KAIST CS492- Homework4 Q2

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Questions

Assume a (p+n+1)-length knot sequence $T = (t_1, t_2, ...t_{n+p+1})$. The B-spline basis function of degree $p \in N_0$ is defined for $i = 1, ...n$ and shown as follows (refer to the slide 37, lecture 8):

$$N_{i,0}(t) \begin{cases} 1 & t_i \leq t \leq t_{i+1} \\ 0 & otherwise \end{cases} \tag{1}$$

$$N_i^p(t) = \frac{t - t_i}{t_{i+p} - t_i} N_i^{p-1}(t) + \frac{t_{i+p+1} - t}{t_{i+p+1} - t_{i+1}} N_{i+1}^{p-1}(t). \tag{2}$$

In this question, we consider the case of open knot sequence, i.e the first p+1 and the last p+1 knot points coincide $t_1 = ... = t_{p+1}$ and $t_{n+1} = ... = t_{n+p+1}$.

- Formulate the basis function of degree p that are created by the open knot sequence with $|T| = K$.
- If the first $p + 1$ repeated knot points are set to 0 while toe last $p + 1$ repeated ones set to 1. How many interior knots (e.g equally distributed in the interval $(0, 1)$) do we need to get a a total n basis functions of degree p?
- Prove that $\sum_{i=1}^{n} N_{i,p}(T) = 1$ on the interval $[t_{p+1}, t_{n+1}]$.

Solutions

- With $K = n + p + 1$, where n is the number of basis function and p the degree, **we have** $n = K - p - 1$ basis functions can be created by from the knot sequence.
- We use k to denote the number of interior knots, we can add the first $p + 1$ and the last $p + 1$ repeated ones to have a total $K = k + 2(p + 1)$. With the formula in (1), **we have** $k + 2(p + 1) = K = n + p + 1$ **and end up getting** $k = n - p - 1$**.**
- we prove it as follows:

- When $p = 0$, the claim holds obviously.
- We assume the claim holds for $p-1$ and do the induction step from $(p-1)$ to p:

Induction step:

$$\sum_{i=1}^{n} N_i^p(t) = \sum_{i=1}^{n} \frac{t-t_i}{t_{i+p}-t_i} N_i^{p-1}(t) + \sum_{i=1}^{n} \frac{t_{i+p+1}-t}{t_{i+p+1}-t_{i+1}} N_{i+1}^{p-1}(t)$$

$$= \frac{t-t_1}{t_{1+p}-t_1} \cdot N_1^{p-1}(t) + \sum_{i=2}^{n} \frac{t-t_i}{t_{i+p}-t_i} N_i^{p-1}(t) + \sum_{i=2}^{n} \frac{t_{i+p}-t}{t_{i+p}-t_i} N_i^{p-1}(t) + \frac{t_{n+p+1}-t}{t_{n+p+1}-t_{n+1}} N_{n+1}^{p-1}(t)$$

$$= \frac{t-t_1}{t_{1+p}-t_1} N_1^{p-1}(t) + \underbrace{\sum_{i=2}^{n} N_i^{p-1}(t)}_{\Downarrow \; = 1 - N_1^{p-1} - N_{n+1}^{p-1}} + \frac{t_{n+p+1}-t}{t_{n+p+1}-t_{n+1}} N_{n+1}^{p-1}(t)$$

$$= 1 + \underbrace{\left(\frac{t-t_1}{t_{1+p}-t_1} - 1\right)}_{\Downarrow \; = \frac{t-t_{1+p}}{t_{1+p}-t_1}} N_1^{p-1}(t) + \underbrace{\left(\frac{t_{n+p+1}-t}{t_{n+p+1}-t_{n+1}} - 1\right)}_{\Downarrow \; = \frac{t_{n+1}-t}{t_{n+p+1}-t_{n+1}}} N_{n+1}^{p-1}(t)$$

$$= 1 + (t-1) \cdot \frac{N_1^{p-1}(t)}{\underbrace{t_{1+p} - \varepsilon_1}_{\equiv 0}} + \underbrace{(t_{n+1}-t) \cdot \frac{N_{n+1}^{p-1}(t)}{t_{n+p+1} - t_{n+1}}}_{\equiv 0}$$

$$= 1$$