

# NEW INTERPOLATION RESULTS AND APPLICATIONS TO FINITE ELEMENT METHODS FOR ELLIPTIC BOUNDARY VALUE PROBLEMS

CONSTANTIN BACUTA, JAMES H. BRAMBLE, AND JOSEPH E. PASCIAK

ABSTRACT. We consider the interpolation problem between  $H^2(\Omega) \cap H_D^1(\Omega)$  and  $H_D^1(\Omega)$ , where  $\Omega$  is a polygonal domain in  $\mathbb{R}^2$  and  $H_D^1(\Omega)$  is the subspace of functions in  $H^1(\Omega)$  which vanish on the Dirichlet part  $(\partial\Omega)_D$  of the boundary of  $\Omega$ . The main result is that the interpolation spaces  $[H^2(\Omega) \cap H_D^1(\Omega), H_D^1(\Omega)]_s$  and  $H^{1+s}(\Omega) \cap H_D^1(\Omega)$  coincide. An application of this result to a nonconforming finite element problem is presented.

## 1. INTRODUCTION

Let  $\Omega$  be a two dimensional domain with boundary  $\partial\Omega = (\partial\Omega)_D \cup (\partial\Omega)_N$ , where  $(\partial\Omega)_D$  is not of measure zero and  $(\partial\Omega)_D$  and  $(\partial\Omega)_N$  are essentially disjoint, and let  $V := H_D^1(\Omega)$  be the subspace of functions in  $H^1(\Omega)$  which vanish on the Dirichlet part  $(\partial\Omega)_D$  of the boundary of  $\Omega$ . Let  $u \in H_D^1(\Omega)$  be the variational solution of an elliptic boundary value problem and  $u_h \in V_h$  be an approximation of  $u$ , where  $V_h$  is a finite dimensional approximation space which might not be a subspace of  $V$ . Further, let us assume that, for a norm  $\|\cdot\|_h$  defined on  $V + V_h$  and a constant  $c$  one can prove that

$$(1.1) \quad \|u - u_h\|_h \leq c\|u\|_{H^1(\Omega)}, \quad \text{for all } u \in H_D^1(\Omega),$$

and

$$(1.2) \quad \|u - u_h\|_h \leq ch\|u\|_{H^2(\Omega)}, \quad \text{for all } u \in H^2(\Omega) \cap H_D^1(\Omega).$$

By interpolation, from (1.1) and (1.2), we obtain that for a fixed  $s \in [0, 1]$ :

$$\|u - u_h\|_h \leq ch^s\|u\|_{[H^2(\Omega) \cap H_D^1(\Omega), H_D^1(\Omega)]_{1-s}} \quad \text{for all } u \in [H^2(\Omega) \cap H_D^1(\Omega), H_D^1(\Omega)]_{1-s}.$$

If we assume that the variational solution  $u$  belongs to an intermediate space  $H^{1+s}(\Omega) \cap H_D^1(\Omega)$ ,  $s \in (0, 1)$  and  $u$  is not in  $H^2(\Omega)$ , then it is natural to ask: Does  $[H^2(\Omega) \cap H_D^1(\Omega), H_D^1(\Omega)]_{1-s}$  coincide with  $H^{1+s}(\Omega) \cap H_D^1(\Omega)$ ? This type of question arose in [4] and [5]. The paper will give a positive answer to this question for the special case when  $\Omega$  is a polygonal domain in  $\mathbb{R}^2$ .

The remaining part of the paper is organized as follows. In Section 2 general interpolation results and some notation are presented. The proof of the fact that  $[H^2(\Omega) \cap H_D^1(\Omega), H_D^1(\Omega)]_{1-s}$  coincides with  $H^{1+s}(\Omega) \cap H_D^1(\Omega)$  when  $\Omega$  is a polygonal domain is given in Section 3. In Section 5, an application of the interpolation result of Section 3 to a nonconforming finite element problem is given.

---

*Date:* August 21, 2001.

*Key words and phrases.* interpolation spaces, finite element method, extension operator.

This work was partially supported by the National Science Foundation under Grant DMS-9973328.

## 2. ABSTRACT INTERPOLATION RESULTS

In this section we give some basic definitions and results concerning interpolation between Hilbert spaces and subspaces using the real method of interpolation of Lions and Peetre (see [11] and [12]).

Let  $(X, Y)$  be a pair of separable Hilbert spaces with inner products  $(\cdot, \cdot)_X$  and  $(\cdot, \cdot)_Y$  respectively, and satisfying, for some positive constant  $c$ ,

$$(2.1) \quad \begin{cases} X \text{ is a dense subset of } Y \text{ and} \\ \|u\|_Y \leq c\|u\|_X \quad \text{for all } u \in X, \end{cases}$$

where  $\|u\|_X^2 = (u, u)_X$  and  $\|u\|_Y^2 = (u, u)_Y$ .

Let  $D(S)$  denote the subset of  $X$  consisting of all elements  $u$  such that the antilinear form

$$(2.2) \quad v \rightarrow (u, v)_X, \quad v \in X$$

is continuous in the topology induced by  $Y$ . For any  $u$  in  $D(S)$ , the antilinear form (2.2) can be extended to a continuous antilinear form on  $Y$ . Then by the Riesz representation theorem, there exists an element  $Su$  in  $Y$  such that

$$(2.3) \quad (u, v)_X = (Su, v)_Y \quad \text{for all } v \in X.$$

In this way,  $S$  is a well defined operator in  $Y$ , with domain  $D(S)$ . The next result gives some of the properties of  $S$ .

**Proposition 2.1.** *The domain  $D(S)$  of the operator  $S$  is dense in  $X$  and consequently  $D(S)$  is dense in  $Y$ . The operator  $S : D(S) \subset Y \rightarrow Y$  is a bijective, self-adjoint and positive definite operator. The inverse operator  $S^{-1} : Y \rightarrow D(S) \subset Y$  is a bounded symmetric positive definite operator and*

$$(2.4) \quad (S^{-1}z, u)_X = (z, u)_Y \quad \text{for all } z \in Y, \quad u \in X$$

The interpolating space  $[X, Y]_s$  for  $s \in (0, 1)$  is defined using the  $K$  function, where for  $u \in Y$  and  $t > 0$ ,

$$K(t, u, X, Y) = K(t, u) := \inf_{u_0 \in X} (\|u_0\|_X^2 + t^2\|u - u_0\|_Y^2)^{1/2}.$$

Then  $[X, Y]_s$  consists of all  $u \in Y$  such that

$$\int_0^\infty t^{-(2s+1)} K(t, u)^2 dt < \infty.$$

The norm on  $[X, Y]_s$  is defined by

$$\|u\|_{[X, Y]_s}^2 := \mathbf{c}_s^2 \int_0^\infty t^{-(2s+1)} K(t, u)^2 dt,$$

where we have chosen the normalization

$$\mathbf{c}_s := \left( \int_0^\infty \frac{t^{(1-2s)}}{t^2 + 1} dt \right)^{-1/2} = \sqrt{\frac{2}{\pi} \sin(\pi s)}.$$

By definition we take

$$[X, Y]_0 := X \quad \text{and} \quad [X, Y]_1 := Y.$$

The next lemma provides the relation between  $K(t, u)$  and the connecting operator  $S$ .

**Lemma 2.1.** *For all  $u \in Y$  and  $t > 0$ ,*

$$K(t, u)^2 = t^2 \left( (I + t^2 S^{-1})^{-1} u, u \right)_Y.$$

For the proof of this lemma see, for example, [1].

**Remark 2.1.** *Lemma 2.1 gives an alternative expression for the norm on  $[X, Y]_s$ , namely:*

$$(2.5) \quad \|u\|_{[X, Y]_s}^2 := \mathbf{c}_s^2 \int_0^\infty t^{-2s+1} \left( (I + t^2 S^{-1})^{-1} u, u \right)_Y dt.$$

*In addition, by this expression for the norm (see Definition 2.1 and Theorem 15.1 in [11]), it follows that the intermediate space  $[X, Y]_s$  coincides topologically with the domain of the unbounded operator  $S^{1/2(1-s)}$  equipped with the norm of the graph of the same operator. As a consequence we have that  $X$  is dense in  $[X, Y]_s$  for any  $s \in [0, 1]$ .*

**Lemma 2.2.** *Let  $X_0$ , be a closed subspace of  $X$  and let  $Y_0$ , be a closed subspace of  $Y$ . Let  $X_0$  and  $Y_0$  be equipped with the topology and the geometry induced by  $X$  and  $Y$  respectively, and assume that the pair  $(X_0, Y_0)$  satisfies (2.1). Then, for  $s \in [0, 1]$ ,*

$$[X_0, Y_0]_s \subset [X, Y]_s \cap Y_0.$$

*Proof.* For any  $u \in Y_0$  we have

$$K(t, u, X, Y) \leq K(t, u, X_0, Y_0).$$

Thus,

$$(2.6) \quad \|u\|_{[X, Y]_s} \leq \|u\|_{[X_0, Y_0]_s} \quad \text{for all } u \in [X_0, Y_0]_s, \quad s \in [0, 1],$$

which proves the lemma.  $\square$

**Lemma 2.3.** *Let  $H^i, \tilde{H}^i, i = 1, 2$ , be separable Hilbert spaces such that  $H^2$  is a subspace of  $H^1$  and the pair  $(\tilde{H}^2, \tilde{H}^1)$  satisfies (2.1). We assume further that there are linear operators  $E$  and  $R$  such that*

$$(2.7) \quad E : H^i \rightarrow \tilde{H}^i \quad \text{is a bounded operator for } i = 1, 2,$$

$$(2.8) \quad R : \tilde{H}^i \rightarrow H^i \quad \text{is a bounded operator for } i = 1, 2,$$

$$(2.9) \quad REu = u \quad \text{for all } u \in H^1.$$

*Then, the pair  $(H^2, H^1)$  satisfies (2.1) and for  $s \in [0, 1]$ , an equivalent norm on  $[H^2, H^1]_s$  is given by  $\|E(\cdot)\|_{[\tilde{H}^2, \tilde{H}^1]_s}$ , i.e., there are positive constants  $c_1$  and  $c_2$  such that*

$$(2.10) \quad c_1 \|u\|_{[H^2, H^1]_s} \leq \|Eu\|_{[\tilde{H}^2, \tilde{H}^1]_s} \leq c_2 \|u\|_{[H^2, H^1]_s} \quad \text{for all } u \in [H^2, H^1]_s.$$

*Proof.* First, we prove that the pair  $(H^2, H^1)$  satisfies (2.1). Let  $u \in H^1$  and let  $\{w_n\}$  be a sequence in  $\tilde{H}^2$  convergent to  $Eu$  in the norm of  $\tilde{H}^1$ . Then  $\{Rw_n\}$  is a sequence in  $H^2$  which converges to  $u$  in the norm of  $H^1$ . Thus,  $H^2$  is dense in  $H^1$ . For the estimate part of (2.1), from our hypothesis, we have

$$\|u\|_{H^1} = \|REu\|_{H^1} \leq c \|Eu\|_{\tilde{H}^1} \leq c \|Eu\|_{\tilde{H}^2} \leq c \|u\|_{H^2} \quad \text{for all } u \in H^2,$$

where  $c$  is a generic constant which is not be the same at different occurrences.

For the second part of the lemma let  $s \in [0, 1]$  be fixed. From the hypothesis (2.7), by interpolation, we have that

$$\|Eu\|_{[\tilde{H}^2, \tilde{H}^1]_s} \leq c_2 \|u\|_{[H^2, H^1]_s} \quad \text{for all } u \in [H^2, H^1]_s,$$

for some positive constant  $c_2$ .

Next, from (2.8), by interpolation, we obtain that for some positive constant  $c_1$

$$c_1 \|Rv\|_{[H^2, H^1]_s} \leq \|v\|_{[\tilde{H}^2, \tilde{H}^1]_s} \quad \text{for all } v \in [\tilde{H}^2, \tilde{H}^1]_s.$$

Finally, for  $u \in [H^2, H^1]_s$

$$\|u\|_{[H^2, H^1]_s} = \|REu\|_{[H^2, H^1]_s} \leq c_1^{-1} \|Eu\|_{[\tilde{H}^2, \tilde{H}^1]_s}.$$

This completes the proof of the lemma.  $\square$

### 3. INTERPOLATION BETWEEN $H^2(\Omega) \cap H_D^1(\Omega)$ AND $H_D^1(\Omega)$ .

Let  $\Omega$  be a polygonal domain in  $\mathbb{R}^2$  with boundary  $\partial\Omega = (\partial\Omega)_D \cup (\partial\Omega)_N$ , where  $(\partial\Omega)_D$  is not of measure zero, and  $(\partial\Omega)_D$  and  $(\partial\Omega)_N$  are essentially disjoint and consist of a finite number of line segments. Let  $H_D^1(\Omega)$  denote the space of all functions in  $H^1(\Omega)$  which vanish on  $(\partial\Omega)_D$ . Let  $\partial\Omega$  be the polygonal line  $P_1 P_2 \cdots P_m P_1$ . Here we consider that the set  $\{P_1, P_2, \dots, P_m\}$  consists of all vertices of  $\partial\Omega$  and all the points of  $(\partial\Omega)_D \cap (\partial\Omega)_N$ . We will also call the points of  $(\partial\Omega)_D \cap (\partial\Omega)_N$  vertices of  $\partial\Omega$ . The main result of this section is:

**Theorem 3.1.** *Let  $s \in [0, 1]$  be fixed and let  $\Omega \subset \mathbb{R}^2$  be a polygonal domain with Lipschitz boundary. Then*

$$(3.1) \quad [H^2(\Omega) \cap H_D^1(\Omega), H_D^1(\Omega)]_s = [H^2(\Omega), H^1(\Omega)]_s \cap H_D^1(\Omega).$$

In order to prove Theorem 3.1 we introduce first some further notation. For  $j = 1, 2, \dots, m$ , let  $U_j$  be an open disk centered at  $P_j$  such that  $U_j$  contains no vertices other than  $P_j$ . Next we add more disks, say  $U_j$ , centered at  $P_j$ ,  $j = m+1, \dots, M$ , such that  $P_j \in \partial\Omega$  or  $\overline{U_j} \subset \Omega$ , and

$$\overline{\Omega} \subset \bigcup_{j=1}^M U_j.$$

By increasing the number  $M$  of disks and modifying the radii of the disks, we can assume that  $P_k$  is not in  $U_j$ , for  $k \neq j$  and the radii of the disks are equal to some positive number  $r_1$ . Then, there is a partition of unity  $\{\phi_j\}_{j=1}^M$  subordinate to the covering  $\bigcup_{j=1}^M U_j$ , i.e.,

$$(3.2) \quad \text{supp}(\phi_j) \subset U_j, \quad \sum_{j=1}^M \phi_j(x) = 1 \quad \text{for all } x \in \overline{\Omega}.$$

Let us denote  $U_j \cap \Omega$  by  $\Omega_j$ ,  $j = 1, \dots, M$ . We note here that one can find  $r_0 > 0$  such that

$$(3.3) \quad \text{dist}(\Omega \setminus \Omega_j, \text{supp } \phi_j) \geq r_0 \quad j = 1, \dots, M.$$

Further, for  $j = 1, 2, \dots, M$ , we define  $(\partial\Omega_j)_D$  and  $(\partial\Omega_j)_N$  to be

$$(\partial\Omega_j)_N := (\partial\Omega)_N \cap \partial\Omega_j, \quad (\partial\Omega_j)_D := \overline{\partial\Omega_j \setminus (\partial\Omega_j)_N},$$

and denote the space of functions in  $H^1(\Omega_j)$  which vanish on  $(\partial\Omega_j)_D$  by  $H_D^1(\Omega_j)$ . Also we introduce the spaces

$$H_*^2(\Omega_j) := \{u \in H^2(\Omega_j) \cap H_D^1(\Omega_j) : \frac{\partial u}{\partial n} = 0 \text{ on } \overline{\partial\Omega_j \setminus (\partial\Omega)}\}.$$

To prove Theorem 3.1 we assume for the moment the following result.

**Theorem 3.2.** *Let  $\Omega_j$  be one of the domains defined above. Then there exist a positive constant  $c$  such that*

$$(3.4) \quad \|u\|_{[H_*^2(\Omega_j), H_D^1(\Omega_j)]_s} \leq c \|u\|_{[H^2(\Omega_j), H^1(\Omega_j)]_s}$$

*for all  $u \in [H_*^2(\Omega_j), H_D^1(\Omega_j)]_s \cap M_j(r)$ , where*

$$M_j(r) := \{u \in H^1(\Omega_j) : \text{dist}(\Omega \setminus \Omega_j, \text{supp } u) \geq r_0\}.$$

In addition, we need also the following lemma.

**Lemma 3.1.** *Let  $\Omega_0 \subset \Omega$  be domains in  $R^N$  with Lipschitz boundary. Let  $m$  be a nonnegative integer,  $0 < s < 1$  and  $r_0 > 0$ . Define*

$$M(r_0) := \{u \in [H^{m+1}(\Omega), H^m(\Omega)]_s : \text{dist}(\Omega \setminus \Omega_0, \text{supp } u) \geq r_0\}.$$

*Then there is a positive constant  $c = c(s, r_0)$  such that*

$$(3.5) \quad \|u\|_{[H^{m+1}(\Omega), H^m(\Omega)]_s} \leq c \|u\|_{[H^{m+1}(\Omega_0), H^m(\Omega_0)]_s} \quad \text{for all } u \in M(r_0).$$

*Proof.* Since  $\Omega$  has Lipschitz boundary (see, e.g., [3], [6]), an equivalent norm on  $[H^{m+1}(\Omega), H^m(\Omega)]_{1-s} = H^{m+s}(\Omega)$  is the double integral norm

$$\|u\|_{m+s, \Omega}^2 := \|u\|_{H^m(\Omega)}^2 + \sum_{|\alpha|=m} \int_{\Omega} \int_{\Omega} \frac{|D^{\alpha}u(x) - D^{\alpha}u(y)|^2}{|x - y|^{N+2s}} dx dy.$$

A similar statement holds for  $\Omega_0$ . Let  $u \in M(r_0)$ . Then,

$$\|u\|_{H^m(\Omega)} = \|u\|_{H^m(\Omega_0)}$$

and for a fixed multi index  $\alpha$  with  $|\alpha| = m$  we have

$$\begin{aligned} & \int_{\Omega} \int_{\Omega} \frac{|D^{\alpha}u(x) - D^{\alpha}u(y)|^2}{|x - y|^{N+2s}} dx dy = I_1 + 2I_2 = \\ & = \int_{\Omega_0} \int_{\Omega_0} \frac{|D^{\alpha}u(x) - D^{\alpha}u(y)|^2}{|x - y|^{N+2s}} dx dy + 2 \int_{\Omega \setminus \Omega_0} \int_{\Omega_0} \frac{|D^{\alpha}u(x)|^2}{|x - y|^{N+2s}} dx dy. \end{aligned}$$

Next, let  $K := \{x \in \Omega_0 : \text{dist}(x, \Omega \setminus \Omega_0) \geq r_0\}$ . It follows that

$$\begin{aligned} I_2 &= \int_{\Omega \setminus \Omega_0} \int_K \frac{|D^{\alpha}u(x)|^2}{|x - y|^{N+2s}} dx dy = \int_K \int_{\Omega \setminus \Omega_0} \frac{1}{|x - y|^{N+2s}} dy |D^{\alpha}u(x)|^2 dx \\ &\leq c \int_K |D^{\alpha}u(x)|^2 dx \leq c \|u\|_{H^m(\Omega_0)}^2. \end{aligned}$$

Summing up these estimates we obtain that (3.5) holds.  $\square$

Now we can prove Theorem 3.1.

*Proof.* The space  $H^2(\Omega) \cap H_D^1(\Omega)$  is dense in  $H_D^1(\Omega)$  (see for example Theorem 1.6.1 in [10]). Applying Lemma 2.2 with  $X = H^2(\Omega)$ ,  $Y = H^1(\Omega)$ ,  $X_0 = H^2(\Omega) \cap H_D^1(\Omega)$  and  $Y_0 = H_D^1(\Omega)$ , we obtain that

$$(3.6) \quad [H^2(\Omega) \cap H_D^1(\Omega), H_D^1(\Omega)]_s \subset [H^2(\Omega), H^1(\Omega)]_s \cap H_D^1(\Omega).$$

In order to prove the other inclusion of (3.1), we need to show that for a positive constant  $c$ ,

$$(3.7) \quad \|u\|_{[H^2(\Omega) \cap H_D^1(\Omega), H_D^1(\Omega)]_s} \leq c \|u\|_{[H^2(\Omega), H^1(\Omega)]_s},$$

for all  $u \in [H^2(\Omega), H_D^1(\Omega)]_s \cap H_D^1(\Omega)$ . We let  $c$  denote a generic positive constant. Let  $u \in [H^2(\Omega), H_D^1(\Omega)]_s \cap H_D^1(\Omega)$ . For  $j = 0, 1, \dots, M$ , let  $u_j := \phi_j u$ . Then,  $u = \sum_{j=1}^M u_j$  and by applying Lemma 2.2, and Theorem 3.2 we obtain

$$\begin{aligned} \|u\|_{[H^2(\Omega) \cap H_D^1(\Omega), H_D^1(\Omega)]_s} &\leq \sum_{j=1}^M \|u_j\|_{[H^2(\Omega) \cap H_D^1(\Omega), H_D^1(\Omega)]_s} \\ &\leq \sum_{j=1}^M \|u_j\|_{[H_*^2(\Omega_j), H_D^1(\Omega_j)]_s} \leq c \sum_{j=1}^M \|u_j\|_{[H^2(\Omega_j), H^1(\Omega_j)]_s}. \end{aligned}$$

Next, using the fact that multiplication by a smooth function is continuous on  $[H^2(\Omega), H^1(\Omega)]_s$ , we have

$$\|u_j\|_{[H^2(\Omega_j), H^1(\Omega_j)]_s} \leq c \|u_j\|_{[H^2(\Omega), H^1(\Omega)]_s} \leq c \|u\|_{[H^2(\Omega), H^1(\Omega)]_s}.$$

Combining the above estimates we see that (3.7) follows. Finally, from (3.6) and (3.7) we conclude the result.  $\square$

**3.1. Proving Theorem 3.2.** To begin with, we consider the case when  $\Omega_j = U_j$ , i.e.,  $\Omega_j$  is a disk. We assume, without loss of generality, that  $\Omega_j$  is the unit disk  $U$  centered at the origin of a Cartesian system of coordinates. In this case we have  $(\partial\Omega_j)_D := (\partial\Omega_j)$  and

$$H_D^1(\Omega_j) = H_0^1(U), \quad H_*^2(\Omega_j) = H_0^2(U).$$

Let  $E : H_0^1(U) \rightarrow H^1(\mathbb{R}^2)$ , be the extension by zero operator (for  $r > 1$ ), and let  $R : H^1(\mathbb{R}^2) \rightarrow H_0^1(U)$  defined as follows.

First, we introduce a smooth cutoff function  $\eta$  which depends only on the distance  $r$  to the origin and satisfies

$$\eta(r) = 1 \text{ for } 0 < r \leq 1 \text{ and } \eta(r) = 0 \text{ for } r \geq 2.$$

Then, for a function  $v \in H^1(\mathbb{R}^2)$  we define  $Rv \in H_0^1(U)$  by

$$(Rv)(r, \theta) := v_1(r, \theta) - 3v_1(1/r, \theta) + 2v_1(1/r^2, \theta), \quad (r, \theta) \in U,$$

where

$$v_1(r, \theta) := v(r, \theta)\eta(r), \quad (r, \theta) \in \mathbb{R}^2.$$

Note that  $R : H^i(\mathbb{R}^2) \rightarrow H_0^i(U)$  for  $i = 1, 2$ , and we can apply Lemma 2.3 with  $H^2 = H_0^2(U)$ ,  $\tilde{H}^2 = H^2(\mathbb{R}^2)$  and  $H^1 = H_0^1(U)$ ,  $\tilde{H}^1 = H^1(\mathbb{R}^2)$ . Thus,

$$\|u\|_{[H_0^2(U), H_0^1(U)]_s} \leq c \|Eu\|_{[H^2(\mathbb{R}^2), H^1(\mathbb{R}^2)]_s} \quad \text{for all } u \in [H_0^2(U), H_0^1(U)]_s,$$

for some positive constant  $c$ . On the other hand, by Lemma 3.1 we have

$$\|Eu\|_{[H^2(\mathbb{R}^2), H^1(\mathbb{R}^2)]_s} \leq c \|u\|_{[H^2(U), H^1(U)]_s} \quad \text{for all } u \in [H_0^2(U), H_0^1(U)]_s \cap M(r_0),$$

where

$$M(r_0) := \{u \in H^1(U) : \text{dist}(\partial U, \text{supp } u) \geq r_0\}.$$

Using the above two estimates proves Theorem 3.2 in this special case.

Before we consider the remaining cases, let us introduce some new notation. Let  $\alpha, \beta$  be real numbers such that  $\alpha < \beta$  and  $\beta - \alpha < 2\pi$ . Using polar coordinates  $(r, \theta)$  we define the sector domain

$$S_{\alpha, \beta} := \{(r, \theta) : 0 < r < 1, \alpha < \theta < \beta\}$$

and the following spaces:

$$\hat{H}^1(S_{\alpha, \beta}) := \{u \in H^1(S_{\alpha, \beta}) : u = 0 \text{ for } r = 1\},$$

$$\hat{H}^2(S_{\alpha, \beta}) := \{u \in H^2(S_{\alpha, \beta}) : u = \frac{\partial u}{\partial n} = 0 \text{ for } r = 1\},$$

$$\hat{H}_\gamma^i(S_{\alpha, \beta}) := \{u \in \hat{H}^i(S_{\alpha, \beta}) : u = 0 \text{ for } \theta = \gamma\},$$

$$\hat{H}_{\alpha, \beta}^i(S_{\alpha, \beta}) := \{u \in \hat{H}^i(S_{\alpha, \beta}) : u = 0 \text{ for } \theta = \alpha \text{ and } \theta = \beta\},$$

where  $i = 1, 2$ ,  $\gamma = \alpha$  or  $\gamma = \beta$  and the functions are zero on line segments or arcs in the trace sense.

All the remaining cases of Theorem 3.2 can be reduced to the following standard ones: The domain  $\Omega_j$  coincides with  $S_{0, \omega}$  for some real number  $\omega \in (0, 2\pi)$  and

- Case 1. “Free-Free”:  $H_D^1(\Omega_j) = \hat{H}^1(S_{0, \omega})$  and  $H_*^2(\Omega_j) = \hat{H}^2(S_{0, \omega})$  or
- Case 2. “Dirichlet-Free”:  $H_D^1(\Omega_j) = \hat{H}_0^1(S_{0, \omega})$  and  $H_*^2(\Omega_j) = \hat{H}_0^2(S_{0, \omega})$  or
- Case 3. “Dirichlet-Dirichlet”:  $H_D^1(\Omega_j) = \hat{H}_{0, \omega}^1(S_{0, \omega})$  and  $H_*^2(\Omega_j) = \hat{H}_{0, \omega}^2(S_{0, \omega})$ .

Next, we prove Theorem 3.2 in Case 1.

We define the infinite sector domain  $\tilde{S}_{0, \omega}$  by

$$\tilde{S}_{0, \omega} := \{(r, \theta) : r > 0, 0 < \theta < \omega\}.$$

The operators  $E : \hat{H}^1(S_{0, \omega}) \rightarrow H^1(\tilde{S}_{0, \omega})$  and  $R : H^1(\tilde{S}_{0, \omega}) \rightarrow \hat{H}^1(S_{0, \omega})$  defined in the case of the disk, satisfy the hypotheses of Lemma 2.3 with  $H^i = H^i(S_{0, \omega})$ , and  $\tilde{H}^i = \hat{H}^i(\tilde{S}_{0, \omega})$ ,  $i=1,2$ . Thus,  $\hat{H}^2(S_{0, \omega})$  is dense in  $\hat{H}^1(S_{0, \omega})$  and similar arguments used in the case of the disk can be used now to show that

$$(3.8) \quad \|u\|_{[\hat{H}^2(S_{0, \omega}), \hat{H}^1(S_{0, \omega})]_s} \leq c \|u\|_{[H^2(S_{0, \omega}), H^1(S_{0, \omega})]_s}$$

for all  $u \in [\hat{H}^2(S_{0, \omega}), \hat{H}^1(S_{0, \omega})]_s \cap M(r_0)$ , where  $c$  is a positive constant and

$$M(r_0) := \{u \in \hat{H}^1(S_{0, \omega}) : \text{dist}(\partial U, \text{supp } u) \geq r_0\}.$$

Therefore, the proof for Case 1 is complete .

For the Case 2 and Case 3 we will use again Lemma 2.3, but we need to construct operators  $E$  and  $R$  with stronger properties.

In order to prove Theorem 3.2 in Case 2, let us assume for the moment that the following existence result holds.

**Theorem 3.3.** *Let  $\alpha < 0$  be such that  $\omega - \alpha < 2\pi$ . Then, there are linear operators  $E$  and  $R$  such that*

$$(3.9) \quad E : \hat{H}^i(S_{0,\omega}) \rightarrow \hat{H}^i(S_{\alpha,\omega}) \text{ is a bounded operator, } i = 1, 2,$$

$$(3.10) \quad R : \hat{H}^i(S_{\alpha,\omega}) \rightarrow \hat{H}_0^i(S_{0,\omega}) \text{ is a bounded operator, } i = 1, 2,$$

$$(3.11) \quad REu = u \quad \text{for all } u \in \hat{H}_0^1(S_{0,\omega}).$$

First, we observe that from (3.9), we get in particular that

$$E : \hat{H}_0^i(S_{0,\omega}) \rightarrow \hat{H}^i(S_{\alpha,\omega}) \text{ is a bounded operator, } i = 1, 2.$$

From the previous case we have that  $\hat{H}^2(S_{\alpha,\omega})$  is dense in  $\hat{H}^1(S_{\alpha,\omega})$ . Thus, we can apply Lemma 2.3 with  $H^i = \hat{H}_0^i(S_{0,\omega})$ , and  $\tilde{H}^i = \hat{H}^i(S_{\alpha,\omega})$ ,  $i=1,2$  and obtain that for a positive  $c$ ,

$$(3.12) \quad \|u\|_{[\hat{H}_0^2(S_{0,\omega}), \hat{H}_0^1(S_{0,\omega})]_s} \leq c \|Eu\|_{[\hat{H}^2(S_{\alpha,\omega}), \hat{H}^1(S_{\alpha,\omega})]_s},$$

for all  $u \in [\hat{H}_0^2(S_{0,\omega}), \hat{H}_0^1(S_{0,\omega})]_s$ .

From (3.9), by interpolation, we have that for another constant  $c$ ,

$$(3.13) \quad \|Eu\|_{[\hat{H}^2(S_{\alpha,\omega}), \hat{H}^1(S_{\alpha,\omega})]_s} \leq c \|u\|_{[\hat{H}^2(S_{0,\omega}), \hat{H}^1(S_{0,\omega})]_s},$$

for all  $u \in [\hat{H}^2(S_{0,\omega}), \hat{H}^1(S_{0,\omega})]_s$ .

Combining (3.12) and (3.13) we obtain

$$(3.14) \quad \|u\|_{[\hat{H}_0^2(S_{0,\omega}), \hat{H}_0^1(S_{0,\omega})]_s} \leq c \|u\|_{[\hat{H}^2(S_{0,\omega}), \hat{H}^1(S_{0,\omega})]_s},$$

for all  $u \in [\hat{H}_0^2(S_{0,\omega}), \hat{H}_0^1(S_{0,\omega})]_s$ .

Now we can use the proof of Case 1 to finish the proof of Case 2. More precisely, from (3.8) and (3.14), we see that

$$(3.15) \quad \|u\|_{[\hat{H}_0^2(S_{0,\omega}), \hat{H}_0^1(S_{0,\omega})]_s} \leq c \|u\|_{[H^2(S_{0,\omega}), H^1(S_{0,\omega})]_s},$$

for all  $u \in [\hat{H}_0^2(S_{0,\omega}), \hat{H}_0^1(S_{0,\omega})]_s \cap M(r_0)$ . Here,

$$M(r_0) := \{u \in \hat{H}_0^1(S_{0,\omega}) : \text{dist}(\partial U, \text{supp } u) \geq r_0\}.$$

Therefore, we have proved Theorem 3.2 in this case too.

For the Case 3 we assume that we have the following result.

**Theorem 3.4.** *Let  $\alpha < 0$  be such that  $\omega - \alpha < 2\pi$ . Then, there are linear operators  $E$  and  $R$  such that*

$$(3.16) \quad E : \hat{H}_\omega^i(S_{0,\omega}) \rightarrow \hat{H}_\omega^i(S_{\alpha,\omega}) \text{ is a bounded operator, } i = 1, 2,$$

$$(3.17) \quad R : \hat{H}_\omega^i(S_{\alpha,\omega}) \rightarrow \hat{H}_{0,\omega}^i(S_{0,\omega}) \text{ is a bounded operator, } i = 1, 2,$$

$$(3.18) \quad REu = u \quad \text{for all } u \in \hat{H}_{0,\omega}^1(S_{0,\omega}).$$



We can reduce the proof of Case 3 to an estimate which follows from the previous case. The arguments are similar to those of Case 2.

#### 4. PROVING THE EXISTENCE OF THE OPERATORS $E$ AND $R$

The proofs of Theorem 3.3 and Theorem 3.4 are based on the following extension result.

**Lemma 4.1.** *Let  $\Omega$  be a triangular domain in  $\mathbb{R}^2$  with boundary  $\partial\Omega = (\partial\Omega)_D \cup (\partial\Omega)_N$ , where  $(\partial\Omega)_N = \bar{\Gamma}$  is one of the edges of  $\partial\Omega$  ( $\Gamma$  is an open interval in  $R$ ) and  $(\partial\Omega)_D$  consists of the union of the other two edges. Then, there exist a linear operator  $P$  such that*

$$(4.1) \quad P : H_{00}^{i-1/2}(\Gamma) \rightarrow H_D^i(\Omega) \text{ and } P \text{ is a bounded operator, } i = 1, 2.$$

Here,

$$H_{00}^{1/2}(\Gamma) = [H_0^1(\Gamma), L^2(\Gamma)]_{1/2}, \quad H_{00}^{3/2}(\Gamma) = [H_0^2(\Gamma), H_0^1(\Gamma)]_{1/2},$$

and

$$H_D^2(\Omega) = \{u \in H^2(\Omega) : u = \frac{\partial u}{\partial n} = 0 \text{ on } (\partial\Omega)_D\}.$$

*Proof.* For  $v \in H_{00}^{1/2}(\Gamma)$  let  $\tilde{v}$  denote the extension by zero of  $v$  to the rest of  $\partial\Omega$ . Then, for some positive constant  $c$  we have

$$(4.2) \quad \|\tilde{v}\|_{H^{1/2}(\partial\Omega)} \leq c\|v\|_{H_{00}^{1/2}(\Gamma)} \quad \text{for all } v \in H_{00}^{1/2}(\Gamma).$$

For  $v \in C_0^\infty(\Gamma)$  we define  $Pv$  to be the solution of the problem:

Find  $b \in H^2(\Omega)$  such that

$$(4.3) \quad \begin{cases} \Delta^2 b = 0 & \text{in } \Omega, \\ b = \tilde{v} & \text{on } \partial\Omega, \\ \frac{\partial b}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases}$$

It is known that (see, e.g., Proposition 1.3 in [8]) Problem (4.3) has exactly one solution  $b \in H_D^2(\Omega) \subset H_D^1(\Omega)$  and

$$(4.4) \quad \|b\|_{H^2(\Omega)} \leq c\|v\|_{H^{3/2}(\Gamma)} \leq c\|v\|_{H_{00}^{3/2}(\Gamma)} \quad \text{for all } v \in C_0^\infty(\Gamma),$$

where  $c$  is a positive constant. In addition, since  $v \in C_0^\infty(\Gamma)$ , we have  $b \in H^3(\Omega)$  (see, e.g., Section 3.4.2 in [10]).

Next, in order to estimate  $\|b\|_{H^1(\Omega)}$  we consider the following fourth order problem. Find  $w$  such that

$$(4.5) \quad \begin{cases} \Delta^2 w = -\Delta b & \text{in } \Omega, \\ w \in H_0^2(\Omega). \end{cases}$$

The (weak) solution  $w$  of the above problem satisfies  $w \in H^3(\Omega) \cap H_0^2(\Omega)$  (see, e.g., Corollary 3.4.2 in [10]). Then, using Green's first and second identities, we get

$$(\nabla b, \nabla b) = (-\Delta b, b) = (\Delta^2 w, b) = \left\langle \frac{\partial(\Delta w)}{\partial n}, b \right\rangle + (\Delta w, \Delta b),$$

where  $(\cdot, \cdot)$  and  $\langle \cdot, \cdot \rangle$  are the inner products on  $L^2(\Omega)$  and  $L^2(\partial\Omega)$ , respectively. Since  $w \in H_0^2(\Omega)$  and  $\Delta b$  is harmonic it follows from Green's identity that  $(\Delta w, \Delta b) = 0$ . Consequently,

$$(4.6) \quad \|b\|_{H^1(\Omega)}^2 \leq c \left\| \frac{\partial(\Delta w)}{\partial n} \right\|_{H^{-1/2}(\partial\Omega)} \|b\|_{H^{1/2}(\partial\Omega)} \quad \text{for all } v \in C_0^\infty(\Gamma),$$

where  $c$  is a positive constant. Next, we have

$$(4.7) \quad \left\| \frac{\partial(\Delta w)}{\partial n} \right\|_{H^{-1/2}(\partial\Omega)} = \sup_{\varphi \in H^{1/2}(\partial\Omega)} \frac{\langle \frac{\partial(\Delta w)}{\partial n}, \varphi \rangle}{\|\varphi\|_{H^{1/2}(\partial\Omega)}}.$$

Denoting the harmonic extension of  $\varphi \in H^{1/2}(\partial\Omega)$  to  $\Omega$  by the same symbol  $\varphi$ , and applying again Green's identity, we obtain

$$(4.8) \quad \langle \frac{\partial(\Delta w)}{\partial n}, \varphi \rangle = \langle \frac{\partial\varphi}{\partial n}, \Delta w \rangle - (\Delta b, \varphi).$$

In order to estimate the right hand side of (4.8), on the one hand we have

$$(4.9) \quad |(\Delta b, \varphi)| = |(\nabla b, \nabla \varphi)| \leq \|b\|_{H^1(\Omega)} \|\varphi\|_{H^1(\Omega)} \leq c \|b\|_{H^1(\Omega)} \|\varphi\|_{H^{1/2}(\partial\Omega)},$$

and on the other hand we can prove that

$$(4.10) \quad |\langle \Delta w, \frac{\partial\varphi}{\partial n} \rangle| \leq c \|b\|_{H^1(\Omega)} \|\varphi\|_{H^{1/2}(\partial\Omega)}.$$

Indeed, using trace inequalities we have

$$|\langle \Delta w, \frac{\partial\varphi}{\partial n} \rangle| \leq c \|\Delta w\|_{H^{1/2}(\partial\Omega)} \|\partial\varphi/\partial n\|_{H^{-1/2}(\partial\Omega)} \leq c \|\Delta w\|_{H^1(\Omega)} \|\partial\varphi/\partial n\|_{H^{-1/2}(\partial\Omega)}$$

where

$$\left\| \frac{\partial\varphi}{\partial n} \right\|_{H^{-1/2}(\partial\Omega)} = \sup_{\theta \in H^{1/2}(\partial\Omega)} \frac{\langle \frac{\partial\varphi}{\partial n}, \theta \rangle}{\|\theta\|_{H^{1/2}(\partial\Omega)}}.$$

Let us denote the harmonic extension of  $\theta \in H^{1/2}(\partial\Omega)$  to  $\Omega$  by the same symbol  $\theta$ . By applying Green's identity and the fact that  $\varphi$  is a harmonic function, we obtain

$$\langle \frac{\partial\varphi}{\partial n}, \theta \rangle = (\nabla \varphi, \nabla \theta) \leq \|\varphi\|_{H^1(\Omega)} \|\theta\|_{H^1(\Omega)} \leq \|\varphi\|_{H^{1/2}(\partial\Omega)} \|\theta\|_{H^{1/2}(\partial\Omega)}.$$

Next, since  $\Omega$  is convex, the operator  $\Delta^2$  defines an isomorphism from  $H^3(\Omega) \cap H_0^2(\Omega)$  onto  $H^{-1}(\Omega)$  (Corollary 3.4.2 in [10]). Thus, we get

$$\|\Delta w\|_{H^1(\Omega)} \leq c \|w\|_{H^3(\Omega)} \leq c \|\Delta^2 w\|_{H^{-1}(\Omega)} \leq c \|\Delta b\|_{H^{-1}(\Omega)}.$$

From Green's identity and the definition of the negative norm we see that

$$\|\Delta b\|_{H^{-1}(\Omega)} \leq \|b\|_{H^1(\Omega)}.$$

Finally, from the above estimates we conclude that (4.10) is proved.

Combining (4.6)-(4.10) we deduce that

$$(4.11) \quad \|b\|_{H^1(\Omega)} \leq c \|b\|_{H^{1/2}(\partial\Omega)} \quad \text{for all } v \in C_0^\infty(\Gamma),$$

where  $c$  is a constant independent of the function  $v \in C_0^\infty(\Gamma)$ . From (4.2), (4.3) and (4.11) we have

$$(4.12) \quad \|b\|_{H^1(\Omega)} \leq c \|v\|_{H_0^{1/2}(\Gamma)} \quad \text{for all } v \in C_0^\infty(\Gamma),$$

Using (4.4), (4.12) and the density of  $C_0^\infty(\Gamma)$  in both  $H_{00}^{1/2}(\Gamma)$  and  $H_{00}^{3/2}(\Gamma)$ , we can extend the definition of  $P$  so that (4.1) is satisfied.  $\square$

*Proof of Theorem 3.3.* Let  $O$  denote the origin of a polar coordinate system used to describe the sector domain  $S_{\alpha,\omega}$ . Let  $\epsilon > 0$  be fixed, and let  $A, B, C, D$  denote the points with polar coordinates  $(1, 0)$ ,  $(1, \omega)$ ,  $(1, \alpha)$  and  $(\epsilon, \pi)$ , respectively (see Figure 1). Let  $I := (O, A) \equiv (0, 1)$ ,  $I_1 := (D, A) \equiv (-\epsilon, 1)$  and denote by  $T, T_1$  the triangular domains  $O, A, C$  and  $D, A, C$ , respectively.

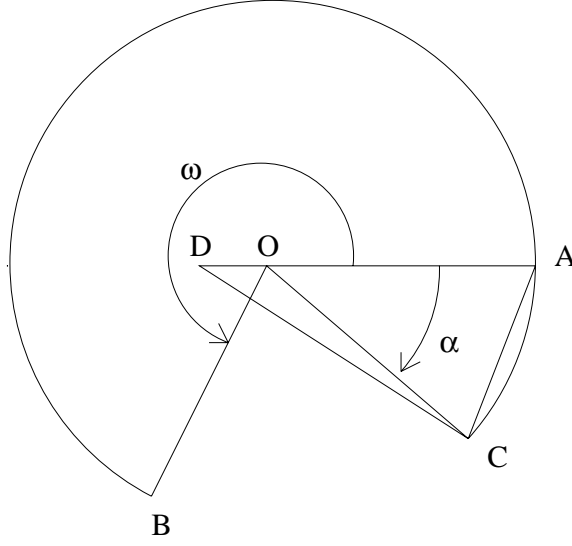


FIGURE 1. Sector domain.

We introduce here two new spaces:

$$H_{00,1}^{1/2}(I) := \{u \in H^{1/2}(I) : \int_0^1 \frac{u^2(x)}{1-x} dx < \infty\},$$

and

$$H_{00,1}^{3/2}(I) := \{u \in H^{3/2}(I) : u(1) = 0, \int_0^1 \frac{u'(x)^2}{1-x} dx < \infty\}.$$

For  $u \in \hat{H}^1(S_{0,\omega})$ , define  $\gamma u$  to be the trace of  $u$  to the interval  $I$ . Thus,

$$(4.13) \quad \|\gamma u\|_{H_{00,1}^{i-1/2}(I)} \leq c \|u\|_{H^i(S_{0,\omega})} \quad \text{for all } u \in \hat{H}^i(S_{0,\omega}), \quad i = 1, 2.$$

Next, we construct an extension operator  $E_1$  which takes functions defined on  $I$  into functions defined on the whole interval  $I_1$  and are zero on the interval  $(-\epsilon, -\epsilon/2)$ . We require that  $E_1$  to satisfy:

$$(4.14) \quad \|E_1 u\|_{H_{00}^{i-1/2}(I_1)} \leq c \|u\|_{H_{00,1}^{i-1/2}(I)} \quad \text{for all } u \in H_{00}^{i-1/2}(I), \quad i = 1, 2,$$

where  $c$  is a positive constant. One way to construct  $E_1$  is the following:

By Theorem 1.4.3.1 [9], one can find an extension operator  $E_2$  which takes functions defined on  $I$  into functions defined on the interval  $J := (-1, 1)$  such that

$$\|E_2 u\|_{H_{00,1}^{i-1/2}(J)} \leq c \|u\|_{H_{00,1}^{i-1/2}(I)}. \quad \text{for all } u \in H_{00,1}^{i-1/2}(I), \quad i = 1, 2.$$

Next, let  $\eta$  be a smooth function on  $J$  which is equal 0 on the interval  $(-1, -\epsilon/2)$  and is equal 1 on the interval  $(0, 1)$ . The operator  $E_3$  which multiplies a function defined on  $J$  with  $\eta$  and then takes the restriction of the new function to the interval  $I_1$  is continuous from  $H_{00,1}^{i-1/2}(J)$  to  $H_{00}^{i-1/2}(I_1)$ . Thus we can define  $E_1$  by

$$E_1(u) := E_3(E_2(u)).$$

By Lemma 4.1, we can extend  $E_1(\gamma u)$  to a function  $b = P(E_1(\gamma u))$  defined on the whole triangular domain  $T_1$  and such that (4.1) is satisfied for  $\Omega = T_1$  and  $\Gamma = I_1$ . Next, we consider the restriction of  $b$  to the triangular domain  $T$  and the extension by zero of the new function to the sector domain  $S_{\alpha,0}$ . Let  $\tilde{b}$  be the function obtained by this process. Then, define an extension operator denoted  $E_b$  mapping functions defined on  $S_{0,\omega}$  into functions defined on  $S_{\alpha,\omega}$  by

$$(E_b u)(x) = \begin{cases} u, & \text{if } x \in S_{0,\omega} \\ \tilde{b}, & \text{if } x \in S_{\alpha,0}. \end{cases}$$

Combining (4.13) and (4.14) with the fact that the operators involved in defining  $\tilde{b}$  are continuous, we get that

$$(4.15) \quad \|E_b u\|_{\hat{H}^i(S_{\alpha,0})} \leq c \|u\|_{\hat{H}^i(S_{0,\omega})}. \quad \text{for all } u \in \hat{H}^i(S_{0,\omega}).$$

Now we introduce another extension operator denoted  $E_e$ , which coincides with the classical even extension operator when  $\omega = -\alpha$ , mapping functions defined on  $S_{0,\omega}$  into functions defined on  $S_{\alpha,\omega}$  by

$$(E_e u)(r, \theta) = \begin{cases} u(r, \theta), & \text{if } (r, \theta) \in S_{0,\omega} \\ u(r, \frac{\omega}{\alpha}\theta), & \text{if } (r, \theta) \in S_{\alpha,0}. \end{cases}$$

Finally, we define the required operators  $E$  and  $R$ , by

$$(Eu)(r, \theta) := \frac{\alpha}{\omega}(E_e u)(r, \theta) + (1 - \frac{\alpha}{\omega})E_b(r, \theta), \quad (r, \theta) \in S_{\alpha,\omega},$$

and

$$(Rv)(r, \theta) := \frac{\omega}{\alpha - \omega}(v(r, \theta) - v(r, \frac{\alpha}{\omega}\theta)), \quad (r, \theta) \in S_{0,\omega}.$$

Simple computations show that  $E$  and  $R$  have the desired properties.

The proof of Theorem 3.4 is similar. The operators  $E$  and  $R$  from Theorem 3.4 are defined in the same manner as in the above proof.

## 5. AN APPLICATION TO A NONCONFORMING FINITE ELEMENT PROBLEM

In this section, we apply the results of the previous to a modified Crouzeix-Raviart nonconforming finite element approximation. Let  $\Omega$  be a polygonal domain in  $\mathbb{R}^2$  with

boundary  $\partial\Omega$ . The  $L^2(\Omega)$ -inner product and the  $L^2(\Omega)$ -norm are denoted by  $(\cdot, \cdot)$  and  $\|\cdot\|$ , respectively. We consider the Dirichlet problem

$$(5.1) \quad \begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

We consider the simple boundary condition above for convenience of notation. All of the results to be given extend to Dirichlet problems with mixed boundary conditions.

The variational formulation of (5.1) is the following:

Find  $u \in V := H_0^1(\Omega)$  such that

$$(5.2) \quad a(u, v) = F(v) \quad \text{for all } v \in H_0^1(\Omega),$$

where  $F(v) = (f, v)$  and

$$a(u, v) = \int_{\Omega} \nabla u \cdot \nabla v \, dx \quad \text{for all } u, v \in H_0^1(\Omega).$$

Let  $\mathcal{T}_h$  be a quasi-uniform triangulation of  $\Omega$  and let  $h = \max_{\tau \in \mathcal{T}_h} \text{diam}(\tau)$ .

Next, we consider the Crouzeix-Raviart finite element nonconforming space

$$V_h := \{v \mid \begin{array}{l} v \text{ is linear on all } \tau \in \mathcal{T}, \\ v \text{ is continuous at the midpoints of the edges} \\ v = 0 \text{ at the midpoints situated on } \partial\Omega \end{array}\},$$

and define on  $V + V_h$  the bilinear form

$$a_h(u, v) := \sum_{\tau \in \mathcal{T}_h} D_{\tau}(u, v), \quad \text{where } D_{\tau}(u, v) = \int_{\tau} \nabla u \cdot \nabla v \, dx$$

and the associated norm

$$\|u\|_h := \sqrt{a_h(u, u)}.$$

It is easy to show that the form  $a_h(\cdot, \cdot)$  is positive definite on  $V_h$ . The Crouzeix-Raviart approximation is: Find  $u_h \in V_h$  such that

$$(5.3) \quad a_h(u_h, v) = F(v) \quad \text{for all } v \in V_h.$$

The next statement is a version of Strang's Lemma [2, 6, 7].

**Proposition 5.1.** *Let  $u \in V$  and  $w \in V_h$  be arbitrary. Then*

$$(5.4) \quad \|u - w\|_h \leq \inf_{v \in V_h} \|u - v\|_h + \sup_{v \in V_h} \frac{a_h(u - w, v)}{\|v\|_h}$$

*Proof.* Let  $\tilde{u} \in V_h$  satisfy

$$a_h(\tilde{u}, v) = a_h(u, v) \quad \text{for all } v \in V_h.$$

Then,  $a_h(\tilde{u} - u, v) = 0$  for all  $v \in V_h$  and consequently,

$$\|u - \tilde{u}\|_h = \inf_{v \in V_h} \|u - v\|_h.$$

Thus,

$$\|u - w\|_h \leq \|u - \tilde{u}\|_h + \|\tilde{u} - w\|_h = \|u - \tilde{u}\|_h + \sup_{v \in V_h} \frac{a_h(\tilde{u} - w, v)}{\|v\|_h}.$$

Moreover,

$$a_h(\tilde{u} - w, v) = a_h(\tilde{u} - u + u - w, v) = a_h(u - w, v).$$

Combining the above estimate and equalities we obtain (5.4).  $\square$

In particular, when  $u$  is the solution of (5.2) and  $w = u_h$  is the solution of (5.3), we obtain the estimate

$$(5.5) \quad \|u - u_h\|_h \leq \inf_{v \in V_h} \|u - v\|_h + \sup_{v \in V_h} \frac{a_h(u - u_h, v)}{\|v\|_h}.$$

If  $u \in H^2(\Omega) \cap H_0^1(\Omega)$ , the first term of the right-hand side of (5.5) can be estimated by  $ch|u|_{H^2}$  using standard approximation properties. For the second term, we can use the following result (see, e.g., [2], [7]).

**Lemma 5.1.** *Let  $u \in H^2(\Omega) \cap H_0^1(\Omega)$  be the solution of (5.2) and  $u_h$  be the solution of the discrete problem (5.3). Then, for some positive constant  $c$*

$$(5.6) \quad \frac{a_h(u - u_h, v)}{\|v\|_h} \leq ch|u|_{H^2}, \quad \text{for all } v \in V_h, u \in H^2(\Omega) \cap H_0^1(\Omega).$$

Consequently,

$$(5.7) \quad \|u - u_h\|_h \leq ch|u|_{H^2} \quad \text{for all } u \in H^2(\Omega) \cap H_0^1(\Omega).$$

The method given by the discretized problem (5.3) has the disadvantage of not being stable when  $F$  is only in  $V' \equiv H^{-1}(\Omega)$ . A modified method, which is stable on  $H^{-1}(\Omega)$ , can be defined as follows.

First, we define  $\mathcal{T}_{\frac{h}{2}}$  to be the triangulation obtained from  $\mathcal{T}_h$  by joining the midpoints of the edges of the triangles in  $\mathcal{T}_h$ . Let  $S_{\frac{h}{2}}$  be the standard conforming finite element space of all functions in  $H_0^1(\Omega)$  which are linear on each triangle  $\tau \in \mathcal{T}_{\frac{h}{2}}$ . Note that  $S_{\frac{h}{2}} \subset V$ .

Next, we define the operator  $T : V_h \longrightarrow S_{\frac{h}{2}}$  by  $Tv = w$ , where

1.  $w(x) = v(x)$  when  $x$  is a midpoint of an edge in  $\mathcal{T}_h$ ,
2.  $w(x) = 0$  when  $x$  is a vertex of  $\partial\Omega$ ,
3.  $w(x) = \frac{1}{n_x} \sum_{j=1}^{n_x} v(y_j)$  when  $x$  is an interior vertex of  $\mathcal{T}_h$ , where  $y_1, y_2, \dots, y_{n_x}$  are the midpoints of those edges in  $\mathcal{T}_h$ , that are adjacent to  $x$ .

Clearly,  $n_x$  is bounded above by a fixed natural number. Let  $M_h$  be the set of all midpoints of the edges in  $\mathcal{T}_h$ . Let  $E_h$  be the set of all line segments connecting in each triangles in  $\mathcal{T}_h$  the mid points of the edges. Finally, let  $E_{h/2}$  be the set of all edges in  $\mathcal{T}_{\frac{h}{2}}$ . Then,

$$\|v\|^2 \approx h^2 \sum_{y_i \in M_h} v^2(y_i), \quad v \in V_h,$$

$$\|v\|_h^2 \approx \sum_{(y_i, y_j) \in E_h} (v(y_i) - v(y_j))^2, \quad v \in V_h,$$

and

$$|w|_{H^1(\Omega)}^2 \approx \sum_{(x_i, x_j) \in E_{h/2}} (w(x_i) - w(x_j))^2, \quad w \in S_{h/2}.$$

From the way we defined  $T$  and by using the above equivalences, it is easy to verify that

$$(5.8) \quad |Tv|_{H^1(\Omega)}^2 \leq c a_h(v, v), \quad \text{for all } v \in V_h$$

and

$$(5.9) \quad \|Tv - v\|^2 \leq ch^2 a_h(v, v), \quad \text{for all } v \in V_h,$$

for some positive constant  $c$ .

Consider the following modified version of Problem 5.3: Find  $\tilde{u}_h \in V_h$  such that

$$(5.10) \quad a_h(\tilde{u}_h, v) = F(Tv), \quad \text{for all } v \in V_h,$$

Note that since  $Tv$  is in  $V$ , (5.10) is well defined for  $F \in V'$ . We will use the interpolation results from the previous two sections to prove an error estimate for the modified method.

**Theorem 5.1.** *Let  $u$  be the solution of (5.2) and let  $\tilde{u}_h$  be the solution of (5.10). Then, for  $s \in [0, 1]$ , we have the following error estimate:*

$$(5.11) \quad \|u - \tilde{u}_h\|_h \leq ch^s \|u\|_{H^{1+s}(\Omega)}, \quad \text{for all } u \in H^{1+s}(\Omega) \cap H_0^1(\Omega).$$

*Proof.* For  $u \in H_0^1(\Omega)$ , we set  $F(v) = (-\Delta u, v)$  and define  $P_h u = \tilde{u}_h$  where  $\tilde{u}_h$  is the solution to (5.10). By taking  $v = \tilde{u}_h$  in (5.10) and by using (5.8), we easily get that

$$\|P_h u\|_h = \|\tilde{u}_h\|_h \leq c \|u\|_{H^1(\Omega)}.$$

This immediately implies that

$$(5.12) \quad \|(I - P_h)u\|_h \leq c \|u\|_{H^1(\Omega)}, \quad \text{for all } u \in H_0^1(\Omega).$$

Next, for  $u \in H^2(\Omega) \cap H_0^1(\Omega)$ , from Proposition 5.1, we obtain

$$\|u - \tilde{u}_h\|_h \leq \inf_{v \in V_h} \|v - u\|_h + \sup_{v \in V_h} \frac{a_h(u - \tilde{u}_h, v)}{\|v\|_h}.$$

Using standard approximation properties gives

$$\inf_{v \in V_h} \|u - v\|_h \leq \inf_{v \in S_h} \|u - v\|_h \leq ch |u|_{H^2(\Omega)}.$$

To estimate the second term in the right-hand side of the above inequality, we note that

$$a_h(u - \tilde{u}_h, v) = a_h(u - u_h, v) + a_h(u_h - \tilde{u}_h, v), \quad v \in V_h.$$

From Lemma 5.1,

$$a_h(u - u_h, v) \leq ch |u|_{H^2(\Omega)} \|v\|_h.$$

On the other hand, with the help of (5.9),

$$\begin{aligned} a_h(u_h - \tilde{u}_h, v) &= a_h(u_h, v) - a_h(\tilde{u}_h, v) = (f, v) - (f, Tv) \\ &\leq \|f\| \|v - Tv\| \leq ch |u|_{H^2(\Omega)} \|v\|_h. \end{aligned}$$

Combining the above estimates gives

$$(5.13) \quad \|(I - P_h)u\|_h \leq ch \|u\|_{H^2(\Omega)}, \quad \text{for all } u \in H^2(\Omega) \cap H_0^1(\Omega).$$

Finally, from (5.12) and (5.13), by using interpolation and the result of Chapter 3, we obtain

$$\|(I - P_h)u\|_h \leq ch^s \|u\|_{[H^2 \cap H_0^1, H_0^1]_{1-s}} = ch^s \|u\|_{H^{1+s}(\Omega)},$$

for all  $u \in H^{1+s}(\Omega) \cap H_0^1(\Omega)$ . This completes the proof of the theorem.  $\square$

## REFERENCES

- [1] C. Bacuta, J.H. Bramble and J. Pasciak. *Using Finite Element Tools in Proving Shift Theorems for Elliptic Boundary Value Problems*. In preparation.
- [2] D. Braess. *Finite Elements. Theory, Fast Solvers, and Applications in Solid Mechanics*. Cambridge University Press, Cambridge, 1997.
- [3] J. H. Bramble. Interpolation between Sobolev spaces in Lipschitz domains with an application to multigrid theory. *Math. Comp.*, 64:1359-1365, 1995.
- [4] J. H. Bramble, R. Lazarov and J. Pasciak. Least-squares for second-order elliptic problems. *Comput. Methods Appl. Mech. Engrg.*, 152:195-210, 1998.
- [5] J. H. Bramble, R. Lazarov and J. Pasciak. A least-squares approach based on a discrete minus one inner product for first order systems. *Math. Comp.*, 66:935-955, 1997. *Math. Comp.*, 60:447-471, 1993. J. H. Bramble, J. Pasciak and P. S. Vassilevski. Computational scales of Sobolev norms with applications to preconditioning. *Math. Comp.*, 69:463-480, 2000.
- [6] S. Brenner and L.R. Scott. *The Mathematical Theory of Finite Element Methods*. Springer-Verlag, New York, 1994.
- [7] P. G. Ciarlet. *The Finite Element Method for Elliptic Problems*. North Holland, Amsterdam, 1978.
- [8] V. Girault and P.A. Raviart. *Finite Element Methods for Navier-Stokes Equations*. Springer-Verlag, Berlin, 1986.
- [9] P. Grisvard. *Elliptic Problems in Nonsmooth Domains*. Pitman, Boston, 1985.
- [10] P. Grisvard. *Singularities in Boundary Value Problems*. Masson, Paris, 1992.
- [11] J. L. Lions and E. Magenes. *Non-homogeneous Boundary Value Problems and Applications, I*. Springer-Verlag, New York, 1972.
- [12] J. L. Lions and P. Peetre. Sur une classe d'espaces d'interpolation. *Institut des Hautes Etudes Scientifique. Publ.Math.*, 19:5-68, 1964.

DEPT. OF MATHEMATICS, TEXAS A & M UNIVERSITY, COLLEGE STATION, TX 77843, USA.  
*E-mail address:* bacuta@math.tamu.edu

DEPT. OF MATHEMATICS, TEXAS A & M UNIVERSITY, COLLEGE STATION, TX 77843, USA.  
*E-mail address:* bramble@math.tamu.edu

DEPT. OF MATHEMATICS, TEXAS A & M UNIVERSITY, COLLEGE STATION, TX 77843, USA.  
*E-mail address:* pasciak@math.tamu.edu