



An Introduction to Differential Evolution

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Synopsis



- Introduction
- Basic Algorithm
- Example
- Performance
- Applications

The Basics of Differential Evolution

- Stochastic, population-based optimisation algorithm
- Introduced by Storn and Price in 1996
- Developed to optimise real parameter, real valued functions
- General problem formulation is:

For an objective function $f : X \subseteq \mathbb{R}^D \rightarrow \mathbb{R}$ where the feasible region $X \neq \emptyset$, the minimisation problem is to find

$$x^* \in X \text{ such that } f(x^*) \leq f(x) \forall x \in X$$

where:

$$f(x^*) \neq -\infty$$

Why use Differential Evolution?



- Global optimisation is necessary in fields such as engineering, statistics and finance
- But many practical problems have objective functions that are non-differentiable, non-continuous, non-linear, noisy, flat, multi-dimensional or have many local minima, constraints or stochasticity
- Such problems are difficult if not impossible to solve analytically
- DE can be used to find approximate solutions to such problems

Evolutionary Algorithms

- DE is an Evolutionary Algorithm
- This class also includes Genetic Algorithms, Evolutionary Strategies and Evolutionary Programming

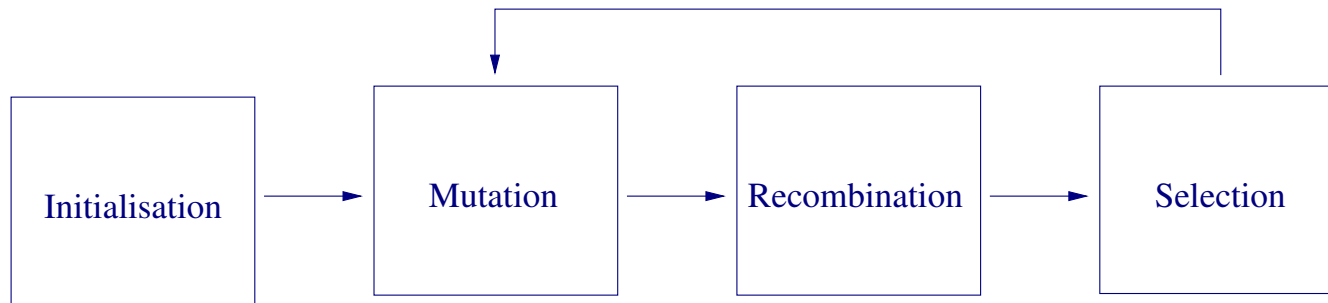


Figure 1: General Evolutionary Algorithm Procedure

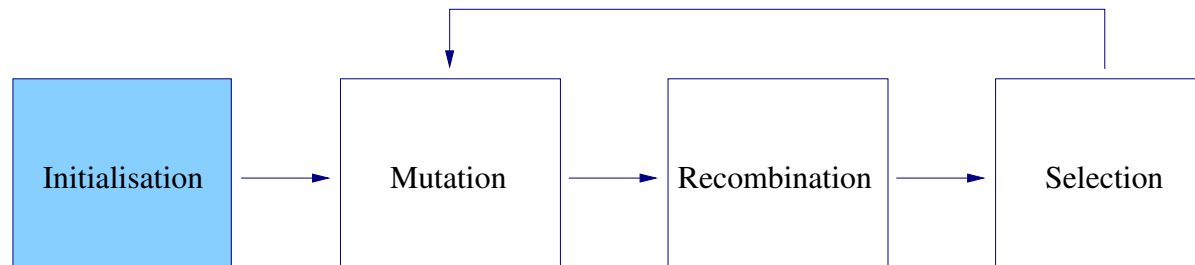
Notation

- Suppose we want to optimise a function with D real parameters
- We must select the size of the population N (it must be at least 4)
- The parameter vectors have the form:

$$x_{i,G} = [x_{1,i,G}, x_{2,i,G}, \dots, x_{D,i,G}] \quad i = 1, 2, \dots, N.$$

where G is the generation number.

Initialisation

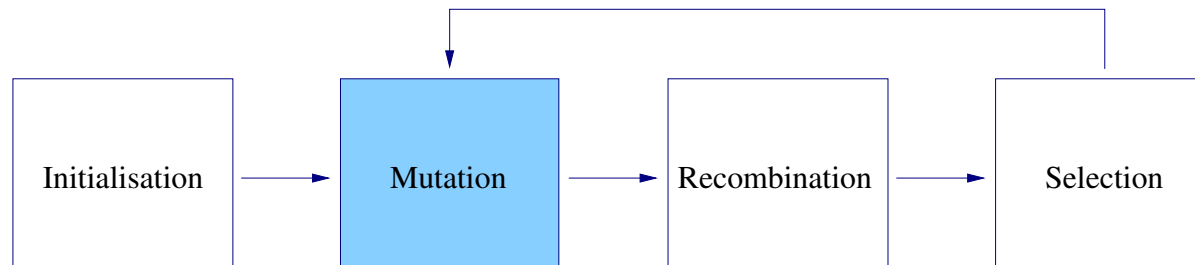


- Define upper and lower bounds for each parameter:

$$x_j^L \leq x_{j,i,1} \leq x_j^U$$

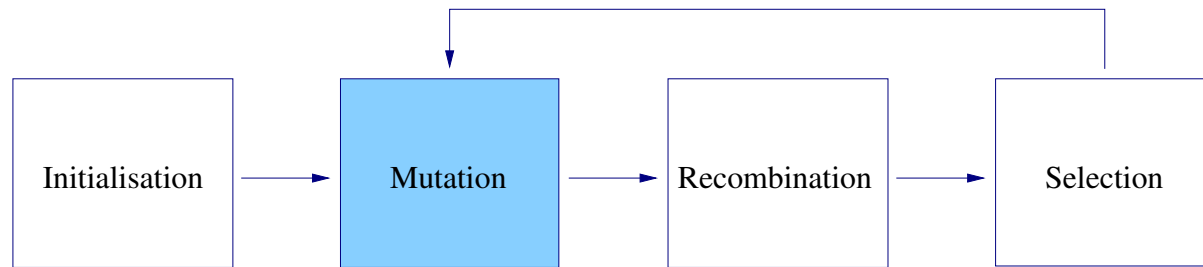
- Randomly select the initial parameter values uniformly on the intervals $[x_j^L, x_j^U]$

Mutation



- Each of the N parameter vectors undergoes mutation, recombination and selection
- Mutation expands the search space
- For a given parameter vector $x_{i,G}$ randomly select three vectors $x_{r1,G}$, $x_{r2,G}$ and $x_{r3,G}$ such that the indices $i, r1, r2$ and $r3$ are distinct

Mutation

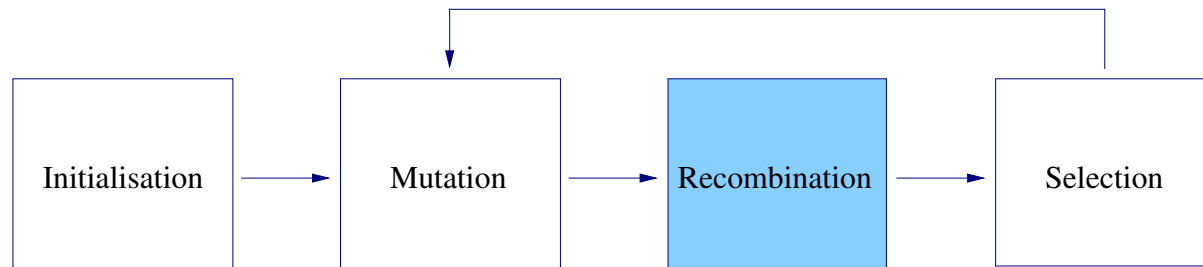


- Add the weighted difference of two of the vectors to the third

$$v_{i,G+1} = x_{r1,G} + F(x_{r2,G} - x_{r3,G})$$

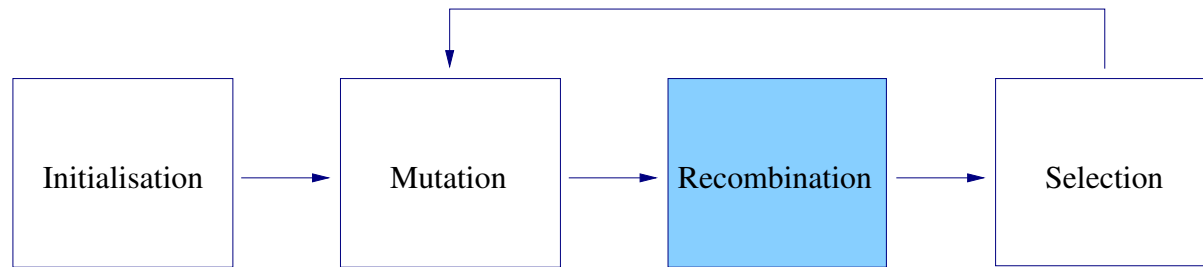
- The mutation factor F is a constant from $[0, 2]$
- $v_{i,G+1}$ is called the donor vector

Recombination



- Recombination incorporates successful solutions from the previous generation
- The trial vector $u_{i,G+1}$ is developed from the elements of the target vector, $x_{i,G}$, and the elements of the donor vector, $v_{i,G+1}$
- Elements of the donor vector enter the trial vector with probability CR

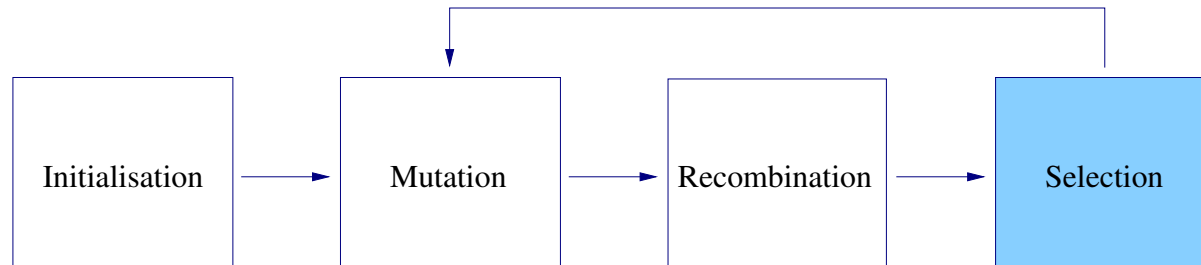
Recombination



$$u_{j,i,G+1} = \begin{cases} v_{j,i,G+1} & \text{if } \text{rand}_{j,i} \leq CR \text{ or } j = I_{\text{rand}} \\ x_{j,i,G} & \text{if } \text{rand}_{j,i} > CR \text{ and } j \neq I_{\text{rand}} \end{cases}$$
$$i = 1, 2, \dots, N; \quad j = 1, 2, \dots, D$$

- $\text{rand}_{j,i} \sim U[0, 1]$, I_{rand} is a random integer from $[1, 2, \dots, D]$
- I_{rand} ensures that $v_{i,G+1} \neq x_{i,G}$

Selection



- The target vector $x_{i,G}$ is compared with the trial vector $v_{i,G+1}$ and the one with the lowest function value is admitted to the next generation

$$x_{i,G+1} = \begin{cases} v_{i,G+1} & \text{if } f(v_{i,G+1}) \leq f(x_{i,G}) \\ x_{i,G} & \text{otherwise} \end{cases} \quad i = 1, 2, \dots, N$$

- Mutation, recombination and selection continue until some stopping criterion is reached

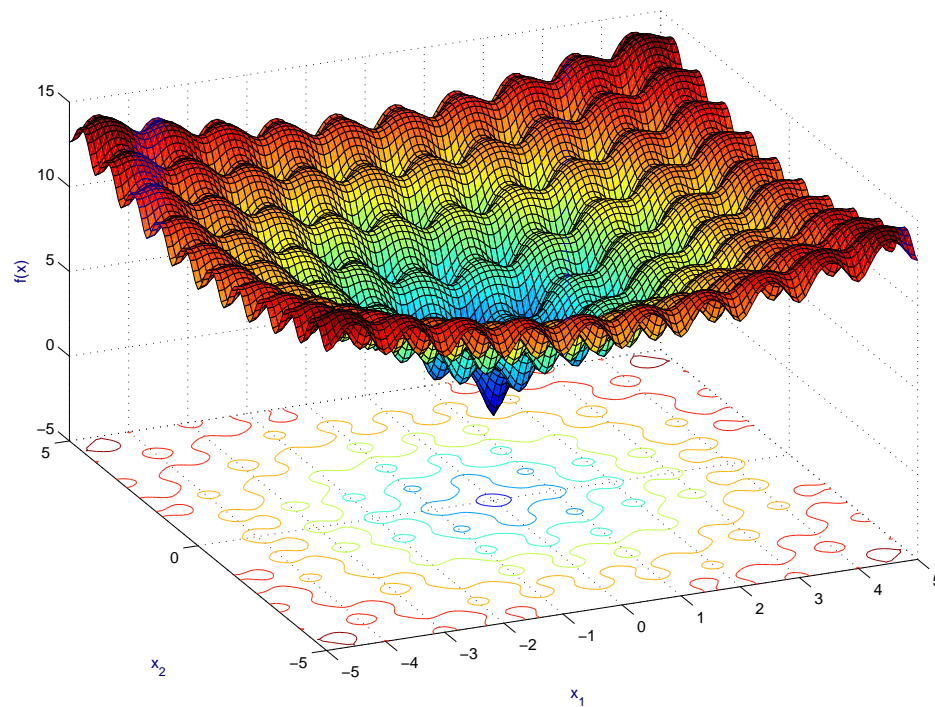
Example: Ackley's function

- DE with $N = 10$, $F = 0.5$ and $CR = 0.1$
- Ackley's function

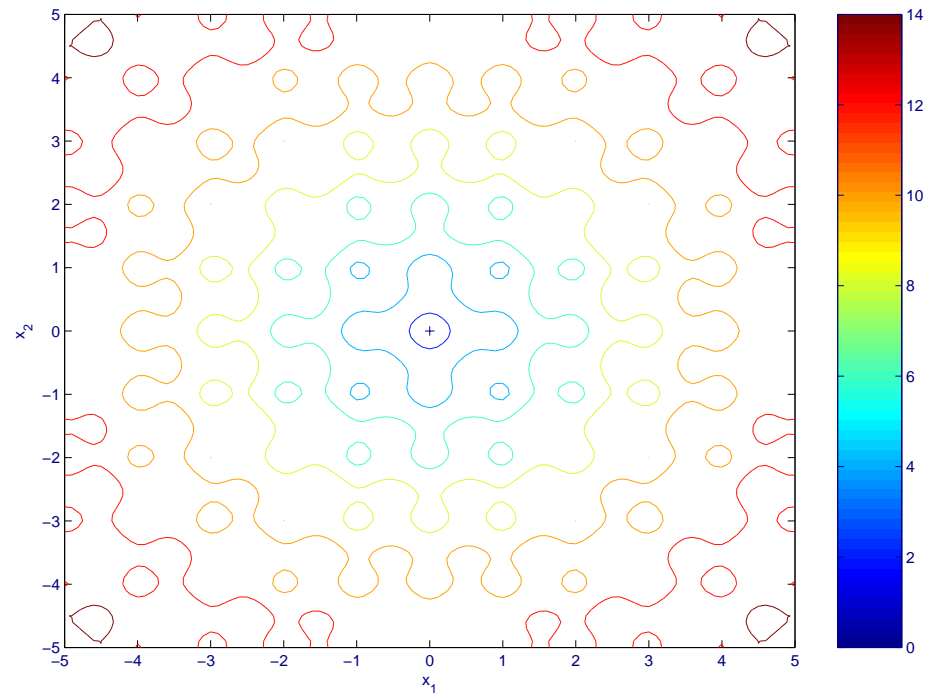
$$f(x_1, x_2) = 20 + e - 20 \exp \left(-0.2 \sqrt{\frac{1}{n} (x_1^2 + x_2^2)} \right) - \exp \left(\frac{1}{n} (\cos(2\pi x_1) + \cos(2\pi x_2)) \right)$$

- Find $x^* \in [-5, 5]$ such that $f(x^*) \leq f(x) \forall x \in [-5, 5]$
- $f(x^*) = 0$; $x^* = (0, 0)$

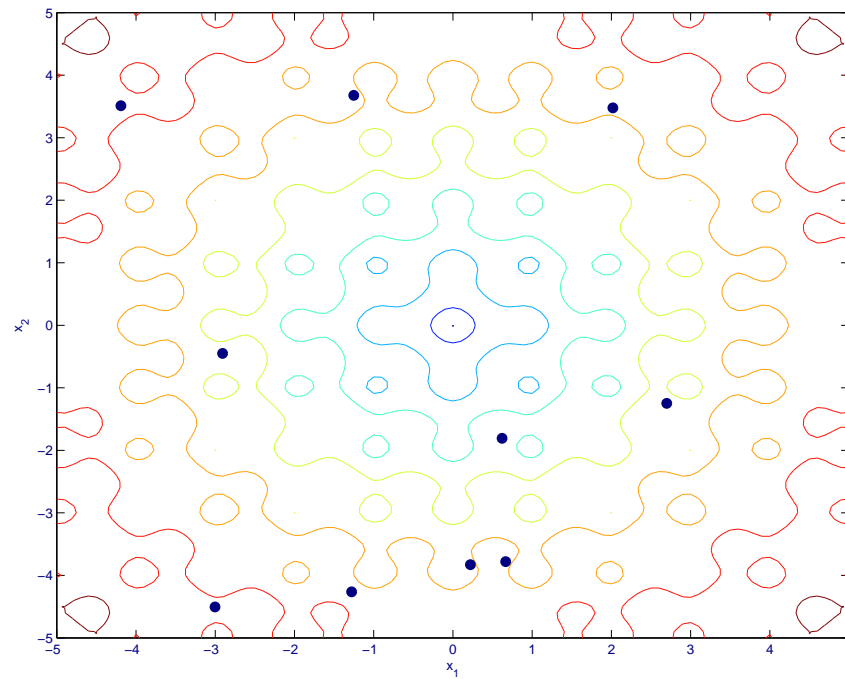
Example: Ackley's function



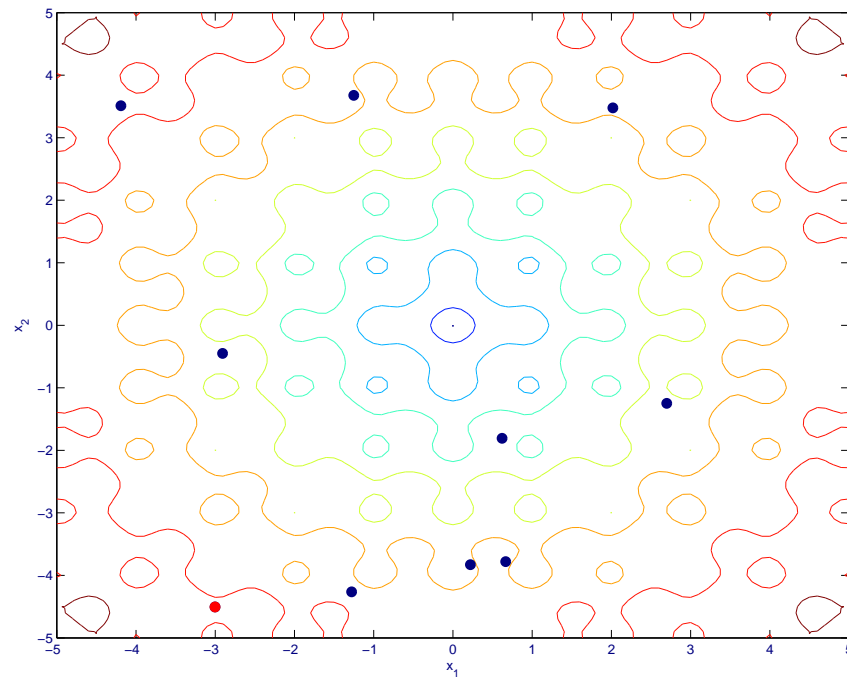
Example: Ackley's function



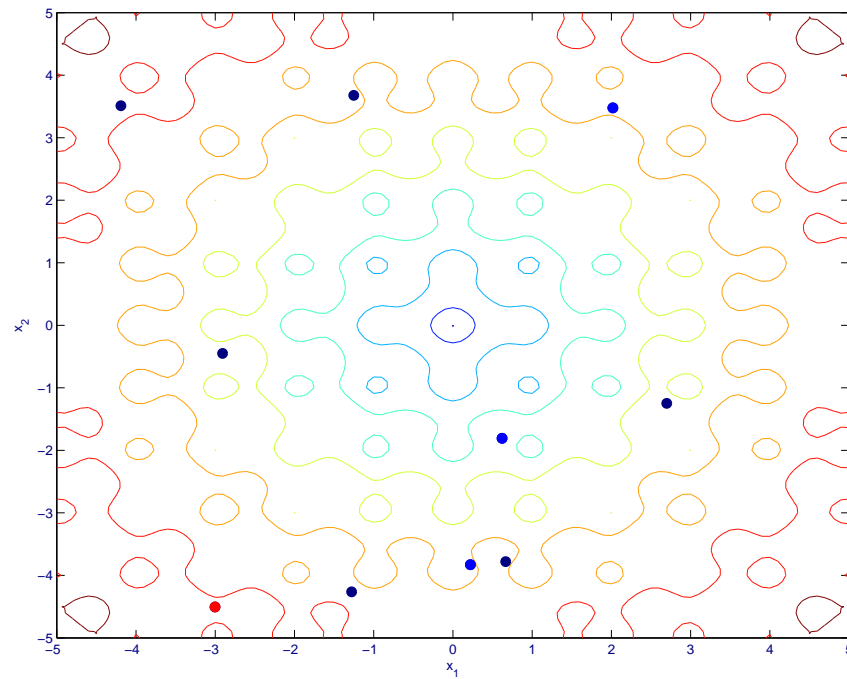
Example: Initialisation



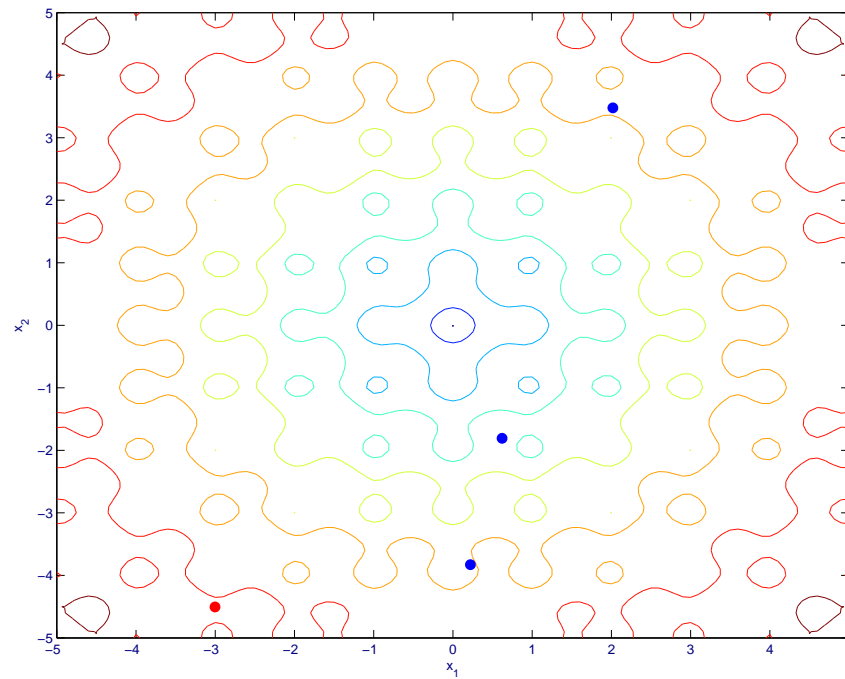
Example: Mutation



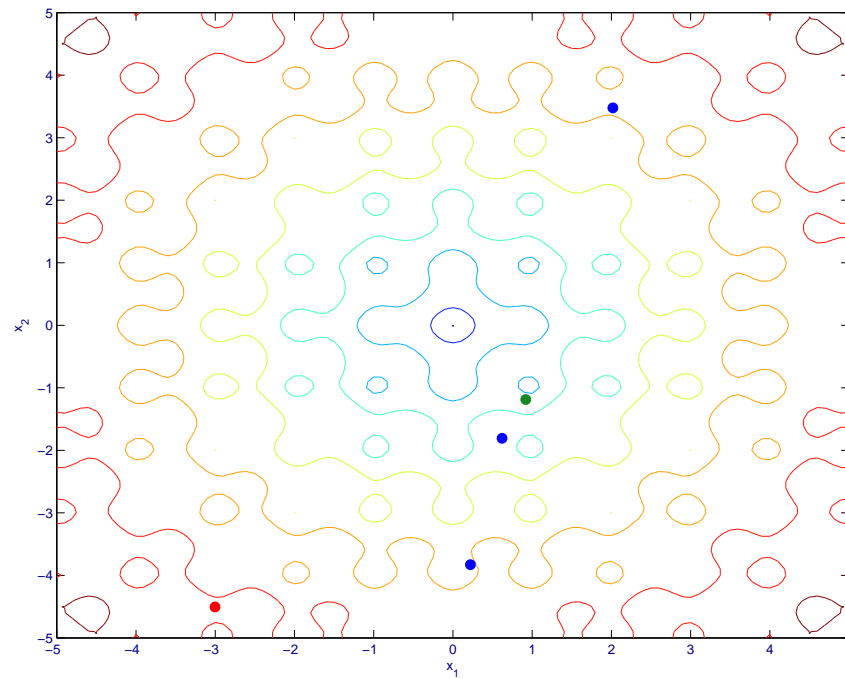
Example: Mutation



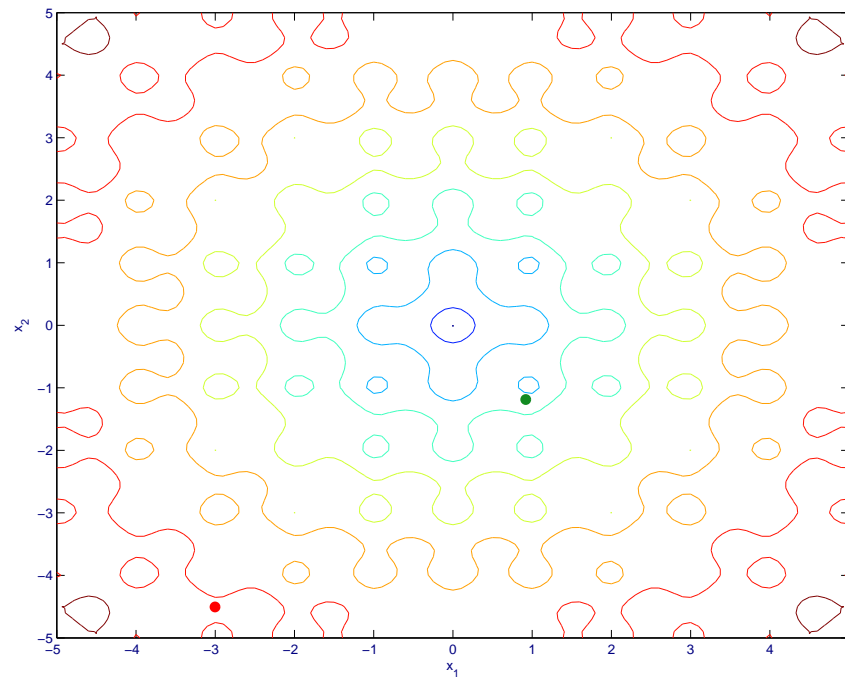
Example: Mutation



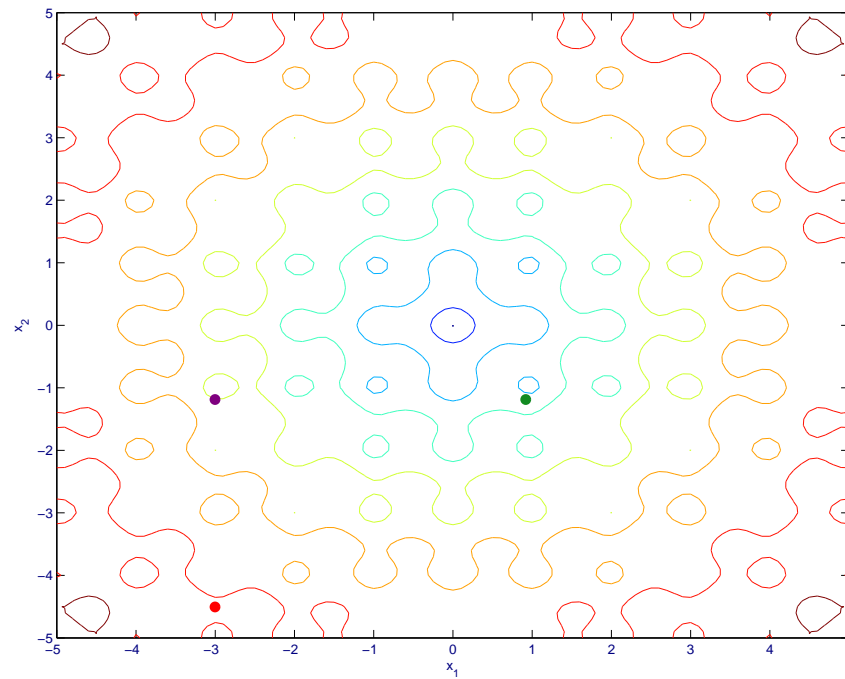
Example: Mutation



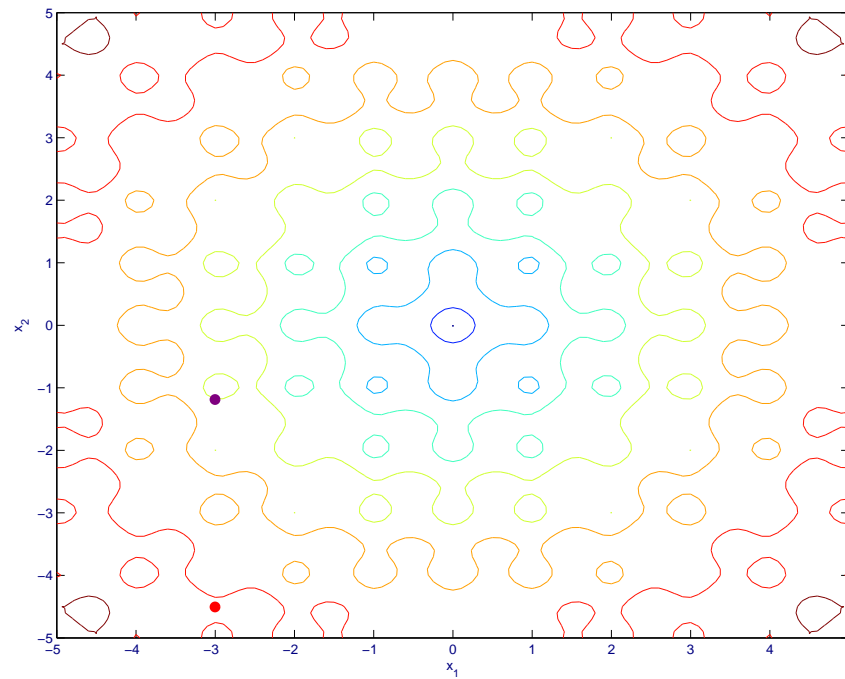
Example: Mutation



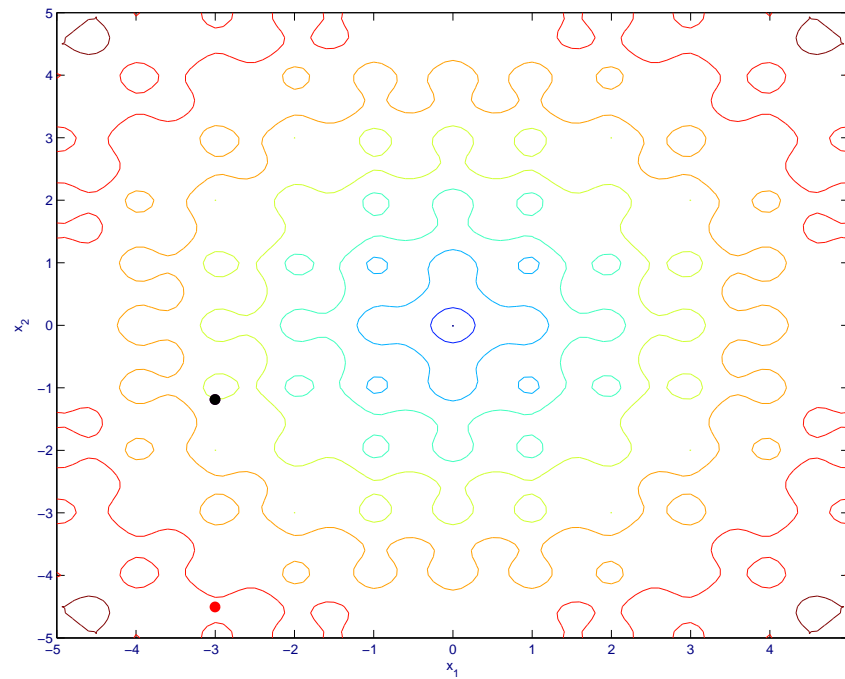
Example: Recombination



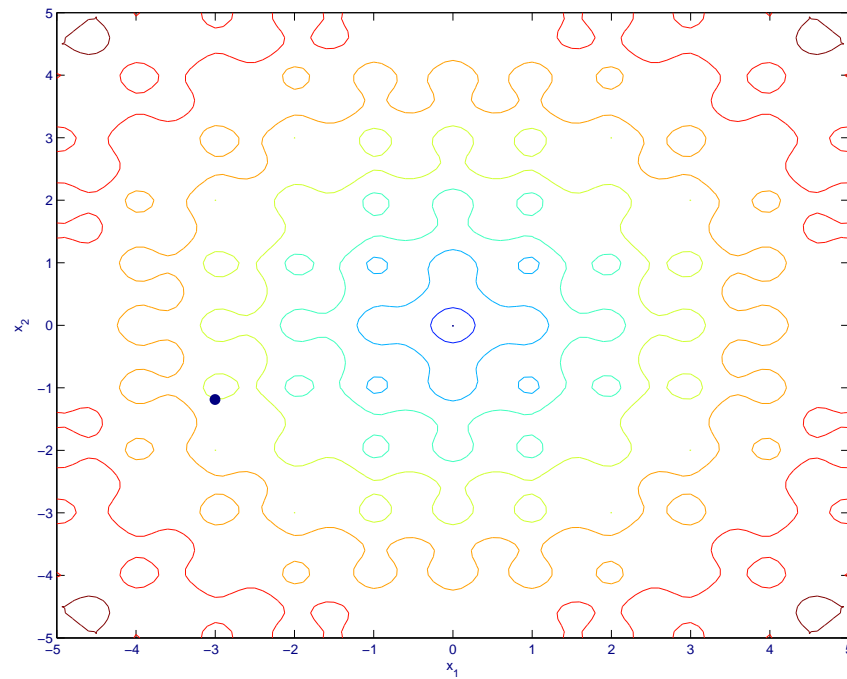
Example: Selection



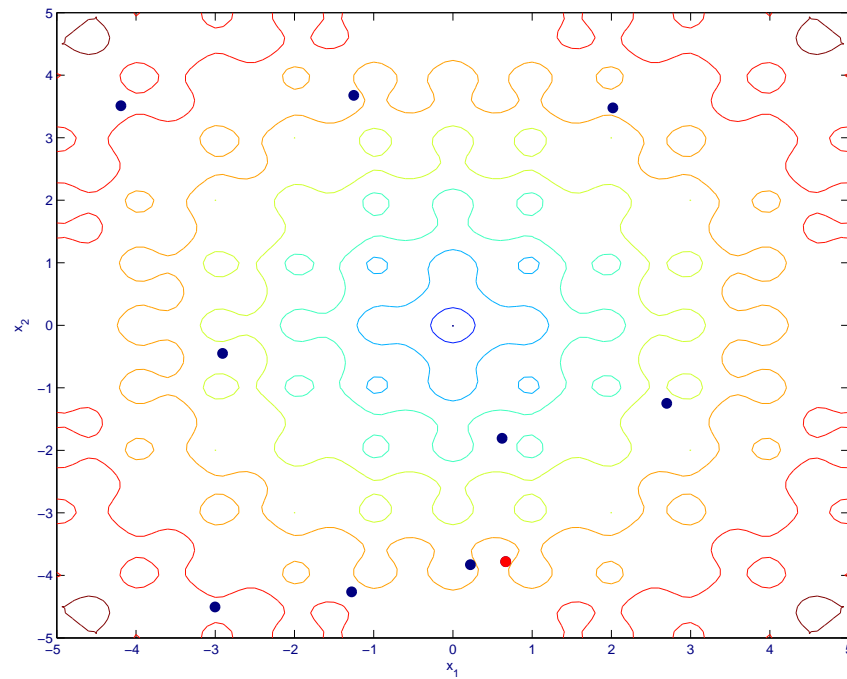
Example: Selection



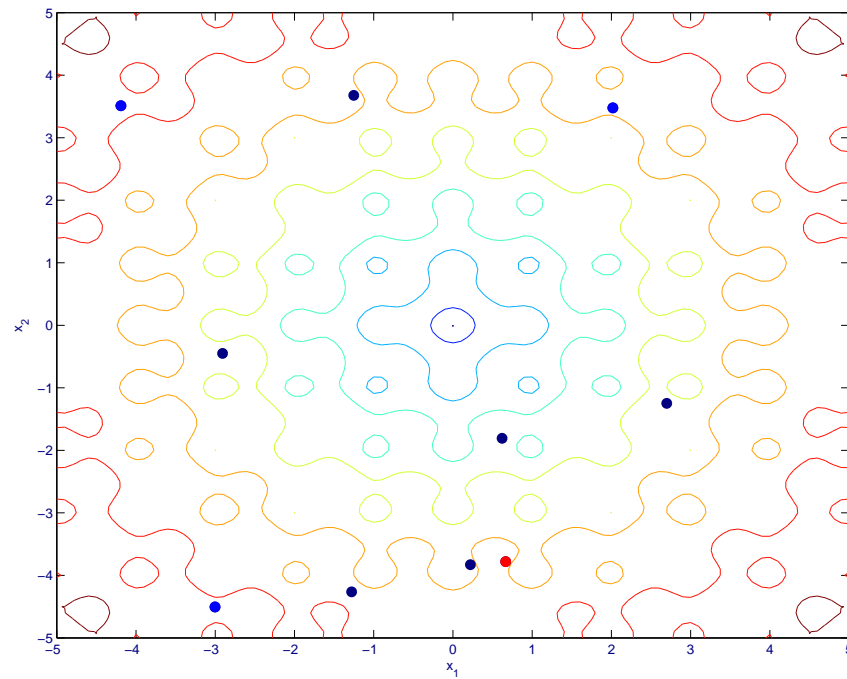
Example: Selection



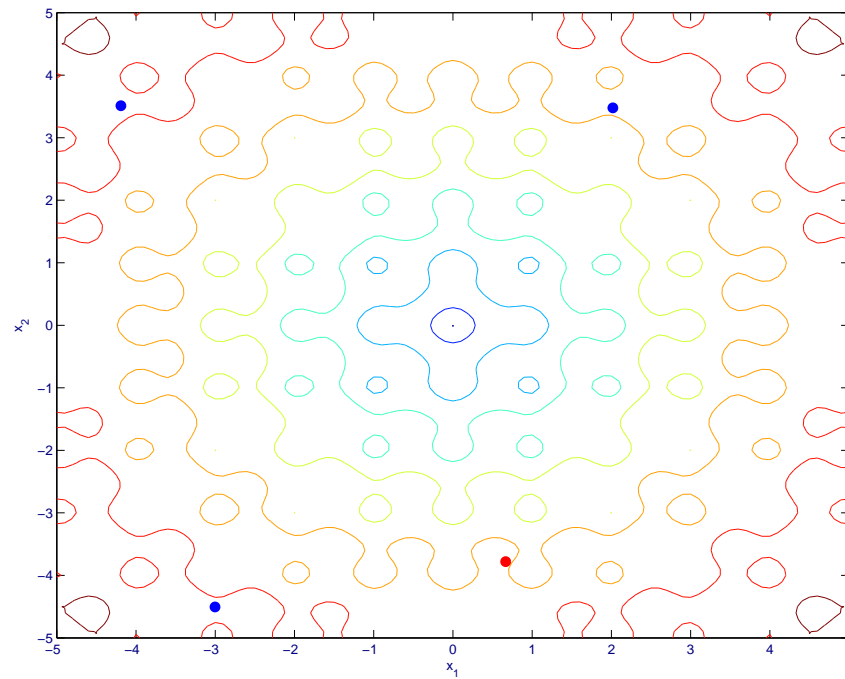
Example: Mutation



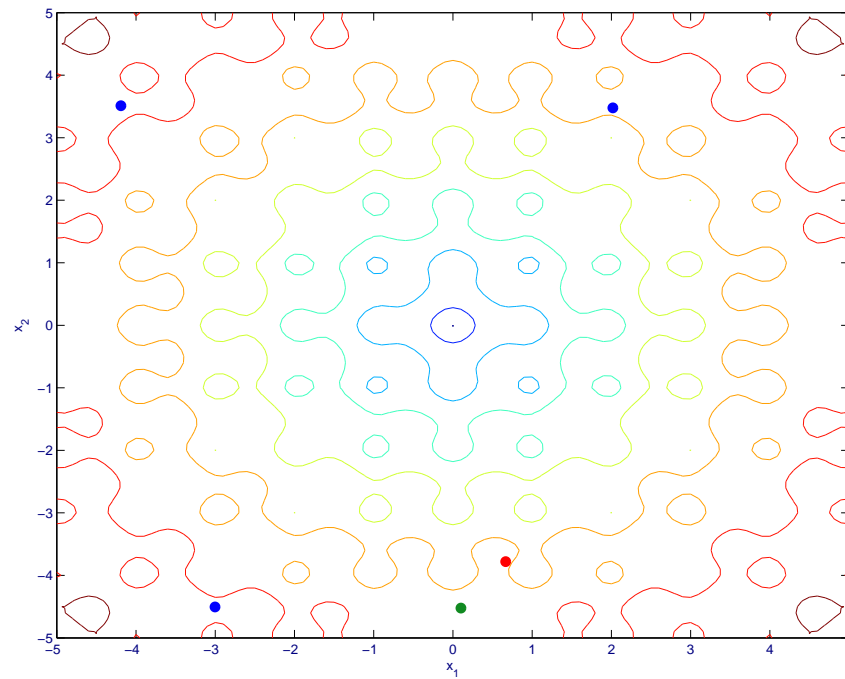
Example: Mutation



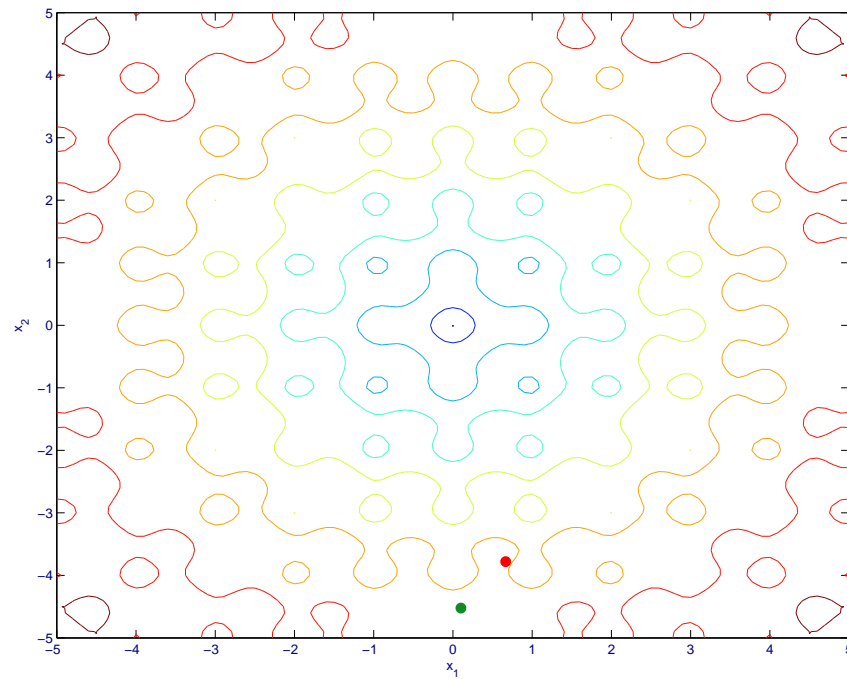
Example: Mutation



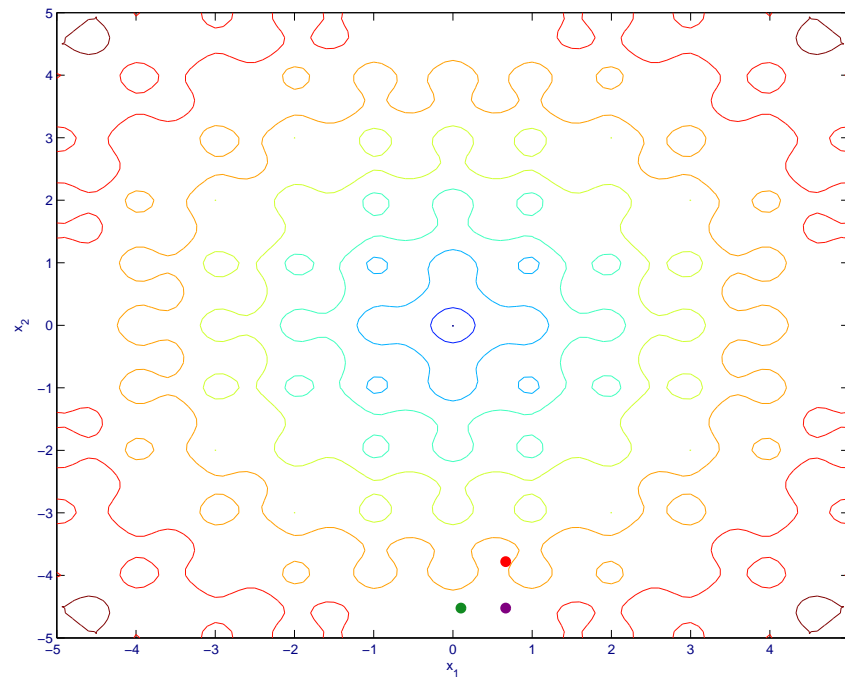
Example: Mutation



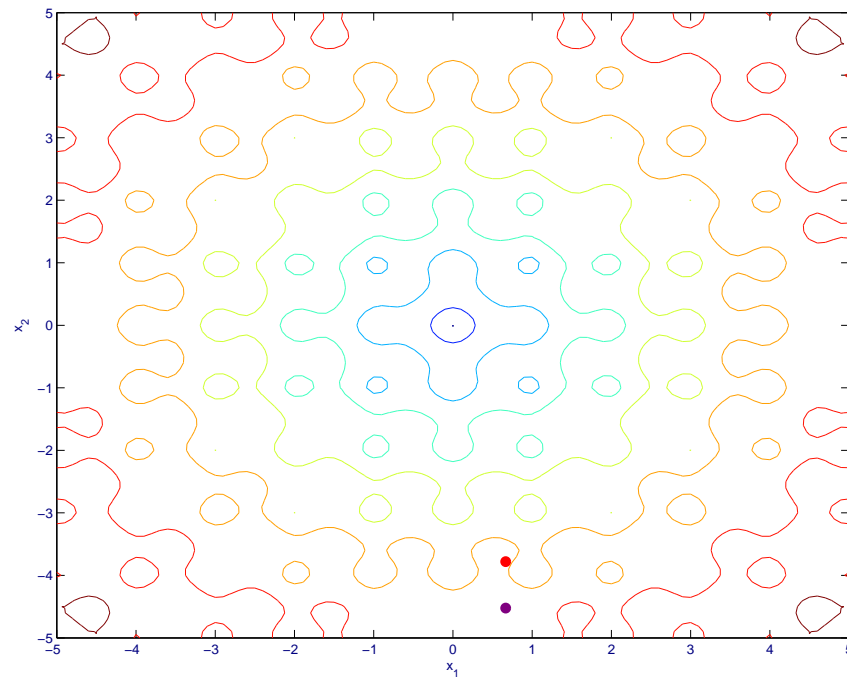
Example: Mutation



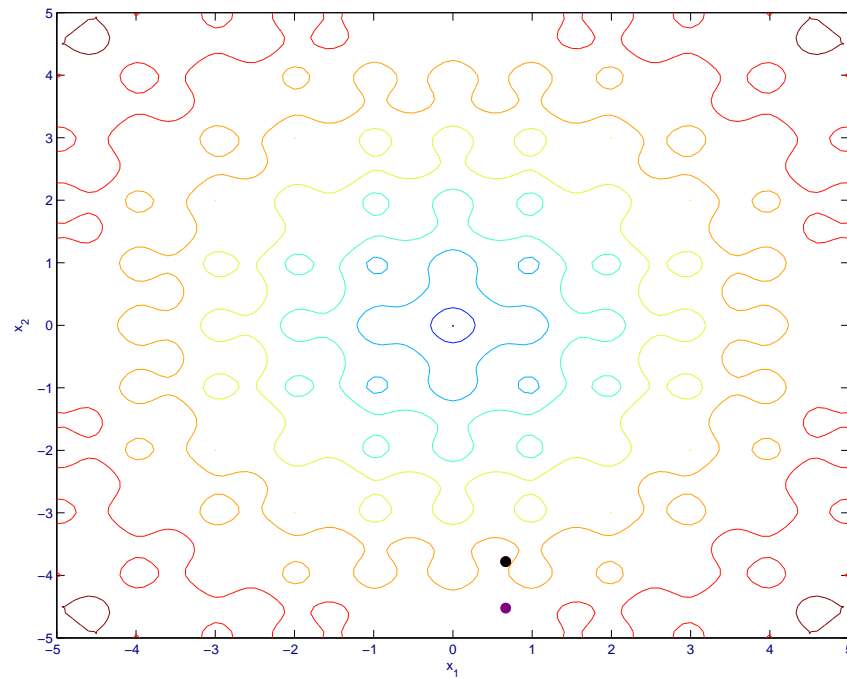
Example: Recombination



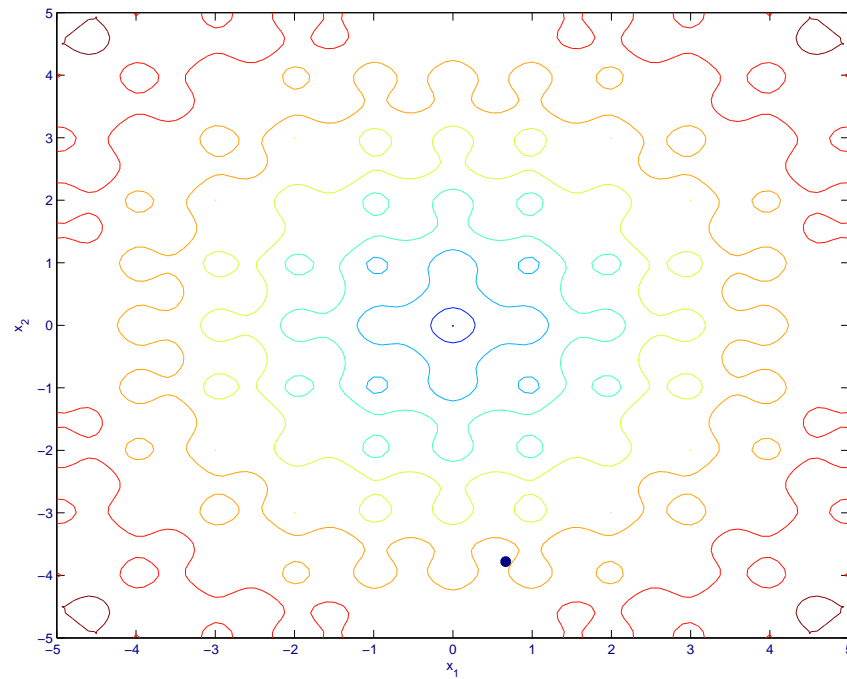
Example: Recombination



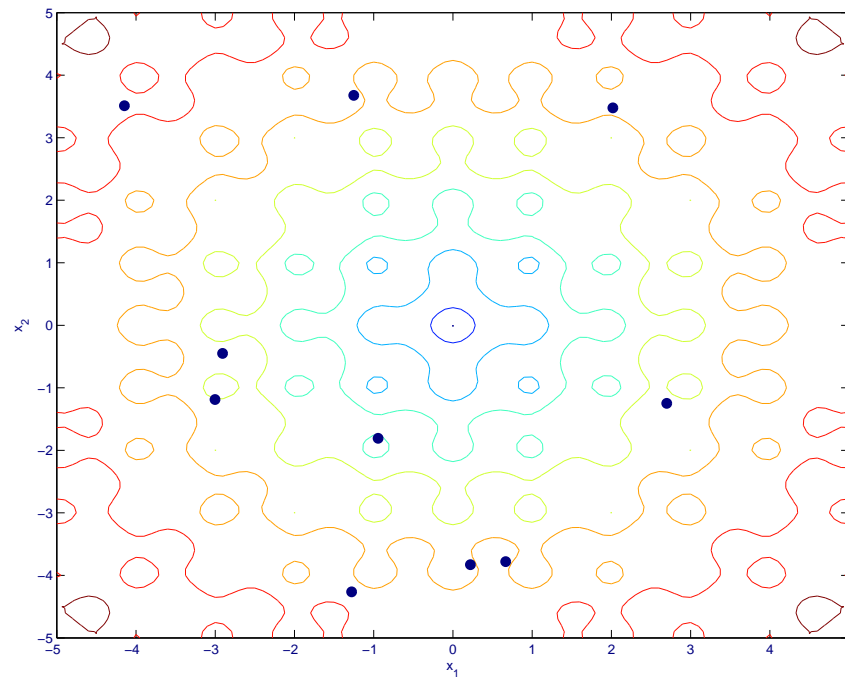
Example: Selection



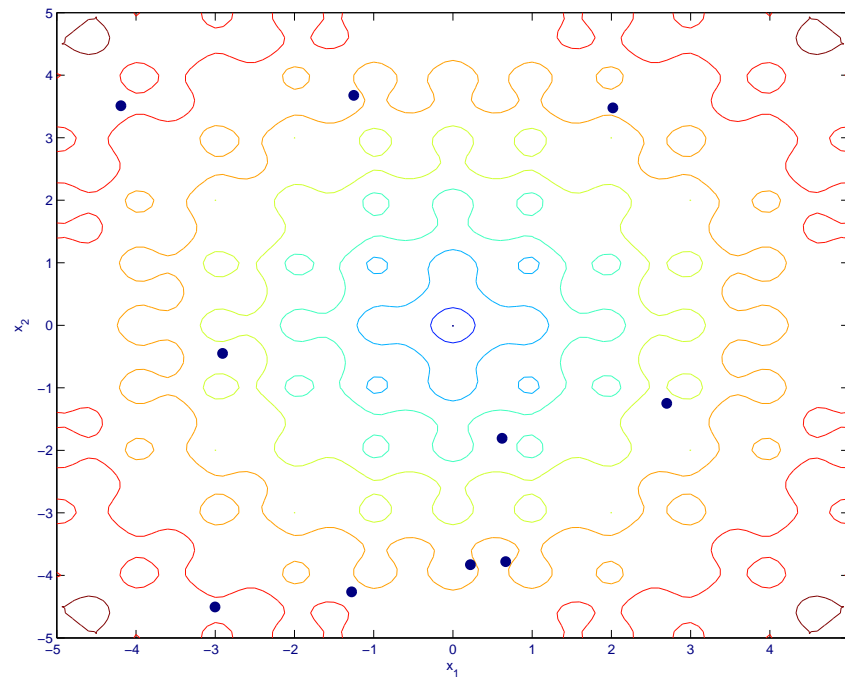
Example: Selection



Example: Generation 2



Example: Generation 1



Example: Movie

- Thirty generations of DE
- $N = 10$, $F = 0.5$ and $CR = 0.1$
- Ackley's function

Performance

- There is no proof of convergence for DE
- However it has been shown to be effective on a large range of classic optimisation problems
- In a comparison by Storn and Price in 1997 DE was more efficient than simulated annealing and genetic algorithms
- Ali and Törn (2004) found that DE was both more accurate and more efficient than controlled random search and another genetic algorithm
- In 2004 Lampinen and Storn demonstrated that DE was more accurate than several other optimisation methods including four genetic algorithms, simulated annealing and evolutionary programming

Recent Applications



- Design of digital filters
- Optimisation of strategies for checkers
- Maximisation of profit in a model of a beef property
- Optimisation of fermentation of alcohol

Further Reading

- Price, K.V. (1999), 'An Introduction to Differential Evolution' in Corne, D., Dorigo, M. and Glover, F. (eds), *New Ideas in Optimization*, McGraw-Hill, London.
- Storn, R. and Price, K. (1997), 'Differential Evolution - A Simple and Efficient Heuristic for Global Optimization over Continuous Spaces', *Journal of Global Optimization*, 11, pp. 341–359.