数 学 作 业 纸

（科目：　　）

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1. (a) $X \sim U[-a,a]$

$\varphi(\theta) = Ee^{i\theta X} = \int_{-\infty}^{+\infty} e^{i\theta x} f(x)\,dx$

$= \int_{-a}^{a} e^{i\theta x}\cdot \frac{1}{2a}\,dx = \frac{1}{2a}\cdot\frac{1}{i\theta}e^{i\theta x}\Big|_{-a}^{a}$

$= \frac{1}{2a}\cdot\frac{1}{i\theta}(e^{i\theta a} - e^{-i\theta a})$

$= \frac{2i\sin(a\theta)}{2a\,i\theta} = \frac{\sin(a\theta)}{a\theta}$

(b) $X \sim \text{Cauchy}(m,a)$

$f(x) = \frac{a}{\pi[(x-m)^2+a^2]}$, $a>0$, $-\infty<x<+\infty$

令 $Y = \frac{X-m}{a}$ 则 $f(y) = \frac{1}{\pi}\frac{1}{y^2+1}$

$I(\theta) = \varphi_Y(\theta) = Ee^{i\theta Y} = \int_{-\infty}^{+\infty} e^{i\theta y} f(y)\,dy$

$= \frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{e^{i\theta y}}{y^2+1}\,dy = \lim_{R\to+\infty}\frac{1}{\pi}\int_{-R}^{R}\frac{e^{i\theta y}}{y^2+1}\,dy$

由留数定理 $\theta>0$ 时

$\frac{1}{\pi}\int_{-R}^{R}\frac{e^{i\theta y}}{y^2+1}\,dy + \frac{1}{\pi}\int_{C_R}\frac{e^{i\theta z}}{z^2+1}\,dz = 2\pi i\,\text{Res}[f(z), i]$

其中C_R是从R到-R的半圆周

由于 $\lim_{R\to+\infty}\int_{C_R}\frac{e^{i\theta z}}{z^2+1}\,dz = 0$

$\therefore \lim_{R\to+\infty}\frac{1}{\pi}\int_{-R}^{R}\frac{e^{i\theta y}}{y^2+1}\,dy = 2\pi i\,\text{Res}\left[\frac{1}{\pi}\frac{e^{i\theta z}}{z^2+1}, i\right]$

$= 2\pi i\,\frac{e^{i\theta i}}{\pi 2i} = e^{-\theta}$

即 $\varphi_Y(\theta) = e^{-\theta} = \frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{\cos\theta y}{y^2+1}\,dy = \frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{\cos(-\theta y)}{y^2+1}\,dy$

$\theta<0$ 时

$\varphi_Y(\theta) = \frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{e^{i\theta y}}{y^2+1}\,dy = \frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{e^{i(-\theta)y}}{y^2+1}\,dy = e^{\theta}$

$\theta=0$ 时，$\varphi_Y(0)=1$

$\therefore \varphi_Y(\theta) = e^{-|\theta|}$

$\therefore \varphi_X(\theta) = \varphi_{aY+m}(\theta) = e^{i\theta m - a|\theta|}$

6. 设X的特征函数为$\varphi(\theta)$

$Y \sim U[-a,a]$

则 $\varphi_Y(\theta) = \frac{\sin(a\theta)}{a\theta}$

且X，Y独立.

则由独立的性质 $Z = X+Y$

的特征函数 $\varphi_Z(\theta) = \varphi(\theta)\frac{\sin(a\theta)}{a\theta}$

$\therefore \varphi_a(\theta) = \varphi(\theta)\frac{\sin a\theta}{a\theta}$

也是特征函数.

7. $X, Y \overset{iid}{\sim} E(\lambda)$

(a) $\varphi_X(\theta) = Ee^{i\theta X}$

$= \int_{-\infty}^{+\infty} e^{i\theta x} f(x)\,dx$

$= \int_0^{+\infty} e^{i\theta x} e^{-x}\,dx$

$= \int_0^{+\infty} e^{(i\theta-1)x}\,dx$

$= \frac{1}{i\theta-1} e^{(i\theta-1)x}\Big|_0^{+\infty}$

$= \frac{1}{i\theta-1}(0-1)$

$= \frac{1}{1-i\theta}$

(b) $\varphi_{-Y}(\theta) = \varphi_Y(-\theta) = \frac{1}{1+i\theta}$

且X与-Y独立.

则 $\varphi_{X-Y}(\theta) = \varphi_X(\theta)\varphi_{-Y}(\theta) = \frac{1}{1+\theta^2}$

(c) $f_{X,Y}(x,y)=f_X(x)f_Y(y)=e^{-(x+y)}$

令 $U=X-Y$, $V=X+Y$

则 $\begin{cases} X=\frac{V+U}{2}\\ Y=\frac{V-U}{2}\end{cases}$

$$f_{U,V}(u,v)=f_{X,Y}(x,y)\left|\frac{\partial(x,y)}{\partial(u,v)}\right|$$

$$=\frac{1}{2}f_{X,Y}(x(u,v),y(u,v))$$

$$=\frac{1}{2}e^{-v}\quad (v>u \text{ 且 } v>-u)$$

$$f_U(u)=\int_{-\infty}^{+\infty}f_{U,V}(u,v)\,dv$$

$$=\int_{|u|}^{+\infty}\frac{1}{2}e^{-v}\,dv$$

$$=\frac{1}{2}e^{-|u|}$$

$$\therefore \varphi_{X-Y}(\theta)=Ee^{i\theta u}=\int_{-\infty}^{+\infty}\frac{1}{2}e^{i\theta u-|u|}\,du$$

$$=\int_{-\infty}^{0}\frac{1}{2}e^{(i\theta+1)u}+\int_{0}^{+\infty}\frac{1}{2}e^{(i\theta-1)u}\,du$$

$$=\frac{1}{1+i\theta}\cdot\frac{1}{2}+\frac{1}{1-i\theta}\frac{1}{2}$$

$=\frac{1}{1+\theta^2}$ 与(b)中求得的相同

若利用 $\varphi_{X-Y}(\theta)=\frac{1}{1+\theta^2}$ 推导 $U=X-Y$ 的密度函数

其实就是对特征函数求一次傅里叶反变换

$$f_U(u)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}e^{-i\theta u}\varphi_{X-Y}(\theta)\,d\theta$$

$$=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{e^{-i\theta u}}{1+\theta^2}\,d\theta$$

由1(b)结论，

$$\text{上式}=\frac{1}{2}\times\int_{-\infty}^{+\infty}\frac{1}{\pi}\frac{e^{-i\theta u}}{1+\theta^2}\,d\theta=\frac{1}{2}\int_{-\infty}^{+\infty}\frac{1}{\pi}\frac{e^{i\theta|u|}}{1+\theta^2}\,d\theta$$

$$=\frac{1}{2}e^{-|u|}$$

与之前求出的结果相同

10. X_i 服从 Cauchy(m,a)

$$\bar{X}=\frac{1}{n}\sum_{i=1}^{n}X_i$$

$$\varphi_{\bar{X}}(\theta)=\varphi_{\sum_{i=1}^{n}\frac{1}{n}X_i}(\theta)=\prod_{i=1}^{n}\varphi_{X_i}\left(\frac{1}{n}\theta\right)$$

$$=\prod_{i=1}^{n}e^{i\frac{\theta}{n}m-a\left|\frac{\theta}{n}\right|}$$

$$=e^{i\theta m-a|\theta|}$$

由唯一性定理

$\bar{X}\sim$ Cauchy(m,a)

11. $\cancel{\boxed{\varphi_Z(\theta)=Ee^{i\theta u}=(1-p)e^{i\theta a}+pe^{i\theta b}}}$

由全期望公式：

$$\varphi_Z(\theta)=\varphi_{\alpha X+(1-\alpha)Y}(\theta)$$

$$=Ee^{i\theta(\alpha X+(1-\alpha)Y)}$$

$$=(1-p)Ee^{i\theta(aX+(1-a)Y)}+pEe^{i\theta(bX+(1-b)Y)}$$

$$=(1-p)\varphi_{aX+(1-a)Y}(\theta)+p\,\varphi_{bX+(1-b)Y}(\theta)$$

$$=(1-p)\varphi_X(a\theta)\varphi_Y((1-a)\theta)+p\,\varphi_X(b\theta)\varphi_Y((1-b)\theta)$$