

# The analysis of multigrid algorithms for cell centered finite difference methods

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In this paper, we examine multigrid algorithms for cell centered finite difference approximations of second order elliptic boundary value problems. The cell centered application gives rise to one of the simplest non-variational multigrid algorithms. We shall provide an analysis which guarantees that the W-cycle and variable V-cycle multigrid algorithms converge with a rate of iterative convergence which can be bounded independently of the number of multilevel spaces. In contrast, the natural variational multigrid algorithm converges much more slowly.

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## 1. Introduction

In recent years, it has become increasingly obvious that iterative methods provide the only feasible technique for solving the large systems of algebraic equations which arise from large scale scientific simulations modeled by partial differential equations. Multigrid methods often represent the most efficient strategy for iteratively solving these systems. For this reason, the multigrid method has been subject to intensive theoretical and computational investigation.

From the point of view of analysis, there are a few basic approaches for providing bounds for the rate of iterative convergence for multigrid algorithms. The initial approach based on discrete Fourier analysis provides sharper convergence rate estimates but its application is limited to constant coefficient operators on only a few special domains [10]. A more general approach based on the approximation properties of the spaces and the elliptic regularity properties of the underlying partial differential equation was pioneered in [1,2,11]. More recently, an analysis for variational multigrid algorithms was provided which was not based on elliptic regularity [6]. This analysis uses a multiplicative representation of the multigrid error propagator and has since been used and refined by other researchers [4,13,14].

The variational multigrid framework provides the most elegant setting for the development and analysis of multigrid algorithms. Here the multilevel spaces are nested and the operators on the spaces are inherited from the operator on the fine grid space. The prolongation operator is the natural imbedding and the restriction operator is its adjoint. Although one might think that the variational multigrid algorithms should always perform better, the example given in this paper shows that they are sometimes worse.

The purpose of this paper is to analyze the simplest example of a non-variational multigrid algorithm, the cell centered finite difference method. This algorithm is non-variational and is analyzed in section 2. It is shown that the W-cycle and variable V-cycle multigrid cycling schemes give rise to iterative methods which converge at a rate independent of the number of levels in the multigrid algorithm.

The cell centered method can be naturally thought to be defined on spaces of piecewise constant functions. These spaces are nested and the natural variational algorithm can be defined by simply using the fine grid operator to define the operator on all coarser grids. Such an approach does not lead to an effective multigrid algorithm. Numerical evidence given in section 3 suggests that the variational algorithm converges at a rate which is bounded from below by  $1 - ch_J/h_j$  where  $h_J$  and  $h_j$  are the mesh sizes on respectively the finest and coarsest grid in the multigrid algorithm.

The cell centered multigrid method developed in this paper is also interesting because it violates some of the accepted multigrid heuristics yet nevertheless results in uniformly convergent iterative schemes. In particular (see remark 2.2), it violates a smoothness condition on the prolongation/restriction pair suggested in [8]. This condition is also required in the theory of [11].

The outline of the remainder of the paper is as follows. We define the cell centered finite difference approximation and the corresponding multigrid

algorithms in section 2. We also prove the “so-called” regularity and approximation assumption there. This result enables the application of the theory in [7] and leads to the above mentioned iterative convergence bounds for the multigrid algorithms. In section 3, we report the results of numerical experiments illustrating the convergence behavior predicted by the theory.

## 2. The cell centered example

We consider a multigrid algorithm for a cell centered discretization of a second order boundary value problems in this section. We start by defining the cell centered method on a simple model problem in two spatial dimensions. We next give the multigrid algorithm. This algorithm is of non-variational nature. By the general theory given in [7], the critical part of the analysis of this algorithm reduces to the proof of the so-called “regularity and approximation” condition. This is verified in theorem 2.1. As we shall see, the cell centered application does not satisfy the usual multigrid heuristics yet nevertheless gives rise to an effective multigrid iteration.

We limit this discussion to a relatively simple model problem. We start by considering the Dirichlet problem

$$\begin{aligned} -\Delta u &= f & \text{in } \Omega, \\ u &= 0 & \text{on } \partial\Omega. \end{aligned} \tag{2.1}$$

Here  $\Omega$  is the unit square in two spatial dimensions and  $\Delta$  denotes the Laplace operator  $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$ .

The cell centered approximation to (2.1) is defined in terms of a regularly spaced grid consisting of smaller squares (or cells) of side length  $1/m$  for some integer  $m \geq 1$ . For our purposes, we shall take  $m = m_k = 2^k$  and index everything in terms of  $k$  instead of  $m$ . Let  $M_k$  denote the set of discontinuous functions on  $\Omega$  which are constant on the smaller squares mentioned above. Let  $\Omega_{i,j}^k$  denote the  $(i, j)$ th square labeled in the natural way ( $i, j = 1, \dots, m_k$ ). Integrating (2.1) over  $\Omega_{i,j}^k$  gives that

$$-\int_{\partial\Omega_{i,j}^k} \frac{\partial u}{\partial n} ds = \int_{\Omega_{i,j}^k} f dx. \tag{2.2}$$

Equation (2.2) does not make sense for functions  $u \in M_k$  since such functions are discontinuous on  $\partial\Omega_{i,j}^k$ . Nevertheless, we still use (2.2) as a basis for the approximation. Let  $U$  be in  $M_k$  and  $U_{i,j}$  denote its value on  $\Omega_{i,j}^k$ . In the following discussion, it may help to think of  $U_{i,j}$  as being the value at the center of  $\Omega_{i,j}^k$ . With this in mind, we assign

$$(U_{i,j+1} - U_{i,j})/h$$

as an approximation to  $\partial U/\partial n$  on the edge between  $\Omega_{i,j}^k$  and  $\Omega_{i,j+1}^k$ . Similarly, we assign

$$(U_{i+1,j} - U_{i,j})/h$$

as an approximation to  $\partial U / \partial n$  on the edge between  $\Omega_{i,j}^k$  and  $\Omega_{i+1,j}^k$ , etc. The above rules are modified when an edge of  $\partial\Omega_{i,j}^k$  coincides with the boundary of  $\Omega$ . If, for example,  $i = 1$ , then the edge corresponding to  $x = 0$  is on  $\partial\Omega$  and we assign  $-2U_{i,j}/h$  as an approximation to  $\partial U / \partial n$  on this edge. Similar definitions are used for the cells which lie along the remaining parts of  $\partial\Omega$ . Substituting the assigned values in (2.2) gives rise to the cell centered finite difference approximation to (2.2), namely,

$$\bar{L}_k \bar{U} = \bar{F}. \quad (2.3)$$

Here  $\bar{L}_k$  is a sparse  $m_k^2 \times m_k^2$  matrix,  $\bar{U}$  is the vector of values  $\{U_{i,j}\}$  and  $\bar{F}$  is the vector with values

$$\int_{\Omega_{i,j}^k} f \, dx.$$

The matrix  $\bar{L}_k$  corresponds to a difference operator with at most a five point stencil. In the case of an interior cell, the stencil is

$$\begin{array}{ccccc} & & -1 & & \\ & & | & & \\ -1 & \text{---} & 4 & \text{---} & -1. \\ & & | & & \\ & & -1 & & \end{array}$$

A typical stencil for a cell on an edge is

$$\begin{array}{ccccc} & & -1 & & \\ & & | & & \\ -1 & \text{---} & 5 & \text{---} & -1 \end{array}$$

while that for a typical corner is

$$\begin{array}{ccc} -1 & & \\ | & & \\ 6 & \text{---} & -1 \end{array}$$

If one thinks of the solution as being reflected as an odd function across the straight line boundaries, then it is clear that these are natural definitions.

The matrix  $\bar{L}_k$  is symmetric and weakly diagonally dominant and hence positive definite.

Let  $V, W$  be in  $M_k$  and define the quadratic form

$$A_k(V, W) = \sum_{i,j=1}^{m_k} (\bar{L}_k \bar{V})_{i,j} W_{i,j}. \quad (2.4)$$

Here  $\bar{V}$  is defined to be the vector with values  $\{V_{i,j}\}$ . It follows from the properties of  $\bar{L}_k$  that  $A_k(\cdot, \cdot)$  is a symmetric positive definite quadratic form on  $M_k$ . An

equivalent formulation of (2.3) is then: Find  $U \in M_k$  satisfying

$$A_k(U, \theta) = (f, \theta) \quad \text{for all } \theta \in M_k. \quad (2.5)$$

Here  $(\cdot, \cdot)$  denotes the  $L^2(\Omega)$  inner product.

Error estimates for cell centered finite differences are well known (see [5]). Let  $Q_k$  denote the  $L^2(\Omega)$  orthogonal projector onto  $M_k$ . If  $u$  solves (2.1) and  $U$  solves (2.5) then

$$A_k(U - Q_k u, U - Q_k u) \leq ch_k^2 \|f\|_2^2. \quad (2.6)$$

Here  $\|\cdot\|_2$  denotes the norm in  $L^2(\Omega)$  and  $h_k = 2^{-k}$  is the mesh size corresponding to  $M_k$ . This estimate depends on full elliptic regularity for solutions of (2.1).

We next set up the multigrid algorithm following [7]. To this end, we first define a sequence of operators on the spaces  $\{M_k\}$ . For  $v \in M_k$  we define  $A_k v = w$  where  $w$  is the unique function in  $M_k$  satisfying

$$(w, \theta) = A_k(v, \theta) \quad \text{for all } \theta \in M_k. \quad (2.7)$$

Our goal is to develop effective iterative procedures for solving the fine grid problem: Find  $U \in M_J$  satisfying

$$A_J(U, \theta) = (F, \theta) \quad \text{for all } \theta \in M_J \quad (2.8)$$

for given  $F \in M_J$ . Here  $J > 1$  defines the fine grid. Note that (p1) is equivalent to

$$A_J U = F.$$

The coarsest grid in our multigrid algorithm will be determined by an integer  $j$  with  $1 \leq j < J$ .

The multigrid algorithm which we shall consider also requires linear smoothing operators  $R_k : M_{k+1} \rightarrow M_k$  for  $k = j+1, \dots, J$ . Let  $R_k'$  denote the adjoint of  $R_k$  with respect to the  $(\cdot, \cdot)$  inner product and define

$$R_k^{(l)} = \begin{cases} R_k & \text{if } l \text{ is odd,} \\ R_k' & \text{if } l \text{ is even.} \end{cases}$$

The multigrid operator  $B_k : M_{k+1} \rightarrow M_k$  is defined by mathematical induction. The operator  $B_k$  can be thought of as a preconditioner for  $A_k$ . An alternative way of presenting the multigrid algorithm is in terms of an iterative process. Both approaches are equivalent and connected in the sense that the multigrid process results in a linear iterative scheme with a reduction operator equal to  $I - B_J A_J$  where  $B_J$  is defined in the following algorithm.

### Algorithm 2.1

Let  $1 \leq j < J$  and  $p$  be a positive integer. Set  $B_j = A_j^{-1}$ . Let  $k$  be greater than  $j$  and assume that  $B_{k-1}$  has been defined. Given  $g \in M_k$ ,  $B_k g$  is defined as follows:

- (1) Set  $x^0 = q^0 = 0$ .
- (2) Define  $x^l$  for  $l = 1, \dots, m(k)$  by

$$x^l = x^{l-1} + R_k^{(l+m(k))}(g - A_k x^{l-1}).$$

(3) Define  $y^{m(k)} = x^{m(k)} + q^p$  where  $q^i$  is defined by

$$q^i = q^{i-1} + B_{k-1}[Q_{k-1}(g - A_k x^{m(k)}) - A_{k-1} q^{i-1}].$$

(4) Define  $y^l$  for  $l = m(k) + 1, \dots, 2m(k)$  by

$$y^l = y^{l-1} + R_k^{(l+m(k))}(g - A_k y^{l-1}).$$

(5) Set  $B_k g = y^{2m(k)}$ .

In the above algorithm,  $m(k)$  gives the number of pre and post smoothing iterations and can vary as a function of  $k$ . The integer  $p$  is usually 1 (a V-cycle multigrid algorithm) or 2 (a W-cycle algorithm). A variable V-cycle algorithm is one in which the number of smoothings  $m(k)$  increase exponentially as  $k$  decreases. The smoothings are alternated following [7] and are put together so that the resulting multigrid preconditioner  $B_k$  is symmetric in the  $(\cdot, \cdot)$  inner product for each  $k$ . The spaces

$$M_1 \subset M_2 \subset \dots \subset M_J$$

are clearly nested so that the addition in step 3 defining  $y^{m(k)}$  makes sense.

We will apply the theory of [7] to algorithm 1. To do this we must show that the smoother satisfies appropriate conditions, check the so-called “regularity and approximation” inequality and estimate the norm of the coarse grid approximation operator.

There is nothing novel about the construction of smoothers for the cell centered application which fit into the theory of [7]. One can use, for example, point Jacobi or Gauss-Seidel smoothing procedures to define  $R_k$ . The smoothing estimates are a consequence of the general smoothing theory in [3].

The remaining conditions are defined in terms of the coarse grid approximation operator  $P_{k-1} : M_k \mapsto M_{k-1}$ . For  $v \in M_k$ ,  $P_{k-1}v = w$  is defined to be the unique function  $w$  in  $M_{k-1}$  satisfying

$$A_{k-1}(w, \theta) = A_k(v, \theta) \quad \text{for all } \theta \in M_{k-1}. \quad (2.9)$$

To apply the theory of [7], we need a bound for the following norm of  $P_{k-1}$ :

$$\|P_{k-1}\|_{M_k \mapsto M_{k-1}} = \sup_{v \in M_k} \frac{A_{k-1}(P_{k-1}v, P_{k-1}v)^{1/2}}{A_k(v, v)^{1/2}}.$$

Let  $I_k$  denote the imbedding of  $M_{k-1}$  into  $M_k$ . From the definition of  $P_{k-1}$ , it follows that  $I_k$  is its adjoint and hence

$$\|P_{k-1}\|_{M_k \mapsto M_{k-1}} = \|I_k\|_{M_{k-1} \mapsto M_k} = \sup_{w \in M_{k-1}} \frac{A_k(w, w)^{1/2}}{A_{k-1}(w, w)^{1/2}}. \quad (2.10)$$

By carefully examining the structure of  $\bar{L}_k$  and  $\bar{L}_{k-1}$ , it is not difficult to see that

$$A_k(v, w) = 2A_{k-1}(v, w) \quad \text{for all } v, w \in M_{k-1}. \quad (2.11)$$

Thus, by (2.10)

$$\|P_{k-1}\|_{M_k \mapsto M_{k-1}} = \|I_k\|_{M_{k-1} \mapsto M_k} = \sqrt{2}. \quad (2.12)$$

To apply the theory of [7], we next need to verify the regularity and approximation condition. This condition is that there is a number  $\alpha \in (0, 1]$  and a constant  $C$  not depending on  $J$  such that for  $k = j + 1, j + 2, \dots, J$ ,

$$|A_k((I - P_{k-1})v, v)| \leq C \left( \frac{\|A_k v\|_2^2}{\lambda_k} \right)^\alpha A_k(v, v)^{1-\alpha} \quad \text{for all } v \in M_k. \quad (2.13)$$

Here  $\lambda_k$  denotes the largest eigenvalue of  $A_k$ . This condition is shown to hold for the cell centered application in the following theorem.

### Theorem 2.1

Let the operator  $A_k$  be defined by (2.7) and  $P_{k-1}$  be defined by (2.9). Then there is a constant  $C$  not depending on  $j$  or  $J$  such that (2.13) holds for  $\alpha = 1/2$ .

#### Proof

Here and in the remainder of this paper,  $C$  with or without subscript will denote a generic positive constant which may take on different values in different occurrences. These constants will always be independent of  $j$  and  $J$  (and hence the dimension of  $M_J$ ).

Fix  $k$  and let  $v$  be an arbitrary element of  $M_k$ . Let  $w$  be the solution of the following boundary value problem:

$$\begin{aligned} -\Delta w &= A_k v & \text{in } \Omega, \\ w &= 0 & \text{on } \partial\Omega. \end{aligned} \quad (2.14)$$

We first note that

$$\begin{aligned} |A_k((I - P_{k-1})v, v)| &\leq |A_k(v - Q_k w, v)| + |A_k(Q_k w - Q_{k-1} w, v)| \\ &\quad + |A_k(Q_{k-1} w - P_{k-1} v, v)|. \end{aligned} \quad (2.15)$$

Note that  $v$  is the cell centered approximation to  $w$  in  $M_k$ . For the first term in (2.15), we apply the Schwarz inequality and (2.4) to get

$$\begin{aligned} |A_k(v - Q_k w, v)| &\leq A_k(v - Q_k w, v - Q_k w)^{1/2} A_k(v, v)^{1/2} \\ &\leq Ch_k \|A_k v\| A_k(v, v)^{1/2}. \end{aligned} \quad (2.16)$$

Similarly,  $P_{k-1} v$  is the cell centered approximation of  $w$  in  $M_{k-1}$ . Thus repeating the above argument and using (2.11) gives

$$\begin{aligned} |A_k(Q_{k-1} w - P_{k-1} v, v)| &\leq \sqrt{2} A_{k-1}(Q_{k-1} w - P_{k-1} v, Q_{k-1} w - P_{k-1} v)^{1/2} A_k(v, v)^{1/2} \\ &\leq Ch_k \|A_k v\| A_k(v, v)^{1/2}. \end{aligned} \quad (2.17)$$

To complete the proof, we need only estimate the middle term of (2.15). We clearly have

$$\begin{aligned} |A_k(Q_k w - Q_{k-1} w, v)| &= |(Q_k w - Q_{k-1} w, A_k v)| \\ &\leq \|Q_k w - Q_{k-1} w\| \|A_k v\| \leq Ch_k \|w\|_1 \|A_k v\|. \end{aligned} \quad (2.18)$$

Here  $\|\cdot\|_1$  denotes the norm in  $H^1(\Omega)$ , the Sobolev space of order one on  $\Omega$ . The last inequality follows from well known approximation properties of  $Q_k$ ,  $Q_{k-1}$  and obvious manipulations.

To complete the proof we need a bound for  $\|w\|_1$ . Let  $A(\cdot, \cdot)$  denote the Dirichlet form defined by

$$A(V, W) = \int_{\Omega} \nabla V \cdot \nabla W \, dx.$$

By the Poincaré inequality,

$$\|w\|_1^2 \leq CA(w, w).$$

Moreover, by the definition of  $w$ , we clearly have that

$$A(w, w) = (A_k v, w).$$

Thus,

$$A(w, w) = (A_k v, Q_k w) \leq A_k(v, v)^{1/2} A_k(Q_k w, Q_k w)^{1/2}.$$

It is not difficult to see that for an arbitrary function  $u \in M_k$ ,  $A_k(u, u)$  is a sum of squares of differences of the nodal values of  $u$  at neighboring cells plus a multiple of the squares of the values of the nodes on the boundary. This multiple is between 2 and 8. The  $(i, j)$ th nodal value of  $Q_k w$  is nothing but the average value of  $w$  over the cell  $\Omega_{i,j}^k$ . It is a simple exercise in calculus to show that the square of the difference of the nodal values of  $Q_k w$  on two neighboring cells can be bounded by the local integral

$$\int |\nabla w|^2 \, dx,$$

where the region of integration is over the two neighboring cells. Similar estimates show that  $(Q_k w)_{i,j}^2$  can be bounded by the above integral over the cell when  $i, j$  correspond to a cell which meets the boundary of  $\Omega$ . Summing these estimates gives that

$$A_k(Q_k w, Q_k w) \leq CA(w, w).$$

Combining the above estimates shows that

$$|A_k(Q_k w - Q_{k-1} w, v)| \leq Ch_k \|A_k v\|_2 A_k(v, v)^{1/2}. \quad (2.19)$$

Combining (2.15)–(2.19) gives

$$|A_k((I - P_{k-1})v, v)| \leq Ch_k \|A_k v\| A_k(v, v)^{1/2}. \quad (2.20)$$



By Gerschgorin's Theorem,

$$\lambda_k \leq 8h_k^{-2}. \quad (2.21)$$

The regularity and approximation condition with  $\alpha = 1/2$  follows from (2.20) and (2.21) and hence completes the proof of the theorem.  $\square$

*Remark 2.1*

We can now apply the results of [7] to algorithm 1 with point Jacobi or Gauss-Seidel smoothing. For example, theorem 6 of [7] implies that the variable V-cycle algorithm ( $p = 1$  and  $m(k) = 2^{J-k}$ ) provides a preconditioner  $B_J$  with a condition number which is bounded independently of the number of levels  $J$ . In addition, theorem 7 of [7] implies that the W-cycle algorithm ( $p = 2$  and  $m(k) = 1$  for all  $k$ ) converges in the norm corresponding to the  $A_J(\cdot, \cdot)$  inner product at a rate which is independent of the number levels  $J$ .

*Remark 2.2*

The convergence achieved by these algorithms is contrary to the popular belief that the sum of the orders of the prolongation and restriction operators should be greater than the order of the differential operator, i.e.,

$$m_p + m_r > 2, \quad (2.22)$$

where  $m_p$  and  $m_r$  denote respectively the order of the prologation and restriction operators. In the above example, the order of both the prolongation and restriction operators is one whereas the underlying differential operator is of second order. The analysis given in [11] requires that (2.22) holds. That we were able to violate this condition and still prove uniform convergence illustrates the power of the theoretical approach provided in [7].

Actually, the cell centered multigrid algorithm described above violates many of the standard heuristics for multigrid algorithms as we shall now demonstrate. Let  $\tilde{P}_{k-1} : M_k \mapsto M_{k-1}$  be defined by  $\tilde{P}_{k-1}v = w$  where  $w$  is the unique function in  $M_{k-1}$  satisfying

$$A_k(w, \theta) = A_k(v, \theta) \quad \text{for all } \theta \in M_{k-1}. \quad (2.23)$$

The definition of  $\tilde{P}_{k-1}$  differs from  $P_{k-1}$  only in that the form  $A_k(\cdot, \cdot)$  is used instead of  $A_{k-1}(\cdot, \cdot)$  on the left hand sides of the definition. Clearly,  $\tilde{P}_{k-1}$  is the  $A_k(\cdot, \cdot)$  orthogonal projector of  $M_k$  onto  $M_{k-1}$ . In addition, (2.11) implies that

$$A_{k-1}(2\tilde{P}_{k-1}v, \theta) = A_k(\tilde{P}_{k-1}v, \theta) = A_k(v, \theta) \quad \text{for all } \theta \in M_{k-1}.$$

Thus,  $P_{k-1} = 2\tilde{P}_{k-1}$ .

For any function  $v \in M_k$ ,

$$v = (I - \tilde{P}_{k-1})v + \tilde{P}_{k-1}v \quad (2.24)$$

is an  $A_k(\cdot, \cdot)$  orthogonal decomposition of  $v$ . Since  $P_{k-1} = 2\tilde{P}_{k-1}$ ,

$$(I - P_{k-1})v = (I - \tilde{P}_{k-1})v - \tilde{P}_{k-1}v. \quad (2.25)$$

Comparing (2.24) and (2.25), we see that  $I - P_{k-1}$  preserves the  $(I - \tilde{P}_{k-1})$  component while changing the sign of the  $\tilde{P}_{k-1}$  component. It follows that  $(I - P_{k-1})^2$  is the identity and

$$A_k((I - P_{k-1})v, (I - P_{k-1})v) = A_k(v, v) \quad \text{for all } v \in M_k,$$

i.e.,  $I - P_{k-1}$  is an isometry of  $M_k$  onto itself.

The classical heuristics for multigrid algorithms suggest that smoothers reduce high frequency errors while the coarse grid correction reduces the low frequency errors. The coarse grid correction operator results in a “reduction” of  $I - P_{J-1}$ . In the cell centered application described above  $I - P_{J-1}$  is an isometry and thus cannot reduce any components of the error. For the above application, a two level slash cycle (step 4 skipped in algorithm 2.1 and  $j = J - 1$ ) gives rise to an error reduction operator of the form

$$E_{new} = (I - P_{J-1})K_J E_{old}.$$

Here  $K_J$  is the reducer associated with the smoothing process. Typical smoothing processes reduce high eigenvalue components (with respect to the  $A_k$  eigenvector decomposition) but make very little change to the low eigenvalue components. Thus, the norm of the reduction process associated with the smoother is close to one. Typically,

$$\|K_J\|_{M_J \rightarrow M_J} \geq 1 - ch_J^2$$

for some constant  $c$  not depending on  $J$ . Since  $I - P_{J-1}$  is an isometry,

$$\|(I - P_{J-1})K_J\|_{M_J \rightarrow M_J} = \|K_J\|_{M_J \rightarrow M_J} \geq 1 - ch_J^2.$$

Thus, the slash cycle algorithm produces a very poor norm reduction.

In contrast, the symmetric cycle in the two level case (algorithm 2.1 with  $j = J - 1$ ) has an error reduction operator of the form

$$E_{new} = K_J^*(I - P_{J-1})K_J E_{old}, \quad (2.26)$$

where  $K_J^*$  is the adjoint of  $K_J$  with respect to the  $A_k(\cdot, \cdot)$  inner product. The above estimates and theorem 7 of [7] guarantee that

$$\|K_J^*(I - P_{J-1})K_J\|_{M_J \rightarrow M_J} \leq \delta, \quad (2.27)$$

with  $\delta < 1$  and independent of  $J$ . Estimate (2.27) shows that the coarse grid correction plays a critical role in the symmetric cycling algorithm even though it preserves norms. The coarse grid correction operator  $I - P_{k-1}$  does not reduce the low eigenfunction components but mixes the high and low eigen-components so that the subsequent application of the adjoint smoothing operator results in an overall procedure which reduces all components.

The fact that the slash cycle algorithm has a very poor norm reduction rate does not necessarily imply that repetitive application of it will lead to a slowly

convergent iterative procedure. If, for example,  $K_J$  is symmetric with respect to the  $A_k(\cdot, \cdot)$  inner product, then two applications of the slash cycle has an error reduction which satisfies

$$\|(I - P_{J-1})K_J(I - P_{J-1})K_J\|_{M_J \mapsto M_J} = \|K_J(I - P_{J-1})K_J\|_{M_J \mapsto M_J} \leq \delta$$

for the same value of  $\delta$  as in (2.27).

### Remark 2.3

Results for cell centered approximations to problems on domains which are non-convex are possible. For example,  $\Omega$  could be the L-shaped domain. For such domains, full elliptic regularity does not hold. However, the above analysis can be carried out with minor modification to show that regularity and approximation still holds but for a smaller value of  $\alpha$ .

### Remark 2.4

The constructions and analysis can be extended to three dimensional problems. The resulting forms still satisfy (2.11) and completely analogous results hold.

There is also a natural variational multigrid algorithm. Indeed, whenever one has a sequence of nested spaces, the variational algorithm can be defined simply by using the fine grid form on all of the levels, i.e., by defining  $A_k(\cdot, \cdot)$  by

$$A_k(v, w) \equiv A_J(v, w) \quad \text{for all } v, w \in M_k.$$

Variational multigrid algorithms often correspond to the most natural and effective multigrid approach. This is not the case in the cell centered application. In fact, numerical evidence suggests that the variational algorithm gives rise to reduction rates which are bounded from below by  $1 - ch_J/h_J$  (see section 3).

The cell centered method is equivalent to the lowest order Raviart-Thomas mixed method when an appropriate quadrature is used (cf. [12]). Thus, the above results provide a multigrid analysis for the Raviart-Thomas method with numerical quadrature.

## 3. Numerical experiments

We report the results of numerical experiments computing the extreme eigenvalues for the operator  $B_J A_J$  corresponding to the cell centered finite difference method studied in the previous section. Similar results have been reported in [9]. The rate of iterative convergence for the multigrid algorithm can be inferred from these eigenvalues. We will first give results for the V-cycle and variable V-cycle non-variational multigrid algorithms. We next report results for the V-cycle and variable V-cycle variational multigrid algorithms. For comparison, we also include analogous results for the multigrid algorithms applied to the piecewise linear finite element approximation.

Table 1

The V-cycle non-variational multigrid algorithm.

$h_J$	$\lambda_{\min}(B_J A_J)$	$\lambda_{\max}(B_J A_J)$	$K(B_J A_J)$
1/8	0.81	1.24	1.53
1/16	0.79	1.34	1.69
1/32	0.79	1.45	1.84
1/64	0.78	1.54	1.96
1/128	0.78	1.61	2.06

In this section, we report the largest and smallest eigenvalues of the operator  $B_J A_J$  as a function of the finest grid size  $h_J$ . For all of the examples, the coarsest grid is of size  $h_J = 1/2$ . For the V-cycle examples, we use one sweep of the point Gauss-Seidel iteration in steps 2 and 4 of algorithm 2.1. In step 4, we sweep through the nodes in the opposite order as that used in step 2. This results in an operator  $B_J$  which is symmetric with respect to the  $(\cdot, \cdot)$  inner product. In the case of the variable V-cycle examples, we use  $m(k) = 2^{J-k+1} - 1$  sweeps of point Gauss-Seidel iteration. The directions of the sweeps are alternated to follow the construction described in algorithm 2.1.

In the case of a non-variational multigrid algorithm, the multigrid error operator  $I - B_J A_J$  may not be a contraction. However, for V-cycle algorithms with arbitrary  $m(k) \geq 1$ ,  $B_J$  is always symmetric and positive definite (cf. [7]). Since we report the eigenvalues of  $B_J A_J$  in tables 1 and 2,  $I - B_J A_J$  fails to be a contraction if and only if the largest eigenvalue is greater than or equal to 2. When  $I - B_J A_J$  is a contraction (as is always the case in the reported examples), the multigrid process produces an iteration with convergence rate equal to the spectral radius of  $I - B_J A_J$ . Alternatively,  $B_J$  can be used as a preconditioner in the conjugate gradient algorithm. In this case, the asymptotic rate of iterative convergence is bounded by

$$\rho = \frac{\sqrt{K(B_J A_J)} - 1}{\sqrt{K(B_J A_J)} + 1}.$$

Here  $K(B_J A_J)$  denotes the condition number of  $B_J A_J$  and is defined to be the ratio of the largest to smallest eigenvalue of  $B_J A_J$ .

In tables 1 and 2, we report results for the V-cycle algorithm and the variable V-cycle algorithm. The analysis of the previous section (see remark 2.1) guarantees

Table 2

The variable V-cycle non-variational multigrid algorithm.

$h_J$	$\lambda_{\min}(B_J A_J)$	$\lambda_{\max}(B_J A_J)$	$K(B_J A_J)$
1/8	0.82	1.19	1.45
1/16	0.80	1.22	1.53
1/32	0.80	1.24	1.55
1/64	0.80	1.25	1.56
1/128	0.80	1.25	1.56

Table 3  
The V-cycle variational multigrid algorithm.

$h_J$	$\lambda_{\min}(B_J A_J)$	$K(B_J A_J)$
1/8	0.53	1.88
1/16	0.32	3.13
1/32	0.18	5.67
1/64	0.09	10.8
1/128	0.05	21.1

Table 4  
The variable V-cycle variational multigrid algorithm.

$h_J$	$\lambda_{\min}(B_J A_J)$	$K(B_J A_J)$
1/8	0.59	1.69
1/16	0.43	2.33
1/32	0.30	3.36
1/64	0.20	5.09
1/128	0.13	7.75

that the condition number of  $B_J A_J$  for the variable V-cycle algorithm can be bounded independently of the number of levels. Although there is no complete theory for the V-cycle algorithm, it can be shown using theorem 2.1 that the smallest eigenvalue is bounded from below by  $C/(J - j)$  for some positive constant  $C$  not depending on  $J$  and  $j$  (cf. [7]). It is of practical interest that the condition numbers for both the V-cycle and variable V-cycle algorithm remain relatively small.

Tables 3 and 4 give the results for the variational multigrid algorithms applied to the cell centered finite difference approximation discussed in the previous section. The variational multigrid algorithm uses the fine grid form to define all of the forms on the coarser grids. In the case of variational multigrid algorithms with appropriately chosen smoothers,  $B_J A_J$  is always positive definite and its largest eigenvalue is always bounded by one. For comparison, the smallest eigenvalue and the condition number (reciprocal of the smallest eigenvalue) is reported. Table 3 (respectively, table 4) corresponds to table 1 (respectively, table 2) in that both algorithms use exactly the same number of smoothings. Note that the condition numbers reported in table 3 grow proportional to the inverse of  $h_J$ .

Table 5  
V-cycle, conforming piecewise linear approximation.

$h_J$	$\lambda_{\min}(B_J A_J)$	$K(B_J A_J)$
1/8	0.78	1.29
1/16	0.75	1.32
1/32	0.74	1.34
1/64	0.74	1.35
1/128	0.74	1.35

Table 6

Variable V-cycle, conforming piecewise linear approximation.

$h_J$	$\lambda_{\min}(B_J A_J)$	$K(B_J A_J)$
1/8	0.79	1.26
1/16	0.78	1.29
1/32	0.77	1.30
1/64	0.77	1.31
1/128	0.76	1.31

We conclude this section by reporting the results for multigrid algorithms applied to the standard piecewise linear finite element approximation to Dirichlet's problem (2.2). This is a variational multigrid approach and we report the lowest eigenvalue and condition numbers for comparison with the above approaches. As we can see, the condition numbers for the non-variational multigrid methods for the cell centered finite difference method compare favorably with these benchmark calculations. Tables 1 and 5 (respectively, tables 2 and 6) correspond to algorithms with the same number of smoothings. In the case of the piecewise linear finite element example, it is possible to prove that the V-cycle algorithm converges with a rate that can be bounded independently of the number of levels and thus there is no theoretical reason for using the variable V-cycle algorithm even though it gives rise to slightly smaller condition numbers.

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