有限元方法4

稳态热传导问题的等价积分形式

$$\begin{cases} \nabla \cdot k\nabla T + Q = 0 & \text{in } \Omega \\ T = \bar{T} & \text{on } \Gamma_1 \\ k\nabla T \cdot n = \bar{q} & \text{on } \Gamma_2 \end{cases}$$

Q 单位体积产生的热量

$\bar{T}$ 已知的表面温度

$\bar{q}$ 表面上单位面积的热流输入

稳态热传导问题的等价积分形式

$$\begin{cases} \nabla \cdot k\nabla T + Q = 0 & \text{in } \Omega \\ T = \bar{T} & \text{on } \Gamma_1 \\ k\nabla T \cdot n = \bar{q} & \text{on } \Gamma_2 \end{cases}$$

$$\Rightarrow$$

$$\int_{\Omega} w(\nabla \cdot k\nabla T + Q)\mathrm{d}\Omega = 0$$

$$\int_{\Gamma_2} \bar{w}_2(k\nabla T \cdot n - \bar{q})\mathrm{d}\Gamma = 0$$

$$T = \bar{T}\left(\text{on } \Gamma_1\right)$$

稳态热传导问题的等价积分形式

$$\int_{\Omega} w(\nabla \cdot k\nabla T + Q)\mathrm{d}\Omega = 0$$

$$\int_{\Gamma_2} \bar{w}_2(k\nabla T \cdot n - \bar{q})\mathrm{d}\Gamma = 0$$

$$\Downarrow$$

$$\int_{\Omega} w(\nabla \cdot k\nabla T + Q)\mathrm{d}\Omega + \int_{\Gamma_2} \bar{w}_2(k\nabla T \cdot n - \bar{q})\mathrm{d}\Gamma = 0$$

稳态热传导问题的等价积分形式

$$\int_{\Omega} \nabla \cdot \boldsymbol{f} \mathrm{d}\Omega = \oint_{\partial\Omega} \boldsymbol{f} \cdot \boldsymbol{n} \mathrm{d}\Omega$$

$$\begin{aligned}\int_{\Omega} w\nabla \cdot k\nabla T \mathrm{d}\Omega &= \int_{\Omega} \left[\nabla \cdot \left(wk\nabla T\right) - k\nabla w \cdot \nabla T\right] \mathrm{d}\Omega \\ &= \oint_{\partial\Omega} kw\nabla T \cdot n \mathrm{d}\Omega - \int_{\Omega} k\nabla w \cdot \nabla T\end{aligned}$$

$$\int_{\Omega} w(\nabla \cdot k\nabla T + Q)\mathrm{d}\Omega + \int_{\Gamma_2} \bar{w}_2 (k\nabla T \cdot n - \bar{q})\mathrm{d}\Gamma =$$

$$-\int_{\Omega} (k\nabla w \cdot \nabla T + wQ)\mathrm{d}\Omega + \oint_{\Gamma_1+\Gamma_2} wk\nabla T \cdot n \mathrm{d}\Gamma + \int_{\Gamma_2} \bar{w}_2 (k\nabla T \cdot n - \bar{q})\mathrm{d}\Gamma = 0$$

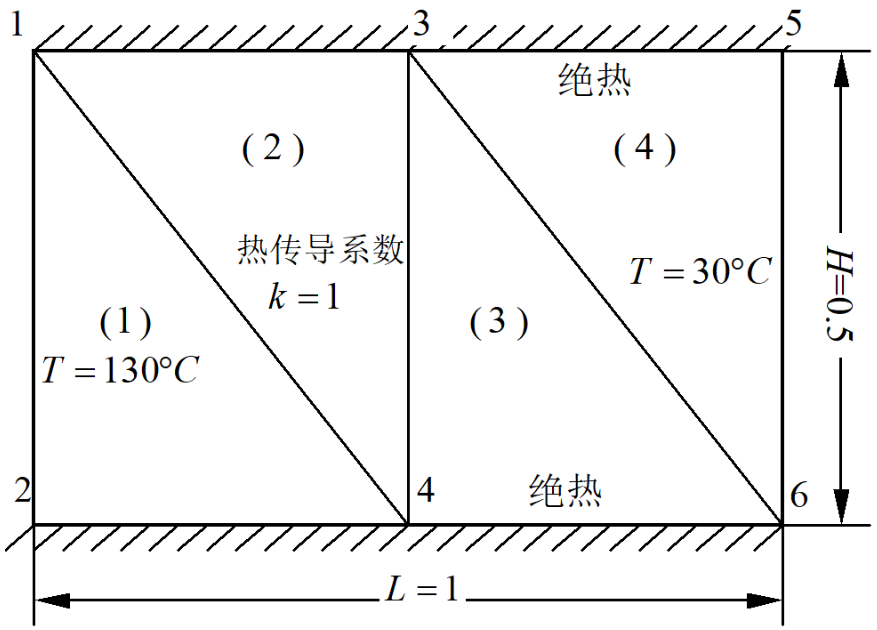
稳态热传导问题的等价积分形式

$$-\int_{\Omega}(k\nabla w\cdot\nabla T+wQ)\mathrm{d}\Omega+\oint_{\Gamma_1+\Gamma_2+\Gamma_3}wk\nabla T\cdot n\mathrm{d}\Gamma+\int_{\Gamma_2}\bar{w}_2(k\nabla T\cdot n-\bar{q})\mathrm{d}\Gamma=0$$

$$w|_{\Gamma_1}=0\quad w|_{\Gamma_2}=-\bar{w}_2$$

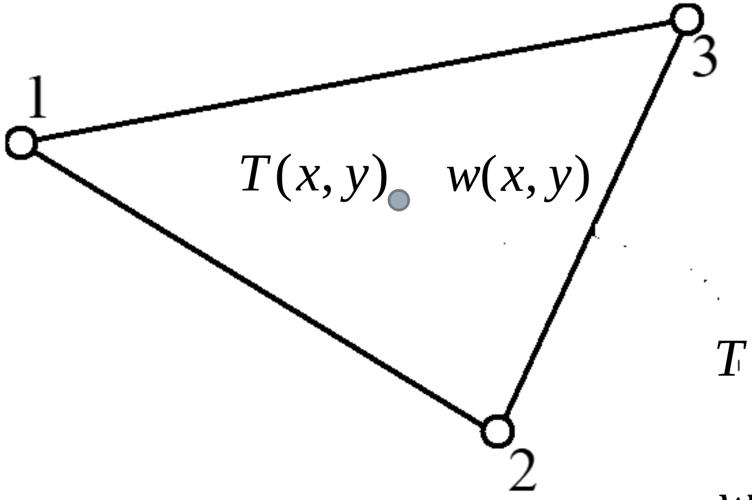
$$\int_{\Omega}(wQ-k\nabla w\cdot\nabla T)\mathrm{d}\Omega+\int_{\Gamma_2}w\bar{q}\mathrm{d}\Gamma=0$$

稳态热传导问题的有限元计算



$$\int_{\Omega}(wQ - k\nabla w \cdot \nabla T)\mathrm{d}\Omega + \int_{\Gamma_2} w\bar{q}\mathrm{d}\Gamma = 0$$

稳态热传导问题的有限元计算



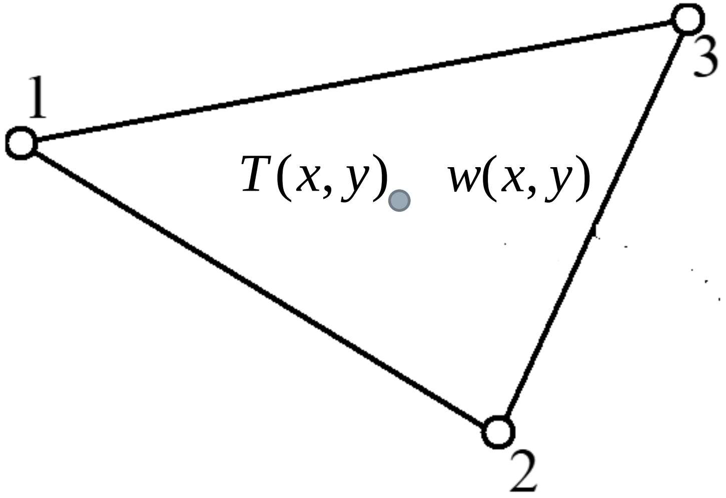
$$\int_{\Omega}\nabla w\cdot\nabla T\mathrm{d}\Omega=\sum_{e=1}^{M}\int_{\Omega_e}\nabla w\cdot\nabla T\mathrm{d}\Omega=0$$

（无内部热源）

$$T=\frac{1}{2\Delta}[(a_1+b_1x+c_1y)T_1+(a_2+b_2x+c_2y)T_2+(a_3+b_3x+c_3y)T_3]$$

$$w=\frac{1}{2\Delta}[(a_1+b_1x+c_1y)w_1+(a_2+b_2x+c_2y)w_2+(a_3+b_3x+c_3y)w_3]$$

稳态热传导问题的有限元计算



$$T=\frac{1}{2\Delta}[(a_1+b_1x+c_1y)T_1+(a_2+b_2x+c_2y)T_2+(a_3+b_3x+c_3y)T_3]$$

$$w=\frac{1}{2\Delta}[(a_1+b_1x+c_1y)w_1+(a_2+b_2x+c_2y)w_2+(a_3+b_3x+c_3y)w_3]$$

$$\{\nabla T\}=\begin{Bmatrix}\partial T/\partial x\\ \partial T/\partial y\end{Bmatrix}=\frac{1}{2\Delta}\begin{Bmatrix}b_1T_1+b_2T_2+b_3T_3\\ c_1T_1+c_2T_2+c_3T_3\end{Bmatrix}=\frac{1}{2\Delta}\begin{bmatrix}b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{bmatrix}\begin{Bmatrix}T_1\\T_2\\T_3\end{Bmatrix}$$

$$\{\nabla w\}=\begin{Bmatrix}\partial w/\partial x\\ \partial w/\partial y\end{Bmatrix}=\frac{1}{2\Delta}\begin{Bmatrix}b_1w_1+b_2w_2+b_3w_3\\ c_1w_1+c_2w_2+c_3w_3\end{Bmatrix}=\frac{1}{2\Delta}\begin{bmatrix}b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{bmatrix}\begin{Bmatrix}w_1\\w_2\\w_3\end{Bmatrix}$$

稳态热传导问题的有限元计算

矢量：

$$\vec{a} = a_x\vec{i} + a_y\vec{j} + a_z\vec{k}$$

$$\vec{b} = b_x\vec{i} + b_y\vec{j} + b_z\vec{k}$$

$$\vec{a}\cdot\vec{b}$$

$$\vec{a}\times\vec{b}$$

向量：

$$\boldsymbol{a} = \{a\} = \begin{Bmatrix} a_x \\ a_y \\ a_z \end{Bmatrix},\quad \boldsymbol{b} = \{b\} = \begin{Bmatrix} b_x \\ b_y \\ b_z \end{Bmatrix}$$

$$\vec{a}\cdot\vec{b} = \boldsymbol{a}^T\boldsymbol{b} = \{a\}^T\{b\}$$

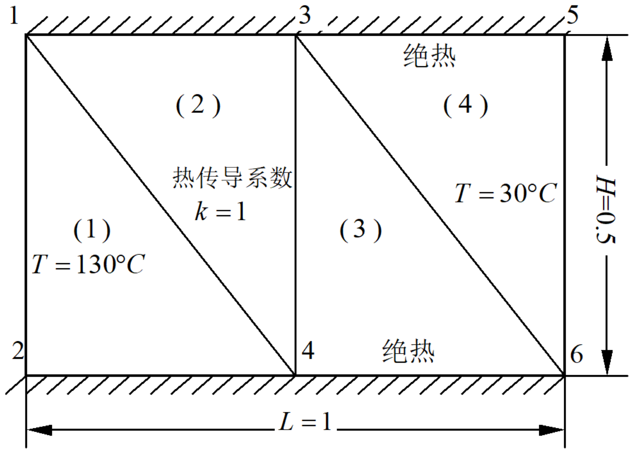
稳态热传导问题的有限元计算

$$\{\nabla T\}=\begin{Bmatrix}\partial T/\partial x\\ \partial T/\partial y\end{Bmatrix}=\frac{1}{2\varDelta}\begin{bmatrix}b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{bmatrix}\begin{Bmatrix}T_1\\ T_2\\ T_3\end{Bmatrix},\quad \{\nabla w\}=\begin{Bmatrix}\partial w/\partial x\\ \partial w/\partial y\end{Bmatrix}=\frac{1}{2\varDelta}\begin{bmatrix}b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{bmatrix}\begin{Bmatrix}w_1\\ w_2\\ w_3\end{Bmatrix}$$

$$\begin{aligned}\int_{\Omega_e}k\nabla w\cdot\nabla T\mathrm{d}\Omega&=k\int_{\Omega_e}\{\nabla w\}^T\{\nabla T\}\mathrm{d}\Omega\\ &=\frac{k}{4\varDelta^2}\{w_1\quad w_2\quad w_3\}\int_{\Omega_e}\begin{bmatrix}b_1 & c_1\\ b_2 & c_2\\ b_3 & c_3\end{bmatrix}\begin{bmatrix}b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{bmatrix}\mathrm{d}\Omega\begin{Bmatrix}T_1\\ T_2\\ T_3\end{Bmatrix}\\ &=\frac{k}{4\varDelta}\{w_1\quad w_2\quad w_3\}\begin{bmatrix}b_1b_1+c_1c_1 & b_1b_2+c_1c_2 & b_1b_3+c_1c_3\\ b_2b_1+c_2c_1 & b_2b_2+c_2c_2 & b_2b_3+c_2c_3\\ b_3b_1+c_3c_1 & b_3b_2+c_3c_2 & b_3b_3+c_3c_3\end{bmatrix}\begin{Bmatrix}T_1\\ T_2\\ T_3\end{Bmatrix}\end{aligned}$$

单元方程的组装

$$\frac{k}{4\Delta}\begin{bmatrix} b_1b_1+c_1c_1 & b_1b_2+c_1c_2 & b_1b_3+c_1c_3 \\ b_2b_1+c_2c_1 & b_2b_2+c_2c_2 & b_2b_3+c_2c_3 \\ b_3b_1+c_3c_1 & b_3b_2+c_3c_2 & b_3b_3+c_3c_3 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix}=\begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{Bmatrix} \Rightarrow \begin{bmatrix} k_{11}^{(e)} & k_{12}^{(e)} & k_{13}^{(e)} \\ k_{21}^{(e)} & k_{22}^{(e)} & k_{23}^{(e)} \\ k_{31}^{(e)} & k_{32}^{(e)} & k_{33}^{(e)} \end{bmatrix}\begin{Bmatrix} T_1^{(e)} \\ T_2^{(e)} \\ T_3^{(e)} \end{Bmatrix}=\begin{Bmatrix} Q_1^{(e)} \\ Q_2^{(e)} \\ Q_3^{(e)} \end{Bmatrix}$$



单元1

$$\begin{bmatrix} k_{11}^{(1)} & k_{12}^{(1)} & 0 & k_{13}^{(1)} & 0 & 0 \\ k_{21}^{(1)} & k_{22}^{(1)} & 0 & k_{23}^{(1)} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ k_{31}^{(1)} & k_{32}^{(1)} & 0 & k_{33}^{(1)} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \end{Bmatrix}=\begin{Bmatrix} Q_1^{(1)} \\ Q_2^{(1)} \\ 0 \\ Q_3^{(1)} \\ 0 \\ 0 \end{Bmatrix}$$

单元2

$$\begin{bmatrix} k_{11}^{(2)} & 0 & k_{13}^{(2)} & k_{12}^{(2)} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ k_{31}^{(2)} & 0 & k_{33}^{(2)} & k_{23}^{(2)} & 0 & 0 \\ k_{21}^{(2)} & 0 & k_{32}^{(2)} & k_{22}^{(2)} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \end{Bmatrix}=\begin{Bmatrix} Q_1^{(2)} \\ 0 \\ Q_3^{(2)} \\ Q_2^{(2)} \\ 0 \\ 0 \end{Bmatrix}$$

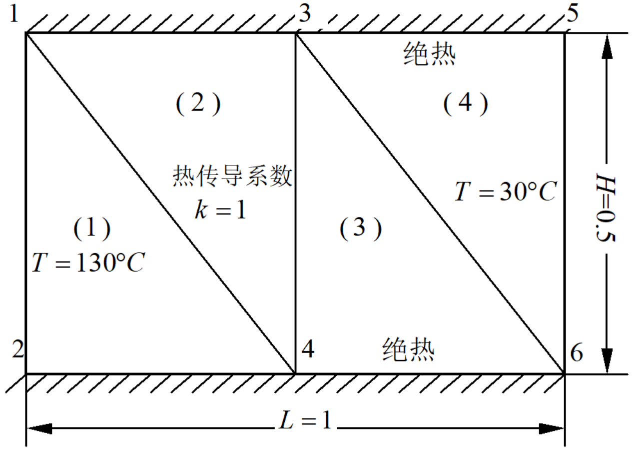
单元3

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & k_{11}^{(3)} & k_{12}^{(3)} & 0 & k_{13}^{(3)} \\ 0 & 0 & k_{21}^{(3)} & k_{22}^{(3)} & 0 & k_{23}^{(3)} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & k_{31}^{(3)} & k_{32}^{(3)} & 0 & k_{33}^{(3)} \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \\ Q_1^{(3)} \\ Q_2^{(3)} \\ 0 \\ Q_3^{(3)} \end{Bmatrix}$$

单元4

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & k_{11}^{(4)} & 0 & k_{13}^{(4)} & k_{12}^{(4)} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & k_{31}^{(4)} & 0 & k_{33}^{(4)} & k_{32}^{(4)} \\ 0 & 0 & k_{21}^{(4)} & 0 & k_{23}^{(4)} & k_{22}^{(4)} \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \\ Q_1^{(4)} \\ 0 \\ Q_3^{(4)} \\ Q_2^{(4)} \end{Bmatrix}$$

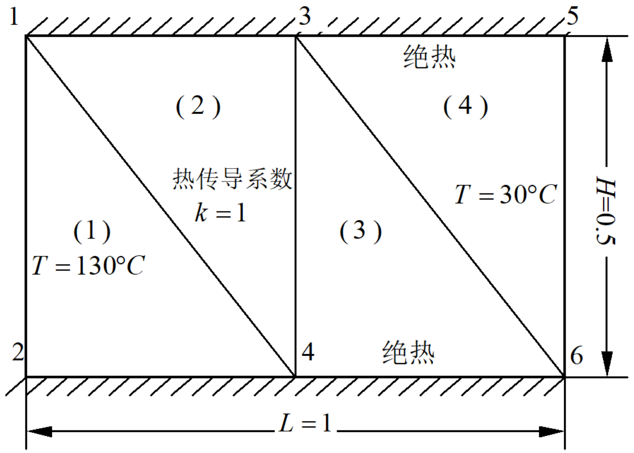
稳态热传导问题的有限元计算



$$\int_{\Omega_3} k\nabla w\cdot\nabla T\mathrm{d}\Omega = \frac{k}{4\Delta}\{w_3 \quad w_4 \quad w_6\}\begin{bmatrix} b_1b_1+c_1c_1 & b_1b_2+c_1c_2 & b_1b_3+c_1c_3 \\ b_2b_1+c_2c_1 & b_2b_2+c_2c_2 & b_2b_3+c_2c_3 \\ b_3b_1+c_3c_1 & b_3b_2+c_3c_2 & b_3b_3+c_3c_3 \end{bmatrix}\begin{Bmatrix} T_3 \\ T_4 \\ T_6 \end{Bmatrix} = \boldsymbol{w}^T(\boldsymbol{L}^{(3)T}[\boldsymbol{k}_e]\boldsymbol{L}^{(3)})\boldsymbol{T}$$

$$\begin{Bmatrix} T_3 \\ T_4 \\ T_6 \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}\begin{Bmatrix} T_1 \\ \vdots \\ T_6 \end{Bmatrix} = \boldsymbol{L}^{(3)}\boldsymbol{T}, \quad \begin{Bmatrix} w_3 \\ w_4 \\ w_6 \end{Bmatrix} = \boldsymbol{L}^{(3)}\boldsymbol{w}$$

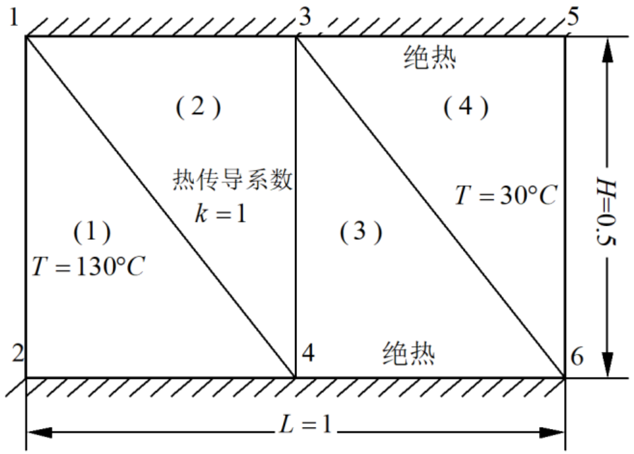
稳态热传导问题的有限元计算



$$\int_{\Omega_e} k\nabla w\cdot\nabla T\mathrm{d}\Omega=\boldsymbol{w}^T(\boldsymbol{L}^{(e)T}[\boldsymbol{k}_e]\boldsymbol{L}^{(e)})\boldsymbol{T}$$

$$\sum_e\int_{\Omega_e} k\nabla w\cdot\nabla T\mathrm{d}\Omega=\boldsymbol{w}^T[\sum_e(\boldsymbol{L}^{(e)T}[\boldsymbol{k}_e]\boldsymbol{L}^{(e)})]\boldsymbol{T}=\boldsymbol{w}^T\boldsymbol{K}\boldsymbol{T}$$

稳态热传导问题的有限元计算

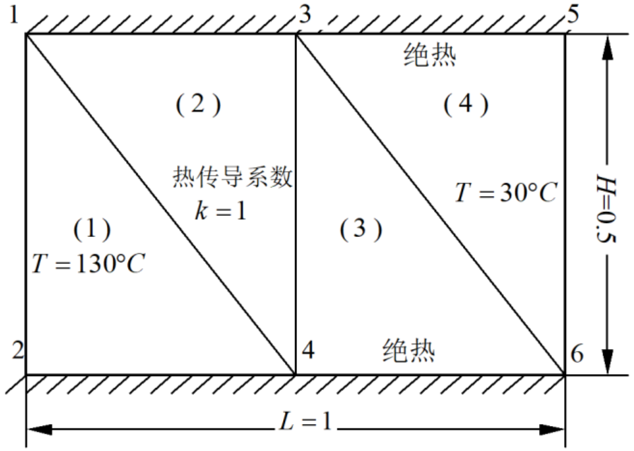


$$w|_{\Gamma_1}=0 \quad w|_{\Gamma_2}=-\bar{w}_2 \quad w|_{\Gamma_3}=-\bar{w}_3$$

$$\sum_e \int_{\Omega_e} k\nabla w\cdot\nabla T \mathrm{d}\Omega = \boldsymbol{w}^T\boldsymbol{K}\boldsymbol{T}=0$$

(只考虑第一类边界条件)

稳态热传导问题的有限元计算



$$\begin{bmatrix} k_{31}^{(2)} & 0 & k_{33}^{(2)}+k_{11}^{(3)}+k_{11}^{(4)} & k_{23}^{(2)}+k_{12}^{(3)} & k_{13}^{(4)} & k_{13}^{(3)}+k_{12}^{(4)} \\ k_{31}^{(1)}+k_{21}^{(2)} & k_{32}^{(1)} & k_{32}^{(2)}+k_{21}^{(3)} & k_{33}^{(1)}+k_{22}^{(2)}+k_{22}^{(3)} & 0 & k_{23}^{(3)} \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

稳态热传导问题的有限元计算

$$\sum_e \int_{\Omega_e} k \nabla w \cdot \nabla T \mathrm{d}\Omega = \boldsymbol{w}^T [\sum_e (\boldsymbol{L}^{(e)T} [\boldsymbol{k}_e] \boldsymbol{L}^{(e)})] \boldsymbol{T} = \boldsymbol{w}^T \boldsymbol{K} \boldsymbol{T}$$

$$\boldsymbol{K} = \sum_e \boldsymbol{L}^{(e)T} [\boldsymbol{k}_e] \boldsymbol{L}^{(e)}$$

$$\boldsymbol{k}_e = \frac{k}{4\Delta} \begin{bmatrix} b_1 & c_1 \\ b_2 & c_2 \\ b_3 & c_3 \end{bmatrix} \begin{bmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} = \frac{k}{4\Delta} \begin{bmatrix} b_1 b_1 + c_1 c_1 & b_1 b_2 + c_1 c_2 & b_1 b_3 + c_1 c_3 \\ b_2 b_1 + c_2 c_1 & b_2 b_2 + c_2 c_2 & b_2 b_3 + c_2 c_3 \\ b_3 b_1 + c_3 c_1 & b_3 b_2 + c_3 c_2 & b_3 b_3 + c_3 c_3 \end{bmatrix}$$

稳态热传导问题的有限元计算

$$\begin{cases} \nabla \cdot k\nabla T + Q = 0 & \text{in } \Omega \\ T = \bar{T} & \text{on } \Gamma_1 \\ k\nabla T \cdot n = \bar{q} & \text{on } \Gamma_2 \end{cases} \Rightarrow \begin{array}{l} \int_\Omega w(\nabla \cdot k\nabla T + Q)\mathrm{d}\Omega = 0 \\ \int_{\Gamma_2} \bar{w}_2(k\nabla T \cdot n - \bar{q})\mathrm{d}\Gamma = 0 \end{array}$$

$$\Downarrow$$

$$\int_\Omega w(\nabla \cdot k\nabla T + Q)\mathrm{d}\Omega + \int_{\Gamma_2} \bar{w}_2(k\nabla T \cdot n - \bar{q})\mathrm{d}\Gamma = 0$$

$$\Downarrow$$

$$-\int_\Omega (k\nabla w \cdot \nabla T + wQ)\mathrm{d}\Omega + \oint_{\Gamma_1+\Gamma_2} wk\nabla T \cdot n\mathrm{d}\Gamma + \int_{\Gamma_2} \bar{w}_2(k\nabla T \cdot n - \bar{q})\mathrm{d}\Gamma = 0$$

$$\Downarrow$$

$$\int_\Omega (wQ - k\nabla w \cdot \nabla T)\mathrm{d}\Omega + \int_{\Gamma_2} w\bar{q}\mathrm{d}\Gamma = 0 \quad \Rightarrow \quad \int_\Omega k\nabla w \cdot \nabla T\mathrm{d}\Omega = 0$$

（只考虑第一类边界条件）

$$\Downarrow$$

$$\sum_e \int_{\Omega_e} k\nabla w \cdot \nabla T\mathrm{d}\Omega = \boldsymbol{w}^T[\sum_e (\boldsymbol{L}^{(e)T}\left[\boldsymbol{k}_e\right]\boldsymbol{L}^{(e)})]\boldsymbol{T} = \boldsymbol{w}^T\boldsymbol{K}\boldsymbol{T}$$

弹性力学问题的等价积分形式

$$\begin{cases} \sigma_{ij,j} + f_i = 0 & \text{平衡方程} \\ \sigma_{ij} = D_{ijkl}\varepsilon_{kl} & \text{本构方程} \\ \varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) & \text{几何方程} \end{cases}$$

$$\begin{cases} u_i = \bar{u}_i & \text{on } \Gamma_1 \\ \sigma_{ij} n_j = \bar{F}_i & \text{on } \Gamma_2 \end{cases}$$

弹性力学问题的等价积分形式

$$\begin{cases} \sigma_{ij,j} + f_i = 0 & \text{in } \Omega \\ \sigma_{ij} n_j = \bar{F}_i & \text{on } \Gamma_2 \end{cases}$$

$$\Downarrow$$

$$\int_\Omega w_i(\sigma_{ij,j} + f_i)\mathrm{d}\Omega + \int_{\Gamma_2} \bar{w}_{2i}(\sigma_{ij} n_j - \bar{F}_i)\mathrm{d}\Gamma = 0$$

$$\left(\begin{aligned} & u_i = \bar{u}_i \\ & \sigma_{ij} = D_{ijkl}\varepsilon_{kl} \\ & \varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \end{aligned} \right)$$

弹性力学问题的等价积分形式

$$\int_{\Omega} w_i \sigma_{ij,j} \mathrm{d}\Omega = \int_{\Omega} [(w_i \sigma_{ij})_{,j} - w_{i,j} \sigma_{ij}] \mathrm{d}\Omega = \int_{\partial\Omega} w_i \sigma_{ij} n_j \mathrm{d}\Gamma - \int_{\Omega} w_{i,j} \sigma_{ij} \mathrm{d}\Omega$$

弹性力学问题的等价积分形式

$$\int_{\Omega} w_i(\sigma_{ij,j} + f_i)\mathrm{d}\Omega + \int_{\Gamma_2} \bar{w}_{2i}(\sigma_{ij}n_j - \bar{F}_i)\mathrm{d}\Gamma = 0$$

$$\int_{\Omega} w_i\sigma_{ij,j}\mathrm{d}\Omega = \int_{\partial\Omega} w_i\sigma_{ij}n_j\mathrm{d}\Gamma - \int_{\Omega} w_{i,j}\sigma_{ij}\mathrm{d}\Omega$$

$$\int_{\Omega}(w_i f_i - w_{i,j}\sigma_{ij})\mathrm{d}\Omega + \oint_{\Gamma} w_i\sigma_{ij}n_j\mathrm{d}\Gamma + \int_{\Gamma_2} \bar{w}_{2i}(\sigma_{ij}n_j - \bar{F}_i)\mathrm{d}\Gamma = 0$$

$$-\int_{\Omega} w_{i,j}\sigma_{ij}\mathrm{d}\Omega + \int_{\Omega} w_i f_i\mathrm{d}\Omega + \int_{\Gamma_2} w_i\bar{F}_i\mathrm{d}\Gamma = 0$$

弹性力学问题的等价积分形式

$$-\int_{\Omega} w_{i,j}\sigma_{ij}\mathrm{d}\Omega - \int_{\Gamma_3} k w_i u_i \mathrm{d}\Gamma + \int_{\Omega} w_i f_i \mathrm{d}\Omega + \int_{\Gamma_2} w_i \bar{F}_i \mathrm{d}\Gamma = 0$$

$$\Downarrow \quad w_i = \delta u_i$$

$$-\int_{\Omega} \delta u_{i,j}\sigma_{ij}\mathrm{d}\Omega + \int_{\Omega} \delta u_i f_i \mathrm{d}\Omega + \int_{\Gamma_2} \delta u_i \bar{F}_i \mathrm{d}\Gamma = 0$$

弹性力学问题的等价积分形式

$$-\int_{\Omega} w_{i,j}\sigma_{ij}\mathrm{d}\Omega + \int_{\Omega} w_i f_i \mathrm{d}\Omega + \int_{\Gamma_2} w_i \bar{F}_i \mathrm{d}\Gamma = 0$$

$$w_i = \delta u_i$$

$$-\int_{\Omega} \delta u_{i,j}\sigma_{ij}\mathrm{d}\Omega + \int_{\Omega} \delta u_i f_i \mathrm{d}\Omega + \int_{\Gamma_2} \delta u_i \bar{F}_i \mathrm{d}\Gamma = 0$$

$$\delta u_{i,j}\sigma_{ij} = \frac{1}{2}(\delta u_{i,j} + \delta u_{j,i})\sigma_{ij} = \delta\varepsilon_{ij} \cdot \sigma_{ij}$$

$$-\int_{\Omega} \delta\varepsilon_{ij}\sigma_{ij}\mathrm{d}\Omega + \int_{\Omega} \delta u_i f_i \mathrm{d}\Omega + \int_{\Gamma_2} \delta u_i \bar{F}_i \mathrm{d}\Gamma = 0$$

弹性力学问题的虚位移原理

$$-\int_{\Omega}\delta\varepsilon_{ij}\sigma_{ij}\mathrm{d}\Omega+\int_{\Omega}\delta u_i f_i\mathrm{d}\Omega+\int_{\Gamma_2}\delta u_i\bar{F}_i\mathrm{d}\Gamma=0$$

弹性力学问题的有限元方法

$$\{\varepsilon\}^T=\begin{bmatrix}\varepsilon_x & \varepsilon_y & \varepsilon_z & 2\gamma_{xy} & 2\gamma_{yz} & 2\gamma_{zx}\end{bmatrix}=\begin{bmatrix}\varepsilon_{11} & \varepsilon_{22} & \varepsilon_{33} & 2\varepsilon_{12} & 2\varepsilon_{23} & 2\varepsilon_{31}\end{bmatrix}$$

$$\{\sigma\}^T=\begin{bmatrix}\sigma_x & \sigma_y & \sigma_z & \tau_{xy} & \tau_{yz} & \tau_{zx}\end{bmatrix}=\begin{bmatrix}\sigma_{11} & \sigma_{22} & \sigma_{33} & \sigma_{12} & \sigma_{23} & \sigma_{31}\end{bmatrix}$$

$$\{u\}^T=\begin{bmatrix}u_x & u_y & u_z\end{bmatrix}=\begin{bmatrix}u_1 & u_2 & u_3\end{bmatrix}$$

$$\{f\}^T=\begin{bmatrix}f_x & f_y & f_z\end{bmatrix}=\begin{bmatrix}f_1 & f_2 & f_3\end{bmatrix}$$

$$\{\bar{F}\}^T=\begin{bmatrix}\bar{F}_x & \bar{F}_y & \bar{F}_z\end{bmatrix}=\begin{bmatrix}\bar{F}_1 & \bar{F}_2 & \bar{F}_3\end{bmatrix}$$

$$\{u_0\}^T=\begin{bmatrix}u_{x0} & u_{y0} & u_{z0}\end{bmatrix}=\begin{bmatrix}u_{10} & u_{20} & u_{30}\end{bmatrix}$$

弹性力学问题的有限元方法

$$-\int_{\Omega}\delta\varepsilon_{ij}\sigma_{ij}\mathrm{d}\Omega+\int_{\Omega}\delta u_i f_i\mathrm{d}\Omega+\int_{\Gamma_2}\delta u_i\bar{F}_i\mathrm{d}\Gamma=0$$

$$\int_{\Omega}\delta\varepsilon_{ij}\sigma_{ij}\mathrm{d}\Omega=\int_{\Omega}\{\delta\varepsilon\}^T\{\sigma\}\mathrm{d}\Omega$$

$$\int_{\Omega}\delta u_i f_i\mathrm{d}\Omega=\int_{\Gamma_2}\{\delta u\}^T\{\bar{F}\}\mathrm{d}\Gamma$$

$$\int_{\Gamma_2}\delta u_i\bar{F}_i\mathrm{d}\Gamma=\int_{\Gamma_2}\{\delta u\}^T\{\bar{F}\}\mathrm{d}\Gamma$$

$$\int_{\Omega}\{\delta\varepsilon\}^T\{\sigma\}\mathrm{d}\Omega-\int_{\Omega}\{\delta u\}^T\{f\}\mathrm{d}\Omega-\int_{\Gamma_2}\{\delta u\}^T\{\bar{F}\}\mathrm{d}\Gamma=0$$

弹性力学问题的有限元方法

$$\int_{\Omega}\{\delta\varepsilon\}^{T}\{\sigma\}\mathrm{d}\Omega-\int_{\Omega}\{\delta u\}^{T}\{f\}\mathrm{d}\Omega-\int_{\Gamma_2}\{\delta u\}^{T}\{\bar{F}\}\mathrm{d}\Gamma=0$$

（只考虑体积力的作用）

$$\int_{\Omega}\{\delta\varepsilon\}^{T}\{\sigma\}\mathrm{d}\Omega-\int_{\Omega}\{\delta u\}^{T}\{f\}\mathrm{d}\Omega=0$$