

# A finite element method for interface problems in domains with smooth boundaries and interfaces

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This paper is concerned with the analysis of a finite element method for nonhomogeneous second order elliptic interface problems on smooth domains. The method consists in approximating the domains by polygonal domains, transferring the boundary data in a natural way, and then applying a finite element method to the perturbed problem on the approximate polygonal domains. It is shown that the error in the finite element approximation is of optimal order for linear elements on a quasiuniform triangulation. As such the method is robust in the regularity of the data in the original problem.

**Keywords:** Finite element method, interface problems.

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## 1. Introduction

In a recent paper [5] we analyzed the overall error in a simple finite element method for nonhomogeneous Dirichlet problems. We showed, when a smooth domain is replaced

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by a polygonal domain and the boundary data are transferred in a natural way, that a particular finite element method applied to the perturbed problem is optimal order accurate. It is shown that the overall approximation is robust in the regularity of the boundary data in the sense that optimal order accuracy is obtained for rough and smooth boundary data alike.

In this paper we provide a similar analysis for an elliptic interface problem. We do not consider the effect of numerical integration in the practical calculation of the quadratic forms. For this aspect we refer the reader to Ciarlet [9], Ciarlet and Raviart [10], and Barrett and Elliott [3]. For the calculation of the data in the nonsmooth case we note that in practical situations this often occurs when the given functions are piecewise smooth and fail to be globally smooth due to point singularities. For example, in a boundary control problem the control may be piecewise linear and therefore would only belong to a boundary Sobolev space provided the index is less than three halves. Such cases are easily handled by appropriate numerical integration schemes. The case of piecewise constant boundary values, although not explicitly covered in theorem 1, could also be handled by piecewise linear approximation with only an arbitrarily small loss in the rate of convergence. These issues will not be further discussed in this paper.

Suppose  $\Omega, \Omega_0 \subset \mathbf{R}^2$  are bounded domains whose boundaries  $\Gamma, \Gamma_0$  are smooth, say of class  $C^\infty$ , with  $\overline{\Omega}_0 \subset \Omega$ . We set  $\Omega_1 = \Omega \setminus \overline{\Omega}_0$  and consider the following interface problem on  $\Omega$ :

$$\begin{aligned} A^1 u^1 &= f & \text{in } \Omega_1, & & A^0 u^0 &= f & \text{in } \Omega_0, \\ u^1 - u^0 &= q_0 & \text{on } \Gamma_0, & & & & \\ \frac{\partial u^1}{\partial \nu_{A^1}} - \frac{\partial u^0}{\partial \nu_{A^0}} &= q_1 & \text{on } \Gamma_0, & & & & \\ u^1 &= g & \text{on } \Gamma, & & & & \end{aligned} \tag{1.1}$$

where  $\partial w / \partial \nu_{A^k}$  denotes the conormal derivative on  $\Gamma_0$ :

$$\frac{\partial w}{\partial \nu_{A^k}}(x) = \sum_{i,j=1}^2 a_{ij}^k(x) \frac{\partial w}{\partial x_i}(x) \nu_j^0, \quad x \in \Gamma_0,$$

and  $\nu^0$  denotes the unit outward (with respect to  $\Omega_0$ ) normal to  $\Gamma_0$ . Here  $A^0, A^1$  are the second order uniformly elliptic operators

$$A^k = - \sum_{i,j=1}^2 \frac{\partial}{\partial x_i} \left( a_{ij}^k(x) \frac{\partial}{\partial x_j} \right), \quad x \in \Omega_k, \quad k = 0, 1,$$

with smooth coefficients  $a_{ij}^k \in C^2(\Omega'_k)$  where  $\overline{\Omega}_k \subset \Omega'_k$ .

Let  $\Omega_h$  be a polygonal domain with quasiuniformly spaced vertices  $x^{(1)}, \dots, x^{(N+1)} \in \Gamma$  with  $x^{(1)} = x^{(N+1)}$ . Denote the "half open" edge from  $x^{(j)}$  to  $x^{(j+1)}$  by

$\Gamma_h^{(j)}$ . Similarly,  $\Gamma^{(j)}$  denotes that part of  $\Gamma$  between  $x^{(j)}$  and  $x^{(j+1)}$ . In the same way let  $\Omega_{0,h}$  be a polygonal domain with quasiuniformly spaced vertices  $y^{(1)}, \dots, y^{(K+1)} \in \Gamma_0$  and corresponding edges  $\Gamma_{0,h}^{(j)}$  and arcs  $\Gamma_0^{(j)}$ . Here  $h$  denotes an upper bound for the length of the longest edge and  $N = N_h$ ,  $K = K_h$  are the respective numbers of boundary edges. For  $h$  sufficiently small, the distance between  $\Gamma$  and  $\Gamma_h$ , the boundary of  $\Omega_h$ , given by

$$d(\Gamma, \Gamma_h) = \max_{x \in \Gamma_h} \{ |t| : x + t\nu_h \in \Gamma \},$$

where  $\nu_h$  denotes the unit outward normal to  $\Gamma_h$ , satisfies

$$d(\Gamma, \Gamma_h) \leq Ch^2.$$

Similarly we assume that  $d(\Gamma_0, \Gamma_{0,h}) \leq Ch^2$  where  $\Gamma_{0,h}$  denotes the boundary of  $\Omega_{0,h}$ . We assume the length of  $\Gamma_h^{(j)}$ ,  $h_j$ , satisfies  $\kappa h \leq h_j \leq h$  where  $\kappa$  is independent of  $h$ , and similarly for the length of  $\Gamma_{0,h}^{(j)}$ . Finally, define  $\Omega_{1,h} := \Omega_h \setminus \overline{\Omega_{0,h}}$ , and note that for  $h$  sufficiently small  $\Omega_{k,h} \subset \Omega'_k$ .

For problem (1.1) we consider a finite element method in which the domains  $\Omega_0$  and  $\Omega_1$  are replaced by polygonal domains  $\Omega_{0,h}$  and  $\Omega_{1,h}$  with  $\overline{\Omega_{k,h}} \subset \Omega'_k$ , the Dirichlet data  $g$  and jump data  $q_0$ ,  $q_1$  are transferred as  $g_h$ , and  $q_{0,h}$ ,  $q_{1,h}$  to the polygonal boundary  $\Gamma_h$  and interface  $\Gamma_{0,h}$  respectively. We determine an approximation  $u_h$  to  $u$  which may be thought of as being obtained by a simple finite element method, using linear elements, applied to the perturbed problem

$$\begin{aligned} A^1 w^1 &= f & \text{in } \Omega_{1,h}, & & A^0 w^0 &= f & \text{in } \Omega_{0,h}, \\ w^1 - w^0 &= q_{0,h} & \text{on } \Gamma_{0,h}, & & & & \\ \frac{\partial w^1}{\partial \nu_{h,A^1}} - \frac{\partial w^0}{\partial \nu_{h,A^0}} &= q_{1,h} & \text{on } \Gamma_{0,h}, & & & & \\ w^1 &= g_h & \text{on } \Gamma_h. & & & & \end{aligned} \tag{1.2}$$

Our approach is somewhat similar to one of the methods considered by Barrett and Elliott [3] wherein isoparametric fitting of the interface is analyzed for problem (1.1) with homogeneous jump data on  $\Gamma_0$  and homogeneous Dirichlet data is imposed on  $\Gamma$ . In [3] the authors also analyze a penalty method for handling the zero jump condition in function values on the approximate interface. Much earlier Babuška [1] and King [16] had analyzed penalty methods for interface problems in the absence of “variational crimes”, that is, when the interface is not replaced by an approximate interface. See also the work of Baker [2].

For exterior interface problems see the work of MacCamy and Marin [18], Goldstein [13], Bramble and Pasciak [6], and the references contained therein. For the treatment of nonlinear elliptic interface problems by finite element methods we refer the reader to Ženíšek [21] and the work of Feistauer and Sobotíková [12].

We now give a brief outline of the paper. In the next section we define the Sobolev spaces pertinent to our problem and give some perturbation estimates for the interface and boundary data. We also introduce certain piecewise polynomial spaces on the interface and outer boundary. We introduce the finite element method that defines our approximate solution,  $u_h$ . Section 3 contains the bulk of our error analysis and involves another finite element approximation,  $v_h$ , that is superclose (see, e.g., lemma 5) to  $u_h$ . Our main results show that optimal order accuracy is attained for rough as well as smooth interface and boundary data. In the appendix we give the somewhat lengthy and technical proofs of key lemmas from section 3.

## 2. Function spaces and the approximate problem

### 2.1. Interface problem: weak formulation

For a bounded open and connected set  $D \subset \mathbf{R}^2$  we denote by  $H^k(D)$  the usual Sobolev space of integer order  $k \geq 0$  with norm  $\|\cdot\|_{k,D}$ . The inner product on  $L_2(D) = H^0(D)$  is given by

$$(v, w)_D = \int_D v(x)w(x) dx.$$

We define the seminorm on  $H^1(D)$  by

$$|w|_{1,D} = \left\{ \sum_{j=1}^2 \left\| \frac{\partial w}{\partial x_j} \right\|_{0,D}^2 \right\}^{1/2}$$

and note that the seminorm is equivalent to the norm  $\|\cdot\|_{1,D}$  for functions that vanish on a subset of  $\partial D$  having positive measure.

For a bounded open set, say  $G = \bigcup_{j=1}^m D_j$ , where  $\{D_j\}$  are the open mutually disjoint components of  $G$ , we denote by  $H^k(G)$  the Sobolev space consisting of functions  $w$  such that  $w|_{D_j} \in H^k(D_j)$  with norm

$$\|w\|_{k,G} = \left( \sum_{j=1}^m \|w\|_{k,D_j}^2 \right)^{1/2}.$$

We denote the seminorm on  $H^1(G)$  as

$$|w|_{1,G} = \left( \sum_{j=1}^m |w|_{1,D_j}^2 \right)^{1/2}.$$

Let  $H^k(\partial D)$  denote the Sobolev space of integer order  $k \geq 0$  on  $\partial D$  with norm denoted by  $|\cdot|_{k,\partial D}$ . The inner product on  $L_2(\partial D)$  is given by

$$\langle v, w \rangle_{\partial D} = \int_{\partial D} vw ds.$$

For real  $r \geq 0$ , the spaces  $H^r(\Omega_k)$ ,  $H^r(\Omega_0 \cup \Omega_1)$ ,  $H^r(\Gamma)$ , and  $H^r(\Gamma_0)$  are defined by interpolation (see Lions and Magenes [17] and Grisvard [14, 15]).

As usual  $H_0^1(\Omega)$  denotes the Sobolev space of order one whose elements have zero trace on  $\Gamma$  and similarly for  $H_0^1(\Omega_h)$ . We define, for  $r > 0$ , the spaces  $H^{-r}(\Gamma)$ ,  $H^{-r}(\Gamma_0)$  as the duals of  $H^r(\Gamma)$  and  $H^r(\Gamma_0)$  respectively. The norm on  $H^{-r}(\Gamma)$  is given by

$$|v|_{-r,\Gamma} = \sup_{\psi \in H^r(\Gamma)} \frac{\langle v, \psi \rangle}{|\psi|_{r,\Gamma}}, \quad r > 0,$$

and analogously for  $H^{-r}(\Gamma_0)$ . Associated with the elliptic operator  $A^k$  is the bilinear form

$$a^k(v, w) = \sum_{i,j=1}^2 \int_{\Omega_k} a_{ij}^k(x) \frac{\partial v}{\partial x_i} \frac{\partial w}{\partial x_j} dx, \quad v, w \in H^1(\Omega_k).$$

The weak form of problem (1.1) is as follows: For  $f \in L_2(\Omega)$ ,  $q_0 \in H^{r-1/2}(\Gamma_0)$ ,  $q_1 \in H^{r-3/2}(\Gamma_0)$  and  $g \in H^{r-1/2}(\Gamma)$  with  $1 \leq r \leq 2$ , find  $u \in H^1(\Omega_0 \cup \Omega_1)$  such that

$$\begin{aligned} a(u, \varphi) &= (f, \varphi)_\Omega - \langle q_1, \varphi \rangle_{\Gamma_0}, \quad \varphi \in H_0^1(\Omega) \quad \text{and} \\ [u] &= q_0 \quad \text{on } \Gamma_0, \quad u = g \quad \text{on } \Gamma \end{aligned} \quad (2.1)$$

where  $[u] = u^1 - u^0$  and  $u^i = u|_{\Omega_i}$ .

From [19] we cite the following estimate for the solution  $u \in H^r(\Omega_0 \cup \Omega_1)$  of (2.1):

$$\|u\|_{r,\Omega_0 \cup \Omega_1} \leq C \{ \|f\|_{0,\Omega} + |g|_{r-1/2,\Gamma} + |q_0|_{r-1/2,\Gamma_0} + |q_1|_{r-3/2,\Gamma_0} \}, \quad 0 \leq r \leq 2. \quad (2.2)$$

In (2.2) and throughout the paper we use  $C$  to denote a generic positive constant that does not depend on  $h$  and we shall always assume that  $f \in L_2(\Omega)$ . The *a priori* estimate (2.2) can be used to define a generalized (very weak) solution of (1.1) for  $g \in H^{r-1/2}(\Gamma)$ ,  $q_0 \in H^{r-1/2}(\Gamma_0)$ , and  $q_1 \in H^{r-3/2}(\Gamma_0)$  for  $0 \leq r < 1$ . Precisely, let  $\{g^n\}$ ,  $\{q_0^n\}$  and  $\{q_1^n\}$  be sequences of smooth functions converging to  $g$  in  $H^{r-1/2}(\Gamma)$ ,  $q_0$  in  $H^{r-1/2}(\Gamma_0)$ , and  $q_1$  in  $H^{r-3/2}(\Gamma_0)$  respectively. Because of (2.2) it follows that  $\{u^n\}$  is a Cauchy sequence. Its limit,  $u \in H^r(\Omega_0 \cup \Omega_1)$  is defined to be the weak solution of (1.1).

## 2.2. The approximate problem and main results

We define a means of transferring data on  $\Gamma_0$  to  $\Gamma_{0,h}$  that is identical to our treatment of inhomogeneous Dirichlet data in [5]. This approach for transferring the boundary and interface data is certainly not new, having been used in Bramble et al. [4], Dupont [11], and Ženíšek [20]. Denote the unit outward normal to  $\Gamma_{0,h}^{(j)}$  by  $\nu_{0,h}^{(j)}$  and let  $x_h(t)$  denote the parameterization of  $\Gamma_{0,h}$  by arc length. This induces the following parametrization on  $\Gamma_0$ :

$$X_h(t) := x_h(t) + \delta_{x_h(t)} \nu_{0,h}^{(j)}, \quad (2.3)$$

where  $|\delta_{x_h(t)}|$  is the distance between  $x_h(t)$  and  $\Gamma_0$  along  $\nu_{0,h}^{(j)}$ . We assume  $h$  is small enough that  $X_h(t)$  is well defined. Then define for a given  $q \in L_2(\Gamma_0)$

$$\tilde{q}(x_h(t)) := q(X_h(t)), \quad x_h(t) \in \Gamma_{0,h}^{(j)}, \quad (2.4)$$

and note that  $\tilde{q}(x) = q(x)$  for  $x = x^{(j)}$  or  $x = x^{(j+1)}$  if  $q$  is defined there. The mapping defined by (2.4) is bounded in  $L_2(\Gamma_0)$  and moreover the inverse mapping is well defined and bounded there. That is, for some constants  $c$  and  $C$ , independent of  $h$ ,

$$c|q|_{0,\Gamma_0} \leq |\tilde{q}|_{0,\Gamma_{0,h}} \leq C|q|_{0,\Gamma_0}. \quad (2.5)$$

The inverse of the “tilde” mapping, called the “hat” map, is given by

$$\hat{q}(X_h(t)) := q(x_h(t)), \quad x_h(t) \in \Gamma_{0,h}^{(j)}, \quad (2.6)$$

where  $q \in L_2(\Gamma_{0,h})$ .

By direct analogy we define the “tilde” mapping from  $L_2(\Gamma)$  to  $L_2(\Gamma_h)$  and the following estimate holds for  $g \in L_2(\Gamma)$ :

$$c|g|_{0,\Gamma} \leq |\tilde{g}|_{0,\Gamma_h} \leq C|g|_{0,\Gamma}. \quad (2.7)$$

We also denote its inverse by the “hat” mapping. This slight abuse of notation should cause the reader no difficulty since the meaning of the tilde (hat) map will be clear from the context in which it is used.

In our finite element method we need the corresponding form,  $a_h^k(\cdot, \cdot)$  on  $\Omega_{k,h}$ , that is,

$$a_h^k(v, w) = \sum_{i,j=1}^2 \int_{\Omega_{k,h}} a_{ij}^k(x) \frac{\partial v}{\partial x_i} \frac{\partial w}{\partial x_j} dx, \quad v, w \in H^1(\Omega_{k,h}),$$

where we assume  $h$  is sufficiently small so that  $\Omega_{k,h} \subset \Omega'_k$  for  $k = 0, 1$ . For  $w, v \in H^1(\Omega_0 \cup \Omega_1)$  we define

$$a(v, w) = a^0(v, w) + a^1(v, w).$$

Similarly,  $a_h(\cdot, \cdot)$  denotes the form:

$$a_h(v, w) = a_h^0(v, w) + a_h^1(v, w),$$

for  $w, v \in H^1(\Omega_{0,h} \cup \Omega_{1,h})$ .

Moreover we need to introduce a form that is a decomposition of the form  $a(\cdot, \cdot)$  in a special way relative to  $\Omega_h$ . Define, for  $v, w \in H^1(G)$ , where  $G$  is the union of the open sets  $\Omega_{0,h} \cap \Omega_0$ ,  $\Omega_0 \setminus \overline{\Omega_{0,h}}$ ,  $\Omega_{1,h} \cap \Omega_1$ ,  $\Omega_{0,h} \setminus \overline{\Omega_0}$ , and  $\Omega_h \setminus \overline{\Omega}$  (note that  $\overline{G} = \overline{\Omega_h}$ )

$$a_{\Omega_{0,h}}(v, w) = a_{\Omega_{0,h} \cap \Omega_0}^0(v, w) + a_{\Omega_{0,h} \setminus \overline{\Omega_0}}^1(v, w),$$

where  $a_D^k(\cdot, \cdot)$  indicates that integration is carried out over  $D$ . Define

$$a_{\Omega_{1,h}}(v, w) = a_{\Omega_{1,h} \cap \Omega_1}^1(v, w) + a_{\Omega_0 \setminus \overline{\Omega}_{0,h}}^0(v, w) + a_{\Omega_h \setminus \overline{\Omega}}^1(v, w),$$

and finally,

$$\tilde{a}(v, w) = a_{\Omega_{0,h}}(v, w) + a_{\Omega_{1,h}}(v, w).$$

Note that the form  $a(\cdot, \cdot)$  is not well defined for functions having jumps on the polygonal interface,  $\Gamma_{0,h}$ , but the form  $\tilde{a}(\cdot, \cdot)$  is well defined for functions having jumps on the smooth interface  $\Gamma_0$  or the polygonal interface,  $\Gamma_{0,h}$ .

In several places in our analysis we need to use bounded linear extension operators (cf. Lions and Magenes [17] and Grisvard [14])  $E_k: H^r(\Omega_k) \rightarrow H^r(\mathbf{R}^2)$ , with  $0 \leq r \leq 2$ , satisfying  $E_k \phi|_{\Omega_k} = \phi$  for  $\phi \in H^r(\Omega_k)$  and

$$\|E_k \phi\|_{r, \mathbf{R}^2} \leq C \|\phi\|_{r, \Omega_k}, \quad k = 0, 1.$$

We shall make the convention that an arbitrary function  $w \in H^r(\Omega_0 \cup \Omega_1)$  has been extended (outside of  $\Omega$ ) to all of  $\mathbf{R}^2$  by the operator  $E_1$  and with a slight abuse of notation also call the extended function  $w$ . We shall denote by  $W \in H^r(\Omega_{0,h} \cup \Omega_{1,h})$  the function defined on  $\Omega_{k,h}$  by

$$W = E_k w|_{\Omega_{k,h}}. \quad (2.8)$$

That is, on  $\Omega_{k,h}$ ,  $W$  is the restriction to  $\Omega_{k,h}$  of the  $E_k$ -extension of  $w$ . Then it is easy to see that

$$\|W\|_{r, \Omega_{0,h} \cup \Omega_{1,h}} \leq C \|w\|_{r, \Omega_0 \cup \Omega_1}, \quad 0 \leq r \leq 2. \quad (2.9)$$

Recall that  $\partial w / \partial \nu_{A^k}$  denotes the conormal derivative on  $\Gamma_0$ :

$$\frac{\partial w}{\partial \nu_{A^k}}(x) = \sum_{i,j=1}^2 a_{ij}^k(x) \frac{\partial w}{\partial x_i}(x) \nu_j^0, \quad x \in \Gamma_0,$$

where  $\nu^0$  denotes the unit outward (with respect to  $\Omega_0$ ) normal to  $\Gamma_0$ . The conormal derivative on  $\Gamma$  is denoted by  $\partial w / \partial \eta_{A^1}$ , on  $\Gamma_h$  by  $\partial w / \partial \eta_{h,A^1}$  and on  $\Gamma_{0,h}$  we use the notation  $\partial w / \partial \nu_{h,A^k}$  where  $\nu_h$  denotes the unit outward (with respect to  $\Omega_{0,h}$ ) normal to  $\Gamma_{0,h}$ .

To define and analyze our finite element method we now introduce certain spaces of piecewise polynomial functions on  $\Gamma_0$ ,  $\Gamma_{0,h}$ ,  $\Gamma$  and  $\Gamma_h$ . Let  $S_h(\Gamma_{0,h})$  consist of piecewise linear functions of  $t$  (arc length on  $\Gamma_{0,h}$ ) on  $\Gamma_{0,h}$ , continuous on  $\Gamma_{0,h}$ , and linear on each edge,  $\Gamma_{0,h}^{(j)}$ . We define the space  $S_h(\Gamma_0)$  to consist of all functions of the form

$$\hat{\phi}(X_h(t)) = \phi(x_h(t)),$$

where  $X_h(t)$  is given by (2.3) and  $\phi \in S_h(\Gamma_{0,h})$ . Clearly  $S_h(\Gamma_0)$  is the space of continuous piecewise linear (with respect to parameter  $t$ ) functions on  $\Gamma_0$ . That is,

$\phi \in S_h(\Gamma_0)$  if  $\phi$  is continuous and on each arc,  $\Gamma_0^{(j)}$ , is a polynomial of degree less than or equal to one in  $t$  (arc length on  $\Gamma_{0,h}$ ). We define the corresponding orthogonal projectors  $Q_{0,h}: L_2(\Gamma_{0,h}) \rightarrow S_h(\Gamma_{0,h})$  and  $\widehat{Q}_{0,h}: L_2(\Gamma_0) \rightarrow S_h(\Gamma_0)$  by

$$\langle Q_{0,h}\psi, \chi \rangle_{\Gamma_{0,h}} = \langle \psi, \chi \rangle_{\Gamma_{0,h}}, \quad \chi \in S_h(\Gamma_{0,h}),$$

and

$$\langle \widehat{Q}_{0,h}\psi, \chi \rangle_{\Gamma_0} = \langle \psi, \chi \rangle_{\Gamma_0}, \quad \chi \in S_h(\Gamma_0).$$

Using the analogous construction on  $\Gamma_h$  we define the spaces,  $S_h(\Gamma_h)$  and  $S_h(\Gamma)$ , of piecewise linear functions. We denote the corresponding orthogonal projectors by  $Q_h: L_2(\Gamma_h) \rightarrow S_h(\Gamma_h)$  and  $\widehat{Q}_h: L_2(\Gamma) \rightarrow S_h(\Gamma)$ .

Let  $\{\mathcal{T}_h\}$  denote a family of triangulations of  $\Omega_h$  such that for disjoint index sets,  $I$  and  $J$ ,  $\mathcal{T}_h^0 = \bigcup_{i \in I} \tau_h^{(i)}$  and  $\mathcal{T}_h^1 = \bigcup_{j \in J} \tau_h^{(j)}$  with:

$$\mathcal{T}_h = \mathcal{T}_h^0 \cup \mathcal{T}_h^1 \quad \text{with } \mathcal{T}_h^0 = \Omega_{0,h} \text{ and } \mathcal{T}_h^1 = \Omega_{1,h}.$$

We assume the families of polygonal domains,  $\{\Omega_{0,h}\}$  and  $\{\Omega_{1,h}\}$ , and corresponding family of triangulations,  $\{\mathcal{T}_h\}$ , satisfy the usual sort of quasiuniformity condition. The only vertices on  $\Gamma_h$  or  $\Gamma_{0,h}$  of a triangle  $\tau_h \in \mathcal{T}_h$  are vertices of  $\Gamma_h$  or  $\Gamma_{0,h}$  respectively, and every triangle  $\tau_h \in \mathcal{T}_h$  is affine equivalent to a reference triangle. Define the space  $V_h(\Omega_{k,h})$  to consist of continuous piecewise linear functions on  $\Omega_{k,h}$  relative to the triangulation  $\mathcal{T}_h$ . We denote by  $V_h$  the set of functions  $\phi \in H^1(\Omega_{0,h} \cup \Omega_{1,h})$  such that  $\phi|_{\Omega_{k,h}} \in V_h(\Omega_{k,h})$ . Note that a function  $\varphi \in V_h$  may have a jump across the interface  $\Gamma_{0,h}$ . On occasion we shall denote the restriction of  $\varphi \in V_h$  to  $\Omega_{k,h}$  by  $\varphi^k$ , that is,  $\varphi^k = \varphi|_{\Omega_{k,h}}$ . The boundary spaces  $V_h(\Gamma_h)$  and  $V_h(\Gamma_{0,h})$  denote the restrictions of  $V_h \cap H^1(\Omega_h)$  to  $\Gamma_h$  and  $\Gamma_{0,h}$  and coincide with  $S_h(\Gamma_h)$  and  $S_h(\Gamma_{0,h})$  respectively.

We define the approximate boundary data,  $g_h$ , and approximate interface jump,  $q_{0,h}$  by:

$$g_h := Q_h \widetilde{g}, \quad q_{0,h} := Q_{0,h} \widetilde{q}_0.$$

The approximate solution  $u_h$  of (1.1) is defined as follows:

**The Approximate Problem.** In  $\Omega_h$  let  $z_h \in V_h$  be the solution of

$$\begin{aligned} a_h(z_h, \phi) &= (f, \phi)_{\Omega_h} - \langle \widetilde{q}_1, \phi \rangle_{\Gamma_{0,h}}, \quad \phi \in V_h^0, \quad \text{with} \\ [z_h] &= q_{0,h} \text{ on } \Gamma_{0,h} \quad \text{and} \quad z_h = g_h \text{ on } \Gamma_h, \end{aligned} \tag{2.10}$$

where  $V_h^0 = V_h \cap H_0^1(\Omega_h)$ ,  $[z_h] = z_h^1 - z_h^0$ , and  $f = 0$  outside  $\Omega$ . Note that  $\widetilde{q}_1$  can be replaced by  $Q_{0,h} \widetilde{q}_1$  in (2.10).

Further define our approximate solution  $u_h$  in  $\Omega$  as follows. In  $\Omega_0 \cap \Omega_{0,h}$  and  $\Omega_1 \cap \Omega_{1,h}$  take  $u_h = z_h$ . We define  $u_h$  in  $\Omega_0 \setminus \Omega_{0,h}$ ,  $\Omega_1 \setminus \Omega_{1,h}$ , and  $\Omega \setminus \Omega_h$  as follows. Let  $\Omega_{0,h}^{(j)}$  denote a typical region in  $\Omega_0 \triangle \Omega_{0,h}$  bounded by  $\Gamma_{0,h}^{(j)}$  and  $\Gamma_0^{(j)}$ . For



$\Omega_{0,h}^{(j)} \subset \Omega_0 \setminus \Omega_{0,h}$  with  $\tau_h^{(j)} \in \mathcal{T}_h$ , the triangle in  $\Omega_{0,h}$  having  $\Gamma_{0,h}^{(j)}$  as one of its sides,  $u_h$  is the linear extension of  $z_h^0$  from  $\tau_h^{(j)}$  to  $\Omega_{0,h}^{(j)}$ . For  $\Omega_{0,h}^{(j)} \subset \Omega_{0,h} \setminus \Omega_0$  with  $\tau_h^{(j)} \in \mathcal{T}_h$  the triangle in  $\Omega_{1,h}$  having  $\Gamma_{0,h}^{(j)}$  as one of its sides,  $u_h$  is the linear extension of  $z_h^1$  from  $\tau_h^{(j)}$  to  $\Omega_{0,h}^{(j)}$ . For  $\Omega_h^{(j)} \subset \Omega \setminus \Omega_h$  a typical region in  $\Omega \setminus \Omega_h$  bounded by  $\Gamma_h^{(j)}$  and  $\Gamma^{(j)}$  we define  $u_h$  in the analogous way, that is, as the linear extension of  $z_h$  from the appropriate triangle in  $\Omega_h$ . Note that our definition of  $u_h$  is the most natural definition on all of  $\Omega$  relative to the interface  $\Gamma_0$ .

We shall denote by  $V_h(\Omega_0 \cup \Omega_1)$  the space of piecewise linear functions obtained from  $V_h$  by the natural extension (given above in the definition of  $u_h$ ).

Our main result states that  $u_h$  is an optimal order approximation to  $u$  on all of  $\Omega$  that is robust in the regularity of problem (1.1).

**Theorem 1.** There exists a constant  $C$ , independent of  $h$ , such that

$$\begin{aligned} \|u - u_h\|_{0,\Omega_0 \cup \Omega_1} &\leq C(h^2\|f\|_{0,\Omega} + h^{\ell+1/2}|q_1|_{\ell-1,\Gamma_0} + h^{s+1/2}|q_0|_{s,\Gamma_0} \\ &\quad + h^{r+1/2}|g|_{r,\Gamma}), \end{aligned} \quad (2.11)$$

where  $0 \leq r, s \leq 3/2$  and  $1 \leq \ell \leq 3/2$ . Moreover, for  $1/2 \leq r, s \leq 3/2$ , and  $1 \leq \ell \leq 3/2$

$$\begin{aligned} \|u - u_h\|_{1,\Omega_0 \cup \Omega_1} &\leq C(h\|f\|_{0,\Omega} + h^{\ell-1/2}|q_1|_{\ell-1,\Gamma_0} + h^{s-1/2}|q_0|_{s,\Gamma_0} \\ &\quad + h^{r-1/2}|g|_{r,\Gamma}). \end{aligned} \quad (2.12)$$

### 2.3. Preliminary estimates

We shall need an estimate for the difference between the trace on  $\Gamma_{0,h}$  and the tilde map of the trace on  $\Gamma_0$  for functions other than  $u$ . Thus we state the following result from [5] for an arbitrary function.

**Lemma 1.** Suppose  $w \in H^r(\Omega_k)$  (extended by  $E_k$ ) for  $k = 0$  or  $1$  and  $\gamma w = q$  denotes the trace of  $E_k w$  on  $\Gamma_0$ . Then for some constant  $C$ , independent of  $h$  and  $w$ ,

$$|\gamma_h w - \tilde{q}|_{0,\Gamma_{0,h}} \leq Ch^r \|w\|_{r,\Omega_k}, \quad 1 \leq r \leq 2, \quad (2.13)$$

where  $\gamma_h w$  is the trace of  $w$  on  $\Gamma_{0,h}$ . Similarly,

$$|\gamma^h w - \tilde{g}|_{0,\Gamma_h} \leq Ch^r \|w\|_{r,\Omega_1}, \quad 1 \leq r \leq 2, \quad (2.14)$$

where  $\gamma^h w$  is the trace of  $w$  on  $\Gamma_h$  and  $g$  is the trace of  $w$  on  $\Gamma$ .

The next estimates for conormal derivatives (cf. Dupont [11]) are useful.

**Lemma 2.** Suppose  $w \in H^r(\Omega_k)$  (extended by  $E_k$ ) for  $k = 0$  or  $1$ . Then for some constant  $C$ , independent of  $h$  and  $w$ ,

$$\left| \frac{\partial w}{\partial \nu_{h,A^k}} \right|_{0,\Gamma_{0,h}} \leq C \|w\|_{2,\Omega_k}$$

and

$$\left| \frac{\partial w}{\partial \eta_{h,A^1}} \right|_{0,\Gamma_h} \leq C \|w\|_{2,\Omega_1}.$$

We shall also need some estimates on the “skin” around  $\Gamma_h$  or  $\Gamma_{0,h}$ . First we introduce the symmetric difference of two sets  $D, G \subset \mathbf{R}^2$ :  $D \Delta G = (D \setminus G) \cup (G \setminus D)$ .

**Lemma 3.** Suppose  $w \in H^1(\Omega_0 \cup \Omega_1)$ . Then there exists a constant  $C$ , independent of  $h$  and  $w$ , such that

$$\|w\|_{0,\Omega_0 \Delta \Omega_{0,h}} \leq C (h |\gamma w|_{0,\Gamma_0} + h^2 |w|_{1,\Omega_0 \cup \Omega_1}). \quad (2.15)$$

Moreover, if  $w \in H^r(\Omega_0 \cup \Omega_1)$ , then

$$|w|_{1,\Omega_0 \Delta \Omega_{0,h}} \leq C h^{r-1} \|w\|_{r,\Omega_0 \cup \Omega_1}, \quad 1 \leq r \leq 2. \quad (2.16)$$

*Proof.* The first estimate is essentially proved in [5]. For the second estimate, consider a typical piece of  $\Omega_{0,h} \Delta \Omega_0$ , say  $\Omega_{0,h}^{(j)}$  bounded by  $\Gamma_0^{(j)}$  and  $\Gamma_{0,h}^{(j)}$ . Then by inequality (2.9) of [5] we have

$$|w|_{1,\Omega_{0,h}^{(j)}} \leq C \left( h \sum_{i=1}^2 \left| \frac{\partial w}{\partial x_i} \right|_{0,\Gamma_0^{(j)}} + h^2 \|w\|_{2,\Omega_{0,h}^{(j)}} \right)$$

and hence we obtain, for  $w \in H^2(\Omega_0 \cup \Omega_1)$ , by summing over the appropriate indices

$$|w|_{1,\Omega_{0,h} \Delta \Omega_0} \leq C \left( h \sum_{i=1}^2 \left| \frac{\partial w}{\partial x_i} \right|_{0,\Gamma_0} + h^2 \|w\|_{2,\Omega_0 \cup \Omega_1} \right).$$

Using the trace inequality

$$\sum_{i=1}^2 \left| \frac{\partial w}{\partial x_i} \right|_{0,\Gamma_0} \leq C \|w\|_{2,\Omega_0},$$

gives the estimate

$$|w|_{1,\Omega_{0,h} \Delta \Omega_0} \leq C h \|w\|_{2,\Omega_0 \cup \Omega_1}.$$

We also have the trivial estimate for  $w \in H^1(\Omega_0 \cup \Omega_1)$

$$|w|_{1,\Omega_{0,h} \Delta \Omega_0} \leq C \|w\|_{1,\Omega_0 \cup \Omega_1}$$

and hence by interpolation between the last two inequalities we obtain the required estimate.  $\square$

The following result is essentially proved in [5].

**Lemma 4.** Suppose  $w \in H^1(\Omega_1)$  and is extended by  $E_1$ . Then there exists a constant  $C$ , independent of  $h$  and  $w$ , such that

$$\|w\|_{0,\Omega_h \setminus \bar{\Omega}} \leq C(h|w|_{0,\Gamma} + h^2|w|_{1,\Omega_1}). \quad (2.17)$$

Moreover, if  $w \in H^2(\Omega_1)$  and extended by  $E_1$ , then

$$|w|_{1,\Omega_h \setminus \bar{\Omega}} \leq Ch\|w\|_{2,\Omega_1}. \quad (2.18)$$

#### 2.4. Properties of the finite element subspaces

The space  $S_h(\Gamma_0)$  satisfies the following approximation property. For  $\psi \in H^r(\Gamma_0)$

$$\inf_{\varphi \in S_h(\Gamma_0)} (|\psi - \varphi|_{0,\Gamma_0} + h|\psi - \varphi|_{1,\Gamma_0}) \leq Ch^r|\psi|_{r,\Gamma_0}, \quad 1 \leq r \leq 2. \quad (2.19)$$

From (2.19) we have

$$|(I - \hat{Q}_{0,h})\psi|_{0,\Gamma_0} = \inf_{\varphi \in S_h(\Gamma_0)} |\psi - \varphi|_{0,\Gamma_0} \leq Ch^2|\psi|_{2,\Gamma_0} \quad (2.20)$$

and the trivial estimate

$$|(I - \hat{Q}_{0,h})\psi|_{0,\Gamma_0} \leq |\psi|_{0,\Gamma_0}. \quad (2.21)$$

It follows, by interpolation between (2.20) and (2.21) that

$$|(I - \hat{Q}_{0,h})\psi|_{0,\Gamma_0} \leq Ch^r|\psi|_{r,\Gamma_0}, \quad 0 \leq r \leq 2. \quad (2.22)$$

It also follows from (2.19) that

$$\inf_{\varphi \in S_h(\Gamma_0)} (|\psi - \varphi|_{0,\Gamma_0} + h^{1/2}|\psi - \varphi|_{1/2,\Gamma_0}) \leq Ch^r|\psi|_{r,\Gamma_0}, \quad 1 \leq r \leq 2. \quad (2.23)$$

The following inverse property is standard:

$$|\varphi|_{1,\Gamma_0} \leq Ch^{-1}|\varphi|_{0,\Gamma_0}, \quad \varphi \in S_h(\Gamma_0),$$

and consequently, by interpolation,

$$|\varphi|_{s,\Gamma_0} \leq Ch^{-s}|\varphi|_{0,\Gamma_0}, \quad \varphi \in S_h(\Gamma_0), \quad 0 \leq s \leq 1. \quad (2.24)$$

Finally, using (2.22), (2.23), and (2.24) it follows that

$$|(I - \hat{Q}_{0,h})\psi|_{1/2,\Gamma_0} \leq Ch^{s-1/2}|\psi|_{s,\Gamma_0}, \quad 1/2 \leq s \leq 2, \quad (2.25)$$

and

$$|(I - \hat{Q}_{0,h})\psi|_{-p,\Gamma_0} \leq Ch^{s+p}|\psi|_{s,\Gamma_0}, \quad 0 \leq s \leq 2, \quad 0 \leq p \leq 3/2. \quad (2.26)$$

The space  $S_h(\Gamma)$  and corresponding orthogonal projector satisfy all of approximation and inverse properties analogous to those for  $S_h(\Gamma_0)$  and  $\hat{Q}_{0,h}$ . We do not list all of these properties since our main emphasis in this work concerns the treatment of the interface.

The following perturbation estimate (see [5, lemma 4]) plays an important role in our error analysis.

**Lemma 5.** Let  $q \in L_2(\Gamma_0)$ , then

$$|Q_{0,h}\tilde{q} - \widetilde{Q_{0,h}q}|_{0,\Gamma_{0,h}} \leq Ch^2 |(I - \widehat{Q}_{0,h})q|_{0,\Gamma_0},$$

and for  $g \in L_2(\Gamma)$

$$|Q_h\tilde{g} - \widetilde{Q_hg}|_{0,\Gamma_h} \leq Ch^2 |(I - \widehat{Q}_h)g|_{0,\Gamma},$$

where  $C$  is independent of  $h$ .

We shall need the following inverse property.

**Lemma 6.** There exists a constant  $C$ , independent of  $h$ , such that for  $q \in H^\ell(\Gamma_0)$

$$|\widehat{Q_{0,h}\tilde{q}}|_{1/2,\Gamma_0} \leq Ch^{\ell-1/2} |q|_{\ell,\Gamma_0}, \quad 0 \leq \ell \leq 1/2.$$

*Proof.* By the inverse property (2.24) we have

$$|\widehat{Q_{0,h}\tilde{q}}|_{1/2,\Gamma_0} \leq Ch^{-1/2} |\widehat{Q_{0,h}\tilde{q}}|_{0,\Gamma_0},$$

and hence by (2.5)

$$\begin{aligned} |\widehat{Q_{0,h}\tilde{q}}|_{1/2,\Gamma_0} &\leq Ch^{-1/2} |Q_{0,h}\tilde{q}|_{0,\Gamma_{0,h}} \\ &\leq Ch^{-1/2} |\tilde{q}|_{0,\Gamma_{0,h}} \leq Ch^{-1/2} |q|_{0,\Gamma_0}. \end{aligned} \quad (2.27)$$

We also have for  $q \in H^{1/2}(\Gamma_0)$ , using (2.5) and (2.24) again

$$\begin{aligned} |\widehat{Q_{0,h}\tilde{q}}|_{1/2,\Gamma_0} &\leq |\widehat{Q_{0,h}\tilde{q}} - \widehat{Q_{0,h}q}|_{1/2,\Gamma_0} + |\widehat{Q_{0,h}q}|_{1/2,\Gamma_0} \\ &\leq Ch^{-1/2} |\widehat{Q_{0,h}\tilde{q}} - \widehat{Q_{0,h}q}|_{0,\Gamma_0} + |\widehat{Q_{0,h}q}|_{1/2,\Gamma_0} \\ &\leq Ch^{-1/2} |Q_{0,h}\tilde{q} - \widetilde{Q_{0,h}q}|_{0,\Gamma_{0,h}} + |\widehat{Q_{0,h}q}|_{1/2,\Gamma_0}, \end{aligned}$$

and hence by lemma 5 and (2.25)

$$\begin{aligned} |\widehat{Q_{0,h}\tilde{q}}|_{1/2,\Gamma_0} &\leq Ch^{3/2} |(I - \widehat{Q}_{0,h})q|_{0,\Gamma_0} + |\widehat{Q_{0,h}q}|_{1/2,\Gamma_0} \\ &\leq Ch^2 |q|_{1/2,\Gamma_0} + |(I - \widehat{Q}_{0,h})q|_{1/2,\Gamma_0} + |q|_{1/2,\Gamma_0} \\ &\leq C|q|_{1/2,\Gamma_0}. \end{aligned} \quad (2.28)$$

The result follows by interpolation between the estimates (2.27) and (2.28).  $\square$

The next estimate gives an approximation property for the projector  $Q_{0,h}$ .

**Lemma 7.** There exists a constant  $C$ , independent of  $h$ , such that for  $q \in H^{p-3/2}(\Gamma_0)$  with  $3/2 \leq p \leq 2$  and  $i = 0, 1$

$$|q - \widehat{Q_{0,h}q}}|_{i-3/2, \Gamma_0} \leq Ch^{p-i} |q|_{p-3/2, \Gamma_0}.$$

*Proof.* By the triangle inequality, (2.5), and lemma 5,

$$\begin{aligned} |q - \widehat{Q_{0,h}q}}|_{i-3/2, \Gamma_0} &\leq |\widehat{Q_{0,h}q} - \widehat{Q_{0,h}\tilde{q}}|_{0, \Gamma_0} + |(I - \widehat{Q_{0,h}})q|_{i-3/2, \Gamma_0} \\ &\leq Ch^2 |(I - \widehat{Q_{0,h}})q|_{0, \Gamma_0} + |(I - \widehat{Q_{0,h}})q|_{i-3/2, \Gamma_0}, \end{aligned}$$

from which the result follows by properties (2.22), (2.26) of  $\widehat{Q_{0,h}}$ .

We also find it convenient to introduce other finite element spaces on  $\Gamma_0$  and  $\Gamma$  that are useful in our analysis. Let  $S_h^1(\Gamma_0)$  be the space of functions that are cubic polynomials with respect to arc length on each  $\Gamma_0^{(j)}$  and are continuously differentiable on  $\Gamma_0$ . Define the orthogonal projector  $Q_{0,h}^1 : L_2(\Gamma_0) \rightarrow S_h^1(\Gamma_0)$  by

$$\langle Q_{0,h}^1 \psi, \chi \rangle_{\Gamma_0} = \langle \psi, \chi \rangle_{\Gamma_0}, \quad \chi \in S_h^1(\Gamma_0).$$

It is well known that  $S_h^1(\Gamma_0)$  is a subspace of  $H^2(\Gamma_0)$  and that the following inverse, approximation, and boundedness properties are valid. For  $0 \leq s \leq 2$  and  $\varphi \in S_h^1(\Gamma_0)$ ,

$$|\varphi|_{s, \Gamma_0} \leq Ch^{-s} |\varphi|_{0, \Gamma_0},$$

$$|(I - Q_{0,h}^1)\psi|_{-1/2, \Gamma_0} + h^{1/2} |(I - Q_{0,h}^1)\psi|_{0, \Gamma_0} \leq Ch^{s+1/2} |\psi|_{s, \Gamma_0}$$

and

$$|Q_{0,h}^1 \psi|_{s, \Gamma_0} \leq C |\psi|_{s, \Gamma_0}.$$

Let  $S_h^1(\Gamma)$  denote the analogous space of piecewise cubic polynomials on  $\Gamma$ . The corresponding orthogonal projector is denoted by  $Q_h^1$ . This space satisfies the corresponding analogous inverse, approximation, and boundedness properties. The spaces  $S_h^1(\Gamma_0)$  and  $S_h^1(\Gamma)$  are not used in our method but only in the error analysis.

As a consequence of the quasiuniformity assumptions the space  $V_h$  has the following simultaneous approximation property. For  $w \in H^r(\Omega_{0,h} \cup \Omega_{1,h})$ ,  $1 \leq r \leq 2$ , there exists  $w_h \in V_h$  such that

$$\|w - w_h\|_{0, \Omega_{0,h} \cup \Omega_{1,h}} + h \|w - w_h\|_{1, \Omega_{0,h} \cup \Omega_{1,h}} \leq Ch^r \|w\|_{r, \Omega_{0,h} \cup \Omega_{1,h}}, \quad (2.29)$$

where  $C$  is independent of  $h$  and  $w$ . This property can be established using a triangle argument that is given in Bramble and Xu [7].  $\square$

The following approximation result will be useful in our analysis.

**Lemma 8.** Let  $w \in H^2(\Omega_{0,h} \cup \Omega_{1,h})$  and  $\varphi_h \in V_h$ , then

$$\begin{aligned} & \inf_{\chi \in V_h^0} \{ \|w - \varphi_h - \chi\|_{0,\Omega_{0,h} \cup \Omega_{1,h}} + h \|w - \varphi_h - \chi\|_{1,\Omega_{0,h} \cup \Omega_{1,h}} \} \\ & \leq C (h^2 \|w\|_{2,\Omega_{0,h} \cup \Omega_{1,h}} + h^{1/2} |w - \varphi_h|_{0,\Gamma_h} + h^{1/2} |[w] - [\varphi_h]|_{0,\Gamma_{0,h}}). \end{aligned}$$

*Proof.* Let  $w_h \in V_h$  satisfy (2.29), then by the triangle inequality we have

$$\begin{aligned} & \|w - \varphi_h - \chi\|_{0,\Omega_{0,h} \cup \Omega_{1,h}} + h \|w - \varphi_h - \chi\|_{1,\Omega_{0,h} \cup \Omega_{1,h}} \\ & \leq Ch^2 \|w\|_{2,\Omega_{0,h} \cup \Omega_{1,h}} + \|w_h - \varphi_h - \chi\|_{0,\Omega_{0,h} \cup \Omega_{1,h}} \\ & \quad + h \|w_h - \varphi_h - \chi\|_{1,\Omega_{0,h} \cup \Omega_{1,h}}. \end{aligned}$$

Now choose  $\chi \in V_h^0$  equal to  $w_h - \varphi_h$  at the interior nodes of  $\Omega_{0,h}$  and  $\Omega_{1,h}$ . Then a straightforward calculation gives

$$\begin{aligned} & \|w_h - \varphi_h - \chi\|_{0,\Omega_{0,h} \cup \Omega_{1,h}} + h \|w_h - \varphi_h - \chi\|_{1,\Omega_{0,h} \cup \Omega_{1,h}} \\ & \leq Ch^{1/2} \left\{ \left( h \sum_{j=1}^N |w_h(x^{(j)}) - \varphi_h(x^{(j)})|^2 \right)^{1/2} \right. \\ & \quad + \left( h \sum_{j=1}^K |w_h^1(y^{(j)}) - \varphi_h^1(y^{(j)}) - \chi(y^{(j)})|^2 \right)^{1/2} \\ & \quad \left. + \left( h \sum_{j=1}^K |w_h^0(y^{(j)}) - \varphi_h^0(y^{(j)}) - \chi(y^{(j)})|^2 \right)^{1/2} \right\}, \end{aligned}$$

where  $w_h^i = w_h|_{\Omega_{i,h}}$ . Since the terms on the right hand side are equivalent to the  $L_2$  norms on  $S_h(\Gamma_h)$  and  $S_h(\Gamma_{0,h})$  respectively we obtain

$$\begin{aligned} & \|w_h - \varphi_h - \chi\|_{0,\Omega_{0,h} \cup \Omega_{1,h}} + h \|w_h - \varphi_h - \chi\|_{1,\Omega_{0,h} \cup \Omega_{1,h}} \\ & \leq Ch^{1/2} \{ |w_h - \varphi_h|_{0,\Gamma_h} + |w_h^1 - \varphi_h^1 - \chi|_{0,\Gamma_{0,h}} + |w_h^0 - \varphi_h^0 - \chi|_{0,\Gamma_{0,h}} \}. \end{aligned}$$

Choosing  $\chi$  equal to  $1/2(w_h^1 + w_h^0) + 1/2(\varphi_h^1 + \varphi_h^0)$  on  $\Gamma_{0,h}$  gives

$$\begin{aligned} & \|w - \varphi_h - \chi\|_{0,\Omega_{0,h} \cup \Omega_{1,h}} + h \|w - \varphi_h - \chi\|_{1,\Omega_{0,h} \cup \Omega_{1,h}} \\ & \leq Ch^2 \|w\|_{2,\Omega_{0,h} \cup \Omega_{1,h}} + C(h^{1/2} |w_h - \varphi_h|_{0,\Gamma_h} \\ & \quad + h^{1/2} |[w_h] - [\varphi_h]|_{0,\Gamma_{0,h}}). \end{aligned} \tag{2.30}$$

Moreover, for  $v \in H^1(\Omega_{k,h})$ , the following trace inequality (see Dupont [11]) holds:

$$|v|_{0,\Gamma_{0,h}}^2 \leq C \|v\|_{0,\Omega_{k,h}} \|v\|_{1,\Omega_{k,h}}, \tag{2.31}$$

where  $C$  is independent of  $h$ . Consequently by (2.31) with  $v = w - w_h$  and the triangle inequality

$$\begin{aligned} h^{1/2} |[w_h] - [\varphi_h]|_{0,\Gamma_{0,h}} &\leq h^{1/2} |[w] - [w_h]|_{0,\Gamma_{0,h}} + h^{1/2} |[w] - [\varphi_h]|_{0,\Gamma_{0,h}} \\ &\leq C(\|w - w_h\|_{0,\Omega_{0,h} \cup \Omega_{1,h}} + h\|w - w_h\|_{1,\Omega_{0,h} \cup \Omega_{1,h}}) \\ &\quad + h^{1/2} |[w] - [\varphi_h]|_{0,\Gamma_{0,h}}. \end{aligned} \quad (2.32)$$

In a similar way using the estimate  $|v|_{0,\Gamma_h}^2 \leq C\|v\|_{0,\Omega_{1,h}}\|v\|_{1,\Omega_{1,h}}$  there follows

$$\begin{aligned} h^{1/2} |w_h - \varphi_h|_{0,\Gamma_h} &\leq h^{1/2} |w - \varphi_h|_{0,\Gamma_h} \\ &\quad + C(\|w - w_h\|_{0,\Omega_{0,h} \cup \Omega_{1,h}} + h\|w - w_h\|_{1,\Omega_{0,h} \cup \Omega_{1,h}}). \end{aligned} \quad (2.33)$$

The lemma now follows by (2.29) and the estimates (2.30), (2.32), and (2.33).  $\square$

**Remark 1.** A consequence of the previous result is as follows. Suppose  $w = 0$  and  $\varphi_h$  is discrete  $A^k$  harmonic, i.e.,

$$a_h(\varphi_h, \chi) = 0, \quad \chi \in V_h^0.$$

Then it is easy to see that for all  $\chi \in V_h^0$

$$\begin{aligned} a_h(\varphi_h, \varphi_h) &\leq a_h(\varphi_h - \chi, \varphi_h - \chi) \\ &\leq C\|\varphi_h - \chi\|_{1,\Omega_{0,h} \cup \Omega_{1,h}} \\ &\leq Ch^{-1/2}(|\varphi_h|_{0,\Gamma_h} + |[\varphi_h]|_{0,\Gamma_{0,h}}). \end{aligned} \quad (2.34)$$

We will need to know that functions vanishing on  $\Gamma$  and having no jumps across the interface,  $\Gamma_0$ , can be approximated well by functions in  $V_h^0$ . This is given in the next lemma.

**Lemma 9.** Suppose  $w \in H^2(\Omega_0 \cup \Omega_1)$  satisfies  $[w] = 0$  on  $\Gamma_0$  and  $w = 0$  on  $\Gamma$ , and is extended by (2.8). Then there exists a constant  $C$ , independent of  $h$  and  $w$ , such that

$$\inf_{\chi \in V_h^0} \{ \|W - \chi\|_{0,\Omega_{0,h} \cup \Omega_{1,h}} + h\|W - \chi\|_{1,\Omega_{0,h} \cup \Omega_{1,h}} \} \leq Ch^2 \|w\|_{2,\Omega_0 \cup \Omega_1}.$$

*Proof.* Take  $\varphi_h = 0$  in lemma 8. Then

$$\begin{aligned} \inf_{\chi \in V_h^0} \{ \|W - \chi\|_{0,\Omega_{0,h} \cup \Omega_{1,h}} + h\|W - \chi\|_{1,\Omega_{0,h} \cup \Omega_{1,h}} \} \\ \leq C(h^2 \|w\|_{2,\Omega_0 \cup \Omega_1} + h^{1/2} |W|_{0,\Gamma_h} + h^{1/2} |[W]|_{0,\Gamma_{0,h}}). \end{aligned}$$

Now it follows from lemma 1 that

$$|W|_{0,\Gamma_h} \leq Ch^2 \|w\|_{2,\Omega_1}$$

and

$$|[W]|_{0,\Gamma_{0,h}} \leq Ch^2 \|w\|_{2,\Omega_0 \cup \Omega_1}.$$

Combining these three inequalities proves the lemma.  $\square$

We shall need the following estimate.

**Lemma 10.** There exists a constant  $C$ , independent of  $h$ , such that for  $v \in H^1(\Omega_{0,h} \cup \Omega_{1,h})$

$$\|v\|_{1,\Omega_{0,h} \cup \Omega_{1,h}} \leq C(|v|_{1,\Omega_{0,h} \cup \Omega_{1,h}} + |[v]|_{0,\Gamma_{0,h}} + |v|_{0,\Gamma_h}). \quad (2.35)$$

*Proof.* Clearly it suffices to show that (2.35) holds for  $\|v\|_{0,\Omega_h}$ . Let  $\varphi \in C_0^\infty(\Omega_h)$  and define  $w \in H^2(\Omega_0 \cup \Omega_1)$ , and extended outside of  $\Omega$  by  $E_1$ , by

$$A^0 w^0 = \varphi \quad \text{in } \Omega_0, \quad A^1 w^1 = \varphi \quad \text{in } \Omega_1$$

$$\text{with } [w] = 0 \text{ on } \Gamma_0 \quad \text{and} \quad w = 0 \text{ on } \Gamma,$$

with  $\partial w^0 / \partial \nu_{h,A^0} = \partial w^1 / \partial \nu_{h,A^1}$  on  $\Gamma_0$ . Then using Greens identities we find that

$$(v, \varphi)_{\Omega_{0,h} \cup \Omega_{1,h}} = \tilde{a}(v, w) + \left\langle [v], \frac{\partial w}{\partial \nu_{h,A}} \right\rangle_{\Gamma_{0,h}} - \left\langle v, \frac{\partial w}{\partial \eta_{h,A^1}} \right\rangle_{\Gamma_h},$$

where  $\partial w / \partial \nu_{h,A} = \partial w^0 / \partial \nu_{h,A^0}$  on edges  $\Gamma_{0,h}^{(j)} \subset \overline{\Omega}_0$  and  $\partial w / \partial \nu_{h,A} = \partial w^1 / \partial \nu_{h,A^1}$  on edges  $\Gamma_{0,h}^{(j)} \subset \overline{\Omega}_1$ .

Thus,

$$\begin{aligned} |(v, \varphi)_{\Omega_{0,h} \cup \Omega_{1,h}}| &\leq |v|_{1,\Omega_{0,h} \cup \Omega_{1,h}} |w|_{1,\Omega_0 \cup \Omega_1} + |[v]|_{0,\Gamma_{0,h}} \left| \frac{\partial w}{\partial \nu_{h,A}} \right|_{0,\Gamma_{0,h}} \\ &\quad + |v|_{0,\Gamma_h} \left| \frac{\partial w}{\partial \eta_{h,A^1}} \right|_{0,\Gamma_h}. \end{aligned}$$

By lemma 2

$$\left| \frac{\partial w}{\partial \nu_{h,A}} \right|_{0,\Gamma_{0,h}} \leq C \|w\|_{2,\Omega_0 \cup \Omega_1}$$

and similarly,

$$\left| \frac{\partial w}{\partial \eta_{h,A^1}} \right|_{0,\Gamma_h} \leq C \|w\|_{2,\Omega_0 \cup \Omega_1}.$$

Consequently,

$$|(v, \varphi)_{\Omega_{0,h} \cup \Omega_{1,h}}| \leq C \|w\|_{2,\Omega_0 \cup \Omega_1} (|v|_{1,\Omega_{0,h} \cup \Omega_{1,h}} + |[v]|_{0,\Gamma_{0,h}} + |v|_{0,\Gamma_h}).$$

Therefore, since  $\|w\|_{2,\Omega_0 \cup \Omega_1} \leq C \|\varphi\|_{0,\Omega_h}$ , and

$$\|v\|_{0,\Omega_h} = \sup_{\varphi \in C_0^\infty(\Omega_h)} \frac{|(v, \varphi)_{\Omega_{0,h} \cup \Omega_{1,h}}|}{\|\varphi\|_{0,\Omega_h}},$$

the lemma follows. □



### 3. Error analysis

We are now in a position to begin the error analysis. Our goal is to obtain optimal order error estimates in  $L_2(\Omega)$  and in  $H^1(\Omega_0 \cup \Omega_1)$  for the approximate solution,  $u_h$ . The general outline of the error analysis is as follows. We shall obtain optimal order estimates for  $z_h$  in  $L_2(\Omega_h)$  and in  $H^1(\Omega_{0,h} \cup \Omega_{1,h})$ . We find it necessary to perform this part of the analysis indirectly by first estimating the error for  $v_h \in V_h$  defined by

$$\begin{aligned} a_h(v_h, \phi) &= (f, \phi)_{\Omega_h} - \langle \tilde{q}_1, \phi \rangle_{\Gamma_{0,h}}, \quad \phi \in V_h^0, \\ \text{with } [v_h] &= \widetilde{\widehat{Q}_{0,h} q_0} \text{ on } \Gamma_{0,h} \text{ and } v_h = \widetilde{\widehat{Q}_{1,h} g} \text{ on } \Gamma_h. \end{aligned} \quad (3.1)$$

Note that  $v_h$  differs from  $z_h$  only in the jump on the polygonal interface and the approximate Dirichlet boundary condition. Consequently,  $v_h - z_h$  is discrete  $A^k$ -harmonic. We show that  $v_h$  is an optimal order approximation and using this show that  $z_h$  is also optimal order accurate. Having demonstrated this we prove that the natural extension of  $z_h$  to  $\Omega$ ,  $u_h$ , is an optimal order accurate approximation in  $L_2(\Omega)$  and in  $H^1(\Omega_0 \cup \Omega_1)$ .

#### 3.1. Error analysis for $v_h$

Unless stated to the contrary we assume, in this subsection, that  $q_1 \in H^{\ell-1}(\Gamma_0)$ ,  $f \in L_2(\Omega)$ ,  $q_0 \in H^s(\Gamma_0)$ , and  $g \in H^r(\Gamma)$  with  $1/2 \leq r, s \leq 3/2$  and  $1 \leq \ell \leq 3/2$ . This implies that  $u \in H^1(\Omega_0 \cup \Omega_1)$ .

We shall make use of the following estimate.

**Lemma 11.** Suppose  $w \in H^r(\Omega_0 \cup \Omega_1)$ , with  $1 \leq r \leq 2$ , and  $W \in H^r(\Omega_{0,h} \cup \Omega_{1,h})$  is defined by (2.8). Then there exists a constant  $C$ , independent of  $h$  and  $w$ , such that for any  $\chi \in V_h^0$

$$|a_h(W, \chi) - a(w, \bar{\chi})| \leq Ch^{r-1} (|\chi|_{1, \Omega_{0,h} \triangle \Omega_0} + |\chi|_{1, \Omega_h \setminus \bar{\Omega}}) \|w\|_{r, \Omega_0 \cup \Omega_1},$$

where  $\bar{\chi}$  denotes the extension of  $\chi$  by zero.

*Proof.* By definition of the forms  $a(\cdot, \cdot)$  and  $a_h(\cdot, \cdot)$  it follows immediately that

$$\begin{aligned} a_h(W, \chi) - a(w, \bar{\chi}) &= a_{\Omega_{0,h} \setminus \bar{\Omega}_0}^0(W, \chi) - a_{\Omega_0 \setminus \bar{\Omega}_{0,h}}^0(w, \chi) + a_{\Omega_{1,h} \setminus \bar{\Omega}_1}^1(W, \chi) \\ &\quad - a_{\Omega_1 \setminus \bar{\Omega}_{1,h}}^1(w, \bar{\chi}). \end{aligned}$$

But

$$a_{\Omega_1 \setminus \bar{\Omega}_{1,h}}^1(w, \bar{\chi}) = a_{\Omega_{0,h} \setminus \bar{\Omega}_0}^1(w, \chi)$$

and

$$a_{\Omega_{1,h} \setminus \bar{\Omega}_1}^1(W, \chi) = a_{\Omega_0 \setminus \bar{\Omega}_{0,h}}^1(W, \chi) + a_{\Omega_h \setminus \bar{\Omega}}^1(W, \chi).$$

Hence,

$$\begin{aligned} |a_h(W, \chi) - a(w, \bar{\chi})| &\leq |a_{\Omega_h \setminus \bar{\Omega}}^1(W, \chi)| + \sum_{k=0}^1 |a_{\Omega_{0,h} \Delta \Omega_0}^k(W, \chi)| + |a_{\Omega_{0,h} \Delta \Omega_0}^k(w, \chi)| \\ &\leq C \{ |\chi|_{1, \Omega_h \setminus \bar{\Omega}} |w|_{1, \Omega_h \setminus \bar{\Omega}} \\ &\quad + |\chi|_{1, \Omega_{0,h} \Delta \Omega_0} (|w|_{1, \Omega_{0,h} \Delta \Omega_0} + |W|_{1, \Omega_{0,h} \Delta \Omega_0}) \}. \end{aligned}$$

Then by lemma 3 we have

$$|w|_{1, \Omega_{0,h} \Delta \Omega_0} \leq Ch^{r-1} \|w\|_{r, \Omega_0 \cup \Omega_1}, \quad 1 \leq r \leq 2.$$

By the same analysis, using the definition of  $W$ , it follows that

$$|W|_{1, \Omega_{0,h} \Delta \Omega_0} \leq Ch^{r-1} \|w\|_{r, \Omega_0 \cup \Omega_1}, \quad 1 \leq r \leq 2,$$

and the proof is complete.  $\square$

Let  $\widehat{u}^h$  denote the solution of (1.1) with  $q_0$  replaced by  $\widehat{Q}_{0,h} q_0$ ,  $q_1$  replaced by  $\widehat{Q}_{0,h} \widetilde{q}_1$ , and  $g$  replaced by  $\widehat{Q}_{0,h} g$ . Let  $\widehat{U}^h$  denote the corresponding function defined by (2.8). We shall estimate the difference,  $\widehat{U}^h - v_h$ , in the  $H^1(\Omega_{0,h} \cup \Omega_{1,h})$  seminorm. First we define  $u^h$  as the solution of (1.1) with  $q_0$  replaced by  $Q_{0,h}^1 q_0$ ,  $q_1$  replaced by  $Q_{0,h}^1 \widetilde{q}_1$ , and  $g$  replaced by  $Q_{0,h}^1 g$ . Let  $U^h$  denote the corresponding function defined by (2.8). The reason for introducing  $S_h^1(\Gamma_0) \subset H^2(\Gamma_0)$  and  $S_h^1(\Gamma)$  is to guarantee that  $Q_{0,h}^1 q_0 \in H^{3/2}(\Gamma_0)$  and  $Q_{0,h}^1 g \in H^{3/2}(\Gamma)$ , thereby ensuring that  $u^h \in H^2(\Omega_0 \cup \Omega_1)$ . For future reference we note that (2.2) gives

$$\begin{aligned} \|u^h\|_{p, \Omega_0 \cup \Omega_1} &\leq C (\|f\|_{0, \Omega_h} + |\widehat{Q}_{0,h} \widetilde{q}_1|_{p-3/2, \Gamma_0} + |Q_{0,h}^1 q_0|_{p-1/2, \Gamma_0} \\ &\quad + |Q_{0,h}^1 g|_{p-1/2, \Gamma}). \end{aligned} \quad (3.2)$$

By properties of  $Q_{0,h}^1$ ,  $Q_h^1$ , and lemma 6

$$\begin{aligned} \|u^h\|_{p, \Omega_0 \cup \Omega_1} &\leq C (\|f\|_{0, \Omega_h} + h^{\ell-p+1/2} |q_1|_{\ell-1, \Gamma_0} \\ &\quad + h^{s-p+1/2} |q_0|_{s, \Gamma_0} + h^{r-p+1/2} |g|_{r, \Gamma}), \end{aligned} \quad (3.3)$$

where  $0 \leq r, s \leq p - 1/2$ ,  $1 \leq \ell \leq 3/2$  and  $3/2 \leq p \leq 2$ . Whereas if  $1 \leq p \leq 3/2$ ,  $0 \leq r, s \leq p - 1/2$  and  $1 \leq \ell \leq 3/2$

$$\|u^h\|_{p, \Omega_0 \cup \Omega_1} \leq C (\|f\|_{0, \Omega_h} + |q_1|_{0, \Gamma_0} + h^{s-p+1/2} |q_0|_{s, \Gamma_0} + h^{r-p+1/2} |g|_{r, \Gamma}). \quad (3.4)$$

To estimate  $\widehat{U}^h - v_h$  in the  $H^1$  seminorm, it is sufficient to estimate  $U^h - v_h$ . To see this we note that by (2.9) for  $i = 0, 1$

$$\|\widehat{U}^h - U^h\|_{i, \Omega_{0,h} \cup \Omega_{1,h}} \leq C \|\widehat{u}^h - u^h\|_{i, \Omega_0 \cup \Omega_1}.$$

By (2.2)

$$\|\widehat{u}^h - u^h\|_{i, \Omega_0 \cup \Omega_1} \leq C \{ |(\widehat{Q}_{0,h} - Q_{0,h}^1)q_0|_{i-1/2, \Gamma_0} + |(\widehat{Q}_h - Q_h^1)g|_{i-1/2, \Gamma} \},$$

and by properties of  $\widehat{Q}_{0,h}$ ,  $Q_{0,h}^1$ ,  $\widehat{Q}_h$  and  $Q_h^1$

$$\|\widehat{u}^h - u^h\|_{i, \Omega_0 \cup \Omega_1} \leq C \{ h^{s+1/2-i} |q_0|_{s, \Gamma_0} + h^{r+1/2-i} |g|_{r, \Gamma} \}, \quad (3.5)$$

for  $q_0 \in H^s(\Gamma_0)$  and  $g \in H^r(\Gamma)$  with  $0 \leq r, s \leq 3/2$ . Thus, for  $i = 0, 1$

$$\|\widehat{U}^h - U^h\|_{i, \Omega_{0,h} \cup \Omega_{1,h}} \leq C \{ h^{s+1/2-i} |q_0|_{s, \Gamma_0} + h^{r+1/2-i} |g|_{r, \Gamma} \}, \quad 0 \leq r, s \leq 3/2.$$

**Lemma 12.** There exists a constant  $C$ , independent of  $h$ , such that

$$\begin{aligned} \|U^h - v_h\|_{1, \Omega_{0,h} \cup \Omega_{1,h}} &\leq C (h^{p-1} \|u^h\|_{p, \Omega_0 \cup \Omega_1} + \|\widehat{u}^h - u^h\|_{1, \Omega_0 \cup \Omega_1} \\ &\quad + h^{-1} \|\widehat{u}^h - u^h\|_{0, \Omega_0 \cup \Omega_1} + h |q_1|_{0, \Gamma_0}), \end{aligned} \quad (3.6)$$

where  $1 \leq p \leq 2$ .

For the proof see the appendix.

By splitting the data  $\{f, q_0, q_1, g\}$ , using lemma 12, linearity and the *a priori* estimate (2.2) the following is an easy consequence.

**Proposition 1.** For  $f \in L_2(\Omega)$ ,  $q_0 \in H^s(\Gamma_0)$ ,  $q_1 \in H^{\ell-1}(\Gamma_0)$ , and  $g \in H^r(\Gamma)$  with  $1/2 \leq r, s \leq 3/2$ , and  $1 \leq \ell \leq 3/2$ , there exists a constant  $C$ , independent of  $h$ , such that

$$\begin{aligned} \|\widehat{U}^h - v_h\|_{1, \Omega_{0,h} \cup \Omega_{1,h}} &\leq C (h \|f\|_{0, \Omega} + h^{\ell-1/2} |q_1|_{\ell-1, \Gamma_0} + h^{s-1/2} |q_0|_{s, \Gamma_0} \\ &\quad + h^{r-1/2} |g|_{r, \Gamma}). \end{aligned}$$

**Remark 2.** It follows from proposition 1 and lemma 10 that

$$\begin{aligned} \|\widehat{U}^h - v_h\|_{1, \Omega_{0,h} \cup \Omega_{1,h}} &\leq C (h \|f\|_{0, \Omega} + h^{\ell-1/2} |q_1|_{\ell-1, \Gamma_0} + h^{s-1/2} |q_0|_{s, \Gamma_0} \\ &\quad + h^{r-1/2} |g|_{r, \Gamma}), \end{aligned} \quad (3.7)$$

where  $1/2 \leq r, s \leq 3/2$  and  $1 \leq \ell \leq 3/2$ .

Next we obtain an  $L_2$  estimate similar to proposition 1. First we need a preliminary estimate similar to lemma 12 whose proof is given in the appendix.

**Lemma 13.** There exists a constant  $C$ , independent of  $h$ , such that

$$\begin{aligned} \|\widehat{U}^h - v_h\|_{0, \Omega_{0,h} \cup \Omega_{1,h}} &\leq C (h^p \|u^h\|_{p, \Omega_0 \cup \Omega_1} + \|\widehat{u}^h - u^h\|_{0, \Omega_0 \cup \Omega_1} \\ &\quad + h \|\widehat{u}^h - u^h\|_{1, \Omega_0 \cup \Omega_1} + h^2 |q_1|_{0, \Gamma_0}). \end{aligned}$$

By splitting the data for problem (1.1), using linearity and the *a priori* estimate (2.2) the following result is obtained from lemma 13.

**Proposition 2.** For  $f \in L_2(\Omega)$ ,  $q_0 \in H^s(\Gamma_0)$ ,  $q_1 \in H^{\ell-1}(\Gamma_0)$ , and  $g \in H^r(\Gamma)$  with  $0 \leq r, s \leq 3/2$  and  $1 \leq \ell \leq 3/2$ , there exists a constant  $C$ , independent of  $h$ , such that

$$\begin{aligned} \|\widehat{U}^h - v_h\|_{0,\Omega_{0,h} \cup \Omega_{1,h}} &\leq C(h^2\|f\|_{0,\Omega} + h^{\ell+1/2}|q_1|_{\ell-1,\Gamma_0} + h^{s+1/2}|q_0|_{s,\Gamma_0} \\ &\quad + h^{r+1/2}|g|_{r,\Gamma}). \end{aligned}$$

### 3.2. The main results

In this section we use propositions 1 and 2 to establish optimal order estimates for  $U - z_h$ , where  $U$  is the extension of  $u$  defined by (2.8) and  $u$  is the solution of (1.1).

**Theorem 2.** There exists a constant  $C$ , independent of  $h$  and  $u$ , such that

$$\begin{aligned} \|U - z_h\|_{0,\Omega_{0,h} \cup \Omega_{1,h}} &\leq C(h^2\|f\|_{0,\Omega} + h^{\ell+1/2}|q_1|_{\ell-1,\Gamma_0} + h^{s+1/2}|q_0|_{s,\Gamma_0} \\ &\quad + h^{r+1/2}|g|_{r,\Gamma}), \end{aligned}$$

where  $0 \leq r, s \leq 3/2$  and  $1 \leq \ell \leq 3/2$ . Moreover,

$$\begin{aligned} \|U - z_h\|_{1,\Omega_{0,h} \cup \Omega_{1,h}} &\leq C(h\|f\|_{0,\Omega} + h^{\ell-1/2}|q_1|_{\ell-1,\Gamma_0} + h^{s-1/2}|q_0|_{s,\Gamma_0} \\ &\quad + h^{r-1/2}|g|_{r,\Gamma}), \end{aligned}$$

where  $1/2 \leq r, s \leq 3/2$  and  $1 \leq \ell \leq 3/2$ .

*Proof.* By the triangle inequality,

$$\begin{aligned} \|U - z_h\|_{i,\Omega_{0,h} \cup \Omega_{1,h}} &\leq \|U - \widehat{U}^h\|_{i,\Omega_{0,h} \cup \Omega_{1,h}} + \|\widehat{U}^h - v_h\|_{i,\Omega_{0,h} \cup \Omega_{1,h}} \\ &\quad + \|z_h - v_h\|_{1,\Omega_{0,h} \cup \Omega_{1,h}}. \end{aligned} \quad (3.8)$$

We estimate each of the terms in (3.8). By (2.9) and (2.2) we have

$$\begin{aligned} \|U - \widehat{U}^h\|_{i,\Omega_{0,h} \cup \Omega_{1,h}} &\leq C\|u - \widehat{u}^h\|_{i,\Omega_0 \cup \Omega_1} \\ &\leq C(|(I - \widehat{Q}_{0,h})q_0|_{i-1/2,\Gamma_0} + |(I - \widehat{Q}_h)g|_{i-1/2,\Gamma} \\ &\quad + |q_1 - \widehat{Q}_{0,h}\widetilde{q}_1|_{i-3/2,\Gamma_0}), \end{aligned}$$

and hence by properties of  $\widehat{Q}_{0,h}$ ,  $\widehat{Q}_h$ , and lemma 7

$$\|U - \widehat{U}^h\|_{i,\Omega_{0,h} \cup \Omega_{1,h}} \leq C(h^{s-i+1/2}|q_0|_{s,\Gamma_0} + h^{r-i+1/2}|g|_{r,\Gamma} + h^{\ell-i+1/2}|q_1|_{\ell-1,\Gamma_0}).$$

For the next term we apply propositions 1 and 2. Finally, by lemma 10 and remark 1 we have

$$\|z_h - v_h\|_{1,\Omega_{0,h}\cup\Omega_{1,h}} \leq Ch^{-1/2}(|Q_{0,h}\tilde{q}_0 - \widetilde{\hat{Q}_{0,h}q_0}|_{0,\Gamma_{0,h}} + |Q_h\tilde{g} - \widetilde{\hat{Q}_hg}|_{0,\Gamma_h}),$$

and hence by lemma 5 and properties of  $\hat{Q}_{0,h}$  and  $\hat{Q}_h$

$$\begin{aligned} \|z_h - v_h\|_{1,\Omega_{0,h}\cup\Omega_{1,h}} &\leq Ch^{3/2}(|(I - \hat{Q}_{0,h})q_0|_{0,\Gamma_0} + |(I - \hat{Q}_h)g|_{0,\Gamma}) \\ &\leq Ch^{3/2}(h^s|q_0|_{s,\Gamma_0} + h^r|g|_{r,\Gamma}). \end{aligned}$$

Combining these estimates proves the theorem.  $\square$

Next we prove that  $u_h$  is an optimal order accurate approximation to  $u$  on all of  $\Omega_0 \cup \Omega_1$  in  $L_2$  and in the  $H^1$  norm. Since  $U|_{\Omega_{i,h}} = u$  and  $z_h = u_h$  on  $\Omega_i \cap \Omega_{i,h}$  it follows from theorem 2 that we need only estimate the error,  $u - u_h$ , on  $\Omega_0 \triangle \Omega_{0,h}$  and  $\Omega \setminus \overline{\Omega}_h$ . We shall make use of the following estimates.

**Lemma 14.** For  $w \in H^2(\Omega_0 \cup \Omega_1)$  and  $\varphi_h \in V_h(\Omega_0 \cup \Omega_1)$ , there exists a constant  $C$ , independent of  $h$  and  $w$ , such that

$$\begin{aligned} |w - \varphi_h|_{1,\Omega_0 \triangle \Omega_{0,h}} &\leq Ch^{1/2}(|W - \varphi_h|_{1,\Omega_{0,h}\cup\Omega_{1,h}} + h\|w\|_{2,\Omega_0\cup\Omega_1}), \\ \|w - \varphi_h\|_{0,\Omega_0 \triangle \Omega_{0,h}} &\leq Ch^{1/2}(\|W - \varphi_h\|_{0,\Omega_{0,h}\cup\Omega_{1,h}} + h|W - \varphi_h|_{1,\Omega_{0,h}\cup\Omega_{1,h}} \\ &\quad + h^{5/2}\|w\|_{2,\Omega_0\cup\Omega_1}), \\ |w - \varphi_h|_{1,\Omega \setminus \overline{\Omega}_h} &\leq Ch^{1/2}(|W - \varphi_h|_{1,\Omega_{1,h}} + h\|w\|_{2,\Omega_1}), \\ \|w - \varphi_h\|_{0,\Omega \setminus \overline{\Omega}_h} &\leq Ch^{1/2}(\|W - \varphi_h\|_{0,\Omega_{1,h}} + h|W - \varphi_h|_{1,\Omega_{1,h}} + h^{5/2}\|w\|_{2,\Omega_1}), \end{aligned} \quad (3.9)$$

where  $W$  is defined by (2.8).

*Proof.* We shall prove the first two inequalities and the last two are proved in similar fashion. Let  $\Omega_{0,h}^{(j)}$ , say  $\Omega_{0,h}^{(j)} \subset \Omega_0 \setminus \overline{\Omega}_{0,h}$ , denote a typical piece of  $\Omega_0 \triangle \Omega_{0,h}$  bounded by  $\Gamma_0^{(j)}$  and  $\Gamma_{0,h}^{(j)}$ . The case  $\Omega_{0,h}^{(j)} \subset \Omega_{0,h} \setminus \overline{\Omega}_0$  is handled similarly. Then by equation (2.10) of [5] applied to derivatives of  $W - \varphi_h$  we have, since  $\varphi_h$  is linear on  $\Omega_{0,h}^{(j)}$ ,

$$|w - \varphi_h|_{1,\Omega_{0,h}^{(j)}}^2 \leq C \left( h^2 \sum_{i=1}^2 \left| \frac{\partial}{\partial x_i} (W - \varphi_h) \right|_{0,\Gamma_{0,h}^{(j)}}^2 + h^4 \|W\|_{2,\Omega_{0,h}^{(j)}}^2 \right). \quad (3.10)$$

Further, if  $\tau_j \subset \Omega_{0,h}$  denotes the triangle having  $\Gamma_{0,h}^{(j)}$  as one of its sides, then the divergence theorem gives

$$h|v|_{0,\Gamma_{0,h}^{(j)}}^2 \leq C(\|v\|_{0,\tau_j}^2 + h^2|v|_{1,\tau_j}^2). \quad (3.11)$$

Applying (3.11) to the derivatives of  $W - \varphi_h$  in (3.10) gives

$$|w - \varphi_h|_{1, \Omega_{0,h}^{(j)}}^2 \leq C(h|W - \varphi_h|_{1, \tau_j}^2 + h^3 \|W\|_{2, \Omega_{0,h}^{(j)} \cup \tau_j}^2),$$

and hence by summing over appropriate indices  $j$  there results

$$|w - \varphi_h|_{1, \Omega_0 \setminus \overline{\Omega}_{0,h}} \leq Ch^{1/2}(|W - \varphi_h|_{1, \Omega_{0,h} \cup \Omega_{1,h}} + h\|W\|_{2, \Omega_0 \cup \Omega_{0,h}}). \quad (3.12)$$

In the case  $\Omega_{0,h}^{(j)} \subset \Omega_{0,h} \setminus \overline{\Omega}_0$  we find

$$|W - \varphi_h|_{1, \Omega_{0,h} \setminus \overline{\Omega}_0} \leq Ch^{1/2}(|W - \varphi_h|_{1, \Omega_{0,h} \cup \Omega_{1,h}} + h\|W\|_{2, \Omega_{0,h} \cup \Omega_1}), \quad (3.13)$$

and hence the first estimate follows by using (2.9). To prove the second estimate we apply equation (2.10) of [5] applied to  $W - \varphi_h$ , and use (3.11) to obtain by summing over appropriate indices  $j$

$$\|w - \varphi_h\|_{0, \Omega_0 \setminus \overline{\Omega}_{0,h}} \leq Ch^{1/2}(\|W - \varphi_h\|_{0, \Omega_{0,h} \cup \Omega_{1,h}} + h|W - \varphi_h|_{1, \Omega_{0,h} \cup \Omega_{1,h}}). \quad (3.14)$$

Combining (3.12), (3.13) and (3.14) gives

$$\begin{aligned} \|w - \varphi_h\|_{0, \Omega_0 \setminus \overline{\Omega}_{0,h}} &\leq Ch^{1/2}(\|W - \varphi_h\|_{0, \Omega_{0,h} \cup \Omega_{1,h}} + h|W - \varphi_h|_{1, \Omega_{0,h} \cup \Omega_{1,h}} \\ &\quad + h^{5/2}\|w\|_{2, \Omega_0 \cup \Omega_1}), \end{aligned} \quad (3.15)$$

where we have used (2.9). By the same analysis we find that  $\|w - \varphi_h\|_{0, \Omega_{0,h} \setminus \overline{\Omega}_0}$  is also bounded by the right hand side of (3.15) and this proves the second inequality in (3.9) and hence the lemma.  $\square$

Now we can prove the main result of the paper.

*Proof of theorem 1.* By the triangle inequality

$$\begin{aligned} \|u - u_h\|_{i, \Omega_0 \cup \Omega_1} &\leq \|U - z_h\|_{i, (\Omega_0 \cap \Omega_{0,h}) \cup (\Omega_1 \cap \Omega_{1,h})} + \|u - u_h\|_{i, \Omega_0 \triangle \Omega_{0,h}} \\ &\quad + \|u - u_h\|_{i, \Omega \setminus \overline{\Omega}_h} \end{aligned}$$

and theorem 2 it suffices to estimate  $\|u - u_h\|_{i, \Omega_0 \triangle \Omega_{0,h}}$  and  $\|u - u_h\|_{i, \Omega \setminus \overline{\Omega}_h}$ . Clearly,

$$\begin{aligned} \|u - u_h\|_{i, \Omega_0 \triangle \Omega_{0,h}} + \|u - u_h\|_{i, \Omega \setminus \overline{\Omega}_h} &\leq \|u^h - u_h\|_{i, \Omega_0 \triangle \Omega_{0,h}} + \|u^h - u_h\|_{i, \Omega \setminus \overline{\Omega}_h} \\ &\quad + C\|u^h - u\|_{i, \Omega_0 \cup \Omega_1}. \end{aligned}$$

By (2.2) we have

$$\begin{aligned} \|u^h - u\|_{i, \Omega_0 \cup \Omega_1} &\leq C(|(I - \widehat{Q}_{0,h})q_0|_{i-1/2, \Gamma_0} + |(I - \widehat{Q}_h)g|_{i-1/2, \Gamma} \\ &\quad + |q_1 - \widehat{Q}_{0,h}\widetilde{q}_1|_{i-3/2, \Gamma_0}), \end{aligned}$$

and hence by (2.25) if  $i = 1$ , (2.26) if  $i = 0$  (and the corresponding properties of  $\widehat{Q}_h$ ), and lemma 7 there results

$$\|u^h - u\|_{i, \Omega_0 \cup \Omega_1} \leq C(h^{s-i+1/2}|q_0|_{s, \Gamma_0} + h^{r-i+1/2}|g|_{r, \Gamma} + h^{\ell-i+1/2}|q_1|_{\ell-1, \Gamma_0}).$$

Consequently it suffices to estimate  $\|u^h - u_h\|_{i, \Omega \setminus \overline{\Omega}_h}$  and  $\|u^h - u_h\|_{i, \Omega_0 \triangle \Omega_{0,h}}$ . To handle these terms we use lemma 14 and theorem 2. We give the analysis only for the  $L_2$  norm since the  $H^1$  analysis is similar. By lemma 14 we have

$$\begin{aligned} & \|u^h - u_h\|_{0, \Omega_0 \triangle \Omega_{0,h}} + \|u^h - u_h\|_{0, \Omega \setminus \overline{\Omega}_h} \\ & \leq Ch^{1/2}(\|U^h - z_h\|_{0, \Omega_{0,h} \cup \Omega_{1,h}} + h\|U^h - z_h\|_{1, \Omega_{0,h} \cup \Omega_{1,h}} + h^{5/2}\|u^h\|_{2, \Omega_0 \cup \Omega_1}). \end{aligned}$$

In view of theorem 2 we only need to estimate the last term in the previous inequality. We have by (3.3),

$$h^{5/2}\|u^h\|_{2, \Omega_0 \cup \Omega_1} \leq C(h^{5/2}\|f\|_{0, \Omega} + h^{\ell+1}|q_1|_{\ell-1, \Gamma_0} + h^{s+1}|q_0|_{s, \Gamma_0} + h^{r+1}|g|_{r, \Gamma}),$$

which proves (2.11).  $\square$

## Appendix

In this section we give the somewhat technical proofs of two lemmas used in the error analysis of section 3.

*Proof of lemma 12.* Without loss in generality, because of ellipticity, we may take the seminorm to be  $a_h(\cdot, \cdot) = |\cdot|_{1, \Omega_{0,h} \cup \Omega_{1,h}}^2$ . For  $\varphi_h \in V_h$

$$|U^h - v_h|_{1, \Omega_{0,h} \cup \Omega_{1,h}}^2 = a_h(U^h - v_h, U^h - \varphi_h) + a_h(U^h - v_h, \chi),$$

where  $\chi = \varphi_h - v_h$ . Choose  $\varphi_h$  such that  $[\varphi_h] = [v_h]$  on  $\Gamma_{0,h}$ ,  $\varphi_h = v_h$  on  $\Gamma_h$ , and

$$a_h(U^h - \varphi_h, \xi) = 0, \quad \xi \in V_h^0.$$

Then by the Cauchy-Schwarz inequality

$$|U^h - v_h|_{1, \Omega_{0,h} \cup \Omega_{1,h}}^2 \leq |U^h - \varphi_h|_{1, \Omega_{0,h} \cup \Omega_{1,h}}^2 + 2a_h(U^h - v_h, \chi),$$

where we note that  $\chi \in V_h^0 \cap H^1(\Omega_h)$ . Now we have

$$a_h(U^h - v_h, \chi) = a_h(U^h, \chi) - a(u^h, \bar{\chi}) + a(u^h, \bar{\chi}) - a_h(v_h, \chi),$$

where  $\bar{\chi}$  is the extension by zero. By lemma 11 we have

$$|a_h(U^h, \chi) - a(u^h, \bar{\chi})| \leq Ch^{p-1}(|\chi|_{1, \Omega_{0,h} \triangle \Omega_0} + |\chi|_{1, \Omega_h \setminus \overline{\Omega}})\|u^h\|_{p, \Omega_0 \cup \Omega_1}. \quad (\text{A.1})$$

From the definitions of  $u^h$  and  $v_h$  we find

$$|a(u^h, \bar{\chi}) - a_h(v_h, \chi)| = |\langle \tilde{q}_1, \chi \rangle_{\Gamma_{0,h}} - \langle q_1, \chi \rangle_{\Gamma_0}|.$$

To estimate this term we let  $J(t)$  denote the magnitude of the derivative of the piecewise smooth transformation  $t \mapsto X_h(t)$ . Note that  $J(t)$  is smooth except at the nodes and, from the definition of  $\Gamma_{0,h}$ , it follows that

$$\max_{t \in [0, \ell_h]} |1 - J(t)| \leq Ch^2, \quad (\text{A.2})$$

where  $\ell_h$  is the length of  $\Gamma_{0,h}$ . Moreover we have, for  $\varphi$  defined on  $\Gamma_0$ ,

$$\int_{\Gamma_0} \varphi \, ds = \int_{\Gamma_{0,h}} \tilde{\varphi} J \, dt.$$

Consequently,

$$\begin{aligned} \langle \tilde{q}_1, \chi \rangle_{\Gamma_{0,h}} - \langle q_1, \chi \rangle_{\Gamma_0} &= \langle \tilde{q}_1, \chi \rangle_{\Gamma_{0,h}} - \langle \tilde{q}_1, J\tilde{\chi} \rangle_{\Gamma_{0,h}} \\ &= \langle \tilde{q}_1, (1 - J)\chi \rangle_{\Gamma_{0,h}} - \langle \tilde{q}_1, J(\tilde{\chi} - \chi) \rangle_{\Gamma_{0,h}} \end{aligned}$$

and by (A.2) and the proof of lemma 1 in [5] there results

$$|\langle \tilde{q}_1, \chi \rangle_{\Gamma_{0,h}} - \langle q_1, \chi \rangle_{\Gamma_0}| \leq C|\tilde{q}_1|_{0, \Gamma_{0,h}} \{h^2|\chi|_{0, \Gamma_{0,h}} + h|\chi|_{1, \Omega_0 \triangle \Omega_{0,h}}\}.$$

Since  $\chi \in V_h^0$  we have

$$|\chi|_{0, \Gamma_{0,h}} \leq C\|\chi\|_{1, \Omega_{1,h}} \leq C|\chi|_{1, \Omega_{1,h}},$$

and hence

$$|\langle \tilde{q}_1, \chi \rangle_{\Gamma_{0,h}} - \langle q_1, \chi \rangle_{\Gamma_0}| \leq Ch|q_1|_{0, \Gamma_0}|\chi|_{1, \Omega_{0,h} \cup \Omega_{1,h}}. \quad (\text{A.3})$$

We combine the estimates (A.1) and (A.3) to yield by the Cauchy–Schwarz inequality

$$\begin{aligned} |U^h - v_h|_{1, \Omega_{0,h} \cup \Omega_{1,h}} &\leq C(h^{p-1}\|u^h\|_{p, \Omega_0 \cup \Omega_1} + |U^h - \varphi_h|_{1, \Omega_{0,h} \cup \Omega_{1,h}} \\ &\quad + h|q_1|_{0, \Gamma_0}). \end{aligned} \quad (\text{A.4})$$

We complete the proof by estimation of the second term in (A.4). By the definition of  $\varphi_h$  we have for any  $\chi \in V_h^0$

$$a_h(U^h - \varphi_h, U^h - \varphi_h) \leq a_h(U^h - \varphi_h - \chi, U^h - \varphi_h - \chi)$$

and hence by lemma 5

$$\begin{aligned} |U^h - \varphi_h|_{1, \Omega_{0,h} \cup \Omega_{1,h}} &\leq C(h^{p-1}\|U^h\|_{p, \Omega_{0,h} \cup \Omega_{1,h}} + h^{-1/2}|[U^h] - [\varphi_h]|_{0, \Gamma_{0,h}} \\ &\quad + h^{-1/2}|U^h - \varphi_h|_{0, \Gamma_h}). \end{aligned} \quad (\text{A.5})$$



We next estimate each term on the right hand side of (A.5). By (2.9) we have

$$\|U^h\|_{p, \Omega_0, h \cup \Omega_1, h} \leq C \|u^h\|_{p, \Omega_0 \cup \Omega_1}. \quad (\text{A.6})$$

For the second term we have, since  $[\varphi_h] = [\widetilde{u}^h]$ ,

$$|[U^h] - [\varphi_h]|_{0, \Gamma_{0, h}} \leq |[U^h] - [\widehat{U}^h]|_{0, \Gamma_{0, h}} + |[\widehat{U}^h] - [\widetilde{u}^h]|_{0, \Gamma_{0, h}}.$$

Taking  $v = U^h - \widehat{U}^h$  in (2.31) gives

$$\begin{aligned} h^{-1/2} |[U^h] - [\widehat{U}^h]|_{0, \Gamma_{0, h}} &\leq Ch^{-1} \{ \|U^h - \widehat{U}^h\|_{0, \Omega_0, h \cup \Omega_1, h} + h \|U^h - \widehat{U}^h\|_{1, \Omega_0, h \cup \Omega_1, h} \} \\ &\leq Ch^{-1} \{ \|u^h - \widehat{u}^h\|_{0, \Omega_0 \cup \Omega_1} + h \|u^h - \widehat{u}^h\|_{1, \Omega_0 \cup \Omega_1} \}. \end{aligned} \quad (\text{A.7})$$

Next, we have by lemma 1

$$\begin{aligned} |[\widehat{U}^h] - [\widetilde{u}^h]|_{0, \Gamma_{0, h}} &\leq |[\widehat{U}^h - U^h] - [\widetilde{u}^h - u^h]|_{0, \Gamma_{0, h}} + |[U^h] - [u^h]|_{0, \Gamma_{0, h}} \\ &\leq C (h \|u^h - \widehat{u}^h\|_{1, \Omega_0 \cup \Omega_1} + h^p \|u^h\|_{p, \Omega_1 \cup \Omega_0}). \end{aligned}$$

Lastly, we have

$$\begin{aligned} h^{-1/2} |U^h - \varphi_h|_{0, \Gamma_h} &= h^{-1/2} |U^h - \widetilde{\widehat{Q}_h g}|_{0, \Gamma_h} \\ &\leq h^{-1/2} |\widehat{U}^h - \widetilde{\widehat{u}^h}|_{0, \Gamma_h} + h^{-1/2} |\widehat{U}^h - U^h|_{0, \Gamma_h}, \end{aligned}$$

and by lemma 1 again

$$\begin{aligned} h^{-1/2} |\widehat{U}^h - \widetilde{\widehat{u}^h}|_{0, \Gamma_h} &\leq h^{-1/2} \{ |(\widehat{U}^h - U^h) - (\widetilde{\widehat{u}^h} - \widetilde{u^h})|_{0, \Gamma_h} + |U^h - \widetilde{u^h}|_{0, \Gamma_h} \} \\ &\leq C (h^{1/2} \|\widehat{u}^h - u^h\|_{1, \Omega_1} + h^{p-1/2} \|u^h\|_{p, \Omega_1}). \end{aligned} \quad (\text{A.8})$$

Combining the estimates (A.4)–(A.8) gives the result.  $\square$

Next we give the rather lengthy proof of lemma 13.

*Proof of lemma 13.* Let  $\varphi \in C_0^\infty(\Omega_h)$  and define  $w \in H^2(\Omega_0 \cup \Omega_1)$ , and extended outside of  $\Omega$  by  $E_1$ , by

$$A^0 w^0 = \varphi \quad \text{in } \Omega_0, \quad A^1 w^1 = \varphi \quad \text{in } \Omega_1$$

$$\text{with } [w] = 0 \text{ on } \Gamma_0 \quad \text{and} \quad w = 0 \text{ on } \Gamma,$$

with  $\partial w^0 / \partial \nu_{h,A^0} = \partial w^1 / \partial \nu_{h,A^1}$  on  $\Gamma_0$ . Then using Greens identities we find that

$$\begin{aligned} (\widehat{U}^h - v_h, \varphi)_{\Omega_{0,h} \cup \Omega_{1,h}} &= \tilde{a}(\widehat{U}^h - v_h, w) + \left\langle [\widehat{U}^h] - [v_h], \frac{\partial w}{\partial \nu_{h,A}} \right\rangle_{\Gamma_{0,h}} \\ &\quad - \left\langle \widehat{U}^h - v_h, \frac{\partial w}{\partial \eta_{h,A^1}} \right\rangle_{\Gamma_h}, \end{aligned}$$

where  $\partial w / \partial \nu_{h,A} = \partial w^0 / \partial \nu_{h,A^0}$  on edges  $\Gamma_{0,h}^{(j)} \subset \Omega_0$  and  $\partial w / \partial \nu_{h,A} = \partial w^1 / \partial \nu_{h,A^1}$  on edges  $\Gamma_{0,h}^{(j)} \subset \Omega_1$ . Now it follows for any  $\chi \in V_h^0$

$$\begin{aligned} (\widehat{U}^h - v_h, \varphi)_{\Omega_{0,h} \cup \Omega_{1,h}} &= \tilde{a}(\widehat{U}^h - v_h, w) - a_h(\widehat{U}^h - v_h, W) + a_h(\widehat{U}^h - v_h, W - \chi) \\ &\quad + a_h(\widehat{U}^h - v_h, \chi) + \left\langle [\widehat{U}^h] - [v_h], \frac{\partial w}{\partial \nu_{h,A}} \right\rangle_{\Gamma_{0,h}} \\ &\quad - \left\langle \widehat{U}^h - v_h, \frac{\partial w}{\partial \eta_{h,A^1}} \right\rangle_{\Gamma_h}, \end{aligned} \tag{A.9}$$

where  $W$  is defined by (2.8). We estimate each of the terms on the right hand side of (A.9) in succession. A straightforward calculation shows that

$$\begin{aligned} \tilde{a}(\widehat{U}^h - v_h, w) - a_h(\widehat{U}^h - v_h, W) &= a_{\Omega_0 \setminus \overline{\Omega}_{0,h}}^0(\widehat{U}^h - v_h, w) - a_{\Omega_{0,h} \setminus \overline{\Omega}_0}^0(\widehat{U}^h - v_h, W) \\ &\quad + a_{\Omega_{0,h} \setminus \overline{\Omega}_0}^1(\widehat{U}^h - v_h, w) \\ &\quad - a_{\Omega_0 \setminus \overline{\Omega}_{0,h}}^1(\widehat{U}^h - v_h, W), \end{aligned}$$

and hence

$$\begin{aligned} &|\tilde{a}(\widehat{U}^h - v_h, w) - a_h(\widehat{U}^h - v_h, W)| \\ &\leq C \|\widehat{U}^h - v_h\|_{1, \Omega_0 \triangle \Omega_{0,h}} (\|w\|_{1, \Omega_0 \triangle \Omega_{0,h}} + \|W\|_{1, \Omega_0 \triangle \Omega_{0,h}}). \end{aligned}$$

By lemma 3 we have

$$\|w\|_{1, \Omega_0 \triangle \Omega_{0,h}} + \|W\|_{1, \Omega_0 \triangle \Omega_{0,h}} \leq Ch \|w\|_{2, \Omega_0 \cup \Omega_1},$$

and by the triangle inequality together with (2.9)

$$\begin{aligned} \|\widehat{U}^h - v_h\|_{1, \Omega_0 \triangle \Omega_{0,h}} &\leq \|\widehat{U}^h - U^h\|_{1, \Omega_0 \triangle \Omega_{0,h}} + \|U^h - v_h\|_{1, \Omega_0 \triangle \Omega_{0,h}} \\ &\leq C \|\widehat{u}^h - u^h\|_{1, \Omega_0 \cup \Omega_1} + \|U^h - v_h\|_{1, \Omega_{0,h} \cup \Omega_{1,h}}. \end{aligned}$$

Consequently, by lemma 12 there results

$$\begin{aligned} |\tilde{a}(\widehat{U}^h - v_h, w) - a_h(\widehat{U}^h - v_h, W)| &\leq C\|w\|_{2,\Omega_0\cup\Omega_1} \{h\|\widehat{u}^h - u^h\|_{1,\Omega_0\cup\Omega_1} \\ &\quad + h^2|q_1|_{0,\Gamma_0} + \|\widehat{u}^h - u^h\|_{0,\Omega_0\cup\Omega_1} + h^p\|u^h\|_{p,\Omega_0\cup\Omega_1}\}. \end{aligned} \quad (\text{A.10})$$

For the next term we have

$$|a_h(\widehat{U}^h - v_h, W - \chi)| \leq |\widehat{U}^h - v_h|_{1,\Omega_{0,h}\cup\Omega_{1,h}} |W - \chi|_{1,\Omega_{0,h}\cup\Omega_{1,h}}.$$

We choose  $\chi$  to be the minimizing function in lemma 9 and use lemma 12 to yield

$$\begin{aligned} |a_h(\widehat{U}^h - v_h, W - \chi)| &\leq C\|w\|_{2,\Omega_0\cup\Omega_1} \{h\|\widehat{u}^h - u^h\|_{1,\Omega_0\cup\Omega_1} + h^2|q_1|_{0,\Gamma_0} \\ &\quad + \|\widehat{u}^h - u^h\|_{0,\Omega_0\cup\Omega_1} + h^p\|u^h\|_{p,\Omega_0\cup\Omega_1}\}. \end{aligned} \quad (\text{A.11})$$

For the third term on the right hand side of (A.9) we have

$$a_h(\widehat{U}^h - v_h, \chi) = \{a_h(\widehat{U}^h, \chi) - a(\widehat{u}^h, \bar{\chi})\} + \{a(\widehat{u}^h, \bar{\chi}) - a_h(v_h, \chi)\},$$

where  $\bar{\chi}$  denotes the extension of  $\chi$  by zero. For the first term in the previous equation we have, since  $\widehat{U}^h|_{\Omega_{i,h}} = \widehat{u}^h$ ,

$$|a_h(\widehat{U}^h, \chi) - a(\widehat{u}^h, \bar{\chi})| \leq C\{\|\widehat{u}^h\|_{1,\Omega_0\Delta\Omega_{0,h}} |\chi|_{1,\Omega_0\Delta\Omega_{0,h}} + |\widehat{U}^h|_{1,\Omega_h\setminus\bar{\Omega}} |\chi|_{1,\Omega_h\setminus\bar{\Omega}}\}.$$

Now it is easy to see, using lemma 3, that

$$\begin{aligned} \|\widehat{u}^h\|_{1,\Omega_0\Delta\Omega_{0,h}} &\leq C\|\widehat{u}^h - u^h\|_{1,\Omega_0\cup\Omega_1} + \|u^h\|_{1,\Omega_0\Delta\Omega_{0,h}} \\ &\leq C(\|\widehat{u}^h - u^h\|_{1,\Omega_0\cup\Omega_1} + h^{p-1}\|u^h\|_{p,\Omega_0\cup\Omega_1}). \end{aligned}$$

Similarly,

$$\begin{aligned} |\widehat{U}^h|_{1,\Omega_h\setminus\bar{\Omega}} &\leq |\widehat{U}^h - U^h|_{1,\Omega_h\setminus\bar{\Omega}} + |U^h|_{1,\Omega_h\setminus\bar{\Omega}} \\ &\leq C(\|\widehat{u}^h - u^h\|_{1,\Omega_0\cup\Omega_1} + h^{p-1}\|u^h\|_{p,\Omega_0\cup\Omega_1}). \end{aligned}$$

Therefore,

$$\begin{aligned} |a_h(\widehat{U}^h, \chi) - a(\widehat{u}^h, \bar{\chi})| &\leq C(\|\widehat{u}^h - u^h\|_{1,\Omega_0\cup\Omega_1} + h^{p-1}\|u^h\|_{p,\Omega_0\cup\Omega_1}) \\ &\quad \times (|\chi|_{1,\Omega_0\Delta\Omega_{0,h}} + |\chi|_{1,\Omega_h\setminus\bar{\Omega}}). \end{aligned} \quad (\text{A.12})$$

We estimate the terms involving  $\chi$  in the last inequality as follows. By lemma 3 and our choice of  $\chi$

$$\begin{aligned} |\chi|_{1,\Omega_0\Delta\Omega_{0,h}} &\leq |W - \chi|_{1,\Omega_0\Delta\Omega_{0,h}} + |W|_{1,\Omega_0\Delta\Omega_{0,h}} \\ &\leq |W - \chi|_{1,\Omega_{0,h}\cup\Omega_{1,h}} + Ch\|w\|_{2,\Omega_0\cup\Omega_1} \\ &\leq Ch\|w\|_{2,\Omega_0\cup\Omega_1}. \end{aligned} \quad (\text{A.13})$$

By a similar analysis we find

$$|\chi|_{1,\Omega_h \setminus \bar{\Omega}} \leq Ch \|w\|_{2,\Omega_0 \cup \Omega_1}, \quad (\text{A.14})$$

and hence the estimates (A.12)–(A.14) yield

$$|a_h(\widehat{U}^h, \chi) - a(\widehat{u}^h, \bar{\chi})| \leq C \|w\|_{2,\Omega_0 \cup \Omega_1} (h |\widehat{u}^h - u^h|_{1,\Omega_0 \cup \Omega_1} + h^p \|u^h\|_{p,\Omega_0 \cup \Omega_1}).$$

It remains to estimate  $a(\widehat{u}^h, \bar{\chi}) - a_h(v_h, \chi)$ . By definitions of  $\widehat{u}^h$  and  $v_h$

$$|a(\widehat{u}^h, \bar{\chi}) - a_h(v_h, \chi)| = |\langle q_1, \chi \rangle_{\Gamma_0} - \langle \widetilde{q}_1, \chi \rangle_{\Gamma_{0,h}}|.$$

Clearly,

$$\begin{aligned} \langle q_1, \chi \rangle_{\Gamma_0} - \langle \widetilde{q}_1, \chi \rangle_{\Gamma_{0,h}} &= \langle \widetilde{q}_1, J\widetilde{\chi} \rangle_{\Gamma_{0,h}} - \langle \widetilde{q}_1, \chi \rangle_{\Gamma_{0,h}} \\ &= \langle \widetilde{q}_1, (J-1)\widetilde{\chi} \rangle_{\Gamma_{0,h}} + \langle \widetilde{q}_1, \widetilde{W} - W \rangle_{\Gamma_{0,h}} \\ &\quad + \langle \widetilde{q}_1, \widetilde{\chi} - \widetilde{W} - (\chi - W) \rangle_{\Gamma_{0,h}}, \end{aligned}$$

and hence by lemma 1 and (A.2)

$$\begin{aligned} |a(\widehat{u}^h, \bar{\chi}) - a_h(v_h, \chi)| &\leq C |\widetilde{q}_1|_{0,\Gamma_{0,h}} \{ h^2 |\widetilde{\chi}|_{0,\Gamma_{0,h}} + h |\chi - W|_{1,\Omega_0 \triangle \Omega_{0,h}} \\ &\quad + h^2 \|w\|_{2,\Omega_0 \cup \Omega_1} \}. \end{aligned} \quad (\text{A.15})$$

Now we note that by lemma 1 and (2.31),

$$\begin{aligned} |\widetilde{\chi}|_{0,\Gamma_{0,h}} &\leq |\widetilde{\chi} - \chi|_{0,\Gamma_{0,h}} + |\chi|_{0,\Gamma_{0,h}} \\ &\leq C (h |\chi|_{1,\Omega_0 \triangle \Omega_{0,h}} + \|\chi\|_{0,\Omega_{0,h}}^{1/2} \|\chi\|_{1,\Omega_{0,h}}^{1/2}) \\ &\leq C \|\chi\|_{1,\Omega_{0,h} \cup \Omega_{1,h}}. \end{aligned}$$

Also,

$$\|\chi\|_{1,\Omega_{0,h} \cup \Omega_{1,h}} \leq \|\chi - W\|_{1,\Omega_{0,h} \cup \Omega_{1,h}} + \|W\|_{1,\Omega_{0,h} \cup \Omega_{1,h}} \leq C \|w\|_{2,\Omega_0 \cup \Omega_1},$$

and hence

$$|\widetilde{\chi}|_{0,\Gamma_{0,h}} \leq C \|w\|_{2,\Omega_0 \cup \Omega_1}. \quad (\text{A.16})$$

By our choice of  $\chi$  and (A.15)–(A.16) we have

$$|a(\widehat{u}^h, \bar{\chi}) - a_h(v_h, \chi)| \leq Ch^2 |q_1|_{0,\Gamma_0} \|w\|_{2,\Omega_0 \cup \Omega_1}. \quad (\text{A.17})$$

Finally we estimate the interface and boundary terms in (A.9). For the interface term we have

$$\left| \left\langle [\widehat{U}^h] - [v_h], \frac{\partial w}{\partial \nu_{h,A}} \right\rangle_{\Gamma_{0,h}} \right| \leq |[\widehat{U}^h] - [v_h]|_{0,\Gamma_{0,h}} \left| \frac{\partial w}{\partial \nu_{h,A}} \right|_{0,\Gamma_{0,h}}. \quad (\text{A.18})$$

By lemma 2

$$\left| \frac{\partial w}{\partial \nu_{h,A}} \right|_{0,\Gamma_{0,h}} \leq C \|w\|_{2,\Omega_0 \cup \Omega_1}. \quad (\text{A.19})$$

Next, we write

$$[\widehat{U}^h] - [v_h] = [\widehat{U}^h] - [\widehat{u}^h] = [\widehat{U}^h - U^h] - [\widehat{u}^h - u^h] + [U^h] - [u^h],$$

and hence by lemma 1

$$\begin{aligned} |[\widehat{U}^h] - [v_h]|_{0,\Gamma_{0,h}} &\leq |[\widehat{U}^h - U^h] - [\widehat{u}^h - u^h]|_{0,\Gamma_{0,h}} + |[U^h] - [u^h]|_{0,\Gamma_{0,h}} \\ &\leq C(h \|\widehat{u}^h - u^h\|_{1,\Omega_0 \cup \Omega_1} + h^p \|u^h\|_{p,\Omega_0 \cup \Omega_1}). \end{aligned}$$

From this, (A.18), and (A.19) there follows

$$\begin{aligned} \left| \left\langle [\widehat{U}^h] - [v_h], \frac{\partial w}{\partial \nu_{h,A}} \right\rangle_{\Gamma_{0,h}} \right| &\leq C(h \|\widehat{u}^h - u^h\|_{1,\Omega_0 \cup \Omega_1} + h^p \|u^h\|_{p,\Omega_0 \cup \Omega_1}) \\ &\quad \times \|w\|_{2,\Omega_0 \cup \Omega_1}. \end{aligned} \quad (\text{A.20})$$

By a similar analysis we find

$$\begin{aligned} \left| \left\langle \widehat{U}^h - v_h, \frac{\partial w}{\partial \eta_{h,A^1}} \right\rangle_{\Gamma_h} \right| &\leq C(h \|\widehat{u}^h - u^h\|_{1,\Omega_0 \cup \Omega_1} + h^p \|u^h\|_{p,\Omega_0 \cup \Omega_1}) \\ &\quad \times \|w\|_{2,\Omega_0 \cup \Omega_1}. \end{aligned} \quad (\text{A.21})$$

Combining (A.10), (A.11), (A.17), (A.20), and (A.21) we conclude that

$$\begin{aligned} \|\widehat{U}^h - v_h\|_{0,\Omega_h} &= \sup_{\varphi \in C_0^\infty(\Omega_h)} \frac{|(\widehat{U}^h - v_h, \varphi)_{\Omega_h}|}{\|\varphi\|_{0,\Omega_h}} \\ &\leq C(h^p \|u^h\|_{p,\Omega_0 \cup \Omega_1} + \|\widehat{u}^h - u^h\|_{0,\Omega_0 \cup \Omega_1} \\ &\quad + h \|\widehat{u}^h - u^h\|_{1,\Omega_0 \cup \Omega_1} + h^2 |q_1|_{0,\Gamma_0}), \end{aligned}$$

and the proof of lemma 13 is complete.  $\square$

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