FIT3139
Final Project

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Section 1: Specification table

Base model	The Lotka-Volterra model is a mathematical model that describes the predator-prey relationship between two interacting species, where the predator's population growth depends on the prey's population size and vice versa.
Extension assumptions	Lotka-Volterra model with Allee effects and stochastic growth rate: The Allee effects: Incorporate the notion that population growth rates decrease as population sizes become smaller. This adds a nonlinear element to the dynamics, leading to potential instability and alternative stable states. Stochastic growth rate: Instead of assuming deterministic growth rates for the predator and prey populations, stochasticity is introduced by considering random fluctuations in their growth rates.This can be done by incorporating noise or variability into the parameter in equations that govern population growth. We also explore predator-prey interactions within competitive or cooperative contexts. By considering the strategic decision-making of the predator and prey populations, we can analyse how their interactions and behaviours influence the system's dynamics.
Techniques showcased	• Monte Carlo Simulation:used to incorporate uncertainty and variability in the parameters of the Lotka-Volterra model • Heuristics:used to find optimal maximum (nash equilibrium) and minimum solutions. • Game theory:introduce strategic interactions between the predator and prey species where each species tries to optimise its own fitness or

	survival by choosing a strategy
Modelling question 1	How does the variability in parameter values affect the population dynamics when the Allee effect is introduced?
Modelling question 2	What are the optimal strategies for the prey and predator populations in a game where the prey can choose between 'in a school' or 'alone,' and the predator can choose between 'hunting aggressively' or 'hunting cautiously'?
Modelling question 3 (Extra: extended from modelling question 2)	How do changes in the initial temperature and cooling rate affect the convergence of the objective value and the optimal strategies?

Section 2: Introduction

The Lotka-Volterra model serves as a fundamental mathematical framework for understanding the dynamics of predator-prey relationships in ecological systems. By describing the interactions between two species, where the growth of the predator population depends on the size of the prey population and vice versa, this model provides insights into population dynamics and ecosystem stability.

The Allee effect, first introduced by Allee in 1931, represents a negative density dependence where the population growth rate is reduced at low population size. It may be caused by a number of factors including difficulties in finding mates, social dysfunction at small population sizes, inbreeding depression and food exploitation (Merdan, 2010).

In a real world scenario, the prey and predator populations often exhibit adaptive behaviours in response to the presence and abundance of their counterparts. Therefore, to capture the inherent uncertainties and complexities of real-world ecological systems and to analyse the optimal decision-making strategies for both the predator and prey, extensions to the base Lotka-Volterra model can be introduced.

Therefore, in this project, we aim to extend the Lotka-Volterra model with Allee effects by introducing two important factors: negative density dependence and stochastic growth rates. The Allee effects incorporate the notion that population growth rates decrease as population sizes become smaller. This adds a nonlinear element to the dynamics, leading to potential instability and alternative stable states.

We also incorporate stochastic growth rates for the predator and prey populations. Rather than assuming deterministic growth rates, we introduce variability by considering random fluctuations in their growth rates. This extension allows us to account for the inherent uncertainties and natural variability observed in ecological systems. By incorporating noise into the equations governing population growth, we can explore how stochasticity impacts the dynamics of predator-prey interactions.

Additionally, we go beyond the traditional analysis of predator-prey dynamics by introducing competitive or cooperative contexts using game theory when the Allee effect is present. By considering the strategic decision-making of the predator and prey populations, we can investigate how their interactions and behaviours influence the system's dynamics. Game theory provides a powerful framework to analyse the emergence of stable coexistence, predator-prey cycles, and the evolution of cooperative or competitive strategies.

To address our modelling questions, we employ three key techniques: Monte Carlo simulation, Game Theory and Heuristics. Monte Carlo simulation is utilised to incorporate uncertainty and variability in the parameters of the Lotka-Volterra model. By sampling parameter values from appropriate distributions, we can explore the effects of parameter variability on population dynamics. This technique enables us to assess the robustness and sensitivity of the model to different parameter values.

Furthermore, we incorporate heuristics, specifically the use of simulated annealing, to find optimal maximum (nash equilibrium) and minimum solutions in game theory. By allowing the prey and predator populations to adopt 2 strategies each, we can analyse strategic interactions and decision-making within the context of predator-prey dynamics. The simulated annealing algorithm helps identify the Nash equilibrium strategies that maximise the fitness or utility of each population. This approach allows us to explore the evolutionary outcomes of different strategies and investigate the dynamics of strategy adoption over time.

Our modelling questions revolve around understanding the effects of parameter variability on population dynamics and the evolution of different strategies by the prey and predator populations. Specifically, we aim to answer the following questions: (1) How does the variability in parameter values affect the population dynamics when the Allee effect is introduced? By incorporating stochastic growth rates, we can examine how parameter variability influences the stability, oscillations, and long-term behaviour of predator-prey interactions when the Allee effect is present. (2)What are the optimal strategies for the prey and predator populations in a game where the prey can choose between 'in a school' or 'alone,' and the predator can choose between 'hunting aggressively' or 'hunting cautiously'? and

(3)How do changes in the initial temperature and cooling rate affect the convergence of the objective value and the optimal strategies?

By applying game theory and heuristics, we can analyse the emergence and evolution of cooperative or competitive strategies, shedding light on the strategic behaviours that influence the dynamics of the predator-prey system.

In summary, through the extension of the Lotka-Volterra model with negative density dependence and stochastic growth rates, and the integration of Monte Carlo simulation, heuristics, and game theory, we aim to gain a deeper understanding of the dynamics and complexities of predator-prey interactions in ecological systems. By considering parameter variability and strategic decision-making, our techniques provide valuable insights into the behaviour and evolution of real-world ecosystems, enhancing our ability to study and predict ecological dynamics.

Section 3: Model description

Base model (from the lecture slides):

$$\boxed{\begin{aligned} \frac{dx}{dt} &= (r-f)x - \alpha xy \\ \frac{dy}{dt} &= (s-f)y + \beta xy \end{aligned}}$$

r is the fish rate of growth
s is the shark rate of growth
f is the fraction of shark (fish) caught
α is the prop. constant of fish being eaten by shark
β is the prop. constant of shark surviving by eating fish

The base model we are considering is the Lotka-Volterra model, also known as the predator-prey model. It is a classic mathematical model that describes the dynamics of a predator-prey relationship in an ecological system. The model consists of two interacting populations: the prey population (fish) and the predator population (shark). The model aims to capture the essential dynamics of how these populations interact and how their numbers change over time. Looking at the differential equations above, dx/dt represents the rate of change of the prey population (x) with respect to time (t). It describes how the prey population size evolves over time whereas dy/dt represents the rate of change of the predator population (y) with respect to time (t). It describes how the predator population size evolves over time.

The relationship between the two equations is through the interaction term βxy in the predator equation and the term -αxy in the prey equation. The prey population affects the predator population positively through the term βxy, as more prey availability leads to increased predator growth. On the other hand, the predator population negatively affects the prey population through the term -αxy, as predation leads to a decrease in the prey population.

The behaviour of the equations depends on the parameter values and initial conditions. In general, the LV model exhibits oscillatory behaviour, where the populations of both prey and predator fluctuate over time. The oscillations occur due to the predator-prey interactions, where increases in the prey population lead to an increase in the predator population, which then results in a decrease in the prey population, and so on.

The plot of the LV model typically shows oscillations in the population sizes of the prey and predator over time. The prey population will rise and fall in a cyclic manner, with corresponding rises and falls in the predator population. The specific shape and characteristics of the plot will depend on the parameter values and initial conditions used in the model.

Model extension (Allee effect):

$$\frac{dx}{dt} = rx(1-x) - axy, \quad \frac{dy}{dt} = ay(x-y),$$

(Merdan, 2010)

Model Extension Details:

1. Allee effect

- The parameter 'threshold' represents the population size below which the Allee effect comes into play. If the prey population size is below this threshold, the Allee effect is applied.
- When the **prey** population size is **below the threshold**, the rate of change of the predator/prey (predator/prey growth rate) with respect to time (dy/dt or dx/dt) is

determined by the **model extension** equation. These equations incorporate the Allee effect, where the growth rate decreases as the prey population size decreases.

- When the **prey** population size is **equal** to or **above the threshold**, the rate of change of the predator/prey (predator/prey growth rate) with respect to time (dy/dt or dx/dt) is determined by the **base model** equation. This equation does **not** incorporate the Allee effect and uses the equations in the original model.
- Basically, the growth rate of the predator/prey population is determined based on whether the prey population size is below or above the threshold. If it's below the threshold, then we use equations in the model extension to capture the nonlinear dynamics and potential instability associated with the Allee effect, else we use equations in the base model (equations are given above).

2. Stochastic growth rate

- The original Lotka-Volterra model is extended to incorporate stochasticity in the growth rates of the predator and prey populations. This is achieved with the use of Monte Carlo simulation to explore the behaviour of the Lotka-Volterra model under various parameter sets and account for parameter variability.
- The original Lotka-Volterra model describes the interaction between predator and prey populations while the Monte Carlo simulation adds an additional layer of variability by sampling parameter values from specified distributions.
- The model defines parameter distributions for each of the parameters (r, s, f, alpha, beta) in the base model. These distributions represent the range and variability of the parameter values.
- The model performs a Monte Carlo simulation by iteratively sampling parameter values from the defined distributions n number of times. Each iteration corresponds to a unique parameter set, and the Lotka-Volterra model is simulated with these sampled values.
- The resulting population dynamics for each parameter set are stored and then plotted over time for all the simulations, showing the range of possible outcomes based on the sampled parameter values.

Assumptions for the Original Lotka-Volterra Model (from the slides):

- The change in the shark and fish populations, in isolation, is proportional to the present population of sharks and fish, respectively.
- The number of sharks and fish caught by fishermen is directly proportional to the present population of the shark and fish population.
- The number of fish eaten by sharks is directly proportional to the product of the number of fishes present and the number of sharks present.
- The additional number of sharks surviving is directly proportional to the number of fish eaten.

- The growth rates of the predator and prey populations are deterministic and constant over time.
- The prey is the only source of food for the predator.
- The population dynamics are based on the assumption that the predator and prey populations interact in a non-linear manner, where the predator's growth depends on the prey's population size and vice versa.
- The model assumes a continuous time scale, meaning that population changes are described by continuous differential equations.
- The populations have access to unlimited resources, and external factors such as competition for resources or environmental constraints are not considered in the original model.

Assumptions for the Lotka-Volterra model with Allee effects and stochastic growth rate:

- The parameter in the model extension maintains the same value as the base model to ensure consistency and simplicity. In the model extension, the parameter 'a' represents the conversion efficiency of prey into predators (Merdan, 2010). For the base model, this concept is essentially captured by the parameter 'alpha' since ‘alpha’ denotes the prop. constant of fish being eaten by sharks. To maintain consistency and clarity, we change the parameter 'a' to 'alpha' in the model extension.
- The parameters in the model, such as the intrinsic growth rate of prey (r), the intrinsic death rate of predators (s), the fraction of prey/predator caught (f), the prop. constant of prey being eaten by predator (alpha), and the prop. constant of predator surviving by eating prey (beta), are assumed to be constant over time. This assumption simplifies the model and allows for the examination of the effects of other factors, such as the Allee effect and stochastic growth rates, on population dynamics.
- The model assumes that there are no external factors influencing the population dynamics, such as immigration, emigration, or other ecological interactions. This assumption isolates the effects of the Allee effect and stochastic growth rates on the prey and predator populations without considering additional external influences.
- The model assumes a homogeneous environment, where the ecological conditions and resource availability are constant throughout the system.
- The extended model assumes that the growth rates of the predator and prey populations are subject to random fluctuations or variability.
- The random fluctuations in the growth rates are assumed to be independent of each other and uncorrelated over time.
- The noise affects the population sizes continuously and gradually, modifying the growth rates without sudden jumps or discontinuities.

Class of the Lotka-Volterra model with Allee effect and stochastic growth rate:

The analysis presented for this model is primarily numerical, as it involves solving the system of differential equations using numerical methods such as numerical integration (e.g., solve_ivp function). This approach allows for the exploration of population dynamics over time and the investigation of the effects of parameter variability and strategic decision-making. Analytical methods, on the other hand, involve finding exact solutions or performing mathematical derivations, which may be challenging for this complex model.

The Lotka-Volterra model with Allee effect and stochastic growth rate is non-linear. The interaction terms between the predator and prey populations are multiplied together, resulting in non-linear dynamics. This nonlinearity is a key aspect of the model, contributing to the emergence of population cycles and the potential for alternative stable states.

The model is continuous, as it describes the population dynamics as a continuous change over time. The differential equations capture the rates of change of the predator and prey populations with respect to time, allowing for a continuous representation of their interactions and growth.

The model incorporates stochastic growth rates, introducing randomness and variability into the population dynamics. Stochasticity is introduced by considering random fluctuations in the growth rates of predator and prey populations. This accounts for the inherent uncertainties and natural variability observed in ecological systems, making the model more realistic and capable of capturing real-world dynamics

Modelling question 1: How does the variability in parameter values affect the population dynamics when the Allee effect is introduced?

Algorithms:

1. Monte Carlo simulation using the Lotka-Volterra model with Allee effect:
   - The function LV_with_Allee represents the Lotka-Volterra model with Allee effect. This function takes the time t, the current state y, and the model parameters r, s, f, alpha, beta, and threshold as inputs. It returns the derivatives dx/dt and dy/dt representing the rate of change of the prey and predator populations, respectively, at the given time.
   - The parameter distributions for Monte Carlo simulation in the param_distributions dictionary is defined. Each parameter is assigned a specific distribution from which random values will be sampled. The distributions used include normal, uniform, and exponential distributions. The parameter distribution is not chosen randomly but rather based on prior knowledge or assumptions about the parameters' values and variability. The specific distributions used are as follows:
     -'r': Normal distribution with a mean of 0.8 and a standard deviation of 0.1.
     -'s': Uniform distribution between -1.0 and 0.0.

-'f': Exponential distribution with a scale parameter of 0.0005.
-'alpha': Normal distribution with a mean of 0.045 and a standard deviation of 0.005.
-'beta': Normal distribution with a mean of 0.03 and a standard deviation of 0.005.
-'threshold': Uniform distribution with a value that will be tweaked for analysis.

- Monte Carlo simulation is performed by iterating over a range of n iterations. For each iteration, random parameter values are sampled from the respective distributions defined in param_distributions.
- Solve the differential equations by calling solve_ivp with the defined function LV_with_Allee, time span, initial conditions, and sampled parameter values. The solution is obtained at evenly spaced time points of t=100.
- The Monte Carlo simulation results are plotted. Each iteration represents a different set of parameter values, and the prey and predator populations are plotted over time.

Mathematical techniques:

1. Differential equations: The Lotka-Volterra model with Allee effect is described by a system of two coupled differential equations representing the rate of change of the prey and predator populations over time.
2. Monte Carlo simulation: The simulation technique used to explore the dynamics of the system by sampling parameter values from the defined distributions and solving the differential equations multiple times to capture the variability and uncertainty in the model.
3. Numerical integration: The solve_ivp function from the scipy.integrate module is used to numerically integrate the system of differential equations and obtain the solution for the prey and predator populations at different time points.

<u>Modelling question 2:What are the optimal strategies for the prey and predator populations in a game where the prey can choose between 'in a school' or 'alone,' and the predator can choose between 'hunting aggressively' or 'hunting cautiously'?</u>

Algorithms:

1. Game theory algorithm
    - The payoff matrices for the prey and predator strategies are defined. These matrices represent the expected benefits or costs associated with each combination of strategies. Each matrix represents the payoffs for the strategies chosen by the prey and predator, respectively.
    - E.g. prey_payoff_matrix = np.array([[7, 3], [1, 2]])

Prey || Predator	Hunt Aggressively	Hunt Cautiously

In a school	7	3
Alone	1	2

- E.g. predator_payoff_matrix = np.array([[8, 2], [3, 4]])

Predator ‖ Prey	In a school	Alone
Hunt Aggressively	8	2
Hunt Cautiously	3	4

- The objective_function calculates the objective value for a given combination of prey and predator strategies. It retrieves the corresponding payoff values from the matrices and combines them to form the objective value. In the context of game theory, the objective value represents the overall payoff or benefit obtained by the prey and predator populations when they adopt the given strategy combination. The simulated annealing algorithm aims to find the strategy combination (prey and predator strategies) that maximises the objective value, which corresponds to the Nash equilibrium.
- The simulated_annealing function implements the simulated annealing algorithm to search for the Nash equilibrium. This algorithm is discussed in detail in the second point.
- After running the simulated annealing algorithm, the Nash equilibrium strategies and objective value is returned.
- It then plots the convergence of the objective value and the strategies of prey and predators over iterations.

2. Simulated annealing algorithm to find the Nash equilibrium
    - Start by setting the initial temperature, cooling rate, and maximum number of iterations:
      - ‘initial_temperature’ variable represents the initial temperature value for the simulated annealing algorithm. Temperature is used to control the acceptance of neighbouring solutions. A higher initial temperature allows for a higher probability of accepting worse solutions at the beginning of the algorithm.
      - ‘cooling_rate’ variable determines the rate at which the temperature is reduced in each iteration. It is multiplied by the temperature value to gradually decrease it over time. A lower cooling rate results in a slower decrease in temperature.
    - The initial strategy combination for prey and predator is defined.
    - The current state and objective value are initialised with the initial strategy combination. These variables store the current strategy choices for the prey and predator at each iteration of the algorithm.

- The best state and objective value are also initialised with the current values. These variables store the best strategy choices and corresponding objective value found so far in the algorithm. They are initially set to the current values.
- Temperature, iteration counter, and lists to store objective values and strategies are initialised.
- The main loop performs simulated annealing until the temperature reaches 0 or the maximum number of iterations is reached.
- At each iteration:
  -The current objective value, prey strategy, and predator strategy are added to their respective lists for later analysis.
  -New strategy choices for the prey and predator are randomly generated, respectively. Each strategy is selected from a range of 0 to 1 (inclusive), representing the available strategies.
  -The objective value for the neighbouring state or strategy combination is calculated.
  -The if statement: checks whether the neighbouring objective value is higher than the current objective value. If so, the neighbouring state is accepted, as a higher objective value indicates improvement. The current strategy choices and objective value are updated. (Note that lower objective value is accepted if we want to find minimum optimal solution)
  -The else statement: If the neighbouring objective value is lower than the current objective value, there is a chance that the neighbouring state will still be accepted. The acceptance probability is calculated using the Metropolis criterion, which allows for the acceptance of worse solutions with a decreasing probability as the temperature decreases.
  -Check whether the current objective value is higher than the best objective value found so far. If it is, the best strategy choices and objective value are updated. (updated for lower objective value for minimum optimal solution)
  -The temperature is then reduced by multiplying it by the cooling rate. It gradually decreases the temperature over iterations, allowing for a more focused search towards the end of the algorithm.
  -Lastly, best_prey_strategy, best_predator_strategy, best_objective, objective_values, prey_strategies, and predator_strategies are returned, which represent the best strategy choices and best objective value.
- After running the simulated annealing algorithm, the Nash equilibrium strategies and objective value is returned. Nash equilibrium is a set of strategies (one for each player) where, for every player, the chosen strategy maximises their payoff given the strategies of the other players. It is a solution concept that predicts the likely outcome of a game when players are rational and aim to maximise their own payoff. In this case, the prey (fish) and predator (shark) would choose the strategy that would maximise their own payoff.

- Basically, this algorithm is the simulated annealing algorithm which searches for an optimal strategy combination for a prey and predator game. It explores different strategy choices, evaluates their objective values using the provided payoff matrices, and gradually refines the strategy choices based on the Metropolis criterion and the cooling of the temperature.

Mathematical techniques:

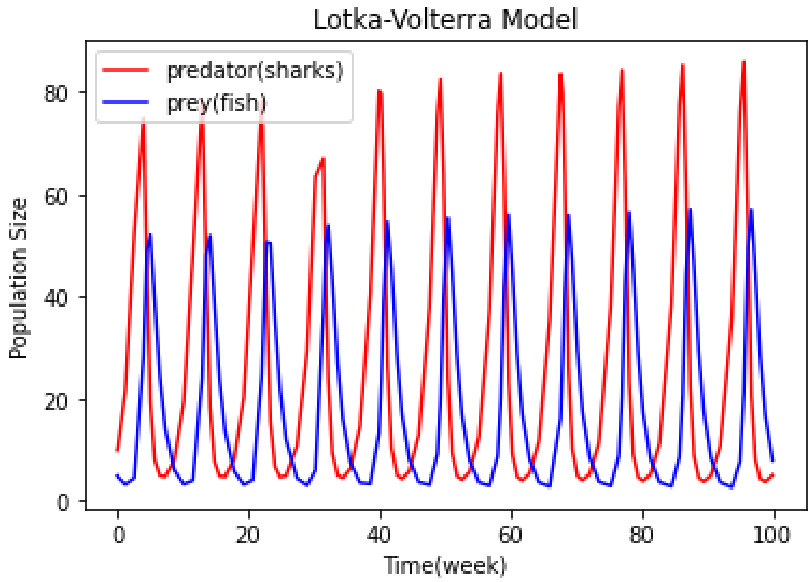
1. Nash Equilibrium: Nash equilibrium is a fundamental concept in game theory, named after John Nash. It represents a set of strategies where no player can unilaterally improve their payoff by changing their strategy, assuming all other players' strategies remain unchanged. Nash equilibrium is often determined by solving systems of equations or through iterative algorithms. In this case, simulated annealing is used to solve for nash equilibrium.
2. Dominant Strategy: A dominant strategy is a strategy that provides a higher payoff regardless of the choices made by other players. Dominant strategies can simplify the analysis of a game by allowing players to focus on the best-response strategy to a dominant strategy.

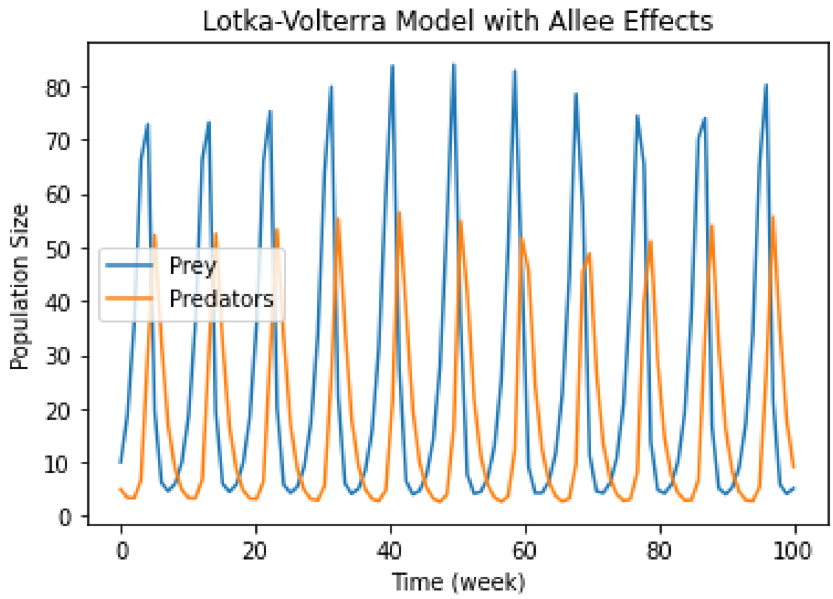
Section 4: Results

Modelling question 1: How does the variability in parameter values affect the population dynamics when the Allee effect is introduced?

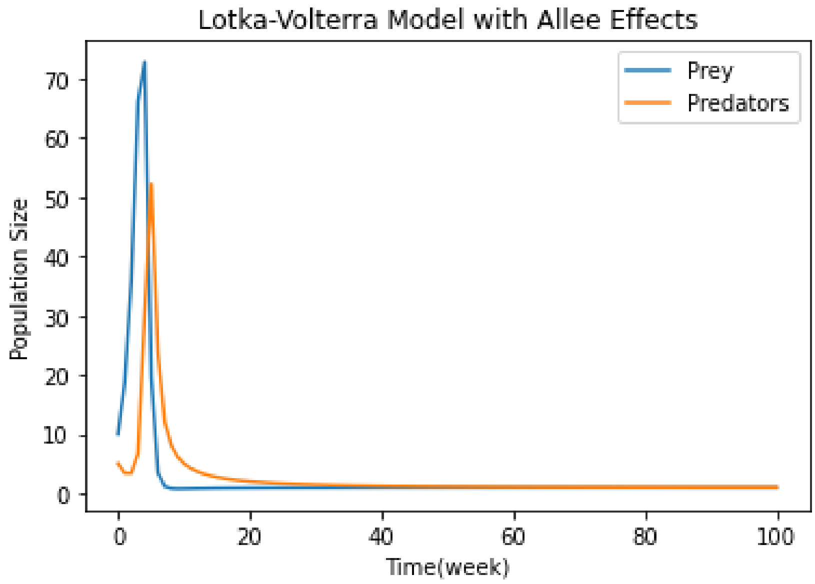
Note: Parameters of model are kept constant unless stated otherwise

Description & parameters of model	Model

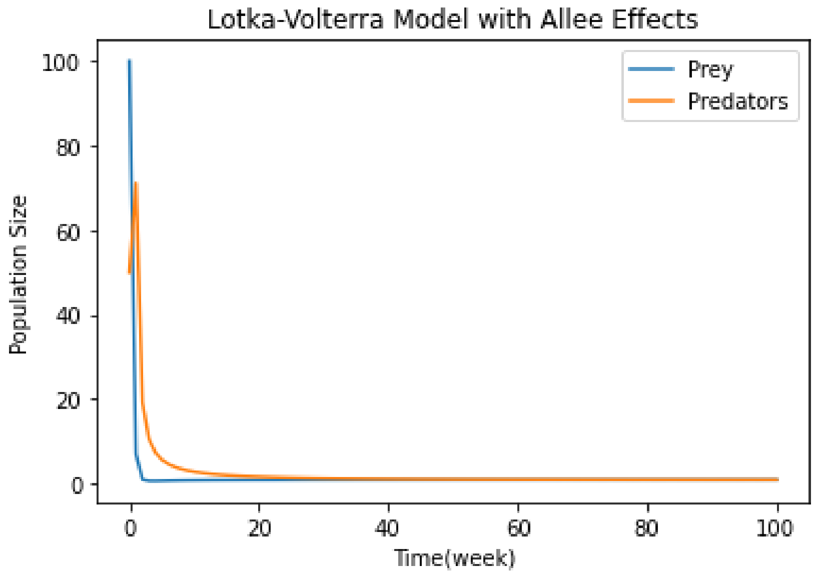
<table>
<tr>
<td>Base model</td>
<td>
</td>
</tr>
<tr>
<td>A:Lotka-Volterra model with Allee effects

-Threshold (Allee effect threshold for prey)= 1
-Prey population=10
-Predator population=5</td>
<td>

The plot of the Lotka-Volterra model with Allee effects above reveals that the Allee effect does not influence the populations of the prey and predator when the threshold value is set to 1. Since the Allee effect is a phenomenon that introduces positive density-dependent effects only at low population densities, in this specific scenario, the prey population size remains way above the threshold value (10), indicating that it is large enough to not be affected by the Allee effect. As a result, the dynamics of the prey and predator populations in this model are not influenced by the Allee effect. In this case, the model above exhibits the same plot as the base model because the Allee effect does not apply to it.</td>
</tr>
</table>

B:Lotka-Volterra model with Allee effects -Threshold (Allee effect threshold for prey)= 10 -Prey population=10 -Predator population=5	 Looking at the model, it is observed that the plot exhibits the same behaviour as that of the base model initially, with oscillatory behaviour observed. However, from week 15 onwards, the Allee effect starts to impact both populations. In this model, a larger threshold value of 10 is chosen, which coincides with the initial prey population. The choice of threshold value indicates that there is a larger chance that the Allee effect would be imposed on the model and that the prey population might need to be sufficiently large to avoid being affected by the Allee effect. As a result, after week 15, both the prey and predator populations experience a continuous decline until they eventually reach zero (extinction). This decline can be attributed to the intensified Allee effect caused by the higher threshold value. Since the prey population serves as the main food source for the predators, the diminishing prey population leads to a corresponding decrease in the predator population. Thus, the plot demonstrates how changes in the threshold value can significantly influence the dynamics and ultimate fate of both populations in the Lotka-Volterra model with Allee effects.

<table>
<tr>
<td>C:Lotka-Volterra model with Allee effects

-Threshold (Allee effect threshold for prey)= 5
-Prey population=100
-Predator population=50</td>
<td>

Upon observing the plot above, it becomes evident that increasing the prey population size to 100 and the predator population size to 50 did not prevent the occurrence of the Allee effect. In this model, with a threshold value set at 5, the Lotka-Volterra system exhibits oscillatory behaviour, characterised by population fluctuations. As the prey population approaches the threshold value, the Allee effect is triggered, influencing both the prey and predator populations. This finding suggests that simply increasing the population size of the prey does not provide immunity against the Allee effect in this particular model due to oscillatory behaviours of the Lotka-Volterra system. Regardless of the prey population size, if the population density falls below the specified threshold, the dynamics of both populations will be affected. Hence, it is apparent that manipulating the prey population size to an excessively large value cannot entirely mitigate the Allee effect in this context. The Lotka-Volterra model with Allee effects demonstrates that the interplay between population dynamics and the threshold value is complex and requires careful consideration.</td>
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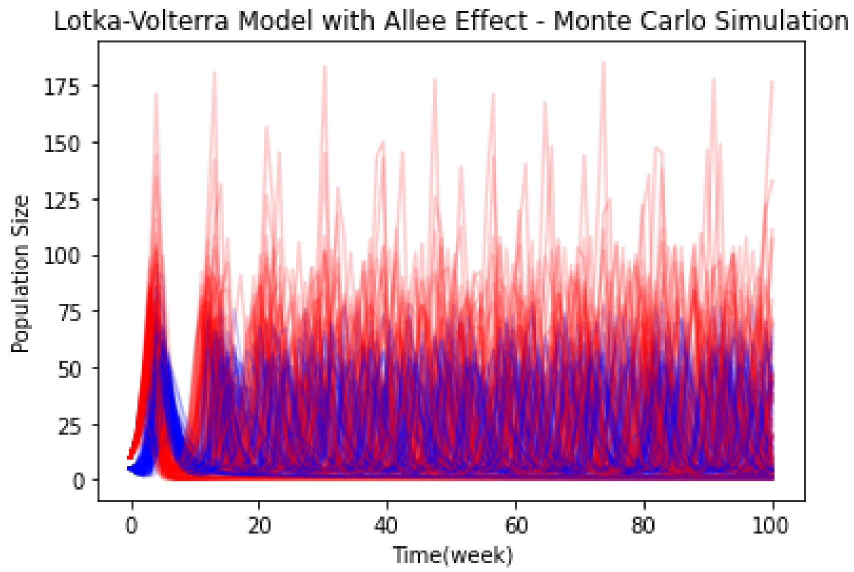
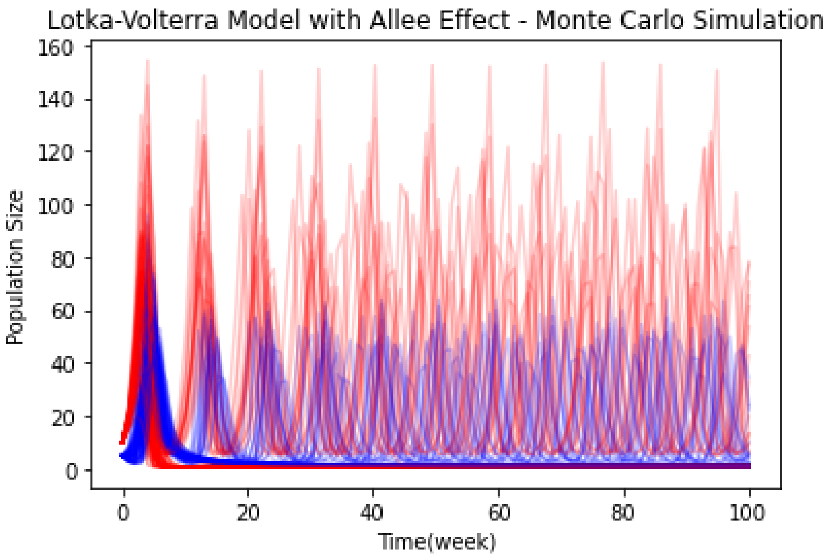
</table>

Now, to answer modelling question 1, we will analyse the plots below which illustrates the Lotka-Volterra model with Allee effects and stochastic growth rate, employing the Monte Carlo technique to account for parameter variability. The inclusion of stochastic growth rate introduces randomness into the model, resulting in multiple trajectories and chaotic behaviour in population dynamics. This stochastic element captures the inherent uncertainty and variability present in real-world ecological systems.

To examine the impact of parameter variability, it is essential to consider the range of parameter values sampled during the Monte Carlo simulation. For instance, the parameter 'r' follows a normal distribution with a mean of 0.8 and a standard deviation of 0.1. Similarly, 's' is uniformly distributed between -1.0 and 0.0, and 'threshold' remains constant throughout the simulation. By sampling from these distributions, we explore a range of parameter values that influence the prey's growth rate, the predator's growth rate, and the mortality rate represented by ‘r’,’s’, and 'f' respectively. The parameters 'alpha' and 'beta' determine the interaction between the prey and predator populations.

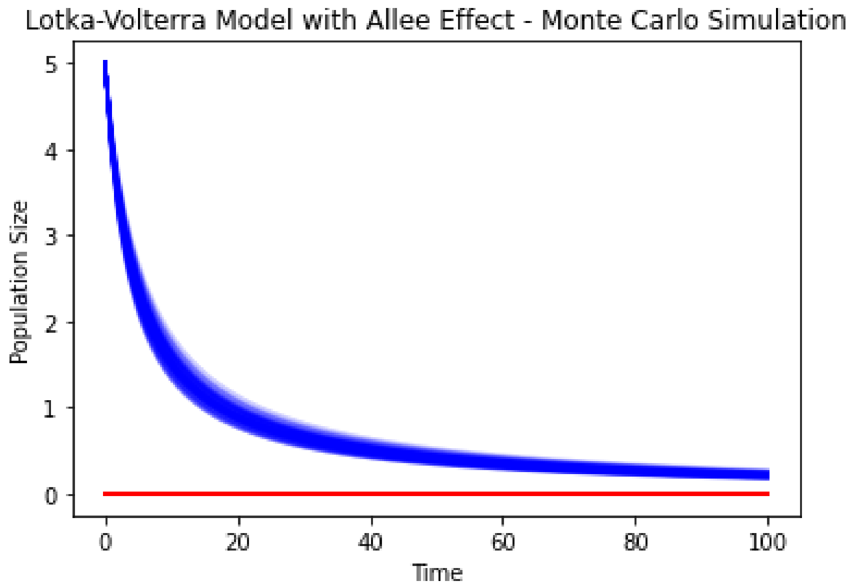
The plot below shows multiple trajectories representing different simulations with varied parameter values. Each trajectory corresponds to the population dynamics of the predator (sharks) and prey (fish) over time. The vertical axis represents the population size, and the horizontal axis represents time. By observing the trajectories, we can analyse the patterns, trends, and fluctuations in the populations.

Description & parameters of model	Model
D:Lotka-Volterra model with Allee effects and stochastic growth rate -Threshold (Allee effect threshold for prey)= 1 -Prey population=10 -Predator population=5 red=prey blue=predator	 Examining the plot, we observe overlapping trajectories and chaotic behaviour, indicating the sensitivity of the model to parameter variations. Although the trajectories share similarities with the base model and model A (due to same threshold value) in terms of oscillatory behaviour, the multiple trajectories showcase the diverse outcomes resulting from uncertainty in the parameter values. The spread, overlap, and density of the trajectories provide insights into the range of potential population dynamics under different parameter combinations. Notably, with a threshold value set to 1 and initial population sizes of 10 for the prey and 5 for the predator, the trajectories display a significant overlap and density throughout time. This indicates a high sensitivity to parameter variations, emphasising the influence of parameter values on the prey and predator populations.

E:Lotka-Volterra model with Allee effects and stochastic growth rate -Threshold (Allee effect threshold for prey)= 5 -Prey population=10 -Predator population=5 red=prey blue=predator	 The plot is more dense before week 15 and becomes less dense after week 15. The trajectories above share a similar pattern as that of the model B. This is because of the same threshold value selected. Since the threshold value is not sampled from a distribution and is kept constant with a value of 5, the Allee effect would be imposed on the model after week 15 when the prey population reaches 5. We can also see a dense trajectory of prey and predator going at a steady decline after the Allee effect is imposed on the populations. In this model however, we can see overlapping and slightly chaotic trajectories after week 15 that are less dense. This indicates a decreasing variability and convergence of the population dynamics after week 15 which makes sense since the Allee effect would be imposed on the populations. The decreasing density of trajectories indicates a reduction in the range of possible outcomes for oscillatory behaviour in both populations and a more focused negative density dependence behaviour as time advances. Therefore, parameter variability also has an impact on this model as we can see overlapping trajectories after week 15 even though they are not dense.

<table>
<tr>
<td>F:Lotka-Volterra model with Allee effects and stochastic growth rate

-Threshold (Allee effect threshold for prey)= 1
-Prey population=0
-Predator population=5

red=prey
blue=predator</td>
<td>

In this plot, our focus is on examining the simulation results when the prey population goes extinct. As expected, we observe a steady decline in the predator population, which is a direct consequence of the prey population's extinction. This decline occurs because predators rely on the availability of prey as their primary food source.

Upon inspecting the plot, we can observe that there are no overlapping or chaotic trajectories. Instead, we see a slightly dense distribution of the predator population, which is consistently decreasing over time. This pattern aligns with our expectations, given the absence of prey and the influence of the Allee effect on the remaining predator individuals.

Therefore, it appears that the variability in parameter values has a limited impact on this particular model scenario, where the prey population has gone extinct. The absence of prey removes the complex interactions and dynamics between predator and prey populations, resulting in a more straightforward and predictable trajectory for the predator population.</td>
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</table>

In your analysis, you can provide a detailed explanation of the model extension, including the assumptions made for both the original and extended Lotka-Volterra models. Specify the class of the model and analysis, highlighting its numerical, non-linear, continuous, and stochastic nature. Additionally, describe the algorithms or mathematical techniques used, such as Monte Carlo simulation and numerical methods for solving differential equations. This will help the reader understand the foundations, assumptions, and computational aspects of your model and analysis.

In conclusion, Monte Carlo simulations allow us to explore the effects of parameter uncertainty and variability on the Lotka-Volterra model. The multiple trajectories illustrate the variability in the population dynamics due to the uncertainty in the parameter values. The spread of the trajectories indicates the range of possible outcomes for the predator and prey populations. The variability can arise from different combinations of parameter values and their interactions, leading to diverse population behaviours.

By observing the plot as a whole, we can assess the overall variability and uncertainty in the model's predictions. The spread, overlap, and density of the trajectories give an impression of the model's sensitivity to parameter variations and the range of potential outcomes.

The plot provides a visual representation of these effects, helping us understand the potential population dynamics and the range of possible scenarios in the predator-prey relationship.

<u>Modelling question 2:What are the optimal strategies for the prey and predator populations in a game where the prey can choose between 'in a school' or 'alone,' and the predator can choose between 'hunting aggressively' or 'hunting cautiously'?</u>

According to Alcott(2015), sticking together helps fish avoid predators by relying on a team of eyes that could potentially detect a predator and then signal to the rest of the group that a predator is nearby. Therefore, the payoff for when fishes are in a school would be higher compared to when they are alone.

prey_payoff_matrix = np.array([[7, 3], [1, 2]])

Prey ll Predator	Hunt Aggressively	Hunt Cautiously
In a school	7	3
Alone	1	2

- The top-left element, 7, represents a higher payoff for the prey when it chooses the strategy "in a school" and the predator chooses the strategy "hunting aggressively." This suggests that being in a school provides better protection against aggressive hunting.
- The top-right element, 3, represents a lower payoff for the prey when it chooses the strategy "in a school" and the predator chooses the strategy "hunting cautiously." This indicates that being in a school may not provide as much advantage against cautious predators.

- The bottom-left element, 1, represents a lower payoff for the prey when it chooses the strategy "alone" and the predator chooses the strategy "hunting aggressively." This suggests that being alone exposes the prey to higher risks from aggressive predators.
- The bottom-right element, 2, represents a higher payoff for the prey when it chooses the strategy "alone" and the predator chooses the strategy "hunting cautiously." This indicates that being alone may allow the prey to avoid or escape from cautious predators more effectively.

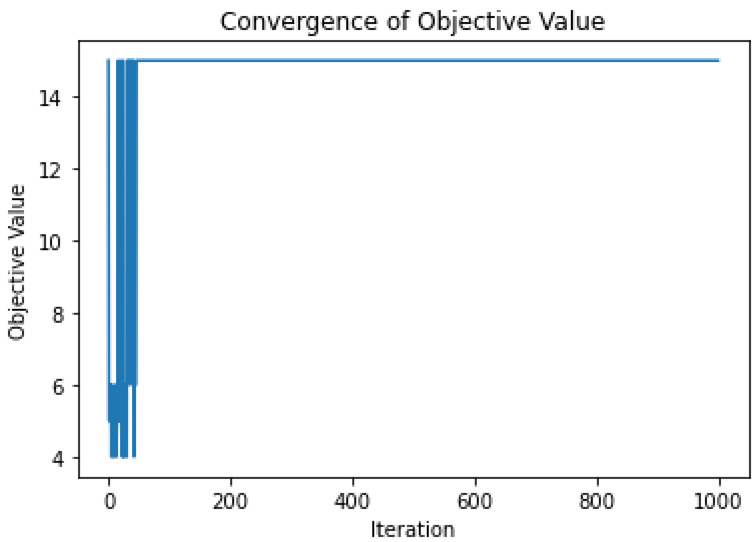
predator_payoff_matrix = np.array([[8, 3], [3, 4]])

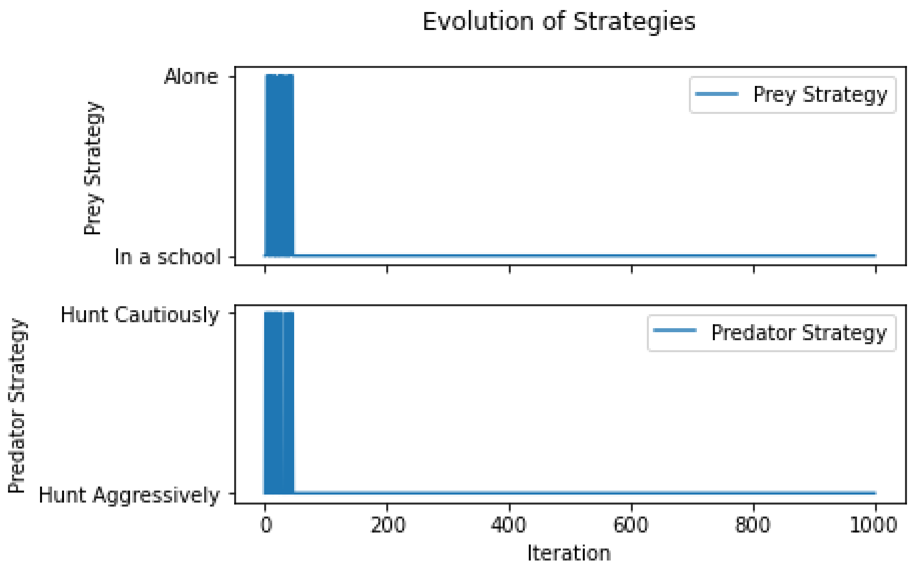
Predator || Prey	In a school	Alone
Hunt Aggressively	8	3
Hunt Cautiously	3	4

- The top-left element, 8, represents a higher payoff for the predator when it chooses the strategy "hunting aggressively" against prey in a school. This suggests that aggressive hunting is more successful in capturing prey in a school.
- The top-right element, 3, represents a lower payoff for the predator when it chooses the strategy "hunting aggressively" against lone prey. This suggests that aggressive hunting may not be as effective when the prey is alone.
- The bottom-left element, 3, represents a lower payoff for the predator when it chooses the strategy "hunting cautiously" against prey in a school. This indicates that cautious hunting may not yield as much success against prey in a school.
- The bottom-right element, 4, represents a higher payoff for the predator when it chooses the strategy "hunting cautiously" against lone prey. This indicates that cautious hunting is more successful in capturing lone prey.

Maximum optimal solution:

```
Nash Equilibrium Strategies:
Prey Strategy: In a school
Predator Strategy: Hunt Aggressively
Objective Value: 15
```





The objective value in this context represents the cumulative payoff or benefit achieved by the prey and predator populations when they adopt specific strategy combinations. The simulated annealing algorithm aims to identify the strategy combination (prey and predator strategies) that maximises the objective value, corresponding to the Nash equilibrium.

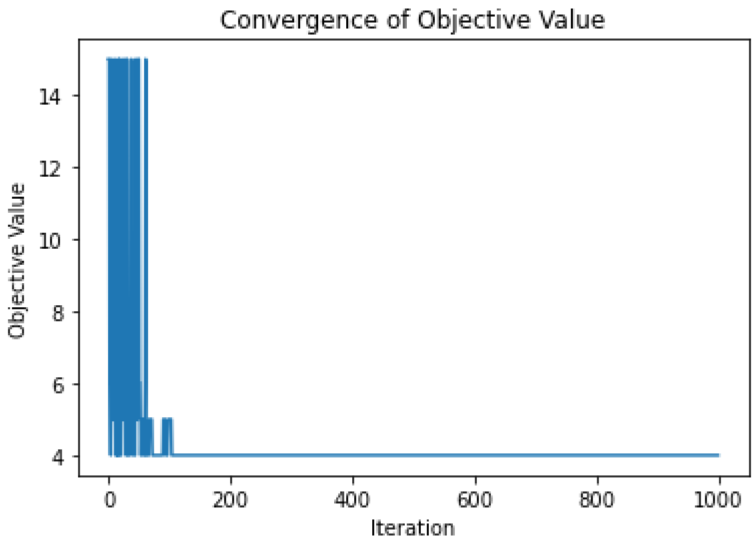
Analysing the plot above, we observe that the simulated annealing algorithm achieves a maximum objective value of 15 within a relatively small number of iterations (less than 100). This indicates that the algorithm efficiently converges to a favourable solution.

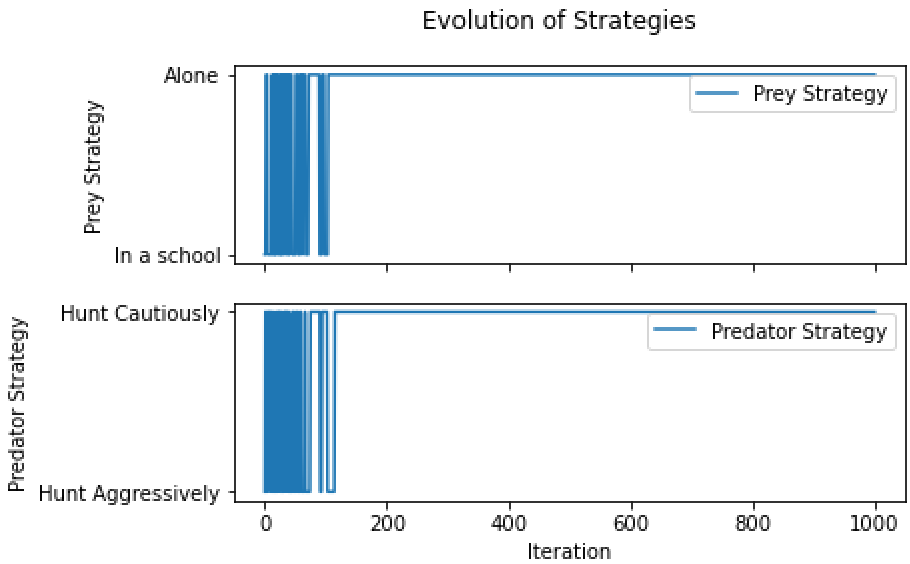
The Nash equilibrium strategies, which represent the optimal choices for both the prey and predator, can be identified based on the maximum objective value. In this case, the optimal strategy for the prey is to be "in a school" (payoff=7), while the optimal strategy for the predator is to "hunt aggressively" (payoff=8). These strategies result in the highest cumulative payoff for both populations (objective value=15).

By identifying the Nash equilibrium strategies, we can conclude that the "in a school" strategy is the dominant strategy for the prey population, as it provides a higher payoff compared to being alone. Similarly, the "hunt aggressively" strategy is the dominant strategy for the predator population, as it yields a higher payoff than hunting cautiously.

Minimum optimal solution:

```
Minimum Optimal Strategies:
Prey Strategy: Alone
Predator Strategy: Hunt Cautiously
Objective Value: 4
```



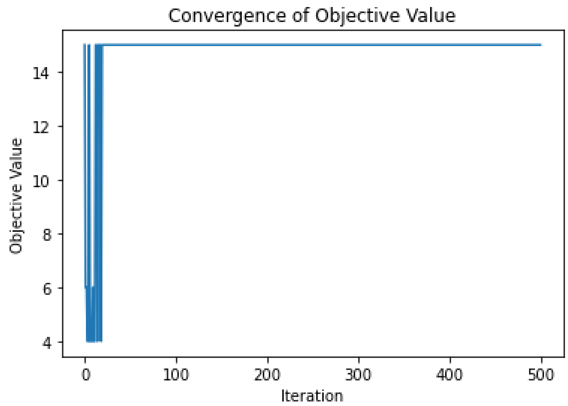
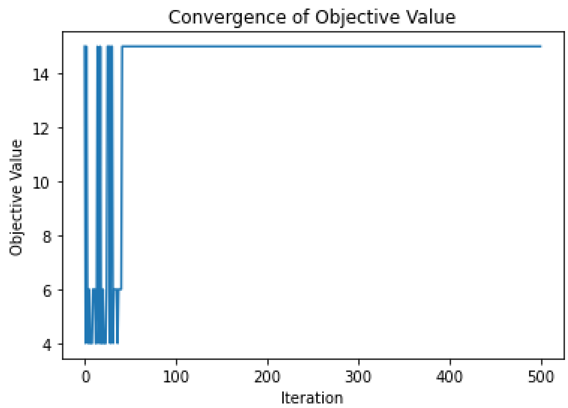


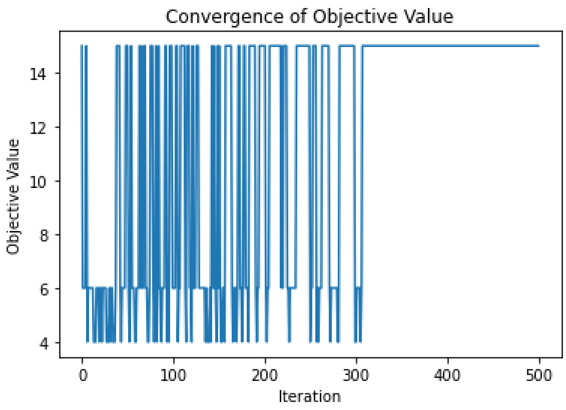
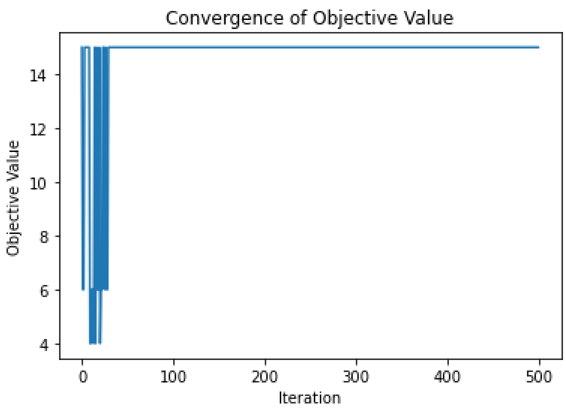
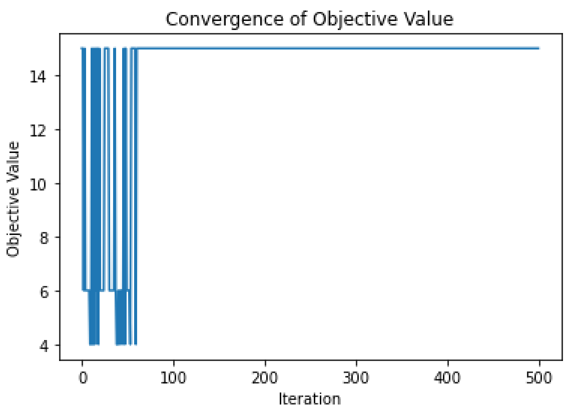
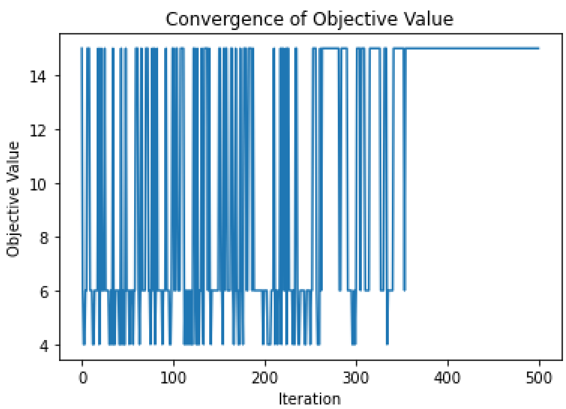
To obtain the minimum optimal solution, we need to minimise the objective value. Therefore, the simulated annealing algorithm is tweaked to identify the strategy combination (prey and predator strategies) that minimises the objective value. Looking at the plot above, we observe that the simulated annealing algorithm achieves a minimum objective value of 4 within a relatively small number of iterations (around 100). This indicates that the algorithm efficiently converges to a favourable solution.

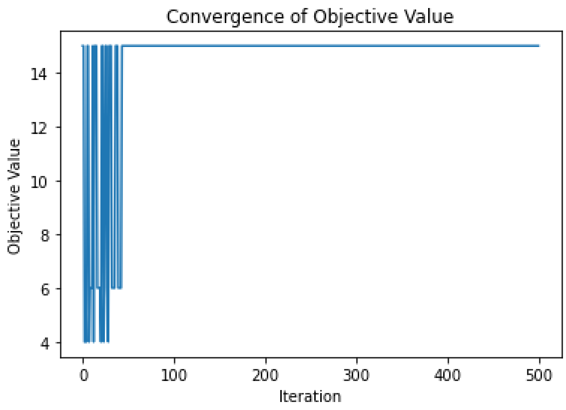
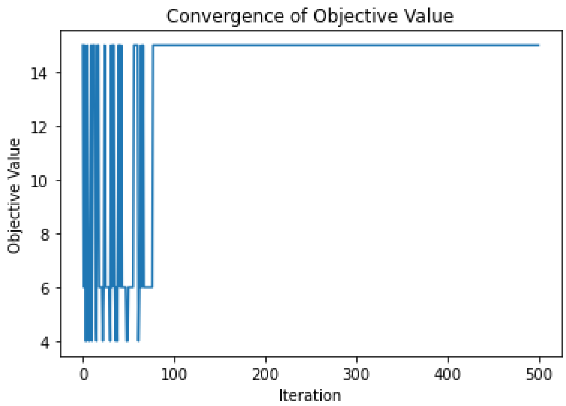
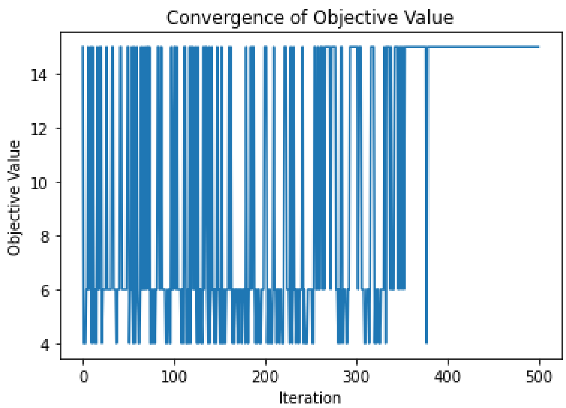
Since we are looking for the least optimal strategies, in this case, the least optimal strategy for the prey is to be "alone", while the least optimal strategy for the predator is to "hunt cautiously" These strategies result in the lowest cumulative payoff for both populations (objective value=4).

Modelling question 3:How do changes in the initial temperature and cooling rate affect the convergence of the objective value and the optimal strategies?

A sensitivity analysis is performed by varying the initial temperature and cooling rate. We observe how changes in these parameters affect the convergence of the objective value and the optimal strategies.

<table>
<tr><td>A:
Initial Temperature: 50.0
Cooling Rate: 0.9
Max Iterations: 500
Nash Equilibrium Strategies:
Prey Strategy: In a school
Predator Strategy: Hunt Aggressively
Objective Value: 15</td><td>
</td></tr>
<tr><td>B:
Initial Temperature: 50.0
Cooling Rate: 0.95
Max Iterations: 500
Nash Equilibrium Strategies:
Prey Strategy: In a school
Predator Strategy: Hunt Aggressively
Objective Value: 15</td><td>
</td></tr>
</table>

C: Initial Temperature: 50.0 Cooling Rate: 0.99 Max Iterations: 500 Nash Equilibrium Strategies: Prey Strategy: In a school Predator Strategy: Hunt Aggressively Objective Value: 15	
D: Initial Temperature: 100.0 Cooling Rate: 0.9 Max Iterations: 500 Nash Equilibrium Strategies: Prey Strategy: In a school Predator Strategy: Hunt Aggressively Objective Value: 15	
E: Initial Temperature: 100.0 Cooling Rate: 0.95 Max Iterations: 500 Nash Equilibrium Strategies: Prey Strategy: In a school Predator Strategy: Hunt Aggressively Objective Value: 15	
F: Initial Temperature: 100.0 Cooling Rate: 0.99 Max Iterations: 500 Nash Equilibrium Strategies: Prey Strategy: In a school Predator Strategy: Hunt Aggressively Objective Value: 15	

G: Initial Temperature: 200.0 Cooling Rate: 0.9 Max Iterations: 500 Nash Equilibrium Strategies: Prey Strategy: In a school Predator Strategy: Hunt Aggressively Objective Value: 15	
H: Initial Temperature: 200.0 Cooling Rate: 0.95 Max Iterations: 500 Nash Equilibrium Strategies: Prey Strategy: In a school Predator Strategy: Hunt Aggressively Objective Value: 15	
I: Initial Temperature: 200.0 Cooling Rate: 0.99 Max Iterations: 500 Nash Equilibrium Strategies: Prey Strategy: In a school Predator Strategy: Hunt Aggressively Objective Value: 15	

Looking at the plots above, we can see how the changes in the parameter values (initial temperature and cooling rate) do not impact the identification of optimal strategies. The algorithm's results consistently converge to the same optimal strategies and objective value.

In simulated annealing, the initial temperature plays a crucial role in the exploration of the search space. A higher initial temperature allows for more exploration, enabling the algorithm to escape local optima and potentially find a global optimum. However, it may also result in slower convergence as the algorithm spends more time exploring the search space which is proven to be true by the plots above. We can compare plots A, D, and G which have initial temperatures of 50, 100, 200 respectively with the same cooling rate. Plot A takes the least number of iterations whereas plot G takes the most number of iterations to converge to the optimal solution. Therefore, a lower initial temperature focuses more on exploitation, increasing the chance of converging to a local optimum, allowing it to converge faster as it quickly narrows down the search space.

A higher cooling rate implies a more rapid decrease in temperature over time. This means that the algorithm moves towards lower temperatures and exploitation of promising solutions at a faster pace. As a result, the algorithm converges more quickly, potentially reaching a stable solution in fewer iterations. However, this faster convergence comes at the cost of limited exploration of the search space, potentially leading to a suboptimal solution if the exploration is insufficient. Conversely, a lower cooling rate slows down the decrease in temperature, allowing the algorithm to explore the search space more extensively. This increased exploration enables the algorithm to potentially find better solutions by escaping local optima and discovering more globally optimal regions. However, the trade-off is that the convergence may take longer since the algorithm spends more time exploring the search space rather than focusing on exploitation.

To observe the effect of cooling rate on convergence, we can compare the plots A, B, and C with different cooling rates (0.9, 0.95, and 0.99) while keeping the initial temperature constant. Plot A, with a cooling rate of 0.9, will show the fastest convergence as the temperature decreases rapidly (least amount of iterations). Plot C, with a cooling rate of 0.99, will exhibit the slowest convergence as the temperature decreases more slowly, allowing for more extensive exploration(most amount of iterations). Plot B, with a cooling rate of 0.95, falls in between these two extremes.

In summary, a higher initial temperature encourages exploration and may lead to finding a global optimum, but it slows down convergence. A lower initial temperature favours exploitation and faster convergence but increases the risk of getting stuck in local optima. The cooling rate influences the convergence speed, as a higher cooling rate exhibits the slowest convergence as the temperature decreases more slowly and a lower cooling rate showing the fastest convergence as the temperature decreases rapidly.

Section 5: List of algorithms and concepts

- Lotka-Volterra Model:A mathematical model used to describe the interactions between predator and prey populations in an ecosystem. It consists of a set of differential equations that represent the population dynamics of the predator and prey species.

- solve_ivp: A function from the SciPy library used to solve initial value problems (IVPs) represented by a system of differential equations. Used in the base model, the Lotka-Volterra model with Allee effects, and the Lotka-Volterra model with Allee effects and stochastic growth rate.
- Allee Effect: A phenomenon in population dynamics where the growth rate of a species decreases as the population size becomes too small.
- Monte Carlo Simulation: A statistical technique used to estimate the outcomes of a complex system by repeatedly sampling from probability distributions of its input parameters. Used to simulate the Lotka-Volterra model with Allee effects and stochastic growth rate.
- np.random.normal: A function from the NumPy library used to generate random numbers from a normal distribution. Used in the Lotka-Volterra model with Allee effects and stochastic growth rate.
- np.random.uniform: A function from the NumPy library used to generate random numbers from a uniform distribution. Used in the Lotka-Volterra model with Allee effects and stochastic growth rate.
- np.random.exponential: A function from the NumPy library used to generate random numbers from an exponential distribution. Used in the Lotka-Volterra model with Allee effects and stochastic growth rate.
- Payoff matrices: Matrices that represent the payoffs (rewards) for different strategy combinations of prey and predator in a game theory setting. Used in Modelling question 2.
- Objective function: A function that quantifies the desirability or performance of a particular solution or strategy combination. Used in Modelling question 2.
- Simulated annealing: A metaheuristic optimization algorithm inspired by the annealing process in metallurgy. It is used to search for the global optimum in a large search space by iteratively exploring and refining solutions based on a cooling schedule and acceptance probability. Used in Modelling question 2 to obtain minimum and maximum optimal solution.
- Nash equilibrium: A concept in game theory where each player's strategy is the best response to the other player's strategy, resulting in a stable solution where no player has an incentive to unilaterally deviate. Mentioned in Modelling question 2.
- Convergence: The process of reaching a stable or optimal solution through iterative refinement. Mentioned in Modelling question 2.
- Sensitivity analysis: A technique used to study the effect of parameter variations on the output of a system or model. Mentioned in Modelling question 3.

(OpenAI, 2023)

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OpenAI. (2023). ChatGPT (Jan 9 version) [Large language model] http://chat.openai.com/chat

I acknowledge the use of [1] ChatGPT (https://chat.openai.com/) to [2] generate materials that were [4] included within my final assessment in modified form. I entered the following prompts on 14 June 2023:

[3] List the concepts and algorithms in the Lotka-Volterra base Model, Lotka-Volterra model with Allee effects, Lotka-Volterra model with Allee effects and stochastic growth rate, Game theory and Heuristics, and Sensitivity analysis for simulated annealing analysis
[4] The output from the generative artificial intelligence was adapted and modified for the final response.