The analytic hierarchy process (AHP)

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Background

To Tsinghua or Peking University?
Publishing Nature or Science?

Atmosphere of learning
Campus view
Proportion of handsome men
Job prospects

Background

	Weight	Tsinghua	Peking
Atmosphere of learning	0.1	0.7	0.3
Campus view	0.1	0.5	0.5
Proportion of handsome men	0.7	0.7	0.7
Job prospects	0.1	0.25	0.75

Essence

AHP is a comprehensive evaluation method for system analysis and decision-making.

The main feature of AHP is that by **establishing a hierarchical structure**, human judgment is transformed into **a comparison of the importance of a number of factors**.

Inspirational example

Xiao Ming wants to travel. He initially selected Nantong, Suzhou and Changsha were selected as the target attractions. Please choose the most suitable scheme for Xiaoming.

- What is **the goal** of our evaluation?

  A: Choose the best tourist attractions for Xiao Ming.

- What are **the options** we have to achieve this goal?

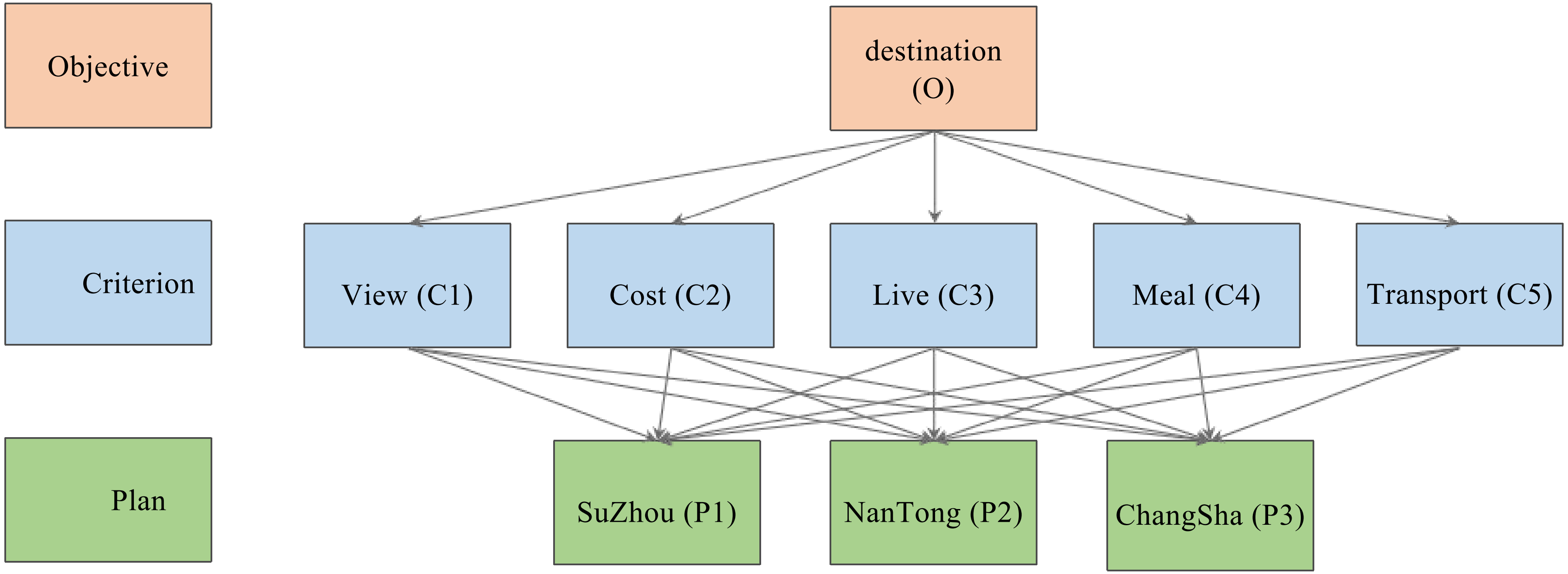
  A: Three, to Nantong, Suzhou and Changsha.

- What are **the evaluation criteria or indicators**? (By what do we judge good or bad?)

  A: The question does not give relevant data support, so we need to confirm.

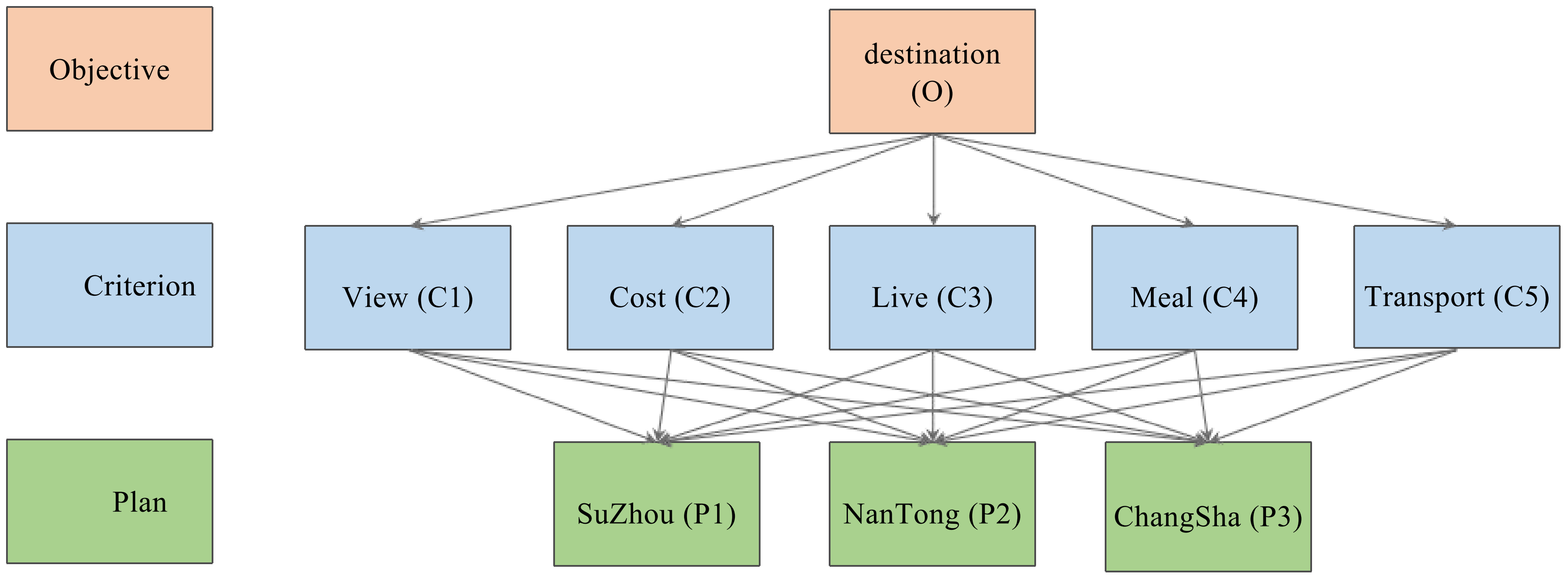
Procedure

Step 1. Analyze the relationship between the factors in the system and establish a **hierarchical structure** for the system.

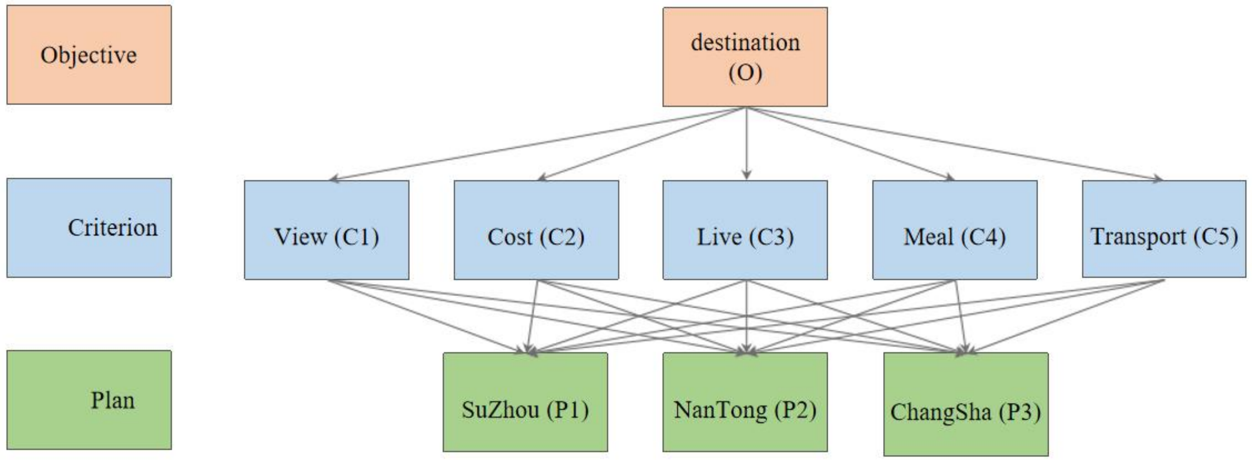


Procedure

Step 2. **A two-by-two comparison** is made between elements of the same level with respect to the importance of a criterion in the previous level, and a two-by-two comparison matrix (**judgment matrix**) is constructed.



Procedure



O	C1	C2	C3	C4	C5
C1	1	1/2	4	3	3
C2	2	1	7	5	5
C3	1/4	1/7	1	1/2	1/3
C4	1/3	1/5	2	1	1
C5	1/3	1/5	3	1	1

The name of the matrix on the left is: **Judgment Matrix** O - C.

Tip: Any evaluation class model is subjective:

Ideal: use expert group judgment

Reality: almost always filled in by yourself

Definition——Judgment Matrix

O	C1	C2	C3	C4	C5
C1	1	1/2	4	3	3
C2	2	1	7	5	5
C3	1/4	1/7	1	1/2	1/3
C4	1/3	1/5	2	1	1
C5	1/3	1/5	3	1	1

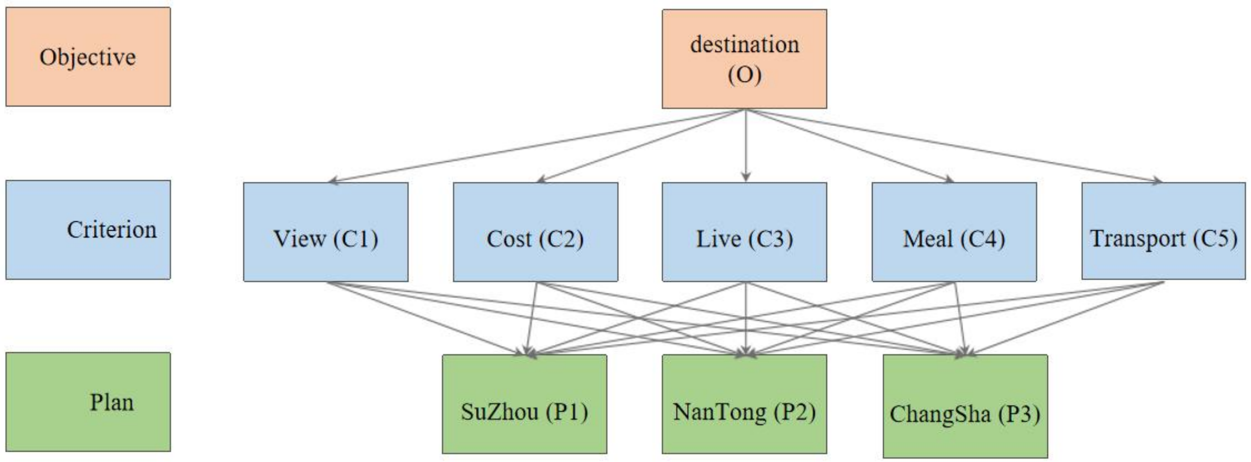
Summarize: we denote the square matrix as A, and the corresponding elements are aij .

(1) aij denotes the significance of the importance of i compared to indicator j.

(2) When i = j, the two indicators are the same, so the equal importance is recorded as 1, which explains that the main diagonal element is 1.

*(3) aij > 0 and satisfy aij*aji = 1 (we call the matrix that satisfies this condition a positive inverse matrix)*

Procedure



C1	P1	P2	P3
P1	1	2	4
P2	1/2	1	2
P3	1/4	1/2	1

JM: C1-P

C2	P1	P2	P3
P1	1	1/3	1/8
P2	3	1	1/3
P3	8	3	1

JM: C2-P

C3	P1	P2	P3
P1	1	1	3
P2	1	1	3
P3	1/3	1/3	1

JM: C3-P

C4	P1	P2	P3
P1	1	3	4
P2	1/3	1	1
P3	1/4	1	1

JM: C4-P

C5	P1	P2	P3
P1	1	1	1/4
P2	1	1	1/4
P3	4	4	1

JM: C5-P

Procedure

Step 3. Calculate the **relative weights** of the elements being compared with respect to the criterion from the judgment matrix and perform **a consistency test** (the weights can only be used if the test is passed).

Three methods are used to calculate the weights: (1) arithmetic mean (2) geometric mean (3) characteristic value method

Note:

(1) the **consistency matrix** does not need consistency test;

(2) The consistency test should be carried out first, and then calculate the weights after passing the test.

Consistency matrix

O	C1	C2	C3	C4	C5
C1	1	1/2	4	3	3
C2	2	1	8	6	6
C3	1/4	1/8	1	3/4	3/4
C4	1/3	1/6	4/3	1	1
C5	1/3	1/6	4/3	1	1

We call a matrix a positive inverse matrix if each element aij > 0 and satisfies aij*aji = 1. In hierarchical analysis, the judgment matrices we construct are all positive inverse matrices.

If the positive inverse matrix satisfies **aij*ajk = aik**, then we call it a consistent matrix.

Observe the characteristics of this matrix above: the rows (columns) are multiplicative of each other

Reference

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \text{为一致矩阵的充要条件:} \begin{cases} a_{ij} > 0 \\ a_{11} = a_{22} = \cdots = a_{nn} = 1 \\ [a_{i1}, a_{i2}, \cdots, a_{in}] = k_i [a_{11}, a_{12}, \cdots, a_{1n}] \end{cases}$$

引理： A为n阶方阵，且$r(A)=1$，则A有一个特征值为$tr(A)$，其余特征值均为0.

因为一致矩阵的各行成比例且不是零矩阵，所以一致矩阵的秩一定为1.

由引理可知：一致矩阵有一个特征值为n，其余特征值均为0.

另外，我们很容易得到，特征值为n时，对应的特征向量刚好为$k\left[\frac{1}{a_{11}}, \frac{1}{a_{12}}, \cdots, \frac{1}{a_{1n}}\right]^T \ (k \neq 0)$

引理: n阶正互反矩阵A为一致矩阵时当且仅当最大特征值$\lambda_{\max} = n$.

且当正互反矩阵A非一致时，一定满足$\lambda_{\max} > n$.

Reference: QingFeng mathematical model

Consistency test

Step 1: Calculate the consistency indicator CI

$$CI = \frac{\lambda_{\max} - n}{n - 1}$$

Step 2: Find the corresponding average random consistency indicator RI

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
RI	0	0	0.52	0.89	1.12	1.26	1.36	1.41	1.46	1.49	1.52	1.54	1.56	1.58	1.59

Step 3: Calculate the consistency ratio CR

$$CR = \frac{CI}{RI}$$

If CR < 0.1, the consistency of the judgment matrix is acceptable; otherwise, the judgment matrix needs to be corrected.

Consistency test

Unfortunately, the CR > 0.1. How can we deal with it?

1	2	5
1/2	1	2
1/5	1/2	1

1	2	4
1/2	1	2
1/4	1/2	1

It's easy, just transform the judgment matrix into a consistency matrix as much as possible.

Calculate weight

How are the weights calculated for the consistency matrix?

C1	P1	P2	P3
P1	1	2	4
P2	1/2	1	2
P3	1/4	1/2	1

$$P1 = 1 /(1+0.5+0.25)$$
$$P2 = 0.5 /(1+0.5+0.25)$$
$$P3 = 0.25 /(1+0.5+0.25)$$

Weights must be normalized!

Calculate weight

How the judgment matrix calculates the weights?

C1	P1	P2	P3
P1	1	2	5
P2	1/2	1	2
P3	1/5	1/2	1

The first column

P1 = 1 /(1+0.5+0.2)= 0.5882

P2 = 0.5 /(1+0.5+0.2)= 0.2941

P3 = 0.2 /(1+0.5+0.2)= 0.1177

The second column

P1 = 2 /(2+1+0.5)= 0.5714

P2 = 1 /(2+1+0.5)= 0.2857

P3 = 0.5 /(2+1+0.5)= 0.1429

The third column

P1 = 5 /(5+2+1)= 0.625

P2 = 2 /(5+2+1)= 0.25

P3 = 1 /(5+2+1)= 0.125

C1	P1	P2	P3
P1	0.5882	0.5714	0.625
P2	0.2941	0.2857	0.25
P3	0.1177	0.1429	0.125

Calculate weight

C1	P1	P2	P3
P1	0.5882	0.5714	0.625
P2	0.2941	0.2857	0.25
P3	0.1177	0.1429	0.125

Method 1. Arithmetic averaging for weights

(0.5882+0.5714+0.625)/3

Method 2. Geometric averaging for weights

(0.5882*0.5714*0.625) ^ (1/3)

Calculate weight

Method 3. Eigenvalue method for finding weights

If the consistency of our judgment matrix is acceptable, then we can follow the method of finding the weights of the consistent matrix.

Step 1: Find the largest eigenvalue of matrix A and its corresponding eigenvector.

Step 2: Normalize the eigenvectors to get our weights.