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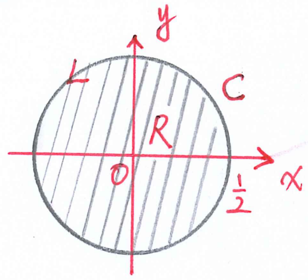
# Tutorial 3 Green's theorem and surface

1. Line integrals: evaluation by Green's theorem (page 438).

Evaluate $\int_C \mathbf{F}(\mathbf{r})\cdot d\mathbf{r}$ counterclockwise around the boundary $C$ of the region $R$ by Green's theorem.

(1) $\mathbf{F} = \langle y, -x \rangle$, $C$ is the circle $x^2 + y^2 = \frac{1}{4}$.




$F_1 = y$. $F_2 = -x$. $\therefore \frac{\partial F_2}{\partial x} = -1$, $\frac{\partial F_1}{\partial y} = 1$.

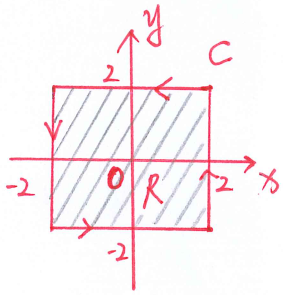
By Green's theorem:

$$\oint_C \vec{F}(\vec{r})\cdot d\vec{r} = \oint_C F_1dx + F_2dy = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dxdy.$$

$$= \iint_R (-1-1)\,dxdy = -2\iint_R dxdy = -2 \times \text{Area of the disk with radius } \tfrac{1}{2}$$

$$= -2 \times \pi \times \left(\tfrac{1}{2}\right)^2 = \boxed{-\tfrac{1}{2}\pi}.$$

(2) $\mathbf{F} = \langle 6y^2, 2x - 2y^4 \rangle$, $R$ is the square with vertices $\pm(2,2), \pm(2,-2)$.



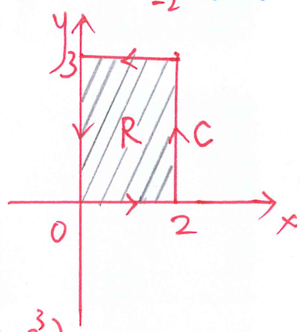

$F_1 = 6y^2$, $F_2 = 2x - 2y^4$. $\therefore \frac{\partial F_2}{\partial x} = 2$, $\frac{\partial F_1}{\partial y} = 12y$

By Green's theorem:

$$\oint_C \vec{F}(\vec{r})\cdot d\vec{r} = \oint_C F_1dx + F_2dy = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dxdy$$

$$= \int_{-2}^{2}\int_{-2}^{2} (2-12y)\,dx\,dy = \int_{-2}^{2}\left[(2-12y)x\right]_{-2}^{2} dy = \int_{-2}^{2} 8-48y\,dy = \left[8y-24y^2\right]_{-2}^{2} = \boxed{32}.$$

(3) $\mathbf{F} = \langle x^2e^y, y^2e^x \rangle$, $R$ is the rectangle with vertices (0,0), (2,0), (2,3), (0,3).




$F_1 = x^2e^y$, $F_2 = y^2e^x$. $\frac{\partial F_2}{\partial x} = y^2e^x$, $\frac{\partial F_1}{\partial y} = x^2e^y$.

By Green's theorem:

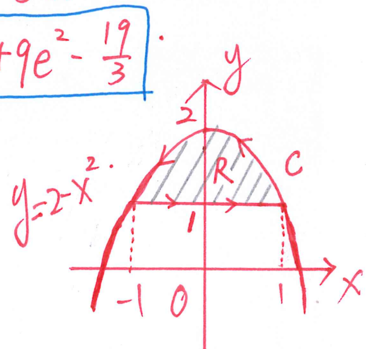
$$\oint \vec{F}(\vec{r})\cdot d\vec{r} = \oint_C F_1dx + F_2dy = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dxdy.$$

$$= \int_0^2\int_0^3 (y^2e^x - x^2e^y)\,dy\,dx = \int_0^2 \left[\tfrac{1}{3}y^3e^x - x^2e^y\right]_0^3 dx$$

$$= \int_0^2 9e^x - x^2e^3 + x^2\,dx = \left[9e^x + (1-e^3)\cdot\frac{x^3}{3}\right]_0^2 = 9(e^2-1) + \tfrac{8}{3}(1-e^3)$$

$$= \boxed{-\tfrac{8}{3}e^3 + 9e^2 - \tfrac{19}{3}}.$$

(4) $\mathbf{F} = \langle x^2 + y^2, x^2 - y^2 \rangle$, $R$: $1 \le y \le 2 - x^2$




$F_1 = x^2 + y^2$. $F_2 = x^2 - y^2$. $\therefore \frac{\partial F_2}{\partial x} = 2x$, $\frac{\partial F_1}{\partial y} = 2y$

By Green's theorem:

$$\oint \vec{F}(\vec{r})\cdot d\vec{r} = \oint_C F_1dx + F_2dy = \iint_R \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\,dxdy$$

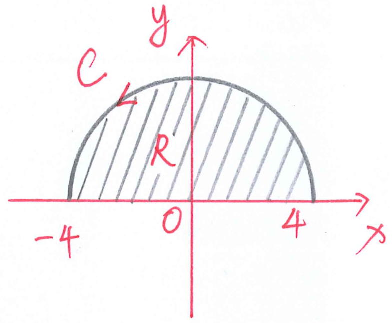
$$= \int_{-1}^{1}\int_{1}^{2-x^2} (2x-2y)\,dy\,dx = \int_{-1}^{1}\left[2xy - y^2\right]_1^{2-x^2} dx$$

$$= \int_{-1}^{1} 2x(2-x^2) - (2-x^2)^2 - 2x + 1\,dx$$

$$= \int_{-1}^{1} -x^4 - 2x^3 + 4x^2 + 2x - 3\,dx$$

$$= \left[-\tfrac{1}{5}x^5 - \tfrac{1}{2}x^4 + \tfrac{4}{3}x^3 + x^2 - 3x\right]_{-1}^{1} = \boxed{-\tfrac{56}{15}}.$$

(5) $\mathbf{F} = \langle -e^{-x}\cos y, -e^{-x}\sin y \rangle$, R is the semidisk $x^2+y^2 \le 16, x \ge 0$.




$F_1 = -e^{-x}\cos y,\ F_2 = -e^{-x}\sin y\ ,\ \therefore \frac{\partial F_2}{\partial x} = e^{-x}\sin y,\ \frac{\partial F_1}{\partial y} = e^{-x}\sin y$

$\therefore \oint_C \vec{F}(\vec{r})\cdot d\vec{r} = \oint_C F_1 dx + F_2 dy$

$= \iint_R \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\, dx\,dy = \iint_R 0\, dx\,dy = \boxed{0}$

2. Parametric surface representation (page 442). Familiarize yourself with parametric representations of important surfaces by deriving a representation as $z = f(x, y)$ or $g(x, y, z) = 0$, by finding the **parameter curves** (curves *u*=constant and *v*=constant) of the surface and a normal vector $\boldsymbol{N} = \boldsymbol{r}_u \times \boldsymbol{r}_v$ of the surface. Show the details of your work.

(1) $xy$-plane $\mathbf{r}(u, v) = (u, v) = u\mathbf{i} + v\mathbf{j}$.

The equation of the xy-plane is: $z = 0$.

$\vec{r}(u,v) = \langle u, v, 0\rangle \therefore \ \vec{r}_u = \langle 1, 0, 0\rangle,\ \vec{r}_v = \langle 0, 1, 0\rangle$

$\therefore$ The normal vector is $\vec{N} = \vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \vec{k}$.

Or, The equation of the plane is $g(x,y,z) = z = 0$, so the normal vector is $\vec{N} = \text{grad } f = \langle 0, 0, 1\rangle = \vec{k}$.

(2) $xy$-plane in polar coordinates $\mathbf{r}(u, v) = \langle u\cos v, u\sin v\rangle$ (thus $u = r, v = \theta$).

$\vec{r}_u = \langle \cos v, \sin v, 0\rangle,\ \vec{r}_v = \langle -u\sin v, u\cos v, 0\rangle$.

$\therefore$ The normal vector of xy-plane is $\vec{N} = \vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos v & \sin v & 0 \\ -u\sin v & u\cos v & 0 \end{vmatrix} = u\vec{k} = \langle 0, 0, u\rangle$,

which is parallel to z-axis.

(3) Cone $\mathbf{r}(u, v) = \langle u\cos v, u\sin v, cu\rangle$.

$x = u\cos v,\ y = u\sin v \rightarrow x^2+y^2 = u^2$, and $z = cu$

So we have $x^2+y^2 = \frac{z^2}{c^2}$ i.e. $z^2 - c^2(x^2+y^2) = 0$.

$\vec{r}_u = \langle \cos v, \sin v, c\rangle,\ \vec{r}_v = \langle -u\sin v, u\cos v, 0\rangle$.

$\therefore \vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos v & \sin v & c \\ -u\sin v & u\cos v & 0 \end{vmatrix} = \vec{i}(-cu\cos v) - \vec{j}(0 + cu\sin v) + \vec{k}u = \langle -cu\cos v, -cu\sin v, u\rangle$.

(4) Elliptic cylinder $\mathbf{r}(u,v)=\langle a\cos v, b\sin v, u\rangle$.

We have $x=a\cos v$, $y=b\sin v$. $z=u$. So we have $(\frac{x}{a})^2+(\frac{y}{b})^2=1$. (Elliptical cylinder).

$\vec{r}_u=\langle 0,0,1\rangle$. $\vec{r}_v=\langle -a\sin v, b\cos v, 0\rangle$. $\therefore \vec{N}=\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 0 & 1 \\ -a\sin v & b\cos v & 0\end{vmatrix}$

$=\vec{i}(-b\cos v)-\vec{j}(a\sin v)+\vec{k}\cdot 0=\langle -b\cos v, a\sin v, 0\rangle$.

(5) Paraboloid of revolution $\mathbf{r}(u,v)=\langle u\cos v, u\sin v, u^2\rangle$.

Let $x=u\sin v$. $y=u\sin v$, $z=u^2$. Then we have. $x^2+y^2=z$.

$\vec{r}_u=\langle \cos v, \sin v, 2u\rangle$. $\vec{r}_v=\langle -u\sin v, u\cos v, 0\rangle$. So the normal vector is

$\vec{N}=\vec{r}_u\times\vec{r}_v=\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos v & \sin v & 2u \\ -u\sin v & u\cos v & 0\end{vmatrix}=\vec{i}(-2u^2\cos v)-\vec{j}(2u^2\sin v)+\vec{k}(u\cos^2 v+u\sin^2 v)$

$=\langle -2u^2\cos v, -2u^2\sin v, u\rangle$.

(6) Hyperbolic paraboloid $\mathbf{r}(u,v)=\langle au\cosh v, bu\sinh v, u^2\rangle$

Let $x=au\cosh v$, $y=bu\sinh v$, $z=u^2$, where $\begin{cases}\cosh x=\frac{e^x+e^{-x}}{2}. \\ \sinh x=\frac{e^x-e^{-x}}{2}.\end{cases}$

Thus we have $\begin{cases}\frac{x}{au}=\cosh v=\frac{e^v+e^{-v}}{2} \\ \frac{y}{bu}=\sinh v=\frac{e^v-e^{-v}}{2}\end{cases} \Rightarrow \begin{cases}\frac{x}{au}+\frac{y}{bu}=e^v \\ \frac{x}{au}-\frac{y}{bu}=e^{-v}.\end{cases}$

$\therefore \frac{x^2}{a^2u^2}-\frac{y^2}{b^2u^2}=1. \rightarrow \frac{x^2}{a^2}-\frac{y^2}{b^2}=u^2=z$.

$\vec{r}_u=\langle a\cosh v, b\sinh v, 2u\rangle$. $\vec{r}_v=\langle au\sinh v, bu\cosh v, 0\rangle$.

$\therefore \vec{N}=\vec{r}_u\times\vec{r}_v=\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a\cosh v & b\sinh v & 2u \\ au\sinh v & bu\cosh v & 0\end{vmatrix}=\langle -2u^2b\cosh v, 2u^2a\sinh v, abu(\cosh^2 v-\sinh^2 v)\rangle$

$=\langle -2u^2b\cosh v, 2u^2a\sinh v, abu\rangle$.