有限元方法

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课程目标

- 1、有限元方法的原理
- 2、有限元方法的实现
- 3、有限元方法的应用

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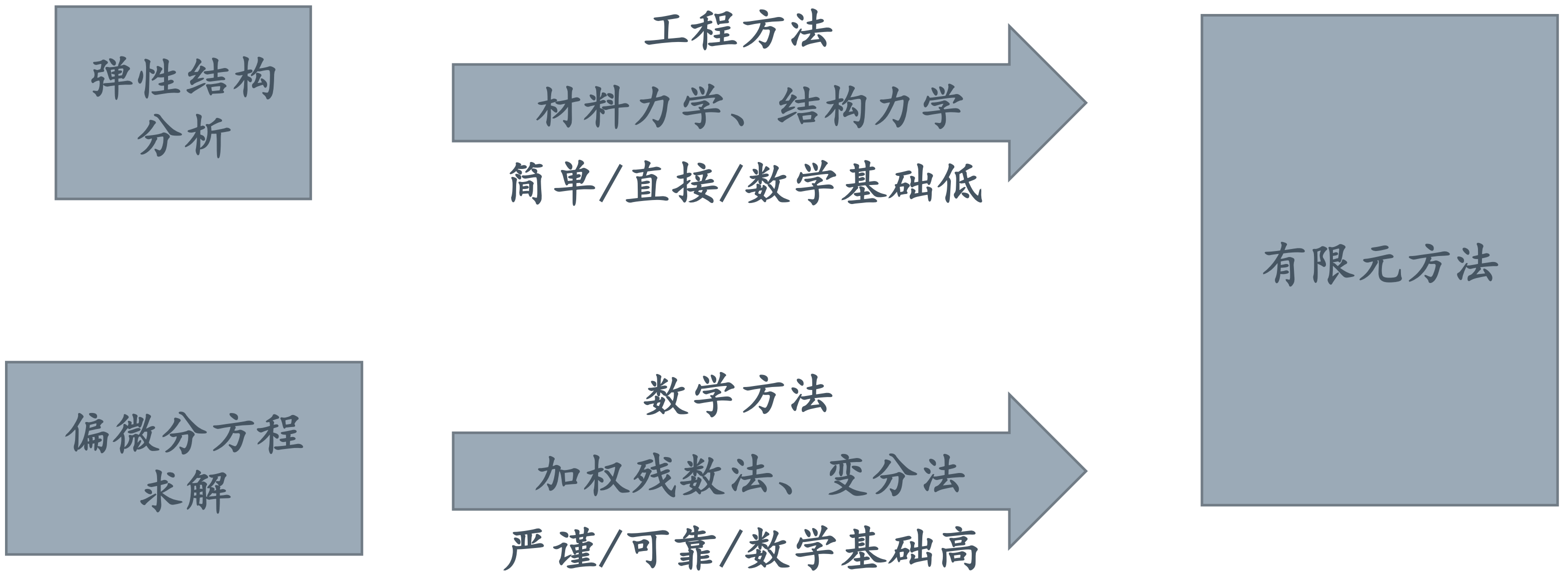
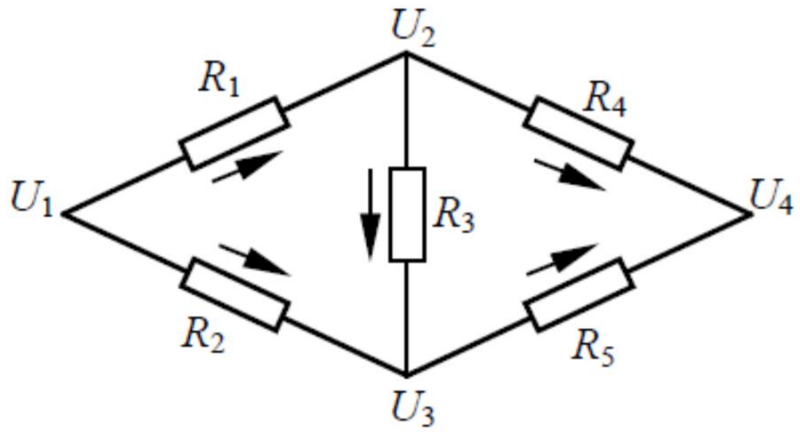
有限元方法的原理


弹性结构分析
工程方法
材料力学、结构力学
简单/直接/数学基础低
偏微分方程求解
数学方法
加权残数法、变分法
严谨/可靠/数学基础高
有限元方法

工程方法

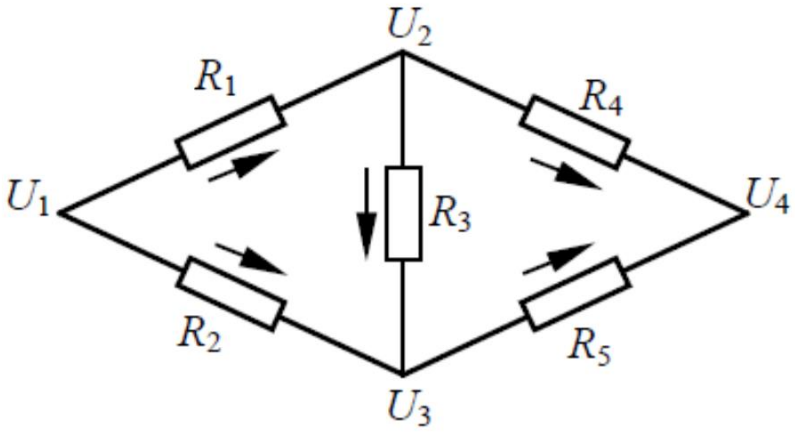
- 电路问题
- 传热问题
- 变形问题

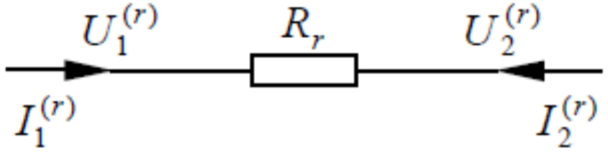


给定U_1和U_4电压差

计算流经各个电阻的电流

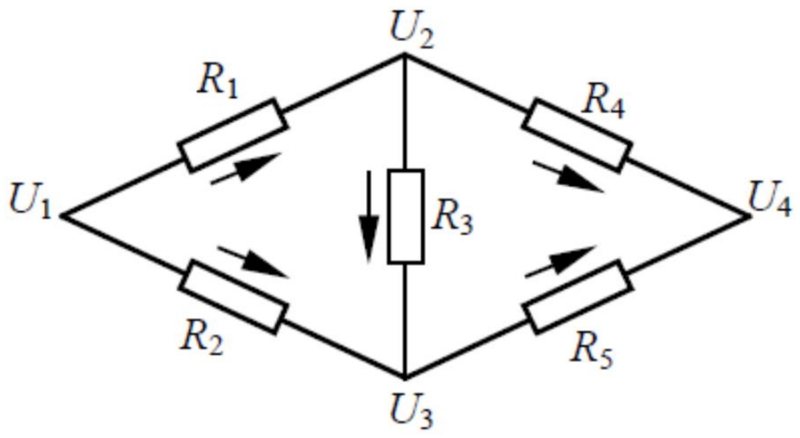
电路问题





$$\frac{U}{R} = I \Rightarrow \begin{cases} \dfrac{U_1^{(r)} - U_2^{(r)}}{R_r} = I_1^{(r)} \\ \dfrac{U_2^{(r)} - U_1^{(r)}}{R_r} = I_2^{(r)} \end{cases}$$

单元方程的建立



$$\frac{1}{R_r}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} U_1^{(r)} \\ U_2^{(r)} \end{Bmatrix} = \begin{Bmatrix} I_1^{(r)} \\ I_2^{(r)} \end{Bmatrix}$$

$$\frac{1}{R_1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix} = \begin{Bmatrix} I_1^{(1)} \\ I_2^{(1)} \end{Bmatrix}$$

$$\frac{1}{R_2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} U_1 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} I_1^{(2)} \\ I_2^{(2)} \end{Bmatrix}$$

$$\frac{1}{R_3}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} I_1^{(3)} \\ I_2^{(3)} \end{Bmatrix}$$

$$\frac{1}{R_4}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} U_2 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} I_1^{(4)} \\ I_2^{(4)} \end{Bmatrix}$$

$$\frac{1}{R_5}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} I_1^{(5)} \\ I_2^{(5)} \end{Bmatrix}$$

单元方程的组装

$$\frac{1}{R_1}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}U_1\\ U_2\end{Bmatrix}=\begin{Bmatrix}I_1^{(1)}\\ I_2^{(1)}\end{Bmatrix}$$

$$\frac{1}{R_1}\begin{bmatrix}1 & -1 & 0 & 0\\ -1 & 1 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\begin{Bmatrix}U_1\\ U_2\\ U_3\\ U_4\end{Bmatrix}=\begin{Bmatrix}I_1^{(1)}\\ I_2^{(1)}\\ 0\\ 0\end{Bmatrix}$$

$$\frac{1}{R_2}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}U_1\\ U_3\end{Bmatrix}=\begin{Bmatrix}I_1^{(2)}\\ I_2^{(2)}\end{Bmatrix}$$

$$\frac{1}{R_2}\begin{bmatrix}1 & 0 & -1 & 0\\ 0 & 0 & 0 & 0\\ -1 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\begin{Bmatrix}U_1\\ U_2\\ U_3\\ U_4\end{Bmatrix}=\begin{Bmatrix}I_1^{(2)}\\ 0\\ I_2^{(2)}\\ 0\end{Bmatrix}$$

$$\frac{1}{R_3}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}U_2\\ U_3\end{Bmatrix}=\begin{Bmatrix}I_1^{(3)}\\ I_2^{(3)}\end{Bmatrix}$$

$$\frac{1}{R_3}\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & 1 & -1 & 0\\ 0 & -1 & 1 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\begin{Bmatrix}U_1\\ U_2\\ U_3\\ U_4\end{Bmatrix}=\begin{Bmatrix}0\\ I_1^{(3)}\\ I_2^{(3)}\\ 0\end{Bmatrix}$$

$$\frac{1}{R_4}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}U_2\\ U_4\end{Bmatrix}=\begin{Bmatrix}I_1^{(4)}\\ I_2^{(4)}\end{Bmatrix}$$

$$\frac{1}{R_4}\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & 1 & 0 & -1\\ 0 & 0 & 0 & 0\\ 0 & -1 & 0 & 1\end{bmatrix}\begin{Bmatrix}U_1\\ U_2\\ U_3\\ U_4\end{Bmatrix}=\begin{Bmatrix}0\\ I_1^{(4)}\\ 0\\ I_2^{(4)}\end{Bmatrix}$$

$$\frac{1}{R_5}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}U_3\\ U_4\end{Bmatrix}=\begin{Bmatrix}I_1^{(5)}\\ I_2^{(5)}\end{Bmatrix}$$

$$\frac{1}{R_5}\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 1 & -1\\ 0 & 0 & -1 & 1\end{bmatrix}\begin{Bmatrix}U_1\\ U_2\\ U_3\\ U_4\end{Bmatrix}=\begin{Bmatrix}0\\ 0\\ I_1^{(5)}\\ I_2^{(5)}\end{Bmatrix}$$

整体方程的建立

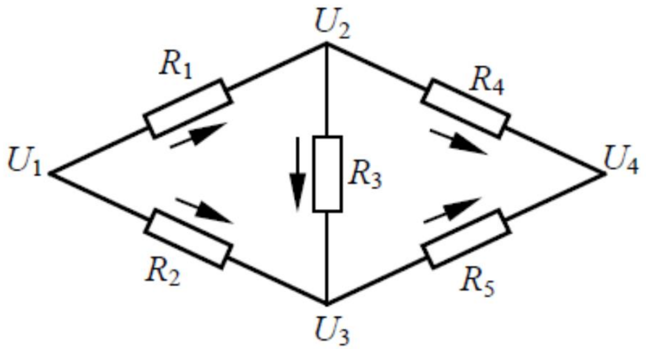
$$\frac{1}{R_1}\begin{bmatrix}1 & -1 & 0 & 0\\ -1 & 1 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\begin{Bmatrix}U_1\\ U_2\\ U_3\\ U_4\end{Bmatrix}=\begin{Bmatrix}I_1^{(1)}\\ I_2^{(1)}\\ 0\\ 0\end{Bmatrix}$$

$$\frac{1}{R_2}\begin{bmatrix}1 & 0 & -1 & 0\\ 0 & 0 & 0 & 0\\ -1 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\begin{Bmatrix}U_1\\ U_2\\ U_3\\ U_4\end{Bmatrix}=\begin{Bmatrix}I_1^{(2)}\\ 0\\ I_2^{(2)}\\ 0\end{Bmatrix}$$

$$\frac{1}{R_3}\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & 1 & -1 & 0\\ 0 & -1 & 1 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\begin{Bmatrix}U_1\\ U_2\\ U_3\\ U_4\end{Bmatrix}=\begin{Bmatrix}0\\ I_1^{(3)}\\ I_2^{(3)}\\ 0\end{Bmatrix}$$

$$\frac{1}{R_4}\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & 1 & 0 & -1\\ 0 & 0 & 0 & 0\\ 0 & -1 & 0 & 1\end{bmatrix}\begin{Bmatrix}U_1\\ U_2\\ U_3\\ U_4\end{Bmatrix}=\begin{Bmatrix}0\\ I_1^{(4)}\\ 0\\ I_2^{(4)}\end{Bmatrix}$$

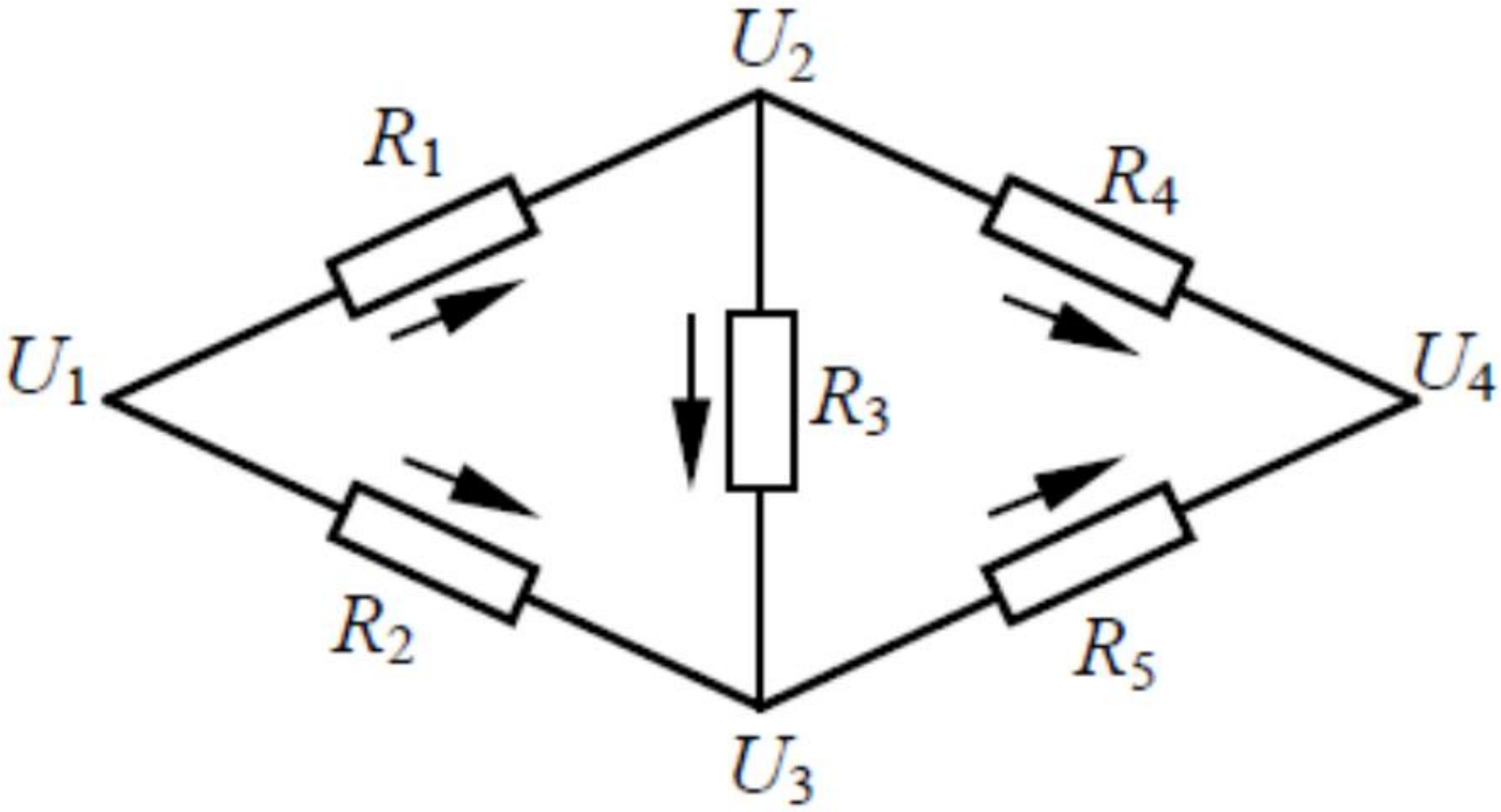
$$\frac{1}{R_5}\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 1 & -1\\ 0 & 0 & -1 & 1\end{bmatrix}\begin{Bmatrix}U_1\\ U_2\\ U_3\\ U_4\end{Bmatrix}=\begin{Bmatrix}0\\ 0\\ I_1^{(5)}\\ I_2^{(5)}\end{Bmatrix}$$



$$\Rightarrow\begin{bmatrix}\frac{1}{R_1}+\frac{1}{R_2} & -\frac{1}{R_1} & -\frac{1}{R_2} & 0\\ -\frac{1}{R_1} & \frac{1}{R_1}+\frac{1}{R_3}+\frac{1}{R_4} & -\frac{1}{R_3} & -\frac{1}{R_4}\\ -\frac{1}{R_2} & -\frac{1}{R_3} & \frac{1}{R_2}+\frac{1}{R_3}+\frac{1}{R_5} & -\frac{1}{R_5}\\ 0 & -\frac{1}{R_2} & -\frac{1}{R_5} & \frac{1}{R_4}+\frac{1}{R_5}\end{bmatrix}\begin{Bmatrix}U_1\\ U_2\\ U_3\\ U_4\end{Bmatrix}=\begin{Bmatrix}I_1^{(1)}+I_1^{(2)}\\ I_2^{(1)}+I_1^{(3)}+I_1^{(4)}\\ I_2^{(2)}+I_2^{(3)}+I_1^{(5)}\\ I_2^{(4)}+I_2^{(5)}\end{Bmatrix}$$

边界条件的处理new

$$\left\{ \begin{array}{c} I_1^{(1)} + I_1^{(2)} \\ \\ I_2^{(1)} + I_1^{(3)} + I_1^{(4)} \\ \\ I_2^{(2)} + I_2^{(3)} + I_1^{(5)} \\ \\ I_2^{(4)} + I_2^{(5)} \end{array} \right\}$$



系统方程的整合

$$\begin{bmatrix} \frac{1}{R_1}+\frac{1}{R_2} & -\frac{1}{R_1} & -\frac{1}{R_2} & 0 \\ -\frac{1}{R_1} & \frac{1}{R_1}+\frac{1}{R_3}+\frac{1}{R_4} & -\frac{1}{R_3} & -\frac{1}{R_4} \\ -\frac{1}{R_2} & -\frac{1}{R_3} & \frac{1}{R_2}+\frac{1}{R_3}+\frac{1}{R_5} & -\frac{1}{R_5} \\ 0 & -\frac{1}{R_2} & -\frac{1}{R_5} & \frac{1}{R_4}+\frac{1}{R_5} \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} I_1^{(1)}+I_1^{(2)} \\ I_2^{(1)}+I_1^{(3)}+I_1^{(4)} \\ I_2^{(2)}+I_2^{(3)}+I_1^{(5)} \\ I_2^{(4)}+I_2^{(5)} \end{Bmatrix} = \begin{Bmatrix} I_1 \\ 0 \\ 0 \\ I_4 \end{Bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} -\frac{1}{R_1} & \frac{1}{R_1}+\frac{1}{R_3}+\frac{1}{R_4} & -\frac{1}{R_3} & -\frac{1}{R_4} \\ -\frac{1}{R_2} & -\frac{1}{R_3} & \frac{1}{R_2}+\frac{1}{R_3}+\frac{1}{R_5} & -\frac{1}{R_5} \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

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系统方程的求解

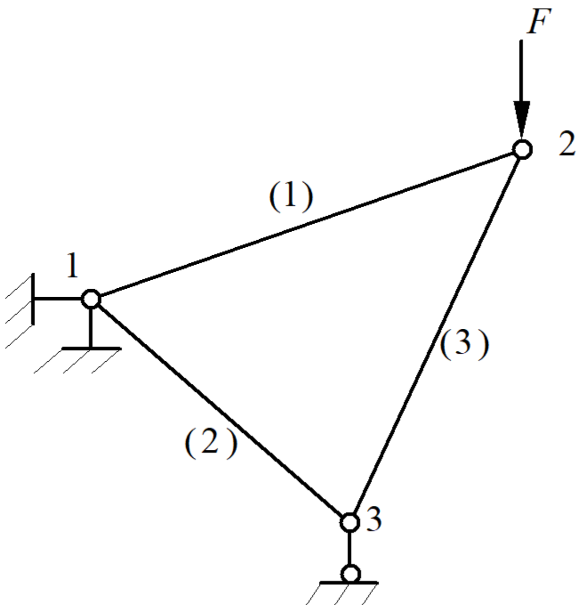
$$\begin{bmatrix} -\frac{1}{R_1} & \frac{1}{R_1}+\frac{1}{R_3}+\frac{1}{R_4} & -\frac{1}{R_3} & -\frac{1}{R_4} \\ -\frac{1}{R_2} & -\frac{1}{R_3} & \frac{1}{R_2}+\frac{1}{R_3}+\frac{1}{R_5} & -\frac{1}{R_5} \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} \frac{1}{R_1}+\frac{1}{R_3}+\frac{1}{R_4} & -\frac{1}{R_3} \\ -\frac{1}{R_3} & \frac{1}{R_2}+\frac{1}{R_3}+\frac{1}{R_5} \end{bmatrix} \begin{Bmatrix} U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} \frac{1}{R_1}U_1+\frac{1}{R_4}U_4 \\ \frac{1}{R_2}U_1+\frac{1}{R_5}U_4 \end{Bmatrix}$$

有限元系统方程的特点

$$\begin{bmatrix} \frac{1}{R_1}+\frac{1}{R_3}+\frac{1}{R_4} & -\frac{1}{R_3} \\ -\frac{1}{R_3} & \frac{1}{R_2}+\frac{1}{R_3}+\frac{1}{R_5} \end{bmatrix} \begin{Bmatrix} U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} \frac{1}{R_1}U_1+\frac{1}{R_4}U_4 \\ \frac{1}{R_2}U_1+\frac{1}{R_5}U_4 \end{Bmatrix}$$

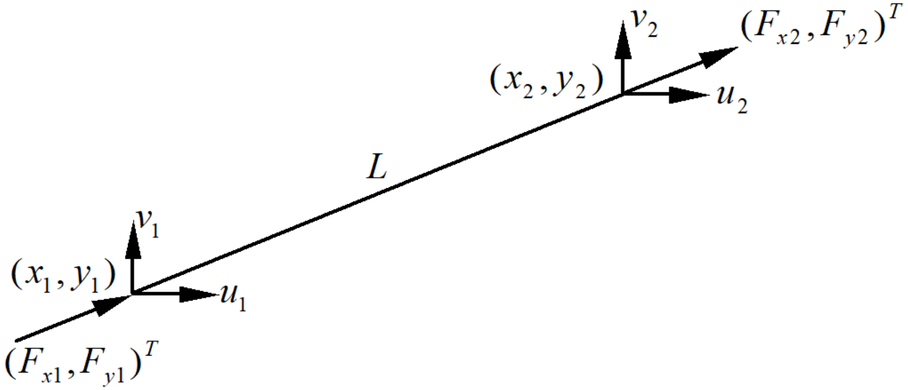
平面桁架的变形问题



有限元计算过程回顾：

1. 单元方程的建立(欧姆定律->胡克定律)
2. 单元方程的组装(方程扩阶)
3. 整体方程的建立(方程累加)
4. 边界条件的处理(电流守恒->力平衡)
5. 系统方程的整合(分离未知量)
6. 系统方程的求解

单元方程的建立



$$\delta = \text{变形后长度} - \text{变形前长度}$$
$$= \sqrt{(x_2+u_2-x_1-u_1)^2+(y_2+v_2-y_1-v_1)^2}$$
$$-\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

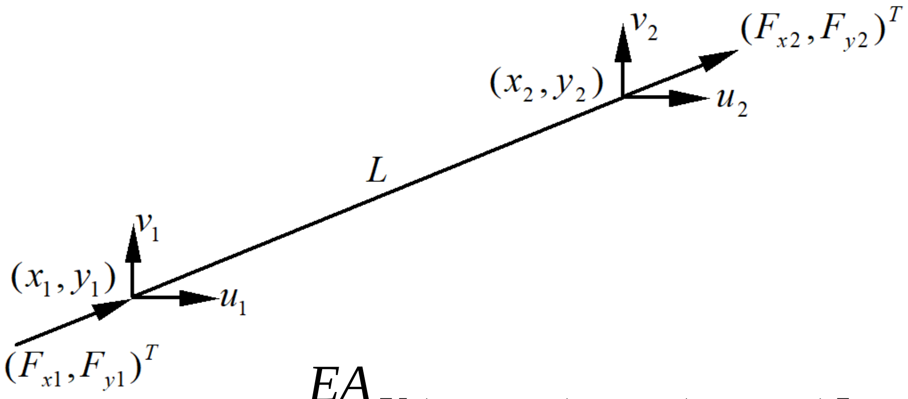
$$\delta \approx \frac{x_2-x_1}{L}(u_2-u_1)+\frac{y_2-y_1}{L}(v_2-v_1)$$
$$=l(u_2-u_1)+m(v_2-v_1)$$

$$T=\frac{EA}{L}[l(u_2-u_1)+m(v_2-v_1)]$$

胡克定律 $\Rightarrow T=\dfrac{EA}{L}\delta$

T:轴力；　δ：轴向伸长

单元方程的建立



$$T = \frac{EA}{L}[l(u_2 - u_1) + m(v_2 - v_1)]$$

$$\frac{1}{R_r}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} U_1^{(r)} \\ U_2^{(r)} \end{Bmatrix} = \begin{Bmatrix} I_1^{(r)} \\ I_2^{(r)} \end{Bmatrix}$$

$$F_{x1} = -Tl = \frac{EA}{L}[l^2(u_1 - u_2) + lm(v_1 - v_2)]$$

$$F_{y1} = -Tm = \frac{EA}{L}[lm(u_1 - u_2) + m^2(v_1 - v_2)]$$

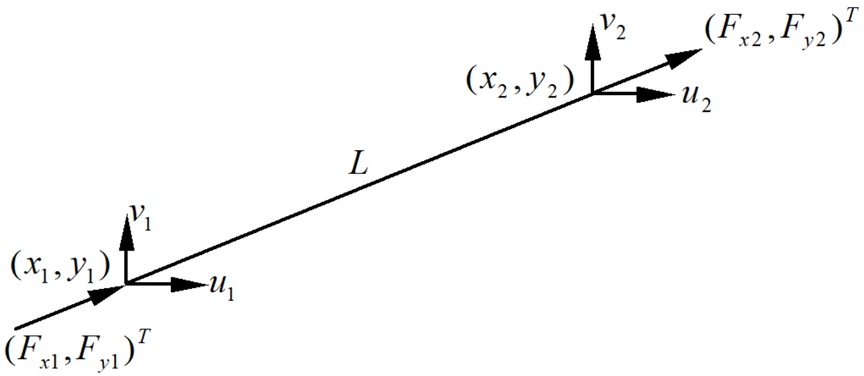
$$F_{x2} = Tl = \frac{EA}{L}[l^2(u_2 - u_1) + lm(v_2 - v_1)]$$

$$F_{y2} = Tm = \frac{EA}{L}[lm(u_2 - u_1) + m^2(v_2 - v_1)]$$

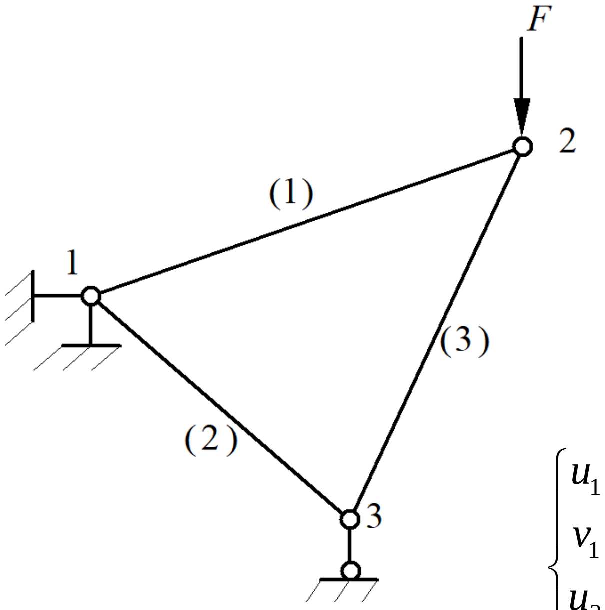
$$\Rightarrow$$

$$\frac{EA}{L}\begin{bmatrix} l^2 & lm & -l^2 & -lm \\ lm & m^2 & -lm & -m^2 \\ -l^2 & -lm & l^2 & lm \\ -lm & -m^2 & lm & m^2 \end{bmatrix}\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix} = \begin{Bmatrix} F_{x1} \\ F_{y1} \\ F_{x2} \\ F_{y2} \end{Bmatrix}$$

单元方程的组装



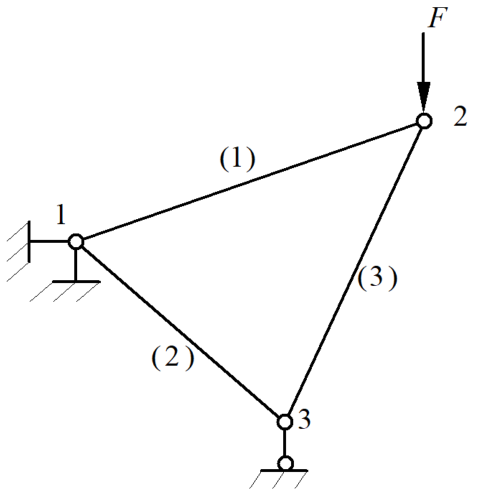
$$\frac{EA}{L}\begin{bmatrix} l^2 & lm & -l^2 & -lm \\ lm & m^2 & -lm & -m^2 \\ -l^2 & -lm & l^2 & lm \\ -lm & -m^2 & lm & m^2 \end{bmatrix}\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix} = \begin{Bmatrix} F_{x1} \\ F_{y1} \\ F_{x2} \\ F_{y2} \end{Bmatrix}$$



$$\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix}^{(3)} = \begin{Bmatrix} U_2 \\ V_2 \\ U_3 \\ V_3 \end{Bmatrix} \qquad \begin{Bmatrix} U_1 \\ V_1 \\ U_2 \\ V_2 \\ U_3 \\ V_3 \end{Bmatrix}$$

$$\frac{E_3A_3}{L_3}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & l_3^2 & l_3m_3 & -l_3^2 & -l_3m_3 \\ 0 & 0 & l_3m_3 & m_3^2 & -l_3m_3 & -m_3^2 \\ 0 & 0 & -l_3^2 & -l_3m_3 & l_3^2 & l_3m_3 \\ 0 & 0 & -l_3m_3 & -m_3^2 & l_3m_3 & m_3^2 \end{bmatrix}\begin{Bmatrix} U_1 \\ V_1 \\ U_2 \\ V_2 \\ U_3 \\ V_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ F_{x1}^{(3)} \\ F_{y1}^{(3)} \\ F_{x2}^{(3)} \\ F_{y2}^{(3)} \end{Bmatrix}$$

单元方程的组装



$$\frac{E_1A_1}{L_1}\begin{bmatrix} l_1^2 & l_1m_1 & -l_1^2 & -l_1m_1 & 0 & 0 \\ l_1m_1 & m_1^2 & -l_1m_1 & -m_1^2 & 0 & 0 \\ -l_1^2 & -l_1m_1 & l_1^2 & l_1m_1 & 0 & 0 \\ -l_1m_1 & -m_1^2 & l_1m_1 & m_1^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} U_1 \\ V_1 \\ U_2 \\ V_2 \\ U_3 \\ V_3 \end{Bmatrix} = \begin{Bmatrix} F_{x1}^{(1)} \\ F_{y1}^{(1)} \\ F_{x2}^{(1)} \\ F_{y2}^{(1)} \\ 0 \\ 0 \end{Bmatrix}$$

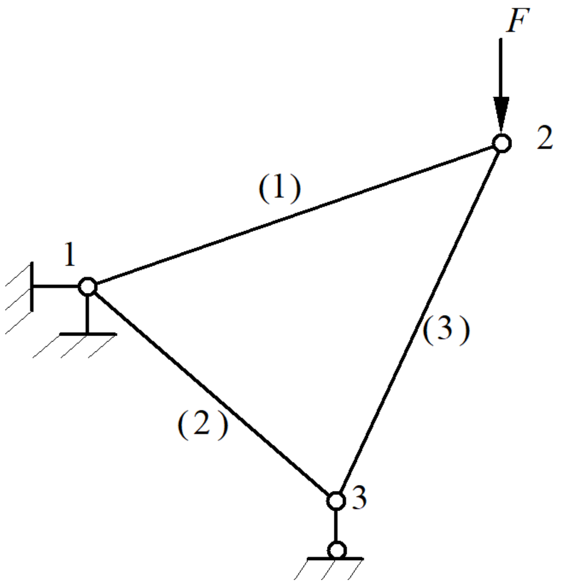
$$\frac{E_2A_2}{L_2}\begin{bmatrix} l_2^2 & l_2m_2 & 0 & 0 & -l_2^2 & -l_2m_2 \\ l_2m_2 & m_2^2 & 0 & 0 & -l_2m_2 & -m_2^2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -l_2^2 & -l_2m_2 & 0 & 0 & l_2^2 & l_2m_2 \\ -l_2m_2 & -m_2^2 & 0 & 0 & l_2m_2 & m_2^2 \end{bmatrix}\begin{Bmatrix} U_1 \\ V_1 \\ U_2 \\ V_2 \\ U_3 \\ V_3 \end{Bmatrix} = \begin{Bmatrix} F_{x1}^{(2)} \\ F_{y1}^{(2)} \\ 0 \\ 0 \\ F_{x2}^{(2)} \\ F_{y2}^{(2)} \end{Bmatrix}$$

$$\frac{E_3A_3}{L_3}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & l_3^2 & l_3m_3 & -l_3^2 & -l_3m_3 \\ 0 & 0 & l_3m_3 & m_3^2 & -l_3m_3 & -m_3^2 \\ 0 & 0 & -l_3^2 & -l_3m_3 & l_3^2 & l_3m_3 \\ 0 & 0 & -l_3m_3 & -m_3^2 & l_3m_3 & m_3^2 \end{bmatrix}\begin{Bmatrix} U_1 \\ V_1 \\ U_2 \\ V_2 \\ U_3 \\ V_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ F_{x1}^{(3)} \\ F_{y1}^{(3)} \\ F_{x2}^{(3)} \\ F_{y2}^{(3)} \end{Bmatrix}$$

整体方程的建立

$$
EA\begin{bmatrix}
\frac{l_1^2}{L_1}+\frac{l_2^2}{L_2} & \frac{l_1m_1}{L_1}+\frac{l_2m_2}{L_2} & -\frac{l_1^2}{L_1} & -\frac{l_1m_1}{L_1} & -\frac{l_2^2}{L_2} & -\frac{l_2m_2}{L_2} \\
\frac{l_1m_1}{L_1}+\frac{l_2m_2}{L_2} & \frac{m_1^2}{L_1}+\frac{m_2^2}{L_2} & -\frac{l_1m_1}{L_1} & -\frac{m_1^2}{L_1} & -\frac{l_2m_2}{L_2} & -\frac{m_2^2}{L_2} \\
-\frac{l_1^2}{L_1} & -\frac{l_1m_1}{L_1} & \frac{l_1^2}{L_1}+\frac{l_3^2}{L_3} & \frac{l_1m_1}{L_1}+\frac{l_3m_3}{L_3} & -\frac{l_3^2}{L_3} & -\frac{l_3m_3}{L_3} \\
-\frac{l_1m_1}{L_1} & -\frac{m_1^2}{L_1} & \frac{l_1m_1}{L_1}+\frac{l_3m_3}{L_3} & \frac{m_1^2}{L_1}+\frac{m_3^2}{L_3} & -\frac{l_3m_3}{L_3} & -\frac{m_3^2}{L_3} \\
-\frac{l_2^2}{L_2} & -\frac{l_2m_2}{L_2} & -\frac{l_3^2}{L_3} & -\frac{l_3m_3}{L_3} & \frac{l_2^2}{L_2}+\frac{l_3^2}{L_3} & \frac{l_2m_2}{L_2}+\frac{l_3m_3}{L_3} \\
-\frac{l_2m_2}{L_2} & -\frac{m_2^2}{L_2} & -\frac{l_3m_3}{L_3} & -\frac{m_3^2}{L_3} & \frac{l_2m_2}{L_2}+\frac{l_3m_3}{L_3} & \frac{m_2^2}{L_2}+\frac{m_3^2}{L_3}
\end{bmatrix}
\begin{Bmatrix} U_1 \\ V_1 \\ U_2 \\ V_2 \\ U_3 \\ V_3 \end{Bmatrix}
=
\begin{Bmatrix}
F_{x1}^{(1)}+F_{x1}^{(2)} \\
F_{y1}^{(1)}+F_{y2}^{(1)} \\
F_{x2}^{(1)}+F_{x2}^{(3)} \\
F_{y1}^{(2)}+F_{y2}^{(3)} \\
F_{x2}^{(2)}+F_{x1}^{(3)} \\
F_{y2}^{(2)}+F_{y1}^{(3)}
\end{Bmatrix}
$$

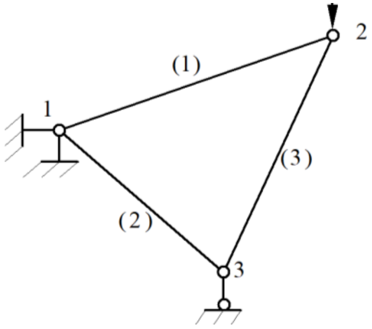
边界条件的处理



单元	节点 1	节点 2
(1)	1	2
(2)	1	3
(3)	3	2

$$\begin{Bmatrix} F_{x1}^{(1)} + F_{x1}^{(2)} \\ F_{y1}^{(1)} + F_{y2}^{(1)} \\ F_{x2}^{(1)} + F_{x2}^{(3)} \\ F_{y1}^{(2)} + F_{y2}^{(3)} \\ F_{x2}^{(2)} + F_{x1}^{(3)} \\ F_{y2}^{(2)} + F_{y1}^{(3)} \end{Bmatrix} = \begin{Bmatrix} R_{x1} \\ R_{y1} \\ 0 \\ -F \\ 0 \\ R_{y3} \end{Bmatrix}$$

系统方程的整合



$$
EA\begin{bmatrix}
\frac{l_1^2}{L_1}+\frac{l_2^2}{L_2} & \frac{l_1m_1}{L_1}+\frac{l_2m_2}{L_2} & -\frac{l_1^2}{L_1} & -\frac{l_1m_1}{L_1} & -\frac{l_2^2}{L_2} & -\frac{l_2m_2}{L_2} \\
\frac{l_1m_1}{L_1}+\frac{l_2m_2}{L_2} & \frac{m_1^2}{L_1}+\frac{m_2^2}{L_2} & -\frac{l_1m_1}{L_1} & -\frac{m_1^2}{L_1} & -\frac{l_2m_2}{L_2} & -\frac{m_2^2}{L_2} \\
-\frac{l_1^2}{L_1} & -\frac{l_1m_1}{L_1} & \frac{l_1^2}{L_1}+\frac{l_3^2}{L_3} & \frac{l_1m_1}{L_1}+\frac{l_3m_3}{L_3} & -\frac{l_3^2}{L_3} & -\frac{l_3m_3}{L_3} \\
-\frac{l_1m_1}{L_1} & -\frac{m_1^2}{L_1} & \frac{l_1m_1}{L_1}+\frac{l_3m_3}{L_3} & \frac{m_1^2}{L_1}+\frac{m_3^2}{L_3} & -\frac{l_3m_3}{L_3} & -\frac{m_3^2}{L_3} \\
-\frac{l_2^2}{L_2} & -\frac{l_2m_2}{L_2} & -\frac{l_3^2}{L_3} & -\frac{l_3m_3}{L_3} & \frac{l_2^2}{L_2}+\frac{l_3^2}{L_3} & \frac{l_2m_2}{L_2}+\frac{l_3m_3}{L_3} \\
-\frac{l_2m_2}{L_2} & -\frac{m_2^2}{L_2} & -\frac{l_3m_3}{L_3} & -\frac{m_3^2}{L_3} & \frac{l_2m_2}{L_2}+\frac{l_3m_3}{L_3} & \frac{m_2^2}{L_2}+\frac{m_3^2}{L_3}
\end{bmatrix}
\begin{Bmatrix} U_1 \\ V_1 \\ U_2 \\ V_2 \\ U_3 \\ V_3 \end{Bmatrix}
=
\begin{Bmatrix} R_{x1} \\ R_{y1} \\ 0 \\ F \\ 0 \\ R_{y3} \end{Bmatrix}
$$

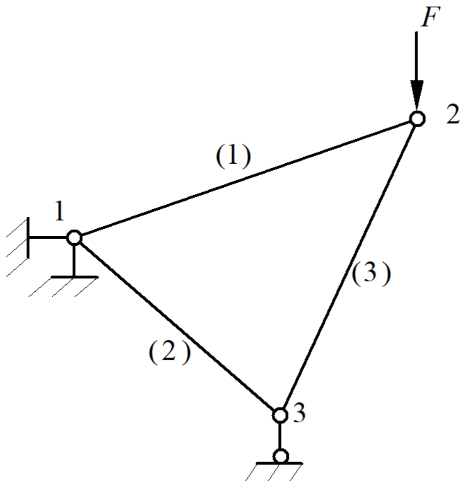
$$\Downarrow$$

$$
EA\begin{bmatrix}
-\frac{l_1^2}{L_1} & -\frac{l_1m_1}{L_1} & \frac{l_1^2}{L_1}+\frac{l_3^2}{L_3} & \frac{l_1m_1}{L_1}+\frac{l_3m_3}{L_3} & -\frac{l_3^2}{L_3} & -\frac{l_3m_3}{L_3} \\
-\frac{l_1m_1}{L_1} & -\frac{m_1^2}{L_1} & \frac{l_1m_1}{L_1}+\frac{l_3m_3}{L_3} & \frac{m_1^2}{L_1}+\frac{m_3^2}{L_3} & -\frac{l_3m_3}{L_3} & -\frac{m_3^2}{L_3} \\
-\frac{l_2^2}{L_2} & -\frac{l_2m_2}{L_2} & -\frac{l_3^2}{L_3} & -\frac{l_3m_3}{L_3} & \frac{l_2^2}{L_2}+\frac{l_3^2}{L_3} & \frac{l_2m_2}{L_2}+\frac{l_3m_3}{L_3}
\end{bmatrix}
\begin{Bmatrix} U_1 \\ V_1 \\ U_2 \\ V_2 \\ U_3 \\ V_3 \end{Bmatrix}
=
\begin{Bmatrix} 0 \\ F \\ 0 \end{Bmatrix}
$$

系统方程的求解

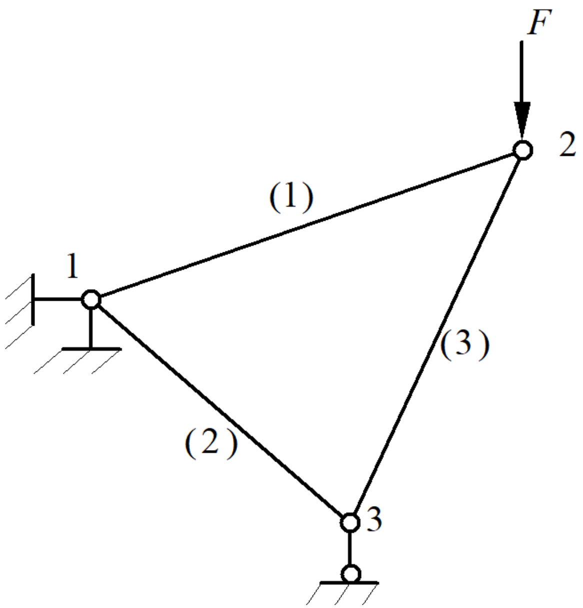
$$EA\begin{bmatrix} -\frac{l_1^2}{L_1} & -\frac{l_1 m_1}{L_1} & \frac{l_1^2}{L_1}+\frac{l_3^2}{L_3} & \frac{l_1 m_1}{L_1}+\frac{l_3 m_3}{L_3} & -\frac{l_3^2}{L_3} & -\frac{l_3 m_3}{L_3} \\ -\frac{l_1 m_1}{L_1} & -\frac{m_1^2}{L_1} & \frac{l_1 m_1}{L_1}+\frac{l_3 m_3}{L_3} & \frac{m_1^2}{L_1}+\frac{m_3^2}{L_3} & -\frac{l_3 m_3}{L_3} & -\frac{m_3^2}{L_3} \\ -\frac{l_2^2}{L_2} & -\frac{l_2 m_2}{L_2} & -\frac{l_3^2}{L_3} & -\frac{l_3 m_3}{L_3} & \frac{l_2^2}{L_2}+\frac{l_3^2}{L_3} & \frac{l_2 m_2}{L_2}+\frac{l_3 m_3}{L_3} \end{bmatrix}\begin{Bmatrix} U_1 \\ V_1 \\ U_2 \\ V_2 \\ U_3 \\ V_3 \end{Bmatrix}=\begin{Bmatrix} 0 \\ F \\ 0 \end{Bmatrix}$$

$$\Downarrow$$



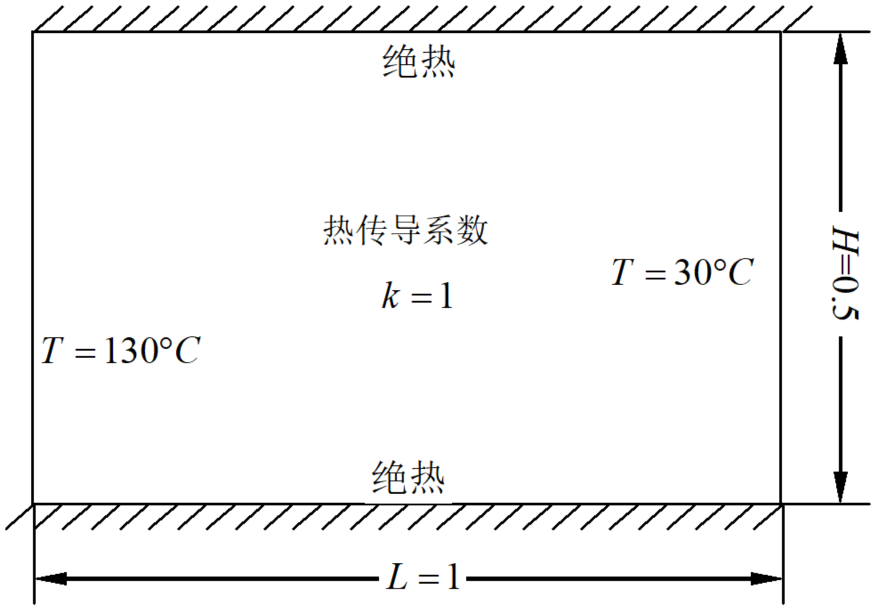
$$EA\begin{bmatrix} \frac{l_1^2}{L_1}+\frac{l_3^2}{L_3} & \frac{l_1 m_1}{L_1}+\frac{l_3 m_3}{L_3} & -\frac{l_3^2}{L_3} \\ \frac{l_1 m_1}{L_1}+\frac{l_3 m_3}{L_3} & \frac{m_1^2}{L_1}+\frac{m_3^2}{L_3} & -\frac{l_3 m_3}{L_3} \\ -\frac{l_3^2}{L_3} & -\frac{l_3 m_3}{L_3} & \frac{l_2^2}{L_2}+\frac{l_3^2}{L_3} \end{bmatrix}\begin{Bmatrix} U_2 \\ V_2 \\ U_3 \end{Bmatrix}=\begin{Bmatrix} 0 \\ F \\ 0 \end{Bmatrix}$$

平面桁架的变形问题



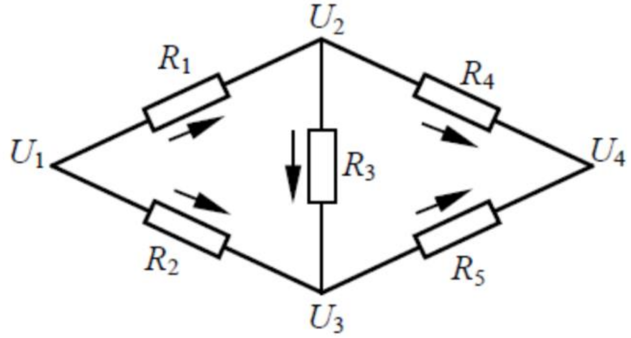
$$EA\begin{bmatrix} \frac{l_1^2}{L_1}+\frac{l_3^2}{L_3} & \frac{l_1m_1}{L_1}+\frac{l_3m_3}{L_3} & -\frac{l_3^2}{L_3} \\ \frac{l_1m_1}{L_1}+\frac{l_3m_3}{L_3} & \frac{m_1^2}{L_1}+\frac{m_3^2}{L_3} & -\frac{l_3m_3}{L_3} \\ -\frac{l_3^2}{L_3} & -\frac{l_3m_3}{L_3} & \frac{l_2^2}{L_2}+\frac{l_3^2}{L_3} \end{bmatrix}\begin{Bmatrix} U_2 \\ V_2 \\ U_3 \end{Bmatrix}=\begin{Bmatrix} 0 \\ F \\ 0 \end{Bmatrix}$$

传热问题

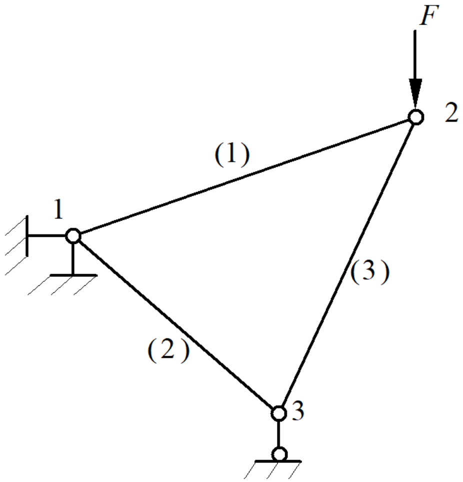


$$
\begin{cases}
\nabla^2 T(x,y)=0 \\
T(0,y)=130,\ T(1,y)=30 \\
\dfrac{\partial T(x,0)}{\partial y}=\dfrac{\partial T(x,0.5)}{\partial y}=0
\end{cases}
$$

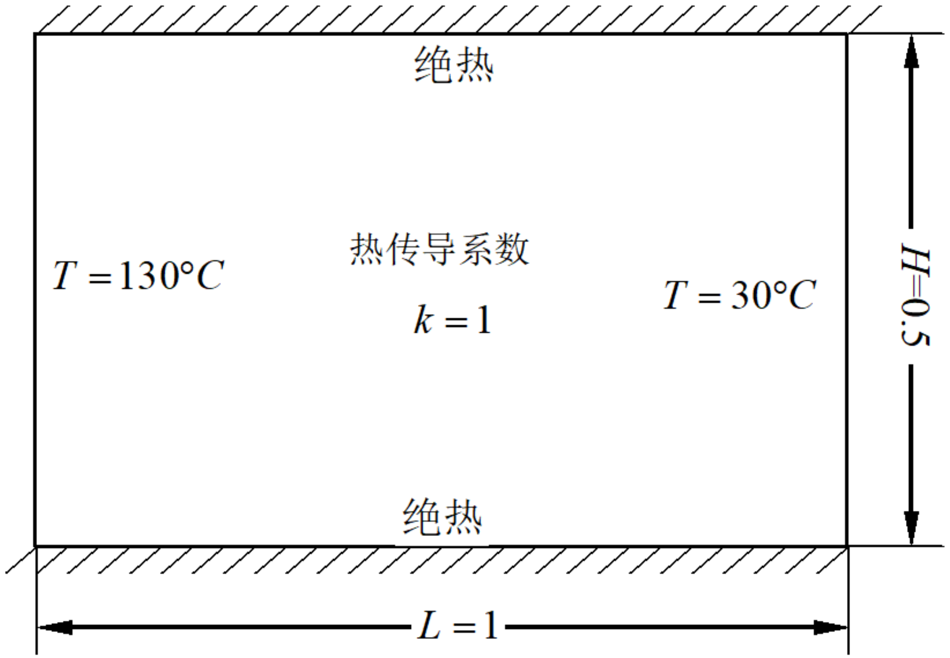
问题的难度



4自由度

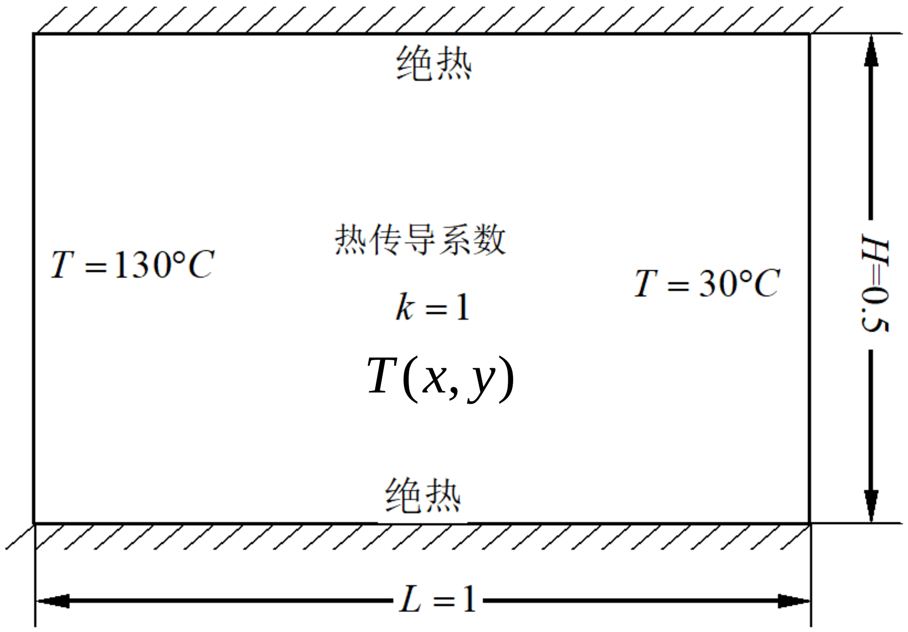


6自由度

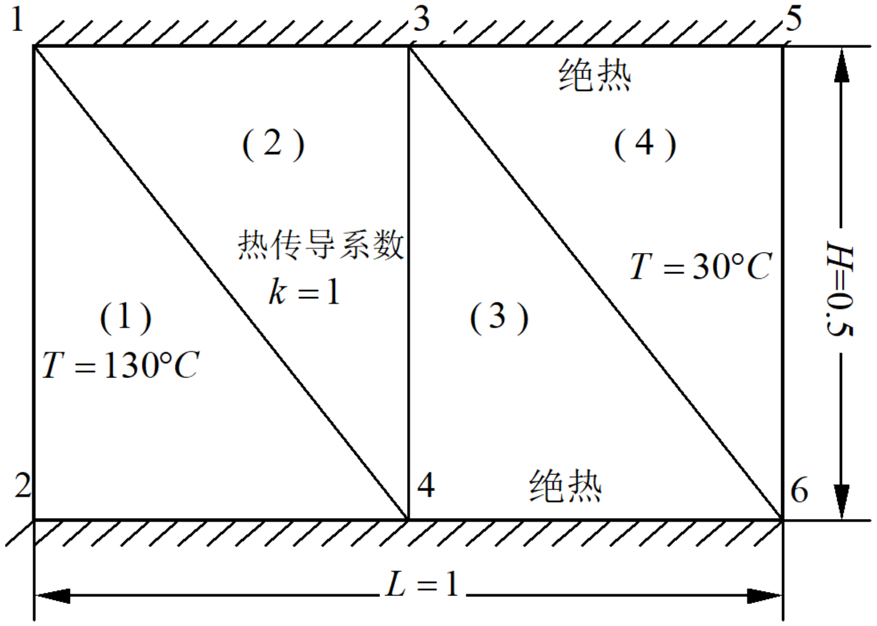


$T(x, y)$无限自由度

离散化

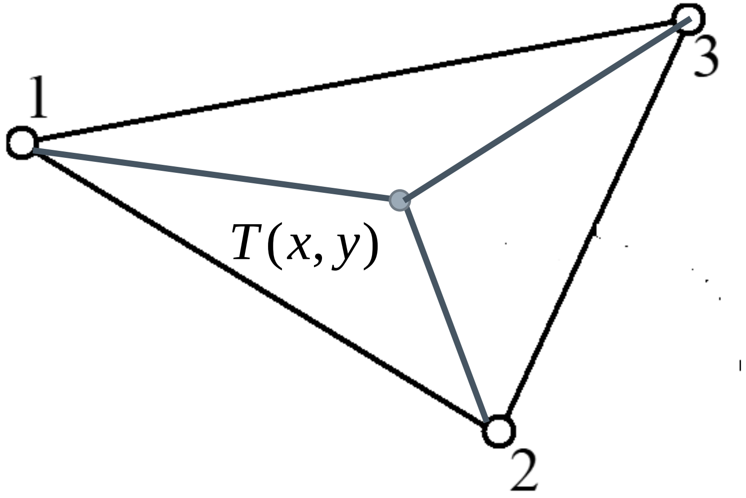


$T(x,y) \Rightarrow$ 无限自由度



$T_i \Rightarrow 6$自由度

温度函数插值



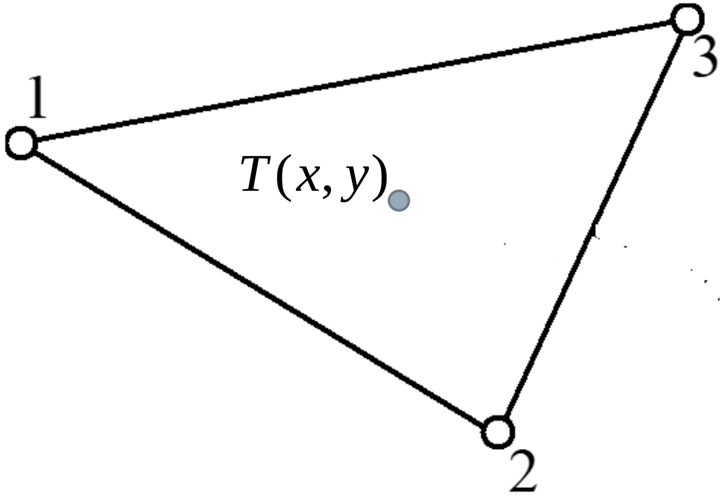
方法I

$$T(x,y)=a+bx+cy \quad \Rightarrow \quad \begin{cases} a+bx_1+cy_1=T_1 \\ a+bx_2+cy_2=T_2 \\ a+bx_3+cy_3=T_3 \end{cases}$$

方法II $$\frac{1}{6}\begin{vmatrix} 1 & x_1 & y_1 & T_1 \\ 1 & x_2 & y_2 & T_2 \\ 1 & x_3 & y_3 & T_3 \\ 1 & x & y & T \end{vmatrix}=0$$

$$-\begin{vmatrix} x_1 & y_1 & T_1 \\ x_2 & y_2 & T_2 \\ x_3 & y_3 & T_3 \end{vmatrix}+x\begin{vmatrix} 1 & y_1 & T_1 \\ 1 & y_2 & T_2 \\ 1 & y_3 & T_3 \end{vmatrix}-y\begin{vmatrix} 1 & x_1 & T_1 \\ 1 & x_2 & T_2 \\ 1 & x_3 & T_3 \end{vmatrix}+T\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}=0$$

温度函数表达式



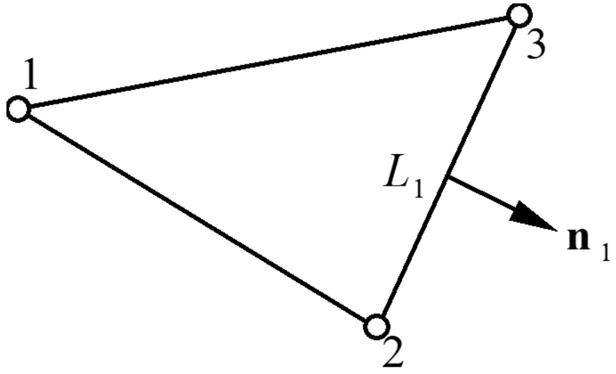
$$-\begin{vmatrix} x_1 & y_1 & T_1 \\ x_2 & y_2 & T_2 \\ x_3 & y_3 & T_3 \end{vmatrix} + x\begin{vmatrix} 1 & y_1 & T_1 \\ 1 & y_2 & T_2 \\ 1 & y_3 & T_3 \end{vmatrix} - y - \begin{vmatrix} 1 & x_1 & T_1 \\ 1 & x_2 & T_2 \\ 1 & x_3 & T_3 \end{vmatrix} + T\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} = 0$$

$$T = \frac{1}{2\Delta}[(a_1 + b_1x + c_1y)T_1 + (a_2 + b_2x + c_2y)T_2 + (a_3 + b_3x + c_3y)T_3]$$

$$\nabla T = \begin{Bmatrix} \partial T/\partial x \\ \partial T/\partial y \end{Bmatrix} = \frac{1}{2\Delta}\begin{pmatrix} b_1T_1 + b_2T_2 + b_3T_3 \\ c_1T_1 + c_2T_2 + c_3T_3 \end{pmatrix}$$

$$a_1 = x_2y_3 - x_3y_2 \quad b_1 = y_2 - y_3 \quad c_1 = x_3 - x_2, \quad \cdots$$

单元及其边界的热流



单元内部的温度梯度：$\nabla T = \dfrac{1}{2\Delta}\begin{pmatrix} b_1T_1 + b_2T_2 + b_3T_3 \\ c_1T_1 + c_2T_2 + c_3T_3 \end{pmatrix}$

单元内部的热流矢量：$q = -k\nabla T$

单元边界的热流密度：$q_n = -k\nabla T \cdot \mathbf{n}$

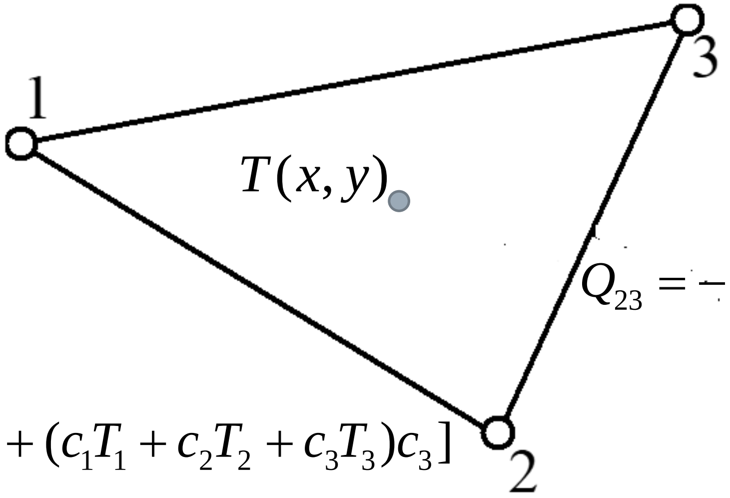
$$L_1 \times n_{x1} = y_3 - y_2 = -b_1$$
$$L_1 \times n_{y1} = x_2 - x_3 = -c_1$$

流入单元边界2-3的热量：$Q_{23} = L_1 \times k\nabla T \cdot \mathbf{n}$

$$Q_{23} = -\frac{k}{2\Delta}[(b_1T_1 + b_2T_2 + b_3T_3)b_1 + (c_1T_1 + c_2T_2 + c_3T_3)c_1]$$

边界的热流

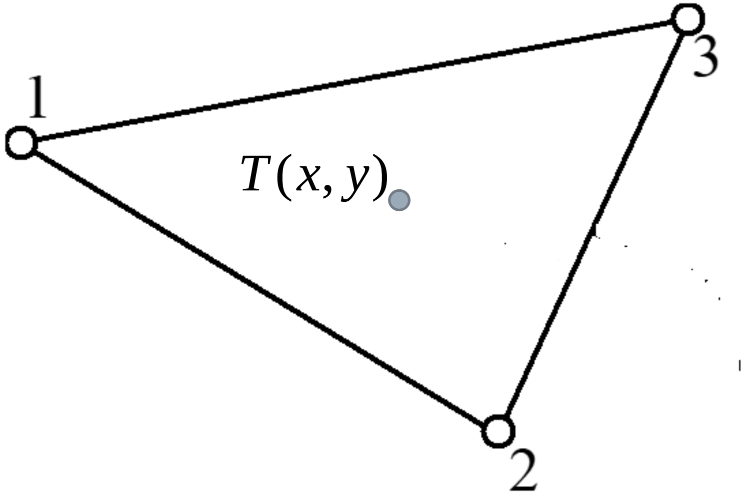
$$Q_{31} = -\frac{k}{2\Delta}[(b_1T_1 + b_2T_2 + b_3T_3)b_2 + (c_1T_1 + c_2T_2 + c_3T_3)c_2]$$



$$Q_{23} = -\frac{k}{2\Delta}[(b_1T_1 + b_2T_2 + b_3T_3)b_1 + (c_1T_1 + c_2T_2 + c_3T_3)c_1]$$

$$Q_{12} = -\frac{k}{2\Delta}[(b_1T_1 + b_2T_2 + b_3T_3)b_3 + (c_1T_1 + c_2T_2 + c_3T_3)c_3]$$

节点的热流



节点的热流

$$
\begin{cases}
Q_{12} = -\dfrac{k}{2\Delta}[(b_1T_1 + b_2T_2 + b_3T_3)b_3 + (c_1T_1 + c_2T_2 + c_3T_3)c_3] \\
Q_{23} = -\dfrac{k}{2\Delta}[(b_1T_1 + b_2T_2 + b_3T_3)b_1 + (c_1T_1 + c_2T_2 + c_3T_3)c_1] \\
Q_{31} = -\dfrac{k}{2\Delta}[(b_1T_1 + b_2T_2 + b_3T_3)b_2 + (c_1T_1 + c_2T_2 + c_3T_3)c_2]
\end{cases}
$$

$$
Q_1 = \frac{1}{2}(Q_{31} + Q_{12}) = \frac{k}{4\Delta}[(b_1b_1 + c_1c_1)T_1 + (b_1b_2 + c_1c_2)T_2 + (b_1b_3 + c_1c_3)T_3]
$$

$$
Q_2 = \frac{1}{2}(Q_{23} + Q_{31}) = \frac{k}{4\Delta}[(b_2b_1 + c_2c_1)T_1 + (b_2b_2 + c_2c_2)T_2 + (b_2b_3 + c_2c_3)T_3]
$$

$$
Q_3 = \frac{1}{2}(Q_{31} + Q_{12}) = \frac{k}{4\Delta}[(b_3b_1 + c_3c_1)T_1 + (b_3b_2 + c_3c_2)T_2 + (b_3b_3 + c_3c_3)T_3]
$$

单元方程的建立

$$\begin{cases} Q_1 = \frac{1}{2}(Q_{31}+Q_{12}) = \frac{k}{4\Delta}[(b_1b_1+c_1c_1)T_1+(b_1b_2+c_1c_2)T_2+(b_1b_3+c_1c_3)T_3] \\ Q_2 = \frac{1}{2}(Q_{23}+Q_{31}) = \frac{k}{4\Delta}[(b_2b_1+c_2c_1)T_1+(b_2b_2+c_2c_2)T_2+(b_2b_3+c_2c_3)T_3] \\ Q_3 = \frac{1}{2}(Q_{31}+Q_{12}) = \frac{k}{4\Delta}[(b_3b_1+c_3c_1)T_1+(b_3b_2+c_3c_2)T_2+(b_3b_3+c_3c_3)T_3] \end{cases}$$

$$\Downarrow$$

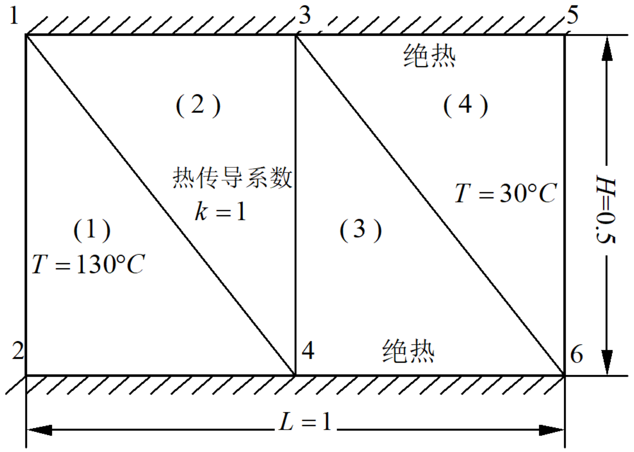
$$\frac{k}{4\Delta}\begin{bmatrix} b_1b_1+c_1c_1 & b_1b_2+c_1c_2 & b_1b_3+c_1c_3 \\ b_2b_1+c_2c_1 & b_2b_2+c_2c_2 & b_2b_3+c_2c_3 \\ b_3b_1+c_3c_1 & b_3b_2+c_3c_2 & b_3b_3+c_3c_3 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{Bmatrix}$$

$$\frac{1}{R_r}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} U_1^{(r)} \\ U_2^{(r)} \end{Bmatrix} = \begin{Bmatrix} I_1^{(r)} \\ I_2^{(r)} \end{Bmatrix}$$

$$\frac{EA}{L}\begin{bmatrix} l^2 & lm & -l^2 & -lm \\ lm & m^2 & -lm & -m^2 \\ -l^2 & -lm & l^2 & lm \\ -lm & -m^2 & lm & m^2 \end{bmatrix}\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix} = \begin{Bmatrix} F_{x1} \\ F_{y1} \\ F_{x2} \\ F_{y2} \end{Bmatrix}$$

单元方程的组装

$$\frac{k}{4\Delta}\begin{bmatrix} b_1b_1+c_1c_1 & b_1b_2+c_1c_2 & b_1b_3+c_1c_3 \\ b_2b_1+c_2c_1 & b_2b_2+c_2c_2 & b_2b_3+c_2c_3 \\ b_3b_1+c_3c_1 & b_3b_2+c_3c_2 & b_3b_3+c_3c_3 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix}=\begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{Bmatrix} \Rightarrow \begin{bmatrix} k_{11}^{(e)} & k_{12}^{(e)} & k_{13}^{(e)} \\ k_{21}^{(e)} & k_{22}^{(e)} & k_{23}^{(e)} \\ k_{31}^{(e)} & k_{32}^{(e)} & k_{33}^{(e)} \end{bmatrix}\begin{Bmatrix} T_1^{(e)} \\ T_2^{(e)} \\ T_3^{(e)} \end{Bmatrix}=\begin{Bmatrix} Q_1^{(e)} \\ Q_2^{(e)} \\ Q_3^{(e)} \end{Bmatrix}$$



单元1

$$\begin{bmatrix} k_{11}^{(1)} & k_{12}^{(1)} & 0 & k_{13}^{(1)} & 0 & 0 \\ k_{21}^{(1)} & k_{22}^{(1)} & 0 & k_{23}^{(1)} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ k_{31}^{(1)} & k_{32}^{(1)} & 0 & k_{33}^{(1)} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \end{Bmatrix}=\begin{Bmatrix} Q_1^{(1)} \\ Q_2^{(1)} \\ 0 \\ Q_3^{(1)} \\ 0 \\ 0 \end{Bmatrix}$$

单元2

$$\begin{bmatrix} k_{11}^{(2)} & 0 & k_{13}^{(2)} & k_{12}^{(2)} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ k_{31}^{(2)} & 0 & k_{33}^{(2)} & k_{23}^{(2)} & 0 & 0 \\ k_{21}^{(2)} & 0 & k_{32}^{(2)} & k_{22}^{(2)} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \end{Bmatrix}=\begin{Bmatrix} Q_1^{(2)} \\ 0 \\ Q_3^{(2)} \\ Q_2^{(2)} \\ 0 \\ 0 \end{Bmatrix}$$

单元3

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & k_{11}^{(3)} & k_{12}^{(3)} & 0 & k_{13}^{(3)} \\ 0 & 0 & k_{21}^{(3)} & k_{22}^{(3)} & 0 & k_{23}^{(3)} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & k_{31}^{(3)} & k_{32}^{(3)} & 0 & k_{33}^{(3)} \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \\ Q_1^{(3)} \\ Q_2^{(3)} \\ 0 \\ Q_3^{(3)} \end{Bmatrix}$$

单元4

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & k_{11}^{(4)} & 0 & k_{13}^{(4)} & k_{12}^{(4)} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & k_{31}^{(4)} & 0 & k_{33}^{(4)} & k_{32}^{(4)} \\ 0 & 0 & k_{21}^{(4)} & 0 & k_{23}^{(4)} & k_{22}^{(4)} \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \\ Q_1^{(4)} \\ 0 \\ Q_3^{(4)} \\ Q_2^{(4)} \end{Bmatrix}$$

整体方程的建立

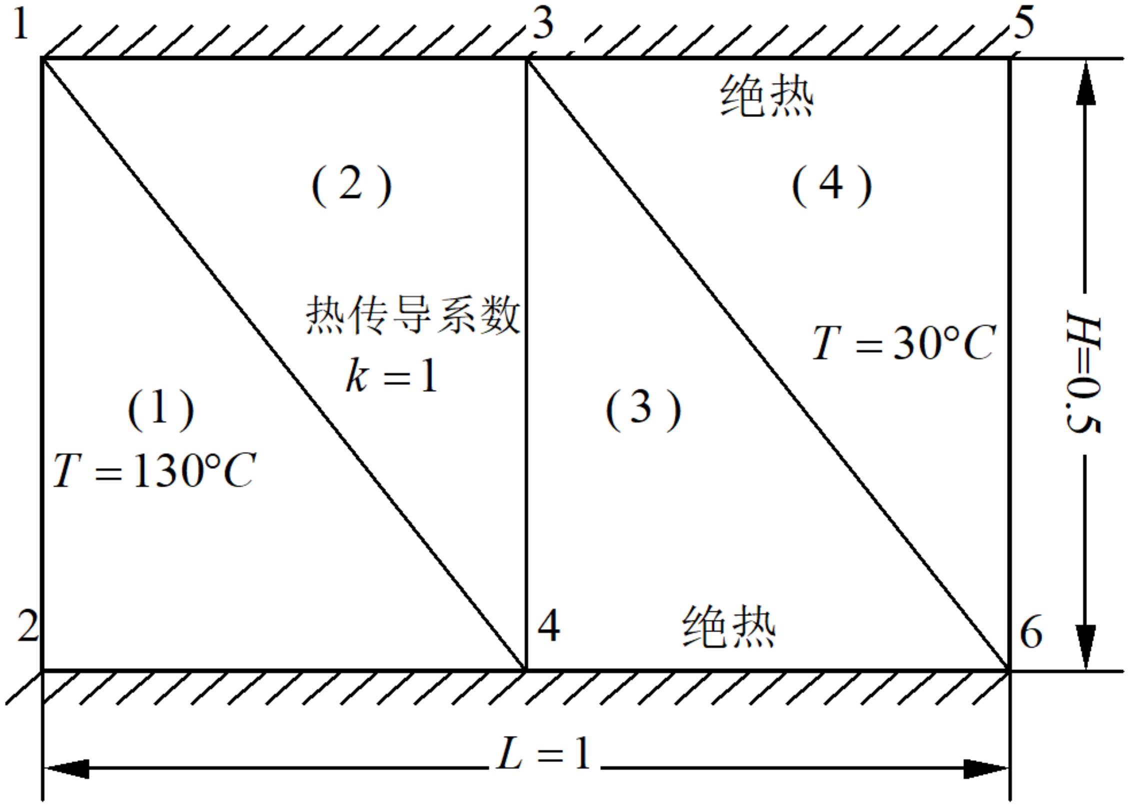
$$\begin{bmatrix} k_{11}^{(1)} & k_{12}^{(1)} & 0 & k_{13}^{(1)} & 0 & 0 \\ k_{21}^{(1)} & k_{22}^{(1)} & 0 & k_{23}^{(1)} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ k_{31}^{(1)} & k_{32}^{(1)} & 0 & k_{33}^{(1)} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \end{Bmatrix} = \begin{Bmatrix} Q_1^{(1)} \\ Q_2^{(1)} \\ 0 \\ Q_3^{(1)} \\ 0 \\ 0 \end{Bmatrix} \quad \begin{bmatrix} k_{11}^{(2)} & 0 & k_{13}^{(2)} & k_{12}^{(2)} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ k_{31}^{(2)} & 0 & k_{33}^{(2)} & k_{23}^{(2)} & 0 & 0 \\ k_{21}^{(2)} & 0 & k_{32}^{(2)} & k_{22}^{(2)} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \end{Bmatrix} = \begin{Bmatrix} Q_1^{(2)} \\ 0 \\ Q_3^{(2)} \\ Q_2^{(2)} \\ 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & k_{11}^{(3)} & k_{12}^{(3)} & 0 & k_{13}^{(3)} \\ 0 & 0 & k_{21}^{(3)} & k_{22}^{(3)} & 0 & k_{23}^{(3)} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & k_{31}^{(3)} & k_{32}^{(3)} & 0 & k_{33}^{(3)} \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ Q_1^{(3)} \\ Q_2^{(3)} \\ 0 \\ Q_3^{(3)} \end{Bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & k_{11}^{(4)} & 0 & k_{13}^{(4)} & k_{12}^{(4)} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & k_{31}^{(4)} & 0 & k_{33}^{(4)} & k_{32}^{(4)} \\ 0 & 0 & k_{21}^{(4)} & 0 & k_{23}^{(4)} & k_{22}^{(4)} \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ Q_1^{(4)} \\ 0 \\ Q_3^{(4)} \\ Q_2^{(4)} \end{Bmatrix}$$

$$\begin{bmatrix} k_{11}^{(1)}+k_{11}^{(2)} & k_{12}^{(1)} & k_{13}^{(2)}+k_{12}^{(2)} & k_{13}^{(1)} & 0 & 0 \\ k_{21}^{(1)} & k_{22}^{(1)} & 0 & k_{23}^{(1)} & 0 & 0 \\ k_{31}^{(2)} & 0 & k_{33}^{(2)}+k_{11}^{(3)}+k_{11}^{(4)} & k_{23}^{(2)}+k_{12}^{(3)} & k_{13}^{(4)} & k_{13}^{(3)}+k_{12}^{(4)} \\ k_{31}^{(1)}+k_{21}^{(2)} & k_{32}^{(1)} & k_{32}^{(2)}+k_{21}^{(3)} & k_{33}^{(1)}+k_{22}^{(2)}+k_{22}^{(3)} & 0 & k_{23}^{(3)} \\ 0 & 0 & k_{31}^{(4)} & 0 & k_{33}^{(4)} & k_{32}^{(4)} \\ 0 & 0 & k_{31}^{(3)}+k_{21}^{(4)} & k_{32}^{(3)} & k_{23}^{(4)} & k_{33}^{(3)}+k_{22}^{(4)} \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \end{Bmatrix} = \begin{Bmatrix} Q_1^{(1)}+Q_1^{(2)} \\ Q_2^{(1)} \\ Q_3^{(2)}+Q_1^{(3)}+Q_1^{(4)} \\ Q_3^{(1)}+Q_2^{(2)}+Q_2^{(3)} \\ Q_3^{(4)} \\ Q_3^{(3)}+Q_2^{(4)} \end{Bmatrix}$$

边界条件的处理

$$\begin{Bmatrix} Q_1^{(1)}+Q_1^{(2)} \\ Q_2^{(1)} \\ Q_3^{(2)}+Q_1^{(3)}+Q_1^{(4)} \\ Q_3^{(1)}+Q_2^{(2)}+Q_2^{(3)} \\ Q_3^{(4)} \\ Q_3^{(3)}+Q_2^{(4)} \end{Bmatrix}$$



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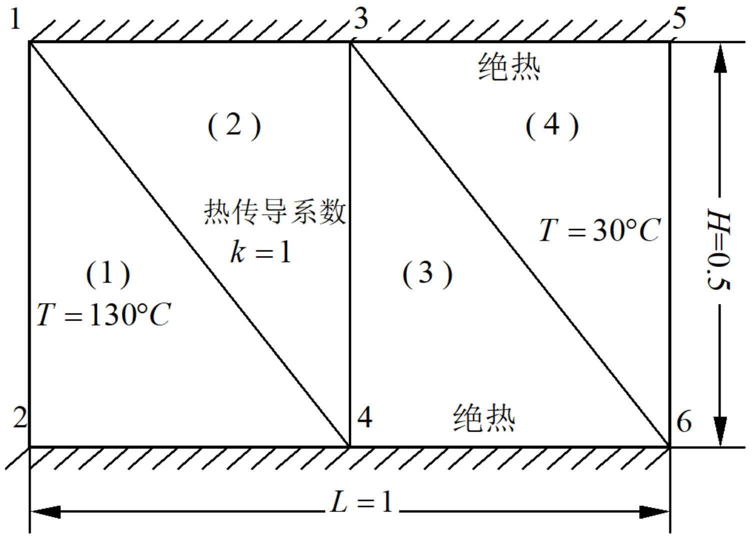
Department of Aeronautics and Astronautics

系统方程的整合

$$\begin{bmatrix} k_{11}^{(1)}+k_{11}^{(2)} & k_{12}^{(1)} & k_{13}^{(2)}+k_{12}^{(2)} & k_{13}^{(1)} & 0 & 0 \\ k_{21}^{(1)} & k_{22}^{(1)} & 0 & k_{23}^{(1)} & 0 & 0 \\ k_{31}^{(2)} & 0 & k_{33}^{(2)}+k_{11}^{(3)}+k_{11}^{(4)} & k_{23}^{(2)}+k_{12}^{(3)} & k_{13}^{(4)} & k_{13}^{(3)}+k_{12}^{(4)} \\ k_{31}^{(1)}+k_{21}^{(2)} & k_{32}^{(1)} & k_{32}^{(2)}+k_{21}^{(3)} & k_{33}^{(1)}+k_{22}^{(2)}+k_{22}^{(3)} & 0 & k_{23}^{(3)} \\ 0 & 0 & k_{31}^{(4)} & 0 & k_{33}^{(4)} & k_{32}^{(4)} \\ 0 & 0 & k_{31}^{(3)}+k_{21}^{(4)} & k_{32}^{(3)} & k_{23}^{(4)} & k_{33}^{(3)}+k_{22}^{(4)} \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \end{Bmatrix} = \begin{Bmatrix} Q_1 \\ Q_2 \\ 0 \\ 0 \\ Q_5 \\ Q_6 \end{Bmatrix}$$



系统方程的整合

$$\begin{bmatrix} k_{31}^{(2)} & 0 & k_{33}^{(2)}+k_{11}^{(3)}+k_{11}^{(4)} & k_{23}^{(2)}+k_{12}^{(3)} & k_{13}^{(4)} & k_{13}^{(3)}+k_{12}^{(4)} \\ k_{31}^{(1)}+k_{21}^{(2)} & k_{32}^{(1)} & k_{32}^{(2)}+k_{21}^{(3)} & k_{33}^{(1)}+k_{22}^{(2)}+k_{22}^{(3)} & 0 & k_{23}^{(3)} \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \end{Bmatrix} = \begin{Bmatrix} Q_3 \\ Q_4 \end{Bmatrix}$$

$$\begin{bmatrix} k_{33}^{(2)}+k_{11}^{(3)}+k_{11}^{(4)} & k_{23}^{(2)}+k_{12}^{(3)} \\ k_{32}^{(2)}+k_{21}^{(3)} & k_{33}^{(1)}+k_{22}^{(2)}+k_{22}^{(3)} \end{bmatrix} \begin{Bmatrix} T_3 \\ T_4 \end{Bmatrix} = \begin{Bmatrix} -k_{31}^{(2)}T_1 - k_{13}^{(4)}T_5 - (k_{13}^{(3)}+k_{12}^{(4)})T_6 \\ -(k_{31}^{(1)}+k_{21}^{(2)})T_1 - k_{32}^{(1)}T_2 - k_{23}^{(3)}T_6 \end{Bmatrix}$$