FIN2020

Foundation of Finance

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有意思的题目：

Quiz： Quiz 1：3

HW ：

L1 Basics of Finance

1. Basics of Finance

① Two aspects

Asset pricing √ : How to price an asset?

Corporate finance : Related to Capital Structure

in real life, C.S. does matter

MM Theory: If no friction, then C.S. doesn't matter

No Tax, No Bankrupcy Cost, No Agency Issue, No Info Asymmetry, No Competition

Gross return $R = 1+r$ (in DM)

② Methodology of analysis

Theoretical √ : Assumptions & Conditions

Empirical : Data

③ Type of Investors

Rational √ : prefer more to less → A well-defined function (utility function)

Behavioral : overconfidence, anchoring, gambler's fallacy

L2-3 Math preliminary

1. Optimization

① Unconstrained Opt

FOC → SOC → Global

② Constrained Opt (Maximum problem)

a. Only equ constraints — Lagrangian Thm

△1 Simplify obj func & constraints

△2 Prove the existence of global optimal sol: EVM / Concavity

△3 Write down $\mathcal{L}$, FOC ($\mathcal{L} = f(x) + \lambda(g(x) - c)$, $g(x) - c \geq 0$)

△4 Check boundary pts

△5 Check the cases that violate CQ (Check all $\nabla g(\vec{x})$)

△6 Compare & Conclude

b. There exists inequ constraints — Kuhn-Tucker Thm

△1 Simplify obj func & constraints

△2 Prove the existence of global optimal sol: EVM / Concavity

△3 Write down $\mathcal{L}$, FOC (Classify + KT conditions)

△4 Check boundary pts

△5 Check the cases that violate CQ

△6 Compare & Conclude

2. Probability

① Jensen's inequality (We cannot exchange the order of expectation & a function operation unless it's linear)

$$\begin{cases} E[g(\tilde{x})] = g(E(\tilde{x})) & : g(x) \text{ is linear} \\ E[g(\tilde{x})] > g(E(\tilde{x})) & : g(x) \text{ is strictly convex} \\ E[g(\tilde{x})] < g(E(\tilde{x})) & : g(x) \text{ is strictly concave} \end{cases}$$

proofs

$E[g(\tilde{x})] = w g(x_1) + (1-w) g(x_2)$

$g(E(\tilde{x})) = g(w x_1 + (1-w) x_2)$

$E[g(\tilde{x})] > g(E(\tilde{x})) \Rightarrow$ strictly convex

② Covariance & Correlation

$Cov(X,Y) = E[(X-EX)(Y-EY)] = E(XY) - E(X)E(Y)$

$\rho = \frac{Cov(X,Y)}{\sqrt{Var(X)}\sqrt{Var(Y)}}$

Ind → Uncorrelated ✓

Ind ⇐ Uncorrelated ✗ Binomial normal distribution: ✓

Correlation can only capture linear dependency between two RVs 相关性⇒线性相关性

③ Iterated expectation conditional

$E(X|Y)$ is a function depending on y $E(X|Y=y)$

$E(E(X|Y)) = \int_{SY} E(X|Y=y) f_Y(y) dy = E(X)$

$E[E(X|Y)|\phi] = E[X]$

The smallest info set matters.

$E(E(E(X|C_1)|C_2)|C_3) = E(X|C_1)$, if $C_1 \subset C_2 \subset C_3$

④ Law of Total variance

$Var(X) = E(Var(X|Y)) + Var(E(X|Y))$ 背诵口诀

proof:

$RHS = E(Var(X|Y)) + Var(E(X|Y))$

$= E(E(X^2|Y) - (E(X|Y))^2) + E(E(X|Y)^2) - (E(E(X|Y)))^2$

$= E(E(X^2|Y)) - E(E(X|Y))^2 + E(E(X|Y))^2 - (E(E(X|Y)))^2$

$= E(X^2) - (E(X))^2$

$= LHS$

3. Calculus

① Taylor Expansion

$f(x+a) = f(x) + \frac{f'(x)}{1!} a + \frac{f''(x)}{2!} a^2 + \cdots$

$= \sum_{n=0}^{\infty} \frac{f^{(n)}(x)}{n!} a^n$

$\approx f(x) + \frac{f'(x)}{1!} a + \frac{f''(x)}{2!} a^2$ (As $a \to 0$)

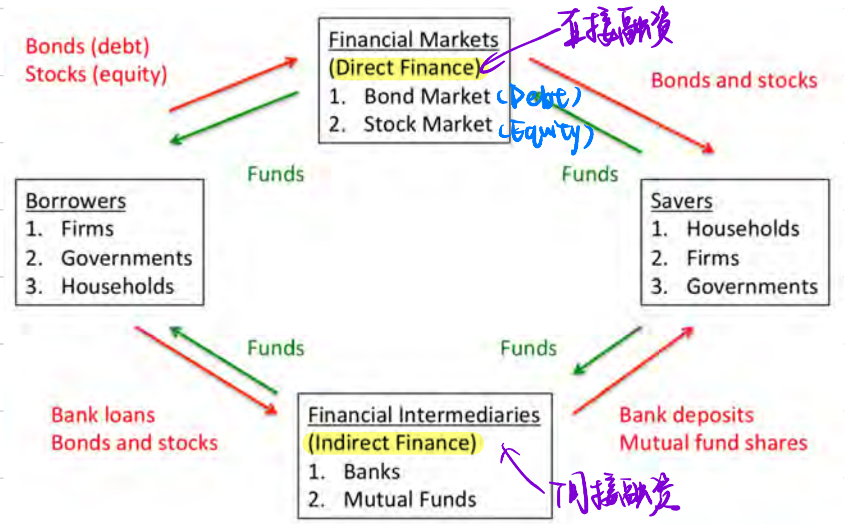
② Elasticity

Elasticity η = $\frac{\text{Percent change in } y}{\text{Percent change in } x} = \frac{\%\Delta y}{\%\Delta x} \approx \frac{\Delta \ln y}{\Delta \ln x}$

L4 Overview of Asset pricing

1. Financial institution

Financial institutions
- Financial Markets
- Financial Intermediaries



- Banks
- Mutual funds
  - Passive　听从大盘数据
  - Active　听从个股数据

2. Assets

① Types

- Safe assets : Consumption goods → Bundle
- Risky assets : Financial assets → Portfolio

a. Bond

Bond
- Term ☆ (Length of maturity)
- Interest rate ☆
  - Credit risk (Default risk) → Two extreme case
    - Govn Bond
    - Junk Bond
  - Term structure (时限)
- Tax treatment : Municipal bonds are exempt from federal income tax

b. Stock

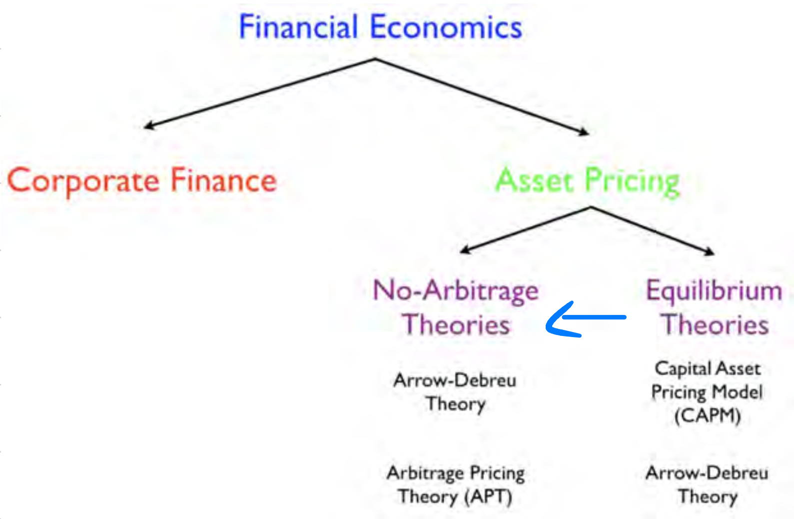
Bonds vs Stocks

	Good Economic Times	Bad Economic Times
Bonds (Debt)	Pays interest and principal.	Still pays interest and principal (except in the rare case of bankruptcy).
Stocks (Equity)	Dividends and hence the share price will increase as the firm earns higher profits.	Dividends and hence the share price may decrease as the firm earns lower profits.

← Safe (Bonds)

← Risky (Stocks)

3. Two method of Asset pricing



L5-6 Competitive equilibrium method — safe assets

1. Concepts

① Competitiveness of the market

- Perfectly competitive mkt → Sellers & Buyers have no impact, they are price takers
- Non-competitive mkt: Monopoly / Eligopoly

② Symbols denoting preference

- $\succ$: (strictly) prefer to — e.g. Asset 1 $\succ$ Asset 2 — $>$
- $\succsim$: weakly prefer to — e.g. Asset 1 $\succsim$ Asset 2 — $\geq$
- $\sim$: indifferent — e.g. Asset 1 $\sim$ Asset 2 — $=$

2. Principle

Overall logic: Optimize utility function ⟶ Determine demand ⟶ Determine price

① Demand – Supply ⇒ Price

- Demand ⟶ $D(p)$ (Unobservable)
- Supply ⟶ $S(p)$ (Observable)

⇒ $D(p) = S(p)$ ⇒ p^*

In financial market,
- demand curve is not necessary to be downward-sloping (e.g. stock market)
- supply curve is also not necessary to be upward-sloping

② Optimize utility function ⇒ Determine demand

a. Quantify the preference (主观与客观共存)

The factor that affects demand is Investors' preference/taste,

then we quantify it as Utility function.

b. Define utility function

△ Assumptions of preference (consumers) (当Preference满足这3个假设，则其可被量化为Utility function)

A1: Completeness: No ambiguity in preference.

A2: Transitivity ≙ Consistency

A3: Continuity: Small change in quantity, then no big change in preference (≙ 无突变; utility function) ⟶ Prefer a_n to b_n ⇒ $\forall n$, $a_n \succ b_n$

特别检查极限位置

Counter example: Lexicographic preference 字典序偏好

$a = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$, $b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ ⇒ $a \succ b$ if $a_1 > b_1$

if $a_1 = b_1$ & $a_2 > b_2$

e.g. $a_n = \begin{bmatrix} 1+\frac{1}{n} \\ 1 \end{bmatrix}$, $b_n = \begin{bmatrix} 1+\frac{1}{n+1} \\ 2 \end{bmatrix}$ ⇒ $a \succ b$ (但是b_n的极限大于a_n ⇒ 不连续)

△ Assumptions of utility function

A1: Prefer more to less : $u'(\cdot)>0$ ⟶ Consumption Smoothness (Consumption 水平不剧变)

A2: Diminishing marginal satisfaction: $u''(\cdot)<0$

△ Result

$$\begin{cases} a \succ b \iff u(a) > u(b) \\ a \sim b \iff u(a) = u(b) \end{cases}$$

△ Property of utility function

Not unique: if $u(\cdot)$ is a utility function, then $F(u(\cdot))$ is also a utility function, where $F(x)$ is a (strictly) increasing function

因为是 safe assets (without uncertainty), 所以此时不要求 $F(u(\cdot))'' < 0$

3. Example

① Two-good Economy

	Apple	Banana	Budget
Price	P_a	P_b	Y
Consumption	C_a	C_b	

Assume that
- Consumers' preference satisfies Completeness, Transitivity, Continuity
- $u(\cdot)$ is defined with $u'(\cdot)>0$ & $u''(\cdot)<0$
- $\beta \in (0,+\infty)$

then we construct opt. problem

$$\max_{C_a, C_b} \quad u(C_a) + \beta u(C_b)$$
$$\text{s.t.} \quad P_aC_a + P_bC_b \le Y$$

graphically,
- Objective function ⟶ Indifference curve (decreasing & convex)
- Constraint ⟶ Triangular region

$P_aC_a + P_bC_b \le Y \Rightarrow C_b \le -\frac{P_a}{P_b}C_a + \frac{Y}{P_b}$

proof:

$u(C_a) + \beta u(C_b) = C$

(i) $C_b'(C_a) = -\dfrac{u'(C_a)}{\beta u'(C_b)} \begin{matrix}\frac{\partial F}{\partial C_a} \\ \frac{\partial F}{\partial C_b}\end{matrix} < 0$

$\Rightarrow$ Decreasing

(ii) $C_b''(C_a) = -\dfrac{\overset{<0}{u''(C_a)}\cdot\beta u'(C_b) - u'(C_a)\cdot \overset{<0}{\beta u''(C_b)}\cdot C_b'(C_a)}{\underset{>0}{[\beta u'(C_b)]^2}} > 0$

非常小心: C_b 是 C_a 的函数

$\Rightarrow$ Convex

To find the opt. sol., we let

$$-\frac{P_a}{P_b} = -\frac{u'(C_a^*)}{\beta u'(C_b^*)} \Rightarrow \frac{P_a}{P_b} = \frac{u'(C_a^*)}{\beta u'(C_b^*)}$$

a 对 a

b 对 b.

Relative price

Marginal rate of substitution

(分子与分母都是 marginal)

$P_a \uparrow \Longrightarrow C_a^* \downarrow$ or $C_b^* \uparrow$

(非常小心: $u'(\cdot)$ 是 Decreasing function)

$$\begin{cases} \dfrac{P_a}{P_b} = \dfrac{u'(C_a^*)}{\beta u'(C_b^*)} \\ P_aC_a^* + P_bC_b^* = Y \end{cases} \Rightarrow \begin{cases} C_a^* \\ C_b^* \end{cases}$$

② Multiple-good Economy

	Apple	Banana	Orange	Budget
Price	P_a	P_b	P_o	Y
Consumption	C_a	C_b	C_o	

$$\max_{C_a, C_b, C_o} \; u(C_a) + \beta u(C_b) + \gamma u(C_o)$$

s.t. $P_a C_a + P_b C_b + P_o C_o \le Y$

$$\Rightarrow \begin{cases} \dfrac{P_a}{P_b} = \dfrac{u'(C_a^*)}{\beta u'(C_b^*)} \\[2ex] \dfrac{P_a}{P_o} = \dfrac{u'(C_a^*)}{\gamma u'(C_o^*)} \\[2ex] \dfrac{P_b}{P_o} = \dfrac{\beta u'(C_b^*)}{\gamma u'(C_o^*)} \end{cases}$$

(两两代替)

③ Two-date Economy

	$t=0$	$t=1$
Endowment	Y_0	Y_1
Consumption	C_0	C_1

a. Without the existence of bonds (即：无利率)

$$\max_{C_0, C_1} \; u(C_0) + \beta u(C_1)$$

(β measures the degree of patience, $0 < \beta < 1$)

$\beta \ge 1$ patient

$\beta < 1$ impatient

s.t. 在$t=0$ $\quad C_0 \le Y_0$

在$t=1$ $\quad C_1 \le Y_1$

不能透支

$$\Rightarrow \begin{cases} C_0^* = Y_0 \\ C_1^* = Y_1 \end{cases}$$

b. With the existence of bonds (Consumers can do savings for returns)

$$\max_{C_0, C_1} \; u(C_0) + \beta u(C_1)$$

s.t. 在$t=0$ $\quad C_0 + b \le Y_0$ $\quad$ (b can be $+$ / $-$)

在$t=1$ $\quad C_1 \le Y_1 + b(1+r)$

$$\Rightarrow \begin{cases} C_0^* + b = Y_0 \\ C_1^* = Y_1 + b(1+r) \end{cases} \Rightarrow \underbrace{C_0^* + \frac{C_1^*}{1+r}}_{\text{PV of total consumption}} = \underbrace{Y_0 + \frac{Y_1}{1+r}}_{\text{PV of total endowment}}$$

⇒ Financial assets help us reallocate resource across time.

④ Comparison of "two-good Economy" & "two-date Economy"

$$\max_{C_a, C_b} \; u(C_a) + \beta u(C_b) \qquad \text{s.t.} \; P_a C_a^* + P_b C_b^* = Y$$

$$\max_{C_0, C_1} \; u(C_0) + \beta u(C_1) \qquad \text{s.t.} \; \underset{\substack{\downarrow \\ P_a}}{1} \cdot C_0^* + \underset{\substack{\downarrow \\ P_b}}{\frac{1}{1+r}} C_1^* = \underset{\substack{\downarrow \\ Y}}{Y_0 + \frac{Y_1}{1+r}}$$

$P_b < P_a \Rightarrow$ 在未来消费的实际价格更低 $\Rightarrow$ 在未来消费更好

$$\frac{P_a}{P_b} = \frac{1}{\frac{1}{1+r}} = \boxed{\frac{u'(C_0^*)}{\beta u'(C_1^*)}} = 1+r$$

Intertemporal marginal rate of substitution

$r \uparrow$, $C_0^* \downarrow$ or $C_1^* \uparrow$

L7-8 Competitive equilibrium method — risky assets

1. Motivation

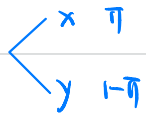
As for Risky assets, the payoff is no longer a constant, but an RV

2. $\tilde{p}$

① Two-value (Simple lottery)

$$\tilde{p} = \left(\begin{array}{ll} x & \pi \\ y & 1-\pi \end{array}\right. \Big/ \ (x, y, \pi)$$

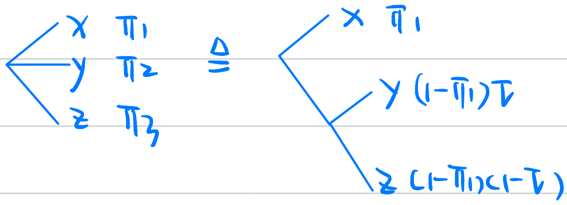
Basic element



② Three-value (Compound lottery)

$$\tilde{p} = \left(\begin{array}{ll} x & \pi_1 \\ y & (1-\pi_1)\tau \\ z & (1-\pi_1)(1-\tau) \end{array}\right. \Big/ \ (x, (y, z, \tau), \pi_1)$$

括号里是括号



3. U(·)

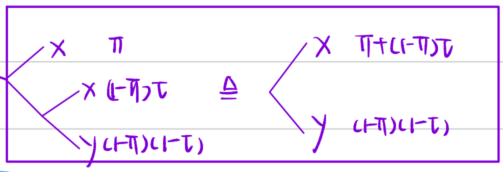
① Assumptions

A1: Equivalent expressions of $\tilde{p}$

$x \sim (x, y, 1) \sim (x, x, \pi)$ π可以是[0,1]间所有值

$(x, y, \pi) \sim (y, x, 1-\pi)$

$(x, (x, y, \tau), \pi) \sim (x, y, \pi + (1-\pi)\tau)$



$y \succ z \Leftrightarrow (x, y, \pi) \succ (x, z, \pi)$ (Independent Axiom)

$x \succ y \succ z \Leftrightarrow \exists \pi,\ s.t.\ y \sim (x, z, \pi)$

$x \succ y$ and $\pi_1 > \pi_2 \Leftrightarrow (x, y, \pi_1) \succ (x, y, \pi_2)$

A2: Assumptions of utility function

- Completeness
- Transitivity
- Continuity

A3: Existence of extreme case

- $\exists$ a best lottery : b
- $\exists$ a worst lottery : w

$(x, y, \pi) \sim ((b, w, \pi_x), (b, w, \pi_y), \pi) \sim (b, w, \pi\pi_x + (1-\pi)\pi_y)$

(总可以用最好与最坏的情况表示)

② Observations

U(·) 的三种自变量取决于最优化的目标
- Wealth
- Consumption
- Payoff

O_1: $U(\cdot)$ depends on $\tilde{p}$, i.e. $U(\cdot) = U(x, y, \pi)$

O_2:
- $x / y \uparrow$, $U(x, y, \pi) \uparrow$
- $\pi \uparrow$, x weighs more

O_3: $U(x, y, \pi)$ is not only related with Expectation, but also related with Variance

即: x & y 之间的差值也有影响

Investors' different attitude

③ Definition

$U(\tilde{p}) = E(u(\tilde{p})) = \pi u(x) + (1-\pi) u(y)$

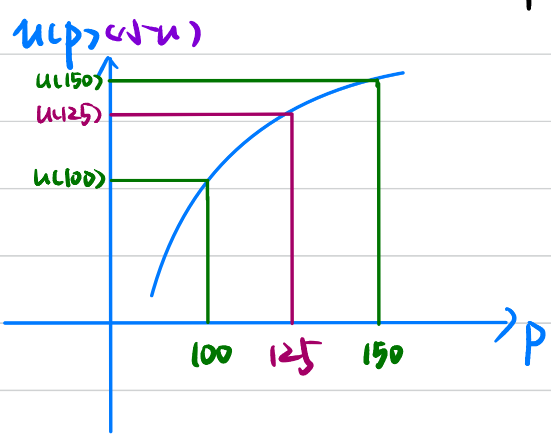
- $U(\tilde{p})$: Expected utility function / nVM
- $u(\tilde{p})$: Bernoulli utility function (increasing)

where $u(x)$ is
- Convex, increasing : for Risk seeker
- Linear, increasing : for Risk neutral
- Concave, increasing : for Risk averse

Proof (Example)

ATT $\begin{cases} 100 & \frac{1}{2} \\ 150 & \frac{1}{2} \end{cases}$, Bond $\begin{cases} 125 & \frac{1}{2} \\ 125 & \frac{1}{2} \end{cases}$ (注意此时 P_{Bond} 不是RV, 而是 Constant)

If $u(\cdot)$ is concave, then
$$\begin{cases} U(P_{Bond}) = E(u(P_{Bond})) = u(E(P_{Bond})) \\ \qquad\qquad\qquad\qquad\qquad\qquad\quad \| \\ U(\tilde{P}_{ATT}) = E(u(\tilde{P}_{ATT})) < u(E(\tilde{P}_{ATT})) \end{cases} \Rightarrow U(P_{Bond}) > U(\tilde{P}_{ATT})$$



$\Rightarrow u(125) > \frac{1}{2}(u(100) + u(150))$

$\Rightarrow E(u(P_{Bond})) > E(u(\tilde{P}_{ATT}))$

$\Rightarrow U(P_{Bond}) > U(\tilde{P}_{ATT})$

④ Properties of u

$u(\cdot) \xrightarrow{\text{Increasing \& Linear Transformation}} F(u(\cdot))$

$F(u(\cdot))$ is also a utility function

比较
- Under certainty (Safe assets): $F(x)$ 只需 increasing
- Under uncertainty (Risky assets): $F(x)$ 需 increasing & linear 因为后面要使用expected utility function

⑤ Limitations

a. The Allais Paradox: Investors are behavioral

b. Early resolution vs Late resolution

L_1
$$\begin{tikzcd} & 100 \arrow[r, dash] & 225 \\ 0 \arrow[ur, dash, "\pi"] \arrow[dr, dash, "1-\pi"'] & & \\ & 100 \arrow[r, dash] & 25 \end{tikzcd}$$
t=0 t=1 t=2

L_2
$$\begin{tikzcd} & & 225 \\ 0 \arrow[r, dash] & 100 \arrow[ur, dash] \arrow[dr, dash] & \\ & & 25 \end{tikzcd}$$
t=0 t=1 t=2

$L_1 \varphi L_2$

(可以更早地了解到收益)

即：忽略了信息的时间价值

L9 Competitive equilibrium method — Risk aversion (Individual)

1. Measure of risk aversion (主观上)

① Absolute risk aversion coefficient

a. Definition

$$R_A(Y) = - \frac{u''(Y)}{u'(Y)}$$

使得这个值为正数

当Risk seeking: 系数为负数

b. Property

Under linear transformation, $R_A(Y)$ stays invariant.

e.g. Let $v = \alpha u + \beta$, then $-\frac{v''(Y)}{v'(Y)} = -\frac{\alpha u''(Y)}{\alpha u'(Y)} = -\frac{u''(Y)}{u'(Y)}$

② Relative risk aversion coefficient

a. Definition

$$R_R(Y) = Y \cdot R_A(Y) = -Y \frac{u''(Y)}{u'(Y)}$$

b. Property

Under linear transformation, $R_R(Y)$ stays invariant.

③ Types of utility (注意+1-常数的变式)

In general, $R_A(Y)$ & $R_R(Y)$ depends on Y.

However, there are some special $u(\cdot)$ that give us a constant $R_A(Y)$ or $R_R(Y)$ (加/减常数之后也是CARA/CRRA)

a. Constant absolute risk aversion utility (CARA) — Exponential utility

$$u(Y) = -\frac{1}{\alpha} e^{-\alpha Y}, \quad \alpha > 0$$

proof: $\begin{cases} u'(Y) = e^{-\alpha Y} \\ u''(Y) = (-\alpha) e^{-\alpha Y} \end{cases} \Rightarrow R_A(Y) = -\frac{u''(Y)}{u'(Y)} = \alpha$ (Constant)

b. Constant relative risk aversion utility (CRRA) — Power utility、Log utility

$\forall \gamma$ | $\gamma = 1$ (特殊情况)

$$u(Y) = \frac{Y^{1-\gamma} - 1}{1-\gamma}, \quad \gamma > 0$$

proof:

If $\gamma \neq 1$, then $\begin{cases} u'(Y) = Y^{-\gamma} \\ u''(Y) = (-\gamma) Y^{-\gamma-1} \end{cases} \Rightarrow R_R(Y) = Y \cdot -\frac{u''(Y)}{u'(Y)} = Y \frac{Y^{-\gamma-1}}{Y^{-\gamma}} = \gamma$ (Constant)

If $\gamma = 1$, then $u(Y) = \frac{0}{0} = \lim_{r \to 1} \frac{\frac{d}{d\gamma}(Y^{1-\gamma} - 1)}{\frac{d}{d\gamma}(1-\gamma)} = \lim_{r \to 1} \frac{\frac{d}{d\gamma}(e^{(1-\gamma)\ln Y} - 1)}{\frac{d}{d\gamma}(1-r)} = \lim_{r \to 1} \frac{-\ln Y e^{(1-\gamma)\ln Y}}{-1} = \ln Y$ (Log utility is a special case of power utility)

then $\begin{cases} u'(Y) = \frac{1}{Y} \\ u''(Y) = -\frac{1}{Y^2} \end{cases} \Rightarrow R_R(Y) = -\frac{Y u''(Y)}{u'(Y)} = 1 = \gamma$ (Constant)

2. Size of bets (客观上)

① Absolute bet (增减具体美元)

$$\begin{cases} \pi & Y+h \\ 1-\pi & Y-h \end{cases}, \text{ where } h>0$$

h is the size of absolute bet

② Relative bet (增减百分比美元)

$$\begin{cases} \pi & Y(1+K) \\ 1-\pi & Y(1-K) \end{cases}, \text{ where } K>0$$

K is the size of relative bet

3. Combination of risk aversion coefficient & size of bet

① $R_A(Y)$ & $h \Rightarrow$ Risky assets (π^*)

π^* that makes safety assets & absolute bet indifferent

is determined by
$$\begin{cases} \text{Absolute risk aversion coefficient } R_A(Y) \text{ (主观上)} \\ \text{Size of absolute bet } \quad h \text{ (客观上)} \end{cases}$$

e.g.

If
$$A_1:\ \begin{cases} \pi^* & Y \\ 1-\pi^* & Y \end{cases} \quad \sim \quad A_2:\ \begin{cases} \pi^* & Y+h \\ 1-\pi^* & Y-h \end{cases}$$

then $E[u(A_1)] = E[u(A_2)]$

Since
$$\begin{cases} E[u(A_1)] = u(A_1) = u(Y) \\ E[u(A_2)] = \pi^* u(Y+h) + (1-\pi^*) u(Y-h) \end{cases}$$

then $u(Y) = \pi^* [u(Y+h) - u(Y-h)] + u(Y-h)$

$$\Rightarrow \pi^* = \frac{u(Y) - u(Y-h)}{u(Y+h) - u(Y-h)}$$

According to Taylor Expansion
$$\begin{cases} u(Y+h) \approx u(Y) + u'(Y)h + \dfrac{u''(Y)}{2} h^2 \\ u(Y-h) \approx u(Y) + u'(Y)(-h) + \dfrac{u''(Y)}{2}(-h)^2 = u(Y) - u'(Y)h + \dfrac{u''(Y)}{2} h^2 \end{cases}$$

then we finally get
$$\pi^* = \frac{u(Y) - u(Y) + u'(Y)h - \frac{u''(Y)}{2} h^2}{2h\,u'(Y)} = \frac{1}{2} + \frac{1}{4}\boxed{\left(-\frac{u''(Y)}{u'(Y)}\right)} h = \frac{1}{2} + \frac{1}{4} R_A(Y) \cdot h > \frac{1}{2} \quad \begin{cases} R_A(Y) \\ h \end{cases}$$

$R_A(Y)$的解释:

Measures the sensitivity of π^* to h

② $R_R(Y)$ & $k \Rightarrow$ Risky assets

π^* that makes safety assets & relative bet indifferent

is determined by $\begin{cases} \text{Relative risk aversion coefficient } R_R(Y) & \text{(主观上)} \\ \text{Size of relative bet } k & \text{(客观上)} \end{cases}$

e.g.

if $\underbrace{\begin{cases} \pi^* & Y \\ 1-\pi^* & Y \end{cases}}_{B_1} \sim \underbrace{\begin{cases} \pi^* & Y(1+k) \\ 1-\pi^* & Y(1-k) \end{cases}}_{B_2}$

then $E[u(B_1)] = E[u(B_2)]$

Since $\begin{cases} E[u(B_1)] = u(B_1) = u(Y) \\ E[u(B_2)] = \pi^* u(Y(1+k)) + (1-\pi^*)\, u(Y(1-k)) \end{cases}$

then $u(Y) = \pi^* u(Y(1+k)) + (1-\pi^*)\, u(Y(1-k))$

$$\Rightarrow \pi^* = \frac{u(Y) - u(Y-kY)}{u(Y+Yk) - u(Y-Yk)}$$

According to Taylor Expansion:

$$\begin{cases} u(Y+Yk) \approx u(Y) + u'(Y)(Yk) + \dfrac{u''(Y)}{2}(Yk)^2 \\ u(Y-Yk) \approx u(Y) - u'(Y)(Yk) + \dfrac{u''(Y)}{2}(Yk)^2 \end{cases}$$

$$\Rightarrow \pi^* = \frac{1}{2} + \frac{1}{4}\left(\boxed{-\frac{u''(Y)}{u'(Y)}} \cdot Y\right) \cdot k = \frac{1}{2} + \frac{1}{4} R_R(Y) \cdot k > \frac{1}{2}$$

$R_R(Y)$的解释:

Measures the sensitivity of π^* to k

③ $R_A(Y)$ & $h \Rightarrow$ Safe assets (Risk premium)

Risk premium is determined by $\begin{cases} \text{Absolute risk aversion coefficient } R_A(Y) \text{ (主观上)} \\ \text{Size of absolute bet } h \text{ (客观上)} \end{cases}$

e.g.

If $\begin{cases} \frac{1}{2} & Y-a \\ \frac{1}{2} & Y-\boxed{a} \end{cases} \sim \tilde{z} = \begin{cases} \frac{1}{2} & Y+h \\ \frac{1}{2} & Y-h \end{cases}$

相当于为了规避风险而牺牲的利益

z (Bond) $\quad \tilde{z}$ (Stock)

Utility $\quad E(\tilde{z}) = E(u(Y))$

Certainty equivalent (确定性等值) 即无风险时 $\quad CE(\tilde{z}) = E(u(Y-a))$ (Constant)

Risk premium $\quad \psi(\tilde{z}) = E(\tilde{z}) - CE(\tilde{z})$ ($\psi(\tilde{z})$ 与 $CE(\tilde{z})$ 反向)

Economic interpretation
Mathematical interpretation

当 Risk averse : $R_p > 0$ $\quad CE(\tilde{z}) < E(\tilde{z})$

当 Risk neutral : $R_p = 0$ $\quad CE(\tilde{z}) = E(\tilde{z})$

当 Risk seeker : $R_p < 0$

在实际生活中, Risk averse 的人是大多数

$$\underset{\text{Constant}}{Y} + \underset{\text{RV}}{\tilde{z}} \sim Y + CE(\tilde{z})$$

$$\Rightarrow E(u(Y+\tilde{z})) = E(u(Y+CE(\tilde{z})))$$

$$= u(Y+CE(\tilde{z}))$$ (因为 $CE(\tilde{z})$ 是个常数)

$$\begin{aligned} \Rightarrow LHS &= E(u(Y+\tilde{z})) \\ &= E[u(Y+E(\tilde{z})-E(\tilde{z})+\tilde{z})] \\ &\approx E[u(Y+E(\tilde{z})) + u'(Y+E(\tilde{z}))(\tilde{z}-E(\tilde{z})) \\ &\quad + \tfrac{1}{2}u''(Y+E(\tilde{z}))(\tilde{z}-E(\tilde{z}))^2] \end{aligned}$$

(Second-order Taylor Expansion)

$$= u(Y+E(\tilde{z})) + \tfrac{1}{2}u''(Y+E(\tilde{z}))\,Var(\tilde{z})$$

$$\begin{aligned} RHS &= u(Y+CE(\tilde{z})) \\ &= u(Y+E(\tilde{z})-E(\tilde{z})+CE(\tilde{z})) \\ &= u(Y+E(\tilde{z})-\psi(\tilde{z})) \\ &\approx u(Y+E(\tilde{z})) - u'(Y+E(\tilde{z}))\cdot\psi(\tilde{z}) \end{aligned}$$ (First-order Taylor Expansion)

当且仅当 $\tilde{z}-E(\tilde{z})$ is small compared to $Y+E(\tilde{z})$

$$\Rightarrow \text{Let } LHS = RHS, \text{ then we have } \psi(\tilde{z}) \approx \frac{1}{2}\left(-\frac{u''(Y+E(\tilde{z}))}{u'(Y+E(\tilde{z}))}\right) Var(\tilde{z})$$

$$= \frac{1}{2} R_A(Y+E(\tilde{z}))\, Var(\tilde{z})$$

u
Y
Distribution of $\tilde{z}$

$$\Rightarrow CE(\tilde{z}) = E(\tilde{z}) - \psi(\tilde{z})$$ (先求 $\psi(\tilde{z})$, 再求 $CE(\tilde{z})$)

因为在这里我们用同一个 $u(\cdot)$ 去打评 $\psi(\tilde{z})$, 所以这里的 Risk premium 是个人层面的.

L10-12 Competitive equilibrium method — Risk aversion (Group)

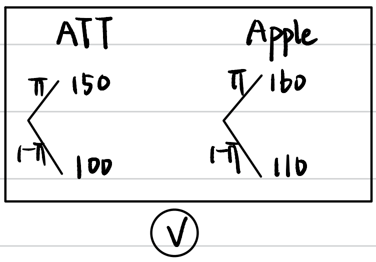
1. Preference-free Comparison

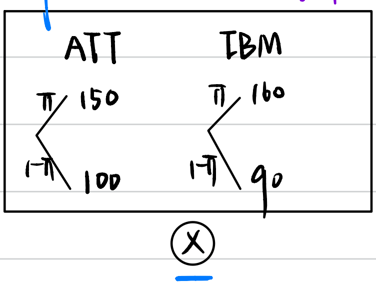
① Definition

Preference-free comparison between two assets means this comparison is robust among a group of investors.
(Different utility function)

② Comparison rule

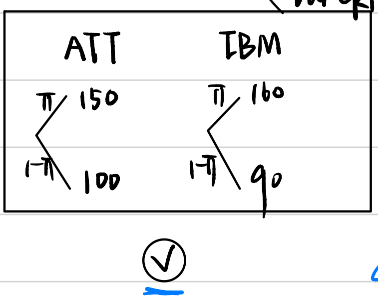
a. State-by-State Dominance (SSD) (即绝对优势) (充分条件)





b. Mean-Variance Dominance (MVD)

$$\tilde{R}_1 \text{ MVD } \tilde{R}_2 \Leftrightarrow \begin{cases} E(\tilde{R}_1) \geq E(\tilde{R}_2) \\ Var(\tilde{R}_1) \leq Var(\tilde{R}_2) \end{cases}$$



MVD is applied wider than SSD

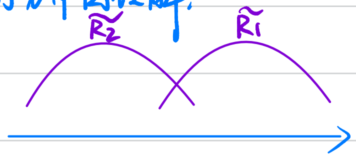
c. Sharpe ratio $= \dfrac{E(\tilde{R})}{\sqrt{Var(\tilde{R})}}$

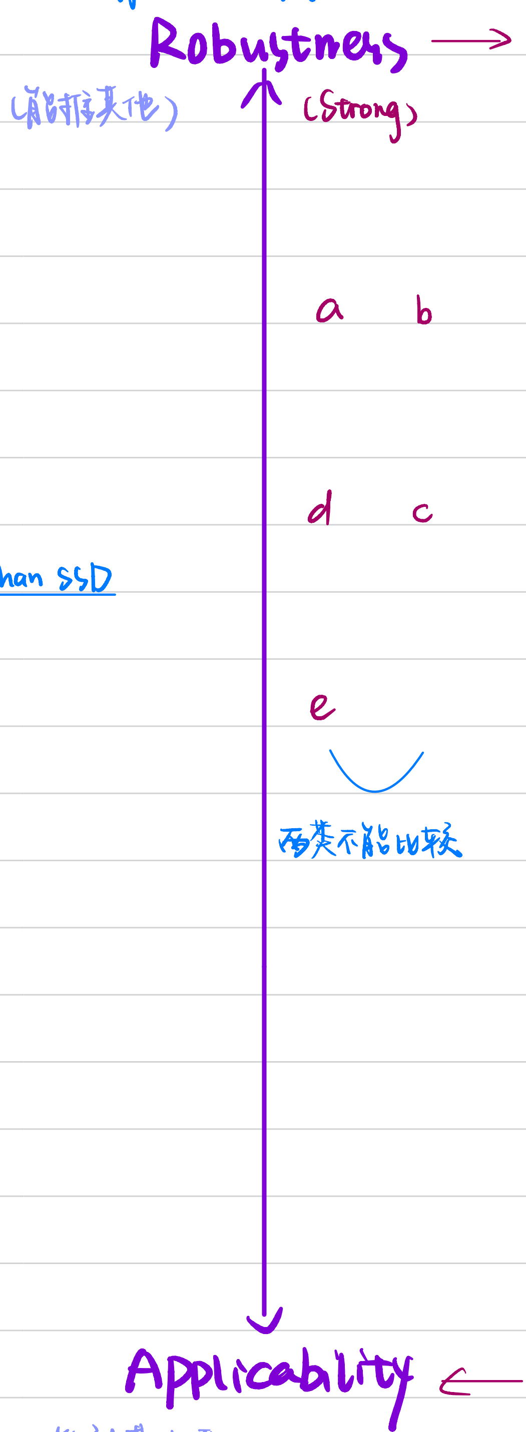
$$\tilde{R}_1 \text{ SRD } \tilde{R}_2 \Leftrightarrow \frac{E(\tilde{R}_1)}{\sqrt{Var(\tilde{R}_1)}} \geq \frac{E(\tilde{R}_2)}{\sqrt{Var(\tilde{R}_2)}}$$

d. First Order Stochastic Dominance (FOSD) (充分必要条件)

$$\tilde{R}_1 \text{ FOSD } \tilde{R}_2 : \forall x,\ F_1(x) \leq F_2(x) \Leftrightarrow \forall u'(\cdot) > 0,\ E[u(\tilde{z}_1)] \geq E[u(\tilde{z}_2)]$$

用分布图理解:





proof:

$\Rightarrow$: Since $E[u(\tilde{z})] = \int_{-\infty}^{+\infty} u(z) f(z) dz = \int_{-\infty}^{+\infty} u(z) dF(z) = u(z)F(z)\Big|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} F(z) du(z) = u(+\infty) - \int_{-\infty}^{+\infty} F(z) u'(z) dz$.

$u'(z) > 0$: prefer more to less

then $F(z)\uparrow$, $E[u(\tilde{z})]\downarrow$, then if $F_1(x) \leq F_2(x)$, then $E[u(\tilde{z}_1)] \geq E[u(\tilde{z}_2)]$

$\Leftarrow$: 同理可得

e. Second order Stochastic Dominance (SOSD)

$\tilde{R}_1$ SOSD $\tilde{R}_2$: $\forall z, \int_{-\infty}^{z} F_1(x)dx \le \int_{-\infty}^{z} F_2(x)dx \Leftrightarrow u'(\cdot)>0$ & $u''(\cdot)<0$, $E[u(\tilde{z}_1)] \ge E[u(\tilde{z}_2)]$

(对cdf再求一次积分⇒二次阶面积下方势)　　　Risk averse

proof:

⇒: Since $E[u(\tilde{z})] = u(+\infty) - \int_{-\infty}^{+\infty} F(z)u'(z)dz$,

let $G(z) = \int_{-\infty}^{z} F(x)dx$,

then $E[u(\tilde{z})] = u(+\infty) - \int_{-\infty}^{+\infty} u'(z)dG(z) = u(+\infty) - u'(z)G(z)\Big|_{-\infty}^{+\infty} + \int_{-\infty}^{+\infty} G(z)du'(z) = u(+\infty) - u'(+\infty)G(+\infty) + \int_{-\infty}^{+\infty} G(z)u''(z)dz$

$G(z)\uparrow$, $E[u(z)]\downarrow$, then if $\int_{-\infty}^{z} F_1(x)dz \le \int_{-\infty}^{z} F_2(x)dx$, then $E[u(\tilde{z}_1)] \ge E[u(\tilde{z}_2)]$

⇐: 同理可证

③ Relationship between dominance

a. $\tilde{A}$ FOSD $\tilde{B}$ ⟶
- $E(\tilde{A}) \ge E(\tilde{B})$ ✓
- $Var(\tilde{A}) \le Var(\tilde{B})$ ✗

b. $\tilde{A}$ SOSD $\tilde{B}$ ⟶
- $E(\tilde{A}) \ge E(\tilde{B})$ ✗
- $Var(\tilde{A}) \le Var(\tilde{B})$ ✗

L13 Saving problem under Uncertainty (One-date)

1. One-date economy with one risky asset & one safe asset

① Background / Setup

	t_0: Y_0	t_1: $\tilde{Y}_1$
Safe assets	$Y_0 - a$	$(Y_0-a)(1+r_f)$
Risky assets	a	$a(1+\tilde{r})$

$r_f \longrightarrow$ Known, constant

$\tilde{r} \longrightarrow$ Unknown, RV

$a<0$: "Short" the risky asset (做空/卖空

看跌 → $t=0$ —— $t=1$

(借别人的,卖出　　买入,还别人

position $\begin{cases} >0: \text{Long} \\ <0: \text{Short} \end{cases}$ (对于safe assets与risky assets都成立)

$a>0$: "Go long" the risky asset (做多/买空

看涨 → $t=0$ —— $t=1$

(借别人的),买入　　卖出,还别人

② Assumptions

a. Rational : $u'(\cdot)>0$

b. Risk averse : $u''(\cdot)<0$

③ Formulating

$$\max_a \ E[u(\tilde{Y}_1)] = E[u(Y_0(1+r_f)+a(\tilde{r}-r_f))]$$

s.t. (Unconstrained)

$\longrightarrow$ FOC (此时因为$E(\cdot)$是个linear function, 求E和求导的顺序可换)

$$\Rightarrow \frac{dE[u(\tilde{Y}_1)]}{da} = 0$$

$$\Rightarrow E\left[\frac{du(\tilde{Y}_1)}{da}\right] = E\left[u'(\tilde{Y}_1)\frac{d\tilde{Y}_1}{da}\right] = E[u'(Y_0(1+r_f)+a(\tilde{r}-r_f))\cdot(\tilde{r}-r_f)] =: W(a) = 0$$

$E[u(\tilde{Y}_1)]$是关于a的方程

$$W(0) = \underbrace{u'(Y_0(1+r_f))}_{\text{Ass1: Rational}}(E[\tilde{r}]-r_f) \quad \text{(i)}$$

$\longrightarrow$ SOC

$$\Rightarrow W'(a) = E[\underbrace{u''(Y_0(1+r_f)+a(\tilde{r}-r_f))}_{\text{Ass2: Risk averse}}\cdot(\tilde{r}-r_f)^2] \le 0 \longrightarrow W(a) \text{ is decreasing} \quad \text{(ii)}$$

因为Expectation是线性方程: $[E(\ )]' = E(\ ')$

$E[u(\tilde{r})]$ & $E[u(r_f)]$
$\#$

→ Results (注意：我们没有认为此时 safe assets ~ risky assets，所以我们不知道 $E(\tilde{r})$ 与 r_f 的关系)

a. if $E(\tilde{r}) > r_f$,

then $\begin{cases} W(0) > 0 \\ W'(a) \le 0 \end{cases}$ $\Rightarrow a^* > 0 \Rightarrow$ Long risky assets

b. if $E(\tilde{r}) = r_f$, (此时 $\tilde{r}$ 是 r_f 的 Mean preserving spread)

then $\begin{cases} W(0) = 0 \\ W'(a) \le 0 \end{cases}$ $\Rightarrow a^* = 0 \Rightarrow$ No investment in risky assets

此时 $\tilde{r}$ SOSD r_f
MVD

→ 不能说明 there will be no investment in $\tilde{r}$

c. if $E(\tilde{r}) < r_f$,

then $\begin{cases} W(0) < 0 \\ W'(a) \le 0 \end{cases}$ $\Rightarrow a^* < 0 \Rightarrow$ Short risky assets

此时 $\tilde{r}$ SOSD r_f
MVD

> if $\tilde{r}_1$ SSD $\tilde{r}_2$,
> then <u>short $\tilde{r}_2$, long $\tilde{r}_1$</u>

需记得还有 Short!!!

④ Other case

Ass 2: Risk averse → Risk neutral (在 SOC 那里改 $W'(a)$ 的符号)

$\begin{cases} \text{if } E(\tilde{r}) > r_f, \text{ then } a^* = +\infty \\ \text{if } E(\tilde{r}) = r_f, \text{ then } a^* = \text{any value} \\ \text{if } E(\tilde{r}) < r_f, \text{ then } a^* = -\infty \end{cases}$

⑤ Property

a^* depends on

- 主观上 How risk averse the investors are $\uparrow, \downarrow$ $(RA(\cdot))$
- 主观上 How weathy the investors are $\uparrow, \sim$ (Y_0)
- 客观上 How risky the assets are $\uparrow, \downarrow$ $(E[\tilde{r}-r_f]^2)$
- 行为上 How large is the risk premium $\uparrow, \uparrow$ $(E(\tilde{r})-r_f)$

proof:

FOC: $W(a)=0$

$\longrightarrow E[u'(Y_0(1+r_f)+a^*(\tilde{r}-r_f))\cdot(\tilde{r}-r_f)]=0$

1^{st} order Taylor expansion: $\approx u'(Y_0(1+r_f))+u''(Y_0(1+r_f))\,a^*(\tilde{r}-r_f)$

$\longrightarrow E[u'(Y_0(1+r_f))\cdot(\tilde{r}-r_f)]+E[u''(Y_0(1+r_f))\,a^*(\tilde{r}-r_f)^2]$

$= u'(Y_0(1+r_f))(E(\tilde{r})-r_f)+u''(Y_0(1+r_f))\,a^*E(\tilde{r}-r_f)^2 \approx 0$

$$\longrightarrow a^* \approx -\frac{u'(Y_0(1+r_f))}{u''(Y_0(1+r_f))}\cdot\frac{E(\tilde{r})-r_f}{E[(\tilde{r}-r_f)^2]} = \boxed{\frac{1}{RA(Y_0(1+r_f))}}\cdot\frac{\boxed{E(\tilde{r})-r_f}}{\boxed{E[(\tilde{r}-r_f)]^2}}$$

Risk premium
- Individual: $E(\tilde{r})-CE(\tilde{r})$
- Group: $E(\tilde{r})-r_f$

proof:

a. 当讨论 a^* 与 Y_0 的单调性分析

$$\frac{da^*}{dY_0} \approx -\frac{E(\tilde{r})-r_f}{E[(\tilde{r}-r_f)^2]}\,\frac{RA'(Y_0(1+r_f))}{RA(Y_0(1+r_f))^2}(1+r_f)$$

$\longrightarrow$
- $E(\tilde{r})-r_f>0$ & $\underline{RA'(Y)>0}$ (越wealthy越risky) : $\downarrow$
- $E(\tilde{r})-r_f>0$ & $\underline{RA'(Y)=0}$ (CARA) : —
- $E(\tilde{r})-r_f>0$ & $\underline{RA'(Y)<0}$ (更现实) : $\uparrow$

b. 当讨论 a^* 与 Y_0 的弹性分析

$$a^* \approx \frac{Y_0(1+r_f)}{R_R(Y_0(1+r_f))} \cdot \frac{E(\tilde{r})-r_f}{E[(\tilde{r}-r_f)^2]}$$

$$\ln a^* \approx \ln Y_0 - \ln R_R(Y_0(1+r_f)) + \ln\left[(1+r_f)\frac{E(\tilde{r})-r_f}{E[(\tilde{r}-r_f)^2]}\right]$$

回忆 $\Delta \ln x$ 的含义：$\Delta \ln x \approx \frac{\Delta x}{x}$

Elasticity η (eta) $= \frac{\Delta \ln a^*}{\Delta \ln Y_0} \approx 1 - \frac{\Delta \ln R_R(Y_0(1+r_f))}{\Delta \ln Y_0} = \boxed{1 - \frac{Y_0}{R_R(Y_0(1+r_f))} \frac{\Delta R_R(Y_0(1+r_f))}{\Delta Y_0}} = 1 - \frac{Y_0}{R_R(Y_0(1+r_f))} R_R'(Y_0(1+r_f))$

$$\longrightarrow \begin{cases} R_R'(Y) > 0 & : \eta < 1 \\ R_R'(Y) = 0 \text{ (CRRA)} & : \eta = 1 \\ R_R'(Y) < 0 & : \eta > 1 \end{cases}$$

L14 Saving problem under Uncertainty (Two-date)

1. Two-date economy with one risky asset

① Background / Setup

	t_0	t_1
Income	Y_0	0
Risky assets	S with $\tilde{R}$ (Gross return) 是个RV	
Consumption	$C_0 = Y_0 - S$	$\tilde{C}_1 = S\tilde{R}$

② Assumptions

a. Rational : $u'(\cdot) > 0$

b. Risk averse : $u''(\cdot) < 0$

③ Formulating

$$\max_{S} E[u(C_0) + \beta u(\tilde{C}_1)] = E[u(Y_0 - S) + \beta u(S\cdot\tilde{R})]$$

s.t. $S > 0$

④ Effects

Eff 1: $u''(\cdot)<0$, Risk averse ↑, $S^* \downarrow$

Eff 2: $u''(\cdot)<0$, Consumption smoothness, $S^* \uparrow$

用FOC证

使得每次的consumption水平相近.

我们接下来研究a与b的强弱关系.

当且仅当CARA | CRRA

$$\begin{cases} \text{Absolute prudence: } P_A(Y) = -\dfrac{u'''(Y)}{u''(Y)} = \alpha \\ \text{Relative prudence: } P_R(Y) = -\dfrac{u'''(Y)\cdot Y}{u''(Y)} = \gamma+1 \quad (\text{Log utility} \longrightarrow \gamma=2) \end{cases}$$

Let $\tilde{\tilde{R}}$ be m.p.s.

$$\begin{cases} \tilde{R}: \ FOC: u'(Y_0-S^*) = \beta E[u'(S^*\tilde{R})\cdot\tilde{R}] \\ \tilde{\tilde{R}}: \ FOC: u'(Y_0-S^{**}) = \beta E[u'(S^{**}\tilde{\tilde{R}})\cdot\tilde{\tilde{R}}] \end{cases}$$

$\longrightarrow$ 接下来，我们通过比较 S^* 与 S^{**} 来比较 eff 1 & eff 2 的强弱关系.

Since

As for LHS: $\dfrac{d\,LHS}{dS} = \dfrac{d\,u'(Y_0-S)}{dS} = -u''(Y_0-S) > 0 \longrightarrow u'(Y_0-S)$ is increasing.

As for RHS: Let $g(R) = u'(S\cdot R)\cdot R$, then $g'(R) = u''(S\cdot R)\cdot S R + u'(S R)$,

then $g''(R) = u'''(S\cdot R)\cdot S^2 R + 2u''(S\cdot R)\cdot S = -u''(S\cdot R)\cdot S\left[-\dfrac{u'''(S\cdot R)\cdot S R}{u''(S\cdot R)} - 2\right]$

$= -u''(S\cdot R)\cdot S\,[P_R(S\cdot R) - 2]$

$$\longrightarrow \begin{cases} P_R(S\cdot R) < 2,\ g''(R) < 0,\ S^{**} < S^*, \ \text{eff 1 dominates} \\ P_R(S\cdot R) = 2,\ g''(R) = 0,\ S^{**} = S^* \\ P_R(S\cdot R) > 2,\ g''(R) > 0,\ S^{**} > S^*, \ \text{eff 2 dominates} \end{cases}$$

2. Two-date economy with one risky asset & one safe asset

① Background / Setup

	t_0	t_1
Income	Y_0	0
Risky asset	a with $\tilde{R}$ (Gross return) 是个RV	
Safe asset	$S-a$	
Consumption	$C_0 = Y_0 - S$	$\tilde{C}_1 = a(1+\tilde{r}) + (S-a)(1+r_f) = S(1+r_f) + a(\tilde{r}-r_f)$

② Assumptions

a. Rational : $u'(\cdot) > 0$

b. Risk averse : $u''(\cdot) < 0$

③ Formulating

$$\max_{S,a} E[u(C_0) + \beta u(C_1)] = E[u(Y_0 - S) + \beta u(S(1+r_f) + a(\tilde{r}-r_f))]$$

$$\frac{dob}{dS} : u'(Y_0 - S^*) = (1+r_f)\beta E[u'(S^*(1+r_f) + a^*(\tilde{r}-r_f))]$$

$$\frac{dob}{da} : \beta E[u'(S^*(1+r_f) + a^*(\tilde{r}-r_f)) \cdot (\tilde{r}-r_f)] = 0$$

L15-18 Modern Portfolio Theory (MPT)

1. Assumption

① Content

Investors are Mean-Variance: Only care about the mean and variance of random return

$\longrightarrow$ For any $\tilde{Y}$, MV investors' $U(\tilde{Y}) = E[u(\tilde{Y})] = f(E(\tilde{Y}), Var(\tilde{Y}))$ $\begin{cases} \frac{\partial U}{\partial E} > 0 \\ \frac{\partial U}{\partial Var} < 0 \end{cases}$ (Prefer higher mean & lower variance)

MV investors is a subset of Risk averse investors

$\tilde{r}_1$ MVD $\tilde{r}_2$ $\underset{\times}{\overset{\checkmark}{\rightleftarrows}}$ All MV investors $\tilde{r}_1 \succ \tilde{r}_2$

$\tilde{r}_1$ SOSD $\tilde{r}_2$ $\underset{\times}{\overset{\checkmark}{\rightleftarrows}}$ All MV investors $\tilde{r}_1 \succ \tilde{r}_2$

② Rational + Risk averse $\xrightarrow{\text{Narrow}}$ MV investors

How to narrow down?

a. Con1: Higher moments part should be removed. (Can be written as a function of E & Var)

$$\begin{cases} LHS = U = E[u(\tilde{Y})] = f(E(\tilde{Y}), Var(\tilde{Y}), \text{higher moments}) \\ RHS = U = E[u(\tilde{Y})] = f(E(\tilde{Y}), Var(\tilde{Y})) \end{cases}$$

b. Con2: $\begin{cases} \frac{\partial U}{\partial E} > 0 \\ \frac{\partial U}{\partial Var} < 0 \end{cases}$ ($\frac{\partial U}{\partial \sigma^2}$ has the same sign as $\frac{\partial U}{\partial \sigma}$)

③ Examples

a. Quadratic utility: $u(\cdot) = ax^2 + bx + c$ $\longleftarrow$ investor's side

b. Normal distribution: $\tilde{Y} \sim N(\mu_Y, \sigma_Y^2)$ $\longleftarrow$ asset's side

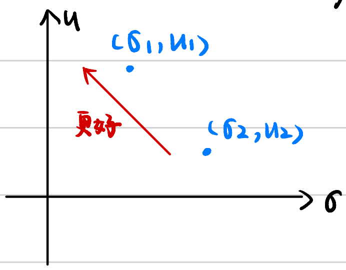
$\tilde{Y} = \mu_Y + \sigma_Y \tilde{\varepsilon}$, where $\tilde{\varepsilon} \sim N(0,1)$

c. Taylor expansion: $(\tilde{Y} - E(\tilde{Y}))$ is small $\longrightarrow$ Then 2^{nd} order Taylor expansion is a good approximation

2. Concepts

① μ-σ graph

Each asset can be denoted by μ-σ graph



② MV efficient portfolio

The portfolio that is not MV dominated by any other assets or portfolio

③ Efficient frontier (E.F.)

The set of MV efficient portfolios

(也就是:我们需要考虑的portfolio范围)

3. Goal

Given assets, try to find the optimal portfolio

也就是说，

$\tilde{r}_i$ (Assets)

ρ_{ij} (Relation between assets)

$\longrightarrow w$

4. Example: Two-assets example

① Setup

$$\begin{cases} \tilde{r}_1 : (\sigma_1, \mu_1) & (w) \\ \tilde{r}_2 : (\sigma_2, \mu_2) & (1-w) \end{cases}$$

可 long 可 short

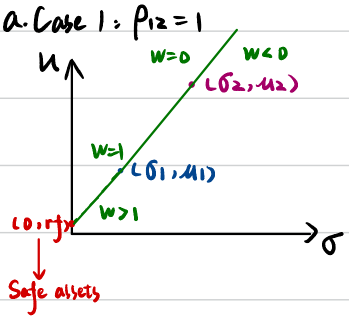
$\tilde{r}_p(w) = w\tilde{r}_1 + (1-w)\tilde{r}_2 \quad w \in (-\infty, +\infty)$

$\tilde{r}_p$ 实际上就是个关于RV的Combination，其本身也是RV.

$$\begin{cases} E[\tilde{r}_p(w)] = w\mu_1 + (1-w)\mu_2 \\ Var[\tilde{r}_p(w)] = w^2\sigma_1^2 + (1-w)^2\sigma_2^2 + 2w(1-w)Cov(\tilde{r}_1, \tilde{r}_2) \end{cases}$$

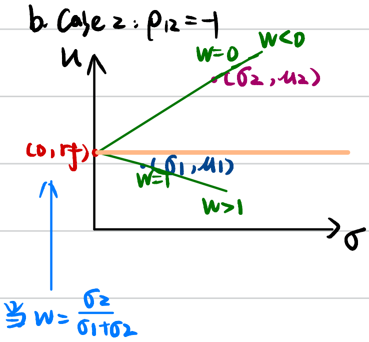
② Cases (绿线代表可能的Combination，即Portfolio)

a. Case 1: $\rho_{12} = 1$

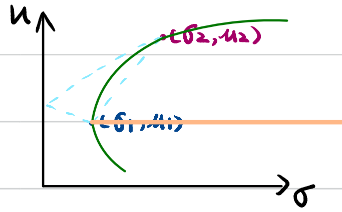


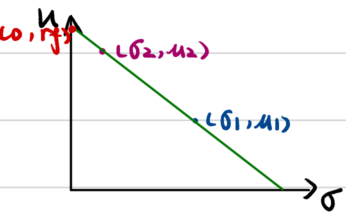
No MVD between $\tilde{r}_1$ & $\tilde{r}_2$

b. Case 2: $\rho_{12} = -1$



c. Case 3: $\rho_{12} \in (-1, 1)$





$\tilde{r}_2$ MVD $\tilde{r}_1$ $\longrightarrow$ Choose red-point portfolio

当且仅当 $\rho = \pm 1$ 时，我们才可能用两个Risky assets 组建出一个Safe asset

5. Example: Three-assets examples

① Assumption

Any pair of risky assets is not perfectly correlated.

⟶ No safe asset / portfolio

② Setup

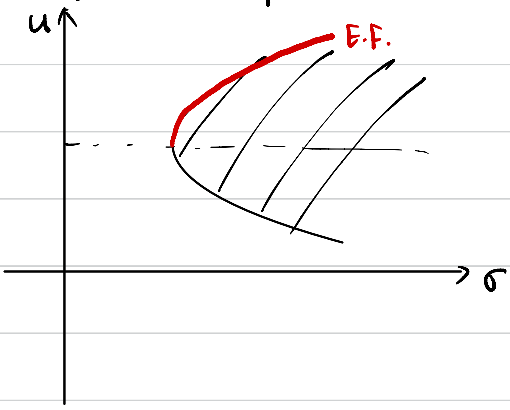
$$\begin{cases} \tilde{r}_1 : (\sigma_1, \mu_1) & (w_1) \\ \tilde{r}_2 : (\sigma_2, \mu_2) & (w_2) \\ \tilde{r}_3 : (\sigma_3, \mu_3) & (1-w_1-w_2) \end{cases}$$

$$\tilde{r}_p(w_1, w_2) = w_1\tilde{r}_1 + w_2\tilde{r}_2 + (1-w_1-w_2)\tilde{r}_3$$

$$\begin{cases} E(\tilde{r}_p) = w_1\mu_1 + w_2\mu_2 + (1-w_1-w_2)\mu_3 \\ Var(\tilde{r}_p) = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + (1-w_1-w_2)^2\sigma_3^2 + 2w_1w_2\rho_{12}\sigma_1\sigma_2 + 2w_1(1-w_1-w_2)\rho_{13}\sigma_1\sigma_3 + 2w_2(1-w_1-w_2)\rho_{23}\sigma_2\sigma_3 \end{cases}$$

③ Case

The combination is a plane at this time. (可通过portfolio+asset方式得到结论)



证明方法 $\begin{cases} \text{纵向} \\ \text{横向} \end{cases}$, 解 constrained optimization problem

6. Example: N-assets example

① Assumption

Any pair of risky assets is not perfectly correlated with each other.

$\longrightarrow$ No safe asset / portfolio

② Setup

$\tilde{r}_i : (\sigma_i, u_i)$, where $u = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$, $\underset{n \times n}{\Sigma}$ (Cov matrix) $= \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \\ \vdots & \ddots & & \vdots \\ & \cdots & & \sigma_{nn} \end{bmatrix} \longrightarrow \Sigma_{ij} = Cov(\tilde{r}_i, \tilde{r}_j) = \rho_{ij}\sigma_i\sigma_j$

(σ_{ij})

No perfect correlation $\longrightarrow$ Σ is invertible (Σ^{-1} exists) (此时 det $\neq 0$)

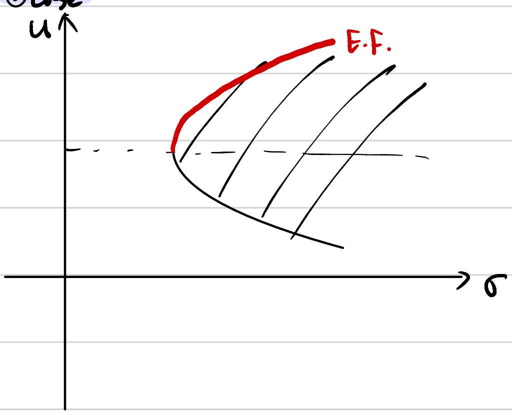
$\det(\Sigma) = 0$

$w = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}$, where $\Sigma w_i = 1$

$$\tilde{r}_p = \sum_{i=1}^{n} w_i \tilde{r}_i$$

$$\begin{cases} E(\tilde{r}_p) = w^T u \\ Var(\tilde{r}_p) = \underline{w^T \Sigma w} \end{cases}$$

③ Case



7. N-assets example (When assumption is violated, i.e. there is a safe asset)

① Setup

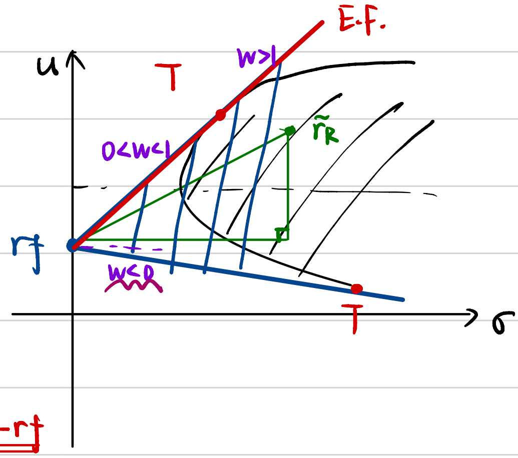
$$\begin{cases} \tilde{r}_R : w \\ r_f : 1-w \end{cases}$$

$$\tilde{r}_p = w\tilde{r}_R + (1-w) r_f$$

$$\begin{cases} E(\tilde{r}_p) = w\mu_R + (1-w) r_f \\ Var(\tilde{r}_p) = w^2 \sigma_R^2 \end{cases}$$

② Derivation

$$\longrightarrow \mu_p = \frac{\sigma_p}{\sigma_R}\mu_R + \left(1-\frac{\sigma_p}{\sigma_R}\right) r_f = r_f + \boxed{\frac{\mu_R - r_f}{\sigma_R}}\,\sigma_p \longrightarrow$$



$\longrightarrow$ Sharpe ratio of $\tilde{r}_p = \dfrac{E(\tilde{r}_R) - r_f}{\sqrt{Var(\tilde{r}_R)}}$

At this time, the portfolios on E.F. are called tangent portfolios.

此时，E.F.从之前的曲线变成一条切线.

③ Property of tangent portfolios

Tangent portfolios have the highest sharpe ratio.

$\longrightarrow$ All M-V investors optimal portfolio must be located on E.F.

④ Choose portfolio on E.F. (To be a fund manager)

Fund manager $\longrightarrow$ E.F. (Tangent portfolio)

Fund buyer $\longrightarrow$ According to their different weights on $\tilde{r}_R$ & r_f, choose the portfolio on E.F.

也就是说，fund manager 只需读懂E.F.（即：找到T作出E.F.的Tangent portfolio

⑤ Two-fund theory

MV investors' optimal portfolios are all located on E.F. with safe asset, and can be constructed by just two funds. r_f & $\tilde{r}_T$, where $\tilde{r}_T$ is constructed only by risky assets & has the highest S.R. in the economy.

即：2 funds $\longrightarrow$ Optimal portfolio

8. Shortcomings of MPT

① Strong assumption

不仅 rational, risk averse, 还要 MV

② Calibration can be huge

N assets, the scale of computation is N^2

L19-20 Capital Asset Pricing Model (CAPM)

1. Derivation

① T = M

CAPM is derived from MPT,

$\longrightarrow$ So, it assumes that investors are MV

T portfolio is optimal $\longrightarrow$ Aggregate demand

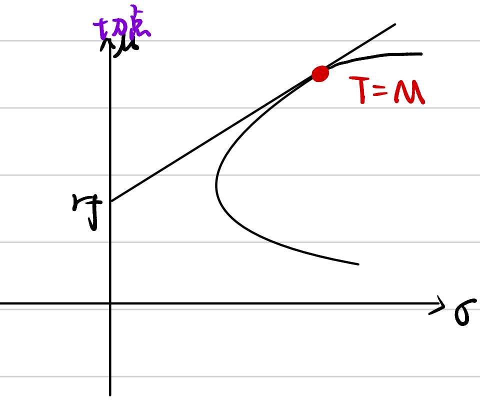
What's more, CAPM goes a further step: It imposes the equilibrium condition for risky assets

$$D = S$$

According to $D = S$

$$\longrightarrow \quad T = M$$

(Tangent portfolio) (Market portfolio)



At this time, the tangent line is called Capital market line

Any portfolio on CML: $\tilde{r_c} = (\sigma_c, \mu_c)$

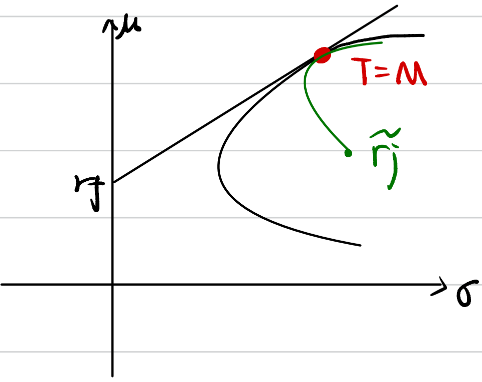
$$E[\tilde{r_c}] = r_f + \frac{E(\tilde{r_T}) - r_f}{\sigma_T} \sigma_c$$

$$= r_f + \boxed{\frac{E(\tilde{r_M}) - r_f}{\sigma_M}} \sigma_c$$

Equilibrium unit price of risk

② Tangency

If we choose another risky asset $\tilde{r}_j : (\sigma_j, \mu_j)$ (定何范围从 Optimal portfolio 转化到 Arbitrary portfolio)



为何与M结合?

无论是否 safe asset 时，M都在E.F.上

then
- the combination line of (σ_p, μ_p) is located on one side of CML,
- the combination line has only one common point with CML : $\tilde{r}_M$

$\longrightarrow$ CML与这条 combination line 相切! ($K_{CML} = K_{CL}$)

As for CML, slope = Sharpe ratio = $\dfrac{E(\tilde{r}_M) - r_f}{\sigma_M}$ ———— ①

As for Combination line,
$$\begin{cases} \mu_p = w \cdot E[\tilde{r}_j] + (1-w) E[\tilde{r}_M] & =: g(w) \\ \sigma_p = \sqrt{w^2 \sigma_j^2 + (1-w)^2 \sigma_M^2 + 2w(1-w)\sigma_{jM}} & =: h(w) \end{cases}$$

then we have $\mu_p = f(\sigma_p) = \underline{f(h(w)) = g(w)}$

$$\longrightarrow f'(\sigma_p) h'(w) = g'(w) \longrightarrow f'(\sigma_p) = \frac{g'(w)}{h'(w)} = \frac{E[\tilde{r}_j] - E[\tilde{r}_M]}{\frac{1}{2}[w^2\sigma_j^2 + (1-w)^2\sigma_M^2 + 2w(1-w)\sigma_{jM}]^{-\frac{1}{2}} \cdot (2w\sigma_j^2 - 2(1-w)\sigma_M^2 + 2(1-w)\sigma_{jM} - 2w\sigma_{jM})}$$

$$\longrightarrow f'(\sigma_p)\big|_{w=0} = \frac{(E[\tilde{r}_j] - E[\tilde{r}_M])\,\sigma_M}{\sigma_{jM} - \sigma_M^2}$$ ———— ②

Let ① = ② (因为相切)

then for any risky asset $\tilde{r}_j$, we get $E(\tilde{r}_j) = r_f + \dfrac{\sigma_{jM}}{\sigma_M^2}(E(\tilde{r}_M) - r_f)$

Let $\beta_j = \dfrac{\sigma_{jM}}{\sigma_M^2}$, (β_j 也 $= \dfrac{Cov(\tilde{r}_j, \tilde{r}_M)}{Var(\tilde{r}_M)}$)

we have CAPM equation ① :

$E(\tilde{r}_j) = r_f + \beta_j (E(\tilde{r}_M) - r_f) \longrightarrow$ SML

某一Risky asset的收益 = Safe asset的收益 + 这一Risky asset对应的β值 × Risk premium when choosing $\tilde{r}_M$

因为此时是 equilibrium condition (针对整个市场)，所以 Risk premium 此时对于所有 investors 都是一样的.

③ β_j & ρ_{jM}

Taking $\beta_j = \frac{\sigma_{jM}}{\sigma^2_M} = \frac{Cov(\tilde{r}_j, \tilde{r}_M)}{Var(\tilde{r}_M)} = \frac{\rho_{jM}\sigma_j}{\sigma_M}$,

we have CAPM equation ② :

$$E(\tilde{r}_j) = r_f + \underline{\frac{E(\tilde{r}_M) - r_f}{\sigma_M}} \rho_{jM}\sigma_j \longrightarrow CML$$

某一Risky asset的收益 = Safe asset的收益 + Adjusted equilibrium unit price of risk × 这一Risky asset的风险

对于CML上的Risky assets: "$\frac{E(\tilde{r}_M) - r_f}{\sigma_M}$" 是 equilibrium unit price of risk : $E(\tilde{r}_j) = r_f + \frac{E(\tilde{r}_M) - r_f}{\sigma_M}\sigma_j$

对于其他任意位置上的Risky assets: "$\frac{E(\tilde{r}_M) - r_f}{\sigma_M}\rho_{jM}$" 是 equilibrium unit price of risk : $E(\tilde{r}_j) = r_f + \frac{E(\tilde{r}_M) - r_f}{\sigma_M}\rho_{jM}\sigma_j$

ρ_{jM}
- $= 1$: All the risk in $\tilde{r}_j$ is priced in by CAPM (CML上)
- $= 0$: None of the risk in $\tilde{r}_j$ is priced in by CAPM
- < 0 : $E(\tilde{r}_j) < r_f \longrightarrow$ 实际上存在: Insurance

2. Type of risk

According to CAPM equation ①:

$$E(\tilde{r}_j) = r_f + \beta_j (E(\tilde{r}_m) - r_f)$$

We can do an OLS estimation on $\tilde{r}_j$:

$$\tilde{r}_j = \alpha + \boxed{\beta_j \tilde{r}_m} + \boxed{\varepsilon_j}$$

Correlated Uncorrelated (with $\tilde{r}_m$)

Aggregate risk 【合计风险】

Idiosyncratic risk 【独特风险】

This part is not held by MV investors

(i.e. this part will be diversified away ⇒ This is why this part is not priced in by CAPM)

Combination

3. Pricing rule

We introduce Time series data to help us price assets.

$$\underset{P_j}{t=0} \longrightarrow \underset{\tilde{c}_j}{t=1} \qquad \longrightarrow \tilde{r}_j = \frac{\tilde{c}_j}{P_j} - 1$$

By CAPM, $E(\tilde{r}_j) = r_f + \beta_j (E(\tilde{r}_m) - r_f) \longrightarrow P_j = \dfrac{E(\tilde{c}_j)}{1 + r_f + \beta_j (E(\tilde{r}_m) - r_f)}$ $\left(P_j = \dfrac{E(\tilde{c}_j)}{1+E(\tilde{r}_j)}\right)$

So we have

$$\begin{cases} \text{For risky assets: } P_j = \dfrac{E(\tilde{c}_j)}{1 + r_f + \beta_j (E(\tilde{r}_m) - r_f)} \\ \text{For safe assets: } P = \dfrac{c}{1 + r_f} \end{cases}$$

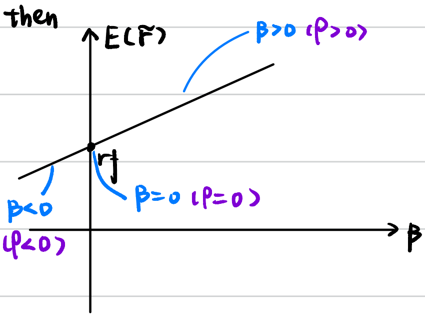
实际上就是用DCF求Price

4. Test of CAPM (Fama MacBeth test)

① Setup

According to CAPM: $E(\tilde{r}_j) = r_f + (E(\tilde{r}_m) - r_f)\beta_j$

then



At this time, this line is called Security market line $(\beta \to E(\tilde{r}))$

② Steps

a. Regress r_{jt} on $\tilde{r}_{M,t}$ to get β_j: $r_{j,t} - r_{f,t} = \alpha_j + \beta_j (r_{M,t} - r_{f,t}) + \beta_{j,t}$

b. Disjoint data to regress $r_{j,t}$ on β_j: $\underbrace{r_{j,t} - r_{f,t}}_{\text{Excess return}} = \gamma_0 + \gamma_1 \beta_j + \eta_{j,t}$

③ Analysis

Upside: CAPM gives us a simple linear pricing rule.

Downside: Strong assumption — MV investors

L21–22 Arbitrage Pricing Theory (APT)

1. Arbitrage opportunity

① Type 1 Arbitrage opportunity

Type 1 Arbitrage opportunity generates $\boxed{\text{future}}$ free cash flows with no risk.

Price — Payoff

cost = 0 — positive payoff always

t_0 — t_1

P_0 — $\tilde{P}_1$

$P_0 = 0$ — $P_1^i \geq 0$ & $\exists\, i,\ P_1^i > 0$

② Type 2 Arbitrage opportunity

Type 2 Arbitrage opportunity generates $\boxed{\text{immediate}}$ free cash flows with no risk.

t_0 — t_1

P_0 — $\tilde{P}_1$

$P_0 < 0$ — $P_1^i \geq 0$

③ No Arbitrage Theory

Any arbitrage opportunity will be eliminated by investors who prefer more to less

(i.e. Arbitrage opportunity can not exist in long run)　只需要 Rational

2. Arbitrage Pricing Theory

① Assumption

a. There exists investors who prefer more to less

b. There are some risk factors (APT lets Data speak) factor $\triangleq$ variable

② Single risk factor model (Market model)

a. Process

The only risk factor is market portfolio $\tilde{r}_M$

For any individual asset $\tilde{r}_j$,

Regress $\underline{\tilde{r}_j - E(\tilde{r}_j)}$ on $\tilde{r}_M - E(\tilde{r}_M)$: $\tilde{r}_j - E(\tilde{r}_j) = \beta_j(\tilde{r}_M - E(\tilde{r}_M)) + \underline{\tilde{\varepsilon}_j}$, $\beta_j = \frac{\sigma_{jM}}{\sigma_M^2}$

de-mean (去均值 → 使得 $E(\tilde{\varepsilon}_j)=0$)

$\tilde{\varepsilon}_j$ is the idiosyncratic risk that is uncorrelated with $\tilde{r}_M$

→ no idiosyncratic risk

For a well-diversified portfolio, we have $\tilde{r}_w = E(\tilde{r}_w) + \beta_w(\tilde{r}_M - E(\tilde{r}_M))$

Same β, same expected return $\xrightarrow[\text{theory}]{\text{No arbitrage}}$ $E(\tilde{r}_w) = r_f + \beta_w(E(\tilde{r}_M) - r_f)$

b. Difference between CAPM & Market model of APT

△ Similarity

Both are single-factor linear pricing rule

△ Differences

- 推导方法不同
  - CAPM: Competitive equalibrium
  - APT: No arbitrage theory
- 假设不同
  - CAPM: MV investors → R^2 更小
  - APT: Rational investors → R^2 更大
- 层次不同
  - CAPM: individual asset / well-defined portfolio
  - APT: well-defined portfolio

③ Two risk factor model

There are two risk factors: $\tilde{r}_M$, $\tilde{r}_V$ (value portfolio).

Regress $(\tilde{r}_j - E(\tilde{r}_j))$ on $(\tilde{r}_M - E(\tilde{r}_M))$ & $(\tilde{r}_V - E(\tilde{r}_V))$: $\tilde{r}_j = E(\tilde{r}_j) + \beta_{jM}(\tilde{r}_M - E(\tilde{r}_M)) + \beta_{jV}(\tilde{r}_V - E(\tilde{r}_V)) + \tilde{\varepsilon}_j$

After diversification, for a well-diversified portfolio $\tilde{r}_w$, $\tilde{r}_w = E(\tilde{r}_w) + \beta_{wM}(\tilde{r}_M - E(\tilde{r}_M)) + \beta_{wV}(\tilde{r}_V - E(\tilde{r}_V))$

Same β, same expected return $\xrightarrow[\text{theory}]{\text{No arbitrage}}$ $E(\tilde{r}_w) = r_f + \beta_{wM}(E(\tilde{r}_M) - r_f) + \beta_{wV}(E(\tilde{r}_V) - r_f)$

④ Three factor model

There are three risk factors:

$$
\begin{cases}
\tilde{r}_M & \text{Market portfolio} \\
\tilde{r}_{SMB} & \text{Size effect (越小收益越大)} \\
\tilde{r}_{HML} & \text{Value Premium}
\end{cases}
$$

In order to determine $\tilde{r}_{SMB}$ & $\tilde{r}_{HML}$,

we sort the stocks into 2×3 categories:

Size \ Value = $\frac{\text{Book}}{\text{Mkt}}$ $\left(\frac{\text{股东权益}}{\text{股数}\times\text{股价}}\right)$	H	N	L
S	S/H	S/N	S/L
B	B/H	B/N	B/L

$$
\begin{cases}
\tilde{r}_{SMB} = \frac{1}{3}(S/H + S/N + S/L) - \frac{1}{3}(B/H + B/N + B/L) \\
\tilde{r}_{HML} = \frac{1}{2}(S/H + B/H) - \frac{1}{2}(S/L + B/L)
\end{cases}
$$

After HT by Fama, r_M、r_{SMB}、$r_{HML} > 0$

$\boxed{\text{Best set of factors?} \longrightarrow \text{Machine Learning}}$

L23 Arrow-Debreu Economy (A-D)

1. A-D Economy Setup

① Two date economy

② One consumption good in the economy, which is perishable

Perishable: The consumption good cannot be stored

③ At $t=1$, there are N future states. In different states, agents receive different endowment.

④ There K agents, who are rational & risk averse.

(No additional assumption on their endowments or preferences)

Without A-D securities:

$t=0$: w_j^0 $\qquad$ $t=1$: $w_j^1, \ldots, w_j^N$

$$\max_{c_j^0, \{c_j^k\}} u_j(c_j^0) + \beta E[u_j(\tilde{c}_j)]$$

s.t. $t=0$: $c_j^0 \le w_j^0$

$t=1$: $c_j^k \le w_j^k$, for $k=1,\cdots,N$ 下标代表人，上标代表state

There is no consumption smoothness, so we need to introduce financial assets into the economy to facilitate the resource reallocation

With A-D securities: AD security = State security = Primitive security $\qquad$ Insurance

State-i AD Security

$t=0$ $\qquad$ $t=1$

Price	Payoff	State
q^i	0	1
	⋮	⋮
	1	i
	⋮	⋮
	0	N

State-i AD security 在 $t=0$ 对应的价格

$$\max_{c_j^0, \{c_j^i\}} u_j(c_j^0) + \beta E[u_j(\tilde{c}_j)]$$

s.t. $c_j^0 + \sum_{i=1}^{k} q^i c_j^i \le w_j^0 + \sum_{i=1}^{k} q^i w_j^i$ Intertemporal budget constraint

$$\longrightarrow \max_{Q_j^i} u_j(w_j^0 - \sum q^i c_j^i) + \beta E[u_j(w_j + Q_j^i)]$$

$+ \sum_j Q_j^i = 0, \forall i$ (对于整个市场)

Allow changing endowment by transaction of A-D securities $\longrightarrow$ Consumption smoothness

2. Process to solve c_j^0 & $\{c_j^i\}$

① Step 1: Write down FOC, solve c_j^0 & c_j^i in the form of q^i

We use KKT theorem to solve this optimization problem:

$$\mathcal{L} = u_j(c_j^0) + \beta E[u_j(\tilde{c}_j)] + \lambda_j(w_j^0 + \sum q^i w_j^i - c_j^0 - \sum q^i c_j^i) = u_j(c_j^0) + \beta \sum \pi_i u_j(c_j^i) + \lambda_j(w_j^0 + \sum q^i w_j^i - c_j^0 - \sum q^i c_j^i)$$

FOC:
$$\begin{cases} \text{For } c_j^0: u_j'(c_j^0) = \lambda_j \\ \text{For } c_j^i: \beta \pi_i u_j'(c_j^i) = \lambda_j q^i, \ i = 1, \cdots, N \end{cases}$$

$$\longrightarrow q^i = \underbrace{\frac{\beta \pi_i u_j'(c_j^i)}{u_j'(c_j^0)}}_{\text{holds for all } j}, \text{ for } i = 1, \cdots, N$$

回忆: Two-date economy without uncertainty: $MRS = \frac{\beta u'(c_1)}{u'(c_0)}$

doesn't depend on j

$$\begin{cases} \beta \uparrow, q^i \uparrow \\ \pi_i \uparrow, q^i \uparrow \\ c_j^i \uparrow, q^i \downarrow \\ c_j^0 \uparrow, q^i \uparrow \end{cases}$$

② Step 2: Using market clearing conditions & Walras' law to solve q^i, then solve c_j^0 & c_j^i

Market clearing conditions: D = S (Aggregate endowments = Aggregate consumptions):

$$\begin{cases} \text{at } t=0 \quad \sum w_j^0 = \sum c_j^0 \\ \text{at } t=1 \quad \sum w_j^i = \sum c_j^i \quad \forall i \end{cases}$$

Walras' Law: If all K consumers' budget constraints are satisfied & N market clearing conditions are satisfied, then the last market clearing condition is automatically satisfied $\longrightarrow$ Unique solution!

(所以只需要从n个mkt clearing conditions中挑出(n-1)个)

L24-25 Social Planner's Economy

1. Definition

No financial market (No trading).

All consumers delegate their consumption decisions to the social planner.

The social planner collects all the resources (endowments) in the economy.

2. Optimization process

① Procedure

$$\max_{c_j^0, \{c_j^i\}} \sum_{j=1}^{K} \theta_j \left(u_j(c_j^0) + \beta E[u_j(\tilde{c}_j)] \right)$$, where θ_j = weight of consumer j's utility in social planner's objective function

$\theta_j \in (0,1)$ & $\sum \theta_j = 1$

s.t. At $t=0$ $\sum_{j=1}^{K} w_j^0 = \sum_{j=1}^{K} c_j^0$

At $t=1$ $\sum_{j=1}^{K} w_j^i = \sum_{j=1}^{K} c_j^i$, $\forall i$

Using FOC to solve c_j^0 & $\{c_j^i\}$

② Pareto optimality (帕累托最优)

A resource allocation is Pareto optimal if

there exists no deviation s.t. the new deviated allocation makes one consumer better off without causing someone else worse off

Efficiency : PO

Fairness : Equality

③ Advantage of social planner's economy

a. Pareto optimal

Social planners' economy guarantees that the resource allocation is Pareto optimal

b. Risk sharing

e.g. Taking an example: There are two type consumers & two state at $t=1$

△ If $w_1^1 + w_2^1 = w_1^2 + w_2^2$ (Without uncertainty in aggregate future endowment) $\longrightarrow \forall i,\ c_i^1 = c_i^2$

For the same-type consumers, they consume the same in both states, no matter whether they receive the same amount of in both stages or not. 宏观环境影响个体行为

△ If $w^1 \neq w^2$ (with ——) $\longrightarrow \exists i,\ s.t.\ c_i^1 \neq c_i^2$

At least one type of consumer…

Since there is an aggregate endowment risk in the economy, which cannot be shared away

Only aggregate risk matters in the social planner's economy

3. A-D market Economy & Social planner Economy

① Transformation

By some transformation, these two mechanism can <u>coincide</u> with each other. Optimal c_j^0 & $\{c_j^i\}$ are the same

② Transformation method: Welfare theory

a. 1^{st} Welfare Theory

$$\left(\begin{array}{l}\text{A-D market Economy} \\ \text{Competitive Equilibrium}\end{array}\right. \xrightarrow{\text{Find } \theta_j} \text{Social planner Economy (Pareto optimal)}$$

Competitive Equilibrium ≜ Mkt clearing

b. 2^{nd} Welfare Theory

$$\text{Social planner Economy (Pareto optimal)} \xrightarrow{\text{Find } w_j^0 \,\&\, w_j^i} \left(\begin{array}{l}\text{A-D market Economy} \\ \text{Competitive Equilibrium}\end{array}\right.$$

c. Summary of two Welfare Theories

$$\text{RA in A-D market Economy} \underset{\text{Thm2 } (w_j^0,\, w_j^i)}{\overset{\text{Thm1 } (\theta_j)}{\rightleftarrows}} \text{RA in Social Planner Economy}$$

<u>Both can be efficient & risk sharing</u>

d. Implications

△ For social planners

They can use the help of tax to redistribute the wealth among consumers to match $\{\theta_j\}$ in their mind, and then let the financial market do the rest thing to achieve the target resource allocation.

A-D market Economy $\xleftarrow{\text{Tax}}$ Social planner Economy

$\{w_j^0, w_j^i\}$ $\Downarrow$ $\downarrow$ $\{\theta_j\}$ $\Downarrow$

$\{q^i\}, \{c_j^0, c_j^i\} \longrightarrow \{c_j^0, c_j^i\}$

Social planner Economy

模型很好（帕累托最优 & 风险分散）

但应用困难（代理人问题 & 交易摩擦）

$\longrightarrow$ Social planner $\xrightarrow{\text{Tax} \to w_j^0 \,\&\, w_j^i}$ A-D

△ For A-D market

Solve opt problem of A-D market Economy $\longrightarrow$ Solve opt problem of Social planner Economy

Find social planner's solution is much easier

$$q_i = \frac{\beta \pi_i u_j'(c_j^i)}{u_j'(c_j^0)} = \frac{\lambda_i}{\lambda_0}$$

Ratio between Lagrangians does not depend on $\{\theta_j\}$

$\longrightarrow$ A-D securities' prices don't depend on individuals' endowment, <u>they only depend on the aggregate endowments</u>.

By trading A-D securities, every consumer share their idiosyncratic risk with others, so only the aggregate risk cannot be shared away, and will be priced in.

L26 Pricing A-D securities by No Arbitrage theory

1. Contengent claim

① Definition

Contengent claim is a claim of a payoff which depends on the realization of certain state of the economy or on the realization of other assets' payoffs.

underlying assets

② Example

a. A-D security

b. Option of a stock

2. Market completeness

① Definition

Market is complete if: there is a full set of A-D securities available in the market

(N states ⟶ N A-D Securities)

② Propositions

a. Prop 1: Any complex security's payoff can be replicated by the existing A-D security payoffs

Be able to construct a portfolio of A-D securities so that the portfolio's payoff is the same as the complex security's payoff.

- A-D security payoff: Simple security payoff
- Other security payoff: Complex security payoff

⟶ The price of the corresponding complex security is $P_c = \Sigma P_s = \Sigma q_s$

Based on No Arbitrage Theory 同收益 ⟶ 同价

⟶
- Payoff 是 Linear combination
- Price 是 Linear combination

⟶ The value additive theory: Payoff 是 LC ⟶ Price 是对应的 LC.

加法的对象是任意portfolio，不局限于A-D security

b. Prop 2: There are N future states, and M available assets

- If $M < N$, then the market is not complete
- If $M \geq N$, N of M assets' payoffs are linearly independent, then the market is complete

⟶ Iff Payoff Matrix is invertible, then the market is complete

$$\begin{bmatrix} z_1^1 & \cdots & z_n^1 \\ \vdots & \ddots & \vdots \\ z_1^n & \cdots & z_n^n \end{bmatrix}$$

c. Prop 3: If there exists an asset / portfolio whose payoff differs across all states, then by introducing call/put options on this asset/portfolio, we can complete the market.

3. Option — European call option

① Definition

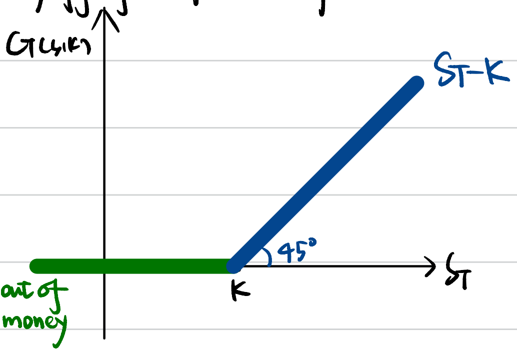
European call option is a contingent claim that gives the buyer the right but not the obligation to purchase
put sell
American

a share of the underlying stock at the strike price K on its expiration date T
on & before

- Call option 看涨期权
- Put option 看跌期权

② Payoff

Payoff of European Call option is denoted by $G_T(S,K)$



If $S_T<K \Rightarrow$ No exercise, Payoff $=0$

If $S_T=K \Rightarrow$ No exercise, Payoff $=0$
(Indifferent)

If $S_T>K \Rightarrow$ Exercise (Buy the stock at K and sell it at S_T), Payoff $= S_T-K$

$\rightarrow G(S,K)=\max\{0, S_T-K\}$

③ Price

Of market is complete, we can price an option on an existing stock.

There is an example: Assume there are two periods $t=0$ & $t=1$ and there are two assets: a stock & a bond.

		Stock (x)	Bond (y)
$t=0$	price	P_0	1
$t=1$	payoff	$\begin{pmatrix} p^G \\ p^B \end{pmatrix}$ $(p^G > p^B)$	$\begin{pmatrix} 1+r_f \\ 1+r_f \end{pmatrix}$

(Full rank) $\rightarrow$ market is complete

We introduce a call option on the stock with strike price K at expiration date $t=1$.

Payoff of this call option is denoted by $C_1(S,K) = \max\{0, S_1 - K\}$, where $S_1 = \begin{cases} p^G & \text{in good state} \\ p^B & \text{in bad state} \end{cases}$

Let V_0 be the price of the call option at $t=0$.

a. Case 1: $K \geq p^G > p^B$

Payoff: $C_1(S,K) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ → For good state / For bad state

Price: $V_0 = 0$

b. Case 2: $p^G > K > p^B$

Payoff: $C_1(S,K) = \begin{bmatrix} p^G - K \\ 0 \end{bmatrix}$

Price: $\begin{cases} \text{Good: } x \cdot p^G + y(1+r_f) = p^G - K \\ \text{Bad: } x \cdot p^B + y(1+r_f) = 0 \end{cases} \longrightarrow \begin{cases} x = \dfrac{p^G - K}{p^G - p^B} \\ y = -\dfrac{K}{1+r_f} \dfrac{p^B(\frac{p^G}{K} - 1)}{p^G - p^B} \end{cases} \longrightarrow V_0 = \dfrac{p^G - K}{p^G - p^B} P_0 - \dfrac{K}{1+r_f} \dfrac{p^B(\frac{p^G}{K} - 1)}{p^G - p^B}$

c. Case 3: $p^G > p^B \geq K$

Payoff: $C_1(S,K) = \begin{bmatrix} p^G - K \\ p^B - K \end{bmatrix}$

Price: $\begin{cases} \text{Good: } x \cdot p^G + y(1+r_f) = p^G - K \\ \text{Bad: } x \cdot p^G + y(1+r_f) = p^B - K \end{cases} \longrightarrow \begin{cases} x = 1 \\ y = -\dfrac{K}{1+r_f} \end{cases} \longrightarrow V_0 = P_0 - \dfrac{K}{1+r_f}$

In sum: $V_0 = N_1 \cdot P_0 - N_2 \cdot \dfrac{K}{1+r_f}$, where N_1 & N_2 are parameters depending on p^G, p^B, K:

$N_1 = \begin{cases} 0 & K > p^G \\ \dfrac{p^G - K}{p^G - p^B} \in (0,1) & \text{Otherwise} \\ 1 & K < p^B \end{cases}$, $N_2 = \begin{cases} 0 & K > p^G \\ \dfrac{p^B(\frac{p^G}{K} - 1)}{p^G - p^B} \in (0,1) & \text{Otherwise} \\ 1 & K < p^B \end{cases}$

		p^B		p^G	
N_1	1		$\dfrac{p^G - K}{p^G - p^B}$		0
N_2	1		$\dfrac{p^B(\frac{p^G}{K} - 1)}{p^G - p^B}$		0

$[P_0, -\dfrac{K}{1+r_f}]$

④ Use options to form a complete market

If there exists an asset / portfolio whose payoff differs across all states, then by introducing call / put options on this asset / portfolio, we can complete the market.

(Proposition 3 of Complete market)

There is an example:

Assume there are two periods $t=0$ & $t=1$; At $t=1$, there are N states. $\exists$ a stock whose payoff $\{P_s^i\}$, $i = 1, 2, \cdots, N$.

where $\{P_s^i\}$ is an Arithmetic sequence $P_s^{i+1} = P_s^i + \delta$, $\delta > 0$

	Stock	$C_1(S, P_s^1)$	$C_1(S, P_s^2)$	$\cdots$	$C_1(S, P_s^{N-1})$
State 1	P_s^1	0	0	$\cdots$	0
2	P_s^2	δ	0	$\cdots$	0
3	P_s^3	2δ	δ	$\cdots$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	0
N	P_s^N	$(N-1)\delta$	$(N-2)\delta$		δ

→ Complete market

By competitive equilibrium, call options are traded in the market and the prices of these options are formed.

Use P_0 to denote the price of the stock at $t=0$

$V_0(S, P_s^i)$ to denote the equilibrium price of $C_1(S, P_s^i)$ at $t=0$

State-i AD Security can be replicated by: $1 \cdot C_1(S, P_s^{i-1}) + 1 \cdot C_1(S, P_s^{i+1}) - 2 \cdot C_1(S, P_s^i) = \delta \cdot \begin{bmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ ← State-i (Payoff)

注意找到对应的$C_1(S, P_s^i)$

By Value Additive Theorem,

$$\delta \cdot q^i = V_0(S, P_s^{i-1}) + V_0(S, P_s^{i+1}) - 2V_0(S, P_s^i)$$

$$\rightarrow q^i = \frac{1}{\delta}\left[V_0(S, P_s^{i-1}) + V_0(S, P_s^{i+1}) - 2V_0(S, P_s^i)\right]$$

L27 Risk Neutral-measure / Q-measure / Martingale-measure

1. Martingale-measure

① Definition

A sequence of RVs $\{X_t\}$ $t=1,2,\cdots$ is called a martingale under P-measure (Physical measure)

if $E_t[X_{t+1}] = E[X_{t+1} \mid F_t] = X_t$, $\forall t$

Infomation up to time t

Super-martingale under P-measure : $E_t[X_{t+1}] \leq X_t$

Sub-martingale under P-measure : $E_t[X_{t+1}] \geqslant X_t$ (更好)

② Property

All the RVs have the same unconditional expectation.

proof:

$E(E_t[X_{t+1}]) = E[X_t]$

$\longrightarrow E[E[X_{t+1} \mid F_t]] = E[X_t]$

$\longrightarrow E[X_{t+1}] = E[X_t]$

③ Example

A sequence of lotteries: $q_t = \begin{cases} 1, & \frac{1}{2} \\ -1, & \frac{1}{2} \end{cases}$, $\{q_t\}$ i.i.d

$X_t = \sum_{i=1}^{t} q_i \longrightarrow X_{t+1} = X_t + q_{t+1}$

$\Rightarrow \{X_t\}$ is a martingale under P-measure, since $E_t[X_{t+1}] = E_t[X_t + q_{t+1}] = X_t + E_t[q_{t+1}] = X_t$

2. Risk neutral measure / Q-measure

① Motivation / Definition

For risk averse investors, when pricing risky assets:

$$P_{risky} = \frac{E(\tilde{c})}{1+r_f+\varphi}, \quad \frac{\boxed{E(\tilde{c})-\phi}}{1+r_f} \quad (\varphi>0, \phi>0)$$

$$= \frac{\boxed{E^Q(\tilde{c})}}{1+r_f}$$

$\longrightarrow \underline{E^Q(\tilde{c}) < E(\tilde{c})}$

Under Q-measure, investors act like risk neutral, so Q-measure is also called risk neutral measure.

② Example

Price of a stock S_t, $t=1,\dots,n$, and let's define $X_t = \frac{S_t}{(1+r_f)^t}$.

By definition of Q-measure: $S_t = \frac{E_t^Q(S_{t+1})}{1+r_f}$

then $\{X_t\}$ is a martingale under Q-measure: $E_t(X_{t+1}) = E_t\left(\frac{S_{t+1}}{(1+r_f)^{t+1}}\right) = \frac{S_t(1+r_f)}{E_t(1+r_f)^{t+1}} = \frac{S_t}{(1+r_f)^t} = X_t$

$\longrightarrow \left\{\frac{S_t}{(1+r_f)^t}\right\}$ is a martingale under Q-measure

$\longrightarrow$ Q-measure is also called martingale measure

③ Construct Q-measure by AD securities

$t=0,1$, at $t=1$, $\exists N$ states

Assume $\exists$ a full set of AD securities $\{q^i\}$, $i=1,2,\dots,N$

For any asset: $t=0$ P_0

		Probability
$t=1$	$\tilde{P}_1 = P_1^1$	π_1
	P_1^2	π_2
	$\vdots$	$\vdots$
	P_1^N	π_N

If $\exists$ Q-measure, $\{\pi_i^Q\}$: $\underbrace{P_0 = \sum_{i=1}^N q^i p^i}_{AD} = \underbrace{\frac{E^Q(\tilde{p}_1)}{1+r_f}}_{Q\text{-measure}} = \frac{\sum_{i=1}^N \pi_i^Q p^i}{1+r_f} = \sum_{i=1}^N \frac{\pi_i^Q}{1+r_f} p^i$

$\longrightarrow q^i = \frac{\pi_i^Q}{1+r_f} \longrightarrow \pi_i^Q = (1+r_f)q^i$ ——— (1)

In order to make $\{\pi_i^Q\}$ a measure, we need $\underbrace{\sum_{i=1}^N \pi_i^Q = 1}_{⚠1}$ & $\underbrace{0<\pi_i^Q<1}_{⚠2}$

Next, we investigate r_f by looking at Bond.

For the bond: $t=0$ b_0

		State
$t=1$	$b_1 = 1 = 1$	1
	1	2
	⋮	⋮
	1	N

$$\longrightarrow b_0 = \sum_{i=1}^{N} q^i \cdot 1 = \sum_{i=1}^{N} q^i = \frac{b_1}{1+r_f} = \frac{1}{1+r_f} \quad \text{———— (ii)}$$

We plug (ii) into (i), then we get $\pi_j^Q = \frac{1}{\sum q^i} \cdot q^i$ ⟶ $\boxed{⚠1 \text{ is satisfied}}$

With no arbitrage theory, $\forall i, q^i > 0$ ⟶ $\boxed{⚠2 \text{ is satisfied}}$ Existence of $\{\pi^Q\}$ depends on No arbitrage theory

If $q^i < 0$ Type 2 Arbitrage opportunity
If $q^i = 0$ Type 1 Arbitrage opportunity

3. Difference between P-measure & Q-measure

① Example

		P-measure	Q-measure
$\tilde{C}$	C_H	π	π^Q
	C_L	$1-\pi$	$1-\pi^Q$

$\pi^Q < \pi$:

Q-measure assigns higher probability in bad state

4. Stochastic Discount Factor (SDF) / Pricing Kernel

① Derivation

$$P_0 = \sum_{i=1}^{N} q^i p^i = \sum_{i=1}^{N} \boxed{\frac{\pi_i \beta u_j'(c_j^i)}{u_j'(c_j^0)}} p^i$$

两种定价方式

$$\text{AD: } P_0 = \sum_{i=1}^{n} q^i p^i = \sum_{i=1}^{n} \frac{\beta \pi_i u'(c_j^i)}{u'(c_j^0)} p^i$$

$$\text{Q-measure: } P_0 = \frac{E^Q(\tilde{p}_i)}{1+r_f} = \frac{\sum_{i=1}^{n} \pi_i^Q p^i}{1+r_f}$$

Should not depend on j, so we let $m_i = \frac{\beta u_j'(c_j^i)}{u_j'(c_j^0)} = \frac{q^i}{\pi^i}$ m^i is higher in Bad state

We let RV $\tilde{m} = \begin{cases} m^1 & \text{State } 1 \\ \vdots & \vdots \\ m^N & N \end{cases}$

$$\to P_0 = \sum_{i=1}^{N} \pi_i m^i p^i = E[\tilde{m} \tilde{p}_i]$$

SDF / Pricing kernel

$$\to P_0 = \begin{cases} \text{Under P-measure: } E(\tilde{m} \tilde{p}_i) & \leftarrow \text{取决于state} \\ \text{Under Q-measure: } E^Q\left(\frac{1}{1+r_f} \tilde{p}_i\right) & \leftarrow \text{不取决于state} \end{cases}$$

5. Fundamental Theorem of Asset Pricing

① Theorems

a. Thm 1

Arbitrage $\longleftrightarrow$ $\nexists$ Q-measure: $\{\pi_i^Q\}$

b. Thm 2

No arbitrage & Complete market $\longleftrightarrow$ $\exists$ a unique Q-measure $\{\pi_i^Q\}$

c. Thm 3

No arbitrage & Incomplete market $\longleftrightarrow$ $\exists$ multiple Q-measure $\{\pi_i^Q\}$

② Examples

a. Example 1: Arbitrage & Complete Market

	Bond	Stock 1	Stock 2	Portfolio (1 Bond + 1 Stock 1)
t=0	1	2	3	3
t=1	1.1	1	3	2.1
	1.1	2	4	3.1
	1.1	3	5	4.1

→ Arbitrage

Full rank → Complete market

Mkt completeness cannot guarantee No arbitrage

In order to check No arbitrage, we need to show the existence of Q-measure

Bond: $1 = \frac{1.1}{1+r_f} \Leftarrow \pi_1^Q + \pi_2^Q + \pi_3^Q = 1$

Stock1: $2 = \frac{E^Q(\tilde{p}_1)}{1+r_f} = \frac{\pi_1^Q + 2\pi_2^Q + 3\pi_3^Q}{1.1}$

Stock 2: $3 = \frac{E^Q(\tilde{p}_1)}{1+r_f} = \frac{3\pi_1^Q + 4\pi_2^Q + 5\pi_3^Q}{1.1}$

$\longrightarrow \exists\, \pi_i^Q \leq 0 \longrightarrow \exists$ Arbitrage

b. Example 2: No arbitrage & Complete market

$$\begin{array}{c|ccc} & \text{Bond} & \text{Stock1} & \text{Stock 2} \\ t=0 & 1 & 2 & 3 \\ t=1 & \left[\begin{matrix} 1.1 & 3 & 1 \\ 1.1 & 2 & 4 \\ 1.1 & 1 & 6 \end{matrix}\right] \end{array}$$

Full rank $\longrightarrow$ Complete market

Bond: $1 = \frac{1.1}{1+r_f} \Leftarrow \pi_1^Q + \pi_2^Q + \pi_3^Q = 1$

Stock1: $2 = \frac{E^Q(\tilde{p}_1)}{1+r_f} = \frac{3\pi_1^Q + 2\pi_2^Q + \pi_3^Q}{1.1}$

Stock 2: $3 = \frac{E^Q(\tilde{p}_1)}{1+r_f} = \frac{\pi_1^Q + 4\pi_2^Q + 6\pi_3^Q}{1.1}$

$\longrightarrow \begin{cases} \pi_1^Q = 0.3 \\ \pi_2^Q = 0.6 \\ \pi_3^Q = 0.1 \end{cases} \longrightarrow$ Unique $\{\pi_i^Q\} \longrightarrow$ No arbitrage

c. Example 3: No arbitrage & Incomplete market

$$\begin{array}{c|cc} & \text{Bond} & \text{Stock 1} \\ t=0 & 1 & 2 \\ t=1 & \left[\begin{array}{c} 1.1 \\ 1.1 \\ 1.1 \end{array}\right. & \left.\begin{array}{c} 3 \\ 2 \\ 1 \end{array}\right] \end{array}$$

Incomplete market

Bond: $1 = \frac{1.1}{1+r_f} \Leftarrow \pi_1^Q + \pi_2^Q + \pi_3^Q = 1$

Stock 1: $2 = \frac{E^Q(\tilde{p}_1)}{1+r_f} = \frac{3\pi_1^Q + 2\pi_2^Q + \pi_3^Q}{1.1}$

$\longrightarrow \begin{cases} \pi_1^Q \in (0,1) \\ \pi_2^Q = 1.2 - 2\pi_1^Q \in (0,1) \\ \pi_3^Q = \pi_1^Q - 0.2 \in (0,1) \end{cases} \longrightarrow \pi_1^Q \in (0.2, 0.6) \longrightarrow$ Multiple Q-measure $\longrightarrow$ No arbitrage

$\longrightarrow$ When No arbitrage, $q^1 = \frac{\pi_1^Q}{1+r_f} \in \left(\frac{0.2}{1.1}, \frac{0.6}{1.1}\right)$