



Analysis of a Cartesian PML approximation to acoustic scattering problems in \mathbb{R}^2 and \mathbb{R}^3



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ABSTRACT

We consider the application of a perfectly matched layer (PML) technique applied in Cartesian geometry to approximate solutions of the acoustic scattering problem in the frequency domain. The PML is viewed as a complex coordinate shift ("stretching") and leads to a variable complex coefficient equation for the acoustic wave posed on an infinite domain, the complement of the bounded scatterer. The use of Cartesian geometry leads to a PML operator with simple coefficients, although, still complex symmetric (non-Hermitian). The PML reformulation results in a problem whose solution coincides with the original solution inside the PML layer while decaying exponentially outside. The rapid decay of the PML solution suggests truncation to a bounded domain with a convenient outer boundary condition and subsequent finite element approximation (for the truncated problem).

This paper provides new stability estimates for the Cartesian PML approximations both on the infinite and the truncated domain. We first investigate the stability of the infinite PML approximation as a function of the PML strength σ_0 . This is done for PML methods which involve continuous piecewise smooth stretching as well as piecewise constant stretching functions. We next introduce a truncation parameter M which determines the size of the PML layer. Our analysis shows that the truncated PML problem is stable provided that the product of $M\sigma_0$ is sufficiently large, in which case the solution of the problem on the truncated domain converges exponentially to that of the original problem in the domain of interest near the scatterer. This justifies the simple computational strategy of selecting a fixed PML layer and increasing σ_0 to obtain the desired accuracy. The results of numerical experiments varying M and σ_0 are given which illustrate the theoretically predicted behavior.

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1. Introduction

In this paper, we consider the application of PML techniques for approximating the solutions of frequency domain acoustic scattering problems. These problems are posed on unbounded domains with a far field boundary condition given by the Sommerfeld radiation condition. The PML technique which we shall study is the one based on Cartesian geometry where each variable is transformed independently.

We consider the exterior Helmholtz scattering problem:

$$\begin{aligned} \Delta u + k^2 u &= 0, \quad \text{in } \Omega^c, \\ u &= g, \quad \text{on } \Gamma = \partial\Omega^c, \\ \lim_{r \rightarrow \infty} r^{(n-1)/2} \left(\frac{\partial u}{\partial r} - iku \right) &= 0. \end{aligned} \tag{1.1}$$

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Here Ω is a bounded domain in \mathbb{R}^n for $n = 2, 3$ and Ω^c denotes the complement of its closure. The limit is taken to mean uniformly on the sphere of radius r as r tends to infinity.

An early paper of Bérenger [1] introduced a PML method for Maxwell's equations in the time domain. This approach was based on constructing a fictitious absorbing layer designed so that plane waves passed into the layer without reflection. The technique involved the introduction of additional variables and equations in the “fictitious material” region. For more analysis on PML applied to time domain problems see [2–5] and the included references. PML type techniques were also developed in terms of a complex change of variable or stretching [6,7]. This approach was especially well suited for frequency domain problems and led to simpler PML formulations more amenable to analysis. Perhaps the simplest and most widely used of the PML variants is that based on Cartesian stretching.

By scaling, we may assume, without loss of generality, that $\Omega \subset (-1, 1)^n$. Cartesian PML can be thought of as a formal complex change of variables. In particular, we consider a change of variables of the form

$$\tilde{u}(x) = u(\tilde{x}), \quad x \in \mathbb{R}^n, \quad n = 2, 3 \quad (1.2)$$

where the variable change \tilde{x} comes from an even function $\sigma(x)$ defined on \mathbb{R} satisfying $\sigma(t) = 0$ for $|t| \leq 1$. We consider two cases:

(Type 1) For $v_1 > 1$, $\sigma(t)$ is piecewise linear and continuous with respect the nodes $\{-v_1, -1, 1, v_1\}$ with $\sigma(t) = \sigma_0$ for $t \geq v_1$.

(Type 2) In this case, $v_1 = 1$ and σ is piecewise constant with $\sigma(t) = \sigma_0$ for $|t| > 1$.

The positive constant σ_0 is bounded away from zero ($\sigma_0 \geq \underline{\sigma}_0 > 0$) and defines the PML strength.

We then set $d(t) = 1 + z\sigma(t)$ and

$$\tilde{x}(t) = \int_0^t d(s)ds.$$

Here z is a complex parameter. The change of variables is given by

$$\tilde{x}(x) = (\tilde{x}(x_1), \dots, \tilde{x}(x_n)), \quad x \in \mathbb{R}^n.$$

For real z with $z > -1/\sigma_0$, the above transformation is a C^1 or C^0 diffeomorphism of \mathbb{R}^n onto \mathbb{R}^n and Ω^c onto Ω^c . Our PML will correspond to complex z with values either i or $1 + i$. In this case, the change of variables is formal since $u(\tilde{x})$ is not well defined and (1.2) is only meant to be heuristic.

In [8], it was essentially shown that the Cartesian PML reformulation on \mathbb{R}^n for a fixed value of σ_0 (σ of Type 1) was stable on \mathbb{R}^n . This fact, combined with the exponential decay of solutions, gives rise to stability of the truncated PML approximation provided that the size of the computational domain is sufficiently large.

The first goal of the current paper is to investigate stability of the PML formulation on \mathbb{R}^n as a function of the parameter σ_0 . We do this by first providing bounds on a solution operator given in terms of a fundamental solution and develop stability estimates as a map $L^2(\mathbb{R}^n) \rightarrow H^1(\mathbb{R}^n)$. This is a fundamental step in showing variational stability, i.e., stability as a map of $H^{-1}(\mathbb{R}^n) \rightarrow H^1(\mathbb{R}^n)$. In both cases, we provide bounds for the stability constant as a function of σ_0 . We note that the solution of the original scattering problem (1.1) and the PML formulation on the infinite domain coincide on the region of interest $\{x \in \mathbb{R}^n, \|x\|_{L^\infty} \leq 1\}$.

The next goal of the paper is to investigate the stability properties of the truncated problem. We study the interplay of the domain size M and σ_0 . In particular, we show that stability of the truncated problem is attained if the product $M\sigma_0$ is sufficiently large. In this case, we show that the error on the region of interest between the truncated problem and the full PML problem decays like $Ce^{-\alpha M\sigma_0}$ for positive constants α and C which are independent of both M and σ_0 . In practice, one has to tune the computational domain and σ_0 to obtain the desired accuracy. Our results show that it is not necessary to change the computational domain, i.e., build a new mesh, but rather it is sufficient to simply adjust the parameter σ_0 .

We note that the stability analysis of a PML reformulation depends in a significant way on the characteristics of the PML stretching. Stability estimates for spherical and polar PML were given in [9–11] for scattering problems in acoustics, electromagnetics and elasticity. Chen and Zheng [12] proved stability estimates for Cartesian PML in the PML region for the piecewise constant case. Their analysis was limited to problems on \mathbb{R}^2 . Their results are most closely related to those of this paper except that we derive stability estimates over the full computational domain. Such a stability result seems necessary for the analysis of the resulting finite element approximation.

The outline of the remainder of the paper is as follows. Section 2 provides some notation and definitions of the PML bilinear forms. A key ingredient in our analysis is that the PML problem, shifted by a multiple of the identity, is stable. This result is also given in Section 2. The analysis of the PML problem on \mathbb{R}^n involves a fundamental solution which is defined in terms of a “complexified distance”. Estimates involving this function are given in Section 3. Estimates on integral operators resulting from the fundamental solution are given in Section 4. A solution operator for the PML equations defined in terms of the integral operators is given in Section 5. Section 6 develops stability estimates for the sequence of domains, \mathbb{R}^n , Ω^c , and the computational domain Ω_M . Finally, Appendix reports the results of numerical calculations which illustrate the convergence behavior of the PML approximations.

2. Preliminaries and notation

We denote $\Omega_M = (-M, M)^n$ and Γ_M to be its boundary. Let Ω'_M denote $\Omega_M \setminus \bar{\Omega}$. Our minimal computational domain will be Ω'_{M_0} with a fixed constant $M_0 > v_1$. Our general computational domain will be Ω'_M where M will always be greater than or equal to M_0 .

Remark 2.1. We restrict to the simple form of σ for convenience. The results of this paper trivially extend to piecewise smooth increasing functions in C^1 or C^0 corresponding to Type 1 or Type 2.

Remark 2.2. For convenience, we only consider the case where the same stretching function is used in each direction. In an application where the domain more naturally fits into a rectangle or brick shaped domain, it is more reasonable (and computationally more efficient) to use direction dependent PML stretching functions. For example, for Type 2, $\sigma = \sigma_{(j)}$ jumps at $\pm v_j$ where $v_j, j = 1, \dots, n$ is chosen from the geometry of the application. The theory to be presented here extends to the more general situation with only minor changes.

The Cartesian PML technique can be thought of as a formal complex shift in the Cartesian coordinate system. We shall use the following notation when $n = 2$:

$$\begin{aligned} J(x) &\equiv d(x_1)d(x_2), \\ H(x) &\equiv \begin{bmatrix} d(x_2)/d(x_1) & 0 \\ 0 & d(x_1)/d(x_2) \end{bmatrix} \quad \text{for } x = (x_1, x_2) \in \mathbb{R}^2. \end{aligned} \quad (2.1)$$

We shall take $z = i$ or $z = 1 + i$ with σ of Type 1 and $z = 1 + i$ with σ of Type 2. Using $z = 1 + i$ as opposed to $z = i$ in the second case avoids a serious theoretical constraint (see, Remark 2.7 below).

In the three dimensional case, all definitions remain the same except

$$\begin{aligned} J(x) &\equiv d(x_1)d(x_2)d(x_3) \quad \text{and} \\ H(x) &\equiv \begin{pmatrix} d(x_2)d(x_3)/d(x_1) & 0 & 0 \\ 0 & d(x_1)d(x_3)/d(x_2) & 0 \\ 0 & 0 & d(x_1)d(x_2)/d(x_3) \end{pmatrix} \end{aligned}$$

for $x = (x_1, x_2, x_3) \in \mathbb{R}^3$.

The planes of constant $x_k = 1$ and $x_k = v_1$, for $k = 1, 2, \dots, n$ partition \mathbb{R}^n into regions $\mathbb{R}^n = \{\bar{S}_\ell\}$ with S_ℓ open. We note that all of the coefficients appearing above are smooth when restricted to Ω_ℓ for any ℓ .

Even though we seek solutions in complex Sobolev spaces, it will be more convenient to work with bilinear as opposed to sesquilinear forms. Accordingly, we define the “stretched” bilinear form

$$A(u, \phi) = \int_{\mathbb{R}^n} (H \nabla u) \cdot \nabla \phi \, dx \quad \text{for all } u, \phi \in H^1(\mathbb{R}^n).$$

We also denote

$$[u, v] = \int_{\mathbb{R}^n} Juv \, dx \quad \text{for all } u, v \in L^2(\mathbb{R}^n).$$

We shall still use this notation for functions in $H^1(D)$ for domains $D \subset \mathbb{R}^n$. In this case, the integration is over D .

The basic goal of this paper is to analyze the truncated domain PML approximation to (1.1). This approximation is defined to be the function $u_M \in H^1(\Omega'_M)$ satisfying $u_M = g$ on Γ , $u_M = 0$ on Γ_M and

$$A(u_M, \phi) - k^2[u_M, \phi] = 0, \quad \text{for all } \phi \in H_0^1(\Omega'_M). \quad (2.2)$$

Our analysis of this problem involves the study of two auxiliary PML problems involving \mathbb{R}^n and Ω^c . The first is the source problem on \mathbb{R}^n whose solution $u \in H^1(\mathbb{R}^n)$ satisfies

$$A(u, \phi) - k^2[u, \phi] = F(\phi), \quad \text{for all } \phi \in H^1(\mathbb{R}^n), \quad (2.3)$$

for a given linear functional $F \in (H^1(\mathbb{R}^n))^*$. The second is the source problem on Ω^c whose solution $u \in H_0^1(\Omega^c)$ satisfies

$$A(u, \phi) - k^2[u, \phi] = \langle F, \phi \rangle, \quad \text{for all } \phi \in H_0^1(\Omega^c), \quad (2.4)$$

for a given linear functional $F \in (H_0^1(\Omega^c))^*$.

We note that stability of (2.4) (which we prove in Section 5) implies the existence of a unique solution to the PML scattering problem on Ω^c , specifically, $\tilde{u} \in H^1(\Omega^c)$ satisfying $\tilde{u} = g$ on Γ and

$$A(\tilde{u}, \phi) - k^2[\tilde{u}, \phi] = 0 \quad \text{for all } \phi \in H_0^1(\Omega^c). \quad (2.5)$$

We shall see in the proof of [Theorem 5.8](#) that \tilde{u} , as defined by (2.5), and the solution u of (1.1) coincide on Ω'_1 . In addition, this theorem will show the exponential convergence of u_M to u , as a function of $(\sigma_0 M)$, on Ω'_1 .

To analyze the above problems, it is convenient to use the following equivalent norm on $H^1(\mathbb{R}^n)$:

$$\|w\|_H^2 = \int_{\mathbb{R}^n} \left[\sum_{j=1}^n |H_{jj}| \left| \frac{\partial w}{\partial x_j} \right|^2 + |J| |w|^2 \right] dx. \quad (2.6)$$

We also define $|\cdot|_H$ to be the weighted $H^1(\mathbb{R}^n)$ semi-norm and $\|\cdot\|_J$ to be the weighted $L^2(\mathbb{R}^n)$ norm appearing in (2.6). Similarly we use the weighted dual norm: for $F \in (H^1(\mathbb{R}^n))^*$ (the space of bounded linear functionals on $H^1(\mathbb{R}^n)$), given by

$$\|F\|_{H^*} = \sup_{\phi \in H^1(\mathbb{R}^n)} \frac{|F(\phi)|}{\|\phi\|_H}, \quad \text{for all } \phi \in H^1(\mathbb{R}^n).$$

These weighted norms will be also used on subdomains D of \mathbb{R}^n and we shall often use the same notation, especially when the function belongs to $H_0^1(D)$. We clearly have

$$|A(u, \phi) - k^2[u, \phi]| \leq \max\{1, k^2\} \|u\|_H \|\phi\|_H.$$

We conclude this section with stability lemmas corresponding to the two types of σ . Note that the constants appearing in these lemmas and many of the results of this paper have different asymptotic behavior in σ_0 depending on n and z and the type of σ . Subsequently, they shall also depend on the size of the computational domain M to be introduced later. These constants have at most polynomial growth in σ_0 and M . We shall denote such constants by a generic constant \tilde{C} (with or without subscript) where

$$\tilde{C} \leq Cp(M, \sigma_0).$$

The constant C and the polynomial p appearing above may vary with different occurrences of \tilde{C} .

Remark 2.3. Note that \tilde{C} , at each occurrence, is bounded by a polynomial in the single variable $M\sigma_0$ with coefficients which may depend on k, σ_0 and n but not σ_0 or M .

Also, here and in the remainder of this paper, C or c , with or without subscript, are generic positive constants which can depend on k and n but not σ_0 or M .

Lemma 2.4. Let σ be of Type 1 and $z = i$. Then for $\gamma = 1/2 + C_1^2 \sigma_0^2 (1 + \sigma_0^2)/2 = \tilde{C}_0$ (with C_1 appearing in (2.10) below),

$$\|u\|_H \leq \tilde{C}_1 \sup_{v \in H^1(\mathbb{R}^n)} \frac{|A(u, v) + \gamma[u, v]|}{\|v\|_H} \quad \text{for all } u \in H^1(\mathbb{R}^n). \quad (2.7)$$

Proof. Let u be in $H^1(\mathbb{R}^n)$ and set $v = e^{i\theta} \tilde{u}$ where $\theta = \arg((1 + i\sigma_0)J^{-1}) = \text{Im}(\ln((1 + i\sigma_0)J^{-1}))$. Here we take the branch cut for the logarithm to be the negative real axis and note that $-\pi < \arg(\theta) < \pi/2$. The assumptions on σ imply that v is in $H^1(\mathbb{R}^n)$. Moreover, except where σ' is not continuous,

$$\frac{\partial \theta}{\partial x_j} = -\text{Im} \left(i \frac{\sigma'(x_j)}{d(x_j)} \right) \quad (2.8)$$

and hence

$$|v|_H^2 \leq \int_{\mathbb{R}^n} \sum_{j=1}^n |H_{jj}| (\eta |u| + |u_{x_j}|)^2 dx$$

where

$$\eta = \|\sigma'/d\|_{L^\infty(\mathbb{R})} = \frac{\sigma_0}{v_1 - 1}.$$

Now $|H_{jj}| \leq |J|$ implies that

$$\|v\|_H \leq C\sigma_0 \|u\|_H. \quad (2.9)$$

We then have,

$$A(u, v) = \int_{\mathbb{R}^n} \left(\sum_{j=1}^n |H_{jj}| |u_{x_j}|^2 e^{i\theta_j} \right) dx + b(u, u)$$

where (by (2.8))

$$b(u, u) = - \sum_{j=1}^n \int_{\mathbb{R}^n} H_{jj} u_{x_j} \bar{u} e^{i\theta_j} \operatorname{Im} \left(\frac{i\sigma'(x_j)}{d(x_j)} \right) dx$$

and $\theta_j = \arg((1 + i\sigma_0)/d(x_j)^2)$. We clearly have

$$|b(u, u)| \leq C_1 \sigma_0 |u|_H \|u\|_J. \quad (2.10)$$

Moreover, it is easy to see that

$$\operatorname{Re}(e^{i\theta_j}) \geq (1 + \sigma_0^2)^{-1/2}$$

and hence

$$\operatorname{Re}(A(u, v)) \geq (1 + \sigma_0^2)^{-1/2} |u|_H^2 - C_1 \sigma_0 |u|_H \|u\|_J.$$

We also have

$$\operatorname{Re}([u, v]) = \operatorname{Re} \left(\int_{\mathbb{R}^n} |J| |u|^2 e^{i \arg(1 + i\sigma_0)} dx \right) = (1 + \sigma_0^2)^{-1/2} \|u\|_J^2.$$

Thus, for γ as above,

$$\begin{aligned} |A(u, v) + \gamma[u, v]| &\geq \operatorname{Re}(A(u, v) + \gamma[u, v]) \geq (1 + \sigma_0^2)^{-1/2} [|u|_H^2 + \gamma \|u\|_J^2] - C_1 \sigma_0 |u|_H \|u\|_J \\ &\geq ((1 + \sigma_0^2)^{-1/2} - C_1 \sigma_0 \beta / 2) |u|_H^2 + (\gamma (1 + \sigma_0^2)^{-1/2} - C_1 \sigma_0 / (2\beta)) \|u\|_J^2. \end{aligned}$$

The last inequality holds for any positive β . Taking $\beta = ((1 + \sigma_0^2)^{1/2} C_1 \sigma_0)^{-1}$ gives

$$|A(u, v) + \gamma[u, v]| \geq (2(1 + \sigma_0^2)^{1/2})^{-1} \|u\|_H^2.$$

The lemma immediately follows from this and (2.9). \square

Lemma 2.5. Let σ be of either type and $z = 1 + i$. Then for all $u \in H^1(\mathbb{R}^n)$,

$$\|u\|_H \leq \cos(n\pi/8) \sup_{v \in H^1(\mathbb{R}^n)} \frac{|A(u, v) + e^{-i\pi/4}[u, v]|}{\|v\|_H}. \quad (2.11)$$

Proof. We consider the case of $n = 3$. The first diagonal entry of H is $d(x_2)d(x_3)/d(x_1)$. Clearly, $0 \leq \arg(d(x_j)) < \pi/4$ and hence $-\pi/4 < \arg(H_{11}) < \pi/2$. This clearly holds for the other diagonal entries of H as well. Moreover, $0 \leq \arg(J) < 3\pi/4$. Thus,

$$\operatorname{Re}(e^{-i\pi/8} H_{kk}) \geq \cos(3\pi/8) |H_{kk}| \quad \text{and} \quad \operatorname{Re}(e^{-3i\pi/8} J) \geq \cos(3\pi/8) |J|.$$

Taking $v = e^{-i\pi/8} \bar{u}$ gives

$$|A(u, v) + e^{-i\pi/4}[u, v]| \geq \operatorname{Re}(A(u, v) + e^{-i\pi/4}[u, v]) \geq \cos(3\pi/8) \|u\|_H^2.$$

This completes the proof of the lemma when $n = 3$. The proof for $n = 2$ is similar except that we take $v = \bar{u}$ in that case. \square

Remark 2.6. Even though Lemmas 2.4 and 2.5 were stated for \mathbb{R}^n , their proofs are valid for any domain contained in \mathbb{R}^n .

Remark 2.7. The above proof works for the case of $z = i$ with $n = 2$ and σ of Type 2 but the inf-sup constant grows like σ_0 . When $n = 3$, $z = i$ and Type 2, we get the severe constraint $\arg(1 + i\sigma_0) < \pi/3$ or $\sigma_0 < \sqrt{3}$.

3. Bounds for PML integral operators

In this section, we provide a preliminary investigation of some integral operators involving a fundamental solution for the PML operators. These are defined in terms of a “complexified distance” \tilde{r} and the fundamental solution to the Helmholtz equation in \mathbb{R}^n .

The PML stretching takes $x = (x_1, \dots, x_k)$ to $\tilde{x} = (\tilde{x}(x_1), \dots, \tilde{x}(x_n))$. The complexified distance between \tilde{x} and $\tilde{y} = (\tilde{x}(y_1), \dots, \tilde{x}(y_n))$ is defined by

$$\tilde{r}(x, y) = \left(\sum_{j=1}^n (\tilde{x}(x_j) - \tilde{x}(y_j))^2 \right)^{1/2}.$$

We take the negative real axis as a branch cut for the square root. For $z = i$ and $z = 1 + i$, the argument appearing in the square root above stays away from the branch cut as we shall see below. Thus $\tilde{r}(x, y)$ is a well defined complex number.

The following two lemmas are critical to the development of this paper. Their proofs use elementary arguments involving the behavior of $\tilde{r}(x, y)$, for $x, y \in \mathbb{R}^n$ and are given in the [Appendix](#).

Lemma 3.1. *Let x, y be in \mathbb{R}^n with $x \neq y$. For $z = i$ or $z = 1 + i$, $0 \leq \arg(\tilde{r}^2(x, y)) < \pi$ and hence $0 \leq \arg(\tilde{r}(x, y)) < \pi/2$. Furthermore,*

$$\tilde{C}_2^{-1}r \leq |\tilde{r}| \leq \tilde{C}_3r.$$

Here $r = |x - y|$ and $\tilde{r} = \tilde{r}(x, y)$.

Lemma 3.2. *Let x, y be in \mathbb{R}^n . Suppose that either*

- (a) $\|x - y\|_\infty \geq 2\alpha v_1$, with $\alpha > 1$ or
- (b) $\|x\|_\infty \geq \alpha v$ and $\|y\|_\infty \leq v$, with $\alpha > 1$ and $v \geq v_1$.

Then there is a constant $c(\alpha, v_1)$ such that

$$\operatorname{Im}(\tilde{r}) \geq c(\alpha, v_1)\sigma_0r. \quad (3.1)$$

Here $r = |x - y|$ and $\tilde{r} = \tilde{r}(x, y)$. This holds for either $z = i$ or $z = 1 + i$.

For positive k and r , we consider the fundamental solution of the Helmholtz equation,

$$\begin{aligned} \Phi(r) &= \frac{i}{4}H_0^{(1)}(kr), \quad \text{for } n = 2, \\ \Phi(r) &= \frac{e^{ikr}}{4\pi r}, \quad \text{for } n = 3. \end{aligned}$$

Here $H_0^{(1)}(\cdot)$ denotes the Hankel function of type 1 and order 0. Our fundamental solution for the PML equations will involve replacing r by \tilde{r} above and hence we shall be interested in the behavior of Φ for complex arguments.

The function Φ satisfies the following asymptotic properties near 0:

$$\begin{aligned} |\Phi(z)| &\leq C|\ln|z||, \quad \text{for } n = 2, \\ |\Phi(z)| &\leq C|z|^{-1}, \quad \text{for } n = 3, \\ |\Phi'(z)| &\leq C|z|^{1-n}, \quad \text{for } n = 2, 3, \\ |\Phi''(z)| &\leq C|z|^{-n}, \quad \text{for } n = 2, 3. \end{aligned}$$

The results are obvious for $n = 3$ and, for $n = 2$, are a consequence of 9.1.12 and 9.1.13 of [13] and differentiation. In addition, there is a positive β_1 such that for $|z| \geq \beta_1$ with $|\arg(z)| \leq \pi - \varepsilon$ for a fixed $\varepsilon > 0$, we have

$$\begin{aligned} \Phi(z) &= \left(c_n^0 \frac{e^{iz}}{z^{(n-1)/2}} (1 + O(|z|^{-1})) \right), \\ \Phi'(z) &= \left(c_n^1 \frac{e^{iz}}{z^{(n-1)/2}} (1 + O(|z|^{-1})) \right), \\ \Phi''(z) &= \left(c_n^2 \frac{e^{iz}}{z^{(n-1)/2}} (1 + O(|z|^{-1})) \right). \end{aligned} \quad (3.2)$$

These estimates are trivial when $n = 3$ and are easily derived from (5.11.4) of [14] and 9.1.31 of [13] when $n = 2$.

The above estimates yield: for all z with $0 \leq \arg(z) < \pi - \varepsilon$,

$$\begin{aligned} |\Phi(z)| &\leq C(1 + |\ln|z||), \quad \text{for } n = 2, \\ |\Phi(z)| &\leq C(1 + |z|^{-1}), \quad \text{for } n = 3, \\ |\Phi'(z)| &\leq C(1 + |z|^{1-n}), \quad \text{for } n = 2, 3, \\ |\Phi''(z)| &\leq C(1 + |z|^{-n}), \quad \text{for } n = 2, 3. \end{aligned} \quad (3.3)$$

For $f \in C_0^\infty(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, we consider the integral operators:

$$\begin{aligned} I_0(f)(x) &= \int_{\mathbb{R}^n} Jf\Phi(\tilde{r}) \, dy \quad \text{and} \\ I_j(f)(x) &= -d(x_j) \int_{\mathbb{R}^n} Jd(y_j)^{-1} f \frac{\partial \Phi(\tilde{r})}{\partial y_j} \, dy, \quad \text{for } j = 1, \dots, n. \end{aligned}$$

Proposition 3.3. For $z = i$ or $z = 1 + i$, $I_0(f)$ defines a bounded operator from $L_2(\mathbb{R}^n)$ into $H^1(\mathbb{R}^n)$ with weak derivatives $D_j I_0(f) = I_j(f)$, $j = 1, \dots, n$. Moreover,

$$\|I_0(f)\|_H \leq \tilde{C}_4 \|f\|_J. \quad (3.4)$$

Proof. We first apply a simple technique due to Holmgren [15]. For $f, g \in L_2(\mathbb{R}^n)$ with $f, g \neq 0$,

$$\begin{aligned} \left| \int_{\mathbb{R}^n} U |I_0(f)g| dx \right| &\leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} U(x) J(y) f(y) g(x) \Phi(\tilde{r}) dy dx \\ &\leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (|U(x)|^{1/2} |J(y)|^{1/2} |\Phi(\tilde{r})|) \left(\frac{\beta}{2} |U(x)| |g(x)|^2 + \frac{1}{2\beta} |J(y)| |f(y)|^2 \right) dy dx. \end{aligned} \quad (3.5)$$

This inequality holds for any positive β . Now if

$$\int_{\mathbb{R}^n} |U(x)|^{1/2} |J(y)|^{1/2} |\Phi(\tilde{r})| dx \leq B \quad \text{for all } y \in \mathbb{R}^n, \quad (3.6)$$

then the double integral of (3.5) is bounded by

$$B \left(\frac{\beta}{2} \|g\|_J^2 + \frac{1}{2\beta} \|f\|_J^2 \right).$$

This shows that the integrand appearing in the double integral of (3.5) is in $L_1((\mathbb{R}^n)^2)$ and taking $\beta = \|f\|_J / \|g\|_J$ gives

$$\left| \int_{\mathbb{R}^n} U |I_0(f)g| dx \right| \leq B \|f\|_J \|g\|_J.$$

That I_0 is a bounded operator on $L_2(\mathbb{R}^n)$ will follow once we prove (3.6).

We clearly have

$$|U(x)|^{1/2} |J(y)|^{1/2} \leq C \sigma_0^n.$$

Let $\beta_2 = \max(4\sqrt{n}v_1, \beta_1 \tilde{C}_2)$ where \tilde{C}_2 appears in Lemma 3.1. Then,

$$\int_{\mathbb{R}^n} |\Phi(\tilde{r})| dx \leq \int_{|x-y| < \beta_2} |\Phi(\tilde{r})| dx + \int_{|x-y| > \beta_2} |\Phi(\tilde{r})| dx. \quad (3.7)$$

Now $|x-y| > \beta_2$ implies that $\|x-y\|_\infty > 4v_1$. Applying Lemmas 3.1, 3.2(a) with $\alpha = 2$ and (3.2) gives

$$\int_{|x-y| > \beta_2} |\Phi(\tilde{r})| dx \leq C \int_{|x-y| > \beta_2} e^{-c\sigma_0 r} dx \leq \tilde{C}.$$

Applying Lemma 3.1 and (3.3) gives

$$|\Phi(\tilde{r})| \leq \begin{cases} C(1 + \ln(\tilde{C}_2) + |\ln(r)|) : & \text{if } n = 2, \text{ and } |\tilde{r}| \leq 1, \\ C(1 + \ln(\tilde{C}_3) + |\ln(r)|) : & \text{if } n = 2, \text{ and } |\tilde{r}| \geq 1, \\ C(1 + \tilde{C}_2 r^{-1}) : & \text{if } n = 3. \end{cases} \quad (3.8)$$

Using this and the fact that $\beta_2 \leq \tilde{C}$ gives

$$\int_{|x-y| < \beta_2} |\Phi(\tilde{r})| dx \leq \tilde{C}.$$

Combining the above estimates gives the bound $B \leq \tilde{C}$ and shows that I_0 is bounded operator on $L_2(\mathbb{R}^n)$.

Note that Fubini's Theorem and the above analysis shows that the double integral of (3.5) can be evaluated in any order when $f, g \in L_2(\mathbb{R}^n)$. Similarly, for $\phi \in C_0^\infty(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} I_0(f) \phi_{x_j} dx = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} J(y) f(y) \phi_{x_j}(x) \Phi(\tilde{r}) dx dy.$$

We write the inner integral as a limit as $\epsilon \rightarrow 0$ of the same integral excluding the ball of radius ϵ centered at y (i.e., $B_\epsilon(y)$). Integrating by parts and evaluating the limit (using (3.3)) shows that

$$\begin{aligned} \int_{\mathbb{R}^n} I_0(f) \phi_{x_j} dx &= - \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} J(y) f(y) \phi(x) \frac{\partial \Phi(\tilde{r})}{\partial x_j} dx dy \\ &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} J(y) d(y_j)^{-1} f(y) d(x_j) \phi(x) \frac{\partial \Phi(\tilde{r})}{\partial y_j} dx dy \end{aligned} \quad (3.9)$$

where we used

$$\frac{\partial \Phi(\tilde{r})}{\partial x_j} = \Phi'(\tilde{r}) \frac{\tilde{x}_j - \tilde{y}_j}{\tilde{r}} d(x_j) = -\frac{d(x_j)}{d(y_j)} \frac{\partial \Phi(\tilde{r})}{\partial y_j} \quad (3.10)$$

for the second equality above. It follows from the Holmgren technique (using the derivative estimates in (3.2) and (3.3)) that the last integral of (3.9) can be evaluated in any order, i.e.,

$$\begin{aligned} \int_{\mathbb{R}^n} I_0(f) \phi_{x_j} dx &= \int_{\mathbb{R}^n} \left(d(x_j) \int_{\mathbb{R}^n} J(y) d(y_j)^{-1} f(y) \frac{\partial \Phi(\tilde{r})}{\partial y_j} dy \right) \phi(x) dx \\ &= - \int_{\mathbb{R}^n} I_j f(x) \phi(x) dx. \end{aligned}$$

Note that we have shown that the weak derivative of $I_0(f)$ in the direction x_j is given by $I_j(f)$.

We finally develop the estimate for $I_j(f)$ in (3.4) using again the Holmgren technique. Note that

$$\tilde{x}(x_j) - \tilde{x}(y_j) = \int_{y_j}^{x_j} (1 + z\sigma(t)) dt = (x_j - y_j)(1 + z\sigma_j), \quad (3.11)$$

where σ_j denotes the average of σ on the interval with endpoints y_j and x_j and is clearly in $[0, \sigma_0]$. We start by applying (3.10), Lemma 3.1 and (3.11) and obtain

$$\left| \frac{\partial \Phi(\tilde{r})}{\partial y_j} \right| = |\Phi'(\tilde{r})| \frac{|\tilde{x}_j - \tilde{y}_j|}{|\tilde{r}|} |d(y_j)| \leq \tilde{C} |\Phi'(\tilde{r})| |d(y_j)|.$$

As in (3.5),

$$\begin{aligned} \left| \int_{\mathbb{R}^n} |H_{jj}| I_j(f) g dx \right| &\leq \tilde{C} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |H_{jj}(x)| |d(x_j)| |J(y)| |f(y)| |g(x)| |\Phi'(\tilde{r})| dy dx \\ &\leq \tilde{C} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left(|J(x)|^{1/2} |J(y)|^{1/2} |\Phi'(\tilde{r})| \right) \left(\frac{\beta}{2} |H_{jj}(x)| |g(x)|^2 + \frac{1}{2\beta} |J(y)| |f(y)|^2 \right) dy dx. \end{aligned} \quad (3.12)$$

We are left to bound the integral

$$\int_{\mathbb{R}^n} |J(x)|^{1/2} |J(y)|^{1/2} |\Phi'(\tilde{r})| dx \leq B_1 \quad \text{for all } y \in \mathbb{R}^n. \quad (3.13)$$

The argument is completely analogous to that used to bound B above except that (3.8) is replaced by

$$|\Phi'(\tilde{r})| \leq C(1 + \tilde{C}|r|^{1-n}).$$

This leads to

$$\int_{\mathbb{R}^n} |H_{jj}| |I_j(f)|^2 dx \leq \tilde{C} \|f\|_J^2$$

and the proposition immediately follows. \square

Proposition 3.4. *Let ϕ be in $\mathbf{H}_{\text{div}}(\mathbb{R}^n)$. Then*

$$I_0(J^{-1} \nabla \cdot \phi) = - \int_{\mathbb{R}^n} \phi \cdot \nabla_y \Phi(\tilde{r}) dy. \quad (3.14)$$

Proof. By Proposition 3.3, the operator on the left hand side of (3.14) is a bounded operator from $\mathbf{H}_{\text{div}}(\mathbb{R}^n)$ into $L_2(\mathbb{R}^n)$. Proposition 3.3 shows that I_j is a bounded operator on $L^2(\mathbb{R}^n)$ from which it easily follows that the right hand side of (3.14) is also a bounded operators from $\mathbf{H}_{\text{div}}(\mathbb{R}^n)$ into $L_2(\mathbb{R}^n)$. Moreover, a limiting argument applied to the right hand integral excluding integration over $B_\epsilon(x)$ and integration by parts shows that (3.14) holds for $\phi \in (C_0^\infty(\mathbb{R}^n))^n$. The proposition follows from the density of $(C_0^\infty(\mathbb{R}^n))^n$ in $\mathbf{H}_{\text{div}}(\mathbb{R}^n)$. \square

4. The PML solution operator on \mathbb{R}^n

In this section, we derive a solution operator for the PML source problem (2.3). Let u be in $C_0^\infty(\mathbb{R}^n)$. For $x \in \mathbb{R}^n$ and $z = i$ or $z = 1 + i$, we consider the integral operator

$$I_*(u)(x) = \int_{\mathbb{R}^n} (H \nabla_y u(y)) \cdot \nabla_y \Phi(\tilde{r}) dy - k^2 \int_{\mathbb{R}^n} Ju \Phi(\tilde{r}) dy.$$

Proposition 3.3 and its proof imply that I_* is a bounded map of $H^1(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$. The following theorem shows that it coincides with the identity.

Theorem 4.1. Assume that $z = i$ or $z = 1 + i$. Then for $u \in H^1(\mathbb{R}^n)$, $I_*(u) = u$ (in $L^2(\mathbb{R}^n)$).

Before proving the theorem, we introduce some additional notation. We consider the differential operator

$$\tilde{\Delta}u = J^{-1}\nabla \cdot (H\nabla u)$$

with domain

$$D(\tilde{\Delta}) = \{w \in H^1(\mathbb{R}^n) : (H\nabla w) \in \mathbf{H}_{\text{div}}(\mathbb{R}^n)\}.$$

We also set $L = (-\tilde{\Delta} - k^2I)$. To avoid confusion, we shall use L^y to denote this operator with differentiation with respect to the y variable (when applied to a function of $x, y \in \mathbb{R}^n$). In the case of smoother $\tilde{\sigma}$, L maps functions in $H^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$.

Proof of Theorem 4.1. We first observe that for σ of Type 1, Theorem 3.4 of [16] shows that for $u \in C_0^\infty(\mathbb{R}^2)$,

$$u(x) = - \int_{\mathbb{R}^2} J(\tilde{\Delta} + k^2)u\Phi(\tilde{r}) dy. \quad (4.1)$$

Moreover, the proof given in [16] for Theorem 3.4 immediately carries over to $d = 3$. We then have

$$\begin{aligned} u(x) &= - \lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}^n \setminus B_\epsilon(x)} JL^y(u)\Phi(\tilde{r}) dy \\ &= \lim_{\epsilon \rightarrow 0} \left[\int_{\mathbb{R}^n \setminus B_\epsilon(x)} [(H \cdot \nabla u) \cdot \nabla \Phi(\tilde{r}) - k^2 Ju\Phi(\tilde{r})] dy + \int_{\partial B_\epsilon(x)} ((H\nabla u) \cdot \mathbf{n})\Phi(\tilde{r}) ds \right] \end{aligned}$$

where we have taken the normal \mathbf{n} pointing outward from $B_\epsilon(x)$. It follows from (3.3) and Lemma 3.1 that the limit of the surface integral is zero, i.e., $I_*(u) = u$ for all $u \in C_0^\infty(\mathbb{R}^n)$.

We now consider the case of jumping σ . To avoid the jump interfaces, we fix $x \in S_m$ for some m . A direct computation using (3.10) shows that provided that we avoid $y = x$,

$$L^y\Phi(\tilde{r}) = 0 \quad \text{for all } y \in S_j \quad (4.2)$$

and any j . Moreover, for y on the interior of a common face of two subdomains S_j and S_k , similar computations give

$$[(H(y)\nabla_y\Phi(\tilde{r})) \cdot \mathbf{n}] = 0.$$

Here $[\cdot]$ denotes the jump across the interface. Thus, for $u \in C_0^\infty(\mathbb{R}^n)$,

$$\begin{aligned} I_*(u)(x) &= \lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}^n \setminus B_\epsilon(x)} (H\nabla_y u(y)) \cdot \nabla_y \Phi(\tilde{r}) dy - k^2 \int_{\mathbb{R}^n \setminus B_\epsilon(x)} Ju\Phi(\tilde{r}) dy \\ &= \lim_{\epsilon \rightarrow 0} \int_{B_\epsilon(x)} u(y)(H\nabla_y\Phi(\tilde{r})) \cdot \mathbf{n} ds \\ &= u(x) \lim_{\epsilon \rightarrow 0} \int_{\partial B_\epsilon(x)} (H\nabla_y\Phi(\tilde{r})) \cdot \mathbf{n} ds. \end{aligned} \quad (4.3)$$

Given a jumping σ , it is easy to construct a σ_* of Type 1 which satisfies:

- (1) For each value t taken on by σ , there is an interval where $\sigma_* = t$.
- (2) $\sigma_* = 0$ in a neighborhood of the origin.

Such a function can be constructed by convolving σ with an appropriate multiple of a characteristic function supported on a small interval around 0. By construction, for any $x \in D_j$, for some j , there is a point x^* which satisfies

$$d \equiv (d(x_1), \dots, d(x_d)) = (d^*(x_1^*), \dots, d^*(x_d^*)) \equiv d^*. \quad (4.4)$$

Here the superscript $*$ indicates that the quantity depends on the smoother function σ_* . Moreover x^* can be chosen so that d^* is constant in a neighborhood of x^* , say on $B_{\epsilon_0}(x^*)$ for some $\epsilon_0 > 0$. By possibly making ϵ_0 smaller, we have that $B_{\epsilon_0}(x) \subset S_j$ so d on $B_{\epsilon_0}(x)$ equals d^* on $B_{\epsilon_0}(x^*)$.

We can apply the above result with σ_* and the corresponding $\tilde{\sigma}_*$ given by

$$\tilde{\sigma}_*(t) = x^{-1} \int_0^t \sigma_*(v) dv$$

to obtain

$$u(x) = I_*^*(u)(x)$$

for all $u \in C_0^\infty(\mathbb{R}^n)$. The limit and integration by parts arguments in (4.3) work for σ_* from which we conclude that

$$1 = \lim_{\epsilon \rightarrow 0} \int_{\partial B_\epsilon(x^*)} (H^* \nabla_y \Phi(\tilde{r}^*)) \cdot \mathbf{n} \, ds. \quad (4.5)$$

For $y \in B_\epsilon(x)$, we set $y^* = x^* + (y - x) \in B_\epsilon(x^*)$. Then for $\epsilon \leq \epsilon_0$, $H(y) = H^*(y)$ and

$$\tilde{r}(x, y)^2 = \sum_{j=1}^n (x_j - y_j)^2 d(x_j)^2 = \sum_{j=1}^n (x_j^* - y_j^*)^2 d^*(x_j^*)^2 = (\tilde{r}^*(x^*, y^*))^2.$$

Thus, we get the same limit in (4.5) if replace $B_\epsilon(x^*)$, H^* and \tilde{r}^* by $B_\epsilon(x)$, H and \tilde{r} . This means that the limit appearing in (4.3) is also 1 and hence $u(x) = I_*(u)(x)$ almost everywhere even for the case of jumping coefficients.

Now by Proposition 3.3, I_* is a bounded map of $H^1(\mathbb{R}^n)$ into $L_2(\mathbb{R}^n)$ and the theorem follows from the density of $C_0^\infty(\mathbb{R}^n)$ in $H^1(\mathbb{R}^n)$. \square

The inf-sup conditions given by Lemmas 2.4 and 2.5 and the symmetry of the coefficients defining $A(\cdot, \cdot)$ and $[\cdot, \cdot]$ imply the stability of the corresponding variational problem. Thus, if f is in $L_2(\mathbb{R}^n)$ there is a unique solution $u \in H^1(\mathbb{R}^n)$ satisfying

$$A(u, \phi) + \gamma [Ju, \phi] = (\phi, f) \quad \text{for all } \phi \in H^1(\mathbb{R}^n). \quad (4.6)$$

It is immediate that u is in $D(\tilde{\Delta})$. On the other hand, for $w \in H^1(\mathbb{R}^n)$, the functional

$$F(\phi) = A(w, \phi) + \gamma [Jw, \phi] \quad \text{for all } \phi \in H^1(\mathbb{R}^n)$$

is in $H^{-1}(\mathbb{R}^n)$. As $L^2(\mathbb{R}^n)$ is dense in $H^{-1}(\mathbb{R}^n)$, given F in $H^{-1}(\mathbb{R}^n)$, there is a sequence $f_n \in L^2(\Omega)$ converging to F in $H^{-1}(\mathbb{R}^n)$. Let w_n solve (4.6) with f replaced by f_n . Then sequence $\{w_n\}$ converges to w in $H^1(\mathbb{R}^n)$. This shows that $D(\tilde{\Delta})$ is dense in $H^1(\mathbb{R}^n)$.

Theorem 4.2. Let f be in $L_2(\mathbb{R}^n)$ and $z = i$ or $z = 1 + i$. Then $u = I_0 f$ is a solution to

$$A(u, \phi) - k^2[u, \phi] = [f, \phi] \quad \text{for all } \phi \in H^1(\mathbb{R}^n). \quad (4.7)$$

Proof. By Proposition 3.3, $u = I_0 f$ is in $H^1(\mathbb{R}^n)$. For $\phi \in D(A)$ and $u_m \in C_0^\infty(\mathbb{R}^n)$ converging to $u \in H^1(\mathbb{R}^n)$,

$$A(u_m, \phi) - k^2[u_m, \phi] = - \int_{\mathbb{R}^n} Ju_m(\tilde{\Delta} + k^2 I) \phi \, dx$$

and hence taking the limit as $m \rightarrow \infty$ gives

$$\begin{aligned} A(u, \phi) - k^2[u, \phi] &= - \int_{\mathbb{R}^n} J \left(\int_{\mathbb{R}^n} Jf \Phi(\tilde{r}) \, dy \right) (\tilde{\Delta} + k^2 I) \phi \, dx \\ &= - \int_{\mathbb{R}^n} Jf \left(\int_{\mathbb{R}^n} J \Phi(\tilde{r}) (\tilde{\Delta} + k^2 I) \phi \, dx \right) dy. \end{aligned}$$

The inner integral on the right hand side is

$$I_0(J^{-1} \nabla \cdot (H \nabla \phi)) + k^2 \int_{\mathbb{R}^n} J \phi \Phi(\tilde{r}) \, dx.$$

Applying Proposition 3.4 to the first term above and combining with the preceding identities gives

$$A(u, \phi) - k^2[u, \phi] = \int_{\mathbb{R}^n} Jf \Pi_*(\phi) \, dx.$$

The equality (4.7) then follows applying Theorem 4.1 and the density of $D(\tilde{\Delta})$ in $H^1(\mathbb{R}^n)$. \square

5. Inf-sup conditions

In this section, we prove inf-sup conditions and variational stability of PML problems on \mathbb{R}^n , Ω^c and our computational domain Ω'_M .

Remark 5.1. We first observe that since our forms are symmetric, the inf-sup condition

$$\|u\|_H \leq \tilde{C} \sup_{\phi \in H^1(\mathbb{R}^n)} \frac{|A(u, \phi) - k^2[u, \phi]|}{\|\phi\|_H}$$

and stability of the resulting variational problem follows if we can construct for each $F \in (H^1(\mathbb{R}^n))^*$, a function $u \in H^1(\mathbb{R}^n)$ solving

$$A(u, \phi) - k^2[u, \phi] = \langle F, \phi \rangle, \quad \text{for all } \phi \in H^1(\mathbb{R}^n)$$

and satisfying the *a priori* estimate

$$\|u\|_H \leq \tilde{C} \|F\|_{H^*}.$$

The analogous statement holds also for the corresponding variational problems on $H_0^1(\Omega^c)$ and $H_0^1(\Omega'_M)$.

Theorem 5.2. *Let $z = i$ or $z = 1 + i$. Then for all $u \in H^1(\mathbb{R}^n)$,*

$$\|u\|_H \leq \tilde{C}_5 \sup_{\phi \in H^1(\mathbb{R}^n)} \frac{|A(u, \phi) - k^2[u, \phi]|}{\|\phi\|_H}.$$

Proof. Let γ be given as in Lemma 2.4 when $z = i$ or $\gamma = e^{-i\pi/4}$ when $z = 1 + i$. Given $F \in H^{-1}(\mathbb{R}^n)$, let $v \in H^1(\mathbb{R}^n)$ solve

$$A(v, \phi) + \gamma[v, \phi] = \langle F, \phi \rangle, \quad \text{for all } \phi \in H^1(\mathbb{R}^n).$$

That this is a well posed problem follows from Lemmas 2.4 or 2.5. Moreover,

$$\|v\|_H \leq \tilde{C} \|F\|_{H^*}.$$

Set $w = (\gamma + k^2)I_0(v)$. A simple computation using Theorem 4.2 shows that $u = w + v$ is in $H^1(\mathbb{R}^n)$ and solves

$$A(u, \phi) - k^2[u, \phi] = \langle F, \phi \rangle, \quad \text{for all } \phi \in H^1(\mathbb{R}^n).$$

Applying Proposition 3.3 gives

$$\begin{aligned} \|u\|_H &\leq \|v\|_H + (|\gamma| + k^2)\|I_0(v)\|_H \\ &\leq \tilde{C} \|F\|_{H^*}. \end{aligned}$$

The theorem now follows from Remark 5.1. \square

Let $v_0 < 1$ be such that $\bar{\Omega}$ is contained in Ω_{v_0} . Suppose that u is in $H^3(D)$ for some neighborhood D of Γ_{v_0} with \bar{D} contained in Ω'_1 . For x in $(\bar{\Omega}_{v_0})^c$ define

$$\mathcal{I}(u)(x) = \int_{\Gamma_{v_0}} \left(\frac{\partial u}{\partial \mathbf{n}_y} \Phi(\tilde{r}) - u \frac{\partial \Phi(\tilde{r})}{\partial \mathbf{n}_y} \right) ds_y.$$

The following proposition shows that $\mathcal{I}(u)$ is an H^1 function and gives some of its decay properties in the norm

$$\|u\|_{H_{\Gamma_M}^{1/2}} = \inf_{\tilde{u}} \|\tilde{u}\|_{H^1(\Omega_M)}. \quad (5.1)$$

Here the infimum is taken over functions \tilde{u} in $H^1(\Omega_M)$ which equal u on Γ_M .

Proposition 5.3. *For u as above, $\mathcal{I}(u)$ is in $H^1((\bar{\Omega}_1)^c)$. Moreover, for $M \geq M_0$ and $M\sigma_0$ sufficiently large*

$$\|\mathcal{I}(u)\|_{H_{\Gamma_M}^{1/2}} \leq \tilde{C}_6 e^{-c\sigma_0 M} \|u\|_{H^3(D)}.$$

Proof. We first observe that $\mathcal{I}(u)$ is a continuous function and derivatives of $\mathcal{I}(u)$ on Ω_1^c can be computed by differentiation under the integral as long as we avoid the interfaces between subdomains $\{S_m\}$. It follows that $\mathcal{I}(u)$ is in H^1 if it is in H^1 of each of the subdomains.

Let $M_1 \geq M_0$ and set $\zeta = (M_0 - v_1)/2$. Suppose that $y \in \Gamma_{v_0}$ and $x \in \Omega_{M_1} \setminus \Omega_{[M_1-\zeta]}$. We take $v = v_1$ and note that for $y \in \Gamma_{v_0}$, $\|y\|_\infty \leq v_1$ while

$$\|x\|_\infty \geq M_1 - (M_0 - v_1)/2 \geq (M_0 + v_1)/2 = \alpha v_1$$

with $\alpha > 1$. We can thus apply Lemma 3.2(b) to conclude that for $x \in \Omega_{M_1} \setminus \Omega_{[M_1-\zeta]}$ and $y \in \Gamma_{v_0}$,

$$\text{Im}(\tilde{r}) = \text{Im}(\tilde{r}(x, y)) \geq c\sigma_0 M_1. \quad (5.2)$$

Now, we assume $M_1 \geq M$ with M large enough so that

$$c\sigma_0 M \geq \beta_1$$

and hence (3.2) holds.

Applying (5.2), Lemma 3.2(b), (3.10) and Lemma 3.1 gives

$$\begin{aligned} |\Phi(\tilde{r})| &\leq Ce^{-c\sigma_0 M_1} \\ |\Phi_{y_j}(\tilde{r})| &= |\Phi'(\tilde{r})(y_j - \tilde{x}(x_j))/\tilde{r}| \leq \tilde{C}e^{-c\sigma_0 M_1} \end{aligned} \quad (5.3)$$

for $x \in \Omega_{M_1} \setminus \Omega_{[M_1-\zeta]}$ and $y \in \Gamma_{v_0}$. These estimates immediately imply that

$$\int_{\Omega_{M_1} \setminus \Omega_{M_1-\zeta}} |J| |\mathcal{I}(u)(x)|^2 dx \leq \tilde{C}e^{-c\sigma_0 M_1} \|u\|_{H^3(D)}^2.$$

Here we used the fact that $u \in H^3(D)$ implies that u is in $W^{1,\infty}(D)$.

Applying this result for $M_1 = M + k\zeta$, $k = 1, 2, \dots$ and summing shows that

$$\int_{\mathbb{R}^n \setminus \tilde{\Omega}_M} |J| |\mathcal{I}(u)(x)|^2 dx < \infty.$$

That the corresponding integral over $x \in \Omega_M \setminus \Omega_1$ is bounded follows from (3.3) and Lemma 3.1.

For the derivatives of $\mathcal{I}(u)(x)$, we differentiate under the integral for x not on the boundary of any of the subdomains $\{S_j\}$. As in (5.3), we find that for $x \in \Omega_{M_1} \setminus \Omega_{[M_1-\zeta]}$ and $y \in \Gamma_{v_0}$,

$$|\Phi_{x_j}(\tilde{r})| = |\Phi'(\tilde{r})d(x_j)(\tilde{x}(x_j) - y_j)/\tilde{r}| \leq \tilde{C}e^{-c\sigma_0 M_1}.$$

Similarly,

$$|\Phi_{y_j, x_k}| \leq \tilde{C}e^{-c\sigma_0 M_1}.$$

Integrating these estimates gives

$$\int_{\Omega_{M_1} \setminus \Omega_{M_1-\zeta}} |H| |\nabla \mathcal{I}(u)|^2 dx \leq \tilde{C}e^{-c\sigma_0 M_1} \|u\|_{H^3(D)}^2.$$

The bound for the integral outside of Ω_1 proceeds similarly to that for the integral involving J above. This completes the first part of the theorem.

For the second part, we assume that $M \geq M_0$ and σ_0 are related so that (5.2) holds with M_1 replaced by M . Then using a cut-down function χ which equals one on Γ_M and vanishes on $\Omega_{M-\zeta}$, we find

$$\|\mathcal{I}(u)\|_{H_M^{1/2}} \leq \|\chi \mathcal{I}(u)\|_{H^1(\Omega_M)} \leq C \|\mathcal{I}(u)\|_{H^1(\Omega_M \setminus \Omega_{M-\zeta})}.$$

Applying the above bounds gives

$$\|\mathcal{I}(u)\|_{H_M^{1/2}} \leq \tilde{C}e^{-c\sigma_0 M} \|u\|_{H^3(D)}.$$

This completes the proof of the proposition. \square

Theorem 5.4. Let $z = i$ or $z = 1 + i$. For all $u \in H_0^1(\Omega^c)$,

$$\|u\|_H \leq \tilde{C}_7 \sup_{\phi \in H_0^1(\Omega^c)} \frac{|A(u, \phi) - k^2[u, \phi]|}{\|\phi\|_H}.$$

Proof. Given $F \in (H^1(\Omega^c))^*$, we denote \tilde{F} to be an extension of F in $(H^1(\mathbb{R}^n))^*$ guaranteed by the Hahn–Banach theorem. A consequence of the previous theorem is that there is a unique solution $\tilde{u} \in H^1(\mathbb{R}^n)$ solving

$$A(\tilde{u}, \phi) - k^2[\tilde{u}, \phi] = \langle \tilde{F}, \phi \rangle \quad \text{for all } \phi \in H^1(\mathbb{R}^n).$$

Let u be the unique solution of (1.1) with $g = \tilde{u}$ on Γ . Next, define v on \mathbb{R}^n by

$$v(x) = \begin{cases} \tilde{u}(x) & \text{for } x \in \Omega, \\ u(x) & \text{for } x \in (-1, 1)^n \setminus \Omega, \\ \mathcal{I}(u)(x) & \text{for } x \notin (-1, 1)^n. \end{cases} \quad (5.4)$$

Note that for $x \in [-1, 1]^n \setminus [-v_0, v_0]^n$ and $y \in \Gamma_{v_0}$, $\tilde{r} = r$. Thus, $u(x)$ and $\mathcal{I}(u)(x)$ coincide for $[-1, 1]^n \setminus [-v_0, v_0]^n$ since

$$u(x) = \int_{\Gamma_{v_0}} \left(\frac{\partial u}{\partial \mathbf{n}_y} \Phi(r) - u \frac{\partial \Phi(r)}{\partial \mathbf{n}_y} \right) ds_y = I(u(x))$$

for such x . Hence the second transition above is smooth. By construction, the first transition gives rise to a function in $H^1((-1, 1)^n)$. Thus, it follows from Proposition 5.3 that v is in $H^1(\mathbb{R}^n)$.

Except at the boundaries of the subdomains $\{S_m\}$ derivatives can be taken inside the integral and as in (4.2), $L(\mathcal{I}(u)) = 0$. At the boundary points in the interior of the faces, a similar computation shows that $[(H\nabla \mathcal{I}(u)) \cdot \mathbf{n}] = 0$. It follows that

$$A(v, \phi) - k^2[v, \phi] = 0 \quad \text{for all } \phi \in H_0^1(\Omega^c). \quad (5.5)$$

Because of (5.5), the functional

$$\langle G, \phi \rangle = A(v, \phi) - k^2[v, \phi], \quad \text{for all } \phi \in H^1(\mathbb{R}^n)$$

satisfies

$$\|G\|_{H^*} \leq C\|v\|_{H^1(\Omega_{v_0})}$$

and hence Theorem 5.2 implies that

$$\|v\|_H \leq \tilde{C}\|v\|_{H^1(\Omega_{v_0})}.$$

From the definition of v and the fact that u solves (1.1),

$$\|v\|_{H^1(\Omega_{v_0})} \leq C\|\tilde{u}\|_{H^1(\Omega)}.$$

It follows that

$$\|v\|_H \leq \tilde{C}\|\tilde{u}\|_H \leq \tilde{C}\|F\|_{(H^1(\Omega^c))^*}.$$

Finally $w = \tilde{u} - v$ satisfies

$$\|w\|_H \leq \tilde{C}\|F\|_{(H^1(\Omega^c))^*}.$$

Note that w vanishes on Γ and is a solution of

$$A(w, \phi) - k^2[w, \phi] = \langle F, \phi \rangle \quad \text{for all } \phi \in H_0^1(\Omega^c).$$

The theorem follows from Remark 5.1. \square

The next result develops the inf-sup conditions for the square domain $\Omega_M = (-M, M)^n$.

Theorem 5.5. Let $z = i$ or $z = 1 + i$. For all $u \in H_0^1(\Omega_M)$,

$$\|u\|_H \leq \tilde{C}_8 \sup_{\phi \in H_0^1(\Omega_M)} \frac{|A(u, \phi) - k^2[u, \phi]|}{\|\phi\|_H} \quad (5.6)$$

provided that $\sigma_0 M$ is sufficiently large with $M \geq M_0$.

Before proving the theorem, we introduce the following proposition which will be used in its proof.

Proposition 5.6. Let M be as above and $\epsilon > 0$. Suppose that $v \in H^1(\Omega_{M+\epsilon})$ satisfies

$$A(v, \phi) - k^2[v, \phi] = 0 \quad \text{for all } \phi \in H_0^1(\Omega_{M+\epsilon}). \quad (5.7)$$

Then

$$\|v\|_{H_{\Omega_M}} \leq \tilde{C}_9\|v\|_{J_{\Omega_{M+\epsilon}}}. \quad (5.8)$$

Proof. This proof is basically classical except for the fact that for σ of Type 2, the coefficients of $A(\cdot, \cdot)$ are discontinuous. We consider $n = 2$ as the case of $n = 3$ is basically the same.

Suppose that v satisfies (5.7). Let χ be a one dimensional cut-off function which is one on $(-M, M)$ and vanishes outside of $(-M - \epsilon, M + \epsilon)$. One then checks that for $\phi \in H_0^1(\Omega_{M+\epsilon})$,

$$A(\chi(x_1)\chi(x_2)v, \phi) = A(v, \chi(x_1)\chi(x_2)\phi) + \varepsilon(v, \phi)$$

where

$$|\varepsilon(v, \phi)| \leq C\|v\|_{J_{\Omega_{M+\epsilon}}} \|\phi\|_{H_{\Omega_{M+\epsilon}}}.$$

This requires integration by parts (moving the derivative off v) on terms such as

$$\int_{\Omega_{M+\epsilon}} H_{11} \frac{\partial v}{\partial x_1} \chi'(x_1) \chi(x_2) \phi \, dx.$$

This integration by parts does not produce interface terms since, even in the case of σ of Type 2, $\chi'(x_1)$ vanishes where H_{11} jumps.

Then with γ as in Lemma 2.4 or Lemma 2.5, (5.7) gives

$$A(\chi(x_1)\chi(x_2)v, \phi) + \gamma[\chi(x_1)\chi(x_2)v, \phi] = \varepsilon(v, \phi) + (\gamma + k^2)[\chi(x_1)\chi(x_2)v, \phi] \equiv \langle F, \phi \rangle \quad (5.9)$$

for all $\phi \in H_0^1(\Omega_{M+\epsilon})$. Clearly,

$$|\langle F, \phi \rangle| \leq C \|v\|_{J\Omega_{M+\epsilon}} \|\phi\|_{H\Omega_{M+\epsilon}}.$$

The stability of (5.9) follows from Remark 2.6 and implies (5.8). This completes the proof of the proposition. \square

Proof of Theorem 5.5. Fix u in $H_0^1(\Omega_M)$. Define $F \in (H_0^1(\Omega_M))^*$ by

$$A(u, \phi) - k^2[u, \phi] = F(\phi), \quad \text{for all } \phi \in H_0^1(\Omega_M). \quad (5.10)$$

The inf-sup condition (5.6) which we are proving is equivalent to the inequality

$$\|u\|_{H\Omega_M} \leq \tilde{C}_5 \|F\|_{H\Omega_M^*}. \quad (5.11)$$

We clearly have

$$\|F\|_{H\Omega_M^*} \leq (1 + k^2) \|u\|_{H\Omega_M}.$$

Set $\epsilon = (M_0 - v_1)/4$ and for $M \geq M_0$, set $d_j = 2M - j\epsilon - v_1$ for $j = 0, 1, 2$.

Let (x, y) be in $\mathbb{R}^n \times \mathbb{R}^n$ with $\|x\|_\infty \geq d_2$ and $\|y\|_\infty \leq M + \epsilon$. The reason for this choice of x and y will become clear later in the proof. As usual, we get a bound for the imaginary part of $\tilde{r}(x, y)$ by applying Lemma 3.2. In this case, we take $v = M + \epsilon$ and find

$$\begin{aligned} \|x\|_\infty &\geq \frac{2M - 2\epsilon - v_1}{M + \epsilon} v \\ &\geq \frac{2M_0 - 2\epsilon - v_1}{M_0 + \epsilon} v = \frac{M_0 + 2\epsilon}{M_0 + \epsilon} v \equiv \alpha v. \end{aligned}$$

Applying Lemma 3.2(b) gives

$$|\tilde{r}(x, y)| \geq \text{Im}(\tilde{r}(x, y)) \geq c(\alpha)\sigma_0 M. \quad (5.12)$$

For simplicity, we consider the case when $n = 2$ as the case of $n = 3$ is similar. Given $u \in H_0^1(\Omega_M)$, our first step is to do an extension of u to the larger domain Ω_{d_0} . We first do an odd reflection of u across the line $x_1 = M$ to define the extension \tilde{u} on $[M, d_0] \times (-M, M)$. Since $\sigma(x)$ is constant for $|x| > v_1$, the coefficients of $A(\cdot, \cdot)$ and $[\cdot, \cdot]$ at the original point in Ω_M and its reflected image are the same. It follows that this reflection satisfies

$$A(\tilde{u}, \phi) - k^2[\tilde{u}, \phi] = \tilde{F}(\phi) \quad \text{for all } \phi \in H_0^1((-M, d_0) \times (-M, M)).$$

The functional \tilde{F} is given by

$$\tilde{F}(\phi) = F(\phi(x_1, x_2) - \phi(2M - x_1, x_2)).$$

Here ϕ is extended by zero outside of $(-M, d_0) \times (-M, M)$. This makes sense as $\phi(x_1, x_2) - \phi(2M - x_1, x_2)$ is in $H_0^1(\Omega_M)$. We do the analogous odd reflection across the line $x_1 = -M$ to define the extended function and functional on $(-d_0, d_0) \times (-M, M)$. Continuing with these odd reflections but across the lines $x_2 = M$ and $x_2 = -M$, we are led to an extension \tilde{u} and extended functional \tilde{F} on $H_0^1(\Omega_{d_0})$ satisfying

$$\|\tilde{u}\|_{H\Omega_{d_0}}^2 \leq 4\|u\|_{H\Omega_M}^2,$$

$$\|\tilde{F}\|_{H\Omega_{d_0}^*} \leq C\|F\|_{H\Omega_M^*}, \quad \text{and}$$

$$A(\tilde{u}, \phi) - k^2[\tilde{u}, \phi] = \tilde{F}(\phi), \quad \text{for all } \phi \in H_0^1(\Omega_{d_0}).$$

Now applying the Hahn–Banach Theorem gives an extension \tilde{F}_1 of \tilde{F} to a functional on $H^1(\mathbb{R}^n)$ with norm bounded by $C\|F\|_{H\Omega_M^*}$. Let $v \in H^1(\mathbb{R}^n)$ solve

$$A(v, \phi) - k^2[v, \phi] = \tilde{F}_1(\phi), \quad \text{for all } \phi \in H^1(\mathbb{R}^n).$$

That this problem has a unique solution follows from Theorem 5.2 and we have

$$\|v\|_H \leq \tilde{C}\|\tilde{F}_1\|_{H^*} \leq \tilde{C}\|F\|_{H\Omega_M^*}.$$

Let $w = v - \tilde{u}$ in Ω_{d_0} . Set $\chi(z)$ to be a smooth cut-down function which satisfies $\chi(z) = 1$ for $|z| \leq d_1$ and $\chi(z) = 0$ for $|z| \geq d_0$. We define $\tilde{w}(z) = \chi(z)w(z)$ on Ω_{d_0} and extend it by zero to \mathbb{R}^2 . Theorem 4.1 gives (for $x \in \Omega_{M+\epsilon}$)

$$w(x) = \int_{\mathbb{R}^n} (H \nabla \tilde{w}) \cdot \nabla \Phi \, dy - k^2 \int_{\mathbb{R}^n} J \tilde{w} \Phi \, dy, \quad \text{a.e.}$$

Note that by construction,

$$A(\tilde{w}, \phi) - k^2[\tilde{w}, \phi] = 0, \quad \text{for all } \phi \in H_0^1(\Omega_{d_1}). \quad (5.13)$$

This implies that $H \nabla \tilde{w}$ is in $H_{\text{div}}(\Omega_{d_1})$ and the limiting argument in the proof of Proposition 3.4 gives (for $x \in \Omega_{M+\epsilon}$ and x not on a subdomain boundary)

$$\int_{\Omega_{d_1}} [(H \nabla \tilde{w}) \cdot \nabla \Phi - k^2 J \tilde{w} \Phi] \, dy = \int_{\partial \Omega_{d_1}} (H \nabla \tilde{w}) \cdot \mathbf{n} \, \Phi \, ds_y.$$

It follows that

$$w(x) = \int_{\partial \Omega_{d_1}} (H \nabla \tilde{w}) \cdot \mathbf{n} \, \Phi \, ds_y + \int_{\Omega_{d_0} \setminus \Omega_{d_1}} (H \nabla \tilde{w}) \cdot \nabla \Phi \, dy - k^2 \int_{\Omega_{d_0} \setminus \Omega_{d_1}} J \tilde{w} \Phi \, dy. \quad (5.14)$$

Obvious manipulations imply that the second and third terms on the right hand side above are bounded by

$$C \|\tilde{w}\|_{H_{\Omega_{d_0} \setminus \Omega_{d_1}}} \|\Phi(\tilde{r}(x, \cdot))\|_{H_{\Omega_{d_0} \setminus \Omega_{d_1}}}.$$

Letting $\tilde{\chi}$ be a cut-down function which is one at d_1 and vanishes at d_2 , we find

$$\begin{aligned} \left| \int_{\partial \Omega_{d_1}} (H \nabla \tilde{w}) \cdot \mathbf{n} \, \Phi \, ds_y \right| &= \left| \int_{\Omega_{d_1} \setminus \Omega_{d_2}} (H \nabla \tilde{w}) \cdot \nabla (\tilde{\chi}(y_1) \tilde{\chi}(y_2) \Phi) \, dy - k^2 \int_{\Omega_{d_1} \setminus \Omega_{d_2}} J \tilde{w} (\tilde{\chi}(y_1) \tilde{\chi}(y_2) \Phi) \, dy \right| \\ &\leq C \|\tilde{w}\|_{H_{\Omega_{d_1} \setminus \Omega_{d_2}}} \|\Phi(\tilde{r}(x, \cdot))\|_{H_{\Omega_{d_1} \setminus \Omega_{d_2}}}. \end{aligned}$$

Combining the above gives

$$|w(x)| \leq C \|\tilde{w}\|_{H_{\Omega_{d_0} \setminus \Omega_{d_2}}} \|\Phi(\tilde{r}(x, \cdot))\|_{H_{\Omega_{d_0} \setminus \Omega_{d_2}}}.$$

We first choose σ_0 and M so that (5.12) implies

$$\text{Im}(\tilde{r}) \geq c\sigma_0 M \geq \beta_1 \quad (5.15)$$

for $x \in \Omega_{M+\epsilon}$ and $y \in \Omega_{d_0} \setminus \Omega_{d_2}$. Applying (3.2) and (3.10) shows that $|\Phi(\tilde{r}(x, y))|$ and its partials with respect to y_i (for y in the interior of the subdomains) are bounded by $\tilde{C}e^{-c\sigma_0 M}$. This implies that

$$\|\Phi(\tilde{r}(x, \cdot))\|_{H_{\Omega_{d_0} \setminus \Omega_{d_2}}}^2 \leq \tilde{C}e^{-c\sigma_0 M}. \quad (5.16)$$

Thus,

$$\begin{aligned} \|w\|_{J_{\Omega_{M+\epsilon}}}^2 &\leq \tilde{C}e^{-c\sigma_0 M} \|w\|_{H_{\Omega_{d_0} \setminus \Omega_{d_2}}}^2 \\ &\leq \tilde{C}e^{-c\sigma_0 M} \|u\|_{H_{\Omega_M}}^2. \end{aligned}$$

Finally (5.13) and the previous proposition imply that

$$\|w\|_{H_{\Omega_M}} \leq \tilde{C}e^{-c\sigma_0 M} \|u\|_{H_{\Omega_M}}.$$

Applying the triangle inequality now gives

$$\|u\|_{H_{\Omega_M}} \leq \tilde{C}\|F\|_{H_{\Omega_M}^s}^2 + \tilde{C}_{10}e^{-c\sigma_0 M} \|u\|_{H_{\Omega_M}}.$$

Now, using Remark 2.3 we may take σ_0 and M so that

$$\tilde{C}_{10}e^{-c\sigma_0 M} \leq 1/2 \quad (5.17)$$

and (5.11) immediately follows. This completes the proof of the theorem. \square

For stability of the truncated scattering problem, we have the following theorem.

Theorem 5.7. For $\sigma_0 M$ sufficiently large with $M \geq M_0$ and $u \in H_0^1(\Omega'_M)$,

$$\|u\|_H \leq \tilde{C}_{11} \sup_{\phi \in H_0^1(\Omega'_M)} \frac{|A(u, \phi) - k^2[u, \phi]|}{\|\phi\|_H}. \quad (5.18)$$

Proof. Given $\theta \in H^{1/2}(\Gamma_M)$, we define $w_1 = w_1(\theta) \in H^1(\Omega_M)$ satisfying

$$\begin{aligned} A(w_1, \phi) - k^2[w_1, \phi] &= 0, \quad \text{for all } \phi \in H_0^1(\Omega_M), \\ w_1 &= \theta, \quad \text{on } \Gamma_M \end{aligned}$$

and $w_2 = w_2(\theta) \in H^1(\Omega^c)$ by

$$\begin{aligned} A(w_2, \phi) - k^2[w_2, \phi] &= 0, \quad \text{for all } \phi \in H_0^1(\Omega^c), \\ w_2 &= w_1, \quad \text{on } \Gamma. \end{aligned}$$

It follows from Theorems 5.5 and 5.4 that

$$\|w_1\|_{H_{\Omega'_M}} \leq \tilde{C} \|\theta\|_{H_{\Gamma_M}^{1/2}}$$

and

$$\|w_2\|_{H_{\Omega'_M}} \leq \tilde{C} \|w_1\|_{H_{\Omega'_M}}.$$

Thus,

$$\|w_1\|_{H_{\Omega'_M}} + \|w_2\|_{H_{\Omega'_M}} \leq \tilde{C} \|\theta\|_{H_{\Gamma_M}^{1/2}}. \quad (5.19)$$

We define $T\theta$ to be w_2 restricted to Γ_M . We shall show that if σ_0 and/or M is large, $\|T\|_{H_{\Gamma_M}^{1/2}} = \gamma$ can be made less than 1 (depending only on the size of the product of σ_0 and M).

Assume temporarily that γ is less than one. We then have that the Neumann series

$$\sum_{j=0}^{\infty} T^j$$

converges in $\|\cdot\|_{H_{\Gamma_M}^{1/2}}$. Its limit is $(I - T)^{-1}$ and satisfies $\|(I - T)^{-1}\|_{H_{\Gamma_M}^{1/2}} \leq (1 - \gamma)^{-1}$.

Given $F \in H_{\Omega'_M}^*$, we extend F to a bounded linear functional \tilde{F} on H_{Ω^c} (without increase in norm) and let $v \in H_0^1(\Omega^c)$ solve

$$A(v, \phi) - k^2[v, \phi] = \langle \tilde{F}, \phi \rangle, \quad \text{for all } \phi \in H_0^1(\Omega^c).$$

Let χ_0 be the trace of v on Γ_M and set $\chi = (I - T)^{-1}\chi_0$. Note that

$$\begin{aligned} A(w_1(\chi) - w_2(\chi), \phi) - k^2[w_1(\chi) - w_2(\chi), \phi] &= 0, \quad \text{for all } \phi \in H_0^1(\Omega'_M), \\ w_1(\chi) - w_2(\chi) &= 0, \quad \text{on } \Gamma, \\ w_1(\chi) - w_2(\chi) &= (I - T)\chi = \chi_0, \quad \text{on } \Gamma_M. \end{aligned}$$

This means that $u = v - w_1(\chi) + w_2(\chi)$ is in $H_0^1(\Omega'_M)$ and solves

$$A(u, \phi) - k^2[u, \phi] = \langle F, \phi \rangle, \quad \text{for all } \phi \in H_0^1(\Omega'_M).$$

Furthermore

$$\|u\|_{H_{\Omega'_M}} \leq \tilde{C} \|F\|_{H_{\Omega'_M}^*}$$

where we used Theorem 5.4 for the bound on v . Thus, the proof of the theorem will be complete once we verify the norm bound on T .

We note that $w_2 = w_2(\theta)$ can be written as in (5.4) and hence $w_2(x) = \mathcal{I}(w_2)(x)$, for $x \notin [-1, 1]^n$. Proposition 5.3 then implies that

$$\begin{aligned} \|w_2\|_{H_{\Gamma_M}^{1/2}} &\leq \tilde{C} e^{-c\sigma_0 M} \|w_2\|_{H^3(D)} \\ &\leq \tilde{C} e^{-c\sigma_0 M} \|w_1\|_{H^{1/2}(\Gamma)} \leq \tilde{C}_{12} e^{-c\sigma_0 M} \|\theta\|_{H_{\Gamma_M}^{1/2}}. \end{aligned}$$

The theorem follows, using [Remark 2.3](#) and choosing $\sigma_0 M$ large enough so that

$$\tilde{C}_{12} e^{-c\sigma_0 M} = \gamma < 1. \quad \square$$

It immediately follows from [Theorem 5.7](#) that the solution of (2.2) exists and is unique provided that $\sigma_0 M$ is sufficiently large. We then have the following theorem.

Theorem 5.8. Suppose that $\sigma_0 M$ is large enough so that [Theorem 5.7](#) holds. Let g be in $H^{1/2}(\Gamma)$, u solve (1.1) and u_M solve (2.2). Then,

$$\|u - u_M\|_{H^1(\Omega'_1)} \leq \tilde{C}_{13} e^{-c\sigma_0 M} \|g\|_{H^{1/2}(\Gamma)}.$$

Proof. Let \tilde{u} solve (2.5). Then $w = \tilde{u} - u_M$ satisfies

$$\begin{aligned} A(w, \phi) - k^2[w, \phi] &= 0, \quad \text{for all } \phi \in H_0^1(\Omega'_M), \\ w &= 0, \quad \text{on } \Gamma, \quad w = \tilde{u} \quad \text{on } \Gamma_M. \end{aligned}$$

It follows from [Theorem 5.7](#) that w satisfies

$$\|w\|_{H_{\Omega'_M}} \leq \tilde{C} \|\tilde{u}\|_{H_M^{1/2}}.$$

Noting that \tilde{u} has a representation analogous to (5.4) and applying [Proposition 5.3](#) gives

$$\|\tilde{u}\|_{H_M^{1/2}} = \|\mathcal{I}(\tilde{u})\|_{H_M^{1/2}} \leq \tilde{C} e^{-c\sigma_0 M} \|g\|_{H^{1/2}}.$$

The theorem now follows from the fact that by (5.4), $\tilde{u} = u$ on Ω'_1 . \square

6. Numerical results

In this section, we report the results of numerical experiments which illustrate the convergence behavior suggested by the theorems of the previous sections.

For simplicity, we consider a two dimensional problem and take Ω to be the unit square $(-1/2, 1/2)^2$. Specifically, we approximate solutions of (1.1) with data g on Γ given by the trace of

$$u(x) = H_1^{(1)}(kr) \cos(\theta) \quad (6.1)$$

with $k = 10$. We approximate the solution on $(-2, 2)^2$ and use Cartesian PML with a continuous piecewise linear σ which vanishes on $[-2, 2]$, is linear on $[2, 2 + (M - 2)/2]$ and $[-2 - (M - 2)/2, -2]$ and equals σ_0 for $2 + (M - 2)/2 \leq |x| \leq M$.

Initially, we consider $M = 3$ and use a uniform mesh of squares on $\Omega'_M = (-3, 3)^2 \setminus [-1/2, 1/2]^2$. Other meshes corresponding to different M are created in such a way as to maintain the same number of elements in the PML region for a given interior mesh size. For example, for a mesh of size h in $(-2, 2)^2 \setminus [-1/2, 1/2]^2$, we use rectangles of width $(M - 2)h$ when $|x_1| > 2$ and height $(M - 2)h$ when $|x_2| > 2$ (see [Fig. 1](#)). In all of our examples, we take $z = i$.

Our approximation employs piecewise C^0 -bilinear finite elements on the mesh just described. We interpolate the values of g on Γ to define a discrete approximate g_h and compute the piecewise C^0 -bilinear function u_h with $u_h = g_h$ on Γ and

$$A(u_h, \phi) - k^2[u_h, \phi] = 0 \quad (6.2)$$

for all piecewise C^0 -bilinear ϕ vanishing on $\partial\Omega'_M$. As our problem is two dimensional, (6.2) leads to algebraic systems with only a modest number of unknowns which can be solved by direct methods available, for example, in UMFPACK [17].

To gauge the accuracy of the approximate solution, we compute the L^2 error between the approximate solution and the finite element interpolant of the analytical solution on $(-2, 2)^2 \setminus [-1/2, 1/2]^2$.

[Fig. 2](#) shows the real part of the solution and its approximation using $\sigma_0 = 4$, $M = 3$, $h = 1/64$. The L^2 error was 0.0042. Note that the solution goes rapidly to zero in the PML layer $(3, 3)^2 \setminus (2, 2)^2$. To illustrate that good approximation can be achieved by a smaller PML layer with a larger σ_0 , we ran the same h but with $\sigma_0 = 40$ and $M = 2.1$. The solution and its approximation are given in [Fig. 3](#). Note that, in this case, the PML region is significantly smaller than the earlier example. Nevertheless, as suggested by our theory, the accuracy of the approximation is preserved in the region of interest with an L^2 error of 0.0044 in this case.

The next two tables illustrate the need to make the product $M\sigma_0$ sufficiently large in conjunction with the desired computational accuracy. Because the solution of our problem is smooth, we can expect second order convergence in L^2 (even though our domain has re-entrant corners). The first two columns of [Table 1](#), corresponding to $\sigma_0 = 1$ and $\sigma_0 = 0.5$, appear to have sufficient PML and suggest second order convergence as the mesh size decreases down to $1/512$. Of course, it is likely that the product of σ_0 and M would have to be increased for smaller values of h . In contrast, the combination $M = 3$ and $\sigma_0 = 0.1$ shows very little convergence.

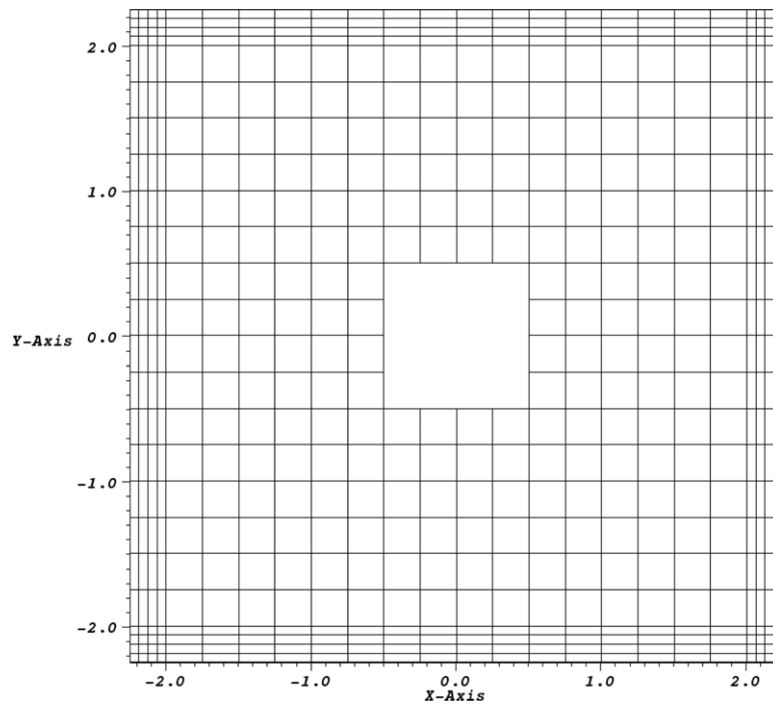


Fig. 1. The grid for $h = 1/4$ and $M = 2.25$.

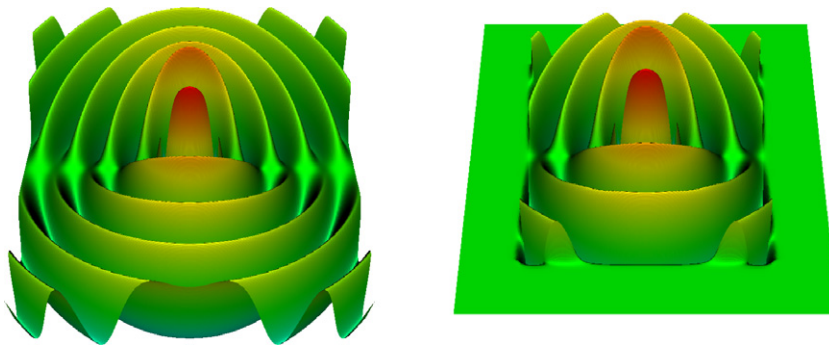


Fig. 2. Solution and approximation, $\sigma_0 = 4$, $M = 3$.

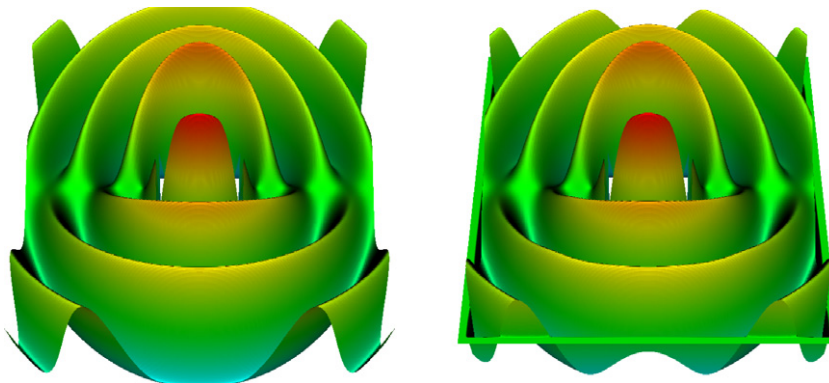


Fig. 3. Solution and approximation, $\sigma_0 = 4$, $M = 2.1$.

Table 2 illustrates that more PML decay can be obtained by keeping $\sigma_0 = 0.1$ fixed while increasing M . The first column is again the case of $M = 3$ while the second and third columns illustrate the change with increasing M . Although $M = 4$

Table 1
Errors as a function of the PML strength σ_0 for $M = 3$.

h	# dofs	$\sigma_0 = 1$	$\sigma_0 = 0.5$	$\sigma_0 = 0.1$
$\frac{1}{2}$	224	0.341	0.342	0.342
$\frac{1}{4}$	1 008	0.402	0.388	0.415
$\frac{1}{8}$	4 256	0.202	0.202	0.234
$\frac{1}{16}$	17 472	0.062	0.062	0.179
$\frac{1}{32}$	70 784	0.016	0.017	0.161
$\frac{1}{64}$	284 928	0.0042	0.0043	0.157

Table 2
Errors as a function of M ($\sigma_0 = 0.1$).

h	# dofs	$M = 3$	$M = 4$	$M = 8$
$\frac{1}{2}$	224	0.342	0.339	0.337
$\frac{1}{4}$	1 008	0.415	0.651	0.626
$\frac{1}{8}$	4 256	0.234	0.269	0.895
$\frac{1}{16}$	17 472	0.179	0.073	0.425
$\frac{1}{32}$	70 784	0.161	0.035	0.050
$\frac{1}{64}$	284 928	0.157	0.028	0.011

Table 3
Errors with $\sigma_0(M - 2) = 1$.

h	# dofs	$\sigma_0 = 1, M = 3$	$\sigma_0 = 4, M = 2.25$	$\sigma_0 = 20, M = 2.05$
$\frac{1}{2}$	224	0.341	0.340	0.340
$\frac{1}{4}$	1 008	0.402	0.423	0.426
$\frac{1}{8}$	4 256	0.202	0.199	0.199
$\frac{1}{16}$	17 472	0.062	0.064	0.064
$\frac{1}{32}$	70 784	0.016	0.017	0.017
$\frac{1}{64}$	284 928	0.0042	0.0043	0.0043

shows some improvement, the convergence rate breaks down at smaller values of h . In contrast, the third column indicates that the combination $M = 8$ and $\sigma = 0.1$ is sufficient to yield better convergence.

Table 3 illustrates that almost identical results are obtained if one keeps the product of the size of the PML layer times σ_0 constant. Here we vary the width of the PML layer between 1 and 0.05. Note that the observed errors across the rows changed only by a negligible amount.

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Appendix

In this appendix, we provide the proofs of Lemmas 3.1 and 3.2.

Proof of Lemma 3.1. We consider first the case of $n = 2$. Let σ_j be defined as in (3.11). We set $d_j = (1 + z\sigma_j)$ so that

$$\tilde{r}^2 = (x_1 - y_1)^2 d_1^2 + (x_2 - y_2)^2 d_2^2. \quad (\text{A.1})$$

Now $\text{Im}(d_j)^2 = 2(1 + \text{Re}(z)\sigma_j)\sigma_j$ and is positive unless $\sigma_j = 0$. That $0 \leq \arg(\tilde{r}^2) < \pi$ follows immediately.

The upper estimate is a consequence of

$$|d_j|^2 = (1 + \text{Re}(z)\sigma_j)^2 + (\sigma_j)^2 \leq (2 + 2|z|^2\sigma_0^2).$$

For the lower estimate, we rewrite (A.1) as

$$\tilde{r}^2 = r^2(t_1 d_1^2 + t_2 d_2^2) = r^2 \psi \quad (\text{A.2})$$

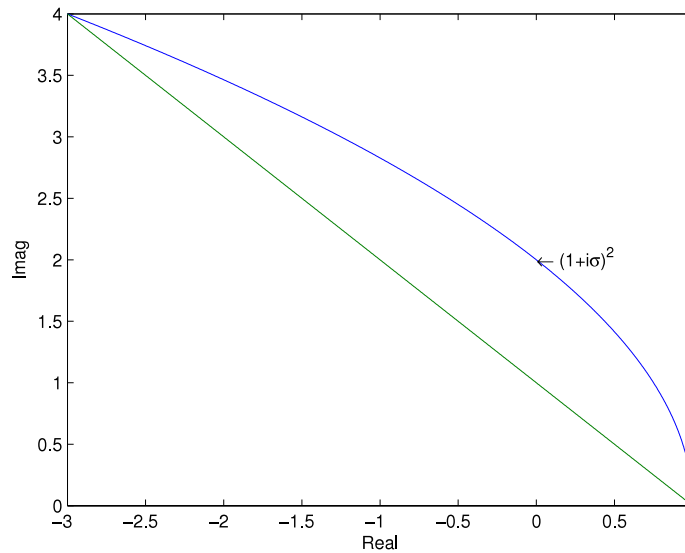


Fig. 4. The curve $(1 + i\sigma)^2$ for $\sigma \in [0, 2]$.

with $t_\ell \geq 0$ and $t_1 + t_2 = 1$. Now when $z = 1 + i$, the real part of $(1 + z\sigma_\ell)^2$ is greater than or equal to one and hence so is the real part of ψ . It follows that $|\tilde{r}| \geq r$ in this case.

When $z = i$, we consider the curve in the complex plane $\gamma(\sigma) = (1 + i\sigma)^2$ for $\sigma \in [0, \sigma_0]$ and the straight line connecting its endpoints. A plot of these are given in the Fig. 4 when $\sigma_0 = 2$.

The tangent of γ points along the direction of i at $\sigma = 0$. In addition, the line connecting 1 and $(1 + i\sigma_0)^2$ is in the direction $(1 + i\sigma_0)^2 - 1$ and

$$\frac{(1 + i\sigma_0)^2 - 1}{i} = \sigma_0(2 + i\sigma_0).$$

This means that the curve γ starts out to the right of the line connecting 1 and $(1 + i\sigma_0)^2$. It is easy to check that the curve γ only intersects the above line at $\sigma = 0$ and $\sigma = \sigma_0$. Accordingly, $\gamma(\sigma)$ is always to the right of the line for $\sigma \in (0, \sigma_0)$.

Now $(t_1(1 + i\sigma_1)^2 + t_2(1 + i\sigma_2)^2)$ is on a line connecting two points of γ and hence must also be to the right of the line connecting 1 and $(1 + i\sigma_0)^2$. A straightforward computation shows that the distance from the origin to this line is $|1 + i\sigma_0/2|^{-1}$ and hence $|\psi|^{1/2} \geq |1 + i\sigma_0/2|^{-1/2}$. The result in the case of $n = 2$ follows.

The proof when $n = 3$ is similar except that in this case

$$\tilde{r}^2 = r^2(t_1(1 + z\sigma_1)^2 + t_2(1 + z\sigma_2)^2 + t_3(1 + z\sigma_3)^2)$$

for nonnegative t_ℓ with $t_1 + t_2 + t_3 = 1$. The case of $z = 1 + i$ is identical with the $n = 2$ argument. For $z = i$,

$$t_1(1 + i\sigma_1)^2 + t_2(1 + i\sigma_2)^2 + t_3(1 + i\sigma_3)^2$$

is in a triangle connecting three points on γ and hence is also to the right of the line from 1 to $(1 + i\sigma_0)^2$ so $|\psi| \geq |1 + i\sigma_0/2|^{-1}$ in this case also. This completes the proof of the lemma. \square

Proof of Lemma 3.2. We first consider case (a), i.e., $\|x - y\|_\infty \geq 2\alpha v_1$ with $\alpha > 1$. Take $j = j(x, y)$ to be an index where $\|x - y\|_\infty = |x_j - y_j|$. Now σ_j is the average of $\sigma(t)$ over the interval with endpoints x_j and y_j and $\sigma(t) = \sigma_0$ on a subinterval of length at least $|x_j - y_j| - 2v_1$. Thus,

$$\sigma_j \geq \frac{|x_j - y_j| - 2v_1}{|x_j - y_j|} \sigma_0 \geq \frac{\alpha - 1}{\alpha} \sigma_0.$$

A similar inequality holds for case (b). In this case, $\|x\|_\infty \geq \alpha v$, $\|y\|_\infty \leq v$ with $\alpha > 1$ and $v \geq v_1$ and we take $j = j(x)$ to be an index where $\|x\|_\infty = |x_j|$. By possibly multiplying x and y by minus one, we may assume that $x_j > 0$. Since $\sigma(t) = \sigma_0$ on $[v, x_j]$, its average over the interval $[y_j, x_j]$ satisfies

$$\sigma_j \geq \frac{x_j - v}{x_j + v} \sigma_0 \geq \frac{\alpha - 1}{\alpha + 1} \sigma_0.$$

Thus, in both cases

$$\sigma_j \geq \frac{\alpha - 1}{\alpha + 1} \sigma_0. \quad (\text{A.3})$$

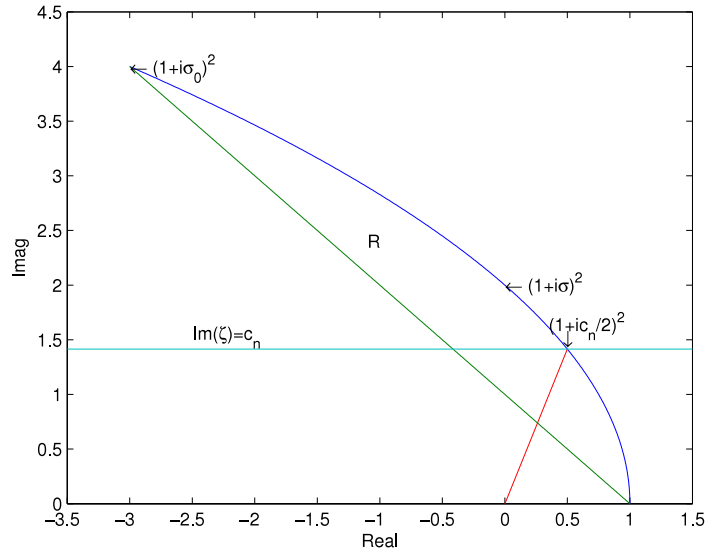


Fig. 5. The sector for ψ when $z = i$.

Set

$$c_n = 2\sigma_0 \frac{\alpha - 1}{(\alpha + 1)n}.$$

We start with $n = 2$. Note that

$$\operatorname{Im}((1 + z\sigma_j)^2) = 2(1 + \operatorname{Re}(z)\sigma_j)\sigma_j \geq 2\sigma_0 \frac{\alpha - 1}{\alpha + 1}.$$

Now $\|x - y\|_\infty = |x_j - y_j|$ implies that $t_j \geq 1/2$ and hence ψ of (A.2) satisfies

$$\operatorname{Im}(\psi) \geq c_n. \quad (\text{A.4})$$

We first consider the case of $z = i$. Note that ψ is in the region $\zeta \in \mathbb{C}$ bounded by $\operatorname{Im}(\zeta) = c_n = \operatorname{Im}((1 + ic_n/2)^2) = \operatorname{Im}(\gamma(c_n/2))$, the curve γ and the line from 1 to $(1 + i\sigma_0)^2$ (see the region R in Fig. 5). The argument of any point in this region is greater than or equal to that of $\gamma(c_n/2)$. By the proof of the previous lemma, ψ is in R and hence $\arg(\sqrt{\psi}) \geq \arg(1 + ic_n/2)$. It follows that $\arg(1 + ic_n/2) \leq \arg(\tilde{r}) < \pi/2$ and so

$$\operatorname{Im}(\tilde{r}) = r \operatorname{Im}(\sqrt{\psi}) \geq r |\sqrt{\psi}| \frac{c_n}{(4 + c_n^2)^{1/2}}.$$

The fraction on the right hand side is an increasing function of c_n and so bounded from below by $c(\alpha)$ since $\sigma_0 \geq \underline{\sigma}_0$. To complete the proof for this case, we need only show that $|\psi| \geq C\sigma_0^2$.

For $\sigma_0^2 \leq 3(\frac{\alpha-1}{\alpha+1})^{-2}$, (A.4) implies that $|\psi| \geq c_2 \geq (\frac{\alpha-1}{\alpha+1})^2 \sigma_0^2 / \sqrt{3}$. We need a similar bound when $\sigma_0^2 \geq 3(\frac{\alpha-1}{\alpha+1})^{-2}$. In this case, (A.3) implies that $(\sigma_j)^2 \geq 3$. Let ψ_0 denote the value of ψ with this value of σ_j but with $\sigma_{3-j} = 0$, i.e.,

$$\psi_0 = t_j \gamma(\sigma_j) + (1 - t_j) \gamma(0) = t_j(1 - (\sigma_j)^2 + 2i\sigma_j) + (1 - t_j).$$

As t_j is greater than or equal to $1/2$, ψ_0 is in the second quadrant of the complex plane. The real part of γ is a decreasing function of σ while the imaginary part is increasing. It follows that

$$\begin{aligned} |\psi| &= |t_j \gamma(\sigma_j) + (1 - t_j) \gamma(\sigma_{3-j})| \geq |\psi_0| \\ &\geq |1/2((\sigma_j)^2 - 2) + i\sigma_j| \geq C(\sigma_j)^2 \geq C\sigma_0^2. \end{aligned}$$

This proves the lemma for $n = 2$ and $z = i$.

The above argument also works for $z = 1 + i$. The curve $\gamma(\sigma) = (1 + z\sigma)^2$ still is to the right of the line from 1 to $(1 + z\sigma_0)^2$ and reasoning as above leads to $\arg(\sqrt{\psi}) \geq \arg(1 + z\tilde{\sigma})$ where $\tilde{\sigma} = (-1 + \sqrt{1 + 2c_n})/2$ satisfies $\operatorname{Im}(\gamma(\tilde{\sigma})) = c_n$. It follows that

$$\operatorname{Im}(\tilde{r}) = r \operatorname{Im}(\sqrt{\psi}) \geq r |\sqrt{\psi}| \frac{\tilde{\sigma}}{|1 + z\tilde{\sigma}|}.$$

We note that the fraction above is an increasing function of $\tilde{\sigma}$ (and hence c_n) and so is bounded from below by its value at the minimal c_n , i.e., $c_n = (\frac{\alpha-1}{\alpha+1})\underline{\sigma}_0$. We are again left to bound $|\psi|$.

For this z , the real and imaginary parts of $\gamma(\sigma)$ are always greater than or equal to zero. It follows that

$$|\psi| \geq \frac{1}{2} |1 + (1+i)\sigma_j|^2 \geq (\sigma_j)^2 \geq \left(\frac{\alpha-1}{\alpha+1}\right)^2 \sigma_0^2.$$

This completes the proof for $n = 2$.

The proof in the case of $n = 3$ is similar. In this case, we write

$$\tilde{r}^2 = r^2(t_1(1+z\sigma_1)^2 + t_2(1+z\sigma_2)^2 + t_3(1+z\sigma_3)^2) \equiv r^2\psi$$

for nonnegative t_j with $t_1 + t_2 + t_3 = 1$. In this case, $t_1 \geq 1/3$ and hence (A.4) follows. The derivation of the bound for $|\psi|$ in the case of $z = 1 + i$ is essentially identical to the $n = 2$ case. The bound for $|\psi|$ in the case of $z = i$ is similar to the argument given above for $n = 2$ and is left as an exercise for the reader. \square

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