Approximation of Solutions of Mixed Boundary Value Problems for Poisson's Equation by Finite Differences

J. H. Bramble and B. E. Hubbard

The University of Maryland, College Park, Maryland†

Abstract. This paper is concerned with the formulation of finite-difference analogues of mixed boundary value problems for Poisson's equation. The normal derivative is approximated in such a way that the matrix of the resulting system is of positive type. The discretization error is shown to be $O(h^2)$, where h is the mesh constant.

I. *Introduction*

In this paper we are concerned with a finite-difference approximation to the solution of the boundary value problem

$$\begin{aligned} -\Delta u &= f, \quad \text{in } R \\ \frac{\partial u}{\partial n} + \alpha u &= g, \quad \text{on } C_1 \\ u &= H, \quad \text{on } C_2 . \end{aligned} \tag{1.1}$$

The region R is a bounded connected open set in the (x, y) plane whose boundary C consists of the two parts C_1, and C_2. The symbol Δ is the Laplace operator $\Delta \equiv (\partial^2/\partial x^2) + (\partial^2/\partial y^2)$, and $\partial/\partial n$ denotes differentiation with respect to the outward-directed normal on C_1. The functions f, g and H are defined to be sufficiently smooth functions on R, C_1 and C_2 respectively. The function α is required to satisfy the following conditions on C_1 : (a) piecewise continuity with a finite number of discontinuities, (b) piecewise differentiability, (c) at all points of continuity, either $\alpha = 0$ (the set $C_1^{(1)}$) or $\alpha \geqq \alpha_m > 0$, where α_m is a constant (the set $C_1^{(2)}$).

We restrict our considerations to the cases in which either the set C_2 or $C_1^{(2)}$ contains a nonempty open subset of C. In these cases (1.1) has a unique solution provided the data and boundary are sufficiently smooth. The case in which C_2 is all of C is just the Dirichlet problem. Results for this special case are contained in [8].

We are interested in formulating a finite-difference analogue of (1.1) which has the following properties: (a) The boundary approximations involve at most three interior points (and one boundary point) (b) The matrix of the system is

This research was supported in part by a grant with the National Science Foundation NSF Grant GP-3 and with the Air Force Office of Scientific Research, Air Research and Development Command AFOSR 62-454.

† Institute for Fluid Dynamics & Applied Mathematics.

Journal of the Association for Computing Machinery, Vol. 12, No. 1 (January, 1965), pp. 114–123

of "positive type" (cf. [4]) (c) The truncation error tends to zero quadratically (as $O(h^2)$, where h is the mesh size).

Section 2 is concerned with the construction of the boundary operator at smooth points of C_1. The construction given there is intended to show that appropriate points can always be chosen. Practically speaking, the choice would simply be made by examining a finite number of possibilities (in as clever a fashion as possible).

In the last section the approximating finite difference problem is defined and the tools with which to study the truncation error are developed. The error is shown to be $O(h^2)$. We wish to emphasize the interesting fact that at points near the boundary the differential operator is approximated only to $O(h)$, while the boundary condition itself is approximated to $O(h^2)$. A direct application of the method of Gershgorin [9] would show only that our method converges as $O(h)$. Our treatment shows that the convergence is, in fact, $O(h^2)$.

Also in the last section a finite-difference analogue involving an $O(h)$ approximation to $\partial u/\partial n$ using two interior points, (cf [11], p. 213) is seen to lead to $O(h)$ convergence.

Among the authors who have studied problems of this type, Batschelet [2] seems to have gone the furthest. He gives an $O(h)$ approximation to the mixed problem and proves convergence by a direct extension of Gershgorin's technique [9].

Methods for setting up boundary operators (to approximate the normal derivative) are given by Kantorovich and Krylov [11], Shaw [13], Allen [1], Viswanathan [16], Forsythe and Wasow [8], Uhlmann [15], and Greenspan [10]. The local boundary approximation of Viswanathan resembles ours in that the differential equation and boundary condition are taken into account. Uhlmann and Greenspan derive higher order formulas by simply involving more points. Only local properties are discussed and no convergence proofs are given in any but the paper of Batschelet.

II. *An $O(h^2)$ Approximation of $\partial u/\partial n$*

We consider an arbitrary point 0 on C at which C is smooth (see Figure 1). Choose 0 to be the origin of a Cartesian coordinate system (x, y) such that the x-axis is tangent to C at 0. The positive y-direction is taken along the inward normal. It can be shown, for any smooth enough function v, that $v_{xy} = -v_{ns} + Kv_s$ at the origin (cf. [14] for the use of geodesic normal coordinates). The subscripts denote the indicated partial differentiation, n being the outward normal direction, s arc length and K the curvature of C. Thus we have

$$v_{xy} = -\frac{\partial}{\partial s}(v_n + \alpha v) + (\alpha + K)v_s + \alpha_s v \tag{2.1}$$

at 0. Also, of course

$$v_{yy} = \Delta v - v_{xx}. \tag{2.2}$$

Now consider the Taylor expansion of v about 0; i.e.

$$v(P) = v(0) + xv_x(0) + yv_y(0) + \tfrac{1}{2}\{x^2v_{xx}(0) + 2xyv_{xy}(0) + y^2v_{yy}(0)\} + O(x^3 + y^3). \tag{2.}$$

We note that $v_x(0) = v_s(0)$ and $v_y(0) = -v_n(0)$. Thus, using (2.1) and (2.

$$\begin{aligned} v(P) = [1 + xy\alpha_s(0)]v(0) &+ x[1 + y(\alpha(0) + K(0)]v_s(0) \\ &- yv_n(0) + \tfrac{1}{2}[x^2 - y^2]v_{xx}(0) + \frac{y^2}{2}\Delta v(0) \\ &- xy[v_n + \alpha v]_s + O(x^3 + y^3). \end{aligned}$$

Let $P_i = (x_i, y_i)$, $i = 1, 2, 3$. We wish to determine three numbers a_i, i 1, 2, 3 such that

$$\begin{aligned} \sum_{i=1}^{3} a_i\{v(P_i) &- [1 + x_i y_i \alpha_s(0)]v(0)\} \\ &= -v_n(0) + \sum_{i=1}^{3} a_i\left\{\frac{y_i^2}{2}\Delta v(0) - x_i y_i[v_n + \alpha v]_s(0)\right\} \\ &\quad + O\left(\sum_{i=1}^{3} a_i[x_i^3 + y_i^3]\right). \end{aligned} \tag{2.}$$

For (2.4) to hold for any v, a_i must satisfy

$$\begin{aligned} \sum_{i=1}^{3} a_i y_i &= 1 \\ \sum_{i=1}^{3} a_i x_i[1 + y_i(\alpha(0) + K(0))] &= 0 \\ \sum_{i=1}^{3} a_i[x_i^2 - y_i^2] &= 0. \end{aligned} \tag{2.}$$

We will show further that the points P_i may be chosen so that $a_i \geqq 0$. This w be useful in the later applications.

To show that we can get a non-negative solution of (2.5), we consider t system

$$\begin{aligned} \sum_{i=1}^{3} \bar{a}_i y_i &= 1 \\ \sum_{i=1}^{3} \bar{a}_i x_i &= 0 \\ \sum_{i=1}^{3} \bar{a}_i(x_i^2 - y_i^2) &= 0. \end{aligned} \tag{2.}$$

Since we are interested in small values of x and y, $\bar{a}_i$ will be close to a_i. T system (2.6) in matrix notation takes the form

Now consider the Taylor expansion of v about 0; i.e.

$$v(P) = v(0) + xv_x(0) + yv_y(0) + \tfrac{1}{2}\{x^2v_{xx}(0) + 2xyv_{xy}(0) + y^2v_{yy}(0)\} + O(x^3 + y^3). \tag{2.3}$$

We note that $v_x(0) = v_s(0)$ and $v_y(0) = -v_n(0)$. Thus, using (2.1) and (2.2),

$$v(P) = [1 + xy\alpha_s(0)]v(0) + x[1 + y(\alpha(0) + K(0)]v_s(0) - yv_n(0) + \tfrac{1}{2}[x^2 - y^2]v_{xx}(0) + \frac{y^2}{2}\Delta v(0) - xy[v_n + \alpha v]_s + O(x^3 + y^3).$$

Let $P_i = (x_i, y_i)$, $i = 1, 2, 3$. We wish to determine three numbers a_i, $i = 1, 2, 3$ such that

$$\sum_{i=1}^{3} a_i\{v(P_i) - [1 + x_i y_i \alpha_s(0)]v(0)\} = -v_n(0) + \sum_{i=1}^{3} a_i\left\{\frac{y_i^2}{2}\Delta v(0) - x_i y_i[v_n + \alpha v]_s(0)\right\} + O\left(\sum_{i=1}^{3} a_i[x_i^3 + y_i^3]\right). \tag{2.4}$$

For (2.4) to hold for any v, a_i must satisfy

$$\begin{aligned} \sum_{i=1}^{3} a_i y_i &= 1 \\ \sum_{i=1}^{3} a_i x_i[1 + y_i(\alpha(0) + K(0))] &= 0 \\ \sum_{i=1}^{3} a_i[x_i^2 - y_i^2] &= 0. \end{aligned} \tag{2.5}$$

We will show further that the points P_i may be chosen so that $a_i \geqq 0$. This will be useful in the later applications.

To show that we can get a non-negative solution of (2.5), we consider the system

$$\begin{aligned} \sum_{i=1}^{3} \bar{a}_i y_i &= 1 \\ \sum_{i=1}^{3} \bar{a}_i x_i &= 0 \\ \sum_{i=1}^{3} \bar{a}_i(x_i^2 - y_i^2) &= 0. \end{aligned} \tag{2.6}$$

Since we are interested in small values of x and y, $\bar{a}_i$ will be close to a_i. The system (2.6) in matrix notation takes the form

$$\begin{bmatrix} y_1 & y_2 & y_3 \\ x_1 & x_2 & x_3 \\ x_1^2 - y_1^2 & x_2^2 - y_2^2 & x_3^2 - y_3^2 \end{bmatrix} \begin{bmatrix} \bar{a}_1 \\ \bar{a}_2 \\ \bar{a}_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}. \tag{2.7}$$

The solution of (2.7) is

$$\begin{aligned} \bar{a}_1 &\equiv \frac{1}{\bar{D}} [x_2(x_3^2 - y_3^2) - x_3(x_2^2 - y_2^2)] \equiv \frac{\bar{D}_1}{\bar{D}} \\ \bar{a}_2 &\equiv \frac{1}{\bar{D}} [x_3(x_1^2 - y_1^2) - x_1(x_3^2 - y_3^2)] \equiv \frac{\bar{D}_2}{\bar{D}} \\ \bar{a}_3 &\equiv \frac{1}{\bar{D}} [x_1(x_2^2 - y_2^2) - x_2(x_1^2 - y_1^2)] \equiv \frac{\bar{D}_3}{\bar{D}} \end{aligned} \tag{2.8}$$

where $\bar{D}$ is the determinant of the system. If the $\bar{D}_i$'s in (2.8) are chosen positive, then clearly the condition that $y_i > 0$ insures that $\bar{D} > 0$ (since $\bar{D} = \sum_{i=1}^{3} y_i \bar{D}_i$) and hence the $\bar{a}_i$'s will be positive. The condition that $y_i > 0$ means essentially that the points P_i are to be taken from R.

Now we need to restrict our attention to only certain points of R. In particular we show that we may always select three *nearby* interior mesh points for which (2.5) is satisfied with $\bar{a}_i > 0$. By nearby we mean that the points (x_i, y_i) lie within a circle about 0 with radius Mh whenever the mesh size, h, is taken sufficiently small. We give now one possible construction to show that this can be done.

In the usual manner we put a square mesh of size h on R and call the crossings *mesh points.* Now let ϵ be a given positive number (which will depend on h). Choose

$$\begin{aligned} &4\epsilon > x_1 > y_1 + \epsilon > 2\epsilon \\ &4\epsilon > -x_2 > y_2 + \epsilon > 2\epsilon \\ &6\epsilon \geqq y_3 > |x_3| + 5\epsilon. \end{aligned} \tag{2.9}$$

Geometrically this means that P_1 lies in I, P_2 in II and P_3 in III of Figure 1. The number $\bar{K}$ denotes the maximum positive curvature of C. It follows that $\bar{D}_i > 12\epsilon^3$, $i = 1, 2, 3$. The triangular regions I, II, III will lie in R provided $\epsilon < 2/17\bar{K}$ and the region is "wide" enough. If on the other hand $\epsilon = (\frac{3}{2})h$, then at least one point of the mesh will lie in each of the regions I, II and III. Hence $h < 4/51\bar{K}$ is a sufficient condition for the existence of $(x_i, y_i) \in R$ for which a solution $\bar{a}_i \geqq 0$ of (2.7) exists. It is also easy to see that the points in question always lie in a region of radius $10h$ so that we have only a finite number of points (independent of h) to consider. Now it is easy to see that $\bar{D} < 768\epsilon^4$. We want finally to relate this to the solution of (2.5). Let $a_i = D_i/D$, where D is the determinant of the system and the D_i's are the appropriate cofactors. Comparing the two systems and using the inequalities (2.9) we have

$$-672\epsilon^4 | \alpha + K |_M + \bar{D}_i \leqq D_i \leqq \bar{D}_i + 672\epsilon^4 | \alpha + K |_M, \tag{2.10}$$

where $| \alpha + K |_M = \text{Max}_{P \in C} | \alpha(P) + K(P) |$. From (2.10) it follows imme-

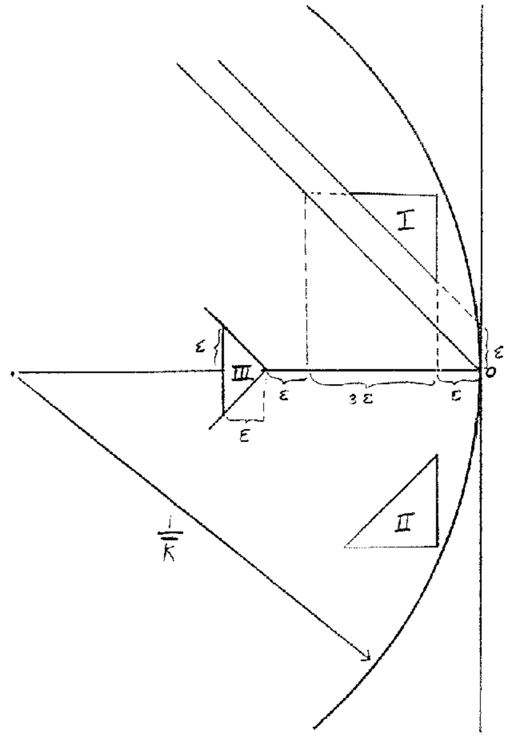

FIG. 1

diately that

$$a_i > h^{-1}\left[\frac{1 - (84 \mid \alpha + K \mid_M)h}{96 + (756 \mid \alpha + K \mid_M)h}\right].$$

Clearly, for sufficiently small h, $a_i > 0$ provided $\mid \alpha + K \mid$ is bounded. In like manner it can be shown that each a_i is bounded above by a term which is $O(h^{-1})$.

Thus we have shown, under these assumptions, that it is always possible to choose three points P_i from a given set of mesh points such that $a_i > 0$. Furthermore the points can be found within a sphere of radius βh where β is a constant independent of h.

Thus we define the boundary operator

$$\delta_n V(P) = \sum_{i=1}^{3} a_i\{[1 + x_i y_i \alpha_s(P)]V(P) - V(P_i)\},$$

where α is a given function on C and the P_i are chosen as mesh points in R such that $a_i \geqq 0$, $i = 1, 2, 3$. The equations (2.5) are assumed to be satisfied. From (2.4) and (2.10) it is easy to see that u of (1.1) satisfies

$$\left| \delta_n u(P) + \alpha(P)u(P) - \left\{g(P) + \sum_{i=1}^{3} a_i\left[\frac{y_i^2}{2} f(P) + x_i\, y_i \frac{\partial g(P)}{\partial s}\right]\right\}\right| \leqq k_1 h^2, \tag{2.11}$$

where k_1 is a constant independent of h.

We remark here that an $O(h)$ approximation to $\partial u/\partial n$ which is of positive type is easily obtained (cf. [11]). Choosing only two interior points P_1 and P_2 in (2.3) we obtain

$$-\bar{\delta}_n V(0) \equiv \sum_{i=1}^{2} b_i[V(P_i) - V(0)] = -V_n(0) + O\left[\sum_{i=1}^{2} b_i(x_i^2 + y_i^2)\right], \quad (2.12)$$

provided

$$\begin{bmatrix} y_1 & y_2 \\ x_1 & x_2 \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \quad (2.13)$$

Clearly $y_1, y_2 > 0, \quad x_1 > 0, \quad x_2 < 0$ will guarantee the existence of non-negative numbers b_1 and b_2. Thus analogous to (2.11) we are led to

$$|\bar{\delta}_n u(P) + \alpha(P)u(P) - g(P)| \leqq \bar{k}_1 h. \quad (2.14)$$

III. *Finite-Difference Analogue*

As mentioned in the last section we place a square mesh of width h on the region R and call the mesh crossings *mesh points*. The set R_h will consist of those mesh points of R whose four nearest neighbors are in R. The intersection of the mesh with C_i will make up the set C_{ih}, $\quad i = 1, 2$. The sets C_{ih}^* will denote those mesh points of R which are at a distance less than or equal to h (along the horizontal or vertical) from C_{ih}, $\quad i = 1, 2$.

We define the following operators. At a point (x, y) of R_h, $\quad \Delta_h V(x, y) \equiv h^{-2}\{V(x + h, y) + V(x - h, y) + V(x, y + h) + V(x, y - h(- 4V(x, y)\}$. It is well known that for $u \in C^{(4)}(\bar{R})$

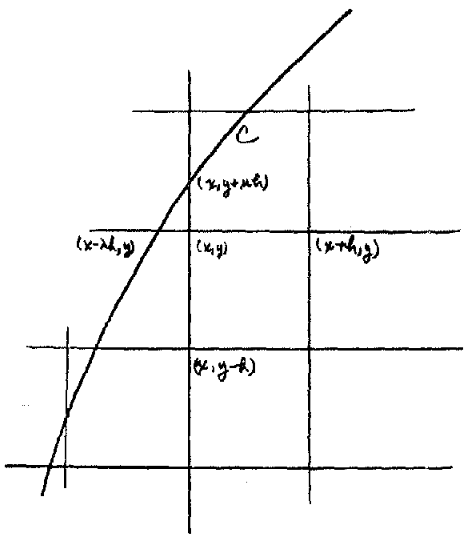

FIG. 2

$$| \Delta u(P) - \Delta_h u(P) | \leqq M_4 h^2, \quad P \in R_h , \tag{3.1}$$

where M_4 is a constant depending on the fourth derivatives of u. On C_{ih}^* we use the operator of Shortley and Weller [12],

$$\Delta_h V(x, y) \equiv 2h^{-2} \left\{ \frac{1}{\lambda(1 + \lambda)} V(x - \lambda h, y) + \frac{1}{1 + \lambda} V(x + h, y) + \frac{1}{\mu(1 + \mu)} V(x, y + \mu h) + \frac{1}{1 + \mu} V(x, y - h) - [(1/\lambda) + (1/\mu)] V(x, y) \right\}$$

For example, if (x, y) is the point (x, y) in Figure 2, then $0 < \lambda, \quad \mu \leqq 1$. The inequality

$$| \Delta u(P) - \Delta_h u(P) | \leqq M_3 h, \quad P \in C_{ih}^* , \tag{3.2}$$

where M_3 depends on the third derivatives of u, is easily verified.

We now pose the following finite-difference analogue of (1.1):

$$\begin{aligned} -\Delta_h U(P) &= f(P), \qquad P \in R_h + C_{ih}^* , \quad i = 1, 2 \\ \delta_n U(P) + \alpha(P) U(P) &= g(P) + \sum_{i=1}^{3} a_i(P) \left[\frac{y_i^2}{2} f(P) + x_i y_i \frac{\partial g}{\partial s}(P) \right], \\ & \qquad P \in C_{1h} \\ U(P) &= H(P), \qquad P \in C_{2h} . \end{aligned} \tag{3.3}$$

In Section 2 it was shown that the operator δ_n, with the desired properties could always be constructed if P is a point where C_1 is smooth and h is small enough. We emphasize, however, that we do not require that the points P_i be chosen as in Section 2. They must simply be chosen so that the corresponding a_i's are non-negative and of course so that (2.11) is satisfied. We assume now that this has been done.

In all cases the matrix of the system (3.3) is of positive type for h sufficiently small that $\sum_{i=1}^{3} a_i x_i y_i \alpha_s + \alpha \geqq 0$ and therefore possesses a non-negative inverse (cf [4]). In the subsequent discussion we assume that this condition on h is satisfied. We thus may introduce Green's function corresponding to (3.3). Let $G_h(P, Q)$ be defined as

$$\begin{aligned} -\Delta_{h,P} G_h(P, Q) &= h^{-2} \delta(P, Q), \qquad P \in R_h + C_{ih}^*, \quad i = 1, \\ \delta_n G_h(P, Q) + \alpha(P) G_h(P, Q) &= \delta(P, Q), \qquad P \in C_{ih} \\ G_h(P, Q) &= \delta(P, Q), \qquad P \in C_{2h} , \end{aligned}$$

for $Q \in R_h + C_{1h}^* + C_{2h}^* + C_{1h} + C_{2h}$. Since $G_h(P, Q)$ is just the inverse of (3.3) multiplied by a diagonal matrix with positive diagonal elements, it follows also that $G_h(P, Q) \geqq 0$. Now, using $G_h(P, Q)$ we have the relation

$$V(P) = h^2 \sum_{Q \in R_h + C^*_{1h} + C^*_{2h}} G_h(P, Q)[-\Delta_h V(Q)] + \sum_{Q \in C_{1h}} G_h(P, Q)[\delta_n V(Q) + \alpha V(Q)] + \sum_{Q \in C_{2h}} G_h(P, Q) V(Q). \tag{3.4}$$

This follows immediately from the uniqueness of solutions of (3.3) for arbitrary right-hand side. Letting $V(P) = 1$ in (3.4), it follows that $\sum_{Q \in C_{2h}} G_h(P, Q) \leqq 1$.

Now suppose R is such that there exists a function $\phi \in C^3(\bar{R})$ (ϕ has continuous third derivatives in the closure of R) satisfying

$$-\Delta\phi \geqq 1 \quad \text{in} \quad R$$
$$\frac{\partial\phi}{\partial n} + \alpha\phi \geqq 1 \quad \text{on} \quad C_1. \tag{3.5}$$

Then for small enough h, $-\Delta_h\phi(P) \geqq \frac{1}{2}$, $P \in R_h + C^*_{1h} + C^*_{2h}$, and $\delta_n\phi(P) + \alpha(P)\phi(P) \geqq \frac{1}{2}$, $P \in C_{1h}$. Taking $V(P) = \phi(P)$ in (3.4) we have

$$\sum_{Q \in C_{1h}} G_h(P, Q) + h^2 \sum_{Q \in R_h + C^*_{1h} + C^*_{2h}} G_h(P, Q) \leqq 4 \mid \phi \mid_M . \tag{3.6}$$

To estimate $\sum_{Q \in C^*_{1h} + C^*_{2h}} G_h(P, Q)$ we observe as in [4] that if W is the function which is zero on C and one in R then $-\Delta_h W(P) \geqq h^{-2}$, $P \in C^*_{1h} + C^*_{2h}$ and $-\Delta_h W(P) = 0$, $P \in R_h$. Letting $V(P) = W(P)$ in (3.4) we have

$$\sum_{Q \in C_{1h} + C^*_{2h}} G_h(P, Q) \leqq 1 + \max_{\bar{Q} \in C_{1h}} \left[\sum_{i=1}^{3} a_i(\bar{Q}) \right] \sum_{Q \in C_{1h}} G_h(P, Q).$$

Now $1 = \sum_{i=1}^3 a_i y_i \geqq [\sum_{i=1}^3 a_i] \min y_i \geqq [\sum_{i=1}^3 a_i] 3h$ for any $P \in C_{1h}$. Thus $\sum_{i=1}^3 a_i \leqq 1/3h$. Hence, using (3.6),

$$h \sum_{Q \in C_{1h^*} + C^*_{2h}} G_h(P, Q) \leqq h + \tfrac{4}{3} \mid \phi \mid_M . \tag{3.7}$$

Actually we can obtain a sharper estimate for that part of the sum in (3.7) for $Q \in C^*_{2h}$. To do this let $\bar{W}(P)$ be the function which has the values zero on C_{2h} and one otherwise. Then $-\Delta_h\bar{W}(P) \geq h^{-2}$, $P \in C^*_{2h}$ and $-\Delta_h\bar{W}(P) = 0$, $P \in R_h + C^*_{1h}$. Letting $V(P) = \bar{W}(P)$ in (3.4) it follows immediately that

$$\sum_{Q \in C^*_{2h}} G_h(P, Q) \leq 1. \tag{3.8}$$

We are now in a position to prove the following theorem.

THEOREM 1. *Let $u \in C^4(\bar{R})$ be the solution of* (1.1). *Suppose that the function ϕ of* (3.5) *exists. Then $\epsilon(P) \equiv u(P) - U(P)$, $P \in R_h + C^*_{1h} + C^*_{2h} + C_{1h} + C_{2h}$, where $U(P)$ is the solution of* (3.3), *satisfies the inequality*

$$\max_P \mid \epsilon(P) \mid \leqq kh^2. \tag{3.9}$$

In (3.9) *k is a constant which depends on u and ϕ but not on h.*

PROOF. Let $V(P) = \epsilon(P)$ in (3.4). Thus

$$\epsilon(P) = h^2 \sum_{Q \in R_h + C^*_{1h} + C^*_{2h}} G_h(P, Q)[-\Delta_h\epsilon(Q)] + \sum_{Q \in C^*_{1h}} G_h(P, Q)[\delta_n\epsilon(Q) + \alpha(Q)\epsilon(Q)].$$

Now since $G_h(P, Q) \geqq 0$ we have

$$\begin{aligned} |\epsilon(P)| \leqq {} & [h^2 \sum_{Q \in R_h} G_h(P, Q)] \max_{Q \in R_h} |\Delta_h \epsilon(Q)| \\ & + [h \sum_{Q \in C^*_{1h}+C_2} G_h(P, Q)] \max_{Q \in C^*_{1h}+C^*_{2h}} |h|\Delta_h\epsilon(Q)| \\ & [\sum_{Q \in C_{1h}} G_h(P, Q)] \max_{Q \in C_{1h}} |\delta_n \epsilon(Q) + \alpha(Q)\epsilon(Q)|. \end{aligned} \tag{3.10}$$

From (3.1), (3.2) and (2.11) we have the estimates

$$\begin{aligned} |\Delta_h \epsilon(P)| &\leqq M_4 h^2, \quad P \in R_h \\ |h\Delta_h \epsilon(P)| &\leqq M_3 h^2, \quad P \in C^*_{1h} + C^*_{2h} \\ |\delta_n \epsilon(P) + \alpha(P)\epsilon(P)| &\leqq k_1 h^2, \quad P \in C_{1h}\,. \end{aligned} \tag{3.11}$$

The result (3.9) now follows by inserting (3.6), (3.7) and (3.11) into (3.10). By reasoning quite parallel to that leading up to, and in, the proof of Theorem 1 it is possible to prove

THEOREM 2. *Let $u \in C^3(\bar{R})$ be the solution of* (1.1). *Suppose that the function ϕ of* (3.5) *exists. Then $\epsilon(P) \equiv u(P) - U(P)$, $P \in R_h + C^*_{1h} + C^*_{2h} + C_{1h} + C_{2h}$, where $U(P)$ is the solution of* (3.3) *with the second equation replaced by $\bar{\delta}_n U(P) + \alpha(P)\, U(P) = g(P)$ (see* (2.12)), *satisfies*

$$\max_P |\epsilon(P)| \leqq \bar{k}h. \tag{3.12}$$

In (3.12) *$\bar{k}$ is a constant which depends on u and ϕ but not on h.*

It is clear that if one is interested only in convergence proofs for either of the two difference methods given above, then the assumption that the second derivatives of u are continuous is a sufficient smoothness requirement.

We note that in the hypothesis of Theorem 1 the local truncation error (i.e. the error in approximating the equations (1.1)) is $O(h^2)$ on the sets C_{1h} and R_h. It was assumed that no error occurred on C_{2h} although it is clear from the development that an error of not worse than $O(h^2)$ would not have changed the final result. On the other hand the error committed on the sets C^*_{1h} and C^*_{2h} was $O(h)$. In spite of this apparent defect the truncation error itself was shown to be $O(h^2)$. Making use of the more refined estimate (3.8) it is easily seen that the contribution to the error from C^*_{2h} was in fact $O(h^3)$, hence on C^*_{2h} cruder approximations could have been made.

It is the opinion of the present authors that under quite general conditions if a truncation error of $O(h^n)$ is desired, it is sufficient to have a local approximation of $O(h^{n-1})$ on C^*_{1h} and $O(h^{n-2})$ on C^*_{2h}, while at most of the other points the defect should be $O(h^n)$.

RECEIVED SEPTEMBER, 1963.

REFERENCES

1. ALLEN, D. N. DEG. *Relaxation Methods.* McGraw-Hill, New York, 1954.
2. BATSCHELET. Über die numerische Anflösung von Randwertproblemen bei elliptischen partiellen Differentialgleichungen. *Z. Angew. Math. Phys. 3,* 1952.

3. Bramble, J.H. Fourth order finite-difference analogues of the Dirichlet problem for Poisson's equation in three and four dimensions. *Math. Comput. 17* (1963), 217–222.
4. Bramble, J. H. and Hubbard, B. E. On the formulation of finite-difference analogues of the Dirichlet problem for Poisson's equation. *Numer. Math. 4* (1962), 313–327.
5. —— and ——. A theorem on error estimation for finite-difference analogues of the Dirichlet problem for elliptic equations. *Contrib. Diff. Eq. 2* (1963), 319–340.
6. —— and ——. A priori bounds on the discretization error in the numerical solution of the Dirichlet problem. *Contrib. Diff. Eq. 2* (1963), 229–252.
7. —— and ——. On a finite-difference analogue of an elliptic boundary problem which is neither diagonally dominant nor of non-negative type. *J. Math. Phys. 43* (1964), 117–135.
8. Forsythe, G., and Wasow, W. *Finite Difference Methods for Partial Differential Equations.* Wiley, New York, 1960.
9. Gerschgorin, S. Fehlerabschätzung für das Differenzenverfahren zur Lösung Partieller Differentialgleichungen. *Z. Angew. Math. Mech. 10* (1930), 373–382.
10. Greenspan, D. On the Numerical Solution of Problems Allowing Mixed Boundary Conditions. *Notices Am. Math. Soc. 10* (1963), 92.
11. Kantorovich, L. and Krylov, V. *Approximate Methods of Higher Analysis.* Noordhoff Ltd., Netherlands, 1958.
12. Shortley, G. and Weller, R. The numerical solution of Laplace's equation. *J. Appl. Phys. 9* (1938), 334–348.
13. Shaw, F. S. *An Introduction to Relaxation Methods.* Dover, New York, 1950.
14. Synge, J. L. and Schild, A. *Tensor Calculus.* U. of Toronto Press, Toronto, 1952.
15. Uhlmann. Differenzenverfahren fur die 2. und 3. Randwertaufgabe mit Rändern bei $\Delta u(x, y) = r(x, y, u)$. *Z. Angew. Math. Mech. 38* (1958).
16. Viswanathan, R. V. Solution of Poisson's equation by relaxation method-normal gradient specified on curved boundaries. *Math. Tables Aids Comput. 11* (1957).