SCHEME THEORETIC NOTIONS

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CONTENTS

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1 SCHEME THEORETIC INTERSECTION AND UNION

Definition 1 (Scheme Theoretic Intersection and Union)**.** Let X be a scheme and Y, Z are closed subschemes corresponding to quasi-coherent ideal sheaves $\mathcal{I}, \mathcal{J} \subset \mathcal{O}_X$. The **scheme theoretic intersection** (S.T.I in brief) of Y and Z is the closed subscheme corresponding to the ideal sheaf $\mathcal{I} + \mathcal{J}$. The **scheme theoretic union** (S.T.U in brief) of Y and Z is the closed subscheme corresponding to the ideal sheaf $\mathcal{I} \cap \mathcal{J}$.

Then we discuss the properties of scheme theoretic intersection and scheme theoretic union.

Lemma 1. *Let X, Y, Z be as previous. Let $Y \cap Z$ be the S.T.I. of Y and Z. Then $Y \cap Z \to Y$ and $Y \cap Z \to Z$ are closed immersions and the following diagram is a cartesian diagram:*

$$\begin{tikzcd} Y \cap Z \arrow[r] \arrow[d] & Z \arrow[d] \\ Y \arrow[r] & X \end{tikzcd}$$

Proof. The fact that morphisms are closed immersions are obvious. Let $U = Spec(A)$ be an affine open and let $Y \cap U$ and $Z \cap U$ correspond to the ideals I, J. Then we have $A/I \otimes_A A/J = A/(I + J)$. Hence by the construction of fibre product we draw the conclusion. □

Lemma 2. *Let X, Y, Z be as previous. Let $Y \cap Z$ be the S.T.I of Y and Z and $Y \cup Z$ be the S.T.U of Y and Z. Then $Y \to Y \cup Z$ and $Z \to Y \cup Z$ are closed immersions. There exists a short exact sequence of $\mathcal{O}_X$ modules:*

$$0 \to \mathcal{O}_{Y \cup Z} \to \mathcal{O}_Y \times \mathcal{O}_Z \to \mathcal{O}_{Y \cap Z} \to 0$$

and the following diagram is cocartesian:

$$\begin{tikzcd} Y \cap Z \arrow[r] \arrow[d] & Y \arrow[d] \\ Z \arrow[r] & Y \cup Z \end{tikzcd}$$

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Proof. The exact sequence comes from the following exact sequence of A-modules:

$$0 \to A/I \cap J \to A/I \times A/J \to A/(I+J) \to 0$$

Given morphisms of schemes $f : Y \to T$ and $Z \to T$ agreeing as $Y \cap Z \to T$. We need to construct the unique morphism $h : Y \cup Z \to T$. Suppose $x \in Y \backslash Z$. Then $Y \to Y \cup Z$ is an isomophism in some neiborhood of x and h can be uniquely defined on such a neiborhood. If $x \in Y \cap Z$. Then there exists an affine open $V = Spec(B) \subset T$ s.t. there exists an affine $U = Spec(A) \subset X$ containing s and $f(Y \cap U) \subset V$, $g(Z \cap U) \subset V$. Then given morphisms $B \to A/I$ and $B \to A/J$ agree as morphisms to $A/(I+J)$. By the exact sequence there exists a unique morphism $B \to A/I \cap J$ as desired. □

2 SCHEME THEORETIC SUPPORT

The support of a quasi-coherent sheaf may not be closed. But it's always closed under specialization.

Lemma 3. *Let X be a scheme and $\mathscr{F} \in QCoh(\mathscr{O}_X)$. The support of $\mathscr{F}$ is closed under specialization.*

Proof. If $x' \rightsquigarrow x$ is a specialization. Then $\mathscr{F}_{x'}$ is a localization of $\mathscr{F}_x$. Hence the conclusion holds. □

But if the quasi-coherent module is of finite type, Then the support of it must be closed.

Lemma 4. *Let $\mathscr{F}$ be a finite type (locally finitely generated) quasi-coherent module on a scheme X. Then the support of $\mathscr{F}$ is closed. And for $x \in X$ we have*

$$x \in Supp(\mathcal{F}) \Leftrightarrow \mathcal{F}_x \neq 0 \Leftrightarrow \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \kappa(x) \neq 0$$

Proof. The support of a local section is always closed in it's corresponding open subset. Hence the first conclusion is obvious. The second conclusion follows from the Nakayama's lemma. □

The property of being finite type is preserved under pullback. And the support of the pullback is exactly the inverse image of the support of the original module.

Lemma 5. *For any morphism of schemes $f : Y \to X$ the pullback $f^*\mathscr{F}$ is of finite type and we have*

$$Supp(f^*\mathcal{F}) = f^{-1}(Supp(\mathcal{F}))$$

Proof. By the definition of f^*, the first conclusion holds obviously. Recall that

$$(f^*\mathcal{F})_y = \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{Y,y}$$

Hence $(f^*\mathscr{F})_y \otimes \kappa(y) = \mathscr{F}_{\otimes}\kappa(x) \otimes_{\kappa(x)} \kappa(y)$. Hence $(f^*\mathcal{F})_y \otimes \kappa(y) \neq 0$ iff $\mathscr{F}_x \otimes \kappa(x)$ is nonzero. Hence it implies that $x \in Supp(\mathscr{F})$ iff $y \in Supp(f^*\mathscr{F})$. □

Before we define the scheme theoretic support, we need the following lemma:

Lemma 6. *Let $i : Z \to X$ be a closed immersion of schemes. Let $\mathcal{I} \subset \mathscr{O}_X$ be the quasi-coherent sheaf of ideals corresponding to Z. The functor*

$$i_* : QCoh(\mathcal{O}_Z) \longrightarrow QCoh(\mathcal{O}_X)$$

is exact, fully faithful, with essential image those quasi-coherent $\mathscr{O}_X$-module $\mathscr{G}$ s.t. $\mathcal{I}\mathscr{G} = 0$.

Proof. A closed immersion is both seperated and quasi-compact. Hence i_* maps quasi-coherent $\mathscr{O}_Z$-module to quasi-coherent $\mathscr{O}_X$-module. The exactness can be checked on the stalks.

Then we show the essential image is exactly that described in the lemma. We have $\mathcal{I}(i_*\mathscr{F}) = 0$ for any quasi-coherent $\mathscr{O}_Z$ module $\mathscr{F}$. Hence it suffice to show that both of the canonical map are isomorphisms

$$\mathscr{G} \to i_*i^*\mathscr{G}$$
$$i^*i_*\mathscr{F} \to \mathscr{F}$$

where $\mathscr{G}$ is a quasi-coherent $\mathscr{O}_X$-module s.t. $\mathcal{I}\mathscr{G} = 0$ and $\mathscr{F}$ is a quasi-coherent $\mathscr{O}_Z$-module. Both of the isomorphisms are direct corrollary of the following algebraic statement: Given a ring R and an ideal I and an R-module N s.t. $IN = 0$. Then the canonical map

$$N \to N \otimes_R R/I$$

is an isomorphism of R-module. □

Now we turn to consider the scheme theoretic support of a finite type quasi-coherent module.

Definition 2 (Scheme Theoretic Support)**.** Let $\mathscr{F}$ be a finite type quasi-coherent module on X. The **scheme theoretic support** (S.T.S in brief) is the minimal closed subscheme $i : Z \to X$ s.t. there exists a quasi-coherent $\mathscr{O}_Z$-module $\mathscr{G}$ with $i_*\mathscr{G} \simeq \mathscr{F}$.

Proposition 1. *The scheme theoretic support always exists. And it satisfies the following properties:*

1. *If $Spec(A) \subset X$ is any affine open and $\mathscr{F}|_{Spec(A)} = \tilde{M}$, then $Z \cap Spec(A) = Spec(A/I)$, where $I = Ann_A(M)$.*
2. *The quasi-coherent sheaf $\mathscr{G}$ is unique up to unique isomorphism.*
3. *The quasi-coherent sheaf $\mathscr{G}$ is of finite type.*
4. *The support of $\mathscr{G}$ and of $\mathscr{F}$ is Z.*

Proof. We define Z by the first property since $\mathrm{Ann}_A(M)_f = \mathrm{Ann}_{A_f}(M_f)$. By the previous lemma we see that there exists a unique quasi-coherent sheaf $\mathscr{G}$ on Z s.t. $\mathscr{F} \simeq i_*\mathscr{G}$. Also, $\mathscr{G}$ is of finite type since such a finite R-module is also a finite R/I-module. The last assertion is trivial. □

3 SCHEME THEORETIC IMAGE

Definition 3 (Scheme Theoretic Image)**.** Let $X \to Y$ be a morphism of schemes. The **scheme theoretic image** of f is the smallest closed subscheme $Z \subset Y$ through which f factors.

Then we show that the scheme theoretic image is always exists. But before the proof we need a lemma.

Lemma 7. *Let X be a scheme and $\mathscr{F}$ a quasi-coherent $\mathscr{O}_X$-module. Let $\mathscr{G} \subset \mathscr{F}$ be a submodule. There exists a unique quasi-coherent submodule $\mathscr{G}' \subset \mathscr{G}$ s.t. For every quasi-coherent $\mathscr{O}_X$-module $\mathcal{H}$*

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{H}, \mathcal{G}') \longrightarrow \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{H}, \mathcal{G})$$

is bijective. In particular, $\mathscr{G}'$ is the maximal quasi-coherent $\mathscr{O}_X$-submodule of $\mathscr{G}$.

Proof. Let $\{G_a\}_{a\in A}$ be the set of all the quasi-coherent submodule of $\mathscr{G}$. Let

$$\mathscr{G}' = Image(\bigoplus_{a\in A} \mathcal{G}_a \longrightarrow \mathcal{F})$$

Since the image of a morphisms between quasi-coherent sheaves is quasi-coherent. $\mathscr{G}'$ is of course the largest quasi-coherent submodule of $\mathscr{G}$.

Let $f : \mathcal{H} \to \mathscr{G}$ be an $\mathscr{O}_X$-module morphism. The image of $\mathcal{H} \to \mathscr{G} \to \mathscr{F}$ is quasi-coherent. Hence it's contained in $\mathscr{G}'$. Thus the formula holds. □

Remark. Let $i : Z \to X$ be a closed immersion of schemes. There is a functor

$$\bigoplus_{a\in A} \mathcal{G}_a \longrightarrow \mathcal{F}$$

defined by $i^! \mathcal{G} = i^*(\mathcal{H}_Z(\mathcal{G})')$ is right adjoint to i_*, where $\mathcal{H}_Z(G)$ are the subsheaf generated by the local sections annihilated by $\mathcal{I}$.

Lemma 8. *For any f, the scheme theoretic image always exists.*

Proof. Let $\mathcal{I} = \mathrm{Ker}(\mathcal{O}_Y \to f_*\mathcal{O}_X)$. There exists a maximal quasi-coherent sheaf of ideals $\mathcal{I}' \subset \mathcal{I}$. Hence we define Z to be the closed subscheme corresponding to the ideal sheaf $\mathcal{I}$. It's obvious the closed subscheme as desired. $\square$

Remark.

1. $\overline{f(X)}$ may not equal to the underlying set of the theoretic image of f.
2. The construction of the scheme theoretic image does not commute with restriction to open subschemes of Y. In other words, suppose $f(X) \subset U \subset V$, where $U, V \subset X$ are open subschemes. Let the scheme theoretic images of $f_1 : X \to V$ and $f_2 : X \to U$ be Z_1 and Z_2. Then it might happen that $Z_2 \cap U \neq Z_1$.

But if f is quasi-compact, things will be very awesome.

Proposition 2. *Let $f : X \to Y$ be a morphism of schemes. Let $Z \subset Y$ be the scheme theoretic image of f. If f is quasi-compact then*

1. *$\mathcal{I} = \mathrm{Ker}(\mathcal{O}_Y \to f_*\mathcal{O}_X)$ is quasi-coherent.*
2. *Z is the closed subscheme determined by $\mathcal{I}$.*
3. *For any open $U \subset Y$, the scheme theoretic image of $f|_{f^{-1}(U)} : f^{-1}(U) \to U$ is equal to $Z \cap U$.*
4. *$\overline{f(X)} = Z$.*

Proof. The fourth assertion follows from the third one. And both the second and the third conclusions follows from the first one. Since the property of being quasi-coherent is local. We may assume Y is affine. Since f is quasi-compact, we can decompose X into finitely many affine opens $X = \bigcup_{i=1,\ldots,n} U_i$. Let $X' = \coprod U_i$, which is affine. Let f' be the composition of

$$X' \to X \to Y$$

Hence we have $f_*\mathscr{O}_X = f'_*\mathscr{O}_{X'}$ and thus $\mathcal{I} = \mathrm{Ker}(\mathcal{O}_Y \to \mathcal{O}_{X'})$. Therefore $\mathcal{I}$ is quasi-coherent. $\square$

More precisely, we only need to adds points which are specializations of points in $f(X)$ to get the scheme theoretic image if f is quasi-compact. We can use the method of valuation to show the conclusion.

Lemma 9. *Let $f : X \to Y$ be quasi-compact morphism and Z the scheme theoretic image of f. Let $z \in Z$. There exists a valuation ring A with fraction field K and*

$$\begin{tikzcd}
\mathrm{Spec}(K) \arrow[rr] \arrow[d] & & X \arrow[dl] \arrow[d] \\
\mathrm{Spec}(A) \arrow[r] & Z \arrow[r] & Y
\end{tikzcd}$$

s.t. the closed point of $Spec(A)$ maps to z. In particular, every point of Z is the specialization of a point of $f(X)$.

Proof. Let $z \in Spec(R) = V \subset Y$ be an affine open. And z corresponds to the prime $\mathfrak{p} \subset R$. The intersection $Z \cap V$ is the scheme theoretic image of $f^{-1}(V) \to V$. Hence WLOG we assume that $Y = Spec(R)$. Let $X = \bigcup_{i=1,\cdots,n} U_i$ be a finite affine open covering, where $U_i = Spec(R)$. Let $I = Ker(R \to A_1 \times \cdots \times A_n)$. Hence Z corresponds to the closed subscheme $Spec(R/I)$.

Now we only need to find a prime $\mathfrak{p}_i \subset \mathfrak{p}$ and some prime $\mathfrak{q}_i \subset A_i$ lying over $\mathfrak{p}_i$. Then we can choose a valuation ring $A \subset K = \kappa(q_i)$ dominating the local ring $R_\mathfrak{p}/\mathfrak{p}_i R_\mathfrak{p} \subset \kappa(p_i) \subset \kappa(q_i)$.

Since we have $I_\mathfrak{p} \subsetneq R_\mathfrak{p}$ because $I \subset \mathfrak{p}$. We see that $R_\mathfrak{p} \to (A_1)_\mathfrak{p} \times \cdots \times (A_n)_\mathfrak{p}$ is not zero. Hence one of the rings $(A_i)_\mathfrak{p}$ is not zero. And there exists an i and a prime $\mathfrak{q} \subset A_i$ lying over a prime $\mathfrak{p}_i \subset \mathfrak{p}$. $\square$

Now let us consider the comma category of morphisms of schemes. The objects of such a category are morphisms of schemes $f : X \to Y$. The morphisms between $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$ is a commutative diagram

$$\begin{tikzcd} X_1 \arrow[r, "f_1"'] \arrow[d] & Y_1 \arrow[d] \\ X_2 \arrow[r, "f_2"] & Y_2 \end{tikzcd}$$

Taking scheme theoretic image is an functor from the category of morphisms of schemes to the category of schemes in fact. In other words, we have the following lemma.

Lemma 10. *Let*

$$\begin{tikzcd} X_1 \arrow[r, "f_1"'] \arrow[d, "g_1"'] & Y_1 \arrow[d, "g_2"] \\ X_2 \arrow[r, "f_2"] & Y_2 \end{tikzcd}$$

be a commutative diagram of schemes. Let $Z_i \subset Y_i$ be the scheme theoretic image of f_i. Then the morphism $Y_1 \to Y_2$ induces a canonical morphism $Z_1 \to Z_2$ and a commutative diagram

$$\begin{tikzcd} X_1 \arrow[r] \arrow[d] & Z_1 \arrow[r] \arrow[d] & Y_1 \arrow[d] \\ X_2 \arrow[r] & Z_2 \arrow[r] & Y_2 \end{tikzcd}$$

Proof. The scheme theoretic image of Z_2 in Y_1 is a closed subscheme of Y_1 through which f_1 factors. In other words, the scheme theoretic image of $g_2 \circ f_1$, denoted by Z', is contained in the scheme theoretic image of f_2. And of course f_1 factors through the scheme theoretic inverse image of Z' by g_2. □

Suppose $f : X \to Y$ is a morphism from a reduced scheme X. Then the scheme theoretic image of f is also the same case as f is quasi-compact.

Lemma 11. *Let $f : X \to Y$ be a morphism of schemes. If X is reduced, then the scheme theoretic image of f is the reduced scheme theoretic structure on $\overline{f(X)}$*

It's a direct corrollary of the following lemma:

Lemma 12. *Let X be a scheme and $Z \subset X$ be a closed subscheme. Let Y be a reduced scheme. A morphism $f : YY \to X$ factors through Z iff $f(Y) \subset Z$ (set theoretically).*

4 SCHEME THEORETIC CLOSURE AND SCHEME THEORETIC DENSE

At first, we consider an immersion $h : Z \to X$. The most interesting cases are h being quasi-compact or Z is reduced.

Proposition 3. *Let $h : Z \to X$ be an immersion. If h is quasi-compact, or Z is reduced, then we can factor $h = i \circ j$ with $j : Z \to \overline{Z}$ an open immersion and $i : \overline{Z} \to X$ a closed immersion.*

Proof. 1. Suppose h is quasi-compact. Since h is an immersion, it's also quasi-separated. Hence $h_*\mathscr{O}_Z$ is a quasi-coherent sheaf of $\mathscr{O}_X$-module. And $\mathcal{I} = \mathrm{Ker}(\mathcal{O}_X \to h_*\mathcal{O}_Z)$ is quasi-coherent. Hence let $\overline{Z} \subset X$ be the closed subscheme corresponding to $\mathcal{I}$. Then h obviously factor through $i : \overline{Z} \to X$ which is a closed immersion. To see $j : Z \to \overline{Z}$ is an open immersion, let $U \subset X$ be an open subscheme s.t. h induces a closed immersion of Z into U. Then it's clear that $\mathcal{I}|_U$ is the sheaf of ideal corresponding to the closed immersion $Z \to U$.

2. Suppose Z is reduced. The assertion is a direct conclusion of *Lemma*12. □

Then we define the scheme theoretic closure for general open subschemes.

Definition 4. Let X be a scheme and $U \subset X$ be an open subscheme.

1. The **scheme theoretic closure** (S.T.C in brief) of U is the scheme theoretic image of $i : U \hookrightarrow X$.
2. We say U is **scheme theoretically dense** (S.T.D in brief) in X if for every open $V \subset X$ the S.T.C of $U \cap V$ in V is equal to V.

Remark. In general, the S.T.C of U being X does not imply U is S.T.D. in X. But if $i : U \to X$ is quasi-compact, then U is S.T.D. in X iff the S.T.C of U is X.

There is an criterion for being S.T.D:

Proposition 4. *Let $j : U \to X$ be an open immersion of schemes. Then U is scheme theoretically dense in X iff $\mathscr{O}_X \to j_*\mathscr{O}_U$ is injective.*

Proof. If $\mathscr{O}_X \to j_*\mathscr{O}_U$ is injective, then the same is ture when restricted to any open V of X. Hence the scheme theoretic closue of $U \cap V$ in V is equal to V. Conversely, suppose $\mathscr{O}_X \to j_*\mathscr{O}_U$ is not injective. Then we can find an affine open $Spec(A) = V \subset X$ and a nonzero element $f \in A$ s.t. f maps to zero in $\gamma(V \cap U, \mathscr{O}_X)$. Hence the scheme theoretic closure of $V \cap U$ in V is contained in $Spec(A/(f))$. □

The intersection of two S.T.D open subschemes is also S.T.D.

Lemma 13. *Let U, V be S.T.D open subschemes of X, then $U \cap V$ is S.T.D in X.*

Proof. Let $W \subset X$ be any open. The composition of the morphisms $\mathcal{O}_X(W) \to \mathcal{O}_X(W \cap V) \to \mathcal{O}_X(W \cap V \cap U)$ is injective. □

Then we return to the case $h : Z \to X$ be an immersion, where h is quasi-compact or Z is reduced.

Lemma 14. *Let $h : Z \to X$ be an immersion. Assume h is quasi-compact or Z is reduced. Let $\overline{Z} \to X$ be the scheme theoretic image of h. Then $Z \to \overline{Z}$ is an open immersion which identifies Z with a S.T.D open subscheme of $\overline{Z}$. And Z is topologically dense in $\overline{Z}$.*

Proof. By *Proposition*3, the underlying set of Z is exactly the topological closure of Z. Furthermore, if Z is reduced, then the theoretic image has also the unique reduced structure on $\overline{Z}$. And if h is quasi-compact, then $\mathscr{O}_{\overline{Z}} \to i_*\mathscr{O}Z$ is an injection. □

Thus in reduced scheme, an open subscheme is topologically dense is exactly the same as S.T.D.

Proposition 5. *Let X be a reduced scheme and $U \subset X$ be an open subscheme. TFAE*

1. *U is topological dense in X*
2. *the S.T.C of U in X is X*
3. *U is S.T.D in X*

Conversely, we have

Lemma 15. *Let X be a scheme and $U \subset X$ be a reduced open subscheme. TFAE*

1. *The S.T.C. of U in X is X*
2. *U is S.T.D. in X*

If this holds then X is also reduced.

At last, we consider the equalizer of morphisms in the category of schemes.

Proposition 6. *Let X, Y be schemes over S and $a, b : X \to Y$ be morphisms of schemes over S. There exists a largest locally closed subscheme $Z \subset X$ s.t. $a|_Z = b|_Z$, namely the equalizer of (a, b). If Y is separated over S, then Z is a closed subscheme.*

Proof. The equalizer of (a, b) is for categorical reasons the fibre product Z in the following diagram

$$\begin{tikzcd} Z = Y \times_{(Y \times_S Y)} X \arrow[r] \arrow[d] & X \arrow[d, "(a,b)"] \\ Y \arrow[r, "\Delta_{Y/S}"] & Y \times_S Y \end{tikzcd}$$

Since being a (closed) immersion is preserved under base change. The proposition follows. □

Two continuous maps that agree on a dense open subset is equal. It has similar generalization in the category of schemes.

Lemma 16. *Let S be a scheme and X, Y be schemes over S. Let $f, g : X \to Y$ be morphisms of schemes over S. Let $U \subset X$ be an open subscheme s.t. $f|_U = g|_u$. If the S.T.C of U is X and $Y \to X$ is separated, then $f = g$.*