第七章 非线性控制系统分析

7.1 导论

7.1.1 线性系统与非线性系统

1. 线性系统的主要特点

- 叠加原理

$$\begin{aligned} c_1(t) &= f\left[r_1(t)\right] \\ c_2(t) &= f\left[r_2(t)\right] \end{aligned} \quad \Rightarrow \quad \begin{aligned} c_1(t)+c_2(t) &= f\left[r_1(t)+r_2(t)\right] \\ ac_1(t) &= f\left[ar_1(t)\right] \end{aligned}$$

- 可用典型的输入获得系统的传递函数
- 分析与设计的数学工具丰富:
  e.g. ODE, Laplace transformation, etc.

2. 非线性系统

- 叠加原理不再适用
- 没有统一的方法

7.1.2 为什么要研究非线性

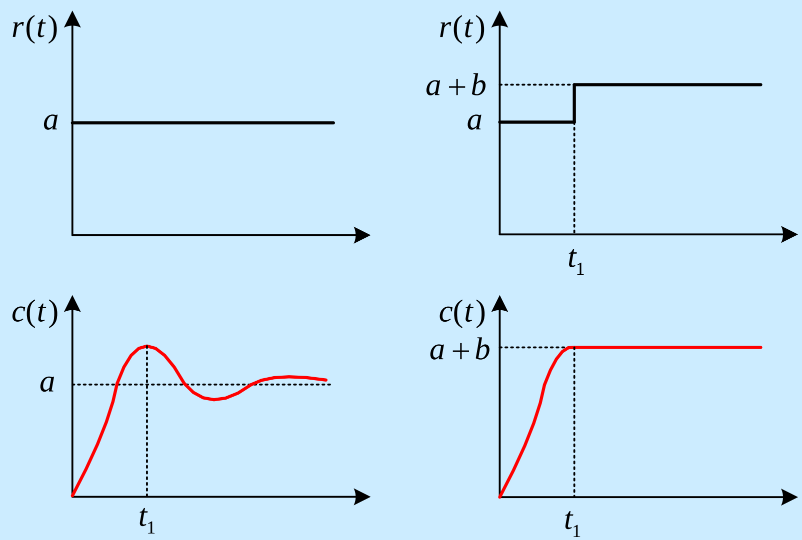
(1) 实际系统基本都是非线性系统

- 线性系统仅仅是对现实非线性系统某种程度上的近似

- 对于某些系统，非线性是不能忽略的

(2) 非线性控制可能取得特殊的效果

- 切换控制



* a, b, t_1 需合理选取

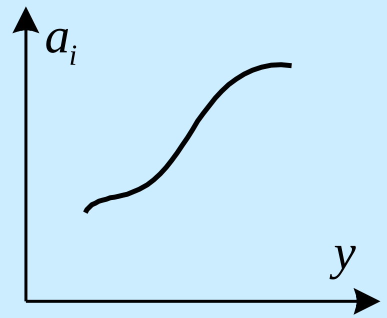
7.1.3 非线性分类

- ♦ **固有非线性:**

  系统或部件持有，在系统中不可避免

- ♦ **人为非线性:**

  为改善性能而故意引入



e.g $a_i = y^2$

$a_i = x^3$

1. 连续非线性

- **非线性微分方程**

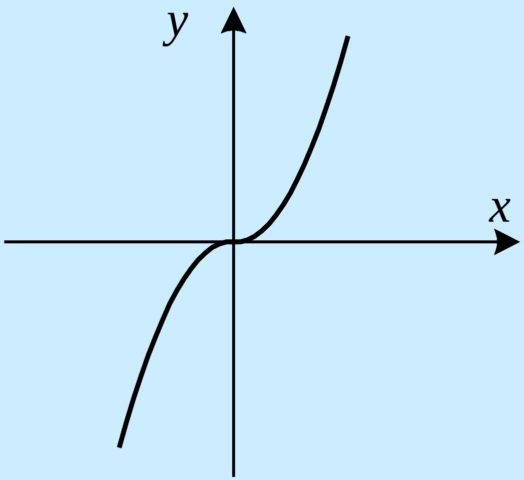
$$a_0 \frac{\mathrm{d}^n y}{\mathrm{d}t^n} + a_1 \frac{\mathrm{d}^{n-1} y}{\mathrm{d}t^{n-1}} + \cdots + a_{n-1} \frac{\mathrm{d}y}{\mathrm{d}t} + a_n y = f(x)$$

- **系数是系统变量的函数**

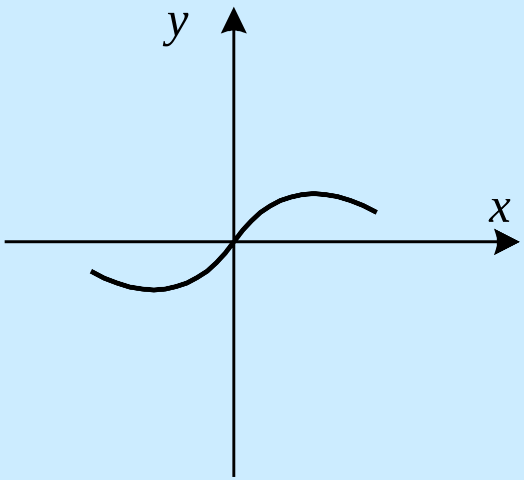
- **非线性弹簧** $y = k_1 x + k_2 x^3 \quad (k_1 > 0)$

* $y = (k_1 + k_2 x^2) x$

x **具有连续的非线性系数**



$k_2 > 0$ **(硬弹簧)**

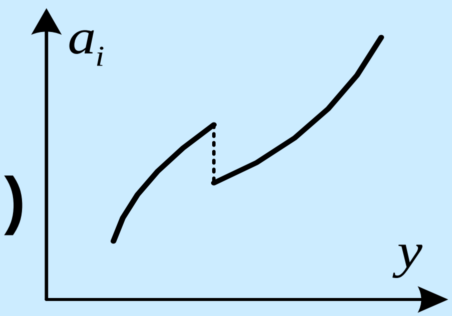


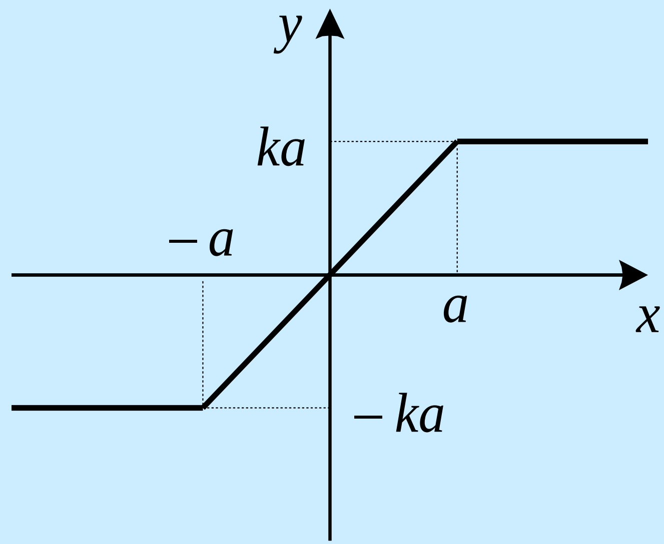
$k_2 < 0$ **(软弹簧)**

2. 不连续非线性

(1) 饱和 Saturation

(磁放大， 电放大， 伺服电机的功率限制)



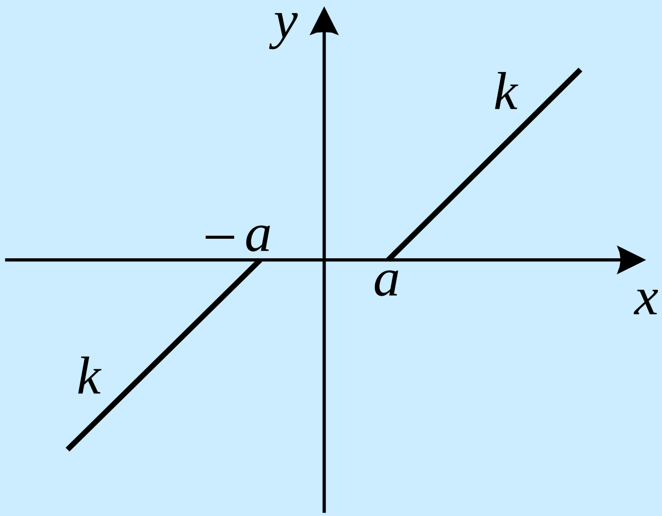


$$y=\begin{cases} -ka & x\le -a \\ kx & -a<x<a \\ ka & x\ge a \end{cases}$$

♦ **特点:**

- **输入大时不能产生足够的输出**
- **调整时间 ↑, 动态误差 ↑**
- **振荡通常不会太强**

(2) 死区 (继电放大器，执行器) Dead Zone



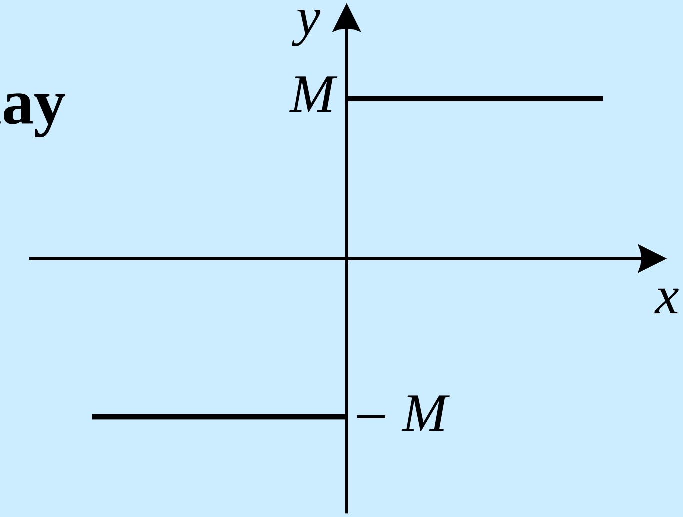
$$y = \begin{cases} k(x+a) & x \le -a \\ 0 & -a < x < a \\ k(x-a) & x \ge a \end{cases}$$

♦ **特点:**

- **输入信号小时没有输出**
- **导致稳态误差**
- **振荡通常削弱**

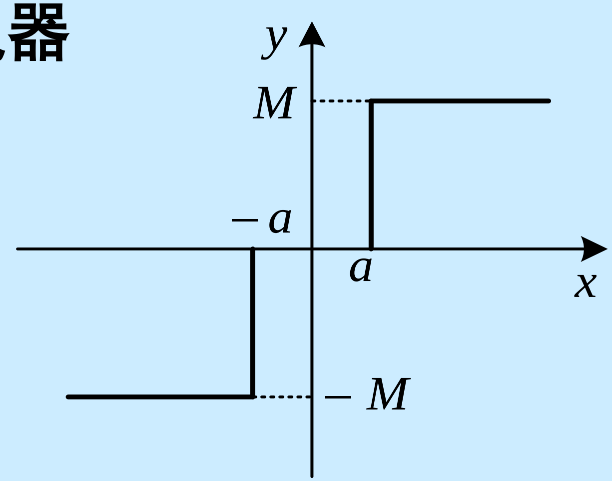
(3) 继电器 Relay

♦ 理想继电器



$$y=\begin{cases}-M & x<0\\ M & x\geq 0\end{cases}$$

♦ 带死区的继电器

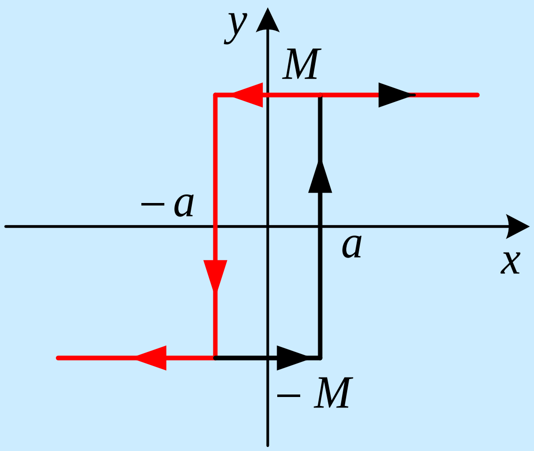


$$y=\begin{cases}-M & x\leq -a\\ 0 & -a<x<a\\ M & x\geq a\end{cases}$$

♦ 特点:

- 快速切换
- 如果使用恰当，可实现快速平滑的调节
- 如果使用不当，会产生振荡

(4) 滞环 Hysteresis



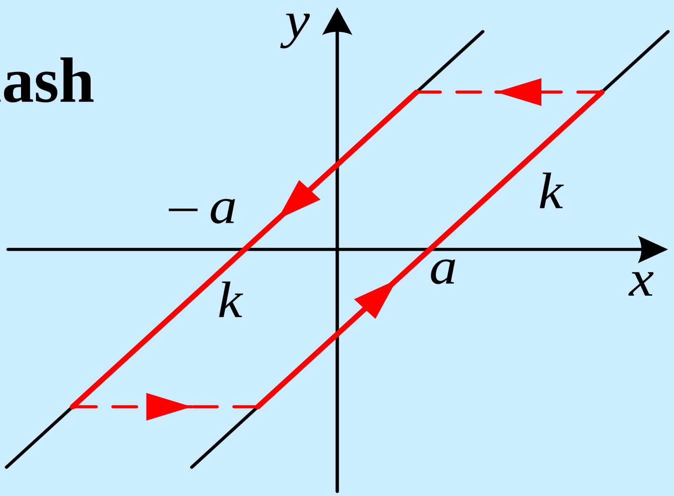
$$y = \begin{cases} -M & x < a \\ M & x \geq a \end{cases} \quad \text{for} \quad \dot{x} \geq 0$$

$$y = \begin{cases} M & x > -a \\ -M & x \leq -a \end{cases} \quad \text{for} \quad \dot{x} < 0$$

♦ **特点:**

- **导致系统响应迟钝**
- **导致振荡**

(5) 间隙 Backlash



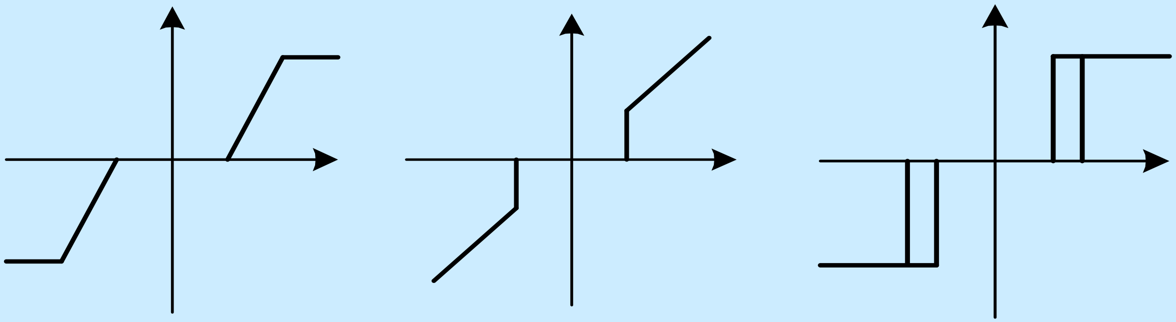
$$y=\begin{cases}k(x+a) & \dot{x}<0\\ k(x-a) & \dot{x}\ge 0\end{cases}$$

♦ **特点:**

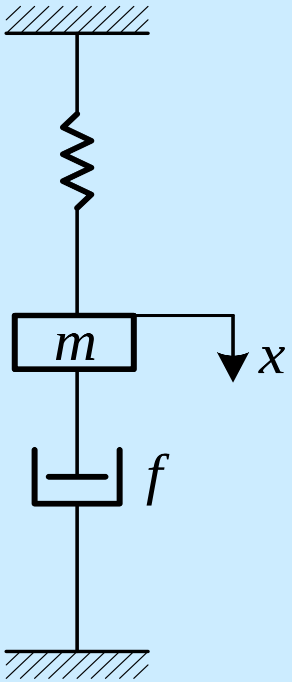
- **操作延迟**
- **会引起振荡**

(6) 组合非线性

譬如：摩擦



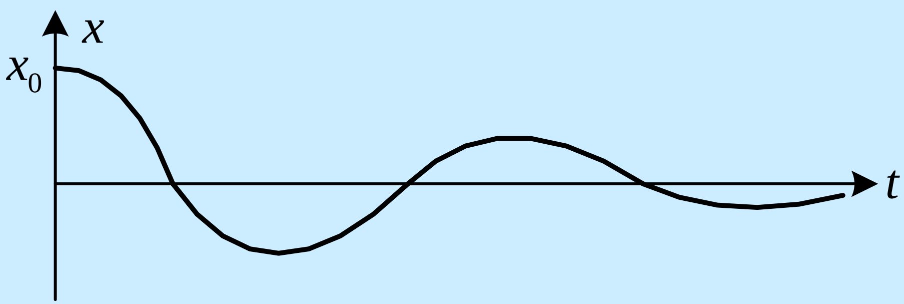
7.1.4 非线性系统的特有现象



1. 频率–幅值相关性

$$m\ddot{x} + f\dot{x} = -k_1 x - k_2 x^3 \qquad k_1 > 0$$

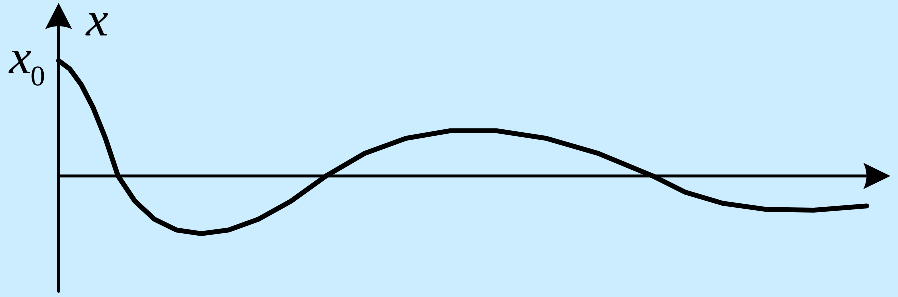
(1) $k_2 = 0$**, 线性弹簧:**



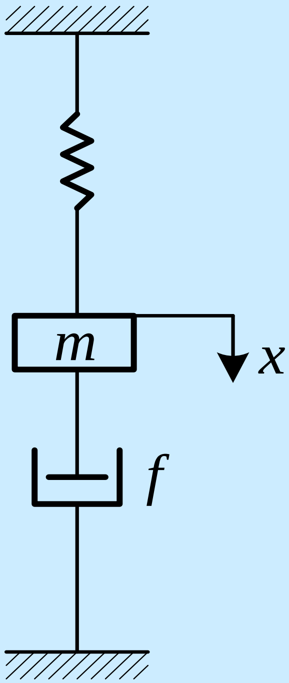
- **频率固定的衰减振荡**

$$m\ddot{x} + f\dot{x} = -k_1 x - k_2 x^3 \qquad k_1 > 0$$

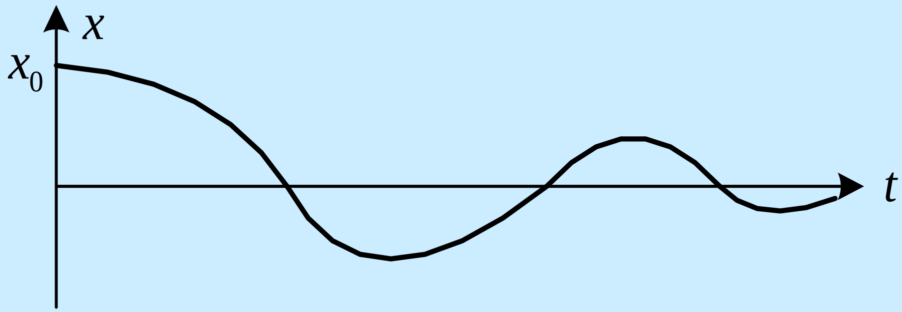
(2) $k_2 > 0$, **硬弹簧:**



- **衰减振荡，频率随幅值减小而减小**



(2) $k_2 < 0$, **软弹簧:**



- **衰减振荡，频率随幅值减小而增大**

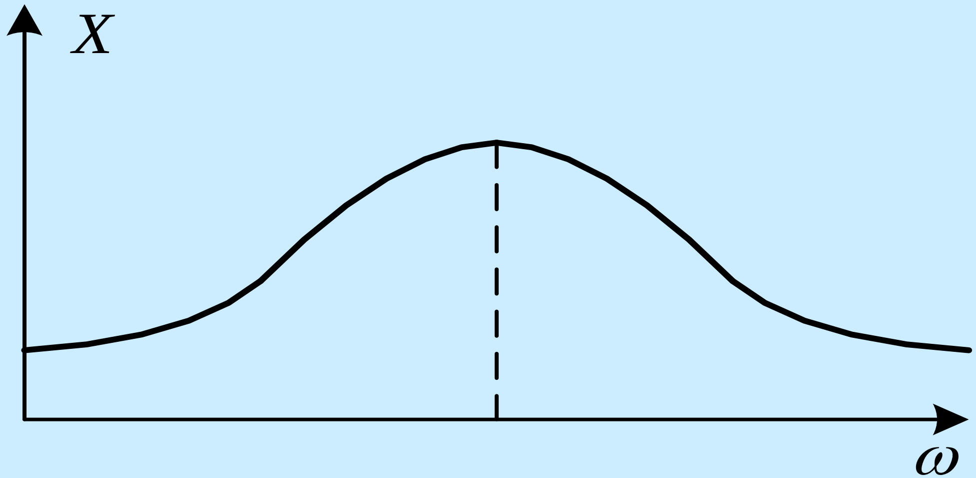
2. 跳跃谐振 Jump Resonance

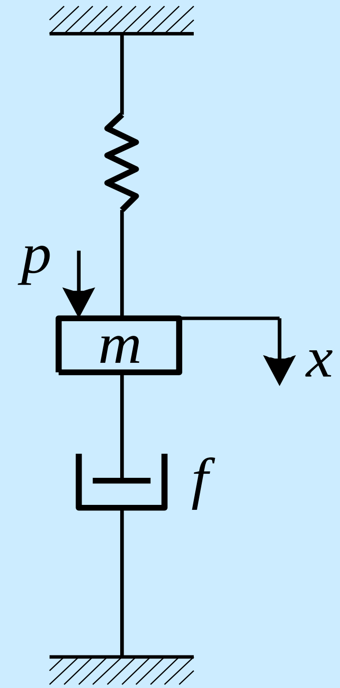
$$m\ddot{x} + f\dot{x} + k_1 x + k_2 x^3 = p$$

$$k_1 > 0$$

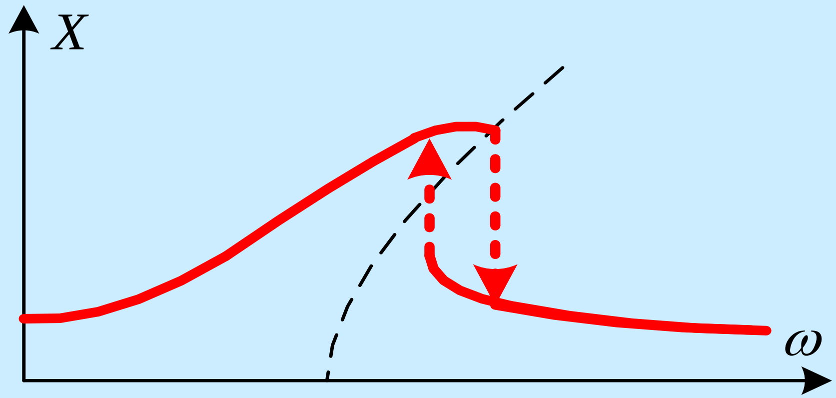
- **施加外力:** $p = P\cos\omega t$

(1) $k_2 = 0$

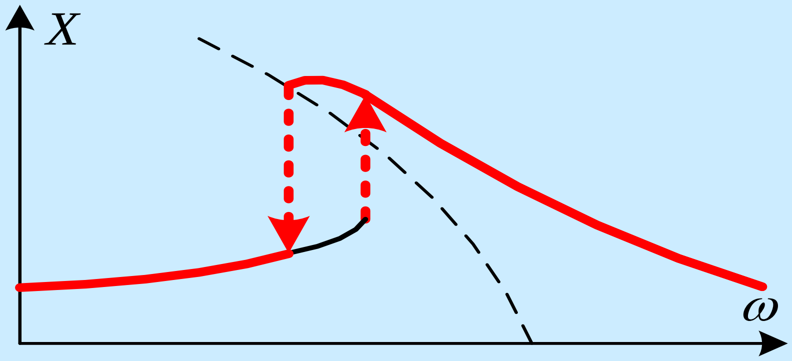


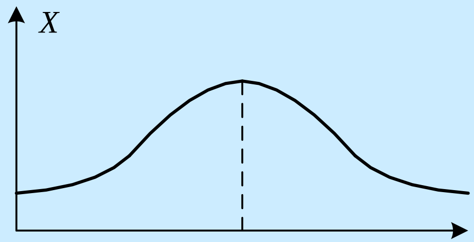


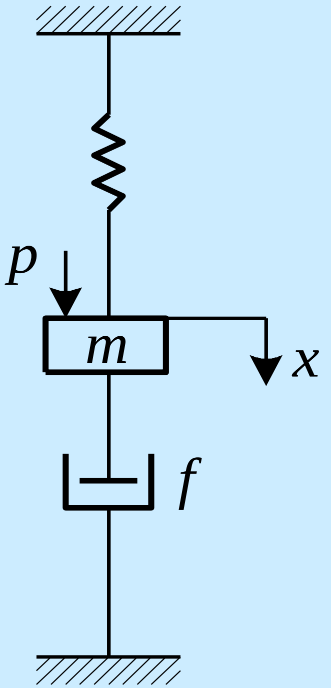
(2) $k_2 > 0$



(3) $k_2 < 0$





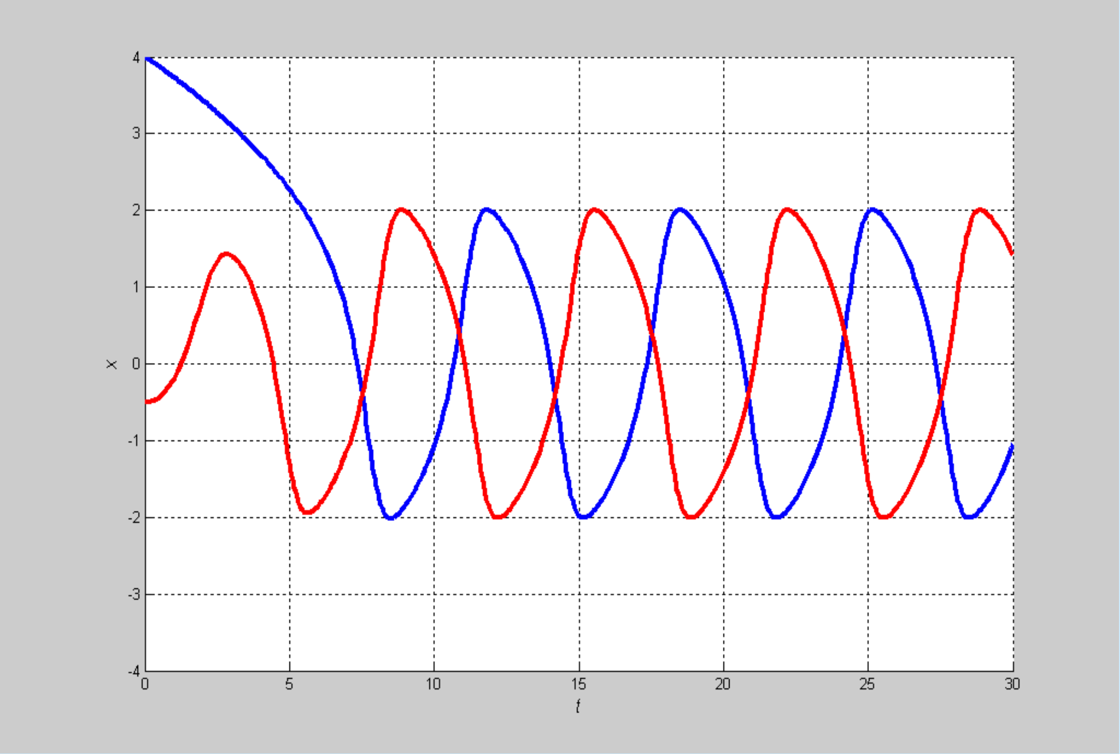


3. 自持振荡 Self-sustained Oscillation

♦ **Van der pol 方程** $m\ddot{x} - f(1-x^2)\dot{x} + kx = 0 \quad (f > 0)$

◊ x **大**

- **阻尼项为正**
- **系统消耗能量**
- **运动衰减**



◊ x **小**

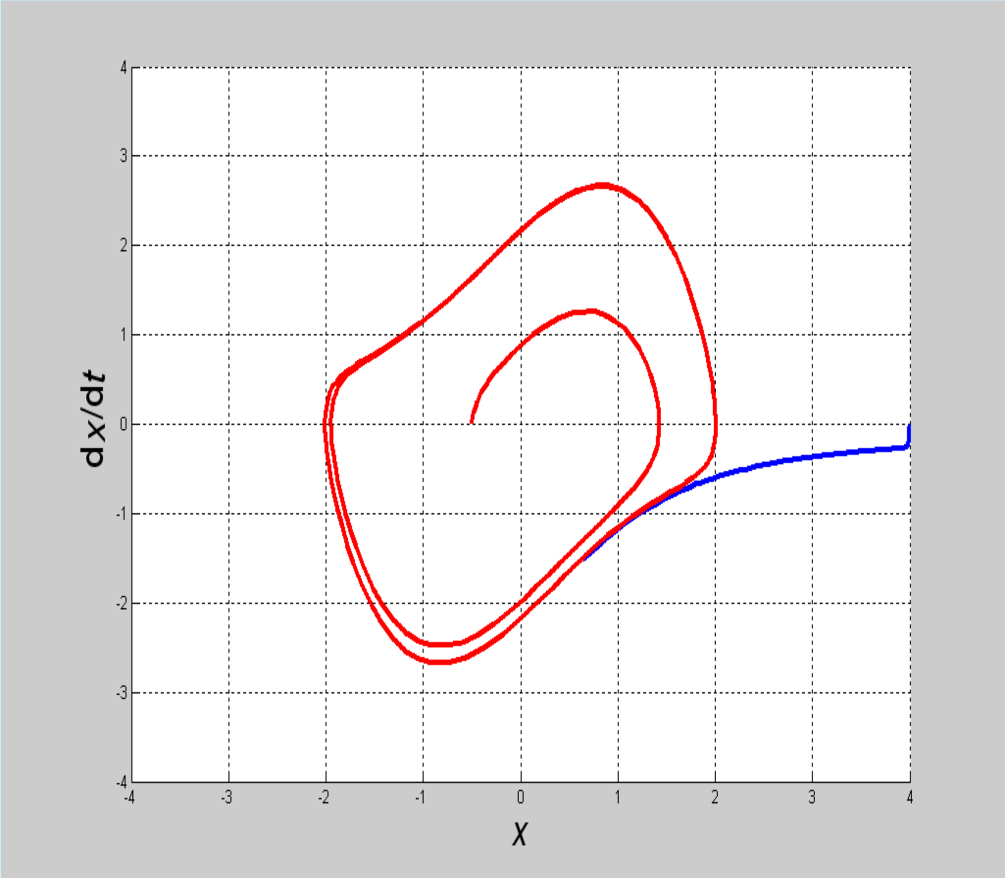
- **阻尼项为负**
- **系统释放能量**
- **运动增强**

♦ 自持振荡 (极限环 Limit Cycle)

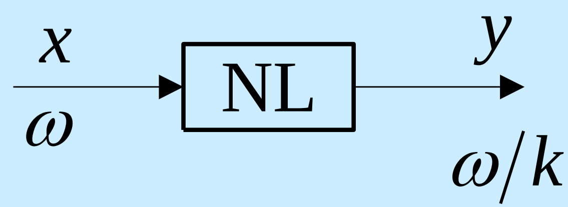
- 即使去除外力后，仍存在固定频率和振幅的自持振荡

- 稳定与不稳定极限环

- 同一系统中可能存在多个极限环



4. 分谐波振荡 Subharmonic Oscillations



5. 稳定性依赖于初始条件

例 7.1.1 考察如下非线性系统状态的稳定性

$$\dot{x} = -x(1-x)$$

Solution:

- **两个平衡状态:** $x = 0$, $x = 1$
- **当** $x \neq 0$ **且** $x \neq 1$**, 有** $\qquad \dfrac{\mathrm{d}x}{x(1-x)} = -\mathrm{d}t$
- **上式积分得**

$$\frac{x}{1-x} = C\mathrm{e}^{-t} \qquad x(t) = \frac{C\mathrm{e}^{-t}}{1+C\mathrm{e}^{-t}}$$

$$\frac{x}{1-x} = Ce^{-t}$$

$$x(t) = \frac{Ce^{-t}}{1+Ce^{-t}}$$

- **若** $x(0) = x_0 \neq 1$ **，则** $C = x_0 / (1 - x_0)$

从而 $$x(t) = \frac{x_0 e^{-t}}{1 - x_0 + x_0 e^{-t}}$$

***** $x_0 > 1$**:**

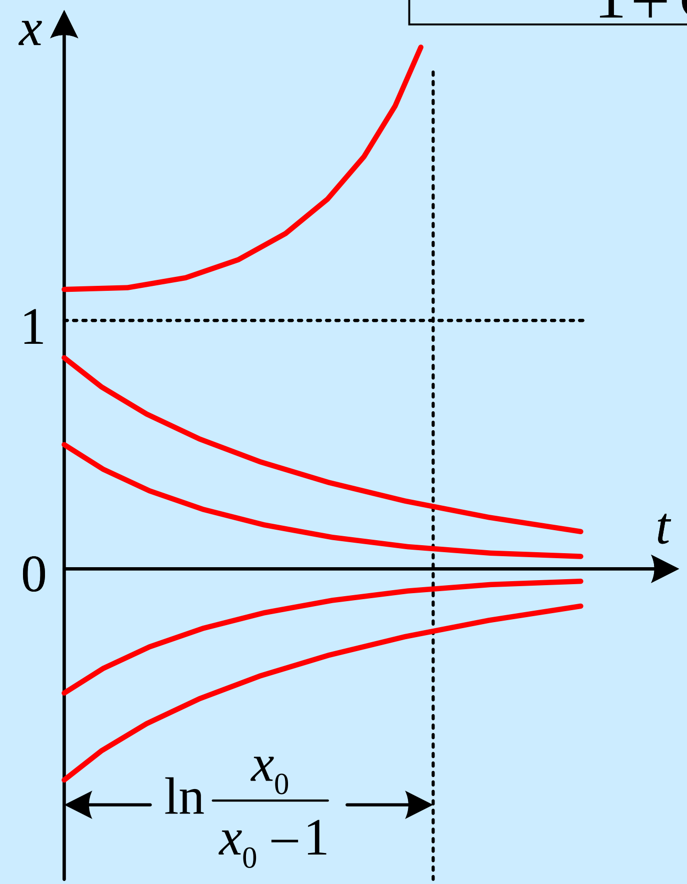
$$x(t) \to \infty \ \textbf{ as } t \to \ln\frac{x_0}{x_0 - 1}$$

***** $x_0 < 1$**:** $x(t) \to 0$ **as** $t \to \infty$

- **结论:**

* $x = 0$ **是稳定的平衡状态**

* $x = 1$ **是不稳定的平衡状态**



6. 分叉 Bifurcation

例 7.1.2 无阻尼 Duffing 方程

$$\ddot{x} + ax + x^3 = 0$$

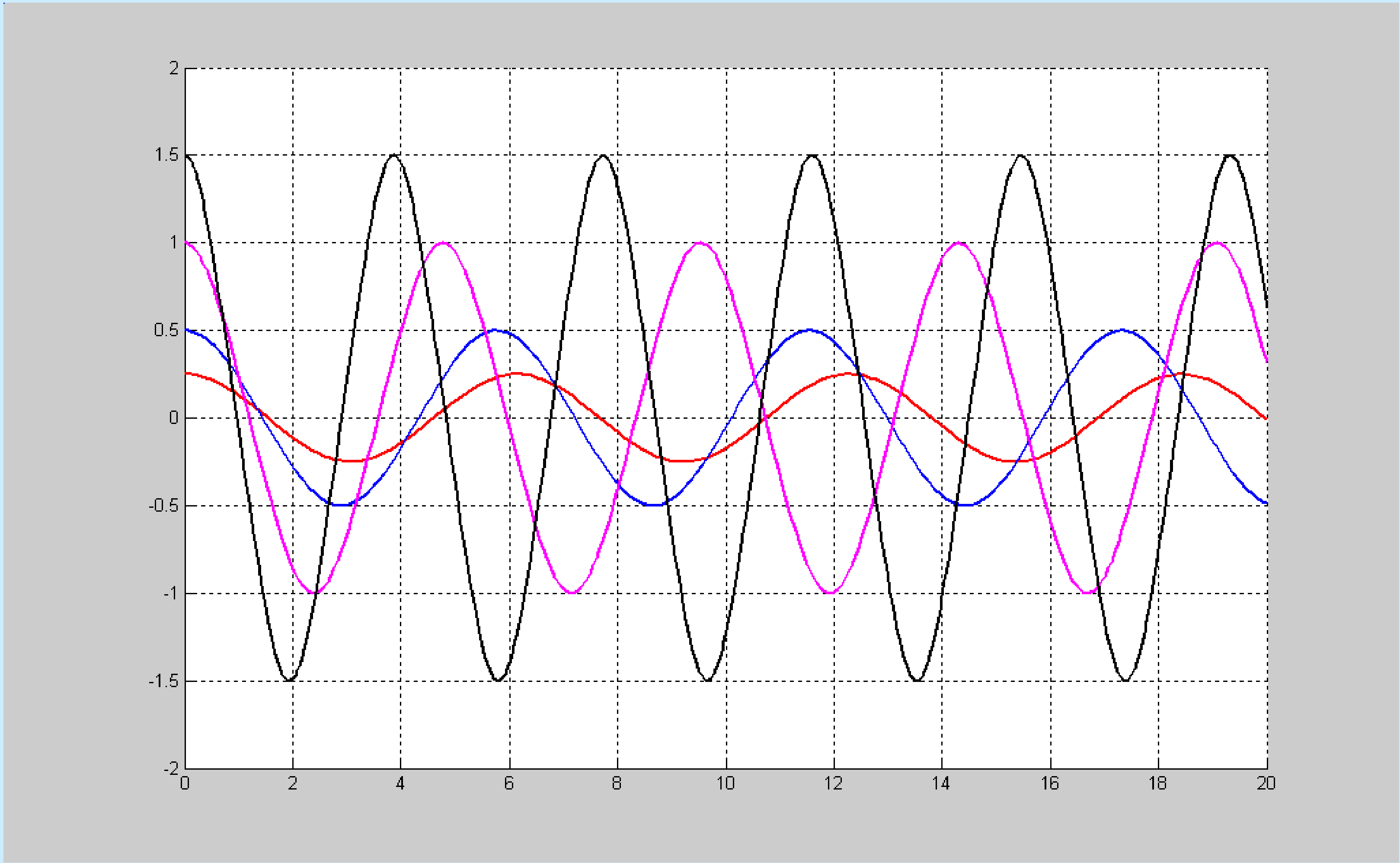
Solution:

- **平衡点的性质可能随a的变化而变化**
- **平衡点的数量也可能随a的变化而变化**

$$\ddot{x} + ax + x^3 = 0$$

$$x + x^3 = 0$$
$$x = 0$$

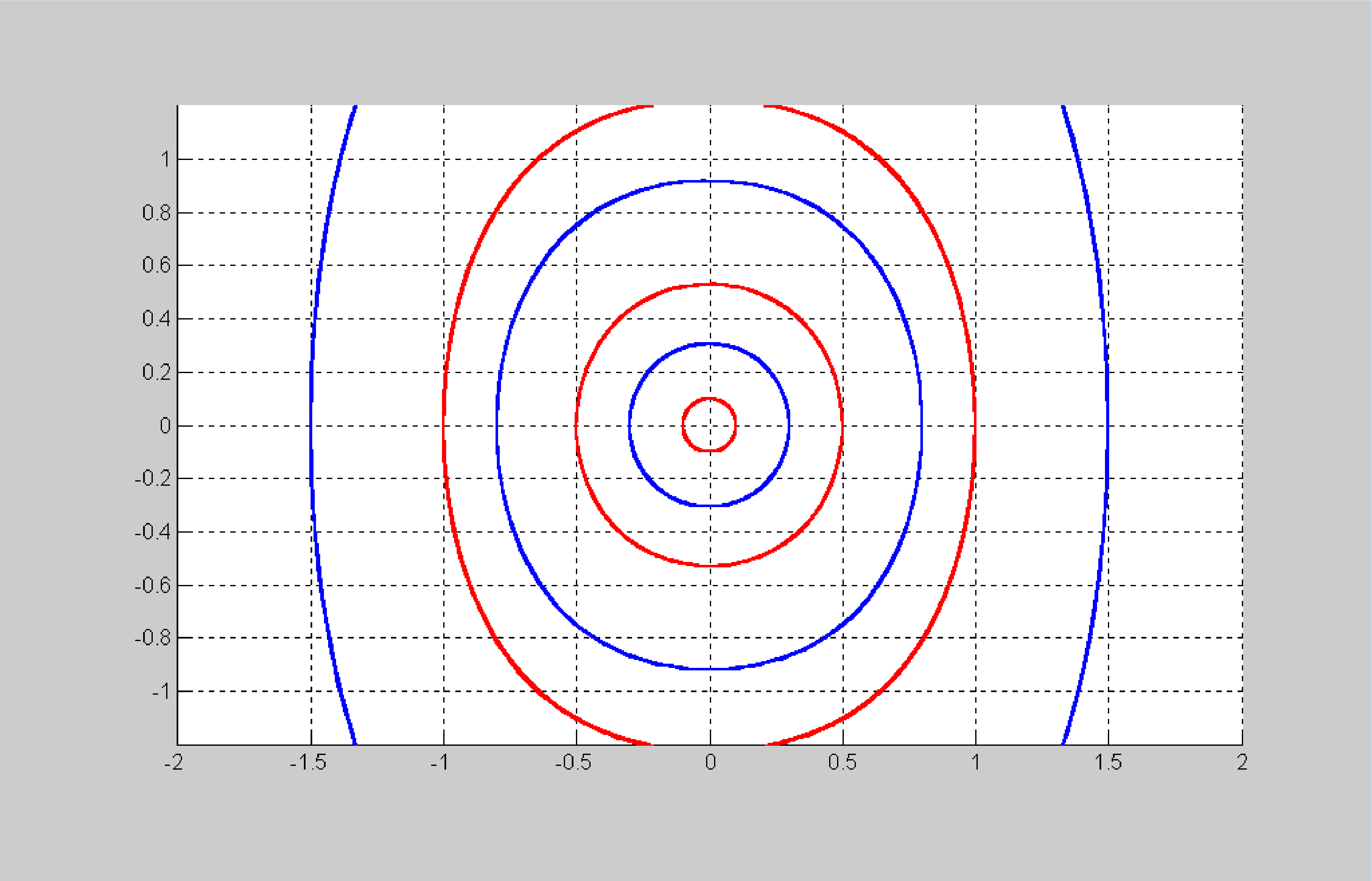
- $a = 1 > 0$



$$\ddot{x} + ax + x^3 = 0 \qquad x + x^3 = 0 \qquad x = 0$$

- $a = 1 > 0$

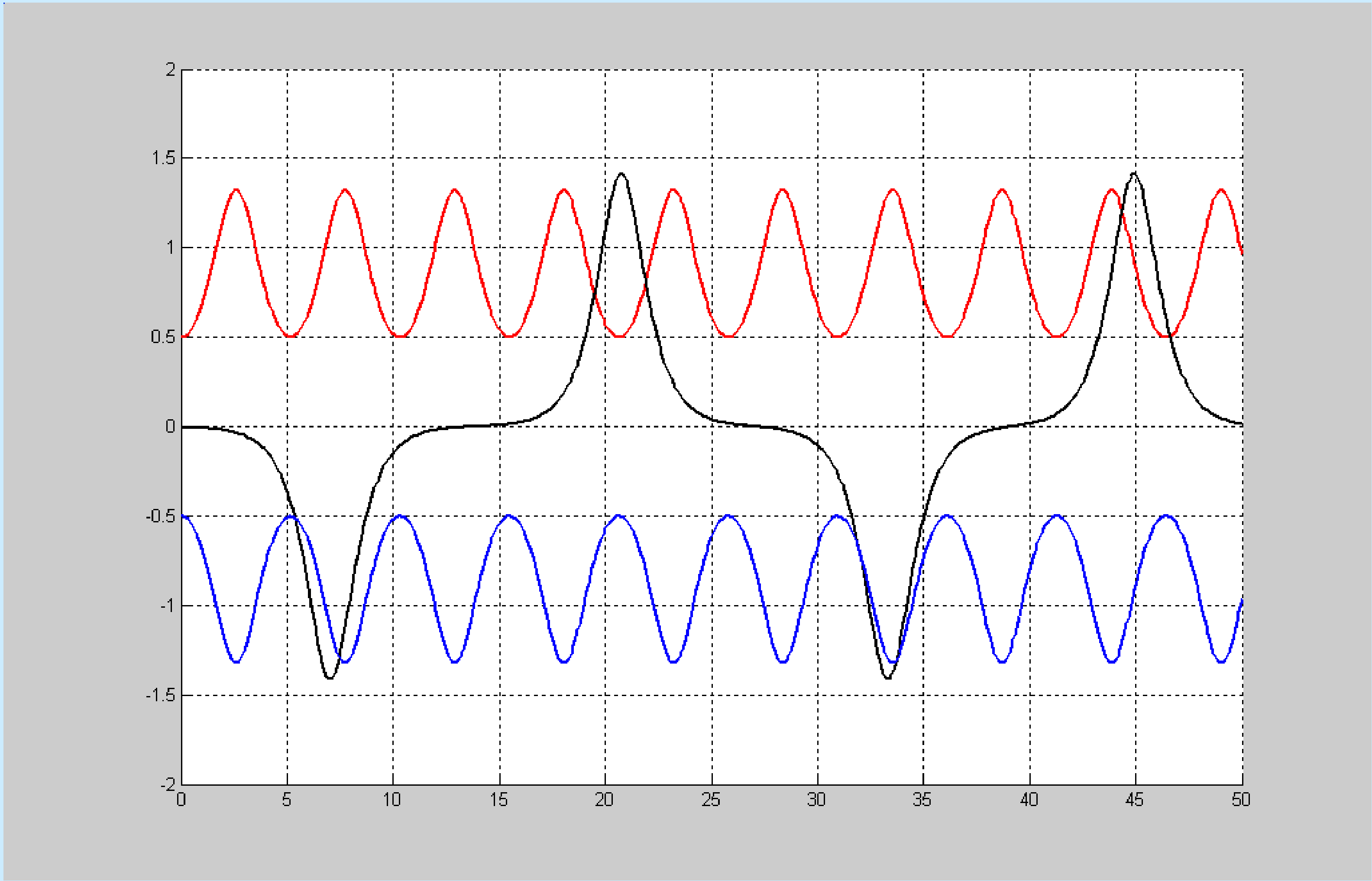
$$x_e = 0$$



$$\ddot{x} + ax + x^3 = 0$$

$$x^3 = x \qquad x = \begin{cases} 0 \\ \pm 1 \end{cases}$$

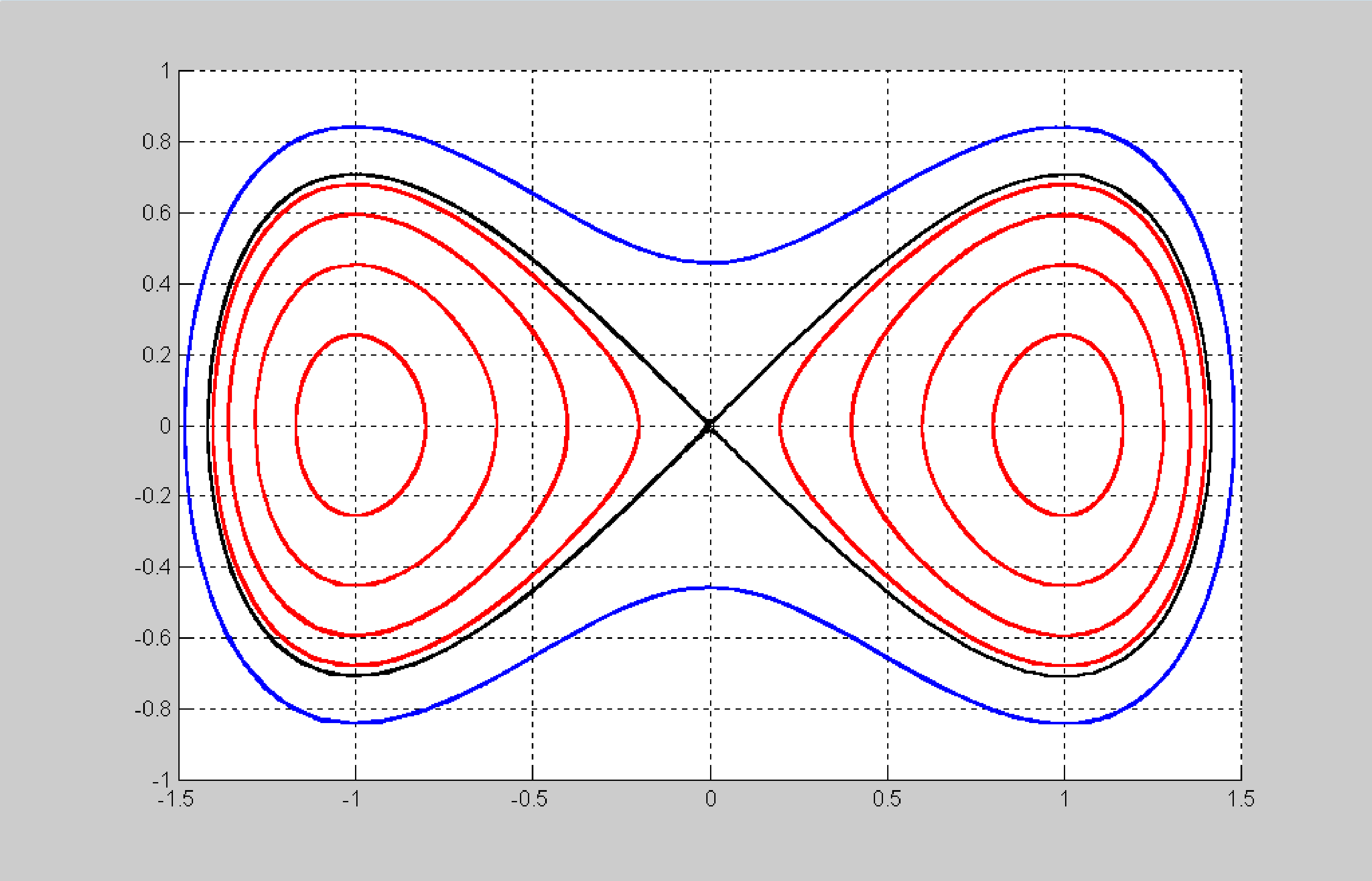
- $a = -1 < 0$



- $a = -1 < 0$

$$\ddot{x} + ax + x^3 = 0 \qquad x^3 = x \qquad x = \begin{cases} 0 \\ \pm 1 \end{cases}$$

$$x_e = \pm 1$$

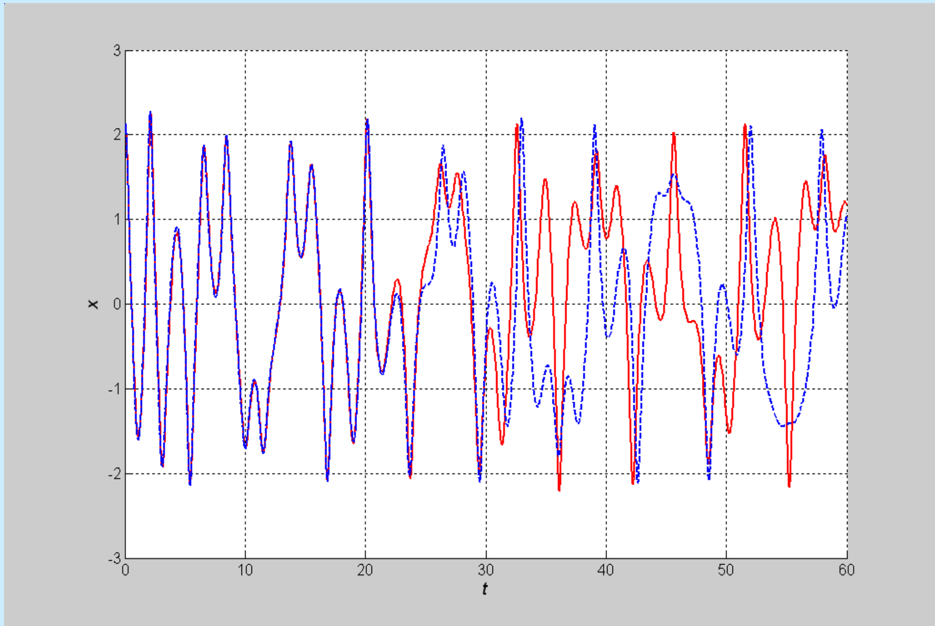


7. 混沌 Chaos

例 7.1.3 考虑如下微分方程

$$\ddot{x} + 0.1\dot{x} + x^5 = 6\sin t$$

- 对于确定性精确模型，通常认为初始条件微小改变会导致输出细微的变化



(实线)

$$x(0) = 2$$
$$\dot{x}(0) = 3$$

(虚线)

$$x(0) = 2.01$$
$$\dot{x}(0) = 3.01$$

- 系统输出相对初始条件的变化极其敏感
- 难以预测初始条件微小变化后的系统输出

7.1.5 分析工具

(1) 线性化

(2) 描述函数

(3) 相平面分析

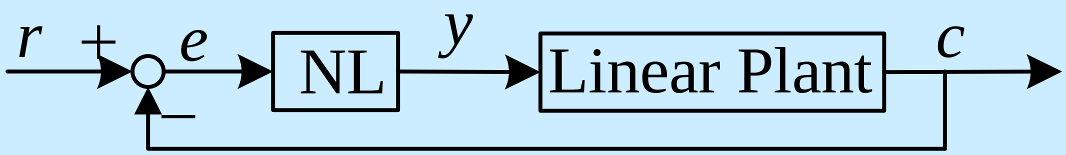
(4) Lyapunov 第二方法

(5) 计算机仿真

7.2 描述函数法

7.2.1 定义

♦ 系统结构



♦ 假设:

- 大多数线性系统属于低通滤波器
- c 主要包含低频成分

- 非线性环节输入输出间的近似关系

$$X\sin\omega t \longrightarrow \boxed{\mathrm{NL}} \longrightarrow Y_1\sin(\omega t+\varphi_1)$$

Y_1：输出基波的振幅

φ_1：输出基波的相移

- 从而，NL系统近似为线性系统

♦ 描述函数

- 令 $x = X\sin(\omega t)$ $\quad N = \dfrac{Y_1}{X}\mathrm{e}^{\mathrm{j}\varphi_1}$

- 近似描述非线性特性的函数

7.2.2 Fourier 级数

1. 定义

$y(t)$: **周期为 T 的有界可积函数**

(一个周期内极大极小点的数量有限)

则

$$y(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty}[A_n \cos(n\omega t) + B_n \sin(n\omega t)]$$

$$= \frac{A_0}{2} + \sum_{n=1}^{\infty} Y_n \sin(n\omega t + \varphi_n)$$

$$A_n = \frac{1}{\pi}\int_0^{2\pi} y(t)\cos(n\omega t)\,\mathrm{d}(\omega t) \qquad Y_n = \sqrt{A_n^2 + B_n^2}$$

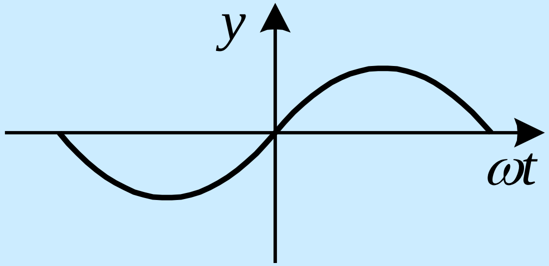
$$\omega = \frac{2\pi}{T}$$

$$B_n = \frac{1}{\pi}\int_0^{2\pi} y(t)\sin(n\omega t)\,\mathrm{d}(\omega t) \qquad \varphi_n = \arctan\frac{A_n}{B_n}$$

2. 性质:

(1) $y(t)$ — 奇函数

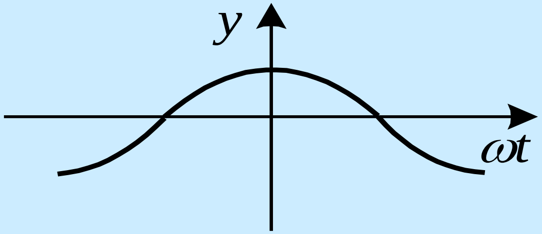
($y(t) = -y(-t)$)



$$A_n = 0$$

(2) $y(t)$ — 偶函数

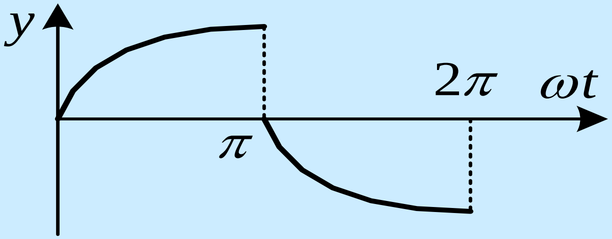
($y(t) = y(-t)$)



$$B_n = 0$$

(3) $y(t)$ — 半波对称函数

($y(\omega t) = -y(\omega t + \pi)$)



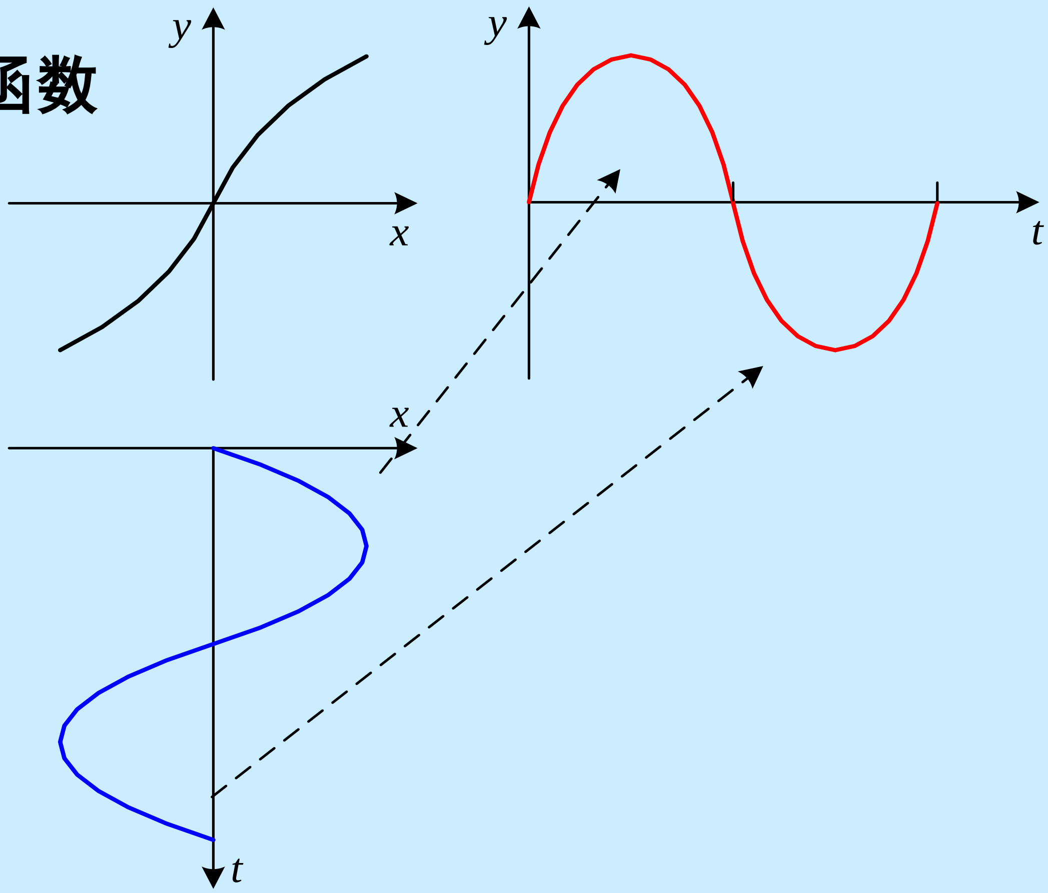
$$A_{2k} = 0 \qquad B_{2k} = 0$$

7.2.3 描述函数的计算

1. 描述函数的类型

(1) 非线性为单值、奇函数

（无记忆性）



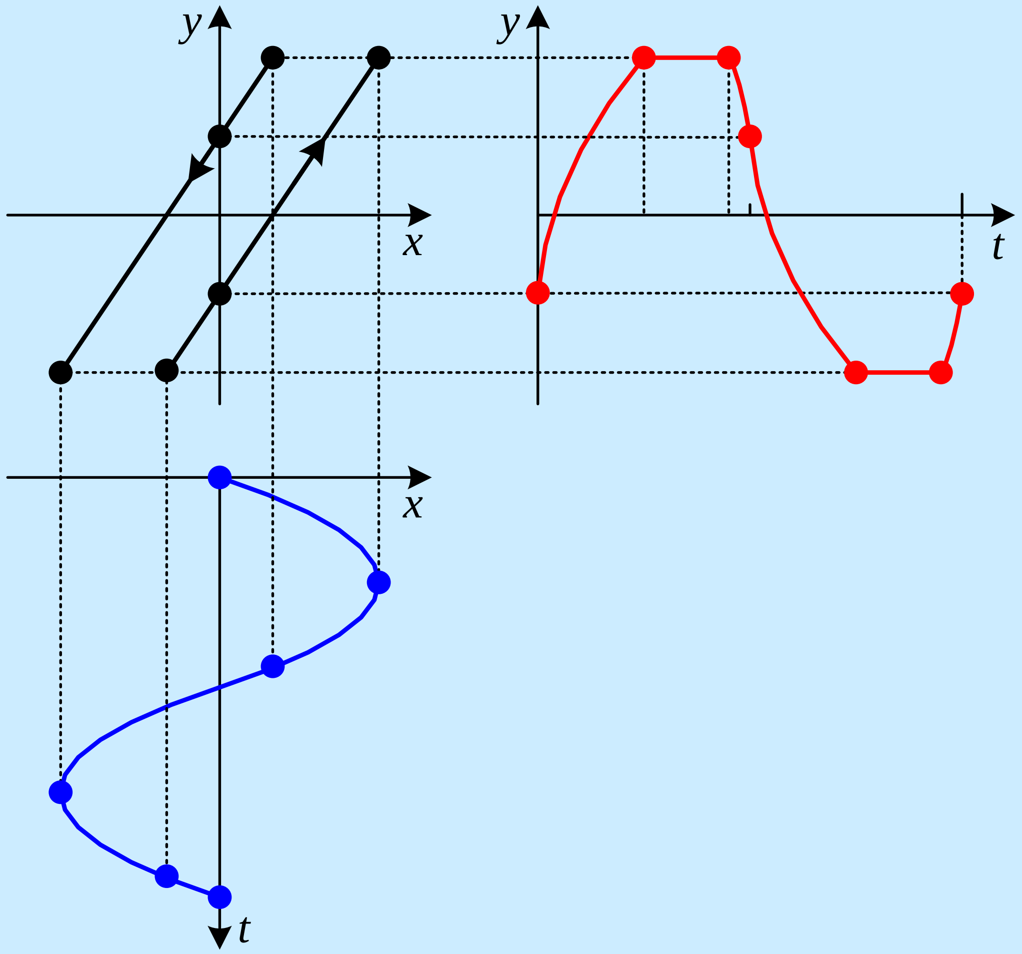
- **输出: 奇函数**

- **则** $A_n = 0$

$$Y_1(\omega) = B_1 \sin \omega t$$

(2) 多值非线性

- **输出：半波对称函数**



- **则**

$$A_0 = A_{2k} = B_{2k} = 0$$

$$Y_1(\omega) = A_1 \cos \omega t \\ + B_1 \sin \omega t$$

(3) 描述函数的一般形式

- 非线性为单值、过于原点对称

$$N = \frac{B_1}{X}$$

- 滞环、间隙等非线性

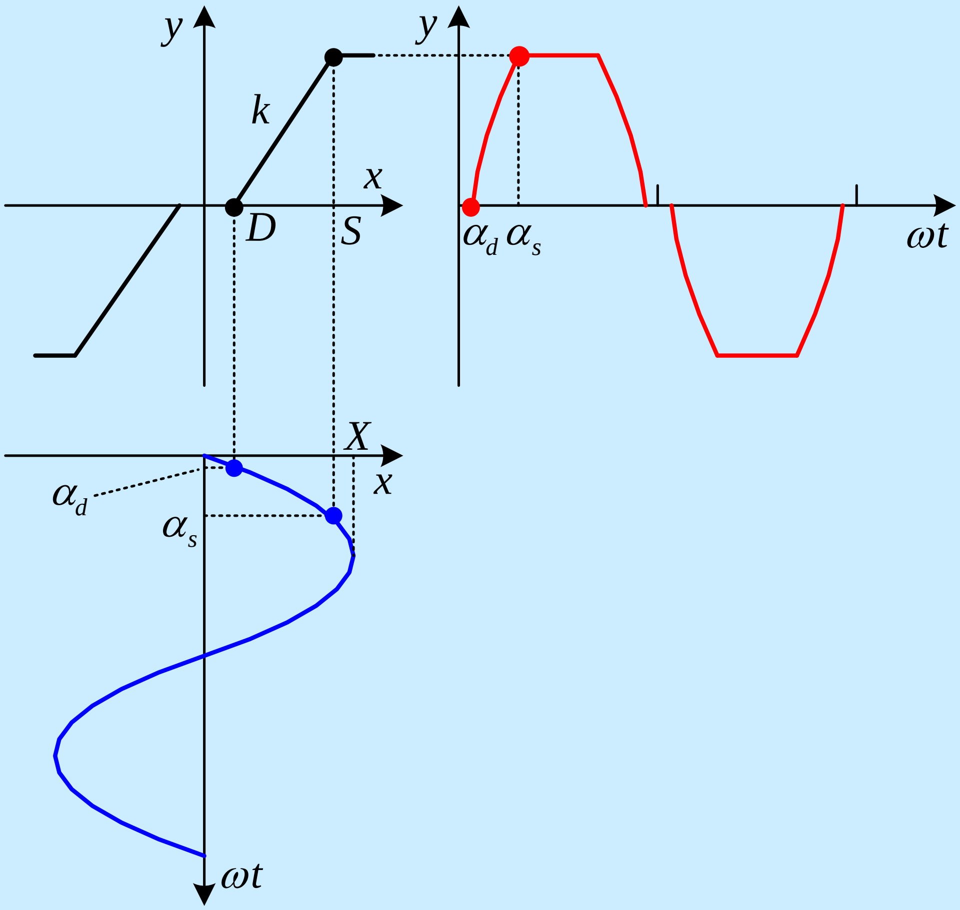
$$N = \frac{Y_1}{X} \mathrm{e}^{\mathrm{j}\varphi_1}$$

2. 描述函数的计算

例 7.2.1 计算兼有死区和饱和的非线性的描述函数

Solution:

(i) 令 $x(t) = X\sin\omega t$



NL – 对称，无记忆性

⇒ 输出 – 奇函数

仅需计算 B_1

(ii) 两个重要的角:

- $$\alpha_{\mathrm{d}} = \arcsin\frac{D}{X}$$

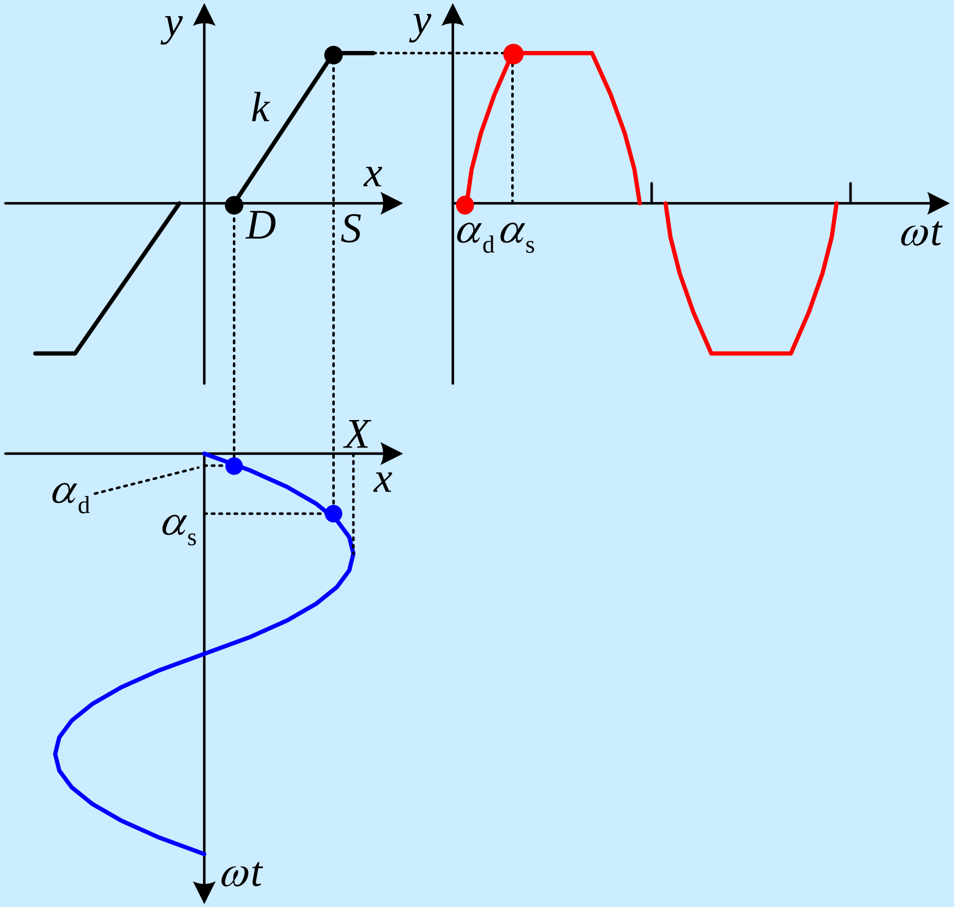
  当 $\omega t > \alpha_{\mathrm{d}}$

  $$y = k(x - D)$$

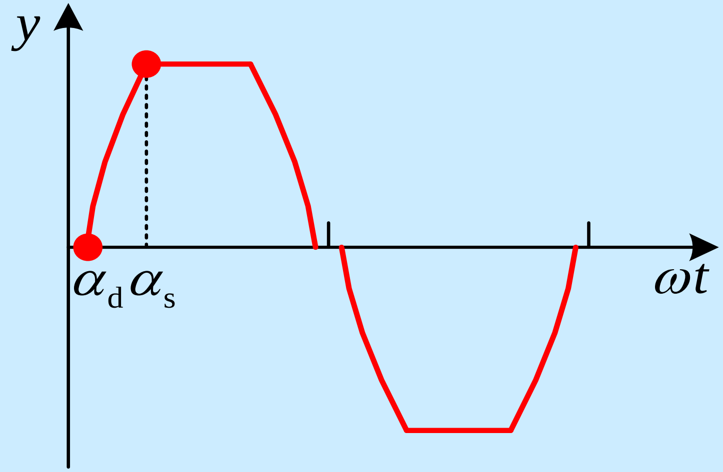
- $$\alpha_{\mathrm{s}} = \arcsin\frac{S}{X}$$

  当 $\omega t > \alpha_{\mathrm{s}}$

  $$y = k(S - D)$$



(iii) 计算 B_1



- $$B_1 = \frac{1}{\pi}\int_0^{2\pi} y(t)\sin\omega t\,\mathrm{d}\omega t$$

$$= \frac{4}{\pi}\int_0^{\pi/2} y(t)\sin\omega t\,\mathrm{d}\omega t$$

$$= \frac{4}{\pi}\left[\int_{\alpha_d}^{\alpha_s} k(X\sin\omega t - D)\sin\omega t\,\mathrm{d}\omega t + k(S-D)\int_{\alpha_s}^{\pi/2}\sin\omega t\,\mathrm{d}\omega t\right]$$

$$= \cdots$$ **(Omitted)**

$$= \frac{2kX}{\pi}\left[\alpha_s - \alpha_d - \frac{1}{2}(\sin 2\alpha_s - \sin 2\alpha_d) + \frac{2S}{X}\cos\alpha_s - \frac{2D}{X}\cos\alpha_d\right]$$

- $\sin\alpha_{\rm d} = \dfrac{D}{X} \quad \Rightarrow$

$$\sin 2\alpha_{\rm d} = 2\sin\alpha_{\rm d}\cos\alpha_{\rm d} = 2\frac{D}{X}\cos\alpha_{\rm d} = 2\frac{D}{X}\sqrt{1-\left(\frac{D}{X}\right)^2}$$

$$\sin\alpha_{\rm s} = \frac{s}{X} \quad \Rightarrow \quad \sin 2\alpha_{\rm s} = 2\frac{s}{X}\cos\alpha_{\rm s} = 2\frac{S}{X}\sqrt{1-\left(\frac{S}{X}\right)^2}$$

- **于是** $B_1 = \dfrac{2kX}{\pi}\left[\alpha_{\rm s} - \alpha_{\rm d} + \dfrac{1}{2}(\sin 2\alpha_{\rm s} - \sin 2\alpha_{\rm d})\right]$

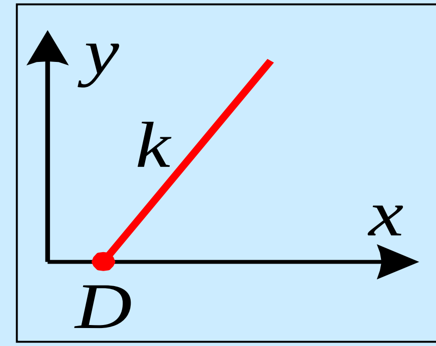
$$= \frac{kX}{\pi}\left[2\alpha_{\rm s} - 2\alpha_{\rm d} + \sin 2\alpha_{\rm s} - \sin 2\alpha_{\rm d}\right]$$

(iv) $N = \dfrac{B_1}{X} = \dfrac{k}{\pi}\left[2\alpha_s - 2\alpha_{\rm d} + \sin 2\alpha_{\rm s} - \sin 2\alpha_{\rm d}\right] = f\left(\dfrac{S}{X}, \dfrac{D}{X}\right)$

$$(X \geq S)$$

(1) 纯死区

- $S \to \infty$ **(or** $X < S$ **),** $\alpha_s = \dfrac{\pi}{2}$ $\sin 2\alpha_s = 0$

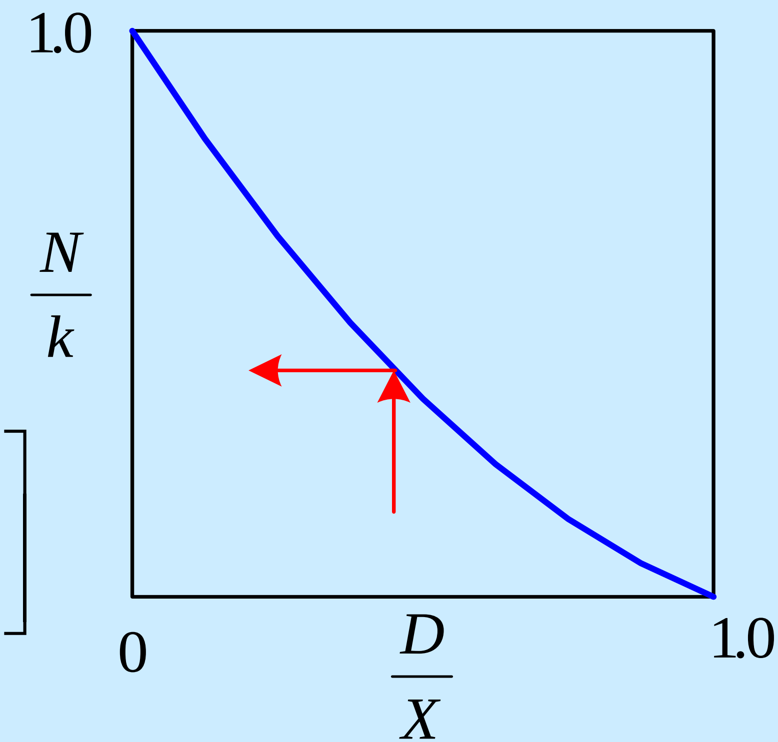


- $$N = \frac{k}{\pi}\left[2\alpha_s - 2\alpha_d + \sin 2\alpha_s - \sin 2\alpha_d\right]$$

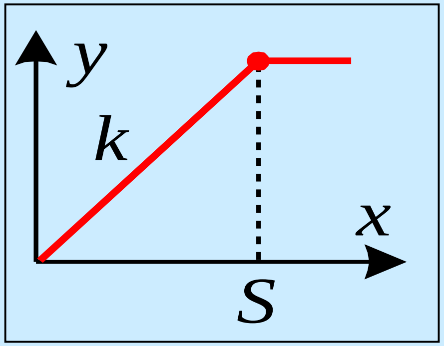
$$= \frac{k}{\pi}\left[\pi - 2\alpha_d - \sin 2\alpha_d\right]$$

$$= k - \frac{2k}{\pi}\left[\arcsin\frac{D}{X} + \frac{D}{X}\sqrt{1-\left(\frac{D}{X}\right)^2}\right]$$

$$(X \ge D)$$



(2) 纯饱和

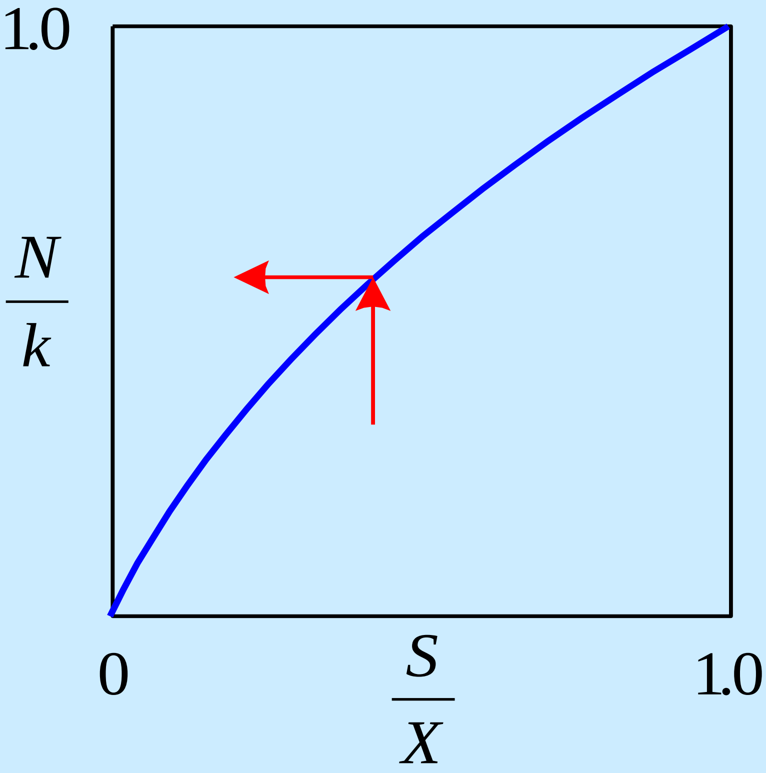


- $\boldsymbol{D = 0},\quad \alpha_{\mathrm{d}} = 0 \qquad \sin 2\alpha_{\mathrm{d}} = 0$
- $N = \dfrac{k}{\pi}[2\alpha_{\mathrm{s}} - 2\alpha_{\mathrm{d}} + \sin 2\alpha_{\mathrm{s}} - \sin 2\alpha_{\mathrm{d}}]$

$$= \frac{k}{\pi}[2\alpha_{\mathrm{s}} + \sin 2\alpha_{\mathrm{s}}]$$

$$= \frac{2k}{\pi}\left[\arcsin\frac{S}{X} + \frac{S}{X}\sqrt{1-\left(\frac{S}{X}\right)^2}\right]$$

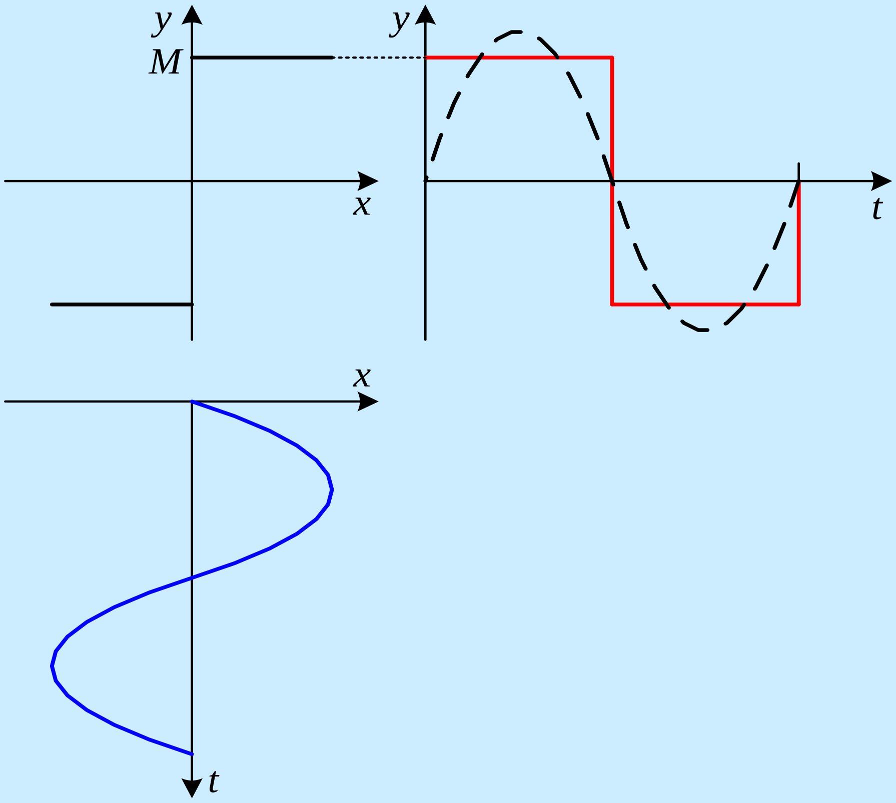
$$(X \geq S)$$



7.2.4 典型非线性的描述函数

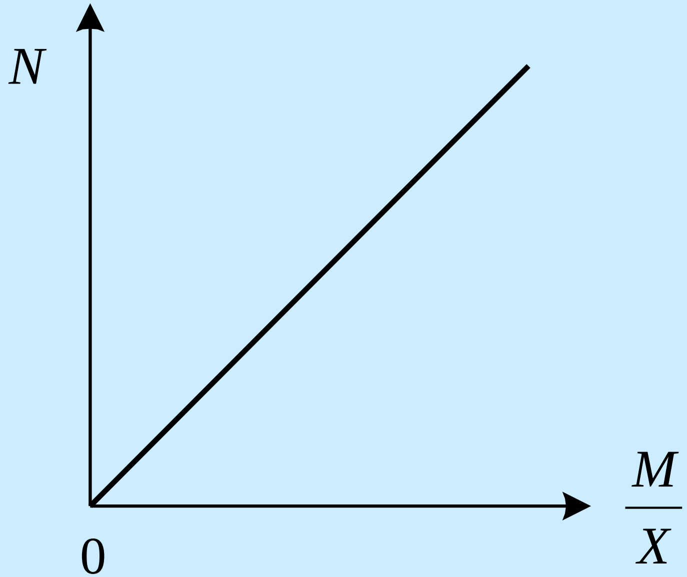
1. 理想继电器 (开关非线性)

- $$B_1 = \frac{1}{\pi}\int_0^{2\pi} y(t)\sin\omega t\,\mathrm{d}\omega t$$
$$= \frac{2}{\pi}\int_0^{\pi} M\sin\omega t\,\mathrm{d}\omega t$$
$$= \frac{2M}{\pi}\int_0^{\pi} \sin\omega t\,\mathrm{d}\omega t$$
$$= \frac{2M}{\pi}\left[-\cos\omega t\right]\Big|_0^{\pi}$$
$$= \frac{4M}{\pi}$$

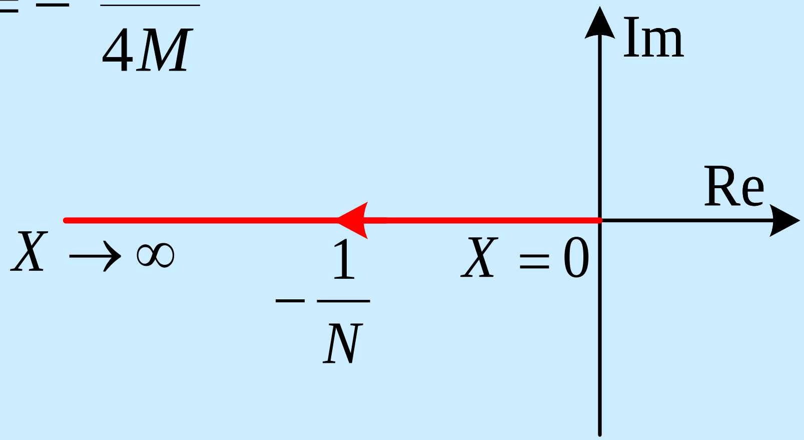


$$\boxed{B_1 = \frac{4M}{\pi}}$$

- $N = \dfrac{4M}{\pi X}$



- $-\dfrac{1}{N} = -\dfrac{\pi X}{4M}$

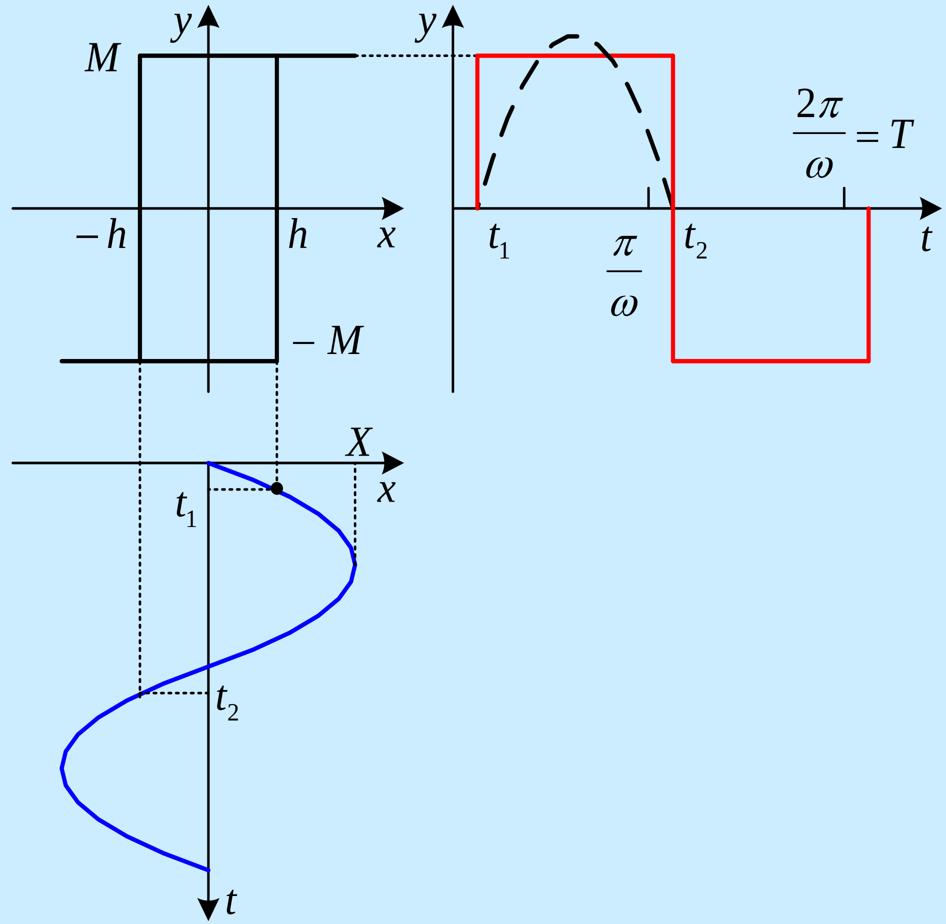


2. 带滞环的继电器

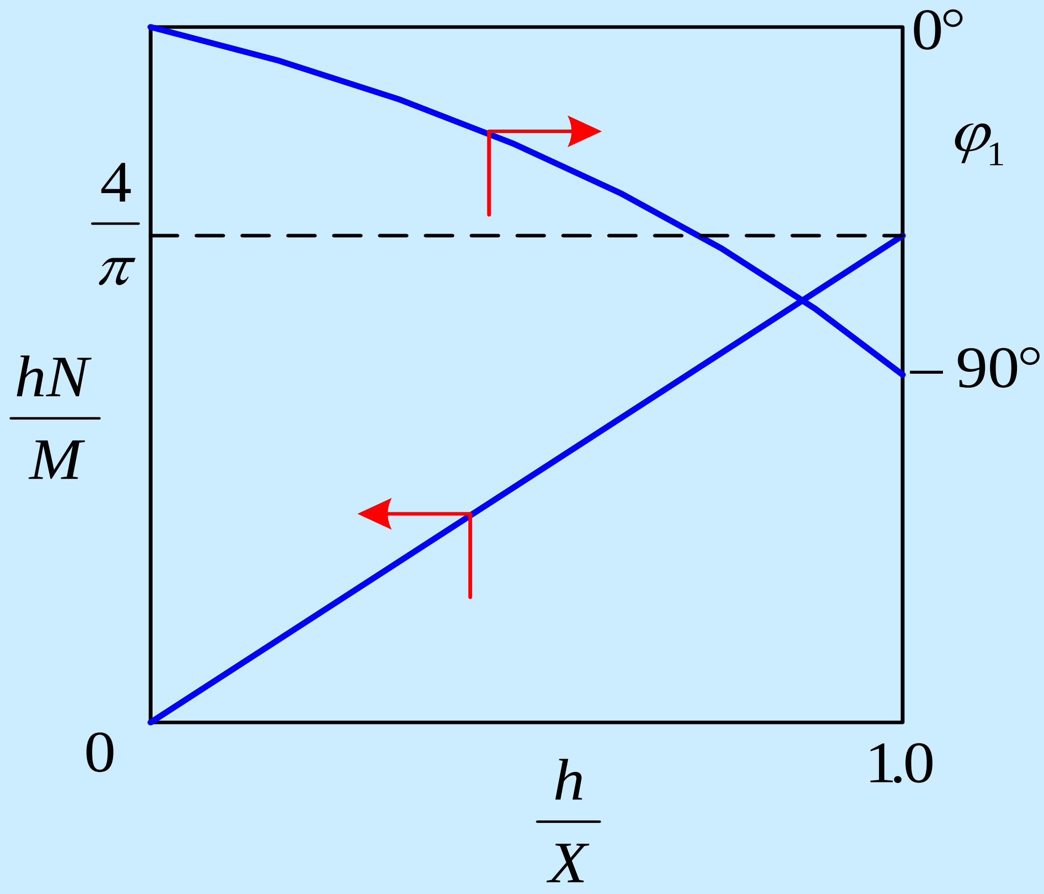
- $x(t) = X\sin\omega t$
- $y = \begin{cases} -M & 0 \le t < t_1, \\ & t_2 < t \le T \\ M & t_1 < t \le t_2 \end{cases}$
- $\omega t_1 = \arcsin\dfrac{h}{X}$

$$\omega t_2 = \arcsin\frac{h}{X} + \frac{\omega T}{2}$$

$$= \omega\left(t_1 + \frac{T}{2}\right)$$



- $N = \dfrac{4M}{\pi X} \mathrm{e}^{-\mathrm{j}\arcsin\frac{h}{X}} \qquad (X \geq h)$



$$\boxed{N = \frac{4M}{\pi X} e^{-\mathrm{j}\arcsin\frac{h}{X}}}$$

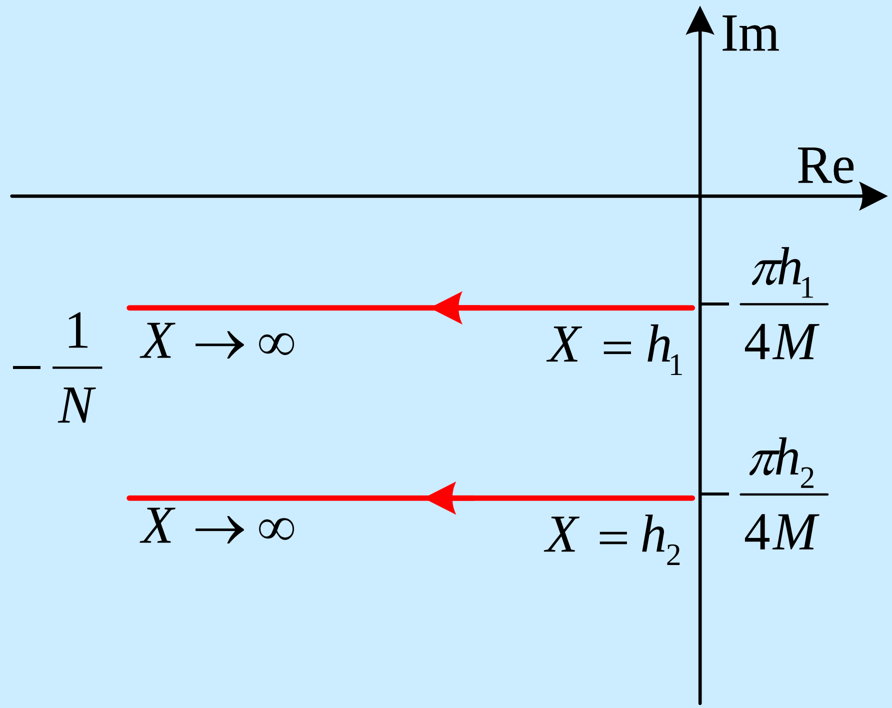
- $$-\frac{1}{N} = -\frac{\pi X}{4M} e^{\mathrm{j}\arcsin\frac{h}{X}} \qquad (X \geq h)$$

$$= -\frac{\pi X}{4M}(\cos\varphi + \mathrm{j}\sin\varphi) = -\frac{\pi X}{4M}\left(\sqrt{1-\left(\frac{h}{X}\right)^2} + \mathrm{j}\frac{h}{X}\right)$$

$$(\ \varphi = \arcsin(h/X)\ )$$

- $$\mathrm{Re}\left(-\frac{1}{N}\right) = -\frac{\pi\sqrt{X^2 - h^2}}{4M}$$

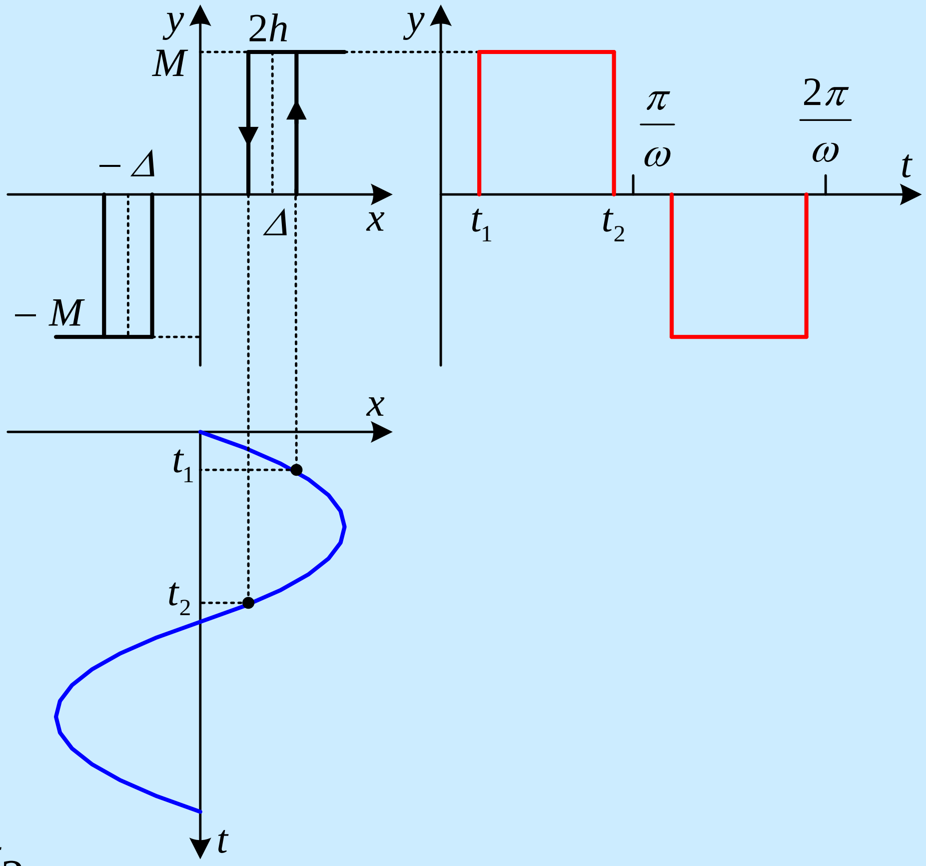
$$\mathrm{Im}\left(-\frac{1}{N}\right) = -\frac{\pi h}{4M}$$



3. 带死区和滞环的继电器

- $x(t) = X\sin\omega t$

- $y = \begin{cases} 0 & 0 \le t < t_1, \\ & t_2 \le t < \dfrac{\pi}{\omega} + t_1, \\ & \dfrac{\pi}{\omega} + t_2 \le t \le \dfrac{2\pi}{\omega} \\ M & t_1 \le t < t_2 \\ -M & \dfrac{\pi}{\omega} + t_1 \le t < \dfrac{\pi}{\omega} + t_2 \end{cases}$



- $$N=\sqrt{\left(\frac{a_1}{X}\right)^2+\left(\frac{b_1}{X}\right)^2}\ \mathrm{e}^{\mathrm{j}\arctan\frac{a_1}{b_1}}$$

$$a_1=-\frac{4hM}{\pi X}$$

$$b_1=\frac{2M}{\pi X}\left[\sqrt{1-\left(\frac{\Delta-h}{X}\right)^2}+\sqrt{1-\left(\frac{\Delta+h}{X}\right)^2}\right]$$

$$(X\geq\Delta+h)$$

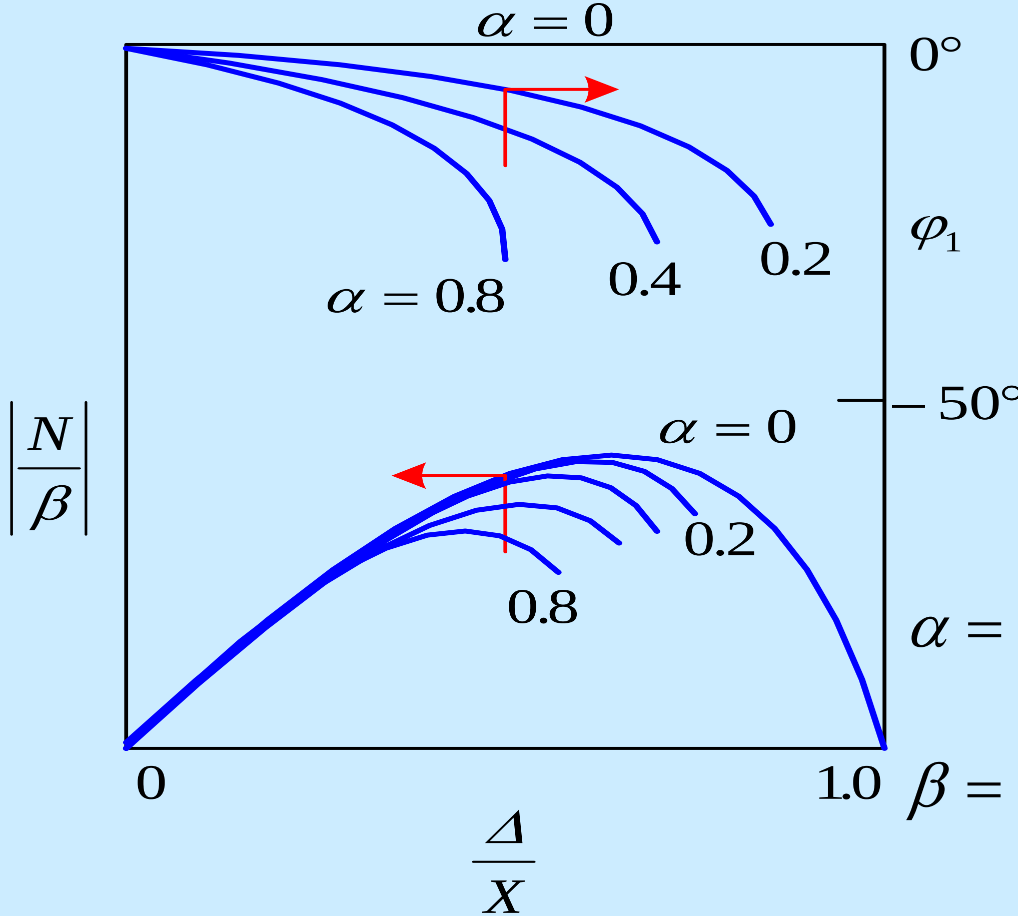
or

$$N=\frac{2M}{\pi X}\left(\mathrm{e}^{\mathrm{j}\theta_2}+\mathrm{e}^{-\mathrm{j}\theta_1}\right)$$

$$(X\geq\Delta+h)$$

$$\theta_1=\arcsin\frac{\Delta-h}{X}$$

$$\theta_2=\arcsin\frac{\Delta+h}{X}$$



$$\alpha=\frac{h}{\Delta}$$

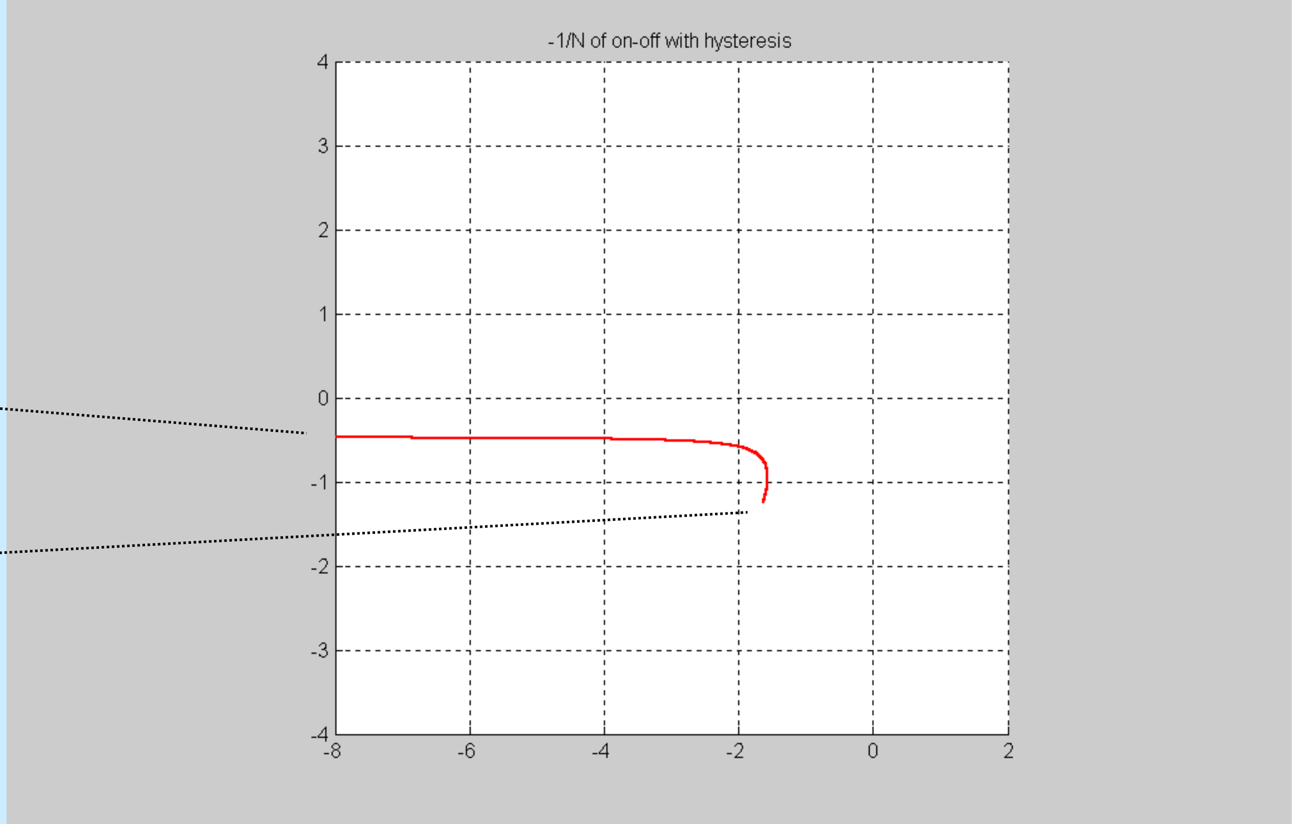
$$\beta=\frac{M}{\Delta}$$

- $-\dfrac{1}{N} \qquad (X \geq \Delta + h)$

$m \to \infty$

$X = \Delta + h$

$m = 1 + \dfrac{h}{\Delta}$

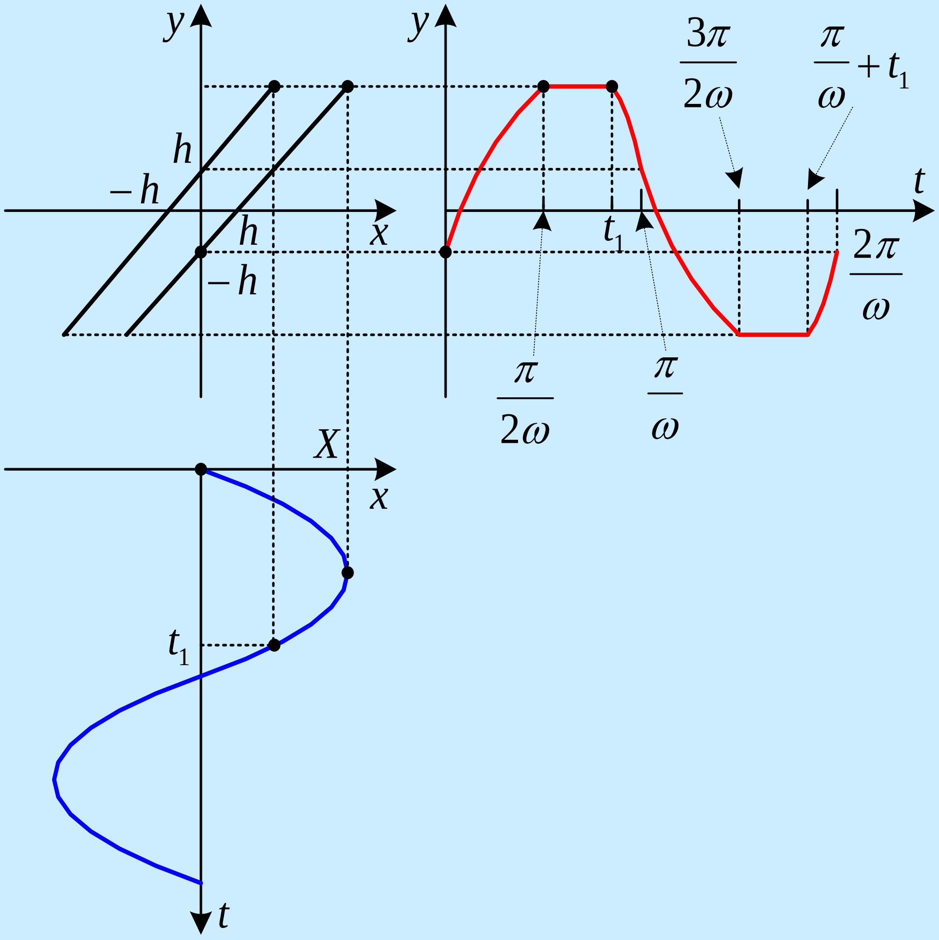


$(h = 0.6,\ M = 1)$

4. 间隙

- $x(t) = X\sin\omega t$

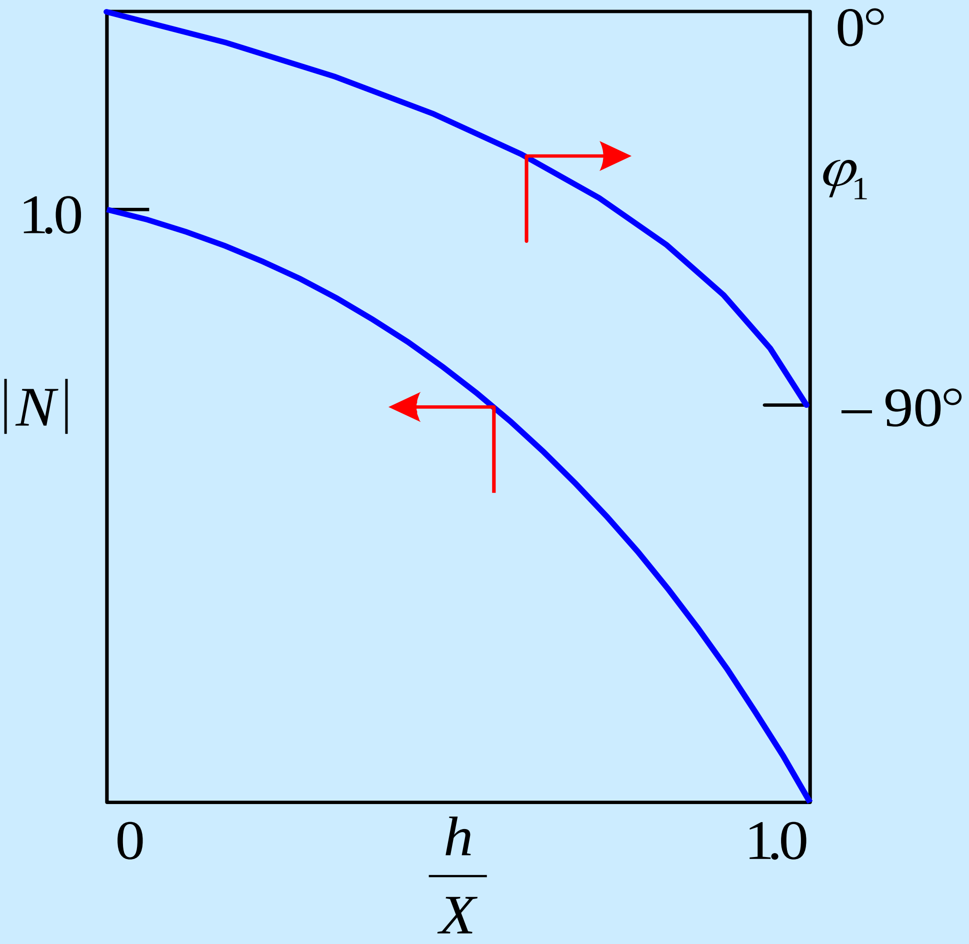
- $$y = \begin{cases} x-h & 0 \le t < \dfrac{\pi}{2\omega} \\ X-h & \dfrac{\pi}{2\omega} \le t < t_1 \\ x+h & t_1 \le t < \dfrac{3\pi}{2\omega} \\ h-X & \dfrac{3\pi}{2\omega} \le t < t_1 + \dfrac{\pi}{\omega} \\ x-h & t_1 + \dfrac{\pi}{\omega} \le t \le \dfrac{2\pi}{\omega} \end{cases}$$



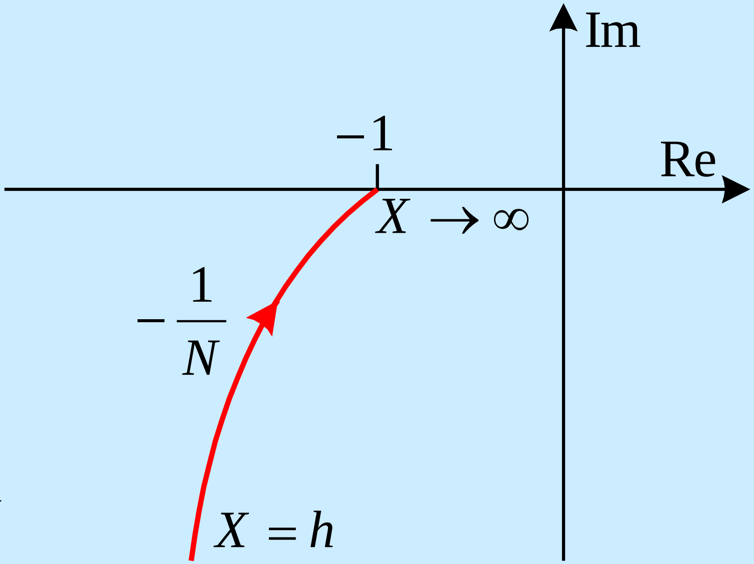
- $N = \dfrac{1}{\pi}\left\{\sqrt{\dfrac{4h}{X}-\left(\dfrac{2h}{X}\right)^2}\left(1-\dfrac{2h}{X}\right)+\pi-\arccos\left(1-\dfrac{2h}{X}\right)\right.$

$$\left.+\,\mathrm{j}\left[\left(\frac{2h}{X}\right)^2-\frac{4h}{X}\right]\right\}$$

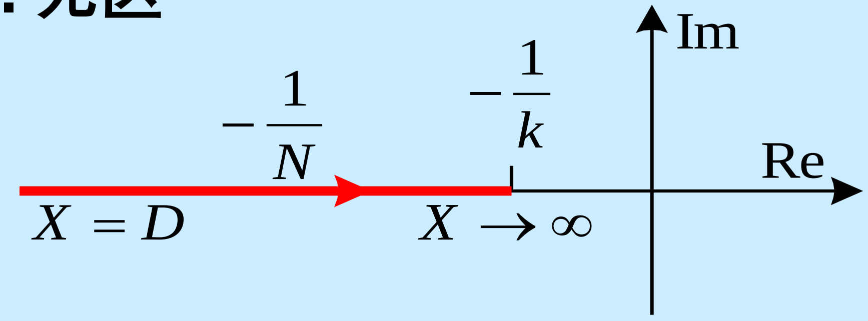
$$(X \geq h)$$



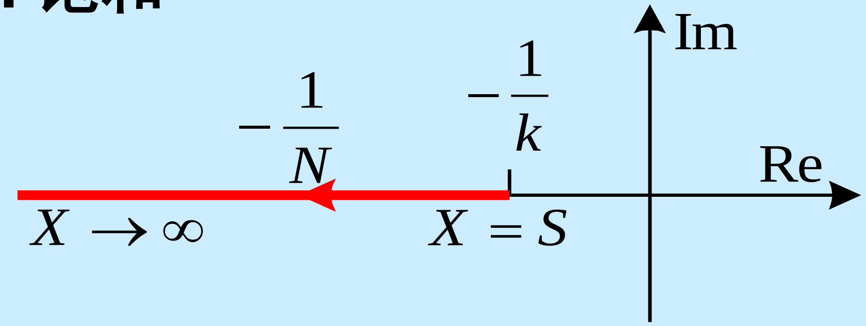
- $-\dfrac{1}{N} \qquad (X \geq h)$



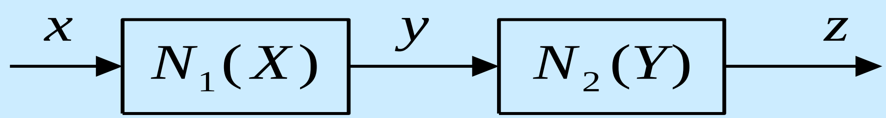
5. 死区



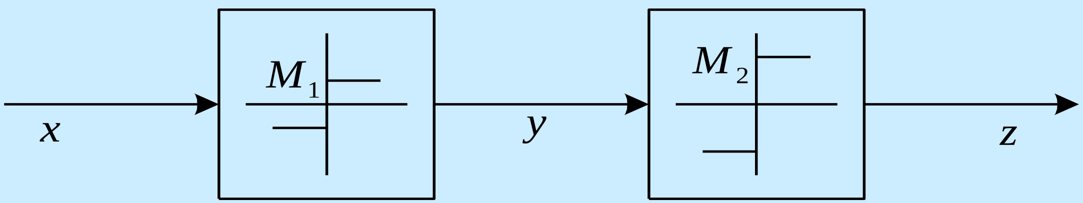
6. 饱和

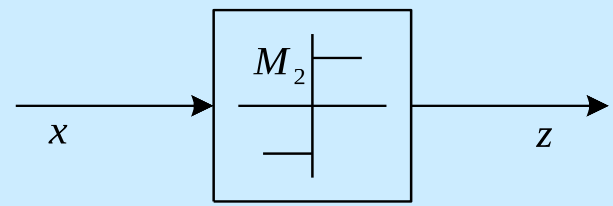


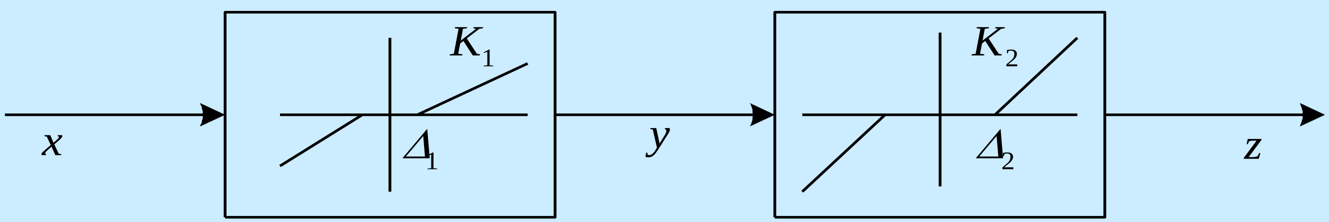
非线性环节的串联

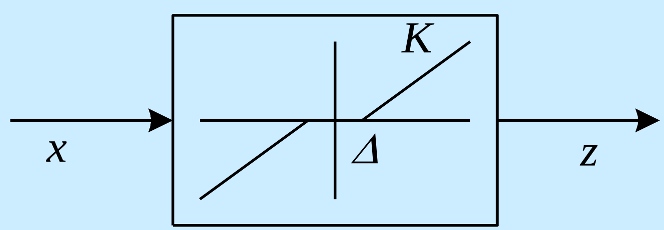


- 通常 $$\frac{Z_1}{X} \neq N_2(Y)N_1(X)$$





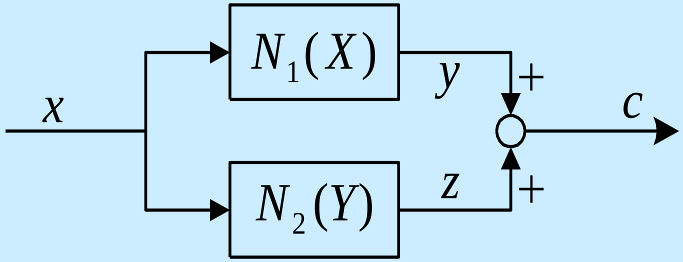




$$K = K_1 K_2$$

$$\varDelta = \varDelta_1 + \frac{\varDelta_2}{K_1}$$

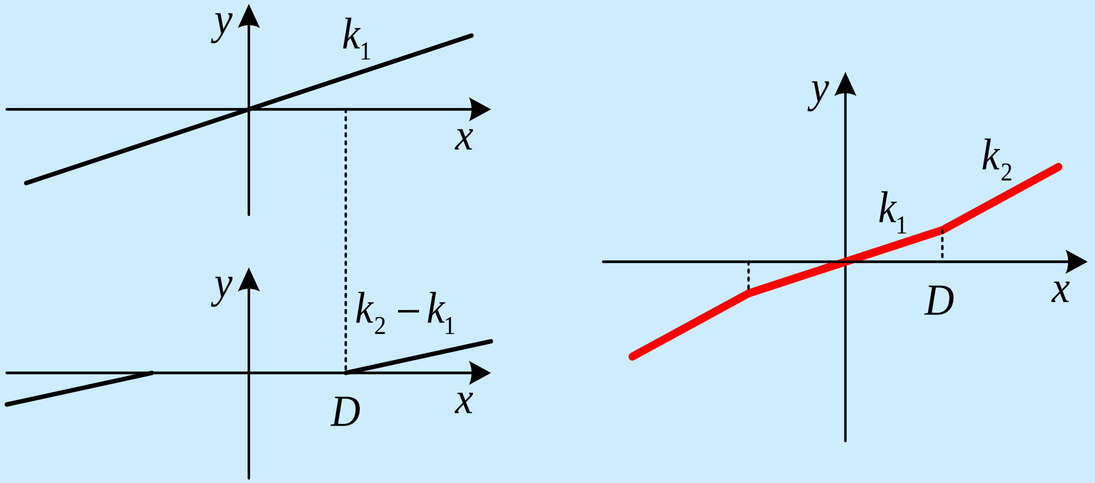
非线性环节的并联



$$c(x) = y(x) + z(x)$$

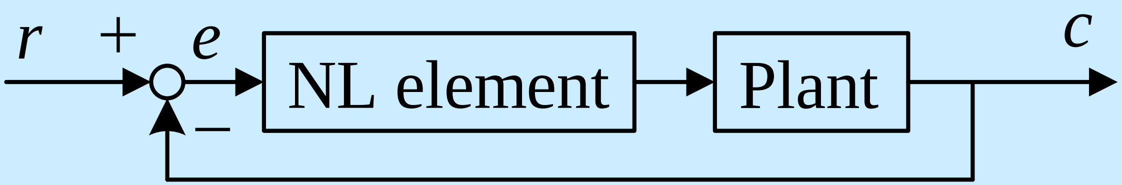
$$C_1 = Y_1 + Z_1$$

于是 $N = \dfrac{C_1}{X} = \dfrac{Y_1}{X} + \dfrac{Z_1}{X} = N_1 + N_2$

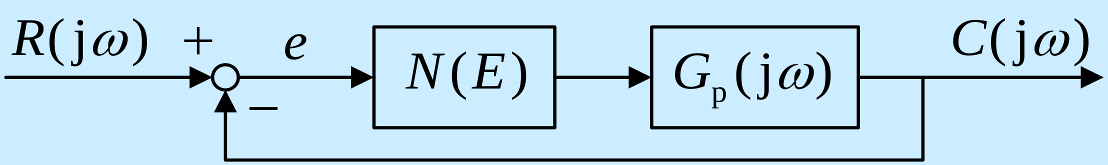


7.2.5 基于描述函数的非线性分析

1. 描述函数法的基础



- 多数对象是低通滤波器
- 回路中主要是低频成分
- 等价框图:



$$1+G(s)=\frac{D(s)+N(s)}{D(s)}$$

2. 稳定性分析

(1) Nyquist 准则 (revision)

- $z - p_0 = N$

  z — RHP闭环极点的数量

  p_0 — 不稳定开环极点的数量

  N — $G(j\omega)$顺时针如围绕（–1, j0）点的圈数

- 闭环系统稳定的条件:

  $N = -p_0$：顺时针绕圈

  $N = p_0$： 逆时针绕圈

(2) 非线性系统的等价闭环频率响应函数

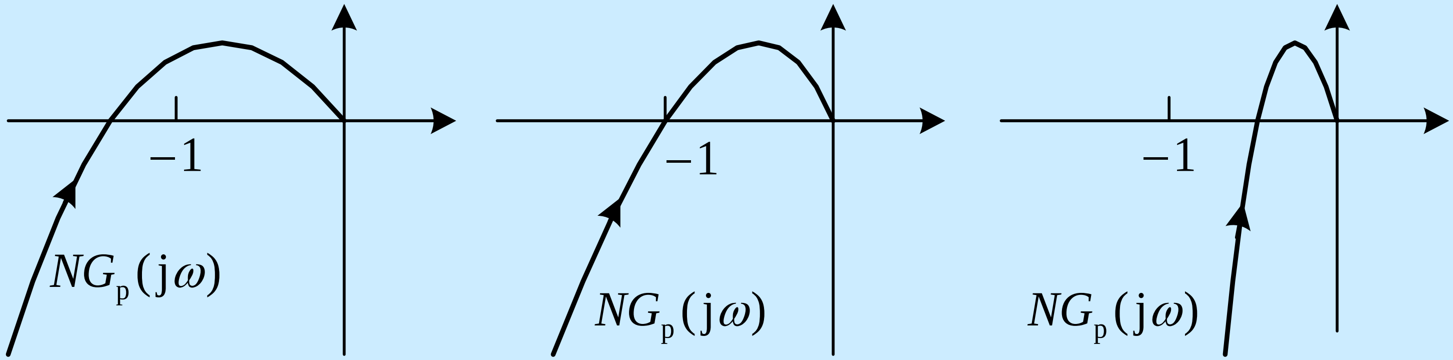
$$\frac{C(\mathrm{j}\omega)}{R(\mathrm{j}\omega)}=\frac{N(X)G_{\mathrm{p}}(\mathrm{j}\omega)}{1+N(X)G_{\mathrm{p}}(\mathrm{j}\omega)}$$

- **特征方程:** $1+N(X)G_{\mathrm{p}}(\mathrm{j}\omega)=0$

(3) 稳定性条件

- **假设 $G_{\mathrm{p}}(\mathrm{j}\omega)$ 为最小相位系统, i.e. $p_0 = 0$**

♦ 通过 $N(X)G_{\mathrm{p}}(\mathrm{j}\omega)$ 判断稳定性



- $NG_{\mathrm{p}}(\mathrm{j}\omega)$ 包围 $(-1, \mathrm{j}\,0) \Rightarrow$ 不稳定
- $NG_{\mathrm{p}}(\mathrm{j}\omega)$ 不包围 $(-1, \mathrm{j}\,0) \Rightarrow$ 稳定
- $NG_{\mathrm{p}}(\mathrm{j}\omega)$ 穿越 $(-1, \mathrm{j}\,0)$) $\Rightarrow$ 临界稳定

  * 存在自持振荡

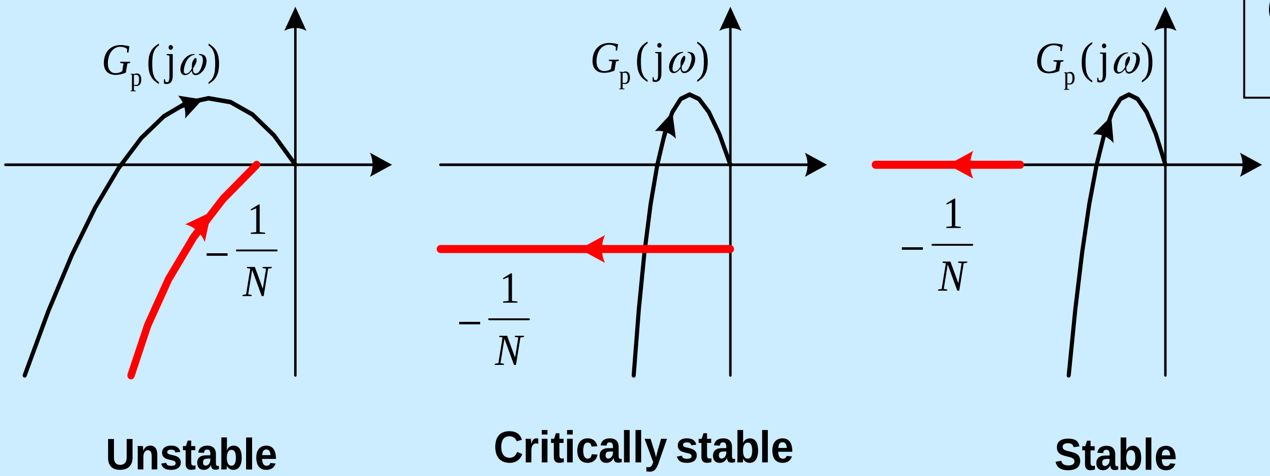
!! 然而，X 未知的情况下无法绘制 $N(X)G_{\mathrm{p}}(\mathrm{j}\omega)$

$$N(X)G_p(j\omega) = -1$$

$$\Downarrow$$

$$G_p(j\omega) = -\frac{1}{N(X)}$$

◆**通过** $G_p(j\omega)$ **和 $-1/N(X)$ 判断稳定性**



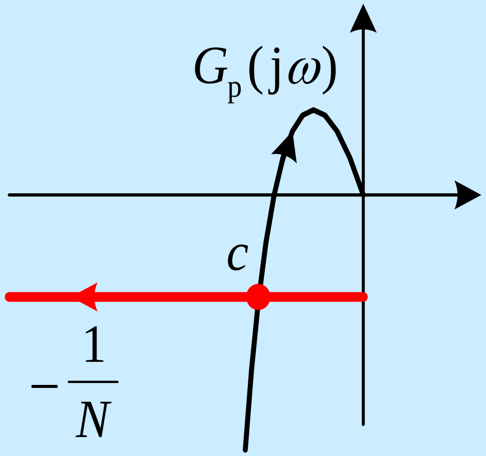
- $G_p(j\omega)$ **穿越** $-1/N \Rightarrow$ **临界稳定　自持振荡**
- $G_p(j\omega)$ **不包围** $-1/N \Rightarrow$ **稳定**
- $G_p(j\omega)$ **包围** $-1/N \Rightarrow$ **不稳定**

3. 自持振荡

(1) 自持振荡的确定

- $G_p(j\omega)$ 的轨迹与 $-1/N$ 的轨迹相交

  $\Rightarrow$ 振荡



- 相交点决定了频率和振幅

(2) 稳定与不稳定的工作点

- a: 不稳定工作点

* 非线性环节输入的幅值 $X = a$

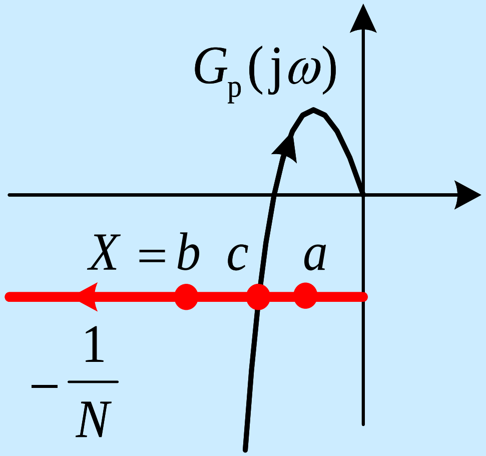
* $G_p(s)$ 包围 $-1 / N(a)$

* $N(a)G_p(s)$ 包围 $-1 + j\,0$

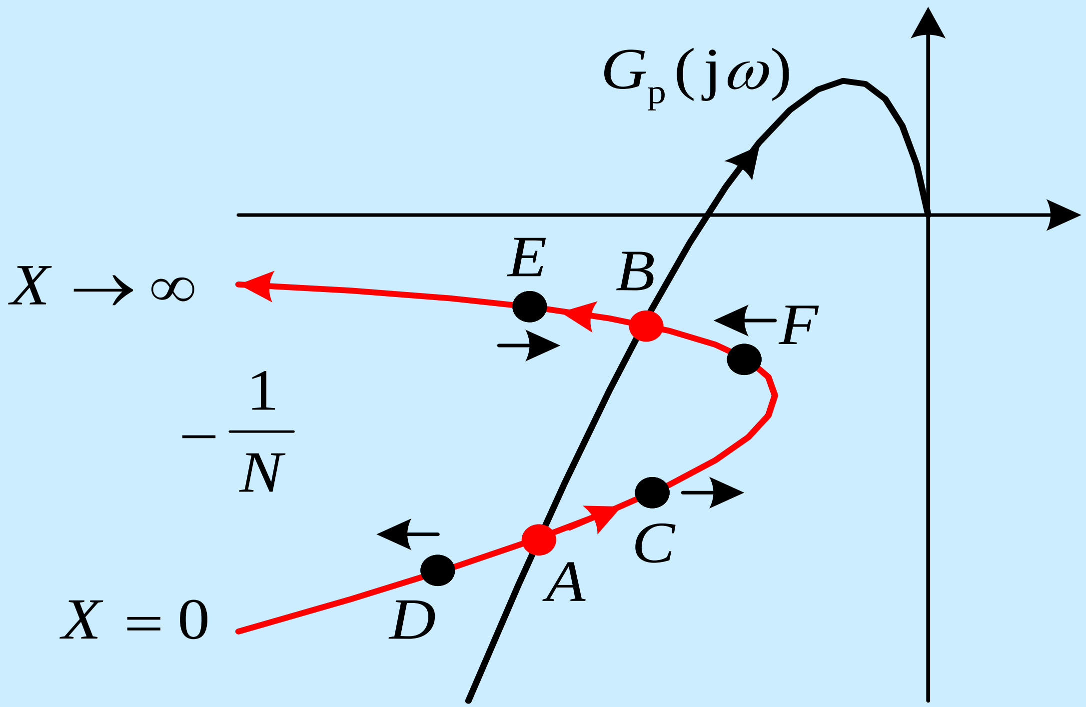
- b: 稳定工作点

- c: $X = c$

  临界稳定工作点



(3) 稳定与不稳定振荡

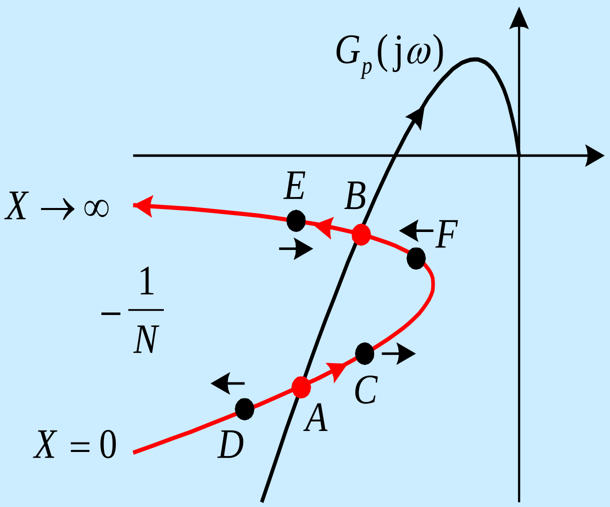


A **不稳定振荡**

B **稳定振荡**

- ***A*** **&** ***B*****: 自持振荡**
- ***C*****,** ***D*****,** ***E*****,** ***F*** **: 幅值为** ***X*****s 的振荡**

- ***C***: $G_p(\mathrm{j}\omega)$ **包围 –1/*N***
  -- 不稳定条件: *X* 增大
  -- 点 *C* 将移向点 *B*
- ***D***: $G_p(\mathrm{j}\omega)$ **不包围 –1/*N***
  -- 稳定条件: *X* 减小
  -- 系统变得更稳定
- ***F***: $G_p(\mathrm{j}\omega)$ **包围 –1/*N***
  --不稳定条件: *X* 增大
  --点 *F* 将移向点 *B*
- ***E***: $G_p(\mathrm{j}\omega)$ **不包围 –1/*N***
  --稳定条件: *X* 减小
  --点 *E* 将移向点 *B*



♦ **结论:**
- ***A* – 不稳定的自持振荡**
- ***B* – 稳定的自持振荡**

(4) 自持振荡的计算

(i) 图形法

(ii) 分析法

- **闭环特征方程**

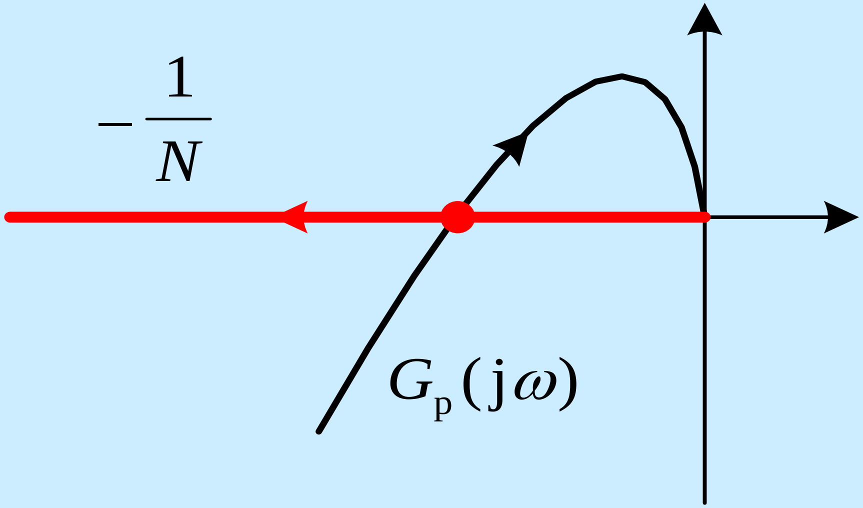
$$1+N(X)G_{\mathrm{p}}(\mathrm{j}\omega)=0$$

- **当 $N(X)$ 是实函数时**

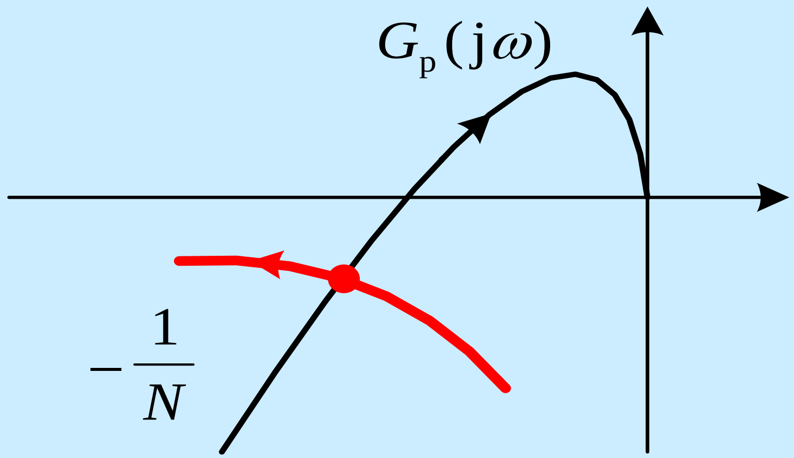
$$\operatorname{Im}G_{\mathrm{p}}(\mathrm{j}\omega)=0 \quad \Rightarrow \quad \omega$$

$$\operatorname{Re}G_{\mathrm{p}}(\mathrm{j}\omega)=-\frac{1}{N(X)}$$

$$\Rightarrow \quad X$$



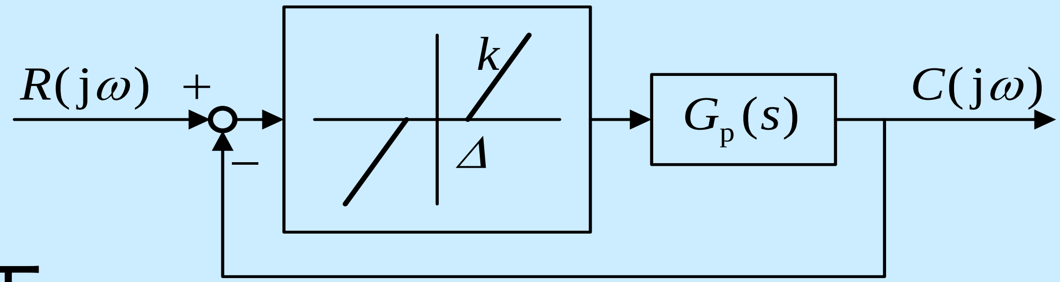
- 当 $N(X)$ 是复函数时



$$\begin{cases} \operatorname{Im} N(X)G_p(j\omega) = 0 \\ \operatorname{Re} N(X)G_p(j\omega) = -1 \end{cases}$$

$$\Rightarrow \quad X, \ \omega$$

例 7.2.2



给定对象如下

$$G_{\mathrm{p}}(\mathrm{j}\omega)=\frac{K}{\mathrm{j}\omega(1+\mathrm{j}\omega)(1+0.5\mathrm{j}\omega)}$$

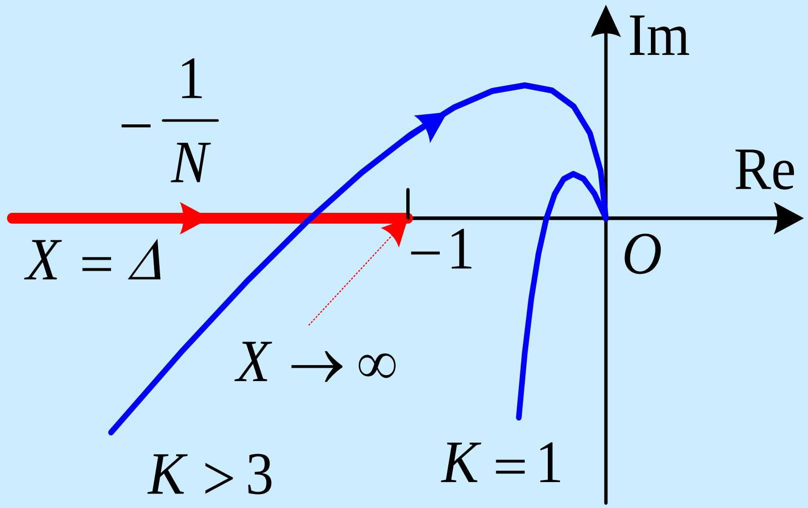
死区非线性如下

$$N(X)=k-\frac{2k}{\pi}\left[\arcsin\frac{\Delta}{X}+\frac{\Delta}{X}\sqrt{1-\left(\frac{\Delta}{X}\right)^2}\right]$$

试确定并分析自持振荡

$$G_p(j\omega)=\frac{K}{j\omega(1+j\omega)(1+0.5j\omega)}$$

Solution:



$$\arg G_p(j\omega)$$
$$=-90°-\arctan\omega-\arctan(0.5\omega)$$
$$=-180°$$

$$0.5\omega^2=1 \qquad \omega=\sqrt{2}$$

$$\left|G_p(j\sqrt{2})\right|=1$$

$$K=\left|j\sqrt{2}(1+j\sqrt{2})(1+0.5j\sqrt{2})\right|$$
$$=3$$

- **系统是否稳定?**

*** $K\geq 3$ 时系统不稳定**

*** 存在自持振荡**

- **自持振荡是否稳定?**

*** 自持振荡不稳定**

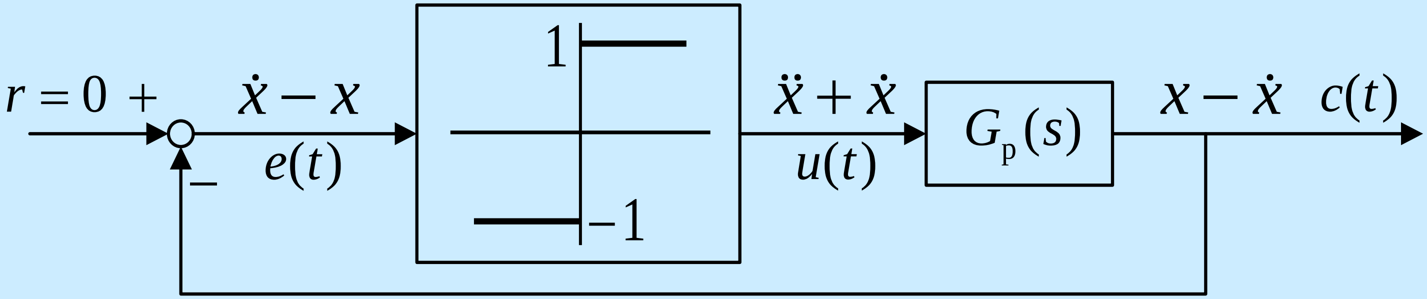
例 7.2.3 给定

$$\ddot{x}+\dot{x}=1 \quad \text{for} \quad \dot{x}-x>0$$
$$\ddot{x}+\dot{x}=-1 \quad \text{for} \quad \dot{x}-x<0$$

试利用描述函数法确定并分析自持振荡

Solution :

(i) 绘制闭环系统框图



- **非线性是啥?**
- **非线性环节的输入和输出是啥?**
- **对象的输入和输出是啥?**

$$u = \ddot{x} + \dot{x}$$

$$c = x - \dot{x}$$

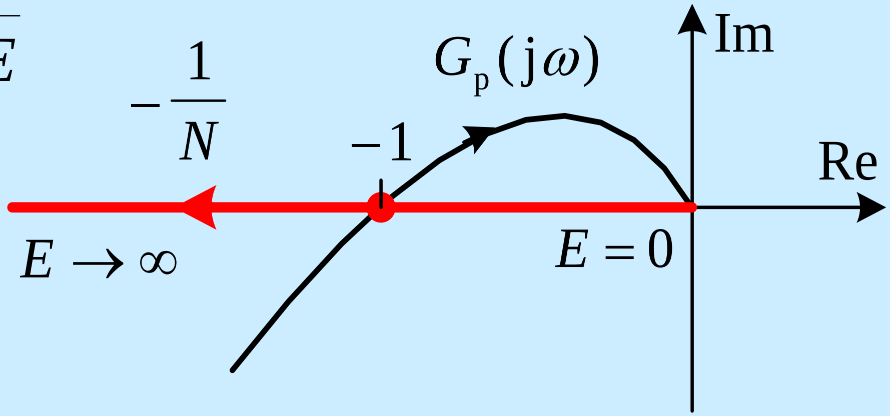
(ii) 计算对象的传递函数

$$G_{\mathrm{p}}(s) = \frac{C(s)}{U(s)} = \frac{(1-s)X(s)}{(s^2+s)X(s)} = \frac{1-s}{s(s+1)}$$

(iii) 继电器的描述函数

$$N(E) = \frac{4M}{\pi E} = \frac{4}{\pi E}$$

(iii) 稳定性分析



- **闭环系统不稳定**
- **自持振荡稳定**

$$G_p(s)=\frac{1-s}{s(s+1)}$$

$$N(E)=\frac{4}{\pi E}$$

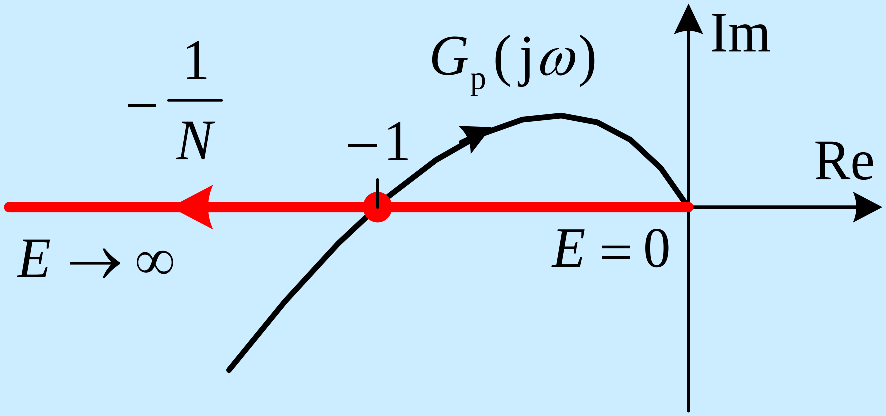
(iv) 频率和振幅的计算

- $G_p(j\omega)=\dfrac{1-j\omega}{j\omega(j\omega+1)}=-\dfrac{2\omega+j(1-\omega^2)}{\omega(1+\omega^2)}$
- $\mathrm{Im}\left[G_p(j\omega)\right]=\dfrac{(1-\omega^2)}{\omega(1+\omega^2)}=0 \quad\Rightarrow\quad \omega=1\,\mathrm{rad/s}$
- $\mathrm{Re}\left[G_p(j1)\right]$

$$=-\frac{2+j(1-1)}{1(1+1)}=-1$$

- $\mathrm{Re}\left[G_p(j1)\right]=-\dfrac{1}{N(E)}$

$$1=\frac{\pi E}{4}\quad\Rightarrow\quad E=\frac{4}{\pi}=1.2733$$

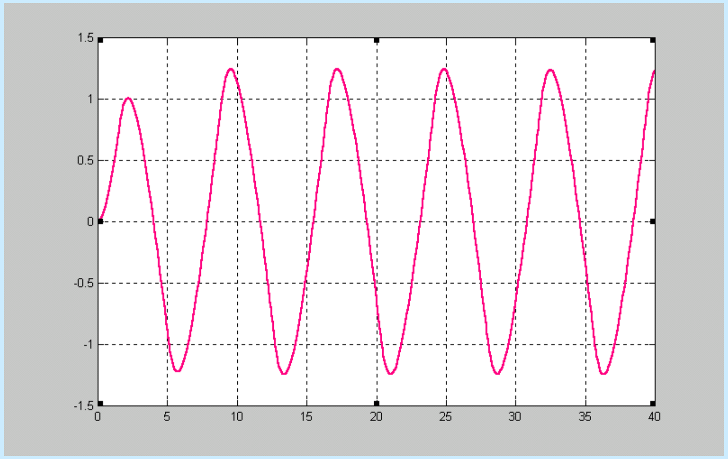


$$\dot{x} - x = e = E\sin\omega t$$

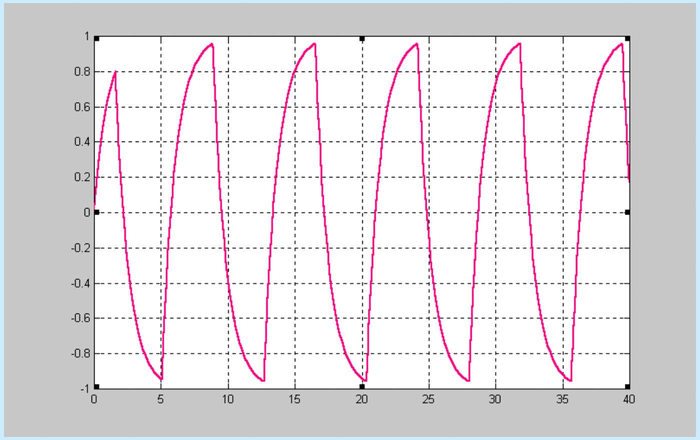
(v) X 的振幅

求x解的微分方程，得 $X = \dfrac{2\sqrt{2}}{\pi} = 0.900$

x 的时间响应



dx/dt 的时间响应

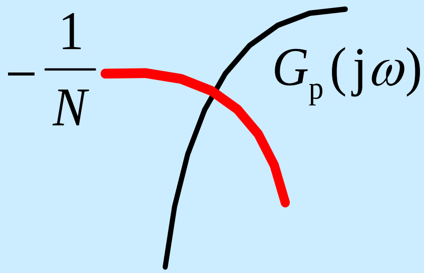


$$x(0) = 0,\ \dot{x}(0) = 0$$

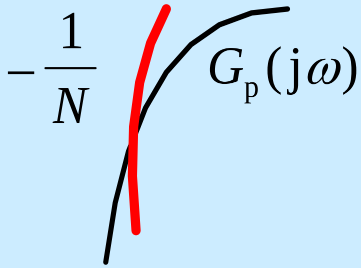
2.6 小结

(1) 描述函数法

- **稳定性分析**
- **无法暂态分析**



越准确



不够准确

(2) 描述函数法是近似法

- $G_p(j\omega)$ **必须是低通滤波器**
- $G_p(j\omega)$ **与 $-1/N$ 正交，结果分析越准确**

(3) 输入为正弦信号时分析越准确

- **其他输入信号，需重新定义描述函数**

(4) 描述函数法分析的难度和准确度主要依赖于非线性环节的复杂性

7.3 相平面

7.3.1 导论

1. 描述函数法的局限性

- 近似法
- 适用于简单非线性
- 不能获取时间响应
- 不适合非周期输入

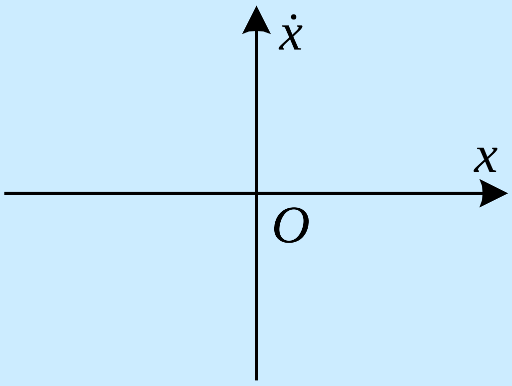
2. 相平面

$$\ddot{x} + f(x, \dot{x}) = 0 \quad (*)$$

- x 和 $\dot{x}$

  — 相变量

- 相平面



$$\boxed{\ddot{x} + f(x,\dot{x}) = 0 \quad (*)}$$

3. 相平面图

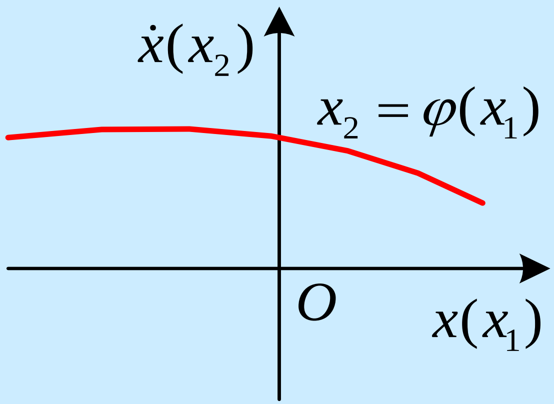
- 令 $x_1 = x,\ x_2 = \dot{x}$
- 式 (*) 变为: $\dfrac{\mathrm{d}x_1}{\mathrm{d}t} = x_2 \qquad \dfrac{\mathrm{d}x_2}{\mathrm{d}t} = -f(x,\dot{x}) = -f(x_1,x_2)$
- 对于二阶时不变系统

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = f_1(x_1,x_2) \qquad \frac{\mathrm{d}x_2}{\mathrm{d}t} = f_2(x_1,x_2)$$

- 相平面上的一条轨迹

$$\frac{\mathrm{d}x_2}{\mathrm{d}x_1} = \frac{f_2(x_1,x_2)}{f_1(x_1,x_2)} \quad \textbf{with } (x_{10},x_{20})$$

$$\Rightarrow \quad x_2 = \varphi(x_1)$$



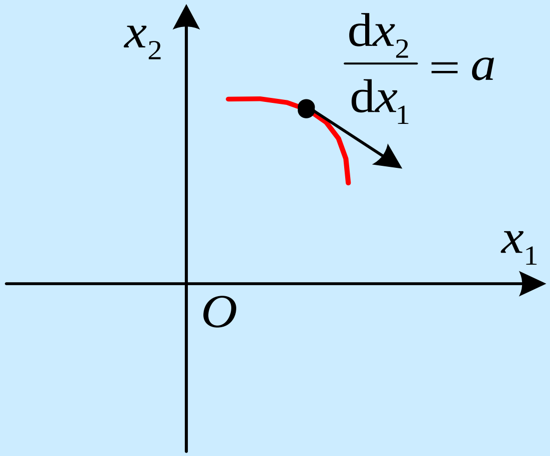
♦ **相平面图**

一组 $(x_{10},x_{20}) \Rightarrow$ **一簇轨迹**

♦ **普通点 Ordinary point**

- **在** (x_1, x_2)

$$\frac{\mathrm{d}x_2}{\mathrm{d}x_1} = a$$ **：** 点 (x_1, x_2) **沿轨迹的运动方向**



- $\boldsymbol{a}$ **是确定值**

  $\Rightarrow$ **在该点的运动方向唯一**

- **始于普通点的轨迹在普通点附近唯一确定**

♦ 奇点 Singular point

若在 (x_1, x_2) $\quad f_1(x_1, x_2) = f_2(x_1, x_2) = 0$

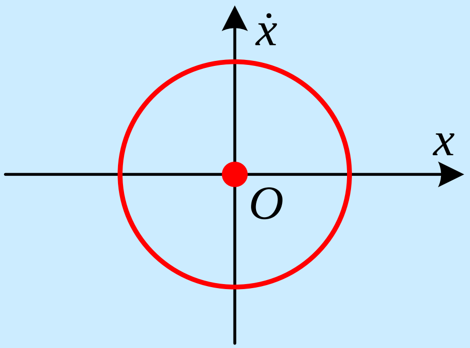
则 $$\frac{\mathrm{d}x_2}{\mathrm{d}x_1} = \frac{0}{0}$$

- 轨迹的斜率 (即运动方向) 是一个不确定值
- 离开该点或到达该点的轨迹无数条
- 奇点是平衡点 $(\dot{x}_1 = 0, \dot{x}_2 = 0)$
- 孤立 (Isolated) 奇点

  * 在其邻域中不存在其他奇点

例 7.3.1 确定如下系统的奇点

$$\ddot{x} + x = 0$$



Solution :

- **令** $x_1 = x,\ x_2 = \dot{x}$ **，则**

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = \dot{x} = x_2 \qquad \frac{\mathrm{d}x_2}{\mathrm{d}t} = \ddot{x} = -x = -x_1$$

$$\Rightarrow \quad (x_1 = 0,\ x_2 = 0)$$ **是平衡点**

- $$\frac{\mathrm{d}x_2}{\mathrm{d}x_1} = -\frac{x_1}{x_2}$$

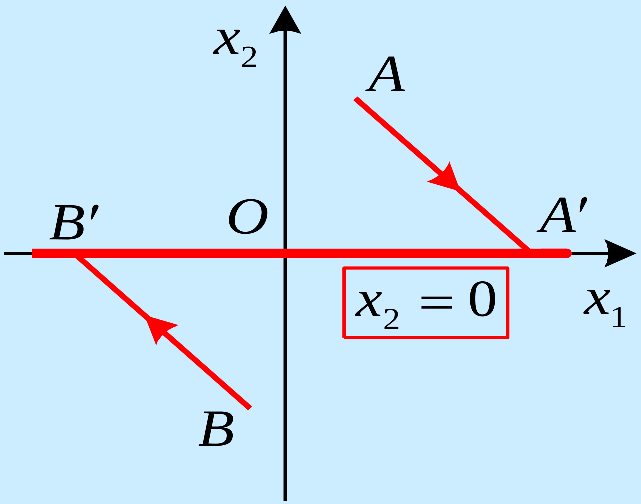
$$\Rightarrow \quad (x_1 = 0,\ x_2 = 0)$$ **是奇点，而且是唯一奇点**

NB: 系统运动是无阻尼振荡，其相轨迹为 $\dot{x}^2 + x^2 = R^2$ ■

例 7.3.2 确定如下系统的奇点

$$\ddot{x}+\dot{x}=0$$

Solution :



- 令 $x_1=x,\ x_2=\dot{x}$ ，则
  $$\frac{\mathrm{d}x_1}{\mathrm{d}t}=\dot{x}=x_2 \qquad \frac{\mathrm{d}x_2}{\mathrm{d}t}=-\dot{x}=-x_2$$
- $$\frac{\mathrm{d}x_2}{\mathrm{d}x_1}=-\frac{x_2}{x_2}$$

  ⇒ 所有满足 $x_2=0$ 的点都是奇点

  i.e. x_1 轴上的所有点均为奇点

NB: 由质点–阻尼器系统，运动方程如下

$$x(t)=[x(0)+\dot{x}(0)]-\dot{x}(0)\mathrm{e}^{-t}$$

■

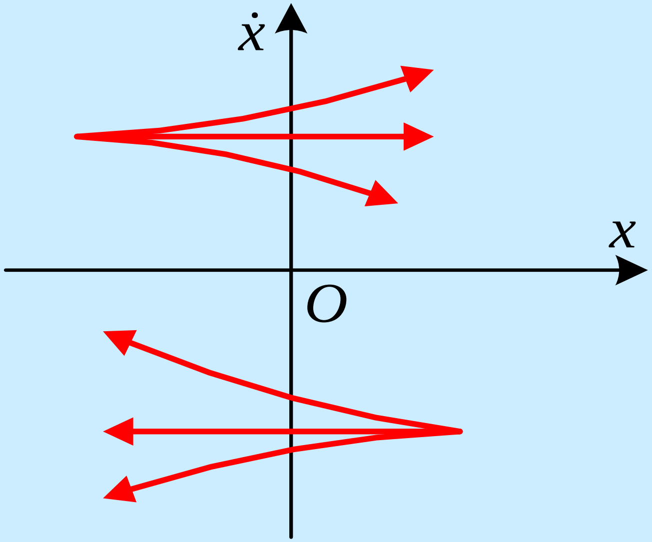
7.3.2 相平面的性质

1. 上半平面与下半平面的运动方向

- 上半平面

$$\dot{x} > 0$$

$$\Rightarrow \boldsymbol{x} \uparrow$$



- 下半平面

$$\dot{x} < 0$$

$$\Rightarrow \boldsymbol{x} \downarrow$$

例 7.3.3 确定如下系统的奇点并绘制相平面图

$$\ddot{x}+\dot{x}+x=0$$

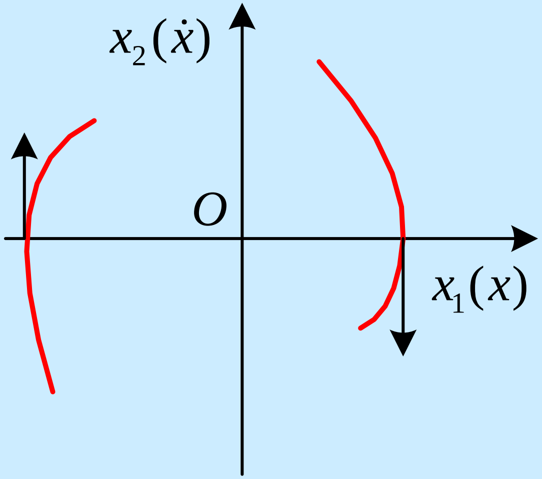
Solution :

- **令** $x_1=x,\ x_2=\dot{x}$ **，则**

$$\frac{\mathrm{d}x_1}{\mathrm{d}t}=\frac{\mathrm{d}x}{\mathrm{d}t}=\dot{x}\qquad \frac{\mathrm{d}x_2}{\mathrm{d}t}=\frac{\mathrm{d}\dot{x}}{\mathrm{d}t}=\ddot{x}=-(\dot{x}+x)\qquad \frac{\mathrm{d}x_2}{\mathrm{d}x_1}=\frac{\ddot{x}}{\dot{x}}=-\frac{\dot{x}+x}{\dot{x}}$$

(i) 当 $\dot{x}=0$ **且** $x\neq 0$

$$\frac{\mathrm{d}x_2}{\mathrm{d}x_1}=-\frac{x}{0}\to\infty$$



- **所有穿越 x 轴的相轨迹均与 x 轴垂直**

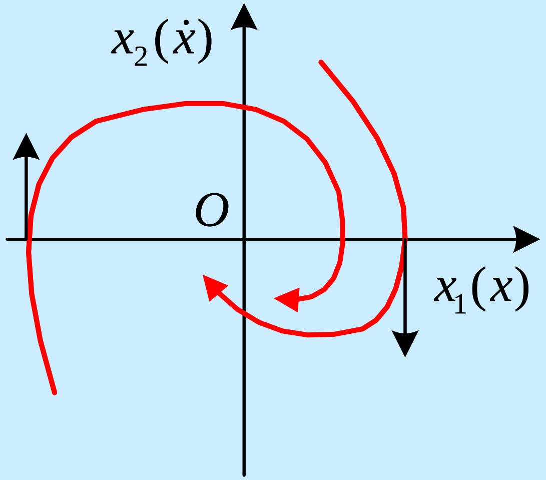
$$\frac{\mathrm{d}x_2}{\mathrm{d}x_1}=\frac{\ddot{x}}{\dot{x}}=-\frac{\dot{x}+x}{\dot{x}}$$

(ii) 当 $\dot{x}=0$ **且** $x=0$

$$\frac{\mathrm{d}x_2}{\mathrm{d}x_1}=-\frac{0}{0}$$

- **原点是奇点**

(iii) $\ddot{x}+\dot{x}+x=0$ **的相平面图**



NB: 对于所有的二阶系统 $\ddot{x}+f(x,\dot{x})=0$

(i) 奇点在 x **轴上**

(ii) 当相轨迹穿越 x **轴时一定与** x **轴垂直**

2. 相平面图的对称性

$$\ddot{x}=\frac{\mathrm{d}\dot{x}}{\mathrm{d}t}=\frac{\mathrm{d}\dot{x}}{\mathrm{d}x}\cdot\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{\mathrm{d}\dot{x}}{\mathrm{d}x}\cdot\dot{x} \qquad \frac{\mathrm{d}\dot{x}}{\mathrm{d}x}=\frac{\ddot{x}}{\dot{x}}$$

- $\ddot{x}+f(x,\dot{x})=0$ **改写如下：**

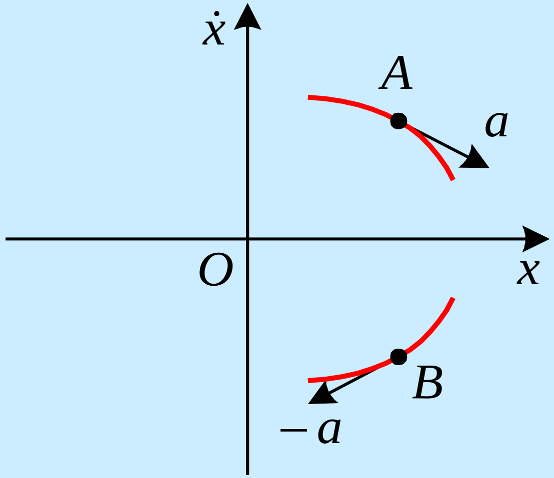
$$\dot{x}\cdot\frac{\mathrm{d}\dot{x}}{\mathrm{d}x}=-f(x,\dot{x}) \qquad \frac{\mathrm{d}\dot{x}}{\mathrm{d}x}=-\frac{f(x,\dot{x})}{\dot{x}}$$

(1) 关于 x 轴对称的情况

- $f(x,\dot{x})$ **是** $\dot{x}$ **的偶函数**

$$A:\frac{\mathrm{d}\dot{x}}{\mathrm{d}x}=-\frac{f(x,\dot{x})}{\dot{x}}=a$$

$$B:\frac{\mathrm{d}\dot{x}}{\mathrm{d}x}=-\frac{f(x,-\dot{x})}{-\dot{x}}=-a$$



$$f(x,\dot{x})=f(x,-\dot{x})$$

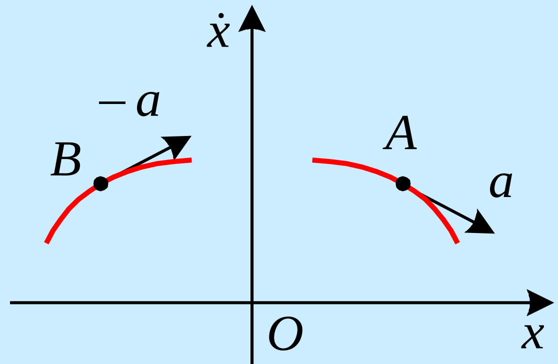
$$\boxed{\frac{\mathrm{d}\dot{x}}{\mathrm{d}x} = -\frac{f(x,\dot{x})}{\dot{x}}}$$

(2) 关于 $\dot{x}$ 轴对称的情况

- $f(x,\dot{x})$ 是 x 的奇函数

$$A:\frac{\mathrm{d}\dot{x}}{\mathrm{d}x} = -\frac{f(x,\dot{x})}{\dot{x}} = a$$

$$B:\frac{\mathrm{d}\dot{x}}{\mathrm{d}x} = -\frac{f(-x,\dot{x})}{\dot{x}} = -a$$



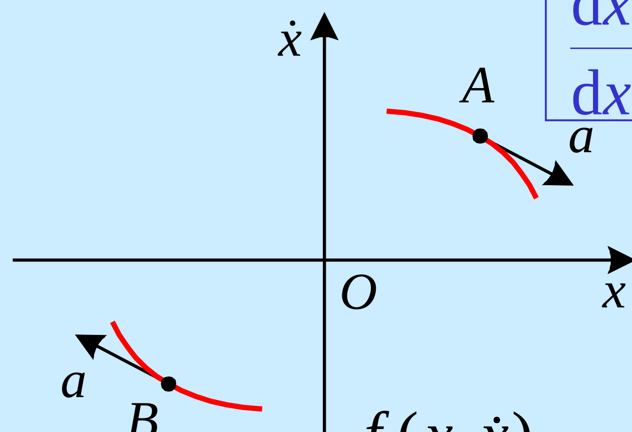
$$f(x,\dot{x}) = -f(-x,\dot{x})$$

$$\boxed{\frac{\mathrm{d}\dot{x}}{\mathrm{d}x} = -\frac{f(x,\dot{x})}{\dot{x}}}$$

(3)关于原点对称的情况

$$A:\frac{\mathrm{d}\dot{x}}{\mathrm{d}x} = -\frac{f(x,\dot{x})}{\dot{x}} = a$$

$$B:\frac{\mathrm{d}\dot{x}}{\mathrm{d}x} = -\frac{f(-x,-\dot{x})}{-\dot{x}} = a$$



$$f(x,\dot{x}) = -f(-x,-\dot{x})$$

$$\ddot{x} = \frac{\mathrm{d}\dot{x}}{\mathrm{d}x}\dot{x}$$

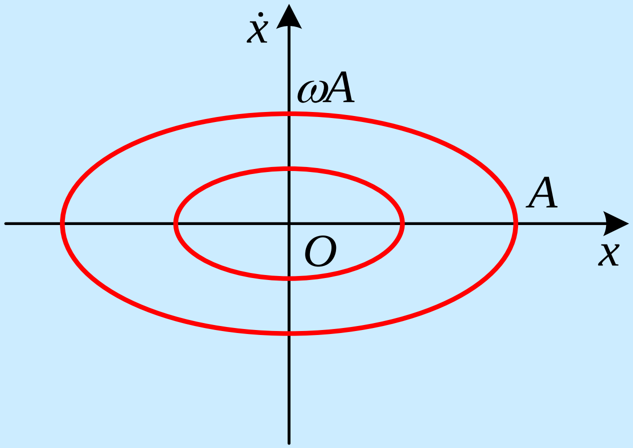
7.3.3 相平面图的绘制

1. 解析法

例 7.3.4 绘制如下系统的相平面图

$$\ddot{x} + \omega^2 x = 0$$

Solution:



- **利用** $\ddot{x} = \dot{x}\frac{\mathrm{d}\dot{x}}{\mathrm{d}x}$ **，得**

$$\dot{x}\frac{\mathrm{d}\dot{x}}{\mathrm{d}x} + \omega^2 x = 0$$

$$\dot{x}\mathrm{d}\dot{x} + \omega^2 x\mathrm{d}x = 0$$

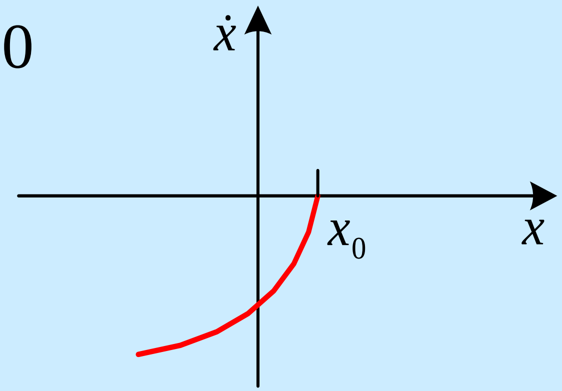
- **等幅振荡**

- **对上式积分得** $\frac{\dot{x}^2}{\omega^2} + x^2 = A^2$

■

例 7.3.5 绘制如下系统的相平面图

$$\ddot{x} = -M \quad x(0) = x_0 \quad \dot{x}(0) = 0$$



Solution:

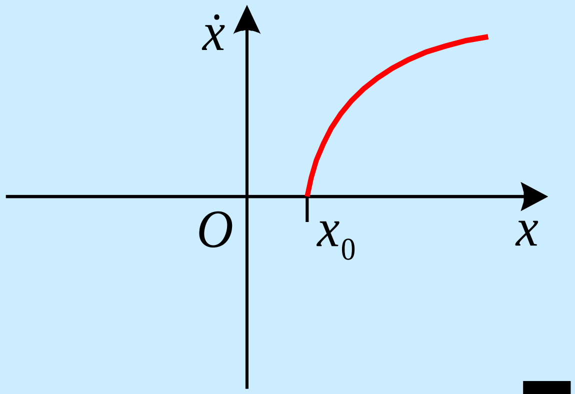
- **由** $\ddot{x} = -M$ **，得**

$$\dot{x} = -Mt + C_1$$

$$x = -\frac{1}{2}Mt^2 + C_1 t + C_2$$

- **根据初始条件** $\Rightarrow$ $C_1 = 0, C_2 = x_0$

$$\dot{x} = -Mt \qquad x = -\frac{1}{2}Mt^2 + x_0$$



i.e. $x = -\dfrac{1}{2M}\dot{x}^2 + x_0$

or $\dot{x}^2 = 2M(x_0 - x) = -2M(x - x_0)$ ■

2. 图解法

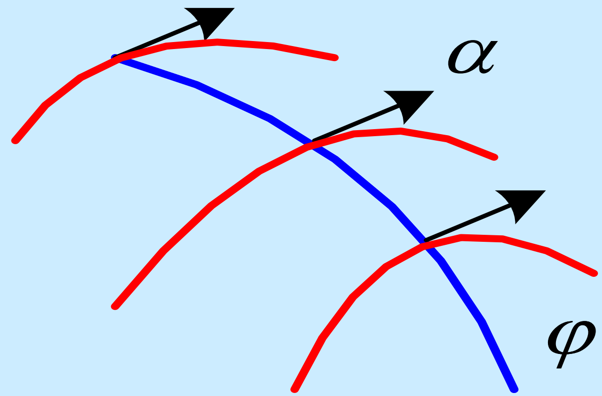
(1) 等倾线法 isocline method

♦ **等倾线**

$$\frac{\mathrm{d}x_2}{\mathrm{d}x_1} = \frac{f_2(x_1, x_2)}{f_1(x_1, x_2)} = \alpha$$

$$\Downarrow \quad f_2(x_1, x_2) = \alpha f_1(x_1, x_2)$$

$$x_2 = \varphi(x_1, \alpha)$$



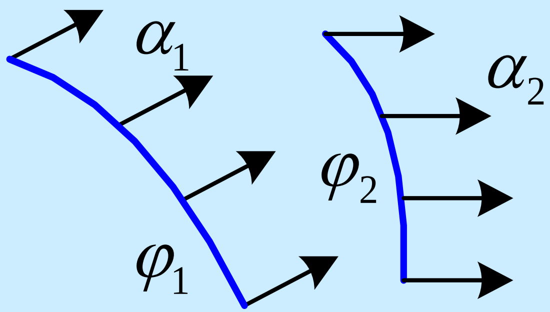
- **对于** $x_2 = \varphi(x_1, \alpha)$**上的点，所有轨迹经过时的斜率均为** α

  i.e. 运动方向相同

- $x_2 = \varphi(x_1, \alpha)$ **称为等倾线**

♦ 方向场

一组不同的 α 值 $\Rightarrow$ 一组不同的等倾线



- **所有这些等倾线给出了轨迹切线的方向场**

例 7.3.6 利用等倾线法绘制如下系统的相平面图

$$\ddot{x}+2\zeta\omega\dot{x}+\omega^2 x=0$$

Solution:

系统方程改写如下

$$\dot{x}\frac{\mathrm{d}\dot{x}}{\mathrm{d}x}+2\zeta\omega\dot{x}+\omega^2 x=0$$

(i) 绘制等倾线

令 $\dfrac{\mathrm{d}\dot{x}}{\mathrm{d}x}=\alpha$ **, 则** $\alpha\dot{x}+2\zeta\omega\dot{x}+\omega^2 x=0$

i.e. $\dfrac{\dot{x}}{x}=\dfrac{-\omega^2}{2\zeta\omega+\alpha}$

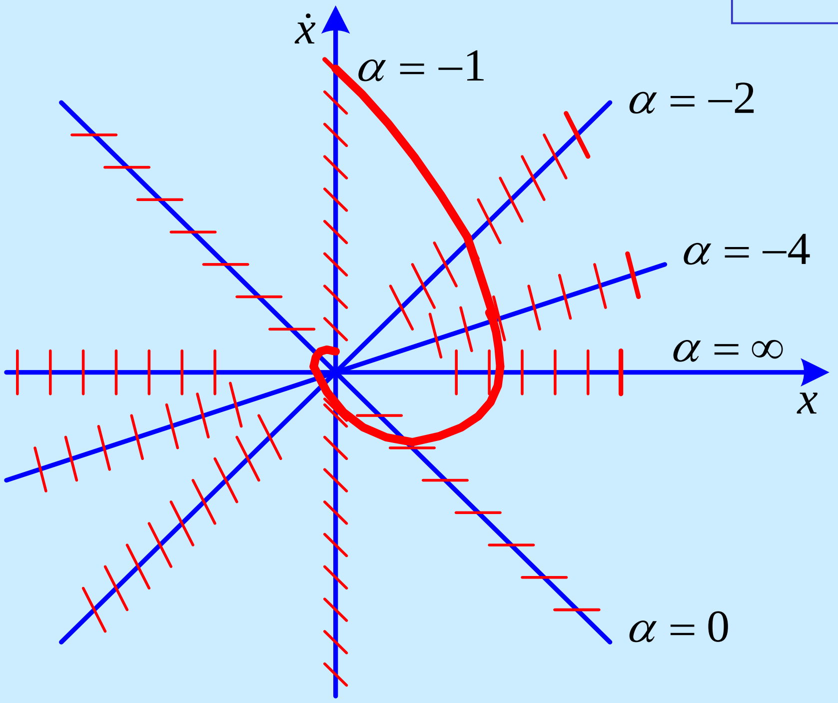
假设 $\zeta=0.5$ **且** $\omega=1$ $\Rightarrow$ $\dot{x}=\dfrac{-1}{1+\alpha}x$

$(\ddot{x}+\dot{x}+x=0)$

$$\dot{x} = \frac{-1}{1+\alpha} x$$

- $\alpha = 0 \quad \Rightarrow \quad \dot{x} = -x$
- $\alpha = \infty \quad \Rightarrow \quad \dot{x} = 0$
- $\alpha = -1 \quad \dot{x}/x = -1/0$

  $\Rightarrow \quad x = -(1+\alpha)\dot{x} = 0$
- $\alpha = -2 \quad \Rightarrow \quad \dot{x} = x$
- $\alpha = -4 \quad \Rightarrow \quad \dot{x} = \frac{1}{3} x$



(ii) 等倾线上绘制短线段表示方向场

(iii) 从普通点开始绘制相轨迹

■

(2) delta 法

- 相轨迹被视为中心在 x 轴上的一组相连的圆弧
- 运动方程写成如下形式

$\ddot{x} = -f(\dot{x}, x)$ $\qquad$ $f(\dot{x}, x)$ 连续单值函数

- 运动方程改写如下

$$\ddot{x} + \omega^2 x = -f(\dot{x}, x) + \omega^2 x$$

令 $\delta = \dfrac{-f(\dot{x}, x) + \omega^2 x}{\omega^2}$

则，在 $(\dot{x}_1, x_1)$ 附近 $\delta_1 = \delta(\dot{x}_1, x_1) \approx \text{const}$

$$\ddot{x} + \omega^2 x = -f(\dot{x}, x) + \omega^2 x$$
$$\delta = \frac{-f(\dot{x}, x) + \omega^2 x}{\omega^2}$$

- $(\dot{x}_1, x_1)$ **附近的运动方程**

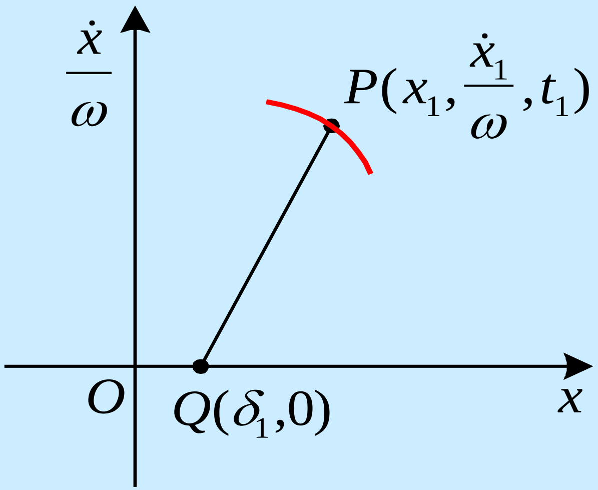
$$\ddot{x} + \omega^2 x = \omega^2 \delta_1$$
$$\ddot{x} + \omega^2 (x - \delta_1) = 0$$

- **其解为**

$$\dot{x}^2 + \omega^2 (x - \delta_1)^2 = A^2$$

- **在归一化的相平面上**

$$\left(\frac{\dot{x}}{\omega}\right)^2 + (x - \delta_1)^2 = B^2$$



- ***P*点附近的圆弧是以 $(\delta_1, 0)$ 为中心的圆的一部分，半径为**

$$|\mathrm{PQ}| = \sqrt{(\dot{x}_1/\omega)^2 + (x_1 - \delta_1)^2}$$

$$\left|PQ\right| = \sqrt{(\dot{x}_1/\omega)^2 + (x_1 - \delta_1)^2}$$

例 7.3.7 采用 δ 法绘制如下系统的相平面图

$$\ddot{x} + 2\zeta\omega\dot{x} + \omega^2 x = 0$$

Solution:

(i) 改写方程 $\ddot{x} + \omega^2 x = -2\zeta\omega\dot{x}$ **令** $\delta = \dfrac{-2\zeta\omega\dot{x}}{\omega^2} = -2\zeta\dfrac{\dot{x}}{\omega}$

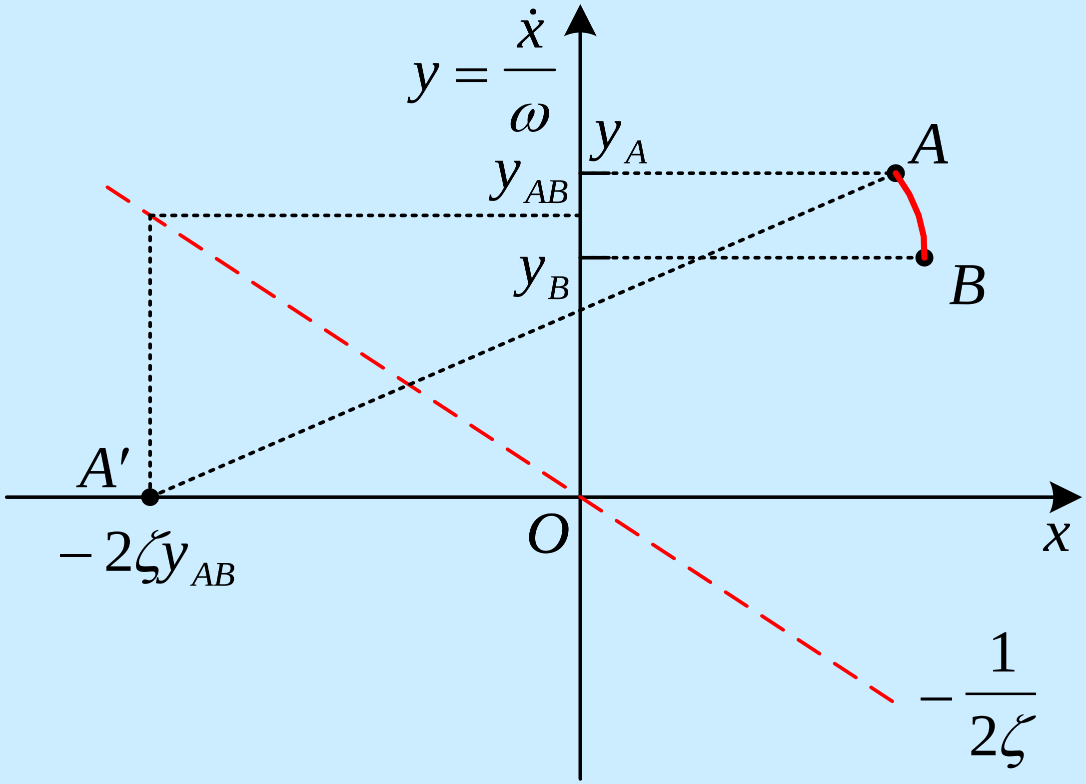
i.e. $\left(\dfrac{\dot{x}}{\omega}\right)^2 + (x-\delta)^2 = R^2$

(ii) 对于点 $(x_1, \dot{x}_1)$

$$R_1 = \sqrt{\left(\frac{\dot{x}_1}{\omega}\right)^2 + \left(x_1 + \frac{2\zeta\dot{x}_1}{\omega}\right)^2}$$

$$\delta_1 = -\frac{2\zeta\dot{x}_1}{\omega}$$

$$R_1 = \sqrt{\left(\frac{\dot{x}_1}{\omega}\right)^2 + \left(x_1 + 2\zeta\frac{\dot{x}_1}{\omega}\right)^2}$$
$$= \sqrt{y^2 + (x_1 + 2\zeta\delta_1)^2}$$



$$\delta_1 = -2\zeta\frac{\dot{x}_1}{\omega}$$
$$= -2\zeta y$$

■

3. 计算机仿真 (MATLAB)

♦ **Van de Pol 方程** $m\ddot{x} - f(1-x^2)\dot{x} + kx = 0$

其中 $m=1, f=1, k=1$

● **令** $x_1 = x, x_2 = \dot{x}$ **，则** $\dot{x}_1 = x_2 \ \dot{x}_2 = (1-x_1^2)x_2 - x_1$

◊ **模型文件: model.m**

```
function [sys,x0]= model(t,x)
   sys=[x(2);(1-x(1)*x(1))*x(2)-x(1)];
```

◊ **仿真程序: simu.m**

```
[t,x]=ode45('model',[0,20],[3;4])
plot(x(:,1),x(:,2),'-b');
```

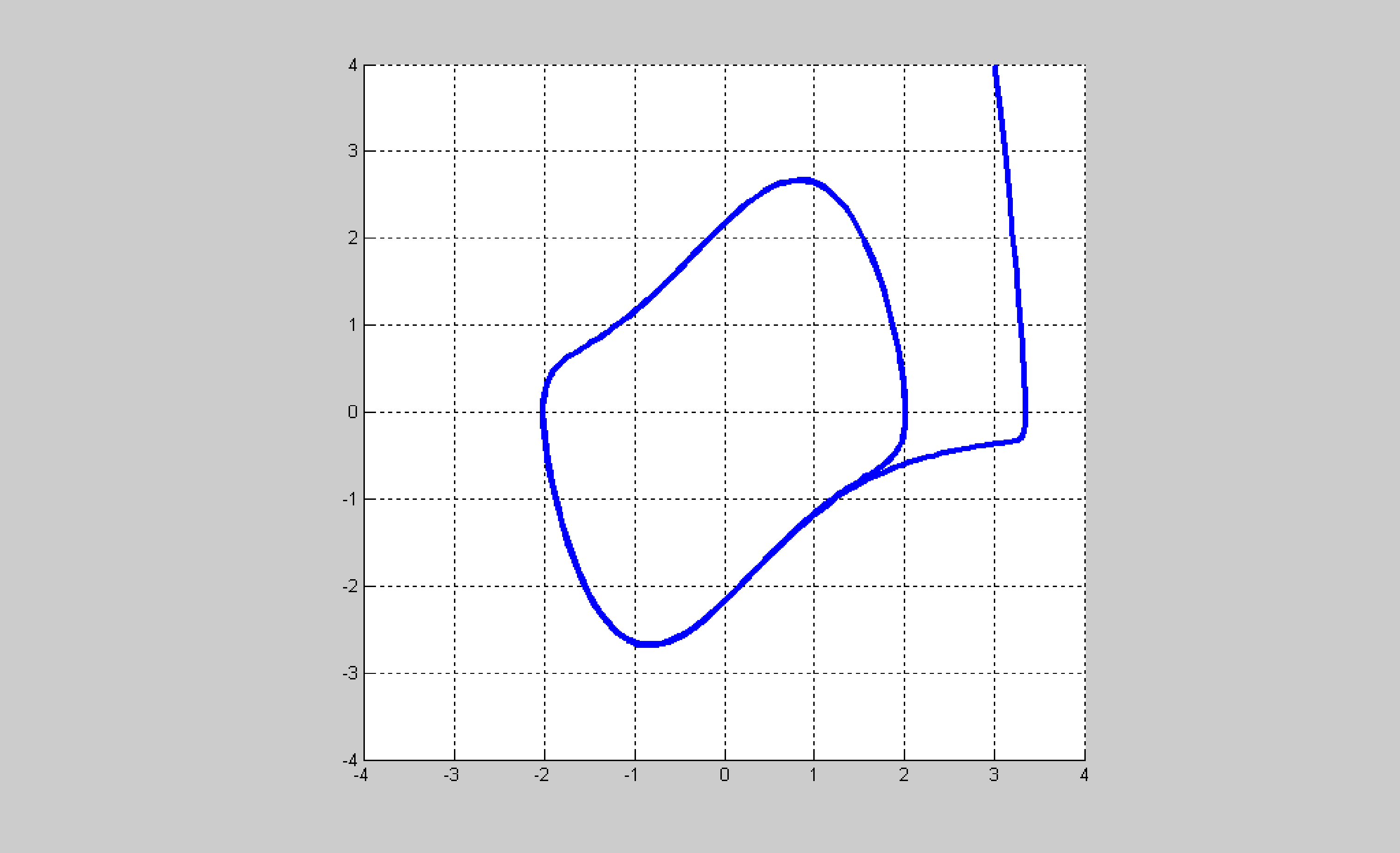
其中: `ode45` – 仿真程序的名称

‘`model`’– 模型函数的名称

`[0,20]`– 仿真的时间间隔

`[3,4]` – 初始条件

◊ 仿真结果



7.3.4 奇点和极限环

1. 奇点

- **满足如下条件的点**

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = f_1(x_1, x_2) = 0 \qquad \frac{\mathrm{d}x_2}{\mathrm{d}t} = f_2(x_1, x_2) = 0$$

- $\dfrac{\mathrm{d}x_2}{\mathrm{d}x_1} = \dfrac{0}{0}$ $\Rightarrow$ **运动不能由** $\dfrac{\mathrm{d}x_2}{\mathrm{d}x_1}$ **确定**

2. 奇点的性质

♦ 运动方程线性化（假设奇点在原点）

• 在原点附近Taylor 级数展开

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = a_1 x_1 + b_1 x_2 \qquad \frac{\mathrm{d}x_2}{\mathrm{d}t} = a_2 x_1 + b_2 x_2$$

• 令 $x = x_1$, 则

$$\begin{aligned} \dot{x} &= \dot{x}_1 = a_1 x_1 + b_1 x_2 = a_1 x + b_1 x_2 \\ \ddot{x} &= \ddot{x}_1 = a_1 \dot{x} + b_1 \dot{x}_2 = a_1 \dot{x} + b_1 (a_2 x_1 + b_2 x_2) \\ &= a_1 \dot{x} + b_1 a_2 x + b_1 b_2 x_2 \end{aligned}$$

由于 $\dot{x} = a_1 x + b_1 x_2$, $b_1 x_2 = \dot{x} - a_1 x$

$$\begin{aligned} \ddot{x} &= a_1 \dot{x} + b_1 a_2 x + b_2 (\dot{x} - a_1 x) \\ &= (a_1 + b_2)\dot{x} + (b_1 a_2 - b_2 a_1) x \end{aligned}$$

$$\ddot{x} = (a_1 + b_2)\dot{x} + (b_1 a_2 - b_2 a_1)x$$

- **于是** $\ddot{x} + a\dot{x} + bx = 0$

  其中 $a = -(a_1 + b_2)$, $b = a_1 b_2 - a_2 b_1$

- **特征方程** $\lambda^2 + a\lambda + b = 0$ **的根如下**

$$\lambda_{1,2} = \frac{-a \pm \sqrt{a^2 - 4b}}{2}$$

* **假设** $\lambda_1 \neq 0$ **且** $\lambda_2 \neq 0$

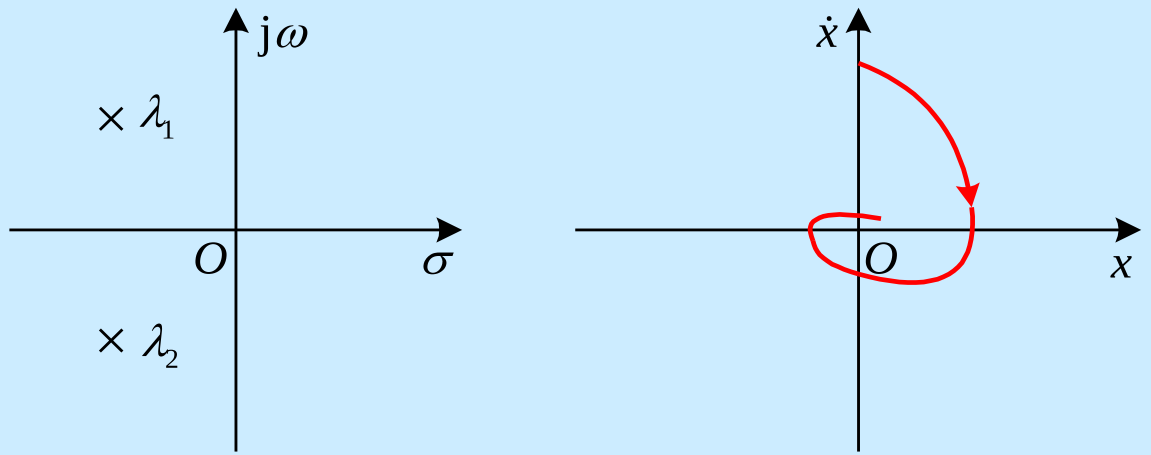
- **线性化后的模型可用于讨论原非线性系统的性质**

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = a_1 x_1 + b_1 x_2 \qquad \frac{\mathrm{d}x_2}{\mathrm{d}t} = a_2 x_1 + b_2 x_2$$

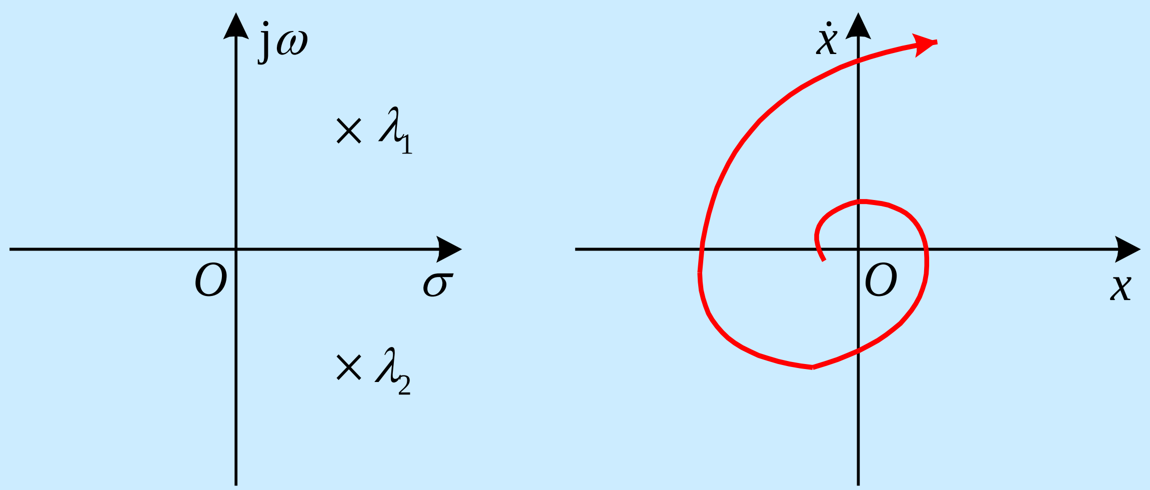
NB: 可采用状态空间法直接计算特征根

♦ 奇点的分类

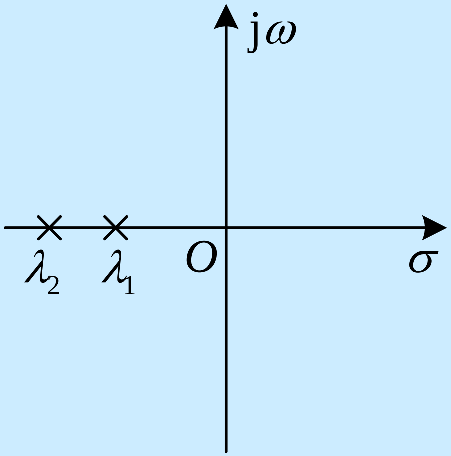
(1) 稳定的焦点 Stable focus

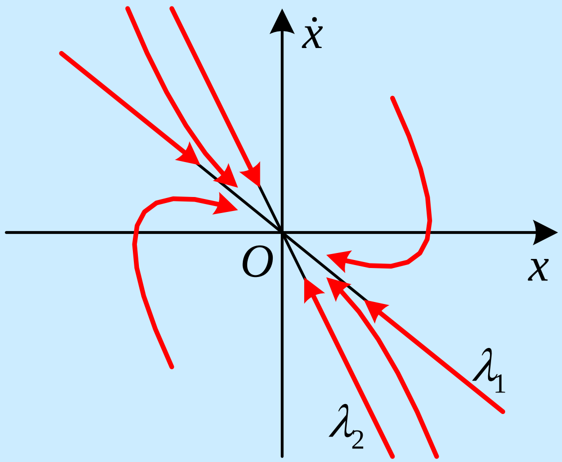


(2) 不稳定的焦点



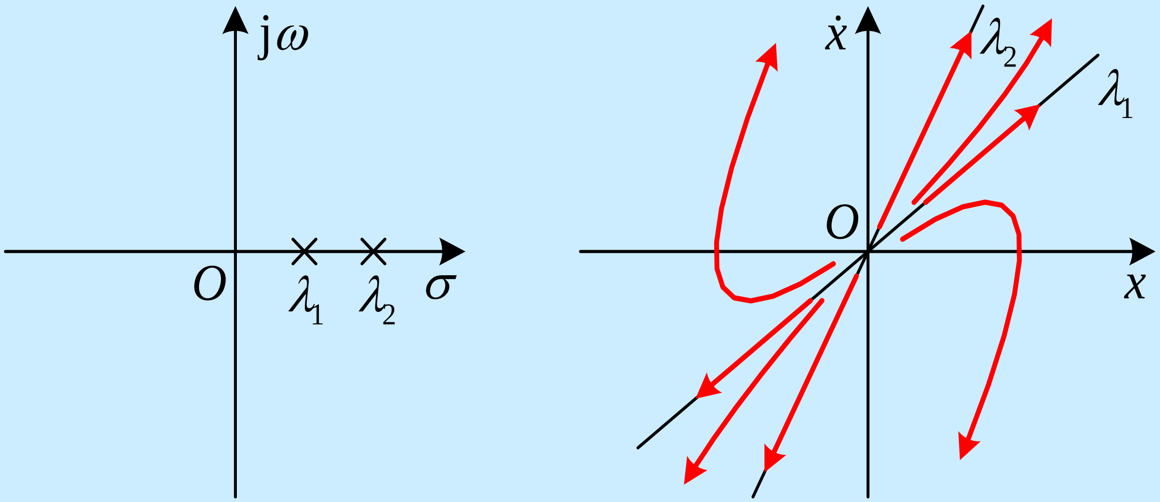
(3) 稳定的节点 Stable node





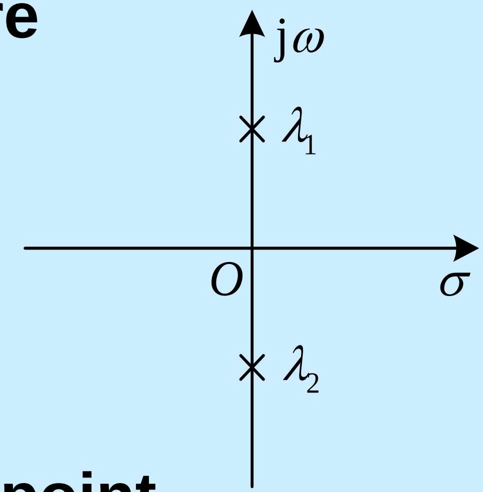
- $x(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$
- **斜率为 λ_1 , λ_2 的直线是相轨迹**
- **斜率为 λ_1 , λ_2 的直线是分隔线**
- **若 $|\lambda_1| < |\lambda_2|$, 所有相轨迹趋于斜率为λ_1的分隔线**

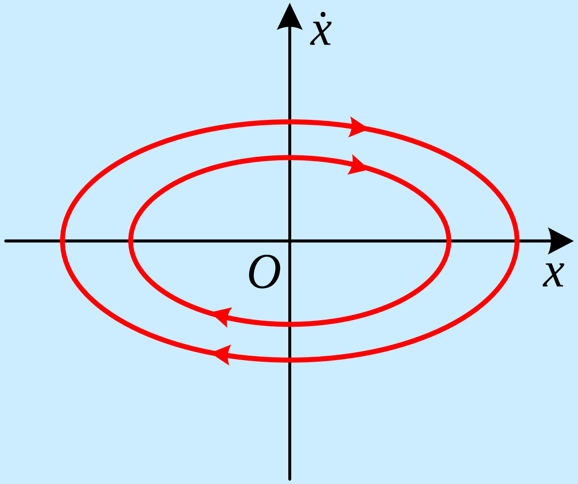
(4) 不稳定的节点



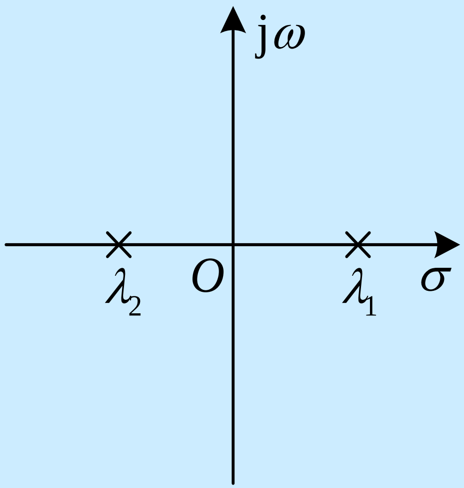
- 若 $|\lambda_1| < |\lambda_2|$, $Ce^{\lambda_2 t}$ 将随时间 t 的增加而主导

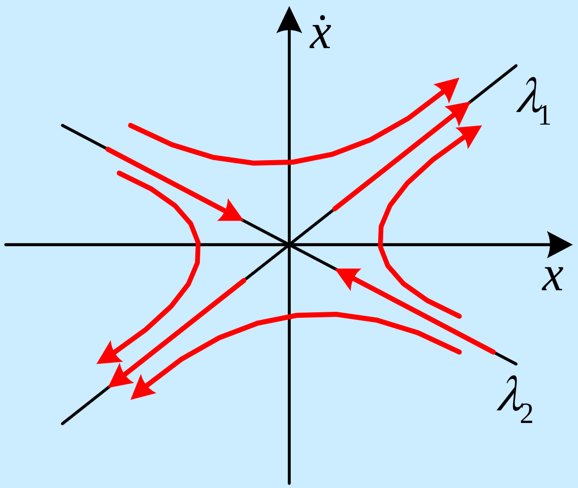
(5) 中心点 Centre





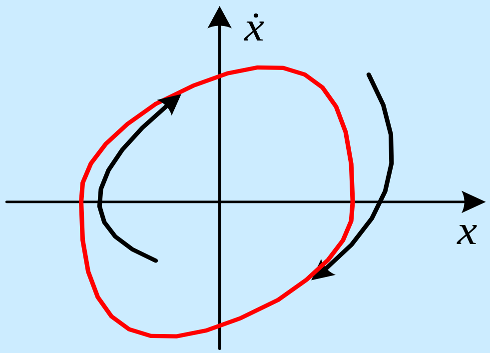
(6) 鞍点 Saddle point



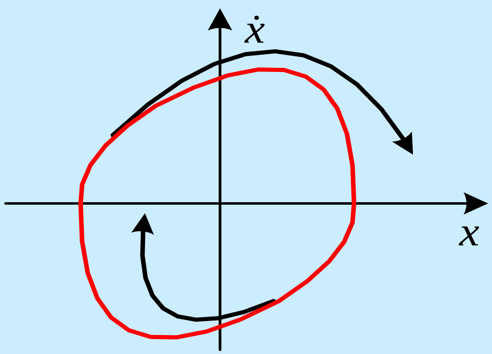


3. 极限环 Limit Cycles

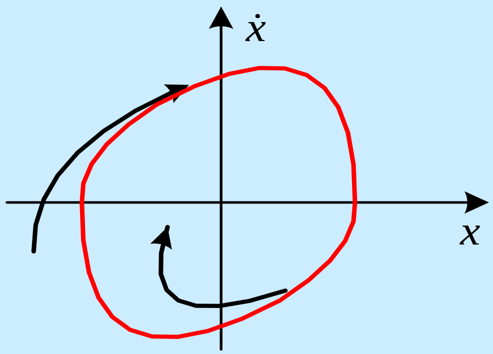
- **定义： 相平面上的一条封闭、孤立的曲线**



稳定的极限环



不稳定的极限环



半稳定的极限环

例 7.3.8 绘制如下系统的相平面图

$$\ddot{x}+0.5\dot{x}+2x+x^2=0$$

Solution:

(i) 标准形式

- **令** $x_1=x,\ x_2=\dot{x}$ **，则**

$$\dot{x}_1=\dot{x}=x_2 \quad = f_1(x_1,x_2)$$

$$\dot{x}_2=\ddot{x}=-0.5\dot{x}-2x-x^2 = -2x_1-0.5x_2-x_1^2 = f_2(x_1,x_2)$$

(ii) 确定奇点

$$\dot{x}_1=x_2=0 \qquad \dot{x}_2=-2x_1-0.5x_2-x_1^2=0$$

- **奇点为**

$$\begin{cases}x_1=0\\ x_2=0\end{cases} \quad \begin{cases}x_1=-2\\ x_2=0\end{cases} \quad \text{等价于} \quad \begin{cases}x=0\\ \dot{x}=0\end{cases} \quad \begin{cases}x=-2\\ \dot{x}=0\end{cases}$$

$$f_1 = x_2$$
$$f_2 = -2x_1 - x_1^2 - 0.5x_2$$

(iii) 奇点的性质

♦ **点 (0, 0)**

- **在 (0, 0) 点线性化**

$$\left.\frac{\partial f_1}{\partial x_1}\right|_{\substack{x_1=0\\x_2=0}} = 0 \qquad \left.\frac{\partial f_1}{\partial x_2}\right|_{\substack{x_1=0\\x_2=0}} = 1 \qquad a_1 = 0,\ b_1 = 1$$

$$\left.\frac{\partial f_2}{\partial x_1}\right|_{\substack{x_1=0\\x_2=0}} = (-2-2x_1)\Big|_{x_1=0} = -2 \qquad \left.\frac{\partial f_2}{\partial x_2}\right|_{\substack{x_1=0\\x_2=0}} = -0.5$$

$$a_2 = -2,\ b_2 = -0.5$$

- **线性化后的方程**

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -2x_1 - 0.5x_2 \end{cases} \quad \textbf{i.e.} \quad \begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = \dot{x} \\ \dfrac{\mathrm{d}\dot{x}}{\mathrm{d}t} = -2x - 0.5\dot{x} \end{cases}$$

$$\boxed{\ddot{x}+0.5\dot{x}+2x+x^2=0}$$

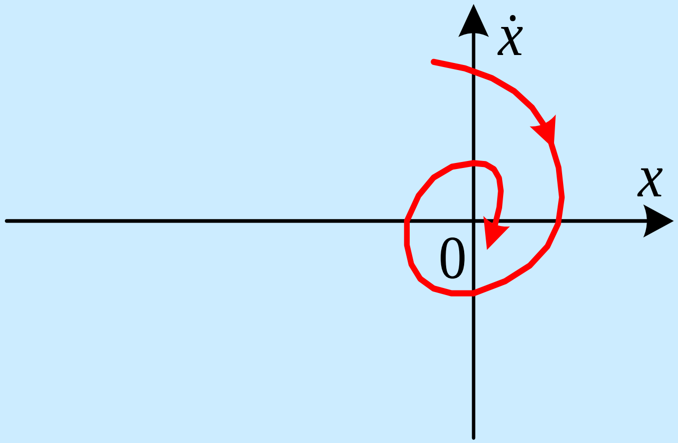
* 由于 $a_1=0,\ b_1=1,\ a_2=-2,\ b_2=-0.5$ ，则

$$\begin{aligned}-(a_1+b_2)&=0.5\\ a_1b_2-b_1a_2&=2\end{aligned}\quad\Rightarrow\quad \ddot{x}+0.5\dot{x}+2x=0$$

* 特征方程

$$\lambda^2+0.5\lambda+2=0$$

$$\lambda_{1,2}=-0.25\pm \mathrm{j}1.987$$



- 结论: (0,0) 是稳定的焦点

$$\ddot{x}+0.5\dot{x}+2x+x^2=0$$

♦ **点 (–2,0)**

*** 令 $y = x + 2$, 则** $\ddot{y}+0.5\dot{y}-2y+y^2=0$

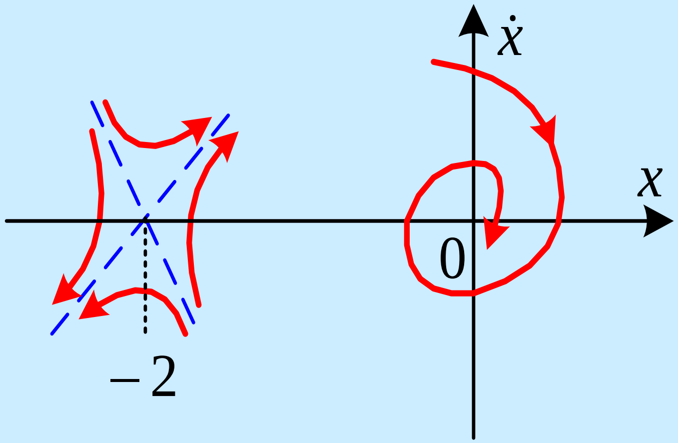
*** 类似可得**

$$\ddot{y}+0.5\dot{y}-2y=0$$

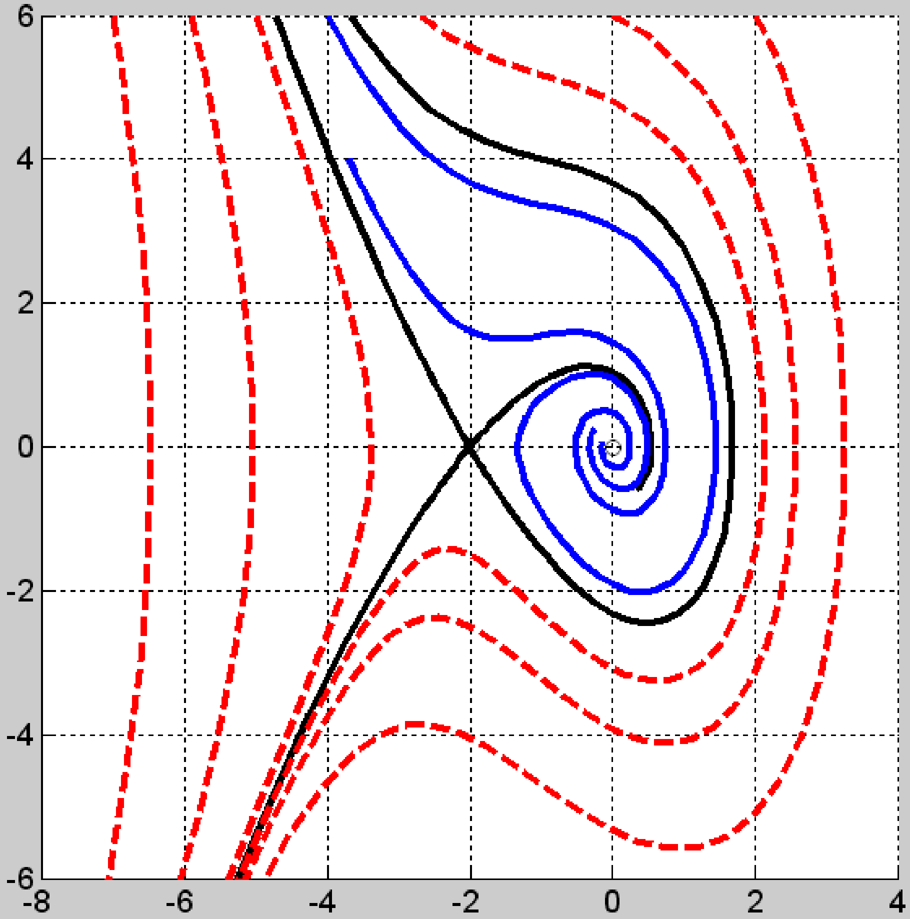
$$\lambda^2+0.5\lambda-2=0$$

$$\lambda_{1,2}=1.186,\ -1.686$$

- **结论: (–2 , 0)是鞍点**



(iv) 真实的相平面图



$$\ddot{x} + 0.5\dot{x} + 2x + x^2 = 0$$

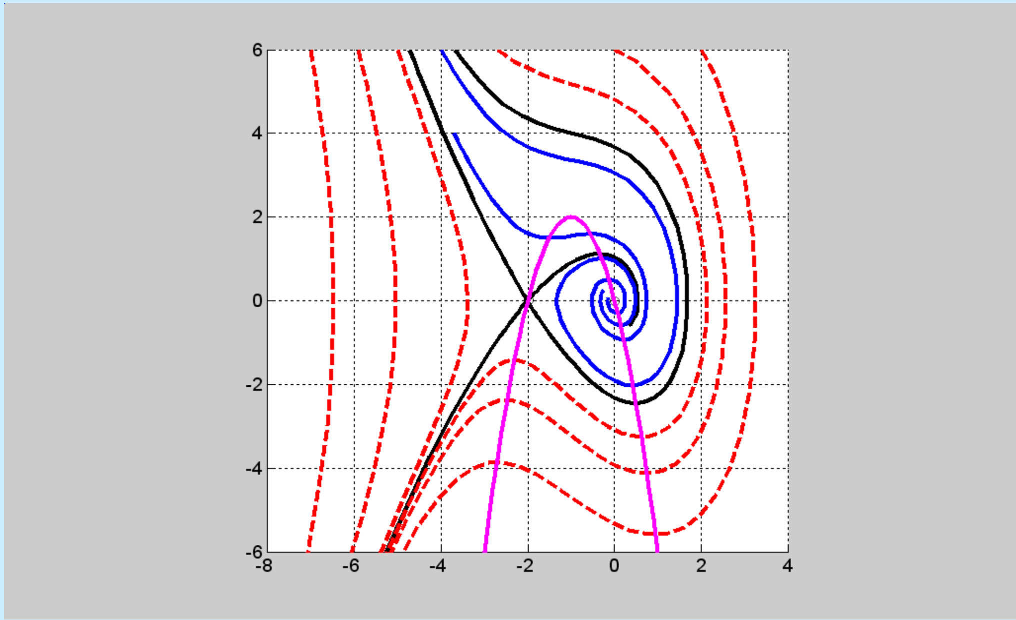
$$\ddot{x} = \frac{\mathrm{d}\dot{x}}{\mathrm{d}x}\dot{x} = \alpha\dot{x}$$

NB: 等倾线法绘制如下

- 等倾线如下

$$\frac{-0.5\dot{x} - 2x - x^2}{\dot{x}} = \alpha \qquad \dot{x} = -\frac{(x+1)^2}{\alpha + 0.5} + \frac{1}{\alpha + 0.5}$$

- 经过点 (–2,0) 和点 (0,0)的抛物线

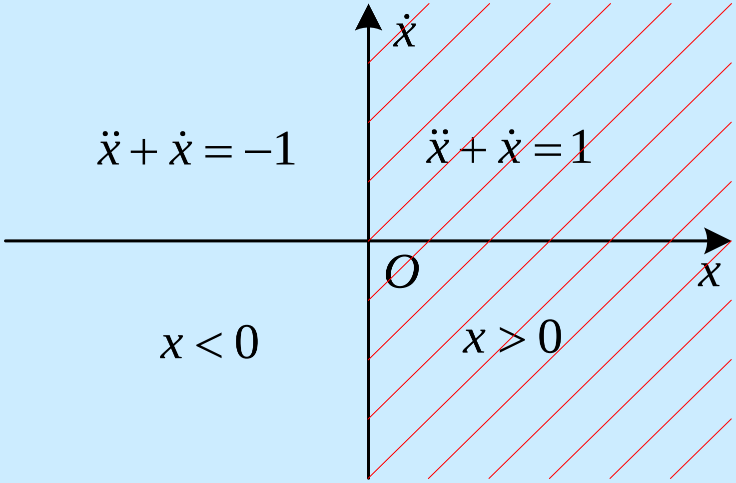


- $\alpha = 0$对应的等倾线

7.4 相平面分析

♦ 一些非线性系统由分片（区）线性模型描述

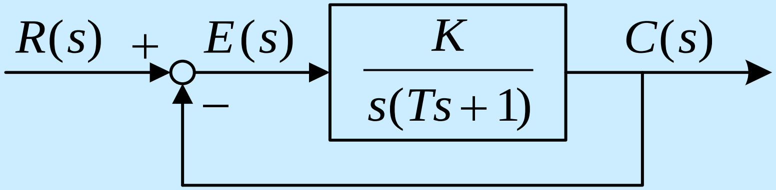
$$\ddot{x}+\dot{x}=\begin{cases}1 & x>0\\ -1 & x<0\end{cases}$$



- 如何对线性系统分析？

7.4.1 线性系统分析

例 7.4.1 试确定如下二阶系统的相轨迹



Solution:

- **基本方程:**

$$\frac{C(s)}{R(s)} = \frac{K}{Ts^2 + s + K} \qquad \Rightarrow \qquad T\ddot{c} + \dot{c} + Kc = Kr$$

$$\frac{E(s)}{R(s)} = \frac{Ts^2 + s}{Ts^2 + s + K} \qquad \Rightarrow \qquad T\ddot{e} + \dot{e} + Ke = T\ddot{r} + \dot{r}$$

$$\boxed{T\ddot{c}+\dot{c}+Kc=Kr \qquad T\ddot{e}+\dot{e}+Ke=T\ddot{r}+\dot{r}}$$

(i) 阶跃响应 $\qquad r(t)=R\cdot 1(t) \quad \Rightarrow \quad \dot{r}=\ddot{r}=0$

$$T\ddot{c}+\dot{c}+Kc=KR \qquad T\ddot{e}+\dot{e}+Ke=0$$

- **对于误差方程**

$$\begin{cases} T\ddot{e}+\dot{e}+Ke=0 \\ e(0)=R,\ \dot{e}(0)=0 \end{cases}$$

令 $x_1=e,\ x_2=\dot{e}$

则

$$\begin{cases} \dfrac{\mathrm{d}x_1}{\mathrm{d}t}=\dot{e} & =x_2 \\ \dfrac{\mathrm{d}x_2}{\mathrm{d}t}=\ddot{e}=-\dfrac{\dot{e}}{T}-\dfrac{K}{T}e & =-\dfrac{x_2}{T}-\dfrac{K}{T}x_1 \end{cases}$$

*** 奇点**

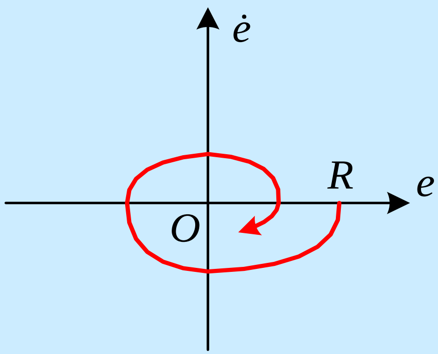
$$\begin{cases} x_1=0 \\ x_2=0 \end{cases} \qquad \textbf{i.e.} \qquad \begin{cases} e=0 \\ \dot{e}=0 \end{cases}$$

$$\boxed{T\ddot{e} + \dot{e} + Ke = 0}$$

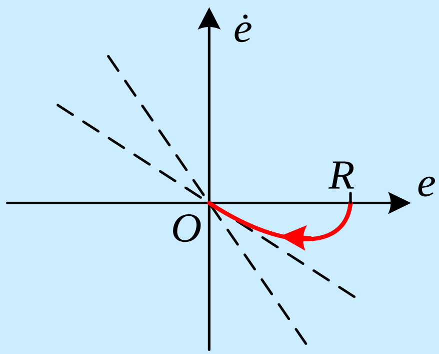
- **(0,0)的性质**

$$T\lambda^2 + \lambda + K = 0$$

若 $1-4KT < 0$ $\Rightarrow$ **稳定的焦点**



若 $1-4KT \geq 0$ $\Rightarrow$ **稳定的节点**



• **对于输出方程**

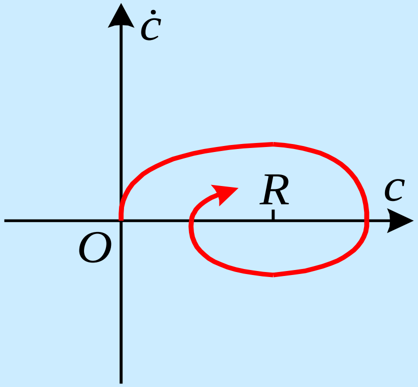
$$\begin{cases} T\ddot{c}+\dot{c}+Kc=KR \\ c(0)=0,\ \dot{c}(0)=0 \end{cases}$$

*** 奇点**

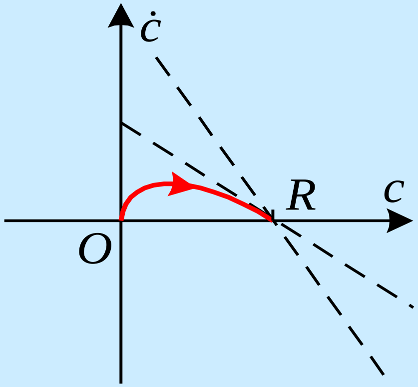
$$\begin{cases} x_1 = R \\ x_2 = 0 \end{cases} \quad \textbf{i.e.} \quad \begin{cases} c = R \\ \dot{c} = 0 \end{cases}$$

*** $(R,0)$的性质**

若 $1-4KT<0$ $\Rightarrow$ **稳定的焦点**



若 $1-4KT\geq 0$ $\Rightarrow$ **稳定的节点**



$$\boxed{T\ddot{c}+\dot{c}+Kc=Kr \qquad T\ddot{e}+\dot{e}+Ke=T\ddot{r}+\dot{r}}$$

(ii) 斜坡响应 $\quad r=Vt+R \qquad \dot{r}=V,\ddot{r}=0$

$$T\ddot{c}+\dot{c}+Kc=KVt+KR \qquad T\ddot{e}+\dot{e}+Ke=V$$

- **对于误差方程** $\begin{cases} T\ddot{e}+\dot{e}+Ke=V \\ e(0)=R,\ \dot{e}(0)=V \end{cases}$

*** 奇点** $\quad e=\dfrac{V}{K},\ \ \dot{e}=0$

*** 奇点的性质**

-- 坐标变换: $\quad x=e-\dfrac{V}{K}$

$$T\ddot{e}+\dot{e}+Ke=V \quad \Rightarrow \quad T\ddot{x}+\dot{x}+Kx=0$$

$$x(0)=e(0)-\frac{V}{K}=R-\frac{V}{K} \qquad \dot{x}(0)=\dot{e}(0)=V$$

$$\begin{cases} T\ddot{e} + \dot{e} + Ke = V \\ e(0) = R,\ \dot{e}(0) = V \end{cases}$$

-- 新的系统方程:

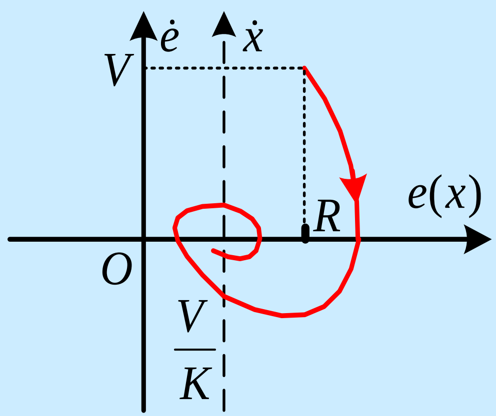
$$\begin{cases} T\ddot{x} + \dot{x} + Kx = 0 \\ x(0) = R - \dfrac{V}{K}, \quad \dot{x}(0) = V \end{cases}$$

$e - \dot{e}$ 平面奇点

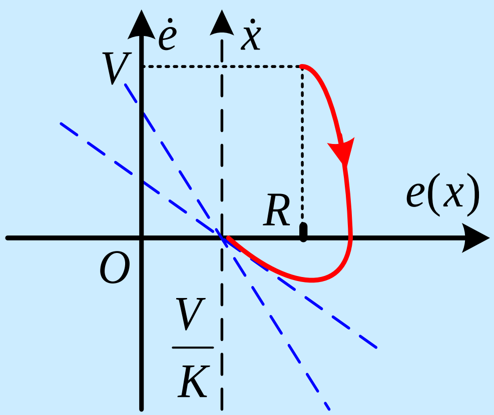
$$e = \frac{V}{K}, \quad \dot{e} = 0$$

-- 奇点的性质

若 $1-4KT < 0$
$\Rightarrow$ 稳定的焦点



若 $1-4KT \geq 0$
$\Rightarrow$ 稳定的节点



■

例 7.4.2 绘制如下系统的相平面图

$$T\ddot{e}+\dot{e}=P$$

Solution:

NB: 没有e !项

(i) 当 $P=0$

$$T\ddot{e}+\dot{e}=0 \qquad T\dot{e}\frac{\mathrm{d}\dot{e}}{\mathrm{d}e}+\dot{e}=0 \qquad \frac{\mathrm{d}\dot{e}}{\mathrm{d}e}=-\frac{\dot{e}}{T\dot{e}}$$

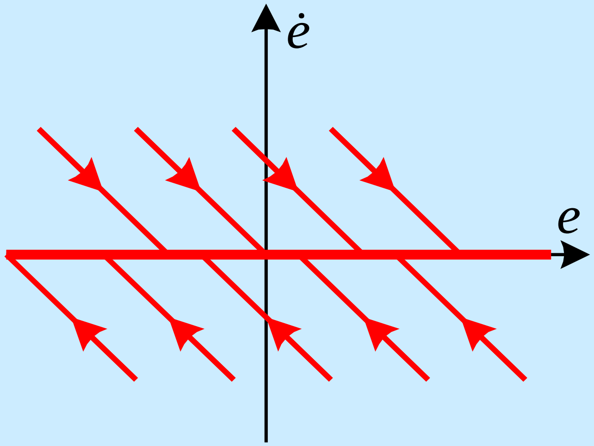
- **若** $\dot{e}=0$

  连续的奇点 $\dot{e}=0$

- **若** $\dot{e}\neq 0$**, 则**

  $$\frac{\mathrm{d}\dot{e}}{\mathrm{d}e}=-\frac{1}{T}$$

- **相平面图**



$$\dot{x}_1 = \dot{e} = x_2$$
$$\dot{x}_2 = \ddot{e} = \frac{P - x_2}{T}$$

(i) 当 $P \neq 0$ $\qquad T\ddot{e} + \dot{e} = P$

- **没有奇点**
- **等倾线:** $\dfrac{\mathrm{d}\dot{e}}{\mathrm{d}e} = \dfrac{P - \dot{e}}{T\dot{e}} = \alpha \qquad \dot{e} = \dfrac{P}{1 + \alpha T}$

$$\ddot{e} = \frac{\mathrm{d}\dot{e}}{\mathrm{d}e}\dot{e} = \alpha\dot{e}$$

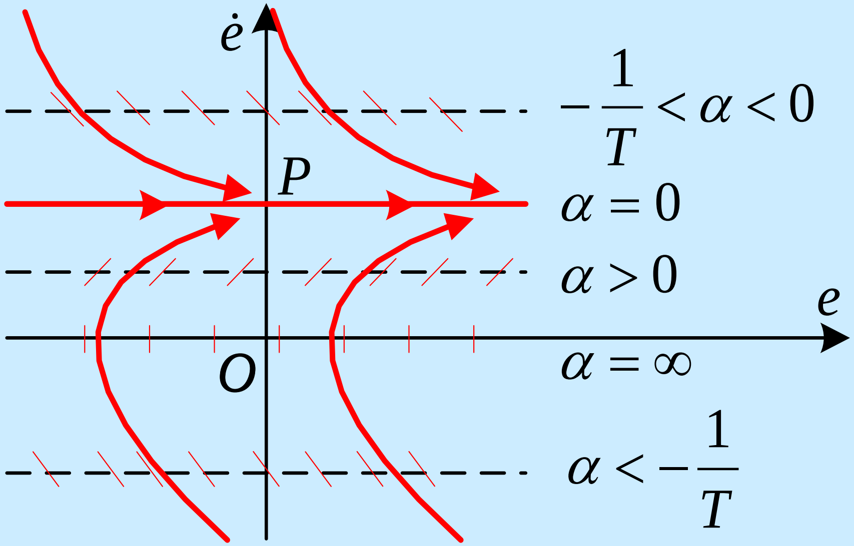
$\Rightarrow$ **一组平行线**

- $P > 0$

* $\alpha = 0 \quad \Rightarrow \quad \dot{e} = P$

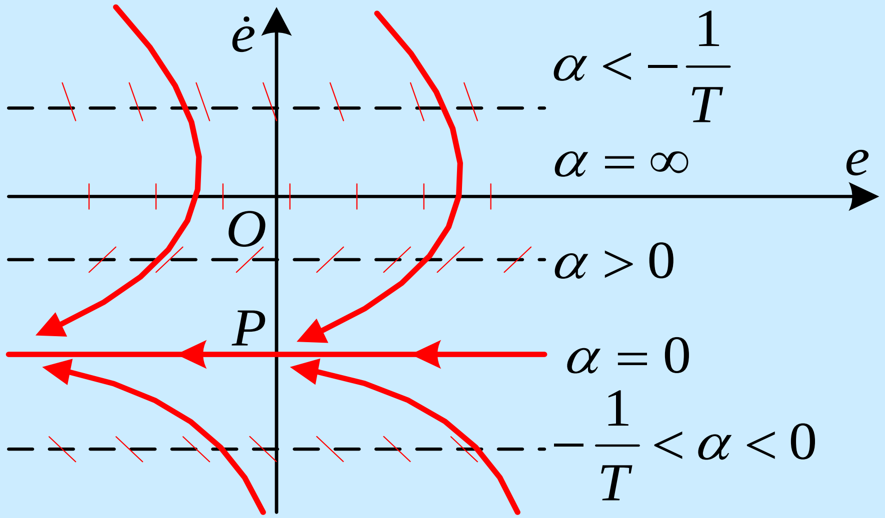
* $\dot{e} = P$ **是一条相轨迹**

* **等倾线法绘制相平面图**



$$\dot{e} = \frac{P}{1+\alpha T}$$

* $P < 0$



* $\dot{e} = P$ **是分界线、所有相轨迹的渐近线**

♦ 已讨论过的二阶系统

$$\ddot{x}+\dot{x}+x=P$$ **7.3.3, 7.4.1**

$$\ddot{x}+\dot{x}=P\begin{cases}=0 \\ \neq 0\end{cases}$$ **7.3.2** / **7.4.2**

$$\ddot{x}+x=0$$ **7.3.1, 7.3.4**

$$\ddot{x}=M$$ **7.3.5**

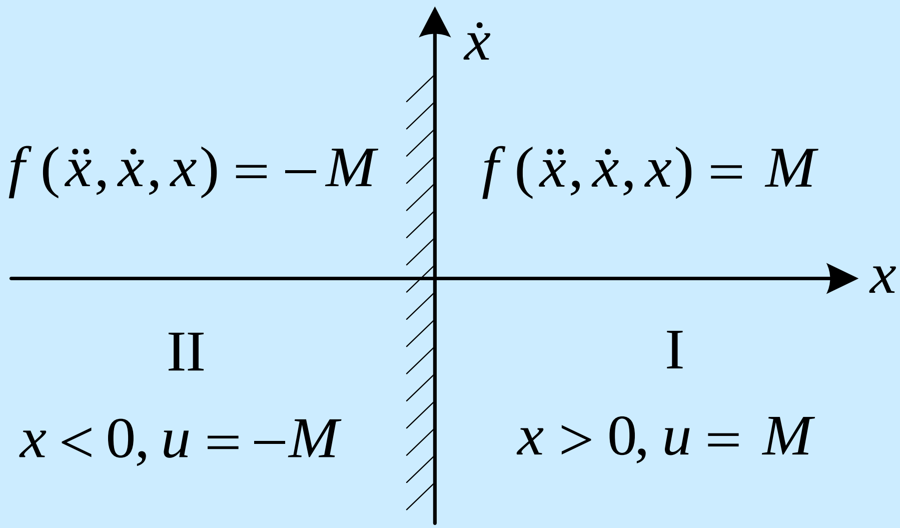
$$\dot{x}+x=P$$

7.4.2 非线性系统分析

1. 分片分析

♦ 示例

$$f(\ddot{x},\dot{x},x)=u=\begin{cases} M & \text{for} \quad x>0 \\ -M & \text{for} \quad x<0 \end{cases}$$

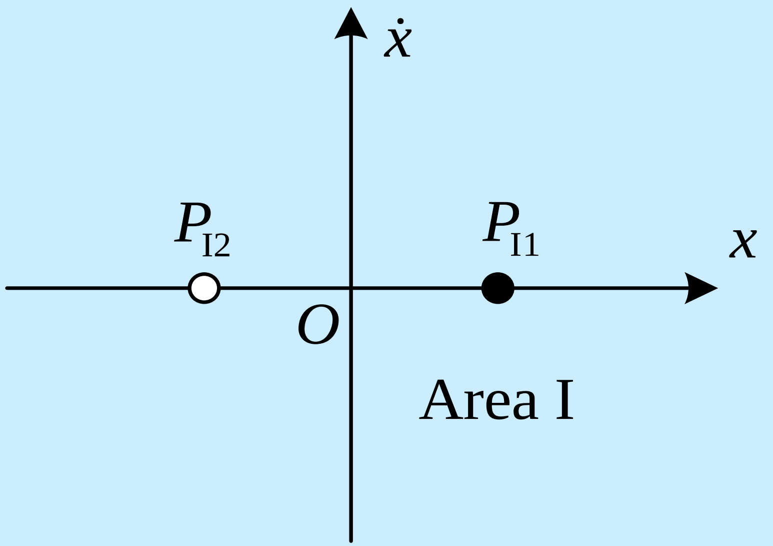


2. 实奇点和虚奇点

♦ 示例

令区域I的运动方程为 $f(\ddot{x},\dot{x},x)=M$

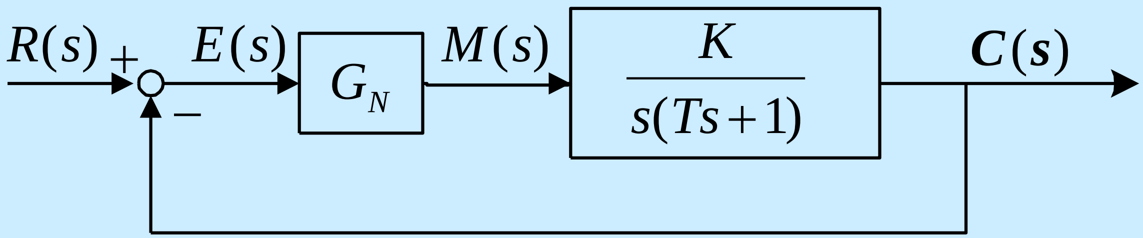
$P_{\mathrm{I}1}$和 $P_{\mathrm{I}2}$ 是该运动方程的奇点



◊ $P_{\mathrm{I}1}$ ：实奇点

◊ $P_{\mathrm{I}2}$ ：虚奇点

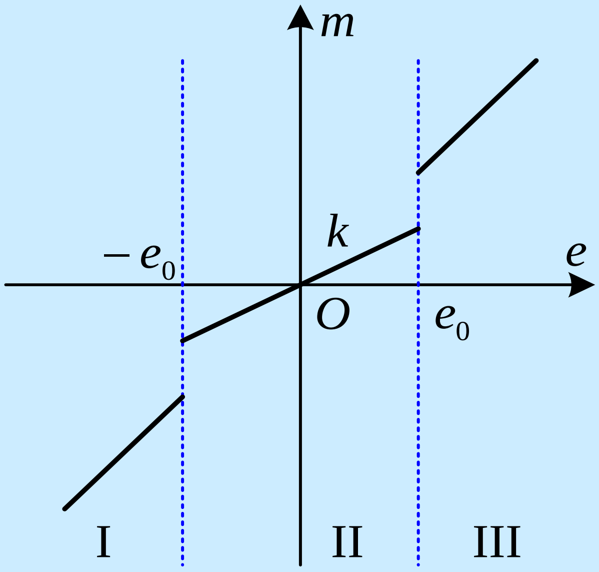
例 7.4.3 绘制如下系统的相平面图

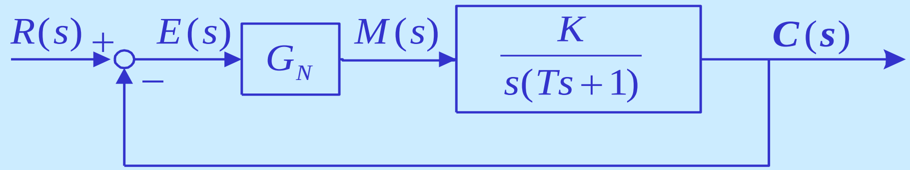


G_N如下

$$m = \begin{cases} e & \text{for } |e| > e_0 \\ ke & \text{for } |e| < e_0 \end{cases}$$

$$k < 1$$





Solution:

- **系统方程:**

$$\frac{C(s)}{M(s)} = \frac{K}{s(Ts+1)} \qquad T\ddot{c} + \dot{c} = Km$$

$$m = \begin{cases} e & \text{for } |e| > e_0 \\ ke & \text{for } |e| < e_0 \end{cases}$$

$$e = r - c \qquad c = r - e$$

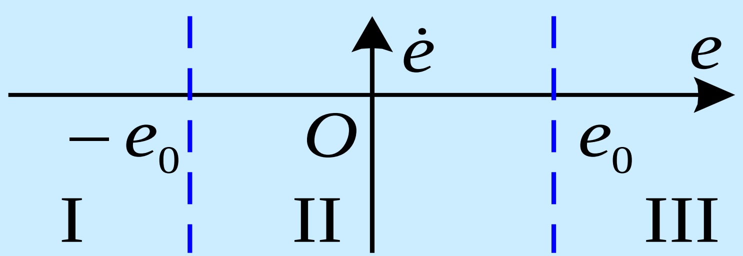
$$\dot{c} = \dot{r} - \dot{e} \qquad \ddot{c} = \ddot{r} - \ddot{e}$$

$$T\ddot{e} + \dot{e} = T\ddot{r} + \dot{r} - Km$$

- **在 $e - \dot{e}$ 平面:**

  3 区域

  2 不同方程



$$T\ddot{e} + \dot{e} = T\ddot{r} + \dot{r} - Km$$

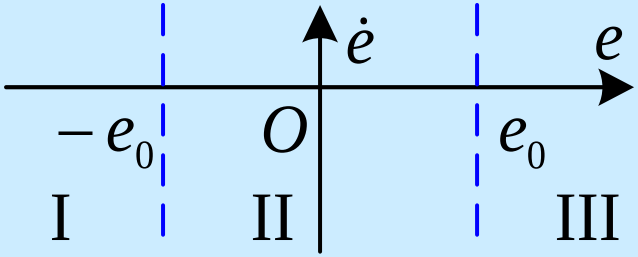
$$m = \begin{cases} e & \text{for } |e| > e_0 \\ ke & \text{for } |e| < e_0 \end{cases}$$

(i) 阶跃响应

$$r(t) = 1(t) \qquad \dot{r} = 0, \ddot{r} = 0$$

$$\begin{cases} T\ddot{e} + \dot{e} + Ke = 0 & \text{Areas I and III} \\ T\ddot{e} + \dot{e} + kKe = 0 & \text{Area II} \end{cases}$$

$$e(0) = E_0 = 1, \ \dot{e}(0) = 0$$



- **奇点:** $e = 0, \dot{e} = 0$

*** 区域II的实奇点**

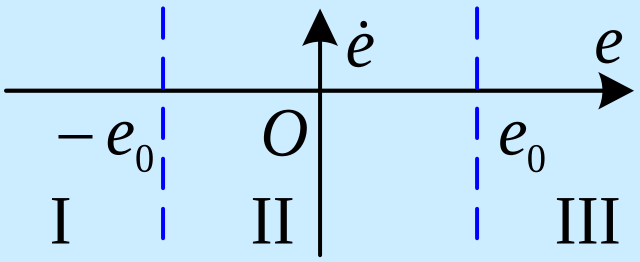
*** 区域I和III的虚奇点**

$$\begin{cases} T\ddot{e}+\dot{e}+Ke=0 & \text{Areas I and III} \\ T\ddot{e}+\dot{e}+kKe=0 & \text{Area II} \end{cases}$$

- **奇点的性质:**

假设 $1-4kKT=0$

由于 $k<1$ **, 则** $1-4KT<0$



*** 小幅误差:** $|e|<e_0$

$$T\ddot{e}+\dot{e}+kKe=0$$

$$\Rightarrow (0,0)$$ **稳定的节点**

*** 大幅误差:** $|e|>e_0$

$$T\ddot{e}+\dot{e}+Ke=0$$

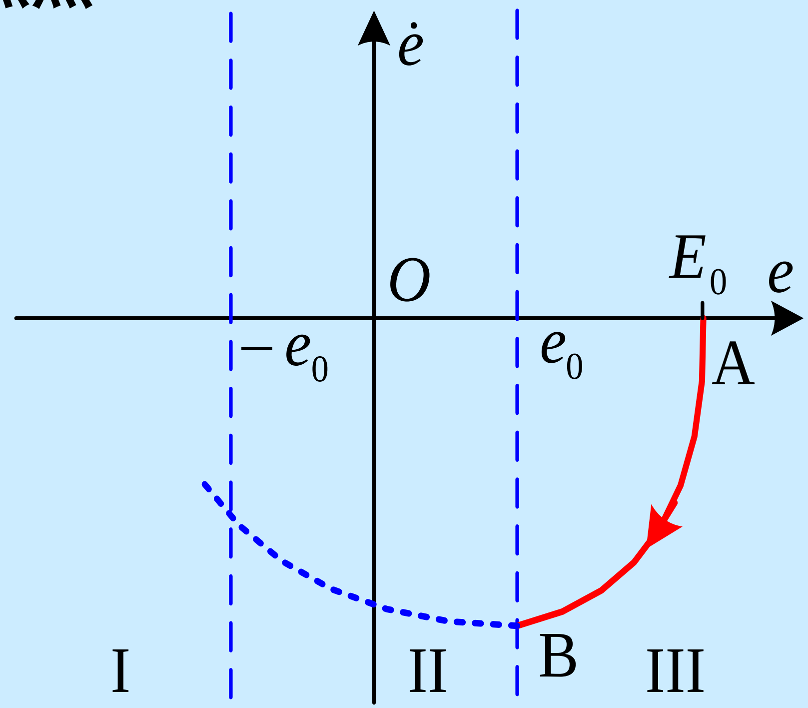
$$\Rightarrow (0,0)$$ **稳定的焦点**

$$\begin{cases} T\ddot{e}+\dot{e}+Ke=0 & \text{Areas I and III} \\ T\ddot{e}+\dot{e}+kKe=0 & \text{Area II} \end{cases}$$

- **相轨迹**

*** 令 A (E_0,0) 是初始点**

*** 对A而言, (0,0) 是稳定的焦点**



$$\begin{cases} T\ddot{e}+\dot{e}+Ke=0 & \text{Areas I and III} \\ T\ddot{e}+\dot{e}+kKe=0 & \text{Area II} \end{cases}$$

- **相轨迹**

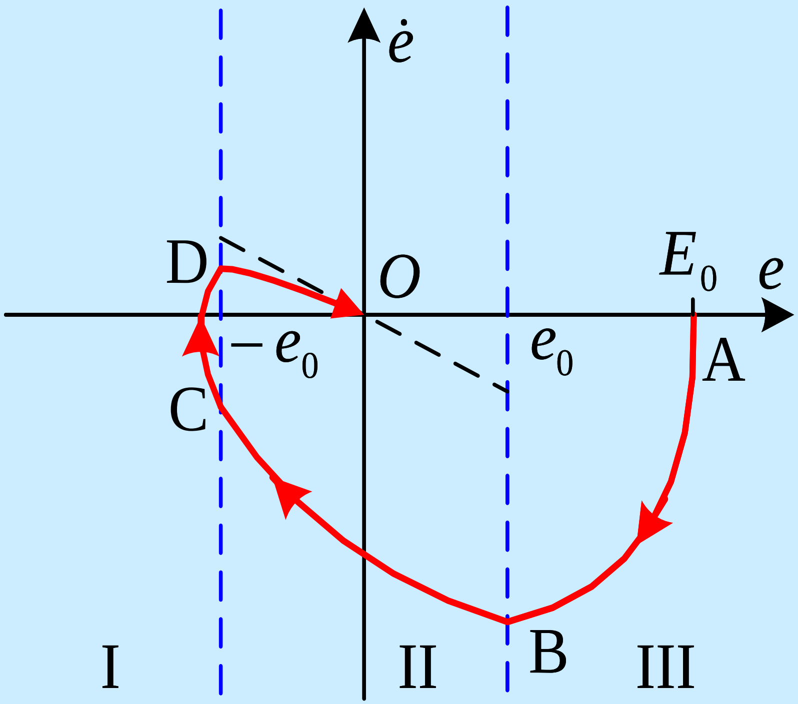
*** 令 A (E_0,0) 是初始点**

*** 对A而言, (0,0)是稳定的焦点**

*** 对B而言, (0,0)是稳定的节点**

*** 对C而言, (0,0)是稳定的焦点**

*** 对D而言, (0,0)是稳定的节点**



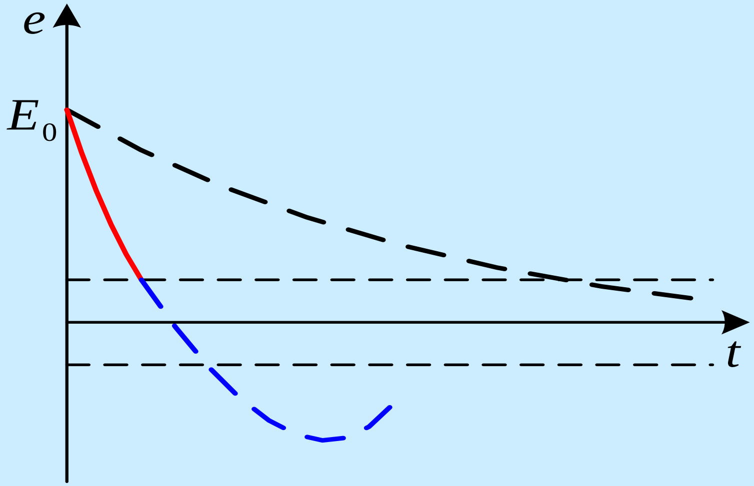
$$T\ddot{e}+\dot{e}+Ke=0$$
$$1-4KT<0$$

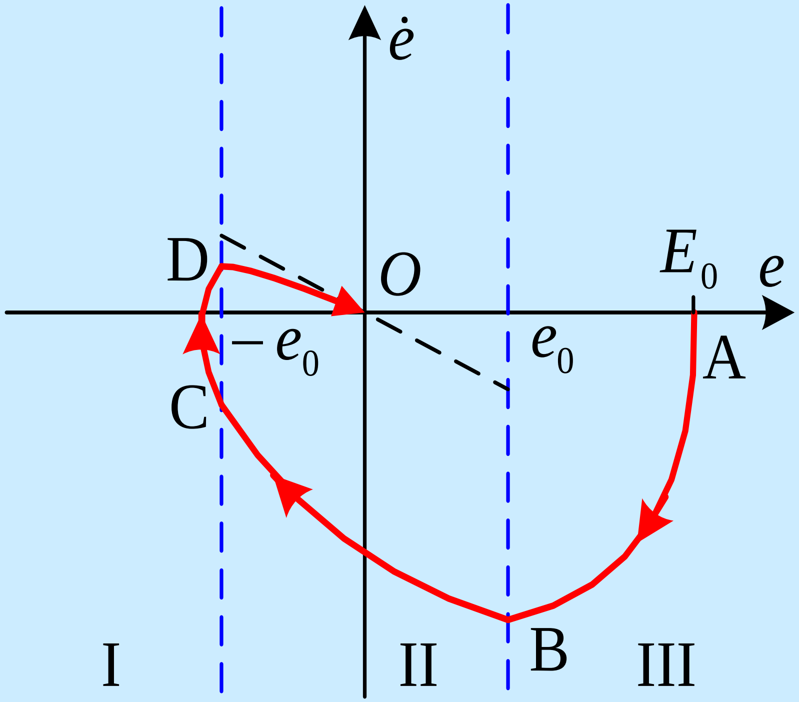
- **特点: 加速调节**

*** 当回路中信号强度大时**

-- 原点是稳定的焦点，运动呈现处欠阻尼特性

-- 误差快速下降





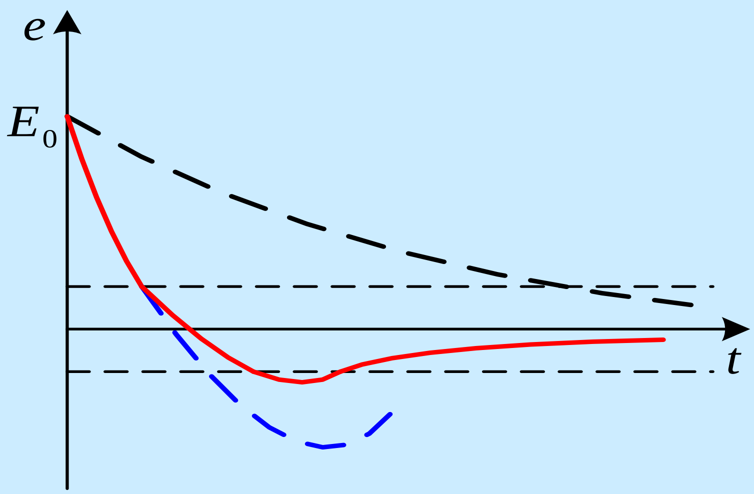
$$T\ddot{e} + \dot{e} + Ke = 0$$
$$1 - 4KT < 0$$

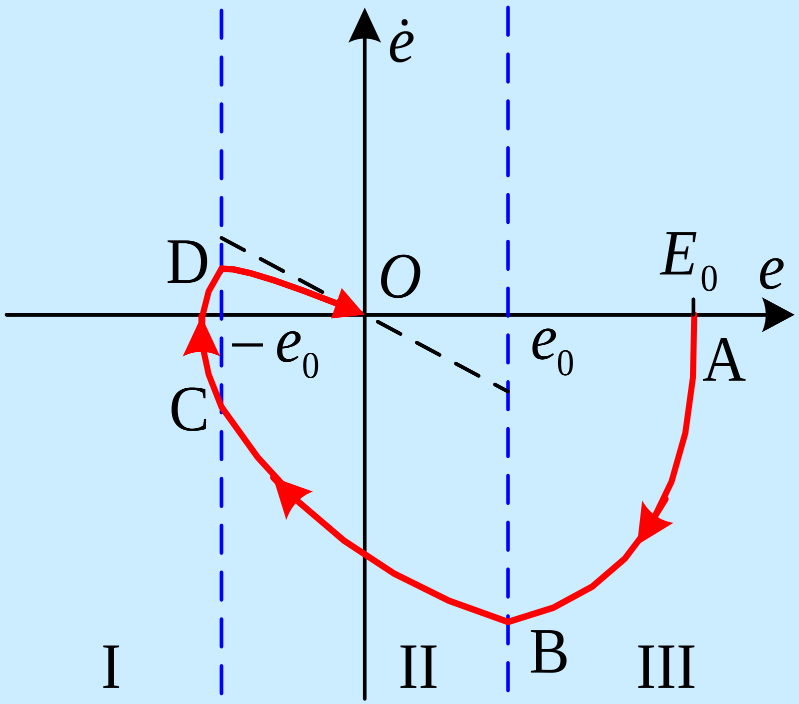
*** 当回路中信号强度小时**

$$T\ddot{e} + \dot{e} + kKe = 0$$
$$1 - 4kKT = 0$$

-- 原点是稳定的节点，运动呈现处临界阻尼特性

-- 避免得以振荡





$$\boxed{T\ddot{e}+\dot{e}=T\ddot{r}+\dot{r}-Km}$$

$$m=\begin{cases} e & \text{for } |e|>e_0 \\ ke & \text{for } |e|<e_0 \end{cases}$$

(ii) 斜坡响应

$$r(t)=R+Vt \qquad \dot{r}=V,\ \ddot{r}=0$$

$$\begin{cases} T\ddot{e}+\dot{e}+Ke=V & \text{Areas I and III} \\ T\ddot{e}+\dot{e}+kKe=V & \text{Area II} \end{cases}$$

$$e(0)=R,\ \dot{e}(0)=V$$

- **奇点**

*** 区域 II:** $P_{\mathrm{II}}: e=\dfrac{V}{kK},\ \dot{e}=0$

*** 区域 I, III:** $P_{\mathrm{I}}:\ e=\dfrac{V}{K},\ \dot{e}=0$

***** $|P_{\mathrm{II}}|>|P_{\mathrm{I}}|$

- **奇点的性质**

*** 假设** $1-4kKT=0$

P_{II} **: 稳定的节点** $\quad P_{\mathrm{I}}$ **: 稳定的焦点**

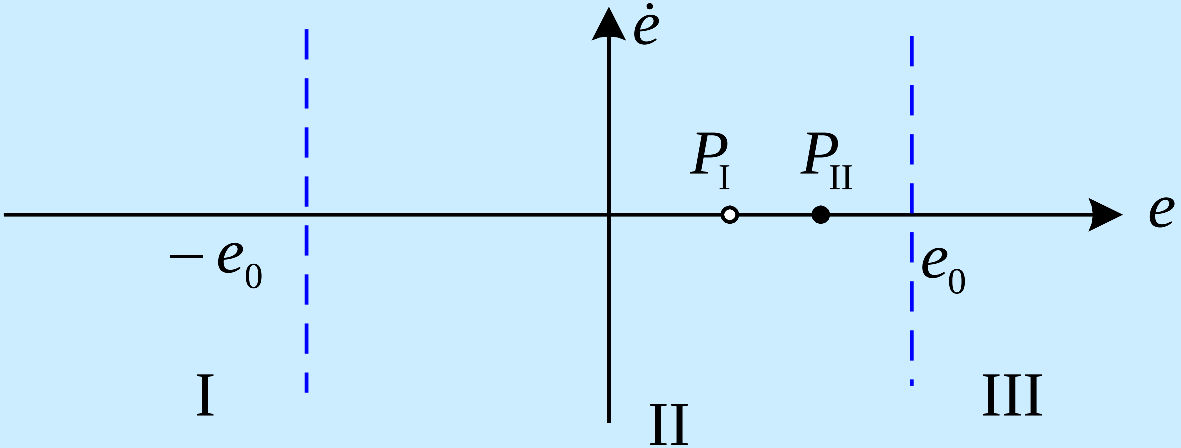
***** P_{I} **和** P_{II} **的位置依赖于参数** k **,** K **,** V

(A) $V < kKe_0$

- **奇点的性质:**

P_{II}: $e = \dfrac{V}{kK} < e_0$ **：实奇点**

P_{I}: $e = \dfrac{V}{K} < ke_0 < e_0$ **：虚奇点**



$$\ddot{e}+\dot{e}+4e=0.04$$
$$\ddot{e}+\dot{e}+0.25e=0.04$$

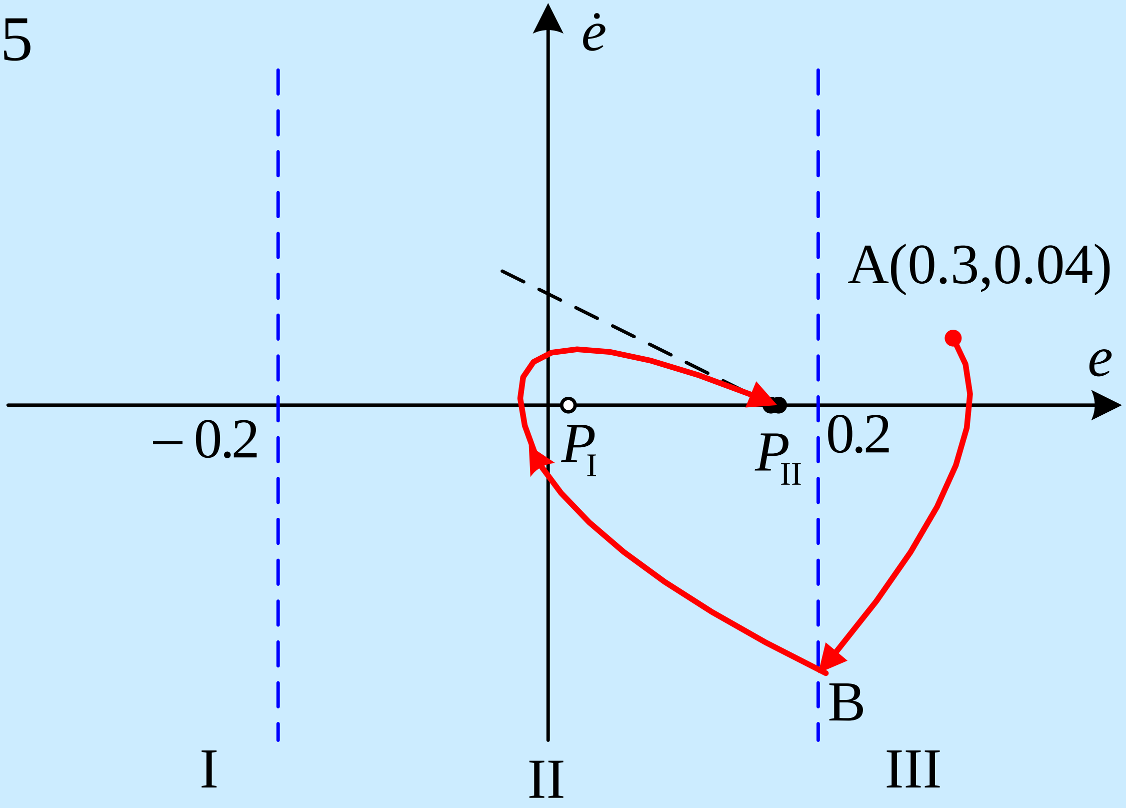
- **相轨迹**

令 $T=1, K=4, k=0.0625, e_0=0.2$ $\quad r(t)=0.3+0.04t$

则 $V<kKe_0=0.05$

P_{II}： $e=\dfrac{V}{kK}=0.16$

P_{I}： $e=\dfrac{V}{K}=0.01$



*** 对A而言:**
P_{I}– **稳定焦点**

*** 对B而言:**
P_{II}– **稳定节点**

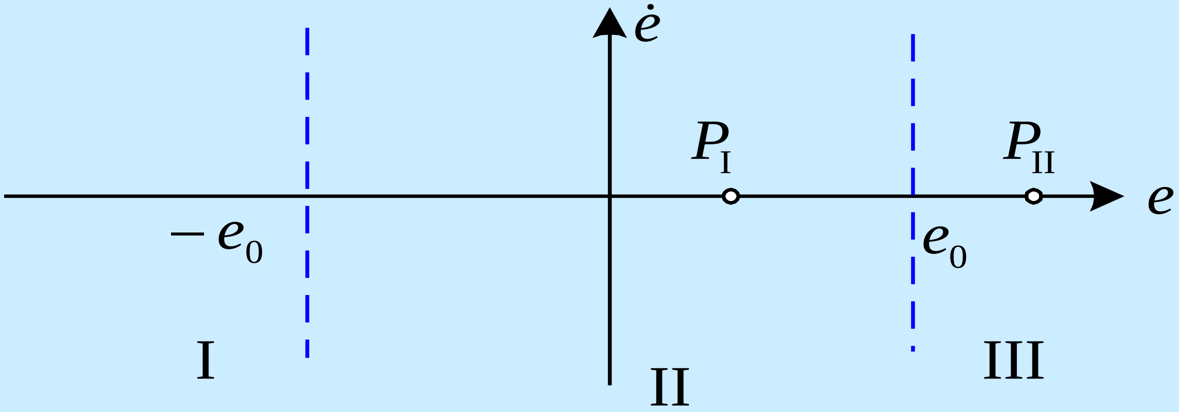
- **特点:** $e_{ss}=P_{\mathrm{II}}$

(B) $kKe_0 < V < Ke_0$

- **奇点的性质:**

$$P_{\mathrm{II}}：\ e = \frac{V}{kK} > e_0$$ **：虚奇点**

$$P_{\mathrm{I}}：\ e = \frac{V}{K} < e_0$$ **：虚奇点**



$$\ddot{e}+\dot{e}+4e=0.4$$
$$\ddot{e}+\dot{e}+0.25e=0.4$$

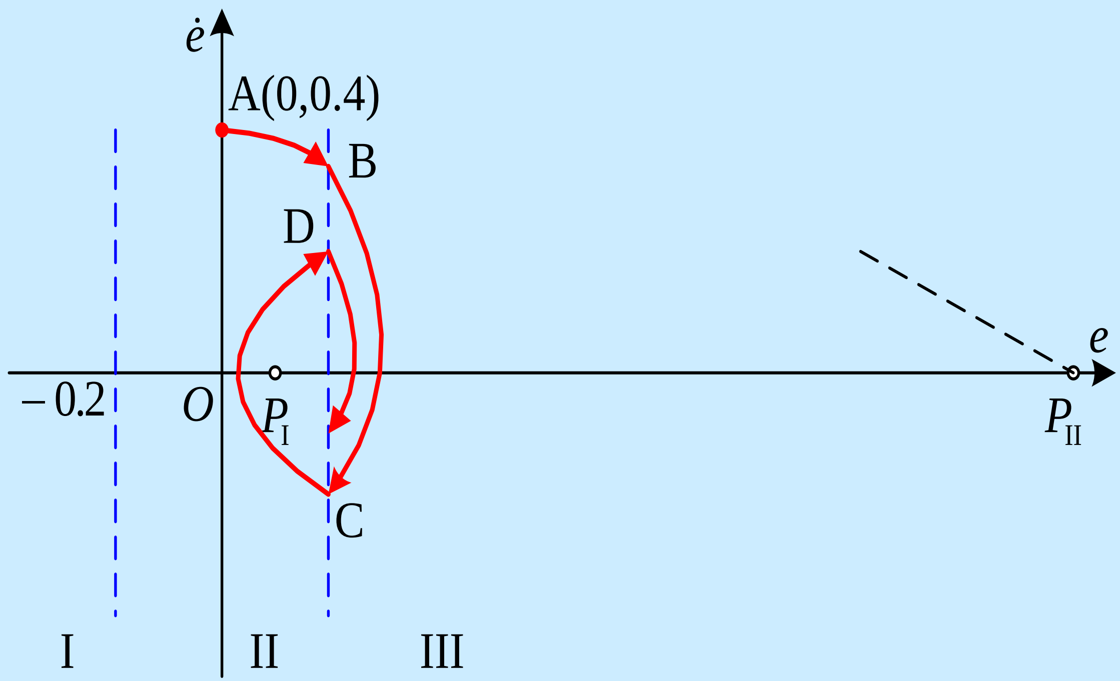
- **相轨迹**

令 $T=1, K=4, k=0.0625, e_0=0.2$ $\qquad r(t)=0.4t$

则 $0.05=kKe_0<V<Ke_0=0.8$

$$P_{\mathrm{II}}:e=\frac{V}{kK}=1.6$$

$$P_{\mathrm{I}}:e=\frac{V}{K}=0.1$$

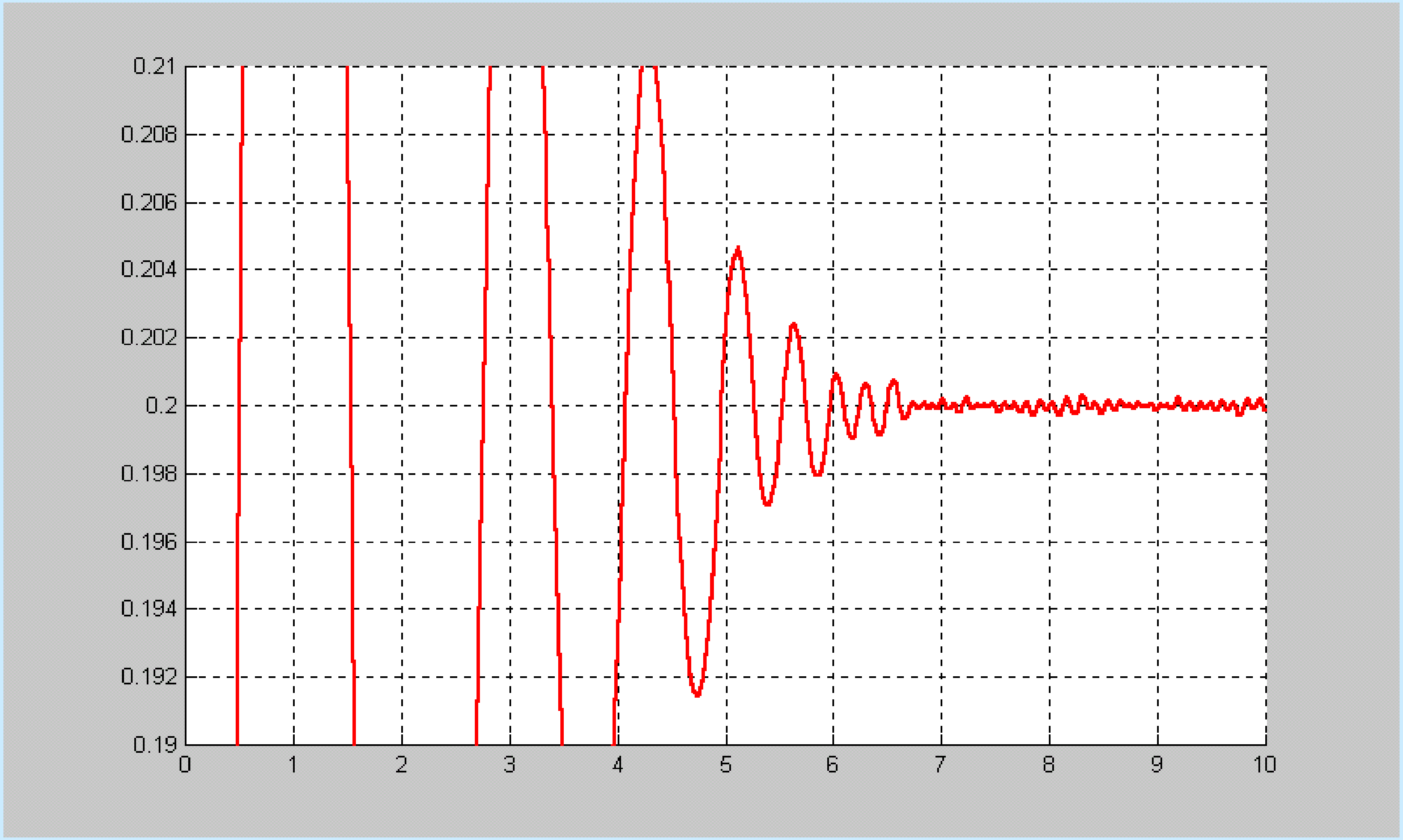


*** 对A & C而言:**
P_{II} **– 稳定节点**

*** 对B & D而言:**
P_{I} **– 稳定焦点**

- **特点:** $e_{ss}=e_0$

- 斜坡输入下误差的时间响应 -- 高频振荡

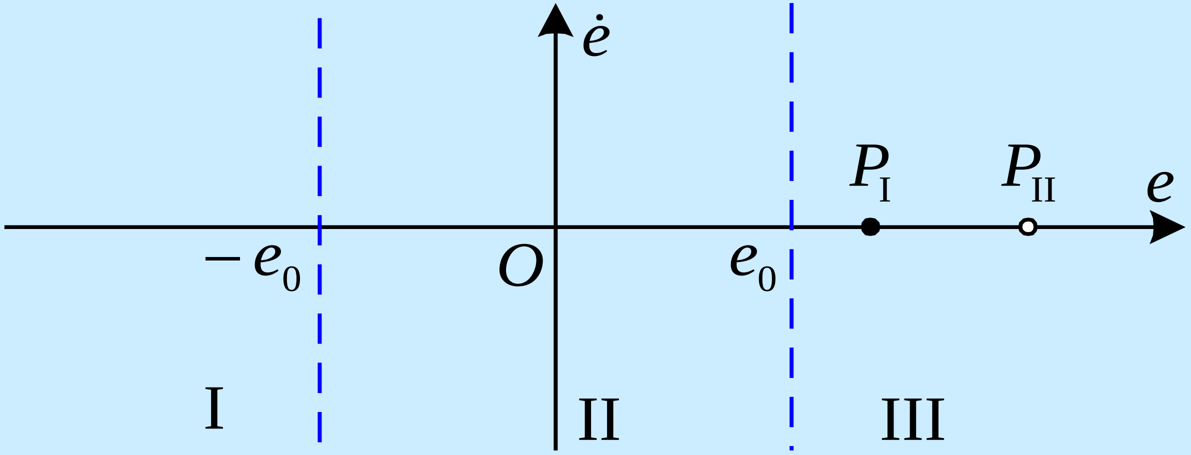


(C) $V > Ke_0$

- **奇点的性质:**

$$P_{\mathrm{II}}:\ \ e=\frac{V}{kK}>\frac{V}{K}>e_0$$ **：虚奇点**

$$P_{\mathrm{I}}:\ \ e=\frac{V}{K}>e_0$$ **：实奇点**



$$\ddot{e}+\dot{e}+4e=1.2$$
$$\ddot{e}+\dot{e}+0.25e=1.2$$

- **相轨迹**

* **令** $T=1, K=4, k=0.0625, e_0=0.2 \qquad r(t)=1.2t$

则 $V>Ke_0=0.8$

$P_{\mathrm{II}}:\quad e=\dfrac{V}{kK}=4.8$

$P_{\mathrm{I}}:\quad e=\dfrac{V}{K}=0.3$

* **对A & C而言:**

P_{II} **– 稳定节点**

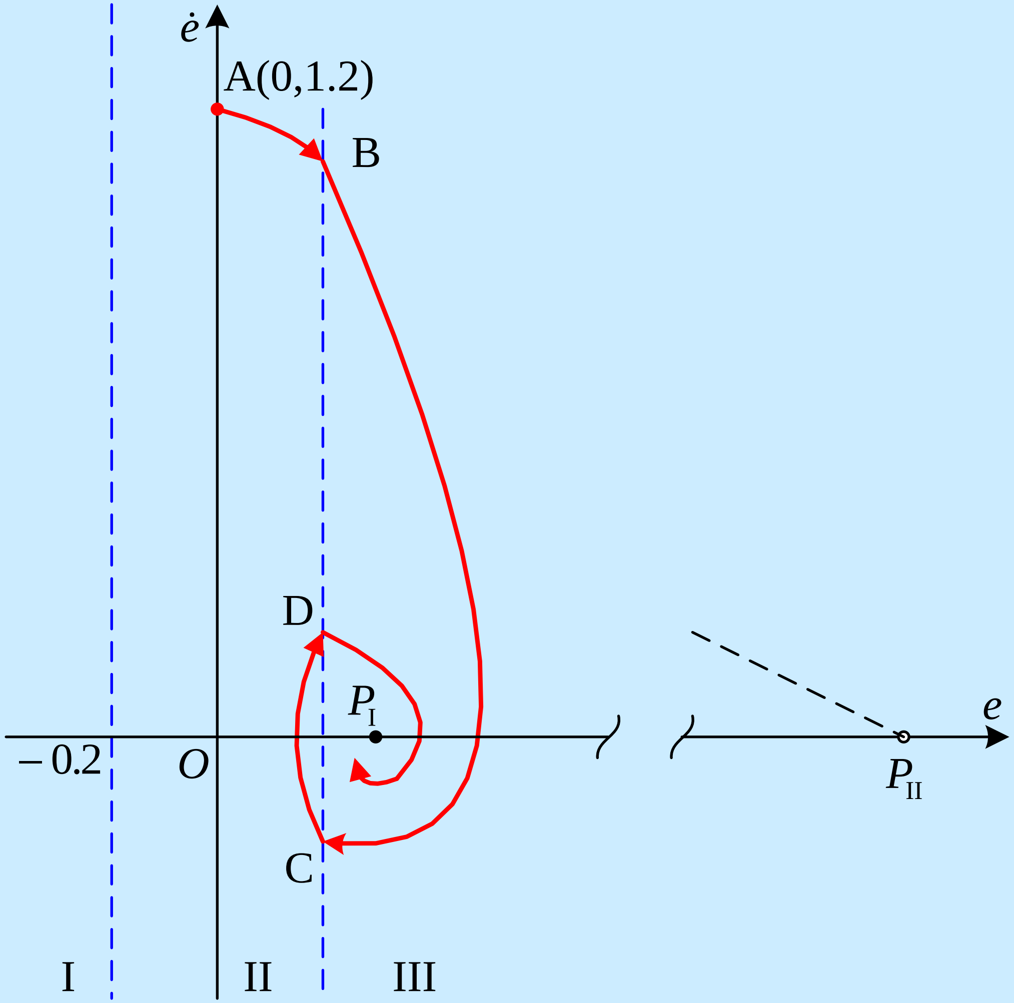
* **对B & D而言:**

P_{I} **-- 稳定焦点**

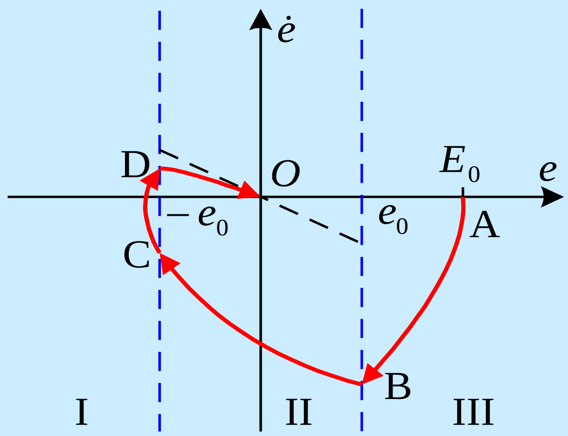
- **特点:**

$e_{ss}\uparrow \quad \Leftarrow e_{ss}=P_{\mathrm{I}}$

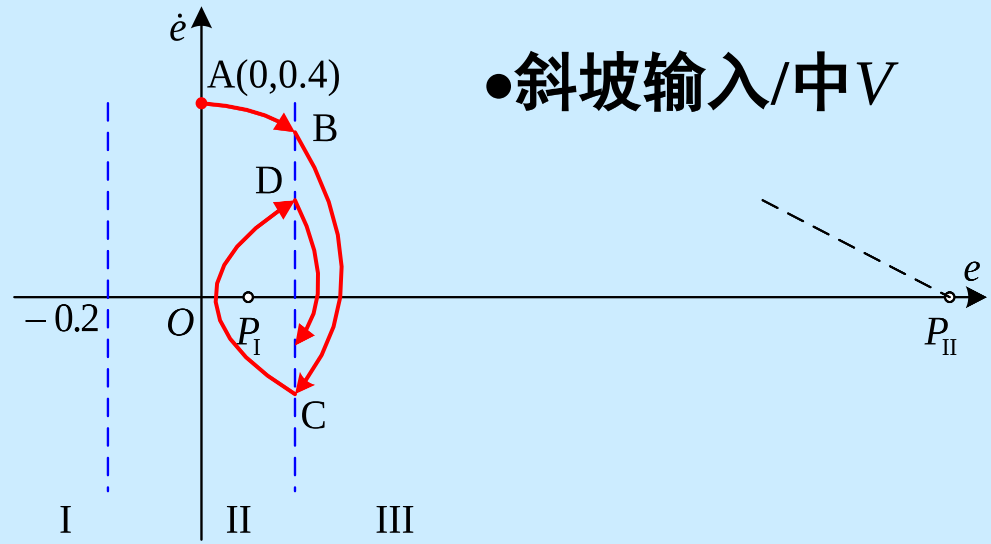
$t_s\uparrow$ **(长时间振荡)**



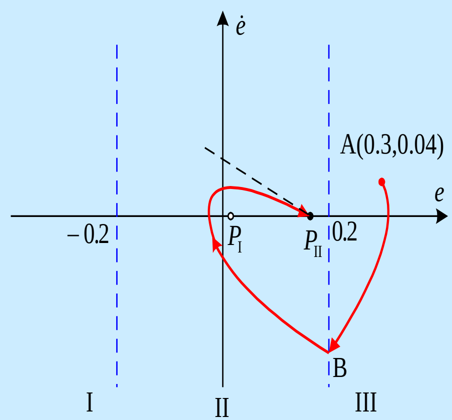
• 阶跃输入



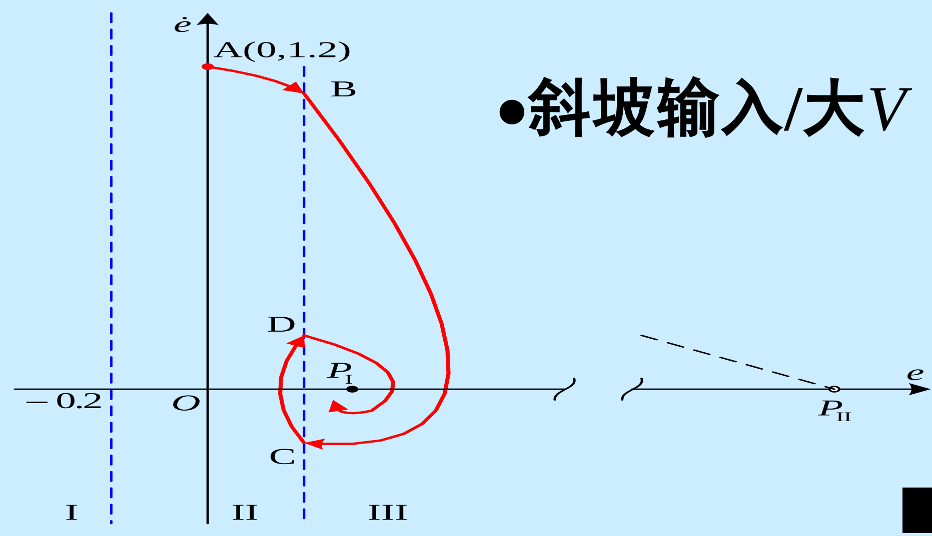
•斜坡输入/中V



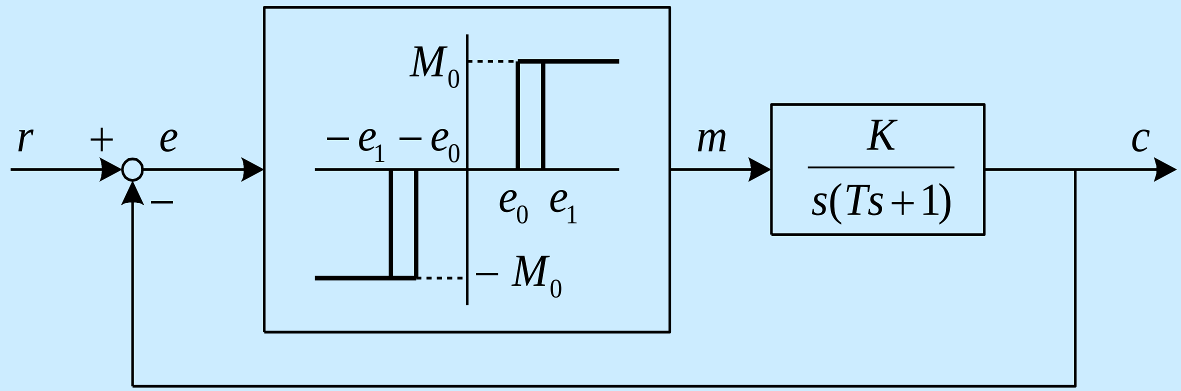
• 斜坡输入/小V



•斜坡输入/大V



例 7.4.4 绘制如下系统的相平面图



$$m=\begin{cases} M_0 & e>e_1 \\ 0 & -e_0<e\le e_1 \\ -M_0 & e\le -e_0 \end{cases} \qquad \text{for } \dot{e}>0$$

$$m=\begin{cases} M_0 & e>e_0 \\ 0 & -e_1<e\le e_0 \\ -M_0 & e\le -e_1 \end{cases} \qquad \text{for } \dot{e}\le 0$$

Solution:

- **系统方程:**

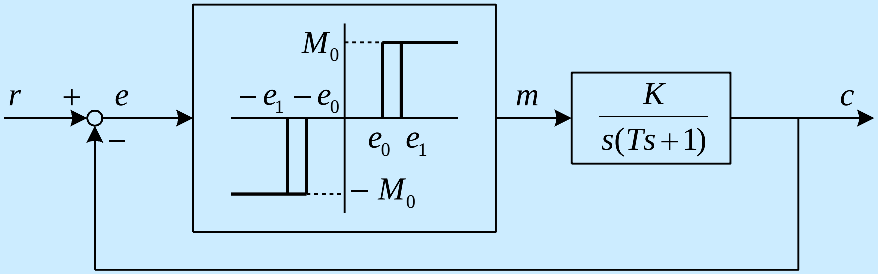
$$\frac{C(s)}{M(s)}=\frac{K}{s(Ts+1)}$$

$$T\ddot{c}+\dot{c}=Km$$

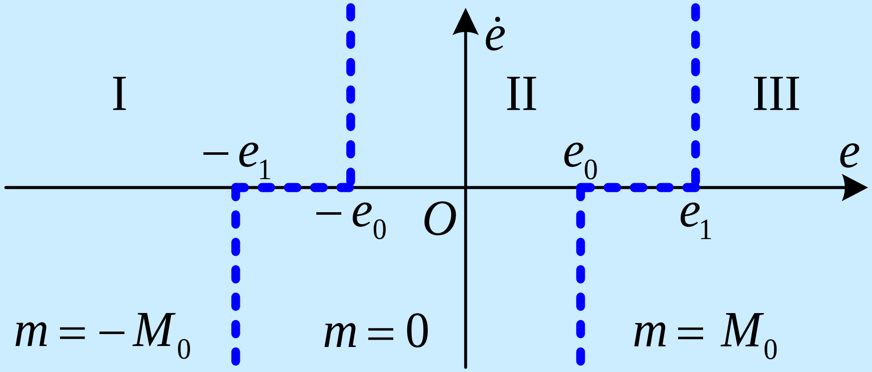
$$e=r-c,\ c=r-e$$

$$\dot{c}=\dot{r}-\dot{e},\ \ddot{c}=\ddot{r}-\ddot{e}$$

$$T\ddot{e}+\dot{e}=T\ddot{r}+\dot{r}-Km$$



- **在 $e-\dot{e}$ 平面:**



3 区域, 3 不同方程

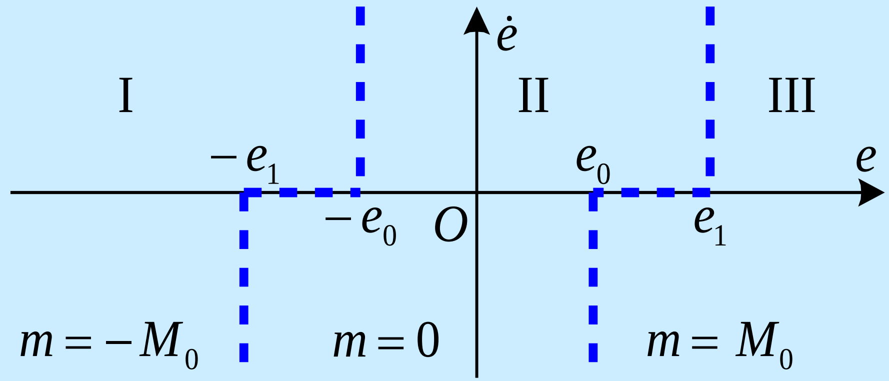
$$\boxed{T\ddot{e}+\dot{e}=T\ddot{r}+\dot{r}-Km}$$

(i) 阶跃响应

$$r(t)=R\cdot 1(t),\ \ \dot{r}=0,\ \ \ddot{r}=0$$

$$T\ddot{e}+\dot{e}=-Km \quad \textbf{(没有 } e \textbf{)}$$

$$e(0)=R,\ \ \dot{e}(0)=0$$



$$\begin{cases} T\ddot{e}+\dot{e}=KM_0 & \text{Area I} \\ T\ddot{e}+\dot{e}=0 & \text{Area II} \\ T\ddot{e}+\dot{e}=-KM_0 & \text{Area III} \end{cases}$$

- **奇点:**

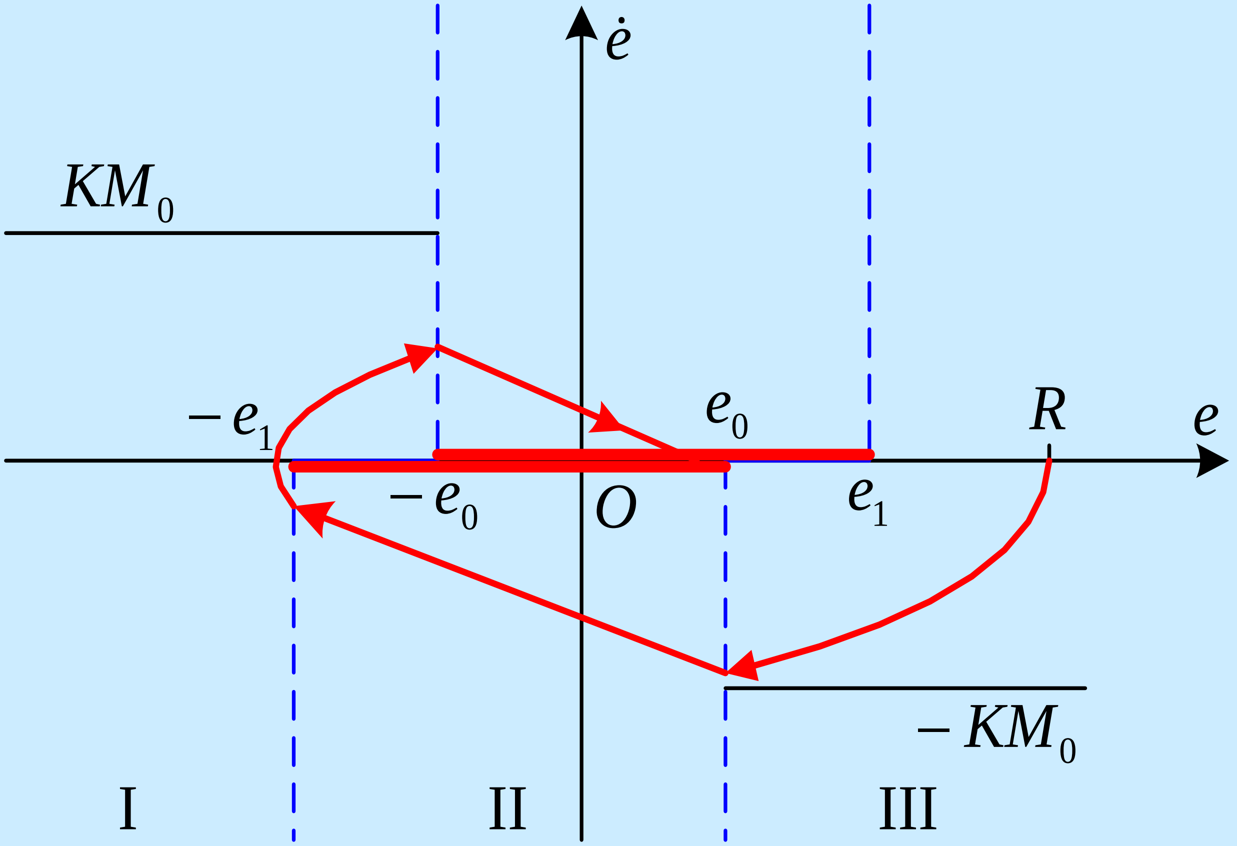
*** Area II: 连续奇点*e*-轴** *** Area I, III: 没有奇点**

-- Areas I & III 渐近线:

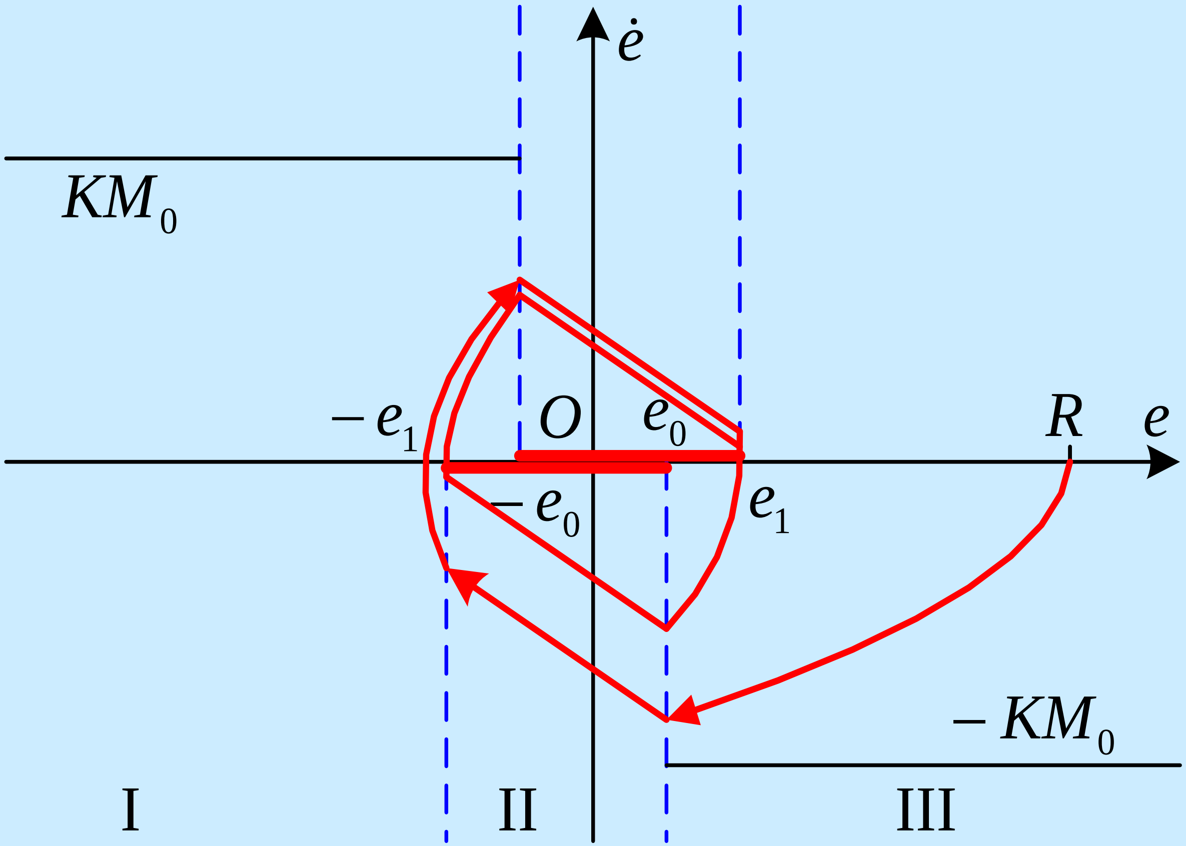
-- Area I: $\dot{e}=KM_0$ **-- Area III:** $\dot{e}=-KM_0$

- **轨迹**

* **大** e_0 **/ 小** KM_0

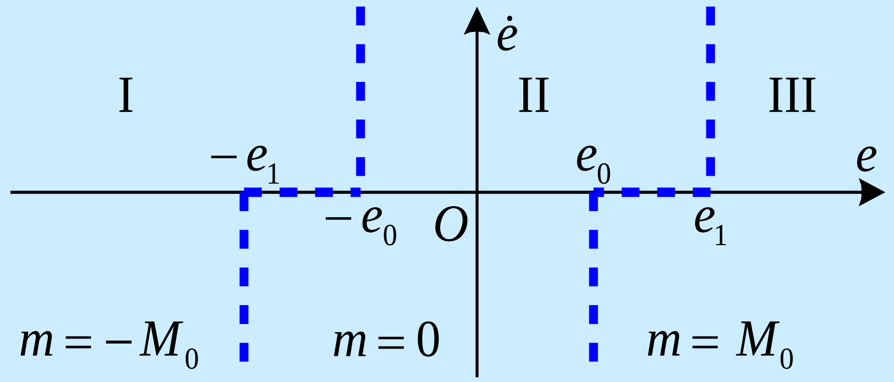


* 小 e_0 / 大 KM_0



e.g. $T=1, KM_0>7, e_1=2, e_0=1, R=4$

$$\boxed{T\ddot{e}+\dot{e}=T\ddot{r}+\dot{r}-Km}$$



(ii) 斜坡响应

$$r(t)=Vt,\ \dot{r}=V,\ \ddot{r}=0$$

$$T\ddot{e}+\dot{e}=V-Km$$

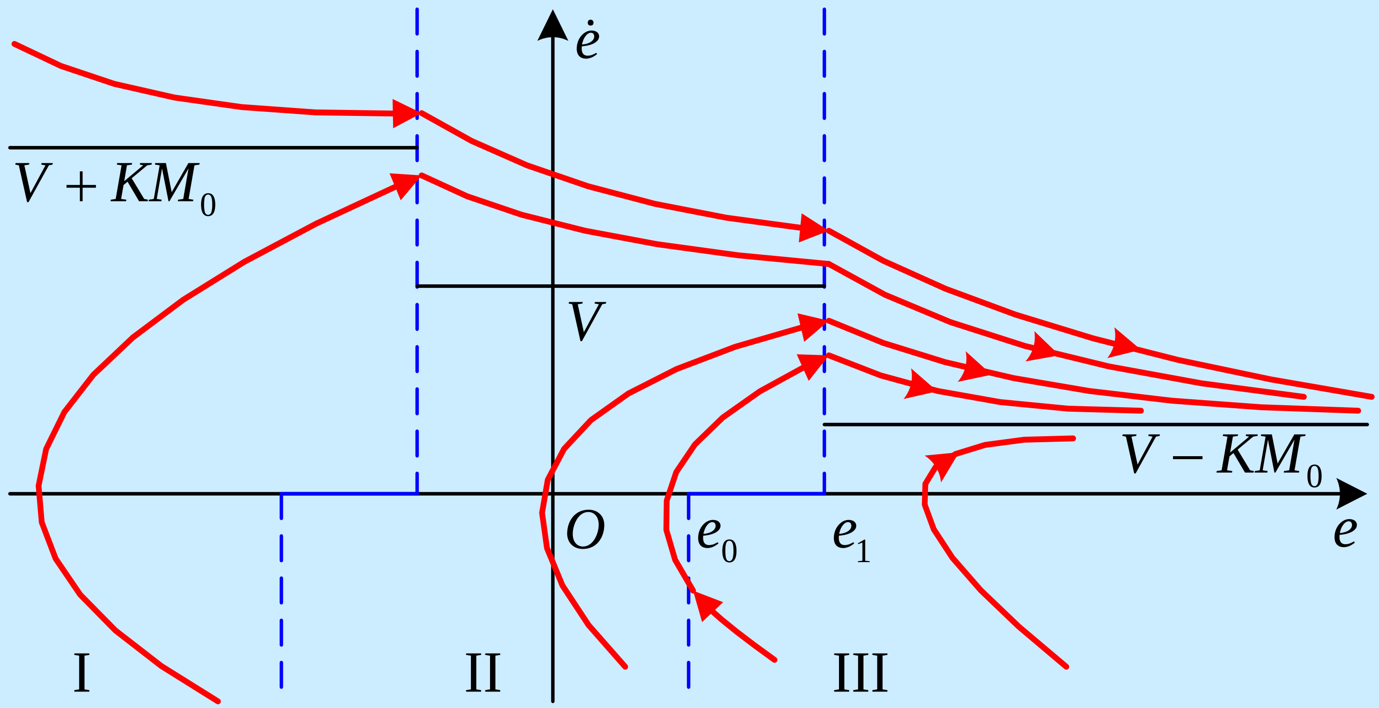
$$e(0)=0,\ \dot{e}(0)=V$$

$$\begin{cases} T\ddot{e}+\dot{e}=V+KM_0 & \text{Area I} \\ T\ddot{e}+\dot{e}=V & \text{Area II} \\ T\ddot{e}+\dot{e}=V-KM_0 & \text{Area III} \end{cases}$$

$$T\ddot{e}+\dot{e}=\begin{cases} V+KM_0 \\ V \\ V-KM_0 \end{cases} \neq 0$$

(A) $V > KM_0$

- **没有奇点**
- **相平面图**

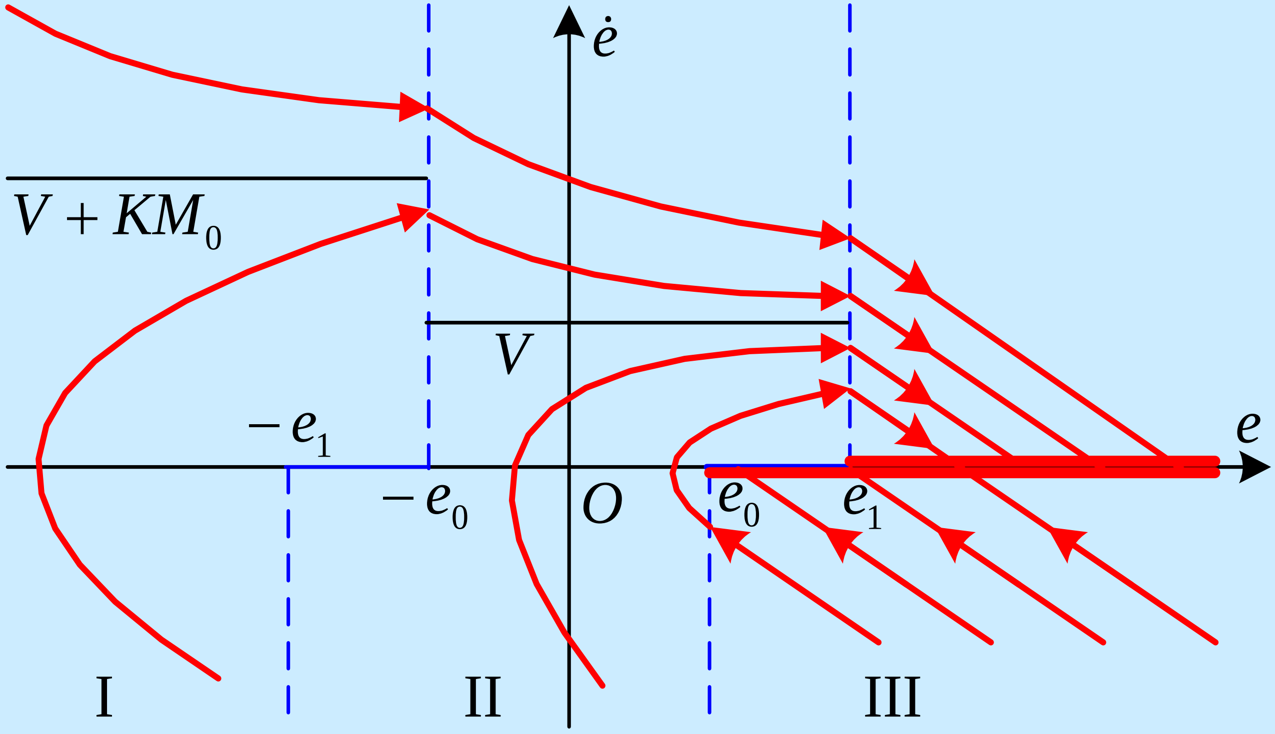


$$T\ddot{e}+\dot{e}=\begin{cases}V+KM_0\\V\\V-KM_0=0\end{cases}$$

(B) $V = KM_0$

*** 连续奇点 e-轴 Area III**

- **相平面图**

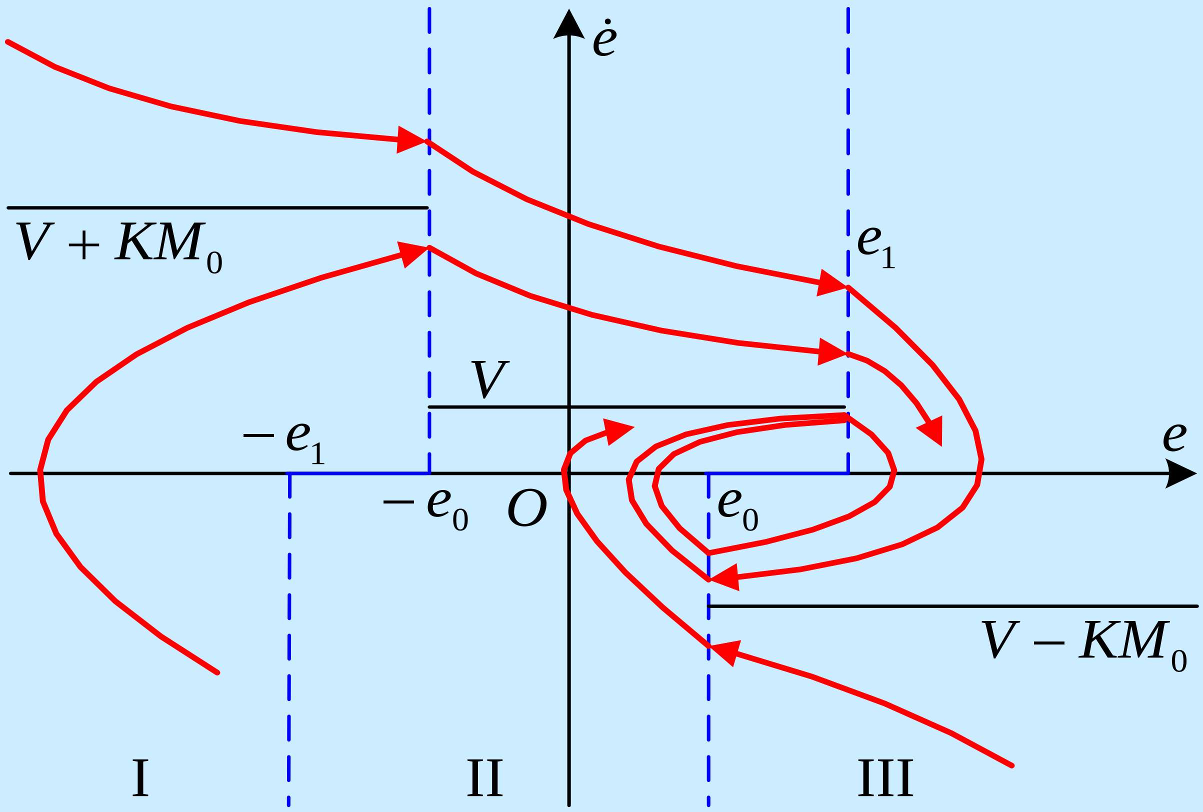


$$T\ddot{e}+\dot{e}=\begin{cases} V+KM_0 \\ V \\ V-KM_0 \end{cases}\neq 0$$

(C) $V < KM_0$

*** 没有奇点**

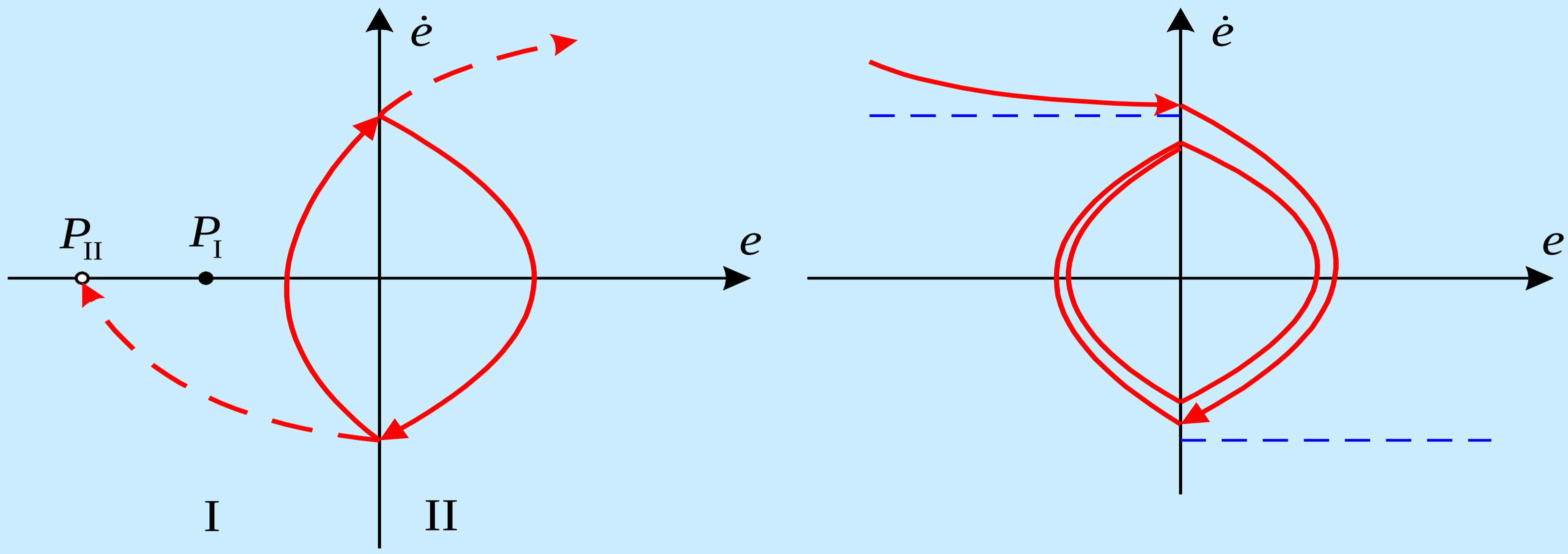
• 相平面图



e.g. $T=1, V=1, KM_0>1.5, e_1=2, e_0=1, R=4$

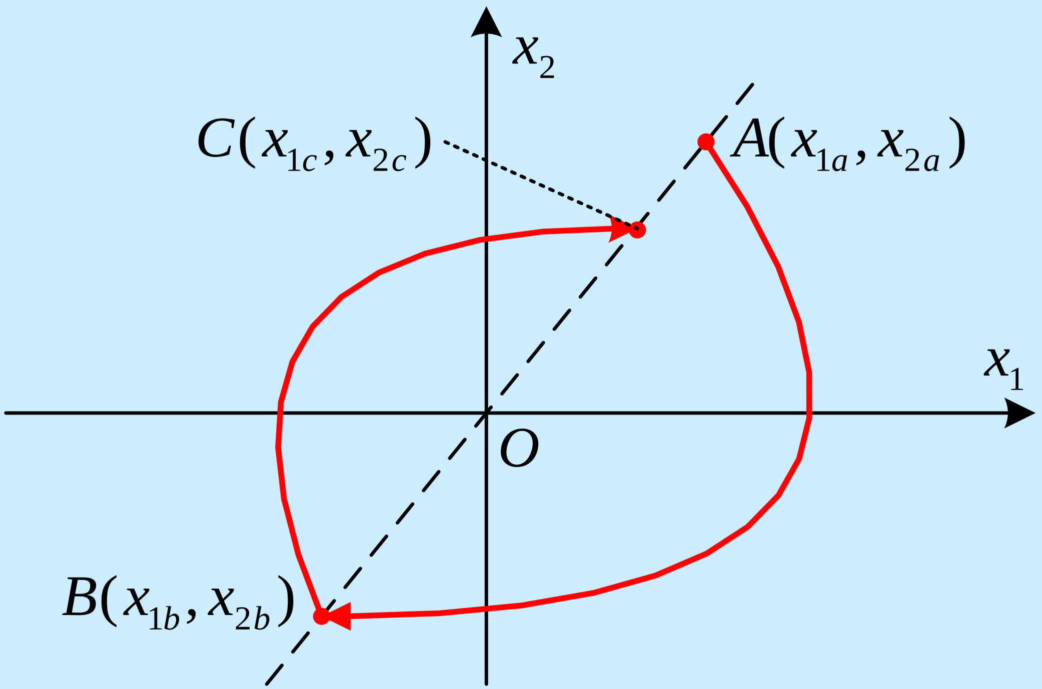
- **形成极限环的情况**

(i) 不稳定实奇点和稳定虚奇点　(ii) 两个不稳定区域相邻



P_{I}: **不稳定实奇点**
P_{II}: **稳定虚奇点**

- **极限环的判断和计算**



$$x_{1c} = x_{1a}$$
$$x_{2c} = x_{2a}$$
$\Rightarrow$ **极限环**

例 7.4.5 利用相平面法分析系统的极限环

$$\ddot{x} + \dot{x} = 1 \qquad (\dot{x} - x > 0)$$

$$\ddot{x} + \dot{x} = -1 \qquad (\dot{x} - x < 0)$$

Solution:

(i) 奇点

$$\ddot{x} + \dot{x} = P \qquad (P \neq 0)$$

- **没有奇点**

$$\ddot{x}+\dot{x}=1$$

(ii) 相平面图

- **区域** $\dot{x}>x$

* **相轨迹的渐近线:** $\dot{x}=1$

* **等倾线:**

$$\ddot{x}+\dot{x}=1$$

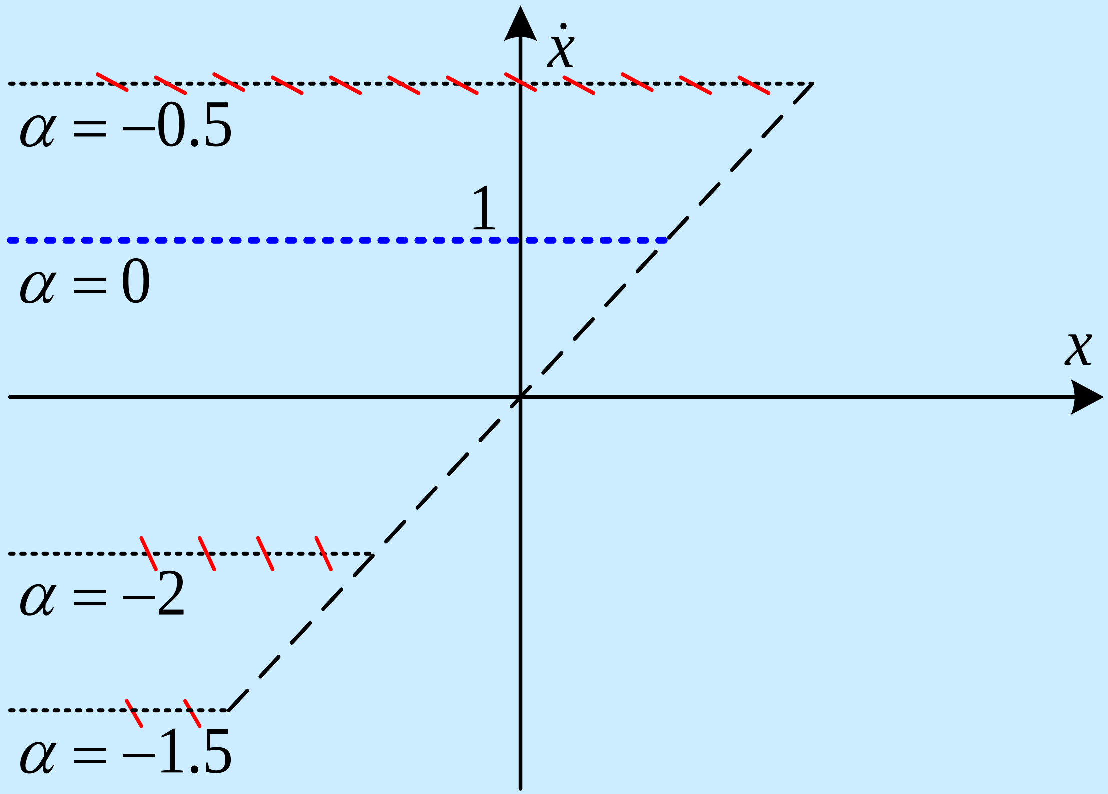
$$\Downarrow$$

$$\dot{x}\frac{\mathrm{d}\dot{x}}{\mathrm{d}x}+\dot{x}=1$$

令 $$\frac{\mathrm{d}\dot{x}}{\mathrm{d}x}=\alpha$$

则 $$\dot{x}(1+\alpha)=1$$

i.e $$\dot{x}=\frac{1}{1+\alpha}$$



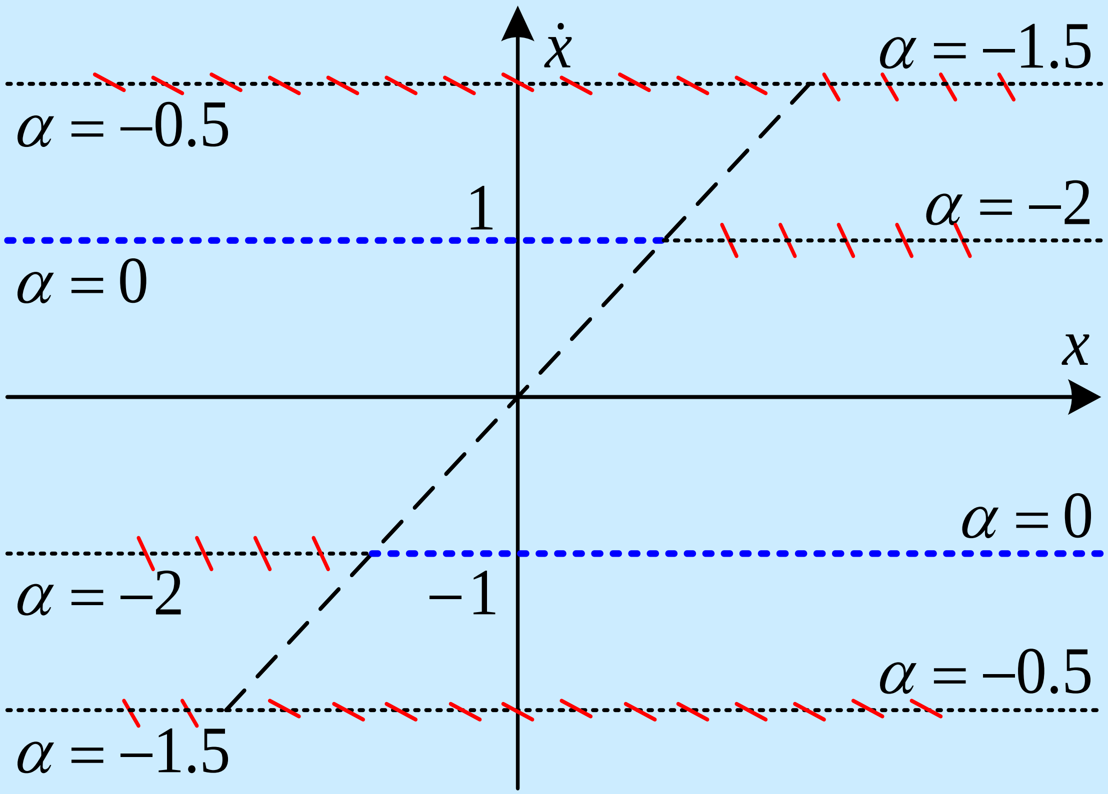
$$\boxed{\ddot{x}+\dot{x}=-1}$$

- **区域** $\dot{x}<x$

* **相轨迹的渐近线:** $\dot{x}=-1$

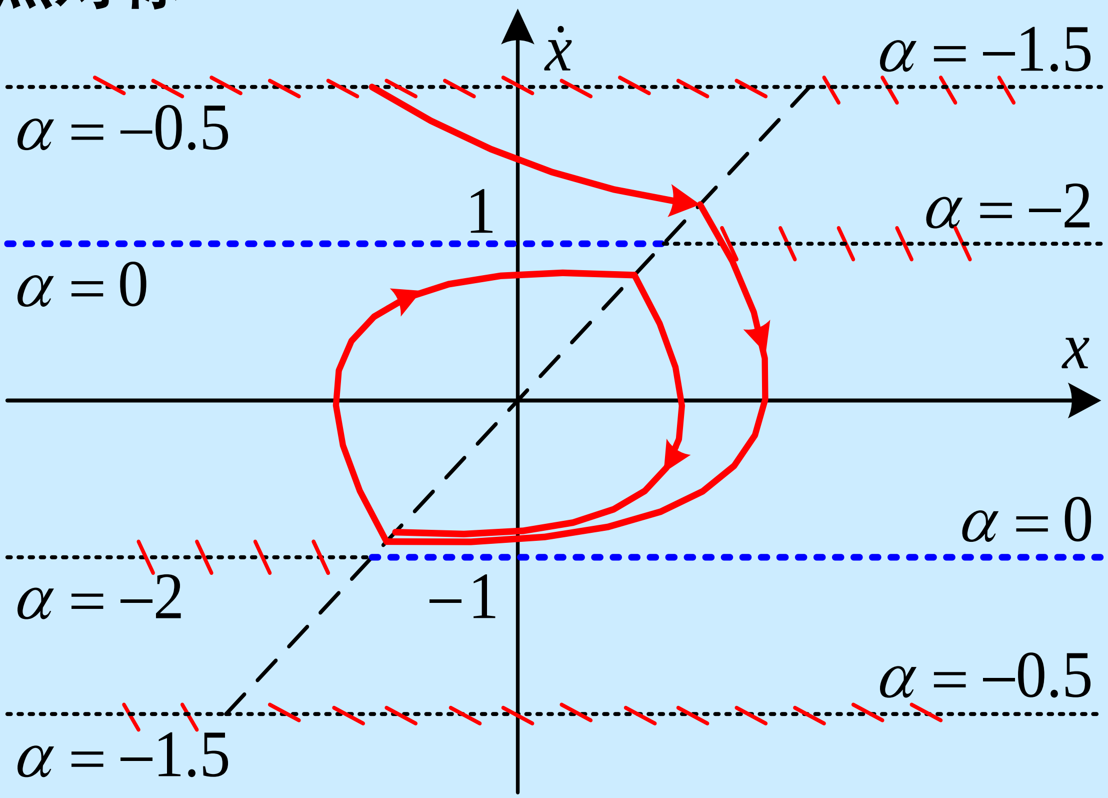
* **等倾线:**

$$\dot{x}=\frac{-1}{1+\alpha}$$



◊ 相平面图

- 极限环可能存在
- 若存在，则关于原点对称



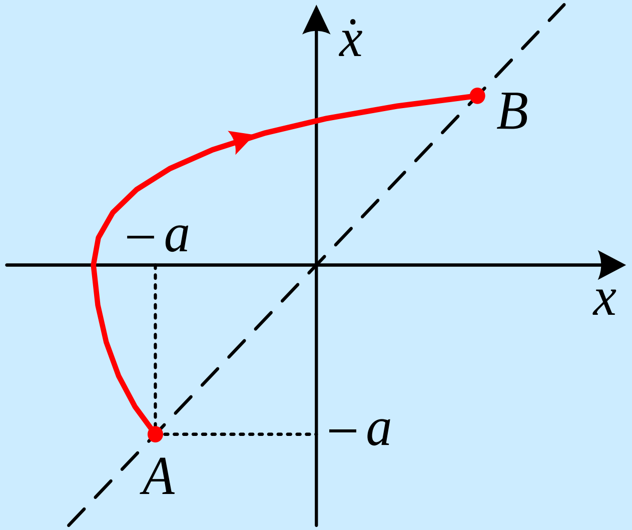
(iii) 极限环的频率和振幅

- **区域** $\dot{x} > x$

  出发点 A：

  $x(0) = -a \qquad \dot{x}(0) = -a$

  终止点 B:



- **求解微分方程**

$$\ddot{x} + \dot{x} = 1$$

$$s^2 X(s) - sx(0) - \dot{x}(0) + sX(s) - x(0) = \frac{1}{s}$$

$$s^2 X(s) - s(-a) - (-a) + sX(s) - (-a) = \frac{1}{s}$$

$$(s^2 + s)X(s) + 2a + as = \frac{1}{s}$$

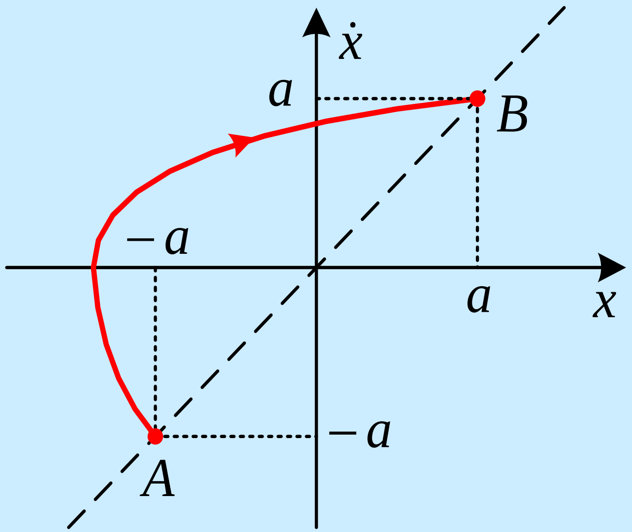
$$\boxed{(s^2+s)X(s)+2a+as=\frac{1}{s}}$$

$$X(s)=\frac{1-2as-as^2}{s^2(s+1)}$$

$$=\frac{1}{s^2}-\frac{2a+1}{s}+\frac{a+1}{s+1}$$

$$x(t)=t-2a-1+(a+1)\mathrm{e}^{-t}$$

$$\dot{x}(t)=1-(a+1)\mathrm{e}^{-t}$$



- **确定终止时间** t_1

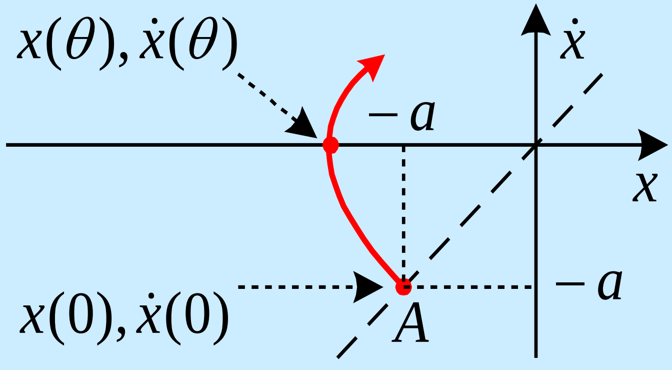
终止点 B**:**

$$x(t_1)=a \qquad \dot{x}(t_1)=a$$

$$\Rightarrow \quad \begin{aligned} a&=t_1-2a-1+(a+1)\mathrm{e}^{-t_1} \\ a&=1-(a+1)\mathrm{e}^{-t_1} \end{aligned}$$

$$a = t_1 - 2a - 1 + (a+1)e^{-t_1}$$
$$a = 1 - (a+1)e^{-t_1}$$

$$t_1 = 4a$$
$$e^{-4a} = \frac{1-a}{1+a} \quad \Rightarrow \quad t_1 = 3.83, \quad a = 0.9575$$



- **频率:**

$$T = 2t_1 = 7.66 \ \text{sec}$$
$$\omega = 0.820 \ \text{rad/s}$$

- **振幅:**

相轨迹与 x–轴垂直， $\dot{x}(\theta) = 0$ **时运动幅值最大**

$$\boxed{x(t)=t-2a-1+(a+1)\mathrm{e}^{-t}} \qquad \boxed{\dot{x}(t)=1-(a+1)\mathrm{e}^{-t}}$$

$$1-(a+1)\mathrm{e}^{-\theta}=0$$

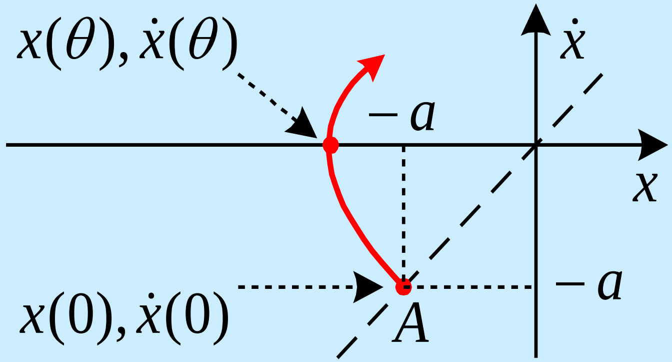
$$\Downarrow \qquad \mathrm{e}^{-\theta}=\frac{1}{a+1}=\frac{1}{1.9575}$$

$$\theta=0.6717$$

$$x(\theta)=\theta-2a-1+(a+1)\mathrm{e}^{-\theta}$$

$$=-1.2433$$

$$X=1.2433$$



(iv) 对比 ×

	DF	**PP**	**Simu**(*Appr*)
T	6.28	7.66	7.65
X	0.90	1.243	1.241

■

§7.5 小结

	DF法	PP分析
方法	等价线性化	图形法
对象复杂性	√	1st & 2nd order system
非线性复杂性	×	Piecewise linearity
时间响应	×	√
稳定性分析	√	√
极限环分析	√	√
准确性	×	√

End of Chapter 7