8803 - Mobile Manipulation: Kinematics II

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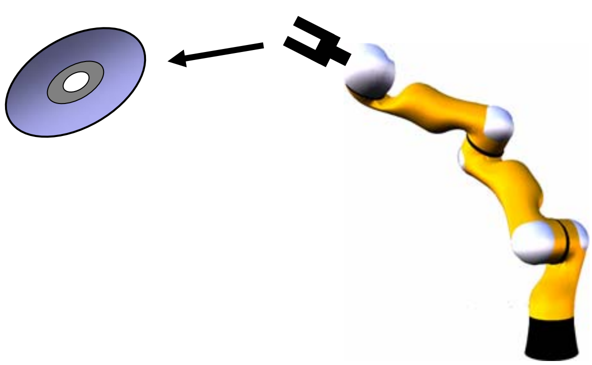
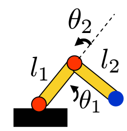
- January 17, 2008

Inverse Kinematics

CD position $\rightarrow$ Robot Joint Angles



Inverse Kinematics



- Equations are nonlinear (lots of trig)

Last time: $\mathbf{T}_3^0 = \begin{bmatrix} c_{12} & -s_{12} & l_1 c_1 + l_2 c_{12} \\ s_{12} & c_{12} & l_1 s_1 + l_2 s_{12} \\ 0 & 0 & 1 \end{bmatrix}$

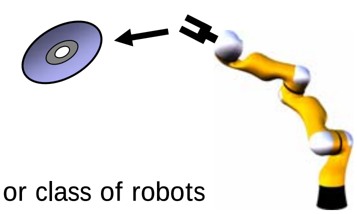
$$\boxed{\begin{aligned} x_e &= l_1 c_1 + l_2 c_{12} \\ y_e &= l_1 c_1 + l_2 c_{12} \end{aligned}}$$

- Multiple Solutions

- Not always possible to find closed-form solution

- Infinite Solutions

Analytical Methods



- Very fast and numerically stable

- No general solution!
  Each solution applies to a particular robot or class of robots

- Requires algebraic/geometric intuition

- Possible for robots with simple kinematics

Analytical Methods

Robots with simple kinematics: 6 DOF has closed-form IK if either:

- Three consecutive revolute axes intersect at a common point
- Three consecutive revolute axes are parallel

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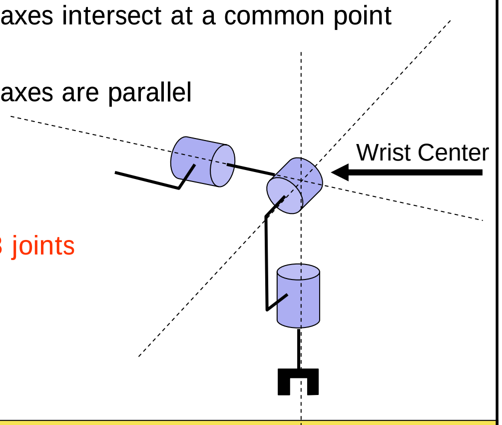
"Spherical Wrist"

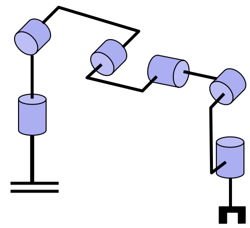
- Wrist position not affected by 3 joints
- Wrist position has 3 DOF



Analytical IK Example

PUMA Class 6 DOF Robot

Goal: $\mathbf{p}_g^0$ and $\mathbf{R}_g^0 = [\mathbf{x}_g \quad \mathbf{y}_g \quad \mathbf{z}_g]$



Analytical IK Example

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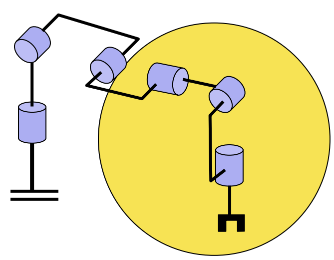
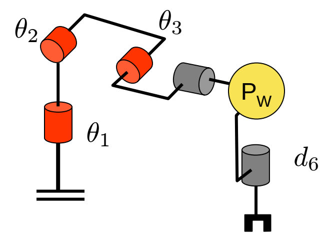


Analytical IK Example

Goal: $\mathbf{p}_G$ and $\mathbf{R}_G^0 = \begin{bmatrix}\mathbf{x}_g & \mathbf{y}_g & \mathbf{z}_g\end{bmatrix}$



$$\mathbf{p}_W = \mathbf{p}_G - d_6\mathbf{z}_g$$

Analytical IK Example

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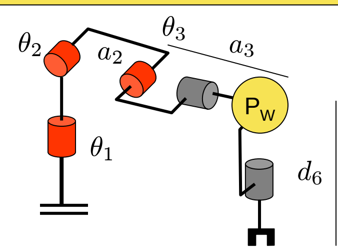

$$\mathbf{p}_W = \mathbf{p}_G - d_6\mathbf{z}_g$$

Forward kinematics for p_W

$$\mathbf{T}_1^0(\theta_1) = \begin{bmatrix} c_1 & 0 & s_1 & 0 \\ s_1 & 0 & -c_1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \mathbf{T}_2^1(\theta_1) = \begin{bmatrix} c_2 & -s_2 & 0 & a_2c_1 \\ s_2 & c_2 & 0 & a_2s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \mathbf{T}_3^2(\theta_1) = \begin{bmatrix} c_3 & -s_3 & 0 & a_3c_3 \\ s_3 & c_3 & 0 & a_3s_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Analytical IK Example

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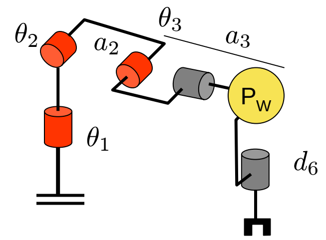

$$\mathbf{p}_W = \mathbf{p}_G - d_6 \mathbf{z}_g$$

$$\mathbf{T}_3^0 = \mathbf{T}_1^0 \mathbf{T}_2^1 \mathbf{T}_3^2 = \begin{bmatrix} c_1 c_{23} & -c_1 s_{23} & s_1 & c_1(a_2 c_2 + a_3 c_{23}) \\ s_1 c_{23} & -s_1 s_{23} & -c_1 & s_1(a_2 c_2 + a_3 c_{23}) \\ s_{23} & c_{23} & 0 & a_2 s_2 + a_3 s_{23} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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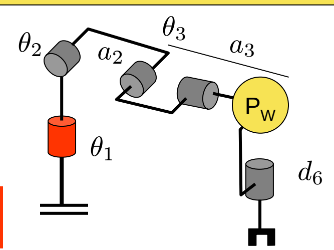

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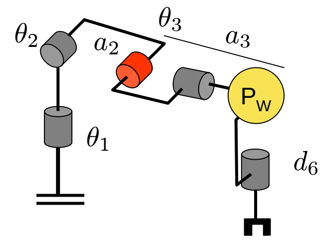
$$\boxed{\theta_1 = \operatorname{atan2}(p_{Wy}, p_{Wx}) \quad \text{or} \quad \theta_1 = \operatorname{atan2}(p_{Wy}, p_{Wx}) + \pi}$$

Analytical IK Example

Goal: $\mathbf{p}_G$ and $\mathbf{R}_G^0 = [\mathbf{x}_g \quad \mathbf{y}_g \quad \mathbf{z}_g]$



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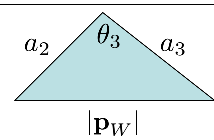
$\theta_1 = \text{atan2}(p_{Wy}, p_{Wx})$ or $\theta_1 = \text{atan2}(p_{Wy}, p_{Wx}) + \pi$

$$\mathbf{p}^T \mathbf{p} = a_2^2 + a_3^2 - 2 a_2 a_3 \cos(\theta_3)$$



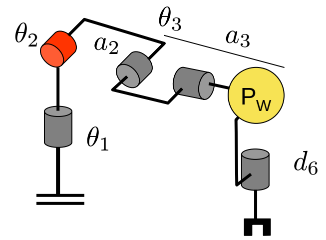
$$c_3 = \frac{p_{Wx}^2 + p_{Wy}^2 + p_{Wz}^2 - a_2^2 - a_3^2}{2 a_2 a_3}$$

$$s_3 = \pm\sqrt{1 - c_3^2}$$

$$\theta_3 = \text{atan2}(s_3, c_3)$$

Analytical IK Example

Goal: $\mathbf{p}_G$ and $\mathbf{R}_G^0 = [\mathbf{x}_g \quad \mathbf{y}_g \quad \mathbf{z}_g]$



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$\theta_1 = \text{atan2}(p_{Wy}, p_{Wx})$ or $\theta_1 = \text{atan2}(p_{Wy}, p_{Wx}) + \pi$

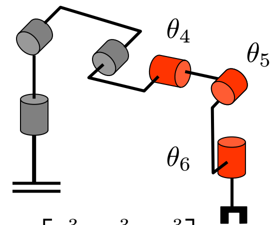
$\theta_3 = \text{atan2}(s_3, c_3)$ $\quad c_3 = \frac{p_{Wx}^2 + p_{Wy}^2 + p_{Wz}^2 - a_2^2 - a_3^2}{2 a_2 a_3}$ $\quad s_3 = \pm\sqrt{1 - c_3^2}$

$$s_2 = \frac{(a_2 + a_3 c_3) p_{Wz} - a_3 s_3 \sqrt{p_{Wx}^2 + p_{Wy}^2}}{p_{Wx}^2 + p_{Wy}^2 + p_{Wz}^2} \qquad c_2 = \frac{(a_2 + a_3 c_3)\sqrt{p_{Wx}^2 + p_{Wy}^2} + a_3 s_3 p_{Wz}}{p_{Wx}^2 + p_{Wy}^2 + p_{Wz}^2}$$

$$\theta_2 = \text{atan2}(s_2, c_2)$$

Analytical IK Example: Wrist

Goal: $\mathbf{p}_G$ and $\mathbf{R}_G^0 = [\mathbf{x}_g \quad \mathbf{y}_g \quad \mathbf{z}_g]$



Given: $\mathbf{R}_3^0$ $\qquad \mathbf{R}_6^3 = \mathbf{R}_3^{0T}\mathbf{R}_G^0$

From Forward Kinematics:

$$\mathbf{R}_6^3 = \begin{bmatrix} c_4c_5c_6 - s_4s_6 & -c_4c_5s_6 - s_4c_6 & c_4s_5 \\ s_4c_5c_6 + c_4s_6 & -s_4c_5s_6 + c_4c_6 & s_4s_5 \\ -s_5c_6 & s_5s_6 & c_5 \end{bmatrix} = \begin{bmatrix} n_x^3 & s_x^3 & a_x^3 \\ n_y^3 & s_y^3 & a_y^3 \\ n_z^3 & s_z^3 & a_z^3 \end{bmatrix}$$

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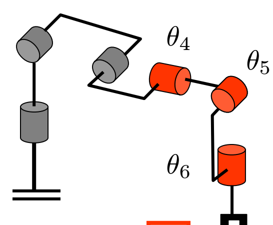

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$$\theta_5 \in (0, \pi)$$

$$\theta_4 \quad = \quad \operatorname{atan2}(a_y^3, a_x^3)$$

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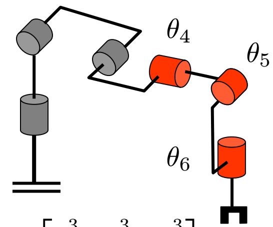


$$\mathbf{R}_6^3 = \begin{bmatrix} c_4c_5c_6 - s_4s_6 & -c_4c_5s_6 - s_4c_6 & \boxed{c_4s_5} \\ s_4c_5c_6 + c_4s_6 & -s_4c_5s_6 + c_4c_6 & \boxed{s_4s_5} \\ -s_5c_6 & s_5s_6 & \boxed{c_5} \end{bmatrix} = \begin{bmatrix} n_x^3 & s_x^3 & a_x^3 \\ n_y^3 & s_y^3 & a_y^3 \\ n_z^3 & s_z^3 & a_z^3 \end{bmatrix}$$

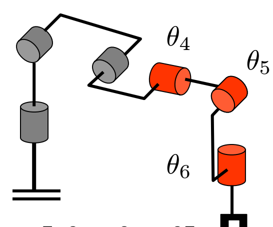
$$\theta_5 \in (0, \pi)$$

$$\theta_4 = \operatorname{atan2}(a_y^3, a_x^3)$$

$$\theta_5 = \operatorname{atan2}(\sqrt{(a_x^3)^2 + (a_y^3)^2}, a_z^3) \qquad c_4^2s_5^2 + s_4^2s_5^2 = (c_4^2 + s_4^2)s_5^2 = s_5^2$$

Analytical IK Example: Wrist

Goal: $\mathbf{p}_G$ and $\mathbf{R}_G^0 = [\mathbf{x}_g \quad \mathbf{y}_g \quad \mathbf{z}_g]$



$$\mathbf{R}_6^3 = \begin{bmatrix} c_4c_5c_6 - s_4s_6 & -c_4c_5s_6 - s_4c_6 & c_4s_5 \\ s_4c_5c_6 + c_4s_6 & -s_4c_5s_6 + c_4c_6 & s_4s_5 \\ \boxed{-s_5c_6} & \boxed{s_5s_6} & c_5 \end{bmatrix} = \begin{bmatrix} n_x^3 & s_x^3 & a_x^3 \\ n_y^3 & s_y^3 & a_y^3 \\ n_z^3 & s_z^3 & a_z^3 \end{bmatrix}$$

$$\theta_5 \in (0, \pi)$$

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$$\theta_6 = \operatorname{atan2}(s_z^3, -n_z^3)$$

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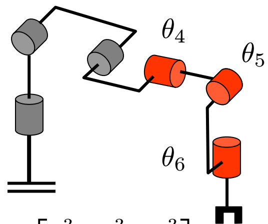

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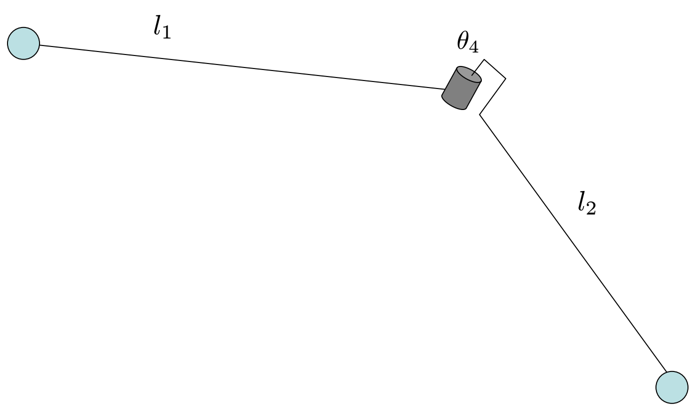
$\theta_5 \in (0, \pi)$	$\theta_5 \in (-\pi, 0)$
$\theta_4 = \text{atan2}(a_y^3, a_x^3)$	$\theta_4 = \text{atan2}(-a_y^3, -a_x^3)$
$\theta_5 = \text{atan2}(\sqrt{(a_x^3)^2 + (a_y^3)^2}, a_z^3)$	$\theta_5 = \text{atan2}(-\sqrt{(a_x^3)^2 + (a_y^3)^2}, a_z^3)$
$\theta_6 = \text{atan2}(s_z^3, -n_z^3)$	$\theta_6 = \text{atan2}(-s_z^3, n_z^3)$

Handling Redundancy

Tolani '2000



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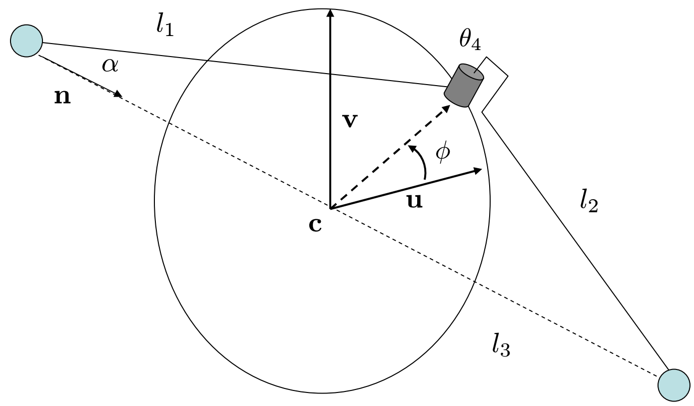

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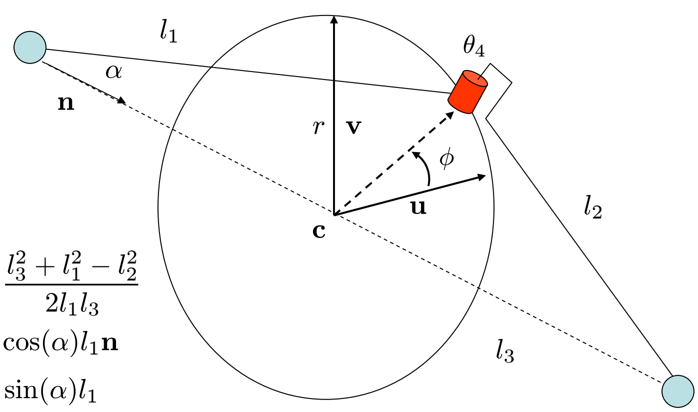

$$\cos(\alpha) = \frac{l_3^2 + l_1^2 - l_2^2}{2 l_1 l_3}$$

$$\mathbf{c} = \cos(\alpha) l_1 \mathbf{n}$$

$$r = \sin(\alpha) l_1$$

$$e(\phi) = \mathbf{c} + r(cos(\phi)\mathbf{u} + sin(\phi)\mathbf{v})$$

Summary

- Difficult to find solution & it does not always exist
- Places constraints on robot construction
- The solution is very fast!

Alternatives

- Iterative Methods
- Set up as optimization problem (distance to goal)
- Follow gradient descent (move angles to minimize distance)

Differential Kinematics

What is the robot Jacobian?

$$\mathbf{J}(\theta_1, ..., \theta_n) = \begin{bmatrix} \frac{\delta x}{\delta \theta_1} & \frac{\delta x}{\delta \theta_2} & & \frac{\delta x}{\delta \theta_n} \\ \frac{\delta y}{\delta \theta_1} & \frac{\delta y}{\delta \theta_2} & & \frac{\delta y}{\delta \theta_n} \\ \frac{\delta z}{\delta \theta_1} & \frac{\delta z}{\delta \theta_2} & \cdots & \frac{\delta z}{\delta \theta_n} \\ \frac{\delta \omega_x}{\delta \theta_1} & \frac{\delta \omega_x}{\delta \theta_2} & & \frac{\delta \omega_x}{\delta \theta_n} \\ \frac{\delta \omega_y}{\delta \theta_1} & \frac{\delta \omega_y}{\delta \theta_2} & & \frac{\delta \omega_y}{\delta \theta_n} \\ \frac{\delta \omega_z}{\delta \theta_1} & \frac{\delta \omega_z}{\delta \theta_2} & & \frac{\delta \omega_z}{\delta \theta_n} \end{bmatrix}$$

Differential Kinematics

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$$\mathbf{J}(\theta_1, ..., \theta_n) = \begin{bmatrix} \frac{\delta x}{\delta \theta_1} & \frac{\delta x}{\delta \theta_2} & & \frac{\delta x}{\delta \theta_n} \\ \frac{\delta y}{\delta \theta_1} & \frac{\delta y}{\delta \theta_2} & & \frac{\delta y}{\delta \theta_n} \\ \frac{\delta z}{\delta \theta_1} & \frac{\delta z}{\delta \theta_2} & \cdots & \frac{\delta z}{\delta \theta_n} \\ \frac{\delta \omega_x}{\delta \theta_1} & \frac{\delta \omega_x}{\delta \theta_2} & & \frac{\delta \omega_x}{\delta \theta_n} \\ \frac{\delta \omega_y}{\delta \theta_1} & \frac{\delta \omega_y}{\delta \theta_2} & & \frac{\delta \omega_y}{\delta \theta_n} \\ \frac{\delta \omega_z}{\delta \theta_1} & \frac{\delta \omega_z}{\delta \theta_2} & & \frac{\delta \omega_z}{\delta \theta_n} \end{bmatrix}$$

Why is it useful?

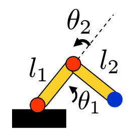
$$\frac{\delta x}{\delta t} = \frac{\delta x}{\delta \theta_1} \frac{\delta \theta_1}{\delta t} \longrightarrow \dot{\mathbf{x}} = \mathbf{J}\dot{\mathbf{q}}$$

$$\dot{\mathbf{q}} = \begin{bmatrix} \dot\theta_1 & \dot\theta_2 & ... & \dot\theta_n \end{bmatrix}^T$$

$$\dot{\mathbf{x}} = \begin{bmatrix} \dot x & \dot y & \dot z & \dot\omega_x & \dot\omega_y & \dot\omega_z \end{bmatrix}^T$$

$\dot{\mathrm{x}} = \mathbf{J}\dot{\mathrm{q}}$ How do we find J? Method 1

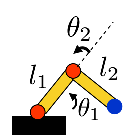
$$\mathbf{T}_3^0 = \begin{bmatrix} c_{12} & -s_{12} & \boxed{l_1 c_1 + l_2 c_{12}} \\ s_{12} & c_{12} & l_1 s_1 + l_2 s_{12} \\ 0 & 0 & 1 \end{bmatrix}$$



$$\begin{array}{ll} & \frac{\delta x}{\delta \theta_1} \\ \mathbf{J} = & \begin{bmatrix} \boxed{-l s_1 - l_2 s_{12}} & \quad \\ & \end{bmatrix} \end{array}$$

$\dot{\mathrm{x}} = \mathbf{J}\dot{\mathrm{q}}$ How do we find J? Method 1

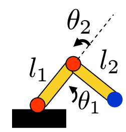
$$\mathbf{T}_3^0 = \begin{bmatrix} c_{12} & -s_{12} & \boxed{l_1 c_1 + l_2 c_{12}} \\ s_{12} & c_{12} & l_1 s_1 + l_2 s_{12} \\ 0 & 0 & 1 \end{bmatrix}$$



$$\begin{array}{lll} & \frac{\delta x}{\delta \theta_1} & \frac{\delta x}{\delta \theta_2} \\ \mathbf{J} = & \begin{bmatrix} -l s_1 - l_2 s_{12} & \boxed{-l_2 s_{12}} \\ & \end{bmatrix} \end{array}$$

$\dot{\mathbf{x}} = \mathbf{J}\dot{\mathbf{q}}$ How do we find J? Method 1

$$\mathbf{T}_3^0 = \begin{bmatrix} c_{12} & -s_{12} & l_1 c_1 + l_2 c_{12} \\ s_{12} & c_{12} & l_1 s_1 + l_2 s_{12} \\ 0 & 0 & 1 \end{bmatrix}$$



$$\begin{array}{ccc} & \frac{\delta x}{\delta \theta_1} & \frac{\delta x}{\delta \theta_2} \\ \boxed{\mathbf{J} =} & \begin{bmatrix} -l s_1 - l_2 s_{12} & -l_2 s_{12} \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} \end{bmatrix} & \\ & \frac{\delta y}{\delta \theta_1} & \frac{\delta y}{\delta \theta_2} \end{array}$$

$\dot{\mathbf{x}} = \mathbf{J}\dot{\mathbf{q}}$ How do we find J? Method 2

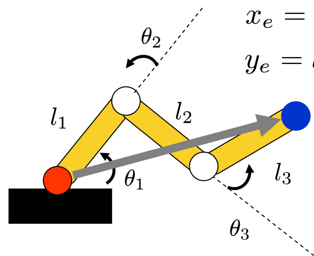
$$\mathbf{T}_i^0 = \begin{bmatrix} \mathbf{x}_i & \mathbf{y}_i & \mathbf{z}_i & \mathbf{p}_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{J} = \begin{bmatrix} \mathcal{J}_{P1} & \dots & \mathcal{J}_{Pn} \\ \mathcal{J}_{O1} & \dots & \mathcal{J}_{On} \end{bmatrix}$$

$$\begin{bmatrix} \mathcal{J}_{Pi} \\ \mathcal{J}_{Oi} \end{bmatrix} = \begin{cases} \begin{bmatrix} \mathbf{z}_i \\ 0 \end{bmatrix} & \text{(prismatic joint)} \\ \begin{bmatrix} \mathbf{z}_i \times (\mathbf{p} - \mathbf{p}_i) \\ \mathbf{z}_i \end{bmatrix} & \text{(revolute joint)} \end{cases}$$

$\dot{\mathbf{x}} = \mathbf{J}\dot{\mathbf{q}}$ How do we find J? Method 2

$$x_e = l_1 c_1 + l_2 c_{12} + l_3 c_{123}$$
$$y_e = l_1 s_1 + l_2 s_{12} + l_3 s_{123}$$



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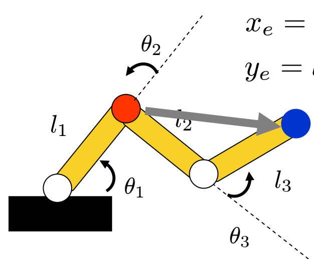

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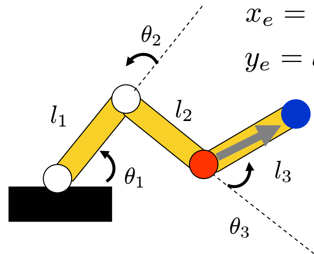

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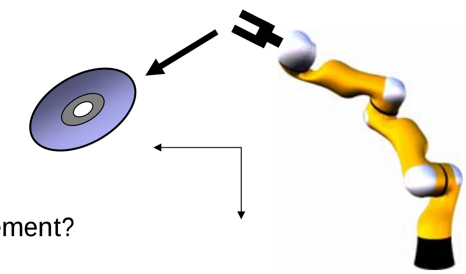
$\dot{\mathbf{x}} = \mathbf{J}\dot{\mathbf{q}}$ How do we find J? Method 3

1. Simulate a small displacement $\Delta\theta_i$ for each joint
2. Observe $\Delta\mathbf{x}$ and place its value into J

- Less accurate, more time consuming
- Sometimes the only option

$\dot{\mathbf{x}} = \mathbf{J}\dot{\mathbf{q}}$ How do we use J?



Workspace goal

How do we get a joint space displacement?

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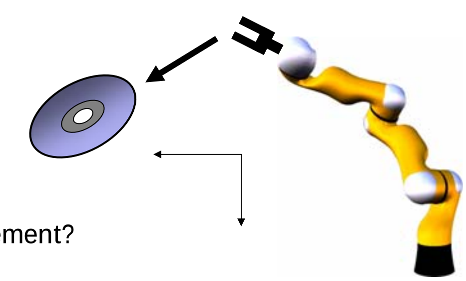
$\dot{\mathbf{x}} = \mathbf{J}\dot{\mathbf{q}}$ How do we use J?



Workspace goal

How do we get a joint space displacement?

Assuming 6 D.O.F and **J** is full rank: $\Delta\mathbf{q} = \mathbf{J}^{-1}\Delta\mathbf{x}$

- Iterate until convergence

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$\dot{\mathbf{x}} = \mathbf{J}\dot{\mathbf{q}}$ How do we use J?

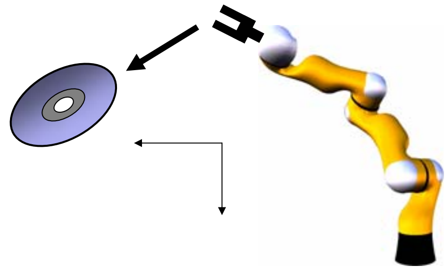
Redundant Robots

Jacobian: 6xn $(n > 6)$

$$\dot{\mathbf{x}} = [\quad J \quad]\dot{\mathbf{q}}$$

Invert: Not full column rank

$$\text{rank}(J) < n$$



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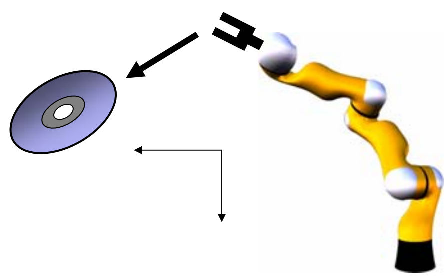

RIGHT Pseudo-inverse:

$$\mathbf{J}^{+} = \mathbf{J}^{T}(\mathbf{J}\mathbf{J}^{T})^{-1}$$ (when J has full row rank – otherwise left or SVD)

$$\Delta q = \mathbf{J}^{+}\Delta x$$ Δq is the unique smallest vector s.t. Δx = J Δq (least norm solution)

Jacobian Gradient IK

- Generally applicable to n-DOF kinematics
- Many existing implementations
- Slower and unstable around singularities

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  - Clamping
  - Use $\mathbf{J}^T$

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