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A DOMAIN DECOMPOSITION TECHNIQUE FOR STOKES PROBLEMS *

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In this paper, we give an analysis for a domain decomposition technique for Stokes problems. The technique involves the application of domain decomposition directly to the Stokes problem and gives rise to an indefinite system for the velocity nodes on the subdomain boundaries and the mean values of the pressure on the subdomains. We analyze the resulting system and show how it can be efficiently solved.

1. Introduction

In this paper, we analyze a domain decomposition technique discussed in [21] for the solution of the discrete systems which arise in finite element approximation to Stokes problems. Specifically, we consider the velocity-pressure formulation of the Stokes equations where the divergence constraint is treated by a Lagrange multiplier technique and the pressure variable corresponds to the multiplier (cf. [15]). The discrete systems which arise are indefinite sytems of a special form. In Section 2, we review some properties of these systems and discuss the construction of effective iteration schemes for their solution. The rate of convergence of the these iterative techniques will be related to the corresponding "inf-sup" condition.

Section 3 defines the model Stokes problem and gives the corresponding weak formulation. The finite element approximation is then defined in terms of this formulation.

In Section 4, we develop and analyze iterative algorithms for Stokes problems by directly applying domain decomposition to the discrete Stokes systems. We develop iterative algorithms for the solution of the original Stokes system which require the solution of discrete Stokes problems on subdomains at each iterative step. The work in [17] provided insight for the development of this technique. Alternative algorithms for domain decomposition applied to Stokes problems can be found in the proceedings of the first and second international symposium on domain decomposition methods for partial differential equations [12,16].

For clarity of presentation we will only consider the simplest applications and approximation techniques. Many generalizations are possible but will not be addressed here.

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2. Iterative methods for multiplier systems

In this section, we consider two techniques used to develop iterative methods for multiplier systems. The first method is well known and the second is a preconditioning technique discussed in [6]. We included this discussion for completeness and continuity of exposition since it explains how the estimates derived in later sections relate to iterative convergence rates of the resulting algorithms.

Let H^1 and H^2 be finite-dimensional Hilbert spaces and consider the problem

$$M\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} F \\ G \end{pmatrix}, \tag{2.1}$$

where $X, F \in H^1$ and $Y, G \in H^2$. We study operators M of the form,

$$M = \begin{pmatrix} A & B^* \\ B & 0 \end{pmatrix}. \tag{2.2}$$

We assume that A is a positive-definite, symmetric operator on H^1 and that B and B^* are adjoints with respect to the inner products in H^1 and H^2. We shall use the notation $(\cdot,\cdot)$ and $\|\cdot\|$ to denote the inner products and norms on H^1 and H^2.

Multiplier systems of the form (2.1) arise in many applications. For example, such systems must be solved for finite element Lagrange multiplier approximations to Dirichlet and interface problems [3,4], velocity-pressure formulations of the equations of Stokes and elasticity [15], and mixed finite element methods [22].

Applying block Gaussian elimination to (2.1) implies that the solution of (2.1) satisfies

$$\begin{pmatrix} A & B^* \\ 0 & BA^{-1}B^* \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} F \\ BA^{-1}F - G \end{pmatrix}. \tag{2.3}$$

Thus, (2.1) is nonsingular if and only if $BA^{-1}B^*$ is invertible. But $BA^{-1}B^*$ is symmetric and nonnegative. Hence, $BA^{-1}B^*$ is invertible if and only if it is definite. A straightforward computation gives

$$(BA^{-1}B^*U, U) = \sup_{\Theta \in H^1} \frac{(B^*U, \Theta)^2}{(A\Theta, \Theta)} \quad \text{for all } U \in H^2, \tag{2.4}$$

and hence solvability of (2.1) will follow if we can verify

$$\sup_{\Theta \in H^1} \frac{(B^*U, \Theta)^2}{(A\Theta, \Theta)} \geqslant c_0 \|U\|^2 \quad \text{for all } U \in H^2, \tag{2.5}$$

holds for some positive constant c_0. Inequality (2.5) is equivalent to the classical LBB (Ladyzhenskaya–Babuška–Brezzi) condition (cf. [15]). In addition to being a sufficient condition for the solvability of (2.1), the constant c_0 in (2.5) will be an ingredient in determining convergence rates for the iterative methods to be subsequently discussed.

By (2.3), we see that the solution of (2.1) can be computed by first solving for Y from

$$BA^{-1}B^*Y = BA^{-1}F - G \tag{2.6}$$

and then backsolving (2.3) for X, i.e., $X = A^{-1}(F - B^*Y)$. For our applications, $BA^{-1}B^*$ is a full matrix and expensive to compute. One alternative is to iteratively solve (2.6), e.g., apply conjugate gradient iteration. The rate of convergence for this iteration is related to the condition number K of $BA^{-1}B^*$. From the above discussion, we cleary have that $K \leqslant c_1/c_0$ where c_0 satisfies (2.5) and c_1 satisfies the reverse inequality,

$$\sup_{\Theta \in H^1} \frac{(B^*U, \Theta)^2}{(A\Theta, \Theta)} \leqslant c_1 \| U \|^2 \quad \text{for all } U \in H^2. \tag{2.7}$$

One gets a rapidly convergent algorithm for the computation of Y if the condition number K is not too large. This is the first iterative technique for solving (2.1) to be considered.

One problem with the iterative technique just developed is that it requires the evaluation of the action of A^{-1} at each step in the iteration. In many applications, the action of A^{-1} is more expensive to compute than that of a suitable preconditioner. We next consider a natural preconditioned conjugate gradient technique for solving (2.1) which does not require the evaluation of the action of A^{-1}. An alternative technique (developed and analyzed in [5]) has similar properties but will not be fully discussed here.

We first consider the block operator

$$M_1 = \begin{pmatrix} A & B^* \\ B & 2BA^{-1}B^* \end{pmatrix} = M \begin{pmatrix} I & 2A^{-1}B^* \\ 0 & -I \end{pmatrix}.$$

Clearly, $MM_1^{-1}M = M_1$. Assume that we are given another symmetric positive-definite operator of the form

$$M_0 = \begin{pmatrix} A_0 & 0 \\ 0 & \mathscr{K} \end{pmatrix},$$

with the action of M_0^{-1} easy to obtain. We further assume that

$$\alpha_0(M_0\Phi, \Phi) \leqslant \left(\begin{pmatrix} A & 0 \\ 0 & BA^{-1}B^* \end{pmatrix} \Phi, \Phi \right)$$
$$\leqslant \alpha_1(M_0\Phi, \Phi) \quad \text{for all } \Phi \in H = H^1 \times H^2, \tag{2.8}$$

with α_1/α_0 not too large. Here $(\cdot,\cdot)$ denotes the sum of the componentwise inner products. The inequalities (2.8) immediately imply that M_0 is comparable to M_1. It then follows from the indentity $MM_1^{-1}M = M_1$ and the Schwarz inequality that

$$C_0(M_0\Phi, \Phi) \leqslant (MM_0^{-1}M\Phi, \Phi) \leqslant C_1(M_0\Phi, \Phi) \tag{2.9}$$

for all $\Phi \in H$. The constants C_0 and C_1 are proportional to α_0^2 and α_1^2 respectively. Note that $M_0^{-1}MM_0^{-1}M$ is a symmetric operator in the inner product $(M_0\cdot,\cdot)$. Moreover, by (2.9), it is positive-definite and well-conditioned provided that α_1/α_0 is not too large. Thus, applying the conjugate gradient method in the $(M_0\cdot,\cdot)$ inner product to the problem

$$M_0^{-1}MM_0^{-1}M\begin{pmatrix} X \\ Y \end{pmatrix} = M_0^{-1}MM_0^{-1}\begin{pmatrix} F \\ G \end{pmatrix} \tag{2.10}$$

leads to a rapidly convergent iterative algorithm for solving (2.1).

Remark 2.1. The condition number of the operator on the left-hand side of (2.10) is proportional to the square of $K = \alpha_1/\alpha_0$. The condition number of the reformulation of (2.1) developed in [5] is linear in K however requires estimation of the smallest eigenvalue of $A_0^{-1}A$. Accordingly, if K is not too large, the above method seems simpler.

3. The model Stokes problem

In this section, we describe the model Stokes problem and its finite element discretization. Let Ω be a domain in N-dimensional Euclidean space for $N=2$ or $N=3$. The velocity-pressure formulation of the steady-state Stokes problem is: Find $\boldsymbol{u}$ and P satisfying

$$\begin{aligned} -\Delta \boldsymbol{u} - \nabla P &= \boldsymbol{F} && \text{in } \Omega, \\ \nabla \cdot \boldsymbol{u} &= 0 && \text{in } \Omega, \\ \boldsymbol{u} &= 0 && \text{on } \partial\Omega, \\ \int_\Omega P &= 0. \end{aligned} \tag{3.1}$$

Here, $\boldsymbol{u}$ is a vector-valued function and P is a scalar-valued function defined on Ω. The first equation is, of course, a vector equality at each $x \in \Omega$ and Δ denotes the componentwise Laplace operator.

We restrict ourselves to the model problem (3.1) for simplicity. Applications to problems with variable coefficients and the equations of linear elasticity are similar.

We consider a weak formulation of problem (3.1). Let $(\cdot,\cdot)$ denote the $L^2(\Omega)$ inner product and $\|\cdot\|$ denote the corresponding norm applied either to scalar or vector functions. Let $H_0^1(\Omega)$ be the Sobolev space of scalar-valued functions defined on Ω which vanish (in an appropriate sense) on $\partial\Omega$ and which along with their first derivatives are square integrable on Ω. Define $\boldsymbol{H} \equiv H_0^1(\Omega) \times H_0^1(\Omega)$ and let $\|\cdot\|_1$ denote the corresponding norm. Let $\Pi = L^2(\Omega)$ and $\Pi/1$ denote the functions in Π with zero mean value on Ω. Multiplying (3.1) by functions in $\boldsymbol{H}$ and Π and integrating by parts when appropriate, it is easy to see that the solution $(\boldsymbol{u}, P)$ satisfies

$$\begin{aligned} D(\boldsymbol{u}, \boldsymbol{v}) + (P, \nabla \cdot \boldsymbol{v}) &= (\boldsymbol{F}, \boldsymbol{v}) && \text{for all } \boldsymbol{v} \in \boldsymbol{H}, \\ (\nabla \cdot \boldsymbol{u}, q) &= 0 && \text{for all } q \in \Pi/1. \end{aligned} \tag{3.2}$$

Here, D is the Dirichlet form on Ω defined by

$$D(\boldsymbol{w}, \boldsymbol{v}) \equiv \sum_{i=1}^{N} \int_\Omega \nabla w_i \cdot \nabla v_i \, \mathrm{d}x.$$

Clearly, (3.2) has the same form as (2.1). The corresponding operator A is unbounded but has a bounded inverse. Moreover, it is well known that the corresponding inf-sup condition:

$$\sup_{\theta \in \boldsymbol{H}} \frac{(p, \nabla \cdot \theta)^2}{D(\theta, \theta)} \geqslant C_0 \| p \|^2 \quad \text{for all } p \in \Pi/1 \tag{3.3}$$

holds for some positive constant C_0 (cf. [15]). It then follows that there is a unique solution $(\boldsymbol{u}, P)$ in $\boldsymbol{H} \times \Pi/1$ to (3.2).

To approximately solve (3.2), we introduce a collection of pairs of approximation subspaces $\boldsymbol{H}_h \subset \boldsymbol{H}$ and $\Pi_h \subset \Pi$ indexed by h in the interval $0 < h < 1$. We will assume that the inf-sup condition holds for the pair of spaces; i.e. we assume that there is a constant c_0 which does not depend upon h such that

$$\sup_{\theta \in \boldsymbol{H}_h} \frac{(p, \nabla \cdot \theta)^2}{D(\theta, \theta)} \geqslant c_0 \| p \|^2 \quad \text{for all } p \in \Pi_h/1. \tag{3.4}$$

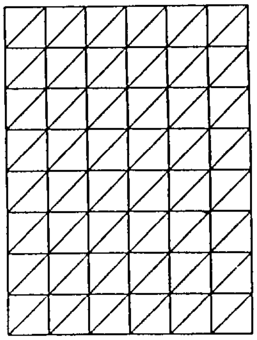

Fig. 1. The triangular mesh.

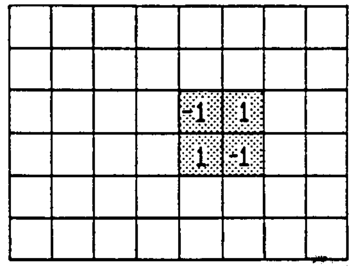

Fig. 2. The rectangular mesh used for $\tilde{\Pi}_h$; the support (shaded) and values for a typical ϕ_{ij}.

Many subspace pairs satisfying (3.4) have been studied and their approximation properties are well known [15,20,23].

The approximations to the functions $(\boldsymbol{u}, P)$ are defined by replacing the spaces in (3.2) by their discrete counterparts. Specifically, the approximations are defined as the functions $\boldsymbol{u}_h \in \boldsymbol{H}_h$ and $P_h \in \Pi_h/1$ satisfying

$$\begin{aligned} &D(\boldsymbol{u}_h, \boldsymbol{v}) + (P_h, \nabla \cdot \boldsymbol{v}) = (\boldsymbol{F}, \boldsymbol{v}) && \text{for all } \boldsymbol{v} \in \boldsymbol{H}_h, \\ &(\nabla \cdot \boldsymbol{u}_h, q) = 0 && \text{for all } q \in \Pi_h/1. \end{aligned} \tag{3.5}$$

Existence and uniqueness for the solution of (3.5) follows from (3.4) and the discussion in Section 2.

We conclude this section with an example of a pair of approximation subspaces. For simplicity of exposition, we shall only describe these spaces when Ω is the unit square. Generalizations to certain more complex domains are possible.

Let $n > 0$ be given. We start by breaking the square into $2n \times 2n$ square subregions and define $h = 1/2n$ (see Fig. 1). Let $x_i \equiv ih$ and $y_j = jh$ for $i, j = 1, \ldots, 2n$. We partition the square subregions into pairs of triangles using one of the square subregions diagonals (for example, the diagonal going from the bottom right corner to the upper left corner of the square subregion). Let H_h be the collection of functions which vanish on the boundary of the square and are piecewise linear and continuous on this triangulation. The subspace $\boldsymbol{H}_h$ is defined to be $H_h \times H_h$.

To define the space Π_h, we first consider the space $\tilde{\Pi}_h$ which is defined to be the space of functions which are piecewise constant on the square subregions (see Fig. 2). It is interesting to note [18] that the subspace pair $\{\boldsymbol{H}_h, \tilde{\Pi}_h/1\}$ is not stable in L^2, i.e., (3.4) fails to hold with c_0 independent of h for the subspace pair. To get a stable pair, we shall consider a somewhat smaller subspace of $\tilde{\Pi}_h$. Let θ_{kl} for $k, l = 1, \ldots, 2n$ be the function which is one on the square subregion $[x_{k-1}, x_k] \times [y_{l-1}, y_l]$ and vanishes elsewhere. We define the functions $\phi_{ij} \in \tilde{\Pi}_h$ for $i, j = 1, \ldots, n$ by (see also, Fig. 2)

$$\phi_{ij} \equiv \theta_{2i-1,2j-1} - \theta_{2i,2j-1} - \theta_{2i-1,2j} + \theta_{2i,2j}. \tag{3.6}$$

We then define Π_h by

$$\Pi_h \equiv \left\{ Q \in \tilde{\Pi}_h \colon \left(Q, \phi_{ij}\right) = 0 \text{ for } i, j = 1, \ldots, n \right\}.$$

An estimate of the form of (3.4) holds with c_0 independent of h for the subspace pair $\{\boldsymbol{H}_h, \Pi_h\}$ [18]. Furthermore, the exclusion of the functions of the form (3.6) does not result in a change in the order of approximation for the space (we obviously still have the subspace of constants on the mesh of size $2h$).

Remark 3.1. The exclusion of functions of the form (3.6) poses no difficulty in practice. In fact, it only affects the definition of the corresponding B in a trivial way. By definition, $B\boldsymbol{v} \equiv Q$ where $Q \in \Pi_h/1$ solves

$$(Q, R) = (\nabla \cdot \boldsymbol{v}, R) \quad \text{for all } R \in \Pi_h/1.$$

It is easy to see that Q is the L^2 orthogonal projection (onto $\Pi_h/1$) of the function $\tilde{Q} \in \tilde{\Pi}_h$ satisfying

$$(\tilde{Q}, R) = (\nabla \cdot \boldsymbol{v}, R) \quad \text{for all } R \in \tilde{\Pi}_h. \tag{3.7}$$

This projection is a trivial local operation since the supports of the functions $\{\phi_{ij}\}$ are essentially disjoint. Furthermore, the computation of $\tilde{Q}$ is straightforward since the gram matrix for (3.7) is diagonal (with the obvious choice of basis).

The discrete Stokes problem can be cast into the form of (2.1). To see this, we introduce the following notation. Let $A : \boldsymbol{H}_h \mapsto \boldsymbol{H}_h$ be defined by

$$(A\boldsymbol{v}, \boldsymbol{w}) = D(\boldsymbol{v}, \boldsymbol{w}) \quad \text{for all } \boldsymbol{w} \in \boldsymbol{H}_h. \tag{3.8}$$

Clearly, (3.8) defines a symmetric positive-definite operator on $\boldsymbol{H}_h$. We define $B : \boldsymbol{H}_h \mapsto \Pi_h/1$ by

$$(B\boldsymbol{w}, q) = (\nabla \cdot \boldsymbol{w}, q) \quad \text{for all } q \in \Pi_h/1,$$

which is nothing more than the divergence followed by L^2 projection into Π_h. Its adjoint, $B^* : \Pi_h/1 \mapsto \boldsymbol{H}_h$ is then defined by

$$(B^*p, \boldsymbol{w}) = (p, \nabla \cdot \boldsymbol{w}) \quad \text{for all } \boldsymbol{w} \in \boldsymbol{H}_h.$$

The discrete solution pair $(\boldsymbol{u}_h, P_h)$ satisfies (2.1).

Remark 3.2. Since the "inf-sup" condition holds for this subspace pair, the second iterative technique of Section 2 (or the technique described in [5]) can be applied by taking

$$M_0 = \begin{pmatrix} A_0 & 0 \\ 0 & I \end{pmatrix},$$

where A_0 is a preconditioner for A. Domain decomposition preconditioners for the general second-order problems have been given in [7–11]. In the remainder of this paper, we consider domain decomposition applied directly to the solution of the discrete Stokes system.

4. A direct domain decomposition approach

In this section, we shall directly apply domain decomposition to the discrete system (3.5). We shall develop algorithms for solving the discrete system (3.5) which only require the solution of smaller discrete Stokes systems on the subdomains and another reduced system. In this case, the reduced system will be of the form of (2.1) and involve the values of $\boldsymbol{u}_h$ on the boundary of the subdomains and the mean value of the pressure on the subdomains.

We assume that $\bar{\Omega}$ has been partitioned into a number of subdomains $\bar{\Omega} = \bigcup_{i=1}^m \bar{\Omega}_i$. We require that the boundary of the subdomains ($\Gamma \equiv \bigcup_{i=1}^m \partial\Omega_i$) align with the mesh in $\boldsymbol{H}_h$ and Π_h. We then define

$$\boldsymbol{H}_h^i = \{ \phi \in \boldsymbol{H}_h \colon \text{support}(\phi) \subset \bar{\Omega}_i \} \tag{4.1}$$

and

$$\Pi_h^i = \{ \phi \in \Pi_h \colon \text{support}(\phi) \subset \bar{\Omega}_i \}. \tag{4.2}$$

We shall assume that the inf-sup condition holds for each subspace pair, i.e.,

$$\sup_{\theta \in \boldsymbol{H}_h^i} \frac{(q, \nabla \cdot \theta)^2}{D(\theta, \theta)} \geqslant c_0 \| q \|_{\Omega_i}^2 \quad \text{for all } q \in \Pi_h^i/1, \tag{4.3}$$

and that the function which is one on Ω_i and vanishes in the remainder of Ω is an element in Π_h. Note that, since the functions in $\boldsymbol{H}_h$ are continuous, the subspace pair ($\boldsymbol{H}_h^i$, Π_h^i) can be used to approximate the Stokes problem with zero boundary conditions on the subdomains.

Because of (4.3), local Stokes problems on the subdomains are solvable. The first step is to solve these local problems and reduce the problem to one which implicitly involves fewer degrees of freedom. To do this, we let $(\boldsymbol{v}_h^i, Q_h^i) \in \boldsymbol{H}_h^i \times \Pi_h^i/1$ be the solution of

$$\begin{aligned} &D(\boldsymbol{v}_h^i, \boldsymbol{w}) + (Q_h^i, \nabla \cdot \boldsymbol{w}) = (\boldsymbol{F}, \boldsymbol{w}) && \text{for all } \boldsymbol{w} \in \boldsymbol{H}_h^i, \\ &(\nabla \cdot \boldsymbol{v}_{h,q}^i) = 0 && \text{for all } q \in \Pi_h^i/1. \end{aligned} \tag{4.4}$$

We set $\boldsymbol{v}_h = \Sigma \boldsymbol{v}_h^i$, $Q_h = \Sigma Q_h^i$ and define $\boldsymbol{w}_h = \boldsymbol{u}_h - \boldsymbol{v}_h$ and $R_h = P_h - Q_h$. Then, $\boldsymbol{w}_h$ and R_h satisfy

$$\begin{aligned} &D(\boldsymbol{w}_h, \boldsymbol{v}) + (R_h, \nabla \cdot \boldsymbol{v}) = F(\boldsymbol{v}) && \text{for all } \boldsymbol{v} \in \boldsymbol{H}_h, \\ &(\nabla \cdot \boldsymbol{w}_h, q) = G(q) && \text{for all } q \in \Pi_h/1. \end{aligned} \tag{4.5}$$

The functionals F and G vanish for functions in $\boldsymbol{H}_h^i$ and $\Pi_h^i/1$ respectively. Thus, the functions $\boldsymbol{w}_h$ and R_h lie in a subspace of $\boldsymbol{H}_h \times \Pi_h/1$ with significantly lower dimension. We shall parameterize this subspace and then derive equations for the parameters corresponding to the solution $\boldsymbol{w}_h$ and R_h.

We shall parameterize the solution ($\boldsymbol{w}_h$, R_h) in terms of parameters $\sigma \in \boldsymbol{H}_h(\Gamma)$ and $\lambda \in \Pi_0$ where

$$\boldsymbol{H}_h(\Gamma) \equiv \{ \phi|_\Gamma, \phi \in \boldsymbol{H}_h \}$$

and

$$\Pi_0 \equiv \{ \phi \in \Pi_h/1 \colon \phi \text{ is constant on } \Omega_i \text{ for each } i \}.$$

To do this, we define the operators $S \colon \boldsymbol{H}_h(\Gamma) \mapsto \Pi_h$ and $T \colon \boldsymbol{H}_h(\Gamma) \mapsto \boldsymbol{H}_h$ satisfying the following:

(1) $S(\gamma)|_{\Omega_i} \in \Pi_h^i/1$,

(2) $T(\gamma)|_\Gamma = \gamma$,

(3) $D(T(\gamma), \phi) + (S(\gamma), \nabla \cdot \phi) = 0 \quad$ for all $\phi \in \boldsymbol{H}_h^i$,

(4) $(\nabla \cdot T(\gamma), q) = 0 \quad$ for all $q \in \Pi_h^i/1$.

It is not difficult to show that the above conditions uniquely define S and T. Moreover, if $\sigma = w_h|_\Gamma$ and $\lambda \in \Pi_0$ is the function which has the same mean values on the subdomains as R_h then it follows directly from the definitions that

$$w_h = T(\sigma), \qquad R_h = S(\sigma) + \lambda. \tag{4.6}$$

Thus, (4.6) gives a parameterization of w_h and R_h in terms of the parameters (σ, λ) in $\boldsymbol{H}_h(\Gamma) \times \Pi_0$. Note that given a value of γ, the evaluation of $S(\gamma)$ and $T(\gamma)$ essentially only involves the solution of discrete Stokes problems on the subdomains.

We next give equations for the determination of σ and λ satisfying (4.6). To do this, we define a quadratic form $E: (\boldsymbol{H}_h(\Gamma) \times \Pi_0)^2 \mapsto R^1$ given by

$$E((\gamma_1, \delta_1), (\gamma_2, \delta_2)) = D(T(\gamma_1), T(\gamma_2)) + (\delta_1, \nabla \cdot T(\gamma_2)) + (\nabla \cdot T(\gamma_1), \delta_2). \tag{4.7}$$

Using the definition of T, it is not difficult to see that

$$E((\sigma, \lambda), (\phi, \psi)) = \tilde{F}(\phi, \psi) = F(\bar{\phi}) + G(\psi), \tag{4.8}$$

where $\bar{\phi}$ is any extension of ϕ in $\boldsymbol{H}_h$. Thus, given local bases for $\boldsymbol{H}_h(\Gamma)$ and Π_0, we can compute the data $\tilde{F}$ satisfying (4.8) using a few operations per basis function.

From the definition of E, it is clear that (4.8) gives rise to a symmetric indefinite system of the form (2.1) which can be used to compute (σ, λ) satisfying (4.6). The form $D(T(\gamma_1), T(\gamma_2))$ corresponds to the operator A in (2.1). The form $(\delta_1, \nabla \cdot T(\gamma_2))$ corresponds to B^*, etc.

Stability properties for the above system are given in the following theorem. We make the further assumption that the velocity subspaces $\boldsymbol{H}_h|_{\Omega_i}$ satisfy a standard extension property: Given a function $\boldsymbol{v} \in \boldsymbol{H}_h(\Gamma)$, there exists $\boldsymbol{w} \in \boldsymbol{H}_h$ which equals $\boldsymbol{v}$ on $\partial\Omega_i$ and satisfies

$$\| \boldsymbol{w} \|_{H^1(\Omega_i)} \leqslant c_2 \| \boldsymbol{v} \|_{1/2,\partial\Omega_i}. \tag{4.9}$$

Here $\| \cdot \|_{1/2,\partial\Omega_i}$ denotes the Sobolev norm of order $\frac{1}{2}$ on $\partial\Omega_i$ and $H^1(\Omega_i)$ denotes the Sobolev norm of order one on Ω_i (cf. [19]). Property (4.9) is known for finite element subspaces defined on quasi-uniform triangulations (cf. [2,7,11]). The constant C depends upon the shape of the subdomains but not on h.

Theorem 4.1. *Assume the extension property holds on the subdomains (see (4.9)). Then, there are positive constants α_0, α_1, C_0 such that*

$$\alpha_0 D(T(\gamma), T(\gamma)) \leqslant \sum_{i=1}^{m} |\gamma|^2_{1/2,\partial\Omega_i} \leqslant \alpha_1 D(T(\gamma), T(\gamma)), \tag{4.10}$$

for all $\gamma \in \mathbf{H}_h(\Gamma)$ and

$$C_0 \| \delta \|^2 \leqslant \sup_{\gamma \in \boldsymbol{H}_h(\Gamma)} \frac{(\delta, \nabla \cdot T(\gamma))^2}{D(T(\gamma), T(\gamma))} \leqslant \| \delta \|^2, \tag{4.11}$$

for all $\delta \in \Pi_0$. Here $|\cdot|_{1/2,\partial\Omega_i}$ denotes the Sobolev seminorm of order $\frac{1}{2}$ on $\partial\Omega_i$. These constants only depend on c_0 in (3.4) and (4.3) and c_2 in (4.9), i.e., not on h or the number of subdomains.

Proof. We first prove (4.10). Given $\gamma \in \boldsymbol{H}_h(\Gamma)$, let u_γ denote its discrete harmonic extension, i.e., u_γ is the unique function in $\boldsymbol{H}_h$ which equals γ on Γ and satisfies

$$D(u_\gamma, \phi) = 0 \tag{4.12}$$

for all $\phi \in \boldsymbol{H}_h$ which vanish on Γ. It is well known [2,7] that if (4.9) holds then on each subdomain

$$cD_i(u_\gamma, u_\gamma) \leqslant |\gamma|^2_{1/2,\Omega_i} \leqslant CD_i(u_\gamma, u_\gamma), \tag{4.13}$$

where D_i denotes the Dirichlet form on Ω_i. Moreover, since u_γ is discrete harmonic,

$$D_i(u_\gamma, u_\gamma) \leqslant D_i(T(\gamma), T(\gamma)). \tag{4.14}$$

Combining (4.13), (4.14) and summing gives the second inequality of (4.10). For the first inequality, we note that, by the definition of S and T,

$$D(T(\gamma), T(\gamma)) = D(T(\gamma), u_\gamma) + (S(\gamma), \nabla \cdot u_\gamma). \tag{4.15}$$

By (4.3),

$$\begin{aligned} \| S(\gamma) \|^2_{\Omega_i} &\leqslant c_0^{-1} \sup_{\zeta \in \boldsymbol{H}_h^i} \frac{(S(\gamma), \nabla \cdot \zeta)^2}{D_i(\zeta, \zeta)} \\ &= c_0^{-1} \sup_{\zeta \in \boldsymbol{H}_h^i} \frac{D_i(T(\gamma), \zeta)^2}{D_i(\zeta, \zeta)} \leqslant c_0^{-1} D_i(T(\gamma), T(\gamma)). \end{aligned} \tag{4.16}$$

Applying the Schwarz inequality to (4.15) and using (4.16) gives

$$D(T(\gamma), T(\gamma)) \leqslant CD(u_\gamma, u_\gamma).$$

The first inequality of (4.10) then follows from (4.13).

We next prove (4.11). The second inequality follows immediately from the Schwarz inequality. For the first, by (3.4),

$$\| \delta \|^2 \leqslant c_0^{-1} \sup_{\zeta \in \boldsymbol{H}_h} \frac{(\delta, \nabla \cdot \zeta)^2}{D(\zeta, \zeta)}. \tag{4.17}$$

Moreover, the above inequalities imply that for $\gamma = \zeta|_\Gamma$,

$$D(T(\gamma), T(\gamma)) \leqslant cD(u_\gamma, u_\gamma) \leqslant cD(\zeta, \zeta). \tag{4.18}$$

In addition, since δ is constant on the subdomains

$$(\delta, \nabla \cdot \zeta) = (\delta, \nabla \cdot T(\gamma)). \tag{4.19}$$

Combining (4.17)–(4.19) proves the first inequality of (4.11). This completes the proof of (4.11). □

Inequalities (4.11) imply that the operator $BA^{-1}B^*$ corresponding to E on the subspace Π_0 is well conditioned independently of h. The boundary form $D(T(\gamma), T(\gamma))$ is not well conditioned but is equivalent to a sum of seminorms on the boundaries of the subdomains. The corresponding form,

$$\langle\langle \gamma_1, \gamma_2 \rangle\rangle_{1/2} \equiv \sum_{i=1}^{m} \langle \gamma_1, \gamma_2 \rangle_{1/2, \partial\Omega_i} \quad \text{for all } \gamma_1, \gamma_2 \in \boldsymbol{H}_h(\Gamma),$$

has been studied in the development of domain decomposition preconditioners for second-order problems. Each domain decomposition technique developed in [1,2,7–11,13,14] gives rise to a computationally effective domain decomposition preconditioner for $\langle\langle\cdot,\cdot\rangle\rangle_{1/2}$. Thus, we can solve (4.8) by using the second iterative technique of Section 2, with preconditioner

$$M_0^{-1}=\begin{pmatrix} A_0 & 0\\ 0 & I\end{pmatrix}^{-1},$$

where A_0 corresponds componentwise to the boundary part of a second-order method developed in [1,2,7–11,13,14]. For example, we can use the technique presented in [8]. This means that the preconditioner for the boundary velocities will involve inverting the $l_0^{1/2}$ operator on the edge segments and the solution of a coarse grid problem with the number of unknowns equal to the number of "cross-points" in the subdomain subdivision. The resulting symmetric positive-definite system (2.10) will have a condition number bounded by $C(1+\ln^4(d/h))$.

It is possible to implement the above technique in such a way that each Stokes subdomain problem need be solved only once per step in the iterative algorithm for the solution of σ, λ. Once these parameters are solved to satisfactory accuracy, w_h and R^h can be computed with one more set of subdomain solves. Moreover, the action of A_0 appearing in the inner product $(M_0\cdot,\cdot)$ need never be explicitly computed in the conjugate gradient algorithm (e.g. see [5, Appendix]).

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