SIAM J. SCI. STAT. COMPUT.
Vol. 13, No. 1, pp. 397–410, January 1992

021

DOMAIN DECOMPOSITION METHODS FOR PROBLEMS WITH PARTIAL REFINEMENT*

JAMES H. BRAMBLE†, RICHARD E. EWING‡,
ROSSEN R. PARASHKEVOV‡, AND JOSEPH E. PASCIAK§

Abstract. In this paper, a flexible mesh refinement strategy for the approximation of solutions of elliptic boundary value problems is considered. The main purpose of the paper is the development of preconditioners for the resulting discrete system of algebraic equations. These techniques lead to efficient computational procedures in serial as well as parallel computing environments. The preconditioners are based on overlapping domain decomposition and involve solving (or preconditioning) subproblems on regular subregions. It is proven that the iteration schemes converge to the discrete solution at a rate which is independent of the mesh parameters in the case of two spatial dimensions. The estimates proved for the iterative convergence rate in three dimensions are somewhat weaker. The results of numerical experiments illustrating the theory are also presented.

Key words. second-order elliptic equation, domain decomposition, overlapping domain decomposition, local mesh refinement, partial refinement, overlapping Schwarz methods, preconditioners

AMS(MOS) subject classifications. 65N30, 65F10

1. Introduction.

To provide the required accuracy in many applications involving large scale scientific computation, it becomes necessary to use local mesh refinement techniques. These techniques allow the use of finer meshes in regions of the computational domain where the solution exhibits large gradients. This remains practical only if efficient techniques for the solution of the resulting discrete systems are available. It is the purpose of this paper to provide such techniques. We will give a flexible scheme for refinement as well as develop and analyze effective iterative methods for the solution of the resulting systems of discrete equations.

In this paper, we shall be interested in techniques for problems with refinements which are not quite local. As an example, one might consider a front passing through a two-dimensional domain. In this case, it might be necessary to refine in the neighborhood of the front.

There are a number of ways of developing preconditioned iterative schemes for the discrete systems resulting from local mesh refinement in the literature. Techniques based on nested multilevel spaces are given in [1], [7], [8], [12]. Techniques based on domain decomposition are given in [2], [10], [13], [14]. The analysis presented there implicitly depends on the shape of the the refinement domain, and hence the resulting algorithms may not be as effective with irregularly shaped refinement regions.

*Received by the editors April 5, 1990; accepted for publication (in revised form) October 5, 1990. This manuscript has been authored under contract number DE-AC02-76CH00016 with the U.S. Department of Energy. Accordingly, the U.S. Government retains a non-exclusive, royalty-free license to publish or reproduce the published form of this contribution, or allow others to do so, for U.S. Government purposes. This work was also supported in part under the National Science Foundation grant DMS84-05352 and by the U.S. Army Research Office through the Mathematical Science Institute, Cornell University. Additional supporters of this work include the Office of Naval Research under contract 0014-88-K-0370 and by the Institute for Scientific Computation at the University of Wyoming through National Science Foundation grant RII-8610680.

†Department of Mathematics, Cornell University, Ithaca, New York 14853.
‡Mathematics Department, University of Wyoming, Laramie, Wyoming 82071.
§Department of Applied Science, Brookhaven National Laboratory, Upton, New York 11973.

These algorithms also require the solution of a subproblem or preconditioner on the refinement regions. We shall provide alternative preconditioned iterative techniques for these problems based on overlapping domain decomposition. Our algorithms are simpler and possibly more effective when implemented since they often lead to preconditioning subproblems defined on either regular subregions or topologically "nice" meshes. The refinement region is the union of the subregions and may be irregularly shaped.

The proposed mesh refinement strategy is important in that it provides a basic approach for implementing dynamic local grid refinement. An example of a refinement strategy involves starting with a uniform coarse grid and refining in small subregions associated with a selected set of coarse-grid vertices. These subregions are allowed to overlap and there are no theoretical restrictions on the resulting refinement region (the union of the subregions). Dynamic refinement is achieved by simply dynamically changing the selected set of coarse-grid vertices.

In addition, the technique can be integrated into existing large scale simulators without a complete redesign of the code. This is because most of the computation involves tasks on either the global coarse grid or the refinement grids associated with the refinement subregions. Choosing the coarse and refinement grid structure to be that already used in the code saves considerable development costs. For example, if one uses regularly structured meshes in the coarse and refinement grids, a substantial part of the resulting algorithm only requires operations on regular grids even though the resulting final approximation space is not regular.

The outline of the remainder of the paper is as follows. In §2, we define some preliminaries and describe the second-order elliptic problems that will be considered. The overlapping domain decomposition algorithms for grids with partial refinement are given in §3. An analysis of the resulting preconditioned algorithms is given in §4. It is shown that the condition number of the preconditioned systems is bounded independently of the mesh parameters for many two-dimensional applications. The results for three dimensions are somewhat weaker and involve logarithms of the mesh parameters. Finally, the results of numerical experiments using these preconditioning techniques are given in §5.

2. The elliptic problem and preliminaries. We shall be concerned with the efficient solution of discrete equations resulting from approximation of second-order elliptic boundary value problems in a polygonal or polyhedral domain Ω contained in Euclidean space R^d, for $d = 2, 3$. We consider the problem of approximating the solution u of

$$\begin{aligned} Lu &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{2.1}$$

Here L is given by

$$Lv = -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i} a_{ij} \frac{\partial v}{\partial x_j},$$

and $\{a_{ij}(x)\}$ is a uniformly positive definite, bounded, piecewise smooth coefficient matrix on Ω. The corresponding bilinear form is denoted by $A(\cdot, \cdot)$ and is given by

$$A(v, w) = \sum_{i,j=1}^{d} \int_{\Omega} a_{ij} \frac{\partial v}{\partial x_i} \frac{\partial w}{\partial x_j} \, dx, \tag{2.2}$$

and is defined for functions $v,w \in H^1(\Omega)$. Here $H^1(\Omega)$ is the Sobolev space of order one on Ω. We denote the $L^2(\Omega)$ inner product by $(\cdot,\cdot)$. The weak solution u of (2.1) is the function $u \in H_0^1(\Omega)$ satisfying

$$A(u,\varphi) = (f,\varphi) \quad \text{for all } \varphi \in H_0^1(\Omega).$$

Here, $H_0^1(\Omega)$ is the subspace of functions in $H^1(\Omega)$ whose trace vanishes on $\partial\Omega$.

We consider the above model problem for convenience. Many extensions of the techniques to be presented are possible; for example, one could consider equations with lower-order terms and different boundary conditions.

In this paper, we shall deal with various domains. These domains will always be open. The closure of a domain θ will be denoted $\bar{\theta}$. In addition, we shall use various positive constants. These will be denoted by the character C, which will take on different values in different places. However, this constant shall always be independent of the mesh parameters in the approximation schemes.

3. The overlapping algorithms. In this section, we shall define iterative methods for problems with partial refinement based on overlapping domain decomposition. Our goal is to illustrate the technique and analysis and hence, for simplicity, we shall not attempt to provide the most general theorems. Many extensions are possible and can be inferred from the analysis presented.

The analysis given in the following section requires the application of techniques from both the theory of overlapping domain decomposition [9], [11] as well as the standard domain decomposition theory [4], [5]. We first give the setup in the two-dimensional case. We start with a coarse mesh $\cup\tau_H^i$ consisting of triangles of quasi-uniform size H. The associated finite element space M_0 is defined to be the set of those continuous piecewise linear functions on the coarse mesh that vanish on $\partial\Omega$. The mesh refinement is defined in terms of a number of coarse grid subdomains $\{\Omega_i\}$, for $i = 1,\cdots,K$. By convention, Ω_i is defined to be the interior of the union of the closures of the coarse grid triangles. The refinement regions will also be referred to as "the subdomains." We assume that they have limited overlap in that any point of Ω is contained in at most a fixed number (not depending on H) of the subdomains. We define the domain of refinement Ω^r to be the union of the subdomains, $\Omega^r = \cup_{1=1}^K \Omega_i$ There are no theoretical restrictions concerning the definition of the refinement subregions except that they are defined in terms of the coarse-grid triangles and satisfy the overlap property as described above.

We provide two examples of the construction in the two-dimensional case. For both examples, the subregions are associated with coarse-grid nodes. The interior and boundary nodes of this mesh will be denoted $\{x_i\}$, for $i = 1,\cdots,N_c$. For the first example, we define the region associated with a coarse-grid node x_i as the subdomain Ω_i that contains the coarse-grid triangles having x_i as a vertex. For the second example, we consider a mesh that is topologically equivalent to a regular rectangular mesh (see Fig. 3.1). In this case, we define Ω_i to be the four quadrilaterals that share the vertex x_i. Some reasons for such a choice will be explained later. In either case, an index set $I \subseteq [1,\cdots,N_c]$ is selected and only those subdomains $\{\Omega_i\}$ with $i \in I$ are used to define the refinement region. By possibly changing the numbering of the coarse-grid nodes, we assume, without loss of generality, that $I = 1,2,\cdots,K$. There are no additional restrictions concerning this set I and hence rather complex refinement regions are possible.

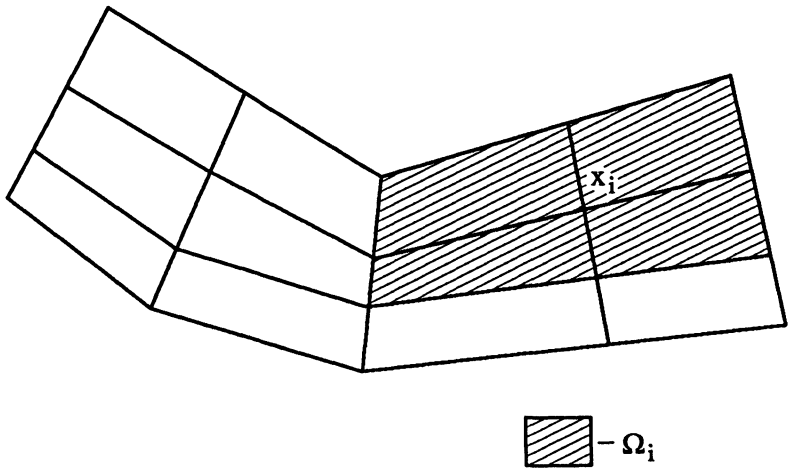

FIG. 3.1. *A distorted rectangular mesh.*

The composite space is defined in terms of a quasi-uniform mesh $\{\tau_h^i\}$ on Ω of size $h < H$ that satisfies

$$\cup_i \partial\tau_H^i \subseteq \cup_i \partial\tau_h^i.$$

The space of continuous piecewise linear functions with respect to this triangulation (which vanish on $\partial\Omega$) will be denoted by $\tilde{M}$. Note that this space is introduced for the construction and analysis of the composite grid space. It is not used in actual computation since it has too many degrees of freedom in Ω/Ω^r. The subspace M_i associated with the subdomain Ω_i is defined by

$$M_i = \{\phi \in \tilde{M} \,|\, \operatorname{supp}\phi \subseteq \Omega_i\}. \tag{3.1}$$

The composite finite element space is then defined to be

$$M = \sum_{i=0}^{K} M_i.$$

Note that the space M provides finer grid approximation in the refinement region Ω^r. An illustrative example of a mesh so generated is given in Fig. 3.2. The nodes on the boundary of the refinement region that are not coarse-grid nodes are slave nodes since, by continuity, the values of functions in M on these points are completely determined by their values on neighboring coarse-grid nodes. The operator $A_i : M_i \mapsto M_i$ is defined for $v \in M_i$ by

$$(A_i v, \phi) = A(v, \phi) \quad \text{for all } \phi \in M_i.$$

Our goal is to efficiently solve the composite grid problem: Given a function $f \in L^2(\Omega)$, find $U \in M$ satisfying

$$A(U, \phi) = (f, \phi) \quad \text{for all } \phi \in M. \tag{3.2}$$

As above, we define $A : M \mapsto M$ by

$$(Av, \phi) = A(v, \phi) \quad \text{for all } \phi \in M.$$

Problem (3.2) can then be rewritten as

$$AU = F, \tag{3.3}$$

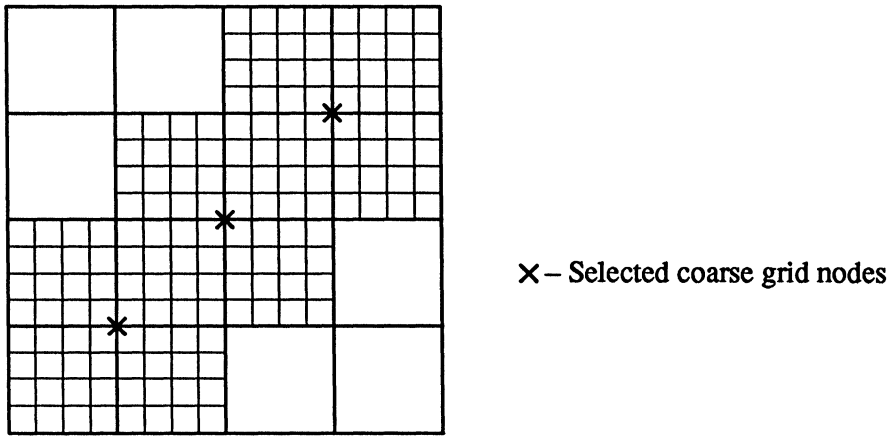

FIG. 3.2. *A composite grid.*

for appropriate $F \in M$. We will develop preconditioners for (3.3) by using overlapping domain decomposition.

There are basically two classes of these preconditioners, the additive and the multiplicative. The additive version defines the preconditioner B_a for A of (3.3) by

$$B_a = \sum_{i=0}^{K} R_i Q_i.$$

Here, Q_i denotes the $L^2(\Omega)$ projection operator onto M_i and R_i is a symmetric positive definite operator on M_i. Explicit choices for R_i will be discussed later; however, we note that it suffices to take R_i to be a preconditioner for A_i.

The multiplicative version is defined by applying the R_i consecutively. The multiplicative preconditioner B_m applied to a function $W \in M$ is defined as follows:

(1) Set $Y_0 = 0$.

(2) For $i = 1, \cdots, K+1$, define Y_i by

$$Y_i = Y_{i-1} + R_{i-1} Q_{i-1}(W - AY_{i-1}). \tag{3.4}$$

(3) For $i = K+2, \cdots, 2K+2$, define Y_i by

$$Y_i = Y_{i-1} + R_{2K+2-i} Q_{2K+2-i}(W - AY_{i-1}). \tag{3.5}$$

(4) Set $B_m W = Y_{2K+2}$.

It is not difficult to see that B_m is a symmetric linear operator on M.

The operators B_a and B_m defined above will be effective as preconditioners A if they satisfy the following:

(1) They are relatively inexpensive to evaluate.

(2) They lead to well-conditioned linear systems.

The first criterion involves implementation issues. The second criterion requires that the condition numbers $K(B_a A)$ and $K(B_m A)$ be small. In the case of the additive algorithms, this is equivalent to the existence of positive constants c_0, c_1 satisfying

$$c_0 A(v,v) \leq A(B_a A v, v) \leq c_1 A(v,v) \quad \text{for all } v \in M, \tag{3.6}$$

with c_1/c_0 small. A similar statement holds for the product algorithm. The goal of the analysis to be presented is to provide estimates for c_0 and c_1.

We note that the subdomains have limited overlap. This immediately implies that $c_1 \leq CN_0$, where C depends only on the preconditioning properties of R_i, and N_0 is the maximum number of subdomains overlapping any point $x \in \Omega$. Thus, to analyze the additive algorithm, we are left to estimate the size of c_0. This will be done in the following section.

To analyze the product algorithm, we apply Theorem 2.2 of [6]. Assume that R_i is scaled so that for a fixed $\omega \in (0, 2)$,

$$A(R_i A_i v, v) \leq \omega A(v, v) \quad \text{for all } v \in M_i. \tag{3.7}$$

Then the product operator defined above satisfies

$$c_2 A(v, v) \leq A(B_m A v, v) \leq A(v, v) \quad \text{for all } v \in M \tag{3.8}$$

where $c_2 \geq Cc_0$. Here c_0 is the constant in (3.6) and C is a positive constant which depends on N_0 and the preconditioning properties of R_i but not on h or H. Thus, to analyze the product algorithm, we are once again left to estimate the size of c_0 in (3.6).

Remark 3.1. It is easy to extend the above ideas to three-dimensional calculations. We consider the case where Ω is the union of rectangular parallelopipeds and the coarse-grid functions are piecewise trilinear on the rectangular parallelopipeds. The refinement subregions are defined to be the interior of the closures of coarse grid parallelopipeds. The composite mesh is defined in terms of a quasi-uniform mesh of parallelopipeds of size $h < H$. This mesh is assumed to be a refinement of the coarse-grid mesh. The space $\tilde{M}$ is defined to be the set of functions which are continuous on Ω, are trilinear with respect to the finer mesh, and vanish on $\partial\Omega$. The construction then proceeds exactly as described above for the two-dimensional case.

4. Convergence analysis. In this section, we provide an analysis for the overlapping domain decomposition preconditioners described in the previous section. As discussed earlier, we are to provide estimates for the constant c_0 appearing in (3.6). The analysis to be presented uses tools from both the theory of overlapping domain decomposition and the standard domain decomposition theory. We shall prove that under suitable hypotheses, the condition numbers $K(B_a A)$ and $K(B_m A)$ remain bounded independently of h and H in the two-dimensional case. The theorem for three dimensions guarantees that the condition numbers grow at most proportional to $(1 + \ln^2(H/h))$.

The first hypothesis for the theorems of this section provides control of the condition number $K(R_i A_i)$. Specifically, we assume that

$$C_0 A(w, w) \leq A(R_i A_i w, w) \leq \omega A(w, w) \quad \text{for all } w \in M_i, \tag{4.1}$$

where the constants C_0 and ω remain fixed independent of h and H. For the product algorithm, we also assume that $0 < \omega < 2$. We have the following theorem.

THEOREM 4.1. *Let $d = 2$ and assume that there are no isolated points on the boundary of Ω^r. Then the condition numbers $K(B_a A)$ and $K(B_m A)$ remain bounded independently of h, H and the choice of subdomains $\{\Omega_i\}$.*

Before proving Theorem 4.1, we review some results from the theory of overlapping and nonoverlapping domain decomposition. These results will play a major role in the subsequent proof.

Let P_i denote the elliptic projection into the subspace M_i, i.e., $P_i v = w$ where w is the unique function in M_i satisfying

$$A(w, \phi) = A(v, \phi) \quad \text{for all } \phi \in M_i.$$

It immediately follows from the definitions that $Q_i A = A_i P_i$ and hence (4.1) implies that, for $v \in M$,

$$\begin{aligned} A(BAv, v) &= \sum_{i=0}^{K} A(R_i Q_i A v, v) \\ &= \sum_{i=0}^{K} A(R_i A_i P_i v, P_i v) \geq C_0 \sum_{i=0}^{K} A(P_i v, v). \end{aligned}$$

Thus, $c_0 \geq \tilde{c}_0 C_0$ for any constant $\tilde{c}_0$ satisfying the inequality

$$\tilde{c}_0 A(v, v) \leq \sum_{i=0}^{K} A(P_i v, v) \quad \text{for all } v \in M. \tag{4.2}$$

It is known (cf. [11]) that (4.2) follows provided that $\tilde{c}_0$ is a constant such that for any $v \in M$ there is a decomposition $v = \sum_{i=0}^{K} v_i$, with $v_i \in M_i$, satisfying

$$\sum_{i=0}^{K} A(v_i, v_i) \leq \tilde{c}_0^{-1} A(v, v). \tag{4.3}$$

We remark that it is easy to prove that statements (4.2) and (4.3) are equivalent.

We shall require a known result concerning overlapping domain decomposition [9], [11]. For each coarse-grid node x_i, we let $\tilde{\Omega}_i$ denote the interior of the union of the closures of the coarse-grid triangles that have x_i as a vertex. We define $\tilde{M}_i$ in terms of $\tilde{\Omega}_i$ as in (3.1). Given $w \in \tilde{M}$, there exists a decomposition $w = \sum_{i=1}^{N_c} \tilde{w}_i$, with $\tilde{w}_i \in \tilde{M}_i$, satisfying

$$\sum_{i=1}^{N_c} A(\tilde{w}_i, \tilde{w}_i) \leq C \left(A(w, w) + H^{-2} \|w\|^2 \right). \tag{4.4}$$

Here C is a constant not depending on h or H. The functions $\{\tilde{w}_i\}$ are defined in terms of a partition of unity with respect to the subdomains $\{\tilde{\Omega}_i\}$. An explicit partition can be defined from the coarse-grid nodal basis functions. These decompositions preserve support in the sense that if w vanishes at a node then every $\tilde{w}_i$ vanishes there also.

We will also need results from the standard domain decomposition theory which we introduce as the following lemma. The proof of this lemma is essentially given in [4].

LEMMA 4.1. *Let $y \in M \cap \tilde{M}_i$ be discrete harmonic on each coarse grid triangle. By this we mean that*

$$A(y, \phi) = 0$$

for all functions $\phi \in M$ which vanish on the coarse-grid mesh. Assume that y vanishes on at least one coarse-grid edge connecting $\partial\tilde{\Omega}_i$ and x_i. Let $\{\tilde{\Gamma}_j\}$ denote the remaining coarse-grid edges connecting $\partial\tilde{\Omega}_i$ and x_i and define y_j to be the function which is

discrete harmonic on the coarse-grid triangles, equals y on $\tilde{\Gamma}_j$ and vanishes on the remaining coarse-grid edges. Then $y = \sum_j y_j$ and

$$\sum_j A(y_j, y_j) \leq CA(y, y). \tag{4.5}$$

Remark 4.1. If the function y in Lemma 4.1 vanishes only on the point x_i (instead of a line from $\partial\tilde{\Omega}_i$ to x_i), then the above decomposition is still defined. However, in such cases, (4.5) only holds with the constant C replaced by $c\ln^2(H/h)$.

Proof of Theorem 4.1. As discussed in the previous section, it suffices to estimate the constant c_0 in (3.6). This in turn follows from the construction of a decomposition $v = \sum_{i=0}^{K} v_i$ with $v_i \in M_i$ satisfying (4.3).

Let Q denote the $L^2(\Omega)$ projection operator onto the subspace M_0.

We note that

$$\|(I - Q)w\|^2 \leq CH^2 A(w, w) \tag{4.6}$$

and

$$A(Qw, Qw) \leq CA(w, w) \tag{4.7}$$

hold for all $w \in H_0^1(\Omega)$.

We define v_0 in terms of Q by

$$v_0(x_i) = \begin{cases} v(x_i) & \text{for coarse-grid nodes } x_i \notin \Omega^r, \\ Qv(x_i) & \text{for coarse-grid nodes } x_i \in \Omega^r. \end{cases}$$

Clearly, we have that

$$\begin{aligned} A(v - v_0, v - v_0) &= A_r(v - v_0, v - v_0) \\ &\leq 2[A((I - Q)v, (I - Q)v) + A_r(Qv - v_0, Qv - v_0)]. \end{aligned} \tag{4.8}$$

Here $A_r(\cdot, \cdot)$ is given by (2.2) but with integration only over the domain Ω^r. Note that $Qv - v_0$ is a function in M_0 which vanishes at all coarse-grid nodes in Ω^r and is equal to $Qv - v$ on the remaining coarse-grid nodes. Consequently,

$$A_r(Qv - v_0, Qv - v_0) \leq C \sum_{x_i \in \partial\Omega^r} |(Qv - v)(x_i)|^2. \tag{4.9}$$

Here, the sum is taken over coarse grid nodes $x_i \in \partial\Omega^r$. There are no isolated points on $\partial\Omega^r$ and hence for each $x_i \in \partial\Omega^r$, there is a coarse grid edge Γ_i contained in $\partial\Omega^r$ ending at x_i. Both functions Qv and v are linear on Γ_i and hence

$$\begin{aligned} |(Qv - v)(x_i)|^2 &\leq cH^{-1} \|(Qv - v)\|_{\Gamma_i}^2 \\ &\leq C[H^{-2} \|Qv - v\|_{\tau_H^i}^2 + A_{\tau_H^i}(Qv - v, Qv - v)]. \end{aligned} \tag{4.10}$$

Here τ_H^i denotes a coarse-grid triangle containing the edge Γ_i and $A_{\tau_H^i}$ denotes the form defined by (2.2) but with integration only over the region τ_H^i. The last inequality in (4.10) is a simple consequence of the divergence theorem and is well known. Combining (4.6)–(4.10) proves that

$$A(v - v_0, v - v_0) \leq CA(v, v). \tag{4.11}$$

A similar argument gives that

$$\|v - v_0\|^2 \leq CH^2 A(v, v). \tag{4.12}$$

We next apply the overlapping domain decomposition result to $w = v - v_0$. Specifically, we decompose $w = \sum_{i=1}^{N_c} \tilde{w}_i$, with $\tilde{w}_i \in \tilde{M}_i$ and satisfying (4.4). Note that this is clearly not the desired decomposition into the refinement subspaces $\{M_i\}$. We will distribute the functions $\tilde{w}_j$, $j = 1, \cdots, N_c$, into these subspaces. We start assigning each coarse grid-node $x_j \in \Omega^r$ to a unique subdomain $\Omega_{J(j)}$ which contains x_j. We then define

$$w_i = \sum_{J(j)=i} \tilde{w}_j.$$

We need to decompose the remaining functions $\tilde{w}_i$ for $x_i \notin \Omega^r$. Note that by the support property, $\tilde{w}_i$ vanishes unless $x_i \in \partial\Omega^r$. Consider a fixed function $\tilde{w}_i$ with $x_i \in \partial\Omega^r$. We write $\tilde{w}_i = y + y_i^0$ where $y = \tilde{w}_i$ on the boundaries of the coarse-grid triangulation and is discrete harmonic on the coarse-grid triangles (as in Lemma 4.1). The function y_i^0 vanishes on the boundaries of the coarse-grid triangulation and is orthogonal (in $A(\cdot,\cdot)$) to y. Note that the function y_i^0 is nonzero only on triangles contained in the refinement region. Thus, we can assign each of these triangles uniquely to a subdomain Ω_j and add the restriction of y_i^0 to the corresponding function w_j. The result of these modifications will still be denoted $\{w_j\}$.

We finally distribute the function y. There are no isolated points in $\partial\Omega^r$ and hence there must be a coarse-grid edge ending at x_i contained in $\partial\Omega^r$. Note, in addition, that both $\tilde{w}_i$ and y vanish on this edge. Thus, by Lemma 4.1,

$$\sum_j A(y_j, y_j) \leq C A(y, y). \tag{4.13}$$

The functions y_j are defined in Lemma 4.1 and the sum over j is taken over the coarse-grid indices corresponding to coarse-grid neighbors of x_i in Ω^r. The functions y_j are assigned to subdomains Ω_k which contain the corresponding edge (where y_j is nonzero) and the functions y_j are added into the corresponding w_k, producing a result which is still denoted w_k. It follows immediately that $w = \sum_{i=1}^{K} w_i$ and

$$\sum_{i=1}^{K} A(w_i, w_i) \leq C(A(w,w) + H^{-2} \|w\|^2). \tag{4.14}$$

Combining (4.11), (4.12), and (4.14) shows that the decomposition $v = \sum_{i=0}^{K} v_i$ with $v_i = w_i$ for $i = 1, \cdots, K$ satisfies (4.3). This completes the proof of the theorem. □

Remark 4.2. The hypothesis concerning isolated points on the boundary of Ω^r is included to provide a uniform bound for c_0. It is possible to show (using of [4], Thm. 1) that the constant c_0 only deteriorates like $C/\ln^2(H/h)$ if the isolated point hypothesis is not satisfied. This sort of decay is actually seen in the last numerical example in §5 where this assumption is violated.

Remark 4.3. There is very little restriction concerning the way that the domains Ω_i are defined. Note that if only one refinement domain is used, then Theorem 4.1 provides a result for the imbedded space case proposed in [2]. Alternatively, one can consider the case where Ω^r is all of Ω and hence $M = \tilde{M}$. In this case, Theorem 4.1 guarantees uniform bounds for the condition numbers without putting restrictions on the shapes of the subdomains $\{\Omega_i\}$. Thus, for example, the subdomains can be taken to be strips as long as the coarse problem is included.

Remark 4.4. There are numerous ways of modifying the above algorithm. One possibility is to include additional subspaces corresponding to the coarse-grid nodes on $\partial\Omega^r$. For such a node x_k, the space M_k would be defined to be the functions in M^r

with support in Ω_k. The same result holds with a somewhat simpler analysis since the standard domain decomposition theory is avoided. However, this algorithm has some practical disadvantages. There are more subproblems and many of them correspond to grids on irregularly shaped domains.

We next provide the result for three-dimensional applications.

THEOREM 4.2. *Let Ω be a domain in R^3 and let the mesh and approximation space be as discussed in Remark* 3.1. *Assume that the hypotheses preceding Theorem* 4.1 *hold and that Ω^r is the interior of its closure. Then*

$$K(B_a A) \leq C(1 + \ln^2(H/h))$$

and

$$K(B_m A) \leq C(1 + \ln^2(H/h)).$$

The constant C above does not depend on H or h.

Proof. The major part of the proof follows the proof of Theorem 4.1. We seek a decomposition of $v \in M$ satisfying (4.3) with $\tilde{c}_0^{-1} \leq C(1+\ln^2(H/h))$. The construction of v_0 is exactly the same as in Theorem 4.1 and still satisfies (4.11) and (4.12). Here we used the assumption that Ω^r was the interior of its closure.

The overlapping domain decomposition $w = \sum_{i=1}^{N_c} \tilde{w}_i$ satisfying (4.4) is also valid in three dimensions. Once again we reduce to the problem of decomposing functions $\tilde{w}_k$ corresponding to coarse-grid nodes $x_k \in \partial\Omega^r$. As in the proof of Theorem 4.1, we write $\tilde{w}_k = y + y_k^0$, where y is discrete harmonic (with respect to the refined mesh) in the interior of the coarse parallelopipeds. The y_k^0 part is added into $\{w_i\}_{i=1}^K$.

Finally, we must take care of the function y. We write

$$y = \sum \bar{y}_{ij} + \sum \tilde{y}_l, \tag{4.15}$$

where:

(1) The functions $\{\bar{y}_{ij}\}$ are discrete harmonic (with respect to the refined mesh) in the interior of the coarse parallelopipeds.

(2) $\bar{y}_{ij} = y$ on the interior nodes on the face between coarse regions τ_H^i and τ_H^j and vanishes on the remaining nodes of $\cup\partial\tau_H^l$.

(3) $\tilde{y}_l$ equals y on the nodes of an edge of $\{\tau_H^l\}$ which is in $\tilde{\Omega}_k \cap \Omega_l$ and vanishes on all of the remaining nodes of the composite grid.

(4) The sums in (4.15) are taken over the appropriate faces and edges.

Applying Lemma 4.3 of [5] gives that

$$A(\bar{y}_{ij}, \bar{y}_{ij}) \leq C(1 + \ln^2(H/h))\{|\tilde{w}_k|^2_{1/2,\partial\tau_H^i} + H^{-1}\,|\tilde{w}_k|^2_{\tau_H^i}\}.$$

Here $|\cdot|_{1/2,\partial\tau_H^i}$ denotes the Sobolev seminorm of order $1/2$ on $\partial\tau_H^i$. We clearly have

$$A(\bar{y}_{ij}, \bar{y}_{ij}) \leq C(1 + \ln^2(H/h))(A(w_k, w_k) + H^{-2}\,\|w_k\|^2).$$

Similar arguments using Lemmas 4.1–4.2 of [5] give

$$A(\tilde{y}_l, \tilde{y}_l) \leq C(1 + \ln^2(H/h))(A(w_k, w_k) + H^{-2}\,\|w_k\|^2).$$

The desired bound for $\tilde{c}_0^{-1}$ follows as in the proof of Theorem 4.1.

5. Numerical results. In this section, we provide the results of numerical examples illustrating the theory developed earlier. We shall consider the model problem

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{5.1}$$

where Δ denotes the Laplacian and Ω is the unit square $[0,1] \times [0,1]$. To define the coarse mesh, the domain Ω is first partitioned into $m \times m$ square subdomains of side length $H = 1/m$. Each smaller square is then divided into two triangles by one of the diagonals (e.g., the diagonal which goes from the bottom left to the upper right-hand corner of the square). The coarse-grid approximation space M_0 is defined to be the set of functions which are continuous on Ω, are piecewise linear with respect to the triangulation, and vanish on $\partial\Omega$. The space $\tilde{M}$ is defined from a similar finer mesh of size $h = H/l$ for some integer $l > 1$.

For our first two examples, we consider an application where it is required to refine along the diagonal connecting the origin with the point $(1,1)$. Such a refinement might be necessary if the function f has large gradients near this diagonal but is well behaved in the remainder of Ω. Accordingly, we select the coarse-grid nodes on the diagonal for refinement. We define the refinement region associated with a refinement node to be the four coarse mesh squares which have that node as a corner.

Note that the refinement region is highly irregular even though the coarse problem and the refinement subproblems involve regular rectangular meshes.

We will illustrate the rate of convergence of preconditioned algorithms for solving (3.2) where $A(\cdot,\cdot)$ is given by the Dirichlet form. To do this, we shall numerically compute the largest and smallest eigenvalue (λ_1 and λ_0, respectively) of the preconditioned operator $B_a A$. As is well known, the rate of convergence of the resulting preconditioned algorithms can be bounded in terms of the condition number $K(B_a A) = \lambda_1/\lambda_0$. We shall not report results for preconditioning with the product operator B_m, although our previous experience [6] suggests that the product version will converge somewhat faster than the additive.

Table 5.1 gives the largest and smallest eigenvalue and the condition number of the system $B_a A$ as a function of h. In this example, we took $R_i = A_i^{-1}$; i.e., we solved exactly on the subspaces $\{M_i\}$. For Table 5.1, $m = 4$, and there are three refinement subdomains $(0,1/2) \times (0,1/2)$, $(1/4,3/4) \times (1/4,3/4)$, and $(1/2,1) \times (1/2,1)$. Note that both the upper and lower eigenvalues appear to be tending to a limit as the ratio $h/H \mapsto 0$. Similar behavior is seen in Table 5.2, which corresponds to $m = 8$ and uses seven smaller refinement subregions.

TABLE 5.1
Condition numbers for 3 overlapping subregions.

h	λ_1	λ_0	$K(B_a A)$
1/8	2.44	0.50	4.9
1/16	2.50	0.41	6.1
1/32	2.51	0.38	6.6
1/64	2.52	0.36	6.9
1/128	2.52	0.35	7.1

In almost all realistic applications, the direct solution of subproblems is much more expensive than the evaluation of a suitable preconditioner. To illustrate the effect on the convergence rate of the preconditioned iteration, we next consider the previous example but with the direct solves on the subspaces replaced by multigrid preconditioners. Specifically, we employ the V-cycle multigrid algorithm (cf. [3]) using one pre- and post-smoothing Jacobi iteration on each grid level. This leads to

TABLE 5.2
Condition numbers for 7 overlapping subregions.

h	λ_1	λ_0	$K(B_aA)$
1/16	2.46	0.47	5.2
1/32	2.52	0.39	6.5
1/64	2.54	0.35	7.2
1/128	2.54	0.34	7.5

a preconditioning operator $R_i : M_i \mapsto M_i$, which satisfies

$$0.4A(v,v) \leq A(R_iA_iv,v) \leq A(v,v) \quad \text{for all } v \in M_i. \tag{5.2}$$

The constant 0.4 above was computed numerically and holds for all of the subspace problems that are required for this application, including M_0.

Tables 5.3 and 5.4 provide the eigenvalues and condition numbers for the above examples when direct solves were replaced by multigrid preconditioners. Note that in all of the reported runs, the condition number with multigrid preconditioners was at most 5/4 times as large as that corresponding to exact solves. Such an increase in condition number is negligible in a preconditioned iteration. In contrast, the computational time required for the multigrid sweep is considerably less than that needed for a direct solve (especially in more general problems with variable coefficients).

TABLE 5.3
Preconditioned subproblems, 3 overlapping subregions.

h	λ_1	λ_0	$K(B_aA)$
1/8	2.37	0.53	4.5
1/16	2.12	0.33	6.4
1/32	2.07	0.27	7.6
1/64	2.04	0.25	8.2
1/128	2.02	0.24	8.4

TABLE 5.4
Preconditioned subproblems, 7 overlapping subregions.

h	λ_1	λ_0	$K(B_aA)$
1/16	2.36	0.40	5.9
1/32	2.11	0.28	7.5
1/64	2.06	0.24	8.8
1/128	2.03	0.22	9.4

As a final example, we consider a case where the isolated point hypothesis of Theorem 4.1 is not satisfied. Specifically, we consider a coarse mesh of size $H = 1/4$ and select the four nodes with (x,y) values (1/4,1/2), (3/4,1/2), (1/2,1/4), and

(1/2,3/4). The refinement region is everything but the subsquares $[0,1/4] \times [0,1/4]$, $[0,1/4] \times [3/4,1]$, $[3/4,1] \times [0,1/4]$, and $[3/4,1] \times [3/4,1]$. Note that, to satisfy the hypotheses of the theorem, it would be necessary to include a refinement region centered at the coarse-grid node $(1/2,1/2)$. Table 5.5 gives the smallest eigenvalue for the operator $B_a A$ as a function of h. The function $(.32 + .36 \log_2(h^{-1}))^{-2}$ is also provided for comparison. These results suggest that smallest eigenvalue λ_0 decays as predicted by the theoretical bound $C/\ln(H/h)^2$ (see Remark 4.2).

TABLE 5.5
A "bad" example in two dimensions.

h	λ_0	$(.32 + .36 \log_2(h^{-1}))^{-2}$
1/8	.50	.51
1/16	.32	.32
1/32	.22	.22
1/64	.16	.16
1/128	.12	.12

REFERENCES

[1] R. E. BANK, T. DUPONT, AND H. YSERENTANT, *The hierarchical basis multigrid method*, Numer. Math., 52 (1988), pp. 427–458.

[2] J. H. BRAMBLE, R. E. EWING, J. E. PASCIAK, AND A. H. SCHATZ, *A preconditioning technique for the efficient solution of problems with local grid refinement*, Comput. Methods Appl. Mech. Engrg., 67 (1988), pp. 149–159.

[3] J. H. BRAMBLE AND J. E. PASCIAK, *New convergence estimates for multigrid algorithms*, Math. Comp., 49 (1987), pp. 311–329.

[4] J. H. BRAMBLE, J. E. PASCIAK, AND A. H. SCHATZ, *The construction of preconditioners for elliptic problems by substructuring*, I, Math. Comp., 47 (1986), pp. 103–134.

[5] ———, *The construction of preconditioners for elliptic problems by substructuring*, IV, Math. Comp., 53 (1989), pp. 1–24.

[6] J. H. BRAMBLE, J. E. PASCIAK, J. WANG, AND J. XU, *Convergence estimates for product iterative methods with applications to domain decomposition*, Math. Comp., 57 (1991), pp. 1–21.

[7] ———, *Convergence estimates for multigrid algorithms without regularity assumptions*, Math. Comp., 57 (1991), pp. 23–45.

[8] J. H. BRAMBLE, J. E. PASCIAK, AND J. XU, *Parallel multilevel preconditioners*, Math. Comp., 55 (1990), pp. 1–22.

[9] M. DRYJA AND O. WIDLUND, *Some domain decomposition algorithms for elliptic problems*, Iterative Methods for Large Linear Systems, L. Hayes and D. Kincaid, eds., Academic Press, New York, 1989.

[10] R.E. EWING, R.D. LAZAROV, AND P.S. VASSILEVSKI, *Local refinement techniques for elliptic problems on cell-centered grids*, II: *Two-grid iterative methods*, Math. Comp., submitted.

[11] P.L. LIONS, *On the Schwarz alternating method*, in Proceedings of the First International Symposium on Domain Decomposition Methods for Partial Differential Equations, R. Glowinski, G. H. Golub, G. A. Meurant, and J. Périaux, eds., Society for Industrial Applied Mathematics, Philadelphia, PA, 1988.

[12] S.F. MCCORMICK, *Multilevel Adaptive Methods for Partial Differential Equations*, Society for Industrial Applied Mathematics, Philadelphia, PA, 1989.

[13] S. MCCORMICK AND J. THOMAS, *The fast adaptive composite grid* (FAC) *method for elliptic equations*, Math. Comp., 46 (1986), pp. 439–456.

[14] O. WIDLUND, *Optimal iterative refinement methods*, Proceedings of the Second International Symposium on Domain Decomposition Methods, T. F. Chan, R. Glowinski, J. Périaux and O.B. Widlund, eds., Society for Industrial and Applied Mathematics, Philadelphia, PA, 1989, pp. 114–125.