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On the Convergence of Difference Approximations to Weak Solutions of Dirichlet's Problem

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I. Introduction

This paper is concerned with the Dirichlet problem for Poisson's equation, generalized in a certain way, and the convergence properties of related finite difference approximations. The problem considered may be written formally as

$$\begin{aligned} \Delta u &= F \quad \text{in} \quad R \\ u &= 0 \quad \text{on} \quad \partial R, \end{aligned} \tag{1.1}$$

where R is a bounded open subset of the N-dimensional euclidian space E_N, ∂R is its boundary, Δ is the Laplace operator and F is an element of $(L_\infty)'$. $\big((L_\infty)'$ is the dual space of L_∞, the space of Lebegue measurable, essentially bounded functions on $R.\big)$ The precise meaning of (1.1) will be given in the next section.

A convergence theorem for difference approximations for the classical problem (1.1) (i.e. F and ∂R sufficiently regular) was given by COURANT, FRIEDRICHS and LEWY [6] and estimates for rates of convergence by GERSHGORIN [7] and others. Recently some rather refined estimates have been given by BRAMBLE, HUBBARD and THOMÉE [2]. CÉA [4] has considered general second order self adjoint elliptic operators, more general boundary conditions but with $F \in L_2$.

The third section contains some definitions and discrete a priori inequalities and in the following section is given an existence and uniqueness theorem for our problem which itself seems to be new. The proof of existence is done by means of a finite difference method since, almost as a biproduct, we can make assertions as to convergence of related sequences of difference approximations. Several convergence theorems are given.

The fifth section contains a study of the generalized Green's function and from the convergence properties of the sequence of "discrete Green's functions" is proved pointwise convergence when $F \in L_q$, $q > N/2$, and uniform convergence if, in addition, the boundary is of class C^2.

II. Formulation of the Problem

In order to formulate precisely our problem we need the following definitions. Let $C_0^\infty(R)$ be the class of real functions, each of which is infinitely differentiable and has its support contained in R. The Hilbert space $\overset{\circ}{H}_1$ is defined as the

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completion of $C_0^\infty(R)$ with respect to the norm

$$\|v\|_{H_1}^2 = \int_R v^2 dx + \sum_{i=1}^{N} \int_R \left(\frac{\partial v}{\partial x_i}\right)^2 dx. \tag{2.1}$$

The class V is defined as follows:

$$V = \{\phi \mid \phi \in \mathring{H}_1;\ \Delta\phi \in C_0^\infty(R)\}. \tag{2.2}$$

If B is a linear topological space and B' is its dual then if $v \in B$ and $f \in B'$ the notation $\langle v, f\rangle$ means the value of f at v.

We now may state our problem.

Problem (D). Given $F \in (L_\infty)'$, find $u \in L_p$, $1 \leqq p < \frac{N}{N-2}$ such that

$$\int_R u \,\Delta\phi \, dx = \langle \phi, F\rangle \tag{2.3}$$

for all $\phi \in V$.

We remark that it will be shown that $V \subset L_\infty$ so that (2.3) is well defined. Also note that when F is a sufficiently smooth function and ∂R is sufficiently regular, problem (D) is just the classical Dirichlet problem.

III. Definitions, Notations and Lemmas

Let E_{Nh} be those points of E_N of the form $(i_1 h, \ldots, i_N h)$ where $h > 0$ and $i_1, \ldots, i_N$ are integers, i.e. the "mesh" points. Let $\overline{R}_h = R \cap E_{Nh}$ and $S_\varrho(x)$ be the sphere with center $x = (x_1, \ldots, x_N)$ and radius ϱ; i.e. $S_\varrho(x) = \{y \mid |x - y| \leqq \varrho\}$. Let

$$R_h = \overline{R}_h \cap \{x \mid S_{\sqrt{N}h}(x) \subset R\}.$$

For any function v defined on E_{Nh} the $2N+1$ point discrete Laplace operator is defined as

$$\Delta_h v(x) = h^{-2} \sum_{i=1}^{N} [v(x + h e_i) + v(x - h e_i) - 2v(x)], \tag{3.1}$$

where e_i is the vector with 1 in the i-th position and 0 in the others. We will need two extensions of v to all of E_N. First we define $\tilde{v}$ to be the following:

$$\tilde{v}(y) = v(x) \quad \text{for} \quad y \in C_h(x) \tag{3.2}$$

where $C_h(x) = \{y \mid x_i - h/2 \leqq y_i < x_i + h/2,\ i = 1, \ldots, N\}$. Let V_i be the forward difference operator defined by $V_i v(x) = \frac{v(x + e_i h) - v(x)}{h}$. Then for each point y of the cube $C'_h(x) = \{y \mid x_i \leqq y_i < x_i + h,\ i = 1, \ldots, N\}$, we define $v'(y)$, $x \in E_{Nh}$ as follows:

$$v'(y) = \left[\prod_{i=1}^{N} \left(1 + (y_i - x_i) V_i\right)\right] v(x). \tag{3.3}$$

This function is continuous in E_N, linear in each of its variables in each cube $C'_h(x)$ and $v'(x) = v(x)$ for $x \in E_{Nh}$ (c.f. COMINCIOLI [5]). It is also obvious that there is a constant C independent of h and v such that

$$\|v'\|_{H_1}^2 \leqq C \left\{\left(h^N \sum_{E_{Nh}} v^2\right) + \sum_{i=1}^{N} \left(h^N \sum_{E_{Nh}} (V_i v)^2\right)\right\}. \tag{3.4}$$

Thus if v has bounded support then $v' \in H_1$. (H_1 is the space obtained by completing the space of infinitely differentiable functions with respect to the norm (2.1).)

Let us now consider the discrete Green's function defined by

$$\begin{aligned} \Delta_{h,x} G_h(x, y) &= -h^{-N}\delta(x, y), \qquad x \in R_h, \\ G_h(x, y) &= 0, \qquad x \in E_{Nh} - R_h \end{aligned}$$

for $y \in E_{Nh}$. It is well known that G_h exists and is unique. It was shown in [3] that if we define $w(x)$ as

$$w(x) = \begin{cases} 1/\gamma_2 \ln\left[\dfrac{d_0^2 + \alpha h^2}{|x|^2 + \alpha h^2}\right], & N = 2 \\ \dfrac{1}{(N-2)\gamma_N}\left[|x|^2 + \alpha h^2\right]^{\frac{2-N}{2}}, & N \geqq 3 \end{cases} \tag{3.5}$$

then

$$0 \leqq G_h(x, y) \leqq w(x - y), \tag{3.6}$$

for a certain choice of γ_N, α and d_0 which is independent of h. From this it is easily seen that the following lemma is true.

Lemma 1. For $1 \leqq p < \frac{N}{N-2}$ there is a constant C which depends only on p and the diameter of R such that

$$\Big(h^N \sum_{y \in R_h} |G_h(x, y)|^p\Big)^{1/p} \leqq C \tag{3.7}$$

for all x.

This lemma follows immediately by applying (3.6) and estimating the result by making comparisons with the analogous integrals.

Now let us consider the function $\phi_h(x)$ defined on E_{Nh} as the solution of

$$\begin{aligned} \Delta_h \phi_h(x) &= (\Delta\phi)_h(x) \equiv \frac{1}{h^N}\int_{C_h(x)} \Delta\phi(y)\,dy, \qquad x \in R_h \\ \phi_h(x) &= 0, \qquad x \in E_{Nh} - R_h, \end{aligned} \tag{3.8}$$

where ϕ is an arbitrary function in the class V. It is well known that

$$\phi_h(x) = -h^N \sum_{y \in R_h} G_h(x, y)\,(\Delta\phi)_h(y) \tag{3.9}$$

and it follows immediately from Hölder's inequality and Lemma 1 that

$$\|\tilde{\phi}_h\|_{L_\infty} \leqq C\Big(h^N \sum_{R_h} |(\Delta\phi)_h|^q\Big)^{1/q}, \qquad q > \frac{N}{2}, \tag{3.10}$$

But from (3.8) we have the following:

Lemma 2. For any $\phi \in V$

$$\|\tilde{\phi}_h\|_{L_\infty} \leqq C\|\Delta\phi\|_{L_q}, \qquad q > \frac{N}{2}.$$

where C does not depend on h or ϕ.

Here and in the sequel we use C for a generic constant not necessarily the same in any two places.

Let us consider $v=\phi_h$ in (3.4). Since, from the definition of ϕ_h, ϕ'_h has support contained in R we have

$$\|\phi'_h\|^2_{H_1} \leqq C\left\{h^N \sum_{R_h} \phi_h^2 + \sum_{i=1}^{N}\left(h^N \sum_{E_{Nh}} (\nabla_i \phi_h)^2\right)\right\}, \tag{3.11}$$

and hence $\phi'_h \in \mathring{H}_1$. But it is well known that there is a constant C independent of h and ϕ'_h such that

$$h^N \sum_{R_h} \phi_h^2 \leqq C\, h^N \sum_{E_{Nh}} (\nabla_i \phi_h)^2. \tag{3.12}$$

Also from partial summation we have

$$h^N \sum (\nabla_i \phi_h)^2 = -h^N \sum_{R_h} \phi_h \Delta_h \phi_h. \tag{3.13}$$

(The notation $\sum$ with nothing written below will always mean summation over E_{Nh}.) An immediate consequence of (3.11)—(3.12), Schwarz's inequality and the definition of ϕ_h is the following lemma.

Lemma 3. There exists a constant C, independent of ϕ and h such that

$$\|\phi'_h\|_{H_1} \leqq C\|\Delta\phi\|_{L_2} \tag{3.14}$$

for all $\phi\in V$.

IV. Existence, Uniqueness and Convergence

We are now in a position to prove the following.

Theorem 1. There exists one and only one solution of problem (D).

Proof. Uniqueness. Let u_1 and u_2 be any two solutions. Then if $v=u_1-u_2$ we have that $\int_R v\Delta\phi\, dx=0$, $\forall\phi\in V$. We want to show that $\int_R v\psi\, dx=0$, $\forall\psi\in C_0^\infty(R)$, i.e. that $\Delta\phi=\psi$ has a solution in V for each $\psi\in C_0^\infty(R)$. Consider the equation

$$-\sum_{i=1}^{N}\int_R \frac{\partial\bar\phi}{\partial x_i}\frac{\partial\chi}{\partial x_i}\, dx=\int\chi\psi\, dx, \qquad \forall\chi\in\mathring{H}_1. \tag{4.1}$$

It is well known that (4.1) has a unique solution $\bar\phi\in\mathring{H}_1$. But by Weyl's lemma there exists ϕ such that $\Delta\phi=\psi$ and $\phi=\bar\phi$ almost everywhere. Thus $\phi\in V$ and we have $\int_R v\psi\, dx=0$, $\forall\psi\in C_0^\infty(R)$. Since $v\in L_p$ for some $p\geqq 1$ it follows that $v=0$ in L_p.

Existence. In order to prove the existence it is sufficient to prove that there exists a constant C depending only on q and R such that

$$\|\phi\|_{L_\infty} \leqq C\|\Delta\phi\|_{L_q}, \quad q>\frac{N}{2} \tag{4.2}$$

for all $\phi\in V$. For, let us consider the linear functional

$$T(\Delta\phi)=\langle\phi, F\rangle \tag{4.3}$$

defined on the linear subspace of L_q, $q<\infty$, defined by $\{\psi|\psi=\Delta\phi,\ \phi\in V\}$. It follows from (4.2) that T is well defined and in fact continuous. Thus from the Hahn-Banach theorem T may be extended as a continuous linear functional to the whole space L_q. Since the linear functionals on L_q can be represented as integrals it follows that a function u exists in L_p, $1<p<\frac{N}{N-2}$ such that

$$\int u\,\Delta\phi\,dx=\langle\phi,F\rangle,\qquad \forall\phi\in V.$$

To prove (4.2) we use the difference method. Let $\phi\in V$. Applying Lemma 2 and using the weak compactness of bounded sets in L_p, $1<p<\infty$ or more generally, reflexive Banach spaces, it follows that there exists a sequence $\{h_n\}$ such that $h_n\to 0$ as $n\to\infty$ and $\tilde\phi_{h_n}\to\tilde\phi$ weakly in L_p for $1<p<\infty$. Clearly

$$\|\tilde\phi\|_{L_\infty}\leqq C\|\Delta\phi\|_{L_q},\qquad q>\frac{N}{2},\tag{4.4}$$

i.e. the inequality of Lemma 2 holds in the limit. Applying Lemma 3 and noting that $\phi_h'\in\mathring{H}_1$, it follows again from the weak compactness of bounded subsets of $\mathring{H}_1$ that there exists $\phi'\in\mathring{H}_1$ such that a subsequence of $\{\phi_{h_n}'\}$ (call it again $\{\phi_{h_n}'\}$) converges weakly in $\mathring{H}_1$ to ϕ'. We will show that $\phi'=\phi$ and that $\tilde\phi=\phi'$.

Now for h sufficiently small we have, for any $\psi\in C_0^\infty(R)$,

$$h^N\sum\phi_h\Delta_h\psi=h^N\sum\psi(\Delta\phi)_h.\tag{4.5}$$

From the uniform boundedness of ϕ_h and $(\Delta\phi)_h$ it follows easily that

$$\int_R\phi_{h_n}'\Delta\psi\,dx=\int_R\psi\Delta\phi\,dx+o(1)\tag{4.6}$$

as $h_n\to 0$. But from the weak convergence of ϕ_{h_n}' to ϕ' we have

$$\int_R\phi'\,\Delta\psi\,dx=\int_R\psi\,\Delta\phi\,dx.\tag{4.7}$$

Clearly for all $\psi\in C_0^\infty(R)$ and $\phi\in V$

$$\int_R\phi\,\Delta\psi\,dx=\int_R\psi\,\Delta\phi\,dx\tag{4.8}$$

and hence

$$\int_R(\phi-\phi')\,\Delta\psi\,dx=0,\qquad\forall\psi\in C_0^\infty(R).\tag{4.9}$$

Now $\phi-\phi'\in\mathring{H}_1$ and applying Weyl's lemma there exists $W=\phi-\phi'$ almost everywhere in R and such that $\Delta W=0$. But this implies that $W=0$. Hence $\phi'=\phi$ almost everywhere.

To show that $\tilde\phi=\phi'=\phi$, almost everywhere, we again consider an arbitrary $\psi\in C_0^\infty(R)$. Now

$$\int_R(\tilde\phi-\phi)\,\psi\,dx=\int_R(\tilde\phi-\tilde\phi_{h_n})\,\psi\,dx+\int_R(\phi_{h_n}'-\phi)\,\psi\,dx+\int_R(\tilde\phi_{h_n}-\phi_{h_n}')\,\psi\,dx.\tag{4.10}$$

The first two terms clearly tend to zero as $n\to\infty$ since $\tilde\phi$ and ϕ are the respective weak limits of $\{\tilde\phi_{h_n}\}$ and $\{\phi_{h_n}'\}$. Because of the smoothness of ψ the last term is easily seen to satisfy

$$\left|\int_R (\tilde{\phi}_{h_n} - \phi'_{h_n}) \psi \, dx\right| \leqq C h \|\Delta\phi\|_{L_2}. \tag{4.11}$$

Thus $\int_R (\tilde{\phi} - \phi) \psi \, dx = 0$, $\forall \psi \in C_0^\infty(R)$, and hence $\tilde{\phi} = \phi$ almost everywhere from which (4.2) follows. This completes the proof.

We shall consider now the case in which $F \in L_1$ so that we may write the Eq. (3.2) in the form

$$\int_R u \Delta\phi \, dx = \int_R \phi F \, dx. \tag{4.12}$$

To define an approximating difference problem we take

$$(F)_h(x) = h^{-N} \int_{C_h(x)} F(y) \, dy, \qquad x \in R_h \tag{4.13}$$

and consider u_h as the solution of

$$\begin{aligned} \Delta u_h(x) &= (F)_h(x), & x &\in R_h \\ u_h(x) &= 0, & x &\in E_{Nh} - R_h. \end{aligned} \tag{4.14}$$

Now

$$u_h(x) = -h^N \sum_{y \in R_h} G_h(x, y) (F)_h(y). \tag{4.15}$$

For any p such that $1 \leqq p < \frac{N}{N-2}$ we obtain

$$|u_h(x)|^p \leqq h^N \sum_{y \in R_h} G_h(x, y) \, |u_h(x)|^{p-1} |(F)_h(y)|. \tag{4.16}$$

Summing both sides of (4.16) with respect to x we have

$$h^N \sum |u_h|^p \leqq h^N \sum_{y \in R_h} |(F)_h(y)| \, h^N \sum_{x \in R_h} G_h(x, y) \, |u_h(x)|^{p-1}. \tag{4.17}$$

Using Hölder's inequality, the symmetry of G_h and Lemma 1 we obtain

$$(h^N \sum |u_h|^p)^{1/p} \leqq C h^N \sum_{y \in R_h} |(F)_h(y)| \tag{4.18}$$

from which follows immediately

$$\|\tilde{u}_h\|_{L_p} \leqq C \|F\|_{L_1}, \qquad 1 \leqq p < \frac{N}{N-2}. \tag{4.19}$$

Again we can extract a sequence, (call it again $\{h_n\}$) such that $\tilde{u}_{h_n} \to u^*$ weakly in L_p, $1 < p < \frac{N}{N-2}$. But

$$\int_R \tilde{u}_{h_n} \Delta\phi \, dx = h_n^N \sum u_{h_n} (\Delta\phi)_{h_n} = h_n^N \sum \phi_{h_n} (F)_{h_n} = \int_R \tilde{\phi}_{h_n} F \, dx. \tag{4.20}$$

Now the left hand side tends to $\int_R u^* \Delta\phi \, dx$ as $n \to \infty$. Since $F \in L_1$ we can approximate F by a sequence $\{F_m\}$ such that $F_m \in C_0^\infty(R)$ for all m and $\lim_{m \to \infty} \int_R |F - F_m| \cdot dx = 0$. Hence

$$\left|\int_R (\tilde{\phi}_{h_n} - \phi) F \, dx\right| \leqq \left|\int (\tilde{\phi}_{h_n} - \phi) F_m \, dx\right| + \left|\int (\tilde{\phi}_{h_n} - \phi)(F - F_m) \, dx\right|. \tag{4.21}$$

Now since $\tilde{\phi}_{h_n}$ and ϕ are bounded we can choose m so large that the last term of (4.21) is as small as we like, say $\varepsilon/2$. Then we may choose n so large that the first term is less than $\varepsilon/2$. Hence it follows that

$$\int_R \tilde{\phi}_{h_n} F\,dx \to \int_R \phi F\,dx \quad \text{as} \quad h_n \to 0. \tag{4.22}$$

Thus

$$\int_R u^* \Delta\phi\,dx = \int_R \phi F\,dx$$

and because of the uniqueness $u^* = u$. Also from the uniqueness it follows that every subsequence co ıverges weakly to u. Thus we have proved

Theorem 2. Let $F \in L_1$. Then $\{\tilde{u}_h\}$ defined by (4.14) converges weakly to u in L_p, for $1 \leqq p < \frac{N}{N-2}$, as $h \to 0$.

In order to show that in fact $\{u_h\}$ converges strongly to u we again approximate F by $\{F_n\}$ in such a way that $F_n \in C_0^\infty(R)$ for each n and

$$\lim_{n\to\infty} \int_R |F - F_n|\,dx = 0.$$

Now let u_n be the solution of problem (D) with F replaced by F_n and u_{nh} be the solution of (4.14) with F replaced by F_n.

Now by the triangle inequality

$$\|\tilde{u}_h - u\|_{L_p} \leqq \|\tilde{u}_h - \tilde{u}_{nh}\|_{L_p} + \|\tilde{u}_{nh} - u_n\|_{L_p} + \|u_n - u\|_{L_p}. \tag{4.23}$$

From (4.19) it certainly follows that

$$\|\tilde{u}_h - \tilde{u}_{nh}\|_{L_p} \leqq C\,\|F - F_n\|_{L_1}$$

and since by Theorem 2, $u - u_n$ is the weak limit of $\tilde{u}_h - \tilde{u}_{nh}$ as $h \to 0$ we also have

$$\|u_n - u\|_{L_p} \leqq C\,\|F - F_n\|_{L_1}, \qquad 1 \leqq p < \frac{N}{N-2}.$$

Hence given $\varepsilon > 0$ we can choose n such that $\|F - F_n\|_{L_1} \leqq \varepsilon/4C$ and hence

$$\|\tilde{u}_h - u\|_{L_p} \leqq \|\tilde{u}_{nh} - u_n\|_{L_p} + \varepsilon/2. \tag{4.24}$$

Now since $F_n \in L_2$ it follows from the results of Céa [4] that

$$\lim_{h\to 0} \|\tilde{u}_{nh} - u_n\|_{L_2} = 0.$$

But $u_n \in V$ so that from Lemma 2 and (4.2) $\tilde{u}_{nh} - u_n$ is bounded. Hence we may choose h so small that

$$\|\tilde{u}_{nh} - u_n\|_{L_p} < \varepsilon/2 \quad \text{for any} \quad p \geqq 1.$$

Thus $\|\tilde{u}_h - u\|_{L_p} < \varepsilon$ and we have

Theorem 3. Let $F \in L_1$. Then $\{\tilde{u}_h\}$ defined by (4.14) converges strongly to u in L_p for $1 \leqq p < \frac{N}{N-2}$ as $h \to 0$.

We want to consider now the more general case in which $F\in(L_\infty)'$. Let $M_h(x)$ be the function of y for each $h>0$ and each $x\in R_h$ defined as

$$M_h(x)=\begin{cases}h^{-N}, & y\in C_h(x)\\ 0, & y\notin C_h(x).\end{cases}$$

In this case we define $u_h(x)$ to be the solution of

$$\begin{aligned}\Delta_h u_h(x) &= \langle M_h(x), F\rangle, && x\in R_h\\ u_h(x) &= 0, && x\notin R_h.\end{aligned} \tag{4.25}$$

By the same techniques as before it is easily seen that

$$\|\tilde u_h\|_{L_p}\leqq C\,\|F\|_{(L_\infty)'},\qquad 1\leqq p<\frac{N}{N-2}. \tag{4.26}$$

As before we obtain a subsequence $\{\tilde u_{h_n}\}$ such that $\tilde u_{h_n}\to u^*$ weakly in L_p, $1\leqq p<\frac{N}{N-2}$. In analogy with (4.20) one easily verifies that

$$\int_R \tilde u_{h_n}\Delta\phi\,dx = h_n^N\sum u_{h_n}(\Delta\phi)_{h\,n} = h_n^N\sum \phi_{h_n}\langle M_{h_n}(x), F\rangle = \langle\tilde\phi_{h_n}, F\rangle. \tag{4.27}$$

As before

$$\lim_{n\to\infty}\int_R \tilde u_{h_n}\Delta\phi\,dx=\int_R u^*\Delta\phi\,dx.$$

The question is: When does $\langle\tilde\phi_{h_n}, F\rangle\to\langle\phi, F\rangle$ as $n\to\infty$? The next two theorems give sufficient conditions.

Theorem 4. Let $F\in(L_\infty)'$ and suppose that F has the property that, for some compact subset Ω of R, $\langle v, F\rangle=0$ for all $v\in L_\infty$ which vanish on Ω. Then $\{\tilde u_h\}$ converges to u weakly in L_p, $1\leqq p<\frac{N}{N-2}$ as $h\to 0$.

In order to complete the proof we need to show that if Ω is any compact subset of R then $\tilde\phi_{h_n}\to\phi$ uniformly on Ω, at least for a subsequence of $\{h_n\}$. But this is not difficult since it is shown in [1] that for any open subset Ω' whose closure is contained in R there is a constant $K(\Omega')$ such that

$$\max_{\Omega'}|\nabla_i\phi_{h_n}|\leqq K(\Omega')\left(\|\tilde\phi_{h_n}\|_{L_\infty}+\max_{i,R}|\nabla_i\Delta\phi|\right). \tag{4.28}$$

The right hand side is clearly bounded so that if we take $\Omega'\supset\Omega$ it follows from the definition of ϕ'_{h_n} that ϕ'_{h_n} has first difference quotients which are uniformly bounded on Ω', the bound not depending on h_n for h_n sufficiently small. Thus by the Ascoli-Arzelà theorem there is a subsequence, call it again $\{\phi'_{h_n}\}$, which converges uniformly (to ϕ) on Ω. But by the definitions of $\tilde\phi_h$ and ϕ'_h and (4.28) it follows that $\sup_\Omega|\tilde\phi_{h_n}-\phi'_{h_n}|\to 0$ and hence $\tilde\phi_{h_n}\to\phi$ uniformly on Ω as $n\to\infty$.

Thus, as before, we see that $\langle\tilde\phi_{h_n}, F\rangle\to\langle\phi, F\rangle$ as $n\to\infty$, that $u^*=u$ and that every subsequence converges to u.

In the next theorem, instead, we impose some regularity on ∂R.

Theorem 5. Let $F\in(L_\infty)'$ and $\partial R\in C^2$. Then $\{\tilde u_h\}$ converges to u weakly in L_p, $1\leqq p<\frac{N}{N-2}$, as $h\to 0$.

To complete the proof it suffices to remark that it was shown recently in a paper by BRAMBLE, HUBBARD and THOMÉE [2] that if $\partial R \in C^2$ the sequence of difference approximations, defined slightly differently from that of (3.8), in fact converges uniformly, with the error tending to zero quadratically. By exactly the same considerations it can be shown that the solution of (3.8) satisfies

$$\sup_R |\tilde{\phi}_h - \phi| \leqq C h. \tag{4.29}$$

Clearly we can state, from the above considerations, the following result.

Theorem 6. Let $F \in (L_\infty)'$ and $\partial R \in C^2$. Then for any $\psi \in C_0^\infty(R)$

$$\left| \int_R (\tilde{u}_h - u) \psi \, dx \right| \leqq C h \|F\|_{(L_\infty)'},$$

where C depends on ψ but not on h.

V. Further Convergence Results

In order to study further convergence properties we introduce the generalized Green's function. For any point x in R, $G(x, y)$ is defined as the solution of

$$\phi(x) = -\int_R G(x, y) \Delta \phi(y) \, dy \tag{5.1}$$

for all $\phi \in V$. From Theorem 1, $G(x, y)$ exists and is unique and as a function of y belong to L_p, $1 \leqq p < \frac{N}{N-2}$. At this point we note the interesting special case of Theorem 4.

Corollary of Theorem 4. $\tilde{G}_h(x, y)$ converges weakly to $G(x, y)$ as a function of y in L_p, $1 \leqq p < \frac{N}{N-2}$ as $h \to 0$, for each fixed x in R. By $\tilde{G}_h(x, y)$ we mean that $\tilde{G}_h(x, y) = G_h(x_0, y_0)$ if $(x, y) \in C_h(x_0) \times C_h(y_0)$, $(x_0, y_0) \in E_{Nh} \times E_{Nh}$.

The functional F in this case is the so called "Dirac delta function", i.e. the linear functional on V defined by $\langle \phi, F \rangle = \phi(x)$ and extended to L_∞ by the Hahn-Banach theorem. Clearly there is an extension F which has properties required in Theorem 4 so the corollary follows. We want to show the symmetry of G. In order to prove this we need the following lemma.

Lemma 4. The Green's function $G(x, y)$ belongs to $L_p(R \times R)$, $1 \leqq p < \frac{N}{N-2}$.

Proof. For $h > 0$, $G_h(x, y)$, defined in Section III satisfies

$$\|\tilde{G}_h\|_{L_p(R \times R)} \leqq C, \quad 1 \leqq p < \frac{N}{N-2}, \tag{5.2}$$

where C does not depend on h. This follows from (3.6). Thus again since for $1 < p < \frac{N}{N-2}$, $L_p(R \times R)$ is a reflexive Banach space we can extract a subsequence $\{\tilde{G}_{h_n}\}$ such that $\tilde{G}_{h_n} \to G^*$ weakly in $L_p(R \times R)$. We shall show that $G^* = G$ (in $L_p(R \times R)$). Let ψ_1 and ψ_2 belong to $C_0^\infty(R)$. We consider

$$\begin{aligned} &\int_R \psi_1(x) \left[\int_R \big(G(x, y) - G^*(x, y)\big) \psi_2(y) \, dy \right] dx \\ &\quad = \int_R \psi_1(x) \left[\int_R \big(G(x, y) - \tilde{G}_{h_n}(x, y)\big) \psi_2(y) \, dy \right] dx \\ &\qquad + \int_{R \times R} \big(\tilde{G}_{h_n}(x, y) - G^*(x, y)\big) \psi_1(x) \psi_2(y) \, dx \, dy, \end{aligned} \tag{5.3}$$

the last integral being written as an integral over $R\times R$. That this is permissable follows from the theorem of FUBINI-TONELLI (c.f. [8], p. 18) since $\widetilde{G}_h - G^* \in L_p(R\times R)$. It is clear from the definition of G^* that the last term on the right of (5.3) tends to zero as $n\to\infty$. The first term has the form

$$\int_R \psi_1(x)\,\big(\phi(x) - \tilde{\phi}_{h_n}(x)\big)\,dx$$

where $\phi\in V$ and ϕ_{h_n} is defined as in (3.8) with $\Delta\phi=\psi_2$. But from the weak convergence in L_p of $\tilde{\phi}_{h_n}$ to ϕ, as was shown in the proof of Theorem 1, it follows that this term also tends to zero as $n\to\infty$. Thus

$$\int_R \psi_1(x)\left[\int_R \big(G(x,y) - G^*(x,y)\big)\,\psi_2(y)\,dy\right]dx = 0$$

for all ψ_1 and ψ_2 in $C_0^\infty(R)$, which implies that $G(x,y)=G^*(x,y)$ for almost all (x,y) in $R\times R$. This completes the proof of Lemma 4. We now can prove the symmetry relation

Lemma 5. $G(x,y) = G(y,x)$.

Proof. $G(x,y)$ is the weak limit in $L_p(R\times R)$, $1<p<\frac{N}{N-2}$, of the sequence $\{\widetilde{G}_{h_n}\}$ as $n\to\infty$ and for each n, $\widetilde{G}_{h_n}(x,y) = \widetilde{G}_{h_n}(y,x)$.

We can now prove the following representation.

Lemma 6. Let $F\in L_1$. Then

$$u(x) = -\int_R G(x,y)\,F(y)\,dy \tag{5.4}$$

is the solution of problem (D).

(We shall only use this lemma here when $F\in L_q$, $q>N/2$ but since it is true for $q=1$ and $L_q\subset L_1$ we prove it in that case.)

Proof. Since $G(x,y)$ is integrable in $R\times R$ it follows from the Fubini-Tonelli theorem that for $\phi\in V$, $\int_R |G(x,y)|\,|\Delta\phi(x)|\,dx$ is integrable, since $G(x,y)$ is the weak limit in L_p, $1\leqq p<\frac{N}{N-2}$ of $G_h(x,y)$ we conclude from (3.7) and the symmetry of G that its L_p norm as a function of x is also bounded. Hence

$$\int_R |F(y)|\left[\int_R |G(x,y)|\,|\Delta\phi(x)|\,dx\right]dy$$

is finite and again, using the Fubini-Tonelli theorem, (5.1) and Lemma 5, we have

$$\begin{aligned}\int_R \phi F\,dx &= -\int_R F(y)\left[\int_R G(x,y)\,\Delta\phi(x)\,dx\right]dy\\ &= \int_R \Delta\phi(x)\left[-\int_R G(x,y)\,F(y)\,dy\right]dx = \int_R u\,\Delta\phi\,dx,\end{aligned}$$

which proves the lemma.

With the preceding lemma we can prove the following convergence theorem.

Theorem 7. Let $F\in L_q$, $q>N/2$. Then $\{\tilde{u}_h\}$ converges pointwise to u in R as $h\to 0$.

Proof. We can write, using Lemma 6 and the definition of u_h

$$u_h(x) - u(x) = \int_R \left(G(x, y) - \tilde{G}_h(x, y)\right) F(y)\, dy.$$

Now for each fixed x in R, $\tilde{G}_h(x, \cdot)$ converges weakly to $G(x, \cdot)$ in L_p for $1 < p < \frac{N}{N-2}$. The theorem follows immediately.

In case the boundary is somewhat regular we can prove

Theorem 8. Let $F \in L_q$, $q > N/2$ and $\partial R \in C^2$. Then $\{\tilde{u}_h\}$ converges uniformly to u as $h \to 0$.

Proof. Let $F_n \in C_0^\infty(R)$ for $n = 1, 2, \ldots$ and $\{F_n\}$ converge strongly to F in L_q, $q > N/2$. Then

$$\begin{aligned}\tilde{u}_h(x) - u(x) = &\int_R \left(G(x, y) - \tilde{G}_h(x, y)\right) F_n(y)\, dy\\ &+ \int_R \left(G(x, y) - \tilde{G}_h(x, y)\right) \left(F(y) - F_n(y)\right) dy.\end{aligned}$$

Hence

$$|\tilde{u}_h(x) - u(x)| \leqq |\tilde{\phi}_{nh}(x) - \phi_n(x)| + \|\tilde{G}_h(x, \cdot) - G(x, \cdot)\|_{L_p} \|F - F_n\|_{L_q} \tag{5.5}$$

$q > \frac{N}{2}$, $\frac{1}{p} + \frac{1}{q} = 1$. In (5.5) $\phi_n(x)$ is defined as $-\int_R G(x, y)\, F_n(y)\, dy$ and ϕ_{nh} correspondingly. Since $F_n \in C_0^\infty(R)$, $\phi_n \in V$ and as is discussed in the proof of Theorem 5, $\phi_{nh} \to \phi_n$ uniformly as $h \to 0$ for each fixed n. Now from (3.7) it follows that $\|\tilde{G}_h(x, \cdot) - G(x, \cdot)\|_{L_p}$, $1 \leqq p < \frac{N}{N-2}$ is uniformly bounded in x and h. Clearly if we take n sufficiently large and then h small we can make the right hand side less than any preassigned $\varepsilon > 0$. This proves Theorem 8.

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