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DISCRETE ANALOGUES OF THE DIRICHLET PROBLEM WITH ISOLATED SINGULARITIES*

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1. Introduction. This paper is a study of the effect on the rate of convergence of the solution of a finite difference analogue to its limiting function in the Dirichlet problem for Poisson's equation resulting from the presence of an isolated singularity.

Let $\mathbf{R}$ be a bounded domain in E_n with boundary, $\partial\mathbf{R}$, and closure $\bar{\mathbf{R}} = \mathbf{R} \cup \partial\mathbf{R}$. The singularity in the solution $u(\mathbf{x})$ of the Dirichlet problem

$$(1.1)\qquad \begin{aligned} -\Delta u(\mathbf{x}) &= f(\mathbf{x}), \qquad \mathbf{x} \in \mathbf{R},\\ u(\mathbf{x}) &= g(\mathbf{x}), \qquad \mathbf{x} \in \partial\mathbf{R}, \end{aligned}$$

is placed at the origin, $\mathbf{o} \in \bar{\mathbf{R}}$. Otherwise we assume that

$$(1.2)\qquad u \in C^M(\bar{\mathbf{R}} - \mathbf{o}),$$

where in the cases discussed $M = 4$ or 6.

If $\mathbf{k} = (k_1, \cdots, k_n)$ and $|\mathbf{k}| = \sum_{i=1}^n k_i$, where k_i are nonnegative integers, then we denote a particular partial derivative of $u(\mathbf{x})$ at the point $\mathbf{x} = (x^1, \cdots, x^n)$ by

$$D^{\mathbf{k}}u(\mathbf{x}) \equiv \frac{\partial^{|\mathbf{k}|}u(\mathbf{x})}{(\partial x^1)^{k_1} \cdots (\partial x^n)^{k_n}}.$$

Since our estimates depend on the behavior of certain of the first M derivatives as $\mathbf{x} \to \mathbf{o}$, we shall assume that for a given integer m and a given real number λ, $0 < \lambda \leqq 1$, subject to the condition

$$(1.3)\qquad \beta = m + \lambda > 2 - n,$$

and for $0 \leqq |\mathbf{k}| \leqq M$,

$$(1.4)\qquad |D^{\mathbf{k}}u(\mathbf{x})| \leqq K \begin{cases} 1 & \text{if } |\mathbf{k}| \leqq m,\\ |\mathbf{x}|^{\beta - |\mathbf{k}|} & \text{if } m + 1 \leqq |\mathbf{k}| \leqq M, \end{cases}$$

where $|\mathbf{x}|$ is the Euclidean distance, $\{\sum_{i=1}^n |x^i|^2\}^{1/2}$, and K is a constant which depends on the data.

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In each of the problems discussed we place an equally spaced mesh on $\mathbf{R}$ with mesh size h and formulate a discrete analogue whose solution we call $u(\mathbf{x}, h)$. The "discretization error" $e(\mathbf{x}, h)$ is then given by

$$e(\mathbf{x}, h) \equiv u(\mathbf{x}, h) - u(\mathbf{x}),$$

where $\mathbf{x}$ ranges over the points of the mesh.

In §§2 and 3 the difference schemes with $O(h^2)$ local errors are posed and discussed. Following this, an inequality (3.5) which is basic to the error estimate is proved and then applied to obtain the rate of convergence. Throughout the paper we make a minor assumption (3.1) on the placement of the mesh which is always satisfied if the singularity is located at the center of a mesh hypercube. The following theorem is then proved.

THEOREM 3.1. *Let $u(\mathbf{x})$ satisfy the conditions (1.2) and (1.4). Then for every $\epsilon > 0$ there exists a constant $K(\epsilon)$ depending on the data and ϵ such that*

$$|e(\mathbf{x}, h)| \leqq K(\epsilon) \begin{cases} h^{\beta+n-2-\epsilon} |\mathbf{x}|^{\epsilon+2-n} & \text{if } 2 - n < \beta \leqq 4 - n, \\ h^2 |\mathbf{x}|^{\beta-2} & \text{if } 4 - n < \beta < 2, \\ h^2 & \text{if } 2 < \beta, \end{cases}$$

where $K(\epsilon)$ may become unbounded as $\epsilon \to 0$. If $n \geqq 3$, then the last inequality becomes $2 \leqq \beta$.

This point dependent estimate of the error shows, in particular, that we can expect convergence on a compact subset excluding the singularity, even if $u(\mathbf{x}) = |\mathbf{x}|^{2-n+\delta}$ for every $\delta > 0$. Hence convergence on compact subsets is lost when uniqueness in the Dirichlet problem is lost. Moreover, it follows from the estimate that $O(h^2)$ convergence in maximum norm is achieved when $u \in C^{2+\delta}$ in some neighborhood of the origin for any $\delta > 0$. Finally, we see that convergence in maximum norm with an error estimate is true when $u \in C^{\lambda}$ in some neighborhood of the singularity, i.e., when u is Hölder continuous there.

In §4 these results are extended to a finite difference analogue whose rate of convergence was shown in [1] to be $O(h^4)$ when $u \in C^6(\bar{\mathbf{R}})$.

In §5 sufficient conditions are given on the data to guarantee that the weak solution of the Dirichlet problem satisfies the hypotheses of the convergence theorem.

In §6 these results are applied to the exterior Dirichlet problem. It is shown how a difference scheme can be given for which estimates of the order of convergence can be derived (Theorem 6.1).

Certain of the results of this paper were given in a preliminary form in a talk by Hubbard [4].

2. Preliminaries. To define a discrete problem we place a square mesh on $\mathbf{R}$ with mesh constant h, and denote by R the set of mesh points in $\mathbf{R}$.

The set of points at which the mesh lines cut $\partial\mathbf{R}$ we call ∂R. We further distinguish between two kinds of mesh points in R. Define $\mathbf{h}_i = (0, \cdots, 0, h, 0, \cdots, 0)$, $i = 1, \cdots, n$, where the only nonzero entry in $\mathbf{h}_i$ is the mesh constant h in the ith position. We call the $2n$ points $\mathbf{x} \pm \mathbf{h}_i$ "neighbors" of $\mathbf{x}$ and denote this set by $N(\mathbf{x})$.

We define the sets

$$R' = \{\mathbf{x} \mid \mathbf{x} \in R,\ N(\mathbf{x}) \cap \mathbf{R} = N(\mathbf{x})\},$$
$$R^* = R - R'.$$

The process of discretizing (1.1) is that of formulating a system of simultaneous linear equations in which we associate one equation with each point of $\bar{R} = R \cup \partial R$. The unknown vector $u(\mathbf{x}, h)$ has a component corresponding to each point of $\bar{R}$. The finite difference analogue which we choose to analyze in detail is only one of several which have been defined and whose errors have been considered by various authors under standard regularity assumptions on $u(\mathbf{x})$ (see [1], [2]).

If $\mathbf{x} \in R'$, the operator Δ_h is defined to be the usual $(2n + 1)$-point approximation to Δ given by

$$\Delta_h V(\mathbf{x}) \equiv \sum_{i=1}^{n} V_{x^i\bar{x}^i}(\mathbf{x}), \tag{2.1}$$

where

$$V_{x^i\bar{x}^i}(\mathbf{x}) = h^{-2}\{V(\mathbf{x} + \mathbf{h}_i) - 2V(\mathbf{x}) + V(\mathbf{x} - \mathbf{h}_i)\}.$$

Then if we define $\mathbf{N}(\mathbf{x})$ to be the set of line segments connecting $\mathbf{x}$ with the points of $N(\mathbf{x})$ and if $u \in C^4(\mathbf{N}(\mathbf{x}) \cap \mathbf{R})$, the "local" error is given by

$$\Delta_h u(\mathbf{x}) - \Delta u(\mathbf{x}) = \frac{h^2}{24}\left\{\frac{\partial^4 u(\xi_i)}{(\partial x^i)^4} + \frac{\partial^4 u(\eta_i)}{(\partial x^i)^4}\right\}, \tag{2.2}$$

where $x^i - h < \xi_i^{\,i} < x^i < \eta_i^{\,i} < x^i + h$ and $x^j = \xi_i^{\,j} = \eta_i^{\,j}$ for $j \neq i$.

If $\mathbf{x} \in R^*$, the use of the preceding definition for Δ_h would necessarily involve points which do not belong to $\bar{R}$. We avoid this by defining Δ_h in terms of nearby points of ∂R as did Shortley and Weller [6]. This leads to a $(2n + 1)$-point difference operator (2.1) where the divided second differences are suitably defined using only points of $\bar{R}$. More precisely, let the set $N(\mathbf{x})$ be composed of the $2n$ nearest neighbors in $\bar{R}$ of a point $\mathbf{x} \in R^*$, i.e., $\mathbf{x} - \beta_i\mathbf{h}_i$ and $\mathbf{x} + \alpha_i\mathbf{h}_i$, where α_i, β_i are real numbers, $0 < \alpha_i, \beta_i \leqq 1$. We then define

$$\begin{aligned} V_{x^i\bar{x}^i}(\mathbf{x}) \equiv \frac{2}{(\alpha_i + \beta_i)h^2}\Bigg\{&\frac{V(\mathbf{x} + \alpha_i\mathbf{h}_i)}{\alpha_i} \\ &- \left(\frac{1}{\alpha_i} + \frac{1}{\beta_i}\right)V(\mathbf{x}) + \frac{V(\mathbf{x} - \beta_i\mathbf{h}_i)}{\beta_i}\Bigg\}. \end{aligned} \tag{2.3}$$

Once again we note that if $u \in C^3(\mathbf{N}(\mathbf{x}) \cap \mathbf{R})$,

$$\Delta_h u(\mathbf{x}) - \Delta u(\mathbf{x}) = \frac{h}{3} \sum_{i=1}^{n} \left\{ \left(\frac{\alpha_i^2}{\alpha_i + \beta_i} \right) \frac{\partial^3 u(\eta_i)}{(\partial x^i)^3} - \left(\frac{\beta_i^2}{\alpha_i + \beta_i} \right) \frac{\partial^3 u(\xi_i)}{(\partial x^i)^3} \right\}, \tag{2.4}$$

where $x^i - \beta_i h < \xi_i^i < x^i < \eta_i^i < x^i + \alpha_i h$ and $x^j = \xi_i^j = \eta_i^j$ for $j \neq i$.

The finite difference analogue of the Dirichlet problem which we shall now analyze is given by

$$\begin{aligned} -\Delta_h u(\mathbf{x}, h) &= f(\mathbf{x}), \qquad \mathbf{x} \in R, \\ u(\mathbf{x}, h) &= g(\mathbf{x}), \qquad \mathbf{x} \in \partial R. \end{aligned} \tag{2.5}$$

The matrix of the above linear system has been examined in some detail by the first two authors [1] and we now summarize the relevant results.

If $W(\mathbf{x})$ is a mesh function with the property

$$\begin{aligned} -\Delta_h W(\mathbf{x}) &\geqq 0, \qquad \mathbf{x} \in R, \\ W(\mathbf{x}) &\geqq 0, \qquad \mathbf{x} \in \partial R, \end{aligned}$$

then

$$W(\mathbf{x}) \geqq 0, \qquad \mathbf{x} \in \bar{R}.$$

This is sometimes called the "discrete maximum principle". From this we immediately deduce the uniqueness and hence the existence of $u(\mathbf{x}, h)$.

We then define the "discrete Green's function", $G(\mathbf{x}, \mathbf{y}, h)$, as the solution of the linear system

$$\begin{aligned} -\Delta_{h,\mathbf{x}} G(\mathbf{x}, \mathbf{y}) &= h^{-n} \delta(\mathbf{x}, \mathbf{y}), \qquad \mathbf{x} \in R, \\ G(\mathbf{x}, \mathbf{y}) &= \delta(\mathbf{x}, \mathbf{y}), \qquad \mathbf{x} \in \partial R, \end{aligned}$$

for each value of the parameter $\mathbf{y} \in \bar{R}$, where

$$\delta(\mathbf{x}, \mathbf{y}) = \begin{cases} 1 & \text{if } \mathbf{x} = \mathbf{y}, \\ 0 & \text{if } \mathbf{x} \neq \mathbf{y}. \end{cases}$$

It follows immediately from the discrete maximum principle that

$$G(\mathbf{x}, \mathbf{y}) \geqq 0. \tag{2.6}$$

An arbitrary mesh function, $W(\mathbf{x})$, defined on $\bar{R}$, satisfies the representation formula

$$W(\mathbf{x}) = h^n \sum_{\mathbf{y} \in R} G(\mathbf{x}, \mathbf{y})[-\Delta_h W(\mathbf{y})] + \sum_{\mathbf{y} \in \partial R} G(\mathbf{x}, \mathbf{y}) W(\mathbf{y}), \tag{2.7}$$

which, in the language of matrices, amounts to representing symbolically the solution of a linear system in terms of the inverse matrix.

The following inequalities were established in [1] for $n = 2$:

$$
\begin{aligned}
&h^n \sum_{\mathbf{y} \in R} G(\mathbf{x}, \mathbf{y}) \leqq K, \\
(2.8) \qquad &h^n \sum_{\mathbf{y} \in R^*} G(\mathbf{x}, \mathbf{y}) \leqq h^2, \\
&\sum_{\mathbf{y} \in \partial R} G(\mathbf{x}, \mathbf{y}) \leqq 1,
\end{aligned}
$$

where, as before, the constant K depends only on $u(\mathbf{x})$ and its derivatives, that is, on the data but not on h. Throughout this paper the constant K will be interpreted in this same manner, and in the various arithmetical processes performed on K we shall not hesitate to replace various functions of K again with K where this is appropriate.

To obtain the classical estimates for the error we substitute $e(\mathbf{x}, h)$ into the representation formula (2.7). It then follows immediately from (2.6), (2.8) and the triangle inequality that

$$
|e(\mathbf{x}, h)| \leqq K \max_{\mathbf{y} \in R'} |\Delta_h u(\mathbf{x}) - \Delta u(\mathbf{x})| + h^2 \max_{\mathbf{y} \in R^*} |\Delta_h u(\mathbf{x}) - \Delta u(\mathbf{x})|.
$$

The local error estimates (2.2) and (2.4) then imply the uniform estimate

$$
|e(\mathbf{x}, h)| \leqq Kh^2, \qquad \mathbf{x} \in \bar{R},
$$

under the assumption that $u \in C^4(\bar{\mathbf{R}})$.

3. The basic estimates. We shall assume that the mesh is placed on $\bar{\mathbf{R}}$ in such a manner that there exists a constant, a, for which

$$
(3.1) \qquad \mathbf{x} \in \bar{R}, \quad \mathbf{y} \in N(\mathbf{x}) \quad \text{implies} \quad |\mathbf{y}| \geqq ah.
$$

If the origin is always placed at the center of a mesh hypercube, then we may take

$$
a = \tfrac{1}{2}\sqrt{n-1}.
$$

The local errors (2.2) and (2.4) now imply the estimates

$$
(3.2) \qquad |\Delta_h u(\mathbf{x}) - \Delta u(\mathbf{x})| \leqq Kh^2(|\mathbf{x}|^{\beta-4} + 1), \qquad \mathbf{x} \in R',
$$

and

$$
(3.3) \qquad
\begin{aligned}
&|\Delta_h u(\mathbf{x}) - \Delta u(\mathbf{x})| \\
&\leqq Kh \begin{cases} 1 & \text{if} \quad m = 3, \\ (|\mathbf{x}|^{\beta-3} + 1) & \text{if} \quad 2 - n \leqq m \leqq 2, \quad \mathbf{x} \in R^*. \end{cases}
\end{aligned}
$$

To obtain the desired estimate of the discretization error in this case we once again substitute $e(\mathbf{x}, h)$ into the representation formula (2.7) and apply the triangle inequality together with the local error estimates (3.2) and (3.3) just obtained, to conclude that

$$(3.4) \qquad |e(\mathbf{x}, h)| \leqq Kh^n \sum_{\mathbf{y}\in R'} G(\mathbf{x}, \mathbf{y})[h^2 |\mathbf{y}|^{\beta-4}] + Kh^n \sum_{\mathbf{y}\in R^*} G(\mathbf{x}, \mathbf{y})[h |\mathbf{y}|^{\beta-3}].$$

In the above inequality we have assumed that $\beta - 3 < 0$ since otherwise one or both of the above sums could be bounded by the techniques of the previous section.

Hence we see that the study of the order of convergence at the point $\mathbf{x}$ in this case involves estimates of certain singular sums over R' and R^*. Accordingly, we shall prove the following inequalities from which the error estimate can be derived.

LEMMA 3.1. *If the mesh is placed on R so that* (3.1) *is satisfied, then for any β in the range* $2 - n < \beta$,

$$(3.5) \qquad \begin{aligned} h^n \sum_{\mathbf{y}\in R'} G(\mathbf{x}, \mathbf{y})|\mathbf{y}|^{\beta-2} &\leqq K|\mathbf{x}|^{\beta}, && \beta < 0, \\ h^n \sum_{\mathbf{y}\in R^*} G(\mathbf{x}, \mathbf{y})|\mathbf{y}|^{\beta} &\leqq Kh^2|\mathbf{x}|^{\beta}, && \beta < 0. \end{aligned}$$

Proof. Define

$$(3.6) \qquad \begin{aligned} \rho(\mathbf{x})^2 &= |\mathbf{x}|^2 + \alpha h^2, \qquad \alpha > 0, \\ \Psi(\mathbf{x}) &= \begin{cases} \gamma\rho(\mathbf{x})^{\beta} & \text{for } \mathbf{x} \in R, \\ 0 & \text{for } \mathbf{x} \in \partial R, \end{cases} \end{aligned}$$

where certain conditions will be imposed on α, γ in the course of the proof.

If $\mathbf{x} \in R$ and $\xi \in \mathbf{N}(\mathbf{x})$, then for a given $k^2 < 1$ it is possible to determine α so that

$$(3.7) \qquad k^{-2}\rho(\mathbf{x})^2 \geqq \rho(\xi)^2 \geqq k^2\rho(\mathbf{x})^2.$$

In fact, if $\xi = (x^1, \cdots, x^i + \delta h, \cdots, x^n)$, $-1 \leqq \delta \leqq 1$, then

$$\begin{aligned} \rho(\xi)^2 &= (x^i + \delta h)^2 + |\mathbf{x}|^2 - (x^i)^2 + \alpha h^2 \\ &= \rho(\mathbf{x})^2 + 2\delta x^i h + \delta^2 h^2. \end{aligned}$$

Hence,

$$\begin{aligned} \rho(\xi)^2 - k^2\rho(\mathbf{x})^2 &\geqq (1 - k^2)|\mathbf{x}|^2 - 2|x^i|h + (1 - k^2)\alpha h^2 \\ &\geqq [(1 - k^2)\alpha - (1 - k^2)^{-1}]h^2 \end{aligned}$$

by the arithmetic-geometric mean inequality. If α is now chosen to satisfy the inequality

$$(3.8) \qquad \alpha \geqq 2(1 - k^2)^{-2},$$

then the second inequality in (3.7) follows. The first inequality is obtained in an analogous manner.

A direct calculation yields

$$
\begin{aligned}
\frac{\partial^4 \rho(\mathbf{x})^\beta}{(\partial x^i)^4} &= 3\beta(\beta - 2)\rho(\mathbf{x})^{\beta-4} + 6\beta(\beta - 2)(\beta - 4)(x^i)^2\rho(\mathbf{x})^{\beta-6} \\
&\qquad + \beta(\beta - 2)(\beta - 4)(\beta - 6)(x^i)^4\rho(\mathbf{x})^{\beta-8} \\
&\leqq \beta(\beta - 2)\rho(\mathbf{x})^{\beta-4}\{3 + 6(\beta - 4)(x^i)^2\rho(\mathbf{x})^{-2} \\
&\qquad + (\beta - 4)(\beta - 6)(x^i)^2\rho(\mathbf{x})^{-2}\} \\
&\leqq \beta(\beta - 2)\rho(\mathbf{x})^{\beta-4}\{3 + \beta(\beta - 4)\} \\
&\leqq \beta(\beta - 1)(\beta - 2)(\beta - 3)\rho(\mathbf{x})^{\beta-4}.
\end{aligned} \tag{3.9}
$$

Also we see that

$$-\Delta[\rho(\mathbf{x})^\beta] = -\beta(n - 2 + \beta)\rho(\mathbf{x})^{\beta-2} + \beta(\beta - 2)\alpha h^2\rho(\mathbf{x})^{\beta-4}. \tag{3.10}$$

Moreover, if $\mathbf{x} \in R'$ we have from (2.2),

$$-\Delta_h[\rho(\mathbf{x})^\beta] = -\Delta[\rho(\mathbf{x})^\beta] - \frac{h^2}{24}\sum_{i=1}^{n}\left\{\frac{\partial^4\rho(\xi_i)^\beta}{(\partial x^i)^4} + \frac{\partial^4\rho(\eta_i)^\beta}{(\partial x^i)^4}\right\}, \tag{3.11}$$

and, consequently, if α satisfies (3.8) it follows from (3.7), (3.9) and (3.10) that

$$
\begin{aligned}
-\Delta_h[\rho(\mathbf{x})^\beta] \geqq &-\beta(n - 2 + \beta)\rho(\mathbf{x})^{\beta-2} \\
&+ \beta(\beta - 2)h^2\rho(\mathbf{x})^{\beta-4}\left\{\alpha - \frac{n(\beta - 1)(\beta - 3)k^{\beta-4}}{12}\right\}.
\end{aligned}
$$

Finally, then, if α satisfies the inequality

$$\alpha \geqq \left\{(1 - k^2)^{-2}\,\frac{n(\beta - 1)(\beta - 3)k^{\beta-4}}{12}\right\}, \tag{3.12}$$

we see that

$$-\Delta_h[\rho(\mathbf{x})^\beta] \geqq -\beta(n - 2 + \beta)\rho(\mathbf{x})^{\beta-2}. \tag{3.13}$$

The inequality (3.1) implies that for $\mathbf{x} \in \bar{\mathbf{R}}$,

$$\rho(\mathbf{x})^2 = |\mathbf{x}|^2 + \alpha h^2 \leqq \left(1 + \frac{\alpha}{a^2}\right)|\mathbf{x}|^2, \tag{3.14}$$

so that if γ satisfies the inequality

$$\gamma\left[-\beta(n - 2 + \beta)\left(1 + \frac{\alpha}{a^2}\right)^{(\beta-2)/2}\right] \geqq 1, \tag{3.15}$$

then (3.13) implies that

$$-\Delta_h\Psi(\mathbf{x}) \geqq |\mathbf{x}|^{\beta-2}, \qquad \mathbf{x} \in R'. \tag{3.16}$$

If $\mathbf{x} \in R^*$, then from (2.3) and (3.7),

$$
\begin{aligned}
-[\rho(\mathbf{x})^\beta]_{x^i\bar{x}^i} &= \frac{2}{(\alpha_i + \beta_i)h^2} \\
&\quad \cdot \left\{ -\frac{\rho(\mathbf{x} + \alpha_i \mathbf{h}_i)^\beta}{\alpha_i} + \left(\frac{1}{\alpha_i} + \frac{1}{\beta_i}\right)\rho(\mathbf{x})^\beta - \frac{\rho(\mathbf{x} - \beta_i \mathbf{h}_i)^\beta}{\beta_i} \right\} \\
&\geqq \left(\frac{2}{\alpha_i + \beta_i}\right) h^{-2}\rho(\mathbf{x})^\beta \left\{ \left(\frac{1}{\alpha_i} + \frac{1}{\beta_i}\right)(1 - k^\beta) \right\} \\
&\geqq \left(\frac{2}{\alpha_i \beta_i}\right)(1 - k^\beta)h^{-2}\rho(\mathbf{x})^\beta.
\end{aligned} \tag{3.17}
$$

It is therefore a consequence of (3.17) and the definition of $\Psi(\mathbf{x})$ that

$$
\begin{aligned}
-\Delta_h \Psi(\mathbf{x}) &= -\Delta_h[\gamma\rho(\mathbf{x})^\beta] + 2h^{-2}\gamma \sum_{\xi_i, \eta_i \in \partial R} \left\{ \frac{\rho(\xi_i)^\beta}{\alpha_i(\alpha_i + \beta_i)} + \frac{\rho(\eta_i)^\beta}{\beta_i(\alpha_i + \beta_i)} \right\} \\
&\geqq 2(1 - k^\beta)\gamma \left[\sum_{i=1}^n \frac{1}{\alpha_i \beta_i} \right] h^{-2}\rho(\mathbf{x})^\beta \\
&\quad + 2\gamma k^{-\beta}h^{-2}\rho(\mathbf{x})^\beta \sum_{i=1}^n \left\{ \frac{1}{\alpha_i + \beta_i} \left[\frac{\bar{\delta}(\mathbf{x}, \xi_i)}{\alpha_i} + \frac{\bar{\delta}(\mathbf{x}, \eta_i)}{\beta_i} \right] \right\},
\end{aligned}
$$

where

$$
\bar{\delta}(\mathbf{x}, \xi_i) = \begin{cases} 1 & \text{for} \quad \xi_i \in \partial R, \\ 0 & \text{for} \quad \xi_i \notin \partial R. \end{cases}
$$

Hence,

$$
\begin{aligned}
-\Delta_h \psi(\mathbf{x}) &\geqq 2\gamma h^{-2}\rho(\mathbf{x})^\beta \left\{ (1 - k^\beta) \left[\sum_{i=1}^n \frac{1}{\alpha_i \beta_i} \right] \right. \\
&\quad \left. + k^{-\beta} \left[\sum_{i=1}^n \left(\frac{1}{\alpha_i + \beta_i}\right) \left(\frac{\bar{\delta}(\mathbf{x}, \xi_i)}{\alpha_i} + \frac{\bar{\delta}(\mathbf{x}, \eta_i)}{\beta_i} \right) \right] \right\}.
\end{aligned} \tag{3.18}
$$

We see that

$$
\begin{aligned}
\sum_{i=1}^n \left(\frac{1}{\alpha_i + \beta_i}\right) \left[\frac{\bar{\delta}(\mathbf{x}, \xi_i)}{\alpha_i} + \frac{\bar{\delta}(\mathbf{x}, \eta_i)}{\beta_i} \right] &\geqq \max_i \left(\frac{1}{2\alpha_i \beta_i}\right) \\
&\geqq \frac{1}{2n} \left[\sum_{i=1}^n \left(\frac{1}{\alpha_i \beta_i}\right) \right]
\end{aligned}
$$

and, consequently, (3.18) takes the form

$$
-\Delta_h \psi(\mathbf{x}) \geqq 2\gamma \left[\sum_{i=1}^n \frac{1}{\alpha_i \beta_i} \right] \left[(1 - k^\beta) + \frac{k^{-\beta}}{2n} \right] h^{-2}\rho(\mathbf{x})^\beta.
$$

We now assume, in addition to the previous condition $k^2 < 1$, the inequality

$$\left[(1 - k^{\beta}) + \frac{k^{-\beta}}{2n}\right] \geqq \frac{1}{4n}, \tag{3.19}$$

which can always be satisfied by choosing k sufficiently close to 1. Hence from (3.14), for $\mathbf{x} \in R^*$, it follows that

$$-\Delta_h \psi(\mathbf{x}) \geqq \frac{\gamma}{2n} h^{-2} \rho(\mathbf{x})^{\beta} \geqq \frac{\gamma}{2n}\left(1 + \frac{\alpha}{a^2}\right)^{\beta} h^{-2} |\mathbf{x}|^{\beta} \geqq h^{-2} |\mathbf{x}|^{\beta} \tag{3.20}$$

if, in addition to (3.15), we require that γ satisfy the inequality

$$\frac{\gamma}{2n}\left(1 + \frac{\alpha}{a^2}\right)^{\beta} \geqq 1. \tag{3.21}$$

In order of choice, then, we pick $k^2 < 1$ to satisfy (3.19). With k thus fixed the number α is then selected subject to the inequality (3.12). Finally, the constant γ is required to satisfy (3.15) and (3.21).

Substituting $\Psi(\mathbf{x})$ into the representation formula (2.7) and using (3.16), (3.20) yields the inequality

$$\Psi(\mathbf{x}) \geqq h^n \sum_{\mathbf{y} \in R'} G(\mathbf{x}, \mathbf{y}) |\mathbf{y}|^{\beta-2} + h^n \sum_{\mathbf{y} \in R^*} G(\mathbf{x}, \mathbf{y}) h^{-2} |\mathbf{y}|^{\beta}. \tag{3.22}$$

Since

$$\Psi(\mathbf{x}) < \gamma |\mathbf{x}|^{\beta},$$

the lemma now follows from the consideration of each positive sum on the right side of (3.22) separately.

Since $\beta = 0$ is not in the range of the preceding lemma, there remains a gap between the inequality (2.8) and (3.5). This gap is partially filled by the following lemma.

LEMMA 3.2. *Under the hypotheses of Lemma* 3.1,

$$h_n \sum_{\mathbf{y} \in R'} G(\mathbf{x}, \mathbf{y}) \, |\mathbf{y}|^{\beta-2} \leqq K \quad \begin{cases} \text{for} \quad 0 \leqq \beta, \quad n \geqq 3, \\ \text{for} \quad 0 < \beta, \quad n = 2. \end{cases}$$

Proof. Since the proof of this lemma is easily carried out by obtaining a majorant for $G(\mathbf{x}, \mathbf{y})$ and bounding the sum by an integral as indicated in [4], we shall not give all of the details here. We shall state the two steps in this process as lemmas, however, and briefly sketch their proofs since these results are themselves of independent interest.

LEMMA 3.3 (Majorant for $G(\mathbf{x}, \mathbf{y})$). *If $v(\mathbf{x})$ is defined by*

$$v(\mathbf{x}) = \begin{cases} \dfrac{1}{\gamma_2} \log \left[\dfrac{d_0^2 + \alpha h^2}{|\mathbf{x}|^2 + \alpha h^2}\right] & \text{for} \quad n = 2, \\[2ex] \dfrac{1}{(n-2)\gamma_n} [|\mathbf{x}|^2 + \alpha h^2]^{(2-n)/2} & \text{for} \quad n \geqq 3, \end{cases} \tag{3.23}$$

where α, γ_n are suitably chosen constants (independent of h) and d_0 is larger than the diameter of $\mathbf{R}$, *then*

$$G(\mathbf{x}, \mathbf{y}) \leqq v(\mathbf{x} - \mathbf{y}), \qquad \mathbf{x}, \mathbf{y} \in R.$$

Proof. The γ_n is chosen so that

$$-\Delta_h v(\mathbf{o}) \geqq -\Delta_{h,\mathbf{x}} G(\mathbf{x}, \mathbf{y}) \big|_{\mathbf{y}=\mathbf{x}} .$$

We verify by a direct calculation, for example, that if $\mathbf{x} \in R'$, γ_n can be chosen to satisfy the inequality

$$\gamma_n \leqq \begin{cases} \log\left(\dfrac{1+\alpha}{\alpha}\right)^4 & \text{for } n = 2, \\ \dfrac{2n}{n-2}\left[\alpha^{(2-n)/2} - (\alpha+1)^{(2-n)/2}\right] & \text{for } n \geqq 3. \end{cases}$$

If $\mathbf{x} \in R^*$, then once again it can be verified that γ_n can be chosen so that it is bounded below by a positive constant independent of $\mathbf{x}$.

We note that in each case $v(\mathbf{x})$ has been defined so that

$$v(\mathbf{x} - \mathbf{y}) \geqq 0, \quad \mathbf{y} \in R, \quad \mathbf{x} \in \partial R.$$

Hence to use the maximum principle we need only show that if $\mathbf{x}, \mathbf{y} \in R$ and $\mathbf{x} \neq \mathbf{y}$, then

$$-\Delta_{h,\mathbf{x}} v(\mathbf{x} - \mathbf{y}) \geqq 0.$$

The steps in the process for $\mathbf{x} \in R'$ are identical to those taken in establishing (3.13). If $\mathbf{x} \in R^*$ we may repeat the same steps for appropriately defined Δ_h and thereby obtain an additional inequality which α is chosen to satisfy.

For each value of the parameter $\mathbf{y} \in R$ the lemma now follows from the maximum principle.

LEMMA 3.4. *If $-n < p, q < 0$ and $\Omega \subset R$ is such that for $\mathbf{y} \in \Omega$,*

$$|\mathbf{x} - \mathbf{y}| \geqq ah > 0, \quad |\mathbf{z} - \mathbf{y}| \geqq ah > 0,$$

then

$$h^n \sum_{\mathbf{y} \in \Omega} |\mathbf{x} - \mathbf{y}|^p |\mathbf{z} - \mathbf{y}|^q \leqq K\{|\mathbf{x} - \mathbf{z}|^{n+p+q} + 1\}.$$

Proof. This lemma is proved by obtaining an inequality of the type

$$h_n \sum_{\mathbf{y} \in \Omega} |\mathbf{x} - \mathbf{y}|^p |\mathbf{z} - \mathbf{y}|^q \leqq K \int_R |\mathbf{x} - \mathbf{y}|^p |\mathbf{z} - \mathbf{y}|^q \, d\mathbf{y},$$

as was done in [4]. The integral itself is well known to satisfy an inequality of the type

$$\int_R |\mathbf{x}-\mathbf{y}|^p\,|\mathbf{z}-\mathbf{y}|^q\,d\mathbf{y} \leqq K\,\{|\mathbf{x}-\zeta|^{n+p+q}+1\}$$

(cf. Friedman, [3, p. 14]).

We are now in a position to prove Theorem 3.1 which was stated in the Introduction.

Proof of Theorem 3.1. Under the assumption (3.1) we see that for $\mathbf{y} \in \bar{R}$,

$$h^2|\mathbf{y}|^{\beta-4} \leqq Kh^{\beta+n-2-\epsilon}|\mathbf{y}|^{\epsilon-n}, \qquad 2-n<\beta\leqq 4-n,$$

$$h|\mathbf{y}|^{\beta-3} \leqq Kh^{\beta+n-4-\epsilon}|\mathbf{y}|^{\epsilon+2-n}, \qquad 2-n<\beta\leqq 5-n.$$

It then follows from Lemmas 3.1 and 3.2 that

$$h^n\sum_{\mathbf{y}\in R'} G(\mathbf{x},\mathbf{y})[h^2|\mathbf{y}|^{\beta-4}]$$

$$(3.24)\qquad \leqq K(\epsilon)\begin{cases} h^{\beta+n-2-\epsilon}|\mathbf{x}|^{\epsilon+2-n} & \text{for } 2-n<\beta\leqq 4-n,\\ h^2|\mathbf{x}|^{\beta-2} & \text{for } 4-n<\beta<2,\\ h^2 & \text{for } 2<\beta.\end{cases}$$

Likewise the sum over the points of R^* satisfies the inequality

$$h^n\sum_{\mathbf{y}\in R^*} G(\mathbf{x},\mathbf{y})[h\,|\mathbf{y}|^{\beta-3}]$$

$$(3.25)\qquad \leqq K(\epsilon)\begin{cases} h^{\beta+n-2-\epsilon}|\mathbf{x}|^{\epsilon+2-n} & \text{for } 2-n<\beta\leqq 5-n,\\ h^3|\mathbf{x}|^{\beta-3} & \text{for } 5-n<\beta<3,\\ h^3 & \text{for } 3<\beta.\end{cases}$$

The conclusion of the theorem can now be obtained by substituting (3.24) and (3.25) into (3.4).

We shall now discuss some implications of the theorem. It will be convenient in what follows to introduce a notation for the L_p norm over an arbitrary point set $\Omega \subset \bar{R}$. If $W(\mathbf{x})$ is a function defined on Ω, we define the discrete L_p norm by

$$|W|_{L_p(\Omega)} \equiv \{h^n\sum_{\mathbf{y}\in\Omega}|W(\mathbf{y})|^p\}^{1/p}.$$

We note that the maximum norm is the special case

$$\max_{\mathbf{y}\in\Omega}|W(\mathbf{y})| = \lim_{p\to\infty}|W|_{L_p(\Omega)} = |W|_{L_\infty(\Omega)}.$$

Although the use of these norms is not necessary to our discussion and, in fact, gives less information about the behavior of the error than the estimates already obtained, we include the appropriate L_p estimates. If nothing else, these show how estimates in L_p norm can conceal rather bad behavior of the discretization error near a point. We now discuss certain special cases of Theorem 3.1 which will aid the reader in understanding its implications and its connection with known results. These special cases will be listed as corollaries for the purpose of easy reference.

COROLLARY 3.1. *If the solution* $u(\mathbf{x})$ *of the Dirichlet problem* (1.1) *satisfies the conditions*

$$u \in C^4(\bar{\mathbf{R}} - \mathbf{o}),$$

$$\left| \frac{\partial^j u(\mathbf{x})}{(\partial x^i)^j} \right| \leqq K |\mathbf{x}|^{2-j}, \qquad j = 3, 4,$$

then the discretization error satisfies the inequality

$$|e(\mathbf{x}, h)|_{L_\infty(R)} \leqq K(\epsilon) \begin{cases} h^{2-\epsilon} & \text{for} \quad n = 2, \\ h^2 & \text{for} \quad n \geqq 3. \end{cases}$$

We recall that the usual convergence theorem is proved under the hypothesis that $u \in C^4(\bar{\mathbf{R}})$, whereas the second order singularity occurring in the fourth derivatives above might arise when $u \in C^{1+1}(\bar{\mathbf{R}})$.

COROLLARY 3.2. *If the solution* $u(\mathbf{x})$ *of the Dirichlet problem satisfies the conditions*:

$$u \in C^4(\bar{\mathbf{R}} - \mathbf{o}),$$

$$\frac{\partial^j u(\mathbf{x})}{(\partial x^i)^j} \leqq K |\mathbf{x}|^{\mu+4-j-n}, \qquad j = 3, 4, \quad 0 < \mu < n - 2,$$

then the discretization error satisfies the inequalities:

$$(3.26) \qquad |e(\mathbf{x}, h)| \leqq Kh^2 |\mathbf{x}|^{2-n+\mu}, \qquad \mathbf{x} \in R,$$

$$(3.27) \qquad |e(\mathbf{x}, h)|_{L_p(R)} \leqq Kh^2, \qquad p < \frac{n}{n - 2 - \mu}.$$

Proof. We obtain (3.27) from (3.26) by means of the following lemma.

LEMMA 3.5. *Let* R *be placed so that for every* $\mathbf{y} \in R$,

$$|\mathbf{y}| \geqq ah > 0.$$

Then

$$h^2 \sum_{\mathbf{y} \in R} |\mathbf{y}|^p \leqq K, \qquad -n < p < 0.$$

Proof. This lemma is similar to Lemma 3.4 and is proved in the same manner by dominating the sum by an integral.

Finally, we consider the question of the rate of convergence in the case where $u(\mathbf{x})$ develops a singularity at the origin which approaches that of the fundamental solution.

COROLLARY 3.3. *If the solution*, $u(\mathbf{x})$, *of the Dirichlet problem satisfies* *the conditions*

$$u \in C^4(\bar{\mathbf{R}} - \mathbf{o}),$$

$$\left| \frac{\partial^j u(\mathbf{x})}{(\partial x^i)^j} \right| \leqq K \,|\, \mathbf{x} \,|^{\mu+2-j-n}, \qquad j = 3, 4, \quad 0 < \mu \leqq 2,$$

then the error satisfies the inequalities

$$|\, e(\mathbf{x}, h) \,| \leqq K(\epsilon) h^{\mu-\epsilon} \,|\, \mathbf{x} \,|^{2-n+\epsilon},$$

$$|\, e(\mathbf{x}, h) \,|_{L_p(R)} \leqq K(\epsilon) h^{\mu-\epsilon}, \qquad p < \frac{n}{n-2}.$$

Since another second order scheme, based on a different discrete operator Δ_h at points of R^*, is very commonly discussed in the literature, we shall consider it here. The method was first proposed by Collatz [2] where he showed that if $u \in C^4(\bar{\mathbf{R}})$, then

$$|\, e(\mathbf{x}, h) \,|_{L_\infty(R)} \leqq K h^2.$$

Our definition of Δ_h which now follows is a variant of that given by Collatz.

If $\mathbf{x} \in R^*$, we define

$$V_{x^i \bar{x}^i}(\mathbf{x}) \equiv h^{-2} \left\{ \frac{V(\mathbf{x} + \alpha_i \mathbf{h}_i)}{\alpha_i} - \left(\frac{1}{\alpha_i} + \frac{1}{\beta_i} \right) V(\mathbf{x}) + \frac{V(\mathbf{x} - \beta_i \mathbf{h}_i)}{\beta_i} \right\}.$$

The operator Δ_h is now given by (2.1) with $V_{x^i \bar{x}^i}$ given above. If $u \in C^2(\mathbf{N}(\mathbf{x}) \cap \mathbf{R})$, then the local error is given by

$$\Delta_h u(\mathbf{x}) - \Delta u(\mathbf{x}) = \sum_{i=1}^{n} \left\{ \beta_i \frac{\partial^2 u(\xi_i)}{(\partial x^i)^2} + \alpha_i \frac{\partial^2 u(\eta_i)}{(\partial x^i)^2} \right\}.$$

The finite difference analogue of (1.1) is once again given by (2.5) with the new definition of Δ_h at points of R^*. We note that the "reduced" matrix of the system (2.5), i.e., the submatrix obtained by striking out rows and columns corresponding to points of ∂R, is symmetric and hence $G(\mathbf{x}, \mathbf{y})$, $\mathbf{x}, \mathbf{y} \in R$, is also symmetric in this case. This property is not surprising since the Green's function $g(\mathbf{x}, \mathbf{y})$ is itself symmetric. Furthermore, since the numerical solution of linear systems with symmetric matrices has been more completely analyzed, this affords the user an additional theoretical advantage.

Since the analysis in this case is almost identical with that given before, we shall comment only upon modifications needed in the discussion of the discretization error. Theorem 3.1 is in fact true without modification.

The first changes in the proof occur in (3.3) and (3.4) which are replaced by

$$|\, \Delta_h u(\mathbf{x}) - \Delta u(\mathbf{x}) \,| \leqq K \begin{cases} 1 & \text{for} \quad m = 2, \\ |\, \mathbf{x} \,|^{\beta-2} & \text{for} \quad 2 - n \leqq m \leqq 1, \end{cases}$$

and

$$|\,e(\mathbf{x}, h)\,| \leqq Kh^n \sum_{\mathbf{y}\in R'} G(\mathbf{x}, \mathbf{y})[h^2|\,\mathbf{y}\,|^{\beta-4}] + Kh^n \sum_{\mathbf{y}\in R^*} G(\mathbf{x}, \mathbf{y})|\,\mathbf{y}\,|^{\beta-2}.$$

The conclusion of Lemma 3.1 is valid as stated and the proof is modified for $\mathbf{x} \in R^*$ in an obvious manner. Lemmas 3.2 and 3.3 also are true if the function $v(\mathbf{x})$ defined in (3.23) is replaced by the function $v(\mathbf{x}, \mathbf{y})$ defined by

$$v(\mathbf{x}, \mathbf{y}) = \begin{cases} v(\mathbf{x} - \mathbf{y}) & \text{for} \quad \mathbf{x} \in R, \\ 0 & \text{for} \quad \mathbf{x} \in \partial R \end{cases}$$

for all $\mathbf{y} \in R$. The steps in the proof of Lemma 3.1 can now be applied to $v(\mathbf{x}, \mathbf{y})$ as a function of $\mathbf{x}$ for an arbitrary, but fixed, value of $\mathbf{y}$ to show that

$$-\Delta_{h,\mathbf{x}} v(\mathbf{x}, \mathbf{y}) \geqq 0, \qquad \mathbf{x} \neq \mathbf{y}, \quad \mathbf{x} \in R,$$

for appropriately chosen k^2 and α. The constant γ_n is again determined by the condition

$$-\Delta_{h,\mathbf{x}} v(\mathbf{x}, \mathbf{y}) \geqq h^{-n}, \quad \mathbf{x} = \mathbf{y}.$$

The proof of Theorem 3.1 is then modified by replacing (3.25) by the inequality

$$h^n \sum_{\mathbf{y}\in R^*} G(\mathbf{x}, \mathbf{y})\,|\,\mathbf{y}\,|^{\beta-2} \leqq K(\epsilon) \begin{cases} h^{\beta+n-2-\epsilon}\,|\,\mathbf{x}\,|^{\epsilon+2-n} & \text{for} \quad 2 - n < \beta \leqq 4 - n, \\ h^2\,|\,\mathbf{x}\,|^{\beta-2} & \text{for} \quad 4 - n < \beta < 2, \\ h^2 & \text{for} \quad 2 < \beta. \end{cases}$$

Since Theorem 3.1 is true as stated, the various corollaries will be valid if the condition on $\partial^3 u(\mathbf{x})/(\partial x^i)^3$ is replaced by the appropriate condition on $\partial^2 u(\mathbf{x})/(\partial x^i)^2$.

We close this part of the discussion with the following comments.

The reader will note that a problem with only one singularity was chosen for our detailed examination. It is clear, however, that the methods described herein can be easily extended to cover the case where $u(\mathbf{x})$ is singular at a finite number of points in $\bar{\mathbf{R}}$. We merely observe in this connection that this introduces additional conditions on the sequence of values through which h approaches zero as well as the manner in which the mesh is placed on $\mathbf{R}$.

4. A fourth order approximation. The reader can easily conjecture from the preceding section how the presence of an isolated singularity will affect the rate of convergence of a difference scheme whose classical error estimates are higher order in h. We shall apply the methods of the preceding sections to study a discrete problem which was first described and analyzed in [1] for $n = 2$. In that paper it was shown that if $u \in C\,(\bar{\mathbf{R}})$, then the discretization error has the property

$$| e(\mathbf{x}, h) |_{L_\infty(R)} \leqq Kh^4.$$

Since we are interested only in indicating how our methods may be generalized to this case, only the situation where $n = 2$ and $\mathbf{x} \in \mathbf{R}$ will be considered.

We introduce the two following subsets of R':

$$R'' = \{\mathbf{x} \mid \mathbf{x} \pm \mathbf{h}_i \in \bar{\mathbf{R}}, \mathbf{x} \pm \mathbf{h}_1 \pm \mathbf{h}_2 \in \bar{\mathbf{R}}\},$$
$$R^{**} = R' - R''.$$

In words, the set R'' contains only those points of R whose eight nearest neighbors also belong to $\bar{\mathbf{R}}$.

If $\mathbf{x} = (x^1, x^2) \in R''$, we define the operator

$$\Delta_h V(\mathbf{x}) \equiv \frac{1}{6h^2}\left\{4 \sum_{i=1}^{2} V(\mathbf{x} \pm \mathbf{h}_i) + \sum_{\substack{i,j=1 \\ i \neq j}}^{2} V(\mathbf{x} \pm \mathbf{h}_i \pm \mathbf{h}_j) - 20V(\mathbf{x})\right\}.$$

It is easily verified that if $u \in C^6(\bar{\mathbf{R}})$, then

$$\left[\Delta_h u(\mathbf{x}) - \left\{\Delta u(\mathbf{x}) + \frac{h^2}{12}\Delta^2 u(\mathbf{x})\right\}\right] = F(u)h^4, \tag{4.1}$$

where $F(u)$ is a linear combination of sixth derivatives of u evaluated at points along the line segments connecting $\mathbf{x}$ with its eight nearest neighbors.

In order to formulate a finite difference problem whose error is $O(h^4)$ when $u \in C^6(\bar{R})$, the operator Δ_h was defined at points of R^* so that the local error is $O(h^2)$. This in turn was accomplished by defining $V_{x^i\bar{x}^i}(\mathbf{x})$ so that

$$u(\mathbf{x})_{x^i\bar{x}^i} - \frac{\partial^2 u(\mathbf{x})}{(\partial x^i)^2} = O(h^2).$$

For example, if $\mathbf{x} - \beta_1\mathbf{h}_1 \in \partial R$ and the points $\mathbf{x}, \mathbf{x} + \mathbf{h}_1, \mathbf{x} + 2\mathbf{h}_1$ all belong to R, then we define

$$V_{x^1\bar{x}^1}(\mathbf{x}) \equiv \frac{1}{h^2}\left\{\left(\frac{\beta_1 - 1}{\beta_1 + 2}\right) V(\mathbf{x} + 2\mathbf{h}_1) + \frac{2(2 - \beta_1)}{\beta_1 + 1} V(\mathbf{x} + \mathbf{h}_1) \right.$$
$$\left. + \left(\frac{\beta_1 - 3}{\beta_1}\right) V(\mathbf{x}) + \frac{6}{\beta_1(\beta_1 + 1)(\beta_1 + 2)} V(\mathbf{x} - \beta_1\mathbf{h}_1)\right\}.$$

As usual, we define Δ_h by

$$\Delta_h V(\mathbf{x}) \equiv \sum_{i=1}^{2} V_{x^i\bar{x}^i}(\mathbf{x}).$$

If $u \in C^4(\bar{\mathbf{R}})$, then the local error is of the form

$$\Delta_h u(\mathbf{x}) - \Delta u(\mathbf{x}) = G(u)h^2,$$

where $G(u)$ is a linear combination of fourth derivatives of u evaluated at points on the line segments connecting $\mathbf{x}$ to certain of its nearby neighbors.

Finally, if $\mathbf{x} \in R^{**}$, then Δ_h is taken to be the usual five-point operator

$$\Delta_h V(\mathbf{x}) \equiv h^{-2}\left\{\sum_{i=1}^{2} V(\mathbf{x} \pm \mathbf{h}_i) - 4V(\mathbf{x})\right\},$$

which, if $u \in C^4(\bar{\mathbf{R}})$, has the local error (2.2).

The finite difference analogue of (1.1) is then defined to be

$$\begin{aligned} -\Delta_h u(\mathbf{x}, h) &= f(\mathbf{x}) + \frac{h^2}{12}\Delta f(\mathbf{x}), && \mathbf{x} \in R'', \\ -\Delta_h u(\mathbf{x}, h) &= f(\mathbf{x}), && \mathbf{x} \in R^* \cup R^{**}, \\ u(\mathbf{x}, h) &= g(\mathbf{x}), && \mathbf{x} \in \partial R. \end{aligned}$$

Once again we assume that the mesh has been placed on $\mathbf{R}$ so that the distance from the origin to the various line segments between $\mathbf{x}$ and the appropriate nearby mesh points on which certain derivatives of u entering into the local error are evaluated is bounded below by some constant times h.

The matrix of this linear system no longer possesses certain desirable algebraic properties which are sufficient for the existence of a nonnegative discrete Green's function. Accordingly an "interior" Green's function is defined by

$$\begin{aligned} -\Delta_{h,\mathbf{x}} G(\mathbf{x}, \mathbf{y}) &= h^{-2}\delta(\mathbf{x}, \mathbf{y}), && \mathbf{x} \in R', \\ G(\mathbf{x}, \mathbf{y}) &= \delta(\mathbf{x}, \mathbf{y}), && \mathbf{x} \in R^*. \end{aligned} \tag{4.2}$$

Once again it is easy to see that the matrix of this system has the proper algebraic properties so that $G(\mathbf{x}, \mathbf{y})$ exists and

$$G(\mathbf{x}, \mathbf{y}) \geqq 0, \qquad \mathbf{x}, \mathbf{y} \in R.$$

Moreover, we have the representation formula

$$V(\mathbf{x}) = h^2 \sum_{\mathbf{y} \in R'} G(\mathbf{x}, \mathbf{y})[-\Delta_h V(\mathbf{x})] + \sum_{\mathbf{y} \in R^*} G(\mathbf{x}, \mathbf{y}) V(\mathbf{y}) \tag{4.3}$$

for an arbitrary mesh function $V(\mathbf{x})$ defined on R. It can be shown that

$$\begin{aligned} h^2 \sum_{\mathbf{y} \in R'} G(\mathbf{x}, \mathbf{y}) &\leqq K, \\ h^2 \sum_{\mathbf{y} \in R^{**}} G(\mathbf{x}, \mathbf{y}) &\leqq h^2, \\ \sum_{\mathbf{y} \in R^*} G(\mathbf{x}, \mathbf{y}) &\leqq 1. \end{aligned} \tag{4.4}$$

If we write the operator Δ_h in the form

$$-\Delta_h V(\mathbf{x}) \equiv \sum_{\mathbf{y}} \sigma(\mathbf{x}, \mathbf{y}) V(\mathbf{y}),$$

then it can be shown that for $\mathbf{x} \in R^*$,

$$\sum_{\substack{\mathbf{y} \in R \\ \mathbf{y} \neq \mathbf{x}}} \left| \frac{\sigma(\mathbf{x}, \mathbf{y})}{\sigma(\mathbf{x}, \mathbf{x})} \right| \leqq \frac{3}{4},$$

$$0 < \sigma(\mathbf{x}, \mathbf{x})^{-1} \leqq 2h^2.$$

Let $W(\mathbf{x})$ be an arbitrary mesh function defined on $\bar{R}$ which vanishes at points of ∂R. Then for $\mathbf{x} \in R^*$,

$$W(\mathbf{x}) - \sum_{\substack{\mathbf{y} \in R \\ \mathbf{y} \neq \mathbf{x}}} \left| \frac{\sigma(\mathbf{x}, \mathbf{y})}{\sigma(\mathbf{x}, \mathbf{x})} \right| W(\mathbf{y}) \equiv \sigma(\mathbf{x}, \mathbf{x})^{-1}[-\Delta_h W(\mathbf{x})].$$

Hence for $\mathbf{x} \in R^*$,

$$|W(\mathbf{x})| \leqq \tfrac{3}{4} |W|_{L_\infty(R)} + 2h^2 |\Delta_h W(\mathbf{x})|. \tag{4.5}$$

Consequently, if we substitute W into (4.3) and use the inequalities (4.4) and (4.5), we see that for $\mathbf{x} \in R$,

$$W(\mathbf{x}) \leqq h^2 \sum_{\mathbf{y} \in R'} G(\mathbf{x}, \mathbf{y}) |\Delta_h W(\mathbf{y})| + \tfrac{3}{4} |W|_{L_\infty(R)} + 2h^2 |\Delta_h W|_{L_\infty(R^*)}$$

and, therefore,

$$|W|_{L_\infty(R)} \leqq K \Big| h^2 \sum_{\mathbf{y} \in R'} G(\mathbf{x}, \mathbf{y}) |\Delta_h W(\mathbf{y})| \Big|_{L_\infty(R)} + Kh^2 |\Delta_h W|_{L_\infty(R^*)}. \tag{4.6}$$

To study the order of convergence we shall once again identify $W(\mathbf{x})$ with $e(\mathbf{x}, \mathbf{y})$ in (4.6). It is clear that the last term in (4.6) is very badly behaved if the singularity is on the boundary. Accordingly, we assume that the origin belongs to $\mathbf{R}$. Then it follows from the assumptions

$$u \in C^6(\bar{\mathbf{R}} - \mathbf{o}),$$

$$|D^{\mathbf{k}} u(\mathbf{x})| \leqq K \begin{cases} 1 & \text{for } |\mathbf{k}| \leqq m, \\ |\mathbf{x}|^{\beta - |\mathbf{k}|} & \text{for } m + 1 \leqq |\mathbf{k}| \leqq 6, \end{cases} \tag{4.7}$$

and (3.1), with $\mathbf{N}(\mathbf{x})$ suitably redefined for the present discrete analogue, that

$$|\Delta_h e|_{L_\infty(R^*)} \leqq Kh^2. \tag{4.8}$$

Likewise we see from (4.4) that

$$\Big| h^2 \sum_{\mathbf{y} \in R^{**}} G(\mathbf{x}, \mathbf{y}) |\Delta_h e(\mathbf{y}, h)| \Big|_{L_\infty} \leqq Kh^4. \tag{4.9}$$

Thus the error estimate can be completed with the aid of Lemma 3.2

which is again true if $G(\mathbf{x}, \mathbf{y})$ is defined by (4.2). In fact, the proof in this case is an obvious modification of that given in 43. We therefore have the following convergence theorem.

THEOREM 4.1. *Let the conditions* (3.1) *and* (4.7) *be satisfied by* $u(\mathbf{x})$ *and* $\bar{R}$ *where* $\mathbf{o} \in \mathbf{R}$. *Then*

$$|e|_{L_\infty(R)} \leqq K(\epsilon) \begin{cases} h^{\beta-\epsilon} & \text{for } 0 < \beta \leqq 4, \\ h^4 & \text{for } 4 < \beta. \end{cases}$$

Proof. We note that from (4.1),

$$h^2 \sum_{\mathbf{y}\in R''} G(\mathbf{x}, \mathbf{y}) |\Delta_h e(\mathbf{y}, h)| \leqq Kh^2 \sum_{\mathbf{y}\in R''} G(\mathbf{x}, \mathbf{y})[h^4 |\mathbf{y}|^{\beta-6}]. \tag{4.10}$$

Now by (3.1), if $\mathbf{y} \in R$,

$$h^4 |\mathbf{y}|^{\beta-6} \leqq K \begin{cases} h^{\beta-\epsilon} |\mathbf{y}|^{\epsilon-2} & \text{for } 0 < \beta \leqq 4, \\ h^4 |\mathbf{y}|^{\beta-6} & \text{for } 4 < \beta, \end{cases} \tag{4.11}$$

so that if we substitute (4.11) into (4.10) and apply Lemma 3.2,

$$h^2 \sum_{\mathbf{y}\in R''} G(\mathbf{x}, \mathbf{y}) |\Delta_h e(\mathbf{y}, h)| \leqq K(\epsilon) \begin{cases} h^{\beta-\epsilon} & \text{for } 0 < \beta \leqq 4, \\ h^4 & \text{for } 4 < \beta. \end{cases} \tag{4.12}$$

The theorem now follows immediately from (4.8), (4.9) and (4.12) if we set $W(\mathbf{x}) = e(\mathbf{x}, h)$ in (4.6).

5. The function $u(\mathbf{x})^{\prime}$ **as a weak solution.** The assumption (1.4) on $u(\mathbf{x})$ may mean that we can no longer interpret (2.1) in the classical sense. In particular, if the origin is an interior point, then the differential equation may not hold there. In this case, the function $u(\mathbf{x})$ may be interpreted as the unique "weak" solution of a corresponding generalized problem.

If we assume that the boundary of $\mathbf{R}$ is such that a local barrier exists at each point, then the Green's function $g(\mathbf{x}, \mathbf{y})$ exists. If, in addition, $\partial\mathbf{R}$ is "regular" so that integration by parts is permitted, then for $f \in C^{0+\lambda}(\mathbf{R})$ the classical solution of (1.1) for $u = 0$ on $\partial\mathbf{R}$ is given by the Poisson formula

$$u(\mathbf{x}) = \int_{\mathbf{R}} g(\mathbf{x}, \mathbf{y}) f(\mathbf{y})\, d\mathbf{y}. \tag{5.1}$$

The function $f(\mathbf{x})$ which we wish to consider, however, may not meet this smoothness requirement, although the function $v(\mathbf{x})$ defined by

$$v(\mathbf{x}) = \int_{\mathbf{R}} g(\mathbf{x}, \mathbf{y}) f(\mathbf{y})\, d\mathbf{y} \tag{5.2}$$

exists. In fact, it is easily proved that the following lemma holds.

LEMMA 5.1. *If* $\partial \mathbf{R}$ *is regular and if* $f \in L_p(\mathbf{R})$, $p > n/2$, *then* $v(\mathbf{x})$ *given by* (5.2) *exists, is continuous in* $\bar{\mathbf{R}}$, *and vanishes on* $\partial \mathbf{R}$.

To describe the weak problem with which we are concerned define the set

$$T = \{\phi \mid \Delta\phi \in C_0(\mathbf{R}),\ \phi \in C(\bar{\mathbf{R}}) \cap C^{2+\alpha}(\mathbf{R}) \text{ for some } \alpha > 0 \text{ and } \phi(\mathbf{x}) = 0 \text{ for } \mathbf{x} \in \partial\mathbf{R}\}.$$

THEOREM 5.1. *If* $f \in L_1(\mathbf{R})$, *there exists a unique function* $u \in L_q(\mathbf{R})$ *for all* $q < n/(n-2)$ *which satisfies the integral relation*

$$\int_{\mathbf{R}} u(\mathbf{y})\Delta\phi(\mathbf{y})\, d\mathbf{y} = \int_{\mathbf{R}} \phi(\mathbf{y})f(\mathbf{y})\, d\mathbf{y}$$

for every $\phi \in T$. *This "weak solution" is given by* (5.1).

Since we shall be interested in giving sufficient conditions on f under which (1.2) and (1.4) hold with $\mathbf{o} \in \mathbf{R}$, it is desirable to remove the smoothness of $\partial\mathbf{R}$ and g as a consideration in the error estimates. Accordingly, we shall consider the problem (1.1) under the assumptions:

$\partial\mathbf{R}$ has a local representation in the neighborhood of each point such that the functions involved have $M + 1$ continuous derivatives;

$$g(\mathbf{x}) = 0, \quad \mathbf{x} \in \partial\mathbf{R};$$

$$f \in C^{M-1}(\bar{\mathbf{R}} - \mathbf{o}); \tag{5.3}$$

$$|D^{\mathbf{k}}f(\mathbf{x})| \leqq K \begin{cases} 1 & \text{for } |\mathbf{k}| \leqq m-2, \\ |\mathbf{x}|^{\beta-|\mathbf{k}|-2} & \text{for } m-1 \leqq |\mathbf{k}| \leqq M-1, \end{cases}$$

where once again $\beta = m + \lambda$.

The following theorem is weaker in many respects than can be proved.

THEOREM 5.2. *Under the assumptions* (1.3) *and* (5.3) *the weak solution of* (1.1) *exists and has the following properties*:

$$\begin{aligned} u &\in C^M(\bar{\mathbf{R}} - \mathbf{o}), \\ u &= 0 \quad on \quad \partial\mathbf{R}, \\ \Delta u &= f \quad in \quad \mathbf{R} - \mathbf{o}. \end{aligned} \tag{5.4}$$

Once again, the proof of the above theorem will be omitted. We note that the only part of any difficulty involves the regularity of u near the boundary. The conclusion (5.4) clearly implies (1.2) and thus we need only show that (1.4) is also valid.

THEOREM 5.3. *Let the hypotheses of Theorem 5.2 be satisfied; then*

$$(5.5)\qquad |D^{\mathbf{k}}u(\mathbf{x})| \leqq K \begin{cases} 1 & \textit{for} \quad |\mathbf{k}| \leqq m, \\ |\mathbf{x}|^{\beta-|\mathbf{k}|} & \textit{for} \quad m+1 \leqq |\mathbf{k}| \leqq M. \end{cases}$$

Proof. We write the Green's function

$$g(\mathbf{x}, \mathbf{y}) = \gamma(\mathbf{x}, \mathbf{y}) + W(\mathbf{x}, \mathbf{y}),$$

where $\gamma(\mathbf{x}, \mathbf{y})$ is the singular part, i.e.,

$$\gamma(\mathbf{x}, \mathbf{y}) = \begin{cases} \dfrac{1}{2\pi} \log |\mathbf{x} - \mathbf{y}|^{-1} & \text{for} \quad n = 2, \\ \dfrac{1}{(n-2)\omega_n} |\mathbf{x} - \mathbf{y}|^{2-n} & \text{for} \quad n \geqq 3, \end{cases}$$

and $W(\mathbf{x}, \mathbf{y})$ is the "regular" part. The factor ω_n is the surface area of the unit sphere in n dimensions.

Since the conditions (1.3) and (5.3) imply that $f \in L_1(\mathbf{R})$, it is clear that

$$\int_{\mathbf{R}} W(\mathbf{x}, \mathbf{y}) f(\mathbf{y})\, d\mathbf{y} \in C^\infty(\mathbf{R})$$

and that an arbitrary derivative of any order can be obtained by differentiation under the integral sign. Hence we need only show that $w(\mathbf{x})$ defined by

$$w(\mathbf{x}) = \begin{cases} \displaystyle\int_{\mathbf{R}} |\mathbf{x} - \mathbf{y}|^{2-n} f(\mathbf{y})\, d\mathbf{y} & \text{for} \quad n \geqq 3, \\ \displaystyle\int_{\mathbf{R}} \log |\mathbf{x} - \mathbf{y}|^{-1} f(\mathbf{y})\, d\mathbf{y} & \text{for} \quad n = 2 \end{cases}$$

satisfies (5.5). We shall indicate the proof for the case $n \geqq 3$. Moreover, we further assume that f has compact support in $\mathbf{R}$ with $\mathbf{o}$ belonging to the support. This is no essential restriction since the inequality (5.5) is easily established for a function $f \in C^{M-1}(\bar{\mathbf{R}})$.

Assume first that $1 \leqq |\mathbf{k}| \leqq m$ and consider

$$D^{\mathbf{k}} w(\mathbf{x}) = \int [D_{\mathbf{x}}^{\mathbf{i}} |\mathbf{x} - \mathbf{y}|^{2-n}] D^{\mathbf{j}} f(\mathbf{y})\, d\mathbf{y},$$

where $\mathbf{k} = \mathbf{i} + \mathbf{j}$ and $|\mathbf{i}| = 1$. Since by assumption,

$$|D^{\mathbf{j}} f(\mathbf{y})| \leqq K |\mathbf{y}|^{\lambda-1},$$

we see that

$$|D^{\mathbf{k}} w(\mathbf{x})| \leqq K \int_{\mathbf{R}} |\mathbf{x} - \mathbf{y}|^{1-n} |\mathbf{y}|^{\lambda-1}\, d\mathbf{y} \leqq K.$$

For $m \geqq |\mathbf{k}| = 0$ we see from (5.3) that

$$|w(\mathbf{x})| \leqq K \int_{\mathbf{R}} |\mathbf{x} - \mathbf{y}|^{\lambda - n} \, d\mathbf{y} \leqq K.$$

Now assume that $M \geqq |\mathbf{k}| \geqq m + 1$ and define $\bar{f}$ and $\bar{w}$ by

$$\bar{f}(\mathbf{x}) = \begin{cases} f(\mathbf{x}) & \text{for} \quad m \leqq 0, \quad \mathbf{k} = 1, \\ D^{\mathbf{m}} f(\mathbf{x}) & \text{for} \quad |\mathbf{m}| = m \geqq 1, \quad \mathbf{k} = 1 + \mathbf{m}, \end{cases}$$

and

$$\bar{w}(\mathbf{x}) = \int |\mathbf{x} - \mathbf{y}|^{2-n} \bar{f}(\mathbf{y}) \, d\mathbf{y}. \tag{5.6}$$

We note that $\bar{w}(\mathbf{x})$ is normalized so that its derivatives are all singular at the origin and our task is to investigate the order of this singularity. In the steps that follow we shall break up the integral (5.6) in such a way that the required number of derivatives can be taken under the integral sign.

First we define the function $\psi \in C^M$ by

$$\psi(r) = \begin{cases} 1 & \text{for} \quad r \leqq \rho, \\ 0 & \text{for} \quad r \geqq 2\rho, \end{cases}$$

where ρ is a prescribed positive number. In fact, we define ψ explicitly in the interval $\rho \leqq r \leqq 2\rho$ to be

$$\psi(r) = \frac{\int_r^{2\rho} (2\rho - \xi)^M (\xi - \rho)^M \, d\xi}{\int_\rho^{2\rho} (2\rho - \xi)^M (\xi - \rho)^M \, d\xi}.$$

It is easily seen that

$$\frac{d^\alpha \psi(r)}{(dr)^\alpha} < K\rho^{-\alpha}, \qquad \alpha \leqq M.$$

We now define

$$\bar{w}_1(\mathbf{x}) = \int |\mathbf{x} - \mathbf{y}|^{2-n} \bar{f}(\mathbf{y}) \psi(|\bar{\mathbf{x}} - \mathbf{y}|) \, d\mathbf{y},$$

$$\bar{w}_2(\mathbf{x}) = \int |\mathbf{x} - \mathbf{y}|^{2-n} \bar{f}(\mathbf{y}) \psi(|\mathbf{y}|) \, d\mathbf{y},$$

$$\bar{w}_3(\mathbf{x}) = \bar{w}(\mathbf{x}) - \bar{w}_1(\mathbf{x}) - \bar{w}_2(\mathbf{x}),$$

where we now restrict $\mathbf{x}$ and ρ to satisfy

$$2|\mathbf{x} - \bar{\mathbf{x}}| \leqq \rho \leqq \tfrac{1}{4}|\bar{\mathbf{x}} - \mathbf{y}|.$$

Clearly, for $1 \leqq |\mathbf{l}| \leqq M - m$,

$$D^{\mathbf{l}}\bar{w}_1(\mathbf{x}) = \int [D_{\mathbf{x}}^{\mathbf{i}}|\mathbf{x} - \mathbf{y}|^{2-n}]D^{\mathbf{j}}[\bar{f}(\mathbf{y})\psi(|\bar{\mathbf{x}} - \mathbf{y}|)]\, d\mathbf{y},$$

where $\mathbf{l} = \mathbf{i} + \mathbf{j}$ and $|\mathbf{i}| = 1$. Now, since

$$D^{\mathbf{j}}[\bar{f}(\mathbf{y})\psi(|\bar{\mathbf{x}} - \mathbf{y}|)] \leqq K \begin{cases} \rho^{m+\lambda-|\mathbf{l}|-1} & \text{for} \quad m \leqq 0, \\ \rho^{\lambda-|\mathbf{l}|-1} & \text{for} \quad m \geqq 1, \end{cases}$$

we see that

$$|D^{\mathbf{l}}\bar{w}_1(\mathbf{x})| \leqq K \begin{cases} \rho^{m+\lambda-|\mathbf{l}|} & \text{for} \quad m \leqq 0, \\ \rho^{\lambda-|\mathbf{l}|} & \text{for} \quad m \geqq 1. \end{cases}$$

In like manner we obtain

$$D^{\mathbf{l}}\bar{w}_2(\mathbf{x}) = \int [D_{\mathbf{x}}^{\mathbf{l}}|\mathbf{x} - \mathbf{y}|^{2-n}]\bar{f}(\mathbf{y})\psi(|\mathbf{y}|)\, d\mathbf{y},$$

and since

$$|[D_{\mathbf{x}}^{\mathbf{l}}|\mathbf{x} - \mathbf{y}|^{2-n}]\psi(|\mathbf{y}|)| \leqq K\rho^{2-n-|\mathbf{l}|},$$

it follows that

$$|D^{\mathbf{l}}\bar{w}_2(\mathbf{x})| \leqq K \begin{cases} \rho^{\beta-|\mathbf{l}|} & \text{for} \quad m \leqq 0, \\ \rho^{\lambda-|\mathbf{l}|} & \text{for} \quad m \geqq 1. \end{cases}$$

Finally, we see that the integrand of $\bar{w}_3(\mathbf{x})$ has no singularities so that

$$D^{\mathbf{l}}\bar{w}_3(\mathbf{x}) = \int [D_{\mathbf{x}}^{\mathbf{l}}\, |\mathbf{x} - \mathbf{y}|^{2-n}]\bar{f}(\mathbf{y})[1 - \psi(|\bar{\mathbf{x}} - \mathbf{y}|) - \psi(|\mathbf{y}|)]\, d\mathbf{y}$$

and, consequently,

$$|D^{\mathbf{l}}\bar{w}_3(\mathbf{x})| \leqq K\rho^{1-|\mathbf{l}|} \begin{cases} \int_{\mathbf{R}} |\mathbf{x} - \mathbf{y}|^{1-n}|\mathbf{y}|^{\beta-2}\, d\mathbf{y} & \text{for} \quad m \leqq 0, \\ \int_{\mathbf{R}} |\mathbf{x} - \mathbf{y}|^{1-n}|\mathbf{y}|^{\lambda-2}\, d\mathbf{y} & \text{for} \quad m \geqq 1. \end{cases}$$

Therefore,

$$|D^{\mathbf{l}}\bar{w}_3(\mathbf{x})| \leqq K \begin{cases} \rho^{\beta-|\mathbf{l}|} & \text{for} \quad m \leqq 0, \\ \rho^{\lambda-|\mathbf{l}|} & \text{for} \quad m \geqq 1. \end{cases}$$

With these estimates we now return to the situation where $M \geqq |\mathbf{k}| \geqq m + 1$ if $m \geqq 1$ and $M \geqq |\mathbf{k}| \geqq 1$ if $m \leqq 0$. Then clearly,

$$D^{\mathbf{k}}\bar{w}(\mathbf{x}) = D^{\mathbf{l}}\bar{w}(\mathbf{x}),$$

from which the conclusion follows for the range of $|\mathbf{k}|$ indicated.

The remaining case $m < 0$, $|\mathbf{k}| = 0$ is treated by examining $w(\mathbf{x})$ directly.

Hence we see that the conditions (5.3) are sufficient for (1.4) to hold in the case of a singularity in $\mathbf{R}$.

6. The exterior Dirichlet problem. Let $\mathbf{R}$ be an unbounded domain whose boundary may itself contain the point at infinity. Once again we wish to approximate the solution of (1.1) by a finite difference scheme. A direct approach would be to discretize a finite portion of the plane and pose a finite difference analogue over this mesh by adjoining equations at the outer boundary which are derived from the asymptotic behavior of u as $x \to \infty$.

As an application of the preceding results we shall instead perform a Kelvin transformation and map $\mathbf{R}$ into a finite domain $\mathbf{D}$. The transformed problem is then discretized and the error estimated by the theorems already proved.

In particular, we shall assume that the origin does not belong to $\mathbf{R}$ and then define the coordinate transformation

$$\bar{x}^i = x^i |\mathbf{x}|^{-2}, \qquad i = 1, \cdots, n.$$

Under this inversion $\mathbf{R}$ goes into the bounded domain $\mathbf{D}$, and, if we define

$$v(\bar{\mathbf{x}}) = |\mathbf{x}|^{n-2} u(\mathbf{x}),$$

then it is easily shown that if $\bar{\Delta}$ is the Laplace operator in the $\bar{x}^i$ coordinate system and $\bar{x} \neq 0$,

$$-\bar{\Delta}\, v(\bar{\mathbf{x}}) = |\bar{\mathbf{x}}|^{-2-n} f(\mathbf{x}) = \bar{f}(\bar{\mathbf{x}}), \quad \bar{\mathbf{x}} \in \mathbf{D},$$
$$v(\bar{\mathbf{x}}) = g(\mathbf{x})|\bar{\mathbf{x}}|^{2-n} = \bar{g}(\bar{\mathbf{x}}), \quad \bar{\mathbf{x}} \in \partial\mathbf{D}.$$

Now if $\mathbf{x} = \infty$ does not belong to $\partial\mathbf{R}$, then the origin will be an interior point of $\mathbf{D}$. In this case we make the following assumptions on $f(\mathbf{x})$:

$$f \in C^{M-1}(\bar{\mathbf{R}}),$$
$$(6.1) \qquad |D^{\mathbf{k}} f(\mathbf{x})| \leqq K|\mathbf{x}|^{-\zeta-2-|\mathbf{k}|} \quad \text{as} \quad |\mathbf{x}| \to \infty,$$

where $\zeta > 0$ is prescribed and $0 \leqq |\mathbf{k}| \leqq M - 1$. We note that if $\bar{D}^{\mathbf{k}}$ denotes partial differentiation with respect to $\bar{\mathbf{x}}$, then (6.1) implies that

$$|\bar{D}^{\mathbf{k}} \bar{f}(\bar{\mathbf{x}})| \leqq K \sum_{\substack{\mathbf{i} \\ |\mathbf{i}| \leqq |\mathbf{k}|}} |\bar{\mathbf{x}}|^{-n-2-|\mathbf{i}|-|\mathbf{k}|} |D^{\mathbf{i}} f(\mathbf{x})|$$
$$\leqq K|\bar{\mathbf{x}}|^{\zeta-n-|\mathbf{k}|}.$$

Hence (5.3) is satisfied for $\beta = \zeta + 2 - n$. The remaining conditions in (5.3) are seen to be satisfied for $\bar{f}(\bar{\mathbf{x}})$, $\bar{g}(\bar{\mathbf{x}})$ and $\partial\mathbf{D}$ if in addition to (6.1) we assume that (5.3) is satisfied for g and $\partial\mathbf{R}$. Then the theorems of §5 hold for $v(\bar{\mathbf{x}})$. In particular, Theorem 5.3 implies that (1.4) is satisfied by $v(\bar{\mathbf{x}})$ and its partial derivatives of order $\leqq M+1$. The same result can be inferred directly from the assumptions:

$$u \in C^M(\bar{\mathbf{R}}),$$
$$(6.2) \qquad |D^{\mathbf{k}}u(\mathbf{x})| \leqq K|\mathbf{x}|^{-\zeta-|\mathbf{k}|} \quad \text{as} \quad |\mathbf{x}| \to \infty, \quad 0 \leqq |\mathbf{k}| \leqq M+1,$$

since

$$|\bar{D}^{\mathbf{k}}v(\bar{\mathbf{x}})| \leqq K \sum_{\substack{\mathbf{i} \\ |\mathbf{i}| \leqq |\mathbf{k}|}} |\bar{\mathbf{x}}|^{2-n-|\mathbf{i}|-|\mathbf{k}|}|D^{\mathbf{i}}u(\mathbf{x})| \leqq K|\bar{\mathbf{x}}|^{2-n+\zeta-|\mathbf{k}|} = K|\bar{\mathbf{x}}|^{\beta-|\mathbf{k}|}$$

as $|\bar{\mathbf{x}}| \to 0$.

Let $v(\bar{\mathbf{x}}, h)$ be the solution of the discrete problem

$$-\Delta_h v(\bar{\mathbf{x}}, h) = \bar{f}(\bar{\mathbf{x}}), \qquad \bar{\mathbf{x}} \in \mathbf{D},$$
$$v(\bar{\mathbf{x}}, h) = \bar{g}(\bar{\mathbf{x}}), \qquad \bar{\mathbf{x}} \in \partial\mathbf{D},$$

and define

$$u(\mathbf{x}, h) \equiv |\mathbf{x}|^{2-n}v(\bar{\mathbf{x}}, h),$$
$$\bar{e}(\bar{\mathbf{x}}, h) \equiv v(\bar{\mathbf{x}}, h) - v(\bar{\mathbf{x}}),$$
$$e(\mathbf{x}, h) \equiv u(\mathbf{x}, h) - u(\mathbf{x}).$$

THEOREM 6.1. *If* (6.2) *is satisfied by* $u(\mathbf{x})$ *and if the mesh on* $\mathbf{D}$ *satisfies* (3.1), *then*

$$|e(\mathbf{x}, h)| \leqq K(\epsilon)\begin{cases} h^{\zeta-\epsilon}|\mathbf{x}|^{-\epsilon} & \textit{for} \quad 0 < \zeta \leqq 2, \\ h^2|\mathbf{x}|^{2-\zeta} & \textit{for} \quad 2 < \zeta < n, \\ h^2|\mathbf{x}|^{2-n} & \textit{for} \quad n < \zeta. \end{cases}$$

Proof. From Theorem 3.1 it is clear that

$$|\bar{e}(\bar{\mathbf{x}}, h)| \leqq K(\epsilon)\begin{cases} h^{\beta+n-2-\epsilon}|\bar{\mathbf{x}}|^{\epsilon+2-n} & \text{for} \quad 2-n < \beta \leqq 4-n, \\ h^2|\bar{\mathbf{x}}|^{\beta-2} & \text{for} \quad 4-n < \beta < 2, \\ h^2 & \text{for} \quad 2 < \beta, \end{cases}$$

where $\beta = \zeta + 2 - n$. Since $e(\mathbf{x}, h) = |\mathbf{x}|^{2-n}\bar{e}(\bar{\mathbf{x}}, h)$, the conclusion follows.

The reader will note that no distinction is made in the theorem between the case where the point at ∞ lies on $\partial\mathbf{R}$ (and hence the origin belongs to $\partial\mathbf{D}$) and the situation where $\partial\mathbf{R}$ is of finite length.

The results of this paper can also be applied to the question of convergence of difference methods for two-dimensional regions with corners.

This application is fully discussed in [4] together with its connection with the results of Laasonen [5].

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