DOWNSAMPLING GRAPHS USING SPECTRAL THEORY

Sunil K Narang and Antonio Ortega

Ming Hsieh Department of Electrical Engineering
University of Southern California
kumarsun@usc.edu, ortega@sipi.usc.edu

ABSTRACT

In this paper we present methods for downsampling datasets defined on graphs (i.e., graph-signals) by extending downsampling results for traditional N-dimensional signals. In particular, we study the spectral properties of k-regular bipartite graphs (K-RBG) and prove that downsampling in these graphs is governed by a Nyquist-like criteria. The results are useful for designing critically sampled filter-banks in various data-domains where the underlying relations between data locations can be represented by undirected graphs. In order to illustrate our results we represent images as a set of k-RBG graphs and apply our downsampling results to them. The results show that common 2-D lattice downsampling methods can be seen special cases of (k-RBG) based downsampling. Further we demonstrate new downsampling schemes for images with non-rectangular connectivity.

***Index Terms*—** Nyquist theorem, bipartite graphs, subsampling

1. INTRODUCTION

Datasets defined on graphs often arise due to irregular sampling of Euclidean spaces, or naturally exist in applications such as data-mining, biology, network analysis [1, 2] and social network analysis [3]. The size of graph datasets can run into millions and billions of nodes (such as in social networks), posing serious obstacles for visualization, compression and analysis of these datasets. In this work we are interesting in providing "coarse" approximations to a given graph, i.e., a graph with fewer nodes and with data representing a smooth approximation to the original graph-signal. In the graph literature a significant amount of work has been done to define transforms and filter-banks on graphs [4, 5] , but much of this work has focused on the transforms, which are often not critically sampled. Our focus in this paper is to analyze downsampling/upsampling methods for graph datasets.

In traditional signal processing applications, downsampling and upsampling are an integral part of critically sampled multi-rate filterbanks. Fig. 1 (top) shows a simple example of downsampling and upsampling by a factor of 2, so that in the resulting signal every other sample is zero. The discrete time Fourier transform of the resulting signal, $f_{ds}(n)$, contains the spectrum of the original signal as well as frequency shifted copies of the original signal, which results in *aliasing* if the signal and shifted copies have overlapping regions of support. Datasets on graphs can be defined as *graph-signals* [4, 5] which have a spectral interpretation given by eigenvalues (and eigenvectors) of the graph Laplacian matrix, similar to Fourier transform for regular signals. Further multirate filterbanks have been proposed for arbitrary graphs which extend the ideas of low-pass and high-pass filtering to graph domain (e.g., [5]). In our present work we seek to provide a spectral interpretation of downsampling and aliasing in graph-signals. We show that downsampling on a special class of graphs known as *k-regular bipartite graphs (k-RBG)* extends most of the downsampling results known for regular signals (such as the Shannon-Nyquist sampling theorem criterion) to these graphs. Although general graphs do not have this form, having a formal approach to analyze downsampling in the k-RBG case provides a tool to address general graphs, by decomposing them into a series of k-RBG. Note that this analogous to approaches for separable filtering of images, where each row is filtered independently before each column is filtered, in that one could perform filtering along selected edges in the graphs (forming a k-RBG) and then complete filtering with other regular graphs considering those edges that were ignored in the first pass.

We choose 2-D images as one such example which can be approximated as k-RBG and for which downsampling/upsampling has a known interpretation. The images can be represented as 4-regular graphs with either rectangular or diamond connectivity. We formulate the problem of downsampling images as bipartizing the underlying graph. Especially we show that common downsampling methods such as rectangular and quincunx sampling can also be understood in terms of spectral properties of these graphs. We then design new downsampling methods for images by way of different bipartization of images into k-RBG. This paper is organized as follows, in Section 2 we discuss downsampling results for k-RBG. In Section 4 we approximate 2-D images as k-RBG and in Section 5, we show results of different graph based downsampling techniques. Finally we conclude our findings in Section 6.

2. PROBLEM FORMULATION

The following notations are used throughout this paper. A signal $\mathbf{f} : \mathcal{V} \rightarrow \mathbb{R}^N$ on a graph of size $|\mathcal{V}| = N$ can be defined as a set of scalars, where each scalar is assigned to one of the vertices of the graph (more generally vectors could be assigned to each vertex). The graphs that we describe are undirected and unweighted (i.e. all links have equal weights). Let $\mathbf{W} = [w_{ij}], w_{ij} \in \{0, 1\} \forall (i, j)$ be the adjacency matrix of the graph, $\mathbf{D} = diag(\{\mathbf{d}_i\}_{i=1}^N)$ be the diagonal degree matrix of graph G, $\mathbf{L} = \mathbf{D} - \mathbf{W}$ be its Laplacian matrix and $\mathbf{I}$ be the identity matrix. Let $\{\lambda_i, \mathbf{u}_i\}_{i=1}^N$ be a system of eigenvalues/vectors of $\mathbf{L}$, with eigen-values in non-decreasing order, which can be grouped into two matrices, $\mathbf{U} = [\mathbf{u}_1, \mathbf{u}_2, ...\mathbf{u}_N]$, and $\Lambda = diag(\{\lambda_i\}$. We denote $< \mathbf{a}, \mathbf{b} >$ as the inner-product between vectors $\mathbf{a}$ and $\mathbf{b}$.

The downsampling of any general sequence $\{x[n]\}_{n \in \mathcal{V}}$ can be defined as choosing a subset $\mathcal{S} \subset \mathcal{V}$ such that all samples in $\{x[n]\}$ with indices not in $\mathcal{S}$ are discarded. Consider a discrete signal de-

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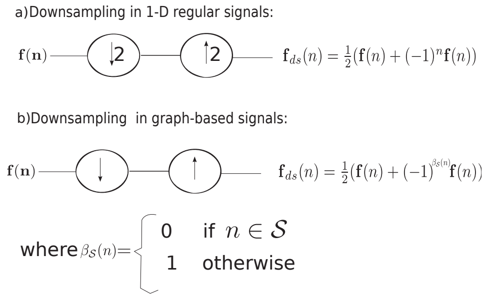


Fig. 1. Downsampling in regular signal (a) can be extended to downsampling in graphs (b) by defining a downsampling function $\beta_{\mathcal{S}}$ which only retains samples in $\mathcal{S}$ and set other samples to 0

fined in the rectangular lattice in M-dimensional space. Subsampling corresponds to using another lattice with generator matrix $\mathbf{V}$ (see [6]) to select some of the points in the rectangular lattice. In this case the set $\mathcal{S}$ is defined as $\mathcal{S} = \{\mathbf{n} : \ s.t.\ \mathbf{V}^{-1}\mathbf{n} \in \mathbb{Z}^M\}$. The upsampling operation that follows this downsampling operation reinstates the original rectangular lattice but inserts zeros at indices not in the set $\mathcal{S}$. The operation as a whole (downsample then upsample) can also be represented algebraically with a *downsampling function* $\boldsymbol{\beta}_{\mathcal{S}}$ for $\mathcal{S} \subset \mathcal{V}$ as explained in Figure 1 and a downsampling matrix $\mathbf{J}_\beta = diag\{(-1)^{\beta_n}\}$. Let $\{\boldsymbol{\psi}_n\}_{n\in\{1,..N\}}$ be an orthogonal set of basis in $\mathbb{R}^N$. Then for any N-dim signal $\mathbf{f}$:

$$\mathbf{f} = \sum_{k=1}^{k=N} \alpha_k \boldsymbol{\psi}_k \tag{1}$$

where $\alpha_k = \boldsymbol{\psi}_k^T\mathbf{f}$. The new signal $\mathbf{f}_{ds}$ obtained as depicted in Fig. 1 is given as $\mathbf{f}_{ds} = 0.5(\mathbf{f}[n] + (-1)^{\beta_n}\mathbf{f}[n])$ with $\beta_n \in \{0, 1\}$. Thus

$$\mathbf{f}_{ds}[n] = \begin{cases} \mathbf{f}[n] & \text{if } \beta_n = 0 \\ 0 & \text{if } \beta_n = 1 \end{cases} \tag{2}$$

Since $\mathbf{f}_{ds} \in \mathbb{R}^N$, it can be represented in terms of basis as:

$$\mathbf{f}_{ds} = \sum_{k=1}^{k=N} \gamma_k \boldsymbol{\psi}_k \tag{3}$$

where

$$\begin{aligned} \gamma_k &= \boldsymbol{\psi}_k^T\mathbf{f}_{ds} = \frac{1}{2}(\boldsymbol{\psi}_k^T\mathbf{f} + \boldsymbol{\psi}_k^T\mathbf{J}_\beta\mathbf{f}) \\ &= \tfrac{1}{2}(\alpha_k + (\mathbf{J}_\beta\boldsymbol{\psi}_k)^T\mathbf{f}) = \tfrac{1}{2}(\alpha_k + \hat{\alpha}_k) \end{aligned} \tag{4}$$

Thus for a given β we obtain a deformed set of basis $\hat{\boldsymbol{\psi}}_k = \mathbf{J}_\beta\psi_v$. Note that $\mathbf{J}_\beta^2 = diag\{(-1)^{2*\beta_n}\} = \mathbf{I}$, hence the deformed basis are also orthonormal and the downsampling operation adds a deformed coefficient $\hat{\alpha}_k = \hat{\boldsymbol{\psi}}_k^T\mathbf{f}$ to each original coefficient α_k in the output γ_k. We are interested in properties of deformed basis function $\mathbf{J}_\beta\boldsymbol{\psi}_k$.

In 1-D periodic signal we find that the normalized basis are DFT basis $\boldsymbol{\psi}_k = \{W_N^k\}$ where $W_N^k[n] = \exp(-2\pi jkn/N)$. The downsampling function $\beta(n) = 0$ if index n is even. The basis functions for this downsampling pattern have the property that

$$\mathbf{J}_\beta W_N^k = W_N^{(k-N/2)_N} \tag{5}$$

where $W_N^{(N/2-k)_N}$ is the $(k - N/2)$ modulo N basis function. That is the deformed basis function is another basis function in $\mathbb{R}^N$ with a different discrete frequency. We would like to consider conditions for this to be true for graphs as well. This is important as it will allow us to design "anti-aliasing" filters such that the distorted signal $\mathbf{J}_\beta\mathbf{f}$ will be band-limited and with spectral response disjoint from that of the original signal (after anti-aliasing has been applied).

3. DOWNSAMPLING IN GRAPHS

For graphs, a natural choice is to use the eigen-vectors of graph Laplacian matrix $\mathbf{L}$ as basis functions. Similar to 1-D periodic signal example we want deformed basis function $\mathbf{J}_\beta\boldsymbol{\psi}_k$ to be some other eigen-vector of Laplacian $\mathbf{L}$, i.e.

$$\mathbf{L}\mathbf{J}_\beta\boldsymbol{\psi}_k = \hat{\lambda}\mathbf{J}_\beta\boldsymbol{\psi}_k \tag{6}$$

This is true if

$$\mathbf{J}_\beta\mathbf{L}\mathbf{J}_\beta\boldsymbol{\psi}_k = \hat{\lambda}\mathbf{J}_\beta^2\boldsymbol{\psi}_k = \hat{\lambda}\boldsymbol{\psi}_k \tag{7}$$

where matrix $\hat{\mathbf{L}} = \mathbf{J}_\beta\mathbf{L}\mathbf{J}_\beta = \mathbf{D} - \mathbf{J}_\beta\mathbf{A}\mathbf{J}_\beta$ is the deformed Laplacian matrix and $\hat{\mathbf{A}} = \mathbf{J}_\beta\mathbf{A}\mathbf{J}_\beta$ is the deformed adjacency matrix given as:

$$\hat{\mathbf{A}}(i,j) = \begin{cases} \mathbf{A}(i,j) & \text{if } (-1)^{\beta_i}(-1)^{\beta_j} > 0 \\ -\mathbf{A}(i,j) & \text{if otherwise} \end{cases} \tag{8}$$

If the graph is *bipartite* (2-colorable) then $(-1)^{\beta_i}(-1)^{\beta_j} < 0$ for all linked nodes and matrices $\hat{\mathbf{A}}$ and $\mathbf{A}$ have the same eigen-vectors. In addition, if the graph is k-regular then matrices $\hat{\mathbf{L}}$ and $\mathbf{L}$ have same set of eigen-vectors. This leads to a natural downsampling pattern for k-RBG graphs based on color-assignment as explained in the section below.

3.1. Downsampling k-RBG

A bipartite graph G is a graph whose vertices can be divided into two disjoint sets $\mathcal{S}_1$ and $\mathcal{S}_2$, such that every link connects a vertex in $\mathcal{S}_1$ to one in $\mathcal{S}_2$. Further, a k-regular bipartite graph is a bipartite graph whose vertices have same degree k (i.e. $\mathbf{D} = k\mathbf{I}$). Following results are known for k-RBG:

Lemma 1 ([7]) *A k-RBG $G = (\mathcal{S}_1, \mathcal{S}_2, \mathbf{E})$ has an even number of nodes $N = 2n$ with $|\mathcal{S}_1| = |\mathcal{S}_2| = n$ nodes in each partition.*

Lemma 2 ([7]) *For all k-RBG if $\mathbf{u} = \begin{bmatrix}\mathbf{u}_1^T & \mathbf{u}_2^T\end{bmatrix}^T$ is an eigenvector of $\mathbf{L}$ with eigenvalue λ then $\hat{\mathbf{u}} = \begin{bmatrix}\mathbf{u}_1^T & -\mathbf{u}_2^T\end{bmatrix}^T$ is also an eigenvector of $\mathbf{L}$ with eigenvalue $2k - \lambda$ which is the $(N-k)^{th}$ eigenvalue of $\mathbf{L}$.*

Given a k-RBG, $G = (\mathcal{S}_1, \mathcal{S}_2, \mathbf{E})$ a downsampling function can be defined such that $\beta_n = 0$ if $n \in \mathcal{S}_1$ and $\beta_n = 1$ if $n \in \mathcal{S}_2$. This downsampling function has the property that $\hat{\mathbf{A}} = \mathbf{J}_\beta\mathbf{A}\mathbf{J}_\beta = -\mathbf{A}$. Thus

$$\begin{aligned} \hat{\mathbf{L}} &= \mathbf{D} + \mathbf{A} = k\mathbf{I} + \mathbf{A} \\ \mathbf{L} &= \mathbf{D} - \mathbf{A} = k\mathbf{I} - \mathbf{A} \end{aligned} \tag{9}$$

In this case both matrices $\mathbf{L}$ and $\hat{\mathbf{L}}$ have the same set of eigenvectors. This leads to following proposition:

Proposition 3 *If $\boldsymbol{\psi}_s$ is an eigen-vector of $\mathbf{L}$ of a k-RBG $G = (\mathcal{S}_1, \mathcal{S}_2, \mathbf{E})$ with eigenvalue λ_s then $\mathbf{J}_{\beta_{\mathcal{S}}}\boldsymbol{\psi}_s$ is also an eigen-vector of $\mathbf{L}$ with eigen-value $2k - \lambda = \lambda_{N-s}$.*

Proof By definition $\mathbf{J}_\beta\mathbf{L}\mathbf{J}_\beta\boldsymbol{\psi}_s = (k\mathbf{I} + \mathbf{A})\boldsymbol{\psi}_s = (2k\mathbf{I} - (k\mathbf{I} - \mathbf{A}))\boldsymbol{\psi}_s = (2k - \lambda_s)\boldsymbol{\psi}_s. \rightarrow \mathbf{L}\mathbf{J}_\beta\boldsymbol{\psi}_s = (2k - \lambda_s)\mathbf{J}_\beta\boldsymbol{\psi}_s$.

Substituting this in (4) we get for k-regular bipartite graphs:

$$\gamma_s = \frac{1}{2}(\boldsymbol{\psi}_s^T \mathbf{f} \pm \boldsymbol{\psi}_{N-s}^T \mathbf{f}) = \frac{1}{2}(\alpha_s \pm \alpha_{N-s}) \tag{10}$$

This result is similar to that of Nyquist Sampling theorem for regular domain signals and can be stated as:

Theorem 4 (Nyquist Theorem) *A graph-signal $\mathbf{f}$ on a k-RBG, $G = (\mathcal{S}_1, \mathcal{S}_2, \mathbf{E})$ can be completely described by only half of its samples in the set $\mathcal{S}_1$ or $\mathcal{S}_2$ if the spectrum of $\mathbf{f}$ is bandlimited by $\lambda_{N/2} = k$.*

Next we describe the properties of resulting graph after downsampling. By some permutation all nodes in sets $\mathcal{S}_1$ can be rearranged to have smaller indices than the nodes in set $\mathcal{S}_2$. The adjacency matrix of a k-RBG can then be represented as

$$\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{A}_1 \\ \mathbf{A}_1^T & \mathbf{0} \end{bmatrix} \tag{11}$$

where matrix $\mathbf{A}_1$ has exactly k ones in each row since the graph is k-regular. Moreover the degree matrix and the Laplacian matrix of this graph can be written as

$$\mathbf{D} = \begin{bmatrix} k\mathbf{I}_n & \mathbf{0} \\ \mathbf{0} & k\mathbf{I}_n \end{bmatrix} \quad \mathbf{L} = \begin{bmatrix} k\mathbf{I}_n & -\mathbf{A}_1 \\ -\mathbf{A}_1^T & k\mathbf{I}_n \end{bmatrix} \tag{12}$$

Now if we construct a new graph by only taking points from nodes in $\mathcal{S}_1$ and linking them by a link with weight equal to number of common neighbors in original graph G, we get a new downsampled graph $G1 = (\mathcal{V}_1, \mathbf{E}_1)$ with adjacency matrix $\mathbf{A}_{\mathcal{S}_1}$ given by:

$$\mathbf{A}_{\mathcal{S}_1}(i,j) = \begin{cases} 0 & \text{if } i = j \\ \mathbf{A}_1\mathbf{A}_1^T(i,j) & \text{if } i \neq j \end{cases} \tag{13}$$

It can be shown that adjacency matrix $\mathbf{A}_{\mathcal{S}_1} = \mathbf{A}_1\mathbf{A}_1^T - k\mathbf{I}_n$ and the degree matrix $\mathbf{D}_{\mathcal{S}_1} = k(k-1)\mathbf{I}_n$. The downsampled Laplacian matrix is given by

$$\mathbf{L}_{\mathcal{S}_1} = \mathbf{D}_{\mathcal{S}_1} - \mathbf{A}_{\mathcal{S}_1} = k^2\mathbf{I}_n - \mathbf{A}_1\mathbf{A}_1^T \tag{14}$$

Proposition 5 *For any k-RBG if $\mathbf{u} = \begin{bmatrix}\mathbf{u}_1^T & \mathbf{u}_2^T\end{bmatrix}^T$ is an eigenvector of $\mathbf{L}$ with eigenvalue λ_s then $\mathbf{u}_1$ is an eigenvector of downsampled Laplacian matrix $\mathbf{L}_{\mathcal{S}_1}$ with eigenvalue $\lambda_s(2k-\lambda_s) = \lambda_s\lambda_{N-s}$.*

Proof We first prove that $\mathbf{u}_1$ is an eigenvector of $\mathbf{A}_1\mathbf{A}_1^T$. Notice that: $\mathbf{L}\mathbf{u} = \lambda\mathbf{u} \rightarrow \mathbf{A}_1\mathbf{u}_2 = (k-\lambda_s)\mathbf{u}_1$ and $\mathbf{A}_1^T\mathbf{u}_1 = (k-\lambda_s)\mathbf{u}_2$ With some algebraic manipulation we get $(\mathbf{A}_1\mathbf{A}_1^T)\mathbf{u}_1 = (k-\lambda_s)^2$. Combining this with (14) we get the result stated in proposition.

This results combined with Lemma 1 and 2 if λ_i proves that **the spectrum of the downsampled graph $G1$ is the spectrum of original graph G folded over in the middle at eigenvalue k.**

This means that for any graph signal $\mathbf{f}$ on the original graph permuted in the same way as in (11) and written as $\mathbf{f} = \begin{bmatrix}\mathbf{f}_1 & \mathbf{f}_2\end{bmatrix}^T$ where

$$\mathbf{f} = \sum_{i=1}^{N} \alpha_i \mathbf{u}^i \;\Rightarrow\; \begin{bmatrix}\mathbf{f}_1 \\ \mathbf{f}_2\end{bmatrix} = \sum_{i=1}^{N} \alpha_i \begin{bmatrix}\mathbf{u}_1^i \\ \mathbf{u}_2^i\end{bmatrix} \tag{15}$$

the downsampled signal $\mathbf{f}_1$ is given as

$$\mathbf{f}_1 = \sum_{i=1}^{N} \alpha_i \mathbf{u}_1^i = \underbrace{\sum_{i=1}^{N/2} \alpha_i \mathbf{u}_1^i}_{original\ signal} + \underbrace{\sum_{i=1}^{N/2} \alpha_{N-i} \mathbf{u}_1^i}_{shifted\ copy} \tag{16}$$

This result is again similar to the downsampling result in regular signal domain where downsampling by a factor by 2 leads to an aliasing term shifted to π.

4. APPLICATIONS: IMAGES AS K-RBG

An $m \times m$ image can be represented by a periodically sampled lattice graph, with $N = m^2$ nodes (pixels) and pixel intensities as node-values. Each node can be linked to k of its neighbors in this graph. In general any k neighbors can be connected to the pixel for as long as the resulting graph is k-RBG Here we connect each pixel to its 4 NWSE or diagonal neighbors. However due to finite nature of graph, the boundary nodes have some missing neighbors and hence the graph is not k-regular. So to make this graph k-RBG we propose extending the boundaries of the graph by adding extra nodes in the graph. A discussion of some of these extensions for 1-D regular signals i.e. *line-graphs* has been given in [8]. In our work we perform a symmetric extension on the boundary pixels, the idea of which is described in Fig. 2 for rectangular and diamond connected graphs. In this extension scheme, extra nodes are added

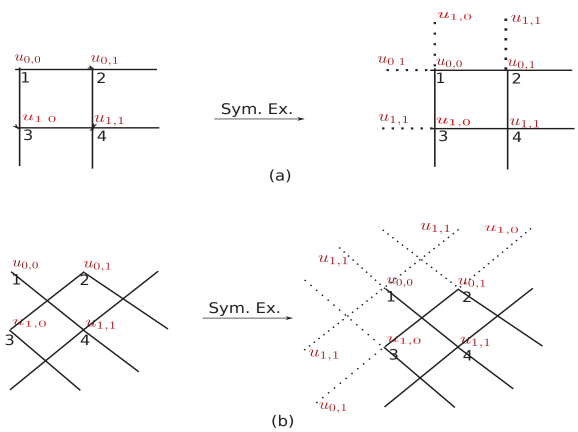
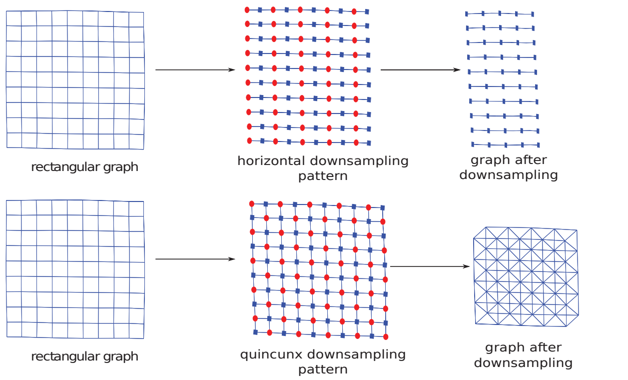


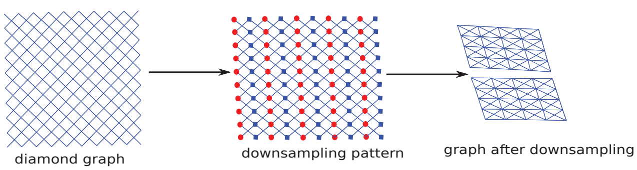
Fig. 2. Symmetric extension of boundary points for (a) rectangular graph and (b) diamond graph

at the boundary points so that the node-values are symmetric around the boundary points. As can be seen from Fig. 2, this is equivalent to adding extra directed links from boundary nodes to their neighbors. Further a similarity transform of the form of $\hat{\mathbf{L}} = \mathbf{D}^{-1/2}\mathbf{L}\mathbf{D}^{1/2}$ is performed on the extended Laplacian matrix to make it symmetric. Note that for large graphs the number of extra edges added at the boundary is very small as compared to the overall size of the graph, therefore the extended graph has a spectrum almost identical to that of the original graph Thus the symmetric extension gives us a k-RBG which is a close approximation of the original graph.

Various lattice-subsampling patterns have been discussed in the image processing literature (e.g., [6]) with known frequency response of downsampled image. In this section we show that these different downsampling methods lead to different realizations of k-RBGs on the given image. Further, since our results are in general based on any k-RBG, the nodes (pixels) can be connected in arbitrary ways (4-connected, 8-connected etc) to form k-RBG and new downsampling patterns can be obtained. As a preliminary analysis we only consider rectangular and diagonal connectivities for each node. We take up a factor 2 rectangular scheme ($\mathbf{V} = [2\ 0; 0\ 1]$) and a quincunx downsampling scheme ($\mathbf{V} = [1\ 1; 1\ -1]$) as our examples. In the factor-2 rectangular subsampling technique (Fig.3(a)), the image is downsampled by 2 in horizontal direction. The more general case of ($\mathbf{V} = [2\ 0; 0\ 2]$) has an extra step of downsampling the resulting graph by 2 in vertical direction. In spectral domain this is equivalent to decomposing the original graph G (4-connected with rectangular neighbors) into a horizontal connected graph G_h and a vertical connected graph G_v, both of which are 2-RBG after boundary extensions. Further both graphs G_h and G_v are disjoint collection of m *line-graphs* of size m each. For line-graphs the eigenvectors of Laplacian matrix are known to be DCT basis [8] with eigenvalues proportional to equally spaced discrete frequencies from 0 to π. Thus the Nyquist criteria of $\mathbf{F}(e^{j\omega}) = 0 \; \forall \; \omega \geq \pi/2$ is equivalent to $\alpha_k = 0 \; \forall \; \lambda_k \geq 2$ given in Theorem 4 in each direction.



(a) Rectangular connectivity



(b) Diagonal connectivity

Fig. 3. (a) Rectangular downsampling scheme can be seen as decomposition and bipartitization of image into two 2-RBG graphs as shown whereas quincunx downsampling scheme results in bipartization of image into a 4-RBG. (b)a diamond graph and its 4-regular bipartization

On the other hand quincunx downsampling is a non-separable scheme, in which the original graph G is downsampled by a factor of 2 into a 4-RBG as shown in Fig. 3(a). We also design a 4-RBG by connecting pixel with their 4 diagonal neighbors. The bipartization scheme has been shown in Fig. 3(b). In this scheme since the pixels are connected along the diagonal axis, which can be useful for images with high frequency response in diagonal directions

5. EXPERIMENTS

In this section we implement the 3 downsampling schemes we described (horizontal, quincunx and diamond) so as to approximate an ideal two-channel spectral filter-bank defined on corresponding k-RBGs. The ideal low-pass and high-pass filters in the filter-bank are spectral transforms on graphs similar to what is proposed in [4]. The spectral transforms in these papers are based on arbitrary bandpass kernel functions in the *spectral domain*. For ideal filters we choose spectral kernels with sharp cutoffs at $\lambda = \lambda_{N/2} = k$. Further we use Chebychev polynomial approximation of these kernels as proposed in [4] for computationally feasible implementation. These spectral filters in the spatial domain are large $N \times N$ matrices whose n^{th} row is the filter applied to node n in the graph. The 2-D Fourier spectrum of the rows (reshaped to image-size) of low-pass transforms is shown in Fig. 4. The results show that for horizontal and quincunx

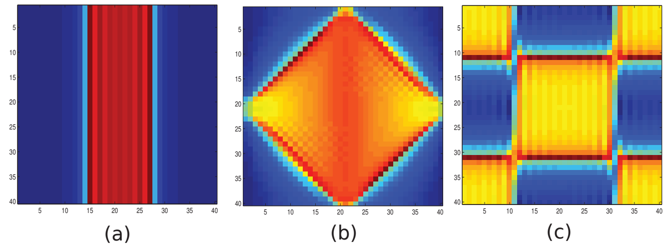


Fig. 4. Frequency responses of anti-aliasing low-pass and high-pass filters for different downsampling cases. The filters are approximation of ideal filters with a $M = 160$ order Chebychev polynomial approximation. Approximated low-pass filter for (a) horizontal downsampling (b) for quincunx downsampling and (c) for diamond graph based downsampling.

sampling the frequency response of our proposed anti-aliasing filters approximately matches the known filters in these cases. For diamond downsampling we obtain Fig. 4(c) which has a wider passband in the diagonal directions. This makes sense as the pixel connectivity is oriented in the diagonal directions.

6. CONCLUSION AND FUTURE WORK

We have proposed a method for downsampling datasets defined on k-RBGs. The results are useful for compressing graph-based data in many domains (networks, meshes, grids etc.) where the underlying graphs can be approximated by k-RBGs. As an example, we approximated images as 2 or 4 RBG and showed that our proposed graph based method provide an alternative way of interpreting downsampling filtering in images. Our future efforts are planned in two directions. On one hand we would like to extend the downsampling results to more general graphs (not just bipartite). On the other hand we will be defining new downsampling schemes that may not be easy to derive with existing lattice sampling based methods.

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