

# Using Finite Element Tools in Proving Shift Theorems for Elliptic Boundary Value Problems

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## SUMMARY

We consider the Laplace equation under mixed boundary conditions on a polygonal domain  $\Omega$ . Regularity estimates in terms of Sobolev norms of fractional order for this type of problem are proved. The analysis is based on new interpolation results and multilevel representation of norms on the Sobolev spaces  $H^\alpha(\Omega)$ . The Fourier transform and the construction of extension operators to Sobolev spaces on  $\mathbb{R}^2$  are avoided in the proofs of the interpolation theorems. Copyright © 2000 John Wiley & Sons, Ltd.

## 1. Introduction

Regularity estimates of the solutions of elliptic boundary value problems in terms of Sobolev norms of fractional order are known as shift theorems or shift estimates. Applications of the shift theorems in the finite element theory can be found for example in Nitsche's duality argument, multigrid convergence theorems, convergence of "mortar" finite element methods, etc.. The shift estimates for the Laplace operator with Dirichlet boundary conditions on nonsmooth domains are well known (see, e.g, [21], [23],[27]). For the second order elliptic boundary value problems with mixed boundary conditions on nonsmooth domains, much less has been done. Even though the shift estimates concerning mixed boundary conditions are often used in the finite element theory, a proof of the regularity estimates is not given, to the best of our knowledge, in any publication. The paper provides a complete proof for the case of polygonal domains. Some of the interpolation results needed for the proof are new and of interest by themselves. The subspace interpolation theory developed in Section 2 can be used for proving regularity estimates for other elliptic boundary value problems for which the associated differential operators are Fredholm operators. The interpolation method used in this paper is the real method of interpolation of Lions and Peetre [2], [24] and [25]. The

interpolation problems to which we are led of the following type. If  $X$  and  $Y$  are Sobolev spaces of integer order and  $X_K$  is a subspace of finite codimension of  $X$  then how can one characterize the interpolation spaces between  $X_K$  and  $Y$ ? The problem was studied by Kellogg, for certain particular cases, in [21] when  $X_K$  was of codimension one.

The rest of the paper is organized as follows. In Section 2, after a short introduction in interpolation between Hilbert spaces and subspaces, a formula connecting the norms on the intermediate subspaces  $[X_K, Y]_s$  and  $[X, Y]_s$  is given for the case when  $X_K$  is of arbitrary finite codimension. The main result of Section 2 is a theorem which provides sufficient conditions (the conditions **(A1)** and **(A2)**) for concluding that the spaces  $[X_K, Y]_s$  and  $[X, Y]_s$  coincide. In Section 3, we use multilevel representations of the norms for the Sobolev spaces on sector domains in order to prove subspace interpolation theorems. The multilevel representations of norms (cf. [13], [15] and [28]) involved in Section 3 allows us to derive a simplified version of the main result of Kellogg [21] concerning the subspace interpolation problem when the subspace has codimension one. Using classical preconditioning techniques ([8]-[14]), a proof of the fact that the multilevel norm on  $H^1$  is equivalent to the standard norm on  $H^1$  is presented in the Appendix. The reader familiar with multilevel theory might skip Section 5.3.

Shift theorems for the Poisson equation (with mixed boundary conditions) on polygonal domains are considered in Section 4.

## 2. Interpolation results

In this section we give some basic definitions and results concerning interpolation between Hilbert spaces and subspaces using the real method of interpolation of Lions and Peetre (see [24]). The results presented in Section 2.2 are new.

### 2.1. Interpolation between Hilbert spaces

Let  $X, Y$  be separable Hilbert spaces with inner products  $(\cdot, \cdot)_X$  and  $(\cdot, \cdot)_Y$ , respectively, and satisfying for some positive constant  $c$ ,

$$\begin{cases} X \text{ is a dense subset of } Y \text{ and} \\ \|u\|_Y \leq c\|u\|_X \quad \text{for all } u \in X, \end{cases} \quad (1)$$

where  $\|u\|_X^2 = (u, u)_X$  and  $\|u\|_Y^2 = (u, u)_Y$ .

Let  $D(S)$  denote the subset of  $X$  consisting of all elements  $u$  such that the antilinear form

$$v \rightarrow (u, v)_X, \quad v \in X \quad (2)$$

is continuous in the topology induced by  $Y$ .

For any  $u$  in  $D(S)$  the antilinear form (2) can be extended to a continuous antilinear form on  $Y$ . Then by Riesz representation theorem, there exists an element  $Su$  in  $Y$  such that

$$(u, v)_X = (Su, v)_Y \quad \text{for all } v \in X. \quad (3)$$

In this way  $S$  is a well defined operator in  $Y$ , with domain  $D(S)$ . The next result illustrates the properties of  $S$ .

**Proposition 2.1.** *The domain  $D(S)$  of the operator  $S$  is dense in  $X$  and consequently  $D(S)$  is dense in  $Y$ . The operator  $S : D(S) \subset Y \rightarrow Y$  is a bijective, self-adjoint and positive definite operator. The inverse operator  $S^{-1} : Y \rightarrow D(S) \subset Y$  is a bounded symmetric positive definite operator and*

$$(S^{-1}z, u)_X = (z, u)_Y \quad \text{for all } z \in Y, u \in X. \quad (4)$$

The interpolating space  $[X, Y]_s$  for  $s \in (0, 1)$  is defined using the  $K$  function, where for  $u \in Y$  and  $t > 0$ ,

$$K(t, u) := \inf_{u_0 \in X} (\|u_0\|_X^2 + t^2 \|u - u_0\|_Y^2)^{1/2}.$$

Then  $[X, Y]_s$  consists of all  $u \in Y$  such that

$$\int_0^\infty t^{-(2s+1)} K(t, u)^2 dt < \infty.$$

The norm on  $[X, Y]_s$  is defined by

$$\|u\|_{[X, Y]_s}^2 := \mathbf{c}_s^2 \int_0^\infty t^{-(2s+1)} K(t, u)^2 dt,$$

where

$$\mathbf{c}_s := \left( \int_0^\infty \frac{t^{(1-2s)}}{t^2 + 1} dt \right)^{-1/2} = \sqrt{\frac{2}{\pi} \sin(\pi s)}$$

By definition we take

$$[X, Y]_0 := X \quad \text{and} \quad [X, Y]_1 := Y.$$

The next lemma provides the relation between  $K(t, u)$  and the operator  $S$ . A proof of the lemma can be found in [24]-Chapter 1.15.

**Lemma 2.1.** *For all  $u \in Y$  and  $t > 0$ ,*

$$K(t, u)^2 = t^2 ((I + t^2 S^{-1})^{-1} u, u)_Y.$$

**Remark 2.1.** *Lemma 2.1 gives another expression for the norm on  $[X, Y]_s$ , namely*

$$\|u\|_{[X, Y]_s}^2 := \mathbf{c}_s^2 \int_0^\infty t^{-2s+1} ((I + t^2 S^{-1})^{-1} u, u)_Y dt. \quad (5)$$

*In addition, using this new expression for the norm (see Definition 2.1 and Theorem 15.1 in [24]), it follows that the intermediate space  $[X, Y]_s$  coincides topologically with the domain of the unbounded operator  $S^{1/2(1-s)}$  equipped with the norm of the graph of the same operator. As a consequence we have that  $X$  is dense in  $[X, Y]_s$  for any  $s \in [0, 1]$ .*

## 2.2. Interpolation between subspaces of a Hilbert space

Let  $\mathcal{K} = \text{span}\{\varphi_1, \dots, \varphi_n\}$  be an  $n$ -dimensional subspace of  $X$  and let  $X_{\mathcal{K}}$  be the orthogonal complement of  $\mathcal{K}$  in  $X$  in the  $(\cdot, \cdot)_X$  inner product. We are interested in determining the interpolation spaces of  $X_{\mathcal{K}}$  and  $Y$ , where on  $X_{\mathcal{K}}$  we consider again the  $(\cdot, \cdot)_X$  inner product. For certain spaces  $X_{\mathcal{K}}$  and  $Y$  and  $n = 1$ , this problem was studied in [21]. To apply the

interpolation results from the previous section we need to check that the density part of the condition (1) is satisfied for the pair  $(X_{\mathcal{K}}, Y)$ .

For  $\varphi \in \mathcal{K}$ , define the linear functional  $\Lambda_{\varphi} : X \rightarrow C$ , by

$$\Lambda_{\varphi} u := (u, \varphi)_X, \quad u \in X.$$

**Lemma 2.2.** *The space  $X_{\mathcal{K}}$  is dense in  $Y$  if and only if the following condition is satisfied:*

$$\begin{cases} \Lambda_{\varphi} \text{ is not bounded in the topology of } Y \\ \text{for all } \varphi \in \mathcal{K}, \varphi \neq 0. \end{cases} \quad (6)$$

*Proof.* First let us assume that the condition (6) does not hold. Then for some nonzero  $\varphi \in \mathcal{K}$  the functional  $L_{\varphi}$  is a bounded functional in the topology induced by  $Y$ . Thus, the kernel of  $L_{\varphi}$  is a closed subspace of  $X$  in the topology induced by  $Y$ . Since  $X_{\mathcal{K}}$  is contained in  $\text{Ker}(L_{\varphi})$  it follows that

$$\overline{X_{\mathcal{K}}}^Y \subset \overline{\text{Ker}(L_{\varphi})}^Y = \text{Ker}(L_{\varphi}).$$

Hence  $X_{\mathcal{K}}$  fails to be dense in  $Y$ .

Conversely, assume that  $X_{\mathcal{K}}$  is not dense in  $Y$ , then  $Y_0 = \overline{X_{\mathcal{K}}}^Y$  is a proper closed subspace of  $Y$ . Let  $y_0 \in Y$  be in the orthogonal complement of  $Y_0$ , and define the linear functional  $\Psi : Y \rightarrow C$ , by

$$\Psi u := (u, y_0)_Y, \quad u \in Y.$$

$\Psi$  is a continuous functional on  $Y$ . Let  $\psi$  be the restriction of  $\Psi$  to the space  $X$ . Then  $\psi$  is a continuous functional on  $X$ . By Riesz Representation Theorem, there is  $v_0 \in X$  such that

$$(u, v_0)_X = (u, y_0)_Y, \quad \text{for all } u \in X. \quad (7)$$

Let  $P_{\mathcal{K}}$  be the  $X$  orthogonal projection onto  $\mathcal{K}$  and take  $u = (I - P_{\mathcal{K}})v_0$  in (7). Since  $(I - P_{\mathcal{K}})v_0 \in X_{\mathcal{K}}$  we have  $((I - P_{\mathcal{K}})v_0, y_0)_Y = 0$  and

$$0 = ((I - P_{\mathcal{K}})v_0, v_0)_X = ((I - P_{\mathcal{K}})v_0, (I - P_{\mathcal{K}})v_0)_X.$$

It follows that  $v_0 = P_{\mathcal{K}}v_0 \in \mathcal{K}$  and, via (7), that  $\psi = \Lambda_{v_0}$  is continuous in the topology of  $Y$ . This is exactly the opposite of (6) and the proof is complete. ■

**Remark 2.2.** *The result still holds if we replace the finite dimensional subspace  $\mathcal{K}$  with any closed subspace of  $X$ .*

For the next part of this section we assume that the condition (6) holds. By the above lemma, the condition (1) is satisfied. It follows from the previous section that the operator  $S_{\mathcal{K}} : D(S_{\mathcal{K}}) \subset Y \rightarrow Y$  defined by

$$(u, v)_X = (S_{\mathcal{K}}u, v)_Y \quad \text{for all } v \in X_{\mathcal{K}}, \quad (8)$$

has the same properties as  $S$ . Consequently, the norm on the intermediate space  $[X_{\mathcal{K}}, Y]_s$  is given by:

$$\|u\|_{[X_{\mathcal{K}}, Y]_s}^2 := \mathbf{c}_s^2 \int_0^\infty t^{-2s+1} ((I + t^2 S_{\mathcal{K}}^{-1})^{-1} u, u)_Y \, dt. \quad (9)$$

Let  $[X, Y]_{s, \mathcal{K}}$  denote the closure of  $X_{\mathcal{K}}$  in  $[X, Y]_s$ . Our aim in this section is to determine sufficient conditions on the functions  $\varphi_i$  such that

$$[X_{\mathcal{K}}, Y]_s = [X, Y]_{s, \mathcal{K}}. \quad (10)$$

First, we note that the operators  $S_{\mathcal{K}}$  and  $S$  are related by the following identity:

$$S_{\mathcal{K}}^{-1} = (I - Q_{\mathcal{K}})S^{-1}, \quad (11)$$

where  $Q_{\mathcal{K}} : X \rightarrow \mathcal{K}$  is the orthogonal projection onto  $\mathcal{K}$ . The proof of (11) follows easily from the definitions of the operators involved.

Next, (11) leads to a formula relating the norms on  $[X_{\mathcal{K}}, Y]_s$  and  $[X, Y]_s$ . Before deriving this formula in Theorem 2.1, we introduce some notation. Let

$$(u, v)_{X,t} := ((I + t^2 S^{-1})^{-1} u, v)_X \quad \text{for all } u, v \in X. \quad (12)$$

and denote by  $M_t$  the Gram matrix associated with the set of vectors  $\{\varphi_1, \dots, \varphi_n\}$  in the  $(\cdot, \cdot)_{X,t}$  inner product, i.e.,

$$(M_t)_{ij} := (\varphi_j, \varphi_i)_{X,t}, \quad i, j \in \{1, \dots, n\}.$$

**Theorem 2.1.** *Let  $u$  be arbitrary in  $X_{\mathcal{K}}$ . Then,*

$$\|u\|_{[X_{\mathcal{K}}, Y]_s}^2 = \|u\|_{[X, Y]_s}^2 + \mathbf{c}_s^2 \int_0^\infty t^{-(2s+1)} \langle M_t^{-1} d_t, d_t \rangle dt, \quad (13)$$

where  $\langle \cdot, \cdot \rangle$  is the inner product on  $\mathbf{C}^n$  and  $d_t$  is the  $n$ -dimensional vector in  $\mathbf{C}^n$  whose components are

$$(d_t)_i := (u, \varphi_i)_{X,t}, \quad i = 1, \dots, n.$$

*Proof.* Let  $u$  be fixed in  $X_{\mathcal{K}}$  and denote

$$w := (I + t^2 S^{-1})^{-1} u \quad \text{and} \quad w_{\mathcal{K}} := (I + t^2 S_{\mathcal{K}}^{-1})^{-1} u. \quad (14)$$

Then the norms on  $[X_{\mathcal{K}}, Y]_s$  and  $[X, Y]_s$  are given by

$$\|u\|_{[X, Y]_s}^2 = \mathbf{c}_s^2 \int_0^\infty t^{-2s+1} (w, u)_Y dt \quad (15)$$

and

$$\|u\|_{[X_{\mathcal{K}}, Y]_s}^2 = \mathbf{c}_s^2 \int_0^\infty t^{-2s+1} (w_{\mathcal{K}}, u)_Y dt \quad (16)$$

respectively. For  $v$  in  $Y$ , using (11), we have

$$S_{\mathcal{K}}^{-1} w_{\mathcal{K}} = S^{-1} w_{\mathcal{K}} - Q_{\mathcal{K}}(S^{-1} w_{\mathcal{K}}) = S^{-1} w_{\mathcal{K}} - \sum_{i=1}^n \alpha_i \varphi_i \quad (17)$$

where  $\alpha_i = (S^{-1} w_{\mathcal{K}}, \varphi_i)_X$ . From (14) it follows that

$$(I + t^2 S_{\mathcal{K}}^{-1}) w_{\mathcal{K}} = u. \quad (18)$$

Combining (17) and (18) we obtain

$$(I + t^2 S^{-1}) w_{\mathcal{K}} = u + t^2 \sum_{i=1}^n \alpha_i \varphi_i.$$

Equivalently, applying  $(I + t^2 S^{-1})^{-1}$  to both sides, we have

$$w_{\mathcal{K}} = w + t^2 \sum_{i=1}^n \alpha_i (I + t^2 S^{-1})^{-1} \varphi_i. \quad (19)$$

We calculate the coefficients  $\alpha_i$  by taking the  $(\cdot, \cdot)_X$  inner product with  $\varphi_j$  on both sides of (19) for  $j = 1, \dots, n$ . From (18) one sees that  $w_{\mathcal{K}} \in X_{\mathcal{K}}$ . Hence

$$\sum_{i=1}^n ((I + t^2 S^{-1})^{-1} \varphi_i, \varphi_j)_X \alpha_i = -t^{-2} (w, \varphi_j)_X \quad j = 1, \dots, n.$$

With the notation adopted in (14) and (12) the system becomes

$$\sum_{i=1}^n (\varphi_i, \varphi_j)_{X,t} \alpha_i = -t^{-2} (u, \varphi_j)_{X,t} \quad j = 1, \dots, n.$$

Let  $\alpha$  be the  $n$ -dimensional vector from  $\mathbf{C}^n$  whose components are  $\alpha_i$ . Then

$$M_t \alpha = -t^{-2} d_t.$$

Since the vectors  $\varphi_1, \dots, \varphi_n$  are linearly independent, the matrix  $M_t$  is invertible and

$$\alpha = -t^{-2} M_t^{-1} d_t.$$

Now, going back to (19), we get

$$\begin{aligned} (w_{\mathcal{K}}, u)_Y &= (w, u)_Y + \sum_{i=1}^n \alpha_i (t^2 (I + t^2 S^{-1})^{-1} \varphi_i, u)_Y \\ &= (w, u)_Y + \sum_{i=1}^n \alpha_i (t^2 S^{-1} (I + t^2 S^{-1})^{-1} \varphi_i, u)_X \\ &= (w, u)_Y + \sum_{i=1}^n \alpha_i ((\varphi_i, u)_X - (I + t^2 S^{-1})^{-1} \varphi_i, u)_X \\ &= (w, u)_Y - \sum_{i=1}^n \alpha_i (\bar{d}_t)_i. \end{aligned}$$

Thus

$$(w_{\mathcal{K}}, u)_Y = (w, u)_Y + t^{-2} \langle M_t^{-1} d_t, d_t \rangle. \quad (20)$$

Combining (15), (16) and (20) completes the proof. ■

For  $n = 1$ , let  $\mathcal{K} = \text{span}\{\varphi\}$  and denote  $X_{\mathcal{K}}$  by  $X_{\varphi}$ . Then, for  $u \in X_{\varphi}$ , the formula (13) becomes

$$\|u\|_{[X_{\varphi}, Y]_s}^2 = \|u\|_{[X, Y]_s}^2 + \mathbf{c}_s^2 \int_0^\infty t^{-(2s+1)} \frac{|(u, \varphi)_{X,t}|^2}{(\varphi, \varphi)_{X,t}} dt. \quad (21)$$

Next theorem gives sufficient conditions for (10) to be satisfied. Before we state the result, we introduce the following conditions:

(A.1)  $[X_{\varphi_i}, Y]_s = [X, Y]_{s, \varphi_i}$  for  $i = 1, \dots, n$ .

(A.2) There exist  $\delta > 0$  and  $\gamma > 0$  such that

$$\sum_{i=1}^n |\alpha_i|^2 (\varphi_i, \varphi_i)_{X,t} \leq \gamma \langle M_t \alpha, \alpha \rangle \quad \text{for all } \alpha = (\alpha_1, \dots, \alpha_n)^t \in \mathbf{C}^n, \quad t \in (\delta, \infty).$$

**Theorem 2.2.** Assume that, for some  $s \in (0, 1)$ , the conditions (A.1) and (A.2) hold. Then

$$[X_{\mathcal{K}}, Y]_s = [X, Y]_{s, \mathcal{K}}.$$

*Proof.* Let  $s$  be fixed in  $(0, 1)$ . Since  $X_{\mathcal{K}}$  is dense in both these spaces, in order to prove (10) it is enough to find, for a fixed  $s$ , positive constants  $c_1$  and  $c_2$  such that

$$c_1 \|u\|_{[X, Y]_s} \leq \|u\|_{[X_{\mathcal{K}}, Y]_s} \leq c_2 \|u\|_{[X, Y]_s} \quad \text{for all } u \in X_{\mathcal{K}}. \quad (22)$$

The function under the integral sign in (13) is nonnegative, so the lower inequality of (22) is satisfied with  $c_1 = 1$ . For the upper part, we notice that, for  $u \in X_{\mathcal{K}}$  and  $w_{\mathcal{K}}$  as defined in the proof of Theorem 2.1,

$$\begin{aligned} (w_{\mathcal{K}}, u)_Y &= ((I + t^2 S_{\mathcal{K}}^{-1})^{-1} u, u)_Y = (u, u)_Y - t^2 (S_{\mathcal{K}}^{-1} (I + t^2 S_{\mathcal{K}}^{-1})^{-1} u, u)_Y \\ &\leq (u, u)_Y \leq c(s) \|u\|_{[X, Y]_s}^2 \end{aligned}$$

Then, using (16), (20) and the above estimate, we have that for any positive number  $\delta$ ,

$$\begin{aligned} \|u\|_{[X_{\mathcal{K}}, Y]_s}^2 &\leq c(\delta, s) \|u\|_{[X, Y]_s}^2 + \int_{\delta}^{\infty} t^{-2s+1} (w_{\mathcal{K}}, u)_Y^2 dt \\ &\leq c(\delta, s) \|u\|_{[X, Y]_s}^2 + \int_{\delta}^{\infty} t^{-2s+1} (w, u)_Y^2 dt + \int_{\delta}^{\infty} t^{-2s+1} \langle M_t^{-1} d_t, d_t \rangle dt. \end{aligned}$$

Hence the upper inequality of (22) is satisfied if one can find a positive  $\delta$  and  $c = c(\delta)$  such that

$$\int_{\delta}^{\infty} t^{-2s+1} \langle M_t^{-1} d_t, d_t \rangle dt \leq c \|u\|_{[X, Y]_s}^2 \quad \text{for all } u \in X_{\mathcal{K}}. \quad (23)$$

From (A.2), there exist  $\delta > 0$  and  $\gamma > 0$  such that

$$\langle M_t^{-1} \alpha, \alpha \rangle \leq \gamma \sum_{i=1}^n |\alpha_i|^2 (\varphi_i, \varphi_i)_{X,t}^{-1}$$

for all  $\alpha = (\alpha_1, \dots, \alpha_n)^t \in \mathbf{C}^n$ ,  $t \in (\delta, \infty)$ . In particular, for  $\alpha_i = (u, \varphi_i)_{X,t}$ ,  $i = 1, \dots, n$ , we obtain

$$\langle M_t^{-1} d_t, d_t \rangle \leq \gamma \sum_{i=1}^n \frac{|(u, \varphi_i)_{X,t}|^2}{(\varphi_i, \varphi_i)_{X,t}} \quad \text{for all } t \in (\delta, \infty), u \in X_{\mathcal{K}}.$$

Thus, using the above estimate, (21) and (A.1) we have

$$\begin{aligned}
\int_{\delta}^{\infty} t^{-2s+1} \langle M_t^{-1} d_t, d_t \rangle dt &\leq \gamma \sum_{i=1}^n \int_{\delta}^{\infty} t^{-2s+1} \frac{|(u, \varphi_i)_{X,t}|^2}{(\varphi_i, \varphi_i)_{X,t}} dt \\
&\leq \gamma \sum_{i=1}^n \int_0^{\infty} t^{-2s+1} \frac{|(u, \varphi_i)_{X,t}|^2}{(\varphi_i, \varphi_i)_{X,t}} dt \\
&\leq \gamma c_s^{-2} \sum_{i=1}^n \|u\|_{[X_{\varphi_i}, Y]_s}^2 \leq \gamma c_s^{-2} n \|u\|_{[X, Y]_s}^2
\end{aligned}$$

Finally, (23) holds, and the result is proved. ■

**Remark 2.3.** By Lemma 2.2, the space  $X_K$  is dense in  $[X, Y]_s$  if and only if the functionals  $L_{\varphi}$ ,  $\varphi \in K, \varphi \neq 0$  are not bounded in the topology induced by  $[X, Y]_s$ .

### 2.3. A subspace interpolation lemma

Let  $\Omega \subset \tilde{\Omega}$  be domains in  $\mathbb{R}^2$  and  $V^1(\Omega)$ ,  $V^1(\tilde{\Omega})$  be subspaces of  $H^1(\Omega)$ ,  $H^1(\tilde{\Omega})$ , respectively. On  $V^1(\Omega)$ ,  $V^1(\tilde{\Omega})$  we consider inner products such that the induced norms are equivalent with the standard norms on  $H^1(\Omega)$ ,  $H^1(\tilde{\Omega})$ , respectively. In addition, we assume that  $V^1(\Omega)$ ,  $V^1(\tilde{\Omega})$  are dense in  $L^2(\Omega)$ ,  $L^2(\tilde{\Omega})$ , respectively. Let's denote the duals of  $V^1(\Omega)$ ,  $V^1(\tilde{\Omega})$  by  $V^{-1}(\Omega)$ ,  $V^{-1}(\tilde{\Omega})$ , respectively. We suppose that there are linear operators  $E$  and  $R$  such that

$$E : L^2(\Omega) \rightarrow L^2(\tilde{\Omega}), \quad E : V^1(\Omega) \rightarrow V^1(\tilde{\Omega}) \quad \text{are bounded operators,} \quad (24)$$

$$R : L^2(\tilde{\Omega}) \rightarrow L^2(\Omega), \quad R : V^1(\tilde{\Omega}) \rightarrow V^1(\Omega), \quad \text{are bounded operators,} \quad (25)$$

$$REu = u \quad \text{for all } u \in L^2(\Omega). \quad (26)$$

Let  $\psi \in L^2(\Omega)$ ,  $\tilde{\psi} = E\psi \in L^2(\tilde{\Omega})$  and  $\theta \in (0, 1)$  be such that

$$L^2(\Omega)_{\psi} := \{u \in L^2(\Omega) : (u, \psi) = 0\} \quad \text{is dense in } [L^2(\Omega), V^{-1}(\Omega)]_{\theta}, \quad (27)$$

$$L^2(\tilde{\Omega})_{\tilde{\psi}} := \{u \in L^2(\tilde{\Omega}) : (u, \tilde{\psi}) = 0\} \quad \text{is dense in } V^{-1}(\tilde{\Omega}), \quad (28)$$

$$[L^2(\tilde{\Omega})_{\tilde{\psi}}, V^{-1}(\tilde{\Omega})]_{\theta} = [L^2(\tilde{\Omega}), V^{-1}(\tilde{\Omega})]_{\theta}. \quad (29)$$

**Lemma 2.3.** Using the above setting, assume that (24)-(29) are satisfied. Then,

$$[L^2(\Omega)_{\psi}, V^{-1}(\Omega)]_{\theta} = [L^2(\Omega), V^{-1}(\Omega)]_{\theta}. \quad (30)$$

The proof of the above lemma is given in Appendix 5.1.



### 3. Subspace interpolation by multilevel norms

Let  $\Omega$  be a domain in  $\mathbb{R}^2$  with boundary  $\partial\Omega = (\partial\Omega)_D \cup (\partial\Omega)_N$ , where  $(\partial\Omega)_D$  is not of measure zero, and  $(\partial\Omega)_D$  and  $(\partial\Omega)_N$  are essentially disjoint. Let  $H_D^1(\Omega)$  denote the space of all functions in  $H^1(\Omega)$  which vanish on  $(\partial\Omega)_D$ . Assume that

$$M_1 \subset M_2 \subset \dots \subset M_k \subset \dots$$

is a sequence of finite dimensional subspaces of  $H_D^1(\Omega)$  whose union is dense in  $H_D^1(\Omega)$ , and assume that an equivalent norm on  $H_D^1(\Omega)$  is given by

$$\|u\|_1^2 := \sum_{k=1}^{\infty} \lambda_k \|(Q_k - Q_{k-1})u\|^2, \quad (31)$$

where  $Q_k$  denotes the  $L^2(\Omega)$  orthogonal projection onto  $M_k$ ,  $\|\cdot\| = \|\cdot\|_{L^2(\Omega)}$ ,  $Q_0 = 0$ , and  $\lambda_k = 4^{k-1}$ . We will prove in the Appendix that for a certain type of polygonal domain  $\Omega$  and  $\{M_k\}$  the standard sequence of piecewise linear functions associated with a sequence of nested meshes, (31) holds. Proofs for the multilevel representation of the norm on  $H^1$  can be found in [28] and [15] also. The goal of this chapter is to solve a codimension one subspace interpolation problem by means of multilevel geometry and topology.

#### 3.1. Scales of multilevel norms

On  $H_D^1(\Omega)$  take the norm given by (31) and define  $H_D^{-1}(\Omega)$  to be the dual of  $H_D^1(\Omega)$ . The elements of  $L^2(\Omega)$  can be viewed as continuous linear functionals on  $H_D^1(\Omega)$  and we have the natural continuous and dense embeddings

$$H_D^1(\Omega) \subset L^2(\Omega) \subset H_D^{-1}(\Omega).$$

One can easily check that

$$\|u\|_{-1}^2 := \sum_{k=1}^{\infty} \lambda_k^{-1} \|(Q_k - Q_{k-1})u\|^2 \quad \text{for all } u \in L^2(\Omega), \quad (32)$$

where  $\|\cdot\|_{-1}$  denotes the norm on  $H_D^{-1}(\Omega)$ . Further, we have that the inner product on  $H_D^\alpha(\Omega)$  is

$$(u, v)_\alpha := \sum_{k=1}^{\infty} \lambda_k^\alpha ((Q_k - Q_{k-1})u, v)_{L^2(\Omega)} \quad \text{for all } u, v \in H_D^\alpha(\Omega) \cap L^2(\Omega), \quad \alpha \in [-1, 1].$$

Then the pairs  $(H_D^1(\Omega), L^2(\Omega))$  and  $(L^2(\Omega), H_D^{-1}(\Omega))$  satisfy the condition (1) and the operator  $S$  associated with each of these pairs is given (in both cases) by

$$Su = \sum_{k=1}^{\infty} \lambda_k (Q_k - Q_{k-1})u, \quad \text{for all } u \in D(S). \quad (33)$$

For any  $\theta \in [0, 1]$ , let

$$H_D^\theta(\Omega) := [H_D^1(\Omega), L^2(\Omega)]_{1-\theta}, \quad H_D^{-\theta}(\Omega) := [L^2(\Omega), H_D^{-1}(\Omega)]_\theta,$$

and let  $\|\cdot\|_\alpha$  be the norm on  $H_D^\alpha(\Omega)$  for  $\alpha \in [-1, 1]$ . By using (5), one can easily check that

$$\|u\|_\alpha^2 := \sum_{k=1}^{\infty} \lambda_k^\alpha \|(Q_k - Q_{k-1})u\|^2, \quad \text{for all } u \in H_D^\alpha(\Omega) \cap L^2(\Omega). \quad (34)$$

Consequently,  $H_D^{-\theta}(\Omega)$  is the dual of  $H_D^\theta(\Omega)$  for  $\theta \in [0, 1]$ .

**Remark 3.1.** For any  $\alpha \in (0, 1]$ , the norm on  $H_D^\alpha(\Omega)$  is given by (34). On the other hand, for  $u \in H_D^\alpha(\Omega)$ ,

$$\sum_{k=1}^J \lambda_k^\alpha \|(Q_k - Q_{k-1})u\|^2 = \|u\|^2 + (4^\alpha - 1) \sum_{k=1}^{J-1} \lambda_k^\alpha \|(I - Q_k)u\|^2 - \lambda_J^\alpha \|(I - Q_J)u\|^2$$

and

$$\lim_{J \rightarrow \infty} \lambda_J^\alpha \|(I - Q_J)u\|^2 = 0.$$

Thus, we obtain that an equivalent norm on  $H_D^\alpha(\Omega)$ , for  $\alpha \in (0, 1]$ , is given by

$$\|u\|_\alpha^2 := \|u\|^2 + \sum_{k=1}^{\infty} \lambda_k^\alpha \|(I - Q_k)u\|^2.$$

### 3.2. Sufficient conditions for (A1)

Let  $X = L^2(\Omega)$  and  $Y = H_D^{-1}(\Omega)$ . For a fixed  $\theta_0$  in the interval  $(0, 1)$ , let  $\phi \in L^2(\Omega)$  satisfy the following conditions:

**(C.0)**  $\phi \notin H_D^{\theta_0}(\Omega)$ .

**(C.1)** There exist  $c_1 > 0$  and  $\delta > 0$  such that

$$(\phi, \phi)_{X,t} = \sum_{k=1}^{\infty} \frac{\lambda_k}{\lambda_k + t^2} \|(Q_k - Q_{k-1})\phi\|^2 \geq c_1 t^{-2\theta_0}, \text{ for } t \geq \delta.$$

**(C.2)** There exist  $c_2 > 0$  such that

$$\|(Q_k - Q_{k-1})\phi\|^2 \leq c_2 \lambda_k^{-\theta_0}, \quad k = 1, 2, \dots$$

Our goal in this section is to characterize the space  $[X_\phi, Y]_\theta$  for  $\theta$  in  $(0, 1)$ ,  $\theta \neq \theta_0$ .

**Remark 3.2.** From **(C.2)** it follows that  $\phi \in H_D^\theta(\Omega)$  for  $\theta < \theta_0$ . Thus, from **(C.0)** and **(C.2)**, by applying Lemma 2.2 (see the proof of (36)), we have that  $X_\phi$  is dense in  $Y$ . Consequently, the space  $[X_\phi, Y]_s$  is well defined.

**Theorem 3.1.** Let  $\phi \in L^2(\Omega)$  and satisfy **(C.0)**-**(C.2)**. Then

$$[L^2(\Omega)_\phi, H_D^{-1}(\Omega)]_\theta = [L^2(\Omega), H_D^{-1}(\Omega)]_{\theta, \phi}, \quad 0 \leq \theta \leq 1, \quad \theta \neq \theta_0. \quad (35)$$

Furthermore, if  $\theta_0 < \theta \leq 1$  then

$$[L^2(\Omega)_\phi, H_D^{-1}(\Omega)]_\theta = [L^2(\Omega), H_D^{-1}(\Omega)]_\theta. \quad (36)$$

*Proof.* Let  $\theta \neq \theta_0$  be fixed. Following the proof of Theorem 2.2 until (23), we see that in order to prove (35), it is enough to show that, for  $\delta$  given by (C.1), there is a positive constant  $c = c(\theta, \delta, c_1, c_2)$  satisfying

$$I := \int_{\delta}^{\infty} t^{-(2\theta+1)} \frac{|(u, \phi)_{X,t}|^2}{(\phi, \phi)_{X,t}} dt \leq c \|u\|_{-\theta}^2 \quad \text{for all } u \in X_{\phi}. \quad (37)$$

Let  $u \in X = L^2(\Omega)$  be fixed. Denote  $Q_k - Q_{k-1}$  by  $q_k$ , with  $Q_0 = 0$ , and for  $u \in L^2(\Omega)$  denote  $\tilde{u}_k := \lambda_k^{-\theta/2} \|q_k u\|$  and  $\tilde{u} := \{u_k\}$ . Then we have

$$\|u\|_{-\theta} = \|\tilde{u}\|_{l_2}.$$

Here  $(\cdot, \cdot)_X$  is simply the  $L^2(\Omega)$  inner product  $(\cdot, \cdot)$ . Then, we have

$$(u, \phi)_{X,t} = ((I + t^2 S^{-1})^{-1} u, \phi) = \sum_{k=1}^{\infty} \frac{\lambda_k}{\lambda_k + t^2} (q_k u, q_k \phi).$$

Using the Cauchy-Schwarz inequality and the estimate given by (C.2) we obtain

$$|(u, \phi)_{X,t}| \leq c_2 \sum_{k=1}^{\infty} \frac{\lambda_k^{1-\theta_0/2}}{\lambda_k + t^2} \|q_k u\|. \quad (38)$$

For  $u \in X_{\phi}$  we have  $(u, \phi) = 0$ . Then

$$\sum_{k=1}^{\infty} (q_k u, \phi) = 0.$$

Thus,

$$(u, \phi)_{X,t} = -t^2 \sum_{k=1}^{\infty} \frac{1}{\lambda_k + t^2} (q_k u, q_k \phi),$$

and hence we also have the estimate

$$|(u, \phi)_{X,t}| \leq c_2 t^2 \sum_{k=1}^{\infty} \frac{\lambda_k^{-\theta_0/2}}{\lambda_k + t^2} \|q_k u\|. \quad (39)$$

Now we are prepared to estimate the integral  $I$ . The constant  $c$ , to be used next, may have different values at different places in which it appears but depends only on the constants  $\theta$ ,  $\delta$ ,  $c_1$  and  $c_2$ . First we will treat the case  $0 < \theta < \theta_0$ . Let  $\theta_1 = \theta_0 - \theta$ . Then, by (C.1) and the estimate (38), we have

$$\begin{aligned} I &\leq c \int_{\delta}^{\infty} t^{-1+2\theta_1} \left( \sum_{k=1}^{\infty} \frac{\lambda_k^{1-\theta_0/2}}{\lambda_k + t^2} \|q_k u\| \right)^2 dt \\ &\leq c \int_{\delta}^{\infty} t^{-1+2\theta_1} \left( \sum_{m,n=1}^{\infty} \frac{(\lambda_m \lambda_n)^{1-\theta_0/2}}{(\lambda_m + t^2)(\lambda_n + t^2)} \|q_m u\| \|q_n u\| \right) dt \\ &= c \sum_{m,n=1}^{\infty} (\lambda_m \lambda_n)^{1-\theta_0/2} \|q_m u\| \|q_n u\| \int_{\delta}^{\infty} \frac{t^{-1+2\theta_1}}{(\lambda_m + t^2)(\lambda_n + t^2)} dt. \end{aligned}$$

Next, we use the formula

$$\int_0^\infty \frac{t^{3-2\theta}}{(a+t^2)(b+t^2)} dt = \frac{1}{\mathbf{c}_\theta^2} \frac{a^{1-\theta} - b^{1-\theta}}{a-b}, \quad 0 < \theta < 2, \quad \theta \neq 1, \quad a, b > 0. \quad (40)$$

The integral can be calculated by elementary calculus methods. If  $a = b$ , then the right side of the above identity is replaced by  $\frac{1-\theta}{\mathbf{c}_\theta^2} a^{-\theta}$ . Thus,

$$\int_\delta^\infty \frac{t^{-1+2\theta_1}}{(\lambda_m + t^2)(\lambda_n + t^2)} dt \leq \int_0^\infty \frac{t^{-1+2\theta_1}}{(\lambda_m + t^2)(\lambda_n + t^2)} dt = \mathbf{c}_{\theta_1}^{-2} (\lambda_m \lambda_n)^{\theta_1-1} \frac{\lambda_m^{1-\theta_1} - \lambda_n^{1-\theta_1}}{\lambda_m - \lambda_n}.$$

Combining the above inequalities, we get

$$I \leq c \sum_{m,n=1}^\infty (\lambda_m \lambda_n)^{\theta_1/2} \frac{\lambda_m^{1-\theta_1} - \lambda_n^{1-\theta_1}}{\lambda_m - \lambda_n} \lambda_m^{-\theta/2} \|q_m u\| \lambda_n^{-\theta/2} \|q_n u\|.$$

Let

$$l_{mn} = (\lambda_m \lambda_n)^{\theta_1/2} \frac{\lambda_m^{1-\theta_1} - \lambda_n^{1-\theta_1}}{\lambda_m - \lambda_n}.$$

Then, the above estimate becomes

$$I \leq c \sum_{m,n=1}^\infty l_{mn} \tilde{u}_m \tilde{u}_n.$$

An elementary calculation gives

$$l_{mn} = \frac{2^{(m-n)(1-\theta_1)} - 2^{-(m-n)(1-\theta_1)}}{2^{(m-n)} - 2^{-(m-n)}} \leq 2^{-|m-n|\theta_1}, \quad m, n = 1, 2, \dots$$

Now we can apply Lemma 5.1 and obtain

$$I \leq c \|\tilde{u}\|_{l_2}^2 = c \|u\|_{-\theta}^2,$$

which proves (37) in this case.

For the remaining part, i.e.,  $\theta_0 < \theta < 1$ , we set  $\theta_1 := \theta - \theta_0$ . The estimate (37) can be made in the same manner. The only difference here is that we use the inequality (39) instead of (38). This completes the proof of (36).

Now let  $\theta_0 < \theta \leq 1$  be fixed. By the previous part, it is enough to show that  $L^2(\Omega)_\phi$  is dense in  $H_D^{-\theta}(\Omega)$ . Using Lemma 2.2, this is equivalent to proving that the functional

$$u \rightarrow (u, \phi), \quad u \in L^2(\Omega), \quad (41)$$

is not continuous in the topology induced by  $H_D^{-\theta}(\Omega)$ . To see that, let  $\{u_n\}$  be the sequence in  $L^2(\Omega)$  defined by

$$u_n := \sum_{k=1}^n \lambda_k^{\theta_0} q_k \phi.$$

From (C.0) we have that

$$(u_n, \phi) = \sum_{k=1}^n \lambda_k^{\theta_0} \|q_k \phi\|^2 \rightarrow \infty,$$

as  $n \rightarrow \infty$ . On the other hand, using (C.2)

$$(u_n, u_n)_{-\theta} = \sum_{k=1}^n \lambda_k^{-\theta+2\theta_0} \|q_k \phi\|^2$$

is uniformly bounded. Therefore, the functional defined in (41) is not continuous and (36) is proved.  $\blacksquare$

#### 4. Applications to shift theorem for the Laplace operator on polygonal domains.

Let  $\Omega$  be a polygonal domain in  $\mathbf{R}^2$  with boundary  $\partial\Omega = (\partial\Omega)_D \cup (\partial\Omega)_N$ , where  $(\partial\Omega)_D$  is not of measure zero, and  $(\partial\Omega)_D$  and  $(\partial\Omega)_N$  are essentially disjoint and consist of a finite number of closed line segments. Let  $\partial\Omega$  be the polygonal arc  $P_1 P_2 \cdots P_m P_1$ . Here we consider that the set  $\{P_1, P_2, \dots, P_m\}$  consists of all vertices of  $\partial\Omega$  and all the points of  $(\partial\Omega)_D \cap (\partial\Omega)_N$ . We will also call the points of  $(\partial\Omega)_D \cap (\partial\Omega)_N$  vertices of  $\partial\Omega$ . At each point  $P_j$ , we denote the measure of  $\angle P_{j-1} P_j P_{j+1}$  (measured from inside  $\Omega$ ) by  $\omega_j$ , where  $P_{m+1} = P_1$  and  $P_0 = P_m$ . For  $j = 1, 2, \dots, m$ , let us define  $\gamma_j := \max\{\omega_j/\pi, 1\}$  if both edges  $[P_j, P_{j-1}]$  and  $[P_j, P_{j+1}]$  belong to the same set  $(\partial\Omega)_D$  or  $(\partial\Omega)_N$ , and  $\gamma_j := \max\{2\omega_j/\pi, 1\}$  if one edge belongs to  $(\partial\Omega)_D$  and the other edge belongs to  $(\partial\Omega)_N$ . Let  $\gamma := \max\{\gamma_j : j = 1, 2, \dots, m\}$ . We consider the boundary value problem for the Poisson equation on  $\Omega$ . Given  $f \in L^2(\Omega)$ , find  $u$  such that

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } (\partial\Omega)_D, \\ \frac{\partial u}{\partial n} = 0 & \text{on } (\partial\Omega)_N. \end{cases} \quad (42)$$

The variational formulation of (42) is : Find  $u \in H_D^1(\Omega)$  such that

$$\int_{\Omega} \nabla u \cdot \nabla v \, dx = \int_{\Omega} f v \, dx \quad \text{for all } v \in H_D^1(\Omega). \quad (43)$$

It is well known that for  $f \in L^2(\Omega)$  the variational problem has a unique solution  $u \in H_D^1(\Omega)$  and

$$\|u\|_{H^1(\Omega)} \leq c \|f\|_{H_D^{-1}(\Omega)} \quad \text{for all } f \in L^2(\Omega), \quad (44)$$

where  $H_D^{-1}(\Omega)$  is the dual of  $H_D^1(\Omega)$ .

Let  $u$  be the solution of (43). By taking  $v$  in  $\mathcal{D}(\Omega)$ , the space of all infinitely differentiable functions with compact support in  $\Omega$ , one has

$$-\Delta u = f$$

in the sense of distributions in  $\Omega$ , so the equality is satisfied pointwise, almost everywhere in  $\Omega$ . Also, the solution  $u$  of (43) satisfies the boundary conditions of (42) (see [20], Chapter 2 therein). In addition, if  $\gamma = 1$  then  $u$  belongs to  $H^2(\Omega) \cap H_D^1(\Omega)$  (see, e.g., [19]), and

$$\|u\|_{H^2(\Omega)} \leq c \|f\|_{L^2(\Omega)} \quad \text{for all } f \in L^2(\Omega). \quad (45)$$

If we define  $T : H_D^{-1}(\Omega) \rightarrow H_D^1(\Omega)$  by  $Tf := u$ , where  $u$  is the solution of (43), then  $T$  is a bounded operator. Moreover, if  $\gamma = 1$ ,  $T$  is a bounded operator from  $L^2(\Omega)$  to  $H^2(\Omega)$ . Thus, by interpolation, we have for any  $s \in [0, 1]$ ,

$$\|u\|_{H^{1+s}(\Omega)} \leq c \|f\|_{H_D^{-1+s}(\Omega)} \quad \text{for all } f \in H_D^{-1+s}(\Omega). \quad (46)$$

Here,  $H^{1+s}(\Omega) := [H^2(\Omega), H^1(\Omega)]_{1-s}$  and  $H_D^{-1+s}(\Omega) := [L^2(\Omega), H_D^{-1}(\Omega)]_{1-s}$ . The main result of this section is given by the following theorem.

**Theorem 4.1.** *Let  $u$  be variational solution of (42) and  $\gamma$  be the number defined above depending only on the largest angle of  $\partial\Omega$  and the type of the boundary conditions. Then for any  $s < 1/\gamma$  there exist a constant  $c$  such tat,*

$$\|u\|_{H^{1+s}(\Omega)} \leq c \|f\|_{H_D^{-1+s}(\Omega)} \quad \text{for all } f \in H_D^{-1+s}(\Omega).$$

#### 4.1. Reduction to sector domains.

For  $j = 1, 2, \dots, m$ , let  $U_j$  be an open disk centered at  $P_j$  such that  $U_j$  contains no vertices other than  $P_j$ . Next we add more disks with centers in  $\partial\Omega$ , say  $U_j$ , centered at  $P_j$ ,  $j = m+1, \dots, M$ , such that  $U_j$  contains no vertices other than  $P_j$ , and

$$\partial\Omega \subset \bigcup_{j=1}^M U_j,$$

By increasing the number  $M$  of disks, we can assume that for some positive numbers  $r_0$  and  $\epsilon$  we have

$$\begin{aligned} U_j \cap \Omega &= \{(r_j, \theta_j) : 0 < r_j < r_0, 0 < \theta_j < \omega_j\} \\ &\subset \{(r_j, \theta_j) : 0 < r_j < (1+\epsilon)r_0, 0 < \theta_j < \omega_j\} := \Omega_j \subset \Omega, \quad j = 1, 2, \dots, M, \end{aligned}$$

where  $(r_j, \theta_j)$  are the polar coordinates with origin at  $P_j$ ,  $\omega_j = \pi$  for  $j = m+1, \dots, M$  and  $P_k$  is not in  $\Omega_j$ , for  $k \neq j$ . Let  $U_0$  and  $\Omega_0$  be two domains with smooth boundaries such that  $\overline{U_0} \subset \Omega_0$  and  $\overline{\Omega_0} \subset \Omega$  and such that

$$\overline{\Omega} \subset \bigcup_{j=0}^M U_j.$$

Then, there is a partition of unity  $\{\phi_j\}_{j=0}^M$  subordinate to the covering  $\bigcup_{j=0}^M U_j$ . Let us denote the restriction of  $\phi_j$  to  $\Omega_j$  by  $\eta_j$  ( $j = 0, 1, \dots, M$ ). Further, we define  $(\partial\Omega_j)_D$  and  $(\partial\Omega_j)_N$  to be

$$(\partial\Omega_j)_N := (\partial\Omega)_N \cap \partial\Omega_j, \quad (\partial\Omega_j)_D := \overline{\partial\Omega_j \setminus (\partial\Omega_j)_N},$$

and denote the space of functions in  $H^1(\Omega_j)$  which vanish on  $(\partial\Omega_j)_D$  by  $H_D^1(\Omega_j)$ , for  $j = 1, 2, \dots, M$ . Also  $(\partial\Omega_0)_D = \partial\Omega_0$ .

We reduce the proof of (46) to the case in which  $\Omega$  is a sector domain. Let us assume for the moment that the following holds.

**Theorem 4.2.** *The variational solution  $u_j$  of (43) relative to  $\Omega_j$ ,  $j = 1, \dots, M$ , satisfies*

$$\|u_j\|_{H^{1+s}(\Omega_j)} \leq c \|f\|_{H_D^{-1+s}(\Omega_j)} \quad \text{for all } f \in L^2(\Omega_j), \quad 0 < s < \gamma_j^{-1}, \quad (47)$$

where we take  $\gamma_j = 1$  for  $j = m+1, \dots, M$ .

Given this result, we can prove that (46) holds for  $\gamma > 1$  and  $s < 1/\gamma$ . Indeed, let  $f \in L^2(\Omega_j)$  and let  $u$  be the solution of (43). For  $j = 0, 1, \dots, M$ , let  $u_j := \eta_j u$ . Then, in the sense of distributions in  $\Omega_j$ , we obtain

$$-\Delta u_j = f \eta_j - u \Delta \eta_j - 2 \nabla u \cdot \nabla \eta_j.$$

Since the boundary conditions of (42) are satisfied on  $(\partial \Omega_j)_D$  and  $(\partial \Omega_j)_N$  for  $u = u_j$ , we have (see [20], Theorem 2.1.1 therein) that  $u_j$  is the unique variational solution of the problem : Find  $u_j \in H_D^1(\Omega_j)$  such that

$$A_j(u_j, v) = \int_{\Omega_j} f_j v \, dx \quad \text{for all } v \in H_D^1(\Omega_j), \quad (48)$$

where  $f_j = f \eta_j - u \Delta \eta_j - 2 \nabla u \cdot \nabla \eta_j$  and

$$A_j(u_j, v) := \int_{\Omega_j} \nabla u_j \cdot \nabla v \, dx.$$

Now,  $f_j$  is a function in  $L^2(\Omega_j)$  and by Theorem 4.2, we get

$$\|u_j\|_{H^{1+s}(\Omega_j)} \leq c \|f_j\|_{H_D^{-1+s}(\Omega_j)}, \quad j = 1, 2, \dots, M. \quad (49)$$

For  $j = 0$  the estimate (49) holds for any  $s \in [0, 1]$ , because the boundary of  $\Omega_0$  is smooth and we can apply the regularity result for domains with smooth boundaries. From the way we have defined the domains  $\Omega_j$  one can find  $r > 0$  such that

$$\text{dist}(\Omega \setminus \Omega_j, \text{supp } u_j) \geq r \quad j = 0, 1, \dots, M.$$

Thus

$$\|u_j\|_{H^{1+s}(\Omega)} \leq c \|u_j\|_{H^{1+s}(\Omega_j)}.$$

Here  $c$  is independent of  $f$  and  $j$ . Since  $u = \sum_{j=0}^M u_j$ , using the triangle inequality, the estimate (49) and the above observation, we obtain

$$\|u\|_{H^{1+s}(\Omega)} \leq c \sum_{j=0}^M \|f_j\|_{H_D^{-1+s}(\Omega_j)}. \quad (50)$$

The estimate of  $\|f_j\|_{H_D^{-1+s}(\Omega_j)}$  is as follows. First,  $L^2(\Omega_j)$  is continuously embedded in  $H_D^{-1+s}(\Omega_j)$ , and multiplication by a smooth function is continuous on  $H_D^{-1+s}(\Omega_j)$ . Thus,

$$\begin{aligned} \|f_j\|_{H_D^{-1+s}(\Omega_j)} &\leq \|f \eta_j\|_{H_D^{-1+s}(\Omega_j)} + c \|u \Delta \eta_j + 2 \nabla u \cdot \nabla \eta_j\|_{L^2(\Omega_j)} \\ &\leq c (\|f\|_{H_D^{-1+s}(\Omega_j)} + \|u\|_{H^1(\Omega_j)}). \end{aligned}$$

Second, the extension by zero operator  $E : H_D^1(\Omega_j) \rightarrow H_D^1(\Omega)$  is continuous. It follows that

$$\|f\|_{H_D^{-1}(\Omega_j)} \leq c \|f\|_{H_D^{-1}(\Omega)} \quad \text{for all } f \in H_D^{-1}(\Omega).$$

Also,

$$\|f\|_{L^2(\Omega_j)} \leq \|f\|_{L^2(\Omega)} \quad \text{for all } f \in L^2(\Omega).$$

By interpolation, we get

$$\|f\|_{H_D^{-1+s}(\Omega_j)} \leq c \|f\|_{H_D^{-1+s}(\Omega)} \quad \text{for all } f \in H_D^{-1+s}(\Omega).$$

Third, we have

$$\|u\|_{H^1(\Omega_j)} \leq \|u\|_{H^1(\Omega)} \leq c \|f\|_{H_D^{-1}(\Omega)} \leq c \|f\|_{H_D^{-1+s}(\Omega)} \quad \text{for all } f \in H_D^{-1+s}(\Omega).$$

Finally, from these inequalities we deduce

$$\|f_j\|_{H_D^{-1+s}(\Omega_j)} \leq c \|f\|_{H_D^{-1+s}(\Omega)} \quad \text{for all } f \in H_D^{-1+s}(\Omega). \quad (51)$$

Thus, from (50) and (51), since  $L^2(\Omega)$  is dense in  $H_D^{-1+s}(\Omega)$ , we obtain that

$$\|u\|_{H^{1+s}(\Omega)} \leq c \|f\|_{H_D^{-1+s}(\Omega)} \quad \text{for all } f \in H_D^{-1+s}(\Omega).$$

Therefore we obtain that (46) holds for  $\gamma > 1$  and all  $s < 1/\gamma$ .

#### 4.2. Solving the problem on sector domains.

Let  $\Omega = S_\omega$  be the sector domain defined by

$$S_\omega := \{(r, \theta) : 0 < r < r_0, 0 < \theta < \omega\}, \quad (52)$$

and let  $(\partial\Omega)_N$  be in one of the possibilities listed below (Case 1, Case 2 or Case 3). We assume, without loss of generality, that  $r_0 = 1$ . Let  $V^2(\Omega) := H^2(\Omega) \cap H_D^1(\Omega)$ . Then, (see e.g., Theorem 2.2.3 in [20]) the Laplace operator  $\Delta : V^2(\Omega) \rightarrow L^2(\Omega)$  is a Fredholm operator. Consequently,

$$\|u\|_{H^2(\Omega)} \leq c \|\Delta u\|_{L^2(\Omega)} \quad \text{for all } u \in V^2(\Omega), \quad (53)$$

and the range of the operator has finite codimension. Grisvard characterized the orthogonal complement  $\mathcal{N}$  of the range of the Laplace operator for the case of a polygonal domain in [19] and [20]. In particular, for our sector domain  $\Omega = S_\omega$  the subspace  $\mathcal{N}$  is described as follows:

- Case 1. 'Dirichlet corner' ;  $(\partial\Omega)_N = \emptyset$ .

- (i)  $0 < \omega \leq \pi$  ;  $\mathcal{N} = \{0\}$ .
- (ii)  $\pi < \omega < 2\pi$  ;  $\mathcal{N} = \text{span}\{\psi\}$ , where

$$\psi(r, \theta) = (r^{-\frac{\pi}{\omega}} - r^{\frac{\pi}{\omega}}) \sin \frac{\pi}{\omega} \theta,$$

- Case 2. 'Neuman corner' ;  $(\partial\Omega)_N = \{(r, \theta) \in \partial\Omega : \theta = 0 \text{ or } \theta = \omega\}$ .

- (i)  $0 < \omega \leq \pi$  ;  $\mathcal{N} = \{0\}$ .
- (ii)  $\pi < \omega < 2\pi$  ;  $\mathcal{N} = \text{span}\{\psi\}$ , where

$$\psi(r, \theta) = (r^{-\frac{\pi}{\omega}} - r^{\frac{\pi}{\omega}}) \cos \frac{\pi}{\omega} \theta$$

- Case 3. 'Mixed corner' ;  $(\partial\Omega)_N = \{(r, \theta) \in \partial\Omega : \theta = \omega\}$ .

- (i)  $0 < \omega \leq \pi/2$  ;  $\mathcal{N} = \{0\}$ .
- (ii)  $\pi/2 < \omega \leq 3\pi/2$  ;  $\mathcal{N} = \text{span}\{\psi_1\}$ .



(iii)  $3\pi/2 < \omega < 2\pi$  ;  $\mathcal{N} = \text{span}\{\psi_1, \psi_2\}$ , where

$$\psi_k(r, \theta) = (r^{-\nu_k} - r^{\nu_k}) \sin(\nu_k \theta), \quad \nu_k = (k - 1/2) \frac{\pi}{\omega}, \quad k = 1, 2.$$

For the (i)-cases, the estimate (47) holds for any  $s \in [0, 1]$ . For the remaining cases we will use the interpolation results of in Section 2.

According to previous notation,  $L^2(\Omega)_{\mathcal{N}}$  denotes the orthogonal complement of the subspace  $\mathcal{N}$  in  $L^2(\Omega)$ . The Laplace operator, from  $V^2(\Omega)$  to  $L^2(\Omega)_{\mathcal{N}}$ , is a bounded operator with a bounded inverse. Thus, the operator  $T : H_D^{-1}(\Omega) \rightarrow H^1(\Omega)$  defined at the beginning of Section 4 is a bounded operator from  $L^2(\Omega)_{\mathcal{N}}$  to  $H^2(\Omega)$ . By interpolation, we obtain

$$\|u\|_{[H^2(\Omega), H^1(\Omega)]_{1-s}} \leq c \|f\|_{[L^2(\Omega)_{\mathcal{N}}, H_D^{-1}(\Omega)]_{1-s}} \quad \text{for all } f \in [L^2(\Omega)_{\mathcal{N}}, H_D^{-1}(\Omega)]_{1-s}. \quad (54)$$

Since  $[H^2(\Omega), H^1(\Omega)]_{1-s} = H^{1+s}(\Omega)$  and  $[L^2(\Omega), H_D^{-1}(\Omega)]_{1-s} = H_D^{-1+s}(\Omega)$ , the only thing which remains to be proved in order to obtain the estimate (46) for  $s < 1/\gamma$  (Theorem 4.2 and Theorem 4.1 as well) is that

$$[L^2(\Omega)_{\mathcal{N}}, H_D^{-1}(\Omega)]_{1-s} = [L^2(\Omega), H_D^{-1}(\Omega)]_{1-s} \quad \text{for } s < 1/\gamma, \quad (55)$$

where  $\gamma = \omega/\pi$  in Case 1 and Case 2, and  $\gamma = 2\omega/\pi$  in Case 3.

Let  $\psi = (r^{-\nu} - r^{\nu})g(\theta)$  be one of the functions which defines the subspace  $\mathcal{N}$ . (Note that  $\nu \in (0, 1)$ ). The next result is of crucial importance in proving (55).

**Theorem 4.3.** *If  $0 < s < \nu$ , then*

$$[L^2(\Omega)_{\psi}, H_D^{-1}(\Omega)]_{1-s} = [L^2(\Omega), H_D^{-1}(\Omega)]_{1-s}. \quad (56)$$

We will give the proof of this theorem later.

When  $\dim(\mathcal{N}) = 1$  we are in one of the (ii) cases listed above. In this case (55) follows directly from Theorem 4.3. Let us consider now the case in which  $\dim(\mathcal{N}) = 2$ , i.e., Case 3 (iii). In order to prove (55) we apply Theorem 2.2. The condition (A.1) of Theorem 2.2 follows easily from the Theorem 4.3. To verify (A.2) for  $X = L^2(\Omega)$ ,  $Y = H_D^{-1}(\Omega)$  and  $\mathcal{K} = \mathcal{N} = \text{span}\{\psi_1, \psi_2\}$ , we start by deriving an eigenfunction representation of the norm on  $H_D^{\alpha}(\Omega)$ . To do this, we consider the following eigenvalue problem.

Find real numbers  $\lambda$  and functions  $u \in H^1(\Omega)$ ,  $u \neq 0$  such that

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega \\ u = 0 & \text{on } (\partial\Omega)_D, \\ \frac{\partial u}{\partial n} = 0 & \text{on } (\partial\Omega)_N. \end{cases} \quad (57)$$

Let  $J_{\nu}$  be the Bessel's function of the first kind, of index  $\nu$ . For  $n = 1, 2, \dots$ , let

$$\nu_n := (n - 1/2) \frac{\pi}{\omega} \quad \text{and} \quad \varphi_n(\theta) := \sqrt{2/\omega} \sin(\nu_n \theta), \quad \theta \in (0, \omega).$$

For each fixed  $n$  and  $k = 1, 2, \dots$  let  $\beta_{k,n}$  be the  $k$ -th positive zero of  $J_{\nu_n}(r) = 0$ , and let  $f_{k,n}(r) := c_{k,n} J_{\nu_n}(\beta_{k,n} r)$ , where  $c_{k,n}^{-2}$  is the positive constant given by

$$c_{k,n}^{-2} := \int_0^1 r J_{\nu_n}(\beta_{k,n} r)^2 dr.$$

Using separation of variables and polar coordinates for the Laplace operator, we find the following set of eigenvalue, eigenvector pairs:

$$(\lambda_{k,n}, \varphi_{k,n}) = (\beta_{k,n}^2, f_{k,n}(r) \varphi_n(\theta)), \quad k, n = 1, 2, \dots$$

Since  $\{\varphi_n\}_{n \geq 1}$  is an orthonormal basis for  $L^2([0, \omega])$  and for each fixed  $n$ ,  $\{f_{k,n}\}_{k \geq 1}$  is an orthonormal basis for  $L^2([0, 1])$  with respect to the inner product with the weight function  $w(r) = r$  (see, e.g., [29]), we obtain that  $\{\varphi_{k,n}\}_{k,n \geq 1}$  is an orthonormal basis for  $L^2(\Omega)$ . Furthermore, each pair  $(\lambda_{k,n}, \varphi_{k,n})$  is a solution of (57), and by Green's formula we have that

$$\int_{\Omega} \nabla \varphi_{k,n} \cdot \nabla v = \lambda_{k,n} \int_{\Omega} \varphi_{k,n} v \quad \text{for all } v \in H_D^1(\Omega).$$

Thus, if  $H_D^1(\Omega)$  is provided with the inner product

$$(u, v)_1 = \int_{\Omega} \nabla u \cdot \nabla v = A(u, v),$$

then  $\{\lambda_{k,n}^{-1/2} \varphi_{k,n}\}_{k,n \geq 1}$  is an orthonormal basis for  $H_D^1(\Omega)$ . Therefore, the norm on  $H_D^1(\Omega)$  is given by

$$\|u\|_1^2 = \sum_{k,n=1}^{\infty} \lambda_{k,n} (u, \varphi_{k,n})^2.$$

Next, the norm on  $H_D^{\alpha}(\Omega)$  for  $\alpha \in [-1, 1]$  is given by

$$\|u\|_{\alpha}^2 = \sum_{k,n=1}^{\infty} \lambda_{k,n}^{\alpha} (u, \varphi_{k,n})^2 \quad \text{for all } u \in H_D^{\alpha}(\Omega) \cap L^2(\Omega), \quad (58)$$

With the notation adopted in Section 2.2, taking  $X = L^2(\Omega)$  and  $Y = H_D^{-1}(\Omega)$  we have

$$(u, v)_{X,t} = \sum_{k,n=1}^{\infty} \frac{\lambda_{k,n}}{\lambda_{k,n} + t^2} (u, \varphi_{k,n}) (v, \varphi_{k,n}) \quad \text{for all } u, v \in X. \quad (59)$$

**Theorem 4.4.** *If  $\dim(\mathcal{N}) = 2$  and  $s < 1/\gamma$ , then (55) holds.*

*Proof.* Let  $s < 1/\gamma = \nu_1$  be fixed. First, we verify the conditions **(A.1)** and **(A.2)** of the Theorem 2.2 for  $n = 2$ ,  $X = L^2(\Omega)$ ,  $Y = H^{-1}(\Omega)$  and  $\mathcal{K} = \mathcal{N}$ . Since  $\psi_k \notin H_D^{1-\nu_k}$ , by Remark 2.3, we have that  $L^2(\Omega)_{\psi_k}$  is dense in  $[L^2(\Omega), H_D^{-1}(\Omega)]_{1-s}$ , for  $k = 1, 2$ . Thus, **(A.1)** is

$$[L^2(\Omega)_{\psi_k}, H_D^{-1}(\Omega)]_{1-s} = [L^2(\Omega), H_D^{-1}(\Omega)]_{1-s}, \quad \text{for } k = 1, 2 \quad (60)$$

which follows from Theorem 4.3.

Checking the condition **(A.2)** is easy in this case. From (59) we have

$$(\psi_1, \psi_2)_{X,t} = \sum_{k,n=1}^{\infty} \frac{\lambda_{k,n}}{\lambda_{k,n} + t^2} (\psi_1, \varphi_{k,n}) (\psi_2, \varphi_{k,n}).$$

Since  $(\psi_1, \varphi_{k,n}) = 0$  for  $n \neq 1$  and  $(\psi_2, \varphi_{k,n}) = 0$  for  $n \neq 2$ , we obtain that  $(\psi_1, \psi_2)_{X,t} = 0$  for all  $t > 0$ . Thus, **(A.2)** is trivially satisfied. By Theorem 2.2 we obtain that

$$[L^2(\Omega)_{\mathcal{N}}, H_D^{-1}(\Omega)]_{1-s} = [L^2(\Omega), H_D^{-1}(\Omega)]_{1-s, \mathcal{N}}.$$

Using again Remark 2.3, one sees that  $L^2(\Omega)_{\mathcal{N}}$  is dense in  $[L^2(\Omega), H_D^{-1}(\Omega)]_{1-s}$ . It follows that

$$[L^2(\Omega), H_D^{-1}(\Omega)]_{1-s, \mathcal{N}} = [L^2(\Omega), H_D^{-1}(\Omega)]_{1-s}.$$

Therefore (55) holds, and the proof is complete.  $\blacksquare$

It remains to prove Theorem 4.3.

#### 4.3. The proof of Theorem 4.3.

Our proof of Theorem 4.3 involves reduction of the problem, via the interpolation result of Section 2.3, to a similar interpolation problem where the domain  $\Omega = S_\omega$  is replaced by a polygonal-sector domain (defined below) containing  $\Omega$ . We say that  $\Omega_s$  is a polygonal-sector domain (see Figure 1) if

$$\overline{\Omega_s} = \bigcup_{i=1}^n \overline{\tau_i},$$

where, for  $i = 1, \dots, n$ ,  $\tau_i$  is a triangular domain with vertices  $S_i$ ,  $O$ ,  $S_{i+1}$  and  $O$  is taken to be the origin of a Cartesian system of coordinates in the plane.

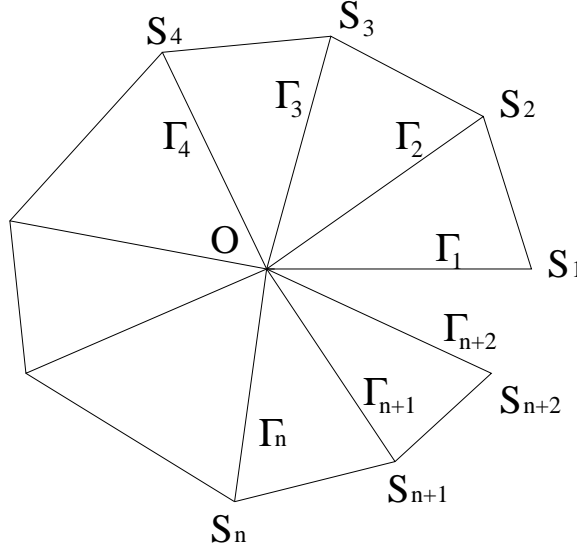


Figure 1. Polygonal-sector domain

We assume, without loss of generality, that  $S_1$  lies on the positive semi-axis. For  $i = 1, \dots, n+1$ , let  $\Gamma_i$  denote the segment  $[O, S_i]$ , and for  $i = 1, \dots, n+1$ , let  $\alpha_i$  be the measure of the angle between  $\Gamma_i$  and  $\Gamma_{i+1}$ , and define the angle  $\omega$  of  $\Omega$  by

$$\omega := \sum_{i=1}^n \alpha_i.$$

For our results concerning interpolation, it is enough to consider only the cases  $(\partial\Omega_s)_N = \emptyset$ ,  $(\partial\Omega_s)_N = \Gamma_{n+1}$  or  $(\partial\Omega_s)_N = \Gamma_1 \cup \Gamma_{n+1}$ . Let  $\mathcal{T}_1 = \{\tau_1, \dots, \tau_n\}$  be the initial triangulation of  $\Omega_s$ . We define multilevel triangulations recursively. For  $k > 1$ , the triangulation  $\mathcal{T}_k$  is obtained from  $\mathcal{T}_{k-1}$  by splitting each triangle in  $\mathcal{T}_{k-1}$  into four triangles by connecting the midpoints of the edges. The space  $M_k$  is defined to be the space of all functions which are piecewise linear with respect to  $\mathcal{T}_k$ , vanish on  $(\partial\Omega_s)_D$  and are continuous on  $\Omega_s$ . Let  $Q_k$  denote the  $L^2(\Omega_s)$  orthogonal projection onto  $M_k$ .

Now let  $S_\omega$  be a sector domain defined by (52) and consider a polygonal-sector domain  $\Omega_s$  with the same angle  $\omega$  and such that

$$S_\omega \subset \{(r, \theta) : 0 < r < 2r_0, 0 < \theta < \omega\} \subset \Omega_s.$$

Note that  $\Omega_s$  is not necessary contained in the original polygonal domain. For  $\Omega_s$  we use the notation given in Figure 1 and for simplicity we take  $r_0 = 1$ .

Assume that the free part  $(\partial\Omega_s)_N$  of  $\partial\Omega_s$  is defined as follows:

$$\begin{aligned} (\partial\Omega_s)_N &= \emptyset \text{ if } S_\omega \text{ is in Case 1,} \\ (\partial\Omega_s)_N &= \Gamma_1 \cup \Gamma_{n+2} \text{ if } S_\omega \text{ is in Case 2 and} \\ (\partial\Omega_s)_N &= \Gamma_{n+2} \text{ if } S_\omega \text{ is in Case 3,} \end{aligned}$$

where Case 1-Case 3 are defined in Section 4.2. Let  $\zeta \in \mathcal{D}(\Omega_s)$  be a cut-off function which depends only on the distance  $r$  to the origin and satisfies

$$\begin{aligned} \zeta(r, \theta) &= 1 \text{ for } 0 < r \leq 1/2 \text{ and } 0 < \theta < \omega, \\ \zeta(r, \theta) &= 0 \text{ for } r \geq 1 \text{ and } (r, \theta) \in \Omega_s. \end{aligned}$$

Let  $\psi$  be one of the functions which defines the subspace  $\mathcal{N}$  (defined in Section 4.2 for  $S_\omega$ ) and let  $\tilde{\psi}$  be the extension by zero of  $\psi$  to  $\Omega_s$ . Then  $\tilde{\psi} = \phi + u^R$  where  $\phi(r, \theta) = \zeta r^{-\nu} g(\theta)$  and  $u^R \in H_D^1(\Omega_s)$ . The next result is a version of Theorem 4.3 for polygonal-sector domains.

**Theorem 4.5.** *Let  $\Omega_s$  be a polygonal-sector domain as defined above. If  $0 < s < \nu$ , then*

$$[L^2(\Omega_s)_{\tilde{\psi}}, H_D^{-1}(\Omega_s)]_{1-s} = [L^2(\Omega_s), H_D^{-1}(\Omega_s)]_{1-s},$$

for any function  $\tilde{\psi} = \phi + u^R$  with  $\phi(r, \theta) = \zeta r^{-\nu} g(\theta)$  and  $u^R$  being arbitrary function in  $H_D^1(\Omega)$ .

*Proof.* From the results of the Appendix 5.3 and Appendix 5.4, we have that an equivalent norm on  $H_D^\alpha(\Omega_s)$  is given by the multilevel norm (34). By Theorem 3.1, it is enough to verify that the function  $\tilde{\psi}$  satisfies the conditions (C.0)-(C.2) defined in Section 3.2 with  $\theta_0 = 1 - \nu$  and  $\theta = 1 - s$ .

To begin with, we will prove that the function  $\phi$  satisfies (C.0)-(C.2). Let  $\overline{M}_k$  be the space of piecewise linear functions with respect to  $\mathcal{T}_k$  defined on  $\Omega_s$ , and let  $\overline{Q}_k$  be the  $L^2(\Omega_s)$  orthogonal projection onto  $\overline{M}_k$ . First step in verifying (C.0) and (C.1) is to prove that there exists a positive constant  $c$  such that

$$\|(I - \overline{Q}_k)\phi\|^2 \geq c\lambda_k^{-\theta_0}, \quad k = 1, 2, \dots \quad (61)$$

We define  $\tau_1^k$  to be the triangle in  $\mathcal{T}_k$  which is the the image of  $\tau_1 \in \mathcal{T}_1$  via the map  $\hat{x} \rightarrow h_k \hat{x}$ . Here, without loss of generality, we assume that  $h_k^2 = \lambda_k^{-1} = 4^{-k+1}$ . Then

$$\|(I - \overline{Q}_k)\phi\|_{L^2(\Omega_s)}^2 \geq \|(I - \overline{Q}_k)\phi\|_{L^2(\tau_1^k)}^2 = \|\phi\|_{L^2(\tau_1^k)}^2 - \|\overline{Q}_k\phi\|_{L^2(\tau_1^k)}^2.$$

The projection  $\overline{Q}_k \phi$  can be estimated on  $\tau_1^k$  in terms of the three nodal functions  $\varphi_1^k, \varphi_2^k, \varphi_3^k$  associated with the three vertices of  $\tau_1^k$ . If  $M^k$  is the  $3 \times 3$  Gram matrix associated with the set  $\{\varphi_1^k, \varphi_2^k, \varphi_3^k\}$ , and  $S^k := (S_{ij}^k)$ ,  $i, j = 1, 2, 3$  is the inverse of  $M^k$ , then

$$\|\phi\|_{L^2(\tau_1^k)}^2 - \|\overline{Q}_k \phi\|_{L^2(\tau_1^k)}^2 = \int_{\tau_1^k} \phi^2 dx - \sum_{i,j=1}^3 S_{ij}^k \int_{\tau_1^k} \phi \varphi_i^k dx \int_{\tau_1^k} \phi \varphi_j^k dx.$$

Further, by making the change of variable  $x = h_k \hat{x}$  in the above integrals, a simple computation shows that

$$\begin{aligned} \|\phi\|_{L^2(\tau_1^k)}^2 - \|\overline{Q}_k \phi\|_{L^2(\tau_1^k)}^2 &= h_k^{2-2\nu} \left( \int_{\tau_1} \phi^2 d\hat{x} - \sum_{i,j=1}^3 S_{ij}^1 \int_{\tau_1} \phi \varphi_i^1 d\hat{x} \int_{\tau_1} \phi \varphi_j^1 d\hat{x} \right) \\ &= \lambda_k^{-\theta_0} \left( \|\phi\|_{L^2(\tau_1)}^2 - \|\overline{Q}_1 \phi\|_{L^2(\tau_1)}^2 \right). \end{aligned}$$

Since  $\phi$  is not linear on  $\tau_1$ , the constant  $\|\phi\|_{L^2(\tau_1)}^2 - \|\overline{Q}_1 \phi\|_{L^2(\tau_1)}^2$  is strictly positive. Combining the above estimates, we have proven that (61) holds.

The second step is to use (61) and the fact that  $M_k$  is a subspace of  $\overline{M}_k$ , in order to obtain

$$\|(I - Q_k)\phi\|^2 \geq c\lambda_k^{-\theta_0}, \quad k = 1, 2, \dots \quad (62)$$

From (62) we see that  $\|\phi\|_{\theta_0}$ , defined in Remark 3.1, is not finite. Hence  $\phi \notin H_D^{\theta_0}(\Omega_s)$ . Using again (62) and the identity

$$\|(Q_k - Q_{k-1})u\|^2 = \|(I - Q_{k-1})u\|^2 - \|(I - Q_k)u\|^2, \quad u \in L^2(\Omega_s),$$

we have

$$\begin{aligned} (\phi, \phi)_{X,t} &= \frac{\lambda_1 \|\phi\|^2}{\lambda_1 + t^2} + t^2 \sum_{k=1}^{\infty} \frac{\lambda_{k+1} - \lambda_k}{(\lambda_{k+1} + t^2)(\lambda_k + t^2)} \|(I - Q_k)\phi\|^2 \\ &\geq ct^2 \sum_{k=1}^{\infty} \frac{(4^k)^{1-\theta_0}}{(4^k + t^2)^2} = t^{-2\theta_0} \sum_{k=1}^{\infty} \frac{(4^k/t^2)^{1-\theta_0}}{(4^k/t^2 + 1)^2}. \end{aligned}$$

Finally, the last sum can be bounded below by a positive constant independent of  $t$  as follows. Let us fix  $t \geq 4$  and let  $k_0$  be the integer such that  $4^{k_0} \leq t^2 < 4^{k_0+1}$ . Then

$$\sum_{k=1}^{\infty} \frac{(4^k/t^2)^{1-\theta_0}}{(4^k/t^2 + 1)^2} > \frac{(4^{k_0}/t^2)^{1-\theta_0}}{(4^{k_0}/t^2 + 1)^2} \geq \inf_{x \in [1/4, 1]} \frac{x^{1-\theta_0}}{(x + 1)^2} > 0.$$

Thus, (C.0) and (C.1) hold for the function  $\phi$ .

To verify (C.2) we first observe that

$$\|(Q_k - Q_{k-1})\phi\|^2 \leq \|(I - Q_{k-1})\phi\|^2.$$

Hence, it is enough to prove that there exists a positive constant  $c$  such that

$$\|(I - Q_k)\phi\|^2 \leq c\lambda_k^{-\theta_0}, \quad k = 1, 2, \dots \quad (63)$$

Let  $\eta_k$  be a cut off function which depends only on  $r$  and satisfies

$$\begin{aligned} \eta_k(r) &= 0 \text{ for } r \leq h_k, \quad \eta_k(r) = 1 \text{ for } r \geq 2h_k, \\ |\eta'_k(r)| &\leq c/h_k, \quad |\eta''_k(r)| \leq c/h_k^2 \quad \text{for all } h_k \leq r \leq 2h_k, \quad k = 1, 2, \dots, \end{aligned}$$

for some positive constant  $c$ . For example, we can take

$$\eta_k(r) = 1/2 + 1/2 \sin \left( (r - 3h_k/2) \frac{\pi}{h_k} \right) \text{ on } [h_k, 2h_k].$$

Then,  $\phi = (1 - \eta_k)\phi + \eta_k\phi$  and  $\eta_k\phi \in H^2(\Omega_s)$ . Let  $\Pi_k : H^2(\Omega_s) \rightarrow M_k$  be the interpolant associated with  $\mathcal{T}_k$ . By applying standard approximation properties and (53) we obtain

$$\|(I - Q_k)\phi\| \leq \|(I - Q_k)(1 - \eta_k)\phi\| + \|(I - Q_k)\eta_k\phi\| \leq \|(1 - \eta_k)\phi\| + \|(I - \Pi_k)\eta_k\phi\|$$

$$\leq \|(1 - \eta_k)\phi\| + ch_k^2 \|\eta_k\phi\|_{H^2(\Omega_s)} \leq \|(1 - \eta_k)\phi\| + ch_k^2 \|\Delta(\eta_k\phi)\|_{L^2(\Omega_s)}.$$

Using a simple computation in polar coordinates, and the estimates for the derivative of  $\eta_k$ , we get

$$\|(1 - \eta_k)\phi\|^2 \leq ch_k^{2\theta_0} \quad \text{for all } k = 1, 2, \dots$$

and

$$h_k^2 \|\Delta(\eta_k\phi)\|_{L^2(\Omega_s)} \leq ch_k^{2\theta_0} \quad \text{for all } k = 1, 2, \dots$$

Combining the above inequalities, we conclude that (63) is valid. Thus, **(C.3)** holds for the function  $\phi$ .

Verifying **(C.0)**-**(C.3)** for the function  $\phi$  is mainly based on finding some positive constants  $c_1, c_2$  such that

$$c_1 \lambda_k^{-\theta_0} \leq \|(I - Q_k)\phi\|^2 \leq c_2 \lambda_k^{-\theta_0}, \quad k = 1, 2, \dots \quad (64)$$

Since the function  $u_R$  belongs to  $H_D^1(\Omega_s)$ , we have

$$\|(I - Q_k)u_R\|^2 \leq c \lambda_k^{-1}, \quad k = 1, 2, \dots$$

Therefore, the function  $\tilde{\psi}$  satisfies an estimate of type (64) and **(C.0)**-**(C.3)** hold for the function  $\tilde{\psi}$  too. The result is now a direct consequence of Theorem 3.1.  $\blacksquare$

*Proof of Theorem 4.3* Let  $E : L^2(S_\omega) \rightarrow L^2(\Omega_s)$ , be the extension by zero operator, and let  $R : L^2(\Omega_s) \rightarrow L^2(S_\omega)$  be defined as follows: First, we introduce a cut-off function  $\eta \in \mathcal{D}(\Omega)$  which depends only on the distance  $r$  to the origin and satisfies

$$\begin{aligned} \eta(r, \theta) &= 1 \text{ for } 0 < r \leq 1 \text{ and } 0 < \theta < \omega, \\ \eta(r, \theta) &= 0 \text{ for } r \geq 2 \text{ and } (r, \theta) \in \Omega. \end{aligned}$$

Then, for a function  $v \in L^2(\Omega_s)$  we define  $Rv \in L^2(S_\omega)$  by

$$(Rv)(r, \theta) := v_1(r, \theta) - v_1(2 - r, \theta), \quad (r, \theta) \in S_\omega,$$

where,

$$v_1(r, \theta) := \eta(r, \theta)v(r, \theta), \quad (r, \theta) \in \Omega.$$

Let  $\tilde{\psi}$  denote the function  $E\psi$ . According to Theorem 4.5

$$[L^2(\Omega_s)_{\tilde{\psi}}, H_D^{-1}(\Omega_s)]_{1-s} = [L^2(\Omega_s), H_D^{-1}(\Omega_s)]_{1-s},$$

It follows that the function  $\psi$  and the operators  $E, R$  satisfy the hypotheses of Lemma 2.3 with  $\theta = 1 - s$ ,  $V^1(\Omega) = H_D^1(S_\omega)$  and  $V^1(\tilde{\Omega}) = H_D^1(\Omega_s)$ . Thus, (56) holds for the sector domain  $S_\omega$  and the proof is complete.

## 5. Appendix

## 5.1. The proof of Lemma 2.3

Using the duality, from (24)-(26) we obtain linear operators  $E^*$ ,  $R^*$  such that

$$E^* : L^2(\tilde{\Omega}) \rightarrow L^2(\Omega), \quad E^* : V^{-1}(\tilde{\Omega}) \rightarrow V^{-1}(\Omega), \quad \text{are bounded operators,} \quad (65)$$

$$R^* : L^2(\Omega) \rightarrow L^2(\tilde{\Omega}), \quad R^* : V^{-1}(\Omega) \rightarrow V^{-1}(\tilde{\Omega}) \quad \text{are bounded operators,} \quad (66)$$

$$E^* R^* u = u \quad \text{for all } u \in L^2(\Omega), \quad (67)$$

$$E^* \text{ maps } L^2(\tilde{\Omega})_{\tilde{\psi}} \text{ to } L^2(\Omega)_{\psi}, \quad (68)$$

$$R^* \text{ maps } L^2(\Omega)_{\psi} \text{ to } L^2(\tilde{\Omega})_{\tilde{\psi}}. \quad (69)$$

From (65) and (68), by interpolation, we obtain

$$\|E^* v\|_{[L^2(\Omega)_{\psi}, V^{-1}(\Omega)]_{\theta}} \leq c \|v\|_{[L^2(\tilde{\Omega})_{\tilde{\psi}}, V^{-1}(\tilde{\Omega})]_{\theta}} \quad \text{for all } v \in L^2(\tilde{\Omega})_{\tilde{\psi}}. \quad (70)$$

For  $u \in L^2(\Omega)_{\psi}$ , let  $v := R^* u$ . Then, using (69), we have that  $v \in L^2(\tilde{\Omega})_{\tilde{\psi}}$ . Taking  $v := R^* u$  in (70) and using (67), we get

$$\|u\|_{[L^2(\Omega)_{\psi}, V^{-1}(\Omega)]_{\theta}} \leq c \|R^* u\|_{[L^2(\tilde{\Omega})_{\tilde{\psi}}, V^{-1}(\tilde{\Omega})]_{\theta}} \quad \text{for all } u \in L^2(\Omega)_{\psi}. \quad (71)$$

Also, from the hypothesis (29), we deduce that

$$\|R^* u\|_{[L^2(\tilde{\Omega})_{\tilde{\psi}}, V^{-1}(\tilde{\Omega})]_{\theta}} \leq c \|R^* u\|_{[L^2(\tilde{\Omega}), V^{-1}(\tilde{\Omega})]_{\theta}} \quad \text{for all } u \in L^2(\Omega)_{\psi}. \quad (72)$$

From (66), again by interpolation, we have in particular

$$\|R^* u\|_{[L^2(\tilde{\Omega}), V^{-1}(\tilde{\Omega})]_{\theta}} \leq c \|u\|_{[L^2(\Omega), V^{-1}(\Omega)]_{\theta}} \quad \text{for all } u \in L^2(\Omega)_{\psi}. \quad (73)$$

Combining (71)-(73), it follows that

$$\|u\|_{[L^2(\Omega)_{\psi}, V^{-1}(\Omega)]_{\theta}} \leq c \|u\|_{[L^2(\Omega), V^{-1}(\Omega)]_{\theta}} \quad \text{for all } u \in L^2(\Omega)_{\psi}. \quad (74)$$

The reverse inequality of (74) holds because  $L^2(\Omega)_{\psi}$  is a closed subspace of  $L^2(\Omega)$ . Thus, the two norms in (74) are equivalent for  $u \in L^2(\Omega)_{\psi}$ . From the assumption (27),  $L^2(\Omega)_{\psi}$  is dense in both spaces appearing in (30). Therefore, we obtain (30).

**Remark 5.1.** *The proof does not change if we consider  $\Omega \subset \tilde{\Omega}$  to be domains in  $R^n$  and  $H^1$  is replaced by any other Sobolev space of positive integer order  $k$ .*

## 5.2. Some results from the multilevel theory.

In this section we present some useful lemmas. (See, e.g., [15] and [28] for a more complete presentation of results concerning multilevel theory.)

**Lemma 5.1.** *Let  $\rho \in (0, 1)$  and let  $\{l_{mn}\}$  be a double sequence of nonnegative real numbers satisfying*

$$l_{mn} \leq \rho^{|m-n|} \quad \text{for all } m, n = 1, 2, \dots$$

*Then for any  $a = \{a_n\}, b = \{b_n\} \in l_2$  with nonnegative entries, we have*

$$\sum_{m,n=1}^{\infty} l_{mn} a_m b_n \leq \frac{1+\rho}{1-\rho} \|a\|_{l_2} \|b\|_{l_2},$$

where  $\|\cdot\|_{l_2}$  denotes the norm on  $l_2$ .

The proof is based on the Cauchy-Schwarz inequality.

**Lemma 5.2.** *Let  $M$  be a Hilbert space with inner product  $(\cdot, \cdot)$ , and let  $\{M_k\}$  be a sequence of nested subspaces of  $M$  ( $M_k \subset M_{k+1}$ ). Denote by  $Q_k$  the orthogonal projection onto  $M_k$  and for any positive integer  $J$  let  $B_J : M_J \rightarrow M_J$ ,  $B_J := \sum_{k=1}^J \lambda_k^{-1} Q_k$ . ( $\lambda_k > 0$ ). Then  $B_J$  is a symmetric positive definite operator and  $B_J^{-1}$  is characterized by*

$$(B_J^{-1}v, v) = \min \left\{ \sum_{k=1}^J \lambda_k \|v_k\|^2, \quad v = \sum_{k=1}^J v_k, \quad v_k \in M_k \right\},$$

where  $\|\cdot\|$  is the norm induced by the inner product  $(\cdot, \cdot)$ .

A proof of the above lemma can be found in [15]. An easy consequence of the above two lemmas is the following.

**Lemma 5.3.** *Assume that the hypotheses of Lemma 5.2 are satisfied and that  $\lambda_k < \rho \lambda_{k+1}$  for some number  $\rho \in (0, 1)$ . Then, there is a constant  $c$  independent of  $J$  such that*

$$(B_J^{-1}v, v) \leq \sum_{k=1}^J \lambda_k \|(Q_k - Q_{k-1})v\|^2 \leq c(B_J^{-1}v, v), \quad \text{for all } v \in M_J. \quad (75)$$

### 5.3. Norm equivalences on $H^1$ by multilevel subspace decomposition

To start with, let  $\Omega$  be a polygonal domain in  $\mathbf{R}^2$  with boundary  $\partial\Omega = (\partial\Omega)_D \cup (\partial\Omega)_N$ , where  $(\partial\Omega)_D$  is not of measure zero, and  $(\partial\Omega)_D$  and  $(\partial\Omega)_N$  are essentially disjoint. Let  $(\mathcal{T}_k)$  be a quasi-uniform sequence of nested triangulations of  $\Omega$  such that the parameter  $h_k$  associated to  $(\mathcal{T}_k)$  is  $h_k = 2^{-k}$ . For  $k \geq 1$  the space  $M_k$  is defined to be the space of all functions which are piecewise linear with respect to  $\mathcal{T}_k$ , vanish on  $(\partial\Omega)_D$  and are continuous on  $\Omega$ . We denote the space  $H_D^1(\Omega)$  by  $M$ . Assume that

$$M_1 \subset M_2 \subset \dots \subset M_J \subset \dots \subset M,$$

is a nested sequence of finite dimensional approximation subspaces of  $M$  defined using a sequence of nested meshes in a way similar to that described at the beginning of Section 4.3. Let  $(\cdot, \cdot)$  denote the  $L^2(\Omega)$  inner product and let  $\|\cdot\|$  be the norm on  $L^2(\Omega)$  induced by  $(\cdot, \cdot)$ . For  $k = 1, 2, \dots$ , we define the operator  $P_k : M \rightarrow M_k$  to be the orthogonal projection with respect to the inner product  $A(\cdot, \cdot)$ , where

$$A(u, v) := \int_{\Omega} \nabla u \cdot \nabla v \, dx \quad \text{for all } u, v \in M,$$



and  $A_k : M_k \rightarrow M_k$  is defined by

$$(A_k u, v) = A(u, v) \quad \text{for all } u, v \in M_k.$$

Let  $\mu_k$  be the largest eigenvalue of  $A_k$ . The sequence  $\{\mu_k\}$  is equivalent to  $\{4^{k-1}\}$ , i.e., there exist positive constants  $\alpha_1, \alpha_2$  such that

$$\alpha_1 4^k \leq \mu_k \leq \alpha_2 4^k, \quad k = 1, 2, \dots, \quad (76)$$

We denote  $4^{k-1}$  by  $\lambda_k$ .

The goal of this section is to show that we have the following.

**(ML.0)** There exist some positive constants  $c_1$  and  $c_2$  such that

$$c_1 A(u, u) \leq \sum_{k=1}^{\infty} \lambda_k \|(Q_k - Q_{k-1})u\|^2 \leq c_2 A(u, u) \quad \text{for all } u \in M.$$

All the considerations of this section remain valid if we replace  $\{\lambda_k\}$  by an equivalent sequence, for example  $\mu_k$ . In order to study the above norm equivalence we start by introducing the following conditions:

**(ML.1)** There is a positive constant  $c$  independent of  $j$  and  $k$  and a number  $\rho \in (0, 1)$  such that

$$|A(u_k, u_j)| \leq c \rho^{|k-j|} A(u_k, u_k)^{1/2} A(u_j, u_j)^{1/2} \quad \text{for all } u \in M.$$

where  $u_k := q_k u := (Q_k - Q_{k-1})u$ .

**(ML.2)** There exists  $c$  independent of  $J$  such that

$$\sum_{k=1}^J \lambda_k \|(Q_k - Q_{k-1})u\|^2 \leq c A(u, u) \quad \text{for all } u \in M_J.$$

**(ML.3)** There exists  $c$  independent of  $k$  and  $J$  such that

$$\|(I - P_{k-1})u\| \leq c \lambda_k^{-1/2} A(u, u)^{1/2}, \quad \text{for all } u \in M_J.$$

**Remark 5.2.** Condition **(ML.1)** is known as a strengthened Cauchy-Schwarz inequality and is satisfied for our sequence of finite dimensional approximation subspaces  $\{M_k\}$  (cf. [15]).

Next, we give some connection between the above conditions. The results are known in the multigrid theory (cf. [14], [15]).

**Proposition 5.1.** *The norm equivalence **(ML.0)** holds whenever Condition **(ML.1)** and Condition **(ML.2)** are satisfied.*

**Remark 5.3.** *It is well known (cf. [8]) that condition **(ML.2)** holds whenever Condition **(ML.3)** holds. The proof is an easy consequence of Lemma 5.2 and Lemma 5.3. Moreover, it is also known that if the domain  $\Omega$  is nice enough (for example  $\Omega$  is convex and  $\partial\Omega = (\partial\Omega)_D$ ), then the regularity condition **(ML.3)** holds.*

In order to prove Condition **(ML.2)**, when  $\Omega$  is an arbitrary polygonal domain, we introduce an overlapping domain decomposition of  $\Omega$  such that on each subdomain Condition **(ML.2)** is satisfied. By the above remark it is enough to verify Condition **(ML.3)** relative

to each subdomain. To get the result on the whole domain, one can use additive Schwarz preconditioning type arguments. We now make this outline more precise.

Let  $M_J = \sum_{i=0}^n M_J^i$  be a splitting of  $M_J$  associated with an overlapping domain decomposition  $\Omega = \bigcup_{i=1}^n \Omega_i$ , i.e.,

$$M_J^i = \{u \in M_J : \text{supp}(u) \subset \overline{\Omega_i}\}. \quad (77)$$

Let  $Q_k^i : L^2(\Omega) \rightarrow M_k^i$ ,  $P_k^i : M \rightarrow M_k^i$  be the orthogonal projections with respect to  $(\cdot, \cdot)$  and  $A(\cdot, \cdot)$ , respectively. We define here a stable decomposition condition with respect to the splitting of  $M_J$ :

**(ML.4)** For each  $u \in M_J$  there exists a partition

$$u = \sum_{i=0}^n u_i, \text{ with } u_i \in M_J^i, \text{ satisfying } \sum_{i=0}^n A(u_i, u_i) \leq cA(u, u),$$

where  $c$  is independent of  $J$  and  $u \in M_J$ .

**Lemma 5.4.** *Assume that Condition (ML.4) is satisfied and that Condition (ML.2) holds on each subdomain, i.e.,*

$$\sum_{k=1}^J \lambda_k \|(Q_k^i - Q_{k-1}^i)u_i\|^2 \leq cA(u_i, u_i) \quad \text{for all } u_i \in M_J^i, \ i = 1, \dots, n. \quad (78)$$

*for some constant  $c$  independent of  $J$  and  $i$ . Then Condition (ML.2) relative to the whole domain  $\Omega$  is also satisfied with constant which may depend on  $n$ . Consequently, (ML.0) holds.*

*Proof.* For any  $u \in M_J$  we consider the decomposition  $u = \sum_{i=0}^n u_i$ , with  $u_i \in M_J^i$  given by Condition (ML.4). Then,

$$\begin{aligned} \sum_{k=1}^J \lambda_k \|(Q_k - Q_{k-1})u\|^2 &\leq n \sum_{k=1}^J \lambda_k \sum_{i=1}^n \|(Q_k - Q_{k-1})u_i\|^2 \\ &= n \sum_{i=1}^n \sum_{k=1}^J \lambda_k \|(Q_k - Q_{k-1})u_i\|^2. \end{aligned}$$

Next, for each fixed  $i$  and  $u_i \in M_J^i \subset M_J$  we have that

$$u_i = \sum_{k=1}^J (Q_k^i - Q_{k-1}^i)u_i,$$

where  $(Q_k^i - Q_{k-1}^i)u_i \in M_k^i \subset M_k$ . Thus, by applying Lemma 5.2, Lemma 5.3 and (78), we obtain that

$$\sum_{k=1}^J \lambda_k \|(Q_k - Q_{k-1})u_i\|^2 \leq c \sum_{k=1}^J \lambda_k \|(Q_k^i - Q_{k-1}^i)u_i\|^2 \leq cA(u_i, u_i).$$

Combining the above estimates with Condition (ML.4), we have

$$\sum_{k=1}^J \lambda_k \|(Q_k - Q_{k-1})u\|^2 \leq cnA(u, u),$$

with  $c$  independent on  $J$ . Therefore Condition **(ML.2)** is satisfied. Finally, from Proposition 5.1, Remark 5.2 and the validity of **(ML.2)** we have that **(ML.0)** holds. ■

#### 5.4. Norm equivalences on $H^1$ by multilevel subspace decomposition for polygonal-sector domains

We restrict our study from the previous section to a simple case when  $\Omega$  is the polygonal-sector domain introduced in Section 4.3 and the free part of  $\partial\Omega$  is  $(\partial\Omega)_N = \emptyset$ ,  $\Gamma_{n+2}$  or  $\Gamma_1 \cup \Gamma_{n+2}$  (see Figure 1). Let  $\{M_k\}$  be the sequence of approximating subspaces defined in Section 4.3. In addition, for  $i = 1, \dots, n$ , we define the subdomain  $\Omega_i$  of  $\Omega$  to be the domain made up by  $\tau_i$  and  $\tau_{i+1}$  ( $\bar{\Omega}_i = \bar{\tau}_i \cup \bar{\tau}_{i+1}$ ), and define the subspaces  $M_k^i$  of  $M_k$  to be

$$M_k^i = \{u \in M_k : \text{supp}(u) \subset \bar{\Omega}_i\}, \quad k = 1, 2, \dots.$$

**Lemma 5.5.** *Let  $\Omega$  be a polygonal-sector domain as defined above and assume that  $(\partial\Omega)_N = \emptyset$  or  $(\partial\Omega)_N = \Gamma_{n+2}$ . Then the splitting  $M_J = \sum_{i=0}^n M_J^i$  satisfies Condition **(ML.4)**.*

*Proof.* For  $i = 2, \dots, n+1$ , let  $\Omega^i$  be the polygonal-sector domain such that

$$\bar{\Omega}^i = \bigcup_{j=1}^{i-1} \bar{\tau}_j.$$

Then  $\Gamma_i$  is a part of  $\partial\Omega^i$  (see Figure 1). We fix  $J$ , and for  $u \in M_J$ , we define  $\gamma_i u$  to be the restriction of  $u$  to  $\Gamma_i$ . By standard results about traces of functions in  $H^1$ , we have

$$\gamma_i u \in H_{00}^{1/2}(\Gamma_i)$$

and

$$\|\gamma_i u\|_{H_{00}^{1/2}(\Gamma_i)} \leq c \|u\|_{H^1(\Omega^i)} \leq c A(u, u) \quad \text{for all } u \in M. \quad (79)$$

Throughout the whole proof of this lemma,  $c$  is a constant independent on  $J$ ,  $i$ , and it might be different at different occurrences. For  $i = 2, \dots, n$ , we extend by zero  $\gamma_i u$  to the rest of  $\partial\tau_i$  and consider an extension of the new function to  $\tau_i$ , denoted by  $\tilde{u}_i$  and satisfying

$$\begin{cases} \tilde{u}_i \in M_J^{i-1,i} := \{v|_{\tau_i} : v \in M_J^{i-1}\}, \\ |\tilde{u}_i|_{H^1(\tau_i)}^2 \leq c \|\gamma_i u\|_{H_{00}^{1/2}(\Gamma_i)}^2 \quad \text{for all } u \in M_J. \end{cases} \quad (80)$$

For example, we can take  $\tilde{u}_i$  to be the discrete harmonic extension of  $\gamma_i u$  to  $\tau_i$ . Define  $u_i \in M_J^i$  by

$$u_1(x) := \begin{cases} u(x) & \text{if } x \in \tau_1 \\ \tilde{u}_2(x) & \text{if } x \in \tau_2 \\ 0 & \text{if } x \in \Omega \setminus \Omega_1, \end{cases}$$

$$u_i(x) := \begin{cases} u(x) - \tilde{u}_i(x) & \text{if } x \in \tau_i \\ \tilde{u}_{i+1}(x) & \text{if } x \in \tau_{i+1} \\ 0 & \text{if } x \in \Omega \setminus \Omega_i, \end{cases}$$

for  $i = 2, \dots, n-1$  and

$$u_n(x) := \begin{cases} u(x) - \tilde{u}_n(x) & \text{if } x \in \tau_n \\ u(x) & \text{if } x \in \tau_{n+1} \\ 0 & \text{if } x \in \Omega \setminus \Omega_n. \end{cases}$$

Clearly,  $u = u_1 + u_2 + \dots + u_n$ . Using (79), (80) and the Cauchy-Schwarz inequality we obtain that

$$A(u_i, u_i) \leq cA(u, u) \quad \text{for all } u \in M_J, \quad i = 1, \dots, n.$$

Therefore,

$$\sum_{i=1}^n A(u_i, u_i) \leq cnA(u, u) \quad \text{for all } u \in M_J,$$

which verifies Condition (ML.4). ■

**Theorem 5.1.** *Let  $\Omega$  be a polygonal-sector domain. Assume that  $(\partial\Omega)_N = \emptyset$  or  $(\partial\Omega)_N = \Gamma_{n+2}$  or  $(\partial\Omega)_N = \Gamma_1 \cup \Gamma_{n+2}$ . Assume that the angles of the polygon  $\partial\Omega$ , excluding the angle at the origin, are not greater than  $\pi$  for those angles contained in  $(\partial\Omega)_D$ , and not greater than  $\pi/2$  for the angles with one edge in  $(\partial\Omega)_D$  and the other edge in  $(\partial\Omega)_N$ . Let the sequence  $(M_k)$  of subspaces of  $H_D^1(\Omega)$  be as described in Section 4.3. Then Condition (ML.0) holds.*

*Proof.* First we consider the case when  $(\partial\Omega)_N = \emptyset$  or  $(\partial\Omega)_N = \Gamma_{n+2}$ . In this case, by using the assumptions about the angles of  $\partial\Omega$ , and eventually by increasing the number  $n$  of subdomains, we have full regularity for the Laplace operator on each subdomain  $\Omega_i$  (defined at the beginning of the section). Thus, Condition (ML.3) is satisfied on each  $\Omega_i$  (see e.g., Theorem 2.3.7 in [20], [15]). On the other hand, from Lemma 5.5 the splitting  $M_J = \sum_{i=0}^n M_J^i$  satisfies Condition (ML.4). Thus, by Lemma 5.4 and Remark 5.3 Condition (ML.0) holds.

Next, we study the case  $(\partial\Omega)_N = \Gamma_1 \cup \Gamma_{n+2}$ . If  $\omega$  is not greater than  $\pi$ , Condition (ML.3) is again fulfilled. Consequently, (ML.2) holds. According to Proposition 5.1 we obtain that Condition (ML.0) holds. Let  $\omega$  be greater than  $\pi$ . Define  $\hat{\Omega}$  to be the polygonal domain  $\text{int}(\bar{\Omega} \cup \bar{\tau}_{n+2})$ , where  $\tau_{n+2} := [S_{n+2}, O, S_1]$ . Let  $\partial\hat{\Omega}$  be the boundary of  $\hat{\Omega}$ , and define  $(\partial\hat{\Omega})_N := \emptyset$  and  $(\partial\hat{\Omega})_D := \partial\hat{\Omega}$ . Assume, without loss of generality, that  $\hat{\Omega}$  is a convex domain. Consider

$$\hat{\mathcal{T}}_1 := \{\tau_0, \dots, \tau_{n+1}, \tau_{n+2}\}.$$

Then we define the multilevel triangulation  $(\hat{\mathcal{T}}_k)$  recursively in the same manner we defined  $(\mathcal{T}_k)$ . For  $k = 1, 2, \dots$ , the space  $\hat{M}_k$  is defined to be the space of all functions which are piecewise linear with respect to  $\hat{\mathcal{T}}_k$ , vanish on  $(\partial\hat{\Omega})_D$  and are continuous on  $\hat{\Omega}$ . The  $L^2(\hat{\Omega})$  orthogonal projection onto  $\hat{M}_k$  is denoted by  $\hat{Q}_k$ . We fix  $J$  and for  $u \in M_J$ , we denote by  $\gamma_N u$  the restriction of  $u$  to  $(\partial\Omega)_N$ . Thus, we have

$$\gamma_N u \in H_{00}^{1/2}((\partial\Omega)_N)$$

and

$$\|\gamma_N u\|_{H_{00}^{1/2}(\Gamma_N)} \leq c\|u\|_{H^1(\Omega)} \leq cA(u, u) \quad \text{for all } u \in M_J, \quad (81)$$

where  $c$  is a constant independent of  $J$ , which might be different at different occurrences. The set  $(\partial\Omega)_N$  is part of the boundary of  $\tau_{n+2}$ . We extend  $\gamma_N u$  by zero to the rest of  $\partial\tau_{n+2}$  and consider an extension of the new function to  $\tau_{n+2}$  denoted  $\tilde{u}_{n+2}$  and satisfying

$$\begin{cases} \tilde{u}_{n+2} \in \hat{M}_J^{n+2} := \{v|_{\tau_{n+2}} : v \in \hat{M}_J\}, \\ |\tilde{u}_{n+2}|_{H^1(\tau_{n+2})}^2 \leq c \|\gamma_N u\|_{H_{00}^{1/2}(\Gamma_N)}^2 \quad \text{for all } u \in M_J. \end{cases} \quad (82)$$

For example, we can take  $\tilde{u}_{n+2}$  to be the discrete harmonic extension of  $\gamma_N u$  to  $\tau_{n+2}$ .

Define  $\hat{u} \in \hat{M}_J$  by

$$\hat{u}(x) := \begin{cases} u(x) & \text{if } x \in \Omega \\ \tilde{u}_{n+2}(x) & \text{if } x \in \tau_{n+2}, \end{cases}$$

Using (81), (82) and the Cauchy-Schwarz inequality we obtain that

$$A(\hat{u}, \hat{u}) \leq cA(u, u) \quad \text{for all } u \in M_J.$$

From Lemma 5.2 and Lemma 5.3, we obtain that

$$\sum_{k=1}^J \lambda_k \|(Q_k - Q_{k-1})u\|^2 \leq c \sum_{k=1}^J \lambda_k \|u_k\|^2 \quad \text{for all } u \in M_J \text{ and} \quad (83)$$

for any partition of  $u$ ,

$$u = \sum_{k=1}^J u_k, \quad \text{with } u_k \in M_k.$$

On the other hand, we have  $\hat{u}|_{\Omega} = u$  and

$$\hat{u} = \sum_{k=1}^J (\hat{Q}_k - \hat{Q}_{k-1})\hat{u}.$$

The restrictions to  $\Omega$  of functions in  $\hat{M}_k$  are in  $M_k$ . Hence, we can take  $u_k := ((\hat{Q}_k - \hat{Q}_{k-1})\hat{u})|_{\Omega}$  in (83). In addition, since  $\hat{\Omega}$  is a convex domain and  $(\partial\hat{\Omega})_N := \emptyset$ , Condition **(ML.3)** is fulfilled for  $\hat{\Omega}$ . Hence, we obtain that Condition **(ML.2)** holds on  $\hat{\Omega}$ . Then

$$\begin{aligned} \sum_{k=1}^J \lambda_k \|(Q_k - Q_{k-1})u\|^2 &\leq c \sum_{k=1}^J \lambda_k \|(\hat{Q}_k - \hat{Q}_{k-1})\hat{u}\|_{L^2(\Omega)}^2 \leq c \sum_{k=1}^J \lambda_k \|(\hat{Q}_k - \hat{Q}_{k-1})\hat{u}\|_{L^2(\hat{\Omega})}^2 \\ &\leq cA(\hat{u}, \hat{u}) \leq cA(u, u), \end{aligned}$$

for all  $u$  in  $M_J$ . Therefore we have proved that Condition **(ML.2)** also holds in this case and, by Proposition 5.1, the proof of the theorem is complete.  $\blacksquare$

The conclusion of this section is that for polygonal-sector domains, as we described above, an equivalent norm on  $H_D^1(\Omega)$  is given by (31).

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