ON THE NUMERICAL SOLUTION OF ELLIPTIC BOUNDARY VALUE PROBLEMS BY LEAST SQUARES APPROXIMATION OF THE DATA*

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1. Introduction

In the approximate solution of boundary value problems arising in the theory of elliptic partial differential equations several rather general approaches have been taken. The method perhaps most extensively used has been the method of finite differences. One theoretical and practical difficulty with this method has been the treatment of the boundary, especially in problems of higher order. A second method which has been considered quite widely is the so-called variational approach. This usually consists in reformulating the boundary value problem as a problem in the minimization of a certain functional (usually an "energy" integral) over an appropriate space of function. The corresponding approximate problem would then be to look for the minimum only over a finite dimensional subspace. Again, except in

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the case of natural boundary conditions, one difficulty has been the selection of an appropriate finite dimensional subspace. Although these methods were first considered many years ago their theoretical investigation has been undertaken only in the last few years, particularly from the point of view of error estimation. The approximation theoretic study of special sets of approximating functions (e.g. piecewise polynomials) has greatly encouraged researchers to investigate more thoroughly variational procedures for the solution of boundary problems since with such choices of trial functions the resulting linear systems retain many of the desirable features of those arising from difference equations. Rather than give extensive references on these subjects we refer the reader to related papers in this volume and references cited therein.

Another approach, though closely related to the second, is the so-called "least squares" method. Roughly speaking, it consists of finding that element of a finite dimensional subspace which best fits the data of a given problem in the least squares sense. The purpose of this paper is to survey some results obtained recently by the authors concerning the behavior of the error resulting from the use of such methods. A great advantage in the use of these methods lies in the fact that the subspaces chosen do not have to satisfy any boundary condition, so that the choice of approximating functions is easy.

Before describing in more detail the outline of this paper we would like to mention some other closely related work. Babuška [4] has introduced a boundary perturbation of the usual energy principle in such a way that his trial functions need not satisfy boundary conditions. His method is a "penalty method" and the penalty for unrestricted trial functions is loss of accuracy. In [6] he uses an analogous approach to treat certain cases in which the coefficients are discontinuous.

A brief outline of the present paper is as follows. Section 2 contains definitions and notations. In the third section we introduce certain classes of finite dimensional subspaces of Sobolev spaces and state a theorem concerning their approximation theoretic properties relative to general boundary problems. Next we state an approximation scheme of least squares type for $2m^{th}$ order elliptic equations and general boundary conditions. Estimates for the error are given. In the fifth section we discuss a second order equation with coefficients discontinuous across an interface. Again an approximation scheme is given together with corresponding error estimates. None of the schemes considered here requires trial functions satisfying boundary conditions and the problems are not required to be self-adjoint. The last section contains examples of specific problems and specific choices of subspaces. Since the purpose of this paper is to outline some recent work of the authors we refer the reader to [12, 13, 14, 15] for detailed proofs.

2. Preliminaries

Let R be a bounded domain in $\mathbb{R}^N$ with boundary ∂R. We shall assume (for convenience) that ∂R is of class C^∞ and shall consider in R the operator A of order $2m$ with infinitely differentiable real coefficients:

$$(2.1) \qquad Au = A(x,D)u = \sum_{|\alpha||\beta|\leq m} (-1)^{|\alpha|} D^\alpha(a_{\alpha\beta}(x)\, D^\beta u)$$

where $\alpha = (\alpha_1,\ldots,\alpha_N)$ and $\beta = (\beta_1,\ldots,\beta_N)$ are multi-indices, $D^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_N}\right)^{\alpha_N}$. We note that A is not necessarily formally self-adjoint. Set

$$A_0(x,\xi) = \sum_{|\alpha|,|\beta|=m} (-1)^m a_{\alpha\beta}(x)\zeta^{\alpha+\beta}$$

where

$$\zeta^{\alpha+\beta} = \zeta_1^{\alpha_1+\beta_1} \cdots \zeta_N^{\alpha_N+\beta_N} .$$

We shall assume that A is uniformly elliptic; i.e. there is a constant $a > 0$ independent of x such that

$$a^{-1}|\xi|^{2m} \leqslant |A_0(x,\xi)| \leqslant a|\xi|^{2m}$$

for all $x \in \bar{R}$ and all $\xi \in R^N$.

We shall consider the boundary value problem

$$(2.2) \qquad \begin{aligned} &Au = f \quad \text{in} \quad R , \\ &B_j u = g_j , \quad j = 0,\ldots,m-1 \quad \text{on} \quad \partial R \end{aligned}$$

where the B_j are boundary differential operators of order m_j, $0 \leqslant m_j \leqslant 2m-1$, $j = 0,\ldots,m-1$, and f and g_j are given. The operators B_j are defined by

$$(2.3) \qquad B_j\phi = \sum_{|\alpha|\leqslant m_j} b_{j\alpha}(x)D^\alpha\phi$$

for $x \in \partial R$ and $D^\alpha\phi$ is defined on ∂R by continuity or if necessary as a trace.

All functions considered in this paper will be real valued. The conditions which we shall place on our problem are as follows.

Condition A:

i) A is uniformly elliptic with coefficients in $C^\infty(\bar{R})$.

ii) The boundary system $\{B_j\}$ is normal and covers the operator A (see e.g. [23]) and has coefficients in $C^\infty(\partial R)$. The order of B_j is m_j , $0 \leqslant m_j \leqslant 2m-1$ where for simplicity we may assume that $m_0 < m_j$ for $j \neq 0$.

iii) The only solution of (2.2) in $C^\infty(\bar{R})$ with zero data is the zero solution.

We shall make use of the following function spaces.

1) $H^s(R)$, for s a non-negative integer, is the Sobolev space of order s on R with the norm $\|\cdot\|_s$ derived from the inner product

$$(u,v)_s = \sum_{|\alpha|\leqslant s} (D^\alpha u, D^\alpha v)_0 ,$$

where

$$(u,v)_0 = \int_R uv dx .$$

For $s > 0$ and not an integer $H^s(R)$ is defined by real interpolation between successive integers (cf. [23]). For $s < 0$, $H^s(R)$ is the completion of $C^\infty(\bar{R})$ with respect to the norm

$$\|u\|_s = \sup_{v \in C^\infty(\overline{R})} \frac{(u,v)_0}{\|v\|_{-s}} .$$

2) $H^s(\partial R)$ for all real s is the Sobolev space of order s on ∂R. For a precise definition we refer the reader to [23]. We shall denote by $|\cdot|_s$ the norm on $H^s(\partial R)$. Note that $H^0(\partial R) = L_2(\partial R)$ and in this case we denote the inner product by

$$\langle u,v\rangle_0 = \int_{\partial R} uv\,d\sigma .$$

In this discussion we shall restrict ourselves to the approximation of solutions u of the boundary value problem (2.2) which have a certain degree of regularity. The conditions that we impose on the data here can be relaxed considerably (see [12], [14]). For simplicity of presentation we shall assume that $u \in H^\beta(R)$ for some $\beta \geq 2m$. The following result, due to a number of authors, gives precise conditions on the data under which such solutions can be found and is essential in what follows.

Theorem (cf. [23],[25]). Suppose that Condition A is satisfied. The mapping $\mathcal{P}u = (Au, B_0u, \ldots, B_{m-1}u)$ of $C^\infty(\overline{R})$ into $C^\infty(R) \times \overbrace{C^\infty(\partial R) \times \cdots \times C^\infty(\partial R)}^{m\text{-times}}$, completed by continuity, is a homeomorphism of $H^\beta(R)$ onto

$H^{\beta-2m}(R) \times \prod_{j=0}^{m-1} H^{\beta-m_j-\frac{1}{2}}(\partial R)$ for all real numbers $\beta \geq 2m$.

The norms $\|u\|_\beta$ and $\|Au\|_{\beta-2m} + \sum_{j=0}^{m-1} |B_j u|_{\beta-m_j-\frac{1}{2}}$ are equivalent.

3. Finite Dimensional Subspaces of $H^k(R)$

We shall now discuss the subspaces of functions which will be used to approximate the solution of (2.2). We shall first state our basic assumption concerning them and then show that as a consequence they have certain other approximation theoretic properties relative to the "data spaces" of the differential operators which we are considering.

Let h , $0 < h < 1$, be a parameter. For any two non-negative integers k and r with $k < r$, let $S^h_{k,r}(R) = S^h_{k,r}$ be any finite dimensional subspace of $H^k(R)$ which satisfies the following condition:

(*) For each $u \in H^r(R)$ there exists a $\bar{u} \in S^h_{k,r}$ and a constant C , independent of h and u , such that

$$\|u-\bar{u}\|_k \leq Ch^{r-k} \|u\|_r \ . \tag{3.1}$$

This is obviously equivalent to the condition that

$$\inf_{\chi \in S^h_{k,r}} \|u-\chi\|_k \leq Ch^{r-k} \|u\|_r \ . \tag{3.2}$$

Subspaces having the property (*) have been constructed by many authors, see for example Hilbert [22], Schultz [27], Aubin [3], DiGuglielmo [18], Babuška [5], Fix and Strang [20] and Bramble and Zlámal [16]. One possible example of such subspaces is the restriction to $\bar{R}$ of "spline functions" defined with respect to a uniform mesh of width h on $\mathbb{R}^N$.

The following approximation theoretic result is crucial to our analysis of approximation schemes for $2m^{th}$ order elliptic boundary value problems which will be discussed in the next section. This result states that the subspaces $S^h_{k,r}$ satisfying (*) have further approximation theoretic properties relative to the "data spaces" occurring in the boundary value problems. It is a special case of a general theorem found in [13].

Theorem 3.1. Let $S^h_{k,r}$ satisfy (*) with $2m \leq k < r$. Then for all $F = (f, g_0, \ldots, g_{m-1}) \in H^{\lambda}(R) \times H^{\lambda_0}(\partial R) \times \cdots \times H^{\lambda_{m-1}}(\partial R)$, $0 \leq \lambda \leq r-2m$, $0 \leq \lambda_j \leq r-m_j - \frac{1}{2}$, $j = 0, \ldots, m-1$, there exists a constant C independent of F and h such that

$$\inf_{\chi \in S^h_{k,r}} \left(\|f - A\chi\|_0 + \sum_{j=0}^{m-1} h^{-(2m-m_j-\frac{1}{2})} |g_j - B_j\chi|_0\right)$$

$$\leq C\left(h^{\lambda}\|f\|_{\lambda} + \sum_{j=0}^{m-1} h^{-(2m-m_j-\frac{1}{2})+\lambda_0} |g|_{\lambda_0}\right).$$

4. Least Squares Methods for $2m^{th}$ Order Boundary Value Problems

Let u be the solution of (2.2). The approximation scheme we shall consider is as follows:

Let $S^h_{k,r}$ be given satisfying (*) with $2m \leq k < r$. Find $w\, S^h_{k,r}$ such that

$$(4.1) \qquad (f-Aw,A\phi)_0 + \sum_{j=0}^{m-1} h^{-2(2m-m_j-\frac{1}{2})}\langle g_j-B_jw,B_j\phi\rangle_0 = 0$$

or

$$\int_R (f-Aw)A\phi dx + \sum_{j=0}^{m-1} h^{-2(2m-m_j-\frac{1}{2})} \int_{\partial R}(g_j-B_jw)B_j\phi\, d\sigma = 0$$

for all $\phi \in S^h_{k,r}$.

Since $\mathcal{P}(S^h_{k,r})$ (the image of $S^h_{k,r}$ under the mapping $\mathcal{P}\phi = (A\phi,B_0\phi,\ldots,B_{m-1}\phi))$ is a finite dimensional subspace of $H^0(R) \times \overbrace{H^0(\partial R) \times \cdots \times H^0(\partial R)}^{m\text{-times}}$, by iii) of Condition A (uniqueness) w exists and is unique. It is determined by solving a linear system of algebraic equations whose coefficients depend only on f, g and $S^h_{k,r}$. An alternative way of stating (4.1) is the following: Among all $\chi \in S^h_{k,r}$ find the one which minimizes the functional

$$(4.2) \qquad \|f-A\chi\|_0^2 + \sum_{j=0}^{m-1} h^{-2(2m-m_j-\frac{1}{2})} |g-B_j\chi|_0^2 .$$

Note that the above scheme is a least squares method involving only L_2 norms of the data. For a given $S^h_{k,r}$ (with h fixed) the weighting factors $h^{-2(2m-m_j-\frac{1}{2})}$ are constants, but the manner in which they depend on h is critical for our estimates.

We shall now consider some results concering the rate of convergence of the approximate solution w determined by (4.1) to the solution u of (2.2). We shall first consider error estimates on R.

Theorem 4.1. Suppose that Condition A is satisfied and u is the solution of (2.2) with $F = (f, g_0, \ldots, g_{m-1}) \in H^{\beta-2m}(R) \times \prod_{j=0}^{m-1} H^{\beta-m_j-\frac{1}{2}}(\partial R)$. For given $S^h_{k,r}$ satisfying (*) with $2m \leqslant k < r$, let w be the solution of the approximate problem (4.1) and set $e = u-w$.

Case 1. Suppose that $4m \leqslant r$. Then there exists a constant C independent of F and h such that

$$\|e\|_\rho \leqslant Ch^{\beta-\rho} \|u\|_\beta \tag{4.3}$$

for each ρ and β satisfying $4m-r \leqslant \rho \leqslant m_0+\frac{1}{2}$ and $2m \leqslant \beta \leqslant r$.

Case 2. Suppose that $2m < r \leqslant 4m$. Then if $4m-r \leqslant m_0+\frac{1}{2}$

$$\|e\|_\rho \leqslant Ch^{\beta-\rho} \|u\|_\beta \tag{4.4}$$

for each ρ and β satisfying $4m-r \leqslant \rho \leqslant m_0+\frac{1}{2}$ and $2m \leqslant \beta \leqslant r$. If $m_0+\frac{1}{2} < 4m-r$ then

$$\|e\|_{m_0+\frac{1}{2}} \leqslant Ch^{(r-2m)+(\beta-2m)} \|u\|_\beta \tag{4.5}$$

for each β satisfying $2m \leqslant \beta \leqslant r$. In (4.4) and (4.5)

C is a constant which is independent of h and F.

Remark. The norm $\|u\|_\beta$ may be replaced by the data norm $\|f\|_{\beta-2m} + \sum_{j=0}^{m-1} |g_j|_{\beta-m_j-\frac{1}{2}}$ in the right hand sides of (4.3), (4.4) and (4.5).

For the proof of Theorem 4.1 the reader is referred to [14]. We just wish to draw attention to the connection between the definition of the approximate problem (4.1) in the form (4.2) and the statement of Theorem 3.1. The relevance of Theorem 3.1 to the proof of Theorem (4.1) is then apparent.

Let us now briefly discuss the results contained in Theorem 4.1. In the case that $r \geq 4m$ the inequality (4.3) says that asymptotically the approximation scheme (4.1) gives the best possible results relative to the assumed approximation theoretic properties of the spaces $S^h_{k,r}$.

In order that we may illustrate this simply, let us consider the case of estimating the error in the $H^0(R) = L_2(R)$ norm. According to our assumptions concerning $S^h_{k,r}$ one can show that the best approximation $\bar{u}$ in $S^h_{k,r}$ to $u \in H^\beta(R)$ in the $L^2(R)$ norm, in general only satisfies the inequality

$$(4.6) \qquad \|u-\bar{u}\|_0 \leq C_1 h^\beta \|u\|_\beta$$

where for simplicity we have taken $2m \leq \beta \leq r-2m$ and where C_1 is independent of u and h. The estimate (4.3) with $\rho = 0$ shows that

$$(4.7) \qquad \|u-w\|_0 \leq Ch^{\beta} \|u\|_{\beta}$$

where w is the solution of the approximate problem (4.1). Hence the property (4.6) is essentially reproduced.

The case when $2m < r < 4m$ is more difficult to discuss. Suppose first that $4m-r \leq m_0+\frac{1}{2}$ (where m_0 is order of the boundary differential operator of lowest order). In this case the inequality (4.4) says if we measure the error in an appropriate norm $\|\cdot\|_{\rho}$, $4m-r \leq \rho \leq m_0+\frac{1}{2}$ then w essentially has the same approximation properties as the best approximation to u in $S^h_{k,r}$ in that norm. In the case when $m_0+\frac{1}{2} < 4m-r$, the inequality (4.5) indicates that the properties of w are not as good as those of the best approximation when measured in any norm up to $m_0+\frac{1}{2}$. Specific examples will be given in Section 6.

Interior estimates. Suppose that R_1 is any compact subset of R. We shall denote by $\|\cdot\|^{R_1}_s$ the norm on the Sobolev space $H^s(R_1)$ which is defined in the same manner as $H^s(R)$ except now with respect to R_1. We shall now state some estimates for the error $e = u-w$ on R_1. The proof may be found in [14].

Theorem 4.2. Suppose that the conditions of Theorem 4.1 are satisfied. Let R_1 be any compact subset of R. Then there exists a constant C independent of h and F such that

$$(4.8) \qquad \|e\|^{R_1}_{\rho} \leq Ch^{\beta-\rho} \|u\|_{\beta}$$

for each ρ and β satisfying $4m-r \leq \rho \leq 2m$, $\rho \leq \beta$ and $2m \leq \beta \leq r$.

We first note that Theorem 4.2 gives estimates for the derivatives of the error up to order $2m$ over R . This is in contrast to Theorem 4.1 where we are restricted to estimates for the derivatives of the error up to order $m_0+\frac{1}{2}$ over R .

In the case when $4m \leq r$ the estimate (4.8) says that we are able to estimate the error in all norms $\|\cdot\|_\rho^{R_1}$ for $0 \leq \rho \leq 2m$ with the best possible power of h . If $2m < r < 4m$, we are also able to do this provided we restrict ourselves to an appropriately high norm $\|\cdot\|_\rho$ where $4m-r \leq \rho \leq 2m$. In contrast to Theorem 4.1 the analogous case of the estimate (4.5) does not occur here since, as mentioned above, we are not restricted to estimates only involving Sobolev norms up to order $m_0+\frac{1}{2}$. A phenomenon similar to that given in Theorem 4.2 has been observed by Fix and Strang [20] in their study of Rayleigh-Ritz-Galerkin methods over R^N . We further note that maximum norm estimates over interior subdomains can be easily obtained from Theorem 4.2 using Sobolev inequalities.

5. Interface Problems - Equations with Discontinuous Coefficients

The approximation scheme which was used to approximate the solution of the boundary value problem can be generalized to yield approximate solutions of so-called interface problems (or equations with discontinuous coefficients). For simplicity of presentation we shall not treat such problems with $2m^{th}$ order operators and general boundary conditions but rather restrict ourselves here to a discussion of a specific example.

Let R_1 be an open domain with C^∞ boundary ∂R_1 (the interface) and suppose that $\overline{R}_1 \subset R$. Let $R_2 = R - \overline{R}_1$. As an example of an interface problem we shall consider the problem

$$(5.1)\qquad \begin{aligned} &\Delta u_i = f_i \quad \text{in } R_i, \quad i = 1, 2, \\ &a_1 u_1 - u_2 = \phi_1 \quad \text{on } \partial R_1, \\ &a_2 \frac{\partial u_1}{\partial n} - \frac{\partial u_2}{\partial n} = \phi_2 \quad \text{on } \partial R_1, \\ &u_2 = g \quad \text{on } \partial R, \end{aligned}$$

where $a_1, a_2 > 0$ are constants and f_1, f_2, ϕ_1, ϕ_2 and g are given functions. Let us denote by $|||u|||_s$ the norm on the product space $H^s(R_1) \times H^s(R_2)$. The following result concerning existence uniqueness and regularity of solutions of (5.1), is basic to our investigation (cf. [25]).

Theorem 5.1. The mapping $\mathcal{P}u = (\Delta u_1, \Delta u_2, a_1 u_1 - u_2, a_2 \frac{\partial u_1}{\partial n} - \frac{\partial u_2}{\partial n}, u_2)$ of $C^\infty(\overline{R}_1) \times C^\infty(\overline{R}_2)$ into $C^\infty(\overline{R}_1) \times C^\infty(\overline{R}_2) \times C^\infty(\partial R_1) \times C^\infty(\partial R_1) \times C^\infty(\partial R)$, completed by continuity, is a homeomorphism of $H^\beta(R_1) \times H^\beta(R_2)$ onto

$$H^{\beta-2}(R_1) \times H^{\beta-2}(R_2) \times H^{\beta-\frac{1}{2}}(\partial R_1) \times H^{\beta-\frac{3}{2}}(\partial R_1) \times H^{\beta-\frac{1}{2}}(\partial R) \quad \text{for all } \beta \geq 2.$$

Then norms $|||u|||_\beta$ and

$$\|\Delta u_1\|^{R_1}_{\beta-2} + \|\Delta u_2\|^{R_2}_{\beta-2} + |u_1-u_2|^{\partial R_1}_{\beta-\frac{1}{2}}$$

$$+ \left|a\frac{\partial u_1}{\partial n} - \frac{\partial u_2}{\partial n}\right|^{\partial R_1}_{\beta-\frac{3}{2}} + |u_1-u_2|^{\partial R}_{\beta-\frac{1}{2}}$$

are equivalent.

We shall consider an approximation scheme which is a generalization of the scheme (4.1). The approximating functions we shall use will be any finite dimensional subspace of $H^k(R_1) \times H^k(R_2)$ of the type $S^h_{k,r}(R_1) \times S^h_{k,r}(R_2) = \tilde{S}^h_{k,r}$ where each $S^h_{k,r}(R_i)$, $i = 1,2$, has the property (*) in their respective domains. Hence we shall assume that for all $u \in H^r(R_1) \times H^r(R_2)$ there exists a constant C independent of u such that

$$(**) \qquad \inf_{\chi\in\tilde{S}^h_{k,r}} |||u-\chi|||_k \leq Ch^{r-k}\,|||u|||_r$$

where $\chi = (\chi_1,\chi_2)$.

One can show, in analogy with Theorem 3.1 that the subspaces $\tilde{S}^h_{k,r}$ have the following approximation theoretic property relative to the "data spaces" of the problem considered here.

<u>Theorem 5.2</u>. Let $\tilde{S}^h_{k,r}$ satisfy (**) with $2 \leq k < r$. Then for all

$$F = (f_1, f_2, \phi_1, \phi_2, g) \in H^{\lambda}(r_1) \times H^{\lambda}(R_2) \times H^{\lambda_1}(\partial R_1) \times H^{\lambda_2}(\partial R_1) \times H^{\lambda_3}(\partial R)$$

satisfying

$$0 \leqslant \lambda \leqslant r-2, \quad 0 \leqslant \lambda_1, \quad \lambda_3 \leqslant r-\tfrac{1}{2}, \quad 0 \leqslant \lambda_2 \leqslant r-\tfrac{3}{2} \quad ,$$

there exists a constant C independent of h and F such that

$$(5.2) \quad \|f_1-\Delta\chi_1\|_0^{R_1} + \|f_2-\Delta\chi_2\|_0^{R_2} + h^{-\frac{3}{2}}|\phi_1-(a_1\chi_1-\chi_2)|_0^{\partial R_1} + h^{-\frac{1}{2}} \left|\phi_2 - \frac{\partial(a_2\chi_1-\chi_2)}{\partial n}\right|_0^{\partial R_1} + h^{-\frac{3}{2}}|g-\chi_2|_0^{\partial R}$$

$$\leqslant C(\|f_1\|_\lambda^{R_1} + \|f_2\|_\lambda^{R_2} + h^{-\frac{3}{2}+\lambda_1}|\phi_1|_{\lambda_1}^{\partial R_1} + h^{-\frac{1}{2}+\lambda_2}|\phi_2|_{\lambda_2}^{\partial R_1} + h^{-\frac{3}{2}+\lambda_3}|g|_{\lambda_3}^{\partial R}) \ .$$

Let u be the solution of (5.1) for given data $F = (f_1, f_2, \phi_1 \phi_2, g)$. The approximation scheme which we shall consider is the following:

Let $\tilde{S}^h_{k,r} = S^h_{k,r}(R_1) \times S^h_{k,r}(R_2)$ be given with $2 \leqslant k < r$. Find $w \in \tilde{S}^h_{k,r}$ such that

$$(5.3) \quad \int_{R_1} (f_1-\Delta w_1)\Delta\chi dx + \int_{R_2} (f_2-\Delta w_2)\Delta\chi dx$$

$$+ h^{-3}\int_{\partial R_1} (\phi_1(a_1 w_1 - w_2))(a_1\chi_1 - \chi_2)\,d\sigma_1$$

$$+ h^{-1}\int_{\partial R_1} \left(\phi_2 - \frac{\partial(a_2 w_1 - w_2)}{\partial n}\right) \frac{\partial(a_2\chi_1 - \chi_2)}{\partial n}\,d\sigma_1$$

$$+ h^{-3}\int_{\partial R} (g - w_2)\chi_2\,d\sigma_2 \quad = \quad 0$$

for all $\chi = (\chi_1, \chi_2) \in \tilde{S}^h_{k,r}$.

It is easily seen that the solution of (5.3) exists and is unique and, as in the scheme (4.1), is determined by solving a linear system of algebraic equations whose coefficients depend only on the data and $\tilde{S}^h_{k,r}$.

Let w be the solution of (5.3). Then using methods analogous to those used in the proof of Theorem 4.1 we can (restricting ourselves for simplicity to estimates for $|||u-w|||_0$) obtain the following error estimates.

<u>Theorem 5.3</u>. Suppose that u is the solution of (5.1) for

$$F = (f_1, f_2, \phi_1, \phi_2, g) \in H^{\beta-2}(R_1) \times H^{\beta-2}(R_2) \times H^{\beta-\frac{1}{2}}(\partial R_1)$$
$$\times H^{\beta-\frac{3}{2}}(\partial R_1) \times H^{\beta-\frac{1}{2}}(\partial R) \ .$$

For given $\tilde{S}^h_{k,r}$, with $2 \leq k < r$, $r \geq 4$, satisfying (**) let w be the solution of the approximate problem (5.3). Then there exists a constant C independent of h and F such that

$$
\begin{aligned}
(5.4) \qquad \|u-w\|_0 &\leq Ch^\beta \|u\|_\beta \\
&\leq Ch^\beta(\|f_1\|_{\beta-2}^{R_1} + \|f_2\|_{\beta-2}^{R_2} + |\phi_1|_{\beta-\frac{3}{2}}^{\partial R_1} \\
&\qquad + |\phi_2|_{\beta-\frac{3}{2}}^{\partial R_2} + |g|_{\beta-\frac{1}{2}}^{\partial R})
\end{aligned}
$$

for each $2 \leq \beta \leq r$.

Analogous schemes for more general problems with discontinuous coefficients can be treated in much the same manner as in the present example. A full account is to be found in [15]. A specific application of the above scheme is given in example 3 in the next section.

6. Examples

In this section we shall illustrate the main results of the previous sections with specific examples in special cases.

Example 1. Consider Dirichlet's problem

$$
(6.1) \qquad \begin{aligned} \Delta u + qu &= f \quad \text{in} \quad R \\ u &= g \quad \text{on} \quad \partial R \end{aligned}
$$

where $q = q(x)$ is any smooth function for which the solution of (5.1) is unique.

For given $S_{k,r}^h$ the approximation scheme (4.1) consists in finding $w \in S_{k,r}^h$ such that

$$
\int_R (f-(\Delta w+qw))(\Delta\phi+q\phi)dx + h^{-3}\int_{\partial R}(g-w)\phi d\sigma = 0
$$

for all $\phi \in S^h_{k,r}$.

Let us first consider error estimates over R .

Case 1. Suppose that we take $k = 2$, $r = 4$ and $S^h_{k,r}$ to be say subic splines or cubic Hermite polynomials (cf. [10],[11]). If w is the solution of the approximate problem (4.1), then (4.3) of Theorem 4.1 yields

$$\|u-w\|_0 \leq Ch^\beta \|u\|_\beta$$

for any $2 \leq \beta \leq 4$. In particular the maximum rate of convergence is given by

$$\|u-w\|_0 \leq Ch^4 \|u\|_4 .$$

If the data are smoother and we take $k = 2$, $r = 6$ and $S^h_{k,r}$ to be say quintic splines then we obtain

$$\|u-w\|_0 \leq Ch^6 \|u\|_6 .$$

In general if $u \in H^r(R)$ then we obtain using any $S^h_{2,r}$ for $r \gtrsim 4$

$$\|u-w\|_0 \leq Ch^r \|u\|_r .$$

If we take $k = 2$, $r = 3$ and $S_{k,r}$ to be say quadratic splines, then (4.5) of Theorem 4.1 yields

$$\|u-w\|_{\frac{1}{2}} \leq Ch^2 \|u\|_3 .$$

Interior estimates for the error may be obtained from Theorem 4.2. We again take w to be the solution of the approximate problem (4.1).

If we take $k = 2$ and $r = 4$ we then have that

$$\|u-w\|_\rho^{R_1} \leqslant Ch^{4-\rho} \|u\|_4$$

for any $0 \leqslant \rho \leqslant 2$. If we were to take $k = 2$ and $r = 3$ then we obtain

$$\|u-w\|_\rho^{R_1} \leqslant Ch^{3-\rho} \|u\|_3$$

where in this case ρ is restricted to $1 \leqslant \rho \leqslant 2$.

We remark that in this example the function q is not required to satisfy a sign condition and hence, for example q could be a positive constant which lies between two fixed membrane eigenvalues. In this case the usual bilinear form associated with the problem (6.1) with $g = 0$ would be

$$\sum_{j=0}^{N} \int_R \frac{\partial\phi}{\partial x_i} \frac{\partial\psi}{\partial x_i} dx - q\int_R \phi\psi dx = 0$$

which is not positive definite. Hence in the classical Rayleigh-Ritz method (for a given h) even the existence is unclear. This difficulty is not present in our method, since, in view of the uniqueness of solutions, the forms associated with (4.1) are always positive definite. The results of this example do not change if instead of $\Delta u + qu$ we take any second order uniformly elliptic operator with smooth coefficients for which the solution of Dirichlet's problem is unique.

Example 2. Let us briefly consider Dirichlet's problem for the biharmonic operator

$$\Delta^2 u = f \quad \text{in} \quad R$$

$$(6.2) \qquad \left.\begin{aligned} u &= g_0 \\ \frac{\partial u}{\partial n} &= g_1 \end{aligned}\right\} \quad \text{on} \quad R\ .$$

In this case the approximation scheme, for given $S^h_{k,r}$, consists in finding a $w \varepsilon S^h_{k,r}$ such that

$$\int_R (f-\Delta^2 w)\Delta^2\phi dx + h^{-7}\int_{\partial R}(g_0-w)\phi d\sigma$$

$$+ h^{-5}\int_{\partial R}\left(g_1 - \frac{\partial w}{\partial n}\right)\frac{\partial \phi}{\partial n}\, d\sigma = 0$$

for all $\phi \varepsilon S^h_{k,r}$.

If we take $k = 4$, $r = 6$ and $S^h_{k,r}$ to be say quintic splines (cf. [10]) we obtain from (4.5) of Theorem 4.1 that

$$\|u-w\|_{\frac{1}{2}} \leqslant Ch^4 \|u\|_6\ .$$

If $k = 4$ and $r = 8$ we obtain from (4.3) that

$$\|u-w\|_0 \leqslant Ch^8 \|u\|_8\ .$$

Interior estimates may be obtained from Theorem 4.1. Let R_1 be any compact subset of R. In the case $k = 4$ and $r = 6$ we obtain

$$\|u-w\|_\rho^{R_1} \leq Ch^{6-\rho} \|u\|_6$$

for any $2 \leq \rho \leq 4$. If we take $k = 4$ and $r = 8$ then

$$\|u-w\|_\rho^{R_1} \leq Ch^{8-\rho} \|u\|_8$$

for any $0 \leq \rho \leq 4$.

Example 3. In the approximation scheme (5.3) for interface problems, we have used two independent sets of trial function, namely $S^h_{k,r}(R_1)$ and $S^h_{k,r}(R_2)$. There are situations which arise in approximating the solution of (2.2) in which the scheme (5.3) will lead to better approximations than those obtained from (4.1). For example consider the Dirichlet problem

$$\begin{aligned} \Delta u &= f \quad \text{in} \quad R \\ u &= 0 \quad \text{on} \quad \partial R \end{aligned} \tag{6.3}$$

for which f is a piecewise smooth function. Say for simplicity that

$$f = \begin{cases} 1 & \text{on} \quad R_1 \\ 0 & \text{on} \quad R_2 \end{cases},$$

where R_1 and R_2 are as described in the interface problem.

Now certainly $f \in H^0(R) = L_2(R)$ and $u \in H^2(R)$. If $w \in S^h_{k,r}(R)$ with $k \geq 2$, $r \geq 4$ is the solution of the approximation scheme (4.1) we obtain the estimate

$$\|u-w\|_0 \leq Ch^2 \|u\|_2 . \tag{6.4}$$

However we have not made full use of the smoothness properties of f. If we set $u = u_i$ in R_i, $i = 1,2$. Then u is the solution of the interface problem

$$\begin{aligned} &\Delta u_1 = 1 \quad \text{in} \quad R_1 , \quad \Delta u = 0 \quad \text{in} \quad R_2 , \\ &u_1 - u_2 = 0 , \ \frac{\partial u_1}{\partial n} - \frac{\partial u_2}{\partial n} = 0 \quad \text{on} \quad \partial R_1 , \\ &u_2 = 0 \quad \text{on} \quad \partial R_2 , \end{aligned} \tag{6.5}$$

where in this case $u_i \in C^\infty(\overline{R}_i)$, $i = 1,2$. If we now use the approximation scheme (5.3) for given $\tilde{S}^h_{k,r}$ with $k \geq 2$ and $r \geq 4$ we obtain

$$|||u-w|||_0 \leq Ch^r |||u|||_r \tag{6.6}$$

where $w = (w_1,w_2)$ is the solution of (5.3). Thus in contrast to (6.4) the order of convergence in (6.6) is limited only by the "goodness" of the subspaces used.

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