有限元方法第四次作业

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课程名称：有限元方法

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目录

第一题：

推导场论公式

$$w\nabla^2 T = \nabla\cdot(w\nabla T) - \nabla w\cdot\nabla T$$

其中 w 和 T 是 x, y（二维问题）或者是 x, y, z（三维问题）的标量函数，∇ 是梯度算子。

解题如下：

$$\begin{aligned}
\nabla\cdot(w\nabla T) &= \nabla\cdot[w(\frac{\partial T}{\partial x}\vec{p}+\frac{\partial T}{\partial y}\vec{q}+\frac{\partial T}{\partial z}\vec{r})] \\
&= \nabla\cdot[w\frac{\partial T}{\partial x}\vec{p}+w\frac{\partial T}{\partial y}\vec{q}+w\frac{\partial T}{\partial z}\vec{r}] \\
&= \frac{\partial w\partial T}{\partial x^2}+\frac{w\partial^2 T}{\partial x^2}+\frac{\partial w\partial T}{\partial x\partial y}+\frac{w\partial^2 T}{\partial x\partial y}+\frac{\partial w\partial T}{\partial x\partial z}+\frac{w\partial^2 T}{\partial x\partial z} \\
&+\frac{\partial w\partial T}{\partial x\partial y}+\frac{w\partial^2 T}{\partial x\partial y}+\frac{\partial w\partial T}{\partial y^2}+\frac{w\partial^2 T}{\partial y^2}+\frac{\partial w\partial T}{\partial y\partial z}+\frac{w\partial^2 T}{\partial y\partial z} \\
&+\frac{\partial w\partial T}{\partial x\partial z}+\frac{w\partial^2 T}{\partial x\partial z}+\frac{\partial w\partial T}{\partial z\partial y}+\frac{w\partial^2 T}{\partial z\partial y}+\frac{\partial w\partial T}{\partial z^2}+\frac{w\partial^2 T}{\partial z^2} \\
&= \frac{\partial w\partial T}{\partial x^2}+\frac{w\partial^2 T}{\partial x^2}+\frac{\partial w\partial T}{\partial y^2}+\frac{w\partial^2 T}{\partial y^2}+\frac{\partial w\partial T}{\partial z^2}+\frac{w\partial^2 T}{\partial z^2}
\end{aligned}$$

同理：

$$\begin{aligned}
\nabla w\cdot\nabla T &= (\frac{\partial w}{\partial x}\vec{p}+\frac{\partial w}{\partial y}\vec{q}+\frac{\partial w}{\partial z}\vec{r})\cdot(\frac{\partial T}{\partial x}\vec{p}+\frac{\partial T}{\partial y}\vec{q}+\frac{\partial T}{\partial z}\vec{r}) \\
&= \frac{\partial w}{\partial x}\frac{\partial T}{\partial x}+\frac{\partial w}{\partial y}\frac{\partial T}{\partial y}+\frac{\partial w}{\partial z}\frac{\partial T}{\partial z}
\end{aligned}$$

可知：

$$\begin{aligned}
&\nabla\cdot(w\nabla T)-\nabla w\cdot\nabla T \\
&= \frac{\partial w\partial T}{\partial x^2}+\frac{w\partial^2 T}{\partial x^2}+\frac{\partial w\partial T}{\partial y^2}+\frac{w\partial^2 T}{\partial y^2}+\frac{\partial w\partial T}{\partial z^2}+\frac{w\partial^2 T}{\partial z^2}-\frac{\partial w}{\partial x}\frac{\partial T}{\partial x}+\frac{\partial w}{\partial y}\frac{\partial T}{\partial y}+\frac{\partial w}{\partial z}\frac{\partial T}{\partial z} \\
&= w(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2})
\end{aligned}$$

可知：

$$w\nabla^2 T = w(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2})$$

得证

第二题：

推导 Gauss 积分定理

$$\int_{\Omega} \nabla \cdot (w \nabla T) \mathrm{d}\Omega = \oint_{\partial\Omega} (w \nabla T \cdot \vec{\mathbf{n}}) \mathrm{d}\Omega$$

其中 Ω 是二维或三维区域，$\partial\Omega$ 是区域 Ω 的边界，$\vec{\mathbf{n}}$ 是边界 $\partial\Omega$ 上的外法线矢量。

解题如下：

$$\int_{\Omega} \nabla \cdot (w \nabla T) \mathrm{d}\Omega = \oint_{\partial\Omega} (w \nabla T \cdot \vec{\mathbf{n}}) \mathrm{d}\Omega$$

$$w \nabla T = w\left(\frac{\partial T}{\partial x} \vec{p} + \frac{\partial T}{\partial y} \vec{q} + \frac{\partial T}{\partial z} \vec{r}\right)$$

$$\nabla \cdot (w \nabla T) = \frac{\partial w \partial T}{\partial x^2} + \frac{w \partial^2 T}{\partial x^2} + \frac{\partial w \partial T}{\partial y^2} + \frac{w \partial^2 T}{\partial y^2} + \frac{\partial w \partial T}{\partial z^2} + \frac{w \partial^2 T}{\partial z^2}$$

$$= \frac{\partial w \partial T}{\partial x^2} + \frac{\partial w \partial T}{\partial y^2} + \frac{\partial w \partial T}{\partial z^2} + w\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right)$$

$$AS: \Phi = w \nabla T$$

$$\nabla \cdot (w \nabla T) = \nabla w \cdot \nabla T + w \nabla^2 T$$

$$\vec{\mathbf{n}} = (\cos\alpha, \cos\beta, \cos\gamma)$$

$$d\Omega = \vec{\mathbf{n}} d\Omega$$

$$\iint_{\partial\Omega} w \nabla T d\Omega = \iiint_{\Omega} \nabla \cdot (w \nabla T) d\Omega = \iiint_{\Omega} (\nabla w \cdot \nabla T + w \nabla^2 T) d\Omega$$

$$\iint_{\partial\Omega} w \nabla T d\Omega = \iint_{\partial\Omega} (w \nabla T) \vec{\mathbf{n}} d\Omega = \iint_{\partial\Omega} w \bullet gradT \bullet \vec{\mathbf{n}} d\Omega = \iint_{\partial\Omega} w \frac{\partial f}{\partial \vec{\mathbf{n}}} d\Omega ......(1)$$

$$\iiint_{\Omega} (\nabla w \cdot \nabla T + w \nabla^2 T) d\Omega = \iint_{\partial\Omega} w \frac{\partial f}{\partial \vec{\mathbf{n}}} d\Omega ......(2)$$

$$\because (1) = (2)$$

$$\therefore \int_{\Omega} \nabla \cdot (w \nabla T) \mathrm{d}\Omega = \oint_{\partial\Omega} (w \nabla T \cdot \vec{\mathbf{n}}) \mathrm{d}\Omega$$

第三题：

整理课堂内容，阐明偏微分方程的边值问题

$$\begin{cases}\nabla\cdot(k\nabla T)+Q=0 & \text{in }\Omega\\ T=\bar{T} & \text{on }\Gamma_1\\ k\nabla T\cdot\vec{\mathbf{n}}=\bar{q} & \text{on }\Gamma_2\end{cases}$$

其中 $\partial\Omega=\Gamma_1\cup\Gamma_2$，等价于如下积分形式的方程

$$\int_\Omega (wQ-k\nabla w\cdot\nabla T)\mathrm{d}\Omega+\int_{\Gamma_2} w\bar{q}\mathrm{d}\Gamma=0$$

对于满足 $w\big|_{\Gamma_1}=0$ 的任意函数 w 均成立。

解题如下：

$$\begin{cases}\nabla\cdot(k\nabla T)+Q=0 & in\ \Omega\\ T=\bar{T} & \text{on }\Gamma_1\\ k\nabla T\cdot\vec{\mathbf{n}}=\bar{q} & \text{on }\Gamma_2\end{cases}$$

$$（一）\Leftrightarrow\begin{cases}\int\limits_\Omega w(\nabla\cdot(k\nabla T)+Q)=0 & in\ \Omega & ①\\ \int\limits_{\Gamma_1} w_1(T-\bar{T})=0 & \text{on }\Gamma_1 & ②\\ \int\limits_{\Gamma_2} w_2(k\nabla T\cdot\vec{\mathbf{n}}-\bar{q})=0 & \text{on }\Gamma_2 & ③\end{cases}$$

$$\int\limits_\Omega w\nabla\cdot(k\nabla T)d\Omega=\int\limits_\Omega k\nabla(w\nabla T)d\Omega-\int\limits_\Omega k\nabla w\bullet\nabla Td\Omega$$

$$=\oint\limits_\Omega wk\nabla T\bullet\vec{\mathbf{n}}d\Gamma-\int\limits_\Omega k\nabla w\bullet\nabla Td\Omega$$

$$=\int\limits_{\Gamma_1+\Gamma_2} wk\nabla T\bullet\vec{\mathbf{n}}d\Gamma-\int\limits_\Omega k\nabla w\bullet\nabla Td\Omega$$

$$w=0\quad(on\ \Gamma_1)$$

$$\nabla w\vec{\mathbf{n}}=\frac{dw}{dn}=-w_1\quad(on\ \Gamma_2)$$

$$（一）\Leftrightarrow ①+②+③$$

$$（一）\Leftrightarrow\int\limits_{\Gamma_1+\Gamma_2} wk\nabla T\bullet\vec{\mathbf{n}}d\Gamma-\int\limits_\Omega(k\nabla w\bullet\nabla T+wQ)d\Omega+\int\limits_{\Gamma_1} w_1(T-\bar{T})+\int\limits_{\Gamma_2} w_2(k\nabla T\cdot\vec{\mathbf{n}}-\bar{q})=0$$