Convergence Estimates for Essentially Positive Type Discrete Dirichlet Problems

By J. H. Bramble, B. E. Hubbard and Vidar Thomée

Abstract. In this paper we consider a class of difference approximations to the Dirichlet problem for second-order elliptic operators with smooth coefficients. The main result is that if the order of accuracy of the approximate problem is ν, and F (the right-hand side) and f (the boundary values) both belong to C^λ for $\lambda < \nu$, then the rate of convergence is $O(h^\lambda)$.

1. Introduction. Let $\mathcal{R}$ with boundary $\partial\mathcal{R}$ be a bounded domain in euclidean N-space E^N. We shall be concerned with the solution of the Dirichlet problem

$$(1.1)\qquad Lu \equiv -\sum_{j,k=1}^{N} a_{jk}(x)\frac{\partial^2 u}{\partial x_j \partial x_k} + \sum_{j=1}^{N} a_j(x)\frac{\partial u}{\partial x_j} + a_0(x)u = F \quad \text{in } \mathcal{R}\,,$$

$$(1.2)\qquad u = f \quad \text{on } \partial\mathcal{R}\,,$$

where L is uniformly elliptic and $a_0 \geqq 0$ in $\mathcal{R}$. For the numerical solution it is common to cover $\mathcal{R}$ by a square mesh with mesh-width h, and for "interior" mesh-points x approximate the Eq. (1.1) by an equation of the form

$$(1.3)\qquad L_h u_h(x) \equiv h^{-2}\sum_{\beta} b_\beta(x,h)u_h(x+\beta h) = M_h F(x)\,,$$

where $\beta = (\beta_1, \cdots, \beta_N)$ has integer components and L_h and M_h are consistent with L and the identity operator, respectively. For mesh-points near $\partial\mathcal{R}$ one considers similarly equations of the form

$$(1.4)\qquad l_h u_h(x) = u_h(x) + \sum_{\beta\neq 0} b_\beta(x,h)u_h(x+\beta h) = m_h(F, f)$$

which take into account both Eqs. (1.1) and (1.2). In much of the literature, it is assumed that L_h is of positive type, or $\sum_\beta b_\beta(x,h) \geqq 0$ and $b_\beta(x,h) \leqq 0$ for $\beta \neq 0$ in (1.3), and this is the case that is considered in this paper. Similarly, the b_β in (1.4) are often assumed $\leqq 0$; we shall assume here that

$$\sum_\beta |b_\beta| \leqq \gamma < 1$$

and shall say then that the pair of operators L_h and l_h is of essentially positive type.

For many special schemes of the type described, convergence results are given in the literature. They are generally of the form that if the discrete problem (1.3), (1.4) approximates the continuous (1.1), (1.2) with order of accuracy ν, then

$$(1.5)\qquad |u(x) - u_h(x)| \leqq Ch^\nu\,.$$

The constant C here depends on the unknown solution u; in general one has had to assume that u, together with its derivatives of orders less than or equal to $\nu + 2$, is bounded in $\mathcal{R}$.

Received May 5, 1969.

Thus in particular, if $N = 2$ and

$$Lu \equiv -\Delta u \equiv -\frac{\partial^2 u}{\partial x_1^{\,2}} - \frac{\partial^2 u}{\partial x_2^{\,2}}$$

a common approximation of (1.1) is the well-known "five-point" formula

$$-\Delta_h^{(5)} u(x) = h^{-2}\Big\{4u(x) - \sum_{|\beta|=1} u(x+\beta h)\Big\} = F(x)\,. \tag{1.6}$$

For this operator, and for the simplest possible boundary approximations, Gerschgorin [7] proved an estimate of the form (1.5) with $\nu = 1$. Later Collatz [6] using linear interpolation near the boundary, improved the result to get (1.5) with $\nu = 2$. Using instead of (1.6) the "nine-point" formula

$$\begin{aligned} -\Delta_h^{(9)} u(x) &= \frac{1}{6h^2}\Big\{20u(x) - 4\sum_{|\beta|=1} u(x+\beta h) - \sum_{|\beta_1|=|\beta_2|=1} u(x+\beta h)\Big\} \\ &= F(x) + \frac{h^2}{12}\,\Delta_h^{(5)} F(x)\,, \end{aligned}$$

Bramble and Hubbard [3] showed that the operator l_h in (1.4) can be chosen in such a manner that (1.5) holds for $\nu = 4$. These authors [4] also constructed operators L_h and l_h in the case of a general L ($N = 2$) such that (1.5) holds with $\nu = 2$.

It was observed by Bahvalov [1] in an important paper, seemingly not well-known outside the Russian literature, that the regularity demands on the solution u of the continuous problem in some cases can be relaxed by essentially two derivatives at the boundary without losing the convergence estimate (1.5) and that for still less regular u one can obtain correspondingly weaker convergence estimates. Bahvalov used his error bounds to estimate the number of arithmetic operations needed to obtain u to a prescribed accuracy. Related results were also obtained in special cases by Wasow [13], Laasonen [8], and by Volkov, cf. [11], [12], and references.

The purpose of this paper is to present a general theory which comprises all the special features mentioned. In doing so we shall express the estimates in terms of the data F and f of the problem rather than in terms of the unknown solution u; the main result will be of the type that if F and f both belong to $\mathcal{C}^\lambda$ for some $\lambda > \nu$, then an inequality of the form (1.5) holds. It will also be shown that if F and f are in $\mathcal{C}^\lambda$ for $\lambda < \nu$, then error bounds of the form $O(h^\lambda)$ can be obtained. Since the effort is concentrated on the dependence of the regularity of F and f, we shall assume that the coefficients and the boundary are infinitely differentiable.

The proofs will be based on new estimates for the discrete Green's function for the operator L_h. This estimate can be thought of as a discrete analogue of the estimate

$$\int_{d(y)=\delta} G(x, y)\,d\delta \leqq C\,\delta\,, \qquad x \in \mathcal{R}$$

for the continuous Green's function, where $d(y)$ is the distance from y to $\partial\mathcal{R}$. (In special cases such results were used by Volkov [12].) The transition between estimates in terms of the solution and the data F and f will be made by means of the

Schauder estimates for second-order elliptic differential operators; at some points it will be convenient to use interpolation properties of Lipschitz spaces. These latter types of techniques also apply to other convergence problems in difference equations (cf. Peetre and Thomée [10] and Bramble, Kellogg, and Thomée [5]).

In a certain sense the results are not optimal as far as the regularity of F is concerned; it will be shown in a forthcoming paper by Bramble [2] that the operator M_h in (1.3) can be chosen in such a manner as to make it possible to further relax the regularity demands on F.

2. Preliminaries. We start by introducing some notation. For $\mathfrak{M} \subset E^N$, let $\mathcal{C}(\mathfrak{M})$ be the set of real-valued continuous functions on $\mathfrak{M}$ and define

$$|u|_{\mathfrak{M}} = \sup_{x \in \mathfrak{M}} |u(x)| .$$

In particular, if $\mathfrak{M}$ is a finite point-set, $\mathcal{C}(\mathfrak{M})$ simply consists of all real-valued functions on $\mathfrak{M}$ and $|u|_{\mathfrak{M}}$ is always finite.

For a domain $\mathcal{R} \subset E^N$ and $u \in \mathcal{C}(\mathcal{R})$, $0 < \sigma \leqq 1$ we set

$$H_{\sigma,\mathcal{R}}(u) = \sup_{x, x+y \in \mathcal{R};\, y \neq 0} \frac{|u(x+y) - u(x)|}{|y|^{\sigma}} .$$

Let $\alpha = (\alpha_1, \cdots, \alpha_N)$ with α_j nonnegative integers and define $D^{\alpha}u = (\partial/\partial x_1)^{\alpha_1} \cdots (\partial/\partial x_N)^{\alpha_N}$. If s is a positive real number and $s = S + \sigma$, where S is an integer and $0 < \sigma \leqq 1$, we say that $u \in \mathcal{C}^s(\bar{\mathcal{R}})$ if $D^{\alpha}u \in \mathcal{C}(\mathcal{R})$ for $|\alpha| = \sum_j \alpha_j \leqq S$ and if

$$|u|_{s,\mathcal{R}} = |u|_{\mathcal{R}} + \sum_{|\alpha|=S} H_{\sigma,\mathcal{R}}(D^{\alpha}u)$$

is finite. We set $\mathcal{C}^{\infty}(\bar{\mathcal{R}}) = \bigcap_{s>0} \mathcal{C}^s(\bar{\mathcal{R}})$.

For $u \in \mathcal{C}(\mathcal{R})$ we also set

$$Z_{\sigma,\mathcal{R}}(u) = \sup_{x, x\pm y \in \mathcal{R};\, y \neq 0} \frac{|u(x+y) - 2u(x) + u(x-y)|}{|y|^{\sigma}}$$

and say again for $s = S + \sigma$, where S is a nonnegative integer and $0 < \sigma \leqq 1$, that $u \in \mathcal{C}_{Z^s}(\bar{\mathcal{R}})$ if $D^{\alpha}u \in \mathcal{C}(\mathcal{R})$ for $|\alpha| \leqq S$ and if

$$|u|_{Z,s,\mathcal{R}} = |u|_{\mathcal{R}} + \sum_{|\alpha|=S} Z_{\sigma,\mathcal{R}}(D^{\alpha}u)$$

is finite. The finiteness of $H_{1,\mathcal{R}}(u)$ or $Z_{1,\mathcal{R}}(u)$ means that u satisfies a Hölder condition or a Zygmund condition, respectively. Under the regularity assumptions below on $\partial\mathcal{R}$ we have $\mathcal{C}_{Z^s}(\bar{\mathcal{R}}) = \mathcal{C}^s(\bar{\mathcal{R}})$ for nonintegral s; for integral s we have $\mathcal{C}^s(\bar{\mathcal{R}}) \subseteq \mathcal{C}_{Z^s}(\bar{\mathcal{R}})$.

Let $\mathcal{R}_{\delta} = \{x; x \in \mathcal{R}; d(x) > \delta\}$ where $d(x)$ is the distance from x to the boundary $\partial\mathcal{R}$ of $\mathcal{R}$. We say that $u \in \mathcal{C}^s(\mathcal{R})$ if $u \in \mathcal{C}^s(\bar{\mathcal{R}}_{\delta})$ for all $\delta > 0$.

We shall always assume that $\partial\mathcal{R} \in \mathcal{C}^{\infty}$, so that each point $x \in \partial\mathcal{R}$ has a neighborhood $\omega_x \subset \partial\mathcal{R}$ which is the homeomorphic map $g(\Omega_x)$ of an open spherical neighborhood Ω_x of the origin in E^{N-1} and $g_j \in \mathcal{C}^{\infty}(\bar{\Omega}_x)$, $j = 1, \cdots, N$ where $g = (g_1, \cdots, g_N)$. Since $\mathcal{R}$ is a bounded domain we can by compactness cover $\partial\mathcal{R}$ by a finite number of the sets ω_{x_j} so that $\partial\mathcal{R} = \bigcup_{j=1}^{J} \omega_{x_j}$. Let $g^{(j)}$ be the mapping corresponding to ω_{x_j}. We say that $f \in \mathcal{C}^s(\partial\mathcal{R})$ if $f(g^{(j)}) \in \mathcal{C}^s(\bar{\Omega}_{x_j})$ for $j = 1, \cdots, J$, and define

$$|f|_{s,\partial\mathcal{R}} = \max_j |f(g^{(j)})|_{s,\Omega_{x_j}} .$$

The definition of $f \in \mathcal{C}_Z{}^s(\partial\mathcal{R})$ and $|f|_{Z,s,\partial\mathcal{R}}$ are analogous.

Consider now in the bounded domain $\mathcal{R}$ the uniformly elliptic operator

$$Lu(x) \equiv -\sum_{j,k=1}^{N} a_{jk}(x) \frac{\partial^2 u}{\partial x_j \partial x_k} + \sum_{j=1}^{N} a_j(x) \frac{\partial u}{\partial x_j} + a_0(x)u(x) \qquad a_{jk}(x) = a_{kj}(x)$$

so that for some constant $\epsilon_0 > 0$, real $\xi = (\xi_1, \cdots, \xi_N)$ and $x \in \mathcal{R}$

$$\sum_{j,k=1}^{N} a_{jk}(x)\xi_j\xi_k \geqq \epsilon_0|\xi|^2 \qquad |\xi| = \left(\sum_{j=1}^{N} \xi_j{}^2\right)^{1/2} .$$

We shall assume for simplicity that all coefficients are in $\mathcal{C}^\infty(E^N)$ and also that $a_0(x) \geqq 0$.

Our aim is to discuss the approximate solution of the Dirichlet problem

$$Lu = F \quad \text{in} \quad \mathcal{R} \tag{2.1}$$

$$u = f \quad \text{on} \quad \partial\mathcal{R} \tag{2.2}$$

by finite difference methods.

We shall study finite difference approximations of L of the form

$$L_h u(x) = h^{-2} \sum_\beta b_\beta(x, h)u(x + \beta h)$$

where $\beta = (\beta_1, \cdots, \beta_N)$ with integral components β_j. We assume that there are positive constants h_0 and B such that $b_\beta \in \mathcal{C}^\infty(E^N \times [0, h_0])$ and $b_\beta = 0$ for $|\beta| > B$. We shall always assume that L_h is consistent with L so that for any x and any u sufficiently smooth

$$\lim_{h\to 0} L_h u(x) = Lu(x) .$$

We shall further assume that L_h is of positive type; i.e. for $h \leqq h_0$ and $x \in \mathcal{R}$ we have

$$\begin{aligned} &\sum_\beta b_\beta(x, h) \geqq 0 , \\ &b_\beta(x, h) \leqq 0 \qquad \beta \neq 0 . \end{aligned} \tag{2.3}$$

Let $E_h{}^N$ be the set of mesh-points $x = (m_1h, \cdots, m_Nh)$ where m_j are integers. For $x \in E_h{}^N$, the set $\{y; y = x + \beta h, b_\beta(x, h) \not\equiv 0 \text{ for } h \leqq h_0\}$ is referred to as the set of neighbors of x; its convex hull in E^N will be denoted by $\mathfrak{N}_x$. We set

$$\begin{aligned} \bar{R}_h &= \bar{\mathcal{R}} \cap E_h{}^N \\ R_h &= \{x; x \in \bar{R}_h, \mathfrak{N}_x \subset \bar{\mathcal{R}}\} \\ \partial R_h &= \bar{R}_h \backslash R_h . \end{aligned}$$

The points in R_h are called interior mesh-points; those of ∂R_h are boundary mesh-points. We denote the set of real-valued functions defined on the above sets by $\bar{\mathfrak{D}}_h$, $\mathfrak{D}_h$, and $\partial\mathfrak{D}_h$, respectively.

In addition to the operator L_h which will be used at interior mesh-points, we introduce an operator l_h which will be related to the boundary values,

$$l_h u(x) = \sum_\beta b_\beta(x, h) u(x + \beta h) \qquad x \in \partial R_h .$$

We shall assume that $b_0(x, h) \equiv 1$ and that $b_\beta(x, h) = 0$ for $|\beta| > B$ and for $x + \beta h \notin \overline{R}_h$. No regularity will be assumed about the coefficients in l_h; instead we shall assume that there exists $\gamma < 1$ such that

$$\sum_{\beta \neq 0} |b_\beta(x, h)| \leqq \gamma , \quad x \in \partial R_h, h \leqq h_0 . \tag{2.4}$$

For the approximate solution of the Dirichlet problem (2.1), (2.2) we now consider a discrete problem

$$L_h u_h = M_h F \quad \text{on } R_h \tag{2.5}$$

$$l_h u_h = m_h \mathfrak{F} \quad \text{on } \partial R_h . \tag{2.6}$$

Here M_h is a bounded linear operator from $\mathcal{C}(\overline{\mathfrak{R}})$ into $\mathfrak{D}_h$, $\mathfrak{F} = (F, f) \in \mathcal{C}(\overline{\mathfrak{R}}) \times \mathcal{C}(\partial \mathfrak{R})$ and m_h is a bounded linear operator from $\mathcal{C}(\overline{\mathfrak{R}}) \times \mathcal{C}(\partial\mathfrak{R})$ into $\partial\mathfrak{D}_h$. We shall prove later (Lemma 5.3) that this problem has a unique solution for small h, and our aim is to study the convergence of this solution u_h to the solution of (2.1), (2.2).

We say that the discrete problem approximates the continuous problem with order of accuracy ν if for any λ, μ with $0 \leqq \lambda \leqq \nu$, $0 \leqq \mu \leqq \nu$ there is a constant C such that

$$|L_h u(x) - M_h L u(x)| \leqq C h^\lambda |u|_{2+\lambda, \mathfrak{N}_x} , \qquad u \in \mathcal{C}^{2+\lambda}(\overline{\mathfrak{N}}_x) , \qquad x \in R_h , \tag{2.7}$$

$$|l_h u - m_h(Lu, \tilde{u})|_{\partial R_h} \leqq C h^\mu (|u|_{\mu, \mathfrak{R}} + |Lu|_{\mathfrak{R}}), \qquad u \in \mathcal{C}^\mu(\overline{\mathfrak{R}}) \tag{2.8}$$

where $\tilde{u}$ denotes the restriction of $u \in \mathcal{C}(\overline{\mathfrak{R}})$ to $\partial\mathfrak{R}$. By the consistency between L_h and L, M_h is then an approximation of the identity operator.

We can now state our main result:

THEOREM. *Assume that the operators L, L_h, l_h, M_h, and m_h satisfy the above assumptions and that the discrete problem* (2.5), (2.6) *approximates the Dirichlet problem* (2.1), (2.2) *with order of accuracy ν. Let u_h and u be the solutions of the discrete and continuous problems, respectively. Then for λ, $\mu \geqq 0$, λ, $\mu \neq \nu$, there is a constant C such that for $F \in \mathcal{C}_Z{}^\lambda(\overline{\mathfrak{R}})$, $f \in \mathcal{C}_Z{}^\mu(\partial\mathfrak{R})$ we have*

$$|u - u_h|_{Rh} \leqq C\{h^{\min(\lambda,\nu)} |F|_{Z,\lambda,\mathfrak{R}} + h^{\min(\mu,\nu)} |f|_{Z,\mu,\partial\mathfrak{R}}\} . \tag{2.9}$$

Further, if $F \in \mathcal{C}^\lambda(\overline{\mathfrak{R}})$ for some $\lambda > 0$, and $f \in \mathcal{C}(\partial\mathfrak{R})$, we have

$$\lim_{h \to 0} |u - u_h|_{R_h} = 0 .$$

The proof of this result will be given in Section 5.

3. Positive Type Operators and Green's Functions. Throughout this section we shall assume that L, L_h, and $\mathfrak{R}$ satisfy the assumptions of Section 2. We start with a lemma concerning the structure of positive type operators.

LEMMA 3.1. *There are positive constants h_0 and $\mathcal{K}$ such that for any $x \in R_h$ and any $\eta \in E^N$ with $|\eta| = 1$ there is a $\beta \in E^N$ with integral components such that for $h \leqq h_0$*

(i) $(\beta, \eta) \geqq \mathcal{K}$,

(ii) $-b_\beta(x, h) \geqq \mathcal{K}$,

where $(\beta, \eta) = \sum_{j=1}^N \beta_j \eta_j$.

Proof. By Taylor's theorem we have for smooth u,

$$L_h u(x) = u(x)h^{-2} \sum_\beta b_\beta(x, h) + \sum_j \frac{\partial u}{\partial x_j} h^{-1} \sum_\beta \beta_j b_\beta(x, h)$$
$$+ \frac{1}{2} \sum_{j,k} \frac{\partial^2 u}{\partial x_j \partial x_k} \sum_\beta \beta_j \beta_k b_\beta(x, h) + o(1) \quad \text{as } h \to 0$$

and so from the consistency we conclude that

$$\begin{aligned} \sum_\beta b_\beta(x, h) &= h^2 a_0(x) + o(h^2) \quad \text{when } h \to 0\,, \\ \sum_\beta \beta_j b_\beta(x, h) &= h a_j(x) + o(h) \quad \text{when } h \to 0\,, \\ \sum_\beta \beta_j \beta_k b_\beta(x, h) &= -2a_{jk}(x) + o(1) \quad \text{when } h \to 0\,. \end{aligned} \tag{3.1}$$

In particular, if $|\eta| = 1$, we obtain after multiplication of (3.1) by $\eta_j \eta_k$ and summation over j and k, and using the ellipticity of L,

$$\begin{aligned} -\sum_\beta (\beta, \eta)^2 b_\beta(x, h) &= 2 \sum_{j,k} a_{jk}(x) \eta_j \eta_k + o(1) \quad \text{when } h \to 0\,, \\ &\geqq 2\epsilon_0 + o(1) \quad \text{when } h \to 0 \end{aligned} \tag{3.2}$$

for sufficiently small h, uniformly in x and η. Similarly,

$$\sum_\beta (\beta, \eta) b_\beta(x, h) = o(1) \quad \text{when } h \to 0\,. \tag{3.3}$$

Since β and b_β are uniformly bounded, to prove the statement it is clearly sufficient to prove

$$\inf_{h \leqq h_0;\, x \in R_h} \left[-\sum_{(\beta,\eta)>0} (\beta, \eta) b_\beta(x, h) \right] > 0\,.$$

But by (3.2) and (3.3) we have for some positive h_0 that for $h \leqq h_0$, $x \in R_h$, and $|\eta| = 1$,

$$\begin{aligned} -\sum_{(\beta,\eta)>0} (\beta, \eta) b_\beta(x, h) &= -\frac{1}{2} \sum_\beta |(\beta, \eta)| b_\beta(x, h) + o(1) \\ &\geqq -(2B)^{-1} \sum_\beta (\beta, \eta)^2 b_\beta(x, h) + o(1) \geqq B^{-1}\epsilon_0 \end{aligned}$$

which thus proves the lemma.

The above lemma tells us that given any $x \in R_h$ and any plane through x, there is a neighbor of x on each side of the plane with distance greater than or equal to $\mathcal{K}h$ from the plane and corresponding to a coefficient with $|b_\beta(x, h)| \geqq \mathcal{K}$.

We can now prove the following maximum principle:

LEMMA 3.2. *Let $h \leqq h_0$ where h_0 is the constant in Lemma 3.1. Then if $v \in \bar{\mathfrak{D}}_h$ satisfies $L_h v \geqq 0$ on R_h, $v \geqq 0$ on ∂R_h, we have $v \geqq 0$ on $\bar{R}_h$.*

Proof. Assume the conclusion is false, that v has a negative minimum $v(x^{(0)})$ on R_h. Since L_h is of positive type we have

$$v(x^{(0)}) \geqq \sum_\beta (-b_0(x, h)^{-1} b_\beta(x, h)) v(x^{(0)} + \beta h)$$

and since the coefficients on the right are nonnegative and have sum at most 1, we conclude that for all neighbors corresponding to nonzero b_β, $v(x^{(0)} + \beta h) \leqq v(x^{(0)})$. Using Lemma 3.1 with $\eta = e_1$, we find that for one such neighbor $x^{(1)}$, $x_1^{(1)} \geqq x_1^{(0)} + \mathcal{K}h$. Iterating this argument we find a sequence of points $x^{(j)}$, $j = 1, 2, \cdots$, such that

$$v(x^{(j)}) \leqq v(x^{(0)}) \qquad x_1^{(j)} \geqq jh\mathcal{K} + x_1^{(0)} .$$

But by the boundedness of $\mathfrak{R}$, after a finite number of steps, $x^{(j)} \in \partial R_h$, and thus $v(x^{(j)}) \leqq 0$, which is a contradiction.

We can now conclude:

LEMMA 3.3. *The discrete problem*

$$\begin{aligned} L_h v &= F \quad \text{on } R_h \\ v &= f \quad \text{on } \partial R_h \end{aligned}$$

has a unique solution $v \in \overline{\mathfrak{D}}_h$ *for any* $F \in \overline{\mathfrak{D}}_h$ *and* $f \in \partial\mathfrak{D}_h$.

Proof. Since for $F = f = 0$, Lemma 3.2 proves that both v and $-v$ are nonpositive, we have the uniqueness. But this implies the existence by Cramer's rule.

We now introduce the discrete Green's function $G_h(x, y)$ defined for each fixed $y \in \overline{R}_h$ by

$$\begin{aligned} L_h G_h(x, y) &= h^{-N}\delta(x, y) \qquad x \in R_h , \\ G_h(x, y) &= \delta(x, y) \qquad x \in \partial R_h , \end{aligned}$$

where $\delta(x, x) = 1$, $\delta(x, y) = 0$ for $x \neq y$. In terms of this function we have the following representation:

LEMMA 3.4. *Let* $v \in \overline{\mathfrak{D}}_h$. *Then for* $x \in \overline{R}_h$ *we have*

$$v(x) = h^N \sum_{y \in R_h} G_h(x, y) L_h v(y) + \sum_{y \in \partial R_h} G_h(x, y) v(y) . \tag{3.4}$$

Proof. This follows immediately from the definition of G_h and the uniqueness part of Lemma 3.3.

We collect some simple properties of G_h in a lemma:

LEMMA 3.5. *The Green's function defined above satisfies*

$$\begin{aligned} G_h(x, y) &\geqq 0 , \qquad x, y \in \overline{R}_h \\ \sum_{y \in \partial R_h} G_h(x, y) &\leqq 1 , \qquad x \in \overline{R}_h \end{aligned} \tag{3.5}$$

and there are positive constants h_0 *and* C *such that for* $h \leqq h_0$,

$$h^N \sum_{y \in R_h} G_h(x, y) \leqq C , \qquad x \in \overline{R}_h . \tag{3.6}$$

Proof. The nonnegativity follows at once from the definition and Lemma 3.2, and (3.5) then follows by setting $v \equiv 1$ in (3.4) and noticing that by (2.3), $L_h 1 \geqq 0$. Because of the assumptions on L there exists a function $\phi \in \mathcal{C}^2(\overline{\mathfrak{R}})$ satisfying $L\phi \geqq 2$, and by consistency for sufficiently small h and $x \in R_h$, we have $L_h\phi(x) \geqq 1$. Setting $v = \phi$ in (3.4) and using (3.5) we therefore obtain (3.6).

In order to give the next lemma which is the crucial lemma for our theorem,we shall need some further notation. Let $d(x)$ denote as before the distance from any point $x \in \mathfrak{R}$ to $\partial\mathfrak{R}$. Since we have assumed $\partial\mathfrak{R} \in \mathcal{C}^\infty$ then if $2\delta_0 > 0$ is less than the minimum over $\partial\mathcal{R}$ of the radius of the osculating sphere, we also have $\partial\mathcal{R}_\delta \in \mathcal{C}^\infty$ for $\delta < 2\delta_0$ and $d \in \mathcal{C}^\infty(\overline{\mathcal{R}} \setminus \mathcal{R}_{2\delta_0})$. For any nonnegative integer j we define

$$P_{h,j} = \{x; x \in R_h, \tfrac{1}{2}\,\mathcal{K}jh < d(x) \leqq \tfrac{1}{2}\,\mathcal{K}(j+1)h\},$$

where $\mathcal{K}$ is the constant in Lemma 3.1.

We shall then have the following (this is the first time any regularity of $\partial\mathcal{R}$ need be assumed):

LEMMA 3.6. *There are positive constants C and h_0 such that when $h \leqq h_0$, $\frac{1}{2}\,\mathcal{K}jh < \delta_0$, we have*

$$h^{N-1} \sum_{y \in P_{h,j}} G_h(x, y) \leqq Cjh .$$

Proof. Let $\frac{1}{2}\,\mathcal{K}jh < \delta_0$ and $Bh < \delta_0$ so that $d \in \mathcal{C}^\infty(\overline{\mathcal{R}}\setminus\mathcal{R}_{\delta_0+Bh})$. Let $\delta = \frac{1}{2}\,\mathcal{K}jh$ and set

$$\begin{aligned}\phi_\delta(x) &= \delta \qquad x \in R_\delta \\ &= d(x) \qquad x \in \overline{\mathcal{R}}\setminus\mathcal{R}_\delta .\end{aligned}$$

We want to apply Lemma 3.4 to the restriction of ϕ_δ to $\overline{R}_h$. We have

$$\begin{aligned}L_h\phi_\delta(y) &= L_hd(y) + h^{-2} \sum_{y+\beta h \in R_h \cap \mathcal{R}_\delta} b_\beta(y, h)[\delta - d(y + \beta h)] \\ &\geqq L_hd(y) , \quad y \in R_h\setminus\mathcal{R}_\delta\end{aligned} \tag{3.7}$$

and since $L_h\delta \geqq 0$ by (2.3)

$$L_h\phi_\delta(y) = L_h\delta + h^{-2} \sum_{y+\beta h \notin R_h \setminus \mathcal{R}_\delta} b_\beta(y, h)[d(y + \beta h) - \delta] \geqq 0 , \qquad y \in R_h \cap \mathcal{R}_\delta . \tag{3.8}$$

We need a stronger result for $y \in P_{h,j}$. To this end let η be the exterior normal at y of $\partial\mathcal{R}_{d(y)}$ and notice that the distance from y to $\partial\mathcal{R}$ is attained in the direction of η. It follows from Lemma 3.1 that there is a β such that

$$(\beta h, \eta) \geqq \mathcal{K}h , \qquad -b_\beta(y, h) \geqq \mathcal{K}$$

and since the distance from y to $\partial\mathcal{R}_\delta$ is at most $\frac{1}{2}\,\mathcal{K}h$ we can conclude that for some positive h_0 depending on the curvature of $\partial\mathcal{R}$ and on B, we have for $h \leqq h_0$ that $d(y + \beta h) \leqq \delta - \frac{1}{4}\,\mathcal{K}h$ and it follows that

$$L_h\phi_\delta(y) \geqq Ch^{-1} , \quad y \in P_{h,j} . \tag{3.9}$$

Using Lemma 3.4 with $v = \phi_\delta$, we now see from (3.7), (3.8), and (3.9) that

$$h^{N-1} \sum_{y \in P_{h,j}} G_h(x, y) \leqq C\left\{\phi_\delta(x) + h^N \sum_{l<j} \sum_{y \in P_{h,l}} G_h(x, y)|L_hd(y)|\right\}$$

and using the definition of ϕ_δ and that fact that $d \in \mathcal{C}^\infty(\overline{\mathcal{R}}\setminus\mathcal{R}_{2\delta_0})$ we have for $\frac{1}{2}\,\mathcal{K}jh < \delta_0$,

$$h^{N-1} \sum_{y \in P_{h,j}} G_h(x, y) \leqq C\left\{\delta + h^N \sum_{l<j} \sum_{y \in P_{h,l}} G_h(x, y)\right\} . \tag{3.10}$$

Since by (3.6) the quantity on the right is bounded independently of j, we get by summation over j and multiplication by h,

$$h^N \sum_{l<j} \sum_{y \in P_{h,l}} G_h(x, y) \leqq C\delta$$

which together with (3.10) proves the result.

4. Some Estimates for the Continuous Problem. We start by quoting some definitions and results on interpolation spaces. For generalities, see [10] and references.

Let B_j, $j = 0, 1$, be two Banach spaces with $B_1 \subseteq B_0$ so that for the corresponding norms,

$$\|u\|_{B_0} \leqq C\|u\|_{B_1} .$$

We set for $t > 0$,

$$K(t, u) = \inf_{v \in B_1} (\|u - v\|_{B_0} + t\|v\|_{B_1})$$

and denote for $0 < \theta < 1$ by $(B_0, B_1)_\theta$ the subspace of B_0 defined by

$$\|u\|_{(B_0, B_1)_\theta} = \sup_{t>0} t^{-\theta} K(t, u) < \infty .$$

We have $B_1 \subseteq (B_0, B_1)_\theta \subseteq B_0$, and for $B_1 = B_0$,

$$\|u\|_{(B_0, B_0)_\theta} = \|u\|_{B_0} .$$

We first state the following interpolation property:

Lemma 4.1. *Let B_j, B_j', $j = 0, 1$, for four Banach spaces with $B_1 \subseteq B_0$, $B_1' \subseteq B_0'$, and let A be a linear operator from B_0 into B_0' such that for $u \in B_1$, $Au \in B_1'$, and*

$$\|Au\|_{B_j'} \leqq C_j\|u\|_{B_j} , \qquad j = 0, 1 .$$

Then for $u \in (B_0, B_1)_\theta = B_\theta$ we have $Au \in (B_0', B_1')_\theta = B_\theta'$ and

$$\|Au\|_{B_\theta'} \leqq C_0^{1-\theta} C_1^{\theta} \|u\|_{B_\theta} , \qquad 0 < \theta < 1 .$$

In our applications, the Banach spaces will be of the type $\mathcal{C}(\bar{\mathcal{R}})$, $\mathcal{C}^\lambda(\bar{\mathcal{R}}_\delta)$, $\mathcal{C}^\mu(\partial\mathcal{R})$, etc. We shall need the following facts:

Lemma 4.2. *With the above notation we have for $0 \leqq p_0 \leqq p_1$ and with $p = p_0 + \theta(p_1 - p_0)$,*

$$(\mathcal{C}^{p_0}(\bar{\mathcal{R}}), \mathcal{C}^{p_1}(\bar{\mathcal{R}}))_\theta = \mathcal{C}_Z^{\,p}(\bar{\mathcal{R}})$$
$$(\mathcal{C}^{p_0}(\partial\mathcal{R}), \mathcal{C}^{p_1}(\partial\mathcal{R}))_\theta = \mathcal{C}_Z^{\,p}(\partial\mathcal{R})$$

where equality signifies equivalence of the respective norms.

We shall now collect some well-known inequalities for the Dirichlet problem (2.1), (2.2). For proofs, see e.g. Miranda [9]. We shall always assume that L and $\mathcal{R}$ satisfy the conditions in Section 2, in particular that $a_0 \geqq 0$ in $\mathcal{R}$. First we have the maximum principle estimate:

Lemma 4.3. *There is a constant C such that for $u \in \mathcal{C}^2(\mathcal{R}) \cap \mathcal{C}(\bar{\mathcal{R}})$ we have*

$$|u|_{\mathcal{R}} \leqq C\{|Lu|_{\mathcal{R}} + |\tilde{u}|_{\partial\mathcal{R}}\} .$$

The following two lemmas contain the interior and up-to-the-boundary Schauder estimates.

LEMMA 4.4. *If* λ *is a positive noninteger, there is a constant* C *such that for* $u \in \mathcal{C}^{2+\lambda}(\mathcal{R}) \cap \mathcal{C}(\overline{\mathcal{R}})$

$$|u|_{2+\lambda,\mathcal{R}_\delta} \leqq C\delta^{-2-\lambda}\{|Lu|_{\lambda,\mathcal{R}} + |u|_{\mathcal{R}}\} .$$

LEMMA 4.5. *If* λ *is a positive noninteger, there is a constant* C *such that for* $u \in \mathcal{C}^{2+\lambda}(\overline{\mathcal{R}})$,

$$|u|_{2+\lambda,\mathcal{R}} \leqq C\{|Lu|_{\lambda,\mathcal{R}} + |\tilde{u}|_{2+\lambda,\partial\mathcal{R}}\} .$$

For a general uniformly elliptic operator there would have been a term $|u|_{\mathcal{R}}$ on the right in this inequality, but here this term can be estimated by Lemma 4.3.

Lemmas 4.3 through 4.5 can be used to prove the following existence and uniqueness result:

LEMMA 4.6. *Let* λ *be a positive noninteger. Then if* $F \in \mathcal{C}^{\lambda}(\overline{\mathcal{R}})$, $f \in \mathcal{C}(\partial\mathcal{R})$ *the Dirichlet problem* (2.1), (2.2) *has a unique solution* $u \in \mathcal{C}^{2+\lambda}(\mathcal{R}) \cap \mathcal{C}(\overline{\mathcal{R}})$. *If in addition* $f \in \mathcal{C}^{2+\lambda}(\partial\mathcal{R})$ *we have* $u \in \mathcal{C}^{2+\lambda}(\overline{\mathcal{R}})$.

Using the above interpolation lemma we shall now derive some auxiliary inequalities for the Dirichlet problem

$$Lu = 0 \quad \text{in } \mathcal{R} , \tag{4.1}$$

$$u = f \quad \text{on } \partial\mathcal{R} . \tag{4.2}$$

LEMMA 4.7. *Let* λ *be a positive noninteger and* $0 \leqq \mu \leqq 2 + \lambda$. *Then there is a constant* C *such that for* $f \in \mathcal{C}_Z{}^{\mu}(\partial\mathcal{R})$ *the solution* u *of* (4.1), (4.2) *satisfies*

$$|u|_{2+\lambda,\mathcal{R}_\delta} \leqq C\delta^{\mu-2-\lambda}|f|_{Z,\mu,\partial\mathcal{R}} .$$

Proof. We have by Lemmas 4.3 and 4.4

$$|u|_{2+\lambda,\mathcal{R}_\delta} \leqq C\delta^{-2-\lambda}|f|_{\partial\mathcal{R}}$$

and by Lemma 4.5,

$$|u|_{2+\lambda,\mathcal{R}_\delta} \leqq |u|_{2+\lambda,\mathcal{R}} \leqq C|f|_{2+\lambda,\partial\mathcal{R}} .$$

The result therefore follows by applying Lemmas 4.1 and 4.2 to the operator which takes f into the solution $u \in \mathcal{C}^{2+\lambda}(\overline{\mathcal{R}}_\delta)$ of (4.1), (4.2).

LEMMA 4.8. *If* μ *is positive there is a constant* C *such that for* $f \in \mathcal{C}_Z{}^{\mu}(\partial\mathcal{R})$ *the solution* u *of* (4.1), (4.2) *belongs to* $\mathcal{C}_Z{}^{\mu}(\overline{\mathcal{R}})$ *and satisfies*

$$|u|_{Z,\mu,\mathcal{R}} \leqq C|f|_{Z,\mu,\partial\mathcal{R}} .$$

Proof. For $2 + \lambda > \mu$ and nonintegral we have again

$$|u|_{2+\lambda,\mathcal{R}} \leqq C|f|_{2+\lambda,\partial\mathcal{R}}$$

and by Lemma 4.3

$$|u|_{\mathcal{R}} \leqq C|f|_{\partial\mathcal{R}} .$$

The result therefore follows by Lemmas 4.1 and 4.2.

5. The Rate of Convergence. In this section we shall establish the unique solvability of the discrete Dirichlet problem (2.5), (2.6) and discuss the rate of convergence of its solution u_h to the solution u of the continuous problem (2.1), (2.2). More precisely we shall prove a sequence of lemmas leading up to the proof of our Theorem as stated in Section 2. Throughout this section we shall assume that the operators L, L_h, l_h, M_h, and m_h satisfy the assumptions of Section 2.

We first have the following two simple estimates:

LEMMA 5.1. *For any mesh-function* $u \in \bar{\mathfrak{D}}_h$ *we have*

$$|u|_{\partial R_h} \leqq \gamma |u|_{\bar{R}_h} + |l_h u|_{\partial R_h}$$

where $\gamma < 1$ *is the constant in* (2.4).

Proof. This is an immediate consequence of the definition of the operator l_h.

LEMMA 5.2. *There are positive constants* h_0 *and* C *such that for* $h \leqq h_0$ *and* $u \in \mathfrak{D}_h$ *we have*

$$|u|_{\bar{R}_h} \leqq C\{|L_h u|_{R_h} + |l_h u|_{\partial R_h}\} . \tag{5.1}$$

Proof. We have by Lemmas 3.4 and 3.5

$$\begin{aligned} |u(x)| &\leqq h^N \sum_{y \in R_h} G_h(x, y)|L_h u(y)| + \sum_{y \in \partial R_h} G_h(x, y)|u(y)| \\ &\leqq C|L_h u|_{R_h} + |u|_{\partial R_h} , \end{aligned} \tag{5.2}$$

and the result therefore follows from Lemma 5.1.

As a consequence we can now prove the existence of a solution of the discrete problem.

LEMMA 5.3. *With the* h_0 *in Lemma* 5.2, *the discrete problem* (2.5), (2.6) *has a unique solution* u_h *for* $h \leqq h_0$ *and arbitrary choice of* F *and* f.

Proof. Uniqueness is an immediate consequence of Lemma 5.2 and as in Lemma 3.3, uniqueness implies existence.

We can now essentially prove the convergence result in the case of homogeneous boundary conditions:

LEMMA 5.4. *Assume that the discrete problem* (2.5), (2.6) *approximates the continuous problem* (2.1), (2.2) *with order of accuracy* ν *and let* $\lambda \geqq 0$, $\lambda \neq \nu$. *Then there is a constant* C *such that if* $F \in \mathcal{C}_Z^{\lambda}(\bar{\mathfrak{R}})$ *and if* u *and* u_h *are the solutions of* (2.1), (2.2) *and* (2.5), (2.6), *respectively, with* $f = 0$, *then*

$$|u - u_h|_{R_h} \leqq Ch^{\min(\lambda,\nu)}|F|_{Z,\lambda,\mathfrak{R}} . \tag{5.3}$$

Proof. Since M_h and m_h are bounded we obtain by Lemmas 4.3 and 5.2

$$|u - u_h|_{R_h} \leqq |u|_{\mathfrak{R}} + |u_h|_{R_h} \leqq C|F|_{\mathfrak{R}}$$

which is (5.3) in the case $\lambda = 0$. For $\lambda > 0$, it is clearly, by Lemmas 4.1 and 4.2, no restriction of the generality to assume that λ is a noninteger. We then have $u \in \mathcal{C}^{2+\lambda}(\bar{\mathfrak{R}})$ by Lemma 4.6. We want to apply Lemma 5.2 to $u - u_h$. We have by (2.7) and Lemma 4.5,

$$\begin{aligned} |L_h(u - u_h)|_{R_h} &= |L_h u - M_h L u|_{R_h} \\ &\leqq Ch^{\min(\lambda,\nu)}|u|_{2+\lambda,\mathfrak{R}} \leqq Ch^{\min(\lambda,\nu)}|F|_{\lambda,\mathfrak{R}} \end{aligned} \tag{5.4}$$

and similarly by (2.8),

$$\begin{aligned} |l_h(u - u_h)|_{\partial R_h} &= |l_h u - m_h(Lu, 0)|_{\partial \mathfrak{R}_h} \\ &\leqq Ch^{\min(\lambda,\nu)}(|u|_{\lambda,\mathfrak{R}} + |Lu|_{\mathfrak{R}}) \leqq Ch^{\min(\lambda,\nu)}|F|_{\lambda,\mathfrak{R}} . \end{aligned} \tag{5.5}$$

Together (5.1), (5.4) and (5.5) prove the lemma.

For the treatment of the homogeneous equation we need an a priori inequality which is somewhat stronger than (5.1). For this purpose we introduce the norm

$$\|u\|_{R_h} = h^2 \sum_{\mathcal{K} jh \leqq 2\delta_0} j|u|_{P_{h,j}} + |u|_{R_{\delta_0,h}}$$

where δ_0 is the positive number introduced in Section 3 and $R_{\delta_0,h} = \mathfrak{R}_{\delta_0} \cap R_h$. We clearly have for some C independent of h,

$$\|u\|_{R_h} \leqq C|u|_{R_h}$$

but the new norm gives less weight to the values of u near ∂R_h; it can be thought of as a discrete analogue of

$$\|u\|_{\mathfrak{R}} = \int_0^{\delta_0} \delta|u|\,\partial\mathfrak{R}_\delta\,\partial\delta + |u|_{\mathfrak{R}_{\delta_0}}.$$

With this norm, we then have

LEMMA 5.5. *There are positive constants h_0 and C such that for $h \leqq h_0$ and $u \in \bar{\mathfrak{D}}_h$ we have*

$$|u|_{\bar{R}_h} \leqq C\{\|L_h u\|_{R_h} + |l_h u|_{\partial R_h}\}.$$

Proof. We have by Lemmas 3.5 and 3.6,

$$h^N \sum_{y \in R_h} G_h(x, y)|L_h u(y)| \leqq h \sum_{\mathcal{K} jh \leqq 2\delta_0} |L_h u|_{P_{h,j}} h^{N-1} \sum_{y \in P_{h,j}} G_h(x, y) + |L_h u|_{R_{\delta_0,h}} h^N \sum_{y \in R_h} G_h(x, y) \leqq C\|L_h u\|_{R_h}.$$

The result therefore follows as above from (5.2).

We can now prove the following convergence result for the homogeneous equation:

LEMMA 5.6. *Assume that the discrete problem* (2.5), (2.6) *approximates the continuous problem* (2.1), (2.2) *with order of accuracy ν and let $\mu \geqq 0$, $\mu \neq \nu$. Then there is a constant C such that if $f \in \mathcal{C}_Z{}^\mu(\partial\mathfrak{R})$ and if u and u_h are the solutions of* (2.1), (2.2) *and* (2.5), (2.6) *respectively, with $F = 0$, then*

$$|u - u_h|_{R_h} \leqq Ch^{\min(\mu,\nu)}|f|_{Z,\mu,\partial\mathfrak{R}}. \tag{5.6}$$

Proof. As in the proof of Lemma 5.4, we first notice that by Lemmas 4.3 and 5.2, (5.6) holds for $\mu = 0$, and that we can then assume without loss of generality that μ is a noninteger. By Lemma 4.8 we have $u \in \mathcal{C}^\infty(\mathfrak{R}) \cap \mathcal{C}^\mu(\bar{\mathfrak{R}})$. We want to apply Lemma 5.5 to $u - u_h$. We have

$$\begin{aligned} \|L_h(u - u_h)\|_{R_h} &= \|L_h u\|_{R_h} \\ &\leqq C\left\{h^2 \sum_{y,\, 4Bh \leqq \mathcal{K} jh \leqq 2\delta_0} j|L_h u|_{P_{h,j}} + h^2|L_h u|_{R_h} + |L_h u|_{R_{\delta_0,h}}\right\}. \end{aligned} \tag{5.7}$$

Consider the second term. By the definition of L_h and Lemma 4.3 it follows that

$$h^2|L_h u|_{R_h} \leqq C|f|_{\partial\mathfrak{R}}.$$

Also for any positive noninteger $\bar{\nu}$ such that $\nu < \bar{\nu} + 2 < \nu + 2$ we have from (2.7) and Lemma 4.8 that

$$h^2|L_h u|_{R_h} \leqq Ch^{2+\bar{\nu}}|f|_{2+\bar{\nu},\partial\mathfrak{R}} .$$

We may now apply Lemma 4.1 to the operator which takes f into $h^2L_hu \in \mathfrak{D}_h$ (with maximum norm) to obtain

$$h^2|L_h u|_{R_h} \leqq Ch^{\mu}|f|_{Z,\mu,\partial\mathfrak{R}} , \qquad 0 \leqq \mu \leqq 2 + \bar{\nu} .$$

Clearly this implies that

$$h^2|L_h u|_{R_h} \leqq Ch^{\min(\mu,\nu)}|f|_{\mu,\partial\mathfrak{R}} \tag{5.8}$$

for all $\mu \geqq 0$.

The last term can be estimated by applying (2.7) and the proof of Lemma 4.7:

$$|L_h u|_{R\delta_0,h} \leqq Ch^{\nu}|u|_{2+\nu,\mathfrak{R}\delta_0/2} \leqq Ch^{\nu}|f|_{\partial\mathfrak{R}} \leqq Ch^{\min(\mu,\nu)}|f|_{\mu,\partial\mathfrak{R}} . \tag{5.9}$$

Consider now the sum on the right in (5.7). For $y \in P_{h,j}$ with $4Bh \leqq \mathcal{K}jh$ we have

$$\mathfrak{N}_y \subseteq \overline{\mathfrak{R}}_{d(y)-Bh} \subseteq \overline{\mathfrak{R}}_{(\mathcal{K}jh-Bh)/2} \subseteq \overline{\mathfrak{R}}_{\mathcal{K}jh/4}$$

and thus by (2.7) and Lemma 4.7

$$|L_h u|_{P_{h,j}} \leqq Ch^{\min(\lambda,\nu)}|u|_{2+\lambda,\mathfrak{R}_{\mathcal{K}jh/4}} \leqq Ch^{\min(\lambda,\nu)}(jh)^{\mu-2-\lambda}|f|_{\mu,\partial\mathfrak{R}} .$$

We obtain

$$h^2 \sum_{4Bh\leqq\mathcal{K}jh\leqq2\delta_0} j|L_h u|_{P_{h,j}} \leqq Ch^{\min(\lambda,\nu)} \sum_{4Bh\leqq\mathcal{K}jh\leqq2\delta_0} (jh)^{-1+\mu-\lambda}h|f|_{\mu,\partial\mathfrak{R}} . \tag{5.10}$$

Since

$$\begin{aligned}\sum_{4Bh\leqq\mathcal{K}jh\leqq2\delta_0} (jh)^{-1+\mu-\lambda}h &\leqq C \quad \text{if } \mu > \lambda ,\\ &\leqq Ch^{\mu-\lambda} \quad \text{if } \mu < \lambda\end{aligned}$$

we can now choose λ between μ and ν and obtain by (5.10)

$$h^2 \sum_{4Bh\leqq\mathcal{K}jh\leqq2\delta_0} j|L_h u|_{P_{h,j}} \leqq Ch^{\min(\mu,\nu)}|f|_{\mu,\delta\mathfrak{R}} . \tag{5.11}$$

Together (5.7), (5.8), (5.9) and (5.11) prove

$$\|L_h(u - u_h)\|_{R_h} \leqq Ch^{\min(\mu,\nu)}|f|_{\mu,\partial\mathfrak{R}} . \tag{5.12}$$

Finally

$$\begin{aligned}|l_h(u - u_h)|_{\partial R_h} = |l_h u - m_h(0, \tilde{u})|_{\partial R_h} &\leqq Ch^{\min(\mu,\nu)}|u|_{\mu,\mathfrak{R}}\\ &\leqq Ch^{\min(\mu,\nu)}\ |f|_{\mu,\partial\mathfrak{R}}\end{aligned} \tag{5.13}$$

and by Lemma 5.5, (5.12) and (5.13) prove the lemma.

We can now complete the proof of the theorem. Let first $\lambda, \mu \geqq 0, \lambda, \mu \neq \nu$, and let $u_h^{(j)}$ and $u^{(j)}, j = 1, 2$, be the solutions of the discrete and continuous problems corresponding to $\mathfrak{F} = (F, 0)$ and $\mathfrak{F}' = (0, f)$, respectively. We then obviously have $u = u^{(1)} + u^{(2)}$ and $u_h = u_h^{(1)} + u_h^{(2)}$ and by Lemmas 5.4 and 5.6 we therefore get

$$\begin{aligned}|u - u_h|_{\bar{R}_h} &\leqq |u^{(1)} - u_h^{(1)}|_{\bar{R}_h} + |u^{(2)} - u_h^{(2)}|_{\bar{R}_h}\\ &\leqq C\{h^{\min(\lambda,\nu)}|F|_{Z,\lambda,\mathfrak{R}} + h^{\min(\mu,\nu)}|f|_{Z,\mu,\partial\mathfrak{R}}\}\end{aligned} \tag{5.14}$$

which is (2.9).

Assume now that $F \in \mathcal{C}^\lambda(\bar{\mathcal{R}})$ and $f \in \mathcal{C}(\partial\mathcal{R})$. Given $\epsilon > 0$ we can find $\tilde{f} \in \mathcal{C}^\lambda(\partial\mathcal{R})$ such that $|f - \tilde{f}|_{\partial\mathcal{R}} < \epsilon$. Let $\tilde{\mathfrak{F}} = (F, \tilde{f})$ and let $\tilde{u}_h$ and $\tilde{u}$ be the solutions of the corresponding discrete and continuous problems, respectively. We then have

$$(5.15) \qquad |u - u_h|_{\bar{R}_h} \leqq |u - \tilde{u}|_{\mathcal{R}} + |u_h - \tilde{u}_h|_{\bar{R}_h} + |\tilde{u} - \tilde{u}_h|_{\bar{R}_h}\,.$$

By Lemmas 4.3 and 5.2, and since M_h and m_h are bounded and linear we have

$$(5.16) \qquad |u - \tilde{u}|_{\mathcal{R}} + |u_h - \tilde{u}_h|_{\bar{R}_h} \leqq C|f - \tilde{f}|_{\partial\mathcal{R}} \leqq C\epsilon\,.$$

Since $\tilde{\mathfrak{F}} \in \mathcal{C}^\lambda(\bar{\mathcal{R}}) \times \mathcal{C}^\lambda(\partial\mathcal{R})$, we have

$$\lim_{h\to 0} |\tilde{u} - \tilde{u}_h|_{\bar{R}_h} = 0$$

by (5.14), and the result therefore follows from (5.15) and (5.16).

Remark. The interpolation technique can also be used to simplify the definition (2.7), (2.8) of the order of accuracy. Assume e.g. that (2.7) holds with $\lambda = \nu$ and that in addition the operator M_h has the property that for some C,

$$|M_h F(x)| \leqq C|F|_{\mathfrak{N}_x}\,, \qquad x \in \mathcal{R}\,.$$

By consistency we find

$$|L_h u(x)| \leqq C|u|_{2,\mathfrak{N}_x}$$

and therefore, (2.7) holds also with $\lambda = 0$. Hence, (2.7) holds for general λ with $0 \leqq \lambda \leqq \nu$ by Lemmas 4.1 and 4.2 (with $\mathfrak{N}_x$ instead of $\mathcal{R}$).

Cornell University
Mathematics Department
Ithaca, N. Y. 14850

University of Maryland
Institute for Fluid Dynamics and Applied Mathematics
College Park, Maryland 20740

Chalmers Institute of Technology
Göteborg, Sweden

1. N. S. Bahvalov, "Numerical solution of the Dirichlet problem for Laplace's equation," *Vestnik Moskov. Univ. Ser. Mat. Meh. Astronom. Fiz. Him.*, v. 3, 1959, no. 5, pp. 171–195. (Russian) MR **22** #6082.

2. J. H. Bramble, "On the convergence of difference approximations for second order uniformly elliptic operators" in *Numerical Solution of Field Problems in Continuum Physics*, SIAM—AMS Proceedings, vol. 2, Amer. Math. Soc., Providence, R. I. (To appear.)

3. J. H. Bramble & B. E. Hubbard, "On the formulation of finite difference analogues of the Dirichlet problem for Poisson's equation," *Numer. Math.*, v. 4, 1962, pp. 313–327. MR **26** #7157.

4. J. H. Bramble & B. E. Hubbard, "A theorem on error estimation for finite difference analogues of the Dirichlet problem for elliptic equations," *Contributions to Differential Equations*, v. 2, 1963, pp. 319–340. MR **27** #2114.

5. J. H. Bramble, R. B. Kellogg & V. Thomée, "On the rate of convergence of some difference schemes for second order elliptic equations," *Nordisk Tidskr. Informations-Behandling*, v. 8, 1968, pp. 154–173.

6. L. Collatz, "Bemerkungen zur Fehlerabschätzung für das Differenzenverfahren bei partiellen Differentialgleichungen," *Z. Angew. Math. Mech.*, v. 13, 1933, pp. 56–57.

7. S. Gerschgorin, "Fehlerabschätzung für das Differenzenverfahren zur Lösung partieller Differentialgleichungen," *Z. Angew. Math. Mech.*, v. 10, 1930, pp. 373–382.

8. P. Laasonen, "On the solution of Poisson's difference equation," *J. Assoc. Comput. Mach.*, v. 5, 1958, pp. 370–382. MR **22** #12726.

9. C. MIRANDA, *Equazioni alle Derivate Parziali di Tipo Ellittico*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Heft 2, Springer-Verlag, Berlin, 1955.

10. J. PEETRE & V. THOMÉE, "On the rate of convergence for discrete initial-value problems," *Math. Scand.*, v. 21, 1967, pp. 159–176.

11. E. A. VOLKOV, "Obtaining an error estimate for a numerical solution of the Dirichlet problem in terms of known quantities," *Ž. Vyčisl. Mat. i Mat. Fiz.*, v. 6, 1966, no. 4, suppl., pp. 5–17. (Russian) MR **35** #2503.

12. E. A. VOLKOV, "Effective estimates of the error in solutions by the method of nets of boundary for the Laplace and Poisson equations on a rectangle and on certain triangles," *Trudy Mat. Inst. Steklov.*, v. 74, 1966, pp. 55–85 = *Proc. Steklov Inst. Math.*, v. 74, 1966, pp. 57–90. MR **37** #1098.

13. W. WASOW, "On the truncation error in the solution of Laplace's equation by finite differences," *J. Res. Nat. Bur. Standards*, v. 48, 1952, pp. 345–348. MR **14**, 93.