数 学 作 业 纸

（科目：随机）

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28.

$$f(x,y)=f_{X|Y}(x|y)f_Y(y) \quad \text{设}D_f=\{(x,y)|0<x<y<1\}$$

$$=\begin{cases}15x^2y & 0<x<y<1\\ 0 & \text{其他}\end{cases}$$

$$P(X>\tfrac{1}{2})=\iint_{\{x>\frac{1}{2}\}\cap D_f}15x^2y\,dx\,dy$$

$$=\int_{\frac{1}{2}}^{1}dy\int_{\frac{1}{2}}^{y}15x^2y\,dx$$

$$=\int_{\frac{1}{2}}^{1}5y^4-\frac{5}{8}y\,dy$$

$$=y^5-\frac{5}{16}y^2\Big|_{\frac{1}{2}}^{1}=\frac{47}{64}$$

$$EX=\iint_{D_f}15x^3y\,dx\,dy$$

$$=\int_0^1dy\int_0^y15x^3y\,dx$$

$$=\int_0^1\frac{15}{4}y^5\,dy$$

$$=\frac{5}{8}$$

$$EXY=\iint_{D_f}xy\cdot15x^2y\,dx\,dy$$

$$=\iint_{D_f}15x^3y^2\,dx\,dy$$

$$=\int_0^1dy\int_0^y15x^3y^2\,dx$$

$$=\int_0^1\frac{15}{4}y^6\,dy=\frac{15}{28}$$

33. $X\sim N(\mu,\sigma^2)$

$$E(X|a<X<b)=\frac{E(X\mathbb{1}_{a<X<b})}{P(a<X<b)}$$

$$P(a<X<b)=\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)$$

$$E(X\mathbb{1}_{a<X<b})=\int_a^b x f(x)\,dx$$

$$=\frac{1}{\sqrt{2\pi}\sigma}\int_a^b x e^{-\frac{(x-\mu)^2}{2\sigma^2}}dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{\frac{a-\mu}{\sigma}}^{\frac{b-\mu}{\sigma}}(\sigma t+\mu)e^{-\frac{1}{2}t^2}dt$$

$$=\frac{1}{\sqrt{2\pi}}\left(\int_{\frac{a-\mu}{\sigma}}^{\frac{b-\mu}{\sigma}}\sigma te^{-\frac{1}{2}t^2}dt+\mu\int_{\frac{a-\mu}{\sigma}}^{\frac{b-\mu}{\sigma}}e^{-\frac{1}{2}t^2}dt\right)$$

$$=-\frac{\sigma}{\sqrt{2\pi}}\left(e^{-\frac{(b-\mu)^2}{2\sigma^2}}-e^{-\frac{(a-\mu)^2}{2\sigma^2}}\right)+\mu\int_{\frac{a-\mu}{\sigma}}^{\frac{b-\mu}{\sigma}}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}t^2}dt$$

$$=\frac{\sigma\left(e^{-\frac{(a-\mu)^2}{2\sigma^2}}-e^{-\frac{(b-\mu)^2}{2\sigma^2}}\right)}{\sqrt{2\pi}}+\mu\left[\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)\right]$$

$\therefore E(X|a<X<b)$

$$=\frac{\sigma\left(e^{-\frac{(a-\mu)^2}{2\sigma^2}}-e^{-\frac{(b-\mu)^2}{2\sigma^2}}\right)}{\sqrt{2\pi}\left[\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)\right]}+\mu$$

34. $$f_Y(y)=\int_{-\infty}^{+\infty}f(x,y)\,dx=\int_y^{+\infty}\lambda^2e^{-\lambda x}dx$$

$$=-\lambda\int_y^{+\infty}e^{-\lambda x}d(-\lambda x)=-\lambda e^{-\lambda x}\Big|_y^{+\infty}=\lambda e^{-\lambda y}\quad y>0$$

$$f_Y(y)=\begin{cases}\lambda e^{-\lambda y} & y>0\\ 0 & \text{其他}\end{cases}$$

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$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}$$

$$= \frac{\lambda^2 e^{-\lambda x}}{\lambda e^{-\lambda y}} = \lambda e^{-\lambda(x-y)} \quad 0<y<\infty$$

$$\therefore f_{X|Y}(x|y) = \begin{cases} \lambda e^{-\lambda(x-y)} & 0<y<x \\ 0 & y>0,\ x\le y \end{cases}$$

$$\therefore E(X|Y=y) = \int_{-\infty}^{+\infty} x f_{X|Y}(x|y)\,dx$$

$$= \int_y^{+\infty} \lambda x e^{-\lambda x}\cdot e^{\lambda y}\,dx$$

$$= e^{\lambda y}\int_y^{+\infty} \lambda x e^{-\lambda x}\,dx$$

$$= \frac{1}{\lambda} e^{\lambda y}\int_y^{+\infty} \lambda x e^{-\lambda x}\,d(\lambda x)$$

$$= \frac{1}{\lambda} e^{\lambda y}\int_{\lambda y}^{+\infty} t e^{-t}\,dt$$

$$= \frac{1}{\lambda} e^{\lambda y}\left(-(t+1)e^{-t}\Big|_y^{+\infty}\right)$$

$$= \frac{1}{\lambda} e^{\lambda y}(\lambda y+1)e^{-\lambda y}$$

$$= \frac{\lambda y+1}{\lambda} = y+\frac{1}{\lambda}$$

$$E(X^2|Y=y) = \lambda\int_y^{+\infty} x^2 e^{-\lambda(x-y)}\,dx$$

$$= \lambda e^{\lambda y}\int_y^{+\infty} x^2 e^{-\lambda x}\,dx$$

$$= \frac{1}{\lambda^2} e^{\lambda y}\int_y^{+\infty} (\lambda x)^2 e^{-\lambda x}\,d(\lambda x)$$

$$= \frac{1}{\lambda^2} e^{\lambda y}\int_{\lambda y}^{+\infty} t^2 e^{-t}\,dt$$

$$= \left(-t^2 e^{-t} - 2(t+1)e^{-t}\Big|_{\lambda y}^{+\infty}\right)\frac{1}{\lambda^2} e^{\lambda y}$$

$$= \frac{\lambda^2 y^2 + 2(\lambda y+1)}{\lambda^2}$$

$$\therefore D(X|Y=y)$$

$$= E(X^2|Y=y) - \left(E(X|Y=y)\right)^2$$

$$= \frac{\lambda^2 y^2 + 2(\lambda y+1)}{\lambda^2} - \frac{(\lambda y+1)^2}{\lambda^2}$$

$$= \frac{\lambda^2 y^2 + 2\lambda y + 1 + 1 - (\lambda y+1)^2}{\lambda^2}$$

$$= \frac{(\lambda y+1)^2 + 1 - (\lambda y+1)^2}{\lambda^2}$$

$$= \frac{1}{\lambda^2}$$

35) $\eta \sim U[0,a]$, $X \sim U[\eta,a]$

$$F_{X,\eta}(x,\eta) = P(X\le x,\ \eta\le \eta)$$

$$= \begin{cases} \frac{\eta}{a}\cdot\frac{x-\eta}{a-\eta} & 0\le\eta\le x<a \\ 0 & \text{其他} \end{cases}$$

$$f_{X,\eta}(x,\eta) = \frac{\partial^2 F}{\partial x\,\partial\eta} = \begin{cases} \frac{1}{(a-\eta)^2} & 0\le\eta\le x<a \\ 0 & \text{其他} \end{cases}$$

$$f_{X|\eta}(x|\eta) = \frac{f_{X,\eta}(x,\eta)}{f_\eta(\eta)} = \frac{\frac{1}{(a-\eta)^2}}{\frac{1}{a}} = \frac{a}{(a-\eta)^2},\quad 0\le\eta\le x<a$$

$$E(X|\eta=\eta) = \int_\eta^a \frac{a}{(a-\eta)^2}x\,dx = \frac{a+\eta}{2(a-\eta)} \quad \therefore E(X|\eta) = \frac{a+\eta}{2(a-\eta)}$$

35. $X|\eta=\eta_0 \sim U[\eta_0, a]$

$\therefore E(X|\eta=\eta_0)=\frac{a+\eta_0}{2}$

$\therefore E(X|\eta)=\frac{a+\eta}{2}$

$\because \eta\sim U[0,a]$

$\therefore \frac{a+\eta}{2}\sim U[\frac{a+0}{2},\frac{a+a}{2}]$

$\therefore E(X|\eta)\sim U[\frac{a}{2}, a]$

41. X_i 记 $E(\lambda_i)$

$$F_{X_1|X_1<X_2}(x)=P(X_1\le x|X_1<X_2)$$

$$=\frac{P(X_1\le x,\ X_1<X_2)}{P(X_1<X_2)}$$

$$P(X_1<X_2)=\int_{-\infty}^{+\infty}P(X_1<X_2|X_2=x)f_{X_2}(x)dx$$

$$=\int_{-\infty}^{+\infty}P(X_1<x)f_{X_2}(x)dx$$

$$=\int_0^{+\infty}(1-e^{-\lambda_1 x})\lambda_2 e^{-\lambda_2 x}dx$$

$$=\frac{\lambda_1}{\lambda_1+\lambda_2}$$

$$P(X_1\le x,\ X_1<X_2)$$

$$=\int_0^x dx_1\int_{x_1}^{+\infty}\lambda_1\lambda_2 e^{-\lambda_1x_1}e^{-\lambda_2x_2}dx_2$$

$$=\int_0^x \lambda_1 e^{-\lambda_1x_1}(-e^{-\lambda_2x_2})\Big|_{x_1}^{+\infty}dx_1$$

$$=\int_0^x\lambda_1e^{-\lambda_1x_1}e^{-\lambda_2x_1}dx_1$$

$$=\int_0^x\lambda_1e^{-(\lambda_1+\lambda_2)x_1}dx_1 \qquad x>0$$

$$\therefore F_{X_1|X_1<X_2}(x)=\int_0^x \frac{\lambda_1e^{-(\lambda_1+\lambda_2)x_1}}{\frac{\lambda_1}{\lambda_1+\lambda_2}}dx_1$$

$$=\begin{cases}\int_0^x(\lambda_1+\lambda_2)e^{-(\lambda_1+\lambda_2)x_1}dx_1 & x>0\\ 0 & x\le 0\end{cases}$$

$$\therefore f_{X_1|X_1<X_2}(x)=(\lambda_1+\lambda_2)e^{-(\lambda_1+\lambda_2)x}$$

$$\therefore X_1|X_1<X_2\sim E(\lambda_1+\lambda_2)$$

$$\therefore E(X_1|X_1<X_2)=\frac{1}{\lambda_1+\lambda_2}$$

49. $\begin{cases}U=X+Y\\ V=\frac{X}{Y}\end{cases} \Leftrightarrow \begin{cases}X=\frac{UV}{1+V}\\ Y=\frac{U}{1+V}\end{cases}$

$$f_{X,Y}(x,y)=e^{-x}e^{-y}=e^{-x+y}$$

$$f_{U,V}(u,v)=f_{X,Y}(x(u,v),y(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right|$$

$$=e^{-u}\frac{u}{(1+v)^2}=\frac{ue^{-u}}{(1+v)^2} \qquad u,v>0$$

$$\therefore f_{U,V}(u,v)=\begin{cases}\frac{ue^{-u}}{(1+v)^2} & u,v>0\\ 0 & \text{其他}\end{cases}$$

$$f_U(u)=\int_{-\infty}^{+\infty}f_{U,V}(u,v)dv=\begin{cases}ue^{-u}(-\frac{1}{1+v})\Big|_0^{+\infty}=ue^{-u} & u>0\\ 0 & u\le 0\end{cases}$$

$$f_V(v)=\int_{-\infty}^{+\infty}f_{U,V}(u,v)du=\begin{cases}\frac{1}{(1+v)^2}(-(1+u)e^{-u})\Big|_0^{+\infty}=\frac{1}{(1+v)^2} & v>0\\ 0 & v\le 0\end{cases}$$

$$\therefore f_{U,V}(u,v)=f_U(u)f_V(v)$$

$\therefore U, V$ 独立.

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50. $\begin{cases} U+V = X+Y \\ V-U = |X-Y| \end{cases}$

$X>Y$ 时，$\begin{cases} V=X \\ U=Y \end{cases} \Rightarrow U<V$ 且 $\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = 1$

$X\le Y$ 时，$\begin{cases} V=Y \\ U=X \end{cases} \Rightarrow U\le V$ 且 $\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = 1$

$\therefore f_{U,V}(u,v) = f_{X,Y}(v,u)\times 1 + f_{X,Y}(u,v)\times 1$

$= \frac{12}{7}v(u+v) + \frac{12}{7}u(u+v)$

$= \frac{12}{7}(u+v)^2 \qquad u<v$

$\therefore f_{U,V}(u,v) = \begin{cases} \frac{12}{7}(u+v)^2 & u<v \\ 0 & 其他 \end{cases}$

51.

$\begin{cases} Y_1 = (-2\ln X_1)^{\frac12}\cos 2\pi X_2 \\ Y_2 = (-2\ln X_1)^{\frac12}\sin 2\pi X_2 \end{cases} \Rightarrow \begin{cases} X_1 = e^{-\frac12(Y_1^2+Y_2^2)} \\ X_2 = \frac{1}{2\pi}\arctan\frac{Y_2}{Y_1} \end{cases}$

$f_{X_1,X_2}(x_1,x_2) = 1 \qquad 0\le x_1,x_2\le 1$

$\therefore f_{Y_1,Y_2}(y_1,y_2) = \left|\frac{\partial(x_1,x_2)}{\partial(y_1,y_2)}\right| = \left\|\begin{matrix} -y_1 e^{-\frac12(y_1^2+y_2^2)} & -y_2 e^{-\frac12(y_1^2+y_2^2)} \\ \frac{1}{2\pi}\frac{-y_2}{y_1^2+y_2^2} & \frac{1}{2\pi}\frac{y_1}{y_1^2+y_2^2} \end{matrix}\right\|$

$= \frac{1}{2\pi}e^{-\frac12(y_1^2+y_2^2)}$

$\therefore Y_1, Y_2 \sim N(0,0,0,1,1)$

$\therefore Y_1 \sim N(0,1)$

$\quad Y_2 \sim N(0,1)$

$\because \rho = 0$

$\therefore Y_1,\ Y_2$ 独立.

53. 设 $Y = \max(X_1,\dots,X_n)$

$F_Y(y) = P(\max(X_1,\dots,X_n)\le y)$

$= P(X_1\le y, X_2\le y, \dots, X_n\le y)$

$= \prod_{i=1}^{n} P(X_i\le y)$

$F_X(x) = \int_{-\infty}^{x} f(x)\,dx$

$= \begin{cases} 0 & x<0 \\ x^2 & 0\le x<1 \\ 1 & x\ge 1 \end{cases}$

$\therefore F_Y(y) = \prod_{i=1}^{n} F_{X_i}(x)$

$= (F_X(x))^n$

$= \begin{cases} 0 & x<0 \\ x^{2n} & 0\le x<1 \\ 1 & x\ge 1 \end{cases}$

$f_Y(y) = \begin{cases} 2nx^{2n-1} & 0<x<1 \\ 0 & 其他 \end{cases}$

$G_n(x) = P(Z_n\le x)$

$= P(n(1-Y)\le x) = P(Y\ge 1-\frac{x}{n})$

$= 1-P(Y<1-\frac{x}{n}) = \begin{cases} 1 & x>n \\ 1-(1-\frac{x}{n})^{2n} & 0<x\le n \\ 0 & x\le 0 \end{cases}$

$n\to+\infty$ 时

$G_n(x) \to \begin{cases} 0 & x\le 0 \\ 1-(1-\frac{x}{n})^{2n} = 1-(1-\frac{x}{n})^{\frac{n}{x}\cdot 2x} \to 1-e^{-2x} \end{cases}$

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$\because \lim_{n\to\infty} G_n(x) = \begin{cases} 0 & x\le 0 \\ 1-e^{-2x} & x>0 \end{cases} = \begin{cases} 0 & x<0 \\ 1-e^{-2x} & x\geq 0 \end{cases}$

$\because G(x) \sim E(2)$

$\therefore G(x) = \begin{cases} 0 & x<0 \\ 1-e^{-2x} & x\geq 0 \end{cases}$

$\therefore \lim_{n\to\infty} G_n(x) = G(x)$

补充题：$(X,Y)\sim N(0,0,\rho,1,1)$ 求 $D(XY)$

$D(XY) = E(X^2Y^2) - (E(XY))^2$

由二维正态分布之性质

$Y|X=x \sim N(\rho x, 1-\rho^2)$

$\therefore E(Y|X) = \rho X \quad D(Y|X) = 1-\rho^2 \quad E(Y^2|X) = D(Y|X) + (E(Y|X))^2 = 1-\rho^2+\rho^2X^2$

$E(XY) = E(E(XY|X)) = E(XE(Y|X)) = E(X\rho X) = \rho EX^2 = \rho(DX+(EX)^2) = \rho DX = \rho$

$E(X^2Y^2) = E(E(X^2Y^2|X)) = E(X^2E(Y^2|X)) = E(X^2(1-\rho^2+\rho^2X^2)) = E((1-\rho^2)X^2+\rho^2X^4)$

$= (1-\rho^2)E(X^2)+\rho^2E(X^4) = 1-\rho^2+3\rho^2 = 1+2\rho^2$