清華大學 TSINGHUA UNIVERSITY　**数学作业纸**

班级：自93　姓名：周义廷　编号：2019010702　科目：随机　第　页

10. 记X_i为第i个骰子的点数

则 $EX_i=\frac{7}{2}$　$i=1,2,3,\dots k$

$X=\sum_{i=1}^{k}X_i$

$EX=E\left(\sum_{i=1}^{k}X_i\right)=\sum_{i=1}^{k}EX_i=\frac{7}{2}k$

17. $P(X=k)=\frac{Ca^k}{(1+a)^{k+1}}$, $a>0$, $k=0,1\dots$

$1=\sum_{k=0}^{+\infty}P(X=k)=\sum_{k=0}^{+\infty}\frac{Ca^k}{(1+a)^{k+1}}$

$=\frac{C}{a}\sum_{k=0}^{+\infty}\left(\frac{a}{1+a}\right)^{k+1}=\frac{C}{a}\frac{\frac{a}{1+a}}{1-\frac{a}{1+a}}=C$

$\therefore C=1$

$P(X=k)=\frac{a^k}{(1+a)^{k+1}}$, $k=0,1,\dots$

设 $Y\sim Ge\left(\frac{1}{1+a}\right)$ 则 $X=Y-1$

$EX=\sum_{k=0}^{+\infty}kP(X=k)$

$=\sum_{k=1}^{+\infty}\frac{ka^k}{(1+a)^{k+1}}$

$=\frac{a}{1+a}\sum_{k=1}^{+\infty}\frac{ka^{k-1}}{(1+a)^k}$

$=\frac{a}{1+a}EY$

$=\frac{a}{1+a}(1+a)$

$=a$

$EX=E(Y-1)=EY-1=1+a-1=a$

$DX=D(Y-1)=DY-D(1)=DY=\frac{\frac{a}{1+a}}{\frac{1}{(1+a)^2}}=a(1+a)$

↓ 此处成立是因为Y与Z（Z=1 a.s.）独立

$\therefore EX=a$，$DX=a(1+a)$

18. 记A_k为第k次摸到白球，$k=1,2,\dots,n$

则 $P(A_k)=P(A_{k-1})P(A_{k-1})+(1-P(A_{k-1}))\left(P(A_{k-1})+\frac{1}{a+b}\right)$

$=P(A_{k-1})-\frac{1}{a+b}P(A_{k-1})+\frac{1}{a+b}$

$=\frac{a+b-1}{a+b}P(A_{k-1})+\frac{1}{a+b}$

$P(A_k)-1=\frac{a+b-1}{a+b}P(A_{k-1})-\frac{a+b-1}{a+b}=\frac{a+b-1}{a+b}(P(A_{k-1})-1)$

$\therefore P(A_k)-1=\left(\frac{a+b-1}{a+b}\right)^{k-1}(P(A_1)-1)=\left(\frac{a+b-1}{a+b}\right)^{k-1}\frac{-b}{a+b}$

$\therefore P(A_k)=1-\left(\frac{a+b-1}{a+b}\right)^{k-1}\frac{b}{a+b}$

记B_k为第k次摸到黑球，$k=1,2,3,\dots,n$

则 $P(B_k)=\left(\frac{a+b-1}{a+b}\right)^{k-1}\frac{b}{a+b}$

设袋中最后白球数为X. 记 $X_k=1_{B_k}=\begin{cases}1 & B_k\\ 0 & \text{否则}\end{cases}$

则 $X=a+\sum_{k=1}^{n}X_k$

$EX=a+\sum_{k=1}^{n}EX_k$

$=a+\frac{b}{a+b}\frac{1-\left(\frac{a+b-1}{a+b}\right)^n}{1-\frac{a+b-1}{a+b}}$

$=a+b\left[1-\left(\frac{a+b-1}{a+b}\right)^n\right]$

$=a+b-b\left(\frac{a+b-1}{a+b}\right)^n$

26. $E(XY)=E\left(\sum_{k_1=1}^{n}\xi_{k_1}\sum_{k_2=1}^{n}\xi_{m+k_2}\right)$

$=E\left(\sum_{k_1=1}^{n}\sum_{k_2=1}^{n}\xi_{k_1}\xi_{m+k_2}\right)=\sum_{k_1=1}^{n}\sum_{k_2=1}^{n}E(\xi_{k_1}\xi_{m+k_2})$

$=\sum_{k_1=1}^{n}\sum_{\substack{k_2=1\\ 且k_1\neq m+k_2}}^{n}E(\xi_{k_1}\xi_{m+k_2})+\sum_{k=m+1}^{n}E(\xi_k^2)$

$=[n^2-(n-m)]E^2(\xi_1)+(n-m)E(\xi_1^2)$

$E(X)E(Y)=\sum_{k_1=1}^{n}\sum_{k_2=1}^{n}E(\xi_{k_1})E(\xi_{m+k_2})=n^2E^2(\xi_1)$

$\therefore Cov(X,Y)=E(XY)-E(X)E(Y)$

$=(n-m)D\xi_1$

$DX = nD\xi_1$

$DY = nD\xi_1$

$\because r_{X,Y} = \dfrac{Cov(X,Y)}{\sqrt{DXDY}}$

$= \dfrac{(n-m)D\xi_1}{nD\xi_1} = \dfrac{n-m}{n}$

27. $X \sim B(n, \frac{1}{6})$ $EX = \frac{n}{6}$ $DX = \frac{5n}{36}$

$Y \sim B(n, \frac{1}{6})$ $EY = \frac{n}{6}$ $DY = \frac{5n}{36}$

$P(X=k) = P(Y=k) = C_n^k (\frac{1}{6})^k (\frac{5}{6})^{n-k}$ $k = 0, 1, \cdots, n$

记 $X_i = \begin{cases} 1 & 第i次出现1点 \\ 0 & 否则 \end{cases}$

$Y_i = \begin{cases} 1 & 第i次出现6点 \\ 0 & 否则 \end{cases}$

$X = \sum_{i=1}^n X_i$ $Y = \sum_{i=1}^n Y_i$

$E(XY) = E(\sum_{i=1}^n X_i \sum_{j=1}^n Y_j)$

$= E(\sum_{i=1}^n \sum_{j=1}^n X_i Y_j)$

$= \sum_{i=1}^n \sum_{j=1}^n E(X_i Y_j)$

$= \sum_{i=1}^n \sum_{\substack{j=1 \\ j\neq i}}^n E(X_i)E(Y_j) + \sum_{i=1}^n \sum_{i=1}^n E(X_i Y_i)$

$= (n^2-n) \times \frac{1}{36} + 0 = \frac{n^2-n}{36}$

$Cov(X,Y) = E(XY) - E(X)E(Y) = -\frac{n}{36}$

$r_{X,Y} = \dfrac{Cov(X,Y)}{\sqrt{DXDY}} = -\frac{n}{36} \times \frac{36}{5n} = -0.2$

补充题1：记白球数量为X

记摸到白球为事件A，由全概率公式

$P(A) = \sum_{k=0}^N P(X=k) \cdot \frac{K}{N}$

$= \frac{1}{N} \sum_{k=0}^N k P(X=k)$

$= \frac{1}{N} EX = \frac{n}{N}$

补充题2：

$Cov[(X-(aY+b)), Y]$

$= E[Y(X-aY-b)] - E[X-(aY+b)]EY$

$= E(XY) - aE(Y^2) - bEY - EXEY + a(EY)^2 + bEY$

$= E(XY) - EXEY - aDY$

$= Cov(X,Y) - a \cdot \frac{Cov(X,Y)}{a}$

$= 0$

补充题3：

记 X_i 为第 i 张卡片上的号码 $EX_i = \frac{n+1}{2}$ $DX_i = EX_i^2 - (EX_i)^2 = \frac{(n+1)(2n+1)}{6} - \frac{(n+1)^2}{4} = \frac{n^2-1}{12}$

① 若放回，则 $EX = E(\sum_{i=1}^k X_i) = \sum_{i=1}^k EX_i = \frac{k(n+1)}{2}$

$DX = D(\sum_{i=1}^k X_i) = \sum_{i=1}^k DX_i = k\frac{n^2-1}{12}$

② 若不放回，则 $EX = E(\sum_{i=1}^k X_i) = \frac{k(n+1)}{2}$

$DX = D(\sum_{i=1}^k X_i) = \sum_{i=1}^k DX_i + 2\sum\sum_{i<j}(E(X_iX_j) - EX_iEX_j)$

$= \sum_{i=1}^k DX_i + 2 \times \frac{k(k-1)}{2} \times [\frac{(3n+2)(n+1)}{12} - \frac{(n+1)^2}{4}]$

$= k\frac{n^2-1}{12} - k(k-1)\frac{n+1}{12} = k\frac{(n+1)(n-k)}{12}$

其中 $E(X_iX_j) = \sum_{k_1=1}^n \sum_{\substack{k_2=1 \\ k_2\neq k_1}}^n k_1k_2 \frac{1}{n}\frac{1}{n-1} = \frac{1}{n(n-1)}[\frac{n^2(n+1)^2}{4} - \frac{n(n+1)(2n+1)}{6}] = \frac{(3n+2)(n+1)}{12}$

$\therefore DX = \frac{k(n+1)(n-k)}{12}$