MULTIGRID METHODS FOR THE BIHARMONIC PROBLEM DISCRETIZED BY CONFORMING C^1 FINITE ELEMENTS ON NONNESTED MESHES*

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Abstract. We consider multigrid algorithms for the biharmonic problem discretized by conforming C^1 finite elements. Most finite elements for the biharmonic equation are nonnested in the sense that the coarse finite element space is not a subspace of the space of similar elements defined on a refined mesh. To define multigrid methods, certain intergrid transfer operators have to be constructed. We construct intergrid transfer operators that satisfy a certain stable approximation property. The so-called regularity-approximation assumption is established by using this stable approximation property of the intergrid transfer operator. Optimal convergence properties of the W-cycle and a uniform condition number estimate for the variable V-cycle preconditioner are established by applying an abstract result of Bramble, Pasciak and Xu. Our theory covers the cases when the multilevel triangulations are nonnested and the spaces on different levels are defined by different finite elements.

1. Introduction. Multigrid methods are among the most efficient methods for solving elliptic partial differential equations discretized by the finite element or finite difference methods; cf. e.g. [5, 13, 10, 4] and the references therein. In this paper, we consider some multigrid algorithms for the biharmonic equation discretized by conforming C^1 finite elements. Most of the finite elements for the biharmonic problem are nonnested in the sense that the finite element space defined on a coarse mesh is not a subspace of the finite element space defined by similar elements on a finer mesh, even when the finer meshes (triangulations) are obtained from the coarser meshes by a uniform refinement.

Because of this lack of "nestness" of multilevel spaces, certain space connection operators, which we refer to as intergrid transfer operators, have to be constructed in order to define multigrid algorithms. In the case in which the multilevel spaces are nested, the natural inclusion operator from coarse to fine space is used as an intergrid transfer operator. For the nonnested case, the most natural intergrid operator seems to be the nodal value interpolation operator, or a simple modification of it when the nodal value interpolant is not well defined. This happens when certain second order derivatives are part of the degrees of

freedom of the finite element space. In such a case, we can either drop the terms associated with the second order derivatives from the interpolation or use a local average for the second order derivatives.

We will assume some minimum relations between the multilevel triangulations so that the cost in each iteration depends linearly on the number of mesh points on the finest triangulation. This, together with the optimal convergence properties of the multigrid algorithms, guarantees that the overall solution process is optimal. Since the multilevel finite element spaces are nonnested in general even on nested meshes, we will assume, throughout this paper, that the triangulations (meshes) are not necessarily nonnested.

We will not assume the multilevel spaces are defined by the same finite elements (the spaces are nonnested anyway). The coarser spaces are used only in the construction of the preconditioner and/or in the correction of the residual, therefore, we can use simpler finite elements on coarser grids to reduce the cost per iteration and to make the algorithm computational more efficient. There seems to be no reason to use more complicated finite elements on coarser grids, although our theory does apply to such cases.

There are some earlier papers on multilevel methods for the biharmonic problem. Peisker [16] studied the W-cycle multigrid methods using a mixed formulation. Peisker and Braess [17] considered the W-cycle for the Morley element. The W-cycle multigrid for some C^1 elements were studied in S. Zhang [20]. In [6], Brenner studied the W-cycle for the Morley elements and simplified the algorithm and analysis of [17]. Hanisch [11, 12] considered the multigrid for mixed formulation as well as Morley element. Oswald [15] studied some additive multilevel methods for bicubic element. X. Zhang [21] studied additive multilevel methods and V-cycle multigrid for bicubic elements. All these papers considered the cases when the multilevel spaces are defined by the same finite element and none of them discussed nonnested meshes.

This paper is organized as follows. In §2, we briefly describe the multigrid algorithms and summarize the basic theory of Bramble, Pasciak and Xu [3]. In §3, we define some intergrid transfer operators and establish a certain stable approximation property of these intergrid transfer operators. Using this stable approximation property of the intergrid transfer operator together with some standard finite element estimates, we prove the *regularity and approximation* assumption in the abstract multigrid theory. We remark that once the stable approximation properties of the intergrid transfer operators are established, the verification of the regularity and approximation assumptions follows in a way similar to that in §7 of [3]

or [19].

Our result is based on the abstract result of Bramble, Pasciak and Xu [3] for the multigrid methods with nonnested spaces.

2. Abstract theory. Let $\{V_k\}$ be a family of spaces which are subspaces of a common Hilbert space V with an inner product $(\cdot, \cdot)$. Denote by $\|\cdot\|$ the norm induced by $(\cdot, \cdot)$. Let $a(\cdot, \cdot)$ be an uniformly bounded and coercive bilinear form on V. Consider the following problem: Find $u_k \in V_k$ such that

$$a(u_k, \chi_k) = (f, \chi_k), \quad \chi_k \in V_k.$$

Define $A_k : V_k \mapsto V_k$ by

$$(A_k u_k, v_k) = a(u_k, v_k), \quad \forall u_k, v_k \in V_k.$$

Let $R_k : V_k \mapsto V_k$ be a linear smoother and set $R_k^{(s)} = R_k$ if s is odd and $R_k^{(s)} = R_k^t$ if s is even. Here R_k^t is the $(\cdot, \cdot)$ adjoint of R_k. The spaces V_{k-1} and V_k are related by "intergrid" transfer operators $\mathcal{I}_k : V_{k-1} \mapsto V_k$. We define $\mathcal{I}_k^t : V_k \mapsto V_{k-1}$ and $\mathcal{I}_k^* : V_k \mapsto V_{k-1}$ to be adjoints of $\mathcal{I}_k$ with respect to $(\cdot, \cdot)$ and $a(\cdot, \cdot)$ respectively. If the spaces $\{V_k\}$ are nested and $\mathcal{I}_k$ is the natural inclusion operator, then $\mathcal{I}_k^t$ and $\mathcal{I}_k^*$ are the projections with respect to $(\cdot, \cdot)$ and $a(\cdot, \cdot)$. We remark that only $\mathcal{I}_k$ and $\mathcal{I}_k^t$ will be used in the multigrid algorithm and $\mathcal{I}_k^*$ is used only in the theoretical analysis.

The multigrid operator $B_k : V_k \mapsto V_k$ is defined by induction as follows.

ALGORITHM 2.1. *Set $B_0 = A_0^{-1}$. Define $B_k g = y^{2m_k}$ in terms of B_{k-1} as follows:*
(1) Set $x^0 = 0$ and $q^0 = 0$ and define

$$x^s = x^{s-1} + R_k^{(s+m_k)}(g - A_k x^{s-1}), \quad s = 1, \ldots, m_k.$$

(2) Define $y^{m_k} = x^{m_k} + \mathcal{I}_k q^p$, where q^i for $i = 1, \ldots, p$ is defined by

$$q^i = q^{i-1} + B_{k-1}[\mathcal{I}_k^t(g - A_k x^{m_k}) - A_{k-1} q^{i-1}].$$

(3) Define y^s for $s = m_k + 1, \ldots, 2m_k$ by

$$y^s = y^{s-1} + R_k^{(s+m_k)}(g - A_k y^{s-1}).$$

Here m_k is the number of smoothing iterations on level k. The cases $p = 1$ and $p = 2$ correspond respectively to the V- and the W-cycle.

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We now summarize the theory of multigrid methods with nonnested spaces [3].

Let $\lambda_k = \lambda_{\max}(A_k)$ be the maximum eigenvalue of A_k. The first assumption relates the regularity of the continuous problem and the approximation properties of the intergrid transfer operator.

A.1 (REGULARITY/APPROXIMATION ASSUMPTION): There exists $0 < \alpha \leq 1$ such that

$$a((\mathcal{I} - \mathcal{I}_k \mathcal{I}_k^*)u, u) \leq C_\alpha^2 \left(\frac{\|A_k u\|^2}{\lambda_k} \right)^\alpha (A_k u, u)^{1-\alpha}, \quad \forall u \in V_k.$$

Let $K_k = \mathcal{I} - R_k A_k$; and $K_k^* = \mathcal{I} - R_k^t A_k$, the adjoint of K_k with respect to $a(\cdot, \cdot)$. Let $\bar{R}_k = (\mathcal{I} - K_k^* K_k)A_k^{-1} \equiv R_k^t + R_k + R_k^t A_k R_k$. The following assumptions regard the properties of the smoother and the number of smoothing in each space.

A.2.1 (SMOOTHER ASSUMPTION):

$$C \lambda_k^{-1}(u, u) \leq (\bar{R}_k u, u), \quad \forall u \in V_k.$$

A.2.2 (SMOOTHER ASSUMPTION): There exist $1 < \beta_0 \leq \beta_1$ such that the smoothings for variable V-cycle satisfy

$$\beta_0 m_k \leq m_{k-1} \leq \beta_1 m_k.$$

Let δ_k or δ be the contraction number for the multigrid algorithm,

$$|a((\mathcal{I} - B_k A_k)u, u)| \leq \delta_k a(u, u).$$

THEOREM 2.1 (W(m, m)-CYCLE). *Assume* **A.1** *and* **A.2.1** *hold. Then there exists* $M > 0$, *independent of* k *such that for* m *large enough, but independent of* k

$$\delta \leq \frac{M}{M + m^\alpha}.$$

THEOREM 2.2 (VARIABLE V-CYCLE). *Assume* **A.1**, **A.2.1** *and* **A.2.2** *hold. Then there exist* $\eta_0, \eta_1 > 0$, *independent of* k, *such that*

$$\eta_0 a(u, u) \leq a(B_k A_k u, u) \leq \eta_1 a(u, u).$$

Remark: Notice that there is no requirement on the number of smoothing steps on the finest level for the variable V-cycle multigrid preconditioner. This is in contrast with the requirement of sufficiently many smoothing steps for the W-cycle multigrid method. Hence, when both algorithms may be applied, the variable V-cycle is more robust.

3. Multigrid methods for biharmonic finite element problems. Consider the weak formulation of the biharmonic Dirichlet problem: Find $u \in H_0^2(\Omega)$ such that

$$(1) \qquad a(u, v) = (f, v), \quad \forall v \in H_0^2(\Omega),$$

where, $(\cdot, \cdot)$ denotes the usual L_2 inner product and $a(u, v) = (u, v)_{H^2} = (\Delta u, \Delta v)$.

In this paper, we consider conforming C^1 finite elements approximations to (1). Some examples of the C^1 elements are the Argyris and Bell elements, the Hsieh-Clough-Tocher (HCT) element, the reduced HCT (RHCT) element, the singular Zienkiewicz (SZ) element, the reduced SZ (RSZ) element, the Birkhoff–Mansfield (BM) triangle, the reduced BM (RBM) triangle, the Powell-Sabin (PS) element, Bogner-Fox-Schmit's (BFS) bicubic element, the Fraeijs de Veubeke-Sander (FdVS) quadrilaterals, and the reduced FdVS (RFdVS) quadrilaterals. The definitions and approximation properties can be found in Ciarlet [8] and Powell and Sabin [18]. We will assume that the finite elements are also in $W^{2,\infty}(\Omega)$. Note that this condition is not part of the definition of C^1 elements, however, all the C^1 elements we know are in fact in $W^{2,\infty}(\Omega)$.

Let $\{\mathcal{T}_k\}$ be a family of quasi-uniform triangulations. We allow nonnested triangulations, however, we assume that the triangulations are essentially nested in the sense that the mesh parameters satisfy $0 < \gamma_1 \leq h_{k+1}/h_k \leq \gamma_2 < 1$. Let V_k be a family of finite element spaces defined by some conforming C^1 elements with respect to $\mathcal{T}_k$. Our theory does not require that the spaces $\{V_k\}$ on different level are defined by the same finite element. It is computationally more efficient to user simpler elements on coarser grids. The finite element solutions $u_k \in V_k$ satisfy

$$(2) \qquad a(u_k, \chi_k) = (f, \chi_k), \quad \forall \chi_k \in V_k.$$

All the finite elements listed above, except the BFS and PS elements, are nonnested even when the triangulations $\{\mathcal{T}_k\}$ are nested. For the BFS and PS elements defined with respect to a family of nested triangulations $\{\mathcal{T}_k\}$, a uniformly convergent theory for multigrid V-cycle can be established along the line of [2]; cf. [21] for more details.

We will denote by $\| \cdot \|_{s,p,D}$ and $\| \cdot \|_{s,D}$ the standard norm on Sobolev spaces $W^{s,p}(D)$ and $H^s(D) = W^{s,2}(D)$, and by $| \cdot |_{s,p,D}$ and $| \cdot |_{s,D}$ the semi-norms.

We will make the following standard assumption on the finite element spaces V_k. The verification of these assumptions is straightforward.

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ASSUMPTION 3.1. The following local inverse properties hold for $v \in V_k$:

$$(3) \qquad |v|_{s,q,\tau} \leq C h_k^{2(\frac{1}{q}-\frac{1}{p})} |v|_{s,p,\tau}, \quad 1 \leq p, q \leq \infty, \; 0 \leq s \leq 2, \; \tau \in \mathcal{T}_k.$$

The basis functions ϕ_i^α are *uniform to order 2*:

$$(4) \qquad |\phi^t|_{s,q,\tau} \leq C h_k^{2/q+t-s}, \quad 1 \leq q \leq \infty, 0 \leq s \leq 2, \tau \in \mathcal{T}_k.$$

Here ϕ^t stands for any basis function associated with a derivative of order t.

Let $A_k : V_k \mapsto V_k$ be defined as

$$(A_k u_k, v_k) = a(u_k, v_k), \quad \forall u_k, v_k \in V_k.$$

The L_2 projections $Q_k : L_2 \mapsto V_k$ are defined by

$$(Q_k u, \chi_k) = (u, \chi_k).$$

It can be shown (cf. e.g. [22]) that

$$|(\mathcal{I} - Q_k)u|_s \leq C h_k^{2-s} |u|_2, \quad \forall u \in H^2(\Omega).$$

It is convenient to use the following discrete Sobolev norms defined by

$$\|u\|_s^2 = (A_k^{s/2} u, u), \quad \forall s \in \mathbb{R}, \quad u \in V_k.$$

It is trivial to see $\|u\|_s \leq \|u\|_{s+\gamma}^{1/2} \|u\|_{s-\gamma}^{1/2}$, $\forall s, \gamma \in \mathbb{R}$. Using eigen-expansion and the Hölder inequality, we have the convexity for the discrete norms (cf. e.g. [4]),

$$\|u\|_s \leq \|u\|_{s_1}^\lambda \|u\|_{s_2}^{1-\lambda}, \quad s = \lambda s_1 + (1-\lambda)s_2, \; 0 \leq \lambda \leq 1.$$

In particular,

$$\|u\|_{2+2\alpha} \leq \|u\|_2^{1-\alpha} \|u\|_4^\alpha, \quad 0 \leq \alpha \leq 1.$$

The norm equivalence

$$\|u\|_s \asymp |u|_s, \quad 0 \leq s \leq 2,$$

is easy to see as follows. The cases $s = 0$ and $s = 2$ follow from the definition of the discrete norms. The result for $0 \leq s \leq 2$ follows by interpolating the operators $\mathcal{I}$ and Q_k; cf. Bank and Dupont[1]. For polynomial or piecewise polynomial elements, the result can be extended to the case $2 < s < 5/2$ based on the same reasoning as in [3], where it is shown that $V_k \subset H^{1+s}$ and $\|u\|_s \asymp |u|_s$ with $0 \leq s < 3/2$ for C^0 polynomial elements V_k. We do not know however whether or not this equivalence still holds for singular elements. We do not even know if $V_h \in H^{2+s}$ for some $s > 0$.

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3.1. Intergrid transfer operators. Let $\mathcal{N}_k$ be the set of nodes associated with the degrees of freedom of V_k. Let ϕ_i^α be the nodal basis functions of V_k at $x_i \in \mathcal{N}_k$, where α indicates the order and directions of the derivative of the corresponding degree of freedom. With a slight abuse of notation, we denote by $|\alpha|$ the order of the corresponding degrees of freedom. Let $\deg(V_k)$ be the maximum order of the derivatives in the degrees of freedom of V_k. Note that for the fourth order problem, $\deg(V_k) = 1$ or 2. Let Λ_i be the index set for the degrees of freedom at x_i. Let $\Lambda_i^0 = \{\alpha; \alpha \in \Lambda_i, |\alpha| = 0\}$, $\Lambda_i^1 = \{\alpha; \alpha \in \Lambda_i, |\alpha| = 1\}$, $\Lambda_i^2 = \{\alpha; \alpha \in \Lambda_i, |\alpha| = 2\}$. Note that if $\deg(V_k) = 1$ then $\Lambda_i^2 = \emptyset$ and $\Lambda_i = \Lambda_i^0 \cup \Lambda_i^1$.

Our first choice for the intergrid transfer operator is the (modified) nodal value interpolation operator: $\mathcal{I}_k : V_{k-1} \mapsto V_k$, defined by

$$(5) \qquad \mathcal{I}_k u = \sum_i \sum_{\alpha \in \Lambda_i^0 \cup \Lambda_i^1} \partial_\alpha u(x_i) \phi_i^\alpha.$$

If $\deg(V_k) = 1$, e.g. V_k defined by the HCT, RHCT, SZ, RSZ, PS, RBM, FdVS or RFdVS element, then the intergrid operator is in fact the standard nodal value interpolation operator.

If all the spaces $\{V_k\}$ are defined by the Argyris, Bell, BFS or BM element, we can also use the following intergrid transfer operator:

$$(6) \qquad \mathcal{I}_k u = \sum_i \sum_{\alpha \in \Lambda_i^0 \cup \Lambda_i^1} \partial_\alpha u(x_i) \phi_i^\alpha + \sum_i \sum_{\alpha \in \Lambda_i^2} u_i^\alpha \phi_i^\alpha,$$

where, $u_i^\alpha = \lambda_1(x_i)\partial_\alpha u(a_1) + \lambda_2(x_i)\partial_\alpha u(a_2) + \lambda_3(x_i)\partial_\alpha u(a_3)$, with $x_i \in \Delta a_1 a_2 a_3 \in \mathcal{T}_{k-1}$ and $\lambda_i(x)$ is the ith barycentric coordinate of x. Alternatively, we could define u_i^α to be the average value of $\partial_\alpha u$ at x_i.

Note that for $\mathcal{I}_k$ defined by (5)–(6), we need to evaluate $\partial_\alpha u(m_i)$ at midpoints m_i of the edges of $\mathcal{T}_{k-1}$, which means that we have to evaluate $\partial_\alpha \Phi_j^\beta(m_i)$ for the basis functions Φ_j^β of V_{k-1}. To avoid that, we can use the following simplified "P_1 preserving interpolation operator" which is determined by

$$(7) \quad \partial_\alpha \mathcal{I}_k u(x_i) = \lambda_1(x_i)\partial_\alpha u(a_1) + \lambda_2(x_i)\partial_\alpha u(a_2) + \lambda_3(x_i)\partial_\alpha u(a_3), \quad \alpha \in \Lambda_i^0 \cup \Lambda_i^1,$$

where $x_i \in \Delta(a_1 a_2 a_3) \in \mathcal{T}_{k-1}$.

Note that (5) and (7) are well defined on $C^1(\Omega)$, and (6) is well defined only on a subset of $C^1(\Omega)$. If in addition, u is also in $H_0^2(\Omega)$, then $\mathcal{I}_k u \in V_k \subset H_0^2(\Omega)$ automatically satisfies the homogeneous boundary conditions.

We note that $\mathcal{I}_k$ defined by (5)–(7) preserves linear functions; i.e.

$$\mathcal{I}_k p|_\tau = p, \quad \forall p \in P_1, \ \tau \in \mathcal{T}_k.$$

By definition, $\mathcal{I}_k$ defined by (6) also preserves quadratic functions, and if $\deg(V_k) = 1$, then $\mathcal{I}_k$ defined by (5) is the standard nodal value interpolation and thus also preserves quadratic functions.

It is well known that $\mathcal{I}_k$ is not bounded in $|\cdot|_2$–norm, however, restricted to the finite element space V_{k-1} (and thus a local inverse property holds), $\mathcal{I}_k$ has the following stable approximation properties.

LEMMA 3.1. *Let $\tau \in \mathcal{T}_k$ and $\hat{\tau} = \cup_{\substack{\tau' \in \mathcal{T}_{k-1} \\ \tau' \cap \tau \neq \emptyset}} \tau'$. There exists $C > 0$ independent of τ such that $\mathcal{I}_k$ defined by (5)–(7) satisfies*

$$|\mathcal{I}_k u - u|_{s,\tau_k} \leq h_k^{2-s} |u|_{2,\hat{\tau}}, \quad \forall u \in V_{k-1}, \ 0 \leq s \leq 2.$$

As a consequence,

(8) $$|\mathcal{I}_k u - u|_{s,\Omega} \leq h_k^{2-s} |u|_{2,\Omega}, \quad \forall u \in V_{k-1}, \ 0 \leq s \leq 2.$$

Proof. Note that $|\partial_\alpha u(x_i)| \leq |u|_{|\alpha|,\infty,\tau}$ and $|u_i^\alpha| \leq |u|_{|\alpha|,\infty,\hat{\tau}}$. By the definitions of $\mathcal{I}_k$ and Assumption 3.1, we have

$$|\mathcal{I}_k u|_{s,\tau} \leq \sum_{r=0}^{\deg(V_k)} Ch_k^{1+r-s} |u|_{r,\infty,\hat{\tau}} \leq \sum_{r=0}^{2} Ch_k^{1+r-s} |u|_{r,\infty,\hat{\tau}}.$$

Since $\hat{\tau}$ is a union of $\tau' \in \mathcal{T}_{k-1}$, the inverse inequality (3) holds for $u \in V_{k-1}$ on $\hat{\tau}$,

$$|u|_{r,\infty,\hat{\tau}} \leq Ch_{k-1}^{-1} |u|_{r,\hat{\tau}} = C(h_k/h_{k-1}) h_k^{-1} |u|_{r,\hat{\tau}} \leq Ch_k^{-1} |u|_{r,\hat{\tau}}.$$

By the triangle inequality

$$|(\mathcal{I} - \mathcal{I}_k)u|_{s,\tau} \leq \sum_{r=0}^{2} Ch_k^{r-s} |u|_{r,\hat{\tau}}.$$

Now using $\mathcal{I}_k p|_\tau = p$ for $p \in P_1$ and the Poincaré inequality, we obtain

$$|(\mathcal{I} - \mathcal{I}_k)u|_{s,\tau} \leq \inf_{p \in P_1} \sum_{r=0}^{2} Ch_k^{r-s} |u + p|_{r,\hat{\tau}} \leq \sum_{r=0}^{2} Ch_k^{r-s} h_{k-1}^{2-r} |u|_{2,\hat{\tau}} \leq Ch_k^{2-s} |u|_{2,\hat{\tau}}.$$

Squaring and summing the above inequality over $\tau \in \mathcal{T}_k$, we obtain

$$|(\mathcal{I} - \mathcal{I}_k)u|_{s,\Omega} \leq Ch_k^{2-s} |u|_{2,\Omega}, \quad s = 0, 1, 2.$$

The result for $0 \leq s \leq 2$ follows from the convexity of the norms. $\square$

If $\{\mathcal{T}_k\}$ are nested, the proof can be slightly simplified, in particular, $\hat{\tau}$ can be replaced by $\tau' \in \mathcal{T}_{k-1}$ which contains τ.

Remark: In the case when $\mathcal{I}_k$ preserves quadratic polynomials, instead of the Poincaré inequality, we can use the Bramble-Hilbert lemma

$$\inf_{p \in P_2} \sum_{r=0}^{2} h_{k-1}^r |u + p|_{r,\hat{\tau}} \leq C h_k^t |u|_{t,\hat{\tau}}, \quad 2 \leq t < 5/2$$

in our proof to obtain a stronger result

$$|\mathcal{I}_k u - u|_{s,\Omega} \leq C h^{t-s} (\sum_{\hat{\tau}} |u|_{t,\hat{\tau}}^2)^{1/2} \leq C h^{t-s} |u|_{t,\Omega}, \quad u \in V_{k-1}, 0 \leq s \leq 2, \ 0 \leq t < \frac{5}{2}.$$

If $\{\mathcal{T}_k\}$ are also nested and V_k as well as V_{k-1} are polynomial elements, then the above inequality also holds for $2 \leq s < 5/2$. This property will not be used in our analysis.

3.2. Multigrid theory for C^1 elements. We now verify the regularity and approximation assumption **A.1**. We assume the following *a priori* estimate (cf. [9])

$$(9) \qquad\qquad |u|_{2+\alpha_0} \leq \|\Delta^2 u\|_{-2+\alpha_0}, \quad \text{for some} \quad \alpha_0 > 0.$$

Note that the estimate also holds for $0 \leq \alpha < \alpha_0$.

We now establish the regularity-approximation assumption in a series of lemmas.

LEMMA 3.2. *Let* $P_{k-1} : H_0^2(\Omega) \mapsto V_{k-1}$ *be the Galerkin projection. Then*

$$(10) \qquad |(\mathcal{I} - P_{k-1})u|_{2-\alpha} \ \leq \ C h_k^\alpha |u|_2, \qquad \forall u \in H_0^2, \ 0 \leq \alpha \leq \alpha_0$$

$$(11) \qquad |(\mathcal{I} - P_{k-1})u|_2 \ \leq \ C h_k^\alpha \|u\|_{2+\alpha}, \quad \forall u \in V_k, \ 0 \leq \alpha \leq \alpha_0$$

Proof. The first inequality follows from the standard finite element error estimate and a duality argument. To prove the second, we note

$$\begin{aligned}
|(\mathcal{I} - P_{k-1})u|_2^2 \ &= \ a(u, (\mathcal{I} - P_{k-1})u) = a(u, P_k(\mathcal{I} - P_{k-1})u) \\
&\leq \ \|u\|_{2+\alpha} \|P_k(\mathcal{I} - P_{k-1})u\|_{2-\alpha} \\
&\leq \ C \|u\|_{2+\alpha} |P_k(\mathcal{I} - P_{k-1})u|_{2-\alpha}
\end{aligned}$$

By the triangle inequality and (10), we have

$$\begin{aligned}
|P_k(\mathcal{I} - P_{k-1})u|_{2-\alpha} \ &\leq \ |(\mathcal{I} - P_k)(\mathcal{I} - P_{k-1})u|_{2-\alpha} + |(\mathcal{I} - P_{k-1})u|_{2-\alpha} \\
&\leq \ C h_k^\alpha |(\mathcal{I} - P_{k-1})u|_2 + C h_{k-1}^\alpha |(\mathcal{I} - P_{k-1})u|_2 \\
&\leq \ C h_k^\alpha |(\mathcal{I} - P_{k-1})u|_2.
\end{aligned}$$

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In the second inequality, we have used obvious facts that $(\mathcal{I} - P_{k-1})^2 = (\mathcal{I} - P_{k-1})$ and $(\mathcal{I} - P_{k-1})u \in H_0^2(\Omega)$. $\square$

LEMMA 3.3. *For* $0 \le \alpha \le \alpha_0$, *we have*

(12)
$$|(\mathcal{I}_k^* - P_{k-1})u_k|_2 \le Ch_k^\alpha \|u_k\|_{2+\alpha}.$$

Proof. By the definition of $\mathcal{I}_k^*$ and P_{k-1}, we have for any $v_{k-1} \in V_{k-1}$,

$$
\begin{aligned}
a((\mathcal{I}_k^* - P_{k-1})u_k, v_{k-1}) &= a(u_k, (\mathcal{I}_k - \mathcal{I})v_{k-1}) = a(u_k, (\mathcal{I}_k - P_k)v_{k-1}) \\
&= \|u_k\|_{2+\alpha}\|(\mathcal{I}_k - P_k)v_{k-1}\|_{2-\alpha} \\
&\le C\|u_k\|_{2+\alpha}|(\mathcal{I}_k - P_k)v_{k-1}|_{2-\alpha}.
\end{aligned}
$$

Using the triangle inequality, Lemma 3.1 and (10),

$$|(\mathcal{I}_k - P_k)v_{k-1}|_{2-\alpha} \le |(\mathcal{I}_k - \mathcal{I})v_{k-1}|_{2-\alpha} + |(\mathcal{I} - P_k)v_{k-1}|_{2-\alpha} \le Ch_k^\alpha|v_{k-1}|_2.$$

Therefore,

$$a((\mathcal{I}_k^* - P_{k-1})u_k, v_{k-1}) \le Ch_k^\alpha\|u_k\|_{2+\alpha}|v_{k-1}|_2.$$

The lemma follows by setting $v_{k-1} = (\mathcal{I}_k^* - P_{k-1})u_k$, $\square$

LEMMA 3.4. *For* $0 \le \alpha \le \alpha_0$, *we have the following estimate for* $\mathcal{I}_k^*$.

$$|(\mathcal{I} - \mathcal{I}_k^*)u_k|_2 \le Ch_k^\alpha\|u_k\|_{2+\alpha}, \quad \forall u_k \in V_k.$$

Proof. The lemma follows trivially from (11) and (12), $\square$

THEOREM 3.5 (REGULARITY-APPROXIMATION). *For* $\alpha = \alpha_0/4$, *we have*

$$a((\mathcal{I} - \mathcal{I}_k\mathcal{I}_k^*)u_k, u_k) \le C\|u_k\|_2^{2(1-\alpha)}\left(\frac{\|u_k\|_4^2}{\lambda_k}\right)^\alpha = C(A_ku_k, u_k)^{1-\alpha}\left(\frac{\|A_ku_k\|_0^2}{\lambda_k}\right)^\alpha.$$

Proof. Let $\alpha = \alpha_0/4$. Then by Lemma 3.4

$$
\begin{aligned}
|a((\mathcal{I} - \mathcal{I}_k\mathcal{I}_k^*)u_k, u_k)| &= |a((\mathcal{I} - \mathcal{I}_k^*)u_k, (\mathcal{I} + \mathcal{I}_k^*)u_k)| \\
&\le |(\mathcal{I} - \mathcal{I}_k^*)u_k|_2|(\mathcal{I} + \mathcal{I}_k^*)u_k|_2 \\
&\le Ch_k^{\alpha_0}\|u_k\|_{2+\alpha_0}|u_k|_2 \\
&\le Ch_k^{4\alpha}\|u_k\|_2^{2-2\alpha}\|u_k\|_4^{2\alpha}.
\end{aligned}
$$

Here we have used the convexity of norms $\|u_k\|_{2+\alpha_0} = \|u_k\|_{2+4\alpha} \le \|u_k\|_2^{1-2\alpha}\|u_k\|_4^{2\alpha}$. $\square$

In the cases when $V_k \subset H^{2+\alpha}(\Omega) \cap H_0^2(\Omega)$ and $\|u\|_{2+\alpha} \approx |u|_{2+\alpha}$, the proof of Theorem 3.5 can be simplified slightly. If the spaces $\{V_k\}$ are nested and $\mathcal{I}_k = \mathcal{I}$, then Theorem 3.5, with $\alpha = \alpha_0/2$, is a direct consequence of (11).

The following is our main result. It is a consequence of Theorems 2.1, 2.2 and 3.5.

THEOREM 3.6. *If the smoother R_k satisfies* **A.2.1***, and the number of smoothing $m_k \equiv m$ is sufficient large, but independent of k, then there exists an $M > 0$ such that the the contraction number for W-cycle multigrid satisfies*

$$\delta \leq \frac{M}{M + m^\alpha}$$

If the smoother R_k satisfies **A.2.1** *and the number of smoothing m_k satisfies* **A.2.2***, then there exist an $M > 0$, such that the variable V-cycle preconditioner satisfies*

$$\frac{m_k^\alpha}{M + m_k^\alpha} a(u, u) \leq a(B_k A_k u, u) \leq \frac{M + m_k^\alpha}{m_k^\alpha} a(u, u).$$

Thus, $\sigma(B_k A_k) \subset [\frac{m_k^\alpha}{M + m_k^\alpha}, \frac{M + m_k^\alpha}{m_k^\alpha}]$ and the preconditioned equations are uniformly well conditioned.

Our algorithms and theory can be generalized easily to the cases when the coarser level triangulations $\mathcal{T}_k$ are defined only on a subregion $\Omega_k \subset \Omega$, with $\mathrm{dist}(\partial\Omega_k, \partial\Omega) = O(h_k)$. In particular, using inequalities

$$(13) \qquad \begin{aligned} \|u\|_{2,\Omega\backslash\Omega_k} &\leq Ch_k^\alpha \|u\|_{2+\alpha}, \quad 0 < \alpha \leq 1/2, \quad u \in H^{2+\alpha}, \\ |u|_{s,\Omega\backslash\Omega_k} &\leq Ch_k^{2-s} |u|_2, \quad\ 0 \leq s \leq 2, \qquad u \in H_0^2(\Omega), \end{aligned}$$

it is easy to see that conclusions in Lemmas 3.1 and 3.2 remain valid. The rest of results follow from Lemmas 3.1 and 3.2. Inequalities similar to (13) can be found, for example, in [14, 2, 4, 7], we refer to Bramble and Pasciak [2] for a proof.

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