一． 绪论

模型误差:数学模型与实际问题之间出现的误差
观测误差:温度\长度等参量观测产生的误差
截断误差/方法误差:近似解与精确解之间的误差
舍入误差/计算误差/存储误差:计算机字长有限

误差与有效数字:

近似解的绝对误差:$e^* = x^* - x$,误差限$\varepsilon^* \ge |x^* - x|$
相对误差$e_r^* = e^*/x = (x^* - x)/x$,
一般取$e_r^* = e^*/x^*$,条件:e^*/x^*较小,相对误差限$\varepsilon_r^* = \varepsilon^*/|x^*|$
有效数字误差限$\varepsilon^* = 1/2 \times 10^{-小数位数}$
有效数字相对误差限$\varepsilon_r^* = \varepsilon$/第一位有效数字

$$\varepsilon(x_1^* \pm x_2^*) = \varepsilon(x_1^*) \pm \varepsilon(x_2^*), \varepsilon(x_1^* x_2^*) = |x_1^*|\varepsilon(x_2^*) + |x_2^*|\varepsilon(x_1^*)$$
$$\varepsilon(x_1^*/x_2^*) = (|x_1^*|\varepsilon(x_2^*) + |x_2^*|\varepsilon(x_1^*))/|x_2^*|^2$$

多元函数误差估计$A = f(x_1, x_2, \dots, x_n)$

$$\Delta A = A - A^* = \left(\frac{\partial f}{\partial x_1}\right)^* (x_1 - x_1^*) + \cdots + \left(\frac{\partial f}{\partial x_n}\right)^* (x_n - x_n^*)$$
$$|\Delta A| \le \left|\left(\frac{\partial f}{\partial x_1}\right)^*\right| \cdot |\Delta x_1| + \cdots + \left|\left(\frac{\partial f}{\partial x_n}\right)^*\right| \cdot |\Delta x_n|$$
$$\le \max\left|\left(\frac{\partial f}{\partial x_1}\right)^*\right| \cdot |\Delta x_1| + \cdots \max\left|\left(\frac{\partial f}{\partial x_n}\right)^*\right| \cdot |\Delta x_n|$$

数值计算基本原则:

1.选择数值稳定性好的公式$I_n = 1 - nI_{n-1} \to (1 - I_{n-1})/n$; 2.防止被除数远大于除数;3.防止相近的数相减;4.防止大数吃小数;5.简化计算步骤;算法的数值稳定性:在计算中舍入误差不增长
病态问题:输入数据微小扰动,输出相对误差很大
问题本身固有不受算法影响,相对误差比值$C_p = |xf'(x)/f(x)|$
迭代法:$x = x + \Delta x$,忽略高阶小量,表示Δx后带回$x = x + \Delta x$

二． 插值:多项式插值模型$P_n(x) = a_0 + a_1 x + \cdots + a_n x^n$

定理:在[a,b]上满足$P_n(x_i) = f(x_i) = y_i$的插值多项式存在唯一
证明:a 系数组成范德蒙德行列式 A,x_j互异→ $detA \ne 0 \to ok$

拉格朗日插值:

n 次多项式→n+1 个未知数→n+1 个插值节点

线性插值基函数$l_k(x) = \frac{\Pi_{i=0,i\ne k}^{n}(x - x_i)}{\Pi_{i=0,i\ne k}^{n}(x_k - x_i)} = \begin{cases}1, i = k\\ 0, i \ne k\end{cases}$

令$\omega_{n+1}(x) = \Pi_{k=0}^{n}(x - x_k), \omega'_{n+1}(x_k) = \Pi_{i=0,i\ne k}^{n}(x_k - x_i)$

$l_i(x) = \frac{\omega_{n+1}(x)}{(x - x_i)\omega'_{n+1}(x_i)}, L_n(x) = \sum_{i=0}^{n} l_i(x) y_i = \sum_{i=0}^{n} \frac{\omega_{n+1}(x)}{(x - x_i)\omega'_{n+1}(x_i)} y_i$

插值余项和误差估计:

插值余项:$R_n(x) = f - L_n = \frac{f^{(n+1)}(\xi)}{(n+1)!}\omega_{n+1}(x)$

令$M_{n+1} = max|f^{(n+1)}(x)|, |R_n(x)| \le \frac{M_{n+1}}{(n+1)!}|\omega_{n+1}(x)|$

若$f(x) \in H_n \to R_n = 0, L_n(x) = f(x)$

均差与牛顿插值:

$\boldsymbol{N_n(x) = a_0 + a_1(x - x_0) + \cdots + a_n(x - x_0)(x - x_1) \dots (x - x_{n-1})}$
一阶:$f[x_0, x_k] = (f(x_k) - f(x_0))/(x_k - x_0)$
二阶:$f[x_0, x_1, x_k] = (f[x_0, x_k] - f[x_0, x_1])/(x_k - x_1)$
K:$f[x_0, x_1, \dots, x_k] = (f[x_0, x_1, \dots, x_{k-2}, x_k] - f[x_0, x_1, \dots, x_{k-1}])/(x_k - x_{k-1})$
$= \sum_{j=0}^{k} \frac{f(x_j)}{(x_j - x_0)(x_j - x_1)\dots(x_j - x_{j-1})(x_j - x_{j+1})\dots(x_j - x_k)} = \sum \frac{f(x_j)}{\omega'_{k+1}(x_j)}$

$$\begin{cases} f(x) = f(x_0) + f[x, x_0](x - x_0) \\ f[x, x_0] = f[x_0, x_1] + f[x, x_0, x_1](x - x_1) \\ f[x, x_0, \dots, x_{n-1}] = f[x_0, \dots, x_n] + f[x, x_0, \dots, x_n](x - x_n) \end{cases}$$

$\boldsymbol{f(x)} = f(x_0) + f[x_0, x_1](x - x_0) + \cdots + f[x_0, \dots, x_n](x - x_0) \dots (x - x_{n-1}) + f[x, x_0, \dots, x_n](x - x_0)(x - x_1) \dots (x - x_{n-1})(x - x_n)$
$\boldsymbol{N_n(x)} = f(x_0) + f[x_0, x_1](x - x_0) + \cdots + f[x_0, \dots, x_n](x - x_0) \dots (x - x_{n-1})$

$$\boldsymbol{R_n(x) = f(x) - N_n(x) = f[x, x_0, \dots, x_n]\omega_{n+1}(x)}$$
$$\boldsymbol{f[x, x_0, \dots, x_n] = f^{(n+1)}(\xi)/((n+1)!)}$$

当$x \in [x_k, x_{k+1}]$,选用x_k附近插值节点:$x_k, x_{k+1}, x_{k-1}, x_{k+2}, \dots$
$f(x) = f(x_k) + f[x_k, x_{k+1}](x - x_k) + f[x_k, x_{k+1}, x_{k-1}](x - x_k)(x - x_{k+1}) + \cdots$

等距节点的牛顿插值公式

$\Delta f_k = f_{k+1} - f_k, \nabla f_k = f_k - f_{k-1}, \delta f_k = f_{k+1/2} - f_{k-1/2}$
$E f_k = f_{k+1}, E^{-1} f_k = f_{k-1}, I f_k = f_k, \Delta = E - I, \nabla = I - E^{-1}$
$\Delta^n f_k = \sum_{j=0}^{n}(-1)^j C_n^j f_{k+n-j}, \nabla^n f_k = \sum_{j=0}^{n}(-1)^j C_n^j f_{k-j}$
$$f[x_k, x_{k+1}, \dots, x_{k+n}] = \Delta^n f_k/(n!\, h^n) = \nabla^n f_{k+n}/(n!\, h^n)$$
$$= f^{(n)}(\xi)/n! \Rightarrow \Delta^n f_k = h^n f^{(n)}(\xi), \xi \in [x_k, x_{k+n}]$$
前插$x = x_0 + th, x_k = x_0 + kh \Rightarrow x - x_k = (t - k)h$
$$f(x) = f(x_0) + \Delta f_0 \cdot t + \Delta^2 f_0 t(t - 1)/2 + \cdots$$
后插$x = x_n + th, x_{n-i} = x_n - ih \Rightarrow x - x_{n-i} = (t + i)h$
$$f(x) = f(x_n) + \nabla f_n \cdot t + \nabla^2 f_n t(t + 1)/2 + \cdots$$
$R_n(x) = \frac{t(t-1)\dots(t-n)}{(n+1)!} h^{n+1} f^{(n+1)}(\xi), \xi \in (x_0, x_n)$

埃尔米特插值:

$H(x_i) = f(x_i) = y_i, H'(x_i) = f'(x_i) = y'_i$,
$2n + 2$个方程→$2n + 2$个未知数→$H_{2n+1}(x)$
定理:存在且唯一
构造法:$H_{2n+1}(x) = \sum_{i=0}^{n} y_i \alpha_i + \sum_{i=0}^{n} y'_i \beta_i(x)$
$$\alpha_i(x_j) = 1, i = j; 0, i \ne j; \alpha'_i(x_i) = 0$$
$$\beta_i(x_j) = 0, \beta'_i(x_j) = 1, i = j; 0, i \ne j$$
$\alpha_i(x) = [1 - 2(x - x_i)\sum_{j=0,j\ne i}^{n} 1/(x_i - x_j)] l_i^2(x)$
$\beta_i(x) = (x - x_i) l_i^2(x_i)$

误差余项:$R_{2n+1}(x_i) = K(x)\omega_{n+1}^2(x) = \frac{f^{(2n+2)}(\xi)}{(2n+2)!}\omega_{n+1}^2(x)$

分段低次插值:

分段线性插值:把[a,b]分为 n 个子区间$[x_k, x_{k+1}]$,分别在区间上进行线性插值$\frac{x - x_{k+1}}{x_k - x_{k+1}} y_k + \frac{x - x_k}{x_{k+1} - x_k} y_{k+1}$
$$|R(x)| \le M_2 h^2/8$$
分段埃尔米特插值:两点三次埃尔米特
$$|R(x)| \le \frac{1}{4!} M_4 \max|(x - x_k)(x - x_{k+1})| \le \frac{h^4}{384} M_4$$

三次样条插值:

插值函数二次可导,均为 n 个区间,三次多项式,共 4n 个未知量
插值节点方程(n+1),区间左右原\一阶导\二阶导(3n-3),缺 2
$$|R(x)| \le \frac{5}{384} M_4 h^4$$

三． 最佳逼近

$||f(x) - P(x)||_\infty = \max_{a \le x \le b} |f(x) - P(x)|$

$||f(x) - P(x)||_2 = \sqrt{\int_a^b [f(x) - P(x)]^2 dx}$

维尔斯特拉斯定理:设$f(x) \in C[a, b], \forall \epsilon > 0, \exists P_n(x)$
使$|f(x) - P_n(x)| < \epsilon$在$[a, b]$上一致成立
$H_n = span\{1, x, x^2, \dots, x^n\}$,求$P_n(x) \in H_n$使范数最小

最佳一致逼近

定义偏差$\Delta(f, P_n) = \max |f(x) - P_n(x)|$,
若$P_n(x_0) - f(x_0) = \Delta(f, P_n)$,称$x_0$为正偏差点。
$\Delta(f, P_n^*) = \min_{P_n \in H_n} \Delta(f, P_n) = E_n \Rightarrow P_n^*$是最佳一致逼近多项式
切比雪夫定理:$P_n^*(x)$是$f(x) \in C[a, b]$的最佳一致逼近多项式⇔$P_n^*(x)$与$f(x)$在$[a, b]$上至少存在 n+2 个正负相间的偏差点⇔$P_n^*(x) - f(x)$至少有 n+1 个零点⇔ $P_n^*(x)$是 n 次多项式
切比雪夫多项式:$T_n(x) = \cos(narccosx), x \in [-1,1]$
$$T_0(x) = 1, T_1(x) = x, T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$$
$T_n(x)$是 n 次多项式,首项系数是2^{n-1},有奇偶性(奇次奇,偶次偶)
$x \in [-1,1], |T_n(x)| \le 1$ 在 $x_k = \cos(k\pi/n)$, $k = 0,1, \dots, n$ 处$T_n(x)$轮流取最大值 1 和最小值-1
$T_n(x)$在$[-1,1]$上有 n 个零点$x_k^* = \cos((2k - 1)\pi/2n)$
在[-1,1]上带权$1/\sqrt{1 - x^2}$正交

$$\int_{-1}^{1} \frac{T_n(x)T_m(x)}{\sqrt{1 - x^2}} = \begin{cases} 0, n \ne m \\ \pi/2, n = m \ne 0 \\ \pi, n = m = 0 \end{cases}$$

局限$f(x)$是仅比$P_n(x)$高 1 次的多项式,$f(x) - P_n(x) = kT_{n+1}(x)$
最小零偏差多项式定理:$\omega_n(x) = T_n(x)/2^{n-1}$与零偏差最小且$\Delta(\omega_n, 0) = 1/2^{n-1}$
拉格朗日插值函数余项最小化:寻找最佳一致逼近多项式=寻找插值节点; $\max|R_{n-1}(x)| \le M_n \max|\omega_n(x)|/n!$
当插值节点$x_k = \cos((2k - 1)\pi/2n)$时,$\omega_n(x) = T_n(x)/2^{n-1}$为最小零偏差多项式, $\max_{-1 \le x \le 1}|f(x) - L_{n-1}(x)| \le M_n/(n!\, 2^{n-1})$
区间变换:
$f(x) \in C[a, b], x \in [a, b] x = (a + b)/2 + (b - a)t/2, t \in [-1,1]$
$\omega_n(x) = \omega_n\left(\frac{a+b}{2} + \frac{t}{2}(b - a)\right) = \omega_n^*(t) = \left(\frac{b-a}{2}\right)\frac{T_n(t)}{2^{n-1}}$

$\max|f(x) - L_{n-1}^*(x)| \le \frac{M_n}{n!}\frac{(b-a)^n}{2^{2n-1}}$

最佳平方逼近:
权函数$\rho(x) \ge 0 \forall x \in [a, b]; \int_a^b \rho(x)g(x)dx = 0 \& g \ge 0 \to g = 0$
$(f, f) \ge 0, \&\&(f, f) = 0 \Leftrightarrow f = 0, ||f||_2 = (f, f)^{0.5}$
定义:设$f(x) \in C[a, b], \Phi = span\{\varphi_0, \varphi_1, \dots, \varphi_n\}, \forall S(x) \in \Phi$,有$\int_a^b \rho(x)[S^*(x) - f(x)]^2 dx \le \int_a^b \rho(x)[S(x) - f(x)]^2 dx$,$S^*$ 最佳平方逼近, $S(x) = \sum_{j=0}^{n} a_j \varphi_j(x)$
最佳平方逼近求解:
令$S(x) = \sum_{j=0}^{n} a_j \varphi(x)$,$k$个方程$\sum_{j=0}^{n} a_j(\varphi_j, \varphi_k) = (f, \varphi_k) = d_k$

$I = \int_a^b \rho(x)\left[\sum a_j \varphi_j(x) - f(x)\right]^2 dx$

均方误差$\int_a^b \rho(x)[S(x) - f(x)]^2 dx = (f - S, f - S) = ||f||_2^2 - \sum_{j=0}^{n} a_j d_j$
按正交多项式展开:
$\Phi = span\{\varphi_0, \varphi_1, \dots, \varphi_n\}$是正交多项式组,$a_k = (f, \varphi_k)/(\varphi_k, \varphi_k)$
均方误差$||f||_2^2 - \sum_{j=0}^{n}(f, \varphi_j)^2/(\varphi_j, \varphi_j)$
按切比雪夫多项式展开$f(x) \in C[-1,1], S^*(x) = \sum a_j T_j(x)$,
$\rho = 1/\sqrt{1 - x^2}, C_k = 2(f, T_k)/\pi, S^*(x) = C_0/2 + \sum C_j T_j(x)$
误差:$f(x) - S^*(x) = C_{n+1}T_{n+1}(x)$
$S^*(x)$既可以看作$f(x)$的 n 次最佳平方逼近多项式,也可以看作它的近似最佳一直逼近多项式
按勒让得多项式展开:

$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}[(x^2 - 1)^n]$,首项系数$\frac{2n(2n-1)\dots(n+1)}{2^n n!}$

记首项系数为 1 的勒让得多项式为$\widetilde{P_n(x)} = \frac{n!}{(2n)!}\frac{d^n}{dx^n}[(x^2 - 1)^n]$

$$P_0(x) = 1, P_1(x) = x, (n + 1)P_{n+1} = (2n + 1)xP_n - nP_{n-1}$$
权函数$\rho(x) = 1, (P_n, P_m) = 2/(2n + 1), n = m; 0, n \ne m$
奇次奇,偶次偶; n 次多项式有 n 个零点; 在$[-1,1]$所有最高次系数为 1 的 n 次多项式中$\widetilde{P_n(x)}$与零函数平方误差最小
$S^*(x) = \sum_{j=0}^{n} a_j P_j(x)$, $a_k = (2k + 1)(f, P_k)/2$
平方误差$||f||_2^2 - \sum_{j=0}^{n}(2j + 1)(f, P_j)^2/2$
曲线拟合:
对已知的$(x_i, y_i), i \in [0, m]$在$\Phi = span\{\varphi_0, \varphi_1, \dots, \varphi_n\}$中求$S^*(x) = \sum_{j=0}^{n} a_j \varphi_j(x)$使$||\delta||_2^2 = \sum_{i=0}^{m}[S^*(x_i) - y_i]^2 = \min_{S(x) \in \Phi} \sum[S(x_i) - y_i]^2$
根据(x_i, y_i)规律确定Φ和权函数$\omega(x)$,求解

$\sum_{j=0}^{m} a_j(\varphi_j, \varphi_k) = (f, \varphi_k) = d_k$

其中$(\varphi_j, \varphi_k) = \sum_{j=0}^{m} \omega(x_i)\varphi_j(x_i)\varphi_k(x_i)$

四． 数值积分与微分

梯形公式$\int_a^b f(x)dx \approx [f(a) + f(b)](b - a)/2$
矩形公式$\int_a^b f(x)\, dx \approx (b - a)f((a + b)/2)$
$I = \int_a^b f(x)dx = \int Ln(x)dx = \int \sum_{k=0}^{n} l_k(x)f(x_k)\, dx = \sum A_k f(x_k)$

其中$A_k = \int_a^b l_k(x)\, dx$,截断误差$R[f] = \int_a^b \frac{f^{(n+1)}(\xi)}{(n+1)!}\omega_{n+1}(x)dx$

代数精度:若关于次数$\le m$的多项式, $\int_a^b f(x)dx = \sum_{k=0}^{n} A_k f(x_k)$均能准确成立, 那么该公式至少具有 m 次代数精度
具有$\int_a^b f(x)dx = \sum_{k=0}^{n} A_k f(x_k)$形式的求积公式如果具有$\ge n$次代数精度, 那么它一定是插值型,即$A_k = \int_a^b l_k(x)\, dx$
牛顿-柯特斯公式:
将区间平分 n 份,$h = (b - a)/n, x_k = a + kh, x = a + th, 0 \le t \le n; x = x_k = (t - k)h$

$A_k = \int_a^b l_k(x)dx = \int_a^b \Pi_{j\ne k}(x - x_j)/\Pi_{j\ne k}(x_k - x_j)\, dx$

$= h\int_0^n \Pi_{j\ne k}(t - j)/(k - j)dt = \frac{(-1)^{n-k}}{k!(n-k)!}\frac{b-a}{n}\int_0^n \Pi_{j=0,j\ne k}^{n}(t - j)dt$

令$C_k^{(n)} = \frac{(-1)^{n-k}}{nk!(n-k)!}\int_0^n \Pi_{j=0,j\ne k}^{n}(t - j)dt$, $A_n = (b - a)C_k^{(n)}$

$I = \sum_{k=0}^{n} A_k f(x_k) = (b - a)\sum_{k=0}^{n} C_k^{(n)} f(x_k)$

一阶:$T = [f(a) + f(b)](b - a)/2$ →梯形公式

二阶:$S = \frac{(b-a)}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$ →辛甫生公式
四阶柯特斯公式
$n \ge 8$时$\sum_{k=1}^{n}|A_k|$渐趋无界,不实用
代数精度:
当阶数 n 为偶数时, 牛顿-柯特斯公式至少有 n+1 次代数精度

截断误差:$\boldsymbol{R[f] = \int_a^b \frac{f^{(n+1)}(\xi)}{(n+1)!}\omega_{n+1}(x)dx}$

梯形公式:$R_T = \int \frac{f''(\xi)}{2}(x - a)(x - b)dx$,

$\frac{\max_y\{f''(y)\}}{2}\int_a^b (x - a)(x - b) \le R_T \le \frac{\min_y\{f''(y)\}}{2}\int_a^b (x - a)(x - b)$

$R_T = -(b - a)^3/12 \cdot f''(\eta)$
辛普生公式$R_s = I - S = \int f'''(\xi)(x - a)(x - \frac{a+b}{2})(x - b)/3!$
由于辛甫生具有三次代数精度,构造两点三次插值多项式 H(x)

$H(a) = f(a), H(b) = f(b), H\left(\frac{a+b}{2}\right) = f(\frac{a+b}{2}), H'(\frac{a+b}{2}) = f'(\frac{a+b}{2})$

$f(x) - H(x) = \frac{f^{(4)}(\xi)}{4!}(x - a)\left(x - \frac{a+b}{2}\right)^2 (x - b)$

$\int_a^b H(x)dx = S(H) = \sum A_k f(x_k) = S(f)$

$R_s = I - S = \int_a^b [f(x) - H(x)]dx = -\frac{(b-a)^5}{2880} f^{(4)}(\eta)$

复化求积公式:
将[a,b]等分, 在每个区间上使用低阶牛顿-柯特斯公式
复化梯形公式:
$I = \frac{h}{2}[f(a) + 2\sum_{k=1}^{n-1} f(x_k) + f(b)]$
$R[f] = \sum_{k=0}^{n-1} -\frac{h^3}{12}f''(\eta_k) = -\frac{nh^3}{12}f''(\eta) = -\frac{(b-a)h^2}{12}f''(\eta)$

$I - T_n = -\frac{h^2}{12}\sum_{k=0}^{n-1} f''(\eta_k) \cdot h, \sum_{k=0}^{n-1} f''(\eta_k) \cdot h \xrightarrow{h\to 0} \int_a^b f''(x)dx = f'(b) - f'(a)$

$I - T_n = o(h^2)$,二阶收敛
复化辛甫生公式:
$I = \frac{h}{6}[f(a) + 2\sum_{k=1}^{n-1} f(x_k) + 4\sum_{k=0}^{n-1} f\left(x_k + \frac{h}{2}\right) + f(b)]$
$R[f] = \sum_{k=0}^{n-1} -\frac{h^5}{2880}f^{(4)}(\eta_k) = -\frac{h^5}{2880}nf^{(4)}(\eta)$

$I - S_n = -\frac{h^4}{2880}\sum_{k=0}^{n-1} hf^{(4)}(\eta_k), \sum_{k=0}^{n-1} hf^{(4)}(\eta_k) \xrightarrow{h\to 0} \int_a^b f^{(4)}(x) = f^{(3)}(b) - f^{(3)}(a)$

$I - S_n = o(h^4)$,四阶收敛
复化柯特斯公式,六阶收敛
龙贝格算法和外推加速法:
复化梯形公式的递推化
$T_1 \to T_2 \to T_4 \to \cdots, T_n = \sum_{k=0}^{n-1}\frac{h}{2}[f(x_k) + f(x_{k+1})]$
即$[x_k, x_{k+1}] \to \left[x_k, x_{k+\frac{1}{2}}\right]\left[x_{k+\frac{1}{2}}, x_{k+1}\right]$
$T_{2n} = \frac{T_n}{2} + \frac{h}{2}\sum_{k=0}^{n-1} f(x + \frac{1}{2})$

龙贝格算法:
$\frac{I - T_{2n}}{I - T_n} = \frac{1}{4} \Rightarrow I = \frac{4}{3}T_{2n} - \frac{1}{3}T_n, I - T_{2n} = \frac{1}{3}(T_{2n} - T_n)$
事后估计法:用$T_{2n} - T_n$的误差判断$I - T_{2n}$的误差
$$\overline{T_n} = T_{2n} + \frac{1}{3}(T_{2n} - T_n) = \frac{4}{3}T_{2n} - \frac{1}{3}T_n$$
例:辛甫生公式
复化辛甫生公式: $S_n = \frac{4}{3}T_{2n} - \frac{1}{3}T_n, \frac{I - S_{2n}}{I - S_n} = \frac{1}{16} \Rightarrow I = \frac{16}{15}S_{2n} - \frac{1}{15}S_n$

柯特斯公式$C_n = \frac{16}{15}S_{2n} - \frac{1}{15}S_n$,龙贝格公式$R_n = \frac{64}{63}C_{2n} - \frac{1}{63}C_n$
$$T_n = I + a_1 h^2 + a_2 h^4 + \cdots$$
李察逊外推加速法:

$T_0^{(k)}$表示二分 k 次的复化梯形公式$(T_{2^k}), T_m^{(k)}$表示$\{T_0^{(k)}\}$的 m 次

加速结果, $T_m^{(k)} = \frac{4^m}{4^m - 1}T_{m-1}^{(k+1)} - \frac{1}{4^m - 1}T_{m-1}^{(k)}$
每加速一次,收敛速度提高 2 阶
高斯求积公式:(插值节点可不等距)
如果选择$x_k, k = 0,1, \dots, n$,使$I = \sum_{k=0}^{n} A_k f(x_k)$具有 2n+1 次代数精度,该公式为高斯公式,$x_k$为高斯点.本质上$2n + 2$个未知参数对应$2n + 2$个方程, 一定是插值型
$I = \int_a^b f(x)dx = \sum_{k=0}^{n} A_k f(x_k) \dashrightarrow \int_a^b Ln(x)dx$
$R[f] = \int_a^b f(x)dx - \sum A_k f(x_k) = \int_a^b f[x, x_0, x_1, \dots, x_n]\omega_{n+1}$
当$f(x)$为$\le 2n + 1$次多项式时,$R[f] = 0$,此时$f[x, x_0, x_1, \dots, x_n]$为$\le n$次的多项式,且与$\omega_{n+1}(x)$正交
若$\omega_{n+1}(x)$与所有$\le n$次多项式正交,那么求积公式具有 2n+1 次代数精度⇒高斯公式。
$\forall g(x)$为$\le n$次多项式,令$f(x) = g(x)\omega_{n+1}(x)$,则$L_n(x) = 0$
$0 = \int_a^b f(x)dx = \int_a^b g(x)\omega_{n+1}(x)dx$
取 $\omega_{n+1}(x) = \widetilde{P_{n+1}}(x)$, $\forall g(x)$ 为 $\le n$ 次多项式, $g(x) = \sum_{k=0}^{n} C_k P_k(x)$, 而$\{P_n(x)\}$是正交多项式组
高斯-勒让得公式:
取x_k为$P_{n+1}(x)$的零点, $I = \int_{-1}^{1} f(x)dx = \sum_{k=0}^{n} A_k f(x_k)$有 2n+1 次代数精度

区间变换: 对于$\int_a^b f(x)dx$,令$x = \frac{a+b}{2} + \frac{t}{2}(b - a), t \in [-1,1]$

$\int_a^b f(x)dx = \int_{-1}^{1} f\left(\frac{a+b}{2} + \frac{t}{2}(b - a)\right)dt \cdot \frac{b-a}{2}$

$= \frac{b-a}{2}\int_{-1}^{1} F(t)ft = \frac{b-a}{2}\sum_{k=0}^{n} A_k F(t_k)$, t_k是$P_{n+1}(t)$的零点
余项:$R[f] = \int_a^b f(x)dx = \sum_{k=0}^{n} A_k f(x_k) = \int_a^b \frac{f^{(n+1)}(\xi)}{(n+1)!}\omega_{n+1}(x)$
构造$\le 2n + 1$次多项式满足$H(x_k) = f(x_k), H'(x_k) = f'(x_k)$

$f(x) - H_{2n+1}(x) = \frac{f^{2n+2}(\xi)}{(2n+2)!}\omega_{n+1}^2(x)$

由$\int_a^b H(x)dx = \sum_{k=0}^{n} A_k H(x_k) = \sum_{k=0}^{n} A_k f(x_k)$
$R[f] = \int_a^b [f(x) - H(x)]dx = \frac{f^{2n+2}(\eta)}{(2n+2)!}\int_a^b \omega_{n+1}^2(x)dx$

对于高斯-勒让德公式$R[f] = \frac{2^{2n+3}[(n+1)!]^4}{(2n+3)[(2n+2)!]^3} f^{(2n+2)}(\eta)$

带权的高斯公式$\int_a^b \rho(x)f(x)dx = \sum_{k=0}^{n} A_k f(x_k) + R[f]$
其中$A_k = \int_a^b \rho(x)l_k(x)dx$

高斯-切比雪夫:$\rho = 1/\sqrt{1 - x^2}$,取$\omega_{n+1} = \widetilde{T_{n+1}}(x)$, 插值节点为

T_{n+1}的零点,$x_k=\cos(\frac{2k+1}{2n+2}\pi)$,$R[f]=\frac{2\pi}{2^{2n+2}(2n+2)!}f^{(2n+2)}(\eta)$

数值微分：

$$f'(a)=\lim_{\delta\to 0}\frac{f(a+\delta)-f(a)}{\delta}$$

用均差来近似导数,向前-向后-中间差商:

$f'(a)=\frac{f(a+h)-f(a)}{h}$,$f'(a)=\frac{f(a)-f(a-h)}{h}$,$f'(a)=\frac{f(a+h)-f(a-h)}{2h}$

向前$=f'(a)+f^{(2)}(a)\cdot\frac{h}{2}+f^{(3)}(a)\cdot\frac{h^2}{3!}+\cdots$

向后$=f'(a)-f^{(2)}(a)\cdot\frac{h}{2}+f^{(3)}(a)\cdot\frac{h^2}{3!}+\cdots$

中间$=f'(a)+f^{(3)}(a)\cdot\frac{h^2}{3!}+\cdots$

用插值多项式$P_n(x)$近似$f(x)$,即$f'(x)\approx P_n'(x)$

误差$E=[f'(x)-P_n'(x)]=[f(x)-P_n(x)]'=R_n'(x)$

$=\left\{\frac{f^{(n+1)}}{(n+1)!}\right\}'\omega_{n+1}(x)+\frac{f^{(n+1)}(\xi)}{(n+1)!}\omega_{n+1}'(x)$

$E(x_k)=\frac{f^{(n+1)}(\xi)}{(n+1)!}\omega_{n+1}'(x_k)$

两点公式：$P_1'(x)=(f(x_1)-f(x_0))/h$

$E(x_k)=f^{(2)}(\xi)(2x_k-x_0-x_1)/2$, $k=0,1$

$E(x_0)=-f^{(2)}(\xi)h/2,E(x_1)=f^{(2)}(\xi)h/2$

三点公式:$x_0,x_1,x_2,P_2(x),P_2'(x_1)$=中间差商公式

五． 常微分方程数值解

$y(x_{n+1})-y(x_n)=\int_{x_n}^{x_{n+1}}f(x,y)dx=I_n\to y_{n+1}=y_n+I_n$

$y_{n+1}=y_n+hy'(x_n)=y_n+hf(x_n,y_n)$

欧拉法:

向前差商$y'(x_n)=\frac{y(x_{n+1})-y(x_n)}{h}\to y_{n+1}=y_n+hf(x_n,y_n)$

局部截断误差$y(x_{n+1})-y_{n+1}=\frac{h^2}{2}y^{(2)}(x_n)=o(h^2)$

累计误差$\Delta_{n+1}=\Delta_n+h\cdot\frac{\partial f}{\partial y}\Delta_n+\frac{h^2}{2}y^{(2)}(x_n)=\Delta_n+\frac{h^2}{2}y^{(2)}(x_n)$

$=\frac{h^2}{2}\sum y^{(2)}(x_i)=\frac{h}{2}[y'(x_{n+1}-y'(x_0)]$

后退欧拉法:

$y_{n+1}=y_n+hf(x_{n+1},y_{n+1})$

隐性公式$y_{n+1}^{(k+1)}=y_n+hf(x_{n+1},y_{n+1}^{(k)}),y_{n+1}=\lim_{k\to\infty}y_{n+1}^{(k)}$

局部截断误差$-\frac{h^2}{2}y^{(2)}(x_{n+1})$,一阶精度

欧拉两步公式：$y_{n+1}=y_{n-1}+2hf(x_n,y_n)$

局部截断误差$\frac{h^3}{3}y^{(3)}(x_n)=o(h^3)$,二阶精度

累计误差$\Delta_{n+1}=\Delta_{n-1}+2h\frac{\partial f}{\partial y}(x_n,y_n)\Delta_n+\frac{h^3}{3}y^{(3)}(x_n)$

$=\begin{cases}0+C_1\Delta_1+C_2h^2,n=2k+1\\ \Delta_1+C_2h^2,n=2k\end{cases}$

改进欧拉法:

预测$\bar y_{n+1}=y_n+hf(x_n,y_n)$

校正$y_{n+1}=y_n+\frac{h}{2}[f(x_n,y_n)+f(x_{n+1},\bar y_{n+1})]$

局部截断误差$-\frac{h^3}{12}y^{(3)}(x_n)=o(h^3)$

龙格-库塔法:

在$[x_n,x_{n+1}]$上多取几点,计算各点近似斜率,加权平均作为区间平均斜率

$K_1=f(x_n,y_n),K_2=f(x_{n+p},y_n+phK_1),K_3=f(x_{n+q},y_n+qh(rK_1+sK_2)),y_{n+1}=y_n+h[\lambda_1K_1+\lambda_2K_2+\cdots]$

$\sum\lambda_i=1$

二阶龙格库塔:$y_{n+1}=y_n+h[\lambda_1K_1+\lambda_2K_2]$,待定常数$p,\lambda_1,\lambda_2$

$\lambda_1+\lambda_2=1,\lambda_2p=0.5$

局部截断误差$\frac{h^3}{24}y^{(3)}(x_n)$

四阶龙格库塔$y_{n+1}=y_n+\frac{h}{6}[K_1+2K_2+2K_3+K_4]$

$K_1=f(x_n,y_n),K_2=f\left(x_n+\frac{h}{2},y_n+\frac{h}{2}K_1\right)$

$K_3=f\left(x_n+\frac{h}{2},y_n+\frac{h}{2}K_2\right),K_4=f(x_{n+1},y_n+hK_3)$

$O(h^5)$四阶代数精度

线性多步法:

在$y(x_{n+1})=y(x_n)+\int_{x_n}^{x_{n+1}}f(x,y(x))dx$中用插值多项式

$P_m(x)=f(x,y(x))$

显性公式:$y_{n+1}=y_n+\sum_{k=0}^m A_kf(x_{n-k},y_{n-k})$,

其中$A_k=\int_{x_n}^{x_{n+1}}l_k(x)dx$,插值结点为$x_n,x_{n-1},\ldots$

隐性公式$y_{n+1}=y_n+\sum_{k=0}^m A_kf(x_{n+1-k},y_{n+1-k})$

其中$A_k=\int_{x_n}^{x_{n+1}}l_k(x)dx$,插值结点为$x_{n+1},x_n,x_{n-1},\ldots$

局部截断误差$y(x_{n+1})-y_{n+1}$=积分公式误差

$m=1$时$\frac12 f^{(2)}(y)\int_0^h x(x-h)dx=-\frac{h^3}{12}f^{(2)}(y)$

$\frac12 f^{(2)}(y)\int_0^h x(x+h)dx=\frac{5h^3}{12}f^{(2)}(y)$,$O(h^3)$

$m=2$时$\frac16 f^{(3)}(y)\int_0^h x(x-h)(x+h)dx=-\frac{h^4}{24}f^{(3)}(y)$

$\frac16 f^{(3)}(y)\int_0^h x(x+h)(x+2h)dx=\frac{9h^4}{24}f^{(3)}(y)$,$o(h^4)$

显性公式用于预测,隐性公式用于校正

使用泰勒展开构造线性多步公式

例$y_{n+1}=\alpha_3y_{n-3}+h[\beta_0f_n+\beta_1f_{n-1}+\beta_2f_{n-2}],f_k=f(x_k,y_k)$

$y_{n-3}=y(x_n)-3hy'(x_n)+\frac32h^2y^{(2)}(x_n)-\frac{27}{6}h^3y^{(3)}(x_n)+\frac{81}{24}h^4y^{(4)}(x_n)+\cdots$

$f_n=y'(x_n),f_{n-1}=y'(x_{n-1})=y'(x_n)-hy^{(2)}(x_n)+\frac{h^2}{2}y^{(3)}(x_n)-\frac{h^3}{6}y^{(4)}(x_n)+\cdots$ 对应项相等可解

方程组与高阶方程:

一阶方程组:类似与状态空间，改写成$Y'=F(X,Y),Y(x_0)=Y_0$之后套用公式(如四阶龙格库塔

高阶方程设$y_1=y,y_2=y'$,化为一阶

边值问题的数值解法:

已知$y^{(2)}=f(x,y,y'),y(a)=\alpha,y(b)=\beta$

求离散点x^*的值,找离x^*最近的x_n近似值插值

例:$y^{(2)}-q(x)y=r(x),y(a)=\alpha,y(b)=\beta$

有$\frac{y_{n+1}-2y_n+y_{n-1}}{h^2}-q_ny_n=r_n,y_0=\alpha,y_N=\beta$

极值定理:当$\forall n,l(y_n)\ge0$,y 正最大值只可能y_0/y_N

当$\forall n,l(y_n)\le0$,y负最小值只可能y_0/y_N

当$q_n\ge0$时，方程解存在且唯一

$|e_n|\le\frac{h^2}{96}(b-a)^2M_4\sim o(h^2)$

六． 方程求根

迭代法:$x_{n+1}=\varphi(x_n)$,核心:构造$\varphi(x_n)$

收敛性:

1：设$\varphi(x_n)$在$[a,b]$上满足$\varphi(x)\in[a,b];\forall x,\bar x\in[a,b],\exists L\in(0,1),|\varphi(x)-\varphi(\bar x)|\le L|x-\bar x|(\Rightarrow|\varphi'(x)|\le L<1)$,$\varphi(x_n)$收敛

2：设x^*是$x=\varphi(x)$的根,如果$\varphi'(x)$在x^*附近连续;$|\varphi'(x^*)|<1$,那么$x_{n+1}=\varphi(x_n)$在x^*附近具有局部收敛性

构造方法:如果求$\sqrt2$,构造$x=a(x^2-2)+x$,先确定区间,推出 a 范围,再根据φ'的限制缩小范围/先利用φ'求 a 范围,再找小邻域

收敛速度与误差分析:

$e_{n+1}=x_{n+1}-x^*=\varphi(x_n)-\varphi(x^*)=\varphi'(x^*)e_n+\varphi^{(2)}(x^*)\cdot\frac{e_n^2}{2}+\cdots+\varphi^{(p)}(x^*)\cdot\frac{e_n^p}{p!}+\cdots$

$\varphi'(x^*)=\varphi^{(2)}(x^*)=\cdots=\varphi^{(p-1)}(x^*)=0,e_{n+1}=ce_n^p$,p 阶收敛

在满足定理一时$|x_{k+p}-x_k|\le\frac{1-L^p}{1-L}|x_{k+1}-x_k|\le\frac{1}{1-L}|x_{k+1}-x_k|$,令$p\to\infty$,有$|x_k-x^*|\le\frac{1}{1-L}|x_{k+1}-x_k|$,可用$|x_{k+1}-x_k|$估计$|x_k-x^*|$误差

事前估计:$|x_k-x^*|\le\frac{L^k}{1-L}|x_1-x_0|$,通过事前估计预估 K;

事后估计:$\frac{1}{1-L}|x_{k+1}-x_k|$是否小于阈值ε,通过时候估计调整 K

牛顿法:$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)},\varphi(x)=x-\frac{f(x_n)}{f'(x_n)}$,

$\varphi'(x^*)=0$二阶收敛

收敛性:一阶收敛$e_{n+1}=\varphi'(\xi_n)e_n,|e_n|\sim L^n$

二阶收敛$e_{n+1}=\frac12\varphi''(t_n)e_n^2,|e_n|\sim L^{2^n}$

定理:$f(x)\in C^2[a,b]$满足$f(a)f(b)<0;f''(x)$在$[a,b]$上不变号;$\forall x\in[a,b],f'(x)\ne0;\left|\frac{f(c)}{b-a}\right|\le f'(c),if\ c=a||b$

那么牛顿迭代法收敛

牛顿下山法:$\bar x_{k+1}=x_k0-f(x_k)/f'(x_k)$

检查$|f(\bar x_{k+1})|<|f(x_k)|$是否成立?成立,$x_{k+1}=\bar x_{k+1}$;

不成立,$x_{k+1}=\lambda\bar x_{k+1}+(1-\lambda)x_k$,$\lambda$从 1 开始逐次减半尝试,直到$|f(x_{k+1})|<|f(x_k)|$

重根问题:

若x^*是$f(x)$的多重根,分母$f'(x^*)=0$,实际上$f(x)=(x-x^*)^mg(x)$,$\varphi(x)=x-\frac{(x-x^*)g(x)}{(x-x^*)g'(x)+mg(x)},\varphi(x^*)=1-\frac1m$线性收敛

改良方法:1.令$\varphi(x)=x-mf(x)/f'(x)$,须知道 m,此时$\varphi'(x^*)=0$;2.令$u(x)=f(x)/f'(x)$,则$u(x)$以x^*为单根

弦截法和抛物线法:

弦截法:$P_1(x)=f(x_n)+f[x_n,x_{n-1}](x-x_n)$,用$P_1(x)$的根近似$x^*$,作为$x_{n+1}$,$x_{n+1}=x_n-f(x_n)/f[x_n,x_{n-1}]$,相当于用$f[]$代替$f'$

$e_{n+1}=x_{n+1}-x^*$

插值余项$f(x)-P_1(x)=\frac{f^{(2)}(\xi_2)}{2}(x-x_n)(x-x_{n-1})$

$P_1(x_{n+1})=f(x_n)+f[x_n,x_{n-1}](x_{n+1}-x_n)$

$P_1(x^*)=f(x_n)+f[x_n,x_{n-1}](x^*-x_n)$

$-p_1(x^*)=f[x_n,x_{n-1}](x_{n-1}-x^*)=f'(\xi_1)e_{n+1},\xi_1\in[x_{n-1},x_n]$

$=\frac{f^{(2)}(\xi_2)}{2}(x^*-x_n)(x^*-x_{n-1})=\frac{f^{(2)}(\xi_2)}{2}e_ne_{n-1}$

$e_{n+1}=\frac{f^{(2)}(\xi_2)}{2f'(\xi_1)}e_ne_{n-1}$,如果x_n和x_{n-1}在x^*的邻域 R:$[x^*-\delta,x^*+\delta]$中,令$M=\frac{\max_{x\in R}|f^{(2)}(x)|}{2\min_{x\in R}|f'(x)|}$则$|e_{n+1}|\le M|e_n||e_{n-1}|$,取δ足够小,使$M\delta<1$,$|e_n|$收敛,收敛速度$\frac{f^{(2)}(\xi_2)}{2f'(\xi_1)}=\frac{f^{(2)}(x^*)}{2f'(x^*)}=K$,则$e_{n+1}=Ke_ne_{n-1}$

假设$e_n=ce_{n-1}^p,e_{n+1}=ce_n^p=Ke_ne_{n-1}=Ke_ne_n^{\frac1p}c^{-\frac1p}\Rightarrow p=1+1/p$

抛物线法:用x_n,x_{n-1},x_{n-2}构造$P_2(x)$,把$P_2(x)$的根作为x_{n+1}

$$x_{n+1}=x_n-\frac{2f(x_n)}{\omega\pm\sqrt{\omega^2-4f(x_n)f[x_n,x_{n-1},x_{n-2}]}}$$

其中$\omega=f[x_n,x_{n-1}]+f[x_n,x_{n-1},x_{n-2}](x_n-x_{n-1})$

七． 线性方程组的数值解

高斯消去法:加减法=$n^3/3$乘除法$n^3/3$，使用条件$a_{kk}^{(k)}\ne0$

各阶顺序主子式均不为 0

矩阵的三角分解:

直接法:$LUx=b\Rightarrow Ly=b,Ux=y$,即高斯消元

追赶法:三对角线方程组,第 i 行角标为 i，从左到右为a,b,c

L 主对角线α_i,下次对角线γ_i,U 主对角线 1,上次对角线β_i

$\alpha_1=b_1,\beta_1=\frac{c_1}{b_1},\gamma_i=a_i,\alpha_i=b_i-\gamma_i\beta_{i-1},\beta_i=\frac{c_i}{b_i-\gamma_i\beta_{i-1}}$

平方根法:适用于对称正定矩阵,$A=A^T,\forall x\ne0,x^TAx>0$

三角分解$A=LU,U=DU_0,A=LDU_0,A^T=U_0^TD^TL^T$

由分解唯一性$L=U_0^T$,故$A=LDL^T$,D主对角矩阵u_{ii},U_0主对角线 1,上三角$ij=u_{ij}/u_{ii}$;平方根法$L_1y=b(L_1=L\sqrt{D}),L_1^Tx=y$;改进平方根法$Lz=b,Dy=z,L^Tx=y$

范数与误差分析:

向量范数:按某种规则将每个$x\in R^n$对应于一个非负实数$||x||$,且满足下列条件$||x||>0(x\ne0),||cx||=|c|||x||,||x+y||\le||x||+||y||$;矩阵范数:$||Ax||\le||A||||x||$,称$||A||$为与向量范数相容的矩阵范数。例$||A||_1=\max_{1\le j\le n}(\sum_{i=1}^n|a_{ij}|)$列范数与$||x||_1$相容;$||A||_\infty=\max_{1\le i\le n}(\sum_{j=1}^n|a_{ij}|)$行范数与$||x||_\infty$相容;$||A||_2=\sqrt{\lambda_{max}(A^TA)}$与$||x||_2$相容

误差来源:$\delta b\ne0\to\frac{||\delta x||}{||x||}\le||A||\cdot||A^{-1}||\cdot||\delta b||/||b||$

$\delta A\ne0\to\frac{||\delta x||}{||x||}=\frac{||\delta x||}{||x+\delta x||}\le||A^{-1}||\cdot||A||\cdot||\delta A||/||A||$

事后估计法:如果$\bar x$为$Ax=b$的一个近似解,x^*为精确解,有$x^*-\bar x=A^{-1}(b-A\bar x),||x^*-\bar x||\le||A^{-1}||||b-A\bar x||$, $||x^*-\bar x||/||x^*||\le||A||||A^{-1}||\cdot||b-A\bar x||/||b||$

迭代法:由$Ax=b\Rightarrow x=Bx+f\Rightarrow x^{(k+1)}=Bx^{(k)}+f$

雅可比迭代法$Ax=b,A=D+L+U$=对角+下三角+上三角

其中 L 和 U 对角线均为 0

$(D+L+U)x=b\Rightarrow x=-D^{-1}(L+U)x+D^{-1}b$

令$B=-D^{-1}(L+U),f=D^{-1}b$,有$x=Bx+f$,

$x_i^{(k+1)}=\frac{1}{a_{ii}}[b_i-\sum_{j=1,j\ne i}^n a_{ij}x_j^{(k)}]$

高斯-塞德尔迭代法(G-S)

利用已计算出的$x_j^{(k+1)},j\in[1,i-1]$来计算$x_i^{(k+1)}$

$x_i^{(k+1)}=\frac1{a_{ii}}[b_i-\sum_{j=1}^{i-1}a_{ij}x_j^{(k+1)}-\sum_{j=i+1}^n a_{ij}x_j^{(k)}]$

$x^{(k+1)}=-(D+L)^{-1}Ux^{(k)}+(D+L)^{-1}b$

收敛性:

充要$\rho(B)<1,\rho(B)=\max|\lambda_i|,\rho(B)$越小收敛越快

充要$||B||<1,because\ \rho(B)\le||B||,||B||$是与向量相容的范数

充要 A 正定对称

构造法:$Ax=b\Rightarrow x=(I-TA)x+Tb$,注意使 TA 是上三角矩阵,甚至是对角形甚至对角线上的元在（0，2）中,使得||I-TA||<1,保证特征值<1

直接法n^3,迭代法kn^2

如果 A 主对角线元>同一行其他元之和,A 为**严格对角优势矩阵**

若 Ax=b 且 A 为严格对角优势矩阵，则两大迭代法均收敛

如果 A 是正定矩阵,那么 G-S 法收敛

逐次超松弛迭代法 SOR：

$$x^{(k+1)}=(D+\omega L)^{-1}[(1-\omega)D-\omega U]x^{(k)}+\omega(D+\omega L)^{-1}b=L_\omega x^{(k)}+f$$

收敛充要条件:$\omega\in(0,2)$且 A 正定对称

ω最佳值$2/(1+\sqrt{1-\rho^2(B_0)})$,$B_0$是雅可比迭代矩阵

迭代法事后估计法$||x^{(k)}-x^*||\le\frac{||B||}{1-||B||}\cdot||x^{(k)}-x^{(k-1)}||$

舍入误差影响$||\delta_{k+1}||\le||B||\cdot||\delta_k||+||\left(\frac12\times10^{-m},\ldots\right)^T||$

八． 矩阵特征值求解

瑞利商:$R(x)=\frac{(A,x)}{(x,x)}=\frac{\sum_{i=1}^n\alpha_i^2\lambda_i}{\sum_{i=1}^n\alpha_i^2},\lambda_{min}\le R(x)\le\lambda_{max}$