有限元方法第五次作业

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课程名称：　有限元方法

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时　　间：　2021 年 4 月 8 日

目录

第一题第一问：

整理课堂内容，梳理清楚弹性力学有限元方法的几个重要步骤。本周先梳理清楚以下两点：

- 弹性力学基本方程和第二类边界条件的等效加权参数形式；
- 平面弹性力学三角形单元上的位移、应变的插值公式。

$$\begin{cases} \sigma_{ij,j} + f_1 = 0 \\ \sigma_{ij} = D_{ijkl}(\varepsilon_{kl} - \varepsilon_{kl}^0) \\ \varepsilon_{ij} = \dfrac{1}{2}(u_{i,j} + u_{j,i}) \end{cases}$$

$$\begin{cases} u_j = \overline{u_i} \\ \sigma_{ij}\eta_j = \overline{F}_i \\ \sigma_{ij}\eta_j = k(u_{i0} - u_i) \end{cases}$$

$$\begin{cases} \sigma_{11,1} + \sigma_{12,2} + \sigma_{13,3} + f_1 = 0 \\ \sigma_{21,1} + \sigma_{22,2} + \sigma_{23,3} + f_2 = 0 \\ \sigma_{31,1} + \sigma_{32,3} + \sigma_{33,3} + f_3 = 0 \end{cases}$$

$$\begin{cases} \int_\Omega (\sigma_{11,1} + \sigma_{12,2} + \sigma_{13,3} + f_1) d\Omega = 0 \\ \int_\Omega (\sigma_{21,1} + \sigma_{22,2} + \sigma_{23,3} + f_2) d\Omega = 0 \\ \int_\Omega (\sigma_{31,1} + \sigma_{32,3} + \sigma_{33,3} + f_3) d\Omega = 0 \end{cases}$$

$$\begin{cases} \int_\Omega w_1(\sigma_{1j,j} + f_1) d\Omega = 0 \\ \int_\Omega w_2(\sigma_{2j,j} + f_2) d\Omega = 0 \\ \int_\Omega w_3(\sigma_{3j,j} + f_3) d\Omega = 0 \end{cases}$$

$$\begin{cases}\int_{\Omega} w_i(\sigma_{ij,j}+f_i)d\Omega=0\\ \int_{\Gamma_2} w_i^{(1)}(\sigma_{ij}\eta_j-\overline{F_i})d\Gamma=0\\ \int_{\Gamma_3} w_i^{(2)}(\sigma_{ij}\eta_j-k(u_{i0}-u_i))d\Gamma=0\end{cases}$$

$$\int_{\Omega} w_i(\sigma_{ij,j})d\Omega=0$$

$$\Leftrightarrow \int_{\Omega}[(w_i\sigma_{ij})_{,j}-w_{i,j}\sigma_{ij}]d\Omega$$

$$=\oint_{\Omega} w_i\sigma_{ij}\eta_j d\Gamma-\int_{\Omega} w_{i,j}\sigma_{ij}d\Omega$$

$$\int_{\Omega}(w_i\frac{\partial\sigma_{ij}}{\partial x_j}+f_i)d\Omega+\int_{\Gamma_2} w_i^{(1)}(\sigma_{ij}\eta_j-\overline{F_i})d\Gamma+\int_{\Gamma_3} w_i^{(2)}(\sigma_{ij}\eta_j-k(u_{i0}-u_i))d\Gamma=0$$

$$u_i=\overline{u_i}$$

$$\sigma_{ij}=D_{ijkl}(\varepsilon_{kl}-\varepsilon_{kl}^{0})$$

$$\varepsilon_{ij}=\frac{1}{2}(u_{i,j}+u_{j,i})$$

$$\begin{cases}\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{xy}}{\partial y}+f_{(x)}=0\\ \frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\sigma_y}{\partial y}+f_{(y)}=0\end{cases}$$

$$\begin{cases}\int w_x(\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{xy}}{\partial y}+f_{(x)})d\Omega=0\\ \int w_y(\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\sigma_y}{\partial y}+f_{(y)})d\Omega=0\end{cases}$$

$$\int[w_x(\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{xy}}{\partial y}+f_{(x)})+w_y(\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\sigma_y}{\partial y}+f_{(y)})]d\Omega=0$$

$$\Leftrightarrow\int_{\Omega}(w_i\frac{\partial\sigma_{ij}}{\partial x_j}+f_i)=0$$

$$w_i\frac{\partial\sigma_{ij}}{\partial x_j}=w_1\frac{\partial\sigma_{1j}}{\partial x_j}+w_2\frac{\partial\sigma_{2j}}{\partial x_j}\text{继续展开}$$

$$w_i\frac{\partial\sigma_{ij}}{\partial x_j}=w_1\frac{\partial\sigma_{11}}{\partial x_1}+w_1\frac{\partial\sigma_{12}}{\partial x_2}+w_2\frac{\partial\sigma_{21}}{\partial x_1}+w_2\frac{\partial\sigma_{22}}{\partial x_2}$$

$$w_i\frac{\partial\sigma_{ij}}{\partial x_j}=\frac{\partial w_i\sigma_{ij}}{\partial x_j}-\frac{\partial w_i}{\partial x_j}\bullet\sigma_{iy}$$

$$\int_\Omega \frac{\partial w_i \sigma_{ij}}{\partial x_j} d\Omega = \oint w_i \sigma_{ij} \eta_j d\Gamma$$

$$\int_\Omega (-\frac{\partial w_i}{\partial x_j} \bullet \sigma_{iy} + w_i f_i) d\Omega - \oint w_i \sigma_{ij} \eta_j d\Gamma = 0$$

$\Leftrightarrow$

$$\begin{cases} \int_\Omega (-\frac{\partial w_i}{\partial x_j} \bullet \sigma_{iy} + w_i f_i) d\Omega + \int_{\Gamma_2} w_i T_i d\Gamma = 0 \\ w_1 |_{\Gamma_1} = 0 \end{cases}$$

定义：

$$[\sigma] = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}, [f] = \begin{bmatrix} f_x \\ f_y \end{bmatrix}, [T] = \begin{bmatrix} T_x \\ T_y \end{bmatrix}, [w] = \begin{bmatrix} w_x \\ w_y \end{bmatrix}$$

$$\frac{\partial w_1}{\partial x_j} \sigma_{1j} + \frac{\partial w_2}{\partial x_j} \sigma_{2j}$$

$$= (\frac{\partial w_1}{\partial x_1} \sigma_{11} + \frac{\partial w_1}{\partial x_2} \sigma_{12}) + (\frac{\partial w_2}{\partial x_1} \sigma_{21} + \frac{\partial w_2}{\partial x_2} \sigma_{22})$$

因为应力对称$\sigma_{12}=\sigma_{21}$

$$= \frac{\partial w_1}{\partial x_1} \sigma_{11} + \frac{\partial w_2}{\partial x_2} \sigma_{22} + (\frac{\partial w_1}{\partial x_2} + \frac{\partial w_2}{\partial x_1}) \sigma_{12} = [\delta\varepsilon]^T [\sigma]$$

$$[\delta\varepsilon] = \begin{bmatrix} \frac{\partial w_x}{\partial x_x} \\ \frac{\partial w_y}{\partial x_y} \\ \frac{\partial w_x}{\partial x_x} + \frac{\partial w_y}{\partial x_y} \end{bmatrix} = [B_{(x,y)}][\delta q]$$

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} N_1^{(e)} & & & N_2^{(e)} & & & N_3^{(e)} & & & N_4^{(e)} & & \\ & N_1^{(e)} & & & N_2^{(e)} & & & N_3^{(e)} & & & N_4^{(e)} & \\ & & N_1^{(e)} & & & N_2^{(e)} & & & N_3^{(e)} & & & N_4^{(e)} \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ w_1 \\ u_2 \\ v_2 \\ w_2 \\ u_3 \\ v_3 \\ w_3 \\ u_4 \\ v_4 \\ w_4 \end{bmatrix}$$

$$= \left[B_{(x,y,z)} \right] \left[q^{(e)} \right]$$

$$\int_\Omega (-\frac{\partial w_i}{\partial x_j} \bullet \sigma_{iy} + w_i f_i) d\Omega + \int_{\Gamma_2} w_i T_i d\Gamma = 0$$

$$\Leftrightarrow \int_\Omega (-\left[\delta_\varepsilon\right]^T \left[\sigma\right] + \left[\delta_u\right]^T \left[f\right]) d\Omega + \int_{\Gamma_2} \left[\delta_u\right]^T \left[\overline{F}\right] d\Gamma = 0$$

$$\int_\Omega \left[\delta_q\right]^T \left[\sigma\right] d\Omega - \int_\Omega \left[\delta_u\right]^T \left[f\right] d\Omega - \int_{\Gamma_2} \left[\delta_u\right]^T \left[\overline{F}\right] d\Gamma = 0$$

$$= \sum_e \int_{\Omega_e} \left[\delta_q\right]^T [B]^T [D][B][q] d\Omega - \int_{\Omega_e} \left[\delta_q\right]^T [N]^T [f] d\Omega - \left[\delta_q\right]^T \int_{\Gamma_{2e}} [N]^T \left[\overline{F}\right] d\Gamma = 0$$

$$= \sum_e \left[\int_{\Omega_e} \left[\delta_q\right]^T [B]^T [D][B][q] d\Omega - \int_{\Omega_e} \left[\delta_q\right]^T [N]^T [f] d\Omega - \left[\delta_q\right]^T \int_{\Gamma_{2e}} [N]^T \left[\overline{F}\right] d\Gamma \right]$$

$$= \sum_e \left[\left[\delta_{q_e}\right]^T [k_e][q_e] - \left[\delta_{q_e}\right]^T [f_e] - \left[\delta_{q_e}\right]^T [\overline{F}_e] \right] = \sum_e \left[\delta_{q_e}\right]^T ([k_e][q_e] - [P_e]) = 0$$

$T = \dfrac{1}{2\Delta}[a_1 + b_1 x + c_1 y)T_1 + (a_2 + b_2 x + c_2 y)T_2 + (a_3 + b_3 x + c_3 y)T_3]$类比之后，得到位移差值

$$\begin{cases} u(x,y,z) = N_1^{(e)}(x,y,z)u_1 + N_2^{(e)}(x,y,z)u_2 + N_3^{(e)}(x,y,z)u_3 + N_4^{(e)}(x,y,z)u_4 \\ v(x,y,z) = N_1^{(e)}(x,y,z)v_1 + N_2^{(e)}(x,y,z)v_2 + N_3^{(e)}(x,y,z)v_3 + N_4^{(e)}(x,y,z)v_4 \\ w(x,y,z) = N_1^{(e)}(x,y,z)w_1 + N_2^{(e)}(x,y,z)w_2 + N_3^{(e)}(x,y,z)w_3 + N_4^{(e)}(x,y,z)w_4 \end{cases}$$

$$
\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix}
=
\begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial w}{\partial z} \\ \dfrac{\partial u}{\partial y}+\dfrac{\partial x}{\partial x} \\ \dfrac{\partial v}{\partial z}+\dfrac{\partial w}{\partial y} \\ \dfrac{\partial w}{\partial z}+\dfrac{\partial u}{\partial z} \end{Bmatrix}
=
\begin{bmatrix}
N_{1,x} & 0 & 0 & * & 0 & 0 & * & 0 & 0 & * & 0 & 0 \\
0 & N_{1,y} & 0 & 0 & * & 0 & 0 & * & 0 & 0 & * & 0 \\
0 & 0 & N_{1,z} & 0 & 0 & * & 0 & 0 & * & 0 & 0 & * \\
N_{N_{1,y}} & N_{N_{1,x}} & 0 & * & * & 0 & * & * & 0 & * & * & 0 \\
0 & N_{N_{1,z}} & N_{N_{1,y}} & 0 & * & * & 0 & * & * & 0 & * & * \\
N_{N_{1,z}} & 0 & N_{N_{1,x}} & * & 0 & * & * & 0 & * & * & 0 & *
\end{bmatrix}
\begin{bmatrix} u_1 \\ v_1 \\ w_1 \\ u_2 \\ v_2 \\ w_2 \\ u_3 \\ v_3 \\ w_3 \\ u_4 \\ v_4 \\ w_4 \end{bmatrix}
= [B][q]
$$

$$-\int_\Omega w_{i,j}\sigma_{ij}d\Omega - \int_{\Gamma_3} kw_iu_id\Gamma + \int_\Omega w_{i,j}f_id\Omega + \int_{\Gamma_2} w_i\overline{F_i}d\Gamma + \int_{\Gamma_3} kw_iu_{i0}d\Gamma = 0$$

$$\delta u_i = w_i$$

$$\delta\varepsilon_{ij} = \frac{1}{2}(w_{i,j} + w_{j,i})$$

$$-\int_\Omega \delta\varepsilon_{ij}d\Omega - \int_{\Gamma_3} k\delta u_iu_id\Gamma + \int_\Omega \delta u_if_id\Omega + \int_{\Gamma_2} \delta u_i\overline{F_i}d\Gamma + \int_{\Gamma_3} k\delta u_iu_{i0}d\Gamma = 0$$

$$\int_\Omega \{\delta\varepsilon\}^T\{\sigma\}\,d\Omega - \int_\Omega \{\delta u\}^T\{f\}\,d\Omega = 0$$