电磁场的四维势

单位制保持和朗道第二卷《场论》一致，这样抄书比较方便
本书的度规选择 $(+,-,-,-)$

$$\mathcal{L} \propto \frac{1}{16\pi} F^{\mu\nu} F_{\mu\nu} + \frac{1}{c} A^{\mu} J_{\mu}$$

其中 $F^{\mu\nu} = \partial^{\mu} x^{\nu} - \partial^{\nu} x^{\mu}, J^{\mu} = (c\rho, \boldsymbol{j})$

根据Eular-Lagrange方程计算电磁场方程

$$\partial_{\mu} \frac{\partial \mathcal{L}}{\partial(\partial_{\mu} A^{i})} - \frac{\partial \mathcal{L}}{\partial A^{i}} = 0$$

我们一项项的计算

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial\left(\partial_{\gamma} A^{i}\right)} &= \frac{\partial}{\partial\left(\partial_{\gamma} A^{i}\right)}\left[\frac{1}{16\pi}(\partial^{\mu} A^{\nu} - \partial^{\nu} A^{\mu})(\partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu})\right] \\
&= \frac{\partial}{\partial\left(\partial_{\gamma} A^{i}\right)}\left[\frac{1}{16\pi}\left(g^{\mu\rho} \partial_{\rho} A^{\nu} - g^{\nu\rho} \partial_{\rho} A^{\mu}\right)\left(g_{\nu\theta} \partial_{\mu} A^{\theta} - g_{\mu\theta} \partial_{\nu} A^{\theta}\right)\right] \\
&= \tfrac{1}{16\pi} \tfrac{\partial}{\partial(\partial_{\gamma} A^{i})}\Big[\left(g^{\mu\rho} g_{\nu\theta} \partial_{\rho} A^{\nu} \partial_{\mu} A^{\theta} - g^{\nu\rho} g_{\nu\theta} \partial_{\rho} A^{\mu} \partial_{\mu} A^{\theta}\right) \\
&- \left(g^{\mu\rho} g_{\mu\theta} \partial_{\rho} A^{\nu} \partial_{\nu} A^{\theta} - g^{\nu\rho} g_{\mu\theta} \partial_{\rho} A^{\mu} \partial_{\nu} A^{\theta}\right)\Big] \\
&= \tfrac{1}{16\pi} \tfrac{\partial}{\partial(\partial_{\gamma} A^{i})}\Big[\left(g^{\mu\rho} g_{\nu\theta} \partial_{\rho} A^{\nu} \partial_{\mu} A^{\theta} - \delta_{\theta}^{\rho} \partial_{\rho} A^{\mu} \partial_{\mu} A^{\theta}\right) \\
&- \left(\delta_{\theta}^{\rho} \partial_{\rho} A^{\nu} \partial_{\nu} A^{\theta} - g^{\nu\rho} g_{\mu\theta} \partial_{\rho} A^{\mu} \partial_{\nu} A^{\theta}\right)\Big] \\
&= \tfrac{1}{16\pi} \tfrac{\partial}{\partial(\partial_{\gamma} A^{i})}\Big[\left(g^{\mu\rho} g_{\nu\theta} \partial_{\rho} A^{\nu} \partial_{\mu} A^{\theta} - \partial_{\rho} A^{\mu} \partial_{\mu} A^{\rho}\right) \\
&- \partial_{\rho} A^{\nu} \partial_{\nu} A^{\rho} + g^{\nu\rho} g_{\mu\theta} \partial_{\rho} A^{\mu} \partial_{\nu} A^{\theta}\Big] \\
&= \frac{1}{8\pi} \frac{\partial}{\partial\left(\partial_{\gamma} A^{i}\right)}\left(g^{\mu\rho} g_{\nu\theta} \partial_{\rho} A^{\nu} \partial_{\mu} A^{\theta} - \partial_{\rho} A^{\mu} \partial_{\mu} A^{\rho}\right) \\
&= \tfrac{1}{8\pi}\Big(g^{\mu\rho} g_{\nu\theta}\left(\delta_{\gamma\rho} \delta^{\nu i}\right) \partial_{\mu} A^{\theta} + g^{\mu\rho} g_{\nu\theta} \partial_{\rho} A^{\nu}\left(\delta_{\gamma\mu} \delta^{i\theta}\right) \\
&- \left(\delta_{\gamma\rho} \delta^{\mu i} \cdot \partial_{\mu} A^{\rho} + \partial_{\rho} A^{\mu} \cdot \delta_{\gamma\mu} \delta^{\rho i}\right)\Big) \\
&= \frac{1}{8\pi}(\partial^{\gamma} A_{i} + \partial^{\gamma} A_{i} - (\partial_{i} A^{\gamma} + \partial_{i} A^{\gamma})) \\
&= \frac{1}{4\pi} g_{ij}\left(\partial^{\gamma} A^{j} - \partial^{j} A^{\gamma}\right) \\
&= \frac{1}{4\pi} g_{ij} F^{\gamma j} = \frac{1}{4\pi} F^{\gamma}{}_{i}
\end{aligned}$$

$$\begin{aligned}
\partial_{\gamma} \frac{\partial \mathcal{L}}{\partial\left(\partial_{\gamma} A^{i}\right)} &= \frac{1}{4\pi} \partial_{\gamma} F^{\gamma}{}_{i} \\
&= \frac{1}{4\pi} g_{ij} \partial_{\gamma}\left(\partial^{\gamma} A^{j} - \partial^{j} A^{\gamma}\right) \\
&= \frac{1}{4\pi} g_{ij}\left(\partial_{\gamma} \partial^{\gamma} A^{j} - \partial^{j} \partial_{\gamma} A^{\gamma}\right) \\
&\text{Lorentz规范}: \partial_{\mu} A^{\mu} = 0 \\
&= \frac{1}{4\pi} \partial_{\mu} \partial^{\mu} A^{i}
\end{aligned}$$

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial A^i} &= \frac{\partial}{\partial A^i} A^\mu J_\mu \\
&= \delta_i^\mu J_\mu \\
&= J_i
\end{aligned}
$$

于是E-L方程导出为

$$
\begin{gathered}
\frac{1}{4\pi}\partial_\mu \partial^\mu A_i - \frac{1}{c} J_i = 0 \\
\Rightarrow \partial_\mu \partial^\mu A^j - \frac{4\pi}{c} J^j = 0
\end{gathered}
$$

将四维电流矢量写成电势和磁势

$$
J^\mu = (c\rho, \boldsymbol{j})
$$

得到给定电荷分布和电流分布的求矢势的方程，我们定义达朗贝尔算符

$$
\begin{gathered}
\Box := \Delta - \frac{\partial}{\partial t^2} = \frac{\partial}{\partial x^2} + \frac{\partial}{\partial y^2} + \frac{\partial}{\partial z^2} - \frac{\partial}{\partial t^2} \\
\Box = \partial_1 \partial_1 + \partial_2 \partial_2 + \partial_3 \partial_3 - \partial_0 \partial_0 = -\partial_\mu \partial^\mu \\
\Rightarrow \Box \varphi = -4\pi\rho, \Box \boldsymbol{A} = -\frac{4\pi}{c} \boldsymbol{j}
\end{gathered}
$$

利用格林函数法根据这对两个方程的求解可以导出库仑定律和比奥-萨法尔定律