APPROXIMATION OF STEKLOV EIGENVALUES OF NON-SELFADJOINT SECOND ORDER ELLIPTIC OPERATORS*

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1. Introduction.

In a recent paper [5] the authors develop a general approximation theory for the eigenvalues and corresponding subspaces of generalized eigenvectors for a certain class of compact operators. Because of the applications treated there the general theory was only presented for the case in which the operators were considered on Sobolev spaces $H^s(\Omega)$ with Ω a bounded domain in R^N with boundary $\partial\Omega$.

In this paper we study the Galerkin method for the approximation of Steklov eigenvalues of a strongly elliptic, non-selfadjoint, second order differential operator L. The Steklov eigenvalues are those complex numbers λ such that for some non-zero u,

$$Lu = 0 \quad \text{in } \Omega$$

and

$$\frac{\partial u}{\partial \nu} = \lambda u \quad \text{on} \quad \partial\Omega,$$

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where $\frac{\partial}{\partial\nu}$ is the conormal derivative (cf. [2]).

For this purpose we restate the main theorems of [5] in a more general setting. Since the proofs may be generalized without effort they will be omitted.

Let λ be a Steklov eigenvalue with algebraic multiplicity m. There will be m Galerkin eigenvalues, $\lambda_1(h),\ldots,\lambda_m(h)$, which converge to λ. Our main result, Theorem 3.2, gives a rate of convergence estimate for the convergence of

$$\Big(\frac{1}{m}\sum_{j=1}^{m} 1/\lambda_j(h)\Big)^{-1}$$

to λ. This estimate depends on the approximability properties of the space of trial functions and gives the same rate as that which can be established in the self-adjoint case by the methods of Birkhoff, de Boor, Swartz, and Wendroff [3].

The applications studied in [5] all involved a compact operator T which was approximated by a family of operators $\{T_h\}$, each T_h being of the form P_hT, where P_h is a projection onto a finite dimensional space. Again, in the Galerkin approximation for the Steklov eigenvalue problem, there arises in a natural way a compact operator T and a family of approximating operators $\{T_h\}$ but, as will be seen in the last section, T_h is not in general of the form P_hT, where P_h is a projection. This is of interest in that much work on the approximation of eigenvalues makes strong use of the assumption that $T_h = P_hT$ (at least on the range of T_h) for some P_h. In particular, Vaĭnikko [11] obtains results similar to our Theorems 3.3 and 3.4 under such an assumption.

2. Preliminaries.

Let $\{H^s\}_{0\leq s<+\infty}$ be a family of Hilbert spaces, with norms $\|\cdot\|_{H^s}$, with the properties

a) H^s is a dense subset of H^0 for all $s>0$,

b) if $0 \leq s_1 < s_2$ then $H^{s_2} \subset H^{s_1}$ and there is a constant C_{s_1,s_2} such that

$$\|\phi\|_{H^{s_1}} \leq C_{s_1,s_2} \|\phi\|_{H^{s_2}}$$

for all $\phi \in H^{s_2}$

and

c) if $0 \leq s_1 < s_2$ then $I_{s_1,s_2}:H^{s_2} \to H^{s_1}$, the injection of H^{s_2} into H^{s_1}, is compact.

Let the inner product in H^0 be denoted by $<\cdot,\cdot>$. For $\psi \in H^0$ and $s < 0$ we define

$$\|\psi\|_{H^s} = \sup_{\phi\in H^{-s}} \frac{|<\psi,\phi>|}{\|\phi\|_{H^{-s}}} .$$

Then we define H^s, for $s < 0$, as the completion of H^0 with respect to this norm (cf. [1]). Each H^s is a Hilbert space. Now we have a family $\{H^s\}_{-\infty<s<+\infty}$ of

Hilbert spaces with the properties

d) if $-\infty < s_1 < s_2 < +\infty$ then $H^{s_2} \subset H^{s_1}$ and

$$\|\phi\|_{H^{s_1}} \leq C_{s_1,s_2} \|\phi\|_{H^{s_2}}$$

for all $\phi \in H^{s_2}$

and

e) if $s_1 < s_2$ then the injection $I_{s_1,s_2} : H^{s_2} \to H^{s_1}$ is compact.

For any $A : H^s \to H^s$ we define, for $s_1 < s < s_2$,

$$\|A\|_{s_1,s_2} = \sup_{\psi \in H^{s_2}} \frac{\|A\psi\|_{H^{s_1}}}{\|\psi\|_{H^{s_2}}} . \tag{2.1}$$

If A is compact then its spectrum consists of a denumerable set of non-zero complex numbers together with zero. Each non-zero μ in the spectrum of A is an eigenvalue; zero may or may not be an eigenvalue.

Let μ be a non-zero eigenvalue of A. The least integer α such that $N((\mu-A)^{\alpha}) = N((\mu-A)^{\alpha+1})$, where N denotes the null space, is called the ascent of $\mu-A$. If A is compact α is finite. The integer $m = \dim N((\mu-A)^{\alpha})$ is called the algebraic multiplicity of μ and is also finite. The vectors in $N((\mu-A)^{\alpha})$ are called generalized eigenvectors of A corresponding to μ. The geometric multiplicity is equal to $\dim N(\mu-A)$ and is less than or equal to the algebraic multiplicity. The two multiplicities are equal if A is self-adjoint.

Let $T^0 : H^0 \to H^0$ be compact and $\{T_h\}_{0<h\leq 1}$ be a family of compact operators $T_h : H^0 \to H^0$ such that

$$\lim_{h\to 0} \|T^0 - T_h\|_{0,0} = 0 .$$

We will need the following well-known eigenvalue convergence result (c.f. [6]). If $\mu^1, \mu^2, \ldots$ are the non-zero eigenvalues of T^0 ordered by decreasing magnitude, taking account of algebraic multiplicities, then for each h there is an ordering (again counting according to algebraic multiplicites) of the eigenvalues of $T_h, \mu^1(h), \mu^2(h), \ldots$, such that $\lim_{h\to 0} \mu^j(h) = \mu^j$ for each j. For our purpose this can be restated as follows. Let μ be a non-zero eigenvalue of T^0 with algebraic multiplicity m. Then exactly m eigenvalues (counting according to algebraic multiplicities) of T_h converge to μ. We denote these eigenvalues by $\mu_1(h), \ldots, \mu_m(h)$. We will denote the space of generalized eigenvectors of T^0 corresponding to μ by $M(\mu)$ and the direct sum of the spaces of generalized eigenvectors of T_h corresponding to $\mu_1(h), \ldots, \mu_m(h)$ by $M_h(\mu)$. For h sufficiently small, $\dim M_h(\mu) = \dim M(\mu)$.

3. Convergence estimates.

In this section we consider a particular type of compact operator T^0 and a family of compact operators $\{T_h\}_{0<h\leq 1}$ which approximates it. We obtain estimates for the rate of convergence of the generalized eigenvectors and

eigenvalues of T_h to the generalized eigenvectors and eigenvalues of T^0, respectively.

Let s_0 be a fixed non-negative real number and let T be an operator from H^{-s_0} to H^{-s_0}. We suppose that there is an $\varepsilon > 0$ such that

$$T: H^s \to H^{s+\varepsilon}$$

as a bounded operator for all $s \geq -s_0$. For $s \geq -s_0$ denote by T^s the restriction of T to H^s considered as a mapping from H^s to H^s. Then, since $I_{s+\varepsilon,s}$ is compact, we see that T^s is compact. Now we shall be interested in the eigenvalues and corresponding spaces of generalized eigenvectors of T^0. Let $\{T_h\}_{0<h\leq 1}$ be a family of compact operators from H^0 to H^0 satisfying

$$\lim_{h\to 0} \|T^0 - T_h\|_{0,0} = 0 .$$

Let μ be a fixed non-zero eigenvalue of T^0 with algebraic multiplicity m. From Section 2 we see that exactly m eigenvalues of T_h, $\mu_1(h),\ldots,\mu_m(h)$, converge to μ as $h \to 0$.

The following results, Theorem 3.1 and its corollary, and Theorems 3.2, 3.3 and 3.4, are direct generalizations of those given by the authors in [5]. There these results were stated and proved for the case $H^0 = L_2(\Omega)$, where Ω is a bounded domain in R^N, and H^s was taken to be a family of Sobolev spaces. The proofs of [5] remain unchanged in this more general setting and, as mentioned above, will not be repeated here.

We can now state our first result. Throughout this section orthonormal will mean orthonormal in H^0 .

Theorem 3.1. Let $\{u_j\}_{j=1}^m$ be an orthonormal basis for $M(\mu)$ and let $0 \leq s \leq s_0$, $0 \leq s_1$. Then there exist constants C_s and h_1 and an orthonormal basis $\{w_j\}_{j=1}^m$ for $M_h(\mu)$ such that

$$(3.1) \qquad \max_{1\leq j\leq m} \|u_j - w_j\|_{H^{-s}} \leq C_s \Big\{ \|T^0 - T_h\|_{-s,s_1} + \|T^0 - T_h\|_{-s,0} \|T^0 - T_h\|_{0,s_1} + \|T^0 - T_h\|_{0,s_1}^2 \Big\}$$

for all $0 \leq h \leq h_1$.

We may also start with an orthonormal basis in $M_h(\mu)$ and construct a close orthonormal basis for $M(\mu)$. This is made precise in the following:

Corollary. For each h with $0 \leq h \leq h_1$ let $\{w_j\}_{j=1}^m$ be an orthonormal basis for $M_h(\mu)$. Then there is an orthonormal basis $\{u_j\}_{j=1}^m$ for $M(\mu)$ such that

$$\max_{1\leq j\leq m} \|u_j - w_j\|_{H^{-s}} \leq C_s \Big\{ \|T^0 - T_h\|_{-s,s_1} + \|T^0 - T_h\|_{-s,0} \|T^0 - T_h\|_{0,s_1} + \|T^0 - T_h\|_{0,s_1}^2 \Big\}$$

holds for all $0 \leq s \leq s_0$, where C_s is a constant which does not depend on h .

Theorem 1 and its corollary show that the H^s-gap between $M(\mu)$ and $M_h(\mu)$, $\hat{\delta}(M(\mu) , M_h(\mu))$, is estimated by the right-hand side of (3.1) (cf. [7]).

As before let $\mu_1(h),\ldots,\mu_m(h)$ be the eigenvalues of T_h which converge to μ (counted according to algebraic multiplicity). Although each of these eigenvalues is close to μ for small h, their arithmetic mean is generally a closer approximation. Thus we define

$$\hat{\mu}(h) = \frac{1}{m} \sum_{j=1}^{m} \mu_j(h) .$$

In the terminology of [7] this is the weighted mean of the μ-group. Our next theorem gives an estimate for $\mu - \hat{\mu}(h)$.

<u>Theorem 3.2</u>. Let $s_1 \geq 0$. Then there exist constants C and h_1 such that for $0 < h \leq h_1$,

$$|\mu-\hat{\mu}(h)| \leq C\Big\{\|T^0 - T_h\|_{-s_0,s_1} + \|T^0 - T_h\|_{-s_0,0}\|T^0 - T_h\|_{0,s_1} + \|T^0 - T_h\|^2_{0,s_1}\Big\} .$$

Next we have an estimate for $\mu - \mu_k(h)$ for each k.

<u>Theorem 3.3</u>. Let α be the ascent of $\mu - T^0$ and let $s_1 \geq 0$. Then there are constants C and h_1 such that for $0 < h \leq h_1$,

$$\max_{1\leq k\leq m} |\mu-\mu_k(h)|^\alpha \leq C\Big\{\|T^0 - T_h\|_{-s_0,s_1} + \|T^0 - T_h\|_{-s_0,0}\|T^0-T_h\|_{0,s_1} + \|T^0 - T_h\|^2_{0,s_1}\Big\} .$$

Our final result in this section gives estimates for the rate of convergence of certain elements of $M_h(\mu)$ to certain elements of $M(\mu)$.

<u>Theorem 3.4</u>. Let $\mu(h)$ be an eigenvalue of T_h such that $\mu(h)\to\mu$ as $h\to 0$ and let $s_1 \geq 0$. Suppose, for each h, w is a unit vector satisfying $\left(\mu(h)-T_h\right)^k w = 0$ for some positive integer $k \leq \alpha$. Then for any integer ℓ with $k \leq \ell \leq \alpha$ there is a unit vector $u \in M(\mu)$ such that $(\mu-T^0)^\ell u = 0$ and

$$\|u-w\|_{H^0} \leq C\|T^0 - T_h\|_{0,s_1}^{(\ell-k+1)/\alpha} . \tag{3.2}$$

Note that in the special case $k = \ell = 1$, (3.2) shows that, for small h, eigenvectors of T_h will be close to eigenvectors of T^0.

4. <u>The Steklov eigenvalue problem</u>.

Let Ω be a bounded domain in N-dimensional Euclidean space with boundary $\partial\Omega$ which we will assume (for convenience) to be of class C^∞. Let $C^\infty(\bar{\Omega})$ be the class of infinitely differentiable complex-valued functions defined on $\bar{\Omega}$, the closure of Ω. For $\phi,\psi \in C^\infty(\bar{\Omega})$ we define the L_2-inner product by

$$(\phi,\psi) = \int_\Omega \phi\, \bar{\psi}\, dx$$

and the corresponding norm by

$$\|\phi\|_0 = \left(\int_\Omega |\phi|^2\, dx\right)^{1/2} .$$

For $\phi \in C^\infty(\bar{\Omega})$ and j a non-negative integer we define the norm

$$\|\phi\|_j = \Big(\sum_{|\alpha| \leq j} \|D^\alpha \phi\|_0^2 \Big)^{1/2}$$

where $\alpha = (\alpha_1, \ldots, \alpha_N)$ is a multi-index,

$$D^\alpha = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_N}}{\partial x_N^{\alpha_N}} ,$$

and $|\alpha| = \alpha_1 + \ldots + \alpha_N$. The completion of $C^\infty(\bar{\Omega})$ with respect to $\|\cdot\|_j$ will be denoted by $H^j(\Omega)$. Note that $H^0(\Omega) = L_2(\Omega)$. This is the Sobolev space of order j and is a Hilbert space. For any positive real s we define $H^s(\Omega)$ by interpolation between successive integers following Definition 2.1 of [8]. For $\phi \in L_2(\Omega)$ and $s < 0$ define the norm

$$\|\phi\|_s = \sup_{\psi \in H^{-s}(\Omega)} \frac{|(\phi,\psi)|}{\|\psi\|_{-s}} .$$

$H^s(\Omega)$, for $s < 0$, is defined as the completion of $L_2(\Omega)$ with respect to $\|\cdot\|_s$ (cf. [1]). Now $H^s(\Omega)$ is a Hilbert space for each real s .

We shall also need the spaces $H^s(\partial\Omega)$, the Sobolev spaces of order s on the boundary. They can be defined as follows. Let $\Delta_{\partial\Omega}$ denote the Laplace-Beltrami operator on $\partial\Omega$. This operator has an increasing sequence of positive eigenvalues $\{\lambda_j\}$ which tend to infinity and corresponding eigenfunctions $\{w_j\}$ with $w_j \in C^\infty(\partial\Omega)$. The w_j are orthonormal with respect to the

L_2 -inner product on $\partial\Omega$,

$$<\phi,\psi> = \int_{\partial\Omega} \phi\bar{\psi}d\sigma \ .$$

Here σ is the surface measure. We use this notation since H^0 will be taken in the sequel to be $L_2(\partial\Omega)$. For any real s , $H^s(\partial\Omega)$ is defined as the completion of $C^\infty(\partial\Omega)$, the space of infinitely differentiable complex-valued functions on $\partial\Omega$, with respect to the norm

$$|u|_s = \left(\sum_{j=1}^{\infty} \lambda_j^s|<u,w_j>|^2\right)^{1/2} .$$

$H^s(\partial\Omega)$ is a Hilbert space for each real s . Note that $H^0(\partial\Omega) = L_2(\partial\Omega)$. Also for $s < 0$ we have

$$(4.1) \qquad |u|_s = \sup_{\phi\in H^{-s}(\partial\Omega)} \frac{|<u,\phi>|}{|\phi|_{-s}} .$$

It is clear that the spaces $H^s(\partial\Omega)$, $s \geq 0$, satisfy conditions a), b) and c) in Section 2 and that $H^s(\partial\Omega)$ for $s < 0$ is constructed from H^0 and H^{-s} as in Section 2.

Let L be a second order partial differential operator given by

$$L\phi = -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}(a_{ij}\frac{\partial\phi}{\partial x_j}) + \sum_{i=1}^{N} b_i \frac{\partial\phi}{\partial x_1} + c\phi$$

where a_{ij}, b_i and c are in $C^\infty(\bar{\Omega})$ and $a_{ij} = a_{ji}$. We assume that L is uniformly strongly elliptic; i.e.

there is a positive constant a_0 such that

$$\operatorname{Re} \sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \geq a_0 \sum_{i=1}^{N} \xi_i^2$$

for all real $\xi_1,\ldots\xi_N$ and all $x \in \Omega$. The sesquilinear form on $H^1(\Omega) \times H^1(\Omega)$ associated with L is given by

$$B(\phi,\psi) = \sum_{i,j=1}^{N} \int_\Omega a_{ij} \frac{\partial\phi}{\partial x_i} \overline{\frac{\partial\psi}{\partial x_j}}\, dx + \sum_{i=1}^{N} \int_\Omega b_i \frac{\partial\phi}{\partial x_i} \bar{\psi}\, dx +$$

$$+ \int_\Omega c\, \phi\, \bar{\psi}\, dx \ .$$

We shall assume that B is coercive over $H^1(\Omega)$; i.e. there is a positive constant C such that

$$\operatorname{Re} B(\phi,\phi) \geq C\|\phi\|_1^2 \tag{4.2}$$

for all $\phi \in H^1(\Omega)$. Since the coefficients in L are bounded it is clear that there is another constant C such that

$$|B(\phi,\psi)| \leq C\|\phi\|_1\|\psi\|_1 \tag{4.3}$$

for all $\phi,\ \psi \in H^1(\Omega)$.

We consider now the Steklov eigenvalue problem for L . The complex number λ is called a Steklov eigenvalue of L if there is a function $u \in H^2(\Omega)$ which is not identically zero such that

$$L\,u = 0 \quad \text{in} \quad \Omega$$

and

$$\frac{\partial u}{\partial \nu} = \lambda u \quad \text{on} \quad \Omega \quad ,$$

where $\frac{\partial}{\partial \nu} = \sum_{i,j=1}^{N} a_{ij}\, n_j \frac{\partial}{\partial x_i}$ and n_j is the jth component of the outward unit normal on $\partial\Omega$. This problem has the following equivalent weak formulation:
λ is a Steklov eigenvalue if

$$B(u,\phi) = \lambda\langle u,\phi\rangle \tag{4.4}$$

for some non-zero $u \in H^1(\Omega)$ and all $\phi \in H^1(\Omega)$.

Next we wish to identify the Steklov eigenvalues of L with the eigenvalues of a certain compact operator. To this end we consider the Neumann problem associated with L ; i.e. the problem of finding $u \in H^1(\Omega)$ such that

$$B(u,\phi) = \langle g,\phi\rangle \tag{4.5}$$

for all $\phi \in H^1(\Omega)$ where g is a given function in $L_2(\partial\Omega)$. Since B is coercive this prolem is uniquely solvable. For $g \in L_2(\partial\Omega)$ the solution u will in fact be in $H^{3/2}(\Omega)$. The solution operator $A: L_2(\partial\Omega) \to H^{3/2}(\Omega)$ is defined by $Ag = u$. Now we define T by $Tg = (Ag)'$, where the prime denotes the restriction to $\partial\Omega$. It follows from the theory of trace (cf.[8]) that $(Ag)' \in H^1(\partial\Omega)$ for each $g \in L_2(\partial\Omega)$. Hence $T: L_2(\partial\Omega) \to H^1(\partial\Omega)$.

The following estimate was proved in [10] (cf. also [9]). For any real s there is a constant C_s such that

$$(4.6) \qquad |Tg|_{s+1} \leq C_s |g|_s$$

for all $g \in H^s(\partial\Omega) \cap L_2(\partial\Omega)$. Inequality (4.6) allows us to extend T from $L_2(\partial\Omega)$ to $H^s(\partial\Omega)$ for any $s < 0$. Now let $s_0 \geq 0$ and consider T as being defined on $H^{-s_0}(\partial\Omega)$. Hence T is an operator of the general type discussed in Section 3 with $\varepsilon = 1$. We write T^s for the restriction of T to $H^s(\partial\Omega)$ for $s \geq -s_0$.

Now suppose μ is an eigenvalue of T^0; i.e. there is a nonzero w in $L_2(\partial\Omega)$ such that $T^0 w = \mu w$. Then $\mu \neq 0$ and

$$\begin{aligned} B(Aw,\phi) &= \langle w,\phi\rangle \\ &= \frac{1}{\mu}\langle T^0 w,\phi\rangle \\ &= \frac{1}{\mu}\langle Aw,\phi\rangle \end{aligned}$$

for all $\phi \in H^1(\Omega)$. Thus $1/\mu$ is an eigenvalue of (4.4) with Aw the corresponding eigenvector. Conversely suppose $B(u,\phi) = \lambda\langle u,\phi\rangle$ with $u \neq 0$ for all $\phi \in H^1(\partial\Omega)$. Then $\lambda \neq 0$ and

$$T^0 u' = (Au')' = \frac{1}{\lambda} u' \ ;$$

i.e. $1/\lambda$ is an eigenvalue of T^0 with u' the corresponding eigenvector. Hence the Steklov eigenvalues are the reciprocals of the eigenvalues of T^0 .

In order to define approximate eigenvalues we must discuss certain classes of finite dimensional spaces of trial functions. Let $\{S_h\}_{0 < h \leq 1}$ be a one parameter family of finite dimensional vector spaces. For given integers k and r with $0 \leq k \leq r$ we say that $\{S_h\}_{0 < h \leq 1}$ is of class $\mathcal{S}_{k,r}$ if $S_h \subset H^k(\Omega)$ for each h and there is a constant C independent of h such that for any $v \in H^t(\Omega)$ with $k \leq t \leq r$,

$$\inf_{\chi \in S_h} \sum_{j=0}^{k} h^j \|v-\chi\|_j \leq C\, h^t \|v\|_t . \tag{4.7}$$

It was proved in [4] that if (4.7) holds for $t = r$ it then holds for $k \leq t \leq r$.

Let $\{S_h\}_{0 < h \leq 1}$ belong to $\mathcal{S}_{1,r}$. For $0 < h \leq 1$ we define the approximate solution of the Neumann problem as the unique $u_h \in S_h$ such that

$$B(u_h, \chi) = \langle g, \chi \rangle \tag{4.8}$$

for all $\chi \in S_h$. Again, since B is coercive, this problem has a unique solution for each h. We define the approximate solution operator $A_h : L_2(\partial\Omega) \to S_h$ by $A_h g = u_h$. Letting δS_h denote the functions defined on $\partial\Omega$ which are restrictions to $\partial\Omega$ of functions in S_h we define $T_h : L_2(\partial\Omega) \to \delta S_h \subset L_2(\partial\Omega)$ by $T_h g = (A_h g)'$.

We want now to consider Galerkin eigenvalues of the Steklov problem; i.e. complex numbers $\lambda(h)$ which satisfy

$$(4.9) \qquad B(w,\chi) = \lambda(h) \langle w,\chi\rangle$$

for some non-zero $w \in S_h$ and all $\chi \in S_h$. These eigenvalues will be considered as approximations to the Steklov eigenvalues. If (4.9) holds then $\lambda(h) \neq 0$ and

$$T_h w' = (A_h w')' = \frac{1}{\lambda(h)} w' ;$$

i.e. $1/\lambda(h)$ is an eigenvalue of T_h and $w' \in \delta S_h$ is a corresponding eigenvector. Conversely, if $T_h w = \mu(h) w$ with $\mu(h) \neq 0$ then

$$B(A_h w,\chi) = \langle w,\chi\rangle = \frac{1}{\mu(h)} \langle T_h w,\chi\rangle = \frac{1}{\mu(h)} \langle A_h w,\chi\rangle$$

for all $\chi \in S_h$; i.e. $1/\mu(h)$ is a Galerkin eigenvalue and $A_h w$ is a corresponding eigenvector. Thus we may compare the eigenvalues of (4.4) with the Galerkin approximations by comparing the eigenvalues of T^0 with those of T_h. In order to apply the theorems of Section 3 to obtain eigenvalue estimates we need two further estimates from [10].

For any real s there is a constant C_s such that

$$(4.10) \qquad \|Ag\|_{s+3/2} \leq C_s |g|_s$$

for all $g \in H^s(\partial\Omega) \cap L_2(\partial\Omega)$. Corresponding to the adjoint problem we have the solution operator $A^*: L_2(\partial\Omega) \to H^{3/2}(\partial\Omega)$ which is characterized by

$$B(\phi, A^*g) = \langle\phi, g\rangle$$

for all $\phi \in H^1(\Omega)$. For A^* we have the same type of estimate, namely

(4.11) $$\|A^*g\|_{s+3/2} \leq C_s^* \ |g|_s$$

for all $g \in H^s(\partial\Omega) \cap L_2(\partial\Omega)$.

We are now ready to derive the required estimates for $T^0 - T_h$. Let $g \in H^{t-1/2}(\partial\Omega)$ where $1/2 \leq t \leq r-1$. Using (4.2), (4.5), (4.8) and (4.3) we have

$$\begin{aligned}\|(A-A_h)g\|_1^2 &\leq C\left|B\Big((A-A_h)g,\ (A-A_h)g\Big)\right| \\ &= C\left|B\Big((A-A_h)g, Ag-\chi_1\Big)\right| \\ &\leq C\|(A-A_h)g\|_1\|Ag-\chi_1\|_1\end{aligned}$$

for any $\chi_1 \in S_h$. Hence

(4.12) $$\|(A-A_h)g\|_1 \leq C \inf_{\chi_1 \in S_h} \|Ag-\chi_1\|_1 \quad .$$

Now let $\psi \in H^{s-1/2}(\partial\Omega)$ be arbitrary where $1/2 \leq s \leq r-1$. Then

(4.13) $$\begin{aligned}\left|\left\langle (T^0 - T_h)g, \psi\right\rangle\right| &= \left|B\Big((A-A_h)g, A^*\psi\Big)\right| \\ &= \left|B\Big((A-A_h)g, A^*\psi-\chi_2\Big)\right| \\ &\leq C\|(A-A_h)g\|_1\|A^*\psi-\chi_2\|_1\end{aligned}$$

for any $\chi_2 \in S_h$. From (4.12), (4.13) and the approximability assumption (4.7) we thus have

$$\left|\left\langle (T^0 - T_h)g, \psi\right\rangle\right| \leq Ch^{s+t}\|Ag\|_{t+1}\|A^*\psi\|_{s+1} \ .$$

Combining this result with (4.10) and (4.11) we obtain

$$\left|\left\langle (T^0 - T_h)g, \psi \right\rangle\right| \leq Ch^{s+t} |g|_{t-1/2} |\psi|_{s-1/2} .$$

Taking $H^s = H^s(\partial\Omega)$ is follows immediately from this inequality, (4.1) and (2.1) that

$$\|T^0 - T_h\|_{1/2-s, t-1/2} \leq Ch^{s+t}$$

for $1/2 \leq s, \ t \leq r-1$.

It remains only to apply the results of Section 3. Notice first, letting $s = t = 1/2$, that $\|T^0 - T_h\|_{0,0} \leq Ch$. Thus all of the results of Section 3 hold for this problem. Now taking $s = t = r-1$ we have

$$\|T^0 - T_h\|_{3/2-r, r-3/2} \leq Ch^{2r-2}$$

taking $s = 1/2$, $t = r-1$ we have

$$\|T^0 - T_h\|_{0, r-3/2} \leq Ch^{r-1/2}$$

and taking $s = r-1$, $t = 1/2$ we have

$$\|T^0 - T_h\|_{3/2-r, 0} \leq Ch^{r-1/2} .$$

Let λ be an eigenvalue with algebraic multiplicity m of (4.4) and let $\mu = 1/\lambda$. The eigenvalues $\mu_1(h), \ldots, \mu_m(h)$ which converge to μ are the reciprocals of the eigenvalues $\lambda_1(h), \ldots, \lambda_m(h)$ of (4.9). From Theorem 2 we thus have the estimate

$$\left|\lambda - \left(\frac{1}{m} \sum_{j=1}^{m} 1/\lambda_j(h) \right)^{-1}\right| \leq Ch^{2r-2} .$$

Using Theorem 3.1 and its corollary we see that the $L_2(\partial\Omega)$-gap between $M(1/\lambda)$ and $M_h(1/\lambda)$, $\hat{\delta}\left(M(1/\lambda), M_h(1/\lambda)\right)$, satisfies

$$\hat{\delta}\left(M(1/\lambda), M_h(1/\lambda)\right) \leq Ch^{r-1/2} .$$

Further, from Theorem 3.3, we obtain the estimate

$$\max_{1 \leq k \leq m} |\lambda - \lambda_k| \leq Ch^{(2r-2)/\alpha} ,$$

where α is the ascent of $1/\lambda - T^0$. Finally for w an eigenvector of T_h in $M_h(1/\lambda)$ we see, using Theorem 3.4, that there is an eigenvector u of T^0 in $M(1/\lambda)$ such that

$$|u-w|_0 \leq Ch^{(r-1/2)/\alpha} .$$

5. On the structure of T_h.

By the term projection onto δS_h we shall mean an operator P_h (not necessarily bounded) defined on a linear manifold in $L_2(\partial\Omega)$, with range equal to δS_h and such that $P_h^2 = P_h$.

We first prove a lemma which gives a necessary condition, when the form B is Hermitian, that $T_h = P_h T^0$ for some projection P_h .

Lemma. Let B be Hermitian and $\{S_h\}_{0<h\leq 1} \in \mathcal{S}_{2,r}$. Suppose there is a projection P_h onto δS_h such that $T_h = P_h T^0$. Let $\tilde{u}_h \in S_h$ and u satisfy

$$L\,u = 0 \text{ in } \Omega$$

and

$$u = \tilde{u}_h \quad \text{on } \partial\Omega.$$

Then $u \in S_h$.

<u>Proof</u>: Set $u_h = A_h \frac{\partial u}{\partial \nu}$. Then

(5.1) $u_h' = T_h \frac{\partial u}{\partial \nu} = P_h T^0 \frac{\partial u}{\partial \nu} = P_h u' = P_h \tilde{u}_h' = \tilde{u}_h' = u'$.

From (4.8) we have

$$B(u_h, \chi) = \langle \frac{\partial u}{\partial \nu}, \chi \rangle$$

for all $\chi \in S_h$ and from (4.5),

$$B(u, \phi) = \langle \frac{\partial u}{\partial \nu}, \phi \rangle$$

for all $\phi \in H^1(\Omega)$. Thus $B(u_h - u, \chi) = 0$ for all $\chi \in S_h$ and hence

(5.2) $$B(u_h, u_h - u) = \overline{B(u_h - u, u_h)} = 0$$

Now $L u = 0$ and from (5.1) we have $u_h - u = 0$ on $\partial\Omega$. By integration by parts,

(5.3) $$B(u, u_h - u) = 0 .$$

Thus from (4.2), (5.2) and (5.3) we see that

$$\|u_h - u\|_1^2 \leq C\, B(u_h - u, u_h - u) = 0$$

which implies that $u = u_h \in S_h$.

Using this result it is easy to see, for many families of spaces $\{S_h\}_{0<h\leq 1}$ which are used in the finite element method, that in general there is no projection P_h onto δS_h such that $T_h = P_h T^0$. More

precisely we have:

Theorem 5.1. Let B be Hermitian, $\{S_h\}_{0<h\leq 1} \in \mathcal{S}_{2,r}$ and suppose that S_h consists of piecewise polynomials and that there is at least one $\chi_0 \in S_h$ such that χ_0' is not the restriction to $\partial\Omega$ of a polynomial. Then there does not exist a projection onto δS_h such that $T_h = P_h T^0$.

Proof: Let $\chi_0 \in S_h$ be such that $\chi_0' \neq p'$ for all polynomials p and let u satisfy

$$Lu = 0 \quad \text{in } \Omega$$

and

$$u = \chi_0 \quad \text{on } \partial\Omega.$$

Suppose now that there exists a projection P_h with $T_h = P_h T^0$. Then by the lemma, $u \in S_h$. Since the coefficients of L are of class C^∞ we see by elliptic regularity theory that $u \in C^\infty(\Omega)$. Thus u must in fact be a polynomial. But this implies that χ_0' is the restriction of a polynomial to $\partial\Omega$ which is a contradiction.

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