Chapter 12 Poisson process

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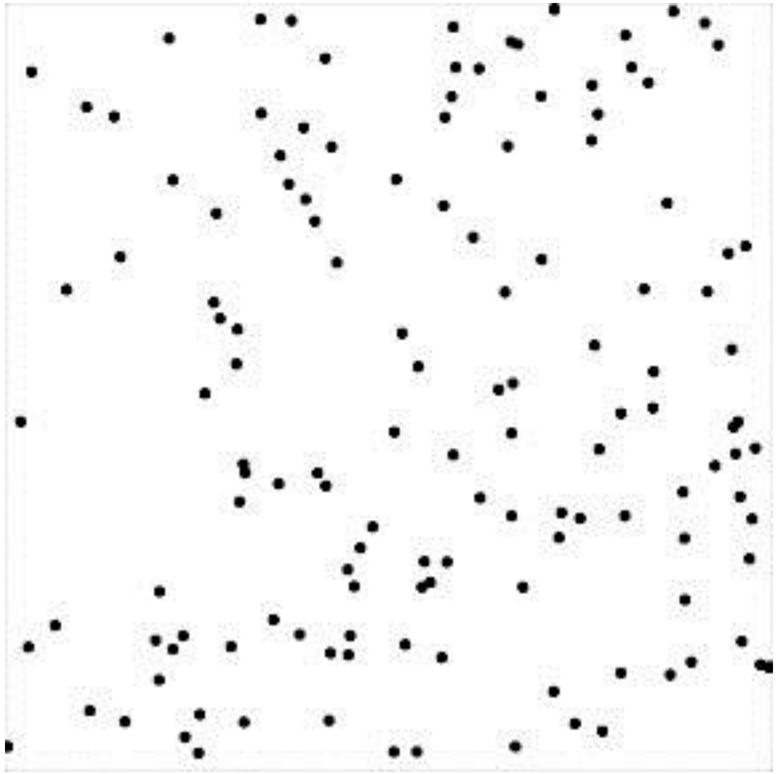
Summary

21 May 2018



12.1 Stochastic processes: definitions

A stochastic (or random) process is a collection of variables indexed with respect to some index.

Usually, variables are indexed with respect to time but they can be indexed with respect to other dimensions, for example space.

The characteristic of a process is how previous values are correlated with the following ones (or near ones).

12.1 Stochastic processes: examples

A stochastic process is described by a variable $X(t)$ or X_t which can take a value in a given range at different times $t = t_1, t_2, t_3, \ldots$

Consider for example the temperature during the day. The readings at different hours can be different, but on a cold day they will all be "low". We don't expect the temperature to change by, say, 10 degrees in one hour. This is time-process, indexed with respect to time.

We can also consider the temperature of the water in the ocean. If we take an origin, the temperature at different distances from that point will not change by much. This is a spatial process indexed with respect to space.

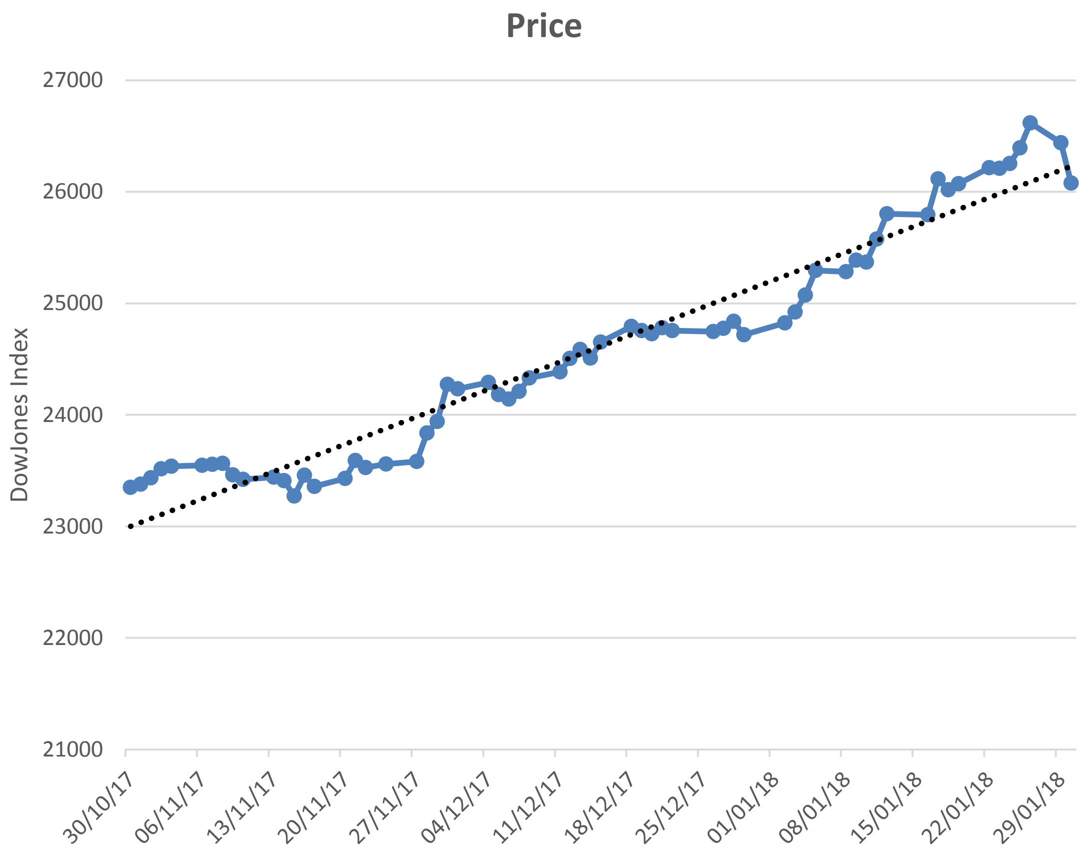
12.1 Stochastic processes: examples

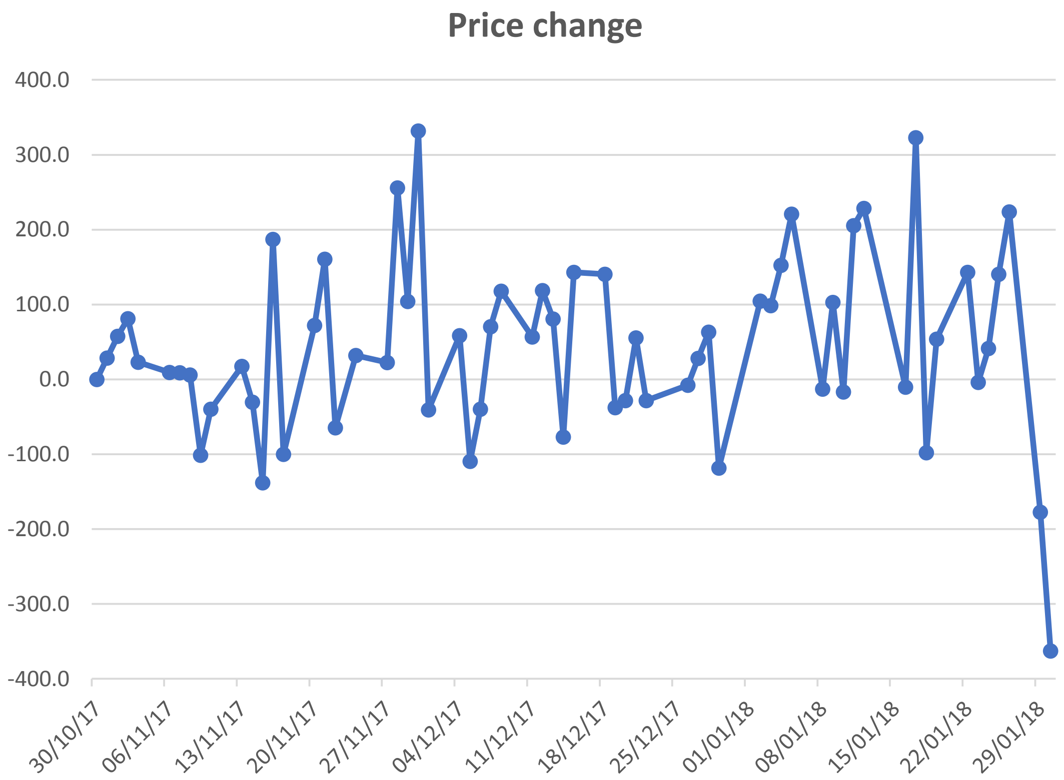
Typical examples of random processes are stock prices, the price at time t, $P(t)$, say day t, depends on the price the day before,
$P(t-1)$. However, we cannot say what the price will be in the future.

For example, we can say that the price tomorrow is
$P(t+1) = P(t) + \theta[P(t) - P(t-1)] + e_{t+1}$. That is, it is equal to price of today plus fraction θ of the change from yesterday plus a random change, e_{t+1}.

If $\theta > 0$ and the price of the stock increased today we can presume that the sequence of changes will all be positive. Or, better, that with “high” probability they will be positive. Say, $P(P(t+1) - P(t) > 0) > 0.5, but\ not\ surely$.

12.1 Stochastic processes: DJ stock index





12.1 Stochastic processes: definition

- A random process is a collection of random variables defined on a set of indices T as
$$\{X(t), t \in T\}$$
- $X(t)$ and T can be either discrete or continuous;
- The process is defined by the collection of joint cumulative distributions
$$F_{X(1),X(2),\dots,X(k)}(x_1, x_2, \dots, x_k)$$
$$= P(X(1) \le x_1, X(2) \le x_2, \dots, X(k) \le x_k)$$

12.1 Stochastic processes: mean and variance

- The mean, $\mu_X(t)$, of a random process is defined as

$$\mu_X(t) = E[X(t)] = \int_{-\infty}^{\infty} X(t) f_{X(t)}(x) dx.$$

In general, $\mu_X(t)$ is a function of time;

- Analogously, the variance is defined as

$$\sigma_X(t) = \int_{-\infty}^{\infty} [X(t) - \mu_X(t)]^2 f_{X(t)}(x) dx.$$

12.1 Stochastic processes: autocovariance and autocorrelation

- The autocovariance (function), $C_{XX}(t,s)$, of a random process is the covariance of the variables in two different times, (t,s)

$$C_{XX}(t,s) = E[(X(t) - \mu_X(t))(X(s) - \mu_X(s)]$$

- The autocorrelation (function), $R_{XX}(t,s)$, is defined as

$$R_{XX}(t,s) = E[(X(t)X(s)] = C_{XX}(t,s) + \mu_X(t)\mu_X(s)$$

These functions are measures of how the different values of a process are linked in time.

Note: this definition of autocorrelation is used in signal processing, other definitions exist.

12.1 Stochastic processes: example

- Consider a process $Y_t = \theta Y_{t-1} + e_t, |\theta| < 1, E(e_t) = 0, V(e_t\ ) = \sigma_e^2$, $E\left(e_t, e_{t-j}\right) = 0, E(Y_t, e_t) = 0$. This is called AR process of order 1.
- We can write $Y_t = \theta Y_{t-1} + e_t = \theta[\theta Y_{t-2} + e_{t-1}] + e_t = \theta^2 Y_{t-2} + \theta e_{t-1} + \ e_t = \theta^2[\theta Y_{t-3} + e_{t-2}] + +\theta e_{t-1} + e_t = \cdots$
$$= e_t + \theta e_{t-1} + \cdots + \theta^{n-1} e_{t-n+1} \left[+ \theta^n Y_{t-n} \xrightarrow[n\to\infty]{} 0\right]$$
- So, $\mu_t = E(Y_t) = \lim_{n\to\infty} [\, E(e_t) + \theta E(e_{t-1}) + \cdots + \theta^{n-1} E(e_{t-n})] = 0$,
- and, $V(Y_t) = \lim_{n\to\infty} V(e_t + \theta e_{t-1} + \cdots + \theta^{n-1} e_{t-n}) = \sum_{i=0}^{\infty} \theta^{2i} \sigma_e^2 = \frac{\sigma_e^2}{1-\theta^2}$.
- The ACF is

$$C(Y_t, Y_{t-\tau}) = C(Y_{s+\tau}, Y_s) = \sum_{i=0}^{\infty} \theta^i \theta^{i+\tau} \sigma_e^2 = \theta^\tau \sigma_e^2 \sum_{i=0}^{\infty} \theta^{2i} = \frac{\theta^\tau \sigma_e^2}{1-\theta^2}$$

12.1.1 Stochastic processes: strong stationarity

- A process is strongly (or strictly) stationary if its distribution does not change over time. That is, if

$$F_{X(t_{1+\tau}),X(t_{2+\tau}),\ldots,X(t_{k+\tau})}(x_1, x_2, \ldots, x_k)$$
$$= F_{X(t_1),X(t_2),\ldots,X(t_k)}(x_1, x_2, \ldots, x_k)$$

This means that the distribution is time invariant.

So, for example, $F\big(X(1), X(3), X(9)\big) = F(X(4), X(6), X(12))$.

So the mean and covariance are the same for if we move along time.

12.1.1 Stochastic processes: weak stationarity

- A process is weakly stationary if

$$\mu_X(t+\tau) = \mu_X(t) = \mu_X$$

And

$$C_{XX}(t, t+\tau) = C_{XX}(s, s+\tau) = C_{XX}(\tau)$$

Mean and covariance are only functions of τ, or $C(t_1, t_2)$ is a function of only $(t_2 - t_1)$. Also $R_{XX}(t, t+\tau) = R_{XX}(s, s+\tau)$

Weak stationarity requires that only some characteristics of the distribution are constant over time. Strong => Weak.

12.1 Stationarity: example

- Consider a process an AR1 process

$Y_t = \theta Y_{t-1} + e_t, |\theta| < 1, E(e_t) = 0, V(e_t\ ) = \sigma_e^2,$
$E\left(e_t, e_{t-j}\right) = 0, E(Y_t, e_t) = 0.$

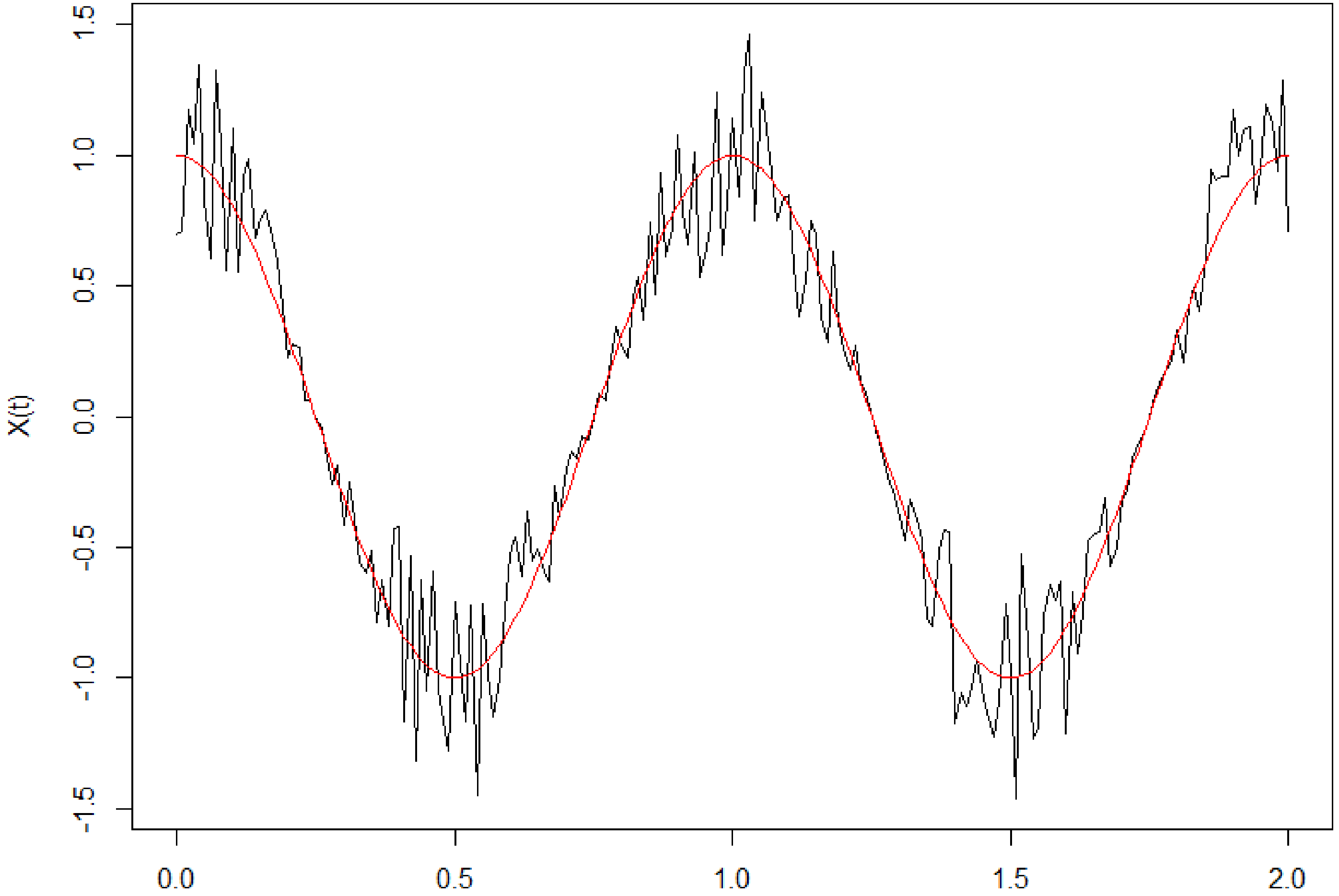
We saw that

$$\mu_t = E(Y_t) = 0\ and\ V(Y_t) = \sum_{i=0}^{\infty} \theta^{2i}\sigma_e^2 = \frac{\sigma_e^2}{1-\theta^2}$$

- The ACF is $C(Y_t, Y_{t-\tau}) = C(Y_{s+\tau}, Y_s) = \frac{\theta^\tau \sigma_e^2}{1-\theta^2}$
- So the process is weakly stationary

12.1 Stochastic processes: Example 1

Sinusoid with random amplitude



12.1 Stochastic processes: Example 1

- Sinusoid with random amplitude

$X(t) = Acos(2\pi t)$, A is some random variable.

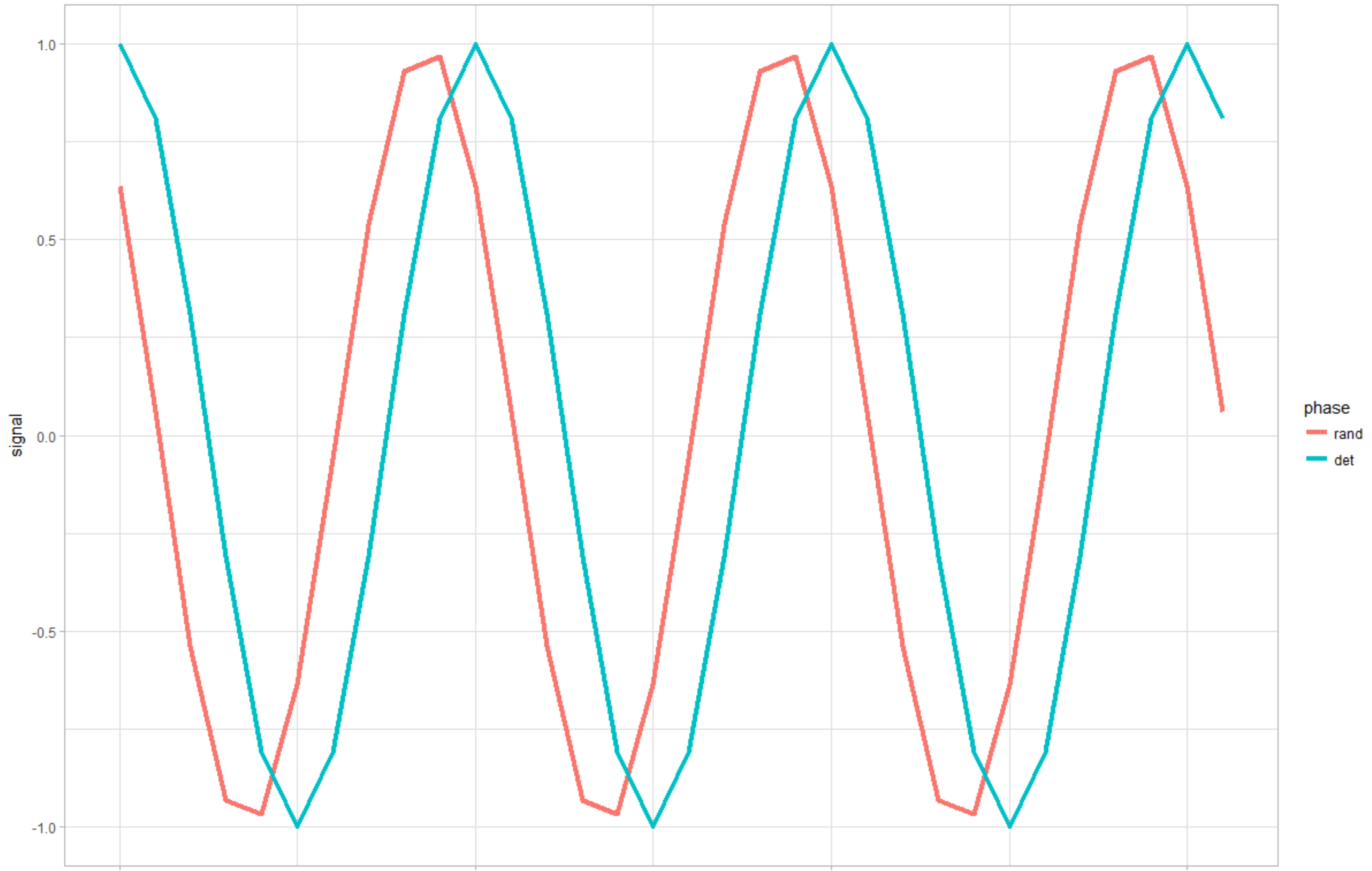
$$E\big(X(t)\big) = \int_{-\infty}^{\infty} Acos(2\pi t)\mathrm{f(A)dA}{=}E(A)cos(2\pi t)$$

$$R(t_1, t_2) = E[Acos(2\pi t_1)Acos(2\pi t_2)] =$$
$$E(A^2)cos(2\pi t_1)cos(2\pi t_2)$$

The process is not stationary, even if $E(A) = 0$, $E(A^2) \neq 0$, so the correlation depends on $t_1$and t_2 (and not on $t_2 - t_1$).

12.1 Stochastic processes: Example 2

Sinusoid with random phase



12.1 Stochastic processes: Example 2

- Sinusoid with random phase

$X(t) = cos(rt + \theta)$, θ uniform in $[-\pi, \pi], r \neq 0$.

$$E\big(X(t)\big) = \frac{1}{2\pi}\int_{-\pi}^{\pi} cos(rt + \theta)\mathrm{d}\theta = 0$$

$$R(t_1, t_2) = C(t_1, t_2) = E[cos(rt_1 + \theta)cos(rt_2 + \theta)] =$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{2}\{\cos[\mathrm{r}(t_1 - t_2)] + \cos[r(t_1 + t_2) + 2r]\}d\theta$$

$$= \frac{1}{2}\cos[r(t_1 - t_2)].$$

The process is stationary, as $E(X(t))$ is constant, and $C(t_1, t_2)$ depends only on $|t_1 - t_2|$.

12.1.2 Stochastic processes: ergodicity

- The definition of ergodicity is quite technical and we skip it.
- Ergodicity (intuitive definition)

Intuitively, a stochastic process

$$\{X(t), t \in T\}$$

Is ergodic if any two collections of random variables partitioned far apart in the sequence are essentially independent.

You can think of this as that the values $Y(t)$ and $Y(t+\tau)$ are independent for large τ.

12.1.2 Stochastic processes: ergodicity in mean

- Ergodicity generally means that certain time averages are asymptotically equal to certain statistical averages
- Consider a stationary process $\{X(t), t \in T\}$ with

$$\mu_X = E(X_t) = \int_{-\infty}^{\infty} X(t) f(X(t)) dX$$

The process is mean-ergodic if the time average of $X(t)$ in the interval $[0, t]$ is

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t X(\tau) d\tau = \mu_X \ a.s. \text{ [for discrete } \frac{1}{n} \sum_{1=0}^{n} X(i) = \mu_X]$$

Note the different integrals

12.1.3 Counting processes

A stochastic process $\{X(t), t \geq 0\}$ is said to be a counting process if X(t)represents the total number of “events” that occur by time t.

Some examples of counting processes are the following:

i. Number of people entering a store;
ii. Number of earthquakes;
iii. Numbers of calls to an emergency centre;
iv. Number of goals in a soccer game;

12.1.3 Counting processes

Properties of a counting process

i. $X(t) \geq 0$;

ii. $X(t) \in \{0, 1, 2, 3, \ldots\}$;

iii. If $s < t$, then $X(s) \leq X(t)$;

iv. If $s < t$, the number of events that occur in $[s, t]$ is called increment and is equal to $X(t) - X(s)$;

12.1.3 Counting processes

If $s < t$, the number of events that occur in $[s, t]$ is called increment and is equal to $X(t) - X(s)$;

Definition: Increments are independent if the number of events that occur in an interval is independent of the number of events occurred in a non-overlapping interval. If $s_1 < t_1 < s_2 < t_2$, $\left(X(t_1) - X(s_1)\right)$ is independent of $\left(X(t_2) - X(s_2)\right)$

Example, number of goals scored in the two halves of a soccer game

12.1.3 Counting processes

If $s < t$, the number of events that occur in $[s, t]$ is called increment and is equal to $X(t) - X(s)$;

Definition: Increments are stationary if the probability of the number of events that occurring in an interval depend only on the length of the interval.
$\mathrm{P}(X(t + s\ ) - X(s) = k) = P(X(t) = k) = P(k \text{ events in } [0, t])$.

Example, number of goals scored in the two halves of a soccer game again (they last 45 minutes each). This means that the process is memoryless: the probability of an event is not affected by how many events have occurred before. [not always realistic]

12.2 Poisson process

A continuous-time $\{X(t): t \geq 0\}$ is a Poisson process with rate $\lambda > 0$ if

i. $X(0) = 0$

ii. It has stationary and independent increments.

iii. The number of events occurring in an interval, $X(t)$, is Poisson with mean λt, that is

$$P(X(t) = k) = \frac{(\lambda t)^k}{k!} e^{-\lambda t}, \quad k = 0,1,2, \dots$$

12.2 Poisson process

Example 1

Defects occur along an undersea cable according to a Poisson process of rate $\lambda = 0.1$ per mile.

i. What is the probability that 1 defect appears in the first mile of cable?

ii. What is the probability that no defects appear in the first two miles of cable?

12.2 Poisson process

Solution

Since $\lambda = 0.1, \mathrm{P}(\text{"}n \text{ defects in } t \text{ miles"}) = P(X(t) = n) = \frac{e^{-0.1t}(0.1t)^n}{n!}$

i. The probability of 1 defect in 1 mile is simply

$$P(X(1) = 1) = \frac{e^{-0.1}(0.1)^{-1}}{1!} = 0.1e^{-0.1} = 0.0905$$

ii. The probability of 0 defects in t =2 miles is

$$\therefore P\{X(2) = 0\} = \frac{e^{-0.1*2}(2*0.1)^0}{0!} = e^{-0.2} = 0.8187 \blacksquare$$

12.2 Poisson process

Example 2, memoryless property

Defects occur along an undersea cable according to a Poisson process of rate $\lambda = 0.1$ per mile.

i. Given that there are no defects in the first two miles of cable, what is the (conditional) probability that no defects between mile points two and three?

12.2 Poisson process

Solution

Recall that

1) increments are independent, so the number of events in the next mile [2, 3] does not depend on how many happened before. Also

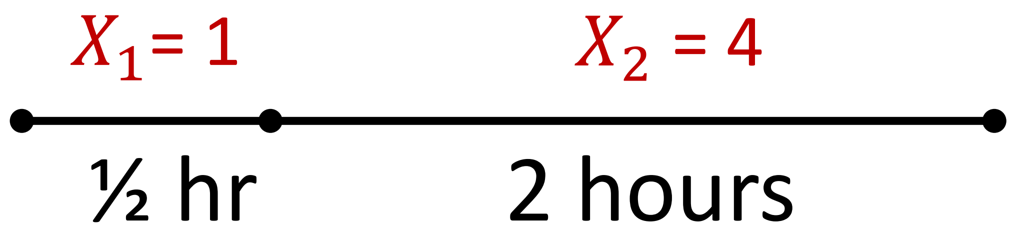
2) the probability of the number of events depends only on the length of the interval. In this case the interval is $t = 3 - 2 = 1$ mile.

So, $P[X(3) - X(2) = 0, X(2) = 0] =$
$P(X(3) - X(2) = 0) = P(X(1) = 0) = e^{-0.1}$ ■

12.2 Poisson process

Example 3, joint probabilities

Customers arrive in a certain store according to a Poisson process of rate $\lambda = 4$ per hour. Given that the store opens at 9.00 AM, what is the probability that exactly 1 customer has arrived by 9.30 AM and a total of 5 have arrived by 11.30 AM?



12.2 Poisson process

Solution

Measuring time t in hours from 9:00 AM, the first interval is $t = \frac{1}{2}h$, the second is from 9:30 to 11:30, $t = 2h$. We find

$$P\{X(\tfrac{1}{2}) = 1, X(\tfrac{5}{2}) = 5\}$$

$$= P\{X(\tfrac{1}{2}) = 1, X(\tfrac{5}{2}) - X(\tfrac{1}{2}) = 4\} = P\{X(\tfrac{1}{2}) = 1, X(4) = 4\}$$

$$= \left\{\frac{e^{-4(1/2)}.(4/2)^1}{1!}\right\}\left\{\frac{e^{-4(2)}.(4*2)^4}{4!}\right\} = \frac{1024}{3}e^{-10} = 0.0155 \quad \blacksquare$$

12.2 Poisson process

Example 4

Let $\{X(t): t \geq 0\}$ be a Poisson process with parameter $\lambda = 1$. Compute the probability $P\{X(1) = 5, X(4) = 9\}$.

Solution

$$P\{X(1) = 5, X(4) = 9\} = P\{X(1) = 5, X(4) - X(1) = 4\} =$$

$$P\{X(1) = 5\}P\{X(4) - X(1) = 4\} = P\{X(1) = 5\}P\{X(3) = 4\}$$

$$= \frac{e^{-\lambda} \cdot \lambda^5}{5!} \cdot \frac{e^{-3\lambda} \cdot (3\lambda)^4}{4!} \quad \text{where } \lambda = 1$$

$$= \frac{27}{8} e^{-4} \quad \blacksquare$$

12.2 Poisson process

In some cases we are interested in conditioning backward

Example 5

Let $\{X(t): t \geq 0\}$ be a Poisson process with parameter $\lambda = 2$.

Compute the conditional probability $P\{X(3) = 1 | X(5) = 2\}$.

12.2 Poisson process

Solution

i. $P\{X(3)=1|X(5)=2\} = \frac{P\{X(3)=1, X(5)=2\}}{P\{X(5)=2\}} =$

$$\frac{P\{X(3)=1, X(5)-X(3)=1\}}{P\{X(5)=2\}} = \frac{P\{X(3)=1\}P\{X(5)-X(3)=1\}}{P\{X(5)=2\}}$$

$$= \frac{P\{X(3)=1\}P\{X(2)=1\}}{P\{X(5)=2\}} = \frac{\frac{e^{-3\lambda}\cdot(3\lambda)^1}{1!}\cdot\frac{e^{-2\lambda}\cdot(2\lambda)^1}{1!}}{\frac{e^{-5\lambda}\cdot(5\lambda)^2}{2!}} \quad \text{(substituting } \lambda = 2\text{)}$$

$$= \frac{12}{25} \blacksquare$$

12.2 Poisson process: example rate

Example

if water flows at a rate of $\lambda = 10Lt/m$, what is the probability that I will fill six 2Lt bottles in 1 minute?

Solution

Let $N_B(t)$ be the number of bottles filled in the interval $[0. t]$. The one minute rate is 5 bottles. Therefore

$$P(N_B > 6) = \frac{e^{-5}5^6}{6!} = 0.1462$$

12.2 Poisson process: problem

Mrs Zou invested 1000\$. As interest she randomly receives 2\$ at a rate of 3 times per month.

What is the probability that after 2 months she received less than 5\$?

12.2 Poisson process: problem

She can only receive even amounts of dollars, so we are asking if she received 4$ or less, that is 2 or less payments, in 2 months. Then the required probability is

$$P(X(2months) \leq 2) = P(X = 0) + P(X = 1) + P(X = 2) =$$

$$e^{-2*3}\left(1 + 6 + \frac{6^2}{2}\right) = 25e^{-6} = 0.062$$

12.2.1. Waiting time for the first event

One important question regarding a Poisson process is: how long will it take for the first event to happen? The answer is simple: Let T_1 be the time at which the first event happens, then

$$P(T_1 \leq t) = P(X(t) > 0) = 1 - P(X(t) = 0) = 1 - e^{-\lambda t}$$

If you remember, this is the Exponential distribution with rate λ and mean $1/\lambda$ and pdf

$$f(t) = \lambda e^{-\lambda t}, t \geq 0; 0\ if\ t < 0$$

12.2.1 Waiting time for the first event

Recall that the Poisson process is memoryless. You will not be surprised to know that also the exponential distribution is.

Proof Let Y$\sim Exp(\lambda\}$

$$P(T_1 > t + s | T_1 > t) = \frac{P(T_1 > t + s)}{P(T_1 > t)} = \frac{e^{-\lambda(t+s)}}{e^{-\lambda t}} = e^{-\lambda s}$$

Therefore, $P(T_1 \leq t + s | T_1 > t) = P(T_1 \leq s) = 1 - e^{-\lambda s}$

which does not depend on t. For example, if nothing happened for 1 hour, the probability that nothing happens for the next hour is the same.

12.2.1 Waiting time for the first event

Example. The number of customers that arrive at a shop is a Poisson process with rate 10 per hour. What is the probability that a customer will arrive in the next 10 minutes? And the probability that the customer will arrive in the next 20 minutes, given that none hasn't arrived in the first ten minutes?

Solution 10 minutes are is $\frac{1}{6}$ hours, so

$$P(T_1 \leq 10) = 1 - e^{-\frac{10}{6}} = 0.8111$$

$$P(T_1 \leq 20 | T_1 > 10) = P(T_1 \leq 20 - 10) = 1 - e^{-\frac{10}{6}} = 0.8111$$

By the memoryless property, the probability is still the same

12.2.2 Distribution of arrival times

Suppose we are told that exactly 1 event of a Poisson process has taken place by time t, and we are asked to determine the distribution of the time at which the event occurred.

Since a Poisson process possesses stationary and independent increments, it seems reasonable that each interval in $[0, t]$ of equal length should have the same probability of containing the event. In other words, the time of the event should be uniformly distributed over $[0, t]$.

12.2.2 Distribution of arrival times

In other words, we expect that on a given interval of time the event is equally likely to happen at any time. So the time of arrival should be uniformly distributed in $[0, t]$ so that $P(T_1 \leq s | X(t) = 1) = \frac{s}{t}$.

This is easily checked since, for $s \leq t$,

$$P(T_1 \leq s | X(t) = 1) = \frac{P(X(s) = 1)P(X(t-s) = 0)}{P(X(t) = 1)} = \frac{e^{-\lambda s}(\lambda s)e^{-\lambda(t-s)}}{(\lambda t)e^{-\lambda t}} = \frac{s}{t}$$

This property is important to show that if the intensity rate is constant, the number of events in an interval are distributed as a Poisson variable.

12.4 Distribution of arrival times: example

Example

Accidents on a road happen at a rate of 2 per hour. We know that between 10 and 12am there has been 1 accident. What is the probability that the accident happened between 10 and 11am?

Solution

$$P(T_1 \leq 1|X(2) = 1) = \frac{P(X(1) = 1, X(1) = 0)}{P(X(2) = 1)} =$$

$$\frac{\{e^{-2}2e^{-2}\}}{\{e^{-4}4\}} = \frac{2}{4} = \frac{1}{2}$$

12.4 distribution of arrival times: example revisited

Example different formulation

Accidents on a road happen at a rate of 2 per hour. What is the probability that the accident happened between 10 and 11am if we know that between 10 and 12am there has been 1 accident?

Solution

$$P(X(1) = 1|X(2) = 1) = \frac{P(X(1) = 1, X(1) = 0)}{P(X(2) = 1)} =$$

$$\frac{\{e^{-2}2e^{-2}\}}{\{e^{-4}4\}} = \frac{2}{4} = \frac{1}{2}$$

12.3 Summary

- Increments are stationary and independent

$$\mathrm{P}(X(t+s)-X(s)=k)=P(X(t)=k)$$

- For a Poisson Process

$$P(X(t)=k)=\frac{(\lambda t)^k}{k!}e^{-\lambda t} \quad ; \quad k=0,1,2,\ldots$$

- The waiting time for the 1^{st} event is exponential with mean $\frac{1}{\lambda}$

$$P(T_1 \leq t) = 1-e^{-\lambda t}$$

- The conditional arrival time of T_1 is uniform

$$P(T_1 \leq s | N(t)=1)=\frac{s}{t}$$