FIN4110: Options and Futures

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L1-2 Introduction

1. Derivatives

① Definition

Derivative = Financial contract whose $\boxed{\text{values}}$ are derived from some other assets

② Types

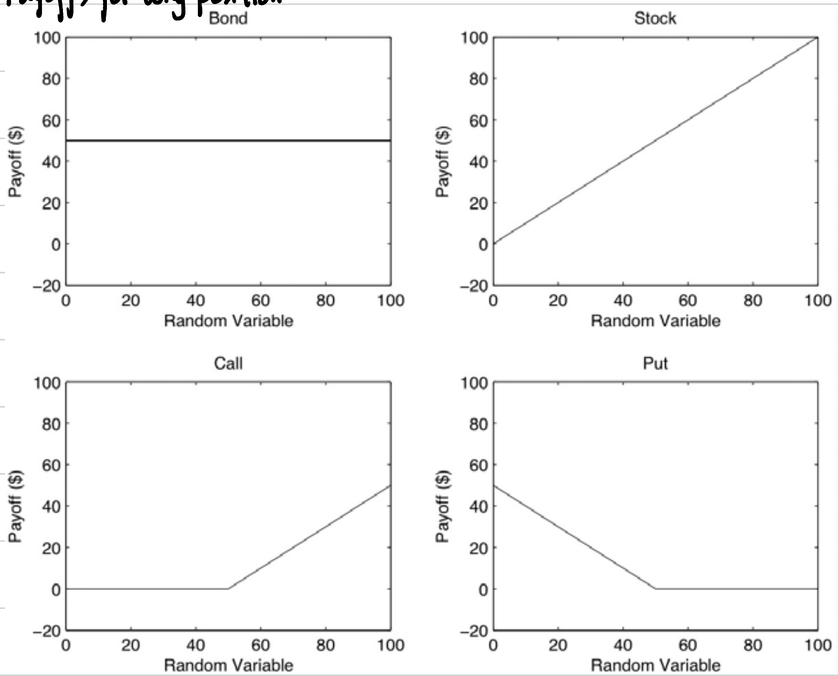
- Futures / Forwards : An agreement to buy or sell an asset at a future date at a price.
- Option
  - Call option : The right to buy an asset on / before (possibly) a future date at a price.
  - Put option : The right to sell an asset on / before (possibly) a future date at a price.

③ Payoff

	Long	Short
Future	$S_T - F_T$	$F_T - S_T$
Call Option	$\max[0, S_T - K]$	$-\max[0, S_T - K]$
Put Option	$\max[0, K - S_T]$	$-\max[0, K - S_T]$

- $F_{0,T}$ Futures / Forwards price
- S_T Future stock price
- K Strike price

Payoffs for long position



Stock price is an RV

④ History

- Future: Used to trade rice in Japan
- Option: Used to trade tulip bulb in Holland
  - Pricing: Black-Scholes & Merton

- Centralized Trading : Stock Exchange — Stock
- Over The Counter (OTC) (Bilateral Trading) — Bond

⑤ Special cases

Swaps: A series of derivatives.

Exotics: Rare derivatives (Like congress election)

⑥ Essence

Transfer future risk to present (Like insurance)

⑦ Examples

a. A tax-exempt institution invests in U.K. stocks

Problem: Dividend is with tax. (Dividend → Investor)

Solution: Give the dividend to a tax-exempt institution, then fixed income is given (Dividend → Tax-exempt Institution → Investor)

b. A CEO has massive exposure to his company's stock price through restricted stock options.

Problem: Legally barred from selling his stocks

Solution: Sell stock index future e.g. Meta is highly correlated with Nasdaq 100 index, then Meta's CEO sells Nasdaq 100 index futures

c. Portfolio manager wants to reduce risk in falling markets

Problem: Sell the stocks proportionally but constant rebalancing is expensive in time and commission (Commission fees)

Solution: Buy long-term put option on portfolio

L3-5 Basics

1. Preliminary Concepts

① Interest rate

$$\begin{array}{lllll} & & \text{1 year} & & \text{T years} \\ \text{Effective annual rate:} & \$1 \rightarrow & \$(1+R)\cdot 1 & \rightarrow & \$(1+R)^T\cdot 1 \\ \text{Continuously compounded rate:} & \$1 \rightarrow & \$e^r & \rightarrow & \$(e^r)^T \end{array}$$

By default

(r is called: annualized continuously compounded rate)

$(1+R) = e^r \longrightarrow R \approx r$

Derivation of e^r

$$CCR = \lim_{n\to\infty}\left(1+\frac{r}{n}\right)^n = \lim_{m\to\infty}\left(1+\frac{1}{m}\right)^{mr} = \left[\lim_{m\to\infty}\left(1+\frac{1}{m}\right)^m\right]^r = e^r$$

② The Bid-Ask Spread

a. Dealer

- Customer: Real trader
- Dealer: Market maker 下场调. Step in and then correct the market price

b. Bid price & Ask price

- Bid price: The price at which a dealer offers to buy
- Ask price: The price at which a dealer offers to sell (= Offer price)

c. The Bid-Ask Spread

The Bid-Ask Spread = Ask price − Bid price (一般大于0)

③ Short sell

a. Process

☆ Absolute level of position = risk
- positive
- negative

Collateral (比stock价值更高, 以降低违约率)

△ Security Lender $\xrightarrow[\text{Lend}]{\text{Stock}}$ Short Seller $\xrightarrow[\text{Sell}]{\text{Stock}}$ Market

△ Security Lender $\xleftarrow[\text{Return}]{\text{Stock}}$ Short Seller $\xleftarrow[\text{Buy}]{\text{Stock}}$ Market

Collateral

包括(div)与(div对应的利息)

b. Objectives

△ Speculation

△ Financing

c. Borrowing rate, Repo rate & Lease rate 由供需决定

Security Lender $\underset{\text{Borrowing rate}}{\overset{\text{Repo rate}}{\rightleftarrows}}$ Short Seller

Repo rate — collateral

Borrowing rate — assets

Fee of borrowing / Lease rate = Borrowing rate − Repo rate Rate = Cost

④ Commission fee 自己承担

2. Forwards and Futures

① Forwards / Forward contract

a. Definition

A forward contract is an agreement between two parties for deferred delivery of an asset.

b. Specifications in the contract

△ The quantity and type of asset

△ Price

△ Delivery time and place

c. Long & Short

Long forwards : Buy forwards
Short forwards : Sell forwards

d. Example

On September 10, two parties enter into a forward contract for the delivery of 100 ounces of gold on December 1 at $450 per ounce.

	Price Increase	Price Decrease
Dec 1 Spot Price	$460	$430
Long Profit	$10	-$20
Short Profit	-$10	$20

Profit = Payoff − Cost (=0 for future; >0 for option)

② Future contract

a. Definition

A future is an exchange-traded forward contract

b. Characteristics affected by "Exchange"

△ Standardized, not tailored to individual needs → increase liquidity (easier to find a counterparty since the # of contracts is limited)

△ Margins are required (Margin: 抵押)

Initial margin

Maintenance margin: If the balance in the margin account falls below a pre-specified amount referred to as the maintenance margin, the trader receives a margin call. < "initial margin" in general

△ The contracts are marked-to-market. This means that gains and loses are realized every day.

c. Example

Suppose a buyer and a seller enter into a futures contract on 7-1 for 5000 bushels of wheat to be delivered on 7-5 at $1/bushel. Suppose the initial margin is $1500 and the maintenance margin is $1200.

			Margin Account			
			Change		Balance	
Date	Transaction	$F_{t,T}$	Buyer	Seller	Buyer	Seller
7-1	Contract initiated. The buyer and the seller deposit the money in their margin accounts.	1.00	1500	1500	1500	1500
7-2	a. Price rises by .10, buyer gains (5000)(.10) and seller loses (5000)(.10). $500 is transferred from the sellers margin account to the buyer's margin account.	1.10	+500	-500	2000	1000
	b. Seller's account balance falls below $1200, so $500 must be deposited into the margin account. If not the Clearing Corporation liquidates the seller's position by buying an offsetting contract with $F_{t,T}$=1.10.			+500		1500
7-3	Price falls by .05, and the buyer loses (.05)(5000) and seller gains (.05)(5000). $250 is transferred from the buyers account to the seller's account.	1.05	-250	+250	1750	1750
7-4	a. Price falls by .15, and the buyer loses (.15)(5000) and seller gains (.15)(5000). $750 is transferred from the buyer's account to the seller's account.	.90	-750	+750	1000	2500
	b. Buyer's margin account falls below the maintenance margin, so $500 must be deposited into the margin account or the buyer's position will be liquidated.		+500		1500	
7-5	Price rises by .05, buyer gains by (.05)(5000) and the seller loses (.05)(5000). $250 is transferred from the seller's account to the buyer's account.	.95	+250	-250	1750	2250
7-5	The contract is settled at $.95. The buyer pays $4750 to the seller and receives5000 bushels of wheat.		-4750	+4750		
Net cash flow excluding margin payments			-5000	5000		
Net margin payments			2000	2000		

每天都依执行

价格上升. Buyer赚

价格下降. Seller赚

→ 最后会返回给双方

③ Forwards v.s. Futures

a. Place

Forwards: OTC

Futures: On the stock exchange

b. Date

Forwards: Only at the end of the contract (Delivery day)

Futures: For a few days

c. Risk

Forwards: Higher

Futures: Lower

d. Payoff

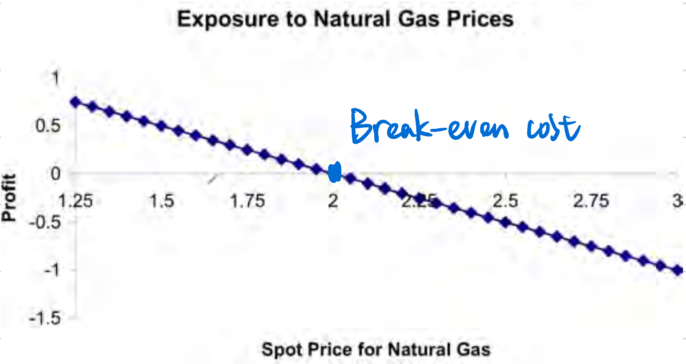
Forwards ≈ ← The effect of random interest rate for each day is minimal

Futures

④ Terminologies

a. Break-even cost of producers

e.g. If the break-even cost is \$2, then: spot price
- < BE cost: make money
- > BE cost: lose money



3. Options

① European option vs American option

European option : Exercised on the pre-specified day (更主流)

American option : Exercised any time prior to expiration

② Notation

C : Call price or premium

P : Put price or premium

S : Price of the underlying asset

T : Expiration date

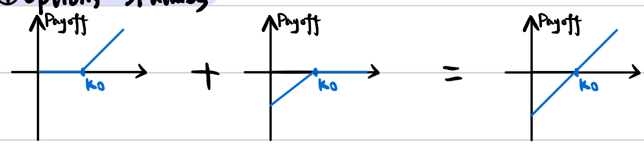
K : Strike price or Exercise price

The call is
- in-the-money : $S_T > K$ make money
- at-the-money : $S_T = K$
- out-of-the-money : $S_T < K$ lose money (Profit < 0, Payoff = 0)

The put is
- in-the-money : $S_T < K$ make money
- at-the-money : $S_T = K$
- out-of-the-money : $S_T > K$ lose money (Profit < 0, Payoff = 0)

4. Conversions between Derivatives

① Options → Futures



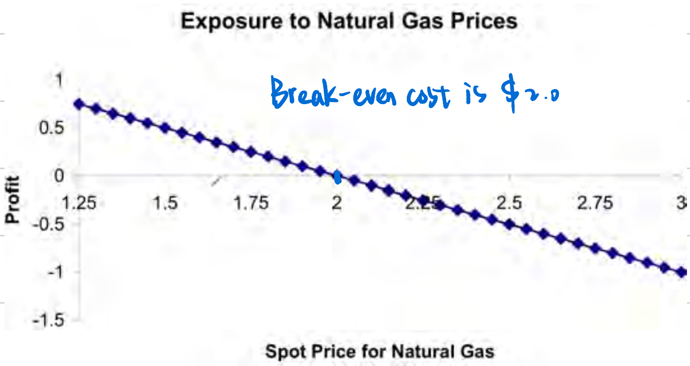
Long call + Short put = Futures

5. Functions of Derivatives

① Hedging

a. Context

We are shorting natural gas futures, and we would like to hedge the exposure.

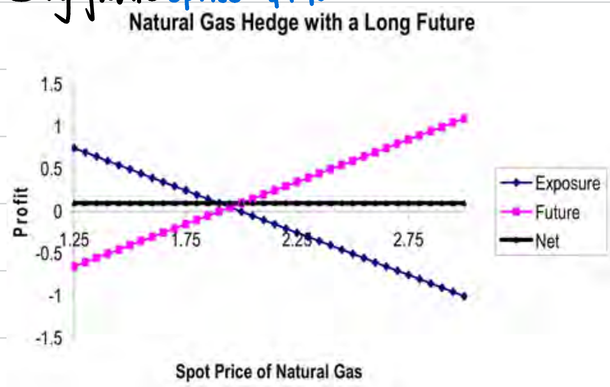


Break-even cost is $2.0

b. Methods (本质上是payoff面积的tradeoff: positive payoff / negative payoff)

△ Long future

Long future (price = $1.9)



$r = r_f$

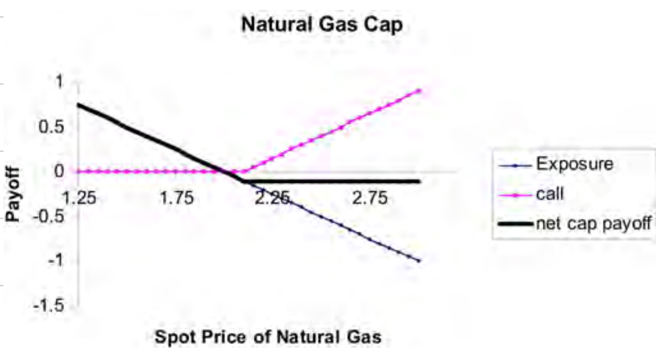
因为 $Corr(r, r_M) = 0$

Pro: Profit > 0, $\forall K_T$

Con: Profit is small

△ Long call

Long call with a strike price of $2.1



Call ≙ Cap
- The cost of gas is capped at $2.1/MMBtu
- The "all in" cost of gas is capped at $2.22

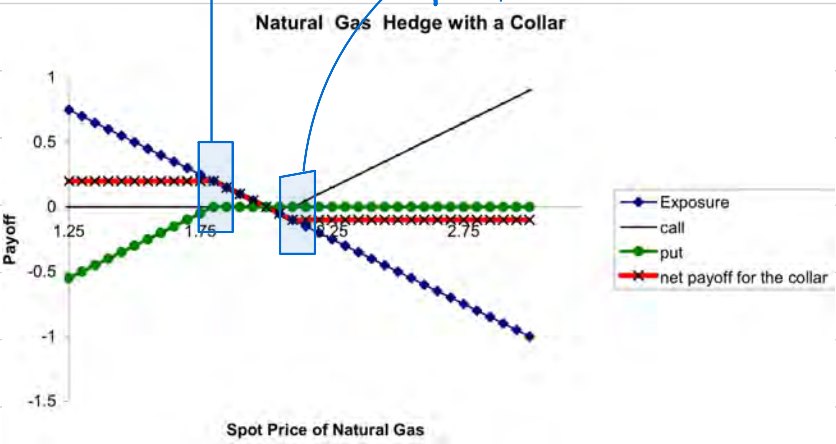
$= 2.1 + 0.12$

Strike price + Call price

△ Short put (+ Long Call) / Collar

A collar = Long call + Short put

Floor = 1.8 + Cap = 2.1 → Collar



注意区别 price & cost (price PV; cost FV)

- Cost of position = Total cost (0.55)
- Cost of collar = Δ (Call & Put)
- The highest all-in cost: 2.1 + 0.55
  The lowest all-in cost: 1.8 + 0.55
- We have exposure to market price between 1.8 & 2.1
- If we want to make the collar less expensive, we can:

or (Raise the floor; Raise the cap)

Cap 头顶
Floor 脚底

② Speculation

a. Vertical spread

A position involving options which have the same expiration date but different exercise prices = strike prices

b. Examples

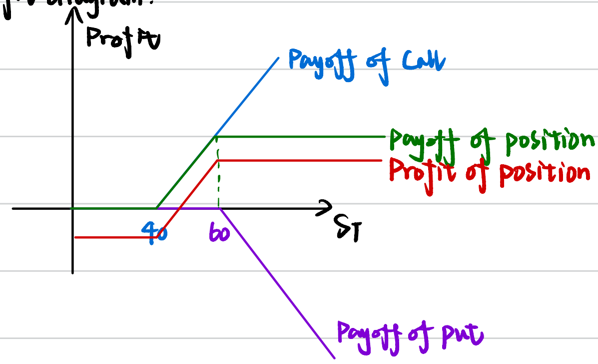
△ Bullish vertical spread ⟵ 认为未来是牛市 (集中在 40-60)

Long 1 call with K = 40 , suppose C(40) = \$2.78 为什么价差这么大?:

Write 1 call with K = 60 C(60) = \$0.11 价差体现人们对于其未来价格分布的估计

= Short

Profit diagram:



Profit = Payoff − Cost of position

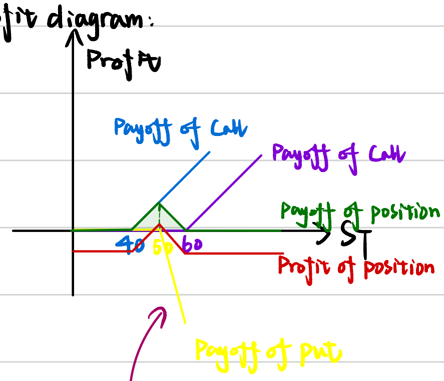
△ Butterfly vertical spread

Buy 1 call with K = 40 C(40) = \$2.78

Write 2 calls with K = 50 , suppose C(50) = \$0.26 positive payoff Δ↓, price↓

Buy 1 call with K = 60 C(60) = \$0.11

Profit diagram:



注意,这里的斜率更大

6. Application in Corporate Finance

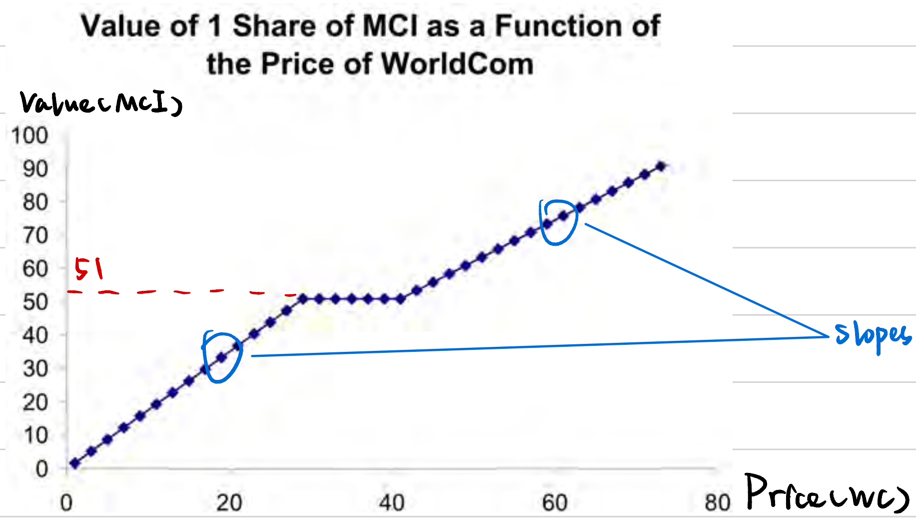
① # of shares

When merging:

WorldCom's offer for MCI depended on the 20-day average of the high and low prices of WorldCom stock prior to the closing of the deal

- If avg price < \$29 : MCI shareholders would receive $\left(\frac{\$51}{\$29}\right)$ shares for each share of MCI
- If avg price ∈ (\$29, \$41) : MCI shareholders would receive shares worth \$51 for each MCI share
- If avg price > \$41 : MCI shareholders would receive $\left(\frac{\$51}{\$41}\right)$ shares for each share of MCI

② Share value

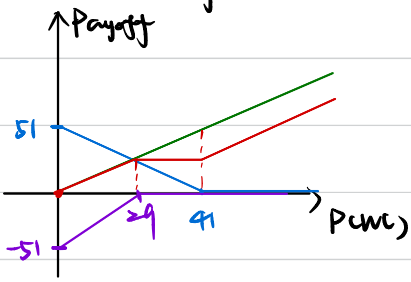


- \$29, \$41 are previously designed in some ways
- The value that MCI shareholder receives = # of shares × share value

③ Valuation of Worldcom's offer with the help of derivatives

a. Combine derivatives to form the same payoff (Same payoff → Same price)

- 1.2439 long puts with K=41
- 1.7586 short puts with K=29
- 1.2439 shares of WorldCom Stock



画图方法：
- (i) 分段
- (ii) 关注每段始末点

b. Evaluate the offer using the Black-Scholes option pricing formula

L6-8 Forwards

1. Pricing Forwards

① Pricing — No dividends

a. Cash-and-Carry Trade

Long the spot
Short the forwards

	CF at t	CF at T
Long the spot	$-S_t$	0
Short the forward	0	$F_{t,T}$
Borrow at rate r	S_t	$-S_t e^{r(T-t)}$
Total	0	$F_{t,T} - S_t e^{r(T-t)}$

$\longrightarrow F_{t,T} - S_t e^{r(T-t)} \leq 0$ —— ①

No arbitrage

b. Reverse Cash-and-Carry Trade

Short the spot
Long the forwards

	CF at t	CF at T
Short the spot	S_t	0
Long the forward	0	$-F_{t,T}$
Lend at rate r	$-S_t$	$S_t e^{r(T-t)}$
Total	0	$S_t e^{r(T-t)} - F_{t,T}$

$\longrightarrow S_t e^{r(T-t)} - F_{t,T} \leq 0$ —— ②

c. Conclusion

$F_{t,T} = S_t e^{r(T-t)}$

r is risk-free rate

t: Assignment date
T: Delivery date

d. Extension: Calendar Spreads

	CF at t	CF at T_1	CF at T_2
Long contract 1	0	$-F_{t,T_1}$	0
Short contract 2	0	0	F_{t,T_2}
Total	0	$-F_{t,T_1}$	F_{t,T_2}

$\longrightarrow F_{t,T_2} = F_{t,T_1} e^{r(T_2-T_1)}$

② Biasedness of forward price (Futures' price is not necessarily future price)

a. Relationship

- If long forwards: $E(S_T) \geq F_{t,T}$
- If short forwards: $E(S_T) \leq F_{t,T}$

b. Derivation

(Taking long position for example)

Let
- α: be the expected return
- R: be the risk free rate

Expected $CF = E_0(S_1 - F_{0,1})$

$= E_0(S_1) - \underline{F_{0,1}}$ 在t=0时, forward price已确定

$= (1+\alpha)S_0 - (1+R)S_0$

$= (\alpha - R)S_0 > 0$

c. Conclusion

△ Forward price is biased ($E(S_T) \neq F_{0,T}$)

$CF(t) = 0$

△ A forward contract requires no investment, so earns only the risk premium $(\alpha - R)$ on the underlying asset

(可以理解为这)forward contract的成本为R(即需要以R的利率借钱(杠杆原理)资金的以R为利率的利息))

- R : Compensation for time
- $\alpha - R$: Compensation for risk
- α : Expected return on the stock

- price − price : R
- price − CF :
  - α Stock
  - $\alpha - R$ Forward

③ Pricing — Considering frictions

a. Types of frictions

- Bid-Ask Spread : $S^b < S^a$, $F^b < F^a$ (买时看高价，卖时看低价)
- Borrowing & Lending rate Spread : $r^b > r^l$
- Per-transaction cost : k (Pay once each to <u>transact</u> in forward and stock)
  - (long / short 都算)

b. Process

△ Cash-and-Carry Trade

	CF at t	CF at T
Long the spot	$-S_t^a - k$	0
Short the forward	$0 - k$	$F_{t,T}^b$
Borrow at rate r	$\boxed{S_t^a + 2k}$ 决定	$-(S_t^a+2k)e^{r^b(T-t)}$
Total	0 使得	$F_{t,T}^b - (S_t^a+2k)e^{r^b(T-t)}$

$$\longrightarrow F_{t,T}^b \le (S_t^a+2k)e^{r^b(T-t)}$$

△ Reverse Cash-and-Carry Trade

	CF at t	CF at T
Short the spot	$S_t^b - k$	0
Long the forward	$0 - k$	$-F_{t,T}^a$
Lend at rate r	$-(S_t^b - 2k)$	$(S_t^b-2k)e^{r^l(T-t)}$
Total	0	$(S_t^b-2k)e^{r^l(T-t)} - F_{t,T}^a$

$$\longrightarrow F_{t,T}^a \ge (S_t^b-2k)e^{r^l(T-t)}$$

△ Conclusion

$$(S_t^b-2k)e^{r^l(T-t)} \le F_{t,T} \le (S_t^a+2k)e^{r^b(T-t)}$$

No-arbitrage price $\xrightarrow{\text{Considering frictions}}$ No-arbitrage region

Transaction cost ↑, region becomes wider.

④ Pricing — Considering dividends

a. One-time dividend

	CF at t	CF at t'	CF at T
Long the spot	$-S_t$	0	0
Short the forward	0	0	$F_{t,T}$
Receive the dividend	0	D	0
Invest the dividend	0	$-D$	$De^{r(T-t')}$
Total	$-S_t$	0	$F_{t,T}+De^{r(T-t')}$

$\longrightarrow F_{t,T}=S_t\cdot e^{r(T-t)}-De^{r(T-t')}$ △

b. Continuous dividend

△ Motivation

If we reinvest the continuous dividend, and we would like to hold 1 share of stock when $t=T$, how many shares should we buy at $t=t$?

$e^{\delta}=(1+d)$

δ: Continuously compounded div rate

d: Effective annual div rate

$e^{\delta\cdot(T-t)}\cdot x=1 \longrightarrow x=e^{-\delta(T-t)}$

△ Process

	CF at t	CF at T
Long the spot	$-S_t\times 1\times e^{-\delta(T-t)}$	0
Short the forward	0	$F_{t,T}$
Total	$-S_te^{-\delta(T-t)}$	$F_{t,T}$

$\longrightarrow F_{t,T}=S_t\cdot e^{-\delta(T-t)}\cdot e^{r(T-t)}=S_t\cdot e^{(r\underline{-}\delta)(T-t)}$ 当考虑连续的分红时，期货价格进一步下降.

2. Pricing Stock Index Futures

① Index

a. Common index at the Chicago Mercantile Exchange

- S&P 500 Index $= \dfrac{\sum_{i=1}^{500} n_i p_i}{\text{constant}}$
- E-Mini S&P 500
- S&P Midcap 400 Index
- S&P 500/BARRA Growth and Value Indices
- Nasdaq 100 Index
- Russell 2000 Index
- Major Market Index
- Nikkei 225 Index
- Goldman Sachs Commodity Index (GSCI)

b. Notes

The value of the index does not take into acount of dividends.

$\longrightarrow$ Decrease in the index $\xrightarrow{\times}$ Bad performance of stocks in the index

This may be caused by ex-dividend date ex-dividend date $\xrightarrow{\text{2 days}}$ record date $\longrightarrow$ send dividend

② Stock index futures

a. Definition

A future whose payoff is based on the value of some index

b. Characteristic

	t	T
short the index buy the futures lend the proceeds	$1400 \times 250 \times e^{-2\%}$ $1400 \times 250 \times e^{-2\%}$	$-250 I_T$ $250(I_T - 1420)$ $1400 \times 250 \times e^{-2\%} \times e^{6\%} = 364283.8$
total	0	$364283.8 - 250 \times 1420 = 9283.77$

$\boxed{\text{Cash settled}}$, no physical delivery.

No index / stock delivered (hard to deliver)

③ Profit

$$\begin{cases} \text{Long: } 250(I_T - F_{t,T}) \\ \text{Short: } 250(F_{t,T} - I_T) \end{cases}$$

Normalization: Contract size.

Contract unit × Contract size = Index unit

④ Pricing

$S_T \to I_T$, the same formulas as before

3. Using Index Futures

① $\rho = +1 / -1$

Be able to perfectly hedge the risk (Future CF = C)

② $\rho \in (-1, 1)$

a. Overview

Unable to perfectly hedge the risk, but we can find H^*, s.t. min Var(Hedged return).

b. Hedged return

r_p: Portfolio return

I_p: Invested dollar in the portfolio → 投资总额

N: Notional amount of futures contract → index of futures × contract size

H: Unit of futures contract

	Cash flows		
Transaction	Today	1 year, $S_1 = 1300$	1 year, $S_1 = 1500$
Own portfolio @ \$1400	-\$1400	\$1300	\$1500
Short futures @ \$1450	0	\$1450 - \$1300	\$1450 - \$1500
Total	-\$1400	\$1450	\$1450

一个正一个负

Hedged return = $r_p I_p + H \cdot N \cdot (r_{s\&p} - r)$ ← 这里以(S&P 500 futures)举例

portfolio 赚 total return r_p

~~futures 赚 risk premium $(r_{s\&p} - r)$~~ !!!

c. Var(Hedged return) & H^* (Variables是 r_p & $r_{s\&p}$)

$$Var(HR) = \sigma_p^2 I_p^2 + H^2 N^2 \sigma_{s\&p}^2 + 2 I_p H N \, Cov(r_p, r_{s\&p})$$

$$\longrightarrow H^* = -\frac{I_p}{N} \frac{Cov(r_p, r_{s\&p})}{\sigma^2_{s\&p}} = -\frac{I_p}{N} \beta_{s\&p}$$ (通过求关于H的一阶导所得) H: Hedge ratio

H^* 可正可负，取决于 ($\rho \to \beta$; I_p (正/负))

$$\longrightarrow Var(H^*) = \sigma_p^2 I_p^2 (1 - \rho^2)$$

$\rho = \pm 1 \longrightarrow Var(H^*) = 0$

$\rho = 0 \longrightarrow Var(H^*)$ is max.

4. Currency Futures

① Motivation

Currency futures = Exchange rate future

- Stock futures: MSF \$100 / share
- Currency futures: Currency \$1.21 €

\$ → Domestic currency → Futures price

€ → Foreign currency (用\$买€) → Underlying asset

② Pricing

- P_t: Spot exchange rate
- r_d: Domestic risk-free rate ⟶ 类比成 r_f (risk-free rate)
- r_f: Foreign risk-free rate ⟶ 类比成 δ (dividend rate)

$$F_{t,T} = P_t e^{(r_d - r_f)(T-t)}$$

把Domestic & Foreign 反过来也成立 (两边同作倒数变换)

- 若 $r_d > r_f$, $F_{t,T} > P_t$ (利率高, 贬值)
- 若 $r_d < r_f$, $F_{t,T} < P_t$

L9-11 Swaps

1. Commodity swaps

① Definition

A strip of volatile price futures ⟷ A strip of fixed price futures (Swaps)

Swaps 本质上是一种交换

② Swap price

PV(A strip of volatile price futures) = PV(A strip of fixed price futures)

⟶ C

③ Market value

a. When the swaps is initiated / assigned, its market value = 0 (Σ PV(Future prices) are the same)

b. When the swaps is in process, its market value can change as the values of underlying assets fluctuate.

$$\begin{cases} \text{Swap price} = C & \text{—— 这里的swap指交换合约内单某期的固定价格} \\ \text{Market value of swaps} = \begin{cases} 0 & \text{initial} \\ \neq 0 & \text{in process} \end{cases} & \text{—— 这里的swap指交换合约} \end{cases}$$

2. Currency Swaps

① Cash flows

Suppose a European firm would like to issue debt in dollars since the CFO thinks that the dollar denominated debt market is more liquid.

The bond is a \$10 million dollar 4 year 3.35% coupon bond.

In order to hedge the exchange rate risk, the firm enters into a swap. (用于储备美金)

Assume:
- the continuously compounded dollar denominated rate $r_{us} = 3.3\%$
- the continuously compounded euro denominated rate $r_{euro} = 4.21\%$
- the current exchange rate = \$0.92/euro

→ According to the criterion of 0 initial market value of swaps,

We have: $10\times 3.35\%\,(e^{-r_{us}}+e^{-2r_{us}}+e^{-3r_{us}})+\underline{10(1+3.35\%)}e^{-4r_{us}}$ 最后一年是本金+利息

$$= \left[C\cdot(e^{-r_{euro}}+e^{-2r_{euro}}+e^{-3r_{euro}})+(C+P)e^{-4r_{euro}}\right]\times \underline{0.92}$$

	0	1	2	3	4
Pay (€)	0	C	C	C	C+P
Receive (\$)	0	10×3.35%	10×3.35%	10×3.35%	10×(1+3.35%)

② Change in the exchange rate

What if the exchange rate in T=1 becomes \$1/euro, what's the market value of the swaps when T=1?

MV(Swaps) = PV(Receive) − PV(Pay)   Receive − Pay

$$= \left[(10\times3.35\%)\cdot(e^{-r_{euro}}+e^{-2r_{euro}})+(10\times(1+3.35\%))\cdot e^{-3r_{euro}}\right]\times \underline{1}$$ Exchange rate 改变

$$-\left[C(e^{-r_{us}}+e^{-2r_{us}})+(C+P)(e^{-3r_{us}})\right]$$ → MV(Swaps) 改变

3. Interest Rate Futures and Swaps

① Interest Rate ⟷ (Zero Coupon) Bond ⟷ Interest rate Futures

名义上：交易Bond (Cash-settled, 和index一样)

实际上：交易Interest rate

a. Interest rate ⟷ (Zero coupon) bond

△ Any bond ⟷ Zero coupon bond

$$\text{Any bond} \underset{\text{Comprise}}{\overset{\text{Decompose}}{\rightleftarrows}} \text{Zero coupon bond}$$

△ Interest rate ⟷ Zero coupon bond

Assume the face value of the zero coupon bond is FV,

then: $P_{0,t} = FV \cdot e^{-r(0,t) \cdot t}$

P、r 都以0为开始（因为便于计算）

P ⟷ r （r也时变会变的）

b. Zero coupon bond ⟷ Interest rate futures

0: Assign date

i: Delivery date

j: Maturity date

	CF at 0	CF at i	CF at j
Long the spot (Bond)	$-P_{0,j}$	0	0
Short the futures	0	$+F_{0,i,j}$	0
Borrow	$+P_{0,j}$	$-P_{0,j} \cdot e^{r(0,i) \cdot i}$	0
Total	0	$F_{0,i,j} - P_{0,j} e^{r(0,i) \cdot i}$	0

$$\longrightarrow F_{0,i,j} = \underbrace{P_{0,j} \cdot e^{r(0,i) \cdot i}}_{\text{含 } r, P} = \underbrace{FV \cdot \frac{P_{0,j}}{P_{0,i}}}_{\text{只含 } P}$$

$$P_{0,j} e^{r(0,i) \cdot i} = FV \cdot e^{-r(0,j) \cdot j} \cdot e^{r(0,i) \cdot i}$$

$$= FV \frac{P_{0,j}}{P_{0,i}}$$

② Example of interest rates — LIBOR Rate

a. Definition

LIBOR rates (London InterBank Offer Rates)

Off-shore dollar market in London   Borrow rate

b. Type

The most popular type is the 3-month LIBOR rate

it's annualized rate

③ Example of interest rate futures — Eurodollar futures Offshore dollar in the euro market

a. Motivation

Buy interest rate futures → Lock in the interest rate.

b. Expiration futures price / Settlement price

$Price = 100(1 - r_{LIBOR})$ (100用于normalization，使得价格整数化)

Annualized LIBOR

c. Payoff at expiration of long position

$Payoff = [100(1 - r_{LIBOR}^{T=t}) - 100(1 - r_{LIBOR}^{T=0})] \times 2500$ $2500 = 1M \times \frac{1}{100} \times \frac{1}{4}$

对于3-month LIBOR rate, the quantity is \$1000000/contract, eurodollar futures的默认值

d. Function

Long / Short → Fix libor → 不同在于对eurodollar / libor的看涨/看跌

这两个负相关

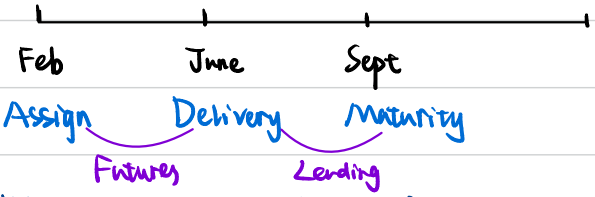
d. Example

In Feb, a firm would like to lock in a 3 month lending rate starting in June on \$1M.

They chose to use Eurodollar futures to do it.

The eurodollar futures price for June delivery was 97.74. (Feb 已知)

The firm expect the spot LIBOR rate in June = 2.1% (Feb 未知)



— What can they do to lock in the 3-month lending rate in June?

Long the Eurodollar futures.

The corresponding lending rate is locked in $(100-97.74)\% = 2.26\%$

— The payoff at expiration?

$\text{Payoff} = [100(1-0.021)-97.74] \times 2500 = 400$

— From the 3-month lending, they receive interest equal to

$\text{Interest} = (0.021) \times \frac{1}{4} \times 1000000 = 5250$

总收入

④ Interest swaps

a. Fixed rate payer & Floating rate payer

Net profit
- Fixed rate payer : $N(r_{floating} - r_{fixed}) \times \frac{1}{n}$ 分n次，每次$\frac{1}{n}$
- Floating rate payer : $N(r_{fixed} - r_{floating}) \times \frac{1}{n}$

b. Scenario: Hedging by swaps

- Receive r_{fixed} (r_{fixed} ⊕) $\longrightarrow$ $r_{fixed}-$, $r_{floating}+$ $\longrightarrow$ Fixed rate payer ☆
- Receive $r_{floating}$ ($r_{floating}$ ⊕) $\longrightarrow$ $r_{fixed}+$, $r_{floating}-$ $\longrightarrow$ Floating rate payer
- Pay r_{fixed} (r_{fixed} ⊖) $\longrightarrow$ $r_{fixed}+$, $r_{floating}-$ $\longrightarrow$ Floating rate payer
- Pay $r_{floating}$ ($r_{floating}$ ⊖) $\longrightarrow$ $r_{fixed}-$, $r_{floating}+$ $\longrightarrow$ Fixed rate payer ☆

(本质上是交换利率)

c. Fixed rate computation

e.g.

rf for 3-month

$$\sum_{i=1}^{n}\left[(1M) \times \frac{90}{360} \times (r_{fixed} - r_{floating})\right] d_i = \sum_{i=1}^{n}\left[2500 \times [100(1-r_{LIBOR}) - F_{0,t,t+1}]\right] d_i$$

$r_{fixed} \to \bigcirc \to r_{floating}$ $\qquad$ $r_{fixed} \to \bigcirc \to r_{floating}$

$\longrightarrow r_{fixed}$.

Long eurodollar futures 本质上就是在进行了一个swap.

☆ PV(息差) = PV(Eurodollar payoff)

L12–13 Commodity

1. Commodity Forwards

① Pricing

a. Formula

$F_{t,T} = P_t \cdot e^{(r+k-\delta)(T-t)}$

- k : Carrying cost (Warehousing, spoilage and insurance)
- δ : Convenience yield (Leasing …)

b. Factors

△ D & S $\rightarrow P_t$ ~

△ Inventory ↑, Carrying cost ↑

△ Inventory ↑, Convenience yield ↓

△ Inventory ↑, P_t less volatile

Inventory 也代表 D & S : Inventory ↑ = D ↓, S ↑

P_t ↑, P_t more volatile

② Types

a. By category

Commodities	GSCI	DJ-UBS	Exchange	Contracts	Start of futures in US	Start of futures in China
				Energy (4 Commodities)		
WTI Crude Oil	40.6	15.0%	NYMEX	Every month	03/30/1983	
Heating Oil	5.3%	4.5%	NYMEX	Every month	11/14/1978	8/25/2004
RBOB Gasoline	4.5%	4.1%	NYMEX	Every month	04/18/2006	
Natural Gas	7.6%	16.0%	NYMEX	Every month	04/04/1990	
				Grains (9 commodities)		
Corn	3.6%	6.9%	CME Group	Mar, May , Jul, Sep & Dec	07/01/1959	09/22/2004
Soybeans	0.9%	7.4%	CME Group	Jan, Mar, May , Jul, Aug, Sep, Nov	07/01/1959	01/04/1999
Chicago Wheat	3.0%	3.4%	CME Group	Mar, May , Jul, Sep, Dec	07/01/1959	01/04/1999
Kansas Wheat	0.7%	0	KCBT[24]	Mar, May, Jul, Sep, Dec	01/05/1970	
Soybean Oil	0	2.9%	CME Group	Jan, Mar, May, Jul, Aug, Sep, Oct, Dec	07/01/1959	
Minn. Wheat	0	0	MGE[25]	Mar, May, Jul, Sep, Dec	01/05/1970	
Soybean Meal	0	0	CME Group	Jan, Mar, May, Jul, Aug, Sep, Oct, Dec	07/01/1959	
Rough Rice	0	0	CME Group	Jan, Mar, May, Jul, Sep, Nov	08/20/1986	
Oats	0	0	CME Group	Mar, May, July, Sep, Dec	07/01/1959	
				Softs (6 Commodities)		
Coffee	0.5%	2.7%	ICE	Mar, May, Jul, Sep, Dec	08/16/1972	
Cotton	0.7%	2.2%	ICE	Mar, May, Jul, Oct, Dec	07/01/1959	06/01/2004
Sugar	2.1%	2.8%	ICE	Mar, May, Jul, Oct	01/04/1961	01/06/2006
Cocoa	0.2%	0	ICE	Mar , May, Jul, Sep, Dec	07/01/1959	
Lumber	0	0	CME Group	Jan, Mar, May, Jul, Sep, Nov	10/01/1969	
Orange Juice	0	0	ICE	Jan, Mar, May, Jul, Sep, Nov	02/01/1967	
				Livestock (4 Commodities)		
Feeder Cattle	0.3%	0.0%	CME Group	Jan, Mar, Apr, May, Aug, Sep, Oct,	11/30/1971	
Lean Hogs[26]	0.8%	2.5%	CME Group	Feb, Apr, May, Jul, Aug, Oct, Dec	02/28/1966	
Live Cattle	1.6%	4.1%	CME Group	Feb, Apr, Jun, Aug, Oct, Dec	11/30/1964	
Pork Bellies	0	0	CME Group	Feb, Mar, May, Jul, Aug	09/18/1961	
				Metals (5 Commodities)		
Gold[27]	1.5%	6.1%	NYMEX	Feb, Apr, Jun, Aug, Oct, Dec	12/31/1974	01/01/2008
Silver	0.2%	2.4%	NYMEX	Jan, Mar, May, Jul, Sep, Dec	06/12/1963	
Copper[28]	2.6%	6.7%	NYMEX	Mar, May, Jul, Sep, Dec	01/03/1989	05/12/1997
Platinum	0	0	NYMEX	Jan, Apr, Jul, and Oct	03/04/1968	
Palladium	0	0	NYMEX	Mar, Jun, Sep, and Dec	01/03/1977	

b. By trend

- Contango : $r+k-\delta > 0$, upward-sloping
- Backwardation : $r+k-\delta < 0$, downward-sloping

Normal case

③ Investment characteristic

a. High Sharpe ratio

Commodity Futures ≈ Stock > Bond

b. Independent before 2004

△ Independent with other financial assets

△ Independent among commodity futures

c. Dependent after 2004.

△ Dependent with other financial assets

△ Dependent among commodity futures

← Financial institution begins to incorporate commodity futures into their financial portfolio

④ Machanics in investing in Commodities

a. Action

Rolling over commodity contracts

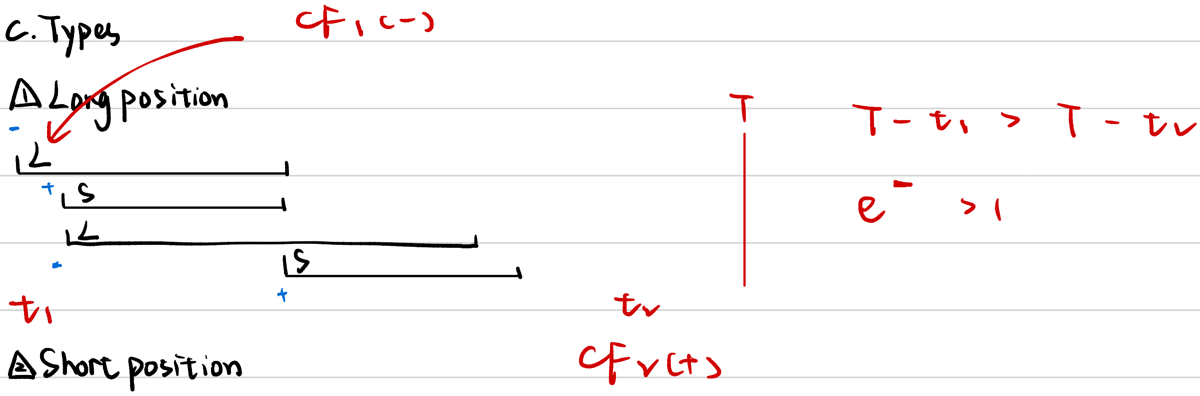
b. Reasons

△ Traders often avoid closing out the commodity futures since physical delivery is costly and inconvenient.

△ The front month contracts are more liquid.

c. Types

△ Long position



△ Short position

d. Return decomposition

Spot return : P ← Spot price fluctuation

Roll return : $e^{r+k-\sigma}$ ← Forward price

For long position:
- Contango : Negative roll return
- Backwardation : Positive roll return

[illegible]

L14-15 Arbitrage restrictions (Equality & Inequality for Options)

1. Put-Call parity

① Formula

	CF(0)	CF(T) $S_T<K$	CF(T) $S_T>K$
Long 1 call	$-C$	0	S_T-K

=

	CF(0)	CF(T) $S_T<K$	CF(T) $S_T>K$
Long 1 put	$-P$	$K-S_T$	0
Long 1 share	$-S$	S_T	S_T
Borrow	$+Ke^{-rT}$	$-K$	$-K$
Total	$-P-S+Ke^{-rT}$	0	S_T-K

Same payoff $\rightarrow$ Same price

$\rightarrow C=P+S-Ke^{-rT}=P+\underline{Se^{-\delta T}}-Ke^{-rT}$ in general (δ here represents the dividend rate)

② Synthetic assets

a. Idea

From the above eqn,

we may form:
- Syn Call
- Syn Put
- Syn Underlying asset (Share, Currency, Futures...)
- Syn Bonds

b. Example: How to employ Put-Call Parity to form a Syn futures?

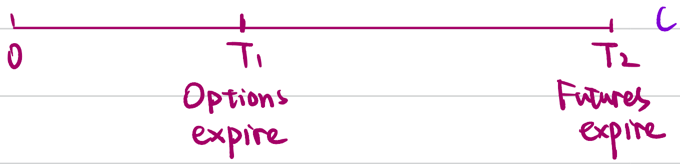
	CF(0)	CF(T_1) $F_T<K$	CF(T_1) $F_T>K$
Long 1 futures	0	$\underline{F_T-F}$	$\underline{F_T-F}$

Marked-to-market

=

	CF(0)	CF(T_1) $F_T<K$	CF(T_1) $F_T>K$
Long 1 Call	$-C$	0	F_T-K
Short 1 put	$+P$	$-(K-F_T)$	0
Lend	$-(K-F)e^{-rT}$	$K-F$	$K-F$
Total	$-C+P-(K-F)e^{-rT}$	F_T-F	F_T-F

(F 期货现价
F_T 期货在T_1时价格



(Options for futures)

$\rightarrow C=P+Fe^{-rT}-Ke^{-rT}$

c. Application

When it is challenging to long / short some certain assets,

we can change to long / short their equivalent Syn assets.

2. Arbitrage Restrictions Inequalities

① S, C, K inequality

a. Inequality

无论如何都不exercise

$S \geq C_{US} \geq C_{Eur} \geq \max[0, S-Ke^{-rT}]$

无论如何都exercise

b. Deduction

△① $S \geq C$

	CF(0)	CF(T) $S_T \leq K$	CF(T) $S_T > K$
Long Stock	$-S$	S_T	S_T
Short Call	$+C$	0	$-S_T+K$
Total	$-S+C$	S_T	K
	$-S+C \leq 0$	≥ 0	≥ 0

$S \geq C$

△② $C_{US} \geq C_{Eur}$

C_{US} 本质上是exercise day T无束的 C_{Eur}

△③ $C \geq 0$

△④ $C \geq S-Ke^{-rT}$

$\rightarrow C \geq \max\{0, S-Ke^{-rT}\}$

	CF(0)	CF(T) $S_T \leq K$	CF(T) $S_T > K$
Long Call	$-C$	0	S_T-K
Short Stock	$+S$	$-S_T$	$-S_T$
Long Bond (Lend)	$-Ke^{-rT}$	K	K
Total	$-C+S-Ke^{-rT}$	$K-S_T$	0
	$-C+S-Ke^{-rT} \leq 0$	≥ 0	

$C \geq S-Ke^{-rT}$

② S, P, K inequality

a. Inequality

$K \geq P_{US} \geq P_{Eur} \geq \max[0, Ke^{-rT} - S]$

3. Early exercising of American Options

① American call

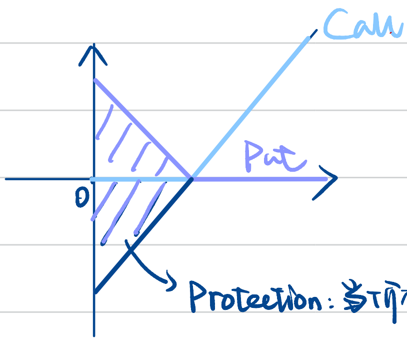
a. Formula

$C_{euro} = P_{euro} + S - Ke^{-rT}$

$= S - PV(K) + P_{euro}$

$= \underbrace{(S-K)}_{\text{Payoff from early exercising}} + \underbrace{(K - PV(K))}_{\text{Save of interest}} + \underbrace{P_{euro}}_{\text{Protection value}}$

Payoff from early exercising　Save of interest　Protection value



Protection：当价格低于K时的保护，此面积与Long put时面积等大

b. Analysis

Early exercise American Call：
- Benefit: Receive Dividend
- Costs：
  - Lose the interest
  - Forsake the protection

② American put

a. Formula

$P_{euro} = C_{euro} - S + PV(K)$

$= \underbrace{(K-S)}_{\text{Payoff from early exercising}} + \underbrace{(PV(K) - K)}_{\text{Loss of interest}} + \underbrace{C_{euro}}_{\text{Protection value}}$

Payoff from early exercising　Loss of interest　Protection value

b. Analysis

Early exercise American Put：
- Benefit: Earn interest
- Costs：
  - The put becomes more valuable after a dividend
  - Forsake the protection

L16 Binomial

1. One-Period Binomial Model

① Setup & Motivation

$$S \begin{cases} uS \\ dS \end{cases}$$

R: Return per period

Let $R = e^{rt}$, then $u > R > d$

Although this tree looks simple, with enough periods, it's a good approximation of reality

② Call pricing (What about Put pricing? → Put-Call Parity)

a. Deduction

$$\begin{cases} S = 50 \\ u = 2 \\ d = 0.5 \\ R = 1.25 \end{cases} \rightarrow 50 \begin{cases} 100 \\ 25 \end{cases}$$

For $K=50$ Call, $C \begin{cases} 50 & C_u \\ 0 & C_d \end{cases}$

In order to price C, we try to construct the payoff with Stock & Bond

Let $\begin{cases} \Delta & \text{denote \# of shares} \\ B & \text{denote dollars of bond} \end{cases}$, then $\Delta S + B \begin{cases} \Delta uS + RB = 100\Delta + 1.25B \\ \Delta dS + RB = 25\Delta + 1.25B \end{cases}$

then $\begin{cases} C_u = \Delta uS + RB \\ C_d = \Delta dS + RB \end{cases}$, then $\begin{cases} \Delta = \dfrac{C_u - C_d}{S(u-d)} = \dfrac{50-0}{100-25} = \dfrac{2}{3} \\ B = \dfrac{uC_d - dC_u}{R(u-d)} = \dfrac{-0.5\times 50}{1.25(2-0.5)} = -13.33 \end{cases}$, then $C = \Delta S + B = 20$

then $C = \dfrac{C_u - C_d}{S(u-d)} S + \dfrac{uC_d - dC_u}{R(u-d)} = \dfrac{1}{R}\left[\dfrac{R-d}{u-d} C_u + \dfrac{u-R}{u-d} C_d\right]$

b. Interpretation

△ Call price

$$C=\frac{1}{R}\left[\frac{R-d}{u-d}C_u+\frac{u-R}{u-d}C_d\right]=\frac{1}{R}\left[pC_u+(1-p)C_d\right]\quad\begin{cases}p=\frac{R-d}{u-d}\\ 1-p=\frac{u-R}{u-d}\end{cases}$$

Call price = PV(Risk-Neutral Expected Payoff)

(i) Investors' attitude toward risk
(ii) Investors' feeling about the prospects for a stock
are reflected in the option price (Specifically, P)

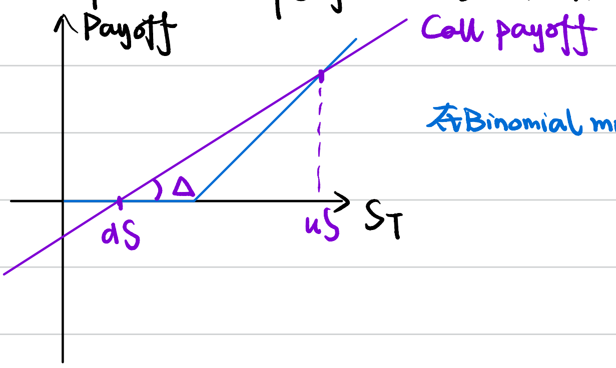
(i) Risk averse ↑ → d ←— R u → $\begin{cases}p\downarrow \\ 1-p\uparrow\end{cases}$ → C↓

(ii) if 99% u, 1% d, then d R —→ u, → $\begin{cases}p\uparrow \\ 1-p\downarrow\end{cases}$ → C↑

△ Δ

Δ: Formula: The ratio of $\Delta C/\Delta S$ → The sensitivity of call price to stock price ($C=\Delta S+B$)
→ To hedge a long position in the call, short Δ shares of the stock

Graph : The slope of the combination line



在Binomial model中, $\begin{cases}\text{Stock}\\ \text{Bond}\end{cases}$ 拟合的本质上是Two-case Payoff 直线

2. Two-Period Binomial Models

① Setup

$$
\begin{array}{lll}
 & & u^2S \\
 & uS & \\
S & & udS \\
 & dS & \\
 & & d^2S
\end{array}
$$

② Call pricing

<u>Backward computation:</u>

$$
\left\{
\begin{array}{l}
C_u = \frac{1}{R}[pC_{uu} + (1-p)C_{ud}] \\
C_d = \frac{1}{R}[pC_{ud} + (1-p)C_{dd}]
\end{array}
\right.
\longrightarrow C = \frac{1}{R}[pC_u + (1-p)C_d]
$$

$C=[pC_u+(1-p)C_d]/R$
$=[p^2C_{uu}+2p(1-p)C_{ud}+(1-p)^2C_{dd}]/R^2$

$\Delta=(C_u-C_d)/(uS-dS)$

$B=C-\Delta S$

$C_u=[pC_{uu}+(1-p)C_{ud}]/R$

$\Delta_u=(C_{uu}-C_{ud})/(u^2S-udS)$

$B_u=C_u-\Delta_u uS$

$C_d=[pC_{ud}+(1-p)C_{dd}]/R$

$\Delta_d=(C_{ud}-C_{dd})/(udS-d^2S)$

$B_d=C_d-\Delta_d dS$

C_{uu}

C_{ud}

C_{dd}

$\longrightarrow$ Adjust $\left\{\begin{array}{l} \Delta \\ B \end{array}\right.$ according to different situations

3. Arbitrage Mispriced Options

- Suppose that the stock price is 120 two periods prior to option expiration. If you close your position at that point, what will you get in the following scenarios:
  - If the call price equals the "theoretical value", $60.50, what is the payoff from closing your position?
  - If the call price is less than 60.50, closing the position yield more than zero since it is cheaper to buy back the call.
  - If the call price is more than 60.50, closing out the position will yield less than zero! What should we do now?

close the position (liquidate)

- Long the call / short the call
- Long the call / short the Syn call

4. Options on Dividend Paying Stocks

① Setup

Let $\theta = e^{\sigma t}$, then units: $\Delta \to \Delta\theta$

$$\Delta S + B \begin{cases} \theta \Delta u S + RB \\ \theta \Delta d S + RB \end{cases}$$

② Call pricing

$$\begin{cases} C_u = \theta \Delta u S + RB \\ C_d = \theta \Delta d S + RB \end{cases} \longrightarrow \begin{cases} \Delta = \dfrac{C_u - C_d}{\theta S(u-d)} \\ B = \dfrac{uC_d - dC_u}{R(u-d)} \end{cases} \longrightarrow C = \Delta S + B = \frac{1}{R}\left[P_{div} C_u + (1-P_{div}) C_d\right]$$

where $P_{div} = \dfrac{\frac{R}{\theta} - d}{u - d}$ $\quad \dfrac{R}{\theta} = \dfrac{e^{rt}}{e^{\sigma t}} = e^{(r-\sigma)t}$ effective rate $\begin{cases} \text{without Div } e^{rt} \\ \text{with Div } e^{(r-\sigma)t} \end{cases}$

5. Binomial Model for American Options

① Criterion

Exercise if the continuation value is less than the immediate exercise payoff

→ Euro Call

or equal to

(For an American Option, we have the choice to exercise at any node on the tree)

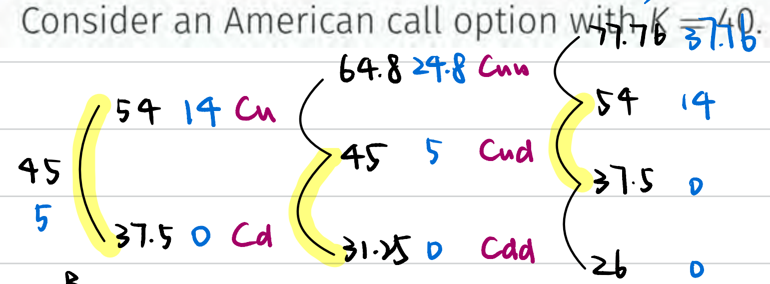
② Pricing

$C_{Amer} = \max\{\text{Immediate Payoff}, C_{Euro}\}$

③ Example

With div payment

An example: $S = 45$, $u = 1.2$, $d = 1/1.2$, $\theta = 1.03$, and $R = 1.025$.
Consider an American call option with $K = 40$.



$p = \dfrac{\frac{R}{\theta} - d}{u - d} = 0.4413$

$C_{uu} = \max\{24.8, \frac{1}{R}[p \times 37.76 + (1-p) \times 14]\} = 24.8$

$C_{ud} = \max\{5, \frac{1}{R}[p \times 14 + (1-p) \times 0]\} = 6.0275$

$C_{dd} = \max\{0, \frac{1}{R}[p \times 0 + (1-p) \times 0]\} = 0$

$C_u = \max\{14, \frac{1}{R}[p \times C_{uu} + (1-p) C_{ud}]\} = 14$ American Options 的价格.

$C_d = \max\{0, \frac{1}{R}[p \times C_{ud} + (1-p) C_{dd}]\} = 2.596$

$C = \max\{5, \frac{1}{R}[p \times C_u + (1-p) C_d]\} = 7.4419$

6. Options on Currencies

$$\begin{cases} \theta S \longrightarrow R_f S \\ R \longrightarrow R_d \end{cases}$$

7. Options on Futures

① Idea

For futures, there is a key difference: Future price is paid at $T=t$

② Setup

$$C \begin{cases} C_u \\ C_d \end{cases} \qquad B \begin{cases} \Delta(uF-F)+RB \\ \Delta(dF-F)+RB \end{cases}$$

③ Call pricing

$$\begin{cases} C_u = \Delta(uF-F)+RB \\ C_d = \Delta(dF-F)+RB \end{cases} \longrightarrow \begin{cases} \Delta = \dfrac{C_u - C_d}{F(u-d)} \\ B = \dfrac{(1-d)C_u + (u-1)C_d}{R(u-d)} \end{cases} \longrightarrow C = B = \frac{1}{R}\left[p_{future} C_u + (1-p_{future}) C_d\right],$$

where $p_{future} = \dfrac{1-d}{u-d}$

$1 \rightarrow = \frac{R}{\theta}$ (where $\theta = R$)

L17-20 Black-Scholes Formula

1. Motivation

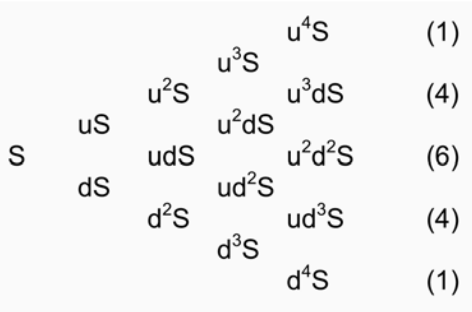
小数点后保留八位数

① Assumption

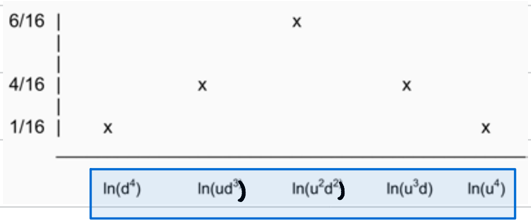
Stock price process is a Binomial tree whose each path is equally likely.

1 2

② Idea

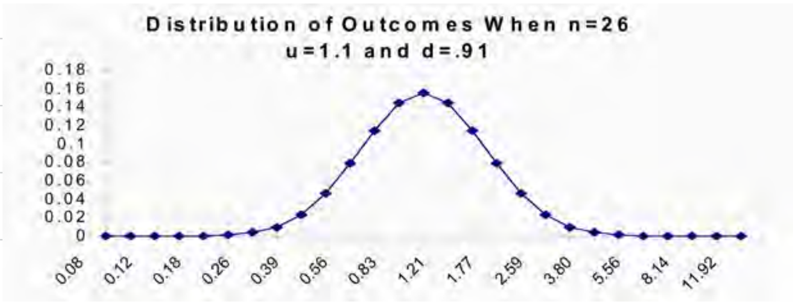


$\longrightarrow$



The CCR is $\ln(\frac{S_T}{S})$

More paths



If we let $n \to \infty$, the distribution of CCR is normal $CCR = \ln(\frac{S_T}{S}) \sim N(\mu, \sigma^2)$

the distribution of S_T is lognormal $\frac{S_T}{S} \sim \log N(\mu, \sigma^2)$

③ Model Consistency (when BS formula can be used based on Binomial model assumption)

Assuming the distribution of returns is stationary, consistency requires that over one step $\begin{cases} \mu' = \frac{T}{n}\mu \\ \sigma'^2 = \frac{T}{n}\sigma^2 \end{cases}$

Since we have $\begin{cases} \mu' = q\ln u + (1-q)\ln d \\ \sigma'^2 = q[\ln u - \mu']^2 + (1-q)[\ln d - \mu']^2 \end{cases}$ (q is the actual probability)

then $\begin{cases} \ln u = \frac{T}{n}\mu + \sqrt{\frac{T}{n}\frac{1-q}{q}}\,\sigma \\ \ln d = \frac{T}{n}\mu - \sqrt{\frac{T}{n}\frac{q}{1-q}}\,\sigma \end{cases}$

$\begin{cases} \frac{1}{2}\ln u + \frac{1}{2}\ln d = 0 \\ \frac{1}{2}(\ln u)^2 + \frac{1}{2}(\ln d)^2 = \frac{T}{n}\sigma^2 \end{cases}$

$\begin{cases} n = \text{number of periods in the tree} \\ T = \text{the time to expiration measured in years} \end{cases}$

Simple version: $\begin{cases} \ln u = \sqrt{\frac{T}{n}}\,\sigma \\ \ln d = -\sqrt{\frac{T}{n}}\,\sigma \end{cases} \longrightarrow \begin{cases} u = e^{\sqrt{\frac{T}{n}}\sigma} \\ d = e^{-\sqrt{\frac{T}{n}}\sigma} = \frac{1}{u} \end{cases}$

只适用于European options

2. Black-Scholes Formula — Original Version

① Call

a. Formulas

△Binomial Formula

$$C=\frac{1}{R^n}\Big[\sum_{j=0}^{n}\underbrace{\frac{n!}{j!(n-j)!}}_{C_n^j}p^j(1-p)^{n-j}\max\{0,u^jd^{n-j}S-K\}\Big]$$

$\begin{cases} j & u\text{的数目} \\ n-j & d\text{的数目}\end{cases}$，如 Cuud: $C_3^2\cdot p^2(1-p)^1$

△BS Formula

$C=SN(d_1)-Ke^{-rT}N(d_2)$, where $\begin{cases} d_1=\frac{1}{\sigma\sqrt{T}}[\ln(\frac{S}{K})+(r+\frac{\sigma^2}{2})T] \\ d_2=d_1-\sigma\sqrt{T}\end{cases}$

$N(\cdot)$ is the cdf of SN Distribution: $N(-x)=1-N(x)$

b. Computation of Δ & B

$\begin{cases} C=SN(d_1)-Ke^{-rT}N(d_2) \\ C=S\Delta_{call}+B_{call}\end{cases}$

$\longrightarrow \begin{cases} \Delta_{call}=N(d_1) & \in[0,1] \\ B_{call}=-Ke^{-rT}N(d_2) & <0\end{cases}$

② Put

a. Formula

$P=C-S+Ke^{-rT}$

$=SN(d_1)-Ke^{-rT}N(d_2)-S+Ke^{-rT}$

$=S(N(d_1)-1)-Ke^{-rT}(N(d_2)-1)$ where $\begin{cases} d_1=\frac{1}{\sigma\sqrt{T}}[\ln(\frac{S}{K})+(r+\frac{\sigma^2}{2})T] \\ d_2=d_1-\sigma\sqrt{T}\end{cases}$

$=-SN(-d_1)+Ke^{-rT}N(-d_2)$ (符号与C完全相反)

b. Computation of Δ & B

$\begin{cases} P=-SN(-d_1)+Ke^{-rT}N(-d_2) \\ P=S\Delta_{put}+B_{put}\end{cases}$

$\longrightarrow \begin{cases} \Delta_{put}=-N(-d_1) & \in[-1,0] \\ B_{put}=Ke^{-rT}N(-d_2) & >0\end{cases}$

3. Black-Scholes Formula — Extension

① With Div Payment

a. Call

d_1 & d_2 也变了

$$C = Se^{-\delta T}N(d_1) - Ke^{-rT}N(d_2), \text{ where } \begin{cases} d_1 = \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{Se^{-\delta T}}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T\right] \\ d_2 = d_1 - \sigma\sqrt{T} \end{cases}$$

$S \rightarrow Se^{-\delta T}$

b. Put

$$P = -Se^{-\delta T}N(-d_1) + Ke^{-rT}N(-d_2)$$

② For options on Currency

a. Call

$$C = Se^{-r_f T}N(d_1) - Ke^{-r_d T}N(d_2), \text{ where } \begin{cases} d_1 = \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{Se^{-r_f T}}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T\right] \\ d_2 = d_1 - \sigma\sqrt{T} \end{cases}$$

b. Put

$$P = -Se^{-r_f T}N(-d_1) + Ke^{-r_d T}N(-d_2)$$

③ For options on futures

a. Call

$$C = Se^{-rT}N(d_1) - Ke^{-rT}N(d_2), \text{ where } \begin{cases} d_1 = \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{Se^{-rT}}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T\right] \\ d_2 = d_1 - \sigma\sqrt{T} \end{cases}$$

$S \rightarrow Se^{-rT}$ $(\delta = r)$

b. Put

$$P = -Se^{-rT}N(-d_1) + Ke^{-rT}N(-d_2)$$

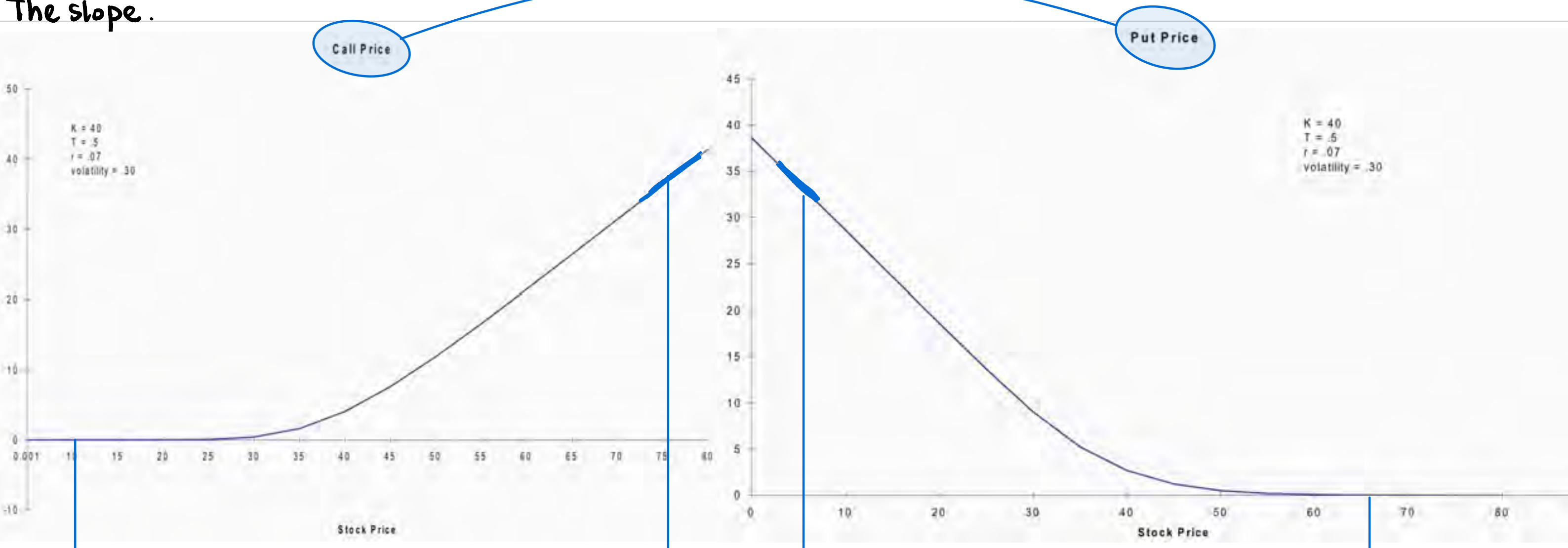
4. Delta Hedging

① Delta

a. Graphic interpretation

The slope.

长得非常像对应的Payoff



$\Delta \to 0$ —— $S_T \uparrow$, Sensitivity $\uparrow$ ——→ $\Delta \to 1$

$\Delta \to -1$ —— $S_T \uparrow$, Sensitivity $\downarrow$ ——→ $\Delta \to 0$

一阶导 ↑

二阶导 ↑, ↓

b. Delta of a portfolio

$$\begin{cases} \Delta_{Stock} = \dfrac{\partial S}{\partial S} = 1 \\ \Delta_{Bond} = \dfrac{\partial B}{\partial S} = 0 \\ \Delta_{call} = \dfrac{\partial C}{\partial S} = N(d_1) \in [0, 1] \\ \Delta_{put} = \dfrac{\partial P}{\partial S} = N(d_1) - 1 \in [-1, 0] \end{cases}$$

$\Delta_{portfolio} = \sum N_i \Delta_i$ $\begin{cases} N_i: \text{the number of units} \\ \Delta_i \end{cases}$

② Delta Neutral

a. Definition

$\sum N_i \Delta_i = 0$

b. Interpretation

Locally

Stock price ↑/↓ (small change in short time) , The value of the position –

③ Delta Hedging

a. Definition

Hedge to make the portfolio Delta Neutral

b. Comparison with Perfect Hedging

Delta hedging ≠ Perfect hedging

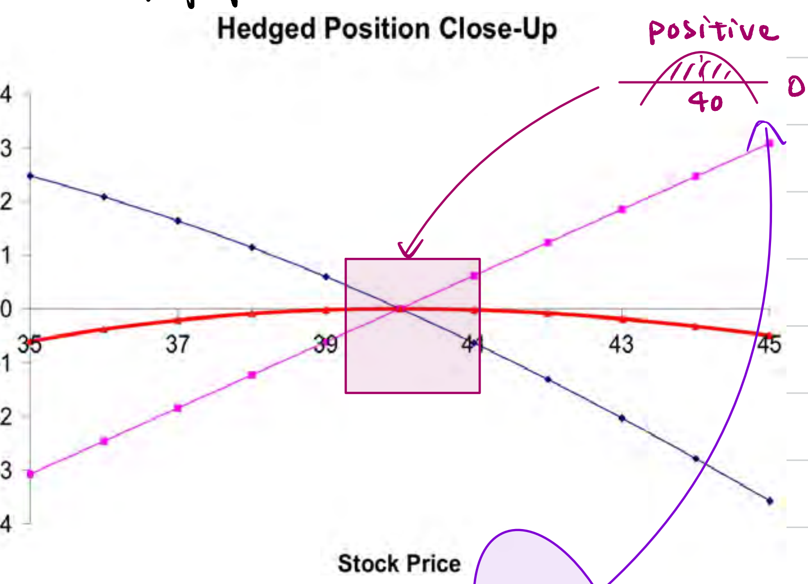
- Delta hedging : Local behavior
- Perfect hedging : Global behavior

Delta Hedging $\underset{\checkmark}{\overset{\times}{\rightleftarrows}}$ Perfect Hedging

c. Significance

① Reduce risk exposure

② Positive profit for HFT



$S=40, K=40, r=7\%, \sigma=0.3, T=0.5$

$S=41/39, K=40, r=7\%, \sigma=0.3, T=0.49$

HFT dynamically changes Δ, B according to small changes in S ⟶ Delta Hedging

↓

Positive profit

Days to Expiration	91	90	89	88	87
Stock Price	40	39.50	42	42.50	40.50
Call Price	2.7295	2.4339	3.9720	4.3112	2.9568
Delta	.5759	.5422	.6980	.7257	.6072
# of Calls	-100	-100	-100	-100	-100
# of Shares	57.59	54.22	69.80	72.57	60.72
Capital Tied Up in the Position	**57.59*40 - 2.7295*100 = 2030.65**	**54.22*39.50 - 2.4339*100 = 1898.30**	**69.80*42 - 3.9720*100 = 2534.40**	**72.57*42.50 - 4.3112*100 = 2653.105**	**60.72*40.50 - 2.9568*100 = 2163.48**
Capital Gain on Call Position*		29.56	-153.81	-33.92	135.44
Capital Gain on the Stock Position*		-28.79	135.55	34.90	-145.14
Interest Expense		-.39	-.36	-.49	-.51
Profit		.38	-18.62	.49	-10.21
Change in Stock Position		-3.37	15.58	2.77	-11.85

Delta hedging
- Decrease the risk.
- Get positive profit if S does not change much

5. Delta-Gamma Hedging

① Gamma

a. Definition

$\Gamma = \frac{\partial \Delta}{\partial S} = \frac{\partial^2 V}{\partial S^2}$ (put, stock, bond ...)

b. Interpretation

Γ is the change of number of shares needed to maintain a delta neutral position

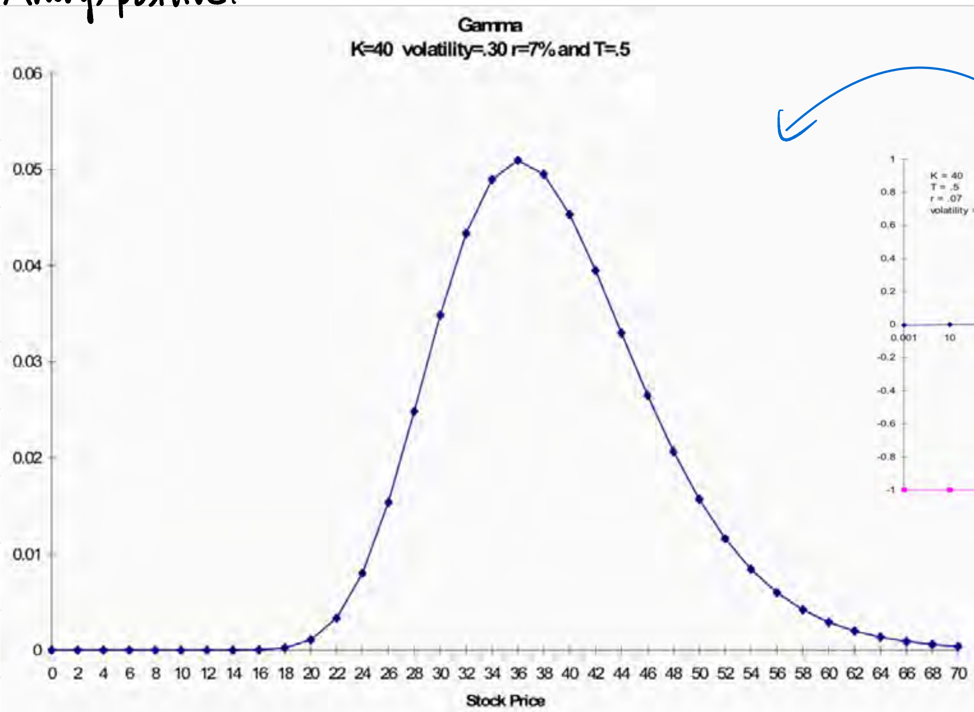
e.g. if S increases by \$1, and $\Gamma = 0.05$, which means Δ will increase 0.05

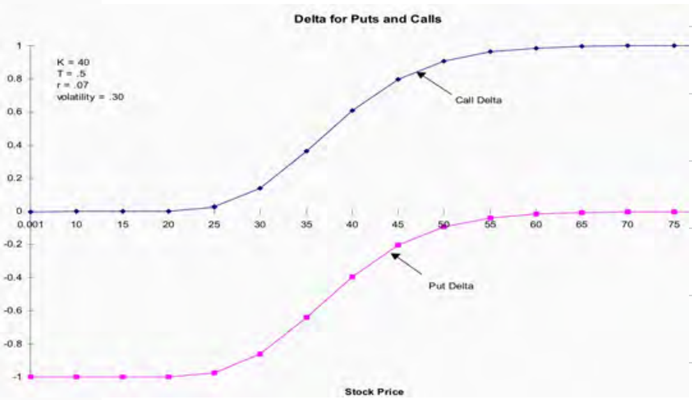
to maintain delta neutral position, I need to short more 0.05 shares.) cancel out

→ Maintain Δ neutral.

c. Graph

Always positive.





② Gamma Neutral

a. Components

$\Gamma_{call} = \Gamma_{put}$ $\quad \Delta_{call} = N(d_1) = 1 - N(-d_1) = 1 + \Delta_{put} \longrightarrow \Gamma_{call} = \Gamma_{put}$

$\Gamma_{stock} = 0$ $\quad \Delta_{stock} = 1$

$\Gamma_{Bond} = 0$ $\quad \Delta_{Bond} = 0$

b. Definition

$\Gamma_{port} = \sum_i N_i \Gamma_i = 0$

③ Delta-Gamma Hedging

a. Process

Step 1: Gamma hedging (Long one options, Short another type of options)

Step 2: Delta hedging

- Example: you write 1 European call with $S = 40$, $K = 40$, $\sigma = 30\%$, $r = 8\%$, $T = 91$ days and $\delta = 0$.
- Here are some information.

	40 strike	45 strike	Sell 40 call, buy 1.2408 45 calls
Price	2.7847	1.3584	-1.0993
Delta	0.5825	0.3285	-0.1749
Gamma	0.0651	0.0524	0.0000
Vega	0.0781	0.0831	0.0250
Theta	-0.0173	-0.0129	0.0013
Rho	0.0513	0.0389	-0.0031

- How could you make your position both delta and gamma neutral?
  - buy $\frac{0.0651}{0.0524} = 1.2408$ 45-strike call;
  - buying 0.1749 shares of stocks.

Step 1 (Gamma) : $-0.0651 + 1.2408 \times 0.0524 = 0$

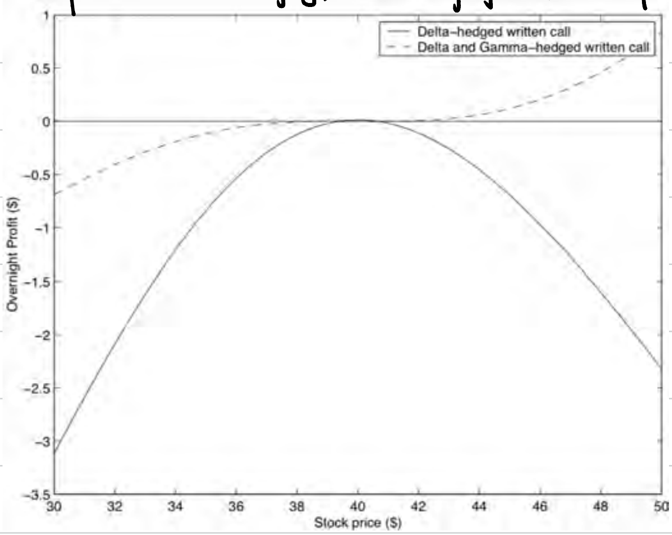
Step 2 (Delta) : $-0.1749 + \Delta = 0 \rightarrow$

Step 3 (Price) : $2.7849 - 1.3584 \times 1.2408 - 0.1749 \times 40 + B = 0 \longrightarrow B$

→ if self-financing (initial investment = 0), then we can derive B.

b. Significance

Compared to D Hedging, DG Hedging makes the portfolio's profit less sensitive to stock price



local behavior

↓

only when S changes little

f: profit

x: Stock price

According to Taylor expansion: $f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2!} f''(x_0)(x - x_0)^2 + \frac{1}{3!} f'''(x_0)(x - x_0)^3 + \cdots$

DH: —— ⟶ 剩2次项

DGH: —— ⟶ 剩3次项

6 Theta

① Definition

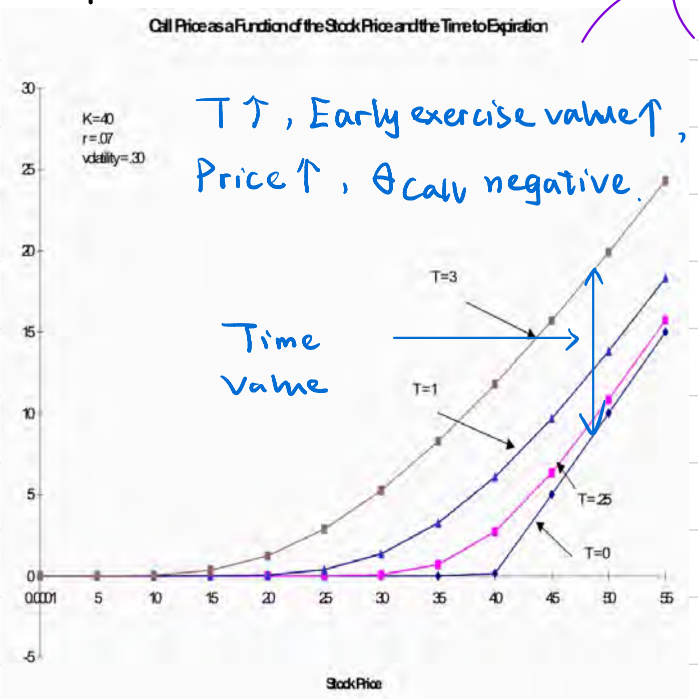
$\theta = \frac{\partial V}{\partial(-T)}$ when time to expiration becomes shorter

② Graph

B: Div
L: Int, Pro

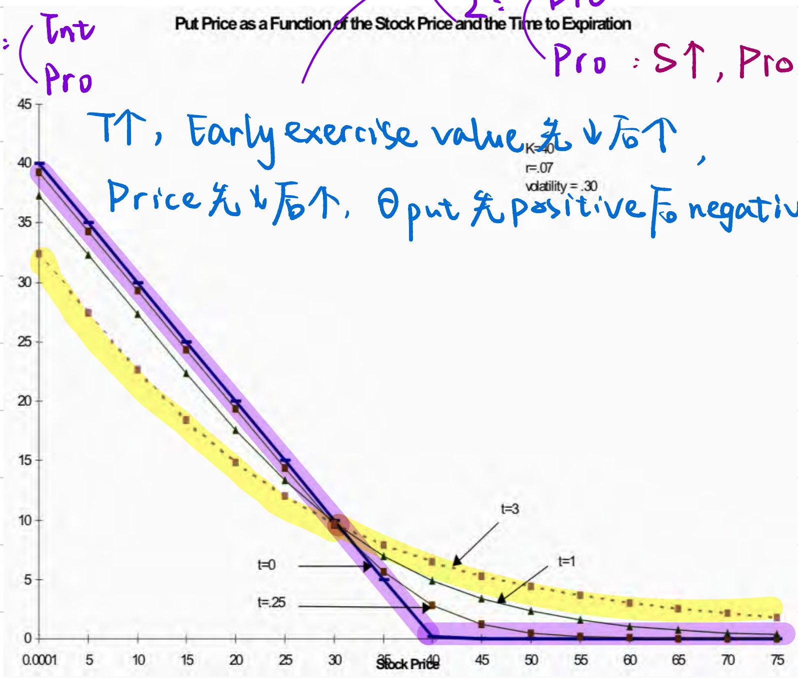
T↑, Early exercise value↑, Price↑, θ_{call} negative.

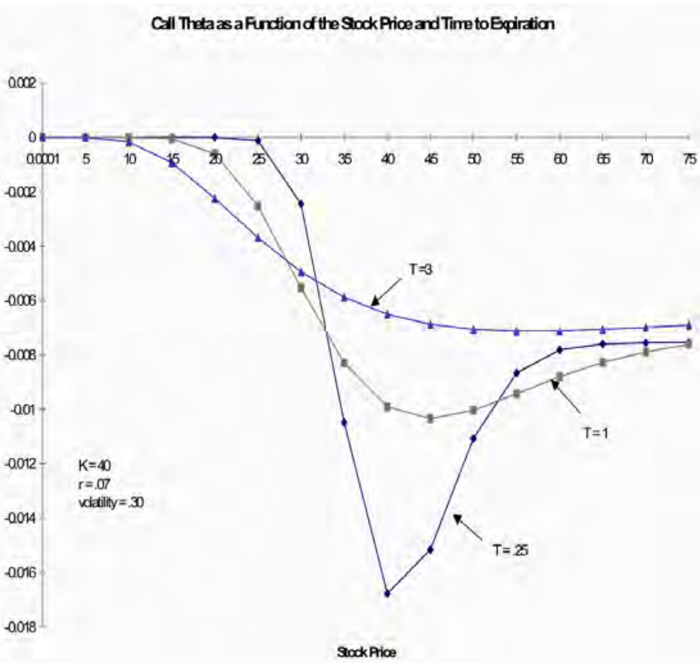
Time value

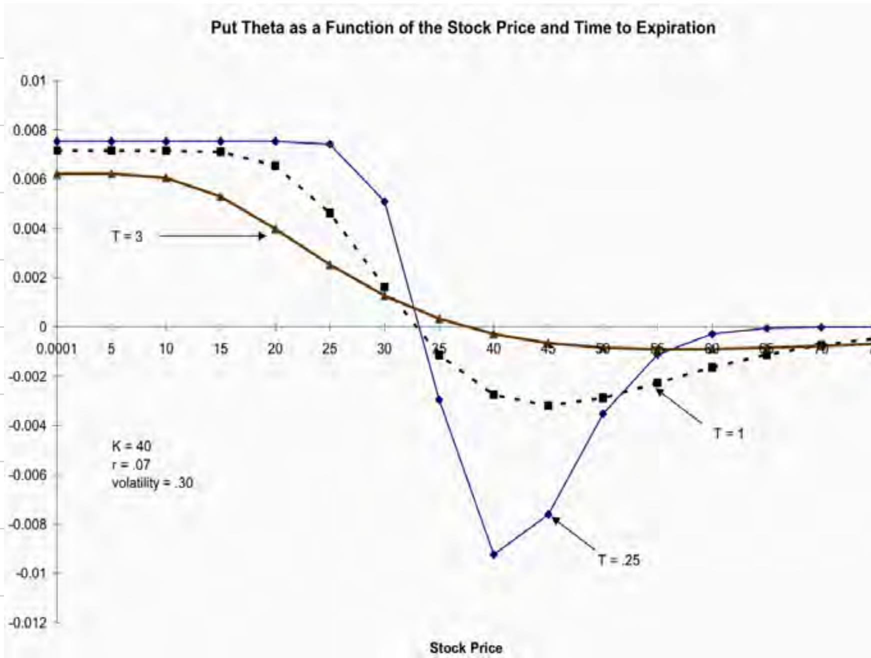


B: Int
L: Div, Pro: S↑, Pro↑

T↑, Early exercise value 先↓后↑, Price 先↓后↑, θ_{put} 先 positive 后 negative







③ Relationship

$\theta_{call} = \theta_{put} - PV(K)\cdot r$

$C = P + S - PV(K)$

$\rightarrow \frac{\partial C}{\partial(-T)} = \frac{\partial P}{\partial(-T)} + 0 - e^{-rT}K\cdot r$ (记着不是-r) $\rightarrow \theta_{call} = \theta_{put} - PV(K)\cdot r$

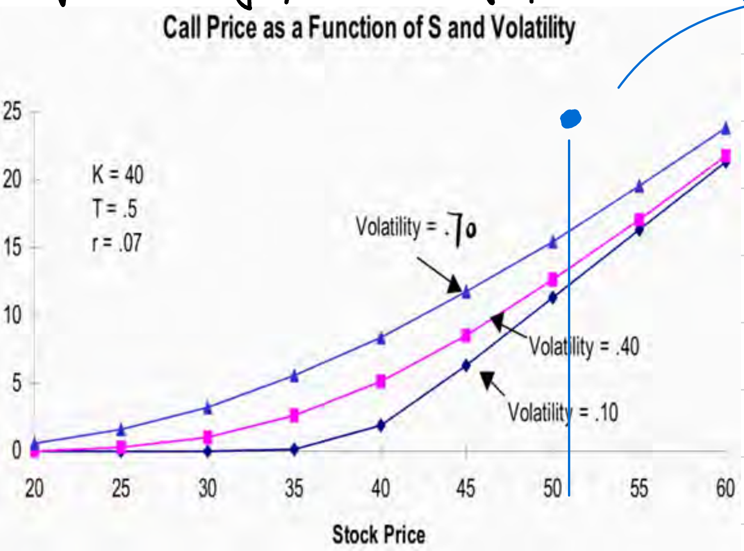
7. Vega

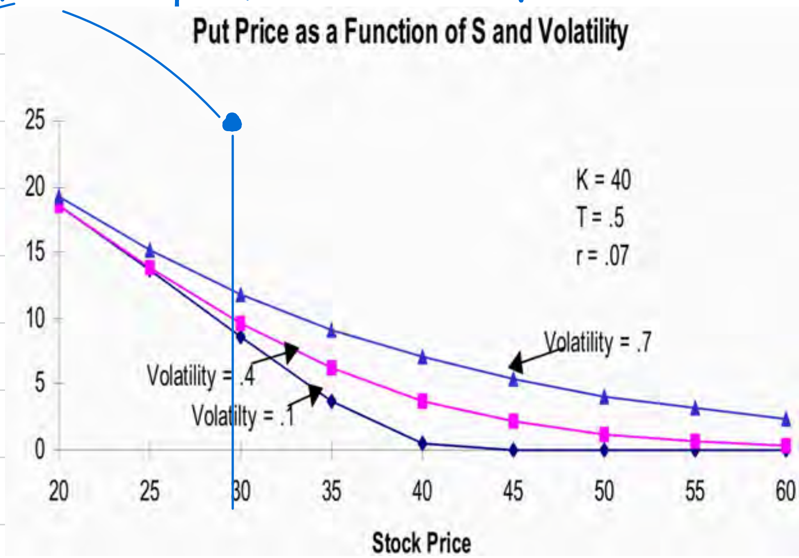
① Definition

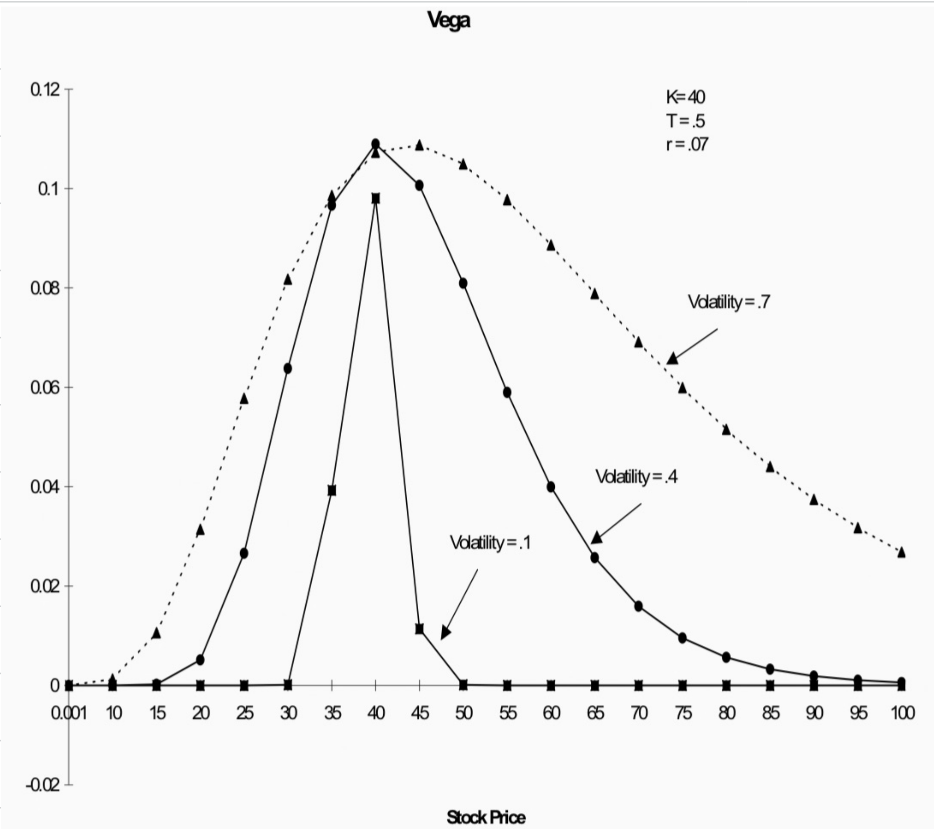
$$\text{Vega} = \frac{\partial V}{\partial \sigma}$$

② Graph

Vega_{call} & Vega_{put} are always positive. ($\sigma\uparrow$, demand for options $\uparrow$, options price $\uparrow$)







③ Relationship

$\text{Vega}_{call} = \text{Vega}_{put}$

$C = P + S - PV(K)$, $\frac{\partial C}{\partial \sigma} = \frac{\partial P}{\partial \sigma} + 0 - 0 \longrightarrow \text{Vega}_{call} = \text{Vega}_{put}$

$V(\underset{\Delta,\Gamma}{S_t}, K, \underset{\theta}{T}, \underset{\rho}{r}, \underset{\nu}{\sigma}) \longrightarrow$ 求对应偏导时，令其他自变量不变。

8. Rho

① Definition

$$\rho = \frac{\partial V}{\partial r}$$

9. Implied Volatility

① Definition

Observed volatility / Market's assessment of the volatility.

$C_{market}, S, T, R \longrightarrow \sigma_{implied}$

$C_{true}, S, T, R \longleftarrow \sigma_{true}$ (When at the money, $\sigma_{implied} \approx \sigma_{true}$)

② Volatility smile / smirk

If $\sigma_{true} < \sigma_{implied}$,

$\boxed{\text{B-S}}$ underprices Call → 对应 σ_{true}

$\boxed{\text{B-S}}$ underprices Put

B-S underprices (out-of-the-money puts & in-the-money calls)

B-S overprices (out-of-the-money calls & in-the-money puts)

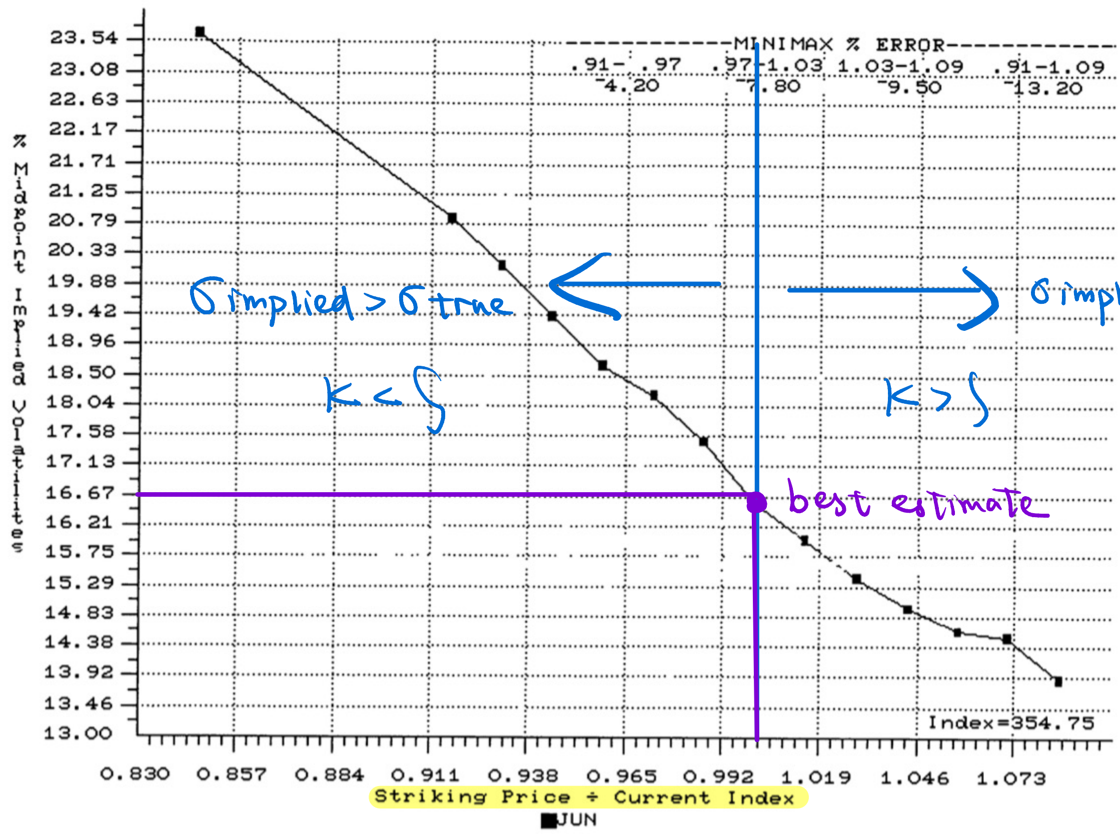
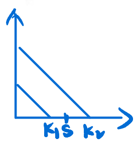


Figure 2. Typical postcrash smile. Implied combined volatilities of S&P 500 index options (January 2, 1990; 10:00 A.M.).

After a crash →



(K<S)的put Demand↑, price↑, $\sigma_{implied}$↑

10. B-S price vs Market price

① Reasons for difference

Most important

a. Assumption violation : Violations of lognormality of stock prices (Security prices can suddenly jump) → Jumps imply fatter tails.

b. Pricing violation : Transaction costs $\xrightarrow{\times}$ Continuously dynamically adjust the portfolio

c. Formula-term violation: Stochastic risk-free rate

11. Extension — Pricing Equity, Debt ⋯

① Motivation

Replicate the payoff using options, then do pricing with the help of option pricing.

② Examples

方法：写Payoff表达式 → 画Payoff Diagram → Replication by options

a. Example 1 : Equity, Senior Bonds and Junior Bonds

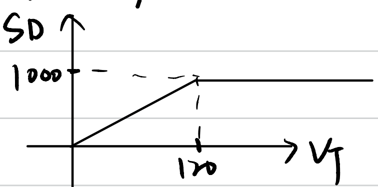
Suppose a holding company H has as its sole assets 1000 shares of firm A. Shares of firm A currently sells for \$127/share. Firm A pays no dividends. Total market value of $H = 127(1000) = \$127,000$. The interest rate is $r = 7\%$ and the volatility of stock A is $\sigma = 40\%$. Firm will be liquidated in six months. Suppose claims on H consist of the following three types of securities:

1. 1000 shares H stock.
2. Senior Debt: 120 bonds (6-month, zero coupon, FV=1000).
3. Junior Debt: 30 bonds (6-month, zero coupon, FV= 1000).

(Let V_T be the asset per shares)

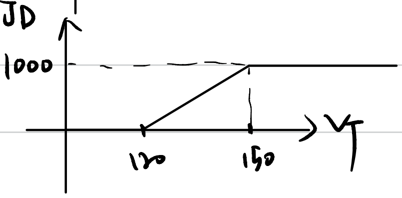
$$\begin{cases} V_T \in [0, 120]: & S_D = \frac{V_T}{120},\ J_D = 0,\ S = 0 \\ V_T \in (120, 120+30]: & S_D = 1000,\ J_D = \frac{1000V_T - 1000\times 120}{30},\ S = 0 \\ V_T \in (120+30, +\infty): & S_D = 1000,\ J_D = 1000,\ S = V_T - 150 \end{cases}$$

Payoff of SD

→ 

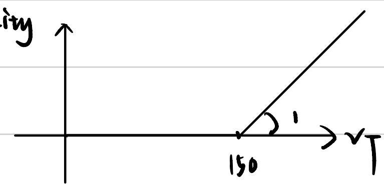
: $PV(1000) - \underbrace{\left(\frac{1000}{120}\right)}_{\text{unit = slope}} P(K=120)$

Payoff of JD

→ 

: $V_0 = E_0 + V_0^{SD} + V_0^{JD}$

Payoff of Equity

→ 

: $C(K=150)$

b. Example 2: Convertible bonds

Convertible bonds are liabilities, and it is given the right to convert into stocks

Example. Suppose holding company H has 1000 shares of stock A as assets and has issued the following claims:

1. 1000 shares of H.
2. 120 convertible bonds (6-mo, zero-coupon, fv=1000) with a conversion ratio of $33\frac{1}{3}$ (at maturity) (i.e. ea. bond can be converted in $33\frac{1}{3}$ shares of stock).

Find the prices of the H equity and bonds.

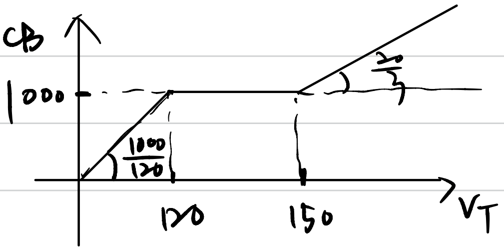
Default, not convert : $V_T \in [0, 120)$: $CB = \frac{1000V_T}{120}$, $S = 0$

Not default, not convert : $V_T \in [120, 150)$: $CB = 1000$, $S = \frac{1000V_T - 120000}{1000} = V_T - 120$

Not default, convert : $V_T \in [150, +\infty)$: $CB = \frac{V_T}{5} \times \frac{100}{3} = \frac{20V_T}{3}$, $S = \frac{1}{5}V_T$

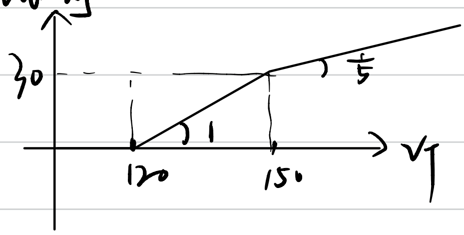
When to convert?: $\frac{1000V_T}{1000+120\times\frac{100}{3}} \times \frac{100}{3} \geq 1000 \rightarrow V_T \geq 150$

→ Payoff of CB



→ $CB = PV(1000) + \frac{20}{3}C(K=150) - \frac{1000}{120}P(K=120)$

→ Payoff of Equity



→ $S = C(K=120) - \frac{4}{5}C(K=150)$

C. Example 4: Warrants

Warrants are liabilities, but similar to options

Example. Suppose a holding company H has 1000 shares of stock A as assets and has issued the following claims:

1. 1000 shares of H
2. 250 European Warrants. Each warrant allows the holder to purchase 1 share for \$120 and the warrants expire in 6 months.

Find the prices of the H shares and warrants assuming A shares trade for \$127 per share and has volatility of 40%, with $r = 7\%$.

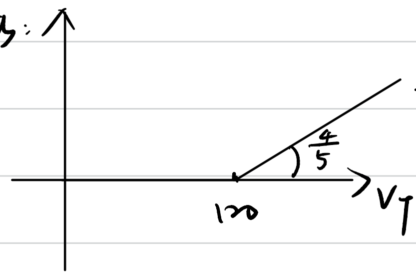
Default, not exercise: Impossible to default

Not default, not exercise: $V_T \in [0, 120)$: $W = 0$; $S = V_T$

Not default, exercise: $V_T \in [120, +\infty)$: $W = \frac{1000V_T + 250\times120}{1250} - 120 = \frac{4}{5}V_T - 96$; $S = \frac{1000V_T + 250\times 120}{1250} = \frac{4}{5}V_T + 24$

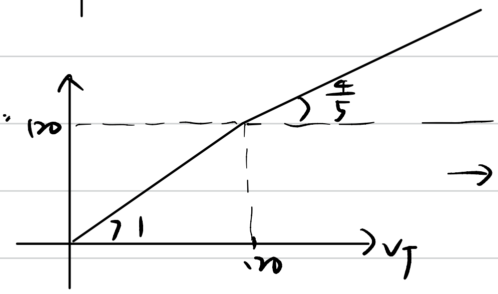
When to exercise?: $\frac{V_T \times 1000 + 250\times120}{1000+250} \geq 120 \to V_T \geq 120$

→ Payoff of warrants:



→ $W = \frac{4}{5} C(K=120)$

→ Payoff of Equity:



→ $S = PV(120) + \frac{4}{5} C(K=120) - P(K=120)$