defined for sufficiently smooth vectors (u_1, u_2, u_3). In (2.1) the comma denotes partial differentiation $\left(u_{j,ji} = \frac{\partial^2 u_j}{\partial x_j \partial x_i}\right)$, and the summation convention is assumed. The constant α is expressible in terms of Poisson's ratio σ as $\alpha = (1 - 2\sigma)^{-1}$, from which we see that the physically interesting values of α satisfy $\alpha > \frac{1}{3}$, *i.e.*, $-1 < \sigma < \frac{1}{2}$. Thus we assume, unless otherwise stated, that α is any constant greater than $\frac{1}{3}$.

The stress components τ_{ij} are defined in terms of the vector field u_i by the expression

$$\tau_{ij} = \mu\{u_{i,j} + u_{j,i} + (\alpha - 1) u_{k,k}\delta_{ij}\} \tag{2.2}$$

where μ, the shear modulus, is a positive constant. The energy $E(u, u)$ is given by

$$E(u, u) = \frac{1}{2}\int\limits_R u_{i,j}\tau_{ij}\,dv = \frac{\mu}{4}\int\limits_R \{(u_{i,j} + u_{j,i})(u_{i,j} + u_{j,i}) + 2(\alpha - 1)u_{k,k}^2\}\,dv. \tag{2.3}$$

We shall establish in this section inequalities (2.19) and (2.29). These relations are valid for a class of vector fields satisfying somewhat artificial normalization conditions. Application of (2.19) and (2.29) are made in Sections 4 and 5 to physically interesting problems.

In order to obtain the preliminary results we proceed as follows. Let the origin be at an arbitrary point of R, and S_a be the interior of a sphere of radius a with center at the origin and such that $S_a \subset R$. The surface of the sphere is denoted as Σ_a. Designate by R_a the region $R - \bar{S}_a$, where $\bar{S}_a$ is the closure of S_a.

If u_i is a sufficiently smooth vector field in $R + C$ and f^i a sufficiently smooth vector field defined in $\bar{R}_a$, then by the divergence theorem we have

$$\oint\limits_C f^k n_k u_i u_i\,ds = -\oint\limits_{\Sigma_a} f^k n_k u_i u_i\,ds + \int\limits_{R_a} f^k_{,k} u_i u_i\,dv + 2\int\limits_{R_a} f^k u_i u_{i,k}\,dv. \tag{2.4}$$

Similarly

$$\oint\limits_C f^k n_i u_i u_k\,ds = -\oint\limits_{\Sigma_a} f^k n_i u_i u_k\,ds + \int\limits_{R_a} f^k_{,i} u_i u_k\,dv + \int\limits_{R_a} f^k u_i u_{k,i}\,dv + \int\limits_{R_a} f^k u_k u_{i,i}\,dv. \tag{2.5}$$

Combining (2.4) and (2.5), we obtain

$$\begin{aligned}\oint\limits_C (f^l n_l \delta_{ik} + 2f^k n_i) u_i u_k\,ds &= -\oint\limits_{\Sigma_a} (f^l n_l \delta_{ik} + 2f^k n_i) u_i u_k\,ds + \\ &+ \int\limits_{R_a} (f^l_{,l}\delta_{ik} + 2f^k_{,i}) u_i u_k\,dv + 2\int\limits_{R_a} f^k u_i (u_{i,k} + u_{k,i})\,dv + 2\int\limits_{R_a} f^k u_k u_{i,i}\,dv.\end{aligned} \tag{2.6}$$

In (2.6) δ_{ik} denotes the Kronecker delta, *i.e.*

$$\delta_{ik} = \begin{cases} 1 & \text{if } i = k \\ 0 & \text{if } i \neq k, \end{cases} \tag{2.7}$$

and n_i denotes the i^{th} component of the unit normal directed outward from D on C. An application of the arithmetic-geometric mean inequality on the right of (2.6) gives:

$$\oint_C (f^l n_l \delta_{ik} + 2 f^k n_i) u_i u_k \, ds \leqq - \oint_{\Sigma_a} (f^l n_l \delta_{ik} + 2 f^k n_i) u_i u_k \, ds +$$

$$(2.8) \qquad + \int_{R_a} \left\{ \left(f^l_{,l} + \frac{1}{\beta_1} f^l f^l \right) \delta_{ik} + 2 f^k_{,i} + \frac{1}{\beta_2} f^k f^i \right\} u_i u_k \, dv +$$

$$+ \int_{R_a} \beta_j (u_{i,k} + u_{k,i})(u_{i,k} + u_{k,i}) \, dv + \int_{R_a} \beta_2 u^2_{l,l} \, dv,$$

where β_1 and β_2 are arbitrary positive functions.

We assume now that f^i, β_1, and β_2 have been so chosen that for every real vector (ξ_1, ξ_2, ξ_3)

$$(f^l n_l \delta_{ik} + 2 f^k n_i) \xi_i \xi_k \geqq K_1 \sum_{i=1}^{3} \xi_i^2 \quad \text{on } C,$$

$$(2.9) \qquad - (f^l n_l \delta_{ik} + 2 f^k n_i) \xi_i \xi_k \leqq K_2 \sum_{i=1}^{3} \xi_i^2 \quad \text{on } \Sigma_a,$$

$$\left\{ \left(f^l_{,l} + \frac{1}{\beta_1} f^l f^l \right) \delta_{ik} + 2 f^k_{,i} + \frac{1}{\beta_2} f^k f^i \right\} \xi_i \xi_k \leqq 0 \quad \text{in } R_a$$

where K_1 and K_2 are positive constants. (In Section 5 we shall construct a vector field f^i satisfying (2.5) for certain domains.) Using conditions (2.9) together with (2.8), we obtain

$$(2.10) \qquad K_1 \oint_C u_i u_i \, ds \leqq K_2 \oint_{\Sigma_a} u_i u_i \, ds + \bar{\beta}_1 \int_{R_a} (u_{i,j} + u_{j,i})(u_{i,j} + u_{j,i}) \, dv + \bar{\beta}_2 \int_{R_a} u^2_{i,i} \, dv,$$

where $\bar{\beta}_1$ and $\bar{\beta}_2$ are upper bounds for β_1 and β_2 respectively.

We suppose now that u_i is normalized so that

$$(2.11) \qquad \oint_{\Sigma_a} u_i \, ds = 0, \qquad \oint_{\Sigma_a} (u_i x_j - u_j x_i) \, ds = 0, \qquad i, j = 1, 2, 3.$$

Then

$$(2.12) \qquad \oint_{\Sigma_a} u_i u_i \, ds \leqq (2\mu q_a)^{-1} \int_{S_a} u_{i,j} \tau_{ij} \, dv$$

where q_a is the first non-zero eigenvalue in the Steklov-type problem considered by BRAMBLE & PAYNE [4], *i.e.*

$$(2.13) \qquad q_a = \min \frac{\int_{S_a} \{ (v_{i,j} + v_{j,i})(v_{i,j} + v_{j,i}) + 2(\alpha - 1) v^2_{k,k} \} \, dv}{4 \oint_{\Sigma_a} v_i v_i \, ds}$$

where the minimum is taken over all sufficiently smooth vector functions in S_a which satisfy (2.11). For a sphere of radius a, q_a is explicitly given (see [4]) by

$$(2.14) \qquad q_a = \frac{1}{a} \left[\min \left\{ \frac{1}{2}, \frac{3(1+\sigma)}{2(2-3\sigma)} \right\} \right].$$

Thus for $\sigma > -\frac{1}{6}$, $q_a = \frac{1}{2} a$.

Combining (2.12) and (2.10), we obtain

$$(2.15) \qquad \begin{aligned} K_1 \oint_C u_i u_i \, ds \leqq {} & K_2 (2\mu q_a)^{-1} \int_{S_a} u_{i,j} \tau_{ij} \, dv + \\ & + \bar{\beta}_1 \int_{R_a} (u_{i,j} + u_{j,i})(u_{i,j} + u_{j,i}) \, dv + \bar{\beta}_2 \int_{R_a} u^2_{k,k} \, dv. \end{aligned}$$

Since

$$u_{k,k}^2 \leqq 3\,(u_{1,1}^2+u_{2,2}^2+u_{3,3}^2) \leqq \tfrac{3}{4}\,(u_{i,j}+u_{i,j})\,(u_{i,j}+u_{i,j})\,, \tag{2.16}$$

it follows that

$$\begin{aligned} K_1 \oint_C u_i u_i\, ds \leqq K_2 (2\mu q_a)^{-1} \int_{S_a} u_{i,j}\tau_{ij}\, dv +{} \\ + (\bar\beta_1 + \tfrac{3}{4}\bar\beta_2) \int_{R_a} (u_{i,j}+u_{j,i})\,(u_{i,j}+u_{j,i})\, dv. \end{aligned} \tag{2.17}$$

Now if $\sigma \geqq 0$, then clearly

$$\int_{R_a} (u_{i,j}+u_{j,i})\,(u_{i,j}+u_{j,i})\, dv \leqq 2/\mu \int_{R_a} u_{i,j}\tau_{ij}\, dv. \tag{2.18}$$

Thus for $\sigma \geqq 0$ we obtain

$$\oint_C u_i u_i\, ds \leqq K_3 (K_1\mu)^{-1} E(u, u) \tag{2.19}$$

where K_3 is given by

$$K_3 = \max\{2K_2 \bar a,\ 4\bar\beta_1 + 3\bar\beta_2\}. \tag{2.20}$$

For $-1<\sigma<0$, we may write instead of (2.17)

$$\begin{aligned} K_1 \oint_C u_i u_i\, ds \leqq K_2 (2\mu q_a)^{-1} \int_{S_a} u_{i,j}\tau_{ij}\, dv +{} \\ + (\bar\beta_1 + \tfrac{3}{4}\bar\beta_2 + \tfrac{3}{4}\gamma) \int_{R_a} (u_{i,j}+u_{j,i})\,(u_{i,j}+u_{j,i})\, dv - \gamma \int_{R_a} u_{k,k}^2\, dv \end{aligned} \tag{2.21}$$

where γ is any positive number. We choose, in particular,

$$\gamma = -(4\bar\beta_1 + 3\bar\beta_2)\,\sigma\,(1+\sigma)^{-1}. \tag{2.22}$$

In this case we have

$$K_1 \oint_C u_i u_i\, ds \leqq K_2 (2\mu q_a)^{-1} \int_{S_a} u_{i,j}\tau_{ij}\, dv + \frac{(4\bar\beta_1+3\bar\beta_2)\,(1-2\sigma)}{2\mu\,(1+\sigma)} \int_{R_a} u_{i,j}\tau_{ij}\, dv. \tag{2.23}$$

Thus for $-1<\sigma<0$ we again have an inequality of the form (2.19). In particular, inequality (2.19) holds for $-1<\sigma<\frac{1}{2}$ with K_3 given by

$$K_3 = \begin{cases} \max\{2K_2 a,\ (4\bar\beta_1+3\bar\beta_2)\}, & \sigma \geqq 0, \\ \max\{2K_2 a,\ (4\bar\beta_1+3\bar\beta_2)\,(1-2\sigma)\,(1+\sigma)^{-1}\}, & -\frac{1}{6} \leqq \sigma < 0, \\ \max\left\{\dfrac{2K_2 a\,(2-3\sigma)}{1+\sigma},\ (4\bar\beta_1+3\bar\beta_2)\,(1-2\sigma)\,(1+\sigma)^{-1}\right\}, & -1<\sigma<-\frac{1}{6}. \end{cases} \tag{2.24}$$

We seek now a bound for $\int_R u_i u_i\, dv$ in terms of $E(u, u)$. From the divergence theorem we have

$$\begin{aligned} &\oint_C (x^l n_l \delta_{ik} + 2x^k n_i)\, u_i u_k\, ds \\ &= 5\int_R u_i u_i\, dv + 2\int_R x^k u_i (u_{i,k}+u_{k,i})\, dv + 2\int_R x^k u_k u_{i,i}\, dv. \end{aligned} \tag{2.25}$$

An application of the arithmetic-geometric mean inequality then gives

$$\begin{aligned} \left[5 - \frac{1}{a_1} - \frac{1}{a_2}\right] \int_R u_i u_i\, dv \leqq 3 r_M \oint_C u_i u_i\, ds +{} \\ + r_M^2 \left\{ a_1 \int_R (u_{i,j}+u_{j,i})\,(u_{i,j}+u_{j,i})\, dv + a_2 \int_R u_{k,k}^2\, dv \right\}, \end{aligned} \tag{2.26}$$

where r_M is the maximum distance from the origin to C, and a_1 and a_2 are arbitrary positive constants. In particular, if $\sigma \geqq 0$ and we take $a_2 = \frac{4}{3} a_1 = \frac{14}{15}$, then we obtain the inequality

$$\int_R u_i u_i \, dv \leqq \frac{6 r_M}{5} \oint_C u_i u_i \, ds + \frac{56 r_M^2}{25 \mu} E(u, u). \tag{2.27}$$

For negative σ we find

$$\int_R u_i u_i \, dv \leqq \frac{6 r_M}{5} \oint_C u_i u_i \, ds + \frac{56 r_M^2}{25 \mu} \frac{(1-2\sigma)}{1+\sigma} E(u, u). \tag{2.28}$$

Combining (2.28) and (2.19), we obtain finally the inequality

$$\int_R u_i u_i \, dv \leqq \mu^{-1} K_4 E(u, u) \tag{2.29}$$

valid for all vectors u_i satisfying (2.11). Here K_4 is given by

$$K_4 = \frac{2}{25} \left[15 \frac{K_3}{K_1} r_M + 28 r_M^2 \lambda \right], \tag{2.30}$$

with

$$\lambda = \begin{cases} 1, & \sigma \geqq 0, \\ \dfrac{1-2\sigma}{1+\sigma}, & -1 < \sigma < 0. \end{cases} \tag{2.31}$$

All of the preceding inequalities depended on the existence of a vector field f^i satisfying (2.9). An explicit representation for f^i in certain cases is given in Section 5. With such an explicit representation, values of K_1, K_2, K_3 and K_4 are easily computed.

In some cases it will be difficult to construct the vector field f^i. We can then make use of an additional inequality to reduce the problem to that of obtaining an inequality of the form (2.29) for a subregion of R.

Let the region R be divided into two subregions R_1 and R_2. These regions are separated by a surface C'. The part of C which is a portion of the bounding surface of R_i is denoted by C_i, $i = 1, 2$. Thus the boundary of R_i is $C_i + C'$. We assume the subdivision to have been made in such a way that on C_1 the condition $h \equiv \frac{x^i n_i}{r} > \frac{1}{2}$ is satisfied. Here the origin is taken at a point not in $R_1 + C'$. By the divergence theorem we have for any constant b,

$$\begin{aligned} &\oint_{C_1+C'} r^{-b} \{x^l n_l \delta_{ij} + 2 x^i n_j\} u_i u_j \, ds + (b - s) \int_{R_1} r^{-b} u_i u_i \, dv + \\ &+ 2b \int_{R_1} r^{-(b+2)} (x^i u_i)^2 dv = 2 \int_{R_1} r^{-b} x^i u_j (u_{i,j} + u_{j,i}) \, dv + 2 \int_{R_1} r^{-b} x^i u_i u_{j,j} \, dv. \end{aligned} \tag{2.32}$$

It is clear that if we choose $b > 1$ and if $\left(\frac{x^l n_l}{r} \delta_{ij} + 2 \frac{x_i}{r} n_j\right) u_i u_j > \overline{C} u_i u_i$ on C_1, where $\overline{C}$ is some positive constant, then we have

$$\overline{C} \int_{C_1} u_i u_i \, ds \leqq k_1 \int_{C'} u_i u_i \, ds + k_2 \int_{R_1} u_{i,j} \tau_{ij} \, dv, \tag{2.33}$$

where k_1 and k_2 are easily determined constants.

We show now see that if $\frac{x^i n_i}{r} > \frac{1}{2}$ on C_1, such a constant $\overline{C}$ can be obtained. Consider the identity

$$(2.34)\quad \begin{aligned}&\int_{C_1} r^{-b}\{x^l n_l u_i u_i + 2x^i n_j u_i u_j\} ds \\ &= \int_{C_1} r^{-b} x^l n_l\{u_i u_i + 2(u_l n_l)^2\} ds + 2\int_{C_1} r^{-b} x^i n_l (u_i n_l - u_l n_i) u_j n_j ds,\end{aligned}$$

which is obtained by decomposing u_i in the second term into its normal and tangential components. The last term in (2.34) may be rewritten as

$$(2.35)\quad 2\int_{C_1} r^{-b} x^i n_l (u_i n_l - u_l n_i) u_j n_j ds = \int_{C_1} r^{-b}(x^i n_l - x^l n_i)(u_i n_l - u_l n_i) u_j n_j ds.$$

Using the arithmetic-geometric mean inequality on the right, we obtain

$$(2.36)\quad \begin{aligned}&\int_{C_1} r^{-b}(x^i n_l - x^l n_i)(u_i n_l - u_l n_i) u_j n_j ds \geqq -\gamma_1 \int_{C_1} r^{-b}(r^2 - [x^i n_i]^2)^{\frac{1}{2}}(u_j n_j)^2 ds - \\ &- \frac{1}{\gamma_1}\int_{C_1} r^{-b}[r^2 - (x^i n_i)^2]^{\frac{1}{2}}[u_i u_i - (u_j n_j)^2] ds\end{aligned}$$

where γ_1 is an arbitrary positive constant. If we choose $\gamma_1 = \frac{\sqrt{1+h}}{\sqrt{1-h}}$, we observe that

$$(2.37)\quad \int_{C_1} r^{-b}\{x^i n_l u_i u_i + 2x^i n_j u_i u_j\} ds \geqq \int_{C_1} r^{-(b-1)}\{2h-1\} u_i u_i ds.$$

Hence we may take

$$(2.38)\quad \overline{C} = \min\{r^{-(b-1)}(2h-1)\}.$$

Now suppose we are able to find a vector field f^i satisfying (2.9) relative to R_2 and obtain the inequality

$$(2.39)\quad \oint_{C_2+C'} u_i u_i ds \leqq k_3 \int_{R_2} u_{i,j}\, \tau_{i,j}\, dv$$

with an explicitly determined k_3. Then by combining (2.33) and (2.39) we could obtain

$$(2.40)\quad \oint_C u_i u_i ds \leqq k_4 E(u, u)$$

where of course u_i is normalized with respect to a sphere S_a in R_2 and k_4 is an explicitly determined constant.

It is clear that the procedure of dividing up R may also be applied to R_2 if the f^i for R_2 are not easily obtained. In fact the procedure could be repeated a finite number of times, with the hope that the region may finally be reduced to one for which the f^i may be more easily constructed. In particular if the procedure is iterated until at each point on the boundary of the n^{th} region $h \equiv \frac{x^i n_i}{r} > \frac{1}{2}$, then, as we shall see in Section 5, a vector field f^i for R_n is easily constructed.

3. Lower bounds for eigenvalues

The first non-zero eigenvalue in the free elastic vibration problem for R satisfies

$$(3.1)\quad L_i(v) + \nu v_i = 0 \quad \text{in} \quad R$$

and

$$(v_{i,j}+v_{j,i})n_j+(\alpha-1)v_{j,j}n_i=0 \quad \text{on} \quad C \tag{3.2}$$

with the normalization conditions

$$\int_R v_i\,dv=\int_R (x^i v_j - x^j v_i)\,dv=0. \tag{3.3}$$

The function v_i is the corresponding eigenvector. It is well known that ν may be characterized by the minimum principle

$$\frac{\mu}{2}\nu=\min\frac{E(\varphi,\varphi)}{\int_R \varphi_i\varphi_i\,dv} \tag{3.4}$$

for sufficiently smooth vectors φ_i satisfying (3.3). But the minimum under this natural normalization is not less than the minimum of the quotient under the normalization over Σ_a. This is easily seen if we set $u_i=v_i+c_i+\varepsilon_{ijk}\hat{c}_j(x^k-\bar{x}^k)$ where ε_{ijk} is the permutation symbol and $\bar{x}^k$ is a constant vector chosen so that $\int_R (x^k-\bar{x}^k)\,dv=0$. Then if the c_i and $\hat{c}_j$ are determined in such a way that u_i satisfies (2.11), it follows that

$$E(u,u)=E(v,v) \tag{3.5}$$

and

$$\int_R u_i u_i\,dv=\int_R v_i v_i\,dv+c_i c_i V+\varepsilon_{ijk}\,\varepsilon_{ilm}\hat{c}_j\hat{c}_k\int_R (x^k-\bar{x}^k)(x^m-\bar{x}^m)\,dv \geqq \int_R v_i v_i\,dv, \tag{3.6}$$

where V denotes the volume of R. Clearly then

$$\frac{\mu}{2}\nu=\frac{E(v,v)}{\int_R v_i v_i\,dv}\geqq\frac{E(u,u)}{\int_R u_i u_i\,dv}\geqq\frac{\mu}{K_4} \tag{3.7}$$

where K_4 is given by (2.30). Hence, we obtain the bound

$$\nu\geqq 2K_4^{-1}. \tag{3.8}$$

A lower bound is also easily obtained for q, the first non-zero eigenvalue in the Steklov-type problem for R. Clearly if w_i is the corresponding eigenvector, then

$$\frac{\mu}{2}q=\frac{E(w,w)}{\oint_C w_i w_i\,ds} \tag{3.9}$$

where w_i satisfies the conditions

$$\oint_C w_i\,ds=0, \qquad \oint_C (w_i x^j - x^i w_j)\,ds=0, \qquad i,j=1,2,3. \tag{3.10}$$

Again it is easily shown that if we choose

$$u_i=w_i+c_i+\varepsilon_{ijk}\hat{c}_j(x^k-\bar{x}^k) \tag{3.11}$$

(with the constant vector $\bar{x}^k$ chosen so that $\oint_C (x^k-\bar{x}^k)\,ds=0$) and adjust the C_i and $\hat{C}_j$ so that (2.11) is satisfied, then

$$E(u,u)=E(w,w) \tag{3.12}$$

and

$$\oint_C u_i u_i \, ds \geqq \oint_C w_i w_i \, ds. \tag{3.13}$$

Hence

$$\frac{\mu}{2} q = \frac{E(w, w)}{\oint_C w_i w_i \, ds} \geqq \frac{E(u, u)}{\oint_C u_i u_i \, ds} \geqq \mu \frac{K_1}{K_3}, \tag{3.14}$$

where K_1 is given by (2.9) and K_3 by (2.24). Thus

$$q \geqq 2K_1/K_3. \tag{3.15}$$

4. Bounds in the second boundary value problem for the equations of elasticity

Let ψ_i be any sufficiently smooth vector function in $R+C$. We seek bounds for the energy $E(\psi, \psi)$, in terms of $L_i(\psi)$ in R and $\tau_{ij} n_j$ on C.

We define $u_i = \psi_i + d_i + \varepsilon_{ijk} \tilde{d}_j x^k$ where the d_i and $\tilde{d}_j$ are so chosen that (2.11) is satisfied. Then by the divergence theorem

$$\begin{aligned} 2\mu^{-1} E(u, u) &= \mu^{-1} \oint_C u_i \tau_{ij}(u) n_j \, ds - \int_R u_i L_i(u) \, dv \\ &= \mu^{-1} \oint_C u_i \tau_{ij}(\psi) n_j \, ds - \int_R u_i L_i(\psi) \, dv. \end{aligned} \tag{4.1}$$

We have used the notation

$$\tau_{ij}(\psi) = \mu [\psi_{i,j} + \psi_{j,i} + (\alpha - 1) \psi_{k,k} \delta_{ij}] \tag{4.2}$$

and the fact that the terms d_i and $\varepsilon_{ijk} \tilde{d}_j x^k$ do not contribute to the stresses (they correspond to rigid-body motions). By Schwarz's inequality

$$\begin{aligned} 2\mu^{-1} E(u, u) = 2\mu^{-1} E(\psi, \psi) \leqq \mu^{-1} &\Big\{ \oint_C u_i u_i \, ds \oint_C \tau_{ij}(\psi) n_j \tau_{ik}(\psi) n_k \, ds \Big\}^{\frac{1}{2}} + \\ &+ \Big\{ \int_R u_i u_i \, dv \int_R L_i(\psi) L_i(\psi) \, dv \Big\}^{\frac{1}{2}}. \end{aligned} \tag{4.3}$$

Making use of (2.19) and (2.29), we obtain

$$\begin{aligned} \{2\mu^{-1} E(\psi, \psi)\}^{\frac{1}{2}} \leqq \mu^{-1} &\Big\{ \frac{K_3}{2K_1} \oint_C \tau_{ij}(\psi) n_j \tau_{ik}(\psi) n_k \, ds \Big\}^{\frac{1}{2}} + \\ &+ \Big\{ \frac{K_4}{2} \int_R L_i(\psi) L_i(\psi) \, dv. \Big\}^{\frac{1}{2}}. \end{aligned} \tag{4.4}$$

The inequalities of this section and of Section 2 together with a mean value inequality given in [*1*] give immediate pointwise bounds for u_i and its derivatives. As an application of these results it is clear that the Rayleigh-Ritz technique may be used in (4.4) to obtain close bounds for the energy in a specific boundary value problem, *cf.* [*5*].

Other methods for obtaining bounds for the strain energy in the second boundary value problem and methods for obtaining strain energy and pointwise bounds in the first boundary value problem, may be found in the literature (see *e.g.* [*7, 8, 11, 13–15*]).

5. Construction of the vector field f^i

In the preceding sections we made use of a vector field satisfying (2.9). Here we illustrate how, in certain cases, this vector field may be constructed.

a) *Strongly star-shaped region* $\left(\frac{x^i n_i}{r} > \frac{1}{2}\right)$

In this case we assume that with respect to some origin the quantity $h \equiv \frac{x^i n_i}{r} > \frac{1}{2}$ at every point on C. We then take our sphere S_a with origin at this point and choose

$$f^i = x^i r^{-b}, \qquad b > 4 \tag{5.1}$$

and

$$\beta_1 = (b-4)\, r^{-(b-2)}, \qquad \beta_2 = b\, r^{-(b-2)}. \tag{5.2}$$

Thus we obtain

$$K_1 = [(2h-1)\, r^{-(b-1)}]_{\min}, \qquad K_2 = 3\, a^{-(b-1)}. \tag{5.3}$$

In this case

$$K_3 = (7b-16)\, \lambda_1 a^{-(b-2)} \tag{5.4}$$

where

$$\lambda_1 = \begin{cases} 1, & \sigma \geqq 0 \\ (1-2\sigma)(1+\sigma)^{-1}, & -1 < \sigma < 0. \end{cases} \tag{5.5}$$

Bounds for ν and q are then given by

$$\nu \geqq 2K_4^{-1} = \frac{25}{\left\{15 \frac{K_3}{K_1} r_M + 56\, r_M^2 \lambda_1\right\}} \tag{5.6}$$

and

$$q \geqq 2K_1/K_3. \tag{5.7}$$

b) *Smooth boundaries*

Let R be such that C has continuous curvature, and let K_M denote the maximum principal curvature on C. At each point P of C we consider the largest sphere of radius not greater than $[K_M(P)]^{-1}$, tangent to C at P and such that the sphere is contained in R. Let the minimum such radius be bounded below by $\overline{K}^{-1}$. We consider the family of parallel surfaces

$$N(x) = N(x^1, x^2, x^3) = \text{Constant} \tag{5.8}$$

with C given by $N(x) = 0$ and

$$0 \leqq N(x) \leqq \overline{K}^{-1}. \tag{5.9}$$

The outward normal vector n_i is defined in the shell characterized by (5.9) and is given by

$$n_i = -\frac{N_{,i}}{\{N_{,j} N_{,j}\}^{\frac{1}{2}}}. \tag{5.10}$$

At a point x on the parallel surface

$$n_{i,i} = J(x) \tag{5.11}$$

where $J(x)$ denotes the average curvature, *cf.* [*16*, p. 3]. We assume that $\bar{K}$ has been chosen so that

$$J(x) \leqq \bar{K}. \tag{5.12}$$

Our conditions and definitions involve the smoothness of C and essentially the thickness of R. We impose a further condition that there be a point in R, which we take as origin, such that

$$h \equiv \frac{x^i n_i}{r} \geqq -m + \frac{1}{2m} + \delta > -m + \frac{1}{2m} \tag{5.13}$$

at each point on C, for some positive constants m and $\delta > 0$ in the shell $0 \leqq N(x) \leqq \bar{K}^{-1}$. In this case f^i may be taken as

$$f^i = \begin{cases} [m n_i (1 - \bar{K} N(x)) + x^i/r]\, r^{-q}, & 0 \leqq N(x) \leqq \bar{K}^{-1} \\ x^i r^{-(q+1)} \end{cases} \tag{5.14}$$

with q yet to be determined.

Suppose S_a has been chosen so that it does not intersect the boundary shell. Then on C

$$\begin{aligned} (f^l n_l \delta_{ik} + 2 f^k n_i)\, \xi_i \xi_k &= (m+h)\, \xi_i \xi_i + 2m (n_i \xi_i)^2 + \frac{2 x^i n_j}{r} \xi_i \xi_j \\ &\geqq (m+h)\, \xi_i \xi_i - \frac{1}{2m} \left(\frac{x^i \xi^i}{r}\right)^2 . \end{aligned} \tag{5.15}$$

Here we have used the arithmetic-geometric mean inequality. It follows then that

$$(f^l n_l \delta_{ik} + 2 f^k n_i)\, \xi_i \xi_k > \left(m + h - \frac{1}{2m}\right) \xi_i \xi_i > \delta \xi_i \xi_i . \tag{5.16}$$

By choosing q sufficiently large and β_1 and β_2 sufficiently large the third of equations (2.9) will be satisfied provided

$$\left\{ [m h (1 - \bar{K} N(x)) + 1]\, \delta_{ik} + 2m \frac{x^k n_i}{r} (1 - \bar{K} N(x)) + 2 \frac{x^i x^k}{r^2} \right\} \xi_i \xi_k \geqq 0 . \tag{5.17}$$

This condition will be met if

$$\{m h (1 - \bar{K} N(x)) + 1\}\, \xi_i \xi_i - \frac{m^2}{2} (n_i \xi_i)^2 \geqq 0 \tag{5.18}$$

or if

$$m h (1 - \bar{K} N(x)) + 1 - \frac{m^2}{2} \geqq 0 . \tag{5.19}$$

In view of (5.13), condition (5.19) is clearly satisfied whenever $h \geqq 0$. Thus (5.17) will be satisfied provided

$$m\left(-m + \frac{1}{2m}\right) + 1 - \frac{m^2}{2} \geqq 0 , \tag{5.20}$$

i.e.

$$m \leqq 1 . \tag{5.21}$$

Thus any $m \leqq 1$ may be used in (5.14) provided q is chosen sufficiently large. This imposes the condition that $h > -\frac{1}{2}$. It is not difficult then to determine bounds for ν and q.

Since our vector field f^i was required only to be piecewise continuously differentiable in R and to have continuous normal component across any surface of discontinuity of the derivatives, it is possible by similar methods to treat more general regions. This may be done by defining f^i as in a) in a portion of R and as in b) in the remaining portion, making sure that the normal components are equal on the common boundaries (see, *e.g.* [*2*, Sect. 5 (c)]).

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