SHIFT THEOREMS FOR THE BIHARMONIC DIRICHLET PROBLEM

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Abstract. We consider the biharmonic Dirichlet problem on a polygonal domain. Regularity estimates in terms of Sobolev norms of fractional order are proved. The analysis is based on new interpolation results which generalizes Kellogg's method for solving subspace interpolation problems. The Fourier transform and the construction of extension operators to Sobolev spaces on R^2 are used in the proof of the interpolation theorem.

1. Introduction

Regularity estimates of the solutions of elliptic boundary value problems in terms of Sobolev-fractional norms are known as shift theorems or shift estimates. The shift estimates are significant in finite element theory.

The shift estimates for the Laplace operator with Dirichlet boundary conditions on nonsmooth domains are studied in [2], [12], [14] and [18]. On the question of shift theorems for the biharmonic problem on nonsmooth domains, there seems to be no work answering this question.

One way of proving shift results is by using the real method of interpolation of Lions and Peetre [3], [15] and [16]. The interpolation problems we are led to are of the following type. If X and Y are Sobolev spaces of integer order and X_K is a subspace of finite codimension of X then characterize the interpolation spaces between X_K and Y.

When X_K is of codimension one the problem was studied by Kellogg in some particular cases in [12]. The interpolation results presented in Section 2 give a natural formula connecting the norms on the intermediate subspaces $[X_K, Y]_s$ and $[X, Y]_s$. The main result of Section 2 is a theorem which provides sufficient conditions to compare the topologies on $[X_K, Y]_s$ and $[X, Y]_s$ and gives rise to an extension of Kellogg's method in proving shift estimates for more complicated boundary value problems.

In proving shift estimates for the biharmonic problem, we will follow Kellogg's approach in solving subspace interpolation problems on sector domains. The method involves reduction of the problem to subspace interpolation on Sobolev spaces defined on all of R^2. This reduction requires construction of "extension" and "restriction" operators connecting Sobolev spaces defined on sectors and Sobolev spaces defined on R^2. The method involves also finding the asymptotic expansion of the Fourier transform of certain singular functions. The remaining part of the paper is organized as follows. In Section 2 we prove a natural formula connecting the norms on the intermediate subspaces $[X_K, Y]_s$ and $[X, Y]_s$. The main result of the section is a theorem which provides sufficient conditions (the **(A1)** and **(A2)** conditions) to compare the topologies on $[X_K, Y]_s$

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and $[X,Y]_s$. A new proof of the main subspace interpolation result presented in [12] and an extension to subspace interpolation of codimension greater than one are given in Section 3. The main result concerning shift estimates for the biharmonic Dirichlet problem is considered in Section 4.

2. Interpolation results

In this section we give some basic definitions and results concerning interpolation between Hilbert spaces and subspaces using the real method of interpolation of Lions and Peetre (see [15]).

2.1. **Interpolation between Hilbert spaces.** Let X, Y be separable Hilbert spaces with inner products $(\cdot,\cdot)_X$ and $(\cdot,\cdot)_Y$, respectively, and satisfying for some positive constant c,

$$\begin{cases} X \text{ is a dense subset of Y and} \\ \|u\|_Y \leq c\|u\|_X \quad \text{for all } u \in X, \end{cases} \tag{2.1}$$

where $\|u\|_X^2 = (u,u)_X$ and $\|u\|_Y^2 = (u,u)_Y$.

Let $D(S)$ denote the subset of X consisting of all elements u such that the antilinear form

$$v \to (u,v)_X, \ v \in X \tag{2.2}$$

is continuous in the topology induced by Y.
For any u in $D(S)$ the antilinear form (2.2) can be extended to a continuous antilinear form on Y. Then by Riesz representation theorem, there exists an element Su in Y such that

$$(u,v)_X = (Su,v)_Y \quad \text{for all } v \in X. \tag{2.3}$$

In this way S is a well defined operator in Y, with domain $D(S)$. The next result illustrates the properties of S .

Proposition 2.1. *The domain $D(S)$ of the operator S is dense in X and consequently $D(S)$ is dense in Y. The operator $S : D(S) \subset Y \to Y$ is a bijective, self-adjoint and positive definite operator. The inverse operator $S^{-1} : Y \to D(S) \subset Y$ is a bounded symmetric positive definite operator and*

$$(S^{-1}z,u)_X = (z,u)_Y \quad \textit{for all } z \in Y, \ u \in X \tag{2.4}$$

If in addition X is compactly embedded in Y, then S^{-1} is a compact operator.

The interpolating space $[X,Y]_s$ for $s \in (0,1)$ is defined using the K function, where for $u \in Y$ and $t > 0$,

$$K(t,u) := \inf_{u_0 \in X} (\|u_0\|_X^2 + t^2\|u-u_0\|_Y^2)^{1/2}.$$

Then $[X,Y]_s$ consists of all $u \in Y$ such that

$$\int_0^\infty t^{-(2s+1)} K(t,u)^2 \, dt < \infty.$$

The norm on $[X,Y]_s$ is defined by

$$\|u\|^2_{[X,Y]_s} := \mathbf{c}_s^2 \int_0^\infty t^{-(2s+1)} K(t,u)^2 \, dt,$$

where

$$\mathbf{c}_s := \left(\int_0^\infty \frac{t^{1-2s}}{t^2+1} dt \right)^{-1/2} = \sqrt{\frac{2}{\pi}\sin(\pi s)}$$

By definition we take

$$[X,Y]_0 := X \quad \text{and} \quad [X,Y]_1 := Y.$$

The next lemma provides the relation between $K(t,u)$ and the connecting operator S.

Lemma 2.1. *For all $u \in Y$ and $t > 0$,*

$$K(t,u)^2 = t^2 \left((I + t^2 S^{-1})^{-1} u, u \right)_Y .$$

Proof. Using the density of $D(S)$ in X, we have

$$K(t,u)^2 = \inf_{u_0 \in D(S)} (\|u_0\|_X^2 + t^2 \|u - u_0\|_Y^2)$$

Let $v = Su_0$. Then

$$K(t,u)^2 = \inf_{v \in Y} ((S^{-1}v, v)_Y + t^2 \|u - S^{-1}v\|_Y^2). \tag{2.5}$$

Solving the minimization problem (2.5) we obtain that the element v which gives the optimum satisfies

$$(I + t^2 S^{-1})v = t^2 u,$$

and

$$(S^{-1}v, v)_Y + t^2 \|u - S^{-1}v\|_Y^2 = t^2 \left((I + t^2 S^{-1})^{-1} u, u \right)_Y .$$

□

Remark 2.1. *Lemma 2.1 gives another expression for the norm on $[X,Y]_s$, namely:*

$$\|u\|^2{}_{[X,Y]_s} := \mathbf{c}_s^2 \int_0^\infty t^{-2s+1} \left((I + t^2 S^{-1})^{-1} u, u \right)_Y dt. \tag{2.6}$$

In addition, by this new expression for the norm (see Definition 2.1 and Theorem 15.1 in [15]*), it follows that the intermediate space $[X,Y]_s$ coincides topologically with the domain of the unbounded operator $S^{1/2(1-s)}$ equipped with the norm of the graph of the same operator . As a consequence we have that X is dense in $[X,Y]_s$ for any $s \in [0,1]$.*

Lemma 2.2. *Let X_0, be a closed subspace of X and let Y_0, be a closed subspace of Y. Let X_0 and Y_0 be equipped with the topology and the geometry induced by X and Y respectively, and assume that the pair (X_0, Y_0) satisfies* (2.1)*. Then, for $s \in [0,1]$,*

$$[X_0, Y_0]_s \subset [X,Y]_s \cap Y_0.$$

Proof. For any $u \in Y_0$ we have

$$K(t,u,X,Y) \le K(t,u,X_0,Y_0).$$

Thus,

$$\|u\|_{[X,Y]_s} \le \|u\|_{[X_0,Y_0]_s} \qquad \text{for all} \;\; u \in [X_0,Y_0]_s, \quad s \in [0,1], \tag{2.7}$$

which proves the lemma. □

2.2. **Interpolation between subspaces of a Hilbert space.**
Let $\mathcal{K} = span\{\varphi_1, \dots, \varphi_n\}$ be a n-dimensional subspace of X and let $X_{\mathcal{K}}$ be the orthogonal complement of $\mathcal{K}$ in X in the $(\cdot,\cdot)_X$ inner product. We are interested in determining the interpolation spaces of $X_{\mathcal{K}}$ and Y, where on $X_{\mathcal{K}}$ we consider again the $(\cdot,\cdot)_X$ inner product. For certain spaces $X_{\mathcal{K}}$ and Y and $n = 1$, this problem was studied in [12]. To apply the interpolation results from the previous section we need to check that the density part of the condition (2.1) is satisfied for the pair $(X_{\mathcal{K}}, Y)$.

For $\varphi \in \mathcal{K}$, define the linear functional $\Lambda_\varphi : X \to C$, by

$$\Lambda_\varphi u := (u, \varphi)_X, \ u \in X.$$

Lemma 2.3. *The space $X_{\mathcal{K}}$ is dense in Y if and only if the following condition is satisfied:*

$$\left\{ \begin{array}{l} \Lambda_\varphi \textit{ is not bounded in the topology of } Y \\ \quad \textit{for all } \varphi \in \mathcal{K}, \ \varphi \neq 0. \end{array} \right. \tag{2.8}$$

Proof. First let us assume that the condition (2.8) does not hold. Then for some $\varphi \in \mathcal{K}$ the functional L_φ is a bounded functional in the topology induced by Y. Thus, the kernel of L_φ is a closed subspace of X in the topology induced by Y. Since $X_{\mathcal{K}}$ is contained in $Ker(L_\varphi)$ it follows that

$$\overline{X_{\mathcal{K}}}^Y \subset \overline{Ker(L_\varphi)}^Y = Ker(L_\varphi).$$

Hence $X_{\mathcal{K}}$ fails to be dense in Y.

Conversely, assume that $X_{\mathcal{K}}$ is not dense in Y, then $Y_0 = \overline{X_{\mathcal{K}}}^Y$ is a proper closed subspace of Y. Let $y_0 \in Y$ be in the orthogonal complement of Y_0, and define the linear functional $\Psi : Y \to C$, by

$$\Psi u := (u, y_0)_Y, \ u \in Y.$$

Ψ is a continuous functional on Y. Let ψ be the restriction of Ψ to the space X. Then ψ is a continuous functional on X. By Riesz Representation Theorem, there is $v_0 \in X$ such that

$$(u, v_0)_X = (u, y_0)_Y, \qquad \text{for all } u \in X. \tag{2.9}$$

Let $P_{\mathcal{K}}$ be the X orthogonal projection onto $\mathcal{K}$ and take $u = (I - P_{\mathcal{K}})v_0$ in (2.9). Since $(I - P_{\mathcal{K}})v_0 \in X_{\mathcal{K}}$ we have $((I - P_{\mathcal{K}})v_0, y_0)_Y = 0$ and

$$0 = ((I - P_{\mathcal{K}})v_0, v_0)_X = ((I - P_{\mathcal{K}})v_0, (I - P_{\mathcal{K}})v_0)_X.$$

It follows that $v_0 = P_{\mathcal{K}} v_0 \in \mathcal{K}$ and, via (2.9), that $\psi = \Lambda_{v_0}$ is continuous in the topology of Y. This is exactly the opposite of (2.8) and the proof is completed. $\square$

Remark 2.2. *The result still holds if we replace the finite dimensional subspace $\mathcal{K}$ with any closed subspace of X.*

For the next part of this section we assume that the condition (2.8) holds. By the above Lemma, the condition (2.1) is satisfied. It follows from the previous section that the operator $S_{\mathcal{K}} : D(S_{\mathcal{K}}) \subset Y \to Y$ defined by

$$(u, v)_X = (S_{\mathcal{K}} u, v)_Y \quad \text{for all } v \in X_{\mathcal{K}}, \tag{2.10}$$

has the same properties as S has. Consequently, the norm on the intermediate space $[X_{\mathcal{K}}, Y]_s$ is given by:

$$\|u\|^2_{[X_{\mathcal{K}},Y]_s} := \mathbf{c}_s^2 \int_0^\infty t^{-2s+1} \left((I + t^2 S_{\mathcal{K}}^{-1})^{-1} u, u\right)_Y \, dt. \tag{2.11}$$

Let $[X, Y]_{s,\mathcal{K}}$ denote the closure of $X_{\mathcal{K}}$ in $[X, Y]_s$. Our aim in this section is to determine sufficient conditions for φ_i's such that

$$[X_{\mathcal{K}}, Y]_s = [X, Y]_{s,\mathcal{K}}. \tag{2.12}$$

First, we note that the operators $S_{\mathcal{K}}$ and S are related by the following identity:

$$S_{\mathcal{K}}^{-1} = (I - Q_{\mathcal{K}}) S^{-1}, \tag{2.13}$$

where $Q_{\mathcal{K}} : X \to \mathcal{K}$ is the orthogonal projection onto $\mathcal{K}$. The proof of (2.13) follows easily from the definitions of the operators involved.

Next, (2.13) leads to a formula relating the norms on $[X_{\mathcal{K}}, Y]_s$ and $[X, Y]_s$. Before deriving this formula in Theorem 2.1 , we introduce some notation. Let

$$(u, v)_{X,t} := \left((I + t^2 S^{-1})^{-1} u, v\right)_X \quad \text{for all } u, v \in X. \tag{2.14}$$

and denote by M_t the Gram matrix associated with the set of vectors $\{\varphi_1, \ldots, \varphi_n\}$ in the $(\cdot,\cdot)_{X,t}$ inner product,i.e.,

$$(M_t)_{ij} := (\varphi_j, \varphi_i)_{X,t}, \ i, j \in \{1, \ldots, n\}.$$

We may assume, without loss, that M_0 is the identity matrix.

Theorem 2.1. *Let u be arbitrary in $X_{\mathcal{K}}$. Then,*

$$\|u\|^2_{[X_{\mathcal{K}},Y]_s} = \|u\|^2_{[X,Y]_s} + \mathbf{c}_s^2 \int_0^\infty t^{-(2s+1)} \left\langle M_t^{-1} d, d \right\rangle \, dt, \tag{2.15}$$

where $< \cdot, \cdot >$ is the inner product on $\mathbf{C}^n$ and d is the n-dimensional vector in $\mathbf{C}^n$ whose components are

$$d_i := (u, \varphi_i)_{X,t}, \ i = 1, \ldots, n.$$

The proof of the of the theorem can be found in [2].

For $n = 1$, let $\mathcal{K} = span\{\varphi\}$ and denote $X_{\mathcal{K}}$ by X_φ. Then, for $u \in X_\varphi$, the formula (2.15) becomes

$$\|u\|^2{}_{[X_\varphi,Y]_s} = \|u\|^2{}_{[X,Y]_s} + \mathbf{c}_s^2 \int_0^\infty t^{-(2s+1)} \frac{|(u,\varphi)_{X,t}|^2}{(\varphi,\varphi)_{X,t}} \, dt. \tag{2.16}$$

Next theorem gives sufficient conditions for (2.12) to be satisfied. Before we state the result we introduce the conditions:

(A.1) $[X_{\varphi_i}, Y]_s = [X, Y]_{s,\varphi_i}$ for $i = 1, \ldots, n$.

(A.2) There exist $\delta > 0$ and $\gamma > 0$ such that

$$\sum_{i=1}^n |\alpha_i|^2 (\varphi_i, \varphi_i)_{X,t} \le \gamma \langle M_t \alpha, \alpha \rangle \qquad \text{for all } \alpha = (\alpha_1, \ldots, \alpha_n)^{\mathbf{t}} \in \mathbf{C^n}, \ \mathbf{t} \in (\delta, \infty).$$

In [2] we give the following result:

Theorem 2.2. *Assume that, for some $s \in (0,1)$, the conditions* **(A.1)** *and* **(A.2)** *hold. Then*

$$[X_{\mathcal{K}}, Y]_s = [X, Y]_{s,\mathcal{K}}.$$

For completness we include the proof.

Proof. Let s be fixed in $(0,1)$. Since $X_{\mathcal{K}}$ is dense in both these spaces, in order to prove (2.12) it is enough to find, for a fixed s, positive constants c_1 and c_2 such that

$$c_1\|u\|_{[X,Y]_s} \le \|u\|_{[X_{\mathcal{K}},Y]_s} \le c_2\|u\|_{[X,Y]_s} \quad \text{for all } u \in X_{\mathcal{K}}. \tag{2.17}$$

The function under the integral sign in (2.15) is nonnegative, so the lower inequality of (2.17) is satisfied with $c_1 = 1$. For the upper part, we notice that, for $u \in X_{\mathcal{K}}$ and $w_{\mathcal{K}} := (I + t^2 S_{\mathcal{K}}^{-1})^{-1}u$

$$(w_{\mathcal{K}}, u)_Y = \left((I + t^2 S_{\mathcal{K}}^{-1})^{-1}u, u\right)_Y = (u,u)_Y - t^2 \left(S_{\mathcal{K}}^{-1}(I + t^2 S_{\mathcal{K}}^{-1})^{-1}u, u\right)_Y$$

$$\le (u,u)_Y \le c(s)\|u\|^2_{[X,Y]_s}$$

It was proved in [2] (Theorem 2.1) that

$$(w_{\mathcal{K}}, u)_Y = (w,u)_Y + t^{-2}\left\langle M_t^{-1}d, d\right\rangle. \tag{2.18}$$

Then, using (2.11), (2.18) and the above estimate, we have that for any positive number δ,

$$\|u\|^2_{[X_{\mathcal{K}},Y]_s} \le c(\delta,s)\|u\|^2_{[X,Y]_s} + \int_\delta^\infty t^{-2s+1}(w_{\mathcal{K}},u)^2_Y \, dt$$

$$\le c(\delta,s)\|u\|^2_{[X,Y]_s} + \int_\delta^\infty t^{-2s+1}(w,u)^2_Y \, dt + \int_\delta^\infty t^{-2s+1}\left\langle M_t^{-1}d, d\right\rangle \, dt.$$

Hence the upper inequality of (2.17) is satisfied if one can find a positive δ and $c = c(\delta)$ such that

$$\int_\delta^\infty t^{-2s+1}\left\langle M_t^{-1}d, d\right\rangle dt \le c\|u\|^2_{[X,Y]_s} \quad \text{for all } u \in X_{\mathcal{K}}. \tag{2.19}$$

From **(A.2)**, there exist $\delta > 0$ and $\gamma > 0$ such that

$$\left\langle M_t^{-1}\alpha, \alpha\right\rangle \le \gamma \sum_{i=1}^n |\alpha_i|^2 (\varphi_i, \varphi_i)^{-1}_{X,t}$$

for all $\alpha = (\alpha_1, \ldots, \alpha_n)^{\mathbf{t}} \in \mathbf{C^n}$, $t \in (\delta, \infty)$. In particular, for $\alpha_i = (u, \varphi_i)_{X,t}$, $i = 1, \ldots, n$, we obtain

$$\left\langle M_t^{-1}d, d\right\rangle \le \gamma \sum_{i=1}^n \frac{|(u,\varphi_i)_{X,t}|^2}{(\varphi_i,\varphi_i)_{X,t}} \quad \text{for all } t \in (\delta,\infty), u \in X_{\mathcal{K}},$$

where $d = (d_1, \ldots, d_n)^{\mathbf{t}}$. Thus, using the above estimate, (2.16) and **(A.1)** we have

$$\int_\delta^\infty t^{-2s+1}\left\langle M_t^{-1}d, d\right\rangle dt \le \gamma \sum_{i=1}^n \int_\delta^\infty t^{-2s+1}\frac{|(u,\varphi_i)_{X,t}|^2}{(\varphi_i,\varphi_i)_{X,t}}dt$$
$$\le \gamma \sum_{i=1}^n \int_0^\infty t^{-2s+1}\frac{|(u,\varphi_i)_{X,t}|^2}{(\varphi_i,\varphi_i)_{X,t}}dt$$
$$\le \gamma c_s^{-2}\sum_{i=1}^n \|u\|^2_{[X_{\varphi_i},Y]_s} \le \gamma c_s^{-2} n \|u\|^2_{[X,Y]_s}$$

Finally, (2.19) holds, and the result is proved. □

Remark 2.3. *By Lemma 2.3, the space $X_{\mathcal{K}}$ is dense in $[X,Y]_s$ if and only if the functionals L_φ, $\varphi \in \mathcal{K}$ are not bounded in the topology induced by $[X,Y]_s$.*

3. Interpolation between subspaces of $H^\beta(R^N)$ and $H^\alpha(R^N)$.

In this section we give a simplified proof of the main interpolation result presented in [12]. An extension to the case when the subspace of interpolation has finite codimension bigger than one is also considered.

Let $\alpha \in R$ and let $H^\alpha(R^N)$ be defined by means of the Fourier transform. For a smooth function u with compact support in R^N, the Fourier transform $\hat{u}$ is defined by

$$\hat{u}(\xi) = (2\pi)^{-N/2}\int u(x)e^{-ix\xi}\,dx,$$

where the integral is taken over the whole R^N. For u and v smooth functions the α -inner product is defined by

$$< u, v >_\alpha = \int (1+|\xi|^2)^\alpha\ \hat{u}(\xi)\overline{\hat{v}(\xi)}\,d\xi.$$

The space $H^\alpha(R^N)$ is the closure of smooth functions in the norm induced by the α -inner product. For α, β real numbers ($\alpha < \beta$), and $s \in [0,1]$ it is easy to check, using Remark 2.1, that

$$\left[H^\beta(R^N), H^\alpha(R^N)\right]_s = H^{s\alpha+(1-s)\beta}(R^N).$$

For $\varphi \in H^\beta(R^N)$, we are interested in determining the validity of the formula

$$\left[H^\beta_\varphi(R^N), H^\alpha(R^N)\right]_s = \left[H^\beta(R^N), H^\alpha(R^N)\right]_{s,\varphi}. \tag{3.1}$$

For certain functions φ the problem is studied by Kellogg in [12]. Next, we give a new proof of Kellogg's result concerning (3.1) and extend it to the case when $H^\beta_\varphi(R^N)$ is replaced by a subspace of finite codimension. First, we consider the case when $0 = \alpha < \beta$. The operator S, associated with the pair $X = H^\beta(R^N)$, $Y = H^0(R^N) = L^2(R^N)$, is given by

$$\widehat{Su} = \mu^{2\beta}\hat{u}, \quad u \in D(S) = H^{2\beta}(R^N),$$

where $\mu(\xi) = (1+|\xi|^2)^{\frac{1}{2}}$, $\xi \in R^N$. For the remaining part of this chapter, H^β denotes the space $H^\beta(R^N)$ and $\hat{H}^\beta$ is the space $\{\hat{u}\ |u \in H^\beta\}$. For $\hat{u}$, $\hat{v} \in \hat{H}^\beta$, we define the inner product and the norm by

$$(\hat{u}, \hat{v})_\beta = \int \mu^{2\beta} \hat{u}\overline{\hat{v}}\ d\zeta, \quad ||\hat{u}||_\beta = (\hat{u}, \hat{u})_\beta^{1/2}.$$

To simplify the notation, we denote the the inner products $(\cdot,\cdot)_0$ and $<\cdot,\cdot>_0$ by $(\cdot,\cdot)$ and $<\cdot,\cdot>$, respectively. The norm $||\cdot||_0$ on H^0 or $\hat{H}^0$ is simply $||\cdot||$.
Let $\phi \in \hat{H}^\beta$ be such that for some constants $\epsilon > 0$ and $c > 0$,

$$\begin{cases} |\phi(\xi) - b(\omega)\rho^{-\frac{N}{2}-2\beta+\alpha_0}| < c\rho^{-\frac{N}{2}-2\beta+\alpha_0-\epsilon} & \text{for all } \rho > 1 \\ 0 < \alpha_0 < \beta, \end{cases} \tag{3.2}$$

where $\rho \geq 0$ and $\omega \in S^{N-1}$ (the unit sphere of R^N) are the spherical coordinates of $\xi \in R^N$, and where $b(\omega)$ is a bounded measurable function on S^{N-1}, which is non zero on a set of positive measure.

Remark 3.1. *From the assumption* (3.2) *about ϕ and by using Lemma 2.3, we have that*

$$\hat{H}_\phi^\beta \text{ is dense in } \hat{H}^\alpha \quad \text{if and only if} \quad \alpha \leq \alpha_0. \tag{3.3}$$

Theorem 3.1. *(Kellogg) Let $\varphi \in H^\beta$ be such that its Fourier transform ϕ satisfies* (3.2), *and let $\theta_0 = \alpha_0/\beta$. Then*

$$\left[H_\varphi^\beta, H^0\right]_s = \left[H^\beta, H^0\right]_{s,\varphi}, \quad 0 \leq s \leq 1,\ 1-s \neq \theta_0, \tag{3.4}$$

Proof. From the way we defined $<\cdot,\cdot>_\beta$, (3.4) is equivalent to

$$\left[\hat{H}_\phi^\beta, \hat{H}^0\right]_s = \left[\hat{H}^\beta, \hat{H}^0\right]_{s,\varphi}, \quad 0 \leq s \leq 1,\ 1-s \neq \theta_0. \tag{3.5}$$

Following the proof of Theorem 2.2, we see that in order to prove (3.5), it is enough to verify (2.19) for some positive constants $c = c(s)$ and δ. Using (2.16), the problem reduces to

$$\int_\delta^\infty t^{-(2s+1)} \frac{|(\hat{u}, \phi)_{X,t}|^2}{(\phi,\phi)_{X,t}} dt \leq c\|\hat{u}\|^2_{[X,Y]_s} \qquad \text{for all } \hat{u} \in X_\phi\ ,$$

where $X = \hat{H}^\beta$ and $Y = \hat{H}^0$. Denoting $1-s = \theta$ and $\Phi(t) = (\phi,\phi)_{X,t}$, this becomes

$$I := \int_\delta^\infty t^{2\theta-3} \frac{\left|\left(\frac{\mu^{4\beta}\hat{u}}{\mu^{2\beta}+t^2}, \phi\right)\right|^2}{\left(\frac{\mu^{4\beta}\phi}{\mu^{2\beta}+t^2}, \phi\right)} dt \leq c\|\hat{u}\|^2_{\theta\beta} \qquad \text{for all } \hat{u} \in \hat{H}_\phi^\beta. \tag{3.6}$$

Using (3.2) it is easy to see that, for a large enough $\delta \geq 1$

$$\left(\frac{\mu^{4\beta}\phi}{\mu^{2\beta}+t^2}, \phi\right) \geq ct^{2(\theta_0-1)} \qquad \text{for all } \ t \geq \delta, \tag{3.7}$$

and (3.2) also implies that

$$|\phi(\xi)| < c|\rho|^{-\frac{N}{2}-2\beta+\alpha_0} \quad \text{for} \quad |\xi| > 1. \tag{3.8}$$

Before we start estimating I, let us observe that by using spherical coordinates

$$\|\hat{u}\|^2_{\theta\beta} = \int_0^\infty U^2(\rho)\, d\rho, \qquad \hat{u} \in \hat{H}^\beta_\phi\ , \tag{3.9}$$

where

$$U(\rho) := \mu(\rho)^{\theta\beta} \rho^{\frac{N-1}{2}} \Bigg(\int\limits_{|\xi|=1} |\hat{u}(\rho,\omega)|^2\, d\omega \Bigg)^{1/2}, \quad \mu(\rho) = (1+\rho^2)^{1/2}.$$

First, we consider the case $0 < \theta < \theta_0$ and set $\theta_1 := \theta_0 - \theta$. For $\hat{u} \in \hat{H}^\beta_\phi$ we have

$$\left| \left(\frac{\mu^{4\beta}\hat{u}}{\mu^{2\beta}+t^2}, \phi \right) \right|^2 = t^4 \left| \left(\frac{\mu^{2\beta}\hat{u}}{\mu^{2\beta}+t^2}, \phi \right) \right|^2.$$

Thus, by this observation and (3.7) we get

$$I \le c \int_\delta^\infty t^{3-2\theta_1} \left(\int \frac{\mu(\xi)^{2\beta}}{\mu(\xi)^{2\beta}+t^2} |\hat{u}(\xi)\phi(\xi)|\, d\xi \right)^2\, dt.$$

Then,

$$\begin{aligned} I_1 &= \int_\delta^\infty t^{3-2\theta_1} \Bigg(\int\limits_{|\xi|<1} \frac{\mu(\xi)^{2\beta}}{\mu(\xi)^{2\beta}+t^2} |\hat{u}(\xi)\phi(\xi)|\, d\xi \Bigg)^2 dt \\ &\le c \int_\delta^\infty \frac{t^{3-2\theta_1}}{t^4} \Bigg(\int\limits_{|\xi|<1} |\hat{u}(\xi)\phi(\xi)|\, d\xi \Bigg)^2 dt \le c \int_\delta^\infty t^{-(1+2\theta_1)}\, dt\ \|\hat{u}\|^2\ \|\phi\|^2 \le\ c(\theta)\|\hat{u}\|^2_{\theta\beta}. \end{aligned}$$

On the other hand, by Fubini's theorem, we have

$$\begin{aligned} I_2 &= \int_\delta^\infty t^{3-2\theta_1} \Bigg(\int\limits_{|\xi|>1} \frac{\mu(\xi)^{2\beta}}{\mu(\xi)^{2\beta}+t^2} |\hat{u}(\xi)\phi(\xi)|\, d\xi \Bigg)^2 dt \\ &= \int_\delta^\infty t^{3-2\theta_1} \Bigg(\int\limits_{|\xi|>1} \frac{\mu(\xi)^{2\beta}}{\mu(\xi)^{2\beta}+t^2} |\hat{u}(\xi)\phi(\xi)|\, d\xi \Bigg) \Bigg(\int\limits_{|\eta|>1} \frac{\mu(\eta)^{2\beta}}{\mu(\eta)^{2\beta}+t^2} |\hat{u}(\eta)\phi(\eta)|\, d\eta \Bigg) dt \\ &= \int\limits_{|\xi|>1} \int\limits_{|\eta|>1} |\hat{u}(\xi)\hat{u}(\eta)\phi(\xi)\phi(\eta)|\, \big(\mu(\xi)\mu(\eta)\big)^{2\beta} \int_\delta^\infty \frac{t^{3-2\theta_1}}{\big(\mu(\xi)^{2\beta}+t^2\big)\big(\mu(\eta)^{2\beta}+t^2\big)}\, dt\, d\eta\, d\xi. \end{aligned}$$

To estimate the last integral we use the formula

$$\int_0^{\ \infty} \frac{t^{3-2\theta}}{(a+t^2)(b+t^2)} dt = \frac{1}{\mathbf{c}^2_\theta} \frac{a^{1-\theta}-b^{1-\theta}}{a-b}, \quad 0 < \theta < 2, \quad \theta \ne 1, \quad a, b > 0. \tag{3.10}$$

The integral can be calculated by standard complex analysis tools. If $a = b$, then the right side of the above identity is replaced by $\frac{1-\theta}{\mathbf{c}^2_\theta} a^{-\theta}$. Next, by using (3.10), (3.8) and spherical coordinates $\xi = (\rho, \omega)$, $\eta = (r, \rho)$, we obtain

$$I_2 \le c(\theta) \int_1^\infty \int_1^\infty (\mu(r)\mu(\rho))^{2\beta - \beta\theta} (r\rho)^{-\frac{1}{2} - 2\beta + \alpha_0} R_{1-\theta_1}(\mu(r)^{2\beta}, \mu(\rho)^{2\beta}) U(r) U(\rho)\; d\rho\; dr,$$

where for $\alpha \in (0,1)$, $x > 0$, $y > 0$, we denote

$$R_\alpha(x, y) = \begin{cases} \frac{x^\alpha - y^\alpha}{x - y}, & for\ x \ne y \\ \alpha x^{\alpha - 1}, & for\ x = y. \end{cases}$$

The function $x \to R_\alpha(x, y)$ is decreasing on $(0, \infty)$ for each $y \in (0, \infty)$ and it is symmetric with respect to x and y.

Using this observation, we get

$$I_2 \le c(\theta) \int_1^\infty \int_1^\infty (r\rho)^{-\frac{1}{2} + \beta\theta_1} R_{1-\theta_1}(r^{2\beta},\ \rho^{2\beta})\ U(\rho) U(r)\ \ dr\ d\rho$$

$$\le c(\theta) \int_0^\infty \int_0^\infty K(r, \rho) U(r) U(\rho)\ dr\ d\rho,$$

where

$$K(r, \rho) = (r\rho)^{-\frac{1}{2} + \beta\theta_1} R_{1-\theta_1}(r^{2\beta},\ \rho^{2\beta}). \tag{3.11}$$

In order to estimate the last integral, we apply the following lemma.

Lemma 3.1. *(Schur) Suppose $K(x, y)$ is nonnegative, symmetric and homogeneous of degree -1, and f, g are nonnegative measurable functions on $(0, \infty)$. Assume that*

$$k = \int_0^\infty K(1, x) x^{-\frac{1}{2}}\ dx\ <\ \infty.$$

Then

$$\int_0^\infty \int_0^\infty K(x, y) f(x) g(y)\ dx\ dy \le k \left(\int_0^\infty f(x)^2\ dx \right)^{\frac{1}{2}} \left(\int_0^\infty g(y)^2\ dy \right)^{\frac{1}{2}}. \tag{3.12}$$

We will prove this lemma later. For the moment, we see that the function $K(x, y)$, given by (3.11), is homogeneous of degree -1, and satisfies

$$k = \int_0^\infty K(x, 1) x^{-\frac{1}{2}}\ dx\ <\ \infty.$$

Indeed

$$k = \int_0^\infty x^{-1 + \beta\theta_1}\ \frac{x^{2\beta(1-\theta_1)} - 1}{x^{2\beta} - 1}\ dx\ \ \overset{x^\beta = t}{=}\ \ \beta \int_0^\infty \frac{t^{1-\theta_1} - t^{\theta_1 - 1}}{t^2 - 1}\ dt\ <\ \infty,\ for\ 0 < \theta_1 < 1.$$

By Lemma 3.1,

$$I_2 \le c(\theta) \int_0^\infty U^2(\rho)\ d\rho\ \le\ c(\theta) \|\hat{u}\|^2_{\beta\theta}$$

and by combining the estimates I_1 and I_2, we obtain (3.6).

Let us consider now the case $\theta_0 < \theta < 1$, and let $\theta_1 = \theta - \theta_0$. Then, by using (3.7), we have

$$I \le c \int_\delta^\infty t^{2\theta_1 - 1} \left(\int \frac{\mu(\xi)^{4\beta}}{\mu(\xi)^{2\beta} + t^2} |\hat{u}(\xi)\phi(\xi)|\ d\xi \right)^2 dt.$$

The remaining part of the proof is very similar to the proof of the first case. The theorem is proved. □

Proof of Lemma 3.1. By Fubini's theorem, it follows

$$
\begin{aligned}
\int_0^\infty\int_0^\infty K(x,y)f(x)g(y)\;dx\;dy \;&=\; \int_0^\infty f(x)\Big(\int_0^\infty K(x,y)g(y)\;dy\Big)\;dx\\
&= \int_0^\infty f(x)\int_0^\infty xK(x,xt)g(xt)\;dt\;dx = \int_0^\infty f(x)\int_0^\infty K(1,t)g(xt)\;dt\;dx\\
&= \int_0^\infty K(1,t)\int_0^\infty f(x)g(xt)\;dx\;dt\\
&\le \int_0^\infty K(1,t)\Big(\int_0^\infty f(x)^2\;dx\Big)^{\frac12}\Big(\int_0^\infty g(xt)^2\;dx\Big)^{\frac12}dt\\
&\le \int_0^\infty K(1,t)t^{-\frac12}\;dt\;\Big(\int_0^\infty f(x)^2\;dx\Big)^{\frac12}\Big(\int_0^\infty g(x)^2\;dx\Big)^{\frac12}.
\end{aligned}
$$

Next we prepare for the generalization of the previous result.

Let $\phi_1,\phi_2,\dots,\phi_n\in\hat H^\beta(R^N)$ such that for some constants $\epsilon>0$ and $c>0$ we have

$$
\left\{\begin{array}{l} |\phi_i(\xi)-\tilde\phi_i(\xi)|<c\rho^{-\frac N2-2\beta+\alpha_i-\epsilon}\;for\;|\xi|>1\\ 0<\alpha_i<\beta,\;\; i=1,\dots,n,\end{array}\right. \tag{3.13}
$$

where

$$
\tilde\phi_i(\xi)=b_i(\omega)\rho^{-\frac N2-2\beta+\alpha_i},\;\;\xi=(\rho,\omega),
$$

and $b_i(\cdot)$ is a bounded measurable function on S^{N-1}, which is non zero on a set of positive measure.

Define

$$
\Phi_{ij}(t)=\Big(\frac{\mu^{4\beta}\phi_i}{\mu^{2\beta}+t^2},\phi_j\Big),\quad \tilde\phi_{ij}(t)=\Big(\frac{|\xi|^{4\beta}\tilde\phi_i}{|\xi|^{2\beta}+t^2},\tilde\phi_j\Big),\quad \theta_i=\frac{\alpha_i}{\beta},
$$

$$
[\tilde\phi_i,\tilde\phi_j]:=\frac1\beta(b_i,b_j)_\sigma\int_0^\infty\frac{x^{\theta_i}x^{\theta_j}}{x(x^2+1)}\;dx,\;i,\;j\;=1,2,\dots,n,
$$

where $(\cdot,\cdot)_\sigma$ is the inner product on $L^2(S^{N-1})$.

Clearly, $[\cdot,\cdot]$ is an inner product on $span\{\tilde\phi_i\mid i=1,2,\dots,n\}$.

Lemma 3.2. *With the above setting we have*

$$
\tilde\Phi_{ij}(t)=[\tilde\phi_i,\tilde\phi_j]t^{\theta_i+\theta_j-2} \tag{3.14}
$$

$$
|\Phi_{ij}(t)-\tilde\Phi_{ij}(t)|\le ct^{\theta_i+\theta_j-2-\eta}\;,\;\;t>\delta, \tag{3.15}
$$

for some constants $c>0$, $\eta>0$ and $\delta\ge1$.

Proof. By using spherical coordinates, we have

$$
\tilde\Phi_{ij}(t)=\int\frac{|\xi|^{4\beta}}{|\xi|^{2\beta}+t^2}\tilde\phi_i\overline{\tilde\phi_j}\;d\xi=\int_0^\infty\frac{\rho^{\alpha_i+\alpha_j-1}}{\rho^{2\beta}+t^2}\;d\rho\int_{|\xi|=1}b_i(\omega)\overline{b_j(\omega)}\;d\omega.
$$

The change of variable $\rho^\beta = tx$ in the first integral completes the proof of (3.14). The proof of (3.15) is straightforward. $\square$

Theorem 3.2. *Let $\varphi_1, \varphi_2, \dots, \varphi_n \in H^\beta$ be such that the corresponding Fourier transforms $\phi_1, \phi_2, \dots, \phi_n$ satisfy* (3.13) *and in addition, the functions $\tilde{\phi}_1, \tilde{\phi}_2, \dots, \tilde{\phi}_n$ are linearly independent.*

Let $\mathcal{K} = span\{\varphi_1, \varphi_2, \dots, \varphi_n\}$. Then

$$[H^\beta_{\mathcal{K}}, H^0]_s \ = \ [H^\beta, H^0]_{s,\mathcal{K}} \ , \ (1-s)\beta \neq \alpha_i, \ for \ i = 1, 2, \dots, n.$$

Proof. We apply the Theorem 2.2 for $X = H^\beta$, $Y = H^0$, $\mathcal{K} = span\{\varphi_1, \dots, \varphi_n\}$ and s such that $(1-s)\beta \neq \alpha_i$, $i = 1, 2, \dots, n$. By using the hypothesis (3.13) and Theorem 3.1, we get

$$[H^\beta_{\varphi_i}, H^0]_s \ = \ [H^\beta, H^0]_{s,\varphi_i}, \ for \ i = 1, 2, \dots, n.$$

So **(A1)** is satisfied. In order to verify the condition **(A2)**, we first observe that $(M_t)_{ij} = \Phi_{ij}(t)$. By denoting $D_t = diag(M_t)$, the condition **(A2)** can be written as follows:

There are $\delta > 0$ and $\gamma > 0$ such that

$$M_t - \gamma D_t \geq 0, \quad for \ all \ t \in (\delta, \infty),$$

where for a square matrix A, $A \geq 0$ means that A is a nonnegative definite matrix. From the previous lemma we obtain the behavior of $(M_t)_{ij}$ for t large:

$$(M_t)_{ij} = \big([\tilde{\phi}_i, \tilde{\phi}_j] + f_{ij}(t)\big) t^{\theta_i - 1} t^{\theta_j - 1}$$

where $|f_{ij}(t)| < ct^{-\eta}$, for $t > \delta$. Denote $\tilde{M}_t$, $\tilde{M}$ the n x n matrices defined by

$$(\tilde{M}_t)_{ij} = [\tilde{\phi}_i, \tilde{\phi}_j] + f_{ij}(t), \ (\tilde{M})_{ij} = [\tilde{\phi}_i, \tilde{\phi}_j]$$

and let $\tilde{D}_t = diag\tilde{M}_t$, $\tilde{D} = diag\tilde{M}$. Next, for $z = (z_1, z_2, \dots, z_n) \in C^n$, we have

$$\big\langle (M_t - \gamma D_t) z, z \big\rangle \ = \ \big\langle (\tilde{M}_t - \gamma \tilde{D}_t) z_t, z_t \big\rangle$$

where $< \cdot, \cdot >$ is the inner product on C^n and $(z_t)_i = z_i \ t^{\theta_i - 1}$, $i = 1, 2, \dots, n$.

Hence, the condition **(A2)** is satisfied if one can find $\gamma > 0$, $\delta > 0$, such that

$$\tilde{M}_t - \gamma \tilde{D}_t \geq 0, \qquad \text{for all } t \in (\delta, \infty).$$

On the other hand, since $\tilde{\phi}_1, \tilde{\phi}_2, \dots, \tilde{\phi}_n$ are linearly independent, $\tilde{M}$ is a symmetric positive definite matrix on C^n and

$$\lim_{\gamma \searrow 0, t \to \infty} (\tilde{M}_t - \gamma \tilde{D}_t) = \tilde{M}.$$

Therefore, there are $\gamma > 0$, $\delta > 0$ such that $\tilde{M}_t - \gamma \tilde{D}_t > 0$, for all $t \in (\delta, \infty)$, and **(A2)** holds. The result is proved by applying Theorem 2.2. $\square$

The corresponding case of interpolation between subspaces of H^β of finite codimensions and H^α, where α, β are real numbers, $\alpha < \beta$, is a direct consequence of the previous theorem.

Let $\alpha < \beta$ and $\varphi_1, \varphi_2, \dots, \varphi_n \in H^\beta$ be such that the corresponding Fourier transform $\phi_1, \phi_2, \dots, \phi_n$ satisfy for some positive constants c and ϵ,

$$\left\{ \begin{array}{l} |\phi_i(\xi) - \tilde{\phi}_i(\xi)| < c\rho^{-\frac{N}{2} - 2\beta + \gamma_i - \epsilon} \ for \ |\xi| > 1 \\ \alpha < \gamma_i < \beta, \ i = 1, \dots, n, \end{array} \right. \tag{3.16}$$

where

$$\tilde{\phi}_i(\xi) = b_i(\omega)\rho^{-\frac{N}{2}-2\beta+\gamma_i},\ \xi = (\rho,\omega),$$

and $b_i(\cdot)$ is a bounded measurable function on S^{N-1}, which is non zero on a set of positive measure.

Theorem 3.3. *Let $\varphi_1, \varphi_2, \dots, \varphi_n \in H^\beta$ be such that the corresponding Fourier transforms $\phi_1, \phi_2, \dots, \phi_n$ satisfy* (3.16)*, and in addition, the functions $\tilde{\phi}_1, \tilde{\phi}_2, \dots, \tilde{\phi}_n$ are linearly independent. Let $\mathcal{L} = span\{\varphi_1, \varphi_2, \dots, \varphi_n\}$. Then*

$$[H^\beta_{\mathcal{L}}, H^\alpha]_s \ = \ [H^\beta, H^\alpha]_{s,\mathcal{L}}\ ,\ s\alpha + (1-s)\beta \neq \gamma_i,\ \ for\ i = 1, 2, \dots, n. \tag{3.17}$$

Furthermore , if $s\alpha + (1-s)\beta < \min\{\gamma_i,\ i = 1, 2, \dots, n\}$, then

$$[H^\beta_{\mathcal{L}}, H^\alpha]_s \ = \ H^{s\alpha+(1-s)\beta}. \tag{3.18}$$

Proof. The first part follows from the main theorem 3.2 and the fact that $T : H^\alpha \to H^0$ defined by $\hat{Tu} = \mu^\alpha \hat{u},\ u \in H^\alpha$ is an isometry from H^α to $H^{\gamma-\alpha}$ for any $\gamma \in [\alpha, \beta]$.

Now let $s < \min\{\gamma_i,\ i = 1, 2, \dots, n\}$. By the first part of the theorem, in order to prove (3.18) we need only to prove that $H^\beta_{\mathcal{L}}$ is dense in $H^{s\alpha+(1-s)\beta}$. By Lemma 2.3, this is equivalent to proving that

$$\begin{cases} H^\beta \ni u \xrightarrow{\Lambda_\varphi} < u, \varphi >_\beta = (\hat{u}, \hat{\varphi})_\beta, \\ \ is\ not\ bounded\ in\ the\ topology\ of\ H^{s\alpha+(1-s)\beta}\ for\ all\ \varphi \in \mathcal{L},\ \varphi \neq 0. \end{cases} \tag{3.19}$$

For a fixed $\varphi \in \mathcal{L}$ we have $\hat{\varphi} = \sum\limits_{i=1}^{n} c_i \phi_i$.

Since $\tilde{\phi}_1, \tilde{\phi}_2, \dots, \tilde{\phi}_n$ are assumed to be linearly independent, φ fails to be a "good" function (better than $\varphi_i,\ i = 1, 2, \dots, n$). More precisely, the asymptotic expansion at infinity of $\hat{\varphi}$ is of the same type (except maybe a different b-part) with one of the functions $\tilde{\phi}_1, \tilde{\phi}_2, \dots, \tilde{\phi}_n$. Thus, it is enough to check (3.19) for $\varphi \in \{\varphi_1, \varphi_2, \dots, \varphi_n\}$.

Assuming that Λ_{φ_i} is continuous, it implies that

$$(\hat{u}, \phi_i)_\beta = (\hat{u}, f_i)_{s\alpha+(1-s)\beta}, u \in H^\beta,$$

for a function $f_i \in \hat{H}^{s\alpha+(1-s)\beta}$.
Thus, by using the density of H^β in H^s, for $s < \beta$, we get that $f_i = \mu^{2\beta}\mu^{-2(s\alpha+(1-s)\beta)}\phi_i$.

On the other hand,

$$\begin{aligned} \int \mu^{2(s\alpha+(1-s)\beta)} |f_i|^2 \, d\xi \ &= \ \int \mu^{2\beta-2s\alpha+2s\beta} |\phi_i|^2 \, d\xi \\ \geq \ c \int_\delta^\infty \rho^{2\beta-2s\alpha+2s\beta} \rho^{-N-4\beta+2\gamma_i} \rho^{N-1} d\rho \\ = \ c \int_\delta^\infty \rho^{-1+2(\gamma_i-(s\alpha+(1-s)\beta))} d\rho \ = \ \infty \end{aligned}$$

for $s\alpha + (1-s)\beta < \min\{\gamma_i,\ i = 1, 2, \dots, n\}$. This completes the proof. □

4. Shift theorem for the Biharmonic operator on polygonal domains.

Let Ω be a polygonal domain in R^2 with boundary $\partial\Omega$. Let $\partial\Omega$ be the polygonal arc $P_1P_2\cdots P_mP_1$. At each point P_j, we denote the measure of the angle P_j (measured from inside Ω) by ω_j. Let $\omega := \max\{\omega_j : j = 1, 2, \ldots, m\}$.

We consider the biharmonic problem Given $f \in L^2(\Omega)$, find u such that

$$\left\{ \begin{aligned} \Delta^2 u = f & \text{ in } \Omega, \\ u = 0 & \text{ on } \partial\Omega, \\ \tfrac{\partial u}{\partial n} = 0 & \text{ on } \partial\Omega. \end{aligned} \right. \tag{4.1}$$

Let $V = H_0^2(\Omega)$ and

$$a(u, v) := \sum_{1\le i,j\le 2} \int_\Omega \frac{\partial^2 u}{\partial x_i \partial x_j} \frac{\partial^2 v}{\partial x_i \partial x_j}\, dx,\ u, v, \in V.$$

The bilinear form a defines a scalar product on V and the induced norm is equivalent to the standard norm on $H_0^2(\Omega)$. The variational form of (4.1) is : Find $u \in V$ such that

$$a(u, v) = \int_\Omega fv\, dx \qquad \text{for all } v \in V. \tag{4.2}$$

Clearly, if u is a variational solution of (4.2), then one has $\Delta^2 u = f$ in the sense of distributions and because $u \in H_0^2(\Omega)$, the homogeneous boundary conditions are automatically fulfilled. As done in [2], the problem of deriving the shift estimate on Ω can be localized by a partition of unity so that only sectors domains or domains with smooth boundaries need to be considered. If Ω is a smooth domain, then it is known that the solution u of (4.2) satisfies

$$\|u\|_{H^4(\Omega)} \le c\|f\|, \qquad \text{for all } f \in L^2(\Omega),$$

and

$$\|u\|_{H^2(\Omega)} \le c\|f\|_{H^{-2}(\Omega)}, \qquad \text{for all } f \in H^{-2}(\Omega).$$

Interpolating these two inequalities yields

$$\|u\|_{2+2s} \le c\|f\|_{-2+2s}, \qquad \text{for all } f \in H^{-2+2s}(\Omega),\ \ 0 \le s \le 1.$$

So we have the shift theorem for all $s \in [0, 1]$. Let us consider the case of a sector domain. The threshold, s_0, below which the shift estimate for a polygonal domain holds is given, as in the Poisson problem, by the largest internal angle ω of the polygon. Thus, it is enough to consider the domain $S\omega$ defined by

$$S_\omega = \{(r, \theta),\ 0 < r < 1, -\omega/2 < \theta < \omega/2\}.$$

We associate to (4.1) and $\Omega = S_\omega$, the characteristic equation

$$\sin^2(z\omega) = z^2 \sin^2 \omega. \tag{4.3}$$

In order to simplify the exposition of the proof, we assume that

$$\sin \sqrt{\frac{\omega^2}{\sin\omega^2} - 1}\ \ne \sqrt{1 - \frac{\sin\omega^2}{\omega^2}} \tag{4.4}$$

and

$$Re z \ne 2 \text{ for any solution } z \text{ of } (4.3).$$

The restriction (4.4) assures that the equation (4.3) has only simple roots. Let $z_1, z_2, \ldots, z_n$ be all the roots of (4.3) such that $0 < Re(z_j) < 2$. It is known (see [7], [10], [13], [17]) that the solution u of (4.2) can be written as

$$u = u_R + \sum_{j=1}^{n} k_j S_j, \tag{4.5}$$

where $u_R \in H^4(\Omega)$ and for $j = 1, 2, \ldots, n$, we have $S_j(r,\theta) = r^{1+z_j} u_j(\theta)$, u_j is smooth function on $[-\omega/2, \omega/2]$ such that $u_j(-\omega/2) = u_j(\omega/2) = u_j'(-\omega/2) = u_j'(\omega/2) = 0$, $k_j = c_j \int_\Omega f \varphi_j \, dx$ and c_j is nonzero and depends only on ω. The function φ_j is called the dual singular function of the singular function S_j and $\varphi_j(r,\theta) = \eta(r)\ r^{1-z_j} u_j(\theta) - w_j$, where $w_j \in V$ is defined for a smooth truncation function η to be the solution of (4.2) with $f = \Delta^2(\eta(r)\ r^{1-z_j} u_j(\theta))$. In addition,

$$\|u_R\|_{H^4(\Omega)} \leq c\|f\|, \qquad \text{for all } f \in L^2(\Omega). \tag{4.6}$$

Next, we define $\mathcal{K} = span\{\varphi_1, \varphi_2, \ldots, \varphi_n\}$. As a consequence of the expansion (4.5) and the estimate (4.6) we have

$$\|u\|_{H^4(\Omega)} \leq c\|f\|, \qquad \text{for all } f \in L^2(\Omega)_{\mathcal{K}}. \tag{4.7}$$

Combining (4.7) with the standard estimate

$$\|u\|_{H^2(\Omega)} \leq c\|f\|_{H^{-2}(\Omega)}, \qquad \text{for all } f \in H^{-2}(\Omega),$$

we obtain, via interpolation

$$\|u\|_{[H^4(\Omega),H^2(\Omega)]_{1-s}} \leq c\|f\|_{[L^2(\Omega)_{\mathcal{K}},H^{-2}(\Omega)]_{1-s}},\ s \in [0,1]. \tag{4.8}$$

Let $s_0 = \min\{Re(z_j) \mid j = 1, 2, \ldots, n\}$. Then, we have

Theorem 4.1. *If* $0 < 2s < s_0$ *and* $\Omega = S_\omega$, *then*

$$[L^2(\Omega)_{\mathcal{K}}, H^{-2}(\Omega)]_{1-s} = [L^2(\Omega), H^{-2}(\Omega)]_{1-s}. \tag{4.9}$$

Proof. First we prove that there are operators E and R such that

$$E : L^2(\Omega) \longrightarrow L^2(R),\ E : H_0^2(\Omega) \longrightarrow H^2(R^2),$$
$$R : L^2(R^2) \longrightarrow L^2(\Omega),\ R : H^2(R^2) \longrightarrow H_0^2(\Omega)$$

are bounded operators, and $REu = u$, for all $u \in L^2(\Omega)$.
Indeed, E can be taken to be the extension by zero operator.

To define R, let $\eta = \eta(r)$ be a smooth function on $(0,\infty)$ such that $\eta(r) \equiv 1$ for $0 < r \leq 1$ and $\eta(r) \equiv 0$ for $r > 2$. Define $\alpha = \frac{\omega}{2}$, $a = \frac{\alpha}{\pi-\alpha}$ and

$$g_1(\theta) = \frac{\alpha - \pi}{\alpha}\theta + \pi, \quad g_2(\theta) = \frac{\pi - \alpha}{\alpha^2}(\alpha - \theta)^2 + \alpha, \quad \theta \in [0, \alpha].$$

Note that $g_i(0) = \pi$ and $g_i(\alpha) = \alpha$, $i = 1, 2$. For a smooth function u defined on R^2 we define $Ru := u_3$, where

Step 1. $u_1 = \eta u$.

Step 2. $u_2(r,\theta) = u_1(r,\theta) + 3u_1(1/r,\theta) - 4u_1(1/2 + 1/(2r),\theta),\ r < 1,\ \theta \in [0, 2\pi)$.

Step 3. For $0 < r < 1$

$$u_3(r,\theta) = \begin{cases} u_2(r,\theta) + au_2(r, g_1(\theta)) - (1+a)u_2(r, g_2(\theta)),\ 0 \leq \theta < \omega/2, \\ u_2(r,\theta) + au_2(r, -g_1(-\theta)) - (1+a)u_2(r, -g_2(-\theta)),\ -\omega/2 < \theta < 0. \end{cases}$$

One can check that, for $u \in H_0^2(R^2)$, $u_3 \in H_0^2(\Omega)$ and $REu = u$. The operator R can be extended by density to $L^2(R^2)$. The extended operator R satisfies all the desired properties.

Next, let ϕ_j be the Fourier transform of $E\varphi_j,\ j = 1, \ldots, n$. Using asymptotic expansion of integrals theory presented in the Appendix 5.2, we have that the functions $\{E\varphi_j,\ j = 1, \ldots, n\}$ satisfy for some positive constants c and ϵ,

$$\begin{cases} |\phi_j(\xi) - \tilde{\phi}_j(\xi)| < c\rho^{-1+(-2+s_j)-\epsilon}\ for\ |\xi| > 1 \\ -2 < -2 + s_i < 0,\ i = 1, \ldots, n, \end{cases} \tag{4.10}$$

where $s_j = Re(z_j)$ and

$$\tilde{\phi}_j(\xi) = b_i(\omega)\rho^{-1+(-2+s_j)},\ \xi = (\rho, \omega) \text{ in polar coordinates,}$$

and $b_j(\cdot)$ is a bounded measurable function on the unit circle, which is non zero on a set of positive measure. Thus, we have that the functions $\{E\varphi_j,\ j = 1, \ldots, n\}$ satisfy the hypothesis (3.16) of Theorem 3.3 with $N = 2$, $\beta = 0$, $\alpha = -2$ and $\gamma_j = -2 + s_j$, $j = 1, \ldots, n$. Denoting $\mathcal{L} := span\{E\varphi_j,\ j = 1, \ldots, n\}$, by Theorem 3.3 applied with $1 - s$ instead of s, we have that

$$[L^2(R^2)_{\mathcal{L}}, H^{-2}(R^2)]_{1-s} = [L^2(R^2), H^{-2}(R^2)]_{1-s} = H^{-2+2s}(R^2), \tag{4.11}$$

for $2s < s_0 := \min\{Re(z_j),\ j = 1, 2, \ldots, n\}$.

Finally, using (4.11), the operators E , R and Lemma 5.1 (adapted to the case when we work with subspaces of codimension $n > 1$), we conclude that (4.9) holds for $2s < s_0$.

From the estimate (4.8) and the interpolation result (4.9) we obtain

$$\|u\|_{2+2s} \le c\|f\|_{-2+2s}, \qquad \text{for all } f \in H^{-2+s}(\Omega),\ \ 0 \le 2s < s_0.$$

The above estimate still holds for the case when Ω is a polygonal domain and s_0 corresponds to the largest inner angle ω of the polygon. Figure 1 (see below) gives the graph of the function $\omega \to 2 + s_0(\omega)$ which represents the regularity threshold for the biharmonic problem in terms of the largest inner angle ω of the polygon. On the same graph we represent the the number of singular (dual singular) functions as function of $\omega \in (0, 2\pi)$. Note that if ω is bigger than 1.43π, which is an approximation for the solution in $(0, 2\pi)$ of the equation $\tan\omega = \omega$, the space $\mathcal{K}$ has the dimension six.

5. Appendix

5.1. **Appendix A. An interpolation result.** Let $\Omega \subset \widetilde{\Omega}$ be domains in R^2 and $V^1(\Omega), V^1(\widetilde{\Omega})$ be subspaces of $H^1(\Omega), H^1(\widetilde{\Omega})$, respectively. On $V^1(\Omega),\ V^1(\widetilde{\Omega})$ we consider inner products such that the induced norms are equivalent with the standard norms on $H^1(\Omega),\ H^1(\widetilde{\Omega})$, respectively. In addition, we assume that $V^1(\Omega),\ V^1(\widetilde{\Omega})$ are dense in $L^2(\Omega), L^2(\widetilde{\Omega})$, respectively. Let's denote the duals of $V^1(\Omega),\ V^1(\widetilde{\Omega})$ by $V^{-1}(\Omega),\ V^{-1}(\widetilde{\Omega})$, respectively. We suppose that there are linear operators E and R such that

$$E : L^2(\Omega) \to L^2(\widetilde{\Omega}), \quad E : V^1(\Omega) \to V^1(\widetilde{\Omega}) \ \text{ are bounded operators,} \tag{5.1}$$

$$R : L^2(\widetilde{\Omega}) \to L^2(\Omega), \quad R : V^1(\widetilde{\Omega}) \to V^1(\Omega), \ \text{ are bounded operators,} \tag{5.2}$$

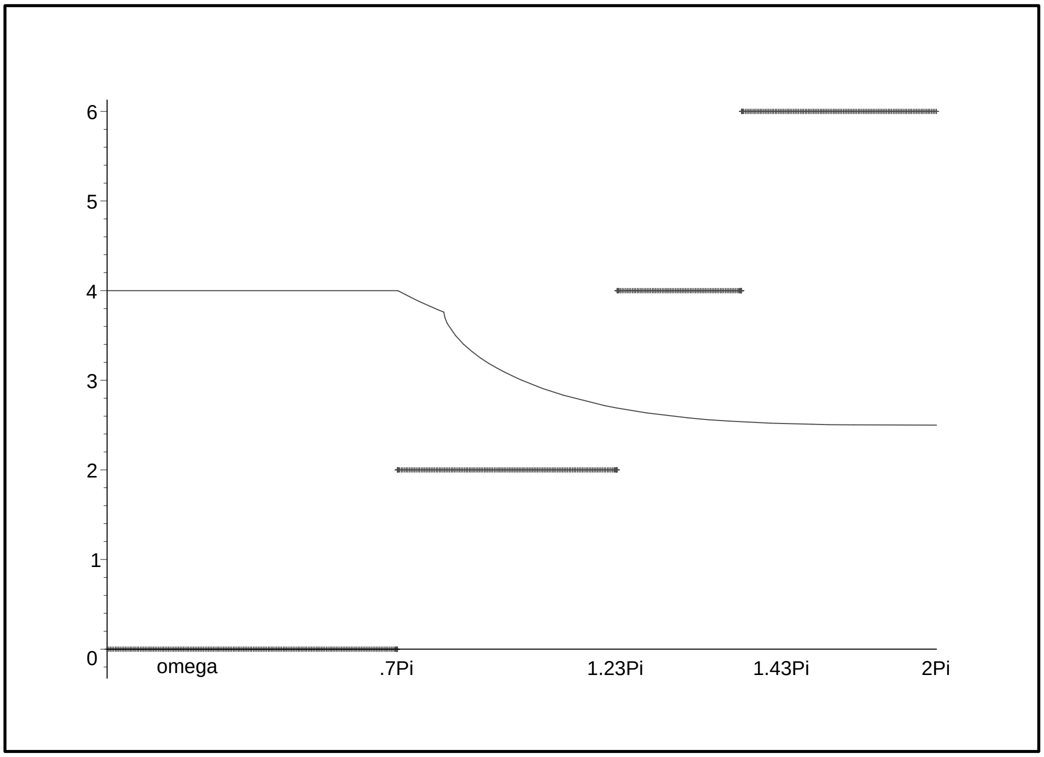

FIGURE 1. Regularity for the biharmonic problem.

$$REu = u \qquad \text{for all } u \in L^2(\Omega). \tag{5.3}$$

Let $\psi \in L^2(\Omega)$, $\widetilde{\psi} = E\psi \in L^2(\widetilde{\Omega})$ and $\theta \in (0,1)$ be such that

$$L^2(\Omega)_\psi := \{u \in L^2(\Omega) : (u,\psi) = 0\} \text{ is dense in } [L^2(\Omega), V^{-1}(\Omega)]_\theta, \tag{5.4}$$

$$L^2(\widetilde{\Omega})_{\widetilde{\psi}} := \{u \in L^2(\widetilde{\Omega}) : (u,\widetilde{\psi}) = 0\} \text{ is dense in } V^{-1}(\widetilde{\Omega}), \tag{5.5}$$

$$[L^2(\widetilde{\Omega})_{\widetilde{\psi}}, V^{-1}(\widetilde{\Omega})]_\theta = [L^2(\widetilde{\Omega}), V^{-1}(\widetilde{\Omega})]_\theta. \tag{5.6}$$

Lemma 5.1. *Using the above setting, assume that* (5.1)-(5.6) *are satisfied. Then,*

$$[L^2(\Omega)_\psi, V^{-1}(\Omega)]_\theta = [L^2(\Omega), V^{-1}(\Omega)]_\theta. \tag{5.7}$$

Proof. Using the duality , from (5.1)-(5.3) we obtain linear operators E^*, R^* such that

$$E^* : L^2(\widetilde{\Omega}) \to L^2(\Omega), \quad E^* : V^{-1}(\widetilde{\Omega}) \to V^{-1}(\Omega), \quad \text{are bounded operators,} \tag{5.8}$$

$$R^* : L^2(\Omega) \to L^2(\widetilde{\Omega}), \quad R^* : V^{-1}(\Omega) \to V^{-1}(\widetilde{\Omega}) \text{ are bounded operators,} \tag{5.9}$$

$$E^*R^*u = u \qquad \text{for all } u \in L^2(\Omega), \tag{5.10}$$

$$E^* \ \text{ maps } \ L^2(\widetilde{\Omega})_{\widetilde{\psi}} \ \text{ to } \ L^2(\Omega)_\psi, \tag{5.11}$$

$$R^* \ \text{ maps } \ L^2(\Omega)_\psi \ \text{ to } \ L^2(\widetilde{\Omega})_{\widetilde{\psi}}. \tag{5.12}$$

From (5.8) and (5.11), by interpolation, we obtain

$$\|E^* v\|_{[L^2(\Omega)_\psi, V^{-1}(\Omega)]_\theta} \le c\|v\|_{[L^2(\widetilde{\Omega})_{\widetilde{\psi}}, V^{-1}(\widetilde{\Omega})]_\theta} \qquad \text{for all } \ v \in L^2(\widetilde{\Omega})_{\widetilde{\psi}}. \tag{5.13}$$

For $u \in L^2(\Omega)_\psi$, let $v := R^* u$. Then, using (5.12), we have that $v \in L^2(\widetilde{\Omega})_{\widetilde{\psi}}$. Taking $v := R^* u$ in (5.13) and using (5.10), we get

$$\|u\|_{[L^2(\Omega)_\psi, V^{-1}(\Omega)]_\theta} \le c\|R^* u\|_{[L^2(\widetilde{\Omega})_{\widetilde{\psi}}, V^{-1}(\widetilde{\Omega})]_\theta} \qquad \text{for all } \ u \in L^2(\Omega)_\psi. \tag{5.14}$$

Also, from the hypothesis (5.6), we deduce that

$$\|R^* u\|_{[L^2(\widetilde{\Omega})_{\widetilde{\psi}}, V^{-1}(\widetilde{\Omega})]_\theta} \le c\|R^* u\|_{[L^2(\widetilde{\Omega}), V^{-1}(\widetilde{\Omega}]_\theta} \qquad \text{for all } \ u \in L^2(\Omega)_\psi. \tag{5.15}$$

From (5.9), again by interpolation, we have in particular

$$\|R^* u\|_{[L^2(\widetilde{\Omega}), V^{-1}(\widetilde{\Omega}]_\theta} \le c\|u\|_{[L^2(\Omega), V^{-1}(\Omega)]_\theta} \qquad \text{for all } \ u \in L^2(\Omega)_\psi. \tag{5.16}$$

Combining (5.14)-(5.16), it follows that

$$\|u\|_{[L^2(\Omega)_\psi, V^{-1}(\Omega)]_\theta} \le c\|u\|_{[L^2(\Omega), V^{-1}(\Omega)]_\theta} \qquad \text{for all } \ u \in L^2(\Omega)_\psi. \tag{5.17}$$

The reverse inequality of (5.17) holds because $L^2(\Omega)_\psi$ is a closed subspace of $L^2(\Omega)$. Thus, the two norms in (5.17) are equivalent for $u \in L^2(\Omega)_\psi$. From the assumption (5.4), $L^2(\Omega)_\psi$ is dense in both spaces appearing in (5.7). Therefore, we obtain (5.7). □

Remark 5.1. *The proof does not change if we consider $\Omega \subset \widetilde{\Omega}$ to be domains in R^N and H^1 is replaced by any other Sobolev space of positive integer order k.*

5.2. **Appendix B. Asymptotic expansion for the Fourier integrals.** For a more general presentation of asymptotic expansion of functions defined by integrals see [4], [8], [19].

Integrals of the form

$$\int_a^b e^{ixt} f(t)\ dt,$$

are called Fourier integrals. We shall present the asymptotic behavior as $x \to \infty$ of the Fourier integrals for a particular type of function f. If ϕ and ψ are two real functions defined on the interval $I = (0, \infty)$ and ψ is a strictly positive function on I, we write $\phi = O(\psi)$ as $x \to \infty$ if ϕ/ψ is bounded on an interval $I = (\delta, \infty)$ for a positive δ, and $\phi = o(\psi)$ as $x \to \infty$ if $\lim\limits_{x\to\infty} \phi/\psi = 0$.

Theorem 5.1. *Let ϕ be a continuously differentiable function on the interval $[a, b]$ and $\lambda \in (0, 1)$.*

a) *If $\phi(b) = 0$ then*

$$\int_a^b e^{ixt}(t-a)^{\lambda-1}\phi(t)\ dt = -\Gamma(\lambda)\phi(a)e^{\frac{\pi}{2}i(\lambda-2)}e^{ixa}x^{-\lambda} + O(x^{-1}).$$

b) *If* $\phi(a) = 0$ *then*

$$\int_a^b e^{ixt}(b-t)^{\lambda-1}\phi(t)\ dt = \Gamma(\lambda)\phi(b)e^{-\frac{\pi}{2}i\lambda}e^{ixb}x^{-\lambda} + O(x^{-1}).$$

Here Γ is the Euler's gamma function.

Remark 5.2. *The result holds for* $\lambda = 1$ *provided* $O(x^{-1})$ *is replaced by* $o(x^{-1})$ *in the above formulas.*

The proof of Theorem 5.1 can be found in [8] Section 2.8.

Next we study the asymptotic behavior of the Fourier transforms of the dual singular functions which appear in Section 4. To this end, let $\eta = \eta(r)$ be a smooth real function on $[0,\infty)$ such that $\eta(r) \equiv 0$ for $r > 3/4$ and let $u = u(\theta)$ be a sufficiently smooth real function on $[0, 2\pi]$. For any non-zero $s \in (-1,1)$ we define

$$u(x) = \eta(r)r^s u(\theta), \qquad x = (r,\theta) \in R^2,$$

and

$$\Phi(\rho,\omega) = 2\pi\bar{\hat{u}}(\xi) = \int_{R^2} e^{ix\cdot\xi}u(x)\ dx, \qquad \xi = (\rho,\omega) \in R^2,$$

where (r,θ) and (ρ,ω) are the polar coordinates of x and ξ, respectively. One can easily see that

$$\Phi(\rho,\omega) = \int_0^1\int_0^{2\pi} \eta(r)r^{1+s}u(\theta)e^{ir\rho\cos(\theta-\omega)}\ d\theta dr. \tag{5.18}$$

To study the asymptotic behavior of Φ for large ρ, we use the technique of [12] to reduce the double integral to a single integral. For a fixed ω, we consider the line $r\cos(\theta-\omega) = t$ in the x plane and denote by $l(t,\omega)$ the intersection of this line with the unit disk. Next, in the (r,t) variables the integral (5.18) becomes:

$$\Phi(\rho,\omega) = \int_{-1}^1 g(t)e^{it\rho}\ dt, \tag{5.19}$$

where

$$g(t) = \int_{l(t,\omega)} \frac{\eta(r)r^{1+s}}{\sqrt{r^2-t^2}}u(\theta)\ dr,$$

$\theta = \omega + \cos^{-1}(t/r)$, if $\theta \in [\omega, \omega+\pi]$ and $\theta = \omega - cos^{-1}(t/r)$, if $\theta \in [\omega-\pi, \omega]$. The function g is continuous differentiable on $[-1,1]$ and $g(-1) = g(1) = 0$. Thus, from (5.19) we have

$$\Phi(\rho,\omega) = \frac{i}{\rho}\int_{-1}^1 g'(t)e^{it\rho}\ dt \tag{5.20}$$

The function g can be described as

$$g(t) = \int_{|t|}^1 \frac{\eta(r)r^{1+s}}{\sqrt{r^2-t^2}}u(\omega+\cos^{-1}(t/r))\ dr + \int_{|t|}^1 \frac{\eta(r)r^{1+s}}{\sqrt{r^2-t^2}}u(\omega-\cos^{-1}(t/r))\ dr,$$

and the integral in (5.20) can be split in $\int_{-1}^0 + \int_0^1$. Thus, the function Φ is defined by a sum of four integrals. We will use Theorem 5.1 in order to find the asymptotic behavior as $\rho \to \infty$ of each of the integrals. We shall present the estimate for only one of them.

Let $s \in (-1, 0)$ be fixed and let h be the function defined by

$$h(t) = \int_t^1 \frac{\eta(r) r^{1+s}}{\sqrt{r^2 - t^2}} u(\theta)\ dt,$$

where $\theta = \omega + cos^{-1}(t/r)$. We apply Theorem 5.1 for the integral

$$\int_0^1 h'(t) e^{it\rho}\ dt. \tag{5.21}$$

To compute $h'(t)$ (by Leibnitz's formula) we set $x = r - t$ to rewrite h as

$$h(t) = \int_{x=0}^{1-t} \frac{\eta(x+t)(x+t)^{1+s}}{\sqrt{x}\sqrt{x+2t}} u(\theta)\ dx.$$

This leads to

$$h'(t) = \int\limits_0^{1-t} \Bigg[\left(\frac{(1+s)\eta(x+t)(x+t)^s}{\sqrt{x}\sqrt{x+2t}} + \frac{\eta'(x+t)(x+t)^{1+s}}{\sqrt{x}\sqrt{x+2t}} - \frac{\eta(x+t)(x+t)^{1+s}}{\sqrt{x}(x+2t)^{3/2}} \right) u(\theta)$$
$$- \frac{\eta(x+t)(x+t)^s}{x+2t} u'(\theta) \Bigg]\ dx$$

Going back to the r variable, via the change $r = x + t$, we get

$$h'(t) = \int\limits_t^1 \Bigg[\left(\frac{(1+s)\eta(r) r^s}{\sqrt{r^2 - t^2}} + \frac{\eta'(r) r^{1+s}}{\sqrt{r^2 - t^2}} - \frac{\eta(r) r^{1+s}}{\sqrt{r^2 - t^2}(r+t)} \right) u(\theta)$$
$$- \frac{\eta(r) r^s}{r+t} u'(\theta) \Bigg]\ dr$$

A new change of variable $r = yt$ leads to the fact that $h'(t) = t^s \phi(t)$, where the function ϕ is continuous differentiable on $[0, 1]$, $\phi(0)$ is in general not zero and $\phi(1) = 0$. According to Theorem 5.1 (with $\lambda = 1 + s$) we have that

$$\int_0^1 h'(t) e^{it\rho}\ dt = b_1(\omega)\rho^{-1-s} + O(\rho^{-1}), \tag{5.22}$$

where the constant in the term $O(\rho^{-1})$ is bounded uniformly in ω. Therefore, from (5.19) and (5.22), for the case $s \in (-1, 0)$ we obtain that

$$\Phi(\rho, \omega) = b(\omega)\rho^{-2-s} + O(\rho^{-2}), \tag{5.23}$$

where the constant in the term $O(\rho^{-2})$ is bounded uniformly in ω. By Remark 5.2, (5.23) holds for $s = 0$ provided $O(\rho^{-2})$ is replaced be $o(\rho^{-2})$. The case $s \in (0, 1)$ can be treated in a similar way. Since $h'(1) = 0$, one can easily see that in fact we have $g'(1) = 0$ and $g'(-1) = 0$. Then, from (5.19) we get

$$\Phi(\rho, \omega) = \frac{-1}{\rho^2} \int_{-1}^1 g''(t) e^{it\rho}\ dt. \tag{5.24}$$

All the considerations for g used in the case $s \in (-1, 0)$ can be reproduced in the case $s \in (0, 1)$ for the functions g' in order to get

$$\Phi(\rho, \omega) = b(\omega)\rho^{-2-s} + O(\rho^{-3}), \tag{5.25}$$

where the constant in the term $O(\rho^{-3})$ is bounded uniformly in ω.

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