Solving PDE’s with FEniCS

Geometry approximation

L. Ridgway Scott

The Institute for Biophysical Dynamics,
The Computation Institute, and the
Departments of Computer Science and Mathematics,
The University of Chicago

Curved boundaries

If the boundary $\partial\Omega$ of a domain Ω is curved, it is often necessary to approximate it in some way.

For simplicity, we will consider the Laplace equation

$$-\Delta u = f \text{ in } \Omega$$

with homogeneous Diriclet boundary conditions

$$u = 0 \text{ on } \partial\Omega.$$

Such boundary conditions easy to satisfy on polygonal boundaries with piecewise polynomials.

But impossible for curved boundaries.

Curved boundaries

There are various ways in which this can be addressed:

- interpolate the boundary conditions [7] (a collocation approach)
- modify the polynomials via a change of coordinates (isoparametric elements)
- incorporate the boundary conditions into the variational form (Nitsche's method).

The name **isoparametric element** was coined by Bruce Irons[1] in part as a play on words, assuming that the audience was familiar with isoperimetric inequalities.

[1] Bruce Irons (1924—1983) is known for many concepts in finite element analysis, including the Patch Test for nonconforming elements [4] and frontal solvers, among others.

Curved boundaries

There are two, interconnected issues related to geometry approximation.

- One is to represent the boundary values accurately.
- Other is to compute required quadrature related to elements with curved sides.

Isoparametric method deals with this together, but requires complex technology to deal with mappings.

Nitsche’s method addresses boundary conditions but not quadrature on curved elements.

However, a modification [2] of Nitsche’s method solves this problem.

Method of Nitsche for polygonal domain:

$$-\Delta u = 2\pi^2(\sin \pi x)(\sin \pi y) \text{ in } \Omega = [0,1]^2, \quad u = 0 \text{ on } \partial\Omega.$$

poly. order	mesh no.	γ	L^2 error
1	32	10	2.09e-03
2	8	10	5.16e-04
4	8	10	1.77e-06
4	128	10	4.96e-12
8	16	10	1.68e-11
16	8	10	2.14e-08

Table 1: L^2 errors for Nitsche's method: effect of varying polynomial order and mesh size for fixed $\gamma = 10$. Here we take $h = N^{-1}$ where N is the mesh number.

Allows use of functions that do not satisfy Dirichlet boundary conditions to approximate solutions which do satisfy Dirichlet boundary conditions. Define

$$\begin{aligned} a_\gamma(u,v) = \int_\Omega \nabla u \cdot \nabla v \, d\mathbf{x} + \gamma h^{-1} \oint_{\partial\Omega} uv \, ds \\ - \oint_{\partial\Omega} \frac{\partial u}{\partial n} v \, ds - \oint_{\partial\Omega} \frac{\partial v}{\partial n} u \, ds, \end{aligned} \tag{1}$$

where $\gamma > 0$ is fixed parameter, h is mesh size [6, 8].

Solution to Laplace's equation satisfies $u \in H_0^1(\Omega)$ and

$$a_\gamma(u,v) = (f,v)_{L^2} \quad \forall v \in H_0^1(\Omega). \tag{2}$$

Reason is that all boundary terms vanish, and

$$a_\gamma(u,v) = a(u,v) \quad \text{for all } u,v \in H_0^1(\Omega)$$

where $a(\cdot,\cdot)$ is usual bilinear form for Laplace operator:

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, d\mathbf{x}.$$

But more generally for all $v \in H^1(\Omega)$,

$$\begin{aligned} \int_\Omega f v \, d\mathbf{x} = \int_\Omega (-\Delta u) v \, d\mathbf{x} &= a(u,v) - \oint_{\partial\Omega} v \frac{\partial u}{\partial n} \, ds \\ &= a_\gamma(u,v), \end{aligned} \tag{3}$$

since $u = 0$ on $\partial\Omega$.

Thus if u solves (2), then it also satisfies $u \in H_0^1(\Omega)$ and

$$a_\gamma(u, v) = (f, v)_{L^2} \quad \forall v \in H^1(\Omega). \tag{4}$$

Now consider the discrete problem

$$\text{find } u_h \in V_h \text{ such that } a_\gamma(u_h, v) = (f, v)_{L^2} \quad \forall v \in V_h, \tag{5}$$

where $V_h \subset H^1(\Omega)$ not required to be subset of $H_0^1(\Omega)$.

Nitsche's method produces results of same quality as are obtained with specifying Dirichlet conditions explicitly, as indicated in Table 1.

However, for particular values of γ, behavior is suboptimal, as indicated in Table 2. Nitsche's method guaranteed to work if γ sufficiently large.

poly. order	mesh no.	γ	L^2 error
1	8	100	3.23e-02
1	8	10	3.10e-02
1	8	2	2.76e-02
1	8	1.5	3.93e-02
1	8	1.1	6.00e-02
1	8	1.0	1.80e-01
1	32	1.0	1.81e-01

Table 2: L^2 errors for Nitsche's method: effect of varying γ. For γ too small, the accuracy is limited.

Now let us analyze Nitsche's method.

As a consequence of (4) and (5), we have

$$a_\gamma(u - u_h, v) = 0 \quad \forall v \in V_h. \tag{6}$$

Nitsche's method is of interest when $V_h \not\subset H_0^1(\Omega)$, for otherwise we could use the usual formulation.

But the bilinear form $a_\gamma(v, v)$ is not defined for general $v \in H^1(\Omega)$.

For example, take $\Omega = [0, 1]^2$ and $v(x, y) = 1 + x^{2/3}$ for $(x, y) \in \Omega$.

Analyzing Nitsche some more

Then

$$v_{,x}(x,y) = \tfrac{2}{3}x^{-1/3} \text{ for all } x, y \in [0,1]^2.$$

Thus, $v_{,x}$ is square integrable on Ω, and since $v_{,y} = 0$, $|\nabla v|$ is square integrable on Ω. So $v \in H^1(\Omega)$.

But, $v_{,x}(x,y) \to \infty$ as $x \to 0$ for all $y \in [0,1]$.

In particular, this means that $\frac{\partial v}{\partial n} = \infty$ on the part of the boundary $\{(0,y)\ :\ y \in [0,1]\}$.

Therefore $a_\gamma(v,v) = \infty$.

Thus we need a new theory to explain the behavior of Nitsche's method.

New norm for Nitsche

Since $a_\gamma(\cdot,\cdot)$ not continuous on $H^1(\Omega)$, cannot use standard approach to analyze behavior of Nitsche's method (5).

Or at least we cannot use the standard norms.

Instead we define

$$|||\, v \,||| = \left(a(v,v) + h \oint_{\partial\Omega} \left|\frac{\partial v}{\partial n}\right|^2 ds + h^{-1} \oint_{\partial\Omega} v^2\, ds \right)^{1/2}. \quad (7)$$

Philosophy of this norm is that it

- penalizes departure from Dirichlet boundary condition
- minimizes impact of normal derivative on boundary.

Continued analysis of Nitsche

Correspondingly, we define V to be the subset of $H^1(\Omega)$ consisting of functions for which this norm is finite.

It is easy to see that this norm matches the different parts of the Nitsche bilinear form, so that

$$|a_\gamma(v,w)| \le C_\gamma ||| \, v \, ||| \; ||| \, w \, ||| \quad \forall v, w \in V. \tag{8}$$

One can show [8] that

$$||| \, v \, ||| \le C_\gamma h^{-1} \|v\|_{L^2(\Omega)} \quad \forall v \in V_h, \tag{9}$$

for any space V_h of piecewise polynomials on a reasonable mesh.

Understanding Nitsche

There exist $\gamma_0 > 0$ and $\alpha > 0$ such that

$$\alpha |||\, v \,|||^2 \le a_\gamma(v,v) \quad \forall v \in V_h, \quad \gamma \ge \gamma_0. \tag{10}$$

Assume that solution u is sufficiently smooth that $|||\, u \,||| < \infty$. In this case, one can prove [8] that

$$\begin{aligned} |||\, u - u_h \,||| &\le \left(1 + \frac{C_\gamma}{\alpha}\right) \inf_{v \in V_h} |||\, u - v \,|||_{L^2(\Omega)} \\ &\le C' h^k \|u\|_{H^{k+1}(\Omega)} \end{aligned} \tag{11}$$

using piecewise polynomials of degree k, and so

$$\|u_h\|_{L^2(\partial\Omega)} \le C h^{k+1/2} \|u\|_{H^{k+1}(\Omega)},$$

so Dirichlet boundary conditions closely approximated.

When boundary is not polygonal, for example, a circle, an error related to approximating $\partial\Omega$ must be made when using piecewise polynomials to approximate the solution of a PDE boundary-value problem.

The simplest approach is to approximate the boundary $\partial\Omega$ by a simplicial surface (in two dimensions, polygonal curve), constructing approximate domain Ω_h.

If the domain Ω is not convex, then the approximate domain Ω_h may not be contained inside Ω.

Circle domain

Consider Nitsche (1) to solve Poisson's equation with $f = 4$ on unit circle, with Dirichlet boundary conditions.

The exact solution is

$$u(x, y) = 1 - x^2 - y^2. \tag{12}$$

To deal with the curved boundary, we use the `mshr` system which includes a circle as a built-in domain type.

What `mshr` does is to approximate the circle by a polygon, as seen on the left side of Figure 1.

Curved domains

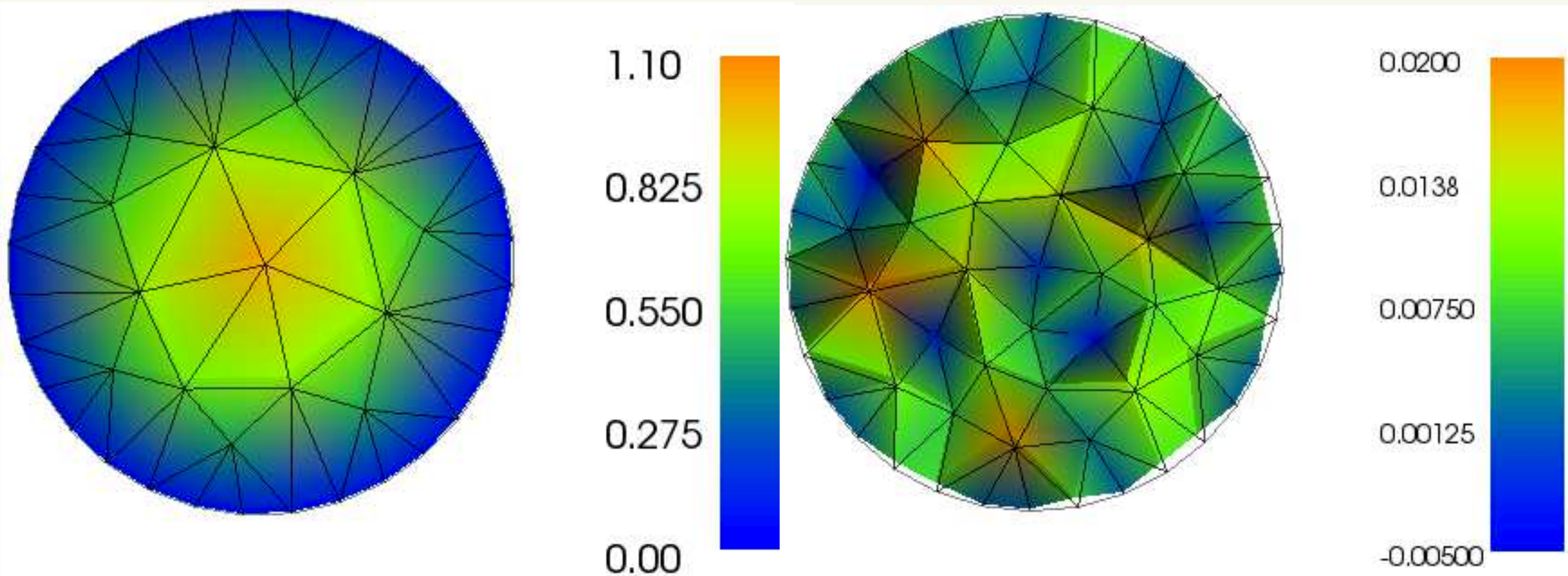

Figure 1: Using `mshr` to generate a mesh for a disc with piecewise linear approximation of the solution (12). The parameter $\gamma = 10$ in Nitche's method. (left) Coarse mesh generated by using the `meshsize` parameter equal to 1. Values of the solution u_h are plotted. (right) Finer mesh generated by using `meshsize` parameter equal to 5. Values of the error $u - u_h$ are plotted.

Curved domains

Instructing `mshr` to use a finer mesh produces an approximation that is uniformly small, as seen on the right side of Figure 1, where the error only is plotted.

The polygonal boundary approximation is a piecewise linear approximation of $\partial\Omega$, and we see that using piecewise linear approximation inside the approximate domain can be effective.

However, we might want more accuracy, and it is natural to consider using piecewise quadratic approximation inside the approximate domain.

More accuracy

Thus we examine the use of piecewise quadratic approximation on a polygonal domain approximating the circle in Figure 2.

We might have expected the error using quadratics for our test problem, whose solution is given in (12), would be essentially zero, since the exact solution is itself a quadratic polynomial.

However, there is a significant geometric error due to the polygonal approximation of the domain, as shown on the left side of Figure 2.

Using an even finer mesh as shown on the right side of Figure 2 indicates that the error is concentrated in a boundary layer around the polygonal boundary approximation.

In the computations for the right side of Figure 2 we specified the `segments` parameter in the `Circle` function to be 18, instead of using the default value 32 as in all other computations.

The corresponding code for this is

```
domain = Circle(dolfin.Point(0.0, 0.0),1.0,segments)
```

Curved domains

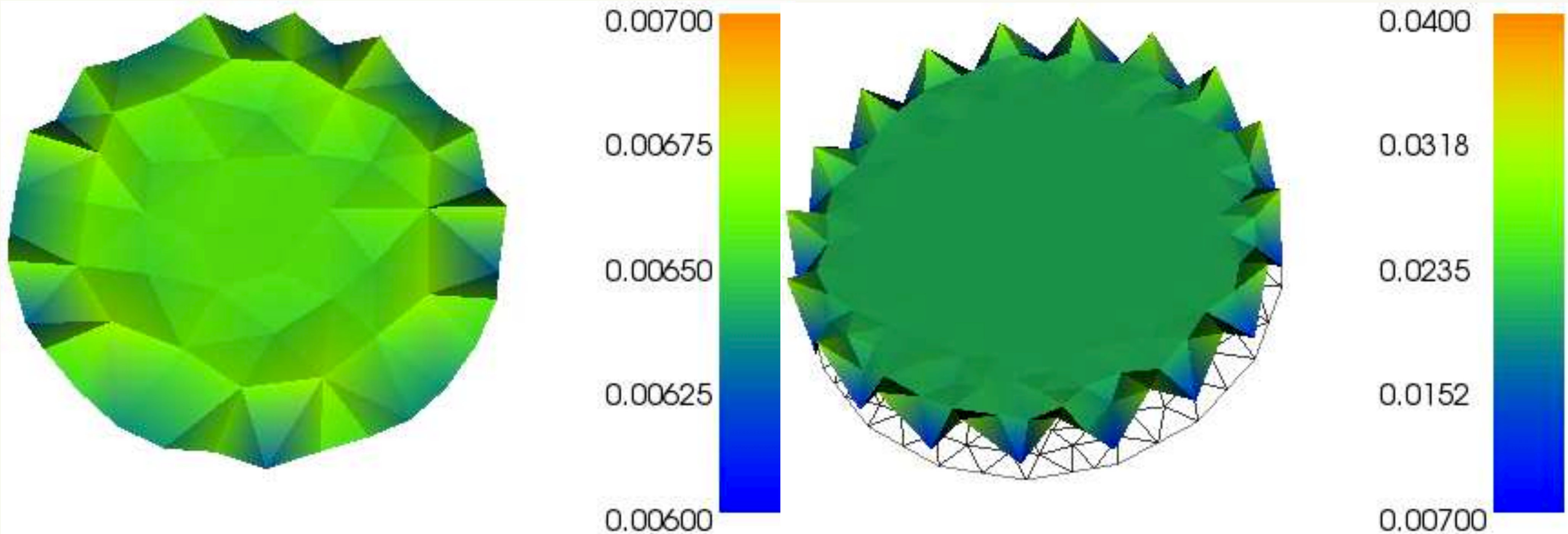

Figure 2: Using `mshr` to generate a mesh for a disc with piecewise quadratic approximation of the solution (12). Values of the error $u - u_h$ are plotted. The parameter $\gamma = 10$ in Nitche's method. (left) Same mesh as used on the right-hand side of Figure 1. Mesh generated by using `meshsize` parameter equal to 5. (right) Even more refined mesh. Mesh generated by using `meshsize` parameter equal to 10 and `segments` parameter set to 18.

Curved domains

polynomial degree	`meshsize`	L^2 error	`segments`
1	1	5.11e-02	default
2	1	1.14e-02	default
1	5	2.88e-02	default
2	5	1.16e-02	default
1	10	1.55e-02	default
2	10	1.16e-02	default
3	10	1.16e-02	default
2	10	3.69e-02	18
2	10	9.17e-03	36
2	10	2.27e-03	72

Table 3: Geometric error as a function of mesh size and polynomial degree for the Laplace problem whose solution is given in (12). The parameter $\gamma = 10$ in Nitche's method. The default value for `segments` in `mshr` is 32.

Table 3 shows reducing mesh size or increasing polynomial degree, reduces L^2 error much.

However, increasing number of `segments` in approximation of circle decreases error as desired.

Limit on accuracy due to geometry approximation known for some time [1].

Order of accuracy for quadratics and higher-order polynomials is restricted, essentially because the geometry approximation is piecewise linear.

Exact solution (12) is quadratic, but finite element approximation with quadratics does not match it.

Accuracy limits

Let us examine more closely the accuracy limits related to polygonal approximation of the boundary.

Continue with circle as test domain Ω, but we choose a higher-order polynomial manufactured solution.

Using the method of manufactured solutions, we take

$$u(x,y) = 1 - (x^2 + y^2)^3 \text{ and } f(x,y) = 36(x^2 + y^2)^2. \quad (13)$$

We start with the standard approximation in which we pose Diriclet boundary conditions on $\partial\Omega_h$ [5].

Computational results are listed in Table 4(a).

Accuracy limits

(a)

k	M	L^2 error	H^1 error
1	16	2.47e-02	5.45e-01
2	16	5.62e-03	5.45e-02
3	16	5.53e-03	3.82e-02

(b)

k	M	γ	L^2 error	H^1 error
1	16	20	2.40e-02	5.42e-01
2	16	20	5.56e-03	3.94e-02
3	16	20	5.51e-03	2.97e-02

Table 4: Errors in $L^2(\Omega)$ and $H^1(\Omega)$ as a function of the `meshsize`, denoted by M, for the polygonal approximation for test problem whose exact solution is given by (13). The number of segments was chosen to be 5 times the `meshsize` in all cases. In (a), the standard finite element method was used with $u_h = 0$ on $\partial\Omega_h$. In (b), Nitsche's method was used on Ω_h.

Accuracy limits

Using Nitsche's method gives comparable results, even slightly better, as indicated in Table 4(b).

But in both cases, using piecewise cubics provides minimal improvement over piecewise quadratics.

To get higher-accuracy, something new has to be done.

We explain one such approach.

But first we look in more detail at the effect of the parameter γ in Nitsche's method.

Consider Nitsche's method as applied in the previous section, but with a variety of values of γ.

Limts on γ

In particular, fix `meshsize` = 16, `seqments` = 80, and $k = 3$ (piecewise cubics).

Errors depicted in Figure 3 for $\gamma = 11$ and $\gamma = 12$.

Errors on left of Figure 3, with $\gamma = 11$, are a hundred times larger than on right, where $\gamma = 12$.

Values of corresponding norms are given in Table 5.

Error with $\gamma = 11$ very localized near part of boundary.

Computations for γ this small are spurious.

Errors very sensitive to exact value of γ, as in Table 5.

Limts on γ

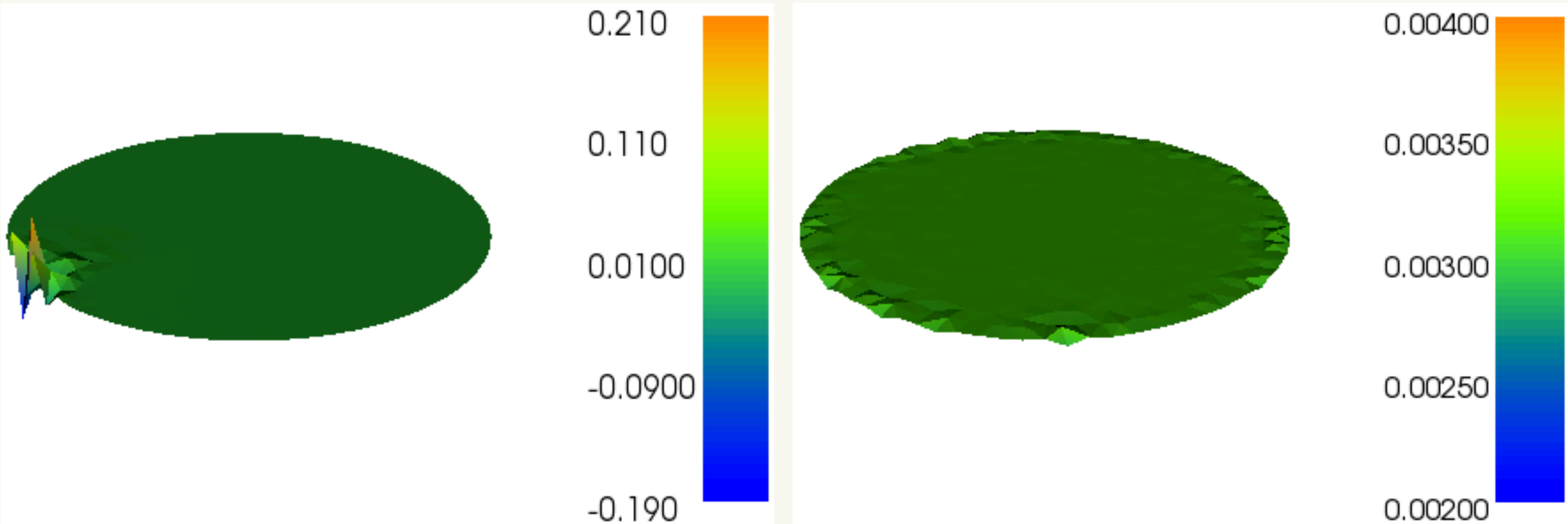

Figure 3: Solution using Nitsche's method with `meshsize` = 16, `seqments` = 80, $k = 3$ (piecewise cubics), for the test problem whose exact solution is given by (13), with (left) $\gamma = 11$ and (right) $\gamma = 12$. The L^2 and H^1 errors are given for these values of γ and others in Table 5.

Limts on γ

γ	L2 error	H1 error
10.9	5.51e-03	6.24e-02
10.99	5.77e-03	3.61e-01
11.00	**2.03e-01**	**4.25e+01**
11.01	5.75e-03	3.59e-01
11.1	5.50e-03	4.88e-02
12.0	5.50e-03	2.92e-02
20.0	5.51e-03	2.97e-02
40.0	5.52e-03	3.35e-02
100	5.52e-03	3.62e-02

Table 5: Errors in $L^2(\Omega)$ and $H^1(\Omega)$ as a function of γ. Computed using Nitsche's method with `meshsize` = 16, `seqments` = 80, $k = 3$ (piecewise cubics). The spatial error distributions for the cases $\gamma = 11$ and $\gamma = 12$ are visualized in Figure 3.

The method [2] of Bramble-Dupont-Thomée (BDT) achieves high-order accuracy by modifying Nitsche's method [6] applied on Ω_h using the bilinear form

$$\begin{aligned} N_{1,h}(u,v) &= a_h(u,v) - \oint_{\partial\Omega_h} \frac{\partial u}{\partial n} v \, ds \\ &\quad - \oint_{\partial\Omega_h} \Big(u + \delta \frac{\partial u}{\partial n} \Big) \Big(\frac{\partial v}{\partial n} - \gamma h^{-1} v \Big) \, ds \end{aligned} \tag{14}$$

is introduced (if δ were 0, this would be Nitsche's method on Ω_h). Here, $\mathbf{n}$ denotes the outward-directed normal to $\partial\Omega_h$ and

$$\delta(\mathbf{x}) = \min \left\{ s > 0 \ : \ \mathbf{x} + s\mathbf{n} \in \partial\Omega \right\}.$$

The BDT approach

Corrections of arbitrary order, involving terms $\delta^\ell \frac{\partial^\ell u}{\partial n^\ell}$ for $\ell > 1$ are studied in [2].

Define W_h^k to be the set of piecewise polynomials of degree k on the mesh $\mathcal{T}_h$ that have no restriction on $\partial\Omega_h$.

It is shown in [2] that the solution $u_h^* \in W_h^k$ of

$$N_{1,h}(u_h^*, v) = \int_{\Omega_h} f v \, dx \text{ for all } v \in W_h^k \tag{15}$$

satisfies

$$|||\, u - u_h^* \,||| \le Ch^k \|u\|_{H^{k+1}(\Omega)} + Ch^{7/2} \|u\|_{W^2_\infty(\Omega)},$$

where $|||\, v \,|||$ is defined in (7).

An example

Consider the case where Ω is the unit circle. We have

$$\mathbf{x} + \delta(\mathbf{x})\mathbf{n} \in \partial\Omega \quad \text{for } \mathbf{x} \in \partial\Omega_h.$$

We can write $\mathbf{x} = (\mathbf{x}\cdot\mathbf{n})\mathbf{n} + (\mathbf{x}\cdot\mathbf{t})\mathbf{t}$, and $(\mathbf{x}\cdot\mathbf{t})^2 = |\mathbf{x}|^2 - (\mathbf{x}\cdot\mathbf{n})^2$. Since $|\mathbf{x} + \delta(\mathbf{x})\mathbf{n}| = 1$, we have

$$1 = (\mathbf{x}\cdot\mathbf{t})^2 + (\mathbf{x}\cdot\mathbf{n} + \delta(\mathbf{x}))^2 = |\mathbf{x}|^2 - (\mathbf{x}\cdot\mathbf{n})^2 + ((\mathbf{x}\cdot\mathbf{n} + \delta(\mathbf{x}))^2.$$

Then

$$\delta(\mathbf{x}) = \sqrt{1 - |\mathbf{x}|^2 + (\mathbf{x}\cdot\mathbf{n})^2} - \mathbf{x}\cdot\mathbf{n}\,.$$

Note that for $\mathbf{x} \in \partial\Omega_h$, $|\mathbf{x}| \le 1$.

Thus $\delta(\mathbf{x}) \ge 0$.

An example

Expression for δ in `dolfin`:

```
x = SpatialCoordinate(mesh)
a = inner(grad(u),grad(v))*dx -
(inner(n,grad(u)))*v*ds \
-(u+(sqrt(1-inner(x,x)+inner(n,x)*inner(n,x)) \
-inner(n,x))*inner(n,grad(u)))* \
((inner(n,grad(v)))-(gamma/h)*v)*ds
```

Computational experiments are summarized in Table 6 for Poisson's equation.

An example

Accuracy in L^2 (resp., H^1) improves as k increases for $k \leq 3$ (resp., $k \leq 4$) but results for $k = 4, 5$ essentially same as for $k = 3$ (resp., for $k = 5$ are essentially the same as for $k = 4$).

We see that a one-line change to Nitsche's method makes a very significant change in accuracy.

The BDT method has been developed and applied in many ways [3].

An example

k	`meshsize`	γ	L2 error	H1 error	`segments`	hmax
1	32	100	6.39e-03	2.77e-01	160	6.88e-02
1	64	100	1.58e-03	1.37e-01	320	3.53e-02
2	32	100	4.77e-05	7.17e-03	160	6.88e-02
2	64	100	5.91e-06	1.79e-03	320	3.53e-02
3	32	100	5.81e-07	9.23e-05	160	6.88e-02
3	64	100	3.57e-08	1.15e-05	320	3.53e-02
4	32	100	5.80e-07	5.90e-06	160	6.88e-02
4	64	100	3.63e-08	5.22e-07	320	3.53e-02
5	32	100	5.80e-07	5.77e-06	160	6.88e-02
5	64	100	3.62e-08	5.12e-07	320	3.53e-02

Table 6: Errors in $L^2(\Omega)$ and $H^1(\Omega)$ as a function of mesh size (hmax) for the the BDT approximation for various polynomial degrees. Key: k is the polynomial degree, `meshsize` and `segments` (the number of boundary edges) are input parameters to the `mshr` function `circle` used to generate the mesh, γ is the parameter used in Nitsche's method, and hmax is the maximum mesh size for the mesh that `mshr` generates.

References

[1] A. Berger, R. Scott, and G. Strang. Approximate boundary conditions in the finite element method. *Symposia Mathematica*, 10:295–313, 1972.

[2] James H. Bramble, Todd Dupont, and Vidar Thomée. Projection methods for Dirichlet's problem in approximating polygonal domains with boundary-value corrections. *Mathematics of Computation*, 26(120):869–879, 1972.

[3] Erik Burman, Peter Hansbo, and Mats G. Larson. A cut finite element method with boundary value correction. *Mathematics of Computation*, 2017.

[4] Ivan Cormeau. Bruce Irons: A non-conforming engineering scientist to be remembered and rediscovered. *International Journal for Numerical Methods in Engineering*, 22(1):1–10, 1986.

[5] T. Dupont, Johnny Guzmán, and L. R. Scott. Obtaining full-order Galerkin accuracy when the boundary is polygonally approximated. *TBA*, 2018.

[6] Mika Juntunen and Rolf Stenberg. Nitsche's method for general boundary conditions. *Mathematics of Computation*, 78(267):1353–1374, 2009.

[7] Ridgway Scott. Interpolated boundary conditions in the finite element method. *SIAM J. Numer. Anal.*, 12:404–427, 1975.

[8] V. Thomee. *Galerkin Finite Element Methods for Parabolic Problems*. Springer Verlag, 1997.