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A LOCAL POST-PROCESSING TECHNIQUE FOR IMPROVING THE ACCURACY IN MIXED FINITE-ELEMENT APPROXIMATIONS*

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This paper is dedicated to Jim Douglas, Jr., on the occasion of his 60th birthday.

Abstract. A general simple post-processing technique is given in this paper. Under certain conditions, if an algorithm approximating a function u also provides a good approximation for its gradient and for some locally defined projection Pu, then by this method, a better approximation for u can be easily obtained. Essentially, the leading term of the error between u and its original approximation can be computed. It is shown how the method may be applied to various mixed type finite-element methods.

Key words. mixed finite element, post-processing

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1. Introduction. Assume u is an exact solution of some partial differential equation on a domain $\Omega \subset \mathbf{R}^2$. Typically a mixed finite-element method is designed to approximate u and ∇u simultaneously where the equation is replaced by a system of equations involving these variables. Associated with some triangulation $\mathcal{T}_h$ of Ω, let W_h and V_h be finite-element spaces in which approximations for u and ∇u are sought. Generally, W_h is a subspace of $L^2(\Omega)$ consisting of discontinuous piecewise polynomials. In many cases, we can prove the following kinds of estimates:

$$\|\nabla u - (\nabla u)_h\| \leqq C h^{r+1} |\log h|^{\mu_1}, \tag{1}$$

$$\|P_m u - u_h\| \leqq C h^{r+2} |\log h|^{\mu_2}. \tag{2}$$

Here $u_h \in W_h$ and $(\nabla u)_h \in V_h$ are the approximations of u and ∇u, respectively, P_m is an operator defined on each element that is invariant on polynomials of degree $m \leqq r$ there, and μ_1 and μ_2 are some nonnegative constants. The norms in (1) and (2) are usually L^p-norms with $1 \leqq p \leqq \infty$.

There are many mixed finite-element methods for which estimates like (1) and (2) are known. Examples are the well-known Raviart–Thomas and Brezzi–Douglas–Marini elements for second-order elliptic problems (see [15], [6]). Also for the elasticity or Stokes equations, the Plane Elasticity Element with Reduced Symmetry (PEERS) method by Arnold, Brezzi, and Douglas [1], the method by Stenberg [18], the pseudo-stress formulation of Arnold and Falk [3], and the method of Bramble and Xu [5] provide further examples. In the case of elasticity or Stokes problems, we should mention that u represents any one of the components of the vector displacement or velocity.

In each of the examples mentioned above, the estimates for $u - u_h$ would never be better than (1), and as a matter of fact we obtain only

$$\|u - u_h\| \leqq C h^l \tag{3}$$

for some $l \leqq r+1$. Therefore it is natural to ask if it is possible to obtain by post-processing a better approximation for u using u_h and $(\nabla u)_h$.

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In [2], Arnold and Brezzi analyze the Raviart–Thomas scheme and show that, if Lagrange multipliers were introduced to replace the constraints on V_h then, using the resulting multipliers, a new and more accurate approximation u_h^* of u could be constructed so that

$$\|u-u_h^*\| \leqq Ch^{r+2}|\log h|^{\mu}, \tag{4}$$

with $\|\cdot\|$ being the L^2 norm and $\mu=0$.

A similar result was obtained in [6] by Brezzi, Douglas, and Marini for the elements they introduced and analyzed. Arnold, Brezzi, and Douglas have made a similar construction for PEERS, and Stenberg also has given an alternative way to construct a new approximation u_h^*.

Another simple idea is to find superconvergence points for $P_m u$; this can be done for rectangular elements (namely the Gaussian points). But there are no superconvergence points for triangular elements except in the case in which $m=0$ (cf. [10]) and W_h consists of piecewise constant functions.

In this paper, we present a general technique that can be used in a straightforward way to construct an improved approximation of u in all the above cases. Essentially our method depends only on estimates of the form (1) and (2). By assuming that (1) and (2) hold in the L^p-norm, u_h^* can be constructed in an elementary fashion so that (4) also holds in the L^p-norm with $\mu=\max\{\mu_1, \mu_2\}$ (see Theorem 2.1 below). We note that all the post-processing results mentioned above are only presented in the L^2-norm and that our results are more general.

Our main result is a pure approximation-theoretic result and hence its application is not restricted to mixed finite-element approximations. Also, for the sake of simplicity, we will state the result in only two dimensions. The generalization to higher dimensions is straightforward. For the mixed finite-element methods in $\mathbf{R}^3$, for example, we refer to [13] and [14].

The rest of the paper is organized as follows. Section 2 starts with a general lemma and ends with a theorem that can be directly applied to finite-element methods; section 3 consists of applications of our result to several mixed finite-element methods.

2. Main results. Let $G \subset \mathbf{R}^2$ be a star-shaped domain with respect to $X_0=(x_0, y_0) \in G$ and $\mathbf{P}_m(G)$ be the space of polynomials of degree m defined on G. Given $u \in \mathbf{P}_m(G)$, the Taylor expansion of u at $X=(x, y) \in G$ gives

$$u(X)=\sum_{k=0}^{m} \frac{1}{k!}\left((x-x_0)\frac{\partial}{\partial x}+(y-y_0)\frac{\partial}{\partial y}\right)^k u(X_0). \tag{5}$$

We would like to rewrite this expansion in a way that is convenient for our later manipulations. For this purpose we denote, for $k \geqq 1$ and $j=0, \cdots, k$,

$$c_{kj}(X)=\binom{k}{j}(x-x_0)^{k-j}(y-y_0)^j,$$

$$D_j^k=\frac{\partial^k}{\partial x^{k-j}\,\partial y^j}.$$

Using this notation, we construct a vector polynomial and a vector differential operator as follows:

$$\vec{c}_k(X)=(c_{kj}(X))_{0\leqq j\leqq k},$$

$$\vec{D}^k=(D_j^k)_{0\leqq j\leqq k}.$$

The formal inner product of these two vectors is a differential operator of order k given by

$$\vec{c}_k \cdot \vec{D}^k = \sum_{j=0}^{k} \binom{k}{j} (x-x_0)^{k-j}(y-y_0)^j \frac{\partial^k}{\partial x^{k-j}\,\partial y^j} = \left((x-x_0)\frac{\partial}{\partial x} + (y-y_0)\frac{\partial}{\partial y} \right)^k.$$

Hence (5) can be rewritten as

$$u(X) = \sum_{k=0}^{m} \frac{1}{k!}[\vec{c}_k \cdot \vec{D}^k u](X_0). \tag{6}$$

For $k \geqq 1$, it is easy to find a $2\times k$ matrix $\mathbf{A}_k$ such that

$$\vec{c}_k \cdot \vec{D}^k = \mathbf{A}_k \vec{D}^{k-1} \cdot \nabla \tag{7}$$

where $\mathbf{A}_k\vec{D}^{k-1}$ is the formal matrix and vector multiplication. Consequently, we can get the Taylor expansion of u in terms of ∇u as follows:

$$u(X) = u(X_0) + \sum_{k=1}^{m} \frac{1}{k!}[(\mathbf{A}_k\vec{D}^{k-1}) \cdot \nabla u](X_0). \tag{8}$$

The choice of $\mathbf{A}_k$ in (7) is obviously not unique; a simple candidate would be

$$\mathbf{A}_k = \begin{pmatrix} c_{k0} & 0 \cdots 0 \\ c_{k1} & c_{k2} \cdots c_{kk} \end{pmatrix}.$$

Now we are in a position to state the main lemma in the paper. We use the standard notation $W^{m,p}(G)$ for the usual Sobolev spaces with the norms denoted by $\|\cdot\|_{m,p,G}$ and $L^p(G) = W^{0,p}(G)$.

LEMMA 2.1. *Assume that m and r are nonnegative integers, $r \geqq m$, and $R_m : L^p(G) \to L^p(G)$ is a linear operator annihilating $\mathbf{P}_m(G)$. Let $\mathbf{A}_k$ be a $2\times k$ matrix satisfying* (7), *and define the vector differential operator $\vec{d}_{r,m}$ by*

$$\vec{d}_{r,m} = \sum_{k=m+1}^{r+1} \sum_{j=0}^{k-1} \sum_{i=0}^{r-k+1} \frac{(-1)^i}{i!\,k!}[R_m \vec{a}^k_{j+1}][\vec{c}_i \cdot (D^{k-1}_j \vec{D}^i)]. \tag{9}$$

Then

$$(R_m - \vec{d}_{r,m} \cdot \nabla) : W^{r,p}(G) \to L^p(G)$$

is a linear operator annihilating $\mathbf{P}_{r+1}(G)$, where $\vec{a}^k_j$ is the jth column of $\mathbf{A}_k$.

Proof. When we take $u \in \mathbf{P}_{r+1}(G)$ and set $\vec{\sigma} = \nabla u$, (8) reads

$$u(X) = u(X_0) + \sum_{k=1}^{r+1} \frac{1}{k!}[\mathbf{A}_k \vec{D}^{k-1} \cdot \vec{\sigma}](X_0). \tag{10}$$

Applying R_m to both sides of (10), we get

$$R_m u(X) = \sum_{k=m+1}^{r+1} \frac{1}{k!}[(R_m \mathbf{A}_k)\vec{D}^{k-1} \cdot \vec{\sigma}](X_0) \tag{11}$$

since by hypothesis

$$R_m\left(u(X_0) + \sum_{k=1}^{m} \frac{1}{k!}[\mathbf{A}_k \vec{D}^{k-1} \cdot \vec{\sigma}](X_0) \right) = 0.$$

By definition

$$[R_m \mathbf{A}_k]\vec{D}^{k-1} \cdot \vec{\sigma} = \sum_{j=0}^{k-1} (R_m \vec{a}^k_{j+1}) \cdot D^{k-1}_j \vec{\sigma}. \tag{12}$$

Since $D_j^{k-1}\vec{\sigma} \in [\mathbf{P}_{r-k+1}(G)]^2$, the Taylor expansion (6) gives

$$D_j^{k-1}\vec{\sigma}(X_0) = \sum_{i=0}^{r-k+1} \frac{(-1)^i}{i!}[\vec{c}_i(X)\cdot \vec{D}^i]D_j^{k-1}\vec{\sigma}(X). \tag{13}$$

Combining (11), (12), and (13), we get

$$R_m u(X) = \sum_{k=m+1}^{r+1}\sum_{j=0}^{k-1}\sum_{i=0}^{r-k+1} \frac{(-1)^i}{i!\,k!}[R_m \vec{a}_{r+1}^k][\vec{c}_i \cdot (D_j^{k-1}\vec{D}^i)]\cdot \vec{\sigma} \tag{14}$$

$$= (\vec{d}_{r,m}\cdot \nabla u)(X) \quad \forall u \in \mathbf{P}_{r+1}(G). \tag{15}$$

This completes the proof.

Remark. The expression for $\vec{d}_{r,m}$ looks a little complicated. In our applications the operator R_m is locally defined, and computing its action on certain low-degree polynomials is trivial.

In the applications given later the following simple special cases will be of importance.

(1) $m = r$:

$$\vec{d}_{r,r} = \frac{1}{(r+1)!}\sum_{j=0}^{r}[R_r \vec{a}_{j+1}^{r+1}]D_j^r.$$

In particular, if $r = m = 0$, and $R_0 = I - P_0$, where P_0 is the $L^2(G)$-projection onto constants, then

$$\vec{d}_{0,0} = \begin{pmatrix} x - \bar{x} \\ y - \bar{y} \end{pmatrix} \tag{16}$$

where $(\bar{x}, \bar{y})$ is the barycenter of G.

(2) $m = r-1$:

$$\vec{d}_{r,r-1} = \frac{1}{(r+1)!}\sum_{j=0}^{r}[R_{r-1}\vec{a}_{j+1}^{r+1}]D_j^r + \frac{1}{r!}\sum_{j=0}^{r-1}\sum_{i=0}^{1}(-1)^i[R_{r-1}\vec{a}_{j+1}^r]\vec{c}_i\cdot(D_j^{r-1}\vec{D}^i).$$

Before proceeding we note that the definition of $\vec{d}_{r,m}$ depends on the domain G. We will, however, suppress this dependence since it will always be clear from the definition of R_m. Hence if $R_m: L^p(G) \to L^p(G)$, then $\vec{d}_{r,m}: W^{r,p}(G) \to L^p(G)$.

Next we apply the above lemma to finite-element methods. The construction is local and will be carried out element by element. Assume $\hat{K}$ is a given reference triangle in $\mathbf{R}^2$ and $\hat{V} \subset [C^\infty(\hat{K})]^2$ is a finite-dimensional space. Let $\mathcal{K}$ be a collection of triangles that satisfies the following properties:

(1) For any $K \in \mathcal{K}$, there is a bijective affine mapping F_K such that

$$F_K: \hat{K} \to K.$$

(2) There exists a positive constant C such that, for any $K \in \mathcal{K}$ and $j \leqq l$, we have

$$\|v\|_{l,p,K} \leqq Ch^{j-l}\|v\|_{j,p,K} \quad \forall v \in V_K \tag{17}$$

where $V_K = F_K(\hat{V})$ and $h = \operatorname{diam}(K)$.

Intuitively we can think of V_K as the restriction of a certain finite-element space to the element K. In what follows, C will denote a generic constant that is independent of any of the variables occurring in the expression in which it appears, and that may be different in different occurrences.

LEMMA 2.2. *Let $K \in \mathcal{K}$ and suppose that R_m is bounded in $L^\infty(K)$, i.e.,*

$$\|R_m v\|_{0,\infty,K} \leqq C\|v\|_{0,\infty,K} \tag{18}$$

for all $v \in L^\infty(K)$.

Then

$$\|\vec{d}_{r,m} \cdot \vec{\sigma}\|_{0,p,K} \leqq C \sum_{k=m}^{r+1} h^{k+1} \|\vec{\sigma}\|_{k,p,K}$$

for all $\vec{\sigma} \in [W^{r+2,p}(K)]^2$.

Proof. It follows from the expression for $\vec{d}_{r,m}$ (9) that

$$\|\vec{d}_{r,m} \cdot \vec{\sigma}\|_{0,p,K} \leqq C \sum_{k=m+1}^{r+1} \sum_{j=0}^{k-1} \sum_{i=0}^{r-k+1} \|R_m \vec{a}^k_{j+1}\|_{0,\infty,K} \|\vec{c}_i\|_{0,\infty,K} \|\vec{\sigma}\|_{k+i-1,p,K}.$$

Obviously $\|\vec{c}_i\|_{0,\infty,K} \leqq Ch^i$, and by hypothesis we have that

$$\|R_m \vec{a}^k_{j+1}\|_{0,\infty,K} \leqq C\|\vec{a}^k_{j+1}\|_{0,\infty,K} \leqq Ch^k.$$

Hence

$$\begin{aligned}\|\vec{d}_{r,m} \cdot \vec{\sigma}\|_{0,p,K} &\leqq C \sum_{k=m+1}^{r+1} \sum_{i=0}^{r-k+1} h^{k+i} \|\vec{\sigma}\|_{k+i-1,p,K} \\ &\leqq C \sum_{k=m}^{r+1} h^{k+1} \|\vec{\sigma}\|_{k,p,K}\end{aligned}$$

as desired.

LEMMA 2.3. *Assume that for any $K \in \mathcal{K}$, we have*

$$\inf_{\chi \in V_K} \sum_{k=0}^{r+1} h^k \|\nabla u - \chi\|_{k,p,K} \leqq Ch^{r+1} |u|_{r+2,p,K}, \tag{19}$$
$$\|R_m w\|_{0,p,K} \leqq C\|w\|_{0,p,K}$$

for all $u \in W^{r+2,p}(K)$ and $w \in L^p(K)$.

Define, with $u_h \in L^p(K)$ and $(\nabla u)_h \in V_K$,

$$u_h^* = u_h + \vec{d}_{r,m} \cdot (\nabla u)_h. \tag{20}$$

Then, with $u \in W^{r+2,p}(K)$ and under the assumptions of Lemmas 2.1 and 2.2,

$$\|u - u_h^*\|_{0,p,K} \leqq \|P_m u - u_h\|_{0,p,K} + C(h\|\nabla u - (\nabla u)_h\|_{0,p,K} + h^{r+2}|u|_{r+2,p,K})$$

where $P_m = I - R_m$.

Proof. It is straightforward to verify that

$$u - u_h^* = P_m u - u_h + \vec{d}_{r,m} \cdot (\nabla u - (\nabla u)_h) + (R_m - \vec{d}_{r,m} \cdot \nabla)u. \tag{21}$$

To bound the second term on the right we use the triangle inequality and inverse property (17). Thus for any $\chi \in V_K$, we have

$$\|\nabla u - (\nabla u)_h\|_{k,p,K} \leqq Ch^{-k} \|\chi - (\nabla u)_h\|_{0,p,K} + \|\nabla u - \chi\|_{k,p,K}. \tag{22}$$

Using Lemma 2.2 and (22), we get

$$\|\vec{d}_{r,m} \cdot (\nabla u - (\nabla u)_h)\|_{0,p,K} \tag{23}$$

$$\leqq Ch\left(\|\chi - (\nabla u)_h\|_{0,p,K} + \sum_{k=m}^{r+1} h^k \|\nabla u - \chi\|_{k,p,K} \right) \tag{24}$$

$$\leqq Ch\left(\|\nabla u - (\nabla u)_h\|_{0,p,K} + \sum_{k=0}^{r+1} h^k \|\nabla u - \chi\|_{k,p,K} \right). \tag{25}$$

Thanks to Lemma 2.1, we can employ the results of Bramble and Hilbert (cf. [4]) to deduce that

$$\|(R_m - \vec{d}_{r,m} \cdot \nabla)u\|_{0,p,K} \leqq Ch^{r+2}|u|_{r+2,p,K}.$$

Combining (21)-(23) and (19) completes the proof.

We note that Lemma 2.3 is a local result. In the following we state a global result that can be applied to finite-element methods more directly. Let us take a domain $\Omega \subset \mathbf{R}^2$ with piecewise smooth boundary. As usual, we say $\mathcal{T}_h$ is a triangulation of $\bar{\Omega}$ if $\mathcal{T}_h$ consists of triangles whose union is $\bar{\Omega}$ and the maximum diameter is h. Triangles meeting the boundary are allowed to have one curved edge. Following Lemma 2.3, we have the main theorem in this paper as follows.

THEOREM 2.1. *Given a triangulation $\mathcal{T}_h$ Ω, assume that*:

(1) *V_h is a subspace of $L^p(\Omega)$ that is piecewise smooth with respect to $\mathcal{T}_h$ and satisfies* (17) (*with $V_K = V_h$ restricted to K*) *and* (19) *for each $K \in \mathcal{T}_h$.*

(2) *$P_m : L^p(\Omega) \to L^p(\Omega)$ is an elementwise, locally defined, bounded linear operator that is invariant on piecewise polynomials of degree m and* (18) *holds with $R_m = I - P_m$ for each $K \in \mathcal{T}_h$.*

For any $u_h \in L^p(\Omega)$ and $(\nabla u)_h \in V_h$, define u_h^* on each element $K \in \mathcal{T}_h$ by (20). *Then, for $u \in W^{r+2,p}(\Omega)$,*

$$\|u - u_h^*\|_{0,p,\Omega} \leqq C[\|P_m u - u_h\|_{0,p,\Omega} + h\|\nabla u - (\nabla u)_h\|_{0,p,\Omega} + h^{r+2}|u|_{r+2,p,\Omega}].$$

3. Applications. As mentioned above, for the application of our result, we must simply see if estimates such as (1) and (2) are available. We will illustrate this through a few examples of some mixed finite-element methods for scalar second-order elliptic problems and Stokes or elasticity equations.

3.1. Scalar second-order elliptic problems. Let us first consider the following simple model problem:

$$-\Delta u = f \quad \text{in } \Omega, \tag{26}$$

$$u = 0 \quad \text{on } \partial\Omega \tag{27}$$

where Δ is the Laplace operator.

In the mixed method $\vec{\sigma} = \nabla u$ is introduced as a new variable. Hence we will have a pair of spaces W_h and V_h, for u and $\vec{\sigma}$, respectively, satisfying

$$W_h \subset L^2(\Omega),$$
$$V_h \subset H(\text{div}, \Omega) \equiv \{\vec{v} \in L^2(\Omega)^2 : \text{div } \vec{v} \in L^2(\Omega)\}.$$

There are two well-known choices for $W_h \times V_h$, namely, the Raviart-Thomas and Brezzi-Douglas-Marini spaces. For the Raviart-Thomas elements $(\text{RT})_k$, we may apply our theorem with $r = m = k$. Here W_h is a subspace of L^p consisting of piecewise polynomials of degree k, and V_h is a subspace of $H(\text{div}, \Omega)$ consisting of some very special kinds of polynomials of degree $k+1$. For details refer to [7], [11], and [15].

For Brezzi-Douglas-Marini elements $(\text{BDM})_k$, we may apply our theorem with $r = m + 1 = k$. Here W_h consists of the piecewise polynomials of degree $k-1$, and V_h consists of piecewise polynomial vectors of degree k that belong to $H(\text{div}, \Omega)$.

Basically both $(\text{RT})_k$ and $(\text{BDM})_k$ elements admit analogous error estimates (except for the lowest-order elements). Estimates such as (1) and (2) are all known in L^p-norms. For L^2 estimates, see [7], [11], [12], and [15], for L^∞ estimates see [8], [16], and [17], and for general L^p estimates see [8].

Here we give some specific examples of some estimates that may be found in the above-cited literature. Assuming that $u_h \in W_h$ and $(\nabla u)_h \in V_h$ are approximations of u

and ∇u, respectively, resulting from $(\mathrm{RT})_k$ or $(\mathrm{BDM})_k$ elements and that P_m is the orthogonal L^2-projection onto W_h, we have the following.

If $k \geqq 1$ for $(\mathrm{RT})_k$ and $k \geqq 2$ for $(\mathrm{BDM})_k$,

$$\|P_m u - u_h\|_{0,p,\Omega} + h\|\nabla u - (\nabla u)_h\|_{0,p,\Omega} \leqq Cp^2 h^{r+2}\|u\|_{r+2,p,\Omega} \qquad (2 \leqq p < \infty),$$

$$\|P_m u - u_h\|_{0,\infty,\Omega} + h\|\nabla u - (\nabla u)_h\|_{0,\infty,\Omega} \leqq Ch^{r+2}|\log h|^2\|u\|_{r+2,\infty,\Omega}.$$

Hence, by Theorem 2.1 (it is easy to see that all of its assumptions are satisfied here),

$$\|u - u_h^*\|_{0,p,\Omega} \leqq Cp^2 h^{r+2}\|u\|_{r+2,p,\Omega} \qquad (2 \leqq p < \infty),$$

$$\|u - u_h^*\|_{0,\infty,\Omega} \leqq Ch^{r+2}|\log h|^2\|u\|_{r+2,\infty,\Omega}.$$

For the lowest-order elements $(\mathrm{RT})_0$ and $(\mathrm{BDM})_1$, the estimates are as follows:

$$\|P_m u - u_h\|_{0,p,\Omega} + h\|\nabla u - (\nabla u)_h\|_{0,p,\Omega} \leqq Cp^2 h^2\|u\|_{3,p,\Omega} \qquad (2 \leqq p < \infty),$$

$$\|P_m u - u_h\|_{0,\infty,\Omega} + h\|\nabla u - (\nabla u)_h\|_{0,\infty,\Omega} \leqq Ch^2|\log h|^2\|u\|_{3,\infty,\Omega}.$$

Again by Theorem 2.1,

$$\|u - u_h^*\|_{0,p,\Omega} \leqq Cp^2 h^2\|u\|_{3,p,\Omega} \qquad (2 \leqq p < \infty),$$

$$\|u - u_h^*\|_{0,\infty,\Omega} \leqq Ch^2|\log h|^2\|u\|_{3,\infty,\Omega}.$$

Remark. For the lowest-order elements $(\mathrm{RT})_0$ and $(\mathrm{BDM})_1$, it follows from (16) and the definition of u_h^* that, for any $K \in \mathscr{T}_h$, if $(\bar{x}, \bar{y})$ is the barycenter of K, then

$$u_h^*(\bar{x}, \bar{y}) = u_h(\bar{x}, \bar{y}).$$

Consequently, as a special case of our results,

$$|(u - u_h)(\bar{x}, \bar{y})| \leqq Ch^2|\log h|^2\|u\|_{3,\infty,\Omega}.$$

This was observed by Gastaldi and Nochetto in [10].

Note that, for $(\mathrm{BDM})_r$ elements with $r \geqq 2$, we have only

$$u - u_h = O(h^r);$$

however,

$$u - u_h^* = O(h^{r+2}).$$

Hence we get a two-order accuracy improvement.

There are many special mixed finite-element methods for elasticity problems; we find that many of them fit into our framework.

1. PEERS. As an example, let us first take a look at the PEERS method introduced by Arnold, Brezzi, and Douglas [1]. In this method, following an idea from [9], the symmetry of the stress tensor is enforced through the introduction of a Lagrange multiplier for the rotation of the displacement. If we let u be a component of the displacement, then it is easy to see that ∇u can be expressed in terms of the stress and rotation variables. Correspondingly an approximation $(\nabla u)_h$ of ∇u can be deduced. Using the error estimates for the stress and rotation obtained in [1, Thm. 4.5], we easily obtain, with $\vec{f}$ being the imposed load,

$$\|\nabla u - (\nabla u)_h\|_0 \leqq Ch\|\vec{f}\|_0 \tag{28}$$

where $\|\cdot\|_0$ is the L^2-norm.

Also Theorem 4.5 of [1] states that

$$\|u_h - P_0 u\|_0 \leqq Ch^2\|\vec{f}\|_1 \tag{29}$$

where P_0 is the L^2-orthogonal projection from L^2 onto the piecewise constant space. Therefore we have all the desired ingredients and can construct u_h^* by (20) so that

$$\|u-u_h^*\|_0 \leqq Ch^2\|\vec{f}\|_1. \tag{30}$$

2. *A method of Stenberg.* Based on the same formulation for the elasticity problem as above, Stenberg [18] proposes some linear mixed finite-element spaces that are of higher order than PEERS. Roughly speaking, with a given triangulation $\mathcal{T}_h$ of a polygonal domain, the spaces for the displacement and rotation variables are the piecewise discontinuous linear polynomials, but the space for the stress tensor consists of some special class of linear functions defined on a further subdivision of the elements of $\mathcal{T}_h$. As before if u denotes a component of the displacement vector, an approximation $(\nabla u)_h^*$ can be obtained so that

$$\|\nabla u-(\nabla u)_h^*\|_0 = O(h^2), \tag{31}$$

$$\|P_1 u - u_h\|_0 = O(h^3) \tag{32}$$

where P_1 is an operator defined in [18]. It is interesting to mention here that P_1 is not the usual L^2-orthogonal projection, but that it does satisfy our assumptions. In particular, its action may be computed element by element.

However, we still cannot apply our result directly here since for each $K \in \mathcal{T}_h$, $(\nabla u)_h^*$ is not smooth on the whole K, but this difficulty can be easily overcome if we modify $(\nabla u)_h^*$ by

$$(\nabla u)_h = Q_1(\nabla u)_h^* \tag{33}$$

where Q_1 is the L^2-orthogonal projection onto the piecewise linear polynomials defined on the elements in $\mathcal{T}_h$. It is not hard to show that

$$\|\nabla u-(\nabla u)_h\|_0 = O(h^2). \tag{34}$$

Consequently the new approximation u_h^* satisfies

$$\|u-u_h^*\|_0 = O(h^3). \tag{35}$$

3. *A pseudostress formulation of Arnold and Falk.* In [3], by introducing the so-called pseudostress tensor, Arnold and Falk formulate the linear elasticity equations so that the standard $(\mathrm{RT})_k$ or $(\mathrm{BDM})_k$ mixed finite-element methods can be adopted directly to approximate the displacement and the pseudostress tensor (hence the gradient of the displacement and the ordinary stress tensor). Consequently this method fits into our framework in the same way as in the first example of this section.

4. *A method of Bramble and Xu.* By introducing the gradient of the velocity as a new variable, Bramble and Xu in [5] propose a new mixed formulation for the Dirichlet problem for the Stokes and elasticity equations. The spaces for the velocity and its gradient are just two copies of the usual $(\mathrm{RT})_k$ or $(\mathrm{BDM})_k$ pair of spaces for the scalar equations, while the pressure space is taken to be piecewise *continuous* polynomials. Because of the existing properties of $(\mathrm{RT})_k$ or $(\mathrm{BDM})_k$ elements, no other inf-sup conditions are required and the method is not restricted to the two-dimensional case. Still, if u is a component of the velocity variable, we can show that

$$\|\nabla u-(\nabla u)_h\|_0 = O(h^{r+1}), \tag{36}$$

$$\|P_m u - u_h\|_0 = O(h^{r+2}) \tag{37}$$

where m, r, and P_m are the same as in the first example of this section. Here, as before, we obtain

$$\|u-u_h^*\|_0 = O(h^{r+2})$$

where r is not of the lowest order.

Remark. Finally we remark that in all the methods above we have, by definition,

$$u-u_h = \vec{d}_{r,m} \cdot (\nabla u)_h + u - u_h^*.$$

Thus, since $u-u_h^*$ is asymptotically smaller than $u-u_h$, the term $\vec{d}_{r,m} \cdot (\nabla u)_h$ represents the leading term of the error $u-u_h$.

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