Comparing Optimization and Estimation Techniques for Low-Thrust Spacecraft Rendezvous

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Low-thrust technologies offer efficient mass usage which can lower spacecraft mass and, thus, mission costs. However, due to the small forces generated by low-thrust engines, long-duration burns are required to significantly alter a spacecraft path. Accordingly, low-thrust trajectory design must incorporate additional variables to describe the thrust vector during the burn duration. The solution to this higher-dimensional trajectory design problem is often obtained via optimal control where the thrust vector constitutes the set of control variables. In this study, a rendezvous scenario between a low-thrust-equipped spacecraft and an uncontrolled "mothership" is considered. Uncertainties in the spacecraft position, velocity, and thrust vectors are mitigated via estimation algorithms and a mass-optimal trajectory is obtained that satisfies the natural dynamics and mission-imposed constraints. Several techniques for optimization and estimation are investigated and their effects on the optimal solution are discussed.

I. Nomenclature

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$\mathbf{\Phi}$	=	state transition matrix
α	=	thrust pointing angle
λ_i	=	costate
θ	=	spacecraft longitude angle
θ'	=	target longitude angle
μ	=	Earth gravitational parameter
$\vec{\psi}$	=	terminal constraint vector
ϕ	=	terminal cost
$\mathbf{A}$	=	Jacobian matrix
$\vec{F}_g$	=	gravity force vector
$\vec{F}_{lt}$	=	low-thrust force vector
H	=	Hamiltonian
I_{sp}	=	specific impulse
J	=	optimization cost
L	=	Lagrangian
T	=	thrust magnitude
$\hat{e}$	=	rotating frame unit vector
g_0	=	Earth mean gravitational acceleration
$\hat{i}$	=	inertial frame unit vector
m	=	spacecraft mass
n'	=	target mean motion
r	=	spacecraft orbital radius
r'	=	target orbital radius
u	=	control state

II. Introduction

Rendezvous of a chaser spacecraft with a target spacecraft in a predetermined orbit is a common problem in spaceflight operations. Spacecraft mass is frequently minimized during such operations to reduce overall mission expenditures. Advancements in low-thrust propulsion technology facilitate these minimizations by transforming

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propellant mass into a propulsive force more efficiently than other technologies. However, due to the small accelerations delivered by low-thrust systems, significant burn durations are required to alter the course of the spacecraft. Accordingly, low-thrust trajectory design must incorporate variables to describe the thrust vector in addition to the usual variables that describe a spacecraft state. The following proposed analysis attempts to blend these two traditional spaceflight problems and determine optimization and estimation techniques that allow rendezvous using low-thrust propulsion systems.

III. Problem Definition

Note: This is likely more verbose than it needs to be; we can trim later

The rendezvous problem includes two vehicles: the target and the "chaser," refered to hereafter simply as the spacecraft. The target moves on a circular orbit about the Earth with an orbital radius r' and is located by angle θ' relative to the inertial x-axis, $\hat{i}_x$, as depicted in Figure 1. As the target orbit is circular, the angle may be represented as a
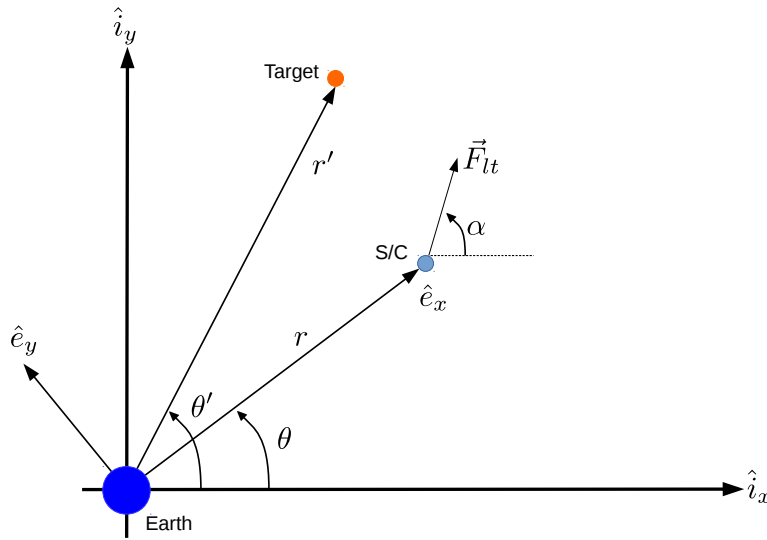
Fig. 1   The spacecraft (S/C) and target are located relative to the Earth

simple function of time,

$$\theta' = \theta'_0 + n't, \tag{1}$$

where θ'_0 locates the initial target position, t is time, n' is the target mean motion,

$$n' = \sqrt{\frac{\mu}{r'^3}}, \tag{2}$$

and μ is the gravitational parameter associated with Earth, $\mu = 3.986 \times 10^5$ km^3/s^2. The chaser vehicle, i.e., the spacecraft, is located in an arbitrary orbit about the Earth with instantaneous radius r and angle θ. Let the $\hat{e}$ frame rotate with the spacecraft such that $\hat{e}_x$ points from the Earth to the spacecraft, $\hat{e}_z = \hat{i}_z$, and $\hat{e}_y$ completes the right-handed set.

A. Equations of Motion

Two forces are modeled to define the spacecraft motion. First, gravity is modeled via the inverse-square relationship,

$$\vec{F}_g = \frac{-\mu m}{r^2} \hat{e}_x, \tag{3}$$

where m is the spacecraft mass. Additionally, a low-thrust force is applied to the spacecraft and is modeled as

$$\vec{F}_{lt} = T \left(\cos \alpha \hat{i}_x + \sin \alpha \hat{i}_y \right), \tag{4}$$

where T represents the thrust magnitude and α orients the thrust vector relative to the inertial x-axis. Apply Newton's law to describe the spacecraft acceleration,

$$m\ddot{\vec{r}} = \sum \vec{F} .\tag{5}$$

Let the spacecraft position vector be $\vec{r} = r\hat{e}_x$. Accordingly, the spacecraft acceleration vector is

$$\ddot{\vec{r}} = \left(\ddot{r} - r\dot{\theta}^2\right)\hat{e}_x + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\hat{e}_y .\tag{6}$$

Substitute this relationship into Newton's law to write two equations of motion,

$$\ddot{r} = r\dot{\theta}^2 - \frac{\mu}{r^2} + \frac{T}{m}\left(C_\alpha C_\theta + S_\alpha S_\theta\right) ,\tag{7}$$

$$\ddot{\theta} = -2\frac{\dot{r}\dot{\theta}}{r} + \frac{T}{mr}\left(S_\alpha C_\theta - C_\alpha S_\theta\right) .\tag{8}$$

An additional equation describes the spacecraft mass,

$$\dot{m} = \frac{-T}{I_{sp}g_0} ,\tag{9}$$

where I_{sp} is the engine specific impulse, and g_0 is the mean Earth gravitational acceleration, $g_0 = 9.80665$ m/s^2. Define a state vector, $\vec{x} = \left\{r \quad \theta \quad \dot{r} \quad \dot{\theta} \quad m\right\}^T$, and let the governing equations be

$$\dot{\vec{x}} = \left\{\begin{array}{c} \dot{r} \\ \dot{\theta} \\ r\dot{\theta}^2 - \frac{\mu}{r^2} + \frac{T}{m}\left(C_\alpha C_\theta + S_\alpha S_\theta\right) \\ -2\frac{\dot{r}\dot{\theta}}{r} + \frac{T}{mr}\left(S_\alpha C_\theta - C_\alpha S_\theta\right) \\ -T/(I_{sp}g_0) \end{array}\right\}\tag{10}$$

. The Jacobian matrix, which relates variations in the state variables to variations in their derivatives, is frequently useful in optimization and estimation processes and facilitates the construction of the state transition matrix. Let the Jacobian be represented by the matrix

$$\mathbf{A} = \frac{\partial\dot{\vec{x}}}{\partial\vec{x}} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \dot{\theta}^2 + 2\mu/r^3 & \frac{T}{m}\left(-C_\alpha S_\theta + S_\alpha C_\theta\right) & 0 & 2r\dot{\theta} & -\frac{T}{m^2}\left(C_\alpha C_\theta + S_\alpha S_\theta\right) \\ \frac{2\dot{r}\dot{\theta}}{r^2} - \frac{T}{mr^2}\left(S_\alpha C_\theta - C_\alpha S_\theta\right) & -\frac{T}{mr}\left(S_\alpha S_\theta + C_\alpha C_\theta\right) & -2\dot{\theta}/r & -2\dot{r}/r & -\frac{T}{m^2 r}\left(S_\alpha C_\theta - C_\alpha S_\theta\right) \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} .\tag{11}$$

Accordingly, the state transition matrix, $\mathbf{\Phi}(t_1, t_0)$, that relates variations in $\vec{x}(t_0)$ to variations at some later state $\vec{x}(t_1)$ is governed by the differential equation,

$$\dot{\mathbf{\Phi}} = \mathbf{A}\mathbf{\Phi} .\tag{12}$$

B. Optimization Problem

The goal of the rendezvous problem is to move the spacecraft to the location of the target while minimizing expended mass or, equivalently, maximizing final mass. Accordingly, the optimization problem is

$$\min_\alpha J = -m(t_f),\tag{13}$$

subject to the dynamics in Equation (10), the initial conditions

$$r(t_0) = r_0, \qquad \theta(t_0) = \theta_0, \qquad \dot{r}(t_0) = \dot{r}_0, \qquad \dot{\theta}(t_0) = \dot{\theta}_0, \qquad m(t_0) = m_0, t_0 = 0,$$

and the terminal constraints

$$\vec{\psi} = \left\{\begin{array}{c} r(t_f) - r' \\ \theta(t_f) - \theta_0' - nt_f \\ \dot{r}(t_f) \\ \dot{\theta}(t_f) - n \end{array}\right\} = \vec{0}\tag{14}$$

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C. Solution via Indirect Optimization

The standard form of the cost function in the Bolza problem is given by

$$J = \phi(x(t_f),\, t_f) + \int_{t_0}^{t_f} L(\vec{x}, \vec{u}, t)\, \mathrm{d}t \,.$$

Thus, the rendezvous problem is formulated such that $\phi = -m(t_f)$, $L = 0$, and $\vec{u} = \{\alpha\}$. Define the Hamiltonian,

$$H \equiv L + \vec{\lambda}^T \dot{\vec{x}}$$

$$H = \lambda_1 \dot{r} + \lambda_2 \dot{\theta} + \lambda_3 \left[r\dot{\theta}^2 - \frac{\mu}{r} + \frac{T}{m}(C_\alpha C_\theta + S_\alpha S_\theta) \right] +$$

$$\lambda_4 \left[-2\frac{\dot{r}\dot{\theta}}{r} + \frac{T}{mr}(S_\alpha C_\theta - C_\alpha S_\theta) \right] - \lambda_5 \frac{T}{I_{sp} g_0} \,. \tag{15}$$

Apply the Euler-Lagrange equations to find the costate derivatives,

$$\dot{\lambda}_1 = -\frac{\partial H}{\partial r} = -\lambda_3 \left[\dot{\theta}^2 + \frac{2\mu}{r^3} \right] - \lambda_4 \left[\frac{2\dot{r}\dot{\theta}}{r^2} - \frac{T}{mr^2}(S_\alpha C_\theta - C_\alpha S_\theta) \right], \tag{16}$$

$$\dot{\lambda}_2 = -\frac{\partial H}{\partial \theta} = -\lambda_3 \frac{T}{m}(-C_\alpha S_\theta + S_\alpha C_\theta) - \lambda_4 \frac{T}{mr}(-S_\alpha S_\theta - C_\alpha C_\theta), \tag{17}$$

$$\dot{\lambda}_3 = -\frac{\partial H}{\partial \dot{r}} = -\lambda_1 + \lambda_4 \frac{2\dot{\theta}}{r}, \tag{18}$$

$$\dot{\lambda}_4 = -\frac{\partial H}{\partial \dot{\theta}} = -\lambda_2 - 2\lambda_3 r\dot{\theta} + \lambda_4 \frac{2\dot{r}}{r}, \tag{19}$$

$$\dot{\lambda}_5 = -\frac{\partial H}{\partial m} = \lambda_3 \frac{T}{m^2}(C_\alpha C_\theta + S_\alpha S_\theta) + \lambda_4 \frac{T}{rm^2}(S_\alpha C_\theta - C_\alpha S_\theta) \,. \tag{20}$$

Similarly, the optimal control law is derived via

$$u^* = u \left| \frac{\partial H}{\partial \alpha} = 0 \right. \quad \rightarrow \quad \frac{\partial H}{\partial u} = \lambda_3 \frac{T}{m}(-S_\alpha C_\theta + C_\alpha S_\theta) + \lambda_4 \frac{T}{mr}(C_\alpha C_\theta + S_\alpha S_\theta) = 0 \,. \tag{21}$$

To simplify notation, let γ represent the thrust pointing angle in the rotating frame, i.e., $\gamma = \alpha - \theta$. Then, assuming T and m are nonzero throughout the optimization, Equation (21) simplifies to

$$\frac{\partial H}{\partial u} = -\lambda_3 S_\gamma + \frac{1}{r}\lambda_4 C_\gamma = 0 \quad \rightarrow \quad r \tan \gamma = \frac{\pm \lambda_4}{\pm \lambda_3} \,. \tag{22}$$

To resolve the sign ambiguity, evaluate the second partial derivative,

$$\frac{\partial^2 H}{\partial \alpha^2} = -\lambda_3 \frac{T}{m}(C_\alpha C_\theta + S_\alpha S_\theta) + \lambda_3 \frac{T}{mr}(-S_\alpha C_\theta + C_\alpha S_\theta) > 0 \tag{23}$$

$$= \frac{T}{m} \left[-\lambda_3 C_\gamma - \lambda_4 \frac{1}{r} S_\gamma \right] > 0 \,. \tag{24}$$

The thrust and mass are always positive ($T/m > 0$), thus, choose the negative coefficients for λ_3 and λ_4. The sine and cosine functions are then rewritten in terms of the costates,

$$S_\gamma = \frac{-\lambda_4}{\sqrt{r^2 \lambda_3^2 + \lambda_4^2}} \,, \tag{25}$$

$$C_\gamma = \frac{-r\lambda_3}{\sqrt{r^2 \lambda_3^2 + \lambda_4^2}} \,. \tag{26}$$

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Boundary conditions are available from the terminal constraints and terminal cost. The variation in each terminal constraint is given by

$$d\psi_i = \frac{\partial \psi_i}{\partial \vec{x}_f} d\vec{x}_f + \frac{\partial \psi_i}{\partial t_f} dt_f \,. \tag{27}$$

Accordingly, the following relationships result,

$$dr_f = 0, \tag{28}$$
$$d\theta_f = n \, dt_f, \tag{29}$$
$$d\dot{r}_f = 0, \tag{30}$$
$$d\dot{\theta}_f = 0 \,. \tag{31}$$

A similar relationship is available for the terminal cost variation,

$$d\phi = \frac{\partial \phi}{\partial \vec{x}_f} d\vec{x}_f + \frac{\partial \phi}{\partial t_f} dt_f \,. \tag{32}$$

Substituting the relationships for this problem yields the equation,

$$d\phi = - \, dm_f \,. \tag{33}$$

This variations are all incorporated into the *transversality condition*,

$$H^*(t_f) \, dt_f - \vec{\lambda}_f^T \, d\vec{x}_f + d\phi = 0 \,. \tag{34}$$

The resulting relationship is given by

$$H_f^* \, dt_f - \lambda_{2,f} \, d\theta_f - \lambda_{5,f} - dm_f = 0$$
$$\left[H_f^* - \lambda_{2,f} n\right] dt_f - \left[1 + \lambda_{5,f}\right] dm_f = 0 \,. \tag{35}$$

Since neither dt_f nor dm_f are zero, each coefficient must independently evaluate to zero to satisfy the transversality condition. Two boundary conditions result:

$$\lambda_{2,f} = H_f^*/n, \qquad \lambda_{5,f} = -1$$

With these boundary conditions, the two-point boundary value problem is well-defined:

$$\dot{r} = \dot{r}$$

$$\dot{\theta} = \dot{\theta}$$

$$\ddot{r} = r\dot{\theta}^2 - \frac{\mu}{r^2} - \frac{T}{m}\frac{r\lambda_3}{\sqrt{r^2\lambda_3^2 + \lambda_4^2}}$$

$$\ddot{\theta} = -2\frac{\dot{r}\dot{\theta}}{r} - \frac{T}{mr}\frac{\lambda_4}{\sqrt{r^2\lambda_3^2 + \lambda_4^2}}$$

$$\dot{m} = -T/(I_{sp}g_0)$$

$$\dot{\lambda}_1 = -\lambda_3\left[\dot{\theta}^2 + \frac{2\mu}{r^3}\right] - \lambda_4\left[\frac{2\dot{r}\dot{\theta}}{r^2} + \frac{T}{mr^2}\frac{\lambda_4}{\sqrt{r^2\lambda_3^2 + \lambda_4^2}}\right],$$

$$\dot{\lambda}_2 = 2\frac{T}{m}\frac{\lambda_3\lambda_4}{\sqrt{r^2\lambda_3^2 + \lambda_4^2}},$$

$$\dot{\lambda}_3 = -\lambda_1 + \lambda_4\frac{2\dot{\theta}}{r},$$

$$\dot{\lambda}_4 = -\lambda_2 - 2\lambda_3 r\dot{\theta} + \lambda_4\frac{2\dot{r}}{r},$$

$$\dot{\lambda}_5 = -\frac{T}{m^2}\frac{r\lambda_3^2}{\sqrt{r^2\lambda_3^2 + \lambda_4^2}} - \frac{T}{rm^2}\frac{\lambda_4^2}{\sqrt{r^2\lambda_3^2 + \lambda_4^2}}.$$

$$r(t_0) = r_0, \qquad \theta(t_0) = \theta_0, \qquad \dot{r}(t_0) = \dot{r}_0, \qquad \dot{\theta}(t_0) = \dot{\theta}_0, \qquad m(t_0) = m_0, \qquad t_0 = 0,$$

$$r_{(t_f)} = f', \qquad \theta(t_f) = \theta_0' + nt_f, \qquad \dot{r}(t_f) = 0, \qquad \dot{\theta}(t_f) = n, \qquad \lambda_2(t_f) = \frac{H_f^*}{n}, \qquad \lambda_5(t_f) = -1$$