14. $\varphi_X(\theta) = Ee^{i\theta^T X}$

$= \int_0^{+\infty}\int_0^{+\infty} e^{i(\theta_1 x_1+\theta_2 x_2)} e^{-(x_1+x_2)} dx_1 dx_2$

$= \int_0^{+\infty} e^{(-1+i\theta_1)x_1} dx_1 \int_0^{+\infty} e^{(-1+i\theta_2)x_2} dx_2$

$= \frac{1}{-1+i\theta_1} \cdot \frac{1}{-1+i\theta_2}$

$= \frac{1}{1-\theta_1\theta_2 - i(\theta_1+\theta_2)}$

17. $X \sim N(\mu, \Sigma)$

$\because \Sigma$ 是一个实对称阵

$\therefore \exists$ 正交阵 Q，对角阵 Λ

s.t. $\Sigma = Q\Lambda Q^T$

令 $A = Q^T$，则

$Y \sim N(0, Q^T\Sigma Q)$

$Q^T\Sigma Q = Q^T Q \Lambda Q^T Q = \Lambda$

$\therefore Y \sim N(0, \Lambda)$

由于 Λ 为对角阵，故 $Y_1, Y_2, \dots, Y_n$ 独立

设 Λ 对角线上的元素为 $d_1, d_2, \dots, d_n$，则

$Y_i \sim N(0, d_i)$

18. (a) 令 $B = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$

则 Y 的协方差阵 $\Sigma' = B\Sigma B^T$

$= \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & \frac{4}{5} \\ \frac{4}{5} & 1 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$

$= \begin{pmatrix} \frac{23}{5} & \frac{22}{5} \\ \frac{22}{5} & \frac{23}{5} \end{pmatrix} \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$

$= \begin{pmatrix} \frac{113}{5} & \frac{112}{5} \\ \frac{112}{5} & \frac{113}{5} \end{pmatrix}$

(b) $\mu' = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$= \begin{pmatrix} 7 \\ 8 \end{pmatrix}$

$\therefore Y \sim \left(\begin{pmatrix} 7 \\ 8 \end{pmatrix}, \begin{pmatrix} \frac{113}{5} & \frac{112}{5} \\ \frac{112}{5} & \frac{113}{5} \end{pmatrix} \right)$

$\rho = \frac{Cov(Y_1, Y_2)}{\sqrt{DY_1}\sqrt{DY_2}}$

$Y_2 | Y_1 = y \sim N(\mu_2 + \rho\frac{\sigma_2}{\sigma_1}(y-\mu_1), \sigma_2^2(1-\rho^2))$

$\mu_2 = 8$

$\rho\frac{\sigma_2}{\sigma_1}(y-\mu_1) = \frac{\frac{112}{5}}{\frac{113}{5}} \cdot 1 \cdot (y-7)$

$= \frac{112}{113}(y-7)$

$\cancel{\therefore Y_2 | Y_1 = y \sim N(8, \frac{112}{113}(y-7))}$

$\therefore E(Y_2 | Y_1 = y) = \rho\frac{\sigma_2}{\sigma_1}(y-\mu_1) + \mu_2$

$= 8 + \frac{112}{113}(y-7)$

$\therefore E(Y_2 | Y_1) = 8 + \frac{112}{113}(Y_1 - 7)$

$Y_1 + Y_2 = (1, 1)\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$

$\therefore E(Y_1 + Y_2) = (1, 1)\begin{pmatrix} 7 \\ 8 \end{pmatrix} = 15$

(c) $\Sigma = \begin{pmatrix} 1 & \frac{4}{5} \\ \frac{4}{5} & 1 \end{pmatrix}$

$\rho = \frac{Cov(X_1, X_2)}{\sqrt{DX_1}\sqrt{DX_2}} = \frac{4}{5}$

$\rho\frac{\sigma_2}{\sigma_1}(x-\mu_1) = \frac{4}{5}(x-1)$

$\sigma_2^2(1-\rho^2) = 1-\frac{16}{25} = \frac{9}{25}$

$\therefore X_2|X_1=x \sim N\left(2+\frac{4}{5}(x-1), \frac{9}{25}\right)$

$\therefore E(X_2|X_1) = 2+\frac{4}{5}(X_1-1)$

$D(X_2|X_1) = \frac{9}{25}$

(d) $E(X_2|X_1) = 2+\frac{4}{5}(X_1-1)$

$X_2 - E(X_2|X_1) = X_2 - \frac{4}{5}X_1 + \frac{6}{5}$

$\because$ Gauss 分布的线性组合还是Gauss分布

$Cov\left(X_2-\frac{4}{5}X_1+\frac{6}{5}, X_1\right)$

$= Cov(X_2, X_1) - \frac{4}{5}Cov(X_1, X_1)$

$= \frac{4}{5} - \frac{4}{5} = 0$

$\therefore X_2 - E(X_2|X_1)$ 与 X_1 协方差矩阵是对角阵

$\therefore X_2 - E(X_2|X_1)$ 与 X_1 独立.

补充题1：

$X \sim N(\mu, \Sigma)$

$Y = AX+B \Rightarrow Y \sim N(A\mu+B, A\Sigma A^T)$

$Z = CX+D \Rightarrow Z \sim N(C\mu+D, C\Sigma C^T)$

$$\begin{pmatrix} Y \\ Z \end{pmatrix} \sim N\left(\begin{pmatrix} A\mu+B \\ C\mu+D \end{pmatrix}, \begin{pmatrix} A\Sigma A^T & \Sigma_{12} \\ \Sigma_{21} & C\Sigma C^T \end{pmatrix}\right)$$

设 $A_{n\times m} = \begin{pmatrix} a_1^T \\ a_2^T \\ \vdots \\ a_n^T \end{pmatrix}$　$C_{l\times m} = \begin{pmatrix} c_1^T \\ c_2^T \\ \vdots \\ c_l^T \end{pmatrix}$

$$\therefore \Sigma_{12} = \begin{pmatrix} Cov(a_1^TX, c_1^TX) & Cov(a_1^TX, c_2^TX) & \cdots & Cov(a_1^TX, c_l^TX) \\ Cov(a_2^TX, c_1^TX) & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ Cov(a_n^TX, c_1^TX) & \cdots & \cdots & Cov(a_n^TX, c_l^TX) \end{pmatrix}$$

$$= \begin{pmatrix} a_1^T\Sigma c_1 & a_1^T\Sigma c_2 & \cdots & a_1^T\Sigma c_l \\ a_2^T\Sigma c_1 & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ a_n^T\Sigma c_1 & - & - & a_n^T\Sigma c_l \end{pmatrix}$$

$= A\Sigma C^T$

同理可得 $\Sigma_{21} = C\Sigma A^T$

$$\therefore \begin{pmatrix} Y \\ Z \end{pmatrix} \sim N\left(\begin{pmatrix} A\mu+B \\ C\mu+D \end{pmatrix}, \begin{pmatrix} A\Sigma A^T & A\Sigma C^T \\ C\Sigma A^T & C\Sigma C^T \end{pmatrix}\right)$$

$\therefore Y, Z$ 独立 $\Longleftrightarrow \begin{cases} A\Sigma C^T = 0 \\ C\Sigma A^T = 0 \end{cases} \Longleftrightarrow A\Sigma C^T = 0$

补充题2：

$$\begin{pmatrix} Y \\ X \\ Z \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 & 2 \\ 1 & 1 & 1 \\ 2 & 1 & 4 \end{pmatrix}\right)$$

记 $\begin{pmatrix} X \\ Z \end{pmatrix} = A$

则 $E(Y|A=a) = \mu_1 + \Sigma_{12}\Sigma_{22}^{-1}(a-\mu_2)$

$\therefore E(Y|X=1, Z=3) = 0 + (1,2)\frac{1}{3}\begin{pmatrix} 4 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \end{pmatrix}$

$= \frac{1}{3}(1,2)\begin{pmatrix} 4 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \end{pmatrix}$

$= \frac{1}{3}(2\ \ 1)\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \frac{1}{3}\times 5 = \frac{5}{3}$