Reinforcement Learning China Summer School

RLChina 2022

Game Theory Basics

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Outline

- Motivation and Normal-form Game
- Extensive-form Game and Imperfect Information
- Bayesian Game and Incomplete Information
- Nash Equilibrium and Variants
- Theoretical Results of Nash Equilibrium
- Repeated Game and Learning Methods
- Evolutionary Game Theory and Coalitional Game Theory

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Collective Decision Intelligence

- Progress of Artificial Intelligence

Recognition Intelligence → Decision Intelligence → Collective Decision Intelligence

Deep Learning → Deep Reinforcement Learning → Multi-agent Reinforcement Learning

Reinforcement Learning → Deep Reinforcement Learning

Game Theory → Multi-agent Reinforcement Learning

Games in Reality

Rock, Scissors, Paper

Auction

Chess

Poker

History of Game Theory

1934, Stackelberg,
Stackelberg Equilibrium[1]
1950, **Nash**,
Mixed Nash Equilibrium[2]

1951, Brown,
Fictitious Play in Repeated game[3]
1965, **Selten**,
Subgame Perfect Equilibrium in Extensive-form Game[4]

1967, **Harsanyi**,
Bayesian Nash Equilibrium in Bayesian game[5]

1973, Smith & Price,
Evolutional Game Theory[6]
1974, **Aumann**,
Correlated Equilibrium[7]

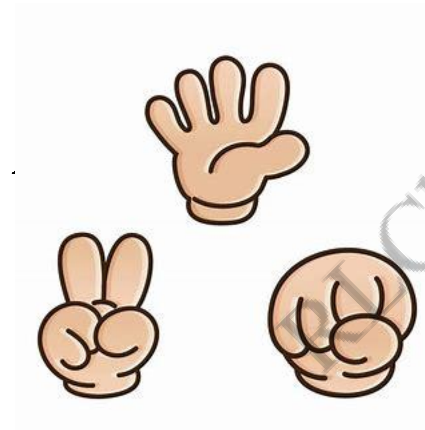
1994,
Papadimitriou,
PPAD[8]

Till now, 18 game theorists received **Nobel Prize in Economics**!

Elements of Game

- Players $N = \{1,2,\ldots,n\}$
  - $N = \{1,2\}$

- Strategies (actions) $A = A_1 \times A_2 \times \ldots \times$
  - $A_1 = \{R, S, P\}$
  - $A_2 = \{R, S, P\}$



- Payoff (utility) $u = (u_1, u_2, \ldots, u_n)$, $u_i: A \to \mathbb{R}$
  - $u_1: A_1 \times A_2 \to \mathbb{R}$
  - $u_2: A_1 \times A_2 \to \mathbb{R}$

Normal-form Game

- Payoff Matrix

Column Player Actions

Row Player Actions

	R	S	P
R	0, 0	1, -1	-1, 1
S	-1, 1	0, 0	1, -1
P	1, -1	-1, 1	0, 0

- More than 2 players

Joint Actions

	p_1	p_2	p_3
R, R, R	0	-1	1
R, R, S	1	1	0
……	……	……	……

Rationality of Players

- Self-interested
  - Preference over game outcome
  - E.g. (paper, rock) is better than (rock, paper) for row player
- Utility
  - Utility of (paper, rock) is 1
  - Utility of (rock, paper) is -1
- Objective
  - Act to maximize (expected) utility

Common Knowledge

- Definition
  - All the players know *p*, they all know that they know *p*, they all know that they all know that they know *p*, and so on

- Example
  - ×
    - Alice knows ‘the weather is good’
    - Bob Knows ‘the weather is good’
  - ×
    - Alice and Bob knows ‘the weather is good’ respectively
    - Alice and Bob knows ‘the opposite knows the weather is good’
  - √
    - Alice and Bob are told ‘the weather is good’ together

Pure Strategy and Mixed Strategy

Matching Pennies

	Heads	Tails
Heads	1, -1	-1, 1
Tails	-1, 1	1, -1

- Pure Strategy
  - $a_1 \in A_1 = \{Heads,\ Tails\}$
  - $a_2 \in A_2 = \{Heads,\ Tails\}$
- Mixed Strategy: Probability Distribution over Pure Strategy
  - $a_1 = (x_H, x_T),\ \ x_H \in [0,1],\ \ x_T \in [0,1],\ \ x_H + x_T = 1$
  - $a_2 = (y_H, y_T),\ \ y_H \in [0,1],\ \ y_T \in [0,1],\ \ y_H + y_T = 1$
- Expected Utility
  - $EU_1 = x_H y_H u_1(H,\ H) + x_H y_T u_1(H,\ T) + x_T y_H u_1(T,\ H) + x_T y_T u_1(T,\ T)$
  - $EU_2 = x_H y_H u_2(H,\ H) + x_H y_T u_2(H,\ T) + x_T y_H u_2(T,\ H) + x_T y_T u_2(T,\ T)$
- Example
  - $a_1 = (0.1,0.9), a_2 = (0.3,0.7)$
  - $EU_1 = 0.32, EU_2 =-0.32$

Classic Games

- Zero-sum Game
  - $u_1(a) + u_2(a) = 0, \forall\, a \in A$
- Cooperative Game
  - $u_i(a) = u_j(a), \forall\, a \in A, i, j \in N$
- Coordination Game
  - Multiple Nash Equilibria Exist
- Social Dilemma [9]
  - Everyone suffers in an NE

Matching Pennies

	Heads	Tails
Heads	1, -1	-1, 1
Tails	-1, 1	1, -1

Road Selection

	Left	Right
Left	1, 1	0, 0
Right	0, 0	1, 1

Battle of Sex

	Party	Home
Party	10, 5	0, 0
Home	0, 0	5, 10

Prisoner’s Dilemma

	Cooperate	Defect
Cooperate	2, 2	0, 3
Defect	3, 0	1, 1

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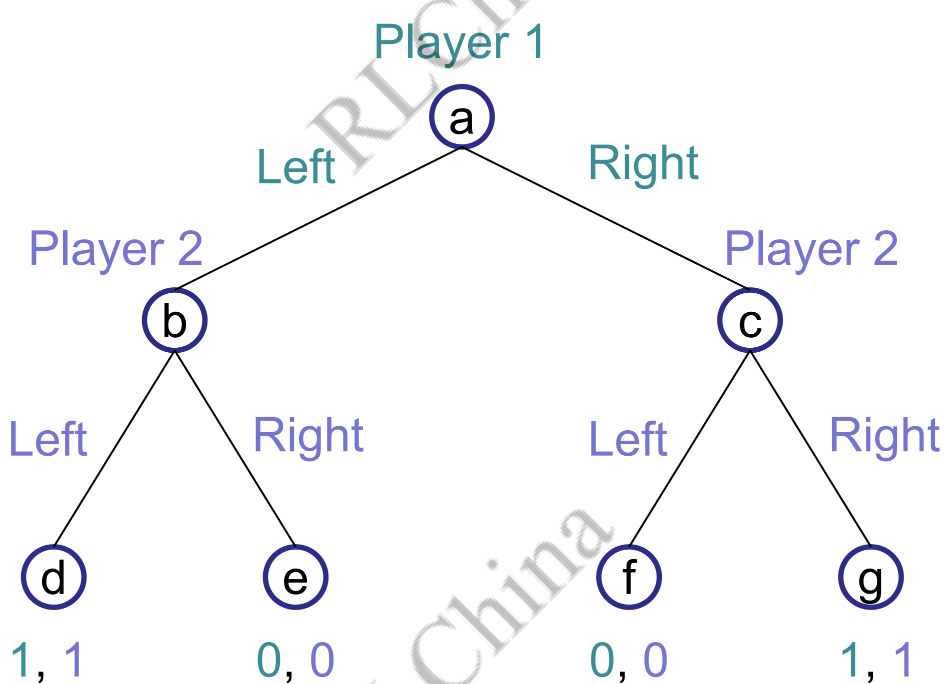
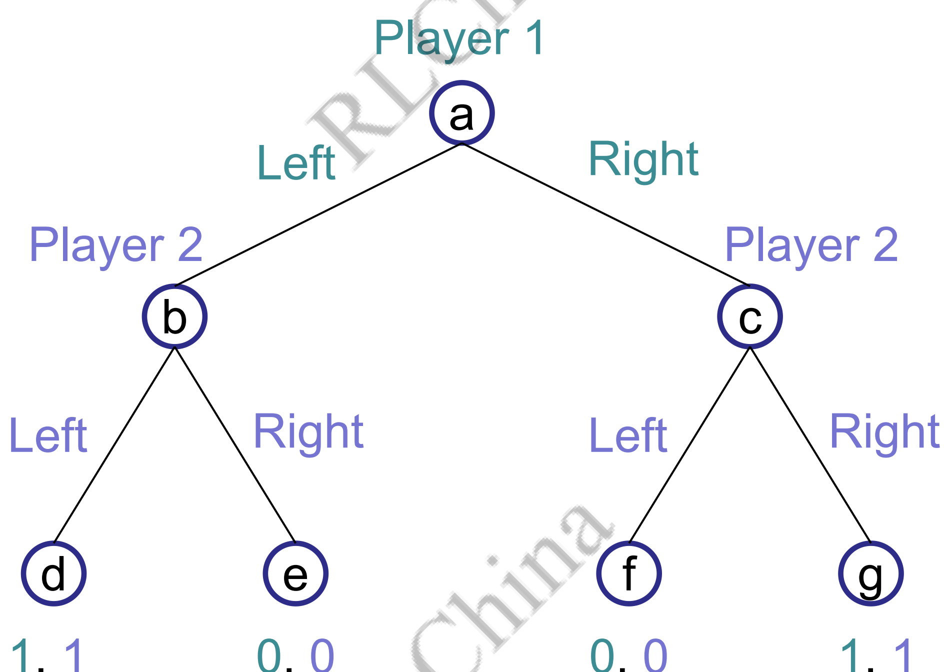
Extensive-form Game

- Game Tree
  - Node: decision point for a specified player
  - Edge: action decided by the player
  - Leaf: outcome of the game with payoff



Strategies in Extensive-form Game

- Strategy Space
  - Player 1: {Left, Right}
  - Player 2: {$\underbrace{\text{(Left, Left)}}_{\text{action in every node}}$, (Left, Right), (Right, Left), (Right, Right)}



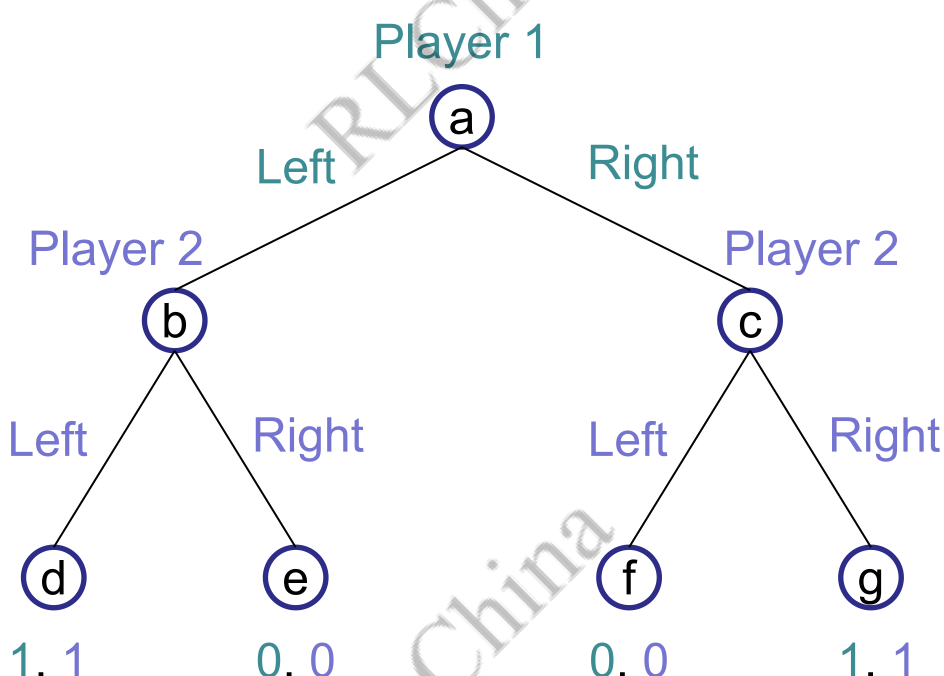
Extensive-form vs. Normal-form

- Equivalent Normal-form Game

single step/state ⇕ **static**

	(Left, Left)	(Left, Right)	(Right, Left)	(Right, Right)
Left	1, 1	1, 1	0, 0	0, 0
Right	0, 0	1, 1	0, 0	1, 1

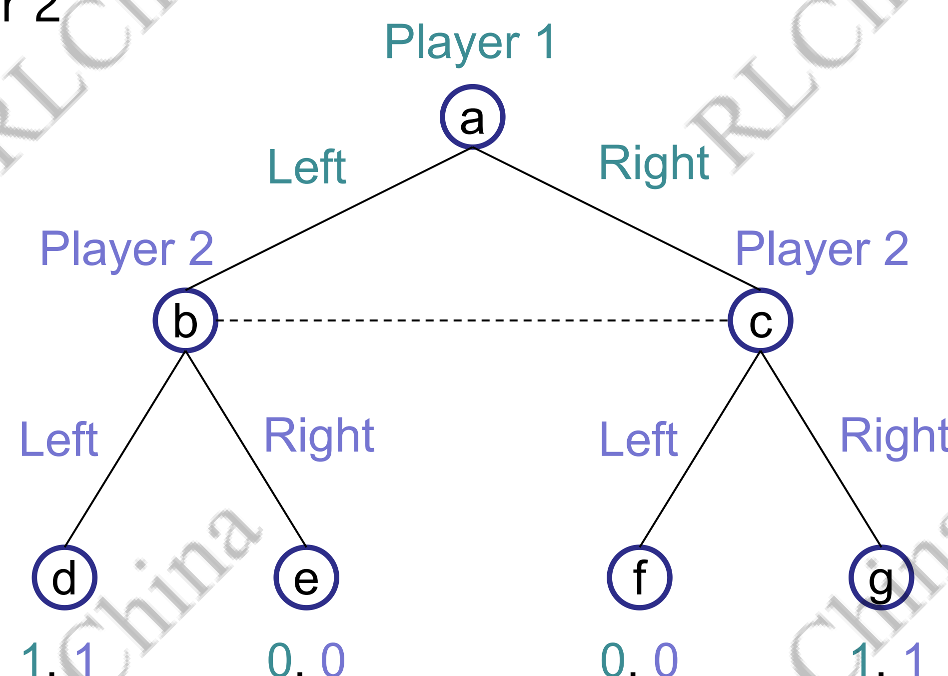
multiple step/state ⇕ **dynamic**



Imperfect Information

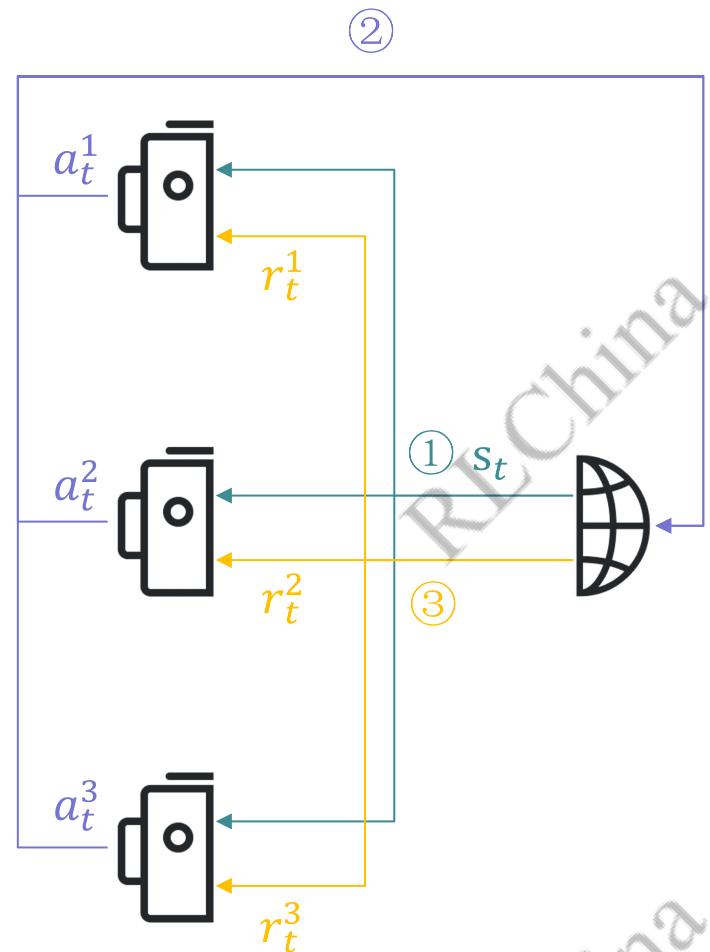
- Imperfect Information Game
  - Some historical actions are invisible by other players
- Information Set
  - A set containing undistinguishable states, e.g. {b, c} is an information set for player 2
- Strategy Space
  - Player 1: {Left, Right}
  - Player 2: {Left, Right}

action in every
information set



Markov Game (or Stochastic Game)

- Game Definition
  - State space S
  - Action space $A = A_1 \times A_2 \times \dots \times A_n$
  - Transition function $p: S \times A \to S$
  - Reward function $r: S \times A \to \mathbb{R}^n$
- Behavioral Strategy
  - Policy $\pi_i: S \times A_i \to [0,1]$
- Properties
  - Simultaneous action (Normal-form)
  - Multiple step/state (Extensive-form)
  - Immediate reward
  - Randomness
  - Cycle



Interaction at time-step t

Summary of Strategy Representation

	Static Game (Single Step/state)	Dynamic Game (Multiple step/state)
Pure Strategy	$a_i \in A_i$	$\pi_i: S \rightarrow A_i$ or $\pi_i \in {A_i}^S$
Mixed Strategy	$a_i: A_i \rightarrow [0,1]$	$\pi_i: {A_i}^S \rightarrow [0,1]$
Behavioral Strategy	$a_i: A_i \rightarrow [0,1]$	$\pi_i: S \times A_i \rightarrow [0,1]$

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Example: Auction

- Game Definition
  - Players has private value v_1, v_2
  - Players decide biddings b_1, b_2
  - Player i with higher bidding b_i has utility $v_i - b_i$
  - The other player has utility 0
- Uncertainty of Private Value
  - $v_1 = 4, v_2 = 4$
  - $b_1 \in \{1,3\}, b_2 \in \{2,4\}$

	$b_2 = 2$	$b_2 = 4$
$b_1 = 1$	0, 2	0, 0
$b_1 = 3$	1, 0	0, 0

  - $v_1 = 4, v_2 = 5$
  - $b_1 \in \{1,3\}, b_2 \in \{2,4\}$

	$b_2 = 2$	$b_2 = 4$
$b_1 = 1$	0, 3	0, 1
$b_1 = 3$	1, 0	0, 1

  - Players don't know the exact payoff matrix of the game!

Incomplete Information

- Recall the Elements of a Game
  - Players $N = \{1,2, \ldots, n\}$
  - Action space $A = A_1 \times A_2 \times \ldots \times A_n$
  - Payoff functions $u = (u_1, u_2, \ldots, u_n)$, $u_i: A \to \mathbb{R}$
- Incomplete Information Game
  - Players know: N and A
  - Players don't completely know: u
  - Criteria: whether players have private information when game starts
- Example
  - Auction
  - Mahjong
  - Werewolves of Miller's Hollow

Bayesian Game

- Basic Idea
  - Payoff function p_i is unknown, but the distribution of p_i is known
- Elements of Bayesian Game
  - Players $N = \{1,2, \ldots, n\}$, action space $A = A_1 \times A_2 \times \ldots \times A_n$
  - Player type space $\Theta = \Theta_1 \times \Theta_2 \times \ldots \times \Theta_n$
  - Distribution over types $d: \Theta \rightarrow [0,1]$
  - Payoff functions $u = (u_1, u_2, \ldots, u_n)$, $u_i: \Theta \times A \rightarrow \mathbb{R}$
- Strategy
  - Pure strategy $\pi_i: \Theta_i \rightarrow A_i$
  - Mixed strategy $\pi_i: \Theta_i \times A_i \rightarrow [0,1]$
- Example
  - $\Theta_1 = \{4\}, \Theta_2 = \{4,5\}$
  - $d(4,4) = 0.3, d(4,5) = 0.7$

$v_2 = 4$
0.3

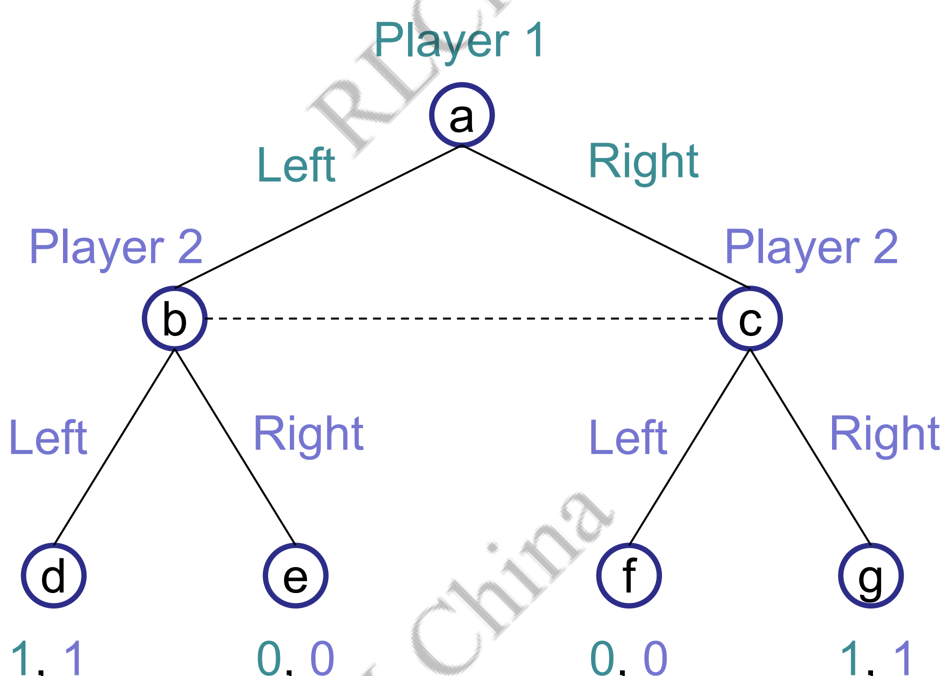
	$b_2 = 2$	$b_2 = 4$
$b_1 = 1$	0, 3	0, 1
$b_1 = 3$	1, 0	0, 1

$v_2 = 5$
0.7

	$b_2 = 2$	$b_2 = 4$
$b_1 = 1$	0, 2	0, 0
$b_1 = 3$	1, 0	0, 0

Dynamic Bayesian Game

- Belief System in Imperfect Information Extensive-form Game
  - Distribution over the states in an information set $b_i: S \rightarrow [0,1]$
- Strategy
  - Pure strategy $\pi_i: S \rightarrow A_i$
  - Behavioral strategy $\pi_i: S \times A_i \rightarrow [0,1]$



Summary of Game Representation

		Complete	Incomplete
Static		Normal-form Game, e.g. Prisoner's Dilemma	Bayesian Game, e.g. Auction
Dynamic	Perfect	Extensive-form Game, e.g. Chess	Texas Hold'em Poker
	Imperfect	StarCraft	Mahjong

Dynamic Bayesian game

- Harsanyi Transformation
  - Incomplete Information → Imperfect Information
  - Introduce a nature player who decides the type of each player

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Game Solution Reasoning

- Best Response (BR)
  - Given $a_{-i} \in A_1 \times \ldots \times A_{i-1} \times A_{i+1} \times \ldots \times A_n$
  - a_i is best response to a_{-i} ⇔ $u_i(a_i, a_{-i}) \geq u_i(a_i', a_{-i}),\ \forall a_i' \in A_i$
- Dominant Strategy (DS)
  - a_i is dominant strategy ⇔ Given any a_{-i}, a_i is best response
- Example

Prisoner's Dilemma

	Cooperate (C)	Defect (D)
Cooperate (C)	2, 2	0, 3
Defect (D)	3, 0	1, 1

$\underbrace{\text{3>2, D is BR to C} \qquad \text{1>0, D is BR to D}}_{\text{D is DS}}$

Game Solution Concept: Nash Equilibrium

- Definition
  - A joint strategy (or strategy profile) $a \in A$ is a Nash Equilibrium $\Leftrightarrow$ a_i is best response to a_{-i} holds for every player i
- Example

Matching Pennies

	Heads	Tails
Heads	1, -1	-1, 1
Tails	-1, 1	1, -1

Battle of Sex

	Party	Home
Party	10, 5	0, 0
Home	0, 0	5, 10

Road Selection

	Left	Right
Left	1, 1	0, 0
Right	0, 0	1, 1

Prisoner's Dilemma

	Cooperate	Defect
Cooperate	2, 2	0, 3
Defect	3, 0	1, 1

Pareto Optimality vs. Nash Equilibrium

- Pareto Optimality (PO)
  - A joint strategy (or strategy profile) $a \in A$ achieves Pareto optimality $\Leftrightarrow \nexists a' \in A \; s.t.$ ① $\forall i, u_i(a') \geq u_i(a)$, ② $\exists i, u_i(a') > u_i(a)$
  - A Pareto optimality is not necessarily a Nash equilibrium
  - A Nash equilibrium is not necessarily a Pareto optimality

Chicken

	C	D
C	3, 3	1, 4
D	4, 1	0, 0

C-D is PO and NE

Stag Hunt

	C	D
C	3, 3	0, 2
D	2, 0	1, 1

D-D is NE but not PO

Prisoner's Dilemma

	C	D
C	2, 2	0, 3
D	3, 0	1, 1

C-C is PO but not NE

Mixed-Strategy Nash Equilibrium

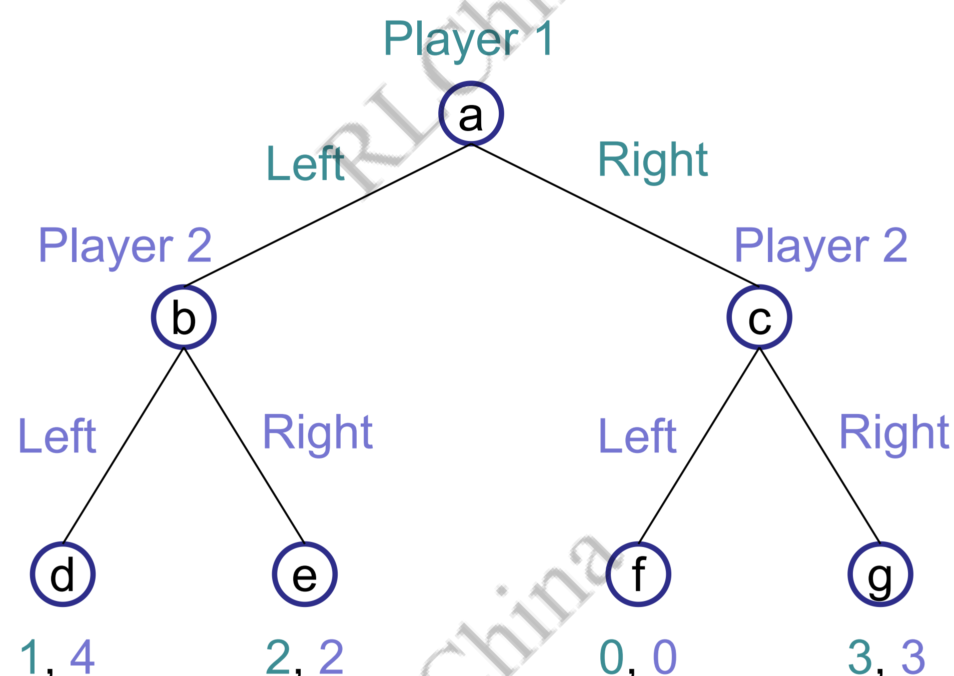
- Definition
  - A mixed-strategy profile $(a_1, a_2, \ldots, a_n)$, $a_i \in \mathrm{PD}(A_i)$ is a Nash Equilibrium ⇔ a_i is best response to a_{-i} holds for every player i
- Example (Rock-Scissors-Paper)
  - $a_1 = ({}^{1}/_{3}, {}^{1}/_{3}, {}^{1}/_{3}), a_2 = ({}^{1}/_{3}, {}^{1}/_{3}, {}^{1}/_{3})$
  - $EU_1(a_1, a_2) = 0 \geq EU_1(a_1{}', a_2) = 0,\ \forall a_1{}' \in A_1$
  - $EU_2(a_1, a_2) = 0 \geq EU_2(a_1, a_2{}') = 0,\ \forall a_2{}' \in A_2$

		${}^{1}/_{3}$	${}^{1}/_{3}$	${}^{1}/_{3}$
		R	S	P
${}^{1}/_{3}$	R	0, 0	1, -1	-1, 1
${}^{1}/_{3}$	S	-1, 1	0, 0	1, -1
${}^{1}/_{3}$	P	1, -1	-1, 1	0, 0

Nash Equilibrium in Extensive-form Game

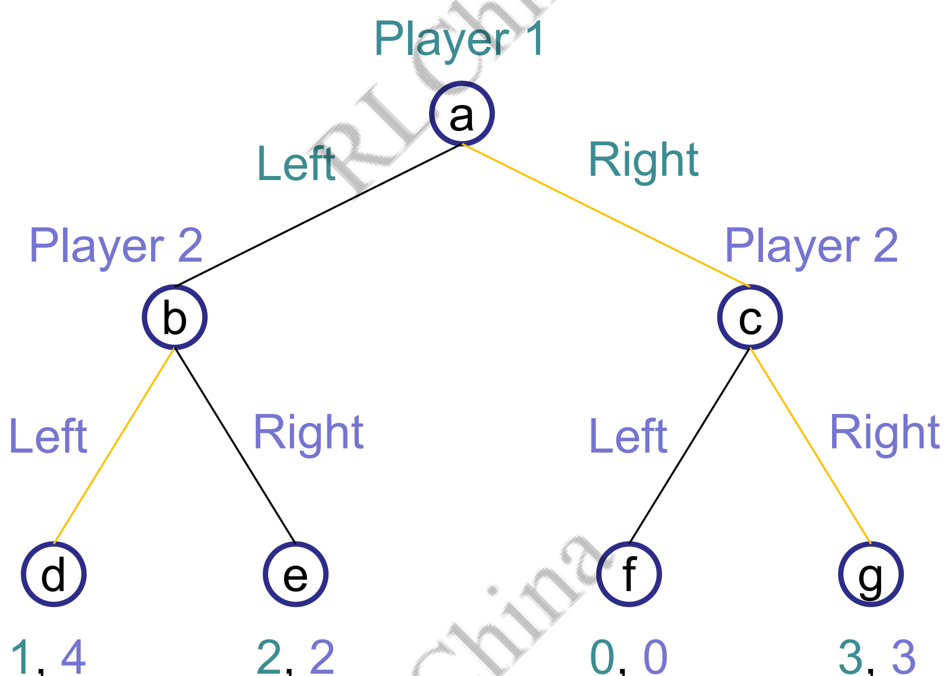
- Incredible Threat

	(Left, Left)	(Left, Right)	(Right, Left)	(Right, Right)
Left	1, 4 ?	1, 4	2, 2	2, 2
Right	0, 0	3, 3	0, 0	3, 3 ?



Subgame Perfect Nash Equilibrium (SPNE)

- Definition
  - An NE is SPNE ⇔ the NE holds in every subgame
- Solution
  - Backward induction: Right - (Left, Right)



Bayesian Nash Equilibrium

- Recall Bayesian Game
  - Player type space $\Theta = \Theta_1 \times \Theta_2 \times \ldots \times \Theta_n$
  - Distribution over types $d: \Theta \rightarrow [0,1]$
  - Payoff functions $u = (u_1, u_2, \ldots, u_n)$, $u_i: \Theta \times A \rightarrow \mathbb{R}$
- Strategy in Bayesian Game
  - Pure strategy $\pi_i: \Theta_i \rightarrow A_i$
  - Mixed strategy $\pi_i: \Theta_i \times A_i \rightarrow [0,1]$
- Bayesian Nash Equilibrium (BNE)
  - Assume each player i knows her own type $\theta_i \in \Theta_i$
  - Set expected utility $\mathbb{E}[u_i|\pi, \theta] = \sum_{a \in A} \left(\prod_{j \in N} \pi_j\left(\theta_j, a_j\right) u_i(\theta, a)\right)$
  - π is BNE $\Leftrightarrow$ $\pi_i \in \text{argmax}_{\pi_i'} \sum_{\theta_{-i} \in \Theta_{-i}} d\,(\theta_i, \theta_{-i}) \mathbb{E}[u_i|\pi_i', \pi_{-i}, \theta_i, \theta_{-i}]$ holds for each player i with her own type θ_i

Bayesian Nash Equilibrium: Example

- Auction
  - $A_1 = \{1,3\}$, $A_2 = \{2,4\}$, $\Theta_1 = \{4\}, \Theta_2 = \{4,5\}$, $d(4,4) = 0.3, d(4,5) = 0.7$
- Strategy
  - $\pi_1(4,1) = \mathrm{x}, \pi_1(4,3) = 1 - \mathrm{x}$
  - $\pi_2(4,2) = y_1, \pi_2(4,4) = 1 - y_1$
  - $\pi_2(5,2) = y_2, \pi_2(5,4) = 1 - y_2$

$v_2 = 4$
0.3

	$b_2 = 2$	$b_2 = 4$
$b_1 = 1$	0, 3	0, 1
$b_1 = 3$	1, 0	0, 1

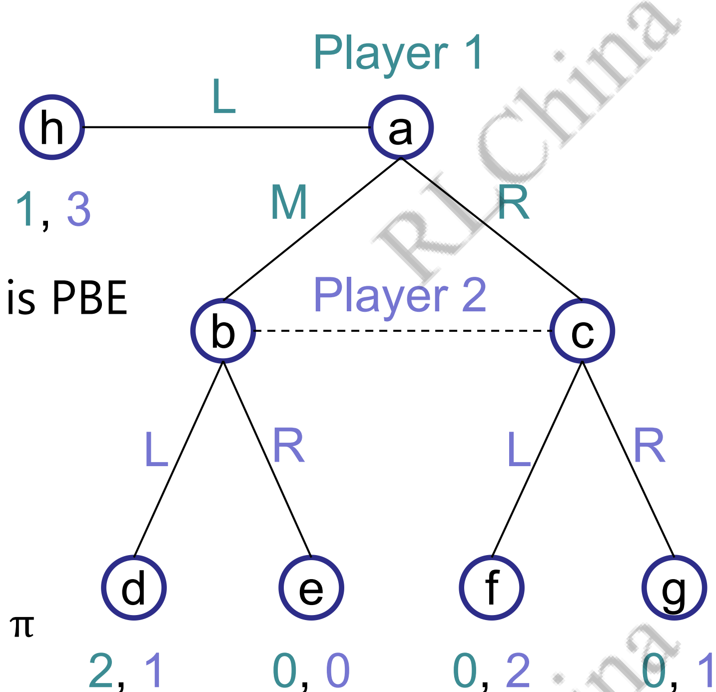
$v_2 = 5$
0.7

	$b_2 = 2$	$b_2 = 4$
$b_1 = 1$	0, 2	0, 0
$b_1 = 3$	1, 0	0, 0

- Equilibrium
  - $\mathbb{E}[u_1|\pi_1, \pi_2, 4,4] = (1 - x)y_1$
  - $\mathbb{E}[u_1|\pi_1, \pi_2, 4,5] = (1 - x)y_2$
  - $\mathbb{E}[u_2|\pi_1, \pi_2, 4,4] = 3xy_1 + x(1 - y_1) + (1 - \mathrm{x})(1 - y_1)$
  - $\mathbb{E}[u_2|\pi_1, \pi_2, 4,5] = 2xy_2$
  - (x, y_1, y_2) satisfies $x = \mathrm{argmax}_x 0.3(1 - x)y_1 + 0.7(1 - x)y_2$ and $y_1 = \mathrm{argmax}_{y_1} 3xy_1 + x(1 - y_1) + (1 - \mathrm{x})(1 - y_1)$ and $y_2 = \mathrm{argmax}_{y_2} 2xy_2$

Perfect Bayesian (Nash) Equilibrium

- Motivation
  - SPNE is not enough for some imperfect information game
  - Example: (L,R) is an SPNE but is incredible
- Recall Dynamic Bayesian Game
  - Belief function $b_i: S \to [0,1]$
  - Behavioral strategy $\pi_i: S \times A_i \to [0,1]$
- Perfect Bayesian Equilibrium (PBE) [10]
  - A strategy profile π with a belief system b is PBE
  - **Sequential rationality**
    - Each player has best expected utility in each information set following b and π
  - **Consistency of beliefs with Strategies**
    - Beliefs b are correct according to strategies π



Summary of Nash Equilibrium

		Complete	Incomplete
Static		Nash Equilibrium	Bayesian Nash Equilibrium
Dynamic	Perfect	Subgame Perfect Nash Equilibrium	Perfect Bayesian Nash Equilibrium
	Imperfect		

- Harsanyi Transformation
  - Incomplete Information → Imperfect Information
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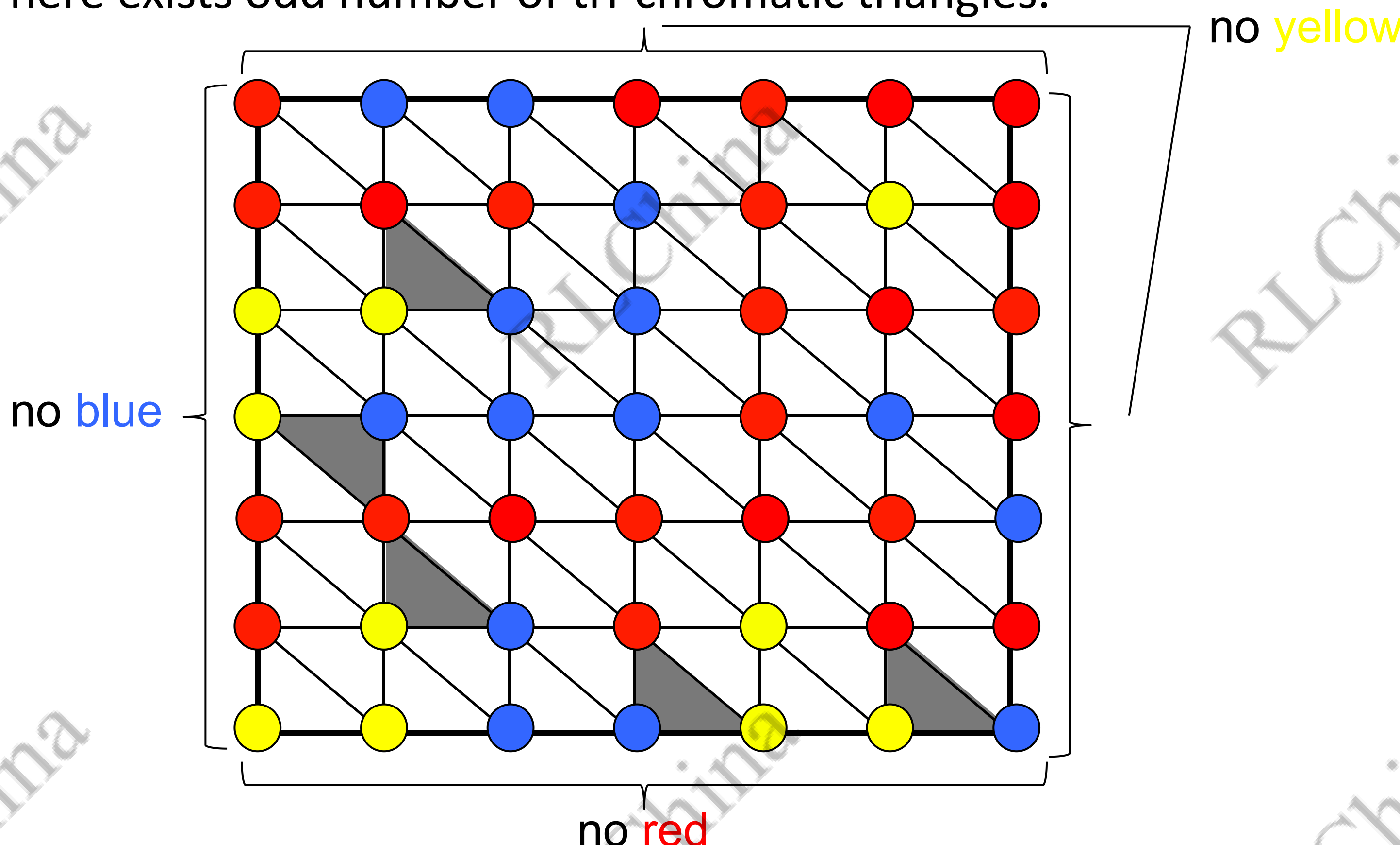
Existence of Nash Equilibrium

- Nash's Theorem [2]
  - Every finite game has a mixed-strategy Nash equilibrium.
  - Proof: apply Brouwer's fixed point theorem.
- Brouwer's Fixed Point Theorem
  - Let D be a convex, compact subset of the Euclidean space. If $f: D \to D$ is continuous, then there exists $x \in D$ such that $f(x) = x$.
  - Proof: apply Sperner's lemma.
- Sperner's Lemma
  - Color the boundary using three colors in a legal way. No matter how the internal nodes are colored, there exists a tri-chromatic triangle. In fact an odd number of those.

Sperner's Lemma (2-D)

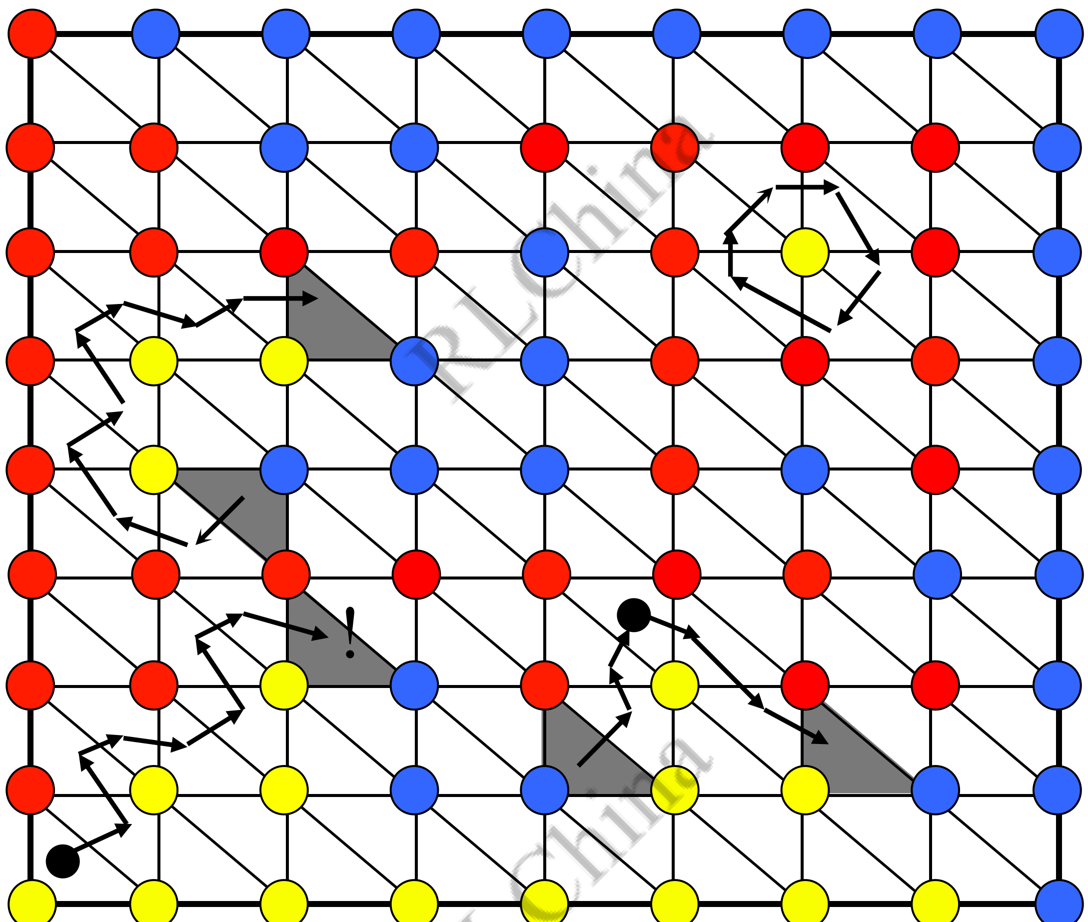
- Lemma
  - There exists odd number of tri-chromatic triangles.



Ref: http://people.csail.mit.edu/costis/6896sp10/

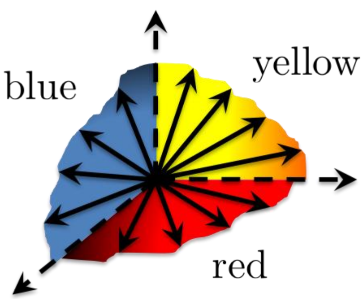
Proof of Sperner's Lemma

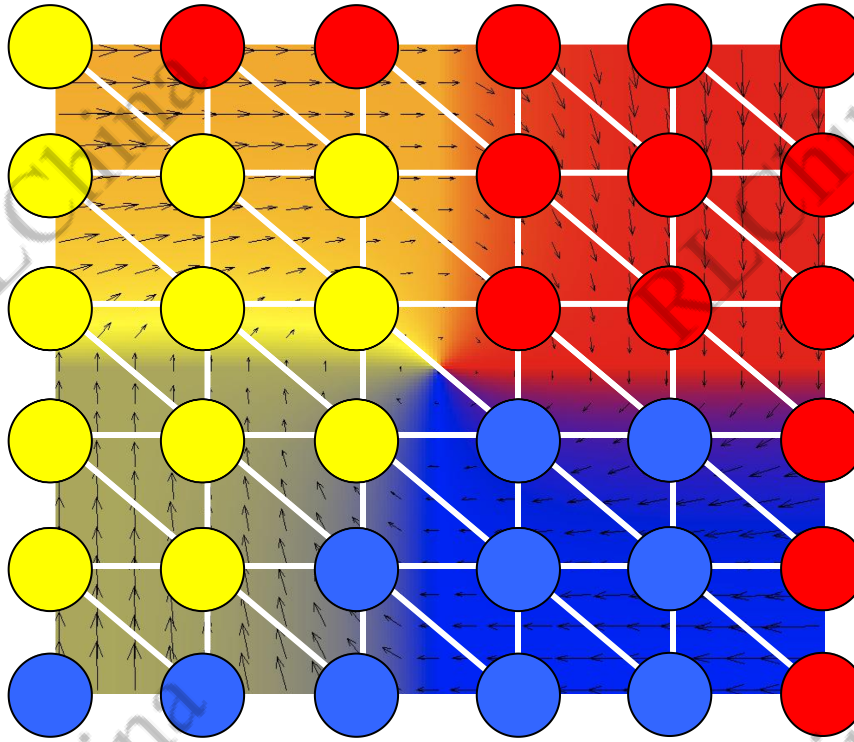
- Proof Sketch



Sperner's Lemma to Brouwer's Theorem

- Proof Sketch

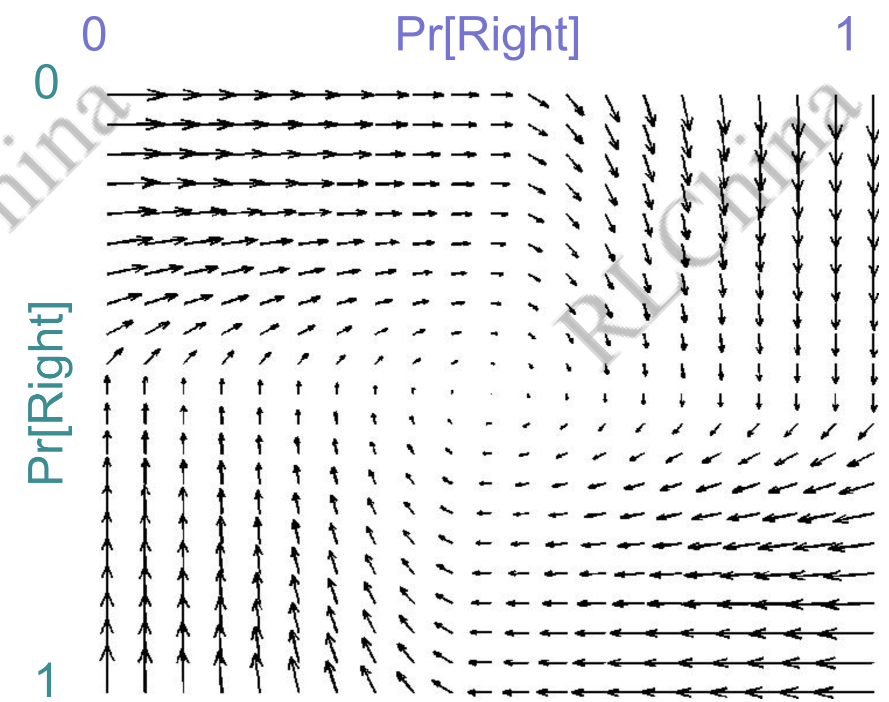




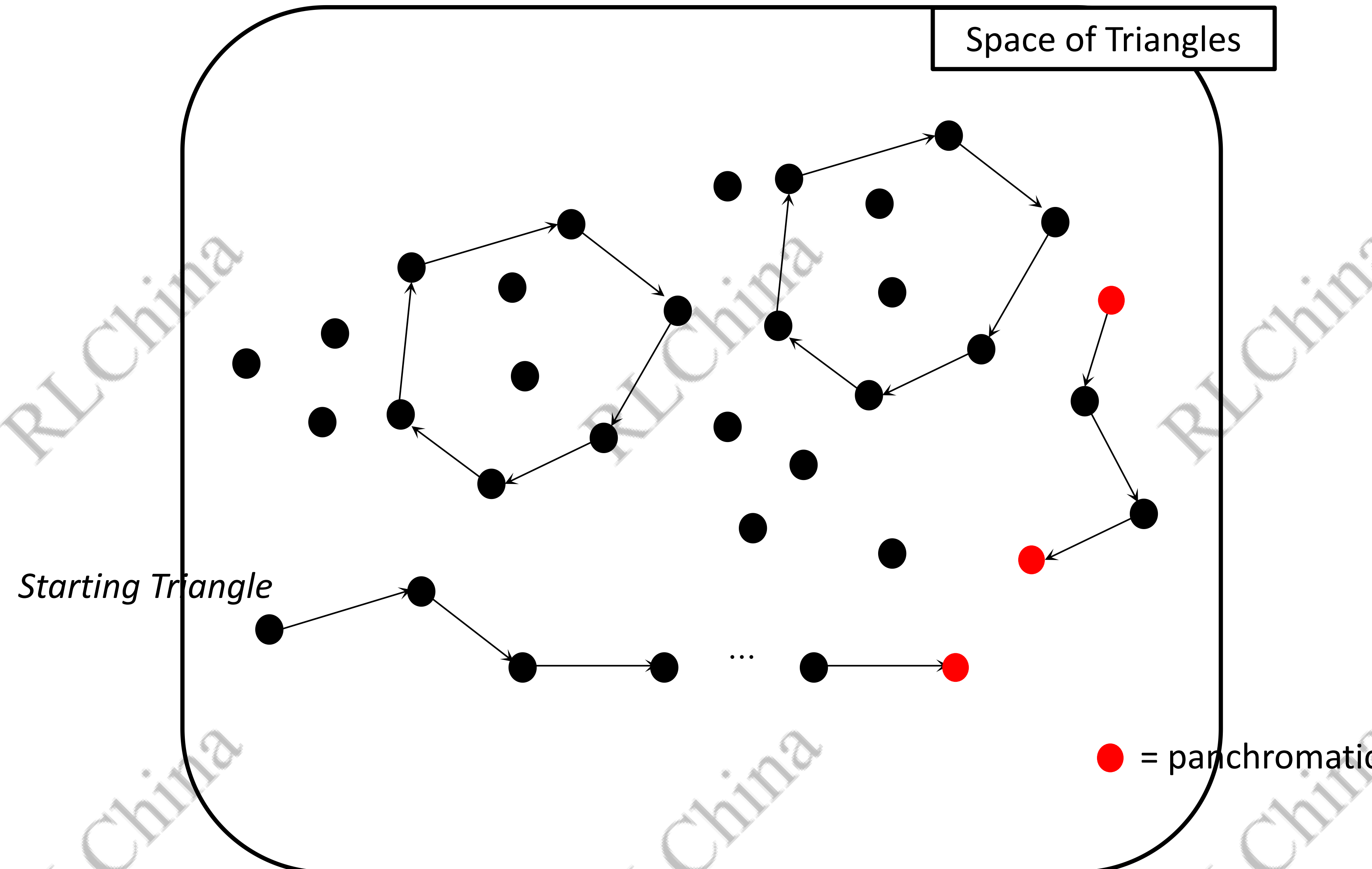
Brouwer’s Theorem to Nash’s Theorem

- Proof Sketch

	Left	Right
Left	1, -1	-1, 1
Right	-1, 1	1, -1

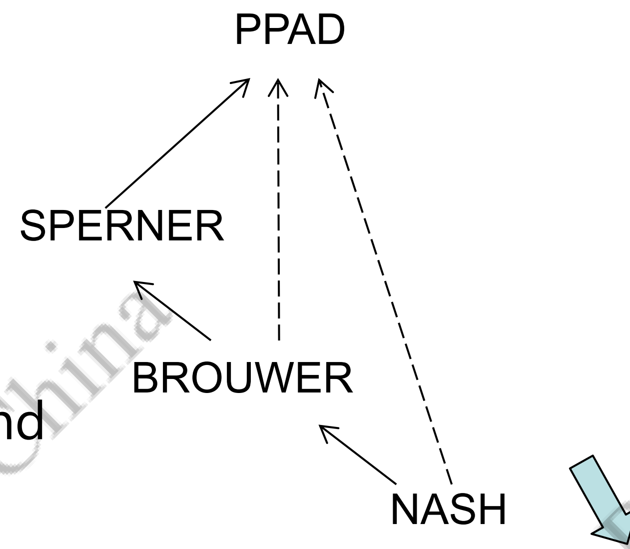


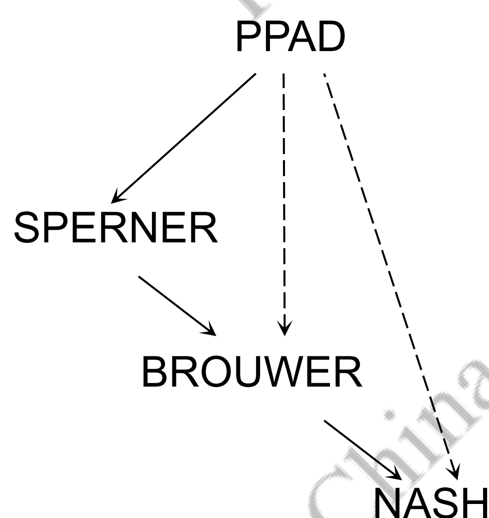
End-of-the-Line to Sperner's Lemma

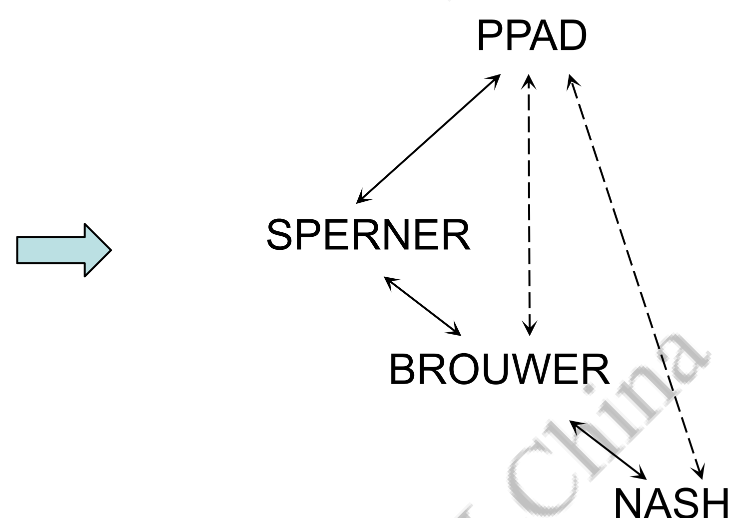


PPAD Complexity Class

- PPAD-complete [11,12,13]
  - End-of-the-line
  - Sperner, Brouwer, Nash
- Computational Complexity
  - Poly-time algorithm not found





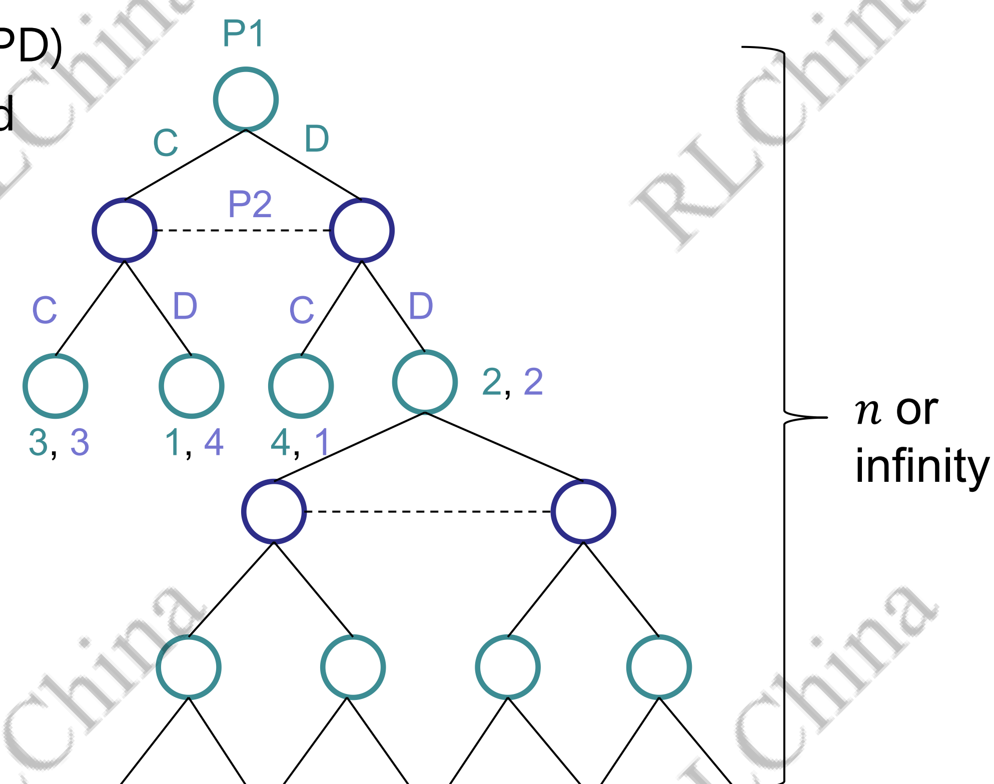


Outline

- Motivation and Normal-form Game
- Extensive-form Game and Imperfect Information
- Bayesian Game and Incomplete Information
- Nash Equilibrium and Variants
- Theoretical Results of Nash Equilibrium
- Repeated Game and Learning Methods
- Evolutionary Game Theory and Coalitional Game Theory

Repeated Game

- Definition
  - A normal-form game is played over and again by the same players
  - The game repeated in each period is referred to as the **stage game**
- Example
  - Iterated Prisoners' Dilemma (IPD)
  - Reward: average or discounted
  - Memory: perfect recall



Memory

- Historical Behavior
  - At stage t, the action profile is a_t
  - Each player remembers the action profiles at last k stages
  - We say the players have k-memory
- Relation to Markov Game
  - Memory is regarded as state

1-memory

	1	2	3	…	m
P1	C	C	D	…	*D*
P2	D	D	C	…	C
…					
Pn	D	C	D	…	C

as state

k-memory

	1	2	3	…	m
P1	C	C	D	…	*D*
P2	D	D	C	…	C
…					
Pn	D	C	D	…	C

as state

Tit-for-tat

- Idea [11]
  - The Tit-for-tat strategy copies what the other player previously choose.
  - Nice: start by cooperating.
  - Clear: be easy to understand and adapt to.
  - Provocable: retaliate against anti-social behavior.
  - Forgiving: cooperate when faced with pro-social play.

	C	D
C	2, 2	0, 3
D	3, 0	1, 1

Prisoner's Dilemma

	1	2	3	4	…
P1	C	C	C	C	…
P2	C	C	C	C	…

cooperate by playing strategy (C,C)

$$\text{payoff} = 2 + 2\gamma + 2\gamma^2 + 2\gamma^3 + \ldots = 2\frac{\gamma^n - 1}{\gamma - 1} = \frac{2}{1-\gamma}$$

	1	2	3	4	…
P1	C	D	C	D	…
P2	D	C	D	C	…

a player deviates to defecting (D)

$$\text{payoff} = 3 + 0\gamma + 3\gamma^2 + 0\gamma^3 + \ldots = \frac{3}{1-\gamma^2}$$

Win-stay, lose-shift

	C	D
C	2, 2	0, 3
D	3, 0	1, 1

Prisoner's Dilemma

- Idea [12]
  - **Repeat** if it was rewarded by 2 or 3
  - **Shift** if it was punished by 0 or 1
  - Advantage: tolerant, one round of mutual defection followed by a return to cooperation
  - Disadvantage: fares poorly against inveterate defectors

	1	2	3	4	…
P1	C	D	C	C	…
P2	D	D	C	C	…

P2 deviates to defecting(D) initially

$$\text{Payoff} > \frac{3}{1-\gamma^2}$$

	1	2	3	4	…
P1	C	C	D	C	…
P2	C	D	D	D	…

P2 is an inveterate defectors

$$\text{payoff}_1 = 2 + 0\gamma + 1\gamma^2 + 0\gamma^3 + \ldots = \frac{1\gamma}{1-\gamma^2} + 1$$

$$\wedge$$

$$\text{payoff}_2 = 2 + 3\gamma + 1\gamma^2 + 3\gamma^3 + \ldots = \frac{1+3\gamma}{1-\gamma^2} + 1$$

Strategies in Iterated Prisoner's Dilemma

	C	D
C	2, 2	0, 3
D	3, 0	1, 1

Prisoner's Dilemma

Action profile at time t	Player 1 strategies at time $t+1$ with 1-memory															
(a_1, a_2)=(C, C)	C	C	C	C	C	C	C	C	D	D	D	D	D	D	D	D
(a_1, a_2)=(C, D)	C	C	C	C	D	D	D	D	C	C	C	C	D	D	D	D
(a_1, a_2)=(D, C)	C	C	D	D	C	C	D	D	C	C	D	D	C	C	D	D
(a_1, a_2)=(D, D)	C	D	C	D	C	D	C	D	C	D	C	D	C	D	C	D

Tit-for-tat strategy

Win-stay, lose-shift strategy

Folk Theorem

- Game Setting
  - n-player infinitely-repeated game $G = (N, A, u)$ with average reward
- Enforceable
  - A payoff profile r is **enforceable** if $r_i \geq v_i$, $v_i = \min_{s_{-i} \in S_{-i}} \max_{s_i \in S_i} u_i(s_{-i}, s_i)$
- Feasible
  - A payoff profile r is **feasible** if there exist rational, non-negative values α_a such that for all i, we can express r_i as $\sum_{a \in A} \alpha_a u_i(a)$, with $\sum_{a \in A} \alpha_a = 1$
- Folk Theorem
  - r is **feasible** and **enforceable** $\Rightarrow$ r is the payoff in some Nash equilibrium

	C	D
C	2, 2	0, 3
D	3, 0	1, 1

Prisoner's Dilemma

(v_1, v_2) =(1, 1)
(-1, -1) is not enforceable, not feasible
(0.5, 2) is not enforceable, **feasible**
(5, 5) is **enforceable**, not feasible
(2, 2) is **enforceable**, **feasible**

Fictitious Play

- Definition
  - Each player plays a best response to **assessed** strategy of the opponent and observe the opponent's actual play and update **beliefs**.

Matching Pennies

	Heads	Tails
Heads	1, -1	-1, 1
Tails	-1, 1	1, -1

Round	1's action	2's action	1's beliefs	2's beliefs
0			(1.5, 2)	(2, 1.5)
1	**T**	T	(1.5, 3)	(2, **2.5**)
2	T	H	(2.5, 3)	(2, 3.5)
3	T	**H**	(**3.5**, 3)	(2, 4.5)
4	H	H	(4.5, 3)	(3, 4.5)
…	…	…	…	…

Convergence of Fictitious Play

- Fictitious Play → Convergence
  - Each of the following are a sufficient conditions for the empirical frequencies of play to converge in fictitious play:
    - The game is zero sum;
    - The game is solvable by iterated elimination of strictly dominated strategies;
    - The game is a potential game;
    - The game is 2 n and has generic payoffs.
- Convergence → Nash Equilibrium
  - If the empirical distribution of each player’s strategies converges in fictitious play, then it converges to a Nash equilibrium.
- Results in Extensive-form Game with Imperfect Information
  - Fictitious self-play converges to approximate Nash equilibrium [13]
  - AlphaStar for StarCraft [14]

No-regret Learning

- Regret
  - Let a^t be the action profile played at time t
  - Regret of player i for not playing action a_i' at time t is $R^t(a_i') = u_i(a_i', a_{-i}^t) - u_i(a^t)$
  - Regret cumulated from time 1 to T is $CR^T(a_i') = \sum_{t=1}^{T} R^t(a_i')$
- Regret Matching
  - At each time step, each action is chosen with probability proportional to its cumulated regret: $\sigma_i^{t+1}(a_i) = \frac{CR^t(a_i)}{\sum_{a_i' \in A_i} CR^t(a_i')}$
  - Converge to correlated equilibrium
- No-regret learning in Extensive-form Game
  - Counterfactual Regret Minimization (CFR)
  - DeepStack for Texas Hold'em poker [15]

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Evolutional Game Theory

- Motivation
  - Nash equilibrium is static, the dynamic of strategy is not described
  - Players are not fully rational
- Basic Idea
  - Strategy is inherent and player can not select strategy by herself
  - Player with high payoff is has more chance to be reproduced
- Evolutionary Stable Strategy (ESS)
  - If almost every member of the population follows a strategy, no mutant (that is, an individual who adopts a novel strategy) can successfully invade.

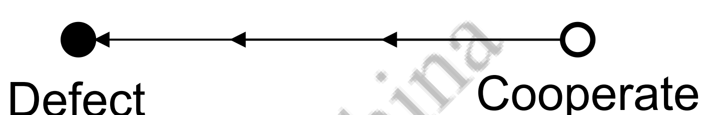
Ref: https://plato.stanford.edu/entries/game-evolutionary/

Replicator Dynamics

- Definition
  - $\dot{x}_i = x_i[f_i(x) - \varphi(x)], \varphi(x) = \sum_{j=1}^{n} x_j f_j(x)$
  - x is distribution of types(strategies) over the population
  - $f_i(x)$ is the fitness for type i in population x
  - $\varphi(x)$ is the average fitness of the population

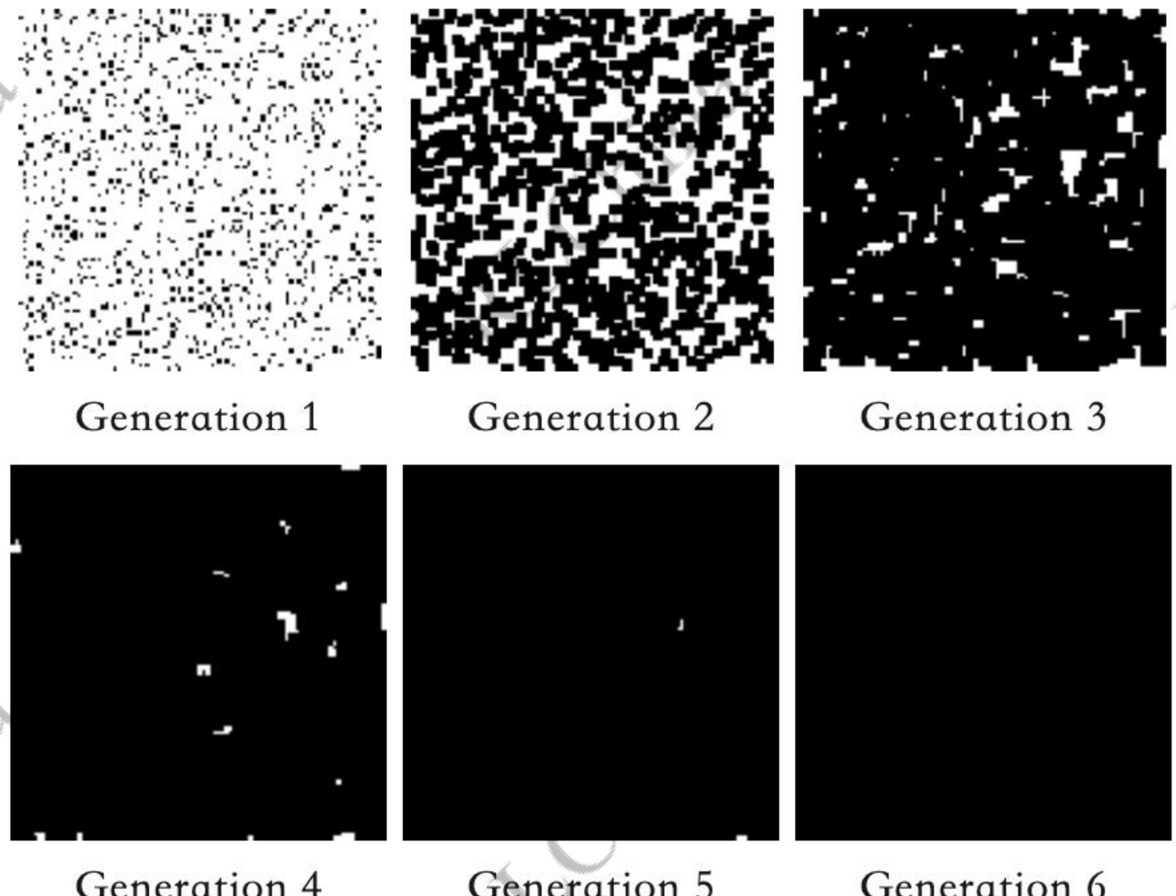
	C	D
C	2, 2	0, 3
D	3, 0	1, 1

Prisoner's Dilemma



Replicator Dynamics: Experiment

- On a local interaction model [17]
  - T = 2.8, R = 1.1, P = 0.1, and S = 0 **All Defect**



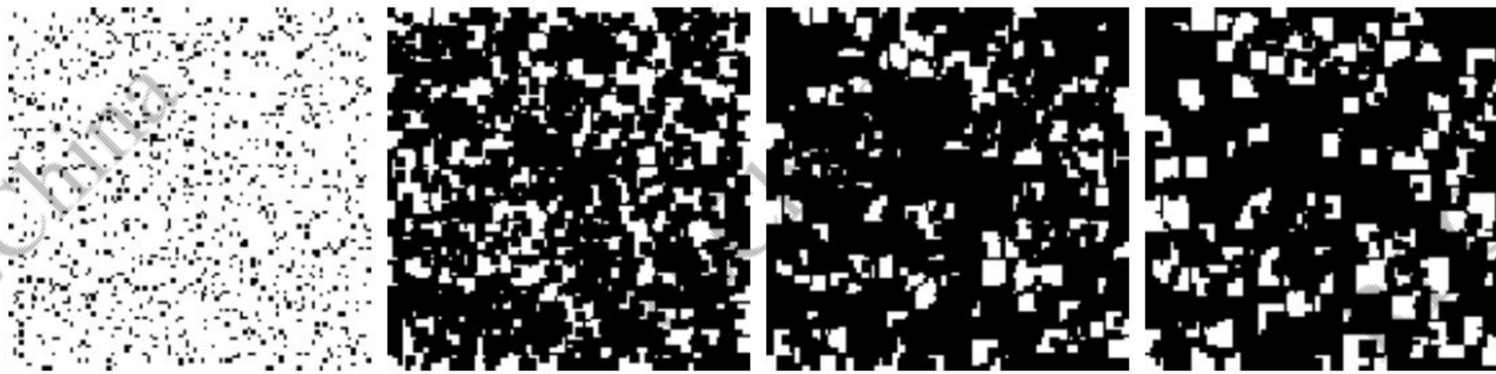
Replicator Dynamics: Experiment

- On a local interaction model
  - T = 1.2, R = 1.1, P = 0.1, and S = 0 **Cooperate**

Generation 1 Generation 2 Generation 19 Generation 20

Replicator Dynamics: Experiment

- On a local interaction model
  - T = 1.61, R = 1.01, P = 0.01, and S = 0 **Chaotic**



Generation 1 Generation 3 Generation 5 Generation 7

Generation 9 Generation 11 Generation 13 Generation 15

Coalitional Game Theory

- Definition
  - N is a set of players indexed by i
  - $v: 2^N \to \mathbb{R}$ associates with each coalition $S \subseteq N$ a payoff $v(S)$, $v(\emptyset) = 0$
- Question
  - Which coalition will form?
  - How to allocate payoff among coalition members?

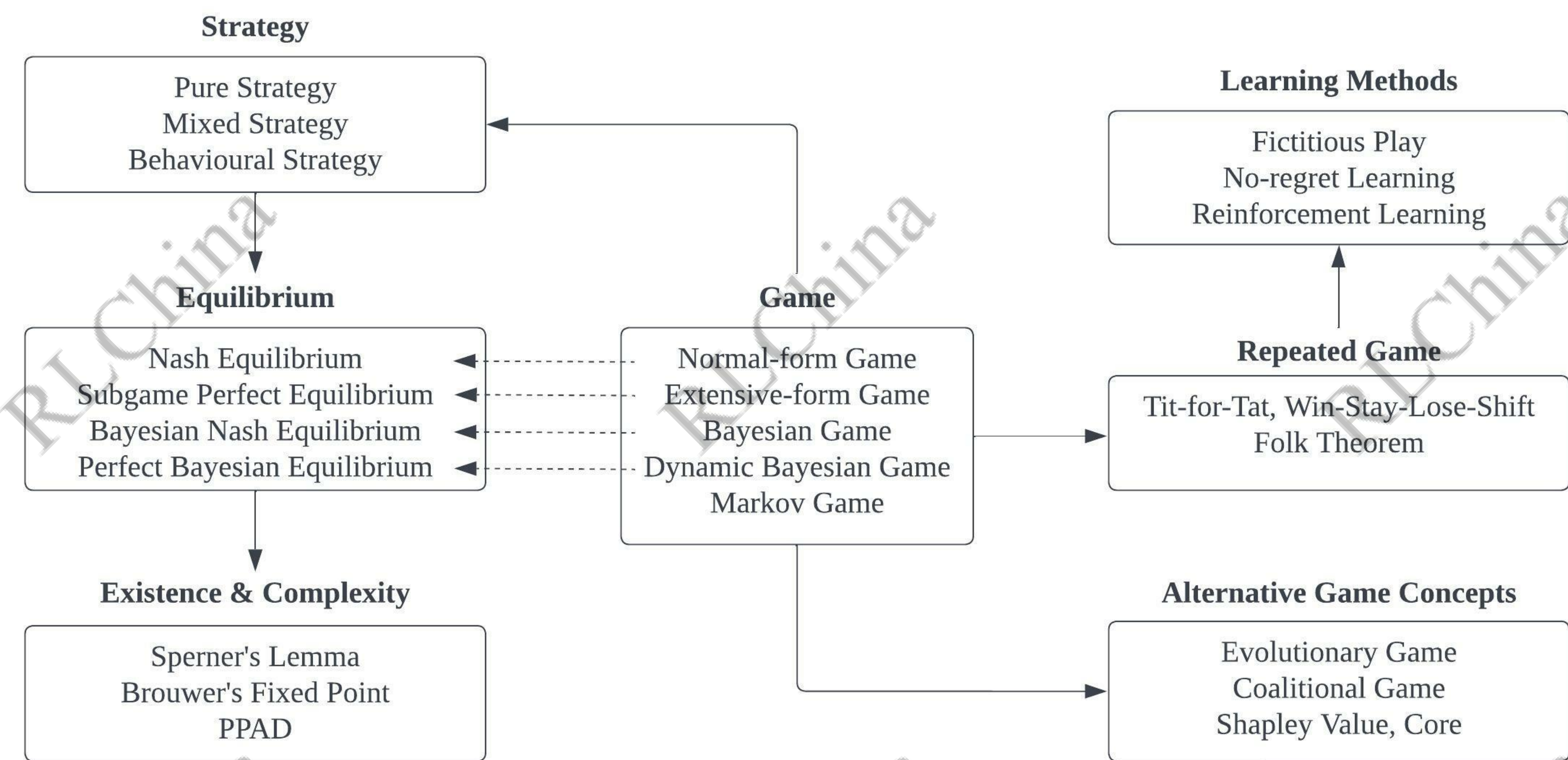
Shapley Value

- Definition
  - $\varphi_i(N, v) = \frac{1}{N!} \sum_{S \subseteq N \setminus \{i\}} |S|! \, (|N| - |S| - 1)! \, [v(S \cup \{i\}) - v(S)]$
- Property
  - Symmetry
    - $\varphi_i(N, v) = \varphi_j(N, v)$ if $v(S \cup \{i\}) = v(S \cup \{j\})$ for $\forall S, i, j \notin S$
  - Dummy Player
    - $\varphi_i(N, v) = 0$ if $v(S \cup \{i\}) = v(S)$ for $\forall S$
  - Additivity
    - $\varphi_i(N, v_1 + v_2) = \varphi_i(N, v_1) + \varphi_i(N, v_2)$ for $\forall i$, where $(v_1 + v_2)(S) = v_1(S) + v_2(S)$ for $\forall S$

Core

- Definition
  - A payoff vector x is in the **core** of a coalition game (N, v), iff $\forall S \subseteq N$, $\sum_{i \in S} x_i \geq v(S)$
- Property
  - A payoff vector in the core is a stable distribution of the grand coalition
  - In some sense, **core** is a more **stable** solution concept than **Nash equilibrium** because every group of the players, rather than every single player, has no intention to deviate

Summary



Strategy
Pure Strategy
Mixed Strategy
Behavioural Strategy
Learning Methods
Fictitious Play
No-regret Learning
Reinforcement Learning
Equilibrium
Nash Equilibrium
Subgame Perfect Equilibrium
Bayesian Nash Equilibrium
Perfect Bayesian Equilibrium
Game
Normal-form Game
Extensive-form Game
Bayesian Game
Dynamic Bayesian Game
Markov Game
Repeated Game
Tit-for-Tat, Win-Stay-Lose-Shift
Folk Theorem
Existence & Complexity
Sperner's Lemma
Brouwer's Fixed Point
PPAD
Alternative Game Concepts
Evolutionary Game
Coalitional Game
Shapley Value, Core

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