# Chapter 4

Approximating functions by Taylor Polynomials.

4.1 Linear Approximations

We have already seen how to approximate a function using its tangent line. This was the key idea in Euler's method. If we know the function value at some point (say $f(a)$) and the value of the derivative at the same point ($f'(a)$) we can use these to find the tangent line, and then use the tangent line to approximate $f(x)$ for other points x. Of course, this approximation will only be good when x is relatively near a. The tangent line approximation of $f(x)$ for x near a is called the first degree Taylor Polynomial of $f(x)$ and is:

$$f(x) \approx f(a) + f'(a)(x-a)$$

f(x)

x
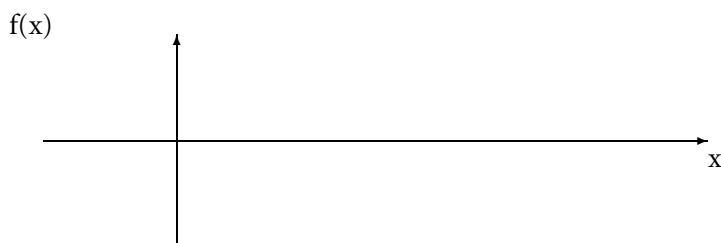
For example, we can approximate the value of $\sin(x)$ for values of x near zero, using the fact that we know $\sin 0 = 0$, the derivative of $\frac{d}{dx} sin(x) = \cos(x)$ and $\cos(0) = 1$

$$\sin(.02) \approx \sin 0 + \cos 0(.02 - 0) = 0 + 1(.02) = +.02$$

This may seem like a useless idea. After all, your calculator will give you an exact(??) value of $\sin x$ for any x you choose. But, did you ever wonder how your calculator knew all those numbers? It hasn't remembered them all, rather it remembers a polynomial approximation for $\sin x$ and uses it to calculate any particular value that you request. This requires much less memory storage space in your calculator.

4.2 Quadratic Approximations

To get a better approximation of our function we try to approximate it using a quadratic polynomial. Another thing we could try is to find a polynomial that has the same value as the function at some point a, the

same derivative at that point a and also the same second derivative there. We do both at once and define the second degree Taylor Polynomial for $f(x)$ near the point $x = a$.

$$f(x) \approx P_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$$

Check that $P_2(x)$ has the same first and second derivative that $f(x)$ does at the point $x = a$.

4.3   Higher Order Taylor Polynomials

We get better and better polynomial approximations by using more derivatives, and getting higher degreed polynomials. The Taylor Polynomial of Degree n, for x near a is given by:
$P_n(x) =$

$$f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{3*2}(x-a)^3 + \frac{f^{(4)}(a)}{4*3*2}(x-a)^4 + \cdots + \frac{f^{(n)}(a)(x-a)^n}{n(n-1)(n-2)\cdots 3*2}$$

The terms in the denominators may be a bit surprising at first. In class we will calculate the first through nth derivatives of $f(x)$ and its Taylor Polynomial $P_n(x)$ to see that they (the derivatives) are the same.

Returning to our example, the second degree Taylor Polynomial for $\sin x$ near 0 is

$$P_2(x) = \sin 0 + \cos 0(x-0) - \frac{\sin 0}{2}(x-0)^2 = 0 + 1(x) + 0$$

It is rather disappointing that this turns out to be no different from P_1 for $\sin x$. The third degree Taylor Polynomial for $\sin x$ near 0 is

$$P_3(x) = \sin 0 + \cos 0(x-0) - \frac{\sin 0}{2}(x-0)^2 - \frac{\cos x}{3*2}(x-0)^3 = 0 + 1(x) - \frac{0}{2}x^2 - \frac{1}{3*2}x^3$$

Check the value of $P_3(.02)$ compared to what your calculator gives you for $\sin .02$.

4.4   Notation

It is helpful to introduce some notation at this point. We have already introduced a new notation for higher order derivatives. That is, we are using the symbol $f^{(i)}(a)$ to mean the ith derivative of the $f(x)$ evaluated at the point a. The factorial notation

$$n! = n(n-1)(n-2)\cdots 3*2*1$$

is useful for the denominators of each term in the Taylor Polynomial. The summation notation, shown below, lets us write the Taylor Polynomial more succinctly.

$$\sum_{i=1}^{5} g(i) = g(1) + g(2) + g(3) + g(4) + g(5)$$

Using these notations we write the nth degree Taylor Polynomial for $f(x)$ near 0 as:

$$f(x) \approx P_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(0)}{i!}(x-0)^i$$

4.5   Important examples

The 8th Taylor Polynomial for e^x for x near $a = 0$:

$$e^x \approx P_8 = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^8}{8!}$$

The nth Taylor Polynomial for $\sin x$ **for** x **near** $a = 0$.   First calculate the derivatives of $\sin x$! You should find a pattern that makes this easy.

	derivative	at $x = 0$
$f(x)$	$= \sin x$	is 0
$f'(x)$	$= \cos x$	is 1
$f''(x)$	$=$	
$f^{(3)}(x)$	$=$	
$f^{(4)}(x)$	$=$	
$f^{(5)}(x)$	$=$	
$f^{(6)}(x)$	$=$	

Now put it together:

$$\sin x \approx P_n(x) = 0 + x + \frac{0}{2} * x^2 + \frac{-1}{3!} * x^3 + \frac{0}{4!} * x^4 + \frac{1}{5!} * x^5 + \cdots$$

The nth Taylor Polynomial for $\cos x$ **for** x **near** $a = 0$:   First calculate the derivatives, again, you should find a pattern that makes this easy.

	derivative	at $x = 0$
$f(x)$	$= \cos x$	is 1
$f'(x)$	$= -\sin x$	is 0
$f''(x)$	$=$	
$f^{(3)}(x)$	$=$	
$f^{(4)}(x)$	$=$	
$f^{(5)}(x)$	$=$	
$f^{(6)}(x)$	$=$	

Now put it together:
$$\cos x \approx P_n(x) =$$

4.6   Taylor Series

You can see that we can make Taylor Polynomial of as high a degree as we'd like. It is often useful to designate the infinite possibilities by what is called the Taylor Series. This is just the Taylor Polynomial with infinite degree.

For example, the Taylor Series for e^x is given by:

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \frac{x^n}{n!} + \cdots = \sum_{i=0}^{\infty} \frac{x^i}{i!}$$

As another example we calculate the Taylor Series of $\frac{1}{x}$. Most of the examples we have seen so far have been Taylor Series centered at $x = 0$. This is not always a good value of a to pick. In this example, it is not even a possibility! (Why???)

We instead find the Taylor Series of $f(x) = \frac{1}{x}$ centered at $x = 1$. As always, we first calculate the derivatives.

$$f'(x) = (-1)x^{-2}$$
$$f''(x) = (+2)x^{-3}$$
$$f'''(x) = (-3 * 2)x^{-4}$$

The Taylor Series is then:

$$P(x) = 1 - 1(x - 1) + \frac{2}{2}(x - 1)^2 + \frac{-3 * 2}{3 * 2}(x - 1)^3 + \cdots$$

$$= 1 - (x - 1) + (x - 1)^2 - (x - 1)^3 + (x - 1)^4 - \cdots$$

Be sure you understand what the next several terms would like for this example and for others we have seen.

4.7   The remainder formula for Taylor Polynomials

In order to use an approximation method intelligently, we need to have an idea of how good our approximation is. That is, how big an error could we be making. The error is the difference between the approximation value and the exact answer. Notice, that we cannot know *exactly* how far off we are (how big the error is), without know the exact answer in the first place! When using the nth degree Taylor Polynomial $P_n(x)$ centered at a to approximate $f(x)$ the error is:

$$E = f(x) - P_n(x)$$

If $E > 0$ that means that our approximation is bigger than the true value. If $E < 0$ then our approximation is too large. Often we are only interested (or can only find) the magnitude of the error $|E|$.

It turns out that there is a simple formula which gives us a bound on the size of the error E. Again, this is only a bound on the size of the error and does not tell us the exact error (why??). This bound is called the remainder formula and is:

$$|R_n| = \left| \frac{f^{(n+1)}(c)}{(n+1)!}(x - a)^{n+1} \right|$$

where c is between a and x.

This formula looks pretty similar to the next term of the Taylor Polynomial itself. The only difference is the c sitting in it. This stands for *some* value between a and x AND WE DON'T KNOW WHICH!

You are probably worrying how on earth we can use this formula to get actual numbers if we don't know what c is. Good question. What we need to do is look at all the values of $f^{(n+1)}(c)$ (for all $a < c < x$) and use the largest of them. Or, pick something that we know is surely larger than all of them.

EXAMPLE 1: Give a bound on the error for when $e^{.5}$ is approximated by the fourth degree Taylor Polynomial of e^x centered at 0.

The fifth derivative of e^x is again e^x. We don't know what that is, but we do know that that e^x is an increasing function between $0 \leq x \leq .5$. So,

$$|f^{(5)}(x)| \leq |e^{.5}| = \sqrt{e} < \sqrt{3} < 2$$

Thus the error can be bounded:

$$|f(.5) - P_n(.5)| \leq |R_4| = \left| \frac{f^{(5)}(c)}{(5)!}(x-0)^5 \right| = \frac{2}{5!}x^5$$

So if we approximate $e^{.5}$ by $P_4(.5) = 1 + (.5) + (.5)^2/2 + (.5)^3/3! + (.5)^4/4!$ our approximation will be within $\frac{2}{120}(.5)^5 < .0006$ of the true value.

EXAMPLE 2: If we take the 5th Taylor Polynomial of $\cos(x)$ centered at $a = \pi$, how big can the error be? The sixth derivative of $\cos(x)$ is $-\sin(x)$. We don't know exactly what that is, since it will depend on x, but we know that no matter what, it will be that $-1 \leq -\sin(x) \leq 1$.

Thus the error can be bounded:

$$|f(x) - P_n(x)| \leq |R_5| = \left| \frac{f^{(6)}(c)}{(6)!}(x-\pi)^6 \right| = \frac{1}{6!}(x-\pi)^6$$

So if we approximate $\cos(3)$ by the 5th Taylor Polynomial centered at π then we will have an error of at most $\frac{1}{720}(\pi - 3)^5 < .000000001119$.

If we approximate $\cos(1)$ by the 5th Taylor Polynomial centered at π then we will have an error of at most $\frac{1}{720}(\pi - 1)^5 < .003$. Note that this is much worse!

4.8   Problems for Chapter 4

Exercise 4.1. Find the 5th degree Taylor Polynomial centered at $x = 0$ for the following functions. Do this directly, by taking the appropriate derivatives etc.
    (a) $\sin(2x)$ (b) e^{5x} (c) $\frac{1}{1+x}$ (d) $\ln(1+x)$

Exercise 4.2. Find the 7th Taylor Polynomial centered at $x = 0$ for the following functions. Do this directly, by taking the appropriate derivatives etc.
    (a) $\cos(-5x)$ (b) xe^x (c) $\frac{1}{2-x}$ (d) $\sqrt{(1+x)}$

Exercise 4.3. Find the 10th degree Taylor Polynomial centered at $x = a$ for the given functions:
    (a) $\sin(x)$, at $x = \pi/2$ (b) $\ln(x)$ at $x = 1$.   (c) e^{-2x}, at $x = .5$ (d) $\ln(x+2)$ at $x = 2$.

Exercise 4.4. Express the 10th degree Taylor Polynomial of the following functions in summation form (using the Σ notation).
    (a) $\sin(x)$ at $x = \pi/2$.   (b) xe^x at $x = 0$.   (c) e^{-2x}, at $x = .5$ (d) $\ln(x)$ at $x = 1$.

Exercise 4.5. We can derive Taylor Polynomials and Taylor Series for one function from another in a variety of ways.

    (a) One useful technique is to *substitute* an expression for a variable. For example to get the Taylor Polynomial of degree 7 for $\sin(2x)$ you could take the Taylor Polynomial of degree 7 for $\sin(u)$ and plug $2x$ in for u. Do this and check that you get the same answer you did for 1.(a).

    (b) Similarly, use the Taylor Polynomial of degree 7 for e^x to get the Taylor Polynomial for e^{x^2}. (Do you get a Taylor Polynomial of degree 7?)

    (c) If $P(x)$ is a Taylor Series for the function $f(x)$ then $\frac{d}{dx}P(x)$ is the Taylor Series for the function $\frac{d}{dx}f(x)$. Similarly, $\int P(x)\,dx$ is the Taylor Series for $\int f(x)\,dx$. Verify that the Taylor Series for $\sin(x)$ and $\cos(x)$ follow these rules.

Exercise 4.6. The Taylor Series for arctangent is

$$\arctan(x) = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \cdots \pm \frac{1}{2n+1}x^{2n+1} + \cdots.$$

Use the fact that $\tan(\pi/4) = 1$ to get a series expansion of π. How far out in the series do you need to go to evaluate the π to three decimal places (you don't have to refer to the remainder formula for this).

Exercise 4.7. In this problem, we estimate the value of $\sin(3)$ using different Taylor Polynomials. All angles are given in Radians!!
    (a) Calculate the value of $\sin(3)$ using the a seventh degree Taylor Polynomial centered at $x = 0$.
    (b) Calculate the value of $\sin(3)$ using a seventh degree Taylor Polynomial centered at $x = \pi$.
    (c) Compare your answers with the answer your calculator gives you.
    (d) Use the error formula, R_n, given in this handout to find a bound on the accuracy of your answers in (a) and (b).

Exercise 4.8. Suppose we want to use the Taylor polynomial of $\sin(x)$ centered at $a = 0$ to approximate $sin(2)$. If we want our answer to have error at most .0000001 how many terms of the Taylor Polynomial should we use?

Exercise 4.9. Consider the Taylor Series centered at $x = 0$ for $(1 + x)^k$.
    (a) What is the series for $k = 2$?

    (b) What is the series for $k = 3$?

    (c) For a positive integer k, can you tell in general how many terms the Taylor Series will have?

    (d) The general formula for the coefficients of these Taylor Series are called the binomial coefficients. They were first discovered by the Twelfth Century Persian poet and mathematician Omar Khayyam, and later generalized by Newton in the form you have just obtained. Write out the first three coefficients for the general positive integer k case.

Exercise 4.10. The *Hyperbolic Trigonometric Functions* $\sinh(x)$ and $cosh(x)$ (pronounced to rhyme with "pinch" and "posh"), are defined by the following formulas.

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \qquad\qquad \cosh(x) = \frac{e^x + e^{-x}}{2}$$

    (a) Use the Taylor series for e^x, centered at $x = 0$, to find Taylor series for each of these functions.

    (b) How are the series you got in (a) similar to and different from the series for $\sin(x)$ and $\cos(x)$?

    (c) Recall that trigonometric functions are sometimes called circular, because for any t, the point $(x, y) = (\cos(t), \sin(t))$ satisfies the equation $x^2 + y^2 = 1$. Show that the point $(\cosh(t), \sinh(t))$ lies on the hyperbola $x^2 - y^2 = 1$.