△ Special Solution

A solution is special if it does not contain any arbitrary constants

A solution can be neither general nor special (if containing $C_1, \cdots, C_n$ but they are dependent)

⑥ Existence interval

Domain of the solution function

Example

Show that: " $y = Ce^{kx}$ " is a solution of ODE $\frac{dy}{dx} = ky$

$$\frac{dy}{dx} = \frac{dCe^{kx}}{dx} = Cke^{kx} = kCe^{kx} = ky \qquad \square$$

Example

Show that: " $y = C_1 + C_2 t - \frac{1}{2}gt^2$ " is the general solution to equation $\frac{d^2y}{dt^2} = -g$.

Step 1: Show that its a solution 关键!

$$y^{(1)} = C_2 - gt, \quad y^{(2)} = -g. \quad \square$$

Step 2: Show that its a general solution

$$\text{Since } \begin{cases} \text{it contains } C_1, C_2 \\ JD = \begin{vmatrix} \frac{\partial y}{\partial C_1} & \frac{\partial y}{\partial C_2} \\ \frac{\partial y^{(1)}}{\partial C_1} & \frac{\partial y^{(1)}}{\partial C_2} \end{vmatrix} = 1 \neq 0 \end{cases} \text{, then its general, } \square$$

Example

Given " $y = C_1 e^x \cos x + C_2 e^x \sin x$ ", find an ODE s.t. y is its general solution

$$\begin{cases} y' = C_1 e^x(\cos x - \sin x) + C_2 e^x(\cos x + \sin x) \\ y'' = C_1 e^x(-2\sin x) + C_2 e^x(2\cos x) \end{cases}$$

Since y is a general solution,

then General: Existence of C_1, C_2

$$\begin{cases} y = C_1 e^x \cos x + C_2 e^x \sin x \\ y' = C_1 e^x(\cos x - \sin x) + C_2 e^x(\cos x + \sin x) \end{cases} \longrightarrow \begin{cases} C_1 = e^{-x}[y(\sin x + \cos x) - y'\sin x] \\ C_2 = e^{-x}[y(\sin x - \cos x) + y'\sin x] \end{cases}$$

Independence of C_1, C_2: $JD = \begin{vmatrix} \frac{\partial y}{\partial C_1} & \frac{\partial y}{\partial C_2} \\ \frac{\partial y'}{\partial C_1} & \frac{\partial y'}{\partial C_2} \end{vmatrix} = \begin{vmatrix} e^x\cos x & e^x \sin x \\ e^x(\cos x - \sin x) & e^x(\cos x + \sin x) \end{vmatrix} = e^{2x} \neq 0$

Solution: Substitude C_1 & C_2 into $y'' = C_1 e^x(-2\sin x) + C_2 e^x(2\cos x)$

$\longrightarrow$ ODE: $y'' - 2y' + 2y = 0$

⑦ Initial condition

An extra bit of information about differential equations that tells us the value of the function at a particular point.

⑧ Initial value problem

$$\begin{cases}\text{Differential equations} \\ \text{Initial conditions}\end{cases}$$

本质上是在 specify c

Example

Compute the solutions of the following initial value problems:

(1) $\begin{cases} y''' = x \\ y(0) = a_0,\ y'(0) = a_0',\ y''(0) = a_0'' \end{cases}$

$$y'' = \frac{x^2}{2} + a_0''$$

$$y' = \frac{x^3}{6} + a_0''x + a_0'$$

$$y = \frac{x^4}{24} + \frac{a_0''x^2}{2} + a_0'x + a_0$$

(2) $\begin{cases} y' = x\sqrt{1+x^2} \\ y(0) = y_0 \end{cases}$

$$y = \int x\sqrt{1+x^2}\,dx + C_1$$

$$= \boxed{\int x(1+x^2)^{\frac{1}{2}}\,dx} + C_1$$

$$= \frac{(1+x^2)^{\frac{3}{2}}}{3} + C_1 = \frac{(1+x^2)^{\frac{3}{2}}}{3} + y_0 - \frac{1}{3}$$

试

L2 Geometric Interpretation

1. Geometric Interpretation

① Solution curve / Integral curve (只针对first-order ODE)

a. Definition

Consider a first-order ODE: $\frac{dy}{dx}=f(x,y)$, where f is continuous in a region $G\subset\mathbb{R}^2$.

Suppose $y=\phi(x), x\in J$ is a solution, $J\subseteq\mathbb{R}$ is an interval.

Then the set $\Gamma=\{(x,y)\mid y=\phi(x), x\in J\}$ is a differentiable curve.

We call Γ a Solution curve / Integral curve

b. Theorem

y_0也需要已知

Let $(x_0,y_0)\in\Gamma$, then the slope of the curve Γ at (x_0,y_0) is $\phi'(x_0)=f(x_0,y_0)$,

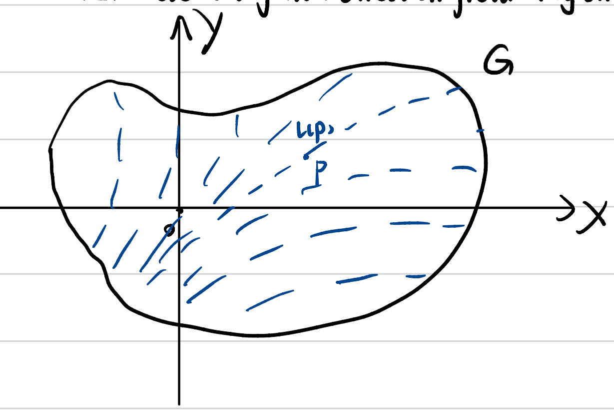
Therefore, we are always able to get the tangent lines without information of solution: $y-y_0=f(x_0,y_0)(x-x_0)$

② Line element & (Line element field / Direction field)

a. Definition

For every point P in the region G, draw a pigment $l(p)$ with the slope $f(p)$. (本质上: Local behaviors of the solution)

Then $\begin{cases}\text{Line element} & : l(p) \\ \text{Line element field / Direction field}: & \text{region } G + \text{all the line elements}\end{cases}$



b. Theorem 「光滑 ⟺ Line element 完美拟合「的切线

A continuous differentiable curve $\Gamma=\{(x,y)\mid y=\phi(x), x\in J\}$ in plane is the integral curve of equation $\frac{dy}{dx}=f(x,y)$ if and only if: For every $(x,y)\in\Gamma$, the tangent line of Γ and the line element field of equation $\frac{dy}{dx}=f(x,y)$ at (x,y) coincide.

Proof

(i) $\rightarrow$: According to ①b, ✓

(ii) $\leftarrow$: $\Gamma: y=\phi(x)$. For every point $(x,\phi(x))\in\Gamma$, the slope of tangent line is $\phi'(x)$.

By condition, we have $\phi'(x)=f(x,\phi(x))$

Since $(x,\phi(x))$ is arbitrary, we have $y=\phi(x)$ is a solution of equation, i.e. Γ is the integral curve. □

③ Isocline

a. Definition

A curve $L_k: f(x,y)=k$ 等倾线 $\frac{dy}{dx}=k$ ("等倾"=同斜率)

b. Usefulness

When draw the line element field, we can first draw isoclines. (作为趋势 / 作为分界线

(isocline → line element field → solution curve)

类似于在分析演译中的水流方向时，我们预先锚定一个既定方向作为baseline.

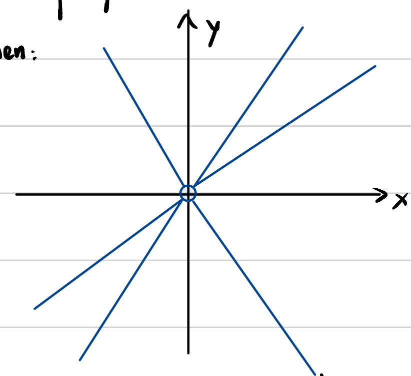
Example

Draw the line element field for $\frac{dy}{dx}=\frac{y}{x}$

Since $\frac{dy}{dx}=f(x,y)=\frac{y}{x}$, then the isocline is $k=\frac{y}{x}$, i.e. $y=kx$

$f(x,y)=k$

then:



← 所有的y落在等倾线上

From the picture, we have $y=kx\ (x\neq 0)$ are solutions.

Example

Draw the line element field for $\frac{dy}{dx}=y(2-y)$ ← 部分的y落在等倾线上

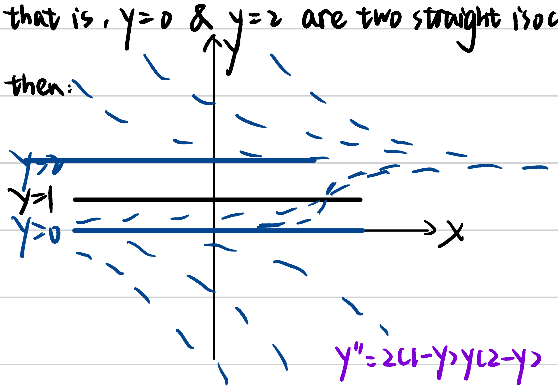
Since $\frac{dy}{dx}=f(x,y)=y(2-y)$, then $y(2-y)=k$

$f(x,y)=k$

Let $k=0$, then $y_1=0$, $y_2=2$, 先赋值 (是为了specify图像)

that is, $y=0$ & $y=2$ are two straight isoclines. 后求零点

then:



$y''=2(1-y)y(2-y)$

Consider
$$\begin{cases} y<0, & \frac{dy}{dx}<0\ \downarrow, \ \frac{d^2y}{dx^2}<0 & \text{Concave} \\ 0<y<2, & \frac{dy}{dx}>0\ \uparrow, \ \frac{d^2y}{dx^2}\begin{cases} 0<y<1: >0 & \text{Convex} \\ 1\le y<2: <0 & \text{Concave}\end{cases} \\ y>2, & \frac{dy}{dx}<0\ \downarrow, \ \frac{d^2y}{dx^2}>0 & \text{Convex} \end{cases}$$

L3 Variable Separation Method

1. Variable Separation Method

① Separable equation

A first order ODE of the form $\underline{\frac{dy}{g(y)} = h(x)dx}$ (即等号左边与右边部分变量完全分离)

② Process

By $\frac{dy}{g(y)} = h(x)dx$, we let $\boxed{\begin{cases} G(y) = \int \frac{dy}{g(y)} \\ H(x) = \int h(x)dx \end{cases}}$, then we get $G(y) = H(x) + C$

构造的时候,注意分母的edge case!!!

"$G(y) = H(x) + C$" is the implicit general solution. to make it explicit, we apply implicit function theorem: $y = G^{-1}(H(x)+C)$

If we have initial condition, we can further specify C:

Consider initial condition $y(x_0) = y_0$, then $C = G(y_0) - H(x_0)$. then $y = G^{-1}(H(x) + G(y_0) - H(x_0))$

Example

Find the general solution to $\frac{dy}{dx} = x(1+y^2)$

Since $\frac{1}{1+y^2}dy = xdx$, then $\int \frac{1}{1+y^2}dy = \int x dx + \underline{C}$

then $\arctan y = \frac{x^2}{2} + C$,

then the general solution is "$y = \tan(\frac{x^2}{2} + C)$"

Example

Find the general solution to $(x+3)\frac{dy}{dx} = 4y$

Since $(\frac{1}{4y})dy = (\frac{1}{x+3})dx$, then $\int \frac{1}{4y}dy = \int \frac{1}{x+3}dx + C$, $(x \neq -3)$

then $\frac{1}{4}\ln|y| = \ln|x+3| + C$, then $\ln|y| = 4\ln|x+3| + C$, $\int \frac{1}{x}dx = \ln|x| + C$

then $|y| = e^{4\ln|x+3|+C} = (x+3)^4 \cdot e^C$

then $y = \pm[(x+3)^4 e^C]$ $(x \neq -3)$

When $x=-3$, $y=0$, it's also the solution. 通过把edge case代回equation进行讨论

then the general solution is "$y = \pm[(x+3)^4 e^C]$"

③ Extension

Some equations are not separable, but they can be solved by variable separation method

a. First order linear ODE : Method of Variation of Constants

$$\frac{dy}{dx} = a(x)y + f(x)$$

When $f(x) = 0$, this equation is called homogeneous linear equation 齐次的对象是$y, y', \cdots, y^{(n)}$

When $f(x) \neq 0$, this equation is called non-homogeneous linear equation

First, we consider homogeneous linear equation : $\frac{dy}{dx} = a(x)y$

Since it's separable, then $\frac{dy}{y} = a(x)dx$ $(y \neq 0)$, then $\int \frac{1}{y} dy = \int a(x)dx + C_1$, then $\ln|y| = \int a(x)dx + C_1$,

then $y = \pm e^{\int a(x)dx + C_1} = C e^{\int a(x)dx}$ (C is non-zero)

When $y = 0$, it's a solution. 返回来检查$y=0$

Therefore, $y = \pm e^{\int a(x)dx + C_1} = C \cdot e^{\int a(x)dx}$ (C is arbitrary) 现在由于$y=0$成立, C也相应变成arbitrary

Second, we consider non-homogeneous linear equation: $\frac{dy}{dx} = a(x)y + f(x)$

We try to find a solution of the form $y = C(x) \cdot e^{\int a(x)dx}$ Homo → Non-Homo : $C \to C(x)$

We substitute this new solution into the equation, we have

LHS: $\frac{dy}{dx} = \frac{dC(x)}{dx} e^{\int a(x)dx} + a(x)C(x)e^{\int a(x)dx}$

RHS: $a(x)y + f(x) = a(x)C(x)e^{\int a(x)dx} + f(x)$

then we have $\frac{dC(x)}{dx} = e^{-\int a(x)dx} f(x)$, then $C(x) = \int e^{-\int a(x)dx} f(x)dx + C$

记住!

then $y = e^{\int a(x)dx}\left(\int e^{-\int a(x)dx} f(x)dx + C\right) = Ce^{\int a(x)dx} + \left(\int e^{-\int a(x)dx} f(x)dx\right)e^{\int a(x)dx}$

This solution is called "Formula of Variation of Constants" = General Sol of HM + Special Sol of Non-HM

这里不需要+C

Example

Find the general solution to $\frac{dy}{dx} = -2xy + 2xe^{-x^2}$

It's a non-homogeneous linear ODE. $\boxed{a(x) = -2x,\quad f(x) = 2xe^{-x^2}}$ 套公式前specify每一公式成分内容

then by the Formula of Variation of Constants,

we have
$$\begin{aligned} y &= Ce^{\int a(x)dx} + \left(\int e^{-\int a(x)dx} f(x)dx\right)e^{\int a(x)dx} \\ &= Ce^{\int -2xdx} + \left(\int e^{\int 2xdx} 2xe^{-x^2} dx\right)e^{\int -2xdx} \\ &= Ce^{-x^2} + \int 2xdx\, e^{-x^2} \\ &= Ce^{-x^2} + x^2e^{-x^2} \end{aligned}$$

then the general solution is : $y = Ce^{-x^2} + x^2e^{-x^2}$,

b. Bernoulli equation

$$\frac{dy}{dx}=a(x)y+f(x)y^{\alpha}$$

When $\alpha=0$ / $\alpha=1$, it's a first-order linear equation

When $\alpha\neq0$ & $\alpha\neq1$, it's not a first-order linear equation

FOLE $\subseteq$ Bernoulli Equation

When $\alpha\neq0$ & $\alpha\neq1$, we multiply both sides by $y^{-\alpha}$, $(y\neq0)$

then we have $y^{-\alpha}\frac{dy}{dx}=a(x)y^{1-\alpha}+f(x)$

Next, we change the variable.

Let $z=y^{1-\alpha}$, then $\frac{dz}{dx}=(1-\alpha)y^{-\alpha}\frac{dy}{dx}$

Since $\frac{dy}{dx}=a(x)y+f(x)y^{\alpha}$, then $\frac{dz}{dx}=(1-\alpha)a(x)y^{1-\alpha}+(1-\alpha)f(x)=(1-\alpha)a(x)z+(1-\alpha)f(x)$

As a result, it has been transformed into a first-order linear ODE.

By the Formula of Variation of Constants $\begin{cases}A(x)=(1-\alpha)a(x)\\F(x)=(1-\alpha)f(x)\end{cases}$

then $z=e^{\int(1-\alpha)a(x)dx}(C+\int(1-\alpha)f(x)e^{-\int(1-\alpha)a(x)dx}dx)$

then $y^{1-\alpha}=e^{\int(1-\alpha)a(x)dx}(\int(1-\alpha)e^{-\int(1-\alpha)a(x)dx}f(x)dx+C)$ Bernoulli Formula of Variation of Constants

Example

Find the general solution to $\frac{dy}{dx}=-\frac{3}{x}y+x^2y^2$ $(x>0)$

This is a Bernoulli equation $\begin{cases}a(x)=-\frac{3}{x}\\f(x)=x^2\\\alpha=2\end{cases}$

We substitude them into Bernoulli Formula of Variation of Constants, then $y^{-1}=e^{\int-(-\frac{3}{x})dx}(C+\int-x^2e^{-\int\frac{3}{x}dx}dx)$,

then $y^{-1}=x^3(C-\ln x)$, then $y=x^{-3}(-\ln x+C)^{-1}$ $(y\neq0)$

When $y=0$, LHS=RHS=0, "$y=0$" is also a solution

Then the general solution is: $y=x^{-3}(-\ln x+C)^{-1}$.

c. 0-th homogeneous equation

n-th homogeneous function: $f(\lambda x, \lambda y) = \lambda^n f(x,y)$

$\frac{dy}{dx} = f(x,y)$, where f is 0-th homogeneous function, i.e. for any $\lambda \neq 0$, $f(\lambda x, \lambda y) = f(x,y)$　零次齐次方程：每一项都是零次

Let $\boxed{y = ux \ (u \text{ is a function of } x)}$, then $\frac{dy}{dx} = x\frac{du}{dx} + u$

$u = \frac{dy}{dx}$，之所以设置是由于 $y \to u \Rightarrow f(x,y) \to f(1,u)$

利用homogeneity的特征

Put it back to the equation, we have $x\frac{du}{dx} + u = f(x, ux) = f(1,u)$ $(x \neq 0)$

i.e. $x\frac{du}{dx} = f(1,u) - u$, it's separable $\longrightarrow$ Original case

$\frac{du}{f(1,u)-u} = \frac{dx}{x}$, then $\int \frac{1}{f(1,u)-u}du = \int \frac{1}{x}dx + C$ $(x \neq 0 \ \& \ f(1,u) - u \neq 0)$

- If $f(1,u) - u = 0$ has a zero point u_0, then $u = u_0$, then $y = u_0 x$ is a solution
- If $f(1,u) - u \equiv 0$, it means $f(x,y) = \frac{y}{x}$, then $\frac{dy}{dx} = \frac{y}{x}$, then it's a separable equation
- If $x = 0$, then $y = 0$, it's a solution

then the general solution is "$y = ux = \ldots$"

Example

Find solutions to $x^2\frac{dy}{dx} - 3xy - 2y^2 = 0$

$\frac{dy}{dx} = \frac{3xy - 2y^2}{x^2} = \frac{3y}{x} - 2\left(\frac{y}{x}\right)^2$

Since $f(x,y) = \frac{3xy-2y^2}{x^2}$ is 0-th homogeneous, then we let $y = ux$,

then $\frac{dy}{dx} = u + x\frac{du}{dx}$, then $x\frac{du}{dx} = 2u(1+u)$, then $\int \frac{2}{x}dx = \int \frac{1}{u(1+u)}du + C$ $(u \neq 0 \ \& \ u \neq -1)$,

then $\ln x^2 = \int \left[\frac{1}{u} - \frac{1}{(1+u)}\right]du + C$, then $\ln x^2 = \ln\left|\frac{u}{1+u}\right| + C$,

then $Cx^2 = \frac{u}{1+u}$ (C is non-zero)

When
- $u = 0$, $y = 0$ is also a solution,
- $u = -1$, $y = -x$ is also a solution

Since $u = \frac{y}{x}$, then
- "$y = Cx^2(x+y)$" is the general solution
- "$y = -x$" is the special solution (u=-1时，解不含C，所以单独写出来成特解)

d. Fraction equation

$\frac{dy}{dx}=f\left(\frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}\right)$, where $a_1,\cdots,c_2$ are given constants

We are able to do transformation to convert it into case c.

Δ When $c_1=c_2=0$, this equation is 0-th homogeneous form equation

Δ When c_1, c_2 are not all zero, we analyze it case by case.

$$\Delta := \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2-a_2b_1$$

(i) $\Delta \neq 0$ $\longrightarrow$ 转化成 0^{th} homogeneous function

Consider $\begin{cases} \boxed{a_1h+b_1k+c_1=0} \\ \boxed{a_2h+b_2k+c_2=0} \end{cases}$, then $\begin{cases} h=\cdots \\ k=\cdots \end{cases}$,

为了换元构造：等式=0 $\rightarrow \begin{cases} h \\ k \end{cases} \rightarrow \begin{cases} x=\varepsilon+h \\ y=\eta+k \end{cases} \rightarrow \begin{cases} \varepsilon \\ \eta \end{cases} \rightarrow \begin{cases} x \\ y \end{cases}$

let $\begin{cases} x=\varepsilon+h \\ y=\eta+k \end{cases}$, then $\underline{\frac{d\eta}{d\varepsilon}=f\left(\frac{a_1\varepsilon+b_1\eta}{a_2\varepsilon+b_2\eta}\right)}$ 通过换元把 c_1、c_2 消掉

(ii) $\Delta=0$ $(a_1b_2=a_2b_1)$ $\longrightarrow$ 都是转化为 separable equation

1 倍数关系：$a_1\neq 0$, $b_1\neq 0$, $\underline{\frac{a_2}{a_1}=\frac{b_2}{b_1}=\lambda\ (\lambda \text{ is constant})}$ (a_2, b_2) 是 (a_1, b_1) 的倍数关系

$\frac{dy}{dx}=f\left(\frac{a_1x+b_1y+c_1}{\lambda(a_1x+b_1y)+c_2}\right)$ 使得 $f(x,y)\rightarrow f(u)$

Let $\underline{u=a_1x+b_1y}$, $\frac{du}{dx}=a_1+b_1\frac{dy}{dx}=a_1+b_1f\left(\frac{u+c_1}{\lambda u+c_2}\right)$, it's separable

2 剩一种变量：$(a_1=a_2=0,\ b_1\neq 0)$ or $(a_1\neq 0,\ b_1=b_2=0)$

$\underline{\frac{dy}{dx}=f\left(\frac{b_1y+c_1}{b_2y+c_2}\right)}$ or $\underline{\frac{dy}{dx}=f\left(\frac{a_1x+c_1}{a_2x+c_2}\right)}$, both are separable

3 剩一边变量：$a_1=0$, $b_1=0$

$\frac{dy}{dx}=f\left(\frac{c_1}{a_2x+b_2y+c_2}\right)$, let $u=a_2x+b_2y$,

then $\frac{du}{dx}=a_2+b_2\frac{dy}{dx}=a_2+b_2f\left(\frac{c_1}{u+c_2}\right)$, it's separable.

Example

Find the general solution to $\frac{dy}{dx} = \frac{2y+4}{x+y-1}$

$\Delta = -2 \neq 0$

Consider $\begin{cases} 2k+4=0 \\ h+k-1=0 \end{cases}$, then $\begin{cases} k=-2, \\ h=3 \end{cases}$

let $\begin{cases} x = \xi+3 \\ y = \eta-2 \end{cases}$, then $\frac{d\eta}{d\xi} = \frac{2\eta}{\xi+\eta}$ (0^{th} homogeneous function)

Let $\eta = u\xi$, then $\begin{cases} \frac{d\eta}{d\xi} = \xi\frac{du}{d\xi} + u \\ \frac{d\eta}{d\xi} = \frac{2u}{1+u} \end{cases}$, then $\xi\frac{du}{d\xi} + u = \frac{2u}{1+u}$ $(u \neq -1)$

then $\frac{(1+u)}{u(1-u)} du = \frac{1}{\xi} d\xi$, $(u \neq 0$ & $u \neq 1$ & $\xi \neq 0)$

then ⋯, (两边同求积分)

then $\begin{cases} \text{"}y+2 = C(x-y-5)^2\text{"} & \text{is a general solution} \\ \text{"}y=-2\text{"} & \text{is a special solution} \end{cases}$

L4 Exact Equation

1. Exact Equation

① Definition

For the first order ODE $M(x,y)dx+N(x,y)dy=0$

if $dU(x,y)=M(x,y)dx+N(x,y)dy$ (即：等号左侧可全微).

then this equation is an exact equation / total differential equation, and we call $U(x,y)$ is the first integral

② Computation of general solution (初级)

Implicit general solution $U(x,y)=C$ $\leftarrow dU(x,y)=0$

Explicit general solution $y=\phi(x)$ s.t. $U(x,\phi(x))=C$

Example

Find solution to $(3x^2y^2+y)dx+(2x^3y+x)dy=0$

Assume $dU(x,y)=(3x^2y^2+y)dx+(2x^3y+x)dy=0$,

then $RHS=3x^2y^2dx+ydx+2x^3ydy+xdy$

注意这一步

$=\underline{(3x^2y^2dx+2x^3ydy)+(ydx+xdy)}$

$=\underline{dx^3y^2+dxy}$ 证明 Total differentiable

$=d(x^3y^2+xy)$ 通过构造 $dU(x,y)$ 来求 $U(x,y)$

then $U(x,y)=x^3y^2+xy=C$

Therefore, the general solution is "$x^3y^2+xy=C$"

③ Judgment of Exactness

If $M(x,y)$ & $N(x,y)$ have continuous partial derivatives on the rectangle domain $G: \alpha < x < \beta, \gamma < y < \delta$,
then Equation is exact if and only if: $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Proof:

$\rightarrow$: Since equation is exact, then $\begin{cases} du(x,y) = M(x,y)dx + N(x,y)dy = 0. \\ du(x,y) = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy \end{cases}$

then $\begin{cases} M = \frac{\partial u}{\partial x} \\ N = \frac{\partial u}{\partial y} \end{cases}$, then $\begin{cases} \frac{\partial M}{\partial y} = \frac{\partial u^2}{\partial y \partial x} \\ \frac{\partial N}{\partial x} = \frac{\partial u^2}{\partial x \partial y} \end{cases}$

Since the partial derivatives of M, N are continuous, then $\frac{\partial u^2}{\partial y\partial x} = \frac{\partial u^2}{\partial x \partial y}$, then $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$\leftarrow$: Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$,

then we need to construct $u(x,y)$ s.t. $M = \frac{\partial u}{\partial x}$, $N = \frac{\partial u}{\partial y}$

Let $u(x,y) = \int_{x_0}^{x} M(x,y)dx + \varepsilon(y)$

In order to make " $N = \frac{\partial u}{\partial y}$ " hold, then $N = \frac{\partial u}{\partial y} = \int_{x_0}^{x} \frac{\partial M(x,y)}{\partial y}dx + \varepsilon'(y) = \int_{x_0}^{x}\frac{\partial N}{\partial x}dx + \varepsilon'(y) = N(x,y) - N(x_0,y) + \varepsilon'(y)$

then $\varepsilon'(y) = N(x_0,y)$, i.e. $\varepsilon(y) = \int_{y_0}^{y} N(x_0,y)dy$

Thus, $u(x,y) = \int_{x_0}^{x} M(x,y)dy + \int_{y_0}^{y} N(x_0,y)dy$,

then the equation is exact

④ Computation of general solution (必背)

The general solution is: " $\int_{x_0}^{x} M(x,y)dx + \int_{y_0}^{y} N(x_0,y)dy = C$ " or " $\int_{x_0}^{x} M(x,y_0)dx + \int_{y_0}^{y} N(x,y)dy = C$ "

where (x_0,y_0) is an arbitrary point

Example

Find the general solution to $(x^2+2xy-y^2)dx + (x^2-2xy-y^2)dy = 0$

Firstly, we check the exactness of the equation:

Let $\begin{cases} M = x^2+2xy-y^2 \\ N = x^2-2xy-y^2 \end{cases}$, then $\begin{cases} \frac{\partial M}{\partial y} = 2x-2y \\ \frac{\partial N}{\partial x} = 2x-2y \end{cases}$, then $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, which means it's exact.

Secondly, we compute the general solution of this exact equation:

注意这里

Since the general solution of an exact equation is $\int_{x_0}^{x} M(x,y)dx + \int_{y_0}^{y} N(x_0,y)dy = C$

Consider $x_0 = y_0 = 0$, then it's: $\int_0^x (x^2+2xy-y^2)dx + \int_0^y (-y^2)dy = C$,

i.e. $\left(\frac{x^3}{3} + x^2y - xy^2\right)\Big|_0^x + \left(-\frac{y^3}{3}\right)\Big|_0^y = \frac{x^3}{3} - \frac{y^3}{3} + x^2y - xy^2 = C$

Therefore, the general solution is " $\frac{x^3}{3} - \frac{y^3}{3} + x^2y - xy^2 = C$ "

2. Special Cases in Non-exact Equation

① Motivation

If the equation is $M(x,y)dx + N(x,y)dy = 0$ but it's non-exact,

we try to find the integral factor $\mu(x,y)$ $(\mu \neq 0)$

s.t. $\mu(x,y)M(x,y)dx + \mu(x,y)N(x,y)dy = 0$ becomes an exact equation

That is, $\dfrac{\partial \mu M}{\partial y} = \dfrac{\partial \mu N}{\partial x}$

② Computation of general solution (初级)

Let $dW(x,y) = \mu(x,y)M(x,y)dx + \mu(x,y)N(x,y)dy$,

then $W(x,y) = C$ is the general solution. (既是原式的，又是变式的)

③ Application range

a. Idea

Since $\dfrac{\partial \mu M}{\partial y} = \dfrac{\partial \mu N}{\partial x}$, then $M\dfrac{\partial u}{\partial y} + u\dfrac{\partial M}{\partial y} = N\dfrac{\partial u}{\partial x} + u\dfrac{\partial N}{\partial x}$

i.e. $N\dfrac{\partial u}{\partial x} - M\dfrac{\partial u}{\partial y} = \left(\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}\right)u$

In general, this PDE is hard to solve. However, if u only depends x / y, then we are able to solve it.

b. Judgment of unique dependence

$$\begin{cases} \text{Integral factor only depends on } x \Leftrightarrow \frac{E}{N} \text{ only depends on } x \\ \text{Integral factor only depends on } y \Leftrightarrow \frac{E}{M} \text{ only depends on } y \end{cases}$$

c. Process

△ u only depends on x

If u only depends on x, then $\dfrac{\partial u}{\partial y} = 0$.

Then $N\dfrac{\partial u}{\partial x} = \left(\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}\right)u$, i.e. $\dfrac{du}{u} = \dfrac{1}{N}\left(\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}\right)dx$

For simplicity, we let $E = \dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}$, then $\int \frac{1}{u}du = \int \frac{E}{N}dx$, then $\ln|u| = \int \frac{E}{N}dx$, then $u = e^{\int \frac{E}{N}dx}$

△ u only depends on y

Similarly, we get $u = e^{-\int \frac{E}{M}dy}$

$\longrightarrow$ Exact equation

Example

Solve: $(3x^3+y)dx+(2x^2y-x)dy=0$

Since $E=\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}=2(1-2xy)$, then this equation is not an exact equation

当不确定是否exact时，直接令 $E=\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}$

However, $\frac{E}{N}=-\frac{2}{x}$, which means this equation has an integral factor $u=e^{\int -\frac{2}{x}dx}=\frac{1}{x^2}$

Multiplying u into the equation, we get an exact equation $3xdx+2ydy+\frac{ydx-xdy}{x^2}=0$, $(x\neq 0)$

By observation, we can directly derive the general solution as "$\frac{3}{2}x^2+y^2-\frac{y}{x}=C$"

In addition, when $x=0$, "$x=0$" is a special solution

Example

Solve: $(e^x+3y^2)dx+2xydy=0$

By observation, LHS $=de^x+3y^2dx+2xydy$

$$=de^x+\frac{1}{x^2}(3x^2y^2dx+2x^3ydy)$$

$$=de^x+\frac{1}{x^2}d(x^3y^2)$$

$$=\frac{1}{x^2}[x^2e^xdx+d(x^3y^2)]$$

$$=\frac{1}{x^2}[d(\int x^2de^x+x^3y^2)]$$

$$=0$$

Then, "$\int x^2de^x+x^3y^2=C$" is the general solution

To be specific, $\int x^2de^x=x^2e^x-\int e^xdx^2=x^2e^x-\int 2xe^xdx=x^2e^x-\int 2xde^x=x^2e^x-2xe^x+\int 2e^xdx=x^2e^x-2xe^x+2e^x$,

then "$x^2e^x-2xe^x+2e^x+x^3y^2=C$" is the general solution

④ **Some useful formulas**

$$\begin{aligned} d(xy) &= ydx+xdy, \\ d(\frac{y}{x}) &= \frac{xdy-ydx}{x^2}, \\ d(x^2+y^2) &= 2(xdx+ydy), \\ d(\arctan\frac{y}{x}) &= \frac{xdy-ydx}{x^2+y^2}, \\ d(\ln\frac{y}{x}) &= \frac{xdy-ydx}{xy}. \end{aligned}$$

L5 Implicit Equation

1. Implicit Equation

① Definition

$\frac{dy}{dx}$ cannot express as a function of x, y: $F(x, y, \frac{dy}{dx}) = 0$ (即 $\frac{dy}{dx}$ 无法被 explicitly 表示)

2. Differential Method

① Case 1

a. Context

y can be expressed as a function of x and $p = \frac{dy}{dx}$: 否则使用 parametric method

$y = f(x, p)$, where f is continuous and differentiable

b. Process

△ Change of variable

Let $p = \frac{dy}{dx}$

② Take derivatives on both sides

Taking derivatives about x in $y = f(x, p)$,

we have $p = \frac{dy}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial p} \frac{dp}{dx}$,

then $\frac{\partial f}{\partial p} \cdot \frac{dp}{dx} = p - \frac{\partial f}{\partial x}$, which is explicit.

a first-order ODE about $x, p, \frac{dp}{dx}$

Example

Solve the following equation: $\left(\frac{dy}{dx}\right)^3+2x\frac{dy}{dx}-y=0$.

Step 1: Obviously, it's implicit.

Step 2: Change of variable.

Let $p=\frac{dy}{dx}$, then $p^3+2px-y=0$

Step 3: Take derivatives on both sides

$$3p^2\cdot\frac{dp}{dx}+2\frac{dp}{dx}x+2p-p=0$$

then $\frac{dp}{dx}(3p^2+2x)+p=0$, which becomes explicit.

Step 4: Solve the explicit equation

Method 1: First-order linear ODE

Method 2: Exact equation

- Method 1: First-order linear ODE

Since $\frac{dp}{dx}=-\frac{p}{3p^2+2x}$, then $\underline{\frac{dx}{dp}=-\frac{2x}{p}-3p}$ $(p\neq0)$ 我们 把x看作因变量 / 把p看作自变量

Since $\begin{cases}a(p)=-\frac{2}{p}\\ f(p)=-3p\end{cases}$, then according to the Formula of Variation of Constants:

$$x=Ce^{\int-\frac{2}{p}dp}+e^{\int-\frac{2}{p}dp}\int-3pe^{\int\frac{2}{p}dp}dp$$

$$=Ce^{\ln|p|^{-2}}+e^{\ln|p|^{-2}}\int-3pe^{\ln|p|^2}dp$$

$$=C\frac{1}{p^2}+\frac{1}{p^2}\int-3p^3dp$$

$$=C\frac{1}{p^2}+\frac{1}{p^2}\left(\frac{-3p^4}{4}+C\right)$$

$$=C\frac{1}{p^2}-\frac{3p^2}{4}$$

Substitute $x=C\frac{1}{p^2}-\frac{3p^2}{4}$ into $p^3+2px-y=0$,

then $y=p^3+2p\frac{C-3p^4}{4p^2}=\frac{C-p^4}{2p}$ $(p\neq0)$

then the general solution is: $\begin{cases}x=\frac{C-3p^4}{4p^2}\\ y=\frac{C-p^4}{2p}\end{cases}$ $(p\neq0)$;

When $p=0$, $y=0$, it's also a special solution

- Method 2: Exact equation

Since $(3p^2+2x)dp + p\,dx = 0$,

and $E = \frac{\partial M}{\partial x} - \frac{\partial N}{\partial p} = 2-1 = 1$

Since $\frac{E}{N} = \frac{1}{p}$, which only depends on p,

then $u = e^{\int \frac{E}{N} dx} = p \quad (p \neq 0)$ is the integral factor.

then we are able to construct an exact equation: $3p^3dp + 2xp\,dp + p^2dx = 0$.

Then the general solution is: "$\int_{p_0}^{p} (3p^3+2xp)dp + \int_{x_0}^{x} p_0^2\,dx = C$"

Taking $p_0 = x_0 = 0$, we have: $\int_0^p (3p^3+2xp)dp = C$,

then $(3\frac{p^4}{4} + xp^2)\Big|_0^p = \frac{3}{4}p^4 + xp^2 = C$

then the general solution is: $\begin{cases} x = \frac{C-3p^4}{4p^2} & (p \neq 0) \\ y = \frac{C-p^4}{2p} \end{cases}$

When $p=0$, $y=0$, it's also a special solution

② Case 2

a. Context

x can be expressed as a function of y and $p=\frac{dy}{dx}$:

$x=f(y,p)$, where f is continuous and differentiable

b. Process

△ Change of variable

Let $p=\frac{dy}{dx}$

△ Take derivatives on both sides

Taking derivatives about x in $x=f(y,p)$,

we have $1=\frac{\partial f}{\partial y}p+\frac{\partial f}{\partial p}\frac{dp}{dx}$,

then $1=\frac{\partial f}{\partial y}p+\frac{\partial f}{\partial p}\frac{dp}{dy}p$,

then $\frac{\partial f}{\partial p}\frac{dp}{dy}p=\frac{\partial f}{\partial y}p-1$, which is explicit.

a first-order ODE of $y, p, \frac{dp}{dy}$

Example

Solve the following equation: $x = y'^3 + y'$

Step 1: Obviously, it's implicit.

Step 2: Change of variable.

Let $p = y'$, then we have $x = p^3 + p$

Step 3: Take derivatives on both sides

$1 = 3p^2 \frac{dp}{dx} + \frac{dp}{dx}$

then $1 = 3p^2 \frac{dp}{dy} p + \frac{dp}{dy} p$

then $\frac{dp}{dy}(3p^3 + p) = 1$

then $\frac{dy}{dp} = 3p^3 + p$

Step 4: Solve the explicit equation

Since the equation above is separable,

then $\int 1 \, dy = \int (3p^3 + p) dp$

then $y = 3\frac{p^4}{4} + \frac{p^2}{2} + C$

then " $\begin{cases} x = p^3 + p \\ y = \frac{3}{4}p^4 + \frac{1}{2}p^2 + C \end{cases}$ " is the general solution

3. Parametric Method

① Motivation

$F(x,y,p)=0$ represents a surface

Suppose $\begin{cases} x=x(u,v) \\ y=y(u,v) \\ p=p(u,v) \end{cases}$

Since $dy=p\,dx$,

then $\frac{\partial y}{\partial u}du+\frac{\partial y}{\partial v}dv=p(u,v)\left(\frac{\partial x}{\partial u}du+\frac{\partial x}{\partial v}dv\right)$

then it can be written as $M(u,v)du+N(u,v)dv=0$, where $\begin{cases} M(u,v)=\frac{\partial y}{\partial u}-p(u,v)\frac{\partial x}{\partial u}, \\ N(u,v)=\frac{\partial y}{\partial v}-p(u,v)\frac{\partial x}{\partial v} \end{cases}$

which is explicit a first order ODE about $u, v, \frac{du}{dv}$

② Process

△ Change of variable $\begin{cases} u=\cdots \\ v=\cdots \end{cases} \longrightarrow \begin{cases} x=\cdots \\ y=\cdots \\ p=\cdots \end{cases}$

△ Expand $dy=p\,dx$

Example

Solve the following equation by parametric method: $\left(\frac{dy}{dx}\right)^2+y-x=0$

Step 1: Obviously, it's implicit

Step 2: Change of variable

Let $\begin{cases} u=x \\ v=\frac{dy}{dx} \end{cases}$, then $\begin{cases} x=u \\ y=u-v^2 \\ p=v \end{cases}$

Step 3: Expand $dy=p\,dx$

$\begin{cases} LHS=d(u-v^2) \\ RHS=v\cdot du \end{cases}$, then $du-2v\,dv=v\,du$,

then $(v-1)du+2v\,dv=0$

Step 4: Solve the explicit equation

$(v-1)du=-2v\,dv$

then $1\,du=-\frac{2v}{v-1}dv \quad (v\neq 1)$

then $\int 1\,du=\int -\frac{2v}{v-1}dv+C \quad (v\neq 1)$

then $u=\int -\frac{2(v-1)+2}{v-1}dv+C \quad (v\neq 1)$

then $u=\int -2dv-\int\frac{2}{v-1}dv+C \quad (v\neq 1)$

then $u=-2v-2\ln|v-1|+C \quad (v\neq 1)$

then "$\begin{cases} x=-2v-2\ln|v-1|+C \\ y=-2v-2\ln|v-1|+C-v^2 \end{cases} \quad (v\neq 1)$" is the general solution

When $v=1$, "$y=x-1$" is a special solution

③ Special cases

a. $F(x, p) = 0$

△ Change of variable

$$\begin{cases} x = A(s) \\ p = B(s) \end{cases}$$

线性关系参数：$x = pt$ | $p = xt$

圆 : $\sin^2 t + \cos^2 t = 1$

椭圆 : $\begin{cases} x = a\sin t \\ p = b\cos t \end{cases}$ $\quad \frac{x^2}{a^2} + \frac{p^2}{b^2} = 1$

双曲线 : $\begin{cases} x = a\frac{e^t + e^{-t}}{2} \\ p = b\frac{e^t - e^{-t}}{2} \end{cases}$ $\quad \frac{x^2}{a^2} - \frac{p^2}{b^2} = 1$

△ Expand $dy = p\,dx$

$dy = B\,dA(s)$

$dy = B\frac{dA}{ds}ds$, which is explicit

Example

Solve the following equation by parametric method: $x^3+(\frac{dy}{dx})^3-3x\frac{dy}{dx}=0$

Step 0: Obviously, it's implicit

Step 1: Change of variable

Since there is no explicit y, then we let $\boxed{\begin{cases} x=A(s)=\frac{3s}{1+s^3} \\ p=B(s)=\frac{3s^2}{1+s^3} \end{cases}}$

then we have $A^3+B^3-3AB=0$

$x^3+p^3-3xp=0$

Let $p=tx$, then $x^3+(tx)^3-3tx^2=0$

then $x+t^3x-3t=0$ $(x\neq 0)$

then $x=\frac{3t}{1+t^3}$, $p=\frac{3t^2}{1+t^3}$

Step 2: Expand $dy=pdx$

$$dy=pdx=B\,dA(s)=B\frac{dA}{ds}ds=\left(\frac{3s^2}{1+s^3}\right)\cdot\frac{3(1+s^3)-3s\cdot 3s^2}{(1+s^3)^2}ds=\frac{9s^2(1-2s^3)}{(1+s^3)^3}ds$$

Step 3: Solve the explicit equation

$$\int 1\,dy=\int\frac{9s^2(1-2s^3)}{(1+s^3)^3}ds+C$$

then $y=\int\frac{3(1-2s^3)}{(1+s^3)^3}ds^3+C$

then let $v=s^3+1$

then $y=\int\frac{-6v+9}{v^3}dv+C$

then $y=\int -6v^{-2}dv+\int 9v^{-3}dv+C$

then $y=\frac{3+12s^3}{2(s^3+1)^2}+C$

then "$\begin{cases} x=\frac{3s}{1+s^3} \\ y=\frac{3+12s^3}{2(s^3+1)^2}+C \end{cases}$" is the general solution

b. $F(y, \frac{dy}{dx}) = 0$

Δ Change of variable

$$\begin{cases} y = A(s) \\ p = B(s) \end{cases}$$

Δ Expand $dy = pdx$

$\frac{dA}{ds} ds = B dx$, which is explicit

Example

Solve the following equation by parametric method: $(\frac{dy}{dx})^3 - y^2(4 - \frac{dy}{dx}) = 0$

Step 0: Obviously, it's implicit

Step 1: Change of variable

Since there is no explicit x, we let $\begin{cases} y = \frac{4t^2}{1+t^2} \\ \frac{dy}{dx} = \frac{4t^3}{1+t^2} \end{cases}$

then $\left(\frac{4t^3}{1+t^2}\right)^3 - \left(\frac{4t^2}{1+t^2}\right)^2 (4 - \frac{4t^3}{1+t^2}) = 0$

Step 2: Expand $dy = pdx$

Step 3: Solve the explicit equation

$\therefore$ " $\begin{cases} x = t + 2\arctan t + C \\ y = \frac{4t^3}{1+t^2} \end{cases}$ $(p \neq 0)$ " is the general solution

When $p = 0$, " $y = 0$ " is a special solution

L6–L7 Some Applications of Elementary Integration Method

1. Singular Solution

① Motivation

Sometimes it's difficult to solve the solution of a first-order ODE,

we can instead find the singular solution.

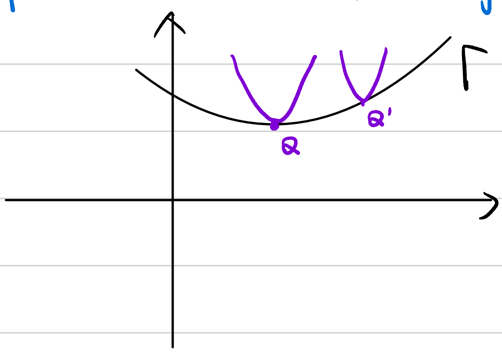
② Definition

For the first-order ODE $F(x, y, \frac{dy}{dx}) = 0$, it has a special solution $\Gamma: y = \varphi(x)$.

If for $\forall$ point $Q \in \Gamma$, on any neighborhood of Q, equation has a solution (different with Γ) tangent with Γ at Q,

then Γ at this time is called singular solution.

Special solution $\xrightarrow{\Gamma\text{任意点与其他解相切}}$ Singular solution (本质上是个切点集合的轨迹)



是否与Γ任意通解都相切?

⟶ 不一定! (通解可能不只一组)

③ Necessary condition

a. p-discriminant

For $F(x, y, \frac{dy}{dx}) = 0$,

p-discriminant is $\begin{cases} \underline{F(x, y, p) = 0} \\ F_p'(x, y, p) = 0 \end{cases}$ → 满足它的条件的不一定是个解

$\triangleq \Delta(x, y) = 0$

b. p-discriminant curve

The curve determined by p-discriminant.

c. Necessary condition

Γ is the singular solution $\xrightarrow{\quad}$ Γ belongs to p-discriminant curve. 本质: special solutions

$\xleftarrow{\otimes}$

Even the p-discriminant curve

can be none of solutions

不局限于 singular solutions

Example

Find singular solution to equation $y^2+\left(\frac{dy}{dx}\right)^2=1$

Step 0: It's implicit

Step 1: Change of variable

Let $\begin{cases} y=\cos t \\ \frac{dy}{dx}=\sin t \end{cases}$ $t\in(-\infty,\infty)$

Then $(\cos t)^2+(\sin t)^2=1$

Step 2: Expand $dy=p\,dx$

$\begin{cases} LHS = d\cos t = -\sin t\,dt \\ RHS = \sin t\,dx \end{cases}$

$\longrightarrow dx=-dt \quad (\sin t\neq 0)$

Step 3: Compute the solution for explicit equation

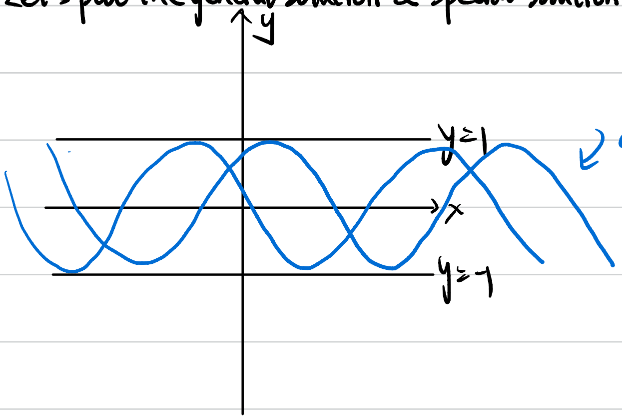
$\int 1\,dx=\int -1\,dt+C$

then $\begin{cases} x=-t+C \\ y=\cos t \end{cases}$ $(y\neq\pm1) \longrightarrow$ "$y=\cos(-x+C)$ $(y\neq\pm1)$" is the general solution

When $y=\pm1$, $\frac{dy}{dx}=0$, then "$y=-1$" & "$y=1$" are two special solutions.

Step 4: Find the singular solution

Let's plot the general solution & special solution



different c (画图的方法不如后面的sufficient condition)

Obviously, "$y=1$" & "$y=-1$" are both tangent with general solution,

therefore, the singular solutions are "$y=1$" & "$y=-1$"

③ Sufficient condition

"$y=\psi(x)$" is the singular solution $\Longleftarrow$
$$\begin{cases} \text{"}y=\psi(x)\text{" satisfies p-discriminant and "}y=\psi(x)\text{" is a solution} \\ \frac{\partial F}{\partial y}(x,\psi(x),\psi'(x)) \neq 0 \\ \frac{\partial^2 F}{\partial p^2}(x,\psi(x),\psi'(x)) \neq 0 \end{cases}$$
Additional conditions

Example

Find the singular solution to equation $[(y-1)y']^2 = ye^{xy}$

Step 1: Solve p-discriminant

$$\begin{cases} F(x,y,p) = (y-1)^2p^2 - ye^{xy} = 0 \\ F_p'(x,y,p) = 2(y-1)^2p = 0 \longrightarrow p=0 \,/\, y=1 \end{cases}$$

Eliminating p, we get $\begin{cases} \text{if } p=0, \text{ then } y=0 \\ \text{if } y=1, \text{ then impossible} \end{cases}$

then "$y=0$" is obtained by p-discriminant.

Step 2: Check additional conditions

$$\begin{cases} \frac{\partial F}{\partial y}(x,\psi(x),\psi'(x)) = \left(2(y-1)p^2 + (y-1)^2\frac{dp}{dy} - e^{xy} - xye^{xy}\right)\Big|_{y=0,p=0} = -1 \neq 0 \\ \frac{\partial^2 F}{\partial p^2}(x,\psi(x),\psi'(x)) = \left(2(y-1)^2 + 4p(y-1)\frac{dy}{dp}\right)\Big|_{y=0,p=0} = 2 \\ \frac{\partial F}{\partial p}(x,\psi(x),\psi'(x)) = 0 \end{cases}$$

Therefore, "$y=0$" is a singular solution

注意:这个方法相比于"解special sol → 解singular sol"的方法更快

2. Envelope

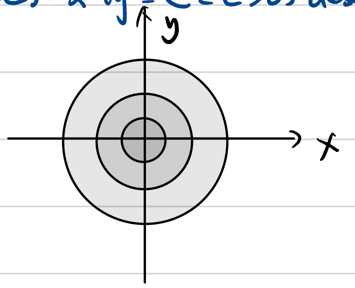
① Definition

a. One-parameter family of curves

$K(C): V(x,y,c)=0$

Example:

$K(C): x^2+y^2=C \ (C>0)$ describes a family of circles



b. Definition

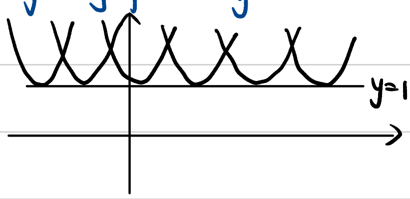
Γ is an envelop of $K(C)$ if

$\forall$ point $Q\in\Gamma$, there is a curve $K(C^*)$ (belong to $K(C)$) tangent with Γ at P.

$K(C^*)$ is different with Γ

Example

The family of curves $y-(x-C)^2=1$ has an envelope $y=1$



② Function

a. Overview

Describe the relationship between general solutions and singular solutions

b. Relationship

If $F(x,y,\frac{dy}{dx})=0$ has the general solutions,

then the envelope is a singular solution.

③ Necessary condition

a. C-discriminant

$$\begin{cases} V(x,y,C)=0 \\ V'_C(x,y,C)=0 \end{cases}$$

b. Necessary condition

Γ is an envelope of $V(x,y,C)=0 \longrightarrow \Gamma$ satisfies C-discriminant

④ Sufficient condition

$\Gamma: \begin{cases} x=\phi(C) \\ y=\psi(C) \end{cases}$ is an envelope of $V(x,y,C)=0 \longleftarrow \begin{cases} \Gamma \text{ satisfies C-discriminant} \\ (V'_x(\phi(C),\psi(C),C), V'_y(\phi(C),\psi(C),C)) \neq (0,0) \\ (\phi'(C),\psi'(C)) \neq (0,0) \end{cases}$

Example

Find the envelope of the Clairaut equation $y=xp+f(p)$, where $f''(p)\neq 0$

Step 1: Find the special solution

It's implicit.

$y-xp-f(p)=0$

Taking derivatives about x, then $p-p-x\frac{dp}{dx}-\frac{df}{dp}\frac{dp}{dx}=0$, then $(x+\frac{df}{dp})\frac{dp}{dx}=0$

then $(\frac{dp}{dx}=0)$ or $(x=-\frac{df}{dp})$

$\begin{cases} \text{when } \frac{dp}{dx}=0 \text{: then } p=C \ (p \overset{y \leftarrow x}{\nwarrow_x} = A(x)) \text{, then } \frac{dy}{dx}=C \text{, then "}y=Cx+f(C)\text{" is the general solution} \\ \text{when } x=-\frac{df}{dp} \text{: } \begin{cases} x=-\frac{df}{dp} \\ y=-p\frac{df}{dp}+f(p) \end{cases} \text{, is a special solution} \end{cases}$

Step 2: Compute C-discriminant

$$\begin{cases} V(x,y,C) = y-xC-f(C)=0 \\ V'_C(x,y,C) = -x-\frac{df}{dC}=0 \end{cases}$$

By eliminating C, we have $\begin{cases} x=-\frac{df}{dC} \\ y=-C\frac{df}{dC}+f(C) \end{cases}$

Step 3: Check additional conditions

$$\begin{cases} (V'_x, V'_y) = (-C,1) \neq (0,0) \\ (x'_C, y'_C) = (-f''(C), -f'(C)-C\cdot f''(C)+f'(C)) = (-f''(C), -Cf''(C)) \neq (0,0) \end{cases}$$

Therefore, "$\begin{cases} x=-\frac{df}{dp} \\ y=-p\frac{df}{dp}+f(p) \end{cases}$" is a singular solution

3. High-order Differential Equations

① Definition

$F(x, y, \frac{dy}{dx}, \cdots, \frac{d^n y}{dx^n}) = 0$

② Key idea of solving high-order differential equations

Reduce the order

(类比：一次微分方程→积分降阶)

③ Methods

a. Case 1: Equation does not contain y

Step 1: Let $p = \frac{d^k y}{dx^k}$, Convert $F(x, \frac{d^k y}{dx^k}, \cdots, \frac{d^n y}{dx^n}) = 0$ to $F(x, p, \cdots, \frac{d^{n-k} p}{dx^{n-k}}) = 0$

Step 2: Integrate p k times, then we can get the general solution.

Example

Solve the following differential equation: $x\frac{d^3y}{dx^3}-2(\frac{d^2y}{dx^2})^2+2\frac{d^2y}{dx^2}=0$

Step 1: Let $p=\frac{d^2y}{dx^2}$, then $x\frac{dp}{dx}-2p^2+2p=0$

Step 2: Solve p

$\frac{1}{(p^2-p)}dp=(\frac{2}{x})dx$ $(p\neq 0$ & $p\neq 1)$

then $\int(\frac{1}{p-1}-\frac{1}{p})dp=\ln x^2+C$

then $\ln|\frac{p-1}{p}|=\ln x^2+C$

then $|\frac{p-1}{p}|=x^2\cdot C$ $(C>0)$

then $1-\frac{1}{p}=\pm Cx^2$

then $p=\frac{1}{1\pm Cx^2}$ $(C>0)$

Step 3: Solve y

$\frac{1}{1\pm Cx^2}=\frac{d^2y}{dx^2}$

① When $\frac{1}{1+Cx^2}=\frac{d^2y}{dx^2}$

Taking integrals on both sides: $\int\frac{1}{1+Cx^2}dx=\frac{dy}{dx}+C_2$

We let $C_1=\sqrt{C}$ $(C_1>0)$, then $\int\frac{1}{1+C_1^2x^2}dx+C_2=\frac{dy}{dx}$

then $\frac{1}{C_1}\arctan C_1x+C_2=\frac{dy}{dx}$

Taking integrals on both sides: $\int(\frac{1}{C_1}\arctan C_1x+C_2)dx=y$

then $y=\frac{x}{C_1}\arctan(C_1x)-\int\frac{x}{1+C_1^2x^2}dx+C_2x+C_3$

$=\frac{x}{C_1}\arctan(C_1x)-\frac{1}{2C_1^2}\ln(1+C_1^2x^2)+C_2x+C_3$

② When $\frac{1}{1-Cx^2}=\frac{d^2y}{dx^2}$

Taking integrals on both sides: $\frac{dy}{dx}=\int\frac{1}{1-C_1^2x^2}dx+C_2$ (Let $C=C_1^2$)

$=\frac{1}{2}\int(\frac{1}{1-C_1x}+\frac{1}{1+C_1x})dx+C_2$

$=\frac{1}{2}(-\frac{1}{C_1}\ln|1-C_1x|+\frac{1}{C_1}\ln|1+C_1x|)+C_2$

Taking integrals on both sides: $y=\frac{x}{2C_1}\ln|\frac{1+C_1x}{1-C_1x}|+\frac{1}{2C_1^2}\ln|1-C_1^2x^2|+C_2x+C_3$

Therefore, "$y=\frac{x}{C_1}\arctan(C_1x)-\frac{1}{2C_1^2}\ln(1+C_1^2x^2)+C_2x+C_3$" & "$y=\frac{x}{2C_1}\ln|\frac{1+C_1x}{1-C_1x}|+\frac{1}{2C_1^2}\ln|1-C_1^2x^2|+C_2x+C_3$"

are general solutions

When $\begin{cases} p=0, \text{"}y=C_1x+C_2\text{"} \\ p=1, \text{"}y=\frac{1}{2}x^2+C_1x+C_2\text{"} \end{cases}$, these two are also general solutions

b. Case 2: Equation does not contain x

Step 1: Let $p=\frac{dy}{dx}$, convert $F(y,\frac{dy}{dx},\cdots,\frac{d^n y}{dx^n})=0$ to $F(y,p,\cdots,\frac{d^{n-1}p}{dx^{n-1}})=0$

Step 2. Solve $F(y,p,\cdots,\frac{d^{n-1}p}{dx^{n-1}})=0$

Example

Solve the following differential equation: $y\frac{d^2y}{dx^2}-\left(\frac{dy}{dx}\right)^2-2\frac{dy}{dx}=0$

Step 1: Let $p=\frac{dy}{dx}$, since $\frac{d^2y}{dx^2}=\frac{d}{dx}(\frac{dy}{dx})=\frac{dp}{dy}\frac{dy}{dx}=p\frac{dp}{dy}$, then $yp\frac{dp}{dy}-p^2-2p=0$

Step 2: Solve the above equation

$y\frac{dp}{dy}-p-2=0\ (p\neq 0)$

then $\frac{1}{p+2}dp=\frac{1}{y}dy\quad (p\neq -2)$

then $\ln|p+2|=\ln|y|+C$

then $|p+2|=|y|\cdot C\ (C>0)$

then $p+2=Cy\quad (C\neq 0)$

then $\frac{dy}{dx}=Cy-2$

then $(\frac{1}{Cy-2})dy=dx$

then $\frac{1}{C}\ln|Cy-2|=x+C_1$

then $\ln|Cy-2|=Cx+CC_1$

then $Cy-2=C_2\cdot e^{Cx}\ (C_2\neq 0)$

then $\therefore y=\frac{C_2e^{Cx}}{C}+\frac{2}{C}$

$=C_3e^{Cx}+\frac{2}{C}\ (C\neq 0, C_3\neq 0)$ ①

When $\begin{cases} p=0, \therefore y=C & ② \\ p=-2, \therefore y=-2x+C & ③ \end{cases}$

Therefore, the above three expressions are general solutions.

C. Case 3: F is a homogeneous function about $y, y', \cdots, y^{(n)}$

Step 1: Since $F(x, y, \frac{dy}{dx}, \cdots, \frac{d^ny}{dx^n}) = F(x, 1, \frac{1}{y}\frac{dy}{dx}, \cdots, \frac{1}{y}\frac{d^ny}{dx^n})$, then we let $p = \frac{1}{y}\frac{dy}{dx}$, then $\begin{cases} y^{(1)} = \cdots \\ y^{(n)} = \cdots \end{cases}$

Step 2: Solve $F(x, \frac{dp}{dx}, \cdots, \frac{d^{n-1}p}{dx^{n-1}}) = 0$

Example

Solve the following differential equation: $x^2 y \frac{d^2y}{dx^2} = (y - x\frac{dy}{dx})^2$

$x^2 y \frac{d^2y}{dx^2} - y^2 - x^2(\frac{dy}{dx})^2 + 2xy\frac{dy}{dx} = 0$ (0-th homogeneous)

Step 1: Let $p = \frac{1}{y}\frac{dy}{dx}$, then $\begin{cases} \frac{dy}{dx} = yp \\ \frac{d^2y}{dx^2} = \frac{dy}{dx}p + y\frac{dp}{dx} = yp^2 + y\frac{dp}{dx} = y(p^2 + \frac{dp}{dx}) \end{cases}$

Then $x^2y^2(p^2 + \frac{dp}{dx}) - y^2 - x^2y^2p^2 + 2xy^2p = 0$

then $x^2y^2\frac{dp}{dx} - y^2 + 2xy^2p = 0$

then $x^2\frac{dp}{dx} - 1 + 2xp = 0$ $(y \neq 0)$

Step 2: Solve the equation

then $x^2 dp + (2xp - 1)dx = 0$

$G = \frac{\partial M}{\partial x} - \frac{\partial N}{\partial p} = 2x - 2x = 0 \longrightarrow$ Exact

then $\int_{p_0}^{p} x^2 dp + \int_{x_0}^{x} (2xp_0 - 1)dx = C$

let $x_0 = p_0 = 0$,

then $\int_0^p x^2 dp + \int_0^x (-1)dx = C$

then $x^2 p - x = C$

then $x^2 \frac{1}{y}\frac{dy}{dx} - x = C$

then $\frac{dy}{dx} = \frac{(x+C)y}{x^2}$

then $\frac{1}{y}dy = (\frac{x+C}{x^2})dx$

then $\ln|y| = \ln|x| + C\frac{x^{-1}}{-1} + C_1$

$= \ln|x| - C\frac{1}{x} + C_1$

then $y = \pm e^{\ln|x| - \frac{C}{x} + C_1}$

$= \pm x \cdot e^{-\frac{C}{x}} \cdot C_2$ $(C_2 > 0)$

$= xe^{-\frac{C}{x}} \cdot C_3$ $(C_3 \neq 0)$

When $y = 0$, it's a solution

Therefore, the general solution can be combined as " $y = x \cdot e^{-\frac{C}{x}} \cdot C_4$ $(C, C_4$ are arbitrary)"

若special solution可与general solution合并表示时,应合并并且调整general solution.

d. Case 4: Total differential equation

Step 1: If total differential : $F(x,y,\frac{dy}{dx},\cdots,\frac{d^n y}{dx^n})=\frac{d}{dx}\phi(x,y,\frac{dy}{dx},\cdots,\frac{d^{n-1}y}{dx^{n-1}})=0$, then $\phi(x,y,\frac{dy}{dx},\cdots,\frac{d^{n-1}y}{dx^{n-1}})=C$

If not total differential : $F(x,y,\frac{dy}{dx},\cdots,\frac{d^n y}{dx^n})\cdot u(x,y,\cdots,\frac{d^{n-1}y}{dx^{n-1}})$ is total differential 不考

Step 2: Solve $\phi(x,y,\frac{dy}{dx},\cdots,\frac{d^{n-1}y}{dx^{n-1}})=C$

Example

Solve the following differential equation: $(1+y^2)\frac{d^2y}{dx^2}-2y(\frac{dy}{dx})^2=0$

Step 1: Let $u=\frac{1}{y'(1+y^2)}$, then $\frac{y''}{y'}-\frac{2yy'}{1+y^2}=0$ $(y'\neq 0)$

Then $(\ln|y'|-\ln(1+y^2))'=0$, then $\ln|y'|-\ln(1+y^2)=C$

Step 2: Solve the equation

$y'=\pm e^{C}(1+y^2)$

then $\frac{dy}{dx}=\pm e^{C}(1+y^2)$

then $\frac{dy}{dx}=C\cdot(1+y^2)$ $(C\neq 0)$

then $\frac{1}{(1+y^2)}dy=C dx$

then $\arctan y=Cx+C_1$ $(C\neq 0)$

then $y=\tan(Cx+C_1)$ $(C\neq 0)$

When $y'=0$, $y=C_2$, it's a solution

When $\begin{cases} C=0 \\ \tan C_1=C_2 \end{cases}$, we can combine two cases

Therefore, the general solution is "$y=\tan(Cx+C_1)$"

4. Riccati Equation

Even if the equation is very simple, we cannot solve it easily by elementary integration method.

Therefore, instead of directly compute the solution, we care about properties of solutions

L8–16 General Theory

1. The existence and uniqueness

① General case & Simple case

a. Equivalence

The solution is the same for two set of equations.

b. Content

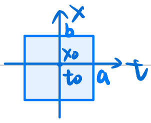
An ODE is equivalent to a system of ODE:

$$\begin{cases} \text{ODE: } \dfrac{d^n y}{dx^n} = F(x, y, \dfrac{dy}{dx}, \cdots, \dfrac{d^{n-1}y}{dx^{n-1}}) \\ \text{System of ODE: } \begin{cases} \dfrac{dy_1}{dx} = y_2 \\ \dfrac{dy_2}{dx} = y_3 \\ \cdots \\ \dfrac{dy_n}{dx} = F(x, y_1, \cdots, y_n) \end{cases} \end{cases} \qquad \begin{cases} y_1 = y \\ y_2 = \dfrac{dy}{dx} \\ \cdots \\ y_n = \dfrac{d^{n-1}y}{dx^{n-1}} \end{cases}$$

The statement and proofs of the corresponding theorems for the general case is similar to this case:

Consider $\begin{cases} \text{the equation: } \dfrac{dx}{dt} = f(t,x) \\ \text{the initial value problem: } \begin{cases} \dfrac{dx}{dt} = f(t,x) \\ x(t_0) = x_0 \end{cases} \end{cases}$, $R = \{(t,x) \in R^2 : |t - t_0| \le a, |x - x_0| \le b\}$

t与x有界



② Existence & Uniqueness of solution

t_0，x_0 是确定的!

a. Lipschitz condition

If $\exists L$, s.t. $\forall (t, x_1), (t, x_2)$, $|f(t,x_1) - f(t,x_2)| \le L|x_1 - x_2|$,

then $\begin{cases} f(t,x) \text{ satisfies Lipschitz condition about } x \\ L \text{ is Lipschitz constant} \end{cases}$

$L = \max\{|\frac{\partial f}{\partial x}|\}$

b. Picard theorem

△ Content

If $f(t,x)$ $\begin{cases} \text{is continuous,} \\ \text{satisfies Lipschitz condition,} \end{cases}$

then there exists a unique solution of initial value problem on the interval $I = [t_0 - h, t_0 + h]$,

where $\begin{cases} h = \min\{a, \frac{b}{M}\} \quad \begin{cases} a: t \\ b: x \end{cases} \\ M = \max\{|f(t,x)| : (t,x) \in R\} \end{cases}$

Peano Theorem

If $f(t,x)$ is continuous and it does not satisfy Lipschitz condition, then there exists a solution.

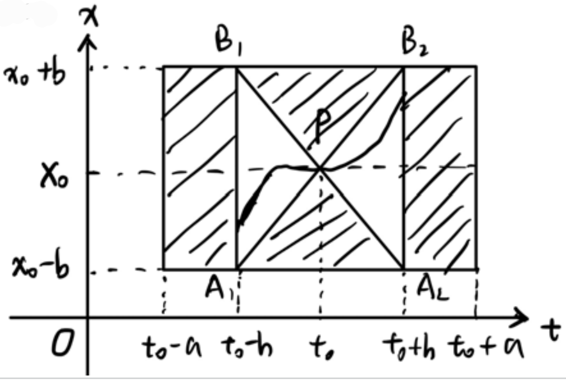
△ Geometric meaning

Consider $h = \frac{b}{M}$, Picard theorem indicates the solution $x = \varphi(t)$ exists on $[t_0 - h, t_0 + h]$.

Since $M = \max\{|f(t,x)| : (t,x) \in R\}$, we have $-M \leq \varphi'(t) \leq M$

Integrating from t_0 to t implies that $x_0 - M(t - t_0) \leq \varphi(t) \leq x_0 + M(t - t_0)$

That means the integral curve will stay in A_1PB_1 & A_2PB_2



△ Approximate solution for IVP by Picard iterative sequence

Let $\begin{cases} \varphi(t) \text{ be the solution} \\ \varphi_n(t) \text{ be the n-th Picard iterative function} \end{cases}$, $\begin{cases} \varphi_0(t) = x_0 \\ \varphi_n(t) = x_0 + \int_{t_0}^{t} f(\tau, \varphi_{n-1}(\tau))\,d\tau, \ (n = 1, 2 \cdots) \end{cases}$, $t \in [t_0, t_0 + h]$

then we have: $|\varphi_n(t) - \varphi(t)| \leq \underline{\frac{ML^n}{(n+1)!}h^{n+1}}$, $t \in [t_0 - h, t_0 + h]$

Error estimate

Example

Consider the IVP: $\begin{cases} \frac{dx}{dt} = x+1 \\ x(0)=0 \end{cases}$, defined on the rectangle $R = \{(t,x)\in\mathbb{R}^2 : |t|\le 1, |x|\le 1\}$

Prove the solution exists and it is unique.

And use Picard iterative sequence to obtain the n-th approximate solution and compute its error estimate.

Step 1: Proof:

Since $\begin{cases} f(t,x)=x+1 \text{ is continuous} \\ \frac{\partial f}{\partial x} = 1 \le 1 \;(L=1) \longrightarrow f(t,x) \text{ satisfies Lipschitz condition} \end{cases}$

then by Picard theorem,

the solution exists and it's unique on $[t_0-h, t_0+h]$

where $\begin{cases} h=\min\{a, \frac{b}{M}\} = \min\{1, \frac{1}{2}\} = \frac{1}{2} \longrightarrow [t_0-h, t_0+h] = [-\frac{1}{2}, \frac{1}{2}] \\ M=\max\{x+1 \mid (t,x)\in R\} = 2 \end{cases}$

Step 2: Compute n-th approximate solution and its corresponding error estimate

$\varphi_0(t) = x_0 = 0$

$\varphi_1(t) = x_0 + \int_{t_0}^{t} f(\tau, \varphi_0(\tau))d\tau$

$= 0 + \int_0^t 1 d\tau = t \qquad = \frac{t^1}{1!}$

$\varphi_2(t) = x_0 + \int_{t_0}^{t} f(\tau, \varphi_1(\tau))d\tau$

$= 0 + \int_0^t (\tau+1)d\tau = \frac{t^2}{2} + t \qquad = \frac{t^2}{2!} + \frac{t^1}{1!}$

$\varphi_3(t) = x_0 + \int_{t_0}^{t} f(\tau, \varphi_2(\tau))d\tau$

$= 0 + \int_0^t (\frac{\tau^2}{2}+\tau+1)d\tau = \frac{t^3}{6} + \frac{t^2}{2} + t \qquad = \frac{t^3}{3!} + \frac{t^2}{2!} + \frac{t}{1!}$

Inductively,

we have "$\varphi_n(t) = \frac{t^n}{n!} + \cdots + \frac{t}{1!}$" as our n-th approximate solution $\quad \frac{t^{n+1}}{(n+1)!} + \cdots + \frac{t^1}{1!} = \frac{t^n}{n!} + \cdots + \frac{t^1}{1!}$

$\varphi_n(t) \xrightarrow{n\to\infty} \underline{e^t - 1}$ Accurate solution

The error is

泰勒展开逆推得. (与n无关, 与t相关)

$|\varphi_n(t) - \varphi(t)| \le \frac{ML^n h^{n+1}}{(n+1)!}$

$= \frac{2\times 1^n \times (\frac{1}{2})^{n+1}}{(n+1)!}$

$= \frac{1}{2^n (n+1)!}$

2. The existence and uniqueness for general linear ODE system

① Setup

$$\frac{d\vec{x}}{dt} = \vec{A}(t)\vec{x} + \vec{f}(t), \text{ where }$$

$$\begin{cases} t \in [\alpha, \beta] \\ \vec{A}, \vec{f} \text{ are continuous} \end{cases}$$

$$\vec{A} = \begin{bmatrix} a_{11}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & & a_{2n}(t) \\ \vdots & & \vdots \\ a_{n1}(t) & \cdots & a_{nn}(t) \end{bmatrix}, \quad \vec{f}(t) = \begin{bmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{bmatrix}, \quad x = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}$$

② Picard theorem application

$$\begin{cases} \frac{d\vec{x}}{dt} = \vec{A}(t)\vec{x} + \vec{f}(t) \\ x(t_0) = x_0 \end{cases}$$ has a unique solution $\vec{x} = \vec{x}(t)$ on $[\alpha, \beta]$ (只与t相关，因为此时L只与t相关)

$$\begin{cases} \text{Continuity: } \checkmark \\ \text{Lipschitz condition: } \checkmark \ (L = \max \vec{A}(t)) \end{cases}$$

3. Linear homogeneous ODE

① Span of solutions

For $\frac{d\vec{x}}{dt} = \vec{A}(t)\vec{x}$,

if $\vec{x_1}(t)$ & $\vec{x_2}(t)$ are two solutions,

then, $\forall C_1, C_2$, $\vec{X}(t) = C_1\vec{x_1}(t) + C_2\vec{x_2}(t)$ is a solution

Let S be the set of all solutions, then S is an N-dim linear space.

② Basic solutions / Basic set of solution

a. Definition

The n linearly independent solution

b. Judgement of linear independence of solutions

△ Solution matrix A basic set of solutions

The matrix consists of solutions $\vec{x_1}(t), \vec{x_2}(t), \cdots, \vec{x_n}(t)$:

$$\vec{X}(t) = \begin{bmatrix} x_{11}(t) & x_{12}(t) & \cdots & x_{1n}(t) \\ x_{21}(t) & x_{22}(t) & \cdots & x_{2n}(t) \\ \vdots & \vdots & & \vdots \\ x_{n1}(t) & x_{n2}(t) & \cdots & x_{nn}(t) \end{bmatrix}$$

Furthermore, $\frac{d\vec{X}(t)}{dt} = \vec{A}(t)\vec{X}(t)$

△ Standard solution matrix

For solution matrix $X(t)$,

if $X(t_0) = I$ for some points $t_0 \in [\alpha, \beta]$, then it is standard

△ Wronsky determinant

$\det(\vec{X}(t))$

△ Linear independence of solutions

The solutions $\{x_i(t) : i = 1, \cdots, n\}$ are linearly independent

if and only if $\det(\vec{X}(t)) \neq 0$, $\forall t \in [\alpha, \beta]$,

equivalently, $\det(\vec{X}(t_0)) \neq 0$, $\exists t_0 \in [\alpha, \beta]$

Liouville formula: $\det(\vec{X}(t)) = \det(\vec{X}(t_0)) \cdot \exp\int_{t_0}^{t} tr(A(\tau))d\tau$

$tr(A(t)) = \sum_{i=1}^{n} a_{ii}(t)$

③ General solution

$x(t) = X(t)\,c$, c is a n-dim constant vector : $c = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$

If we consider $x(t_0) = x_0$, then $c = X^{-1}(t_0)\cdot x(t_0) = X^{-1}(t_0)x_0$,

then $x(t) = \underline{X(t)\cdot X^{-1}(t_0)}\,x_0 = \underline{X(t,t_0)}\,x_0$

where $\underline{X(t,t_0) = X(t)\cdot X^{-1}(t_0)}$ is a standard solution matrix

因为: $X(t_0,t_0) = X(t_0)\cdot X^{-1}(t_0) = I$

4. Linear non-homogeneous ODE

① Definition

$$\frac{d\vec{x}}{dt} = A(t)\vec{x} + \vec{f}(t)$$

② Solution relationship

If x is a solution of LH, x^* is a solution of LNH, then $(x+x^*)$ is a solution of LNH

$\boxed{\text{Linear Homo}} \rightleftarrows \boxed{\text{Linear Non-homo}}$

If x_1^*, x_2^* are solutions of LNH, then $(x_1^* - x_2^*)$ is a solution of LH

③ General solution

a. Original version

$$x(t) = X(t)c + x^*(t)$$

General sol of LNH = (General sol of LH + Special sol of NLH)

b. Formula of variation of constant

$$x(t) = X(t)c + X(t)\int_{t_0}^{t} X^{-1}(\tau) f(\tau)\, d\tau$$

General sol of LNH ← Basic sol of LH

Example

Find the basic matrix solution and the general solution of $\frac{d}{dt}\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 \\ 0 & \frac{1}{t} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

and then find the general solution of $\frac{d}{dt}\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 \\ 0 & \frac{1}{t} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} -1 \\ t \end{bmatrix}$, with initial value $\begin{bmatrix} x_1(-1) \\ x_2(-1) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Step 1: Find the basic matrix solution and general solution of LH

The LH can be decomposed into $\begin{cases} \frac{dx_1}{dt} = x_1 + x_2 & \text{—— ①} \\ \frac{dx_2}{dt} = \frac{1}{t} x_2 & \text{—— ②} \end{cases}$

For equation ②, we have $\frac{1}{x_2} dx_2 = \frac{1}{t} dt$ $(x_2 \neq 0)$, then $\ln|x_2| = \ln|t| + C_0$, then $x_2 = C_1 t$ $(x_2 \neq 0 \,\&\, C_1 \neq 0)$

when $x_2 = 0$, LHS = RHS, it's also a solution of ②, then $x_2 = Ct$ (C arbitrary)

For equation ①, consider $\begin{cases} x_2 = 0 \\ x_2 = t \end{cases}$, then respectively, $\begin{cases} \frac{dx_1}{dt} = x_1 \\ \frac{dx_1}{dt} = x_1 + t \end{cases}$, $\begin{cases} \ln|x_1| = t + C \to x_1 = Ce^t \\ x_1 = e^{\int a(t)dt}\left[\int e^{\int -a(t)dt} f(t)dt + C\right] = Ce^t - (t+1) \end{cases}$

Then in above all, we find two special solutions of LH: $\begin{bmatrix} e^t \\ 0 \end{bmatrix}$, $\begin{bmatrix} -(t+1) \\ t \end{bmatrix}$

Then we have a solution matrix $\begin{bmatrix} e^t & -(t+1) \\ 0 & t \end{bmatrix}$

$\boxed{\text{Since } \det(X) = e^t \cdot t \neq 0}$, then X is a basic solution matrix. 如果此X非Basic，则找其他special solutions

Then the general solution is: $\begin{bmatrix} e^t & -(t+1) \\ 0 & t \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = C_1 \begin{bmatrix} e^t \\ 0 \end{bmatrix} + C_2 \begin{bmatrix} -(t+1) \\ t \end{bmatrix}$

Step 2: Find the general solution of LNH

$x(t) = X(t)c + X(t)\int_{t_0}^{t} X^{-1}(\tau) f(\tau) d\tau$

Since $\begin{cases} X^{-1}(\tau) = \frac{1}{\det(X)} \operatorname{adj}(X) = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{\tau e^\tau} \begin{bmatrix} \tau & \tau+1 \\ 0 & e^\tau \end{bmatrix} \\ f(\tau) = \begin{bmatrix} -1 \\ \tau \end{bmatrix} \end{cases}$

then $\int_{t_0}^{t} X^{-1}(\tau) f(\tau) d\tau = \int_{t_0}^{t} \frac{1}{\tau e^\tau} \begin{bmatrix} \tau & \tau+1 \\ 0 & e^\tau \end{bmatrix} \begin{bmatrix} -1 \\ \tau \end{bmatrix} d\tau$

Let $t_0 = -1$, then $= \int_{-1}^{t} \begin{bmatrix} \tau e^{-\tau} \\ 1 \end{bmatrix} d\tau = \begin{bmatrix} -\tau e^{-\tau} - e^{-\tau} \\ \tau \end{bmatrix}\Bigg|_{-1}^{t} = \begin{bmatrix} -(t+1)e^{-t} \\ t+1 \end{bmatrix}$

Since $c = X^{-1}(t_0) \cdot \vec{x}(t_0) = \begin{bmatrix} e^{-1} & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

then $\vec{x}(t) = \begin{bmatrix} -t^2 - 2t - 1 \\ t^2 \end{bmatrix}$

5. High-order Differential Equation

① Representation of linear high-order ODE

a. Equation representation

Linear high-order ODE $\longrightarrow$ Linear ODE system

$$\frac{d^n x}{dt^n} + a_1(t)\frac{d^{n-1}x}{dt^{n-1}} + \cdots + a_n(t)x = f(t) \longrightarrow \frac{d\vec{x}}{dt} = A(t)\vec{x} + \vec{f}(t)$$

Let $\begin{cases} x_1 = x \\ x_2 = \frac{dx}{dt} \\ \cdots \\ x_n = \frac{d^{n-1}x}{dt^{n-1}} \end{cases}$, then $\boxed{\begin{cases} \frac{dx_1}{dt} = x_2 \\ \frac{dx_2}{dt} = x_3 \\ \cdots \\ \frac{dx_{n-1}}{dt} = x_n \\ \frac{dx_n}{dt} = -a_1(t)x_n - \cdots - a_n(t)x_1 + f(t) \end{cases}}$ 关键

Let $A(t) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & & 1 \\ -a_n(t) & -a_{n-1}(t) & -a_{n-2}(t) & \cdots & -a_1(t) \end{bmatrix}$, $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix}$, $\vec{f}(t) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ f(t) \end{bmatrix}$

then $\frac{d\vec{x}}{dt} = A(t)\vec{x} + \vec{f}(t)$

b. Solution representation

Linear High-order ODE   Linear ODE system

$$x = \phi(t) \longleftrightarrow \vec{x} = \begin{bmatrix} \phi(t) \\ \phi'(t) \\ \vdots \\ \phi^{(n-1)}(t) \end{bmatrix}$$

② General solution

$$x(t) = X(t)c + x^*(t)$$
$$= \sum_{k=1}^{n} c_k x_k(t) + \sum_{k=1}^{n} x_k(t) \int_{t_0}^{t} \frac{W_k(\tau)}{W(\tau)} f(\tau)\, d\tau$$

$W_k(t)$ is the cofactor of $W_{nk}(t)$ (最后一列)

6. Power series solution

① Motivation

Power series is the form of solution of second order linear homogeneous equation:

$$\frac{d^2x}{dt^2} + a(t)\frac{dx}{dt} + b(t)x = 0$$

② Analytic function

f is analytic if and only if:

f can be expanded as a convergent power series.

eg. $f(x) = \sum_{i=0}^{n} a_i (x - x_{i0})^i$

③ Cauchy theorem

a. Original theorem

If $f(t, \vec{x})$ is analytic on R ($R = \{(t, \vec{x}) \in R \times R^n : |t - t_0| < a, |\vec{x} - \vec{x_0}| < b\}$,

then the initial value problem $\begin{cases} \frac{d\vec{x}}{dt} = \vec{f}(t, \vec{x}) \\ \vec{x}(t_0) = \vec{x_0} \end{cases}$ has a unique analytic solution $\vec{x} = \vec{x}(t)$ near t_0,

Analytic solution = Convergent power series solution

i.e. $\vec{x}(t) = \sum_{i=0}^{\infty} \vec{c_i} (t - t_0)^i$

b. Extended theorem (More pratical)

For second-order linear homogeneous equation. $\frac{d^2x}{dt^2} + a(t)\frac{dx}{dt} + b(t)x = 0$ (本质上是个方程组),

and $a(t)$ & $b(t)$ are analytic on the interval $|t - t_0| < r$.

then the equation has an analytic solution: "$x(t) = \sum_{i=0}^{\infty} c_i (t - t_0)^i$" near t_0

where $\boxed{\begin{cases} c_0 = x(t_0) \\ c_1 = x'(t_0) \end{cases}}$, c_is are determined by c_0 & c_1.

当给 $\begin{cases} x_0 \\ x'(t_0) \end{cases}$ 时,才解出 c_0, c_1

Example

Find the power series solution for $x''-2tx'+\lambda x=0$ near $t=0$ (λ is a constant)

Step 1: Write down the formula of power series solution

$$x(t)=\sum_{n=0}^{\infty}C_n(t-t_0)^n=\sum_{n=0}^{\infty}C_nt^n$$

Step 2: Compute C_n

$$\begin{cases} x'(t)=\sum_{n=1}^{\infty}nC_nt^{n-1} \\ x''(t)=\sum_{n=2}^{\infty}n(n-1)C_nt^{n-2}\end{cases}$$

注意，首项应调整至使得最小指数为0

Then we plug them back into the equation:

$$\sum_{n=2}^{\infty}n(n-1)C_nt^{n-2}-2t\sum_{n=1}^{\infty}nC_nt^{n-1}+\lambda\left(\sum_{n=0}^{\infty}C_nt^n\right)=0$$

调整，使得首项为0

Let $n=k+2$, then let $n=k$

$nC_0t^0\big|_{n=0}=0$

$$\sum_{n=0}^{\infty}(n+1)(n+2)C_{n+2}t^n-2\sum_{n=0}^{\infty}nC_nt^n+\lambda\left(\sum_{n=0}^{\infty}C_nt^n\right)=0$$

这里使用不同的构造方法是为了构造t^n

$$\sum_{n=0}^{\infty}t^n\left[(n+1)(n+2)C_{n+2}-2nC_n+\lambda C_n\right]=0$$

$$(n+1)(n+2)C_{n+2}-(2n-\lambda)C_n=0$$

级数之和=0 ⟶ 系数都为0

$$C_{n+2}=\frac{2n-\lambda}{(n+1)(n+2)}C_n$$

Step 2-1: Compute C_{2n}

$$C_2=\frac{-\lambda}{2\times1}C_0,$$

$$C_4=\frac{4-\lambda}{4\times3}\cdot C_2=\frac{(4-\lambda)(-\lambda)}{4\times3\times2\times1}$$

$$C_6=\frac{8-\lambda}{6\times5}C_4=\frac{(8-\lambda)(4-\lambda)(-\lambda)}{6\times5\times4\times3\times2\times1}$$

$$\rightarrow C_{2n}=\frac{(2(2n-2)-\lambda)(2(2n-4)-\lambda)\cdots(-\lambda)}{(2n)!}C_0$$

Step 2-2: Compute C_{2n+1}

$$\boxed{\begin{aligned} C_3&=\frac{2-\lambda}{3\times2}C_1\\ C_5&=\frac{6-\lambda}{5\times4}C_3=\frac{(6-\lambda)(2-\lambda)}{5\times4\times3\times2}C_1\\ C_7&=\frac{10-\lambda}{7\times6}C_5=\frac{(10-\lambda)(6-\lambda)(2-\lambda)}{7\times6\times5\times4\times3\times2}C_1\end{aligned}}$$

可以直接类比.

$$\rightarrow C_{2n+1}=\frac{(2(2n-1)-\lambda)(2(2n-3)-\lambda)\cdots(2-\lambda)}{(2n+1)!}C_1$$

Example

Find the power series solution for $(1-t^2)x''-2tx'+L(L+1)x=0$ near $t=0$ (L is a constant)

Step 1: Write down the formula of power series solution

$$x(t)=\sum_{n=0}^{\infty} C_n(t-t_0)^n=\sum_{n=0}^{\infty} C_n t^n$$

Step 2: Compute C_n

$$\begin{cases} x'(t)=\sum_{n=1}^{\infty} nC_n t^{n-1} \\ x''(t)=\sum_{n=2}^{\infty} n(n-1)C_n t^{n-2} \end{cases}$$

Then we plug them back into the equation:

$$(1-t^2)\sum_{n=2}^{\infty} n(n-1)C_n t^{n-2}-2t\sum_{n=1}^{\infty} nC_n t^{n-1}+L(L+1)\sum_{n=0}^{\infty} C_n t^n=0$$

$$\sum_{n=2}^{\infty} n(n-1)C_n t^{n-2}-\sum_{n=2}^{\infty} n(n-1)C_n t^n-2\sum_{n=1}^{\infty} nC_n t^n+L(L+1)\sum_{n=0}^{\infty} C_n t^n=0$$

$$\sum_{n=0}^{\infty}(n+1)(n+2)C_{n+2}t^n-\sum_{n=0}^{\infty} n(n-1)C_n t^n-2\sum_{n=0}^{\infty} nC_n t^n+L(L+1)\sum_{n=0}^{\infty} C_n t^n=0$$

$$\sum_{n=0}^{\infty} t^n\left[(n+1)(n+2)C_{n+2}-n(n-1)C_n-2nC_n+L(L+1)C_n\right]=0$$

$$(n+1)(n+2)C_{n+2}-n(n-1)C_n-2nC_n+L(L+1)C_n=0$$

分子分母都应该为乘积形式，以便于求通项公式

$$C_{n+2}=\frac{n(n-1)+2n-L(L+1)}{(n+1)(n+2)}C_n=\frac{n(n+1)-L(L+1)}{(n+1)(n+2)}C_n=\frac{(n+L+1)(n-L)}{(n+1)(n+2)}C_n$$

Step 2-1: Compute C_{2n}

$$C_2=\frac{(-L)(L+1)}{2\times1}C_0$$

$$C_4=\frac{(2-L)\times(2+L+1)}{4\times3}C_2=\frac{(2-L)(-L)(L+1)(2+L+1)}{4\times3\times2\times1}C_0$$

$$C_6=\frac{(4-L)(4+L+1)}{6\times5}C_4=\frac{(4-L)(2-L)(-L)(L+1)(L+3)(L+5)}{6\times5\times4\times3\times2\times1}C_0$$

$$\rightarrow C_{2n}=\frac{(-L+2n-2)(-L+2n-4)\cdots(-L)\cdot(L+1)\cdot\cdots\cdot(L+2n-1)}{(2n)!}C_0$$

Step 2-2: Compute C_{2n+1}

$$\rightarrow C_{2n+1}=\frac{(-L+2n-1)(-L+2n-3)\cdots(1-L)\cdot(L+2)\cdots(L+2n)}{(2n+1)!}C_1$$

L17-20 ODE with constant coefficients — Expansion form

1. Homogeneous Second-order ODE with constant coefficients

① Idea

When the ODE is: $ay''+by'+cy=0$ (a, b, c是常数),

then $y(t)=\exp(rt)$ is a solution.

Then $(ar^2+br+c)\exp(rt)=0$, which means $\underline{ar^2+br+c=0}$

Characteristic equation of the ODE

② Different situations

For $ar^2+br+c=0$, $r=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

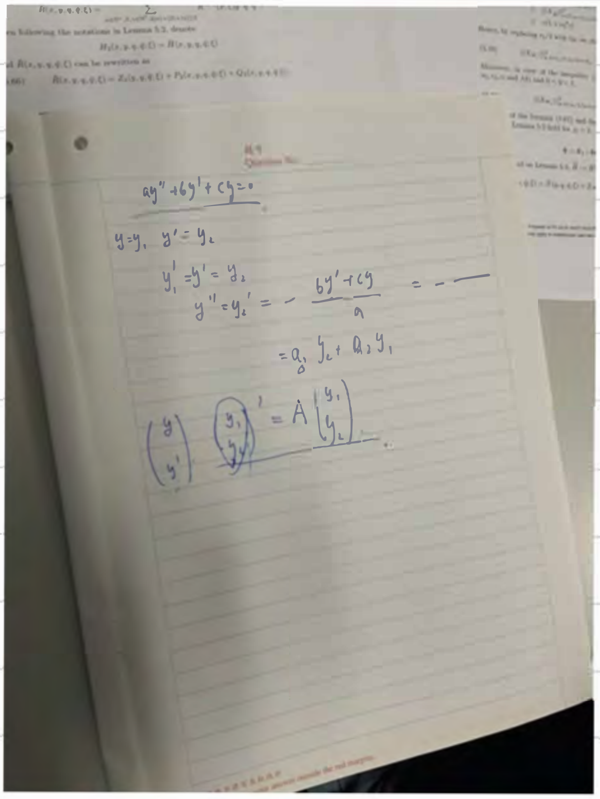
$$\begin{cases} \text{if } b^2-4ac>0, \text{ there are two extinct real roots } r_1, r_2 \\ \text{if } b^2-4ac=0, \text{ there is a repeated real root } r \\ \text{if } b^2-4ac<0, \text{ there are two complex roots } \underline{r_1, \bar{r}_1} \end{cases}$$

(Complex conjugate pairs $\begin{cases} r_1=a+bi \\ \bar{r}_1=a-bi \end{cases}$)

③ Case 1: Two Distinct Real Roots

a. General solution

$\begin{cases} y_1(t)=\exp(r_1t) \\ y_2(t)=\exp(r_2t) \end{cases}$, $\boxed{\text{then "}y(t)=c_1\exp(r_1t)+c_2\exp(r_2t)\text{" is the general solution}}$



proof: $y_1(t)$ & $y_2(t)$ are linearly independent.

Method 1: Prove when $a_1y_1(t)+a_2y_2(t)=0$, $a_1=a_2=0$

$$\begin{cases} a_1\exp(r_1t_1)+a_2\exp(r_2t_1)=0 \\ a_1\exp(r_1t_2)+a_2\exp(r_2t_2)=0 \end{cases}$$

Let $\exp(r_1t_2)\cdot a_1\exp(r_1t_1)=\exp(r_1t_1)a_1\exp(r_1t_2)$

then $a_2[\exp(r_1t_2+r_2t_1)-\exp(r_2t_2+r_1t_1)]=0$

since $r_1\neq r_2$, then $\exp(r_1t_2+r_2t_1)-\exp(r_2t_2+r_1t_1)\neq 0$

then $\begin{cases} a_1=0 \\ a_2=0 \end{cases}$, then $y_1(t)$ & $y_2(t)$ are linearly independent

Method 2: Check Wronsky Det

$$W(y_1,y_2)=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}=\begin{vmatrix} \exp(r_1t) & \exp(r_2t) \\ r_1\exp(r_1t) & r_2\exp(r_2t) \end{vmatrix}\neq 0$$

then $y_1(t)$ & $y_2(t)$ are linearly independent 为何取 y_1', y_2'

b. General solution of IVP

Initial conditions $\longrightarrow$ Specify c_1 & c_2

Example

Find the general solution to the ODE $y''+9y'+20y=0$

The character equation is $r^2+9r+20=0$

then $(r+4)(r+5)=0$, $\begin{cases} r_1=-4 \\ r_2=-5 \end{cases}$

then $y(t)=c_1\exp(-4t)+c_2\exp(-5t)$

Example

Find the general solution to the ODE $y''-y'-42y=0$

The character equation is $r^2-r-42=0$

then $(r-7)(r+6)=0$, $\begin{cases} r_1=7 \\ r_2=-6 \end{cases}$

then $y(t)=c_1\exp(7t)+c_2\exp(-6t)$

④ Case 2: Complex roots

a. General solution (Complex-valued version)

$$\begin{cases} r_1 = a + bi \\ r_2 = a - bi \end{cases} \longrightarrow \begin{cases} y_1(t) = \exp((a+bi)t) \\ y_2(t) = \exp((a-bi)t) \end{cases}$$

Applying Euler's formula $\begin{cases} \exp(ix) = \cos(x) + i\sin(x) \\ \exp(-ix) = \cos(x) - i\sin(x) \end{cases}$

then $\begin{cases} y_1(t) = \exp(at)\cdot(\cos(bt) + i\sin(bt)) \\ y_2(t) = \exp(at)\cdot(\cos(bt) - i\sin(bt)) \end{cases}$ 也是共轭复数,

then the general solution is "$y(t) = c_1\exp(at)\cdot(\cos(bt)+i\sin(bt)) + c_2\exp(at)\cdot(\cos(bt) - i\sin(bt))$

$= \underline{\exp(at)\left[(c_1+c_2)\cos(bt) + i(c_1-c_2)\sin(bt)\right]}$"

b. General solution (Real-valued version)

△ Theorem

For $y'' + p(t)y' + q(t)y = 0$, p, q are cont,

if $y(t) = u(t) + iv(t)$ is a solution,

then $\begin{cases} u(t) \\ v(t) \end{cases}$ are also solutions.

Let $\begin{cases} C_1 = c_1 + c_2 \\ C_2 = c_1 - c_2 \end{cases}$

△ General solution (Real-valued version)

Since $\underline{y(t) = C_1\exp^{at}\cos(bt) + iC_2\exp^{at}\sin(bt)}$ is a solution,

then $\begin{cases} u(t) = \exp^{at}\cos(bt) \\ v(t) = \exp^{at}\sin(bt) \end{cases}$

then "$y(t) = c_1 e^{at}\cos(bt) + c_2 e^{at}\sin(bt)$" is the general solution.

Example

Solve the IVP: $\begin{cases} y'' + y' + 9.25y = 0 \\ y(0) = 2,\ y'(0) = 8 \end{cases}$

The characteristic equation is $r^2 + r + 9.25 = 0$, then $(r+\frac{1}{2})^2 + 9 = 0$, then $(r+\frac{1}{2})^2 = -9$

then $\begin{cases} r_1 = -\frac{1}{2} + 3i \\ r_2 = -\frac{1}{2} - 3i \end{cases}$

then the general solution is "$y(t) = c_1 e^{-\frac{1}{2}t}\cos(3t) + c_2 e^{-\frac{1}{2}t}\sin(3t)$"

⑤ Case 3: One repeated real root

a. General solution

$y_1(t) = y_2(t) = e^{-\frac{b}{2a}t}$ $\longrightarrow$ $\begin{cases} y_1(t) = e^{-\frac{b}{2a}t} \\ y_2(t) = te^{-\frac{b}{2a}t} \end{cases}$

then the general solution is " $y(t) = C_1 e^{-\frac{b}{2a}t} + C_2 t e^{-\frac{b}{2a}t}$ " $\left(W(y_1, y_2)(t) = W(y_1, y_2)(t_0) \exp \int_{t_0}^{t} tr(A(x))dx\right)$

b. Method 1: Wronsky Det

Method 2: Reduction of Order : $y_2(t) = y_1 \int \frac{1}{y_1^2} \exp\left(-\int p\,dt\right)dt$, where $p = \frac{b}{a}$.

2. Non-Homo Second-order ODE with constant coefficients

① Form of equation

$ay'' + by' + cy = r(t)$

② Method: Method of undetermined coefficients

a. Idea

General Sol of Non-Homo = General Sol of Homo + Special Sol of Non-Homo

$= C_1 y_1 + C_2 y_2 + y^*$

(y_1 & y_2 here are ind)

y^* here can be guessed

b. Process

Step 1: Compute the General Sol of Homo by above method

Step 2: Compute y^* based on the form of $r(t)$

c. Guessing

△ Basic type

- Exponential form: $r(t) = e^{at}$ →
  - $r(t)$ is not a multiple of y_1 or y_2 → Try $y^* = Ae^{at}$
  - $r(t)$ is a multiple of y_1 or y_2 → Try $y^* = Ate^{at}$
  - $r(t)$ is a multiple of both y_1 and y_2 → Try $y^* = (At^2)e^{at}$
- Polynomial form: $r(t) = P_n(t)$ → Try $y^* = t^s Q_n(t)$, where $s = \begin{cases} 0 & \text{if } c \neq 0 \\ 1 & \text{if } c = 0,\ b \neq 0 \\ 2 & \text{if } b = c = 0 \end{cases}$

  对应 最高次数 ✗ / 有个次数项 ✓
- Trigonometric form: $r(t)$ = Combination of $\sin(\omega t)$ & $\cos(\omega t)$ → Try $y^* = A\sin(\omega t) + B\cos(\omega t)$

③ Addition

$$r(t) = + \left(\begin{array}{ccc} r_1(t) & \longrightarrow & \text{Try } y^*_1 \\ \cdots & & \cdots \\ r_m(t) & \longrightarrow & \text{Try } y^*_m \end{array} \right) \longrightarrow \text{Try } y^* = y^*_1 + \cdots + y^*_m$$

③ Product

- Poly · Exp : $r(t) = P_n(t) \cdot e^{\alpha t} \longrightarrow$ Try $y^* = t^s Q_n(t) e^{\alpha t}$, where $s = \begin{cases} 0 & \text{if } \alpha \neq r_1, \alpha \neq r_2 \\ 1 & \text{if } \alpha = r_1, \alpha \neq r_2 \\ 2 & \text{if } r_1 = r_2 = \alpha \end{cases}$

  where r_1 & r_2 are the roots to the characteristic equation $ar^2 + br + c = 0$

- Poly · Exp · Trig : $r(t) = P_n(t) e^{\alpha t} \sin(\beta t) / P_n(t) e^{\alpha t} \cos(\beta t) \longrightarrow$ Try $y^* = t^s e^{\alpha t} (Q_n(t) \cos(\beta t) + R_n(t) \sin(\beta t))$,

  where $s = \begin{cases} 0 & \text{if } r_1 \neq \alpha + i\beta \text{ or } r_2 \neq \alpha + i\beta \\ 1 & \text{if } r_1 = \alpha + i\beta \text{ (and thus } r_2 = \alpha - i\beta) \end{cases}$

3. Homogeneous High-order ODE with constant coefficients

① Form of equation

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + a_{n-2} y^{(n-2)} + \cdots + a_1 y' + a_0 y = 0$$

② Method

Decompose the characteristic equation, making $CE = \underline{a_n (r-r_1)(r-r_2)\cdots(r-r_n)}$

确保每一项都是一次项，如 $(r-1)^2 \to (r-1)(r-1)$

Note: $(a-b)^3 = a^3 + (-b)^3 + 3a^2(-b) + 3a(-b)^2$

- $r_i \to y_i = e^{r_i t}$
- 重根 → 根据重的次数，在 $e^{r_i t}$ 前乘 t 乘到不重为止
- 复数根 $r_i = \begin{cases} a+bi \\ a-bi \end{cases} \to y_i = \begin{cases} e^{at}\cdot\cos(bt) \\ e^{at}\cdot\sin(bt) \end{cases}$ 也适用

Example

Find y_i for $(r^2+1)(r-1)^2(r+2) = 0$

$(r^2+1)(r-1)(r-1)(r+2) = 0$

$\to r_1 = i,\ r_2 = -i,\ r_3 = 1,\ r_4 = 1,\ r_5 = -2$

$\to y_1 = e^0 \cos t,\ y_2 = e^0 \sin t,\ y_3 = e^t,\ y_4 = te^t,\ y_5 = e^{-2t}$

$\quad = \cos t \qquad = \sin t$

Example

Find y_i for $(r^2+1)^2(r-1)^3 = 0$

$(r^2+1)(r^2+1)(r-1)(r-1)(r-1) = 0$

$r_1 = i,\ r_2 = -i,\ r_3 = i,\ r_4 = -i,\ r_5 = 1,\ r_6 = 1,\ r_7 = 1$

$y_1 = \cos t,\ y_2 = \sin t,\ y_3 = t\cos t,\ y_4 = t\sin t,\ y_5 = e^t,\ y_6 = te^t,\ y_7 = t^2e^t$

4. Non-homogeneous High-order ODE with constant coefficients

① Method

Find y_{gen}, $y^* \longrightarrow$ Find y

② Find y^*

$$\begin{cases} r(t) = e^{\alpha t} P_m(t) \longrightarrow y^* = t^s e^{\alpha t} Q_m(t) \\ r(t) = e^{\alpha t} P_m(t) \sin(\beta t) \,/\, e^{\alpha t} P_m(t) \cos(\beta t) \longrightarrow y^* = t^s e^{\alpha t} [Q_m(t) \sin(\beta t) + R_m(t) \cos(\beta t)] \end{cases}$$

最高次与$P_m(t)$相同的多项式

t^s 由重了n个决定.

Example

Solve $y''' - 3y'' + 3y' - y = 4e^t$

$r^3 - 3r^2 + 3r - 1 = 0$, $(r-1)^3 = 0$, $(r-1)(r-1)(r-1) = 0$

$r_1 = 1$, $r_2 = 1$, $r_3 = 1$

$y_1 = e^t$, $y_2 = te^t$, $y_3 = t^2 e^t$

$y^* = At^3 e^t$.

then
$$\begin{cases} y^{*\prime} = 3At^2e^t + At^3e^t = Ae^t(3t^2 + t^3) \\ y^{*\prime\prime} = Ae^t(3t^2 + t^3 + 6t + 3t^2) = Ae^t(t^3 + 6t^2 + 6t) \\ y^{*\prime\prime\prime} = Ae^t(t^3 + 6t^2 + 6t + 3t^2 + 12t + 6) = Ae^t(t^3 + 9t^2 + 18t + 6) \end{cases}$$

then $Ae^t(t^3 + 9t^2 + 18t + 6 - 3t^3 - 18t^2 - 18t + 9t^2 + 3t^3 - t^3) = 4e^t$

then $6Ae^t = 4e^t$, $A = \frac{2}{3}$

then $y = C_1 e^t + C_2 te^t + C_3 t^2 e^t + \frac{2}{3} t^3 e^t$

Example

Solve $y^{(4)}+2y''+y=3\sin t$.

$r^4+2r^2+1=0$, $(r^2+1)^2=0$, $(r^2+1)(r^2+1)=0$,

$r_1=i$, $r_2=-i$, $r_3=i$, $r_4=-i$

$y_1=\cos t$, $y_2=\sin t$, $y_3=t\cos t$, $y_4=t\sin t$.

$y^*=t^2(A\sin t+B\cos t)$

$y^{*\prime}=t^2(A\cos t-B\sin t)+2t(A\sin t+B\cos t)$

$y^{*\prime\prime}=t^2(-A\sin t-B\cos t)+2t(A\cos t-B\sin t)+2(A\sin t+B\cos t)+2t(A\cos t-B\sin t)$

$\quad=\sin t(-At^2-2Bt+2A-2Bt)+\cos t(-Bt^2+2At+2B+2At)$

$y^{*\prime\prime\prime}=\sin t(-2At-2B-2B)+\cos t(-At^2-2Bt+2A-2Bt)+\cos t(-2Bt+2A+2A)-\sin t(-Bt^2+4At+2B)$

$\quad=\sin t(-2At-4B+Bt^2-4At-2B)+\cos t(-At^2-4Bt+2A-2Bt+4A)$

$\quad=\sin t(Bt^2-6At-6B)+\cos t(-At^2-6Bt+6A)$

$y^{*(4)}=\sin t(2Bt-6A)+\cos t(Bt^2-6At-6B)+\cos t(-2At-6B)-\sin t(-At^2-6Bt+6A)$

$\quad=\sin t(2Bt-6A+At^2+6Bt-6A)+\cos t(Bt^2-6At-6B-2At-6B)$

$\quad=\sin t(At^2+8Bt-12A)+\cos t(Bt^2-8At-12B)$

Then $\sin t(At^2+8Bt-12A-2At^2-4Bt+4A-4Bt+At^2)$

$\quad+\cos t(Bt^2-8At-12B-2Bt^2+8At+4B+Bt^2)=3\sin t$,

then $\begin{cases}-8A\sin t=3\sin t\\-8B\cos t=0\end{cases}\rightarrow\begin{cases}A=-\frac{3}{8}\\B=0\end{cases}$

$\rightarrow y^*=t^2(-\frac{3}{8})\sin t$

$\rightarrow y=C_1\cos t+C_2\sin t+C_3t\cos t+C_4t\sin t-\frac{3}{8}t^2\sin t$

L21–22 ODE with constant coefficients — Matrix form

1. Homogeneous High-order ODE with constant coefficients

① Form of equation

$$\frac{d\vec{x}}{dt} = \underline{A}\vec{x}$$

② Method

a. Original expression

$\vec{x}(t) = \underline{X}(t)C$

$\underline{X}(t) = e^{\underline{A}t} \longrightarrow \vec{x}(t) = e^{\underline{A}t}C$

called "Matrix exponential"

For Non-Homo ODE with Constant coefficients,

$$\vec{x}(t) = \underline{X}(t)C + \underline{X}(t)\int_{t_0}^{t} \underline{X}^{-1}(s)\vec{f}(s)ds$$
$$= e^{\underline{A}t}C + \int_{t_0}^{t} e^{\underline{A}(t-s)}\vec{f}(s)ds$$

b. Simpler expression (by elementary functions)

△ Lemma 1: Jordan decomposition for square matrix

For $\underline{A} \in M_n$, $\exists$ an invertible matrix P, s.t.

$P^{-1}\underline{A}P = \underline{J} = \mathrm{diag}(\underline{J}_1, \cdots, \underline{J}_s)_{n\times n}$, $n_1 + \cdots + n_s = n$; $\longrightarrow \underline{A} = \underline{P}\,\underline{J}\,\underline{P}^{-1}$

$$\underline{J}_i = \begin{bmatrix} \boxed{\lambda_i} & 1 & & 0 \\ & \lambda_i & 1 & \\ & & \lambda_i & \ddots \\ & & & \ddots & 1 \\ 0 & & & & \lambda_i \end{bmatrix}_{n_i \times \boxed{n_i}}$$

△ Lemma 2

$e^{\underline{A}t} = e^{\underline{P}t\underline{J}\underline{P}^{-1}} = \underline{P}e^{t\underline{J}}\underline{P}^{-1}$

$\rightarrow e^{\underline{A}t}\cdot\underline{P} = \underline{P}e^{t\underline{J}}$

$\left(\begin{array}{l} e^{\underline{A}t} \\ e^{\underline{A}t}\cdot\underline{P} \end{array} \right.$ 都是 Basic solution $\rightarrow \underline{P}e^{t\underline{J}}$ 也是 Basic solution

$\rightarrow$ Rather than $e^{\underline{A}t}\cdot\underline{P}$, we investigate $\underline{P}e^{t\underline{J}}$.

For every column of $\underline{P}e^{t\underline{J}}$,

$$\vec{x}(t) = e^{\lambda_i t}\left[C_0 + tC_1 + \cdots + \frac{t^{n_i-1}}{(n_i-1)!}C_{n_i-1}\right]$$

☆ Result

(i) $\bar{A}$ has n different evalues

Suppose $\bar{A}$ has n different evalues, $\lambda_1, \dots, \lambda_n$, then $\underline{X}(t) = (e^{\lambda_1 t}\vec{v_1}, \dots, e^{\lambda_n t}\vec{v_n})$,

where $\vec{v_i}$ is the evector of evalue λ_i.

then $\vec{x}(t) = \underline{X}(t)C = C_1 e^{\lambda_1 t}\vec{v_1} + \cdots + C_n e^{\lambda_n t}\vec{v_n}$

☆注意: evectors 非零!

If some $\vec{v_i}$ are complex, then there are two ways to get the real solutions.

Way 1:

$\underline{X}(t) = \underline{X}(t) \cdot \underline{X}^{-1}(0) \rightarrow \vec{x}(t) = \underline{X}(t) \cdot C$

$X(0) = \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \Rightarrow \boxed{X^{-1}(0)} = \frac{1}{|X(0)|}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}$

可以先代0,后求逆.

Way 2: ✓

$\vec{x_i} = e^{\lambda_i t}\vec{v_i} = u + v i$

$\rightarrow \vec{x}(t) = C_1\vec{u}(t) + C_2\vec{v}(t)$

(ii) $\bar{A}$ has repeated evalues

$$\begin{cases} \text{非重根部分}: e^{\lambda_i t}\vec{v_i} \\ \text{重根部分}: e^{\lambda_i t}\left(\vec{v_{i0}} + \frac{t}{1!}\vec{v_{i1}} + \frac{t^2}{2!}\vec{v_{i2}} + \cdots\right) \end{cases}$$

如 $M=2$, $\begin{cases} e^{\lambda_1 t}\left(\vec{v_{10}} + \frac{t}{1!}\vec{v_{11}}\right) \\ e^{\lambda_2 t}\left(\vec{v_{20}} + \frac{t}{1!}\vec{v_{21}}\right) \end{cases}$ $\lambda_1 = \lambda_2$ $\vec{v_{11}} = (A - \lambda_1 I)\vec{v_{10}}$.

$M=3$ $\begin{cases} e^{\lambda_1 t}\left(\vec{v_{10}} + \frac{t}{1!}\vec{v_{11}} + \frac{t^2}{2!}\vec{v_{12}}\right) \\ e^{\lambda_2 t}\left(\vec{v_{20}} + \frac{t}{1!}\vec{v_{21}} + \frac{t^2}{2!}\vec{v_{22}}\right) \\ e^{\lambda_3 t}\left(\vec{v_{30}} + \frac{t}{1!}\vec{v_{31}} + \frac{t^2}{2!}\vec{v_{32}}\right) \end{cases}$ $\lambda_1 = \lambda_2 = \lambda_3$

Exercise

Solve the equation

$$\frac{d\vec{x}}{dt} = \begin{bmatrix} 1 & 1 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \vec{x}$$

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

解λ的方法：直接因式分解；先猜出若干根，再分解。

$$\det(A-\lambda I) = \det \begin{bmatrix} 1-\lambda & 1 & 0 \\ 4 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{bmatrix} = (1-\lambda)(1-\lambda)(1-\lambda) - 4(1-\lambda) = 0$$

$$\rightarrow (1-\lambda)\left[(1-\lambda)^2 - 4\right] = (1-\lambda)(1-\lambda+2)(1-\lambda-2) = 0$$

$$\rightarrow \lambda_1 = 1,\ \lambda_2 = 3,\ \lambda_3 = -1$$

For λ_1, $(A-\lambda_1 I)\vec{v_1} = \begin{bmatrix} 0 & 1 & 0 \\ 4 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} b \\ 4a \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow \vec{v_1} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

For λ_2, $(A-\lambda_2 I)\vec{v_2} = \begin{bmatrix} -2 & 1 & 0 \\ 4 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -2a+b \\ 4a-2b \\ -2c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow \vec{v_2} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

For λ_3, $(A-\lambda_3 I)\vec{v_3} = \begin{bmatrix} 2 & 1 & 0 \\ 4 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 2a+b \\ 4a+2b \\ 2c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow \vec{v_3} = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$

$$\rightarrow \vec{X}(t) = \bar{X}(t)\cdot C = c_1 e^{t} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + c_3 e^{-t} \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$$

Exercise

Solve: $\frac{d\vec{x}}{dt} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \vec{x}$

$\det(A-\lambda I) = \begin{bmatrix} 1-\lambda & 1 \\ -1 & 1-\lambda \end{bmatrix} = (1-\lambda)^2+1=0 \rightarrow (1-\lambda)^2 = -1$

$\rightarrow \lambda_1 = 1+i,\ \lambda_2 = 1-i.$

For λ_1, $(A-\lambda_1 I)\vec{v_1} = 0$, $\begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -ai+b \\ -a-bi \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \vec{v_1} = \begin{bmatrix} i \\ -1 \end{bmatrix}$

For λ_2, $(A-\lambda_2 I)\vec{v_2} = 0$, $\begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} ai+b \\ -a+bi \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \vec{v_2} = \begin{bmatrix} i \\ 1 \end{bmatrix}$

$\vec{x_1} = e^{(1+i)t} \begin{bmatrix} i \\ -1 \end{bmatrix} = e^t \begin{bmatrix} e^{it} i \\ -e^{it} \end{bmatrix} = e^t \begin{bmatrix} \cos t\, i - \sin t \\ -\cos t - i\sin t \end{bmatrix} = \left(e^t \begin{bmatrix} -\sin t \\ -\cos t \end{bmatrix}\right) + \left(e^t \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix}\right) i$

Euler formula: $e^{it} = \cos t + i\sin t$

$\rightarrow \begin{cases} u = e^t \begin{bmatrix} -\sin t \\ -\cos t \end{bmatrix} \\ v = e^t \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} \end{cases} \rightarrow \vec{x}(t) = C_1 e^t \begin{bmatrix} -\sin t \\ -\cos t \end{bmatrix} + C_2 e^t \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix}$

Exercise

Solve the equation: $\frac{d\vec{x}}{dt} = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} x$

$\det(A-\lambda I) = \begin{bmatrix} -\lambda & 1 & -1 \\ 1 & 1-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{bmatrix} = (-\lambda)(1-\lambda)^2 + (1-\lambda) - (1-\lambda) = (-\lambda)(1-\lambda)^2 = 0$

$\rightarrow \lambda_1 = 0,\ \lambda_2 = \lambda_3 = 1$

For $\lambda_1 = 0$,

$(A-\lambda_1 I)\vec{v_1} = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} b-c \\ a+b \\ a+c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow \vec{v_1} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$

For $\lambda_2 = \lambda_3 = 1$

$(A-\lambda_2 I)^2 \vec{v_2} = \begin{bmatrix} -1 & 1 & -1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1 & 1 & -1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a-b+c \\ -a+b-c \\ -a+b-c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\rightarrow \vec{v_{20}} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix},\ \vec{v_{30}} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$\rightarrow \begin{cases} \vec{v_{21}} = (A-\lambda_2 I)\vec{v_{20}} = \begin{bmatrix} -1 & 1 & -1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \\ \vec{v_{31}} = (A-\lambda_2 I)\vec{v_{30}} = \begin{bmatrix} -1 & 1 & -1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix} \end{cases}$

$\rightarrow \underline{X}(t) = [e^{\lambda_1 t}\vec{v_1},\ e^{\lambda_2 t}(\vec{v_{20}} + \frac{t}{1!}\vec{v_{21}}),\ e^{\lambda_2 t}(\vec{v_{30}} + \frac{t}{1!}\vec{v_{31}})]$

$= \begin{bmatrix} 1 & e^t & -e^t \\ -1 & e^t(1+t) & e^t(-t) \\ -1 & e^t t & e^t(1-t) \end{bmatrix}$

$\rightarrow \vec{x}(t) = c_1 \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} e^t \\ e^t(1+t) \\ e^t t \end{bmatrix} + c_3 \begin{bmatrix} -e^t \\ e^t(-t) \\ e^t(1-t) \end{bmatrix}$

Exercise

$$\frac{d\vec{x}}{dt} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 2 \end{bmatrix} \vec{x}$$

$$\det(A-\lambda I) = \begin{vmatrix} 1-\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 0 & -1 & 2-\lambda \end{vmatrix} = (1-\lambda)(-\lambda)(2-\lambda) + (1-\lambda) = (1-\lambda)(-2\lambda+\lambda^2+1) = (1-\lambda)^3 = 0$$

$\rightarrow \lambda_1 = \lambda_2 = \lambda_3 = 1$

For $\lambda_1 = \lambda_2 = \lambda_3 = 1$

$$(A-\lambda_1 I)^3 v_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\rightarrow \vec{v_{10}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{v_{20}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{v_{30}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\rightarrow \begin{cases} \vec{v_{11}} = (A-\lambda_1 I)\vec{v_{10}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \\ \vec{v_{12}} = (A-\lambda_1 I)\vec{v_{11}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \end{cases}$$

$$\begin{cases} \vec{v_{21}} = (A-\lambda_1 I)\vec{v_{20}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \\ \vec{v_{22}} = (A-\lambda_1 I)\vec{v_{21}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \end{cases}$$

$$\begin{cases} \vec{v_{31}} = (A-\lambda_1 I)\vec{v_{30}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \\ \vec{v_{32}} = (A-\lambda_1 I)\vec{v_{31}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \end{cases}$$

$$\rightarrow \underline{X}(t) = \left[e^{\lambda_1 t}\left(\vec{v_{10}} + \frac{t}{1!}\vec{v_{11}} + \frac{t^2}{2!}\vec{v_{12}}\right), e^{\lambda_1 t}\left(\vec{v_{20}} + \frac{t}{1!}\vec{v_{21}} + \frac{t^2}{2!}\vec{v_{22}}\right), e^{\lambda_1 t}\left(\vec{v_{30}} + \frac{t}{1!}\vec{v_{31}} + \frac{t^2}{2!}\vec{v_{32}}\right)\right]$$

$$= \begin{bmatrix} e^t & (-\frac{t^2}{2}+t)e^t & (\frac{t^2}{2})e^t \\ 0 & (1-t)e^t & te^t \\ 0 & (-t)e^t & (t+1)e^t \end{bmatrix}$$

$$\rightarrow \vec{x}(t) = c_1\begin{bmatrix} e^t \\ 0 \\ 0 \end{bmatrix} + c_2\begin{bmatrix} (-\frac{t^2}{2}+t)e^t \\ (1-t)e^t \\ (-t)e^t \end{bmatrix} + c_3\begin{bmatrix} (\frac{t^2}{2})e^t \\ te^t \\ (t+1)e^t \end{bmatrix}$$

L23-24 ODE with constant coefficients — Discontinuous part

1. Laplace Transform

① Motivation

$\underline{a y''(t) + b y'(t) + c y(t)} = \underline{F(t)}$

Const-coefficient ODE　　　　Discontinuous function

(无论n阶)

⟶ Laplace method.

② Improper integral

a. Improper integral

Type 1 : $\int_a^{+\infty} f(x)\,dx$ (integral interval 出现 ∞)

Type 2 : $\int_a^b f(x)\,dx$ (integral interval 边界不连续)

b. Type 1 improper integral

$$\int_a^{+\infty} f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx$$

e.g.

$$\int_0^\infty e^{ct}\,dt = \lim_{A\to\infty} \int_0^A e^{ct}\,dt = \lim_{A\to\infty} \left(\frac{e^{ct}}{c}\right)\Big|_0^A = \lim_{A\to\infty} \left(\frac{e^{cA}}{c} - \frac{1}{c}\right)$$

对 $c>0$, diverge

对 $c=0$, $=1$

对 $c<0$, $= -\frac{1}{c}$

③ Definition of Laplace transform

对 $f(t)$ is real defined on $[0, +\infty)$

$f(t)$ is piecewise continuous

$|f(t)| \le Ke^{at}$ ⟶ 使得 $e^{-st} f(t)$ 收敛.

then the Laplace transform $\mathcal{L}\{f(t)\}(s) = F(s)$ defined as:

$$\mathcal{L}\{f(t)\}(s) = F(s) = \int_0^\infty e^{-st} f(t)\,dt$$ $\underline{\text{exists for } s>a}$ 收敛

$s>a \to -s+a<0 \to$ 收敛

本质: 求一函数的某种收敛反常积分

Exercise

Find the Laplace transform for $f(t) = 1,\ t \geq 0$

$$\mathcal{L}\{f(t)\}(s) = F(s) = \int_0^\infty e^{-st}\,1\,dt = \int_0^\infty e^{-st}\,dt = \lim_{A\to\infty}\int_0^A e^{-st}\,dt = \lim_{A\to\infty}\left(\frac{e^{-st}}{-s}\right)\Big|_0^A = \lim_{A\to\infty}\left(\frac{e^{-sA}}{-s} + \frac{1}{s}\right) = \frac{1}{s}$$

Exercise

Find the Laplace transform for $f(t) = e^{at},\ t \geq 0$

$$\mathcal{L}\{f(t)\}(s) = F(s) = \int_0^\infty e^{-st} e^{at}\,dt = \int_0^\infty e^{(a-s)t}\,dt = \lim_{A\to\infty}\int_0^A e^{(a-s)t}\,dt = \lim_{A\to\infty}\left(\frac{e^{(a-s)t}}{a-s}\right)\Big|_0^A$$

$$= \lim_{A\to\infty}\left(\frac{e^{(a-s)A}}{a-s} - \frac{1}{a-s}\right)$$

Since $s > a$,

then $= -\frac{1}{a-s} = \frac{1}{s-a}$

Exercise

Find the Laplace transform for $f(t) = \sin\alpha t,\ t \geq 0$

$$\mathcal{L}\{f(t)\}(s) = F(s) = \int_0^\infty e^{-st}\sin\alpha t\,dt = \lim_{A\to\infty}\int_0^A e^{-st}\sin\alpha t\,dt$$

$$\int e^{-st}\sin\alpha t\,dt = \left(-\frac{1}{s}\right)\int \sin\alpha t\,de^{-st} = \left(-\frac{1}{s}\right)\left[\sin\alpha t\,e^{-st} - \int e^{-st}\,d\sin\alpha t\right] = -\frac{1}{s}(\sin\alpha t)e^{-st} + \frac{\alpha}{s}\int e^{-st}\cos\alpha t\,dt$$

$$= -\frac{1}{s}(\sin\alpha t)e^{-st} - \frac{\alpha}{s^2}\left(\int \cos\alpha t\,de^{-st}\right)$$

$$= -\frac{1}{s}(\sin\alpha t)e^{-st} - \frac{\alpha}{s^2}\left(\cos\alpha t(e^{-st}) - \int e^{-st}\,d\cos\alpha t\right)$$

$$= -\frac{1}{s}(\sin\alpha t)e^{-st} - \frac{\alpha}{s^2}(\cos\alpha t)(e^{-st}) - \frac{\alpha^2}{s^2}\int e^{-st}\sin\alpha t\,dt$$

$$\rightarrow \left(1 + \frac{\alpha^2}{s^2}\right)F(s) = \left[-\frac{1}{s}(\sin\alpha t)(e^{-st}) - \frac{\alpha}{s^2}(\cos\alpha t)(e^{-st})\right]\Big|_0^A = \left[0 + \frac{\alpha}{s^2}\right]$$

$$\frac{s^2+\alpha^2}{s^2}F(s) = \frac{\alpha}{s^2}$$

$$F(s) = \frac{\alpha}{s^2+\alpha^2}$$

④ Definition of Inverse Laplace transform

$\mathcal{L}\{f(t)\}(s) = F(s) = \int_0^\infty e^{-st} f(t)\,dt$

$\longrightarrow \mathcal{L}^{-1}\{F(s)\} = f(t)$

e.g. $\mathcal{L}\{1\} = \frac{1}{s} \longleftrightarrow \mathcal{L}^{-1}\{\frac{1}{s}\} = 1$

$\mathcal{L}\{e^{at}\} = \frac{1}{s-a} \longleftrightarrow \mathcal{L}^{-1}\{\frac{1}{s-a}\} = e^{at}$

$\mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2} \longleftrightarrow \mathcal{L}^{-1}\{\frac{a}{s^2+a^2}\} = \sin at$

input output

Table for Elementary Laplace Transforms

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1. 1	$\frac{1}{s}, s > 0$
2. e^{at}	$\frac{1}{s-a}, s > a$
3. $t^n, n > 0$ integer	$\frac{n!}{s^{n+1}}, s > 0$
4. $t^p, p > -1$	$\frac{\Gamma(p+1)}{s^{p+1}}, s > 0$
5. $\sin at$	$\frac{a}{s^2+a^2}, s > 0$
6. $\cos at$	$\frac{s}{s^2+a^2}, s > 0$
7. $\sinh at$	$\frac{a}{s^2-a^2}, s > \lvert a\rvert$
8. $\cosh at$	$\frac{s}{s^2-a^2}, s > \lvert a\rvert$
9. $e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}, s > a$
10. $e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}, s > a$
11. $t^n e^{at}, n > 0$ integer	$\frac{n!}{(s-a)^{n+1}}, s > a$
12. $u_c(t)$	$\frac{e^{-cs}}{s}, s > 0$
13. $u_c(t)f(t-c)$	$e^{-cs}F(s)$
14. $e^{ct}f(t)$	$F(s-c)$
15. $f(ct)$	$\frac{1}{c}F\left(\frac{s}{c}\right), c > 0$

⑤ Properties

a. Linearity

$\mathcal{L}\{c_1 f_1(t) + c_2 f_2(t)\} = c_1\mathcal{L}\{f_1(t)\} + c_2\mathcal{L}\{f_2(t)\}$. (同样适用于 $\mathcal{L}^{-1}\{\}$)

b. Derivatives

1st-order : $\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$

nth-order : $\mathcal{L}\{f^{(n)}(t)\} = s^n\mathcal{L}\{f(t)\} - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$

c. Shifting

$\mathcal{L}\{e^{ct}f(t)\} = F(s-c)$

$\longrightarrow \mathcal{L}^{-1}\{F(s-c)\} = e^{ct}f(t)$

Exercise

Find the Laplace transform for $f(t) = e^{ct}\sin at,\ t \geq 0$

$\mathcal{L}\{e^{ct}\sin at\} = \mathcal{L}\{e^{ct}f(t)\}$ (where $f(t) = \sin at$) $= F(s-c)$

$= \dfrac{a}{(s-c)^2+a^2}$

Exercise

Find the Inverse transform of $G(s) = \dfrac{1}{s^2-4s+5}$, $s > 2$

$\mathcal{L}^{-1}\left\{\dfrac{1}{s^2-4s+5}\right\} = \mathcal{L}^{-1}\left\{\dfrac{1}{(s-2)^2+1}\right\}$ $\left(\begin{array}{l} c = 2 \\ a = 1 \end{array}\right.$

$= e^{2t}\sin t$

2. Application of Laplace transform

① Process

Consider $ay'' + by' + cy = f(t)$

Step 1: Laplace transform : $a[s^2 Y(s) - sy(0) - y'(0)] + b[sY(s) - y(0)] + cY(s) = F(s)$

Step 2: Solve $Y(s)$: $Y(s) = \frac{(as+b)y(0) + ay'(0)}{as^2+bs+c} + \frac{F(s)}{as^2+bs+c}$

Step 3: Solve $y = \mathcal{L}^{-1}\{Y(s)\}$

Exercise

Find the solution to the IVP:

$y'' - y' - 2y = 0,\ y(0) = 1,\ y'(0) = 0$

$[s^2 Y(s) - sy(0) - y'(0)] - [sY(s) - y(0)] - 2Y(s) = 0$

$(s^2 - s - 2)Y(s) - s + 1 = 0$

$\rightarrow Y(s) = \frac{s-1}{s^2-s-2} = \frac{s-1}{(s-2)(s+1)} = \frac{a}{s-2} + \frac{b}{s+1}$

$a(s+1) + b(s-2) = (a+b)s + (a-2b) = s-1 \rightarrow \begin{cases} a+b=1 \\ a-2b=-1 \end{cases} \rightarrow \begin{cases} a = \frac{1}{3} \\ b = \frac{2}{3} \end{cases}$

$\rightarrow Y(s) = \frac{1}{3} \times \frac{1}{s-2} + \frac{2}{3}\frac{1}{s+1}$

$\rightarrow y = \mathcal{L}^{-1}\{Y(s)\} = \frac{1}{3}\mathcal{L}^{-1}\{\frac{1}{s-2}\} + \frac{2}{3}\mathcal{L}^{-1}\{\frac{1}{s+1}\}$

$= \frac{1}{3}e^{2t} + \frac{2}{3}e^{-t}$

Exercise

Find the solution to the IVP:

$y'' + y = \sin 2t$, $y(0) = 2$, $y'(0) = 1$

$$(s^2Y(s) - sy(0) - y'(0)) + Y(s) = \frac{2}{s^2+4}$$

$$s^2Y(s) - 2s - 1 + Y(s) = \frac{2}{s^2+4}$$

$$Y(s)(s^2+1) = \frac{2}{s^2+4} + 2s + 1$$

$$= \frac{2+2s^3+8s+s^2+4}{s^2+4}$$

$$Y(s) = \frac{2s^3+s^2+8s+6}{(s^2+4)(s^2+1)}$$

Set $Y(s) = \frac{as+b}{s^2+4} + \frac{cs+d}{s^2+1} = \frac{(as+b)(s^2+1)+(cs+d)(s^2+4)}{(s^2+4)(s^2+1)} = \frac{as^3+as+bs^2+b+cs^3+4cs+ds^2+4d}{(s^2+4)(s^2+1)}$

$$\to \left(\begin{array}{l} a+c=2 \\ b+d=1 \\ a+4c=8 \\ b+4d=6 \end{array}\right. \to \left(\begin{array}{l} a=0 \\ b=-\frac{2}{3} \\ c=2 \\ d=\frac{5}{3} \end{array}\right. \to Y(s) = -\frac{2}{3}\frac{1}{s^2+4} + \frac{2s+\frac{5}{3}}{s^2+1} = -\frac{2}{3}\frac{1}{s^2+4} + 2\frac{s}{s^2+1} + \frac{5}{3}\frac{1}{s^2+1}$$

$$\to y = \mathcal{L}^{-1}\{Y(s)\} = -\frac{1}{3}\sin 2t + 2\cos t + \frac{5}{3}\sin t$$

Exercise

Find the solution of the IVP:

$y^{(4)} - y = 0$, $y(0) = 0$, $y'(0) = 1$, $y''(0) = 0$, $y'''(0) = 0$

$(s^4 Y(s) - s^3 y(0) - s^2 y'(0) - s y''(0) - y'''(0)) - Y(s) = 0$

$s^4 Y(s) - s^2 - Y(s) = 0$,

$(s^4 - 1) Y(s) = s^2$

$Y(s) = \frac{s^2}{s^4-1} = \frac{s^2}{(s^2+1)(s^2-1)}$

Let $Y(s) = \frac{as+b}{s^2+1} + \frac{cs+d}{s^2-1}$

then $(as+b)(s^2-1) + (cs+d)(s^2+1) = as^3 - as + bs^2 - b + cs^3 + cs + ds^2 + d$

$\longrightarrow \begin{cases} a+c=0 \\ b+d=1 \\ -a+c=0 \\ -b+d=0 \end{cases} \longrightarrow \begin{cases} a=0 \\ b=\frac{1}{2} \\ c=0 \\ d=\frac{1}{2} \end{cases} \to Y(s) = \frac{1}{2}\frac{1}{s^2+1} + \frac{1}{2}\frac{1}{s^2-1}$

$\to y(t) = \mathcal{L}^{-1}\{Y(s)\} = \frac{1}{2}\sin t + \frac{1}{2}\sinh t$

Exercise

Find the solution of the IVP

$x' = x + 2y$ $\quad x(0) = 0$, $y(0) = 2$

$y' = 2x + y$

$\begin{cases} sX(s) - x(0) = X(s) + 2Y(s) \\ sY(s) - y(0) = 2X(s) + Y(s) \end{cases} \longrightarrow \begin{cases} (s-1)X(s) = 2Y(s) \\ (s-1)Y(s) = 2X(s) + 2 \end{cases} \longrightarrow Y(s) = \frac{s-1}{2} X(s)$

$\longrightarrow (s-1)\frac{s-1}{2} X(s) = 2X(s) + 2$

$\left(\frac{(s-1)^2}{2} - 2\right) X(s) = 2$

$\left(\frac{s^2+1-2s-4}{2}\right) X(s) = 2$

$\begin{cases} X(s) = \frac{4}{s^2-2s-3} = \frac{4}{(s-1)^2-4} = 2\frac{2}{(s-1)^2-4} \\ Y(s) = \frac{2(s-1)}{s^2-2s-3} = \end{cases}$

$x(t) = \mathcal{L}^{-1}\{X(s)\} = 2e^t \sinh(2t)$

$y(t) = \mathcal{L}^{-1}\{Y(s)\} = 2e^t \cosh(2t)$

② Step function → Discont function

a. Definition

$$u_c(t) = \begin{cases} 0, & t < c \\ 1, & t \geq c \end{cases}$$

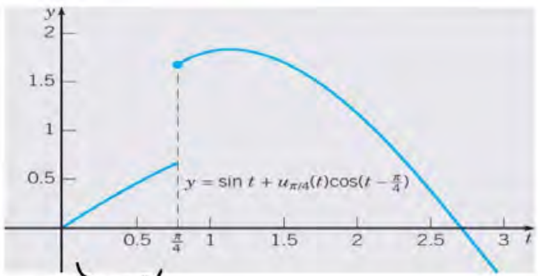
b. Application

if $u_c(t)$ is the step function,

then
$$\begin{cases} \mathcal{L}\{u_c(t) f(t-c)\} = e^{-cs}\mathcal{L}\{f(t)\} = e^{-cs}F(s) \\ \mathcal{L}^{-1}\{e^{-cs}F(s)\} = u_c(t) f(t-c) \end{cases}$$

Exercise

Find $\mathcal{L}\{f(t)\}$, where $f(t) = \begin{cases} \sin t, & 0 \leq t \leq \frac{\pi}{4} \\ \sin t + \cos(t - \frac{\pi}{4}), & t \geq \frac{\pi}{4} \end{cases}$



sin t

$$g(t) = \begin{cases} 0 \\ \cos(t - \frac{\pi}{4}) \end{cases} = u_{\frac{\pi}{4}}(t) \cdot \cos(t - \tfrac{\pi}{4})$$

$\rightarrow f(t) = \sin t + g(t)$

$\rightarrow \mathcal{L}\{f(t)\} = \mathcal{L}\{\sin t\} + \mathcal{L}\{g(t)\}$

$$= \frac{1}{s^2+1} + e^{-\frac{\pi}{4}s}\mathcal{L}\{\cos t\}$$

$$= \frac{1}{s^2+1} + e^{-\frac{\pi}{4}s}\frac{s}{s^2+1}$$

Exercise

Find the inverse Laplace transform of $F(s) = \frac{1-e^{-2s}}{s^2}$

$$\mathcal{L}^{-1}\left\{\frac{1-e^{-2s}}{s^2}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - \mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^2}\right\}$$

$$= t^1 - \mathcal{L}^{-1}\left\{e^{-2s}\frac{1}{s^2}\right\}$$

$$= t - u_2(t)(t-2)$$

Exercise

Find the solution of the IVP:

$2y'' + y' + 2y = g(t)$, $y(0) = 0$, $y'(0) = 0$,

where $g(t) = u_5(t) - u_{20}(t) = \begin{cases} 1, & 5 \le t \le 20, \\ 0, & 0 \le t < 5 \text{ and } t \geq 20 \end{cases}$

$$2(s^2 Y(s) - s y(0) - y'(0)) + (sY(s) - y(0)) + 2Y(s) = \mathcal{L}\{u_5(t)\} - \mathcal{L}\{u_{20}(t)\}$$

$$2s^2 Y(s) + sY(s) + 2Y(s) = (e^{-5s} - e^{-20s})\frac{1}{s}$$

$$Y(s) = \frac{1}{s}\,\frac{e^{-5s} - e^{-20s}}{2s^2 + s + 2}$$

...

L25-27 Stability

1. Stability of Solutions

① phase space & phase portrait

For $\frac{d\vec{x}}{dt} = \vec{f}(\vec{x})$,

- Phase space : Domain of x　　相空间
- Orbit / Trajectory : Solution curve
- Phase portrait : A representative set of orbit / trajectory　　相图

② Equilibrium

a. Definition

The constant solution(s): $\vec{x}(t) \equiv \vec{x_0}$, then $\vec{x_0}$ is called an equilibrium

b. Computation

Let $\vec{f}(\vec{x_0}) = 0$ (Since $\vec{x_0}$ is a constant), then get $\vec{x_0}$.

Exercise

Find the equ to $\frac{dx}{dt} = 1 - x^2$

Let $1 - x^2 = 0$, then equ $= \pm 1$.

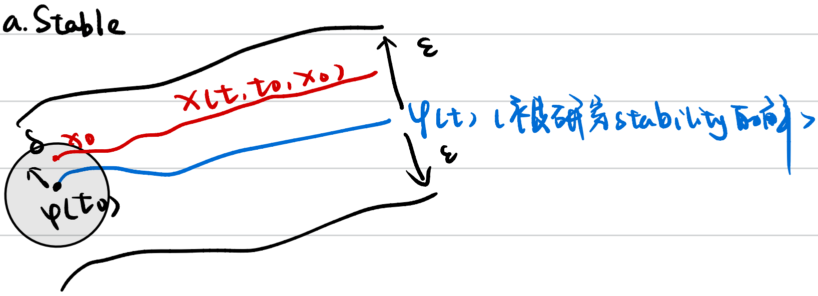
Exercise

Find the equ to $\frac{dx}{dt} = x - y + 1$, $\frac{dy}{dt} = x + y$

$$\begin{cases} x - y + 1 = 0 \\ x + y = 0 \end{cases} \longrightarrow \begin{cases} x = -\frac{1}{2} \\ y = \frac{1}{2} \end{cases}$$ is the equ.

③ Stable

a. Stable



解与另外一个从初始值邻域出发的解在一定的区域内 ⟶ Stable

2. Check the Stability

① Linear approximation method

$$\frac{d\vec{x}}{dt} = \underline{A}\vec{x} + \overrightarrow{R(x)}$$

If
$$\begin{cases} R(0) = 0 \\ \lim\limits_{|x| \to 0} \dfrac{|R(x)|}{|x|} = 0 \end{cases}$$

$$\begin{cases} \text{All evalues of } \underline{A} \text{ satisfy } Re(\lambda_i) < 0 \longrightarrow \underline{\text{Zero sol}} \text{ is asy stable} \\ \exists \text{ one evalue of } \underline{A} \text{ satisfies } Re(\lambda_i) > 0 \longrightarrow \underline{\text{Zero sol}} \text{ is unstable.} \end{cases}$$

其他的向可通过平移zero sol获得.

Exercise

Study the stability of zero sol of
$$\begin{cases} \dfrac{dx}{dt} = -y + x(x^2+y^2-1) \\ \dfrac{dy}{dt} = x + y(x^2+y^2-1) \end{cases}$$

Set $A = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}$, $\overrightarrow{R(x)} = \begin{bmatrix} x(x^2+y^2) \\ y(x^2+y^2) \end{bmatrix}$

$\overrightarrow{R(0)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ & $\lim\limits_{|\vec{x}| \to 0} \dfrac{|\overrightarrow{R(x)}|}{|\vec{x}|} = \dfrac{\sqrt{(x(x^2+y^2))^2 + (y(x^2+y^2))^2}}{\sqrt{x^2+y^2}} = \lim\limits_{|\vec{x}| \to 0} = (x^2+y^2) = 0$

$$\det(A-\lambda I) = \begin{vmatrix} -1-\lambda & -1 \\ 1 & -1-\lambda \end{vmatrix} = (-1-\lambda)^2 + 1 = 1 + \lambda^2 + 2\lambda + 1 = 0 \rightarrow \lambda^2 + 2\lambda + 2 = 0$$

$$\rightarrow (\lambda+1)^2 = -1 \rightarrow \lambda_1 = -1+i$$

$$\lambda_2 = -1-i$$

$Re(\lambda_1) = Re(\lambda_2) = -1 \longrightarrow$ Zero sol is asy stable.