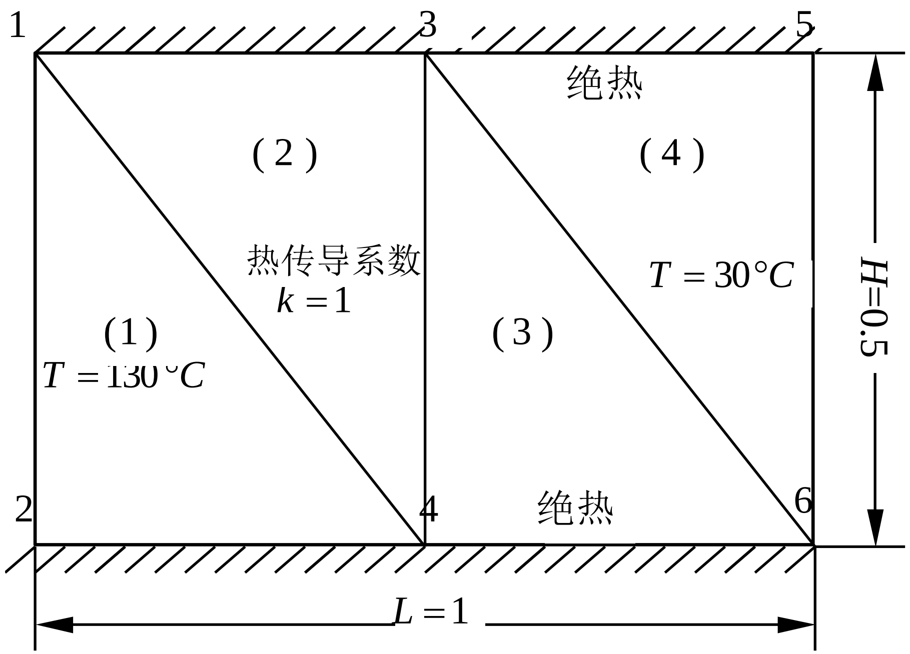
热传导问题有限元计算——初步实现



单元	节点 1	节点 2	节点 3
(1)	1	2	4
(2)	1	4	3
(3)	3	4	6
(4)	3	6	5

$$\frac{k}{4\Delta}\begin{bmatrix} b_1b_1+c_1c_1 & b_1b_2+c_1c_2 & b_1b_3+c_1c_3 \\ b_2b_1+c_2c_1 & b_2b_2+c_2c_2 & b_2b_3+c_2c_3 \\ b_3b_1+c_3c_1 & b_3b_2+c_3c_2 & b_3b_3+c_3c_3 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix}=\begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{Bmatrix}$$

三角形热传导单元

$$\mathbf{K}_e=\frac{k}{4\Delta}\begin{bmatrix} b_1b_1+c_1c_1 & b_1b_2+c_1c_2 & b_1b_3+c_1c_3 \\ b_2b_1+c_2c_1 & b_2b_2+c_2c_2 & b_2b_3+c_2c_3 \\ b_3b_1+c_3c_1 & b_3b_2+c_3c_2 & b_3b_3+c_3c_3 \end{bmatrix}=\frac{k}{4\Delta}\left(\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}\begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix}+\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}\begin{bmatrix} c_1 & c_2 & c_3 \end{bmatrix}\right)$$

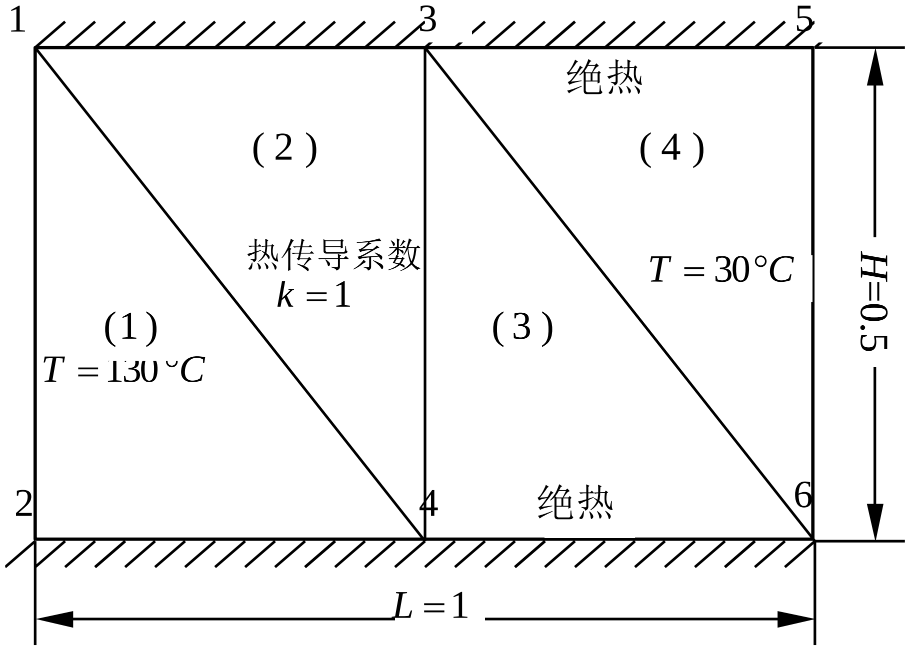
$$\begin{cases} b_1=y_2-y_3 & c_1=x_3-x_2 \\ b_2=y_3-y_1 & c_2=x_1-x_3 \\ b_3=y_1-y_2 & c_3=x_2-x_1 \end{cases}$$

$$\Delta=\frac{1}{2}\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$$

输入参数：$k,x_1,x_2,x_3,y_1,y_2,y_3$

输出参数：$\mathbf{K}_e$

三角形热传导单元



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三角形热传导单元

$$\mathbf{K}_e = \frac{k}{4\Delta}\left(\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix} + \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \begin{bmatrix} c_1 & c_2 & c_3 \end{bmatrix} \right)$$

$$\begin{cases} b_1 = y_2 - y_3 & c_1 = x_3 - x_2 \\ b_2 = y_3 - y_1 & c_2 = x_1 - x_3 \\ b_3 = y_1 - y_2 & c_3 = x_2 - x_1 \end{cases}$$

$$\Delta = \frac{1}{2}\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$$

三角形热传导单元

$$\begin{bmatrix} k_{11}^{(e)} & k_{12}^{(e)} & k_{13}^{(e)} \\ k_{21}^{(e)} & k_{22}^{(e)} & k_{23}^{(e)} \\ k_{31}^{(e)} & k_{32}^{(e)} & k_{33}^{(e)} \end{bmatrix} \begin{Bmatrix} T_1^{(e)} \\ T_2^{(e)} \\ T_3^{(e)} \end{Bmatrix} = \begin{Bmatrix} Q_1^{(e)} \\ Q_2^{(e)} \\ Q_3^{(e)} \end{Bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} k_{11}^{(1)} & k_{12}^{(1)} & 0 & k_{13}^{(1)} & 0 & 0 \\ k_{21}^{(1)} & k_{22}^{(1)} & 0 & k_{23}^{(1)} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ k_{31}^{(1)} & k_{32}^{(1)} & 0 & k_{33}^{(1)} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \end{Bmatrix} = \begin{Bmatrix} Q_1^{(1)} \\ Q_2^{(1)} \\ 0 \\ Q_3^{(1)} \\ 0 \\ 0 \end{Bmatrix}$$

三角形热传导单元

$$\begin{bmatrix} k_{33} & k_{34} \\ k_{43} & k_{44} \end{bmatrix} \begin{Bmatrix} T_3 \\ T_4 \end{Bmatrix} = \begin{Bmatrix} -k_{31}T_1 - k_{32}T_2 - k_{35}T_5 - k_{36}T_6 \\ -k_{41}T_1 - k_{42}T_2 - k_{45}T_5 - k_{46}T_6 \end{Bmatrix}$$

$$= -\begin{bmatrix} k_{31} & k_{32} & k_{35} & k_{36} \\ k_{41} & k_{42} & k_{45} & k_{46} \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_5 \\ T_6 \end{Bmatrix}$$

有限元分析的数学基础

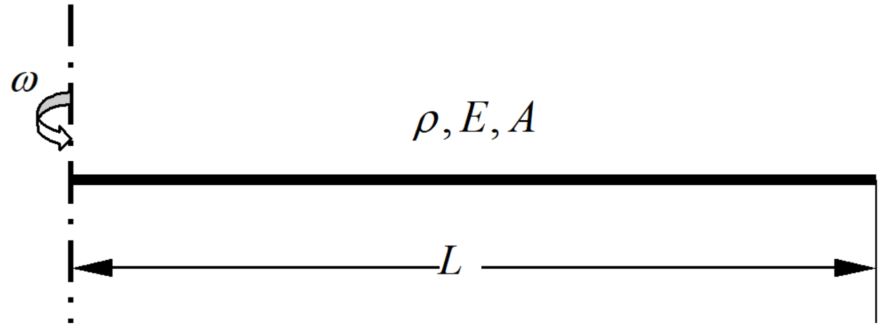
$$\begin{cases} u''(x) + u(x) + x = 0, \quad (0 < x < 1) \\ u(0) = 0 \\ u'(1) = 0 \end{cases}$$

绕固定轴旋转的弹性直杆

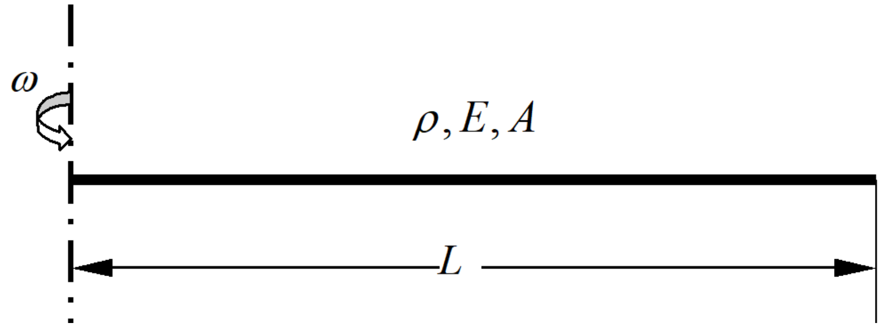
实际问题 $\Rightarrow$ 数学模型

$$\begin{cases} u''(x) + u(x) + x = 0, \quad (0 < x < 1) \\ u(0) = 0 \\ u'(1) = 0 \end{cases}$$

绕固定轴旋转的弹性直杆



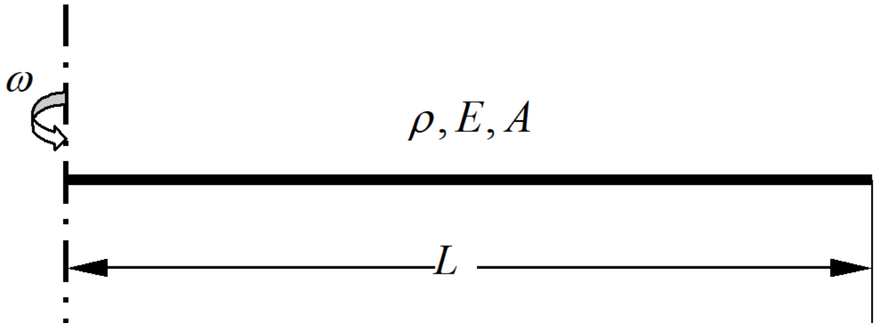
绕固定轴旋转的弹性直杆



$$
\begin{cases}
E\dfrac{d^2u}{dx^2}+\rho(x+u)\omega^2=0 \\
u(0)=0 \\
\sigma(L)=E\varepsilon(L)=E\dfrac{\mathrm{d}u(L)}{\mathrm{d}x}=0
\end{cases}
$$

$$
\begin{cases}
u''(x)+u(x)+x=0, \quad (0<x<1) \\
u(0)=0 \\
u'(1)=0
\end{cases}
$$

绕固定轴旋转的弹性直杆



$$\begin{cases} u''(x) + u(x) + x = 0, \quad (0 < x < 1) \\ u(0) = 0 \\ u'(1) = 0 \end{cases} \Rightarrow \quad u(x) = \frac{\sin x}{\cos 1} - x$$

常微分方程边值问题的近似解

$$\begin{cases} \Delta(u) = u''(x) + u(x) + x = 0, \quad (0 < x < 1) \\ u(0) = 0 \\ u'(1) = 0 \end{cases}$$

$$\Downarrow$$

$$\begin{cases} \Delta(\tilde{u}) \approx 0, \quad (0 < x < 1) \\ \tilde{u}(0) \approx 0 \\ \tilde{u}'(1) \approx 0 \end{cases}$$

近似解的构造

$$\begin{cases} \Delta(\tilde{u}) \approx 0, \quad \left(0 < x < 1\right) \\ \tilde{u}(0) \approx 0 \\ \tilde{u}'(1) \approx 0 \end{cases}$$

近似解的构造

$$\Delta(\tilde{u}) = \tilde{u}''(x) + \tilde{u}(x) + x \approx 0$$

$$\tilde{u}(x) = x(-2a_1 - 3a_2 + a_1 x + a_2 x^2)$$

$$\Delta(\tilde{u}) = a_2 x^3 + 3a_2 x + x + a_1(x^2 - 2x + 2)$$

近似解的构造

$$\tilde{u}(x) = x(-2a_1 - 3a_2 + a_1 x + a_2 x^2)$$

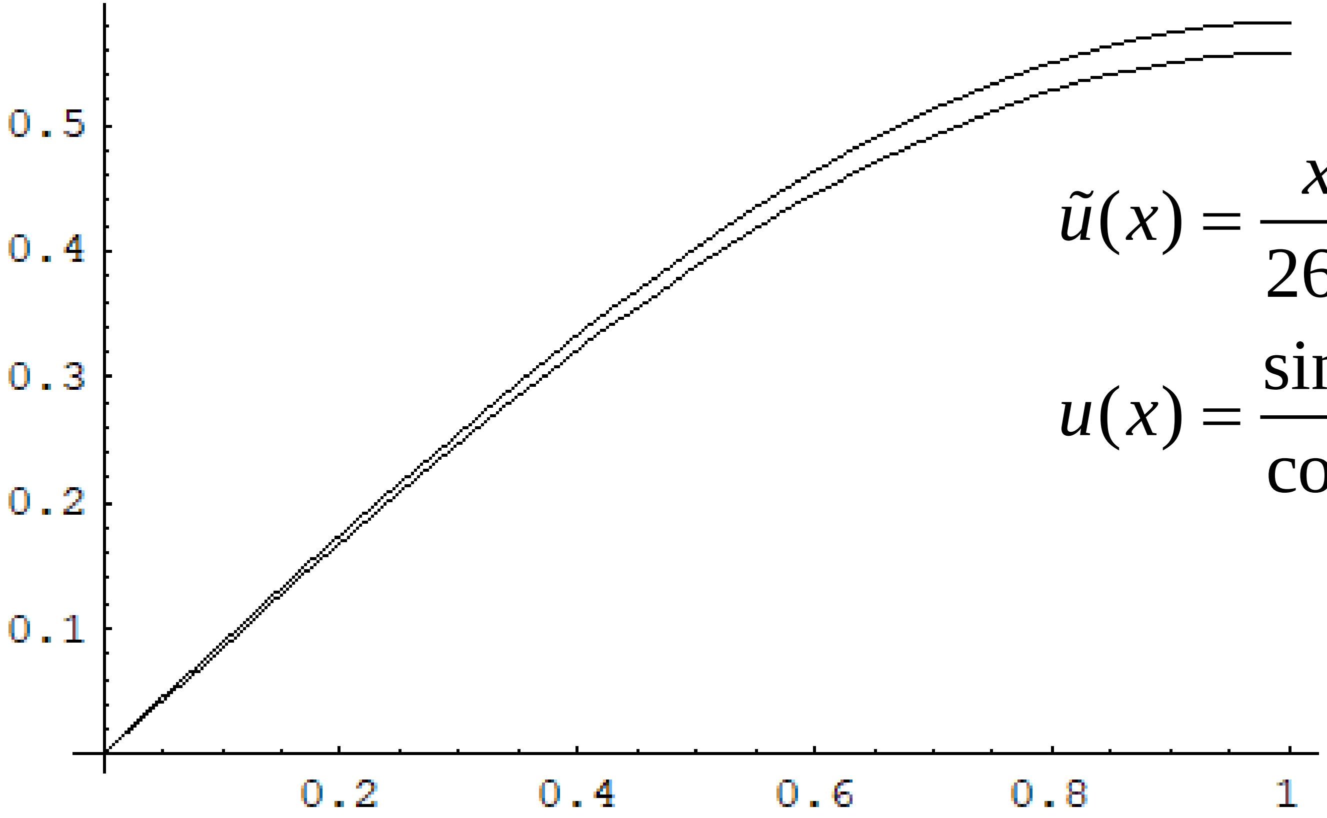
$$\Delta(\tilde{u}) = a_2 x^3 + 3a_2 x + x + a_1(x^2 - 2x + 2)$$

$$\Downarrow$$

$$\Delta(\tilde{u})\Big|_{x=\frac{1}{3}} = \left[a_2 x^3 + 3a_2 x + x + a_1(x^2 - 2x + 2)\right]_{x=\frac{1}{3}} = \frac{1}{3} + \frac{13}{9}a_1 + \frac{28}{27}a_2 = 0$$

$$\Delta(\tilde{u})\Big|_{x=\frac{2}{3}} = \left[a_2 x^3 + 3a_2 x + x + a_1(x^2 - 2x + 2)\right]_{x=\frac{2}{3}} = \frac{2}{3} + \frac{10a_1}{9} + \frac{62a_2}{27} = 0$$

近似解的构造



$$\tilde{u}(x) = \frac{x}{263}(234 - 9x - 72x^2)$$

$$u(x) = \frac{\sin x}{\cos 1}$$

加权残数法

$$\Delta(\tilde{u}) = 0 \quad \Leftrightarrow \quad \int_0^1 w(x)\Delta(\tilde{u})\mathrm{d}x = 0 \quad \big(w(x)\text{是任意的}\big)$$

加权残数法

$$a=0 \quad \Leftrightarrow \quad ab=0 \ \left(b\text{是任意的}\right)$$

$$f(x)=0 \quad \left(x\in(0,1)\right) \quad \Leftrightarrow \quad \int_0^1 f(x)g(x)\mathrm{d}x=0,\ \text{对于任意的}g(x)\left(x\in(0,1)\right)$$

加权残数法

$$\begin{cases} \Delta(\tilde{u}) = \tilde{u}''(x) + \tilde{u}(x) + x = 0 \\ u(0) = 0 \\ u'(1) = 0 \end{cases} \Leftrightarrow \begin{cases} \int_0^1 w(x)\Delta(\tilde{u})\mathrm{d}x = 0 \quad \left(w(x)\text{是任意的}\right) \\ u(0) = 0 \\ u'(1) = 0 \end{cases}$$

加权残数法

$$\begin{cases} \int_0^1 w(x)\Delta(\tilde{u})\mathrm{d}x = 0 \quad \left(w(x)\text{是任意的}\right) \\ u(0) = 0 \\ u'(1) = 0 \end{cases}$$

$$\tilde{u}(x) = x(-2a_1 - 3a_2 + a_1 x + a_2 x^2)$$

$$\Delta(\tilde{u}) = a_2 x^3 + 3a_2 x + x + a_1(x^2 - 2x + 2)$$

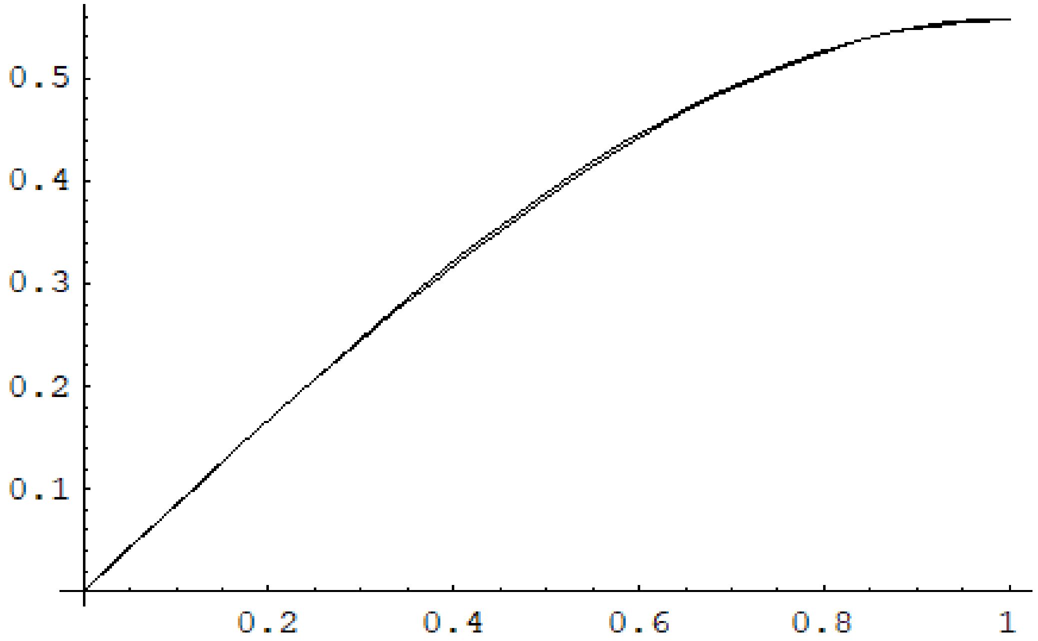
$$\begin{cases} w(x) = 1 \quad \Rightarrow \quad \int_0^1 \Delta(\tilde{u})dx = \frac{4}{3}a_1 + \frac{7}{4}a_2 + \frac{1}{2} = 0 \\ w(x) = x \quad \Rightarrow \quad \int_0^1 x\Delta(\tilde{u})dx = \frac{7}{12}a_1 + \frac{6}{5}a_2 + \frac{1}{3} = 0 \end{cases}$$

加权残数法

$$\begin{cases} \dfrac{4}{3}a_1 + \dfrac{7}{4}a_2 + \dfrac{1}{2} = 0 \\ \dfrac{7}{12}a_1 + \dfrac{6}{5}a_2 + \dfrac{1}{3} = 0 \end{cases}$$

$$a_1 = -\frac{4}{137}, \quad a_2 = -\frac{110}{417}$$

$$\tilde{u}(x) = -\frac{2x}{417}(55x^2 + 6x - 177)$$



加权残数法

$$\Delta(\tilde{u}) = 0 \quad \Leftrightarrow \quad \int_0^1 w(x)\Delta(\tilde{u})\mathrm{d}x = 0 \quad \Leftrightarrow \quad \begin{aligned} &\int_0^1 1\times\Delta(\tilde{u})\mathrm{d}x = 0 \\ &\int_0^1 x\times\Delta(\tilde{u})\mathrm{d}x = 0 \end{aligned}$$