数 学 作 业 纸

（科目：随机数学与统计

班级：自93　　姓名：周义函　　编号：2019010702　　第 1 页

20. (a)

必要性：$s\le t$ 时，

$Cov(X_s,X_t)=Cov(X_0,X_{t-s})$ 只依赖于 $t-s$

令 $n=1$，令 $t_1=0$，$a=t$，则

X_0 与 X_t 有相同分布

$\therefore EX_t=EX_0=const$

充分性：设 $Cov(X_sX_t)=f(t-s)$，$EX_t=EX_0$

$\forall n,a$

$$\begin{pmatrix}X_{t_1+a}\\X_{t_2+a}\\\vdots\\X_{t_n+a}\end{pmatrix}\sim N\left(\begin{pmatrix}EX_0\\EX_0\\\vdots\\EX_0\end{pmatrix},\begin{pmatrix}f(0)&f(t_2-t_1)&\cdots&f(t_n-t_1)\\f(t_2-t_1)&f(0)&&\vdots\\\vdots&&\ddots&\\f(t_n-t_1)&\cdots&&f(0)\end{pmatrix}\right)$$

即 $Cov(X_{t_1+a},X_{t_2+a})=f(t_2-t_1)$

$=Cov(X_{t_1},X_{t_2})$ 且 $EX_t=EX_0$

$\therefore \forall n,a$，$\forall t_1,t_2\cdots t_n$，$X_{t_1},X_{t_2}\cdots X_{t_n}$ 与 $X_{t_1+a},X_{t_2+a}\cdots X_{t_n+a}$ 分布相同

$\therefore$ ~~Gauss过程为平稳过程~~

(b) $\because \{B_t:t\ge0\}$ 是Brown运动

$\therefore \{B_t:t\ge0\}$ 是Gauss过程

$U_t=e^{-\frac{\alpha t}{2}}B_{e^{\alpha t}}$

U_t 的任意有限维分布都是高斯分布

$\therefore U_t$ 是Gauss过程

$EU_t=e^{-\frac{\alpha t}{2}}EB_{e^{\alpha t}}=0=const$

$Cov(U_t,U_s)=Cov(e^{-\frac{\alpha t}{2}}B_{e^{\alpha t}},e^{-\frac{\alpha s}{2}}B_{e^{\alpha s}})$

$=e^{-\frac{\alpha(t+s)}{2}}Cov(B_{e^{\alpha t}},B_{e^{\alpha s}})=e^{-\frac{\alpha(t+s)}{2}}(e^{\alpha t}\wedge e^{\alpha s})$

$=e^{-\frac{\alpha}{2}(t+s)+\alpha(t\wedge s)}=e^{-\frac{\alpha}{2}|t-s|}$ 只与 $t-s$ 有关

由(a) $\{U_t:t\ge0\}$ 是平稳的高斯过程

21.

(a) $B_t\sim N(0,t)$

由高斯分布的线性组合依然是高斯分布

可得 $B_1+B_2+\cdots+B_n$ 服从高斯分布

$E(\sum_{i=1}^n B_i)=\sum_{i=1}^n EB_i=0$

$D(\sum_{i=1}^n B_i)=\sum_{i=1}^n DB_i+2\sum\sum_{0\le i<j\le n}Cov(B_i,B_j)$

$=\sum_{i=1}^n i+2\sum\sum_{0\le i<j\le n} i$

$=\sum_{i=1}^n i+2\sum_{j=1}^n\sum_{i=1}^{j-1} i$

$=\sum_{j=1}^n j+\sum_{j=1}^n j(j-1)$

$=\sum_{j=1}^n j^2=\frac{n(n+1)(2n+1)}{6}$

$\therefore \sum_{i=1}^n B_i\sim N(0,\frac{n(n+1)(2n+1)}{6})$

(b) 由于 $Y_t=B_{t+1}-B_t$

$\therefore Y_t$ 的任意有限维分布均为高斯分布的线性组合，即任意有限维分布均为Gauss分布

$\therefore \{Y_t:t\ge0\}$ 是Gauss过程

由20.(a)的结论，只需证：$\begin{cases}Cov(X_s,X_t)\text{只依赖于}t-s & (s\le t)\\EX_t\text{为常数}\end{cases}$

① $EY_t=EB_{t+1}-EB_t=0=const$

② $\forall s\le t$，$Cov(X_s,X_t)=Cov(B_{s+1}-B_s,B_{t+1}-B_t)$

$=Cov(B_{s+1},B_{t+1})+Cov(B_s,B_t)-Cov(B_{s+1},B_t)$

$+Cov(B_s,B_{t+1})=s+1+s-\min(s+1,t)-s$

$=s+1-\min(s+1,t)=\begin{cases}0 & t-s\ge1\\1-(t-s) & t-s\le1\end{cases}=\max\{0,1-(t-s)\}$

$\because Cov(X_s, X_t)$ 只依赖 $t-s$

由①②$\{Y_t: t\geq 0\}$为平稳过程

22. (1)

平移不变性：$\{B_{t+a}-B_a, t\geq 0\}$ $a\geq 0$为常数是Brown运动

证：$\because \{B_t: t\geq 0\}$是Brown运动

$\therefore \{B_t, t\geq 0\}$是高斯过程

令 $Y_t = B_{t+a} - B_a$

$\because Y_t$的任意有限维分布均为高斯分布的线性变换，

即也是高斯分布

$\therefore \{Y_t: t\geq 0\}$为高斯过程

② $\because Y_0 = B_{0+a} - B_a = 0$

$EY_t = E(B_{t+a}-B_a) = EB_{t+a} - EB_a = 0$

$E(Y_t Y_s) = E((B_{t+a}-B_a)(B_{s+a}-B_a))$

$= EB_{t+a}B_{s+a} - EB_{t+a}B_a - EB_{s+a}B_a + EB_aB_a$

$= (t+a)\wedge(s+a) - a - a + a$

$= (t+a)\wedge(s+a) - a$

$= t\wedge s$

$\therefore \{B_{t+a} - B_a, t\geq 0\}$ $a\geq 0$为常数仍为Brown运动

(2)

尺度不变性：$\{\frac{B_{ct}}{\sqrt{c}}: t\geq 0\}$ $c>0$为常数仍为Brown运动.

证：令 $Y_t = \frac{B_{ct}}{\sqrt{c}}$

$\because \{B_t, t\geq 0\}$为Brown运动

$\therefore \{B_t, t\geq 0\}$为Gauss过程

$\therefore Y_t$的任意有限维分布是高斯分布的线性变换，故也是高斯分布

$\therefore \{Y_t: t\geq 0\}$为Gauss过程

② $\because Y_0 = \frac{B_0}{\sqrt{c}} = 0$

$EY_t = \frac{EB_{ct}}{\sqrt{c}} = 0$

$E(Y_t Y_s) = \frac{1}{c}E(B_{ct}B_{cs})$

$= \frac{1}{c}(ct\wedge cs)$

$= t\wedge s$

$\therefore \{\frac{B_{ct}}{\sqrt{c}}, t\geq 0\}$ $c>0$为常数仍为Brown运动.

23. $B_t \sim N(0, t)$

$s<t$时.

$$\begin{pmatrix} B_s \\ B_t \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} s & s \\ s & t \end{pmatrix}\right)$$

$E(B_s|B_t = x) = \mu_s + \rho\frac{\sigma_s}{\sigma_t}(x-\mu_t)$

$= \rho\frac{\sigma_s}{\sigma_t}x$

$= \frac{Cov(B_s, B_t)}{\sqrt{DB_s}\sqrt{DB_t}}\frac{\sigma_s}{\sigma_t}x$

$= \frac{s}{\sqrt{st}}\cdot\frac{\sqrt{s}}{\sqrt{t}}x$

$= \frac{sx}{t}$

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4. 令 $Y_i=(X_i-\mu)^2 \quad i=1,2,3,\cdots$

则 Y_i 独立同分布且 $EY_i=E(X_i-\mu)^2$

$=EX_i^2-2E\mu X_i+E\mu^2$

$=DX_i+(EX_i)^2-2\mu EX_i+\mu^2$

$=\sigma^2+\mu^2-2\mu^2+\mu^2$

$=\sigma^2$

由 Khintchine 大数定律

$\frac{1}{n}\sum_{k=1}^{n}(X_k-\mu)^2 \xrightarrow{P} \sigma^2$

5. $\because \lim_{|i-j|\to\infty} r(X_i,X_j)=0$

$\therefore \forall \varepsilon_0>0, \exists N>0$, s.t. $\forall |i-j|>N$,

$r(X_i,X_j)<\varepsilon$

$\because Cov(X_i,X_j)=r(X_i,X_j)\sqrt{DX_i}\sqrt{DX_j}$

$DX_n<C\ (n\geq 1)$

$\therefore Cov(X_i,X_j)<C\varepsilon_0$

$\therefore D(\sum_{i=1}^{n}X_i)=\sum_{i=1}^{n}\sum_{j=1}^{n}Cov(X_i,X_j)$

$=\sum\sum_{0<|i-j|\leq N}Cov(X_i,X_j)+\sum\sum_{|i-j|>N}Cov(X_i,X_j)+\sum_{i=1}^{n}DX_i$

$\leq 2N\cdot n\cdot \max_{0<|i-j|\leq N}Cov(X_i,X_j)+n^2C\varepsilon_0+nC$

由柯西不等式 $|Cov(X_i,X_j)|=|EX_iX_j-EX_iEX_j|\leq|EX_iX_j|+|EX_iEX_j|$

$\leq\sqrt{EX_i^2EX_j^2}+\sqrt{EX_i^2EX_j^2}\leq 2\sqrt{DX_iDX_j}\leq 2C$

$\therefore \forall\varepsilon>0, \exists N= \qquad >0$

s.t. $P\left(\left|\frac{1}{n}\sum_{i=1}^{n}X_i-E\left(\frac{1}{n}\sum_{i=1}^{n}X_i\right)\right|\geq\varepsilon\right)$

$\leq\frac{D(\sum_{i=1}^{n}X_i)}{n^2\varepsilon^2}$

$\leq\frac{2N\cdot n\cdot\max_{0<|i-j|\leq N}Cov(X_i,X_j)+n^2C\varepsilon_0+nC}{n^2\varepsilon^2}$

$=\frac{C+2N\max_{0<|i-j|\leq N}Cov(X_i,X_j)}{n\varepsilon^2}+$

$\therefore$ 上式 $\leq 2N\cdot n\cdot 2C+n^2C\varepsilon_0+nC$

$=4NnC+n^2C\varepsilon+nC$

$\therefore \forall\varepsilon>0, \exists N=\left[\frac{2C(4N+1)}{\varepsilon^3}\right]+1>0$

s.t. $P\left(\left|\frac{1}{n}\sum_{i=1}^{n}X_i-E\left(\frac{1}{n}\sum_{i=1}^{n}X_i\right)\right|\geq\varepsilon\right)$

$\leq\frac{D(\sum_{i=1}^{n}X_i)}{n^2\varepsilon^2}$

$\leq\frac{4NnC+n^2C\varepsilon_0+nC}{n^2\varepsilon^2}$

$=\frac{C+4NC}{n\varepsilon^2}+\frac{C\varepsilon_0}{\varepsilon^2}\ \underline{\underline{\text{取}\ \varepsilon_0=\frac{\varepsilon^3}{2C}}}\ \frac{C+4NC}{n\varepsilon^2}+\frac{1}{2}\varepsilon$

$<\varepsilon$

9. 记 $Y_k=\cos kX$

$i\neq j$ 且 $i,j\in N^*$ 时

$Cov(Y_i, Y_j)$

$=Cov(\cos iX, \cos jX)$

$=E(\cos iX\cos jX)-E\cos iX E\cos jX$

$E\cos iX=\int_{-\pi}^{\pi}\cos ix\cdot\frac{1}{2\pi}dx=0$

$E\cos jX=\int_{-\pi}^{\pi}\cos jx\cdot\frac{1}{2\pi}dx=0$

$E(\cos iX\cos jX)$

$=\frac{1}{2\pi}\int_{-\pi}^{\pi}\cos ix\cos jx\,dx$

$=\frac{1}{2\pi}\times\frac{1}{2}\int_{-\pi}^{\pi}(\cos(i+j)x+\cos(i-j)x)dx$

$=0\quad(i\neq j)$

$\therefore Cov(Y_i, Y_j)=0\quad\forall i\neq j$

$DY_k=EY_k^2-(EY_k)^2=EY_k^2$

$=\frac{1}{2\pi}\int_{-\pi}^{\pi}\cos^2 kx\,dx$

$=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\cos 2kx+1}{2}dx$

$=\frac{1}{2}$

$\because\{Y_n\}$ 满足两两不相关且方差一致有界

由 Chebyshev 大数定律：

$\frac{1}{n}S_n=\frac{1}{n}\sum_{i=1}^{n}Y_i-\frac{1}{n}\sum_{i=1}^{n}EY_i\xrightarrow{P}0$

12. $P(|M_n-a|\geq\varepsilon)$

$=P(|\min\{X_1,X_2,\cdots,X_n\}-a|\geq\varepsilon)$

$=P(\min\{X_1,X_2,\cdots,X_n\}-a\geq\varepsilon)$

$=P(\min\{X_1,X_2,\cdots,X_n\}\geq a+\varepsilon)$

$=P(X_1\geq a+\varepsilon, X_2\geq a+\varepsilon,\cdots,X_n\geq a+\varepsilon)$

$=\prod_{i=1}^{n}P(X_i\geq a+\varepsilon)$

$=P(X_1\geq a+\varepsilon)^n$

$P(X_1<a+\varepsilon)=\int_a^{a+\varepsilon}e^{-(x-a)}dx$

$=1-e^{-\varepsilon}$

$\therefore P(X_i\geq a+\varepsilon)=e^{-\varepsilon}$

$\therefore P(|M_n-a|\geq\varepsilon)=e^{-n\varepsilon}=\frac{1}{e^{n\varepsilon}}$

$\forall 0<\varepsilon<1, \exists N=[\frac{1}{\varepsilon}\ln\frac{1}{\varepsilon}]+1$

s.t. $\forall n>N$.

$P(|M_n-a|\geq\varepsilon)<\varepsilon$

$\therefore\lim_{n\to\infty}P(|M_n-a|\geq\varepsilon)=0$

$\therefore M_n\xrightarrow{P}a$.

13. $\ln Y_n = \frac{\sum_{k=1}^{n} \ln X_k}{n}$

令 $V_i = \ln X_i$ $i=1,2,3,\dots$

$\because X_i \overset{iid}{\sim} U[0,1]$

$\therefore V_i$ 独立同分布

且 $EV_i = \int_0^1 \ln x\, dx$ (瑕积分收敛).

$= x\ln x \Big|_0^1 - \int_0^1 x\, d(\ln x)$

$= -\int_0^1 dx = -1$

$\therefore$ 由 Khintchine 大数定律

$\ln Y_n \xrightarrow{P} -1$

$\because e^x$ 是连续函数

$\therefore e^{\ln Y_n} \xrightarrow{P} e^{-1}$

$\therefore Y_n \xrightarrow{P} e^{-1}$

$\therefore c = \frac{1}{e}$

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前面少写了24题，补充在这里：

解：$B_t \sim N(0, t)$

$$f_{B_t}(x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}$$

$$EY_t = \int_{-\infty}^{+\infty} e^x \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} dx$$

$$= \int_{-\infty}^{+\infty} \frac{e^{x - \frac{x^2}{2t}}}{\sqrt{2\pi t}} dx$$

$$= \frac{e^{\frac{t}{2}}}{\sqrt{2\pi t}} \int_{-\infty}^{+\infty} e^{-\frac{1}{2t}(x-t)^2} dx$$

$$= \frac{e^{\frac{t}{2}}}{\sqrt{2\pi t}} \int_{-\infty}^{+\infty} e^{-\frac{1}{2t}x^2} dx$$

$$= \frac{e^{\frac{t}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{x^2}{2t}} d\left(\frac{x}{\sqrt{t}}\right)$$

$$= \frac{e^{\frac{t}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}x^2} dx$$

$$= \frac{e^{\frac{t}{2}}}{\sqrt{2\pi}} \cdot \sqrt{2\pi}$$

$$= e^{\frac{t}{2}}$$

$$EY_t^2 = \int_{-\infty}^{+\infty} e^{2x} \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} dx$$

$$= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi t}} e^{-\frac{1}{2t}(x^2 - 4xt)} dx$$

$$= \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{+\infty} e^{-\frac{1}{2t}(x^2 - 4xt + 4t^2) + 2t} dx$$

$$= \frac{e^{2t}}{\sqrt{2\pi t}} \int_{-\infty}^{+\infty} e^{-\frac{x^2}{2t}} dx = \frac{e^{2t}}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{x^2}{2t}} d\left(\frac{x}{\sqrt{t}}\right)$$

$$= \frac{e^{2t}}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}x^2} dx = \frac{e^{2t}}{\sqrt{2\pi}} \cdot \sqrt{2\pi} = e^{2t}$$

$$\therefore DY_t = EY_t^2 - (EY_t)^2$$

$$= e^{2t} - e^{\frac{t}{2} \cdot 2}$$

$$= e^{2t} - e^{t}$$