有限元方法3

加权残数法

$$\begin{cases} u''(x) + u(x) + x = 0, \quad (0 < x < 1) \\ u(0) = 0 \\ u'(1) = 0 \end{cases}$$

$$\tilde{u}(x) = x(-2a_1 - 3a_2 + a_1 x + a_2 x^2)$$

$$\int_0^1 w(x)\Delta(\tilde{u})\mathrm{d}x = 0$$

1. 在二维和三维问题中，在复杂区域(多连通、曲线/曲面边界)内构造具有连续高阶导数的近似函数是非常困难的。
2. 让近似函数精确满足第二类边界条件也是难以实现的。

加权残数法——第二类边界条件的等价

针对近似函数精确满足第二类边界条件存在困难：

$$\begin{cases} u''(x)+u(x)+x=0, \quad (0<x<1) \\ u'(1)=0 \end{cases} \quad \overset{\text{等价}}{\Longleftrightarrow} \quad \begin{cases} \int_0^1 w(u''+u+x)\mathrm{d}x=0 \\ \bar{w}u'(1)=0 \end{cases}$$

加权残数法——第二类边界条件的等价

$$
\begin{cases}
\int_0^1 w(u'' + u + x)\mathrm{d}x = 0 \\
\bar{w}u'(1) = 0
\end{cases}
$$

等价

$$
\int_0^1 w(u'' + u + x)\mathrm{d}x + \bar{w}u'(1) = 0
$$

加权残数法——第二类边界条件的等价

$$\begin{cases} u''(x)+u(x)+x=0, \quad (0<x<1) \\ u(0)=0 \\ u'(1)=0 \end{cases}$$

$$\begin{cases} \int_0^1 w(u''+u+x)\mathrm{d}x=0 \\ \bar{w}u'(1)=0 \end{cases}$$

$$\int_0^1 w(u''+u+x)\mathrm{d}x+\bar{w}u'(1)=0$$

加权残数法——第二类边界条件的等价

$$\int_0^1 w(u'' + u + x)\mathrm{d}x + \bar{w}u'(1) = 0$$

函数$w(x)$在区间(0,1)上是任意的;

常数$\bar{w}$是任意的。

$w(0)$和$w(1)$没有约束

加权残数法——分步积分后的形式

$$\int_0^1 w(u''+u+x)\mathrm{d}x+\bar{w}u'(1)$$

$$=\int_0^1 wu''\mathrm{d}x+\int_0^1 w(u+x)\mathrm{d}x+\bar{w}u'(1)$$

$$=\int_0^1 [(wu')'-w'u']\mathrm{d}x+\int_0^1 w(u+x)\mathrm{d}x+\bar{w}u'(1)$$

$$=u'(1)w(1)-u'(0)w(0)+\int_0^1 (-u'w'+uw+xw)\mathrm{d}x+\bar{w}u'(1)$$

$$=\int_0^1 (-u'w'+uw+xw)\mathrm{d}x$$

$$\left(\begin{array}{l} w(0)\text{和}w(1)\text{没有约束, 可令} \\ w(0)=0,\ \ w(1)=-\bar{w}(\text{任意}) \end{array}\right)$$

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加权残数法——分步积分后的形式

$$\begin{cases} u''(x)+u(x)+x=0, \quad (0<x<1) \\ u(0)=0 \\ u'(1)=0 \end{cases}$$

等价

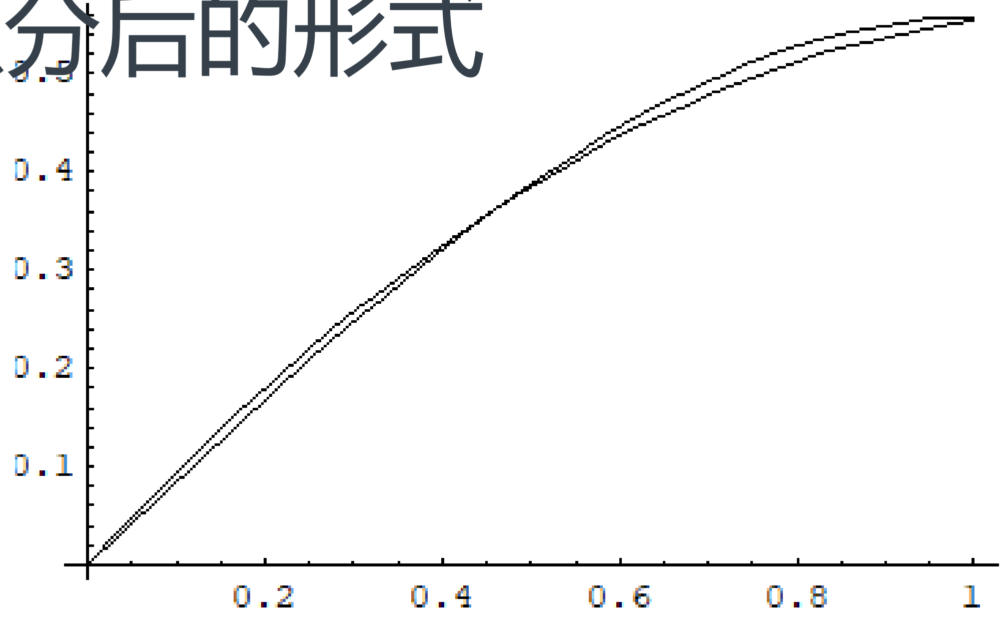
$$\begin{cases} \int_0^1 (-u'w'+uw+xw)\mathrm{d}x=0 \\ u(0)=0 \end{cases}$$

其中$w(x)$在$(0,1]$任意，

且$w(0)=0$

加权残数法——分步积分后的形式

$$\begin{cases} \int_0^1 (-u'w' + uw + xw)\mathrm{d}x = 0 \\ u(0) = 0 \end{cases}$$



$$u(x) = x(a_0 + a_1 x)$$

$$w(x) = x: \quad \int_0^1 (-u'w' + uw + xw)\mathrm{d}x = \frac{1}{12}(4 - 8a_0 - 9a_1)$$

$$w(x) = x^2: \quad \int_0^1 (-u'w' + uw + xw)\mathrm{d}x = \frac{1}{5} - \frac{4a_0}{5} - \frac{4a_1}{3}$$

$$a_0 = \frac{137}{139}, a_1 = -\frac{60}{139}, \ u(x) = \frac{x(137 - 60x)}{139}$$

加权残数法——伽辽金方法

$$\begin{cases} \int_0^1 (-u'w' + uw + xw)\mathrm{d}x = 0 \\ u(0) = 0 \end{cases}$$

$$u(x) = x(a_0 + a_1 x)$$
$$w_1(x) = x$$
$$w_2(x) = x^2$$

更为一般情况

$$u(x) = u_1 N_1(x) + u_2 N_2(x)$$
$$w_1(x) = N_1(x)$$
$$w_2(x) = N_2(x)$$

加权残数法——伽辽金方法(分段函数)

$$\int_0^1 (-u'w' + uw + xw)\mathrm{d}x = \int_0^{\frac{1}{2}} (-u'w' + uw + xw)\mathrm{d}x + \int_{\frac{1}{2}}^1 (-u'w' + uw + xw)\mathrm{d}x$$

$$w(x) = N_1(x)$$

$$(0 < x < \frac{1}{2}): \quad w(x) = \frac{x}{1/2},\ u(x) = u_1 \frac{x}{1/2}, \qquad I_{11} = \int_0^{\frac{1}{2}} (\bullet)\mathrm{d}x = \frac{1 - 22u_1}{12}$$

$$(\frac{1}{2} < x < 1): \quad w(x) = \frac{1-x}{1/2},\ u(x) = u_1 \frac{1-x}{1/2} + u_2 \frac{x - 1/2}{1/2}, \qquad I_{12} = \int_{\frac{1}{2}}^1 (\bullet)\mathrm{d}x = \frac{2 - 22u_1 + 25u_2}{12}$$

$$w(x) = N_2(x)$$

$$(0 < x < \frac{1}{2}): \quad w(x) = 0,\ \cdots, \qquad I_{21} = \int_0^{\frac{1}{2}} (\bullet)\mathrm{d}x = 0$$

$$(\frac{1}{2} < x < 1): \quad w(x) = \frac{x - 1/2}{1/2},\ u(x) = u_1 \frac{1-x}{1/2} + u_2 \frac{x - 1/2}{1/2}, \qquad I_{22} = \int_{\frac{1}{2}}^1 (\bullet)\mathrm{d}x = \frac{5 + 50u_1 - 44u_2}{24}$$

加权残数法——伽辽金方法(分段函数)

$$\int_0^1(-u'w'+uw+xw)\mathrm{d}x=\int_0^{\frac{1}{2}}(-u'w'+uw+xw)\mathrm{d}x+\int_{\frac{1}{2}}^1(-u'w'+uw+xw)\mathrm{d}x$$

$$w(x)=N_1(x):\qquad I_1=I_{11}+I_{12}=\frac{3-44u_1+25u_2}{12}=0$$

$$w(x)=N_2(x):\qquad I_2=I_{21}+I_{22}=\frac{5+50u_1-44u_2}{24}=0$$

$$\Rightarrow\quad u_1=\frac{257}{686},\quad u_2=\frac{185}{343}$$

	加权残数	$\dfrac{\sin x}{\cos 1}-x$
u_1	0.3746	0.3873
u_2	0.5394	0.5574

利用分段线性插值的伽辽金法

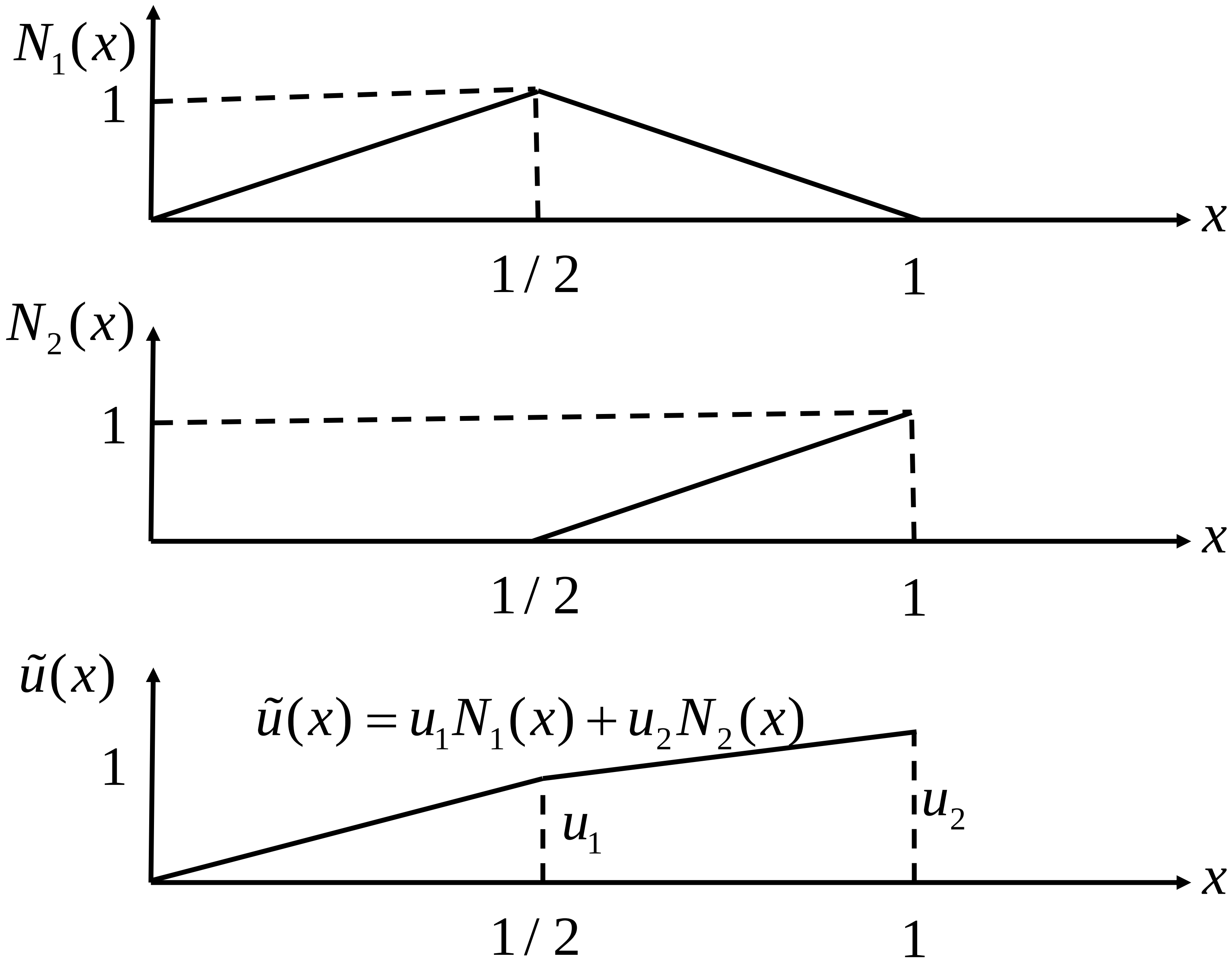
$$\int_0^1 (-u'w' + uw + xw)\mathrm{d}x = 0$$

$$u(x) = u_1 N_1(x) + u_2 N_2(x) \ldots\ldots\ldots\ldots\ldots\ldots (u_1, u_2\text{待定})$$

$$w(x) = w_1 N_1(x) + w_2 N_2(x) \ldots\ldots\ldots\ldots\ldots\ldots (w_1, w_2\text{任意})$$

$$\int_0^1 (-\tilde{u}' N_1' + \tilde{u} N_1 + x N_1)\mathrm{d}x = 0$$

利用分段线性插值的伽辽金法



利用分段线性插值的伽辽金法

$$\int_0^1 (-\tilde{u}' N_1' + \tilde{u} N_1 + x N_1) \mathrm{d}x = 0$$

$$\int_0^1 (-\tilde{u}' N_2' + \tilde{u} N_2 + x N_2) \mathrm{d}x = 0$$

$$\Downarrow$$

$$\int_0^1 (-\tilde{u}'(w_1 N_1' + w_2 N_2') + \tilde{u}(w_1 N_1 + w_2 N_2) + x(w_1 N_1 + w_2 N_2)) \mathrm{d}x = 0$$

$$\Downarrow$$

$$\int_0^1 (-\tilde{u}' \tilde{w}' + \tilde{u} \tilde{w} + x \tilde{w}) \mathrm{d}x = 0$$

利用分段线性插值的加权残数法

$$\int_0^1 (-\tilde{u}' N_1' + \tilde{u} N_1 + x N_1)\mathrm{d}x = 0$$

$$\int_0^1 (-\tilde{u}' N_2' + \tilde{u} N_2 + x N_2)\mathrm{d}x = 0$$

$$\Updownarrow$$

$$\int_0^1 (-\tilde{u}' \tilde{w}' + \tilde{u} \tilde{w} + x \tilde{w})\mathrm{d}x = 0$$

$$\Downarrow$$

$$\sum_e \int_e (-\tilde{u}' \tilde{w}' + \tilde{u} \tilde{w} + x \tilde{w})\mathrm{d}x = 0$$

利用分段线性插值的加权残数法

$$\sum_e \int_e (-\tilde{u}'\tilde{w}' + \tilde{u}\tilde{w} + x\tilde{w})\mathrm{d}x = 0$$

利用分段线性插值的加权残数法

$$\sum_e \int_e (-\tilde{u}'\tilde{w}' + \tilde{u}\tilde{w} + x\tilde{w})\mathrm{d}x = 0$$

$$\tilde{u} = u_1^{(e)} N_1^{(e)}(x) + u_2^{(e)} N_2^{(e)}(x)$$

$$\tilde{w} = w_1^{(e)} N_1^{(e)}(x) + w_2^{(e)} N_2^{(e)}(x)$$

$$N_1^{(e)}(x) = \frac{x - x_1^{(e)}}{x_2^{(e)} - x_1^{(e)}}, \quad N_2^{(e)}(x) = \frac{x_2^{(e)} - x}{x_2^{(e)} - x_1^{(e)}}$$

利用分段线性插值的加权残数法

$$\sum_e \int_e (-\tilde{u}'\tilde{w}' + \tilde{u}\tilde{w} + x\tilde{w})\mathrm{d}x = 0$$

$$\tilde{u} = u_1^{(e)} N_1^{(e)}(x) + u_2^{(e)} N_2^{(e)}(x)$$

$$= \begin{bmatrix} N_1^{(e)}(x) & N_2^{(e)}(x) \end{bmatrix} \begin{Bmatrix} u_1^{(e)} \\ u_2^{(e)} \end{Bmatrix} = \begin{bmatrix} u_1^{(e)} & u_2^{(e)} \end{bmatrix} \begin{Bmatrix} N_1^{(e)}(x) \\ N_2^{(e)}(x) \end{Bmatrix}$$

$$\tilde{w} = w_1^{(e)} N_1^{(e)}(x) + w_2^{(e)} N_2^{(e)}(x)$$

$$= \begin{bmatrix} N_1^{(e)}(x) & N_2^{(e)}(x) \end{bmatrix} \begin{Bmatrix} w_1^{(e)} \\ w_2^{(e)} \end{Bmatrix} = \begin{bmatrix} w_1^{(e)} & w_2^{(e)} \end{bmatrix} \begin{Bmatrix} N_1^{(e)}(x) \\ N_2^{(e)}(x) \end{Bmatrix}$$

利用分段线性插值的加权残数法

$$\sum_e \int_e (-\tilde{u}'\tilde{w}' + \tilde{u}\tilde{w} + x\tilde{w})\mathrm{d}x = 0$$

$$\tilde{u} = \begin{bmatrix} N_1^{(e)}(x) & N_2^{(e)}(x) \end{bmatrix} \begin{Bmatrix} u_1^{(e)} \\ u_2^{(e)} \end{Bmatrix}$$

$$\tilde{w} = \begin{bmatrix} w_1^{(e)} & w_2^{(e)} \end{bmatrix} \begin{Bmatrix} N_1^{(e)}(x) \\ N_2^{(e)}(x) \end{Bmatrix}$$

$$\tilde{u}'\tilde{w}' = \begin{bmatrix} w_1^{(e)} & w_2^{(e)} \end{bmatrix} \begin{Bmatrix} N_1'^{(e)}(x) \\ N_2'^{(e)}(x) \end{Bmatrix} \begin{bmatrix} N_1'^{(e)}(x) & N_2'^{(e)}(x) \end{bmatrix} \begin{Bmatrix} u_1^{(e)} \\ u_2^{(e)} \end{Bmatrix}$$

$$\tilde{u}\tilde{w} = \begin{bmatrix} w_1^{(e)} & w_2^{(e)} \end{bmatrix} \begin{Bmatrix} N_1^{(e)}(x) \\ N_2^{(e)}(x) \end{Bmatrix} \begin{bmatrix} N_1^{(e)}(x) & N_2^{(e)}(x) \end{bmatrix} \begin{Bmatrix} u_1^{(e)} \\ u_2^{(e)} \end{Bmatrix}$$

$$x\tilde{w} = \begin{bmatrix} w_1^{(e)} & w_2^{(e)} \end{bmatrix} \begin{Bmatrix} N_1^{(e)}(x) \\ N_2^{(e)}(x) \end{Bmatrix} x$$

利用分段线性插值的加权残数法

$$\sum_e \int_e (-\tilde{u}'\tilde{w}' + \tilde{u}\tilde{w} + x\tilde{w})\mathrm{d}x = 0$$

$$\int_e \tilde{u}'\tilde{w}'\mathrm{d}x = \int_e \begin{bmatrix} w_1^{(e)} & w_2^{(e)} \end{bmatrix} \begin{Bmatrix} N_1'^{(e)}(x) \\ N_2'^{(e)}(x) \end{Bmatrix} \begin{bmatrix} N_1'^{(e)}(x) & N_2'^{(e)}(x) \end{bmatrix} \begin{Bmatrix} u_1^{(e)} \\ u_2^{(e)} \end{Bmatrix} \mathrm{d}x$$

$$= \begin{bmatrix} w_1^{(e)} & w_2^{(e)} \end{bmatrix} \left(\int_e \begin{Bmatrix} N_1'^{(e)}(x) \\ N_2'^{(e)}(x) \end{Bmatrix} \begin{bmatrix} N_1'^{(e)}(x) & N_2'^{(e)}(x) \end{bmatrix} \mathrm{d}x \right) \begin{Bmatrix} u_1^{(e)} \\ u_2^{(e)} \end{Bmatrix}$$

$$\int_e \tilde{u}\tilde{w}\mathrm{d}x = \begin{bmatrix} w_1^{(e)} & w_2^{(e)} \end{bmatrix} \int_e \begin{Bmatrix} N_1^{(e)}(x) \\ N_2^{(e)}(x) \end{Bmatrix} \begin{bmatrix} N_1^{(e)}(x) & N_2^{(e)}(x) \end{bmatrix} \mathrm{d}x \begin{Bmatrix} u_1^{(e)} \\ u_2^{(e)} \end{Bmatrix}$$

$$= \begin{bmatrix} w_1^{(e)} & w_2^{(e)} \end{bmatrix} \left(\int_e \begin{Bmatrix} N_1^{(e)}(x) \\ N_2^{(e)}(x) \end{Bmatrix} \begin{bmatrix} N_1^{(e)}(x) & N_2^{(e)}(x) \end{bmatrix} \mathrm{d}x \right) \begin{Bmatrix} u_1^{(e)} \\ u_2^{(e)} \end{Bmatrix}$$

$$\int_e x\tilde{w}\mathrm{d}x = \int_e \begin{bmatrix} w_1^{(e)} & w_2^{(e)} \end{bmatrix} \begin{Bmatrix} N_1^{(e)}(x) \\ N_2^{(e)}(x) \end{Bmatrix} x\mathrm{d}x = \begin{bmatrix} w_1^{(e)} & w_2^{(e)} \end{bmatrix} \int_e \begin{Bmatrix} N_1^{(e)}(x) \\ N_2^{(e)}(x) \end{Bmatrix} x\mathrm{d}x$$

利用分段线性插值的加权残数法

$u_0 = w_0 = 0$ $\quad u_1, w_1 \quad u_2, w_2$

$x = 0 \quad e = 1 \quad x = \frac{1}{2} \quad e = 2 \quad x = 1$

$e = 1$

$$w_1^{(e)} = w_0,\ w_2^{(e)} = w_1,\ u_1^{(e)} = 0,\ u_2^{(e)} = u_1,\ x_1^{(e)} = 0,\ x_2^{(e)} = \frac{1}{2}$$

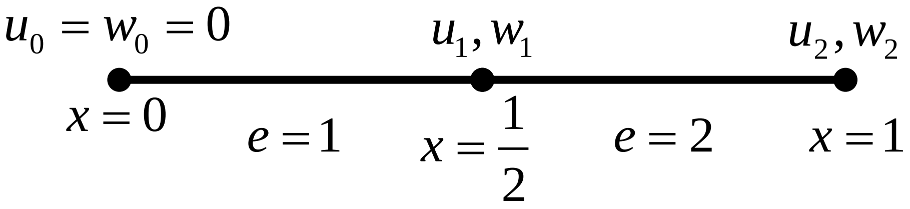
$$N_1^{(e)}(x) = 1 - 2x,\ N_2^{(e)}(x) = 2x,\ N_1'^{(e)}(x) = -2,\ N_2'^{(e)}(x) = 2$$

$$\begin{bmatrix} w_1^{(e)} & w_2^{(e)} \end{bmatrix} \left(\int_e \begin{Bmatrix} N_1'^{(e)}(x) \\ N_2'^{(e)}(x) \end{Bmatrix} \begin{bmatrix} N_1'^{(e)}(x) & N_2'^{(e)}(x) \end{bmatrix} \mathrm{d}x \right) \begin{Bmatrix} u_1^{(e)} \\ u_2^{(e)} \end{Bmatrix} = \begin{bmatrix} 0 & w_1 \end{bmatrix} \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \begin{Bmatrix} 0 \\ u_1 \end{Bmatrix} = w_1 [2] u_1$$

$$\begin{bmatrix} w_1^{(e)} & w_2^{(e)} \end{bmatrix} \left(\int_e \begin{Bmatrix} N_1^{(e)}(x) \\ N_2^{(e)}(x) \end{Bmatrix} \begin{bmatrix} N_1^{(e)}(x) & N_2^{(e)}(x) \end{bmatrix} \mathrm{d}x \right) \begin{Bmatrix} u_1^{(e)} \\ u_2^{(e)} \end{Bmatrix} = \begin{bmatrix} 0 & w_1 \end{bmatrix} \begin{bmatrix} \frac{1}{6} & \frac{1}{12} \\ \frac{1}{12} & \frac{1}{6} \end{bmatrix} \begin{Bmatrix} 0 \\ u_1 \end{Bmatrix} = w_1 \left[\frac{1}{6} \right] u_1$$

$$\begin{bmatrix} w_1^{(e)} & w_2^{(e)} \end{bmatrix} \left\{ \int_e \begin{Bmatrix} N_1^{(e)}(x) \\ N_2^{(e)}(x) \end{Bmatrix} x \mathrm{d}x \right\} = \begin{bmatrix} 0 & w_1 \end{bmatrix} \begin{Bmatrix} \frac{1}{24} \\ \frac{1}{12} \end{Bmatrix} = w_1 \left\{ \frac{1}{12} \right\}$$

利用分段线性插值的加权残数法



$$e = 2$$

$$w_1^{(e)} = w_1,\ \ w_2^{(e)} = w_2,\ \ u_1^{(e)} = u_1,\ \ u_2^{(e)} = u_2,\ \ x_1^{(e)} = \frac{1}{2},\ \ x_2^{(e)} = 1$$

$$N_1^{(e)}(x) = 2(1-x),\ \ N_2^{(e)}(x) = 2x-1,\ \ N_1'^{(e)}(x) = -2,\ \ N_2'^{(e)}(x) = 2$$

$$\begin{bmatrix} w_1^{(e)} & w_2^{(e)} \end{bmatrix} \left(\int_e \begin{Bmatrix} N_1'^{(e)}(x) \\ N_2'^{(e)}(x) \end{Bmatrix} \begin{bmatrix} N_1'^{(e)}(x) & N_2'^{(e)}(x) \end{bmatrix} \mathrm{d}x \right) \begin{Bmatrix} u_1^{(e)} \\ u_2^{(e)} \end{Bmatrix} = \begin{bmatrix} w_1 & w_2 \end{bmatrix} \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$\begin{bmatrix} w_1^{(e)} & w_2^{(e)} \end{bmatrix} \left(\int_e \begin{Bmatrix} N_1^{(e)}(x) \\ N_2^{(e)}(x) \end{Bmatrix} \begin{bmatrix} N_1^{(e)}(x) & N_2^{(e)}(x) \end{bmatrix} \mathrm{d}x \right) \begin{Bmatrix} u_1^{(e)} \\ u_2^{(e)} \end{Bmatrix} = \begin{bmatrix} w_1 & w_2 \end{bmatrix} \begin{bmatrix} \frac{1}{6} & \frac{1}{12} \\ \frac{1}{12} & \frac{1}{6} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$\begin{bmatrix} w_1^{(e)} & w_2^{(e)} \end{bmatrix} \left\{ \int_e \begin{Bmatrix} N_1^{(e)}(x) \\ N_2^{(e)}(x) \end{Bmatrix} x \mathrm{d}x \right\} = \begin{bmatrix} w_1 & w_2 \end{bmatrix} \begin{Bmatrix} \frac{1}{6} \\ \frac{5}{24} \end{Bmatrix}$$

利用分段线性插值的加权残数法

$$\sum_e \int_e (-\tilde{u}'\tilde{w}' + \tilde{u}\tilde{w} + x\tilde{w})\mathrm{d}x$$

$$= w_1[-2]u_1 + w_1\left[\frac{1}{6}\right]u_1 + w_1\left\{\frac{1}{12}\right\}$$

$$+[w_1 \quad w_2]\begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} + [w_1 \quad w_2]\begin{bmatrix} \frac{1}{6} & \frac{1}{12} \\ \frac{1}{12} & \frac{1}{6} \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} + [w_1 \quad w_2]\begin{Bmatrix} \frac{1}{6} \\ \frac{5}{24} \end{Bmatrix}$$

$$= w_1\left[-\frac{11}{6}\right]u_1 + w_1\left\{\frac{1}{12}\right\} + [w_1 \quad w_2]\begin{bmatrix} -\frac{11}{6} & \frac{25}{12} \\ \frac{25}{12} & -\frac{11}{6} \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} + [w_1 \quad w_2]\begin{Bmatrix} \frac{1}{6} \\ \frac{5}{24} \end{Bmatrix}$$

$$= [w_1 \quad w_2]\begin{bmatrix} -\frac{11}{6} & 0 \\ 0 & 0 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} + [w_1 \quad w_2]\begin{bmatrix} -\frac{11}{6} & \frac{25}{12} \\ \frac{25}{12} & -\frac{11}{6} \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} + [w_1 \quad w_2]\begin{Bmatrix} \frac{1}{12} \\ 0 \end{Bmatrix} + [w_1 \quad w_2]\begin{Bmatrix} \frac{1}{6} \\ \frac{5}{24} \end{Bmatrix}$$

利用分段线性插值的加权残数法

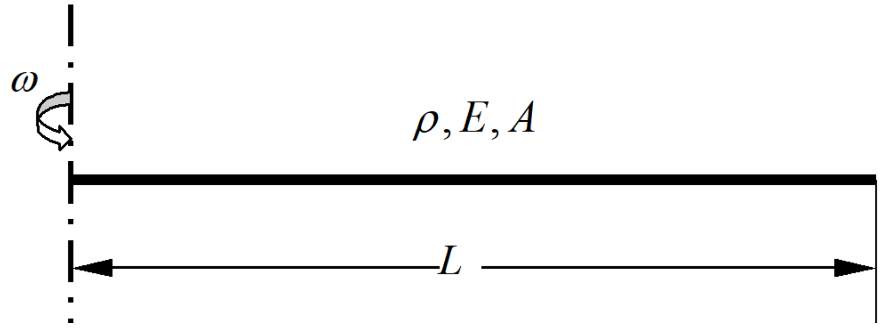
$$\int_0^1(-\tilde{u}'\tilde{w}'+\tilde{u}\tilde{w}+x\tilde{w})\mathrm{d}x=\begin{bmatrix}w_1 & w_2\end{bmatrix}\begin{bmatrix}-\dfrac{11}{3} & \dfrac{25}{12}\\ \dfrac{25}{12} & -\dfrac{11}{6}\end{bmatrix}\begin{Bmatrix}u_1\\ u_2\end{Bmatrix}+\begin{bmatrix}w_1 & w_2\end{bmatrix}\begin{Bmatrix}\dfrac{1}{4}\\ \dfrac{5}{24}\end{Bmatrix}$$

$$=\begin{bmatrix}w_1 & w_2\end{bmatrix}\left(\begin{bmatrix}-\dfrac{11}{3} & \dfrac{25}{12}\\ \dfrac{25}{12} & -\dfrac{11}{6}\end{bmatrix}\begin{Bmatrix}u_1\\ u_2\end{Bmatrix}+\begin{Bmatrix}\dfrac{1}{4}\\ \dfrac{5}{24}\end{Bmatrix}\right)=0$$

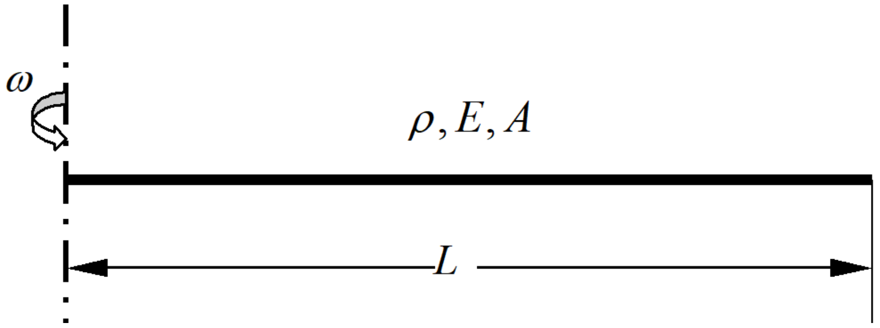
$$\begin{bmatrix}-\dfrac{11}{3} & \dfrac{25}{12}\\ \dfrac{25}{12} & -\dfrac{11}{6}\end{bmatrix}\begin{Bmatrix}u_1\\ u_2\end{Bmatrix}+\begin{Bmatrix}\dfrac{1}{4}\\ \dfrac{5}{24}\end{Bmatrix}=\begin{Bmatrix}0\\ 0\end{Bmatrix}$$

$$\begin{cases}I_1=\displaystyle\int_0^1(\bullet)\mathrm{d}x=\dfrac{3-44u_1+25u_2}{12}=0\\[2ex] I_2=\displaystyle\int_0^1(\bullet)\mathrm{d}x=\dfrac{5+50u_1-44u_2}{24}=0\end{cases}$$

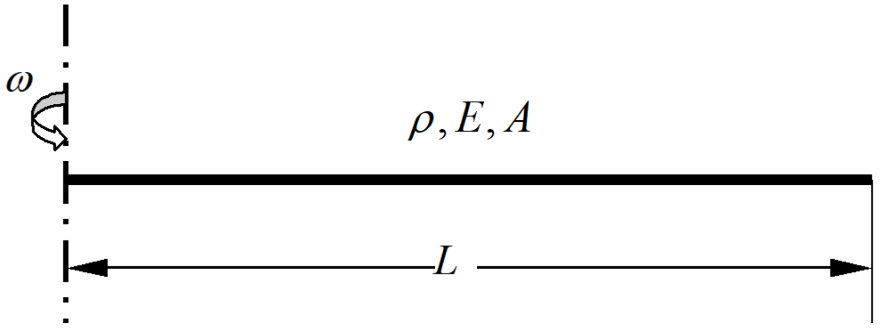
绕固定轴旋转的弹性直杆



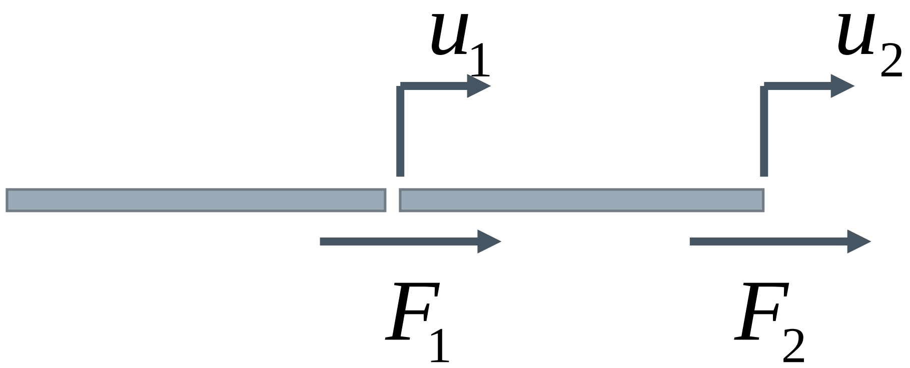
绕固定轴旋转的弹性直杆



绕固定轴旋转的弹性直杆

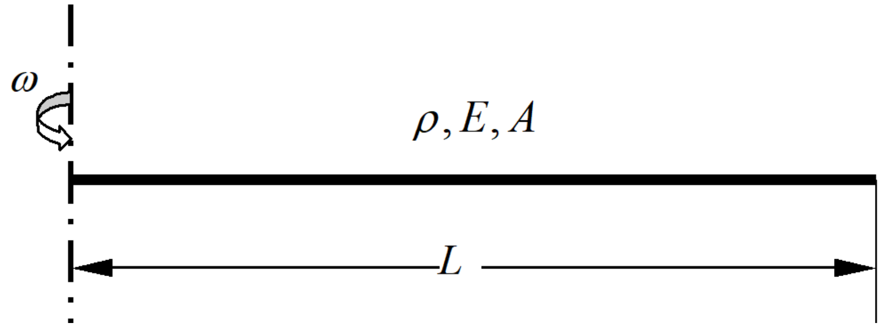


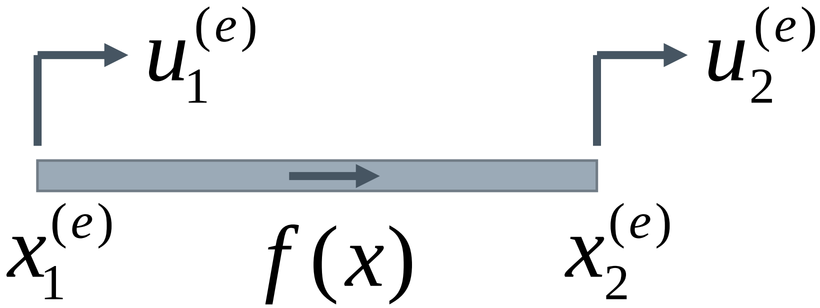
$$\begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$



$$(k = 2)$$

绕固定轴旋转的弹性直杆

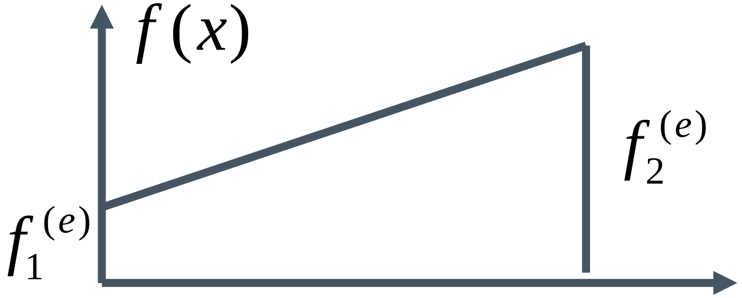




$$
\begin{aligned}
f(x) &= \rho\omega^2 r \\
&= x + u \\
&= x + u_1^{(e)}\frac{x_2^{(e)} - x}{1/2} + u_2^{(e)}\frac{x - x_1^{(e)}}{1/2}
\end{aligned}
$$

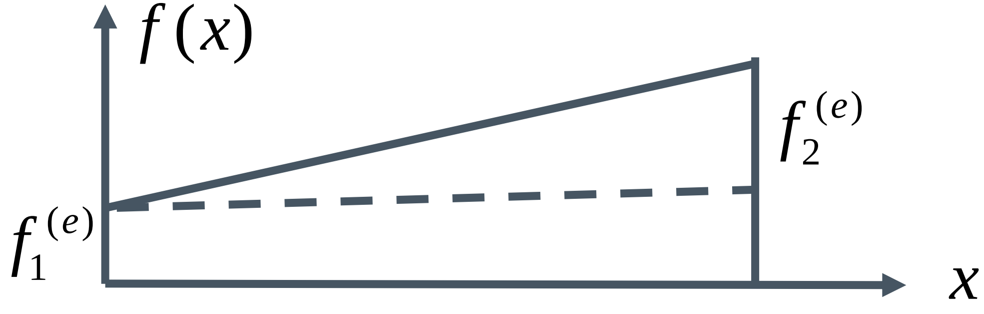
$$f_1^{(e)} = x_1^{(e)} + u_1^{(e)}$$

$$f_2^{(e)} = x_2^{(e)} + u_2^{(e)}$$



绕固定轴旋转的弹性直杆

$$f_1^{(e)} = x_1^{(e)} + u_1^{(e)}$$
$$f_2^{(e)} = x_2^{(e)} + u_2^{(e)}$$



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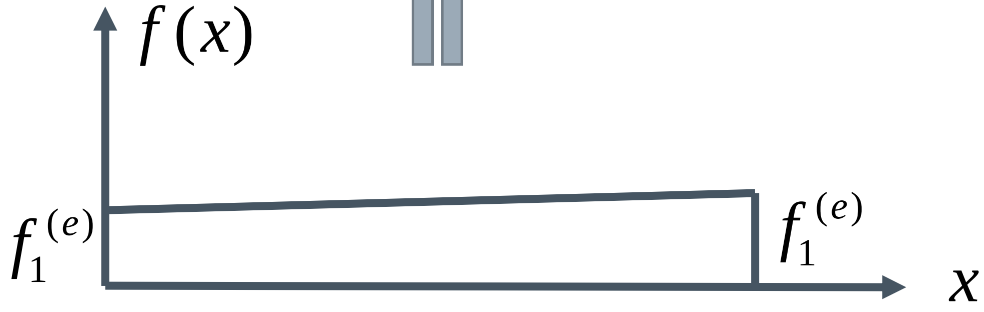
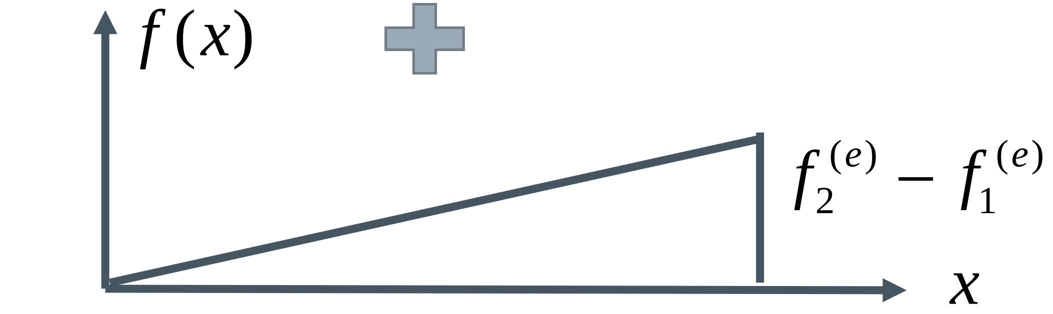
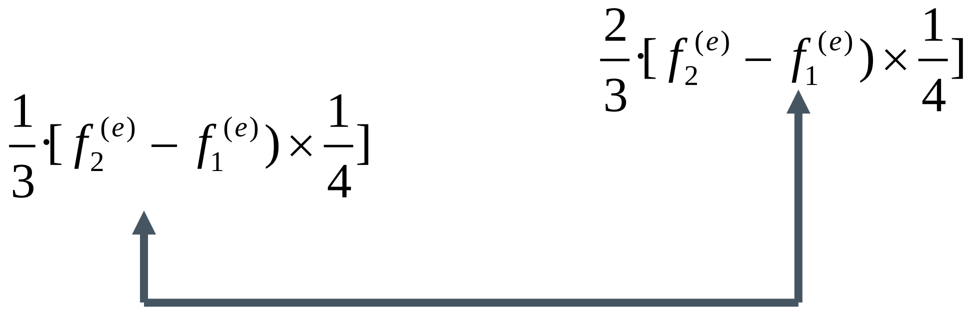


+





绕固定轴旋转的弹性直杆

$$f_1^{(e)} = x_1^{(e)} + u_1^{(e)}$$

$$f_2^{(e)} = x_2^{(e)} + u_2^{(e)}$$

$$F_1^{(1)} = \frac{2f_1^{(1)} + f_2^{(1)}}{12} = \frac{u_1}{12} + \frac{1}{24} \qquad F_1^{(2)} = \frac{2f_1^{(1)} + f_2^{(1)}}{12} = \frac{u_1}{6} + \frac{u_2}{12} + \frac{1}{6}$$

$$F_2^{(1)} = \frac{f_1^{(1)} + 2f_2^{(1)}}{12} = \frac{u_1}{6} + \frac{1}{12} \qquad F_2^{(2)} = \frac{f_1^{(1)} + 2f_2^{(1)}}{12} = \frac{u_1}{12} + \frac{u_2}{6} + \frac{5}{24}$$

$$\frac{2f_1^{(e)} + f_2^{(e)}}{12}、 \qquad \frac{f_1^{(e)} + 2f_2^{(e)}}{12}、$$

$$=$$

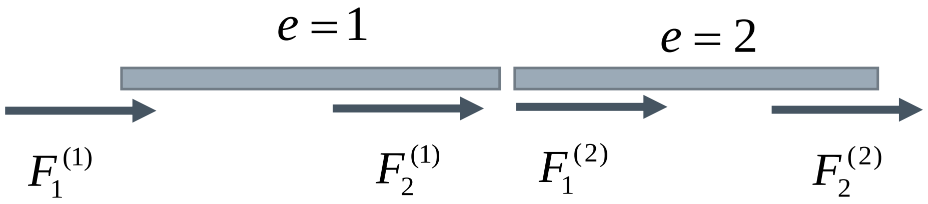
$$\frac{1}{2}(f_1^{(e)} \times \frac{1}{2}) \qquad \frac{1}{2}(f_1^{(e)} \times \frac{1}{2})$$

$$+$$

$$\frac{1}{3}\cdot[f_2^{(e)} - f_1^{(e)}) \times \frac{1}{4}] \qquad \frac{2}{3}\cdot[f_2^{(e)} - f_1^{(e)}) \times \frac{1}{4}]$$

绕固定轴旋转的弹性直杆

$$\frac{2f_1^{(e)}+f_2^{(e)}}{12}、\qquad \frac{f_1^{(e)}+2f_2^{(e)}}{12}、$$

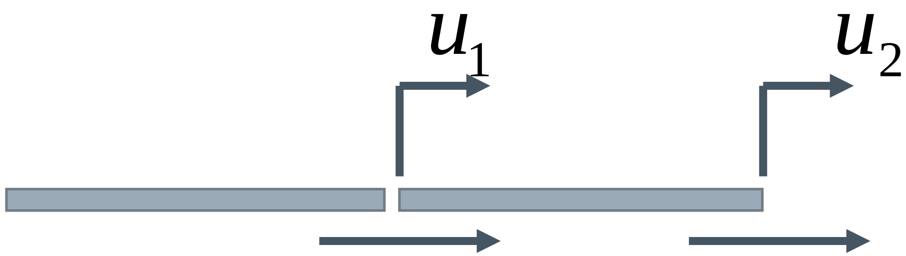


$$F_1^{(1)}=\frac{2f_1^{(1)}+f_2^{(1)}}{12}=\frac{u_1}{12}+\frac{1}{24}$$

$$F_2^{(1)}=\frac{f_1^{(1)}+2f_2^{(1)}}{12}=\frac{u_1}{6}+\frac{1}{12}$$

$$F_1^{(2)}=\frac{2f_1^{(2)}+f_2^{(2)}}{12}=\frac{u_1}{6}+\frac{u_2}{12}+\frac{1}{6}$$

$$F_2^{(2)}=\frac{f_1^{(2)}+2f_2^{(2)}}{12}=\frac{u_1}{12}+\frac{u_2}{6}+\frac{5}{24}$$

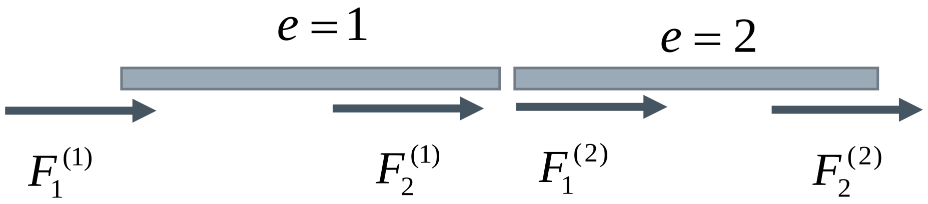


$$F_1=F_2^{(1)}+F_1^{(2)}=\frac{u_1}{3}+\frac{u_2}{12}+\frac{1}{4}$$

$$F_2=F_2^{(2)}=\frac{u_1}{12}+\frac{u_2}{6}+\frac{5}{24}$$

绕固定轴旋转的弹性直杆

$$\frac{2f_1^{(e)}+f_2^{(e)}}{12}、\qquad \frac{f_1^{(e)}+2f_2^{(e)}}{12}、$$

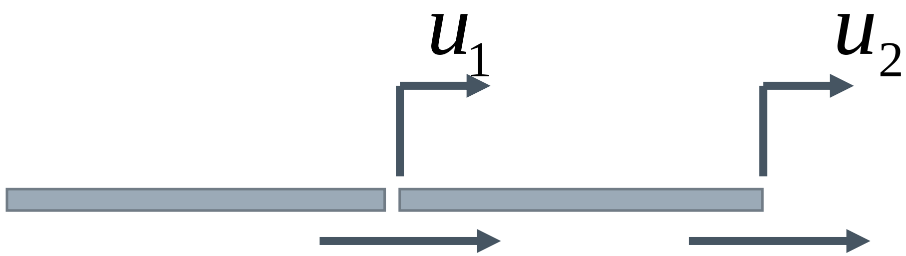


$$F_1^{(1)}=\frac{2f_1^{(1)}+f_2^{(1)}}{12}=\frac{u_1}{12}+\frac{1}{24}$$

$$F_2^{(1)}=\frac{f_1^{(1)}+2f_2^{(1)}}{12}=\frac{u_1}{6}+\frac{1}{12}$$

$$F_1^{(2)}=\frac{2f_1^{(2)}+f_2^{(2)}}{12}=\frac{u_1}{6}+\frac{u_2}{12}+\frac{1}{6}$$

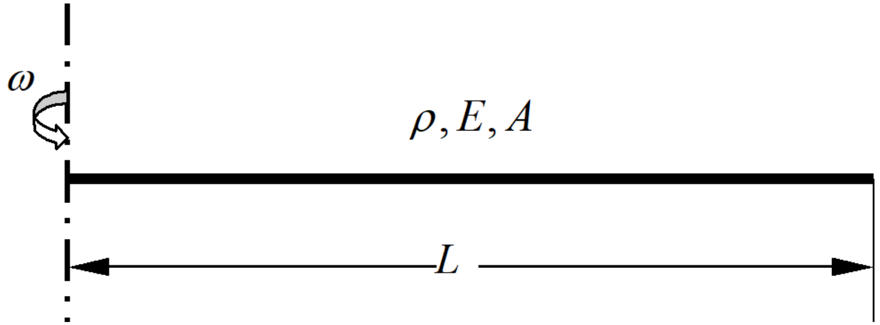
$$F_2^{(2)}=\frac{f_1^{(2)}+2f_2^{(2)}}{12}=\frac{u_1}{12}+\frac{u_2}{6}+\frac{5}{24}$$



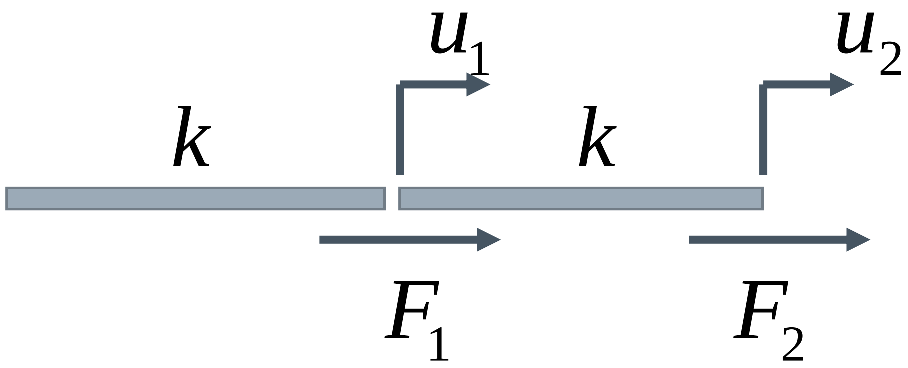
$$F_1=F_2^{(1)}+F_1^{(2)}=\frac{u_1}{3}+\frac{u_2}{12}+\frac{1}{4}$$

$$F_2=F_2^{(2)}=\frac{u_1}{12}+\frac{u_2}{6}+\frac{5}{24}$$

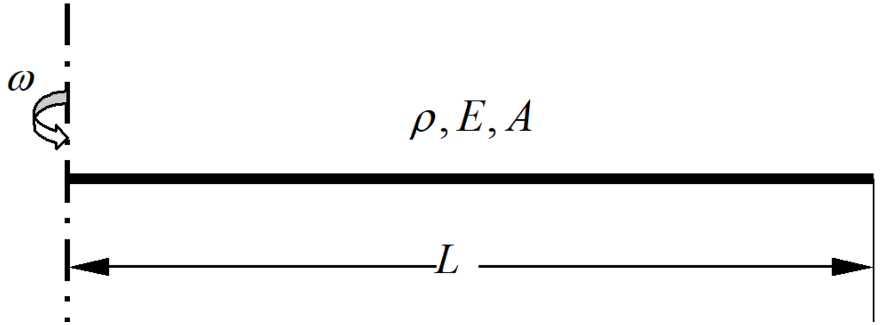
绕固定轴旋转的弹性直杆

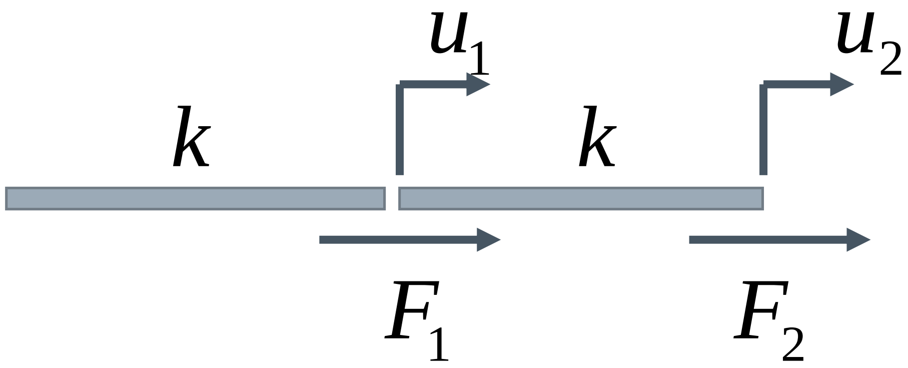


$$\begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \begin{Bmatrix} \dfrac{u_1}{3} + \dfrac{u_2}{12} + \dfrac{1}{4} \\ \dfrac{u_1}{12} + \dfrac{u_2}{6} + \dfrac{5}{24} \end{Bmatrix}$$



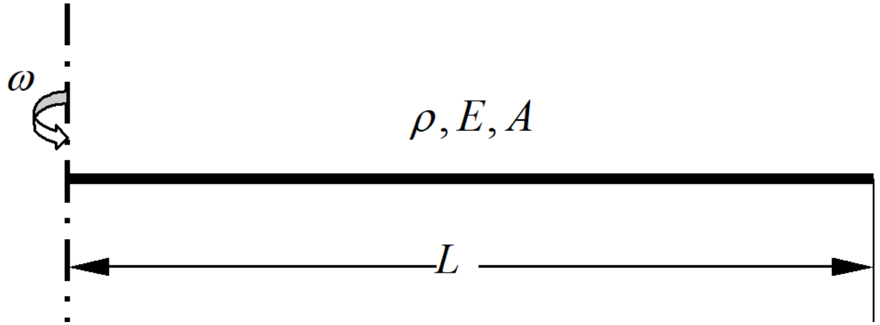
绕固定轴旋转的弹性直杆

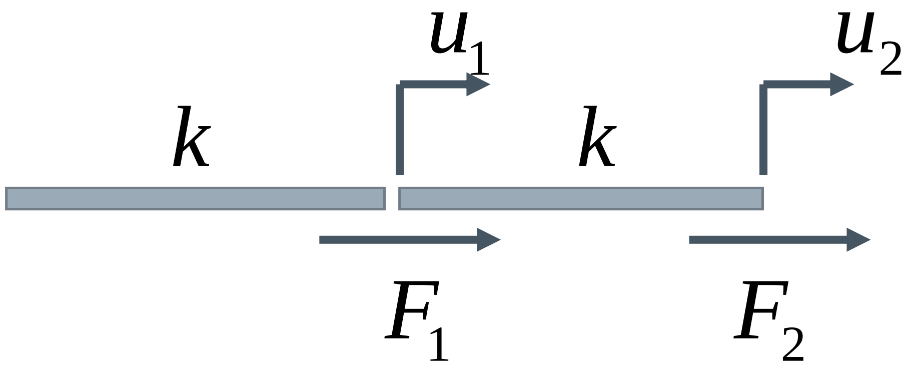




$$\begin{bmatrix} \frac{11}{3} & -\frac{25}{12} \\ -\frac{25}{12} & \frac{11}{6} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} \frac{1}{4} \\ \frac{5}{24} \end{Bmatrix}$$

绕固定轴旋转的弹性直杆





$$\begin{bmatrix} \frac{11}{3} & -\frac{25}{12} \\ -\frac{25}{12} & \frac{11}{6} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} \frac{1}{4} \\ \frac{5}{24} \end{Bmatrix}$$

Fudan University

Department of Aeronautics and Astronautics