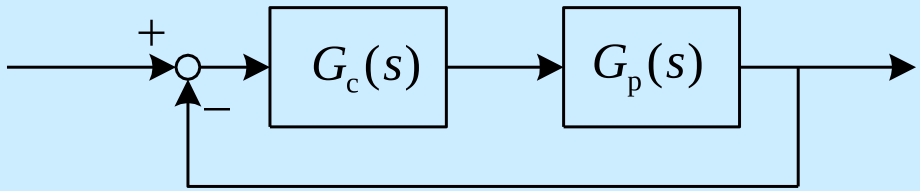
第八章 采样系统

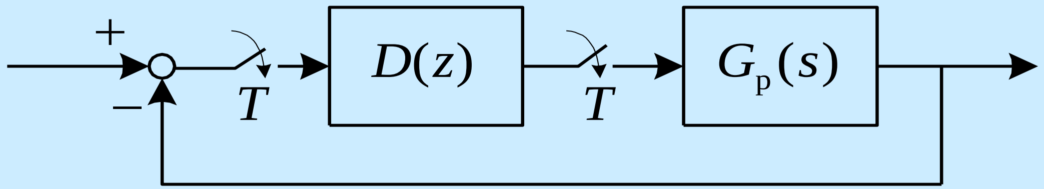
8.1 导论

1. 连续系统



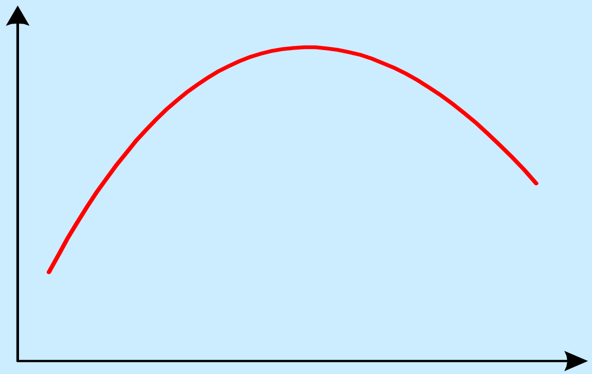
- $G_p(s)$ 连续对象
- $G_c(s)$ 模拟控制器
- 时间变量、系统变量均连续变化

2. 采样系统

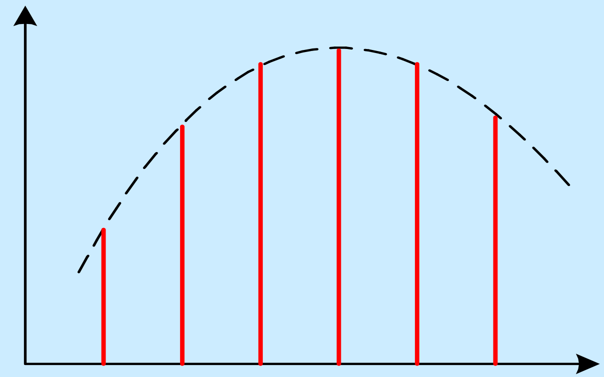


- $G_p(s)$ **连续对象**
- $D(z)$ **数字控制器**
- **连续 (模拟) 信号和离散 (数字) 信号同时存在**
- **采样开关, A/D、D/A 转换器**

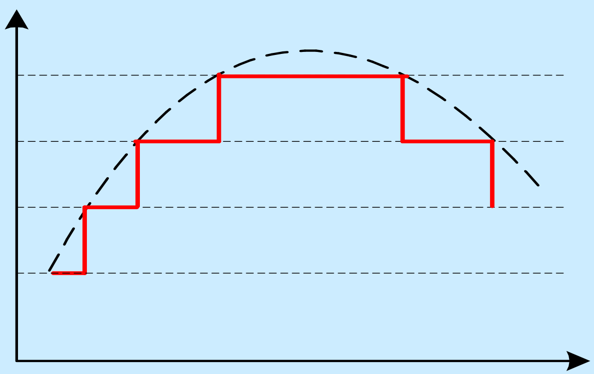
3. 离散时间信号和数字信号



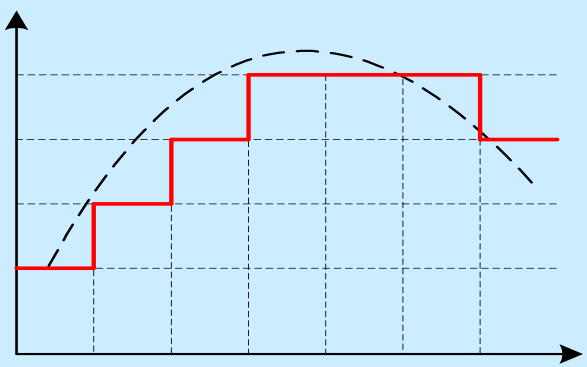
连续信号



离散时间信号



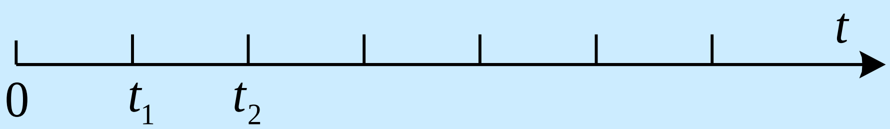
量化信号



离散时间量化信号

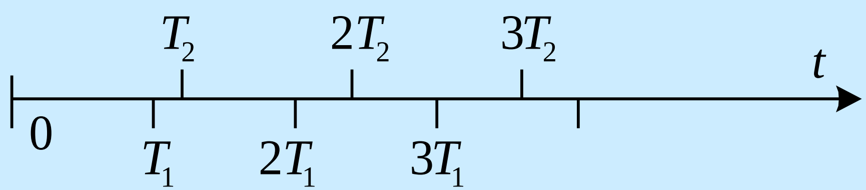
4. 采样类型

(1) 周期采样



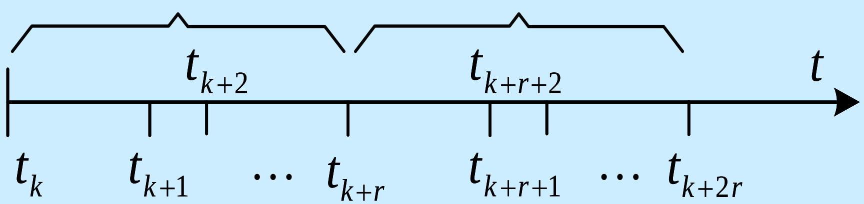
$$t_k = kT \qquad k = 0,1,2,\cdots$$

(2) 多速采样



$$t_k = \begin{cases} pT_1 \\ qT_2 \end{cases} \qquad p, q = 0,1,2,\cdots$$

(3) 多阶采样



$t_{k+1} - t_k \neq t_k - t_{k-1}$，

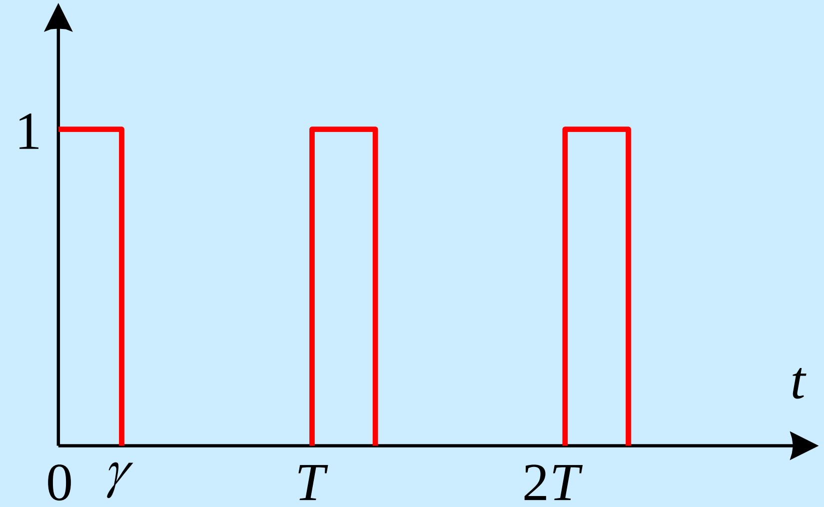
$t_{k+r} - t_k =$ 常值

(4) 随机采样

8.2 采样与保持

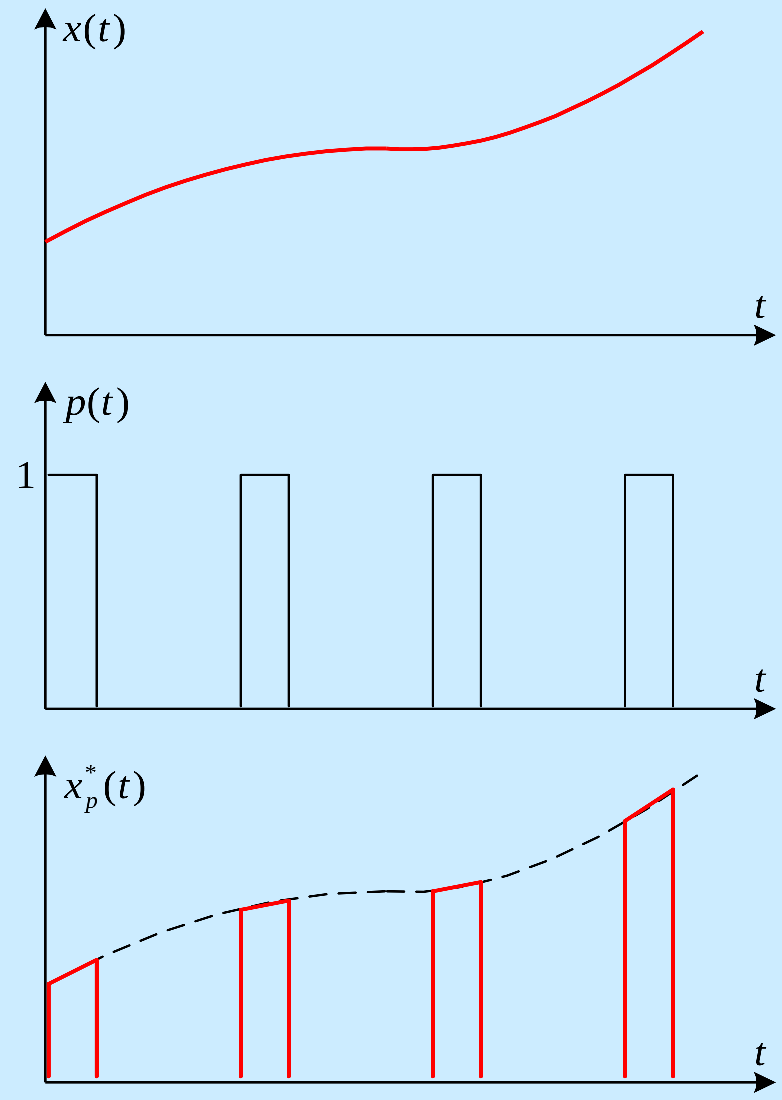
8.2.1 脉冲采样

♦ **采样器:**
单位阶跃响应的开关



- **采样间隔** T = 常数
- **采样时刻** $t_k = kT$
- $\gamma << T$

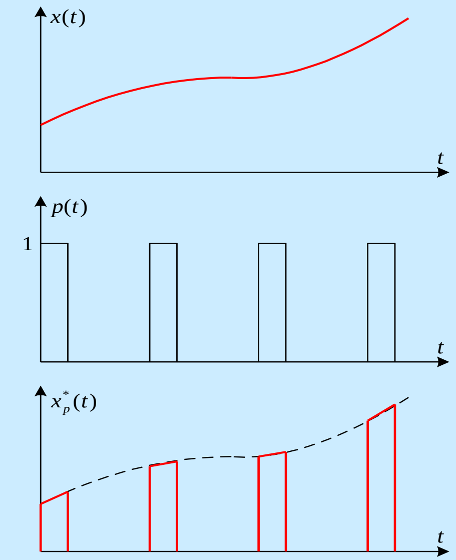
♦ 采样后的信号:



• 采样操作即调幅过程

- $p(t)$：脉冲阵列
- $p(t)$的Fourier级数

$$p(t)=\sum_{k=-\infty}^{\infty} C_k \exp(\mathrm{j}k\omega_{\mathrm{s}}t) \qquad \omega_{\mathrm{s}}=2\pi/T$$



* C_k 的计算很复杂

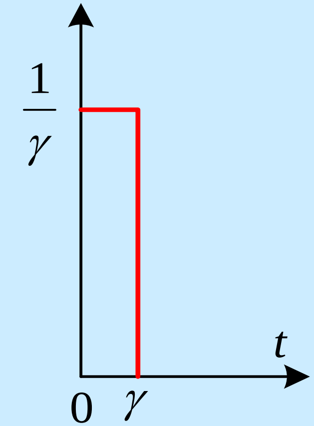
$$C_k=\frac{\gamma}{T}\times\frac{\sin(k\gamma\pi/T)}{k\gamma\pi/T}\times e^{-jk\gamma\pi/T}$$

- $x_{\mathrm{p}}^*(t)=x(t)\cdot p(t) \ =\sum_{k=-\infty}^{\infty} C_k x(t)\exp(\mathrm{j}k\omega_{\mathrm{s}}t)$

* $x_{\mathrm{p}}{}^*(t)$ 的计算也很复杂

8.2.2 冲激采样 (理想采样)

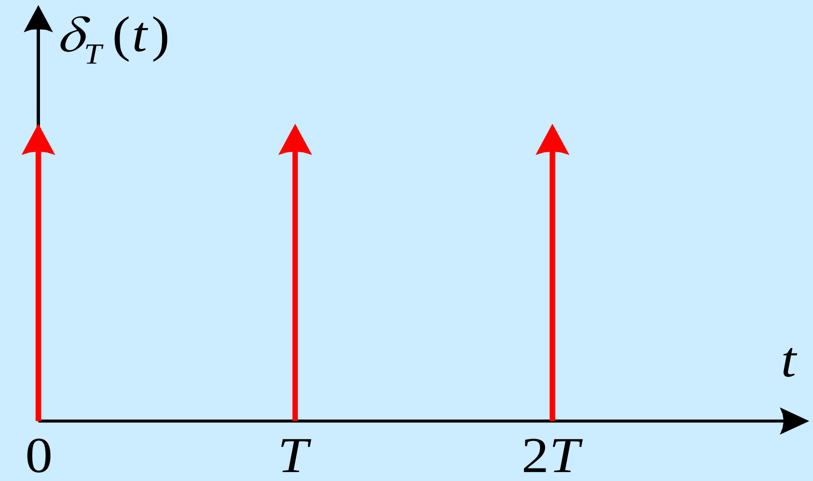
♦ 理想采样器



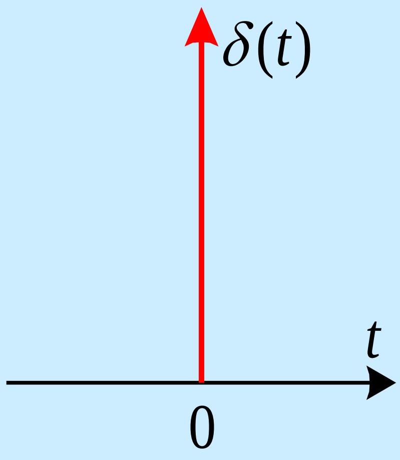
- 脉冲高度为 $1/\gamma$
- $\gamma \to 0 \;\Rightarrow$ 理想采样器

♦ 冲激阵列

- $\delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$



- **Dirac δ-函数:** $\delta(t)$



(i) $\delta(0) = \infty$

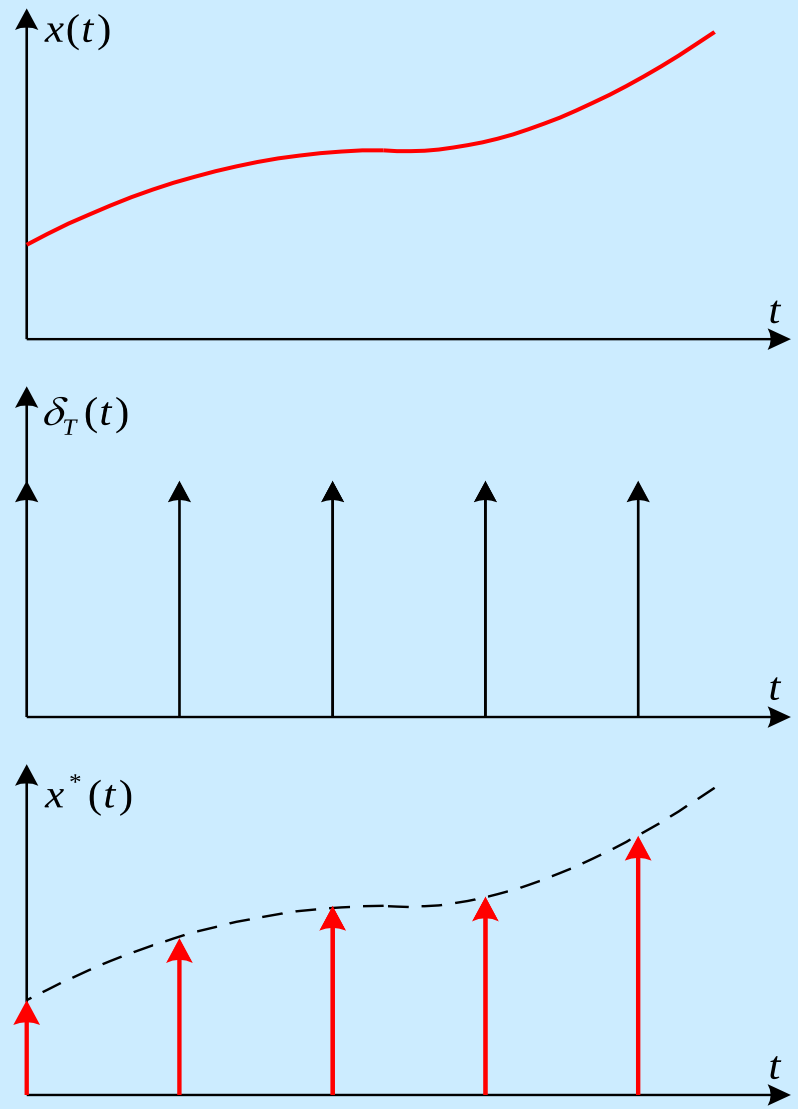
(ii) $\delta(t) = 0$ **for all** $t \neq 0$

(iii) $\int_{-\infty}^{\infty} \delta(t)\,\mathrm{d}t = 1$

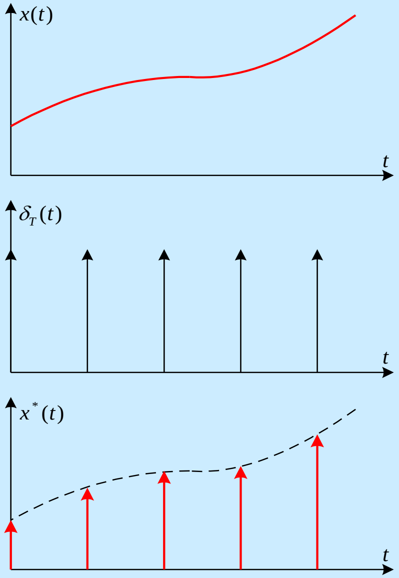
i.e. 冲激强度为1

(iv) $\int_{-\infty}^{\infty} \delta(t)\varphi(t)\,\mathrm{d}t = \varphi(0)$ **对任意函数** $\varphi(t)$

♦ 采样后的信号



$$\delta_T(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$$



- $x^*(t) = x(t) \cdot \delta_T(t)$

$$= \sum_{n=-\infty}^{\infty} C_n \cdot \delta(t - nT)$$

- $C_n = \int_{nT^-}^{nT^+} x(t)\delta(t - nT)\,\mathrm{d}t$

$$= x(nT) = x^*(nT)$$

- **$x^*(t)$的Fourier级数**

$$\delta_T(t) = \frac{1}{T}\sum_{k=-\infty}^{\infty} \mathrm{e}^{\mathrm{j}k\omega_{\mathrm{s}}t} \qquad x^*(t) = \frac{1}{T}\sum_{k=-\infty}^{\infty} x(t)\,\mathrm{e}^{\mathrm{j}k\omega_{\mathrm{s}}t}$$

$$x^*(t) = \frac{1}{T}\sum_{k=-\infty}^{\infty} x(t)\mathrm{e}^{\mathrm{j}k\omega_s t}$$

- $x^*(t)$**的频谱**

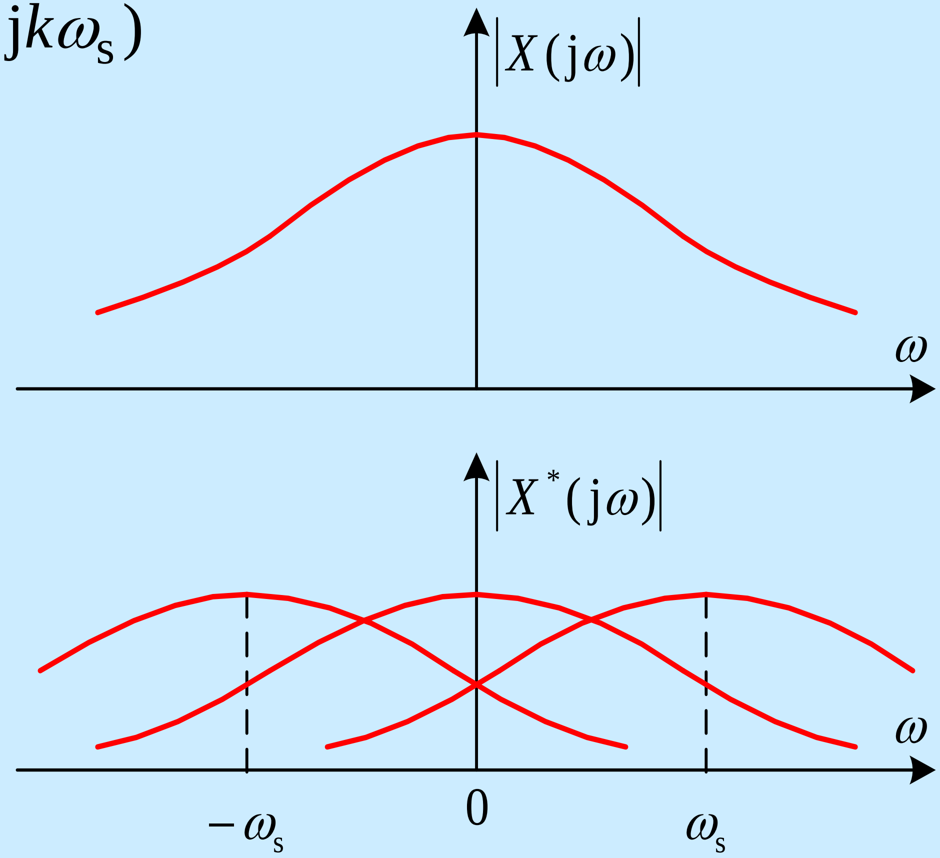
采用Laplace变换

$$X^*(s) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X(s-\mathrm{j}k\omega_s)$$

令 $s = \mathrm{j}\omega$ **，则**

$$X^*(\mathrm{j}\omega) =$$

$$\frac{1}{T}\sum_{k=-\infty}^{\infty} X[\mathrm{j}(\omega - k\omega_s)]$$

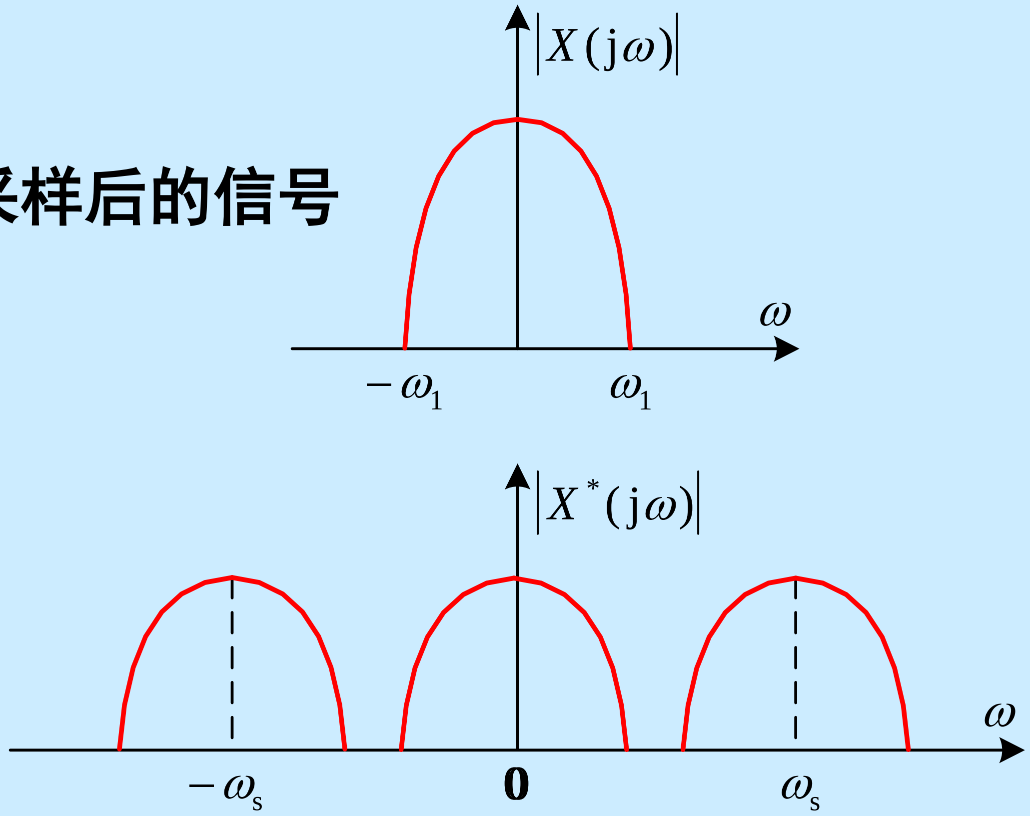


8.2.3 Shannon采样定理

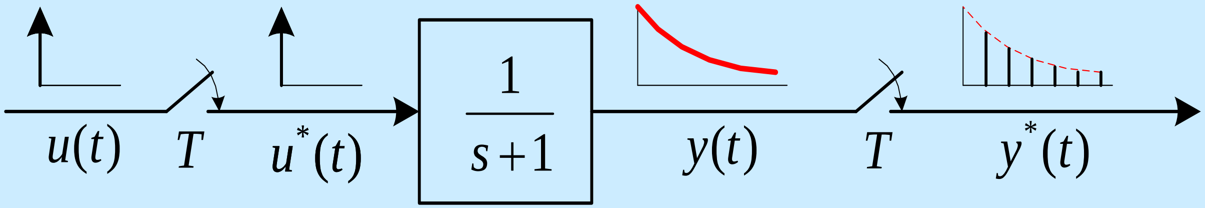
令 $2\omega_1$ 为连续信号$x(t)$的频谱宽度

- 若 $\omega_s \geq 2\omega_1$，

  则信号 $x(t)$ 可利用采样后的信号 $x^*(t)$完全重构



例 8.2.1 确定如下系统采样输入到采样输出之间的传递函数



Solution:

$$u^*(t) = \delta(t) \qquad U^*(s) = 1$$

$$y(t) = \mathrm{e}^{-t} \qquad y^*(t) = \delta(t) + \mathrm{e}^{-T}\delta(t-T) + \mathrm{e}^{-2T}\delta(t-2T) + \cdots$$

$$Y^*(s) = 1 + \mathrm{e}^{-T}\mathrm{e}^{-Ts} + \mathrm{e}^{-2T}\mathrm{e}^{-2Ts} + \cdots = \frac{1}{1-\mathrm{e}^{-T(s+1)}}$$

$$G^*(s) = \frac{Y^*(s)}{U^*(s)} = \frac{1}{1-\mathrm{e}^{-T(s+1)}}$$

- 采样系统的分析与设计看起来似乎并不复杂

  * *s*-域的传递函数

$$\boxed{\frac{1}{1-e^{-T(s+1)}}}$$

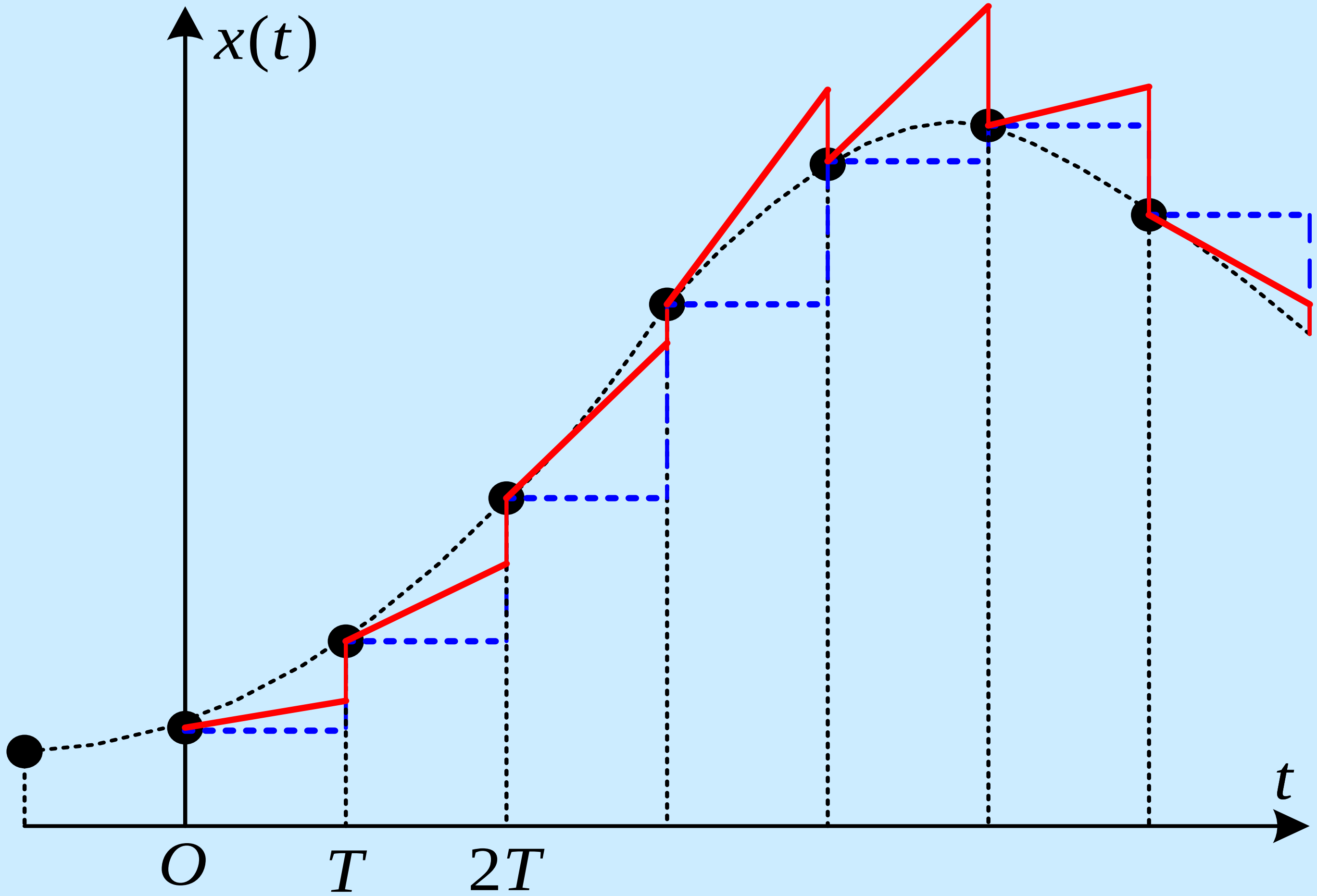
- 请看看传递函数的形式

* 超越函数

* 不便于分析和设计

* 需寻求新的分析工具

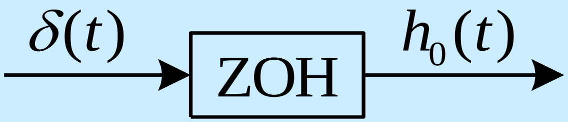
8.2.4 保持器 (Holder)

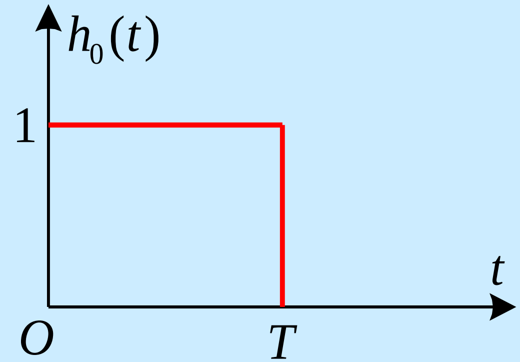


零阶保持

一阶保持

♦ **零阶保持器Zero-order hold:**





- $h_0(t) = 1(t) - 1(t-T)$

- **ZOH的传递函数:** $$H_0(s) = \frac{1-\mathrm{e}^{-Ts}}{s}$$

♦ **ZOH的频率响应:**

令 $s = \mathrm{j}\omega$， $\omega_s = \dfrac{2\pi}{T}$ **，则**

$$H_0(\mathrm{j}\omega) = \frac{1-\mathrm{e}^{-\mathrm{j}\omega T}}{\mathrm{j}\omega} = \frac{2}{\omega}\cdot \mathrm{e}^{-\frac{\mathrm{j}\omega T}{2}} \cdot \frac{\mathrm{e}^{\frac{\mathrm{j}\omega T}{2}} - \mathrm{e}^{-\frac{\mathrm{j}\omega T}{2}}}{2\mathrm{j}} = \frac{2}{\omega}\cdot \sin\frac{\omega T}{2}\cdot \mathrm{e}^{-\frac{\mathrm{j}\omega T}{2}}$$

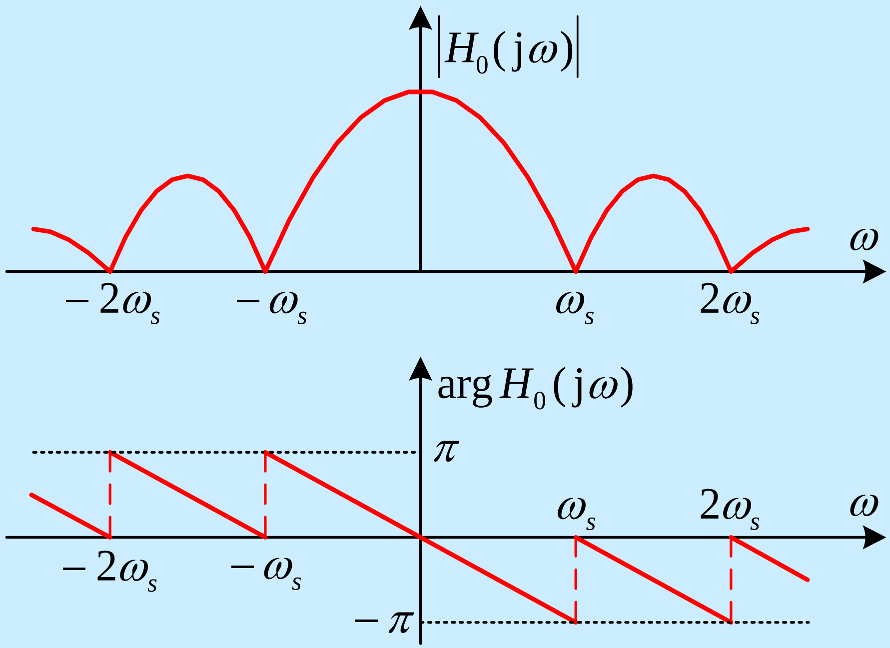
$$\boxed{T=\frac{2\pi}{\omega_{\mathrm{s}}}}$$

$$\boxed{H_0(\mathrm{j}\omega)=\frac{2}{\omega}\cdot\sin\frac{\omega T}{2}\cdot\mathrm{e}^{-\frac{\mathrm{j}\omega T}{2}}}$$

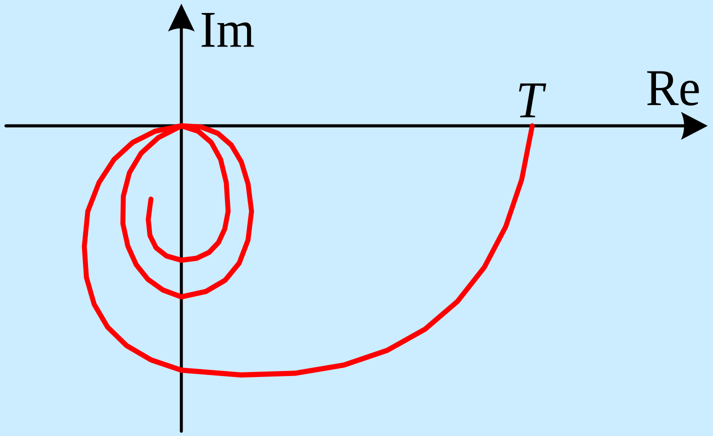
$$=\frac{2}{\omega}\cdot\sin\frac{\omega\pi}{\omega_{\mathrm{s}}}\cdot\mathrm{e}^{-\mathrm{j}\frac{\omega\pi}{\omega_{\mathrm{s}}}}=\frac{2\pi}{\omega_{\mathrm{s}}}\cdot\frac{\sin\frac{\omega\pi}{\omega_{\mathrm{s}}}}{\frac{\omega\pi}{\omega_{\mathrm{s}}}}\cdot\mathrm{e}^{-\mathrm{j}\frac{\omega\pi}{\omega_{\mathrm{s}}}}$$

$$H_0(\mathrm{j}\omega)=\frac{2\pi}{\omega_s}\cdot\frac{\sin\frac{\omega\pi}{\omega_s}}{\frac{\omega\pi}{\omega_s}}\cdot\mathrm{e}^{-\mathrm{j}\frac{\omega\pi}{\omega_s}}$$

- **Bode图**



- **Nyquist图**



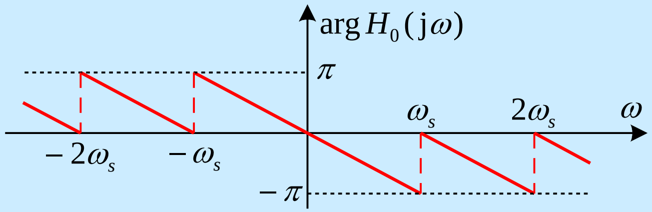
♦ ZOH的特点

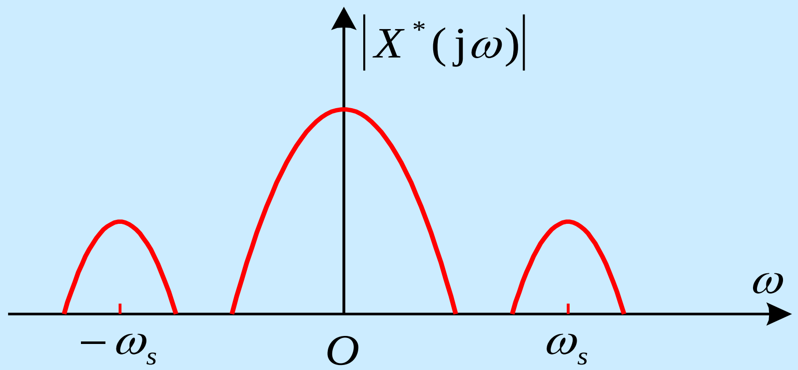
- **仅利用最近的一个数据**
- **当** $\omega = k\omega_s$

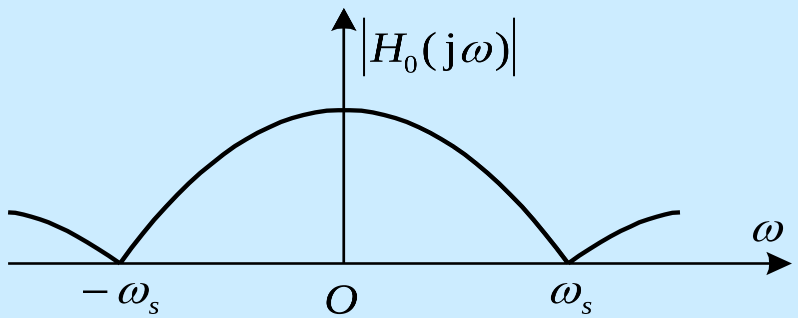
$$|H_0(\mathrm{j}\omega)| = 0$$

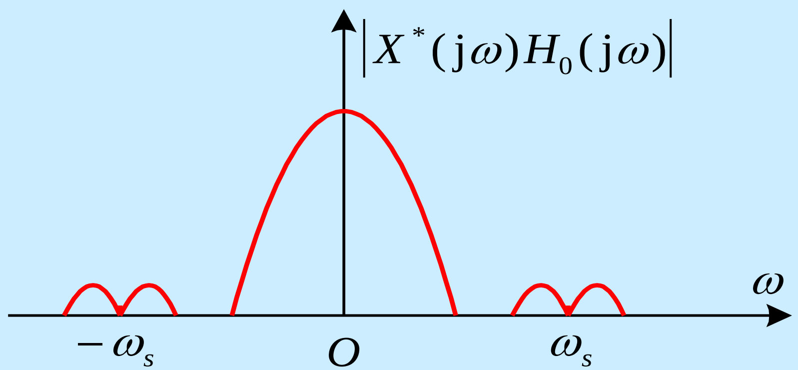
$$X^*(\mathrm{j}\omega)H_0(\mathrm{j}\omega) \approx X(\mathrm{j}\omega)$$

- **相角滞后小于** π







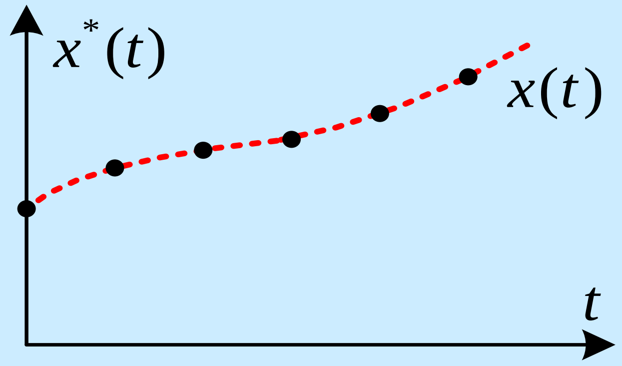


8.3 z-变换法

8.3.1 z-变换

1. 定义

♦ 采样后的信号



- 单边信号 $x(t)$

$$(x(t)=0 \quad \text{for} \quad t<0)$$

- 采样后的信号:

$$x^*(t)=x(t)\delta_T(t) = \sum_{n=0}^{\infty} x(kT)\delta(t-kT)$$

$$x^*(t) = \sum_{n=0}^{\infty} x(kT)\delta(t-kT)$$

- **时间函数的z-变换**

$$X(z) = \mathcal{Z}\{x^*(t)\}$$

$$= x(0) + x(T)z^{-1} + x(2T)z^{-2} + \cdots + x(kT)z^{-k} + \cdots$$

$$= \sum_{k=0}^{\infty} x(kT)z^{-k}$$

$$\begin{tikzcd}
x(t) \arrow[r] \arrow[dr, bend right, "\mathcal{Z}"] & x^*(t) \arrow[d, "\mathcal{Z}"'] \\
 & X(z)
\end{tikzcd}$$

- **对于离散时间信号** $x(k)$

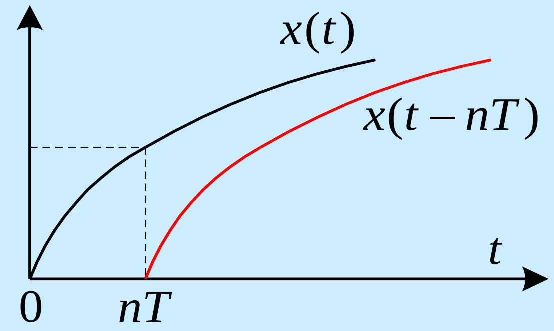
$$X(z) = \mathcal{Z}\{x(k)\} = \sum_{k=0}^{\infty} x(k)z^{-k}$$

- **常用函数可查阅z-变换表**

8.3.2 *z*-变换的性质

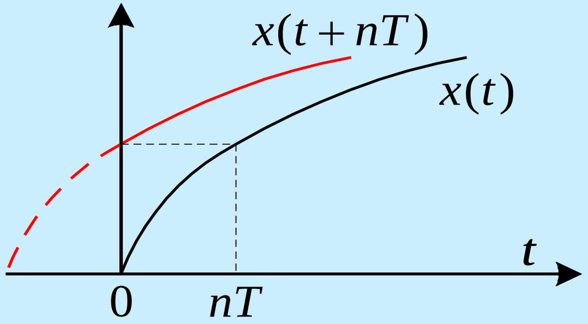
(1) 平移定理

$$\mathcal{Z}\{x(t-nT)\}=z^{-n}X(z)$$



$$\mathcal{Z}\{x(t+nT)\}=$$

$$z^{n}X(z)-\sum_{i=0}^{n-1}x(iT)z^{n-i}$$



(2) 初值定理

若 $\lim\limits_{z\to\infty} X(z)$ 存在，则 $\lim\limits_{z\to\infty} X(z) = x(0)$

(3) 终值定理

若 $x(\infty)$ 存在，即 $X(z)$ 没有 $|z| \geq 1$ 的极点 $(z=1$除外$)$

则 $x(\infty) = \lim\limits_{z\to 1}\left[(z-1)X(z)\right]$

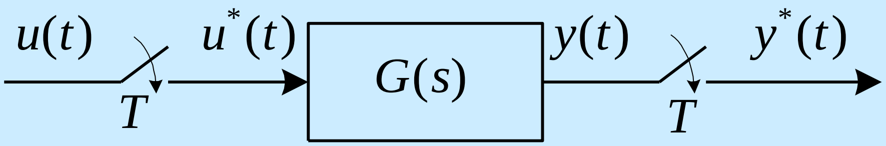
- 更多性质请参阅教材

8.3.3 *z*-传递函数

- **定义：** $G(z)=\dfrac{Y(z)}{U(z)}$

- **也称脉冲传递函数**

1. 根据输入与输出信号求*z*-传递函数



♦ **采样后的输入信号**

$$u^*(t) = u(0)\delta(t) + u(T)\delta(t-T) + u(2T)\delta(t-2T) + \cdots + u(nT)\delta(t-nT) + \cdots = \sum_{n=0}^{\infty} u(nT)\delta(t-nT)$$

- $u^*(t)$**的z-变换**

$$U(z) = \sum_{n=0}^{\infty} u(nT) z^{-n}$$

- $u^*(t)$**的Laplace变换**

$$U^*(s) = \sum_{n=0}^{\infty} u(nT) \mathrm{e}^{-nTs}$$

- **可见** $z = \mathrm{e}^{Ts}$

♦ 采样后的输出信号

$$y^*(t) = y(0)\delta(t) + y(T)\delta(t-T) + y(2T)\delta(t-2T) + \cdots$$
$$+ y(kT)\delta(t-kT) + \cdots$$
$$= \sum_{k=0}^{\infty} y(kT)\delta(t-kT)$$

$$y^*(t)=\sum_{k=0}^{\infty} y(kT)\delta(t-kT)$$

- **卷积和 (Convolution summation)**

  当 $t=kT$

$$y(kT)=g(t)u(0)+g(t-T)u(T)+$$
$$g(t-2T)u(2T)+\cdots+g(t-kT)u(kT)$$
$$=\sum_{n=0}^{k} g(t-nT)u(nT)=\sum_{n=0}^{k} g(kT-nT)u(nT)$$
$$\underset{(n=k-m)}{\overset{m=k-n}{=}}\sum_{m=0}^{k} g(mT)u(kT-mT)=u(kT)*g(kT)$$

- $y^*(t)$**的z-变换**

$$Y(z)=\sum_{k=0}^{\infty} y(kT)z^{-k}=\sum_{k=0}^{\infty}\sum_{n=0}^{k} g(kT-nT)u(nT)z^{-k}$$

$$Y(z)=\sum_{k=0}^{\infty}\sum_{n=0}^{k} g(kT-nT)u(nT)z^{-k}$$

* 由于

$$\sum_{n=0}^{\infty} g(kT-nT)u(nT)$$

$$= g(kT)u(0)+g(kT-T)u(T)+\cdots+g(T)u(kT-T)$$

$$+g(0)u(kT)+g(-T)u(kT+T)+\cdots$$

$$= g(kT)u(0)+g(kT-T)u(T)+\cdots+g(T)u(kT-T)$$

$$+g(0)u(kT)$$

$$=\sum_{n=0}^{k} g(kT-nT)u(nT)= y(kT)$$

$$Y(z) = \sum_{k=0}^{\infty} \sum_{n=0}^{k} g(kT - nT) u(nT) z^{-k}$$

$$\sum_{n=0}^{k} g(kT - nT) u(nT) = \sum_{n=0}^{\infty} g(kT - nT) u(nT)$$

$$G(z) = \frac{Y(z)}{U(z)}$$

*** 因此**

$$Y(z) = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} g(kT - nT) u(nT) z^{-k}$$

$$\overset{m=k-n}{=} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} g(mT) u(nT) z^{-(n+m)}$$

$$\underset{n\,\&\,m}{\overset{\text{independent}}{=}} \sum_{m=0}^{\infty} g(mT) z^{-m} \sum_{n=0}^{\infty} u(nT) z^{-n}$$

$$= \sum_{m=0}^{\infty} g(mT) z^{-m} \cdot U(z) = G(z) \cdot U(z)$$

$$\sum_{m=0}^{\infty} g(mT)z^{-m} \cdot U(z) = G(z)U(z)$$

- *z*-传递函数

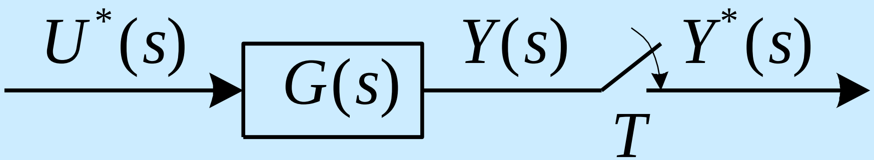
$$G(z) = \sum_{m=0}^{\infty} g(mT)z^{-m}$$

- 根据定义求z-传递函数的流程

$$G(s) \xrightarrow{\mathcal{L}^{-1}} g(t) \longrightarrow g^{*}(t) \xrightarrow{\mathcal{Z}} G(z)$$

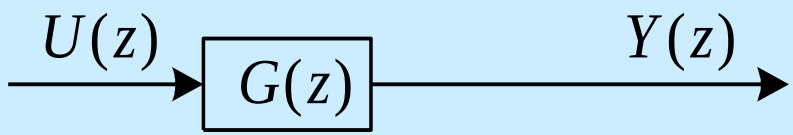
2. 根据框图变换求z-传递函数

- 对于如下框图



- $Y(s) = G(s) \cdot U^*(s)$ $\qquad\qquad$ $Y(z) = \mathcal{Z}\left[G(s) \cdot U^*(s)\right]$

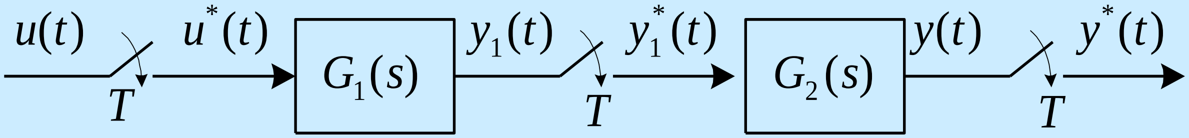
- z-变换表达式



$$Y(z) = G(z) \cdot U(z)$$

- $\mathcal{Z}\left[G(s) \cdot U^*(s)\right] = G(z) \cdot U(z)$

例 8.3.1 确定如下系统的脉冲传递函数



Solution:

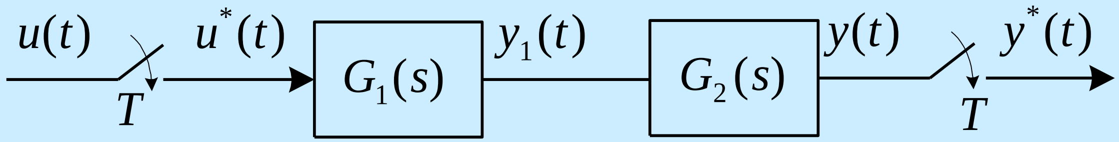
- $Y_1(s) = G_1(s)U^*(s) \quad \Rightarrow \quad Y_1(z) = G_1(z)U(z)$
- $Y(s) = G_2(s)Y_1^*(s) \quad \Rightarrow \quad Y(z) = G_2(z)Y_1(z)$

$$Y(z) = G_2(z)G_1(z) \cdot U(z)$$
$$= G(z) \cdot U(z)$$

- $G(z) = G_2(z)G_1(z)$

$\blacksquare$

例 8.3.2 确定如下系统的脉冲传递函数



Solution:

- **令** $G(s) = G_1(s)G_2(s)$ **，则**

$$Y(s) = G_2(s)G_1(s)U^*(s) = G(s)\cdot U^*(s)$$

$$Y(z) = G(z)\cdot U(z)$$

- **其中** $$G(z) = \sum_{m=0}^{\infty} g(mT)z^{-m}$$

$$g(t) = \mathcal{L}^{-1}[G_1(s)G_2(s)]$$

- **N.B: 一般而言** $\mathcal{Z}\left[G_1(s)G_2(s)\right] \neq G_1(z)G_2(z)$

$$G_1(s)=\frac{1}{s} \qquad G_1(z)=\frac{z}{z-1}$$

$$G_2(s)=\frac{1}{s+1} \qquad G_2(z)=\frac{z}{z-\mathrm{e}^{-T}} \qquad G_1(z)G_2(z)=\frac{z^2}{(z-1)(z-\mathrm{e}^{-T})}$$

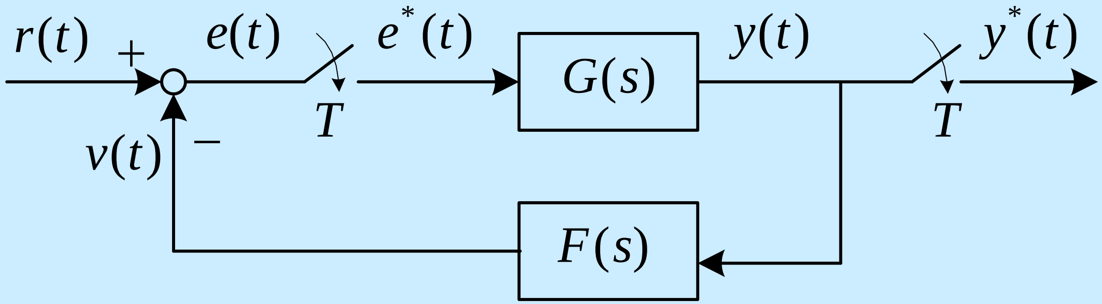
$$G(s)=G_1(s)G_2(s)=\frac{1}{s(s+1)}$$

$$\mathcal{Z}\left[G_1(s)G_2(s)\right]=\frac{(1-\mathrm{e}^{-T})z}{(z-1)(z-\mathrm{e}^{-T})}$$

- **记** $\mathcal{Z}\left[G_1(s)G_2(s)\right]=G_1G_2(z)$

■

例 8.3.3 确定如下系统的脉冲传递函数



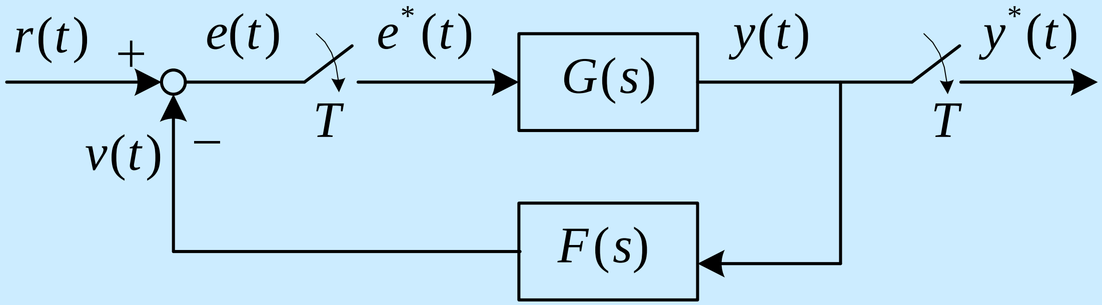
Solution:

- $e(t) = r(t) - v(t) \quad \Rightarrow \quad E(z) = R(z) - V(z)$
- $V(s) = F(s)Y(s) = F(s)G(s)\cdot E^*(s)$

  $V(z) = FG(z)\cdot E(z)$
- $E(z) = R(z) - FG(z)\cdot E(z)$

$$E(z) = \frac{R(z)}{1 + FG(z)}$$

$$E(z)=\frac{R(z)}{1+FG(z)}$$

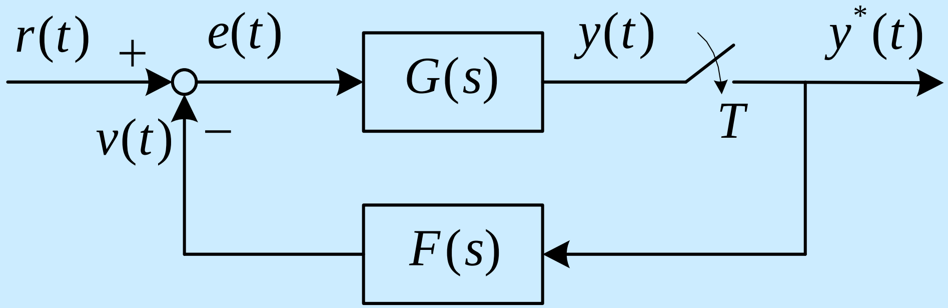


- $Y(s)=G(s)\cdot E^*(s)$
- $Y(z)=G(z)\cdot E(z)=G(z)\cdot\dfrac{R(z)}{1+FG(z)}$
- **闭环脉冲传递函数:**

$$\frac{Y(z)}{R(z)}=\frac{G(z)}{1+FG(z)}$$

■

例 8.3.4 确定如下系统的脉冲传递函数



Solution:

- $Y(s) = G(s)E(s) = G(s)\left[R(s) - V(s)\right]$

  $= G(s)R(s) - G(s)F(s)\cdot Y^*(s)$

- $Y(z) = GR(z) - GF(z)\cdot Y(z)$

  $$Y(z) = \frac{GR(z)}{1+GF(z)}$$

- $R(z)$ 到 $Y(z)$ 没有脉冲传递函数

■

8.3.4 计算z-传递函数的一般流程

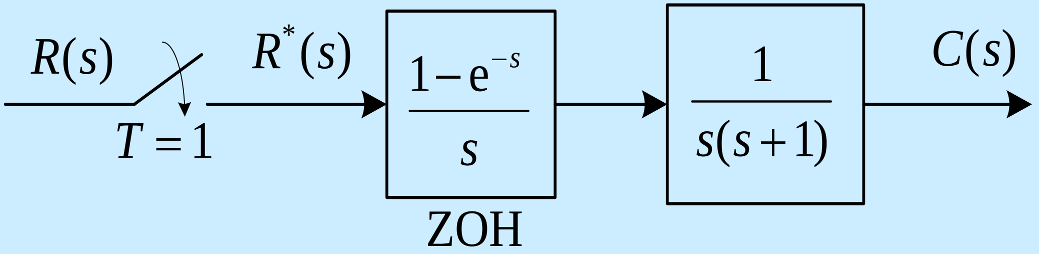
(1) 确定两个采样器之间的 $G(s)$

- **如果存在 ZOH, 则** $G(s) = G_p(s)H_0(s)$
- $G(s)$ **须包括所有串联的传递函数**

(2) 计算 $g(t) = \mathcal{L}^{-1}[G(s)]$

(3) 计算 $G(z) = \sum_{k=0}^{\infty} g(kT)z^{-k}$

例 8.3.5 确定如下系统的脉冲传递函数

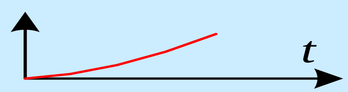


Solution:

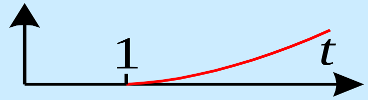
(i) 方法 A: 根据定义

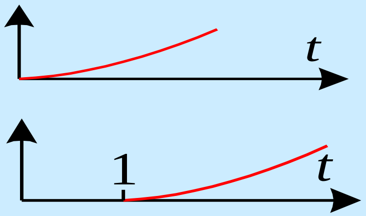
- $$G(s)=\frac{C(s)}{R^*(s)}=\frac{1-e^{-s}}{s^2(s+1)}=(1-e^{-s})\cdot\left[\frac{1}{s^2}-\frac{1}{s}+\frac{1}{s+1}\right]$$

- $$g(t)=\mathcal{L}^{-1}[G(s)]=\left[t-1+e^{-t}\right]\cdot 1(t)$$



$$-\left[(t-1)-1+e^{-(t-1)}\right]\cdot 1(t-1)$$





- $g(0)=0$
- **对于** $k = 1, 2, \cdots$

$$g(k)=\left[k-1+\mathrm{e}^{-k}\right]-\left[k-1-1+\mathrm{e}^{-(k-1)}\right]$$

$$=1+\mathrm{e}^{-k}-\mathrm{e}^{-(k-1)}$$

- $G(z)=\sum_{k=0}^{\infty} g(k)z^{-k}=\sum_{k=1}^{\infty} g(k)z^{-k}$

$$=\sum_{k=1}^{\infty}\left[1+\mathrm{e}^{-k}-\mathrm{e}^{-(k-1)}\right]z^{-k}$$

$$=\sum_{k=1}^{\infty}\left[z^{-k}+(1-\mathrm{e})(\mathrm{e}z)^{-k}\right]$$

$$= \frac{\frac{1}{z}}{1-\frac{1}{z}} + \frac{(1-\mathrm{e})\frac{1}{\mathrm{e}z}}{1-\frac{1}{\mathrm{e}z}}$$

$$= \frac{1}{z-1} + \frac{\mathrm{e}^{-1}-1}{z-\mathrm{e}^{-1}} \quad = \frac{\mathrm{e}^{-1}z+1-2\mathrm{e}^{-1}}{z^2-(1+\mathrm{e}^{-1})z+\mathrm{e}^{-1}}$$

$$= \frac{0.368z+0.264}{z^2-1.368z+0.368}$$

$$\boxed{G(z) = \sum_{k=1}^{\infty} \left[z^{-k} + (1-\mathrm{e})(\mathrm{e}z)^{-k} \right]}$$

$$G(s)=(1-\mathrm{e}^{-s})\left[\frac{1}{s^2}-\frac{1}{s}+\frac{1}{s+1}\right]$$

(ii) 方法 B: 根据z-变换表

- 若 $G(s)=\left(1-\mathrm{e}^{-Ts}\right)X(s)=X(s)-\mathrm{e}^{-Ts}X(s)$

  则 $g(t)=x(t)-x(t-T)\cdot 1(t-T)$

- 根据叠加和平移定理知

$$G(z)=X(z)-z^{-1}X(z)=\left(1-z^{-1}\right)X(z)$$

- 根据对应关系

$$G(s)=\left(1-\mathrm{e}^{-Ts}\right)X(s)$$

$$G(z)=\left(1-z^{-1}\right)X(z)$$

可得到计算z-传递函数的如下步骤

- **带零阶保持器的传递函数**

$$G(s)=(1-\mathrm{e}^{-s})\cdot\frac{G_{\mathrm{p}}(s)}{s}$$

$$X(s)=\frac{G_{\mathrm{p}}(s)}{s}=\frac{1}{s^2}-\frac{1}{s}+\frac{1}{s+1}$$

- **根据 *z*- 变换表**

$$X(z)=\mathcal{Z}\left[X(s)\right]=\mathcal{Z}\left[\frac{G_{\mathrm{p}}(s)}{s}\right]=\frac{z}{(z-1)^2}-\frac{z}{z-1}+\frac{z}{z-\mathrm{e}^{-1}}$$

$$=\frac{z(2-z)}{(z-1)^2}+\frac{z}{z-\mathrm{e}^{-1}}$$

$$X(z)=\frac{z(2-z)}{(z-1)^2}+\frac{z}{z-e^{-1}}$$

- $G(z)=(1-z^{-1})X(z)$

$$=\frac{z-1}{z}\cdot\left[\frac{z(2-z)}{(z-1)^2}+\frac{z}{z-e^{-1}}\right]$$

$$=\frac{2-z}{z-1}+\frac{z-1}{z-e^{-1}}$$

$$=\frac{e^{-1}z-2e^{-1}+1}{z^2-(1+e^{-1})z+e^{-1}} \qquad \blacksquare$$

8.4 采样系统分析

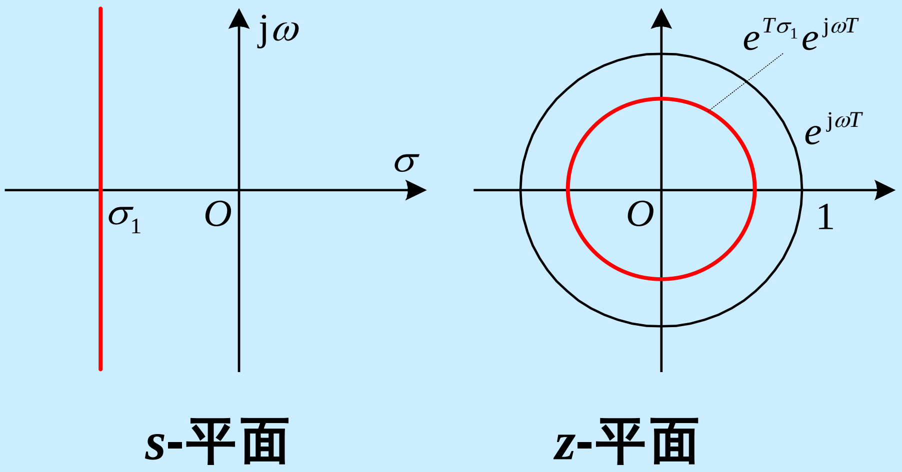
8.4.1 *z*-平面上的稳定性分析

♦ ***z*-平面上的稳定域**

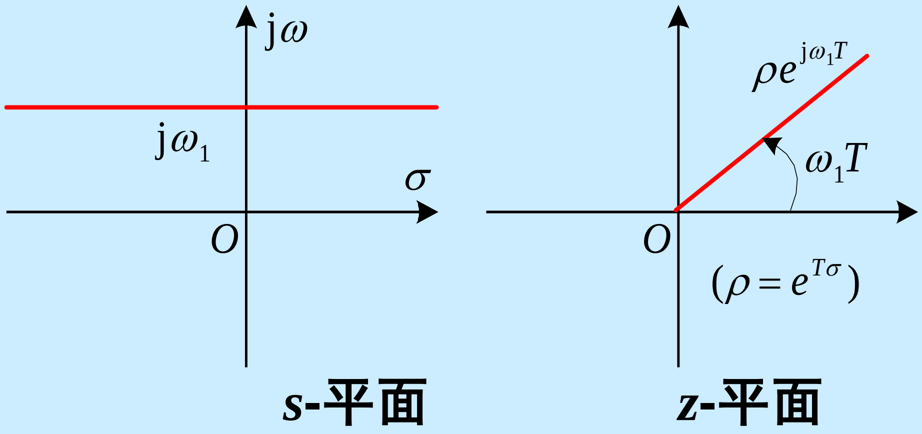
◊ ***z* 和 *s* 之间的映射**

- $z = \mathrm{e}^{Ts}$
- $s = \sigma + \mathrm{j}\omega \quad \Rightarrow \quad z = \mathrm{e}^{T(\sigma+\mathrm{j}\omega)} = \mathrm{e}^{T\sigma}\mathrm{e}^{\mathrm{j}T\omega}$

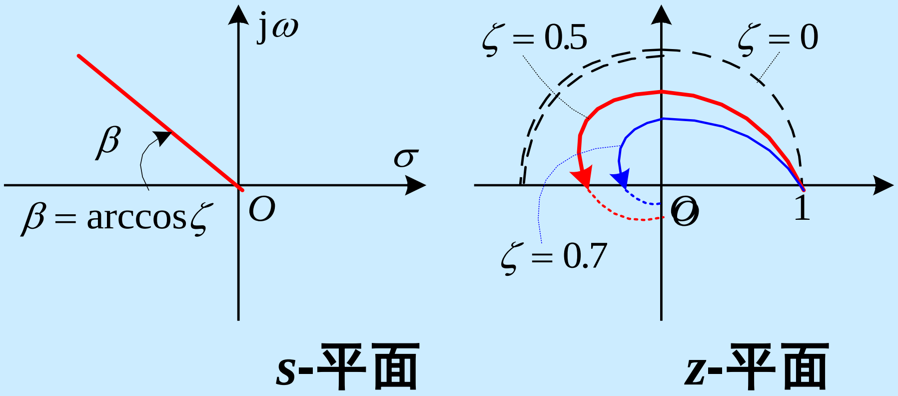
- **定常 σ**



- **定常** ω



- **定常** ζ



◊ *z*-平面上的稳定域

$$z = \mathrm{e}^{Ts} \overset{s=\sigma+\mathrm{j}\omega}{=} \mathrm{e}^{T\sigma}\mathrm{e}^{\mathrm{j}T\omega} \qquad \mathrm{Re}\{s\}<0 \quad \Leftrightarrow \quad |z|<1$$

♦ 渐近稳定条件:

- OL *z*-传递函数: $GH(z) = \dfrac{K\psi(z)}{\varphi_{\mathrm{o}}(z)}$
- CL 特征方程: $\varphi_{\mathrm{c}}(z) = \varphi_{\mathrm{o}}(z) + K\psi(z)$

◊ 稳定条件: $\varphi_{\mathrm{c}}(z) = 0$所有的根都在单位圆内

1. Jury稳定性判据 (代数准则)

- **CL 特征多项式**

$$\varphi_c(z) = z^n + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \cdots + a_1 z + a_0$$

- **Jury阵列**

$$
\begin{array}{lllllll}
1 & a_{n-1} & a_{n-2} & \cdots & \cdots & a_1 & a_0 \\
a_0 & a_1 & a_2 & \cdots & \cdots & a_{n-1} & 1 \\
\hline
b_{21} & b_{22} & b_{23} & \cdots & \cdots & b_{2n} & \\
c_{21} = b_{2n} & c_{22} = b_{2,n-1} & c_{23} = b_{2,n-2} & \cdots & \cdots & c_{2n} = b_{21} & \\
\hline
b_{31} & b_{32} & b_{33} & \cdots & b_{3,n-1} & & \\
c_{31} = b_{3,n-1} & c_{32} = b_{3,n-2} & c_{33} & \cdots & c_{3n} = b_{31} & & \\
\hline
\cdots & \cdots & \cdots & \cdots & & & \\
\cdots & \cdots & \cdots & \cdots & & & \\
\hline
c_{n+1,1} & & & & & &
\end{array}
$$

其中

$$b_{21} = \det\begin{bmatrix} 1 & a_{n-1} \\ a_0 & a_1 \end{bmatrix} \qquad b_{22} = \det\begin{bmatrix} 1 & a_{n-2} \\ a_0 & a_2 \end{bmatrix} \qquad \cdots$$

$$b_{31} = \det\begin{bmatrix} b_{21} & b_{22} \\ c_{21} & c_{22} \end{bmatrix} \qquad b_{32} = \det\begin{bmatrix} b_{21} & b_{23} \\ c_{21} & c_{23} \end{bmatrix} \qquad \cdots$$

…

- **稳定性的充分和必要条件:**

$$
\begin{array}{ccc}
1 & a_{n-1} & \cdots \\
a_0 & a_1 & \cdots \\
\hline
b_{21} & \cdots & b_{2n} \\
c_{21}=b_{2n} & \cdots & \\
\hline
\cdots & &
\end{array}
$$

$$\varphi_c(1)>0 \qquad (-1)^n\varphi_c(-1)>0$$

$$c_{21}>0$$

$$c_{i1}<0 \quad \textbf{for} \quad i=3,4,\cdots,n+1$$

N.B. 存在多种代数准则

E. I. Jury.
Theory and Application of the z- transform Method.
(2nd ed), Malabar, Florida: Krieger, 1982)

例 8.4.1 试确定如下闭环特征多项式表征的系统的稳定性

$$\varphi_c(z)=z^3+\frac{2}{3}z^2-\frac{1}{4}z-\frac{1}{6}$$

Solution:

(i) 书写Jury阵列

$$\begin{array}{lcccc}
 & 1 & \frac{2}{3} & -\frac{1}{4} & -\frac{1}{6} \\
 & -\frac{1}{6} & -\frac{1}{4} & \frac{2}{3} & 1 \\
\hline
b_{21}= & -\frac{5}{36} & \frac{5}{8} & \frac{35}{36} & \\
c_{21}= & \frac{35}{36} & \frac{5}{8} & -\frac{5}{36} & \\
\hline
b_{31}= & -\frac{25}{36} & -\frac{25}{27} & & \\
c_{31}= & -\frac{25}{27} & -\frac{25}{36} & & \\
\hline
\end{array}$$

$$b_{41}=-0.3750$$

$$c_{41}=-0.3750$$

$$\boxed{\varphi_c(z) = z^3 + \frac{2}{3}z^2 - \frac{1}{4}z - \frac{1}{6}}$$

$$\boxed{\varphi_c(z) = 0 \quad z = -0.667, \pm 0.5}$$

(ii) 稳定条件

$$\varphi_c(1) = \frac{15}{12} > 0$$

$$(-1)^3 \varphi_c(-1) = \frac{3}{12} > 0$$

$$c_{21} = \frac{35}{36} > 0$$

$$c_{31} < 0$$

$$c_{41} < 0$$

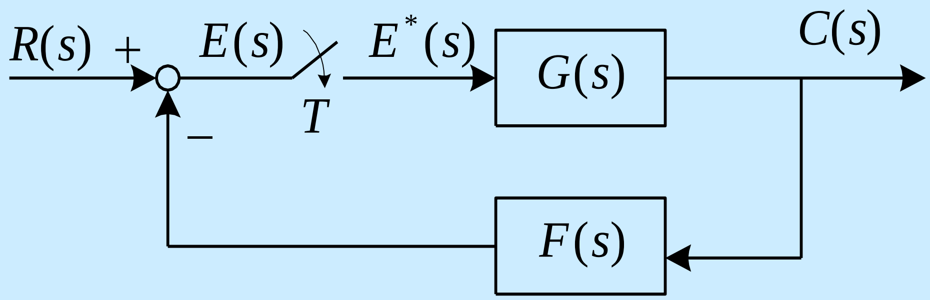
$\Rightarrow$ **系统稳定**

$$\begin{array}{lcccc}
 & 1 & \frac{2}{3} & -\frac{1}{4} & -\frac{1}{6} \\
 & -\frac{1}{6} & -\frac{1}{4} & \frac{2}{3} & 1 \\
\hline
b_{21} = & -\frac{5}{36} & \frac{5}{8} & \frac{35}{36} & \\
c_{21} = & \frac{35}{36} & \frac{5}{8} & -\frac{5}{36} & \\
\hline
b_{31} = & -\frac{25}{36} & -\frac{25}{27} & & \\
c_{31} = & -\frac{25}{27} & -\frac{25}{36} & & \\
\hline
b_{41} = & -0.3750 & & & \\
c_{41} = & -0.3750 & & &
\end{array}$$

■

2. 根轨迹法

例 8.4.2 确定如下系统的稳定性



其中 $G(s)=\dfrac{K}{s(s+a)}$

$K>0 \quad a=1 \quad F(s)=1$

Solution:

(i) OL *z*-传递函数

$$C(z)=\frac{G(z)}{1+GF(z)}R(z)$$

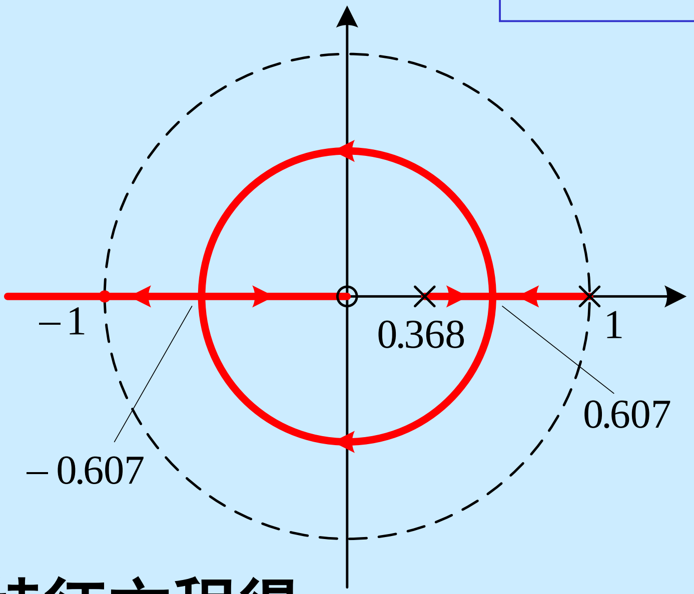
$$G(z)=GF(z)=\frac{0.632Kz}{(z-1)(z-0.368)}$$

$$G(z)=\frac{0.632Kz}{(z-1)(z-0.368)}$$

$$G(s)=\frac{K}{s(s+a)}$$

(ii) 根轨迹

$z=-1$ **是临界点**



(iii) 稳定条件

- **将** $z=-1$**代入闭环特征方程得**

$$z^2-(1.368-0.632K)z+0.368=0 \quad\Rightarrow\quad K=4.329$$

- **当** $K>4.329$**, 系统不稳定**

N.B: 连续闭环系统当 $K>0$ **时一直稳定** ■

8.4.2 双线性变换 (Bilinear Transformation)

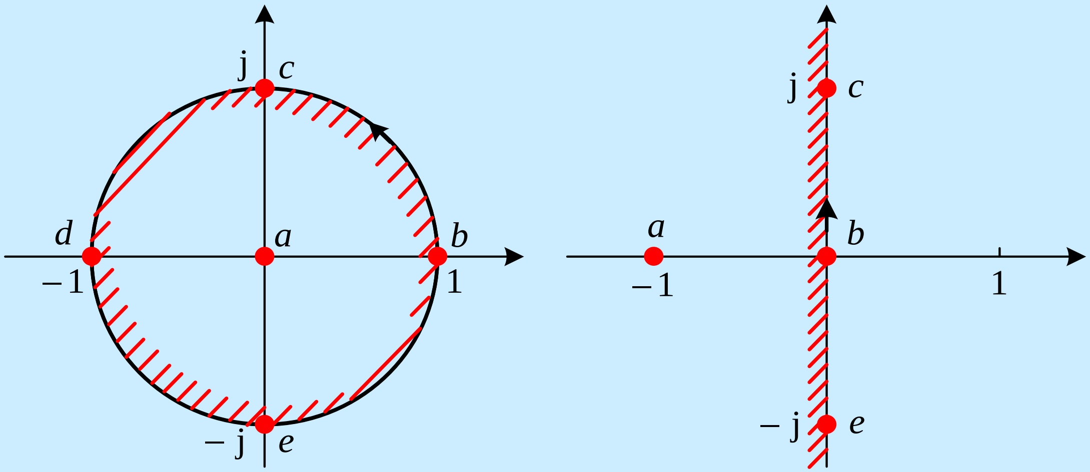
- $w = \dfrac{z-1}{z+1}$ **i.e.** $z = \dfrac{1+w}{1-w}$

- 其他方式

$$z = \frac{w+1}{w-1}$$

i.e.

$$w = \frac{z+1}{z-1}$$



***z*-plane** ***w*-plane**

◊ *z*-平面上单位圆内 ⇔ *w*-平面上左半平面

Routh准则

- $\varphi_c(z) = 0$ 的根都在z-平面上的单位圆内

$$\Leftrightarrow \quad (1-w)^n \varphi_c\left(\frac{1+w}{1-w}\right) = 0$$ 的根都在w-平面上的左半平面

例 8.4.3 考虑如下开环z-传递函数，确定闭环系统的稳定性

$$GH(z) = \frac{0.632Kz}{(z-1)(z-0.368)}$$

Solution:

$$GH(z)=\frac{0.632Kz}{(z-1)(z-0.368)}$$

Solution:

$$z=\frac{1+w}{1-w}$$

(i) CL 特征方程

- $\varphi_c(z) = z^2 - (1.368 - 0.632K) z +.368 = 0$
- **通过双线性变换**

$$(1+w)^2+(0.632K-1.368)(1+w)(1-w)+0.368(1-w)^2=0$$

$$(2.736-0.632K)w^2+1.264w+0.632K=0$$

(ii) CL 稳定条件

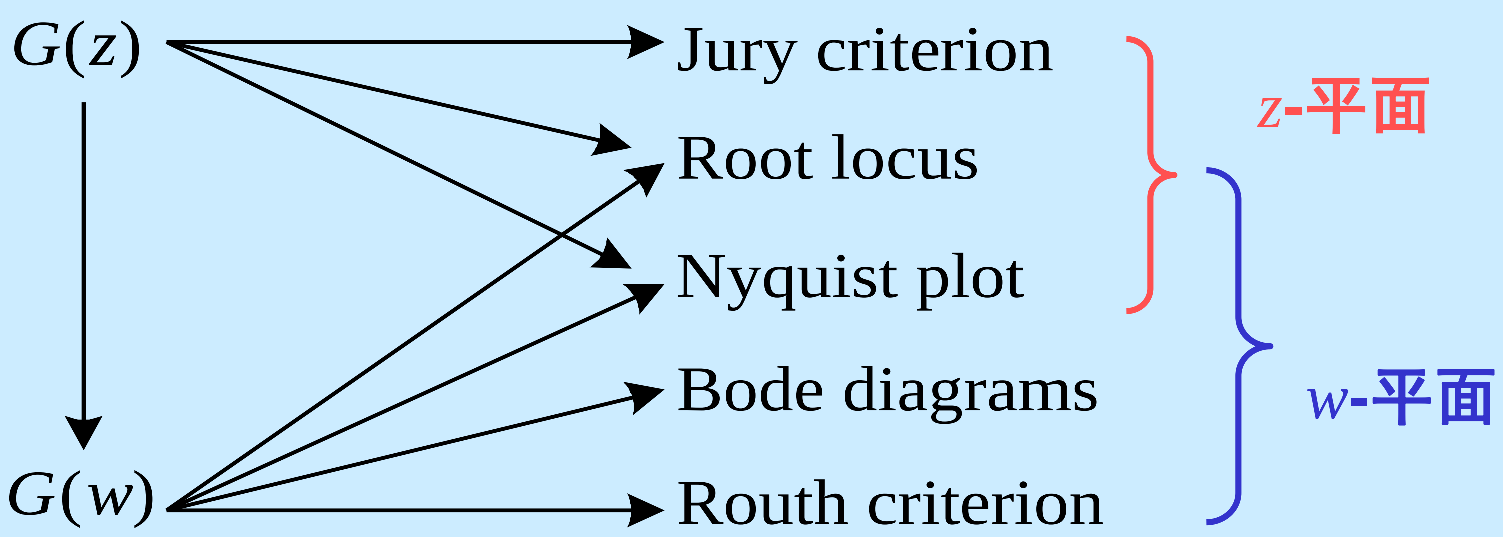
- $2.736 - 0.632K > 0$ **且** $0.632K > 0$
- **即** $0 < K < 4.329$

N.B: 通过如下双线性变换将得到相同的结果

$$z=\frac{w+1}{w-1} \quad \textbf{i.e.} \quad w=\frac{z+1}{z-1}$$

■

♦ 稳定性分析方法小结


$G(z)$
$G(w)$
Jury criterion
Root locus
Nyquist plot
Bode diagrams
Routh criterion
z-平面
w-平面

8.4.3 时间响应

1. 有限拍响应

令 $$G_{\mathrm{CL}}(z)=\frac{a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n}{z^n}$$

$$= a_0 + a_1 z^{-1} + \cdots + a_{n-1} z^{-n+1} + a_n z^{-n}$$

- **单位脉冲响应**

$$g^*(t) = a_0\delta(t) + a_1\delta(t-T) + \cdots + a_{n-1}\delta[t-(n-1)T] + a_n\delta(t-nT)$$

- **CL 响应在有限时间内达到稳态值**

例 8.4.7 考虑如下闭环z-传递函数，试确定系统的时间响应

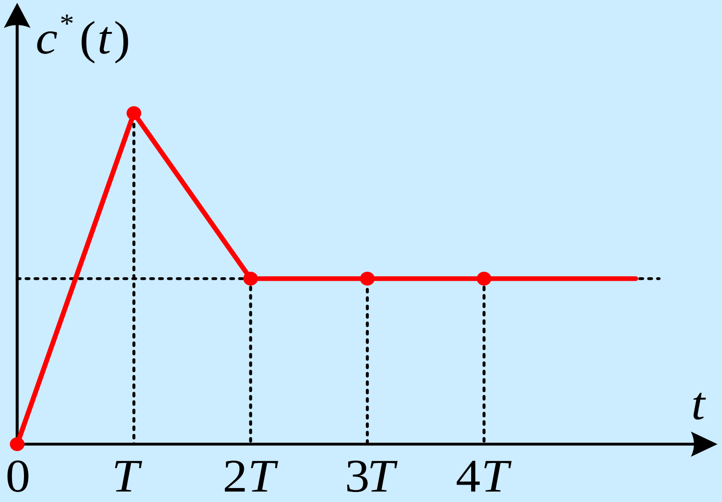
$$G_{\mathrm{CL}}(z)=\frac{2z-1}{z^2}$$

Solution:

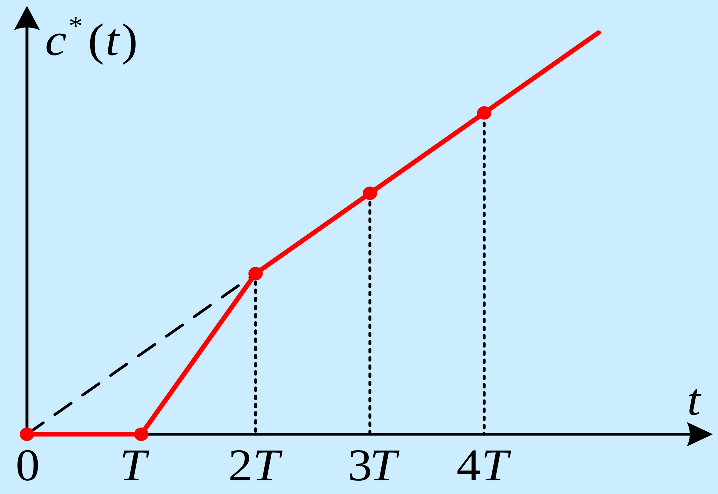
(1) 单位阶跃响应

$$R(z)=\frac{z}{z-1}$$

$$C(z)=G_{\mathrm{CL}}(z)R(z)$$
$$=\frac{2z-1}{z^2}\cdot\frac{z}{z-1}$$
$$=\frac{2z-1}{z^2-z}$$
$$=2z^{-1}+z^{-2}+z^{-3}+z^{-4}+\cdots$$



(2) 单位斜坡响应



$$R(z)=\frac{Tz}{(z-1)^2}$$

$$C(z)=G_{\mathrm{CL}}(z)R(z)$$

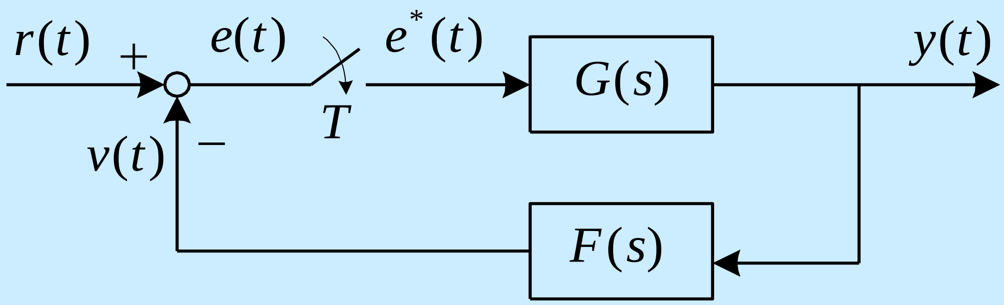
$$=\frac{2z-1}{z^2}\cdot\frac{Tz}{(z-1)^2}$$

$$=\frac{T(2z-1)}{z^3-2z^2+z}=2Tz^{-2}+3Tz^{-3}+4Tz^{-4}+\cdots$$

■

2. 稳态响应

- 考虑稳定的闭环系统



- 误差 $E(z)=\dfrac{R(z)}{1+GF(z)}$

$$E(z)=\frac{R(z)}{1+GF(z)}$$

(1) 单位阶跃输入 $r(t)=1(t)$

$$R(z)=\frac{z}{z-1} \qquad E(z)=\frac{1}{1+GF(z)}\cdot\frac{z}{z-1}$$

- **根据终值定理**

$$\lim_{t\to\infty} e(t)=\lim_{z\to 1}(z-1)E(z)=\lim_{z\to 1}\frac{z}{1+GF(z)}$$

$$=\frac{1}{1+\lim\limits_{z\to 1} GF(z)}$$

- $K_{\mathrm{p}}=\lim\limits_{z\to 1} GF(z)$ **：静态位置误差系数**

- $e_{\mathrm{ss}}=\dfrac{1}{1+K_{\mathrm{p}}}$

$$E(z)=\frac{R(z)}{1+GF(z)}$$

(2) 斜坡输入 $r(t)=t$

$$R(z)=\frac{Tz}{(z-1)^2} \qquad E(z)=\frac{1}{1+GF(z)}\cdot\frac{Tz}{(z-1)^2}$$

- **根据终值定理**

$$\lim_{t\to\infty} e(t)=\lim_{z\to 1}(z-1)E(z)=\lim_{z\to 1}\frac{Tz}{(z-1)[1+GF(z)]}$$

$$=\lim_{z\to 1}\frac{T}{(z-1)GF(z)}$$

- $K_{\mathrm{v}}=T^{-1}\lim\limits_{z\to 1}(z-1)GF(z)$ **：静态速度误差系数**

- $e_{\mathrm{ss}}=\dfrac{1}{K_{\mathrm{v}}}$

$$E(z)=\frac{R(z)}{1+GF(z)}$$

(3) 抛物线输入 $r(t)=\frac{1}{2}t^2$

$$R(z)=\frac{T^2z(z+1)}{2(z-1)^3} \qquad E(z)=\frac{1}{1+GF(z)}\cdot\frac{T^2z(z+1)}{2(z-1)^3}$$

- **根据终值定理**

$$\lim_{t\to\infty}e(t)=\lim_{z\to1}(z-1)E(z)=\lim_{z\to1}\frac{T^2z(z+1)}{2(z-1)^2[1+GF(z)]}$$

$$=\lim_{z\to1}\frac{T^2}{(z-1)^2GF(z)}$$

- $K_{\mathrm{a}}=T^{-2}\lim\limits_{z\to1}(z-1)^2GF(z)$ **：静态加速度误差系数**

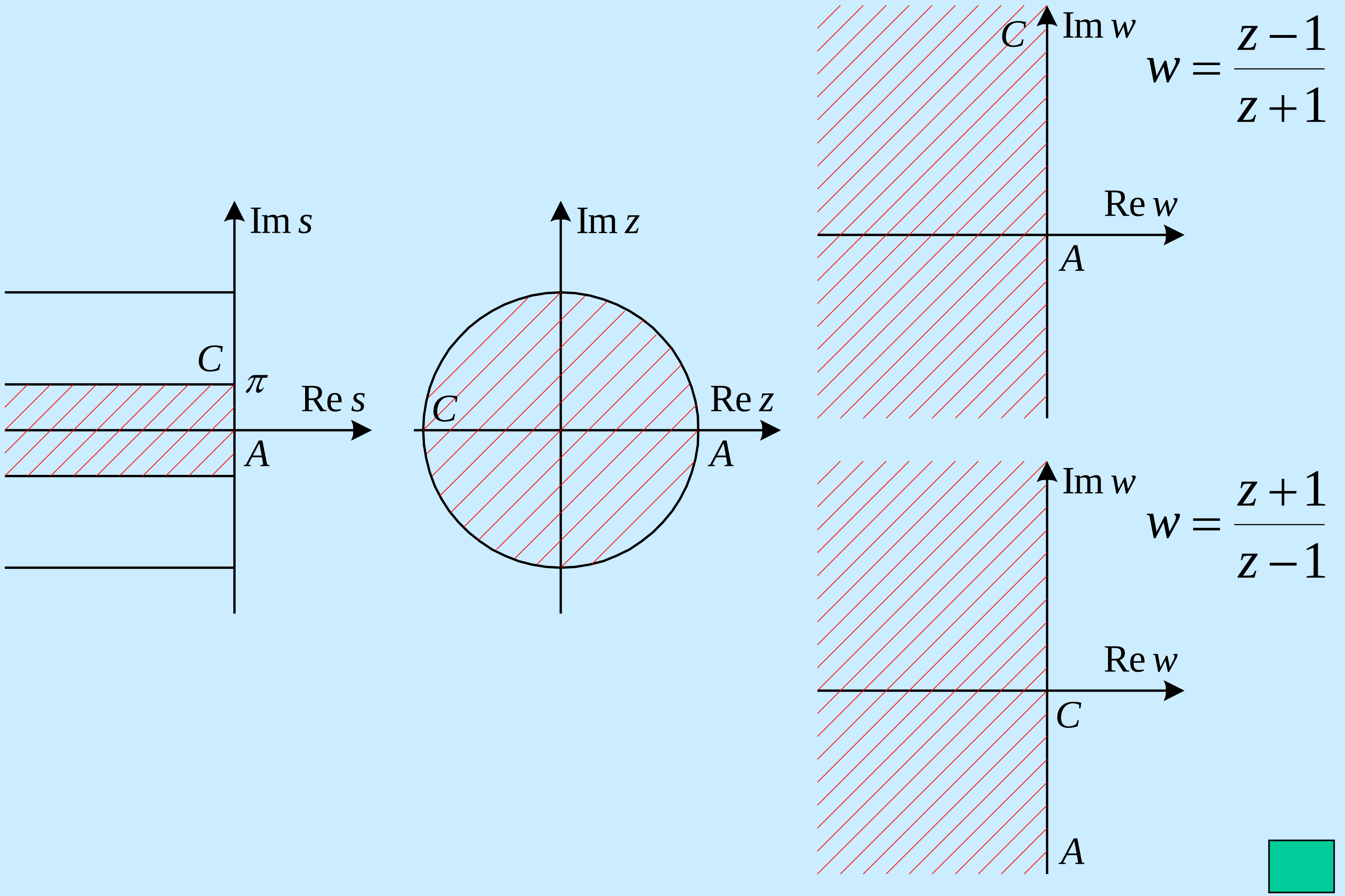
- $e_{\mathrm{ss}}=\dfrac{1}{K_{\mathrm{a}}}$

输入	单位阶跃	单位斜坡	单位抛物线
Type 1 **单个极点** $z = 1$	$e_{ss} \to 0$ $K_p \to \infty$	$e_{ss} = \frac{1}{K_v}$	$e_{ss} \to \infty$
Type 2 **两个极点** $z = 1$		$e_{ss} \to 0$ $K_v \to \infty$	$e_{ss} = \frac{1}{K_a}$
Type 3 **三个极点** $z = 1$			$e_{ss} \to 0$ $K_a \to \infty$

§ 8.5 小结

1. 脉冲采样、冲激采样、保持
2. z-传递函数
3. z-传递函数的计算
4. 采样系统的稳定性分析
5. 采样系统的时间响应

End of Chapter 8


Im s
C
π
Re s
A
Im z
Re z
C
A
Im w
C
$w = \frac{z-1}{z+1}$
Re w
A
Im w
$w = \frac{z+1}{z-1}$
Re w
C
A