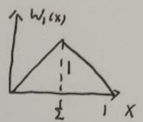
QUESTION 1　　　　张进　2021029015

对 $\begin{cases} u''(x)+u(x)+x=0 & x\in(0,1) \\ u(0)=0 \\ u'(1)=0 \end{cases}$

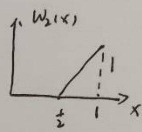
设 $u(x) = a_1 w_1(x) + a_2 w_2(x)$

伽辽金法：对 $\forall w(x)$，均有 $\int_0^1 (u'w' - wu - wx)dx = 0$.

不妨设 $w_1(x) = \begin{cases} 2x & 0<x<\frac{1}{2} \\ -2x+2 & \frac{1}{2}\le x<1 \end{cases}$



$w_2(x) = \begin{cases} 0 & 0<x<\frac{1}{2} \\ 2x-1 & \frac{1}{2}\le x<1 \end{cases}$



这样，易有 $u_1 = u(\frac{1}{2}) = a_1$　　$u_2 = u(1) = a_2$

对于 $\int_0^1 \{ w_i'(x)[a_1 w_1'(x) + a_2 w_2'(x)] - w_i(x)\cdot[a_1 w_1(x) + a_2 w_2(x)] - w_i x \} dx = 0$.

化简后，为 $a_{i1}\cdot a_1 + a_{i2}\cdot a_2 = b_i$

$a_{i1} = \int_0^1 [w_i'(x)\, w_1'(x) - w_i(x)\cdot w_1(x)]dx$

$a_{i2} = \int_0^1 [w_i'(x)\, w_2'(x) - w_i(x)\, w_2(x)]dx$

$b_i = \int_0^1 w_i(x)\cdot x\, dx$

且需对积分分段求积分.

例如 $a_{11} = \int_0^{\frac{1}{2}} [w_1'(x)\cdot w_1'(x) - w_1(x)\cdot w_1(x)]dx + \int_{\frac{1}{2}}^1 [w_1'\cdot w_1' - w_1\cdot w_1]dx$

$= \cancel{\int_0^{\frac{1}{2}} [4-4x^2]dx + \int_{\frac{1}{2}}^1 [4-4x^2]dx = 4x - \frac{4}{3}x^3 \Big|_0^1 = \frac{8}{3}}$

$= \int_0^{\frac{1}{2}} [4-4x^2]dx + \int_{\frac{1}{2}}^1 [4-(2x-2)^2]dx = 4x - \frac{4}{3}x^3\Big|_0^{\frac{1}{2}} + (4x - \frac{1}{6}(2x-2)^3)\Big|_{\frac{1}{2}}^1$

$= 2 - \frac{1}{6} + 4 - (2 + \frac{1}{6}) = \frac{11}{3}$

$a_{12} = \int_0^1 [w_2'(x)\cdot w_1'(x) - w_2(x)\cdot w_1(x)] = \int_0^{\frac{1}{2}} 0\cdot dx + \int_{\frac{1}{2}}^1 [-4 - (2x-1)\cdot(-2x+2)]dx$

$= \int_{\frac{1}{2}}^1 [-4 + 4x^2 - 6x + 2]dx = \frac{4}{3}x^3 - 3x^2 - 2x \Big|_{\frac{1}{2}}^1 = \frac{4}{3} - 5 - \frac{1}{6} + \frac{3}{4} + 1 = -\frac{25}{12}$

a_{12}与a_{21}对称. $\therefore a_{21} = a_{12} = -\frac{25}{12}$

$$a_{22} = \int_0^1 [w_2'(x)\cdot w_2'(x) - w_2(x)\cdot w_2(x)]dx$$
$$= \int_0^{\frac{1}{2}} 0\cdot dx + \int_{\frac{1}{2}}^1 [2\cdot 2 - (2x-1)^2]\,dx = 4x - \frac{1}{6}(2x-1)^3\Big|_{\frac{1}{2}}^1$$
$$= 4 - \frac{1}{6} - (2-0) = \frac{11}{6}$$

$\therefore$ 整理新得系数矩阵为 $\begin{bmatrix} \frac{11}{3} & -\frac{25}{12} \\ -\frac{25}{12} & \frac{11}{6} \end{bmatrix}$

$$b_1 = \int_0^1 w_1(x)\cdot x\,dx = \int_0^{\frac{1}{2}} 2x^2dx + \int_{\frac{1}{2}}^1 -2x^2+2x\,dx = \frac{2}{3}x^3\Big|_0^{\frac{1}{2}} + \left(-\frac{2}{3}x^3+x^2\right)\Big|_{\frac{1}{2}}^1$$
$$= \frac{1}{12} + \left(-\frac{2}{3}+1\right) - \left(-\frac{2}{3}\cdot\frac{1}{8}+\frac{1}{4}\right)$$
$$= \frac{1}{6} + \frac{1}{3} - \frac{1}{4} = \frac{1}{4}$$

$$b_2 = \int_0^1 w_2(x)\cdot x\,dx = \int_0^{\frac{1}{2}} 0\cdot dx + \int_{\frac{1}{2}}^1 2x^2 - x\,dx = \frac{2}{3}x^3 - \frac{1}{2}x^2\Big|_{\frac{1}{2}}^1 = \frac{2}{3} - \frac{1}{2} - \frac{1}{12} + \frac{1}{8}$$
$$= \frac{5}{24}$$

得线性方程为 $\begin{bmatrix} \frac{11}{3} & -\frac{25}{12} \\ -\frac{25}{12} & \frac{11}{6} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} \\ \frac{5}{24} \end{bmatrix}$

解得 $\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 0.375 \\ 0.539 \end{Bmatrix}$

QUESTION 2

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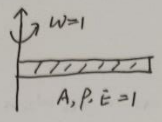
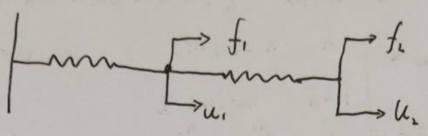
$$\begin{cases} u'' + u + x = 0 \\ u(0) = 0 \\ u(1) = 0 \end{cases}$$

可看作 

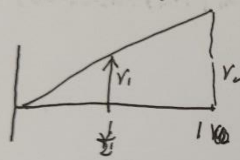
求解 u(x) 的物理问题

将杆看作两段弹簧

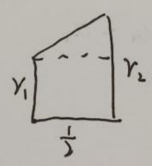


$k = \frac{EA}{\Delta L} = \frac{1 \cdot 1}{1 \cdot \frac{1}{2}} = 2$　则 $\begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix}$ (I)

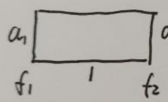
现将连续力等效为 f_1, f_2. 原理为使集中力与连续力合力相同，合力矩相同.



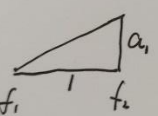
对 $(\frac{1}{2}, 1)$ 区段：



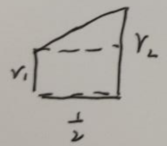
对于：



矩形等效力. $f_1 = \frac{a_1 + a_1}{2}$　$f_2 = \frac{a_1 + a_1}{2}$

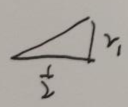


三角形等效力. $f_1 = \frac{1}{3} \frac{1}{2} a_1$　$f_2 = \frac{2}{3} \cdot a_1 \cdot \frac{1}{2}$

∴ 对 

矩形 $f_1' = \frac{1}{2} \cdot r_1 \cdot \frac{1}{2}$　$f_2' = \frac{1}{2} \cdot r_1 \cdot \frac{1}{2}$

三角形 $f_1'' = \frac{1}{3} \cdot \frac{1}{4} \cdot (r_2 - r_1)$　$f_2'' = \frac{2}{3} \cdot \frac{1}{4} \cdot (r_2 - r_1)$

对 

$f_1''' = \frac{1}{2} \cdot r_1 \cdot \frac{2}{3}$

$$\therefore \begin{cases} f_1 = f_1' + f_1'' + f_1''' = \frac{1}{2} r_1 - \frac{1}{12} r_2 \\ f_2 = f_2' + f_2'' = \frac{1}{6} r_2 - \frac{1}{12} r_1 \end{cases}$$

从而 $\begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{12} \\ -\frac{1}{12} & \frac{1}{6} \end{bmatrix} \begin{Bmatrix} r_1 \\ r_2 \end{Bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{12} \\ -\frac{1}{12} & \frac{1}{6} \end{bmatrix} \begin{Bmatrix} \frac{1}{2} + u_1 \\ 1 + u_2 \end{Bmatrix}$ (II)

(II) 代入 (I) 式. ⇒ 有 $\begin{bmatrix} -\frac{11}{3} & \frac{25}{12} \\ \frac{25}{12} & -\frac{11}{6} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} + \begin{Bmatrix} \frac{1}{4} \\ \frac{5}{24} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$