Problem 1:

① Find a biquadratic, rational parameterization of this torus.

$\Rightarrow$ Let $Q = \tan\frac{a}{2}$ $\quad T = \tan\frac{\beta}{2}$

$$\begin{cases} X = (R + r\cos a)\cos\beta \\ \quad = \left(R + r\cdot\frac{1-Q^2}{1+Q^2}\right)\cdot\left(\frac{1-T^2}{1+T^2}\right) \\ Y = (R + r\cos a)\cdot\sin\beta \\ \quad = \left(R + r\cdot\frac{1-Q^2}{1+Q^2}\right)\cdot\left(\frac{2T}{1+T^2}\right) \\ Z = r\cdot\sin a \\ \quad = r\cdot\frac{2Q}{1+Q^2} \end{cases}$$

$\therefore$ For any point $V(1; X, Y, Z)$ in the torus, we can get its parameterization:

$$\boxed{\begin{cases} W(Q;T) = (1+Q^2)(1+T^2) \\ X(Q;T) = [(1+Q^2)\cdot R + r(1-Q^2)]\cdot(1-T^2) = (1-T^2)[R+r+Q^2(R-r)] \\ Y(Q;T) = ((1+Q^2)\cdot R + r(1-Q^2))\cdot 2T = [R+r+Q^2(R-r)]\cdot 2T \\ Z(Q;T) = 2Qr\cdot(1+T^2) \end{cases}}$$

② Homogenize and polarize your parameterization.

1) We first polarize it:

$$W(Q,T) = 1+Q^2T^2+Q^2+T^2$$

$$\Rightarrow W((Q_1,Q_2),(T_1,T_2)) = 1+Q_1Q_2T_1T_2+Q_1Q_2+T_1T_2$$

$$X(Q,T) = (1-T^2)[R(1+Q^2)+r(1-Q^2)]$$

$$\Rightarrow X((Q_1,Q_2),(T_1,T_2)) = (1-T_1T_2)[R(1+Q_1Q_2)+r(1-Q_1Q_2)]$$

$$Y(Q,T) = 2T[R(1+Q^2)+r(1-Q^2)]$$

$$\Rightarrow Y((Q_1,Q_2),(T_1,T_2)) = (T_1+T_2)[R(1+Q_1Q_2)+r(1-Q_1Q_2)]$$

$$Z(Q,T) = 2Qr(1+T^2)$$

$$\Rightarrow Z((Q_1,Q_2),(T_1,T_2)) = (Q_1+Q_2)\cdot r(1+T_1T_2)$$

2) Next, we homogenize the terms:

$$W((P_1;q_1),(P_2;q_2);(S_1;t_1),(S_2;t_2)) = 1+\frac{q_1}{P_1}\cdot\frac{q_2}{P_2}\cdot\frac{t_1}{S_1}\cdot\frac{t_2}{S_2}+\frac{q_1}{P_1}\cdot\frac{q_2}{P_2}+\frac{t_1}{S_1}\cdot\frac{t_2}{S_2}$$

$$X((P_1,q_1),(P_2,q_2);(S_1;t_1),(S_2;t_2)) = \left(1-\frac{t_1}{S_1}\cdot\frac{t_2}{S_2}\right)\cdot\left[R+r+\frac{q_1}{P_1}\cdot\frac{q_2}{P_2}(R-r)\right]$$

$$Y((P_1,q_1),(P_2,q_2);(S_1;t_1),(S_2;t_2)) = \left(\frac{t_1}{S_1}+\frac{t_2}{S_2}\right)\left[R+R\cdot\frac{q_1}{P_1}\cdot\frac{q_2}{P_2}+r-r\cdot\frac{q_1}{P_1}\cdot\frac{q_2}{P_2}\right]$$

$$Z((P_1,q_1),(P_2,q_2);(S_1,t_1),(S_2,t_2)) = r\cdot\left(\frac{q_1}{P_1}+\frac{q_2}{P_2}\right)\cdot\left(1+\frac{t_1}{S_1}\cdot\frac{t_2}{S_2}\right)$$

To summarize, we get:

$$W = (S_1S_2+t_1t_2)(q_1q_2+P_1P_2)$$

$$X = (S_1S_2-t_1t_2)\cdot[R(P_1P_2+q_1q_2)+r(P_1P_2-q_1q_2)]$$

$$Y = (S_2t_1+S_1t_2)\cdot[R(P_1P_2+q_1q_2)+r(P_1P_2-q_1q_2)]$$

$$Z = r\cdot(P_2q_1+P_1q_2)(S_1S_2+t_1t_2)$$

③ Describe this portion and give the coordinates.

$\Rightarrow$ Aim: $V([0 \ldots \infty] \times [0, \ldots \infty]))$

$$\begin{cases} \Rightarrow Q \in [0, \infty] & \alpha \in [0, \pi] \\ \Rightarrow T \in [0, \infty] & \beta \in [0, \pi] \end{cases}$$

Nine coordinates:

$\Rightarrow$

$f((0,0);(0,0))$	$f((0,\infty);(0,0))$	$f((\infty,\infty);(0,0))$
$f((0,0);(0,\infty))$	$f((0,\infty);(0,\infty))$	$f((\infty,\infty);(0,\infty))$
$f((0,0);(\infty,\infty))$.	$f((0,\infty);(\infty,\infty))$	$f((\infty,\infty);(\infty,\infty))$.

$\Rightarrow$ We can first get four corner coordinates

$$f((0,0);(0,0)) = F(0,0) = (1; R+r, 0, 0)$$
$$f((\infty,\infty);(0,0)) = F(\infty,0) = (1; R-r, 0, 0)$$
$$f((0,0);(\infty,\infty)) = F(0,\infty) = (1; -(R+r), 0, 0)$$
$$f((\infty,\infty);(\infty,\infty)) = F(\infty,\infty) = (1; -(R-r), 0, 0)$$

$\Rightarrow$ We calculate other five points:

① $f((0,\infty);(0,0)) = f((1;0),(0;1);(1;0)(1;0))$

$$\begin{cases} p_1=1 \quad p_2=0 \quad s_1=1 \quad s_2=1 \\ q_1=0 \quad q_2=1 \quad t_1=0 \quad t_2=0 \end{cases}$$

$$\Rightarrow \begin{cases} x = (1\times1 - 0\times0)\cdot[R(1\times0+0\times1)+r(1\times0-0\times1)] = 0 \\ y = (1\times0+0\times1)\cdot[\cdots] = 0 \\ z = r\cdot(0\times0+1\times1)(1\times1+0\times0) = r \\ w = (1\times1+0\times0)\cdot(0\times0+1\times0) = 0 \end{cases}$$

$\Rightarrow f((0,\infty);(0,0)) = (0; 0, 0, r)$

Similarly, we get other points:

$$\begin{cases} f((0,0);(0,\infty)) = (0; 0, R+r, 0) \\ f((0,\infty);(0,\infty)) = (0; 0, 0, 0) \\ f((\infty,\infty);(0,\infty)) = (0; 0, R-r, 0) \\ f((0,\infty);(\infty,\infty)) = (0; 0, 0, r) \end{cases}$$

2(c) Determine the control points for the other three quarters

Similarly, we calculate points for other three quarters:

① Negative y and positive z: $V([0 \dots \infty] \times [-\infty \dots 0])$

$f((0,0);(-\infty,-\infty)) = (1; -(R+r), 0, 0)$

$f((0,0);(0,0)) = (1; R+r, 0, 0)$

$f((\infty,\infty);(0,0)) = (1; R-r, 0, 0)$

$f((\infty,\infty);(-\infty,-\infty)) = (1; -(R-r), 0, 0)$

$f((0,\infty);(-\infty,-\infty)) = (0; 0, 0, r)$

$f((0,0);(-\infty,0)) = (0; 0, -(R+r), 0)$

$f((0,\infty);(-\infty,0)) = (0; 0, 0, 0)$

$f((\infty,\infty);(-\infty,0)) = (0; 0, -(R-r), 0)$

$f((0,\infty);(0,0)) = (0; 0, 0, r)$

② Positive y and negative z: $V([-\infty, 0] \times [0, \dots \infty])$

$f((-\infty,-\infty);(0,0)) = (1; R-r, 0, 0)$

$f((0,0);(0,0)) = (1; R+r, 0, 0)$

$f((-\infty,-\infty);(\infty,\infty)) = (1; -(R-r), 0, 0)$

$f((0,0);(\infty,\infty)) = (1; -(R+r), 0, 0)$

$f((-\infty,0);(0,0)) = (0; 0, 0, -r)$

$f((-\infty,-\infty);(0,\infty)) = (0; 0, R-r, 0)$

$f((-\infty,0);(0,\infty)) = (0; 0, 0, 0)$

$f((0,0);(0,\infty)) = (0; 0, R+r, 0)$

$f((-\infty,0);(\infty,\infty)) = (0; 0, 0, -r)$

③ Negative Y and negative Z: $V([-\infty, \dots 0] \times [-\infty, \dots 0])$

$f((-\infty,-\infty);(-\infty,-\infty)) = (1; -(R-r), 0, 0)$

$f((-\infty,-\infty);(0,0)) = (1; R-r, 0, 0)$

$f((0,0);(-\infty,-\infty)) = (1; -(R+r), 0, 0)$

$f((0,0);(0,0)) = (1; R+r, 0, 0)$

$f((-\infty,0);(-\infty,-\infty)) = (0; 0, 0, -r)$

$f((-\infty,0);(-\infty,0)) = (0; 0, 0, 0)$

$f((-\infty,-\infty);(-\infty,0)) = (0; 0, -(R-r), 0)$

$f((0,0);(-\infty,0)) = (0; 0, -(R+r), 0)$

$f((-\infty,0);(0,0)) = (0; 0, 0, -r)$

Problem 3:

a) Show that N is the kernel of M.

$\Rightarrow$ We know that N is unrelated to θ

$$M\cdot N = \frac{1}{2\pi}\int_{-\pi}^{\pi} k_\theta (T_\theta T_\theta^T)\cdot N\, d\theta$$

$\Rightarrow$ For $T_\theta \cdot T_\theta^T$, we get its property (with $T_\theta = [T_1, T_2]\cdot\begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}$)

$$\text{Rank}(T_\theta T_\theta^T) \le \min\{\text{rank}(T_\theta), \text{rank}(T_\theta^T)\} = 1$$

$\Rightarrow$ For any matrix A of rank 1, it can be represented as $A = vw^T$

and $AU = (vw^T)U = (w^T U)\cdot v$

$$\therefore M\cdot N = \frac{1}{2\pi}\int_{-\pi}^{\pi} k_\theta \cdot \underbrace{T_\theta^T N}_{=0}\cdot T_\theta\, d\theta$$

$\Rightarrow$ Since T_θ is tangent to S at P.

$\therefore M\cdot N = 0 \Rightarrow M\cdot N = 0\cdot N$

$\Rightarrow$ N is an eigen vector of M with eigenvalue 0.

b) Show that $T_1^T M T_2 = 0$.

① We first write it as:

$$T_1^T M T_2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} T_1^T k_\theta T_\theta T_\theta^T T_2\, d\theta$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} k_\theta\, T_1^T T_\theta T_\theta^T T_2\, d\theta$$

when A of rank 1 $\quad AU = (vw^T)V = (w^T U)v$

$$\Rightarrow T_1^T M T_2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} k_\theta\, T_1^T T_\theta T_\theta^T T_2\, d\theta.$$

② We represent T_θ as:

$$T_\theta = [T_1, T_2]\cdot\begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}$$

$$\Rightarrow T_1^T T_\theta T_\theta^T T_2 = [T_1 T_1,\ T_1 T_2]\begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}\cdot[\cos\theta, \sin\theta]\cdot\begin{bmatrix}T_1 T_2\\ T_2 T_2\end{bmatrix}$$

$\Rightarrow T_1, T_2$ are orthogonal $\Rightarrow T_1\cdot T_2 = 0 \quad T_1\cdot T_1 = 1,\ T_2\cdot T_2 = 0.$

$$\Rightarrow T_1^T T_\theta T_\theta^T T_2 = [1, 0]\begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}\cdot[\cos\theta, \sin\theta]\cdot\begin{bmatrix}0\\ 1\end{bmatrix} = \sin\theta\cos\theta$$

③ We show:

$$T_1^T M T_2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} k_\theta \cdot \sin\theta\cos\theta\, d\theta$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} (k_1\cos^2\theta + k_2\sin^2\theta)\sin\theta\cos\theta\, d\theta$$

$$= \frac{k_1}{2\pi}\int_{-\pi}^{\pi} \cos^3\theta\sin\theta\, d\theta + \frac{k_2}{2\pi}\int_{-\pi}^{\pi}\sin^3\theta\cos\theta\, d\theta$$

$\Rightarrow$ both integrands are odd functions.

$\Rightarrow T_1^T M T_2 = 0.$

c) Show that $T_1^T M T_1 = \frac{3}{8}k_1 + \frac{1}{8}k_2$.

$\Rightarrow$ Similarly, we represent M_p'' as:

$$M_p'' = T_1^t M_p T_1$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} k_\theta T_1^T \cdot T_\theta \cdot T_\theta^T \cdot T_1 \, d\theta$$

$$\Rightarrow T_1^T \cdot T_\theta \cdot T_\theta^T \cdot T_1 = T_1 \cdot [T_1, T_2] \cdot \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \cdot [\cos\theta, \sin\theta] \cdot \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} \cdot T_1$$

$$= \cos\theta \cdot \cos\theta = \cos^2\theta$$

$$\Rightarrow M_p'' = \frac{1}{2\pi}\int_{-\pi}^{\pi} (k_1\cos^2\theta + k_2^2\sin^2\theta)\cdot\cos^2\theta \, d\theta$$

$$= \frac{k_1}{2\pi}\int_{-\pi}^{\pi} \cos^4\theta \, d\theta + \frac{k_2}{2\pi}\int_{-\pi}^{\pi} \cos^2\theta\sin^2\theta \, d\theta.$$

$$= \frac{k_1}{2\pi}\int_{-\pi}^{\pi} \left[\frac{1}{2}(1+\cos 2\theta)\right]^2 d\theta + \frac{k_2}{2\pi}\int_{-\pi}^{\pi} \left(\frac{1+\cos 2\theta}{2}\right)\left(\frac{1-\cos 2\theta}{2}\right) d\theta$$

$$= \frac{k_1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{4}\cdot[1+\cos^2 2\theta + 2\cos 2\theta]\cdot d\theta + \frac{k_2}{2\pi}\int_{-\pi}^{\pi} \frac{1}{4}\cdot[1-\cos^2 2\theta)\, d\theta$$

$$= \frac{k_1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{4}\cdot\left[\frac{3}{2} + 2\cos 2\theta + \frac{1}{2}\cos 4\theta\right]\cdot d\theta + \frac{k_2}{2\pi}\int_{-\pi}^{\pi} \frac{1}{8}[1-\cos 4\theta]\, d\theta$$

$$= \frac{k_1}{2\pi}\cdot\left[\frac{3}{8}x + \frac{1}{4}\sin 2\theta + \frac{1}{2}\sin 4\theta\right]_{-\pi}^{\pi} + \frac{k_2}{2\pi}\cdot\frac{1}{8}\cdot\left[x - \frac{1}{4}\sin 4\theta\right]_{-\pi}^{\pi}$$

$$= \frac{3}{8}k_1 + \frac{1}{8}k_2$$

d) find k_1 and k_2, and argue that T_1 and T_2 are eigenvectors of M with eigenvalues m_{11} and m_{22}.

1) $\Rightarrow$ From (c), we get:

$$M = \frac{1}{2\pi}\int_{-\pi}^{\pi} k_\theta T_\theta \cdot T_\theta^T d\theta$$
$$= T_{12}^t \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \cdot T_{12}$$

$$\Rightarrow \begin{cases} m_{11} = \frac{3}{8}k_1 + \frac{1}{8}k_2 \\ m_{22} = \frac{1}{8}k_1 + \frac{3}{8}k_2 \\ m_{21} = 0 \\ m_{12} = 0 \end{cases} \Rightarrow \begin{cases} k_1 = 3m_{11} - m_{22} \\ k_2 = 3m_{22} - m_{11} \end{cases}$$

2)
$$M\cdot T_1 = T_{12}^t \begin{pmatrix} m_{11} & 0 \\ 0 & m_{22} \end{pmatrix} \cdot T_{12} \cdot T_1$$

$$= [T_1, T_2] \cdot \begin{pmatrix} m_{11} & 0 \\ 0 & m_{22} \end{pmatrix} \begin{bmatrix} T_1, \\ T_2 \end{bmatrix} \cdot T_1$$

$$= [m_{11}T_1, m_{22}T_2] \cdot \begin{bmatrix} T_1, \\ T_2 \end{bmatrix} \cdot T_1 = m_{11} \cdot T_{11}$$

$\Rightarrow$ Similarly, we can get:

$$M \cdot T_2 = m_{22} \cdot T_2$$

$\Rightarrow$ We show T_1 and T_2 are eigenvectors of M with eigenvalues of m_{11} and m_{22}.

3(e) To derive the relationship

$$k(T) \approx \frac{2N^T(r(s)-p)}{\|r(s)-p\|^2}$$

$\Rightarrow$ We can approximate r with

$$r(s) \approx r(0) + r'(0)\cdot s + \frac{1}{2}\underbrace{r''(0)}_{\substack{\| \\ k(T)\cdot N}}\cdot s^2$$

$$r(s) \approx p + T\cdot s + \frac{1}{2}\cdot k(T)\cdot N\cdot s^2$$

$$\Rightarrow 2(r(s)-p-T\cdot s) \approx k(T)\cdot N\cdot s^2$$

$$\Rightarrow 2N(r(s)-p-Ts) \approx k(T)\cdot N\cdot N\cdot s^2$$

$$\Rightarrow N\cdot T\cdot s = 0$$

$$\Rightarrow \quad 2N(r(s)-p) \approx k(T)\cdot s^2$$

$\Rightarrow$ Then we approximate $\|r(s)-p\|^2$ with the inner product of r with N

$$\begin{aligned}
\langle r(s), N\rangle &\approx \langle p + T\cdot s + \tfrac{1}{2}k(T)\cdot N\cdot s^2, N\rangle \\
&\approx \langle p, N\rangle + \langle T\cdot s, N\rangle + \tfrac{1}{2}\langle k(T)\cdot N\cdot s^2, N\rangle \\
&= 0 + 0 + \tfrac{1}{2}\cdot \times 2\cdot s^2 \\
&= s^2
\end{aligned}$$

$\Rightarrow$ With $\begin{cases} 2N(r(s)-p) \approx k(T)\cdot s^2 \\ \|r(s)-p\|^2 \approx s^2 \end{cases}$, We get

$$k(T) \approx \frac{2N(r(s)-p)}{\|r(s)-p\|^2}$$