机器人大作业

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0. 问题分析

设关节位置为 p , 角度为q, 关节对角度的雅可比矩阵为J，于是有:

$$\dot{p} = J\dot{q}$$

由题意: 关节处在静止状态, 故 $\dot{p}=0$, $\dot{q}=0$, 不难分析得出:

$$\ddot{p} = J\ddot{q}$$

由此, 参考课间中求解最小关节速度方法,建立拉格朗日方程:

$$\begin{aligned} G(\ddot{q}) &= \ddot{q}^\top W \ddot{q} \\ G(q,\lambda) &= \ddot{q}^\top W \ddot{q} + \lambda^\top (\ddot{p} - J\ddot{q}) \end{aligned}$$

由极值约束有:

$$\left.\begin{aligned} \frac{\partial G}{\partial \ddot{q}} &= 0 \to 2W\ddot{q} - J^\top \lambda = 0 \\ \frac{\partial G}{\partial \lambda} &= 0 \to \ddot{p} - J\ddot{q} = 0 \end{aligned}\right\} \longrightarrow \left\{\begin{aligned} \ddot{q} &= \tfrac{1}{2} w^{-1} J' \lambda \\ (JW^{-1}J^\top)\lambda &= 2\ddot{p} \end{aligned}\right.$$

$$\begin{aligned} \lambda &= 2\left(JW^{-1}J^\top\right)^{-1}\ddot{p} \\ \ddot{q} &= W^{-1}J^\top\left(JW^{-1}J^\top\right)^{-1}\ddot{p} \end{aligned}$$

由此，我们只需要将 $\ddot{q}$ 代入由牛顿欧拉法建立的动力学方程即可解得力矩:

$$H(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = \tau$$

根据上述分析，可以知道 $\dot{q}=0$，所以有:

$$\tau = H(q)\ddot{q} + G(q)$$

用matlab求解该三连杆模型得到（为了简化表达，这里在计算τ时直接代入了姿态q):

$$H = \begin{pmatrix} L^2\, m\,(\cos(q_2+q_3) + 3\,\cos(q_2) + \cos(q_3) + 4) & \sigma_1 & \sigma_2 \\ \sigma_1 & \frac{L^2\, m\,(3\,\cos(q_3)+5)}{3} & \frac{L^2\, m\,(3\,\cos(q_3)+2)}{6} \\ \sigma_2 & \frac{L^2\, m\,(3\,\cos(q_3)+2)}{6} & \frac{L^2\, m}{3} \end{pmatrix}$$

where

$$\sigma_1 = \frac{L^2\, m\,(3\,\cos(q_2+q_3)+9\,\cos(q_2)+6\,\cos(q_3)+10)}{6}$$

$$\sigma_2 = \frac{L^2\, m\,(3\,\cos(q_2+q_3)+3\,\cos(q_3)+2)}{6}$$

$$G = \begin{pmatrix} \frac{L\,g\,m\,(3\ \cos(q_1+q_2+q_3)+9\ \cos(q_1+q_2)+15\ \cos(q_1))}{6} \\ \frac{L\,g\,m\,(3\ \cos(q_1+q_2+q_3)+9\ \cos(q_1+q_2))}{6} \\ \frac{L\,g\,m\ \cos(q_1+q_2+q_3)}{2} \end{pmatrix}$$

$$\tau = \begin{pmatrix} \frac{L\,m\left(60\,L\,\mathrm{ddq}_1+26\,L\,\mathrm{ddq}_2+10\,L\,\mathrm{ddq}_3+27\,\sqrt{2}\,g\right)}{12} \\ \frac{L\,m\left(13\,L\,\mathrm{ddq}_1+10\,L\,\mathrm{ddq}_2+2\,L\,\mathrm{ddq}_3+6\,\sqrt{2}\,g\right)}{6} \\ \frac{L\,m\left(10\,L\,\mathrm{ddq}_1+4\,L\,\mathrm{ddq}_2+4\,L\,\mathrm{ddq}_3+3\,\sqrt{2}\,g\right)}{12} \end{pmatrix}$$

接下来考虑题目中的几种代价函数:

1. $H_A = \frac{1}{2}|\ddot{q}|^2$

该代价函数的物理意义为：在自己的关节参考系中末端关节速度最小

即取上述问题分析中的

$$W = I$$

代入上述式子解出:

$$\ddot{q} = \begin{pmatrix} -\frac{\sqrt{2}}{6\,l} \\ \frac{2\,\sqrt{2}}{3\,l} \\ -\frac{5\,\sqrt{2}}{6\,l} \end{pmatrix}$$

$$\tau = \begin{pmatrix} \frac{l\,m\left(27\,\sqrt{2}\,g-\sqrt{2}\right)}{12} \\ \frac{l\,m\left(36\,\sqrt{2}\,g+17\,\sqrt{2}\right)}{36} \\ \frac{l\,m\left(9\,\sqrt{2}\,g-7\,\sqrt{2}\right)}{36} \end{pmatrix}$$

2. $H_B = \frac{1}{2}|\ddot{q}_a|^2$

该代价函数的物理意义为：在原参考系关节速度最小

设变换矩阵A的表达式如下:

$$A = \begin{bmatrix} 1 & & \\ 1 & 1 & \\ 1 & 1 & 1 \end{bmatrix}$$

即取上述问题分析中的:

$$W = A^T A$$

代入上述式子解出:

$$\ddot{q} = \begin{pmatrix} -\frac{\sqrt{2}}{4\,l} \\ \frac{3\,\sqrt{2}}{4\,l} \\ -\frac{3\,\sqrt{2}}{4\,l} \end{pmatrix}$$

$$\tau = \begin{pmatrix} \frac{l\,m\,\left(18\,\sqrt{2}\,g-2\,\sqrt{2}\right)}{8} \\ \frac{l\,m\,\left(24\,\sqrt{2}\,g+11\,\sqrt{2}\right)}{24} \\ \frac{l\,m\,\left(6\,\sqrt{2}\,g-5\,\sqrt{2}\right)}{24} \end{pmatrix}$$

3. $H_c = \frac{1}{2}\ddot{q}^\top M(q)\ddot{q}$

该代价函数的物理意义为：关节转动能量最小

即取上述问题分析中的:

$$W = M(q)$$

其中，$M(q)$为牛顿欧拉法求解出的惯性矩阵。
代入上述式子解出：

$$\ddot{q} = \begin{pmatrix} \frac{\sqrt{2}}{20\,l} \\ \frac{9\,\sqrt{2}}{20\,l} \\ -\frac{21\,\sqrt{2}}{20\,l} \end{pmatrix}$$

$$\tau = \begin{pmatrix} \frac{l\,m\,\left(90\,\sqrt{2}\,g+14\,\sqrt{2}\right)}{40} \\ \frac{l\,m\,\left(120\,\sqrt{2}\,g+61\,\sqrt{2}\right)}{120} \\ \frac{l\,m\,\left(30\,\sqrt{2}\,g-19\,\sqrt{2}\right)}{120} \end{pmatrix}$$

4. 力矩最小代价

我们定义力矩最小的代价函数为：

$$\min(H\ddot{q}+G)^\top W(H\ddot{q}+G)$$

其中取W $= I$

该代价函数的物理意义为：关节驱动力矩最小

$$(H\ddot{q}+G)^\top I(H\ddot{q}+G) = \ddot{q}^\top H^\top H\ddot{q} + 2G^\top H\ddot{q} + G^\top G$$

建立广义目标函数:

$$G(q,\lambda) = (H\ddot{q}+G)^\top I(H\ddot{q}+G) + \lambda^\top(\ddot{p} - J\ddot{q})$$

极值满足:

$$\frac{\partial G}{\partial \dot{\theta}} = 0$$
$$\frac{\partial G}{\partial \lambda} = 0$$

得到(另取$W = H^T H$):

$$2W\ddot{q} + 2H^\top G - J^\top \lambda = 0$$
$$\ddot{p} - J\ddot{q} = 0$$

整理得到:

$$\ddot{q} = \frac{1}{2} W^{-1} \left(J^\top \lambda - 2H^\top G \right)$$
$$\ddot{p} = \frac{1}{2} J W^{-1} \left(J^\top \lambda - 2H^\top G \right)$$
$$\lambda = 2 \left(J W^{-1} J^\top \right) \left(\ddot{p} + J W^{-1} H^\top G \right)$$

所以有:

$$\ddot{q} = W^{-1} J^\top \left(J W^{-1} J^\top \right)^{-1} \left(\ddot{p} + J W^{-1} H^\top G \right) - W^{-1} H^\top G$$

代入上式可以解得:

$$\ddot{q} = \begin{pmatrix} -\frac{678\,\sqrt{2}\,g+80\,\sqrt{2}}{584\,l} \\ \frac{678\,\sqrt{2}\,g+372\,\sqrt{2}}{584\,l} \\ \frac{678\,\sqrt{2}\,g-504\,\sqrt{2}}{584\,l} \end{pmatrix}$$

$$\tau = \begin{pmatrix} -\frac{\sqrt{2}\,l\,m\,(42\,g+14)}{584} \\ \frac{\sqrt{2}\,l\,m\,(1413\,g+836)}{1752} \\ \frac{\sqrt{2}\,l\,m\,(99\,g-332)}{1752} \end{pmatrix}$$

注：上述涉及到的具体计算过程均在附件的matlab代码中