1. 原问题 $\min \sum_{i,j} c_{ij} x_{ij}$

$$
\begin{aligned}
\text{s.t. } & \sum_{j=1}^{n} x_{ij} = a_i \quad (i=1,2,\cdots,m) \\
& \sum_{i=1}^{m} x_{ij} = b_j \quad (i=1,2,\cdots,n) \\
& x_{ij} \geq 0 \quad (i=1,2,\cdots m;\ j=1,2,\cdots,n)
\end{aligned}
$$

$$
\begin{aligned}
L(X,U,V,W) &= \sum_{i,j} c_{ij} x_{ij} + \sum_{i=1}^{m} u_i \Big(\sum_{j=1}^{n} x_{ij} - a_i\Big) + \sum_{j=1}^{n} v_j \Big(\sum_{i=1}^{m} x_{ij} - b_j\Big) - \sum_{i,j} w_{ij} x_{ij} \\
&= \sum_{i,j} (c_{ij} + u_i + v_j - w_{ij}) x_{ij} - \sum_{i=1}^{m} u_i a_i - \sum_{j=1}^{n} v_j b_j
\end{aligned}
$$

记 $\varphi(U,V,W) = \min\limits_{X\in R} L(X,U,V,W)$

则 $\varphi(U,V,W) = \begin{cases} -\Big(\sum_{i=1}^{m} u_i a_i + \sum_{j=1}^{n} v_j b_j\Big) & c_{ij} + u_i + v_j - w_{ij} = 0 \\ -\infty & c_{ij} + u_i + v_j - w_{ij} \neq 0 \end{cases}$

对偶问题: $\max -\Big(\sum_{i=1}^{m} u_i a_i + \sum_{j=1}^{n} v_j b_j\Big)$

$$
\begin{aligned}
\text{s.t. } & c_{ij} + u_i + v_j - w_{ij} = 0, \quad i=1,2,\cdots m;\ j=1,2,\cdots n \\
& w_{ij} \geq 0, \quad i=1,2,\cdots m;\ j=1,2,\cdots n
\end{aligned}
$$

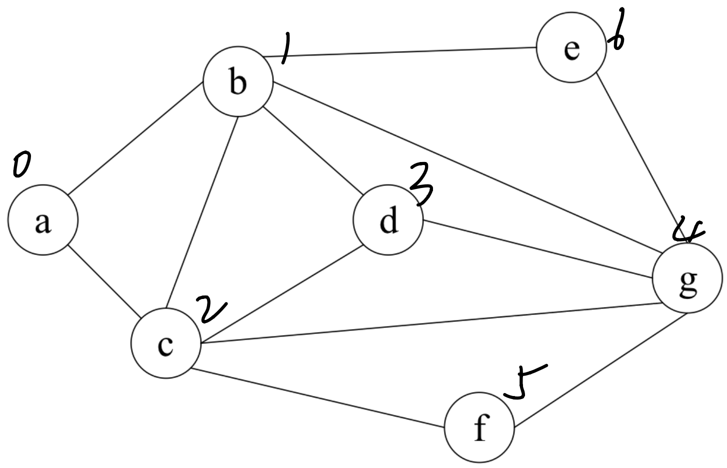
等价表示: $\max -\Big(\sum_{i=1}^{m} u_i a_i + \sum_{j=1}^{n} v_j b_j\Big)$

$$
\text{s.t. } -(u_i + v_j) \leq c_{ij} \quad i=1,2,\cdots m;\ j=1,2,\cdots n
$$

记 $\hat{u}_i = -u_i$, $\hat{v}_j = -v_j$, 则: $\max \sum_{i=1}^{m} \hat{u}_i a_i + \sum_{j=1}^{n} \hat{v}_j b_j$

$$
\text{s.t. } \hat{u}_i + \hat{v}_j \leq c_{ij}, \quad i=1,2,\cdots m;\ j=1,2,\cdots n
$$

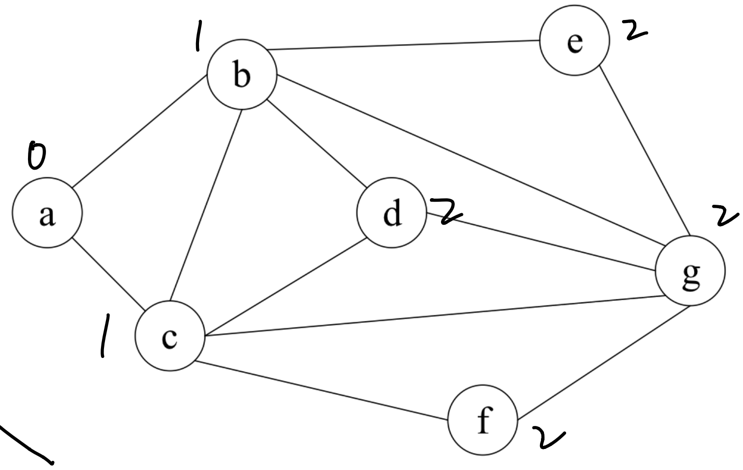
2. 深探法:

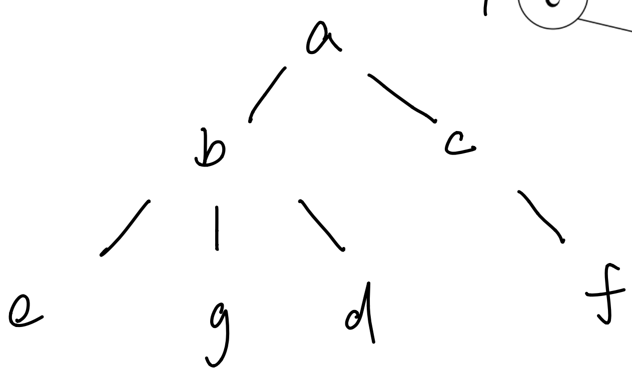


a — b — c — d — g — f

\ e

广探法:





3. 对偶目标函数: $\rho(u,v)=\min_{x\in R^n} L(x,u,v)$

$$=\min_{x\in R^n}\left(f(x)+\sum_{j=1}^{m} h_j(x)u_j+\sum_{i=1}^{l} g_i(x)v_i\right)$$

根据对偶函数的定义:

$\rho(u,v)$是把u和v看做常量,求x变化时L的最小值

所以，可把$\rho(U,V)$表示为如下形式：

$$\rho(U,V)=\min\{L(x_1,U,V),\ L(x_2,U,V),\ \cdots L(x_n,U,V)\}$$

$$n=+\infty$$

记$\gamma=(U,V)$，对任意$L(x_i,\gamma)$，$i=1,2,3,\cdots$

由于其x_i的值已确定，U,V前系数确定，故L可看做γ的仿射函数

而仿射函数既凸又凹，所以有：

$$\rho(\theta\gamma_1+(1-\theta)\gamma_2)$$

$$=\min\{L(x_1,\theta\gamma_1+(1-\theta)\gamma_2),\ L(x_2,\theta\gamma_1+(1-\theta)\gamma_2),\ \cdots,\ L(x_n,\theta\gamma_1+(1-\theta)\gamma_2)$$

$$\geqslant\min\{\theta L(x_1,\gamma_1)+(1-\theta)L(x_1,\gamma_2),\ \theta L(x_2,\gamma_1)+(1-\theta)L(x_2,\gamma_2),\ \cdots,$$
$$\theta L(x_n,\gamma_1)+(1-\theta)L(x_n,\gamma_2)\}$$

$$\geqslant\theta\min\{L(x_1,\gamma_1),\ L(x_2,\gamma_1),\ \cdots,\ L(x_n,\gamma_1)\}$$
$$+(1-\theta)\min\{L(x_1,\gamma_2),\ L(x_2,\gamma_2),\ \cdots,\ L(x_n,\gamma_2)\}$$

$$=\theta\rho(\gamma_1)+(1-\theta)\rho(\gamma_2)$$

∴对偶问题是凹函数，

而最大化凹函数是凸优化问题，

∴拉格朗日对偶问题是凸优化问题