清華大學 TSINGHUA UNIVERSITY 数学作业纸

班级：自93 姓名：周文涵 编号：2019010702 科目：随机数学与统计 页

19. 记 $\begin{cases} X_1: \text{第一份报纸卖出时的过路人数} \\ X_i: \text{第 } i-1 \text{ 份报纸卖出后到第 } i \text{ 份报纸卖出时的过路人数} \end{cases}$

则由题意可知 X_i 服从 $Ge(\frac{1}{3})$ $EX_i=\frac{1}{p}=3$ $DX_i=\frac{q}{p^2}=6$

令 $Y=\sum_{i=1}^{100}X_i$ $EY=100EX_i=300$, $DY=600$

由 Levy-Lindeberg 中心极限定理

$$S_n^*=\frac{Y-EY}{\sqrt{D(Y)}}\xrightarrow{D}N(0,1)$$

$$\because P(|Y-300|\le 20)=P\left(\left|\frac{Y-300}{\sqrt{600}}\right|\le\frac{20}{\sqrt{600}}\right)$$

$$=P\left(\left|\frac{Y-EY}{\sqrt{DY}}\right|\le\frac{2}{\sqrt{6}}\right)\doteq 2\Phi\left(\frac{2}{\sqrt{6}}\right)-1\doteq 0.588$$

20. (a) $Y_i \overset{iid}{\sim}\begin{pmatrix}1 & -1\\ \frac{1}{2} & \frac{1}{2}\end{pmatrix}$ $EY_i=0$, $DY_i=EY_i^2=1$

$X_n=\sum_{i=1}^{n}Y_i$ $EX_n=nEY_n=0$.

$DX_n=D(\sum_{i=1}^{n}Y_i)=nDY_i=n$

(b) 由 Levy-Lindeberg 中心极限定理：

$n\to\infty$ 时，$\frac{X_n-EX_n}{\sqrt{DX_n}}\longrightarrow N(0,1)$

$$\therefore \lim_{n\to\infty}P\left(\frac{X_n}{\sqrt{n}}\le x\right)=\lim_{n\to\infty}P\left(\frac{X_n-EX_n}{\sqrt{DX_n}}\le x\right)$$

$$=\Phi(x)=\int_{-\infty}^{x}\frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}}dt=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}e^{-\frac{t^2}{2}}dt.$$

8. $X\sim U[-1,1]$ X_i 服从 $U[-1,1]$

$$E\bar{X}=E\left(\frac{1}{n}\sum_{i=1}^{n}X_i\right)=\frac{1}{n}\sum_{i=1}^{n}EX_i=0$$

$$D\bar{X}=D\left(\frac{1}{n}\sum_{i=1}^{n}X_i\right)$$

$$=\frac{DX_i}{n}=\frac{1}{n}\cdot\frac{(1+1)^2}{12}$$

$$=\frac{1}{3n}$$

10. $$S^2=\frac{1}{n-1}\sum_{i=1}^{n}(x_i-\bar{x})^2$$

$$=\frac{1}{n-1}\sum_{i=1}^{n}(x_i^2-2\bar{x}x_i+\bar{x}^2)$$

$$=\frac{1}{n-1}\left(\sum_{i=1}^{n}x_i^2-2\bar{x}\sum_{i=1}^{n}x_i+\sum_{i=1}^{n}\bar{x}^2\right)$$

$$=\frac{1}{n-1}\left(\sum_{i=1}^{n}x_i^2-2n\bar{x}\cdot\frac{1}{n}\sum_{i=1}^{n}x_i+n\bar{x}^2\right)$$

$$=\frac{1}{n-1}\left(\sum_{i=1}^{n}x_i^2-2n\bar{x}^2+n\bar{x}^2\right)$$

$$=\frac{1}{n-1}\left(\sum_{i=1}^{n}x_i^2-n\bar{x}^2\right) \quad ①$$

$$\because \frac{1}{n(n-1)}\sum_{i<j}(x_i-x_j)^2$$

$$=\frac{1}{n(n-1)}\sum_{j<i}(x_j-x_i)^2$$

$$=\frac{1}{n(n-1)}\sum_{i>j}(x_i-x_j)^2$$

且 $\sum_{i=j}(x_i-x_j)^2=0$

$$\therefore \frac{1}{n(n-1)}\sum_{i<j}(x_i-x_j)^2$$

$$=\frac{1}{n(n-1)}\cdot\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(x_i-x_j)^2$$

$$=\frac{1}{2n(n-1)}\sum_{i=1}^{n}\sum_{j=1}^{n}(x_i^2-2x_ix_j+x_j^2)$$

$$= \frac{1}{2n(n-1)} \sum_{i=1}^{n} \left(n x_i^2 - 2x_i \sum_{j=1}^{n} x_j + \sum_{j=1}^{n} x_j^2 \right)$$

$$= \frac{1}{2n(n-1)} \left(n \sum_{i=1}^{n} x_i^2 - 2 \sum_{i=1}^{n} x_i \sum_{j=1}^{n} x_j + n \sum_{j=1}^{n} x_j^2 \right)$$

$$= \frac{1}{2n(n-1)} \left(2n \sum_{i=1}^{n} x_i^2 - 2 \left(\sum_{i=1}^{n} x_i \right)^2 \right)$$

$$= \frac{1}{n-1} \left(\sum_{i=1}^{n} x_i^2 - n \left(\frac{1}{n} \sum_{i=1}^{n} x_i \right)^2 \right)$$

$$= \frac{1}{n-1} \left(\sum_{i=1}^{n} x_i^2 - n\bar{x}^2 \right) \quad ②$$

由①②得

$$S^2 = \frac{1}{n(n-1)} \sum_{i<j} (x_i - x_j)^2$$

24. (1) ~~令 $y_i = \frac{x_i - 8}{2}$ $i = 1, 2, \dots, 16$~~

~~$\therefore y_1, y_2, \dots, y_{16}$ 是来自 $N(0,1)$ 的样本.~~

$$P(X_{(16)} > 10) = 1 - P(X_{(16)} \le 10)$$

$$= 1 - P(X_1 \le 10, X_2 \le 10, \dots, X_{16} \le 10)$$

$$= 1 - \prod_{i=1}^{16} P(X_i \le 10)$$

$$= 1 - \prod_{i=1}^{16} P\left(\frac{X_i - 8}{2} \le 1 \right)$$

$$= 1 - (\Phi(1))^{16}$$

$$\doteq 1 - (0.8413)^{16}$$

$$\doteq 0.9370$$

(2) $P(X_{(1)} > 5)$

$$= P(X_1 > 5, X_2 > 5, \dots, X_{16} > 5)$$

$$= \prod_{i=1}^{16} P(X_i > 5)$$

$$= \prod_{i=1}^{16} (1 - P(X_i \le 5))$$

$$= \prod_{i=1}^{16} \left(1 - P\left(\frac{X_i - 8}{2} \le -\frac{3}{2} \right)\right)$$

$$= \prod_{i=1}^{16} \left(1 - \Phi\left(-\frac{3}{2}\right)\right)$$

$$= \prod_{i=1}^{16} \Phi\left(\frac{3}{2}\right) = \Phi^{16}\left(\frac{3}{2}\right)$$

$$\doteq 0.3308$$

29.

(1) $f_{X_{(6)}}(x) = \frac{10!}{5!4!} [F(x)]^5 [1 - F(x)]^4 f(x)$

记 $Y = F(X_{(6)})$ 则 $X_{(6)} = F^{-1}(Y)$

$$f_Y(y) = \frac{10!}{5!4!} [F(F^{-1}(y))]^5 [1 - F(F^{-1}(y))]^4 f(x) \cdot \frac{1}{f(x)}$$

$$= \frac{10!}{5!4!} y^5 (1-y)^4$$

$$= \frac{\Gamma(11)}{\Gamma(6)\Gamma(5)} y^{6-1} (1-y)^{5-1} \quad 0 < y < 1$$

$\therefore Y = F(X_{(6)}) \sim \text{Beta}(6, 5)$

$\therefore EY = \frac{a}{a+b} = \frac{6}{11}$ $DY = \frac{ab}{(a+b)^2(a+b+1)} = \frac{30}{11^2 \times 12} = \frac{5}{242}$

(2) $F_Y(0.15) = \int_0^{0.15} \frac{10!}{5!4!} y^5 (1-y)^4 \, dy \doteq 0.00138$

32. $f_{X_{(4)}}(x) = \frac{5!}{3!1!1!} (F(x))^3 (1 - F(x)) p(x)$

$$f_{X_{(2)}, X_{(4)}}(x, y) = 5! \, F(x) [F(y) - F(x)] [1 - F(y)] p(x) p(y) \quad x \le y$$

令 $\begin{cases} U \triangleq \frac{X_{(2)}}{X_{(4)}} \\ V = X_{(4)} \end{cases} \Rightarrow \begin{cases} X_{(2)} = UV \\ X_{(4)} = V \end{cases}$ $\frac{\partial(X_{(2)}, X_{(4)})}{\partial(U, V)} = \begin{vmatrix} V & U \\ 0 & 1 \end{vmatrix} = V$,

$f_{U,V}(u,v) = 5!\,F(uv)[F(v)-F(uv)][1-F(v)]\,p(uv)p(v)\,v \qquad uv \le v$

$p(x) = \begin{cases} 3x^2 & 0<x<1 \\ 0 & \text{其他} \end{cases} \qquad F(x) = \begin{cases} 0 & x<0 \\ x^3 & 0\le x<1 \\ 1 & x\ge 1 \end{cases}$

$f_{X(4)}(x) = \begin{cases} 20x^9(1-x^3)\,3x^2 = 60x^{11}(1-x^3) & 0\le x<1 \\ 0 & \text{其他} \end{cases}$

$\because f_{U,V}(u,v) = 120\,(uv)^3[v^3-(uv)^3][1-v^3]\,3u^2v^2 \cdot 3v^2 \cdot v$

$= 1080\,u^5(1-u^3)\cdot v^{11}(1-v^3) \qquad \begin{cases} 0<v<1 \\ 0<u<1 \end{cases}$

$f_U(u) = \int_0^1 1080\,u^5(1-u^3)\,v^{11}(1-v^3)\,dv$

$= 1080\,u^5(1-u^3)\int_0^1 v^{11}(1-v^3)\,dv$

$= 1080\,u^5(1-u^3)\,\frac{1}{60}$

$= 18u^5(1-u^3) \qquad 0<u<1$

$\therefore \frac{X_{(2)}}{X_{(4)}} \sim 18u^5(1-u^3) \qquad 0<u<1$

$X_{(4)} \sim 60v^{11}(1-v^3) \qquad 0<v<1$

且联合分布 $\left(\frac{X_{(2)}}{X_{(4)}}, X_{(4)}\right) \sim 1080\,u^5(1-u^3)v^{11}(1-v^3) \quad \begin{cases} 0<v<1 \\ 0<u<1 \end{cases}$

$\because f_{\left(\frac{X_{(2)}}{X_{(4)}}, X_{(4)}\right)}(u,v) = f_{\frac{X_{(2)}}{X_{(4)}}}(u)\cdot f_{X_{(4)}}(v)$

$\therefore \frac{X_{(2)}}{X_{(4)}}$ 与 $X_{(4)}$ 独立.

8. $Z = \frac{n}{m}X \Big/ \left(1+\frac{n}{m}X\right) = \frac{1}{1+\frac{m}{n}\cdot\frac{1}{X}}$

令 $Y = \frac{1}{X}$ 则 $Y \sim F(m,n)$. 故 $Z = \frac{1}{1+\frac{m}{n}Y}$ $\quad Y = \frac{n}{m}\left(\frac{1}{z}-1\right)$ $\quad \frac{dy}{dz} = -\frac{n}{m}\cdot\frac{1}{z^2}$

由F分布的密度函数 $f_Y(y) = \frac{\Gamma\left(\frac{m+n}{2}\right)\left(\frac{m}{n}\right)^{\frac{m}{2}}}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)} y^{\frac{m}{2}-1}\left(1+\frac{m}{n}y\right)^{-\frac{m+n}{2}}$, $y>0$

$\therefore f_Z(z) = \dfrac{\Gamma(\frac{m+n}{2})(\frac{m}{n})^{\frac{m}{2}}}{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})} \left(\frac{n}{m}\right)^{\frac{m}{2}-1} \left(\frac{1}{z}-1\right)^{\frac{m}{2}-1} z^{\frac{m+n}{2}} \cdot \frac{n}{m} \cdot \frac{1}{z^2} \quad 0<z<1$

$= \dfrac{\Gamma(\frac{m+n}{2})}{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})} \frac{m}{n} (1-z)^{\frac{m}{2}-1} z^{\frac{n}{2}+1} \cdot \frac{n}{m} \cdot \frac{1}{z^2} \quad 0<z<1$

$= \dfrac{\Gamma(\frac{m+n}{2})}{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})} z^{\frac{n}{2}-1} (1-z)^{\frac{m}{2}-1} \quad 0<z<1$

$\therefore Z \sim Beta(\frac{n}{2}, \frac{m}{2})$

19. $T = -2\sum_{i=1}^{n} \ln F(X_i) = \sum_{i=1}^{n} \ln \frac{1}{F^2(X_i)}$

令 $Y = \ln \frac{1}{F^2(X)}$，则

$F_Y(Y) = P(Y \le y) = P(\ln \frac{1}{F^2(X)} \le y) = P(\frac{1}{F^2(X)} \le e^y) = P(F^2(X) \ge e^{-y})$

$= P(F(X) \ge e^{-\frac{y}{2}}) = 1 - P(F(X) < e^{-\frac{y}{2}}) = 1 - P(X < F^{-1}(e^{-\frac{y}{2}})) = 1 - F(F^{-1}(e^{-\frac{y}{2}}))$

$= 1 - e^{-\frac{y}{2}} \quad \therefore Y \sim E(\frac{1}{2})$ 即 $Y \sim G(1, \frac{1}{2})$

由 Gamma 分布的可加性（可由特征函数证明）且 $X_1, \cdots X_n$ 相互独立

$T = \sum_{i=1}^{n} \ln \frac{1}{F^2(X_i)} \sim G(n, \frac{1}{2})$

$\therefore T \sim \chi^2(2n)$