# Week 5 Fourier Series

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Corresponding to textbook Chapter 3.0--3.4

# Convolution integral

$\delta(t)$
$a\delta(t-\tau)$
$x(t)$

$\longrightarrow$ **LTI** $S\{.\}$ $\longrightarrow$

$S\{\delta(t)\} = \mathbf{h(t)}$
$S\{\delta(t-\tau)\} = \mathbf{ah(t-\tau)}$
$y(t)=S\{x(t)\} = h(t)*x(t)$

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau \quad \Longrightarrow \quad \boxed{y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau}$$

Convolution Integral

# System analysis

## Strategy:

- Decompose input signal into a linear combination of basic signals.
- Choose basic signals so that response easy to compute

LTI System:

delayed impulses ⇔ convolution

complex exponentials ⇔ Fourier analysis

# Fourier analysis

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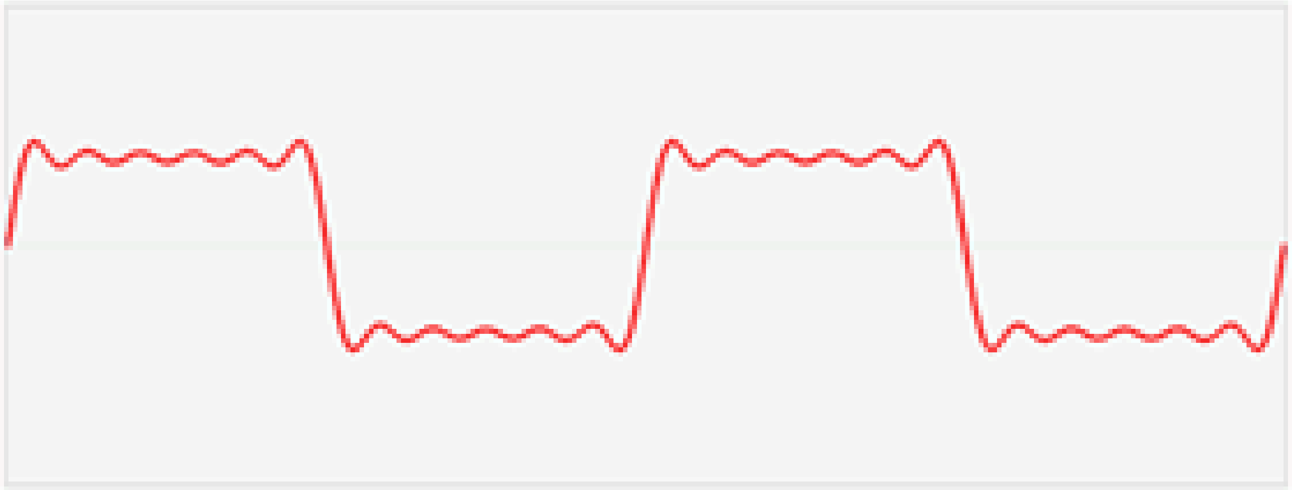
Jean Baptiste Joseph Fourier(1768-1830)



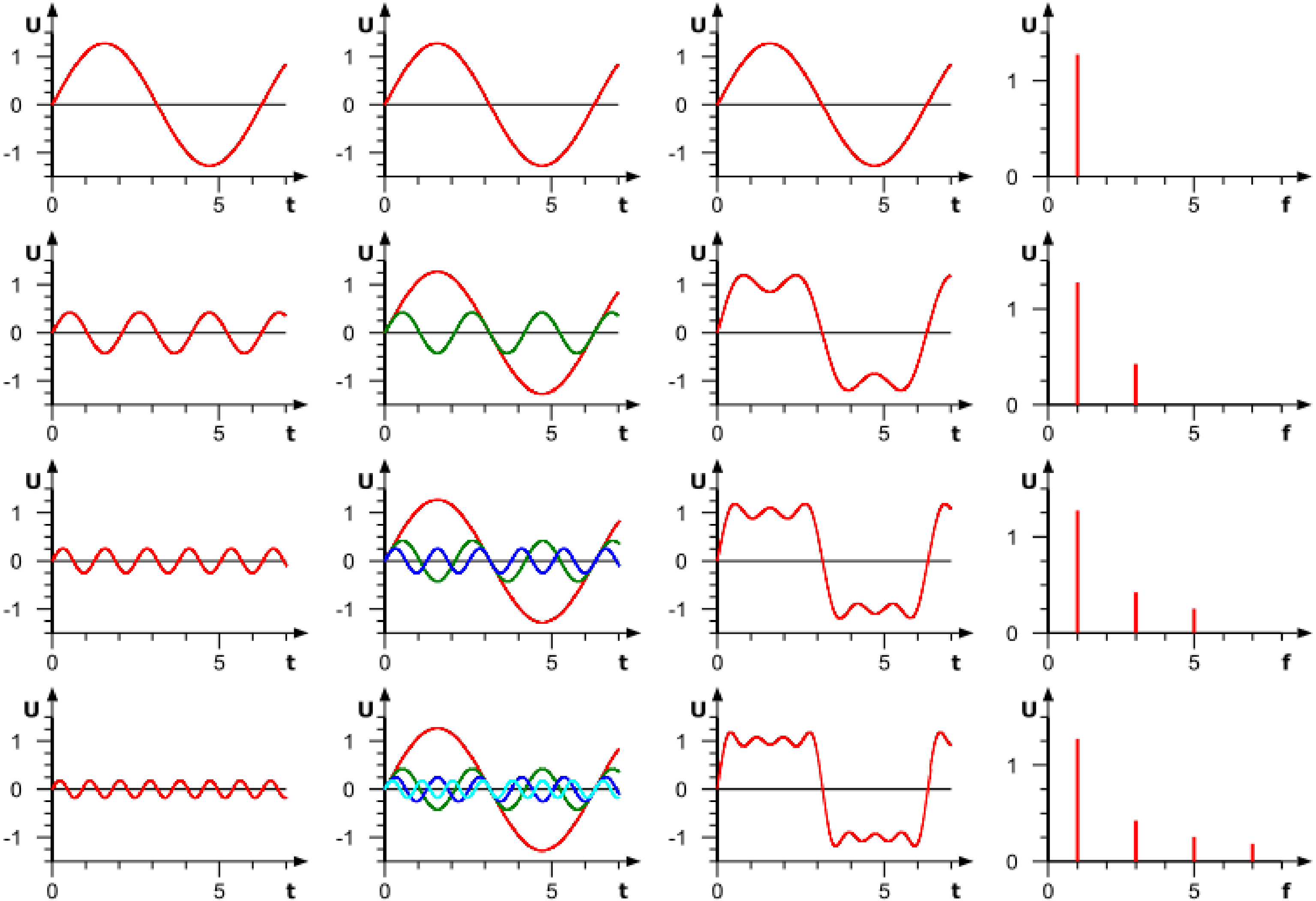
First published in 1807

http://open.163.com/movie/2011/8/8/A/M8AROL7GG_M8ARQCE8A.html

http://en.wikipedia.org/wiki/Fourier_series

# Synthesis of square wave






http://zh.wikipedia.org/wiki/%E5%82%85%E9%87%8C%E5%8F%B6%E7%BA%A7%E6%95%B0

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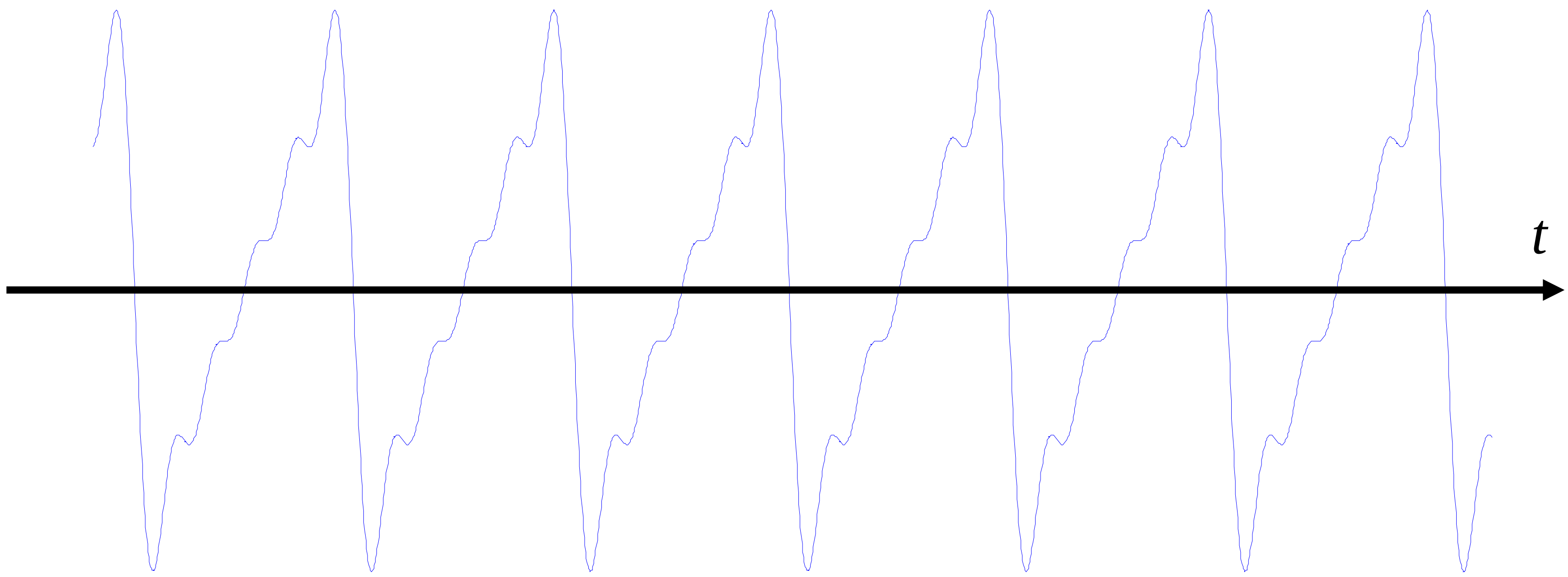
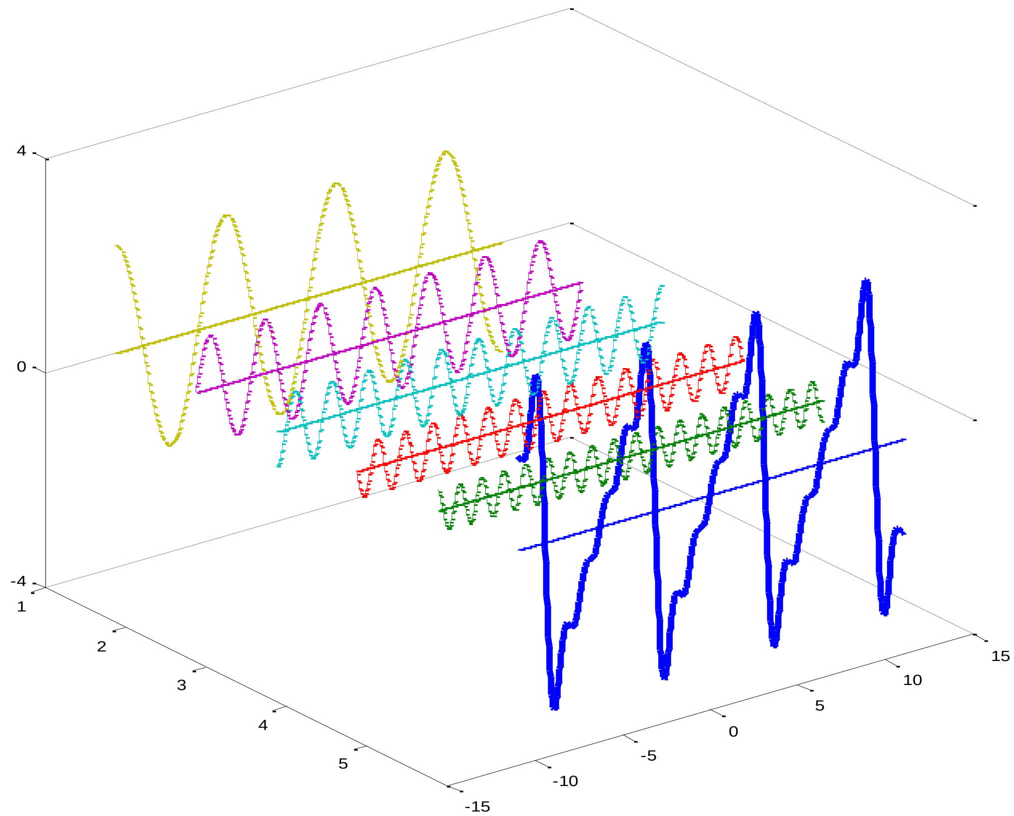
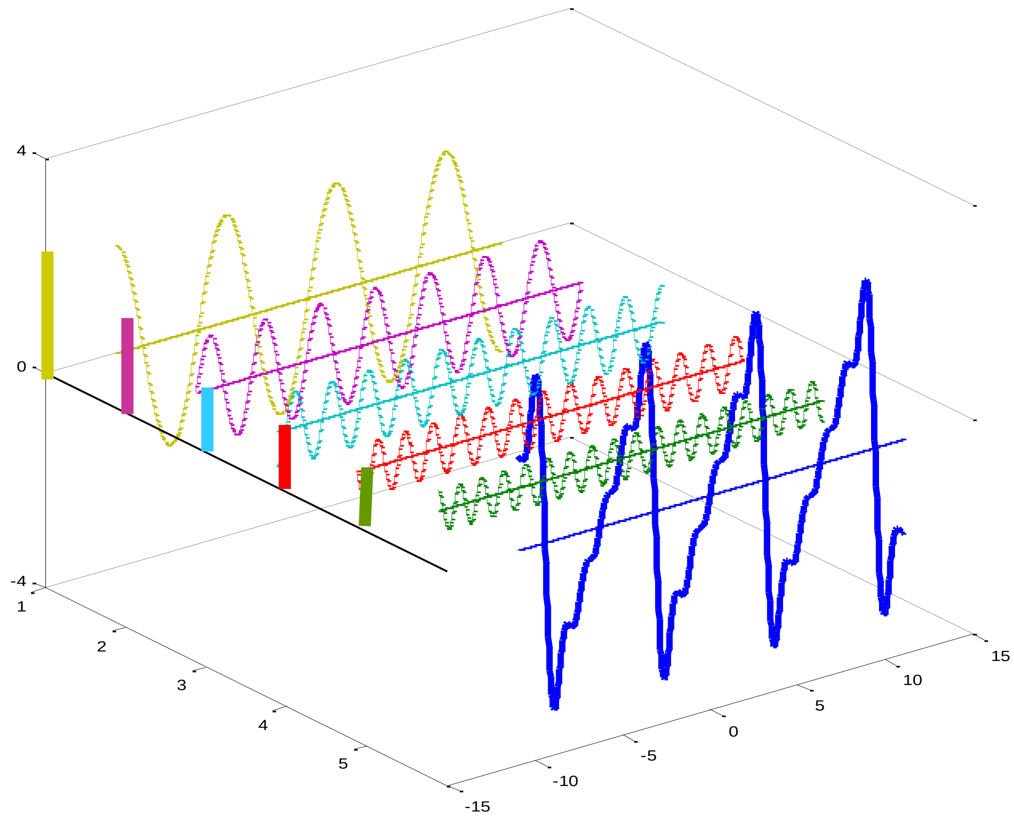
- **What is the Fourier series of the following periodic signal ("nearly sawtooth" signal)?**



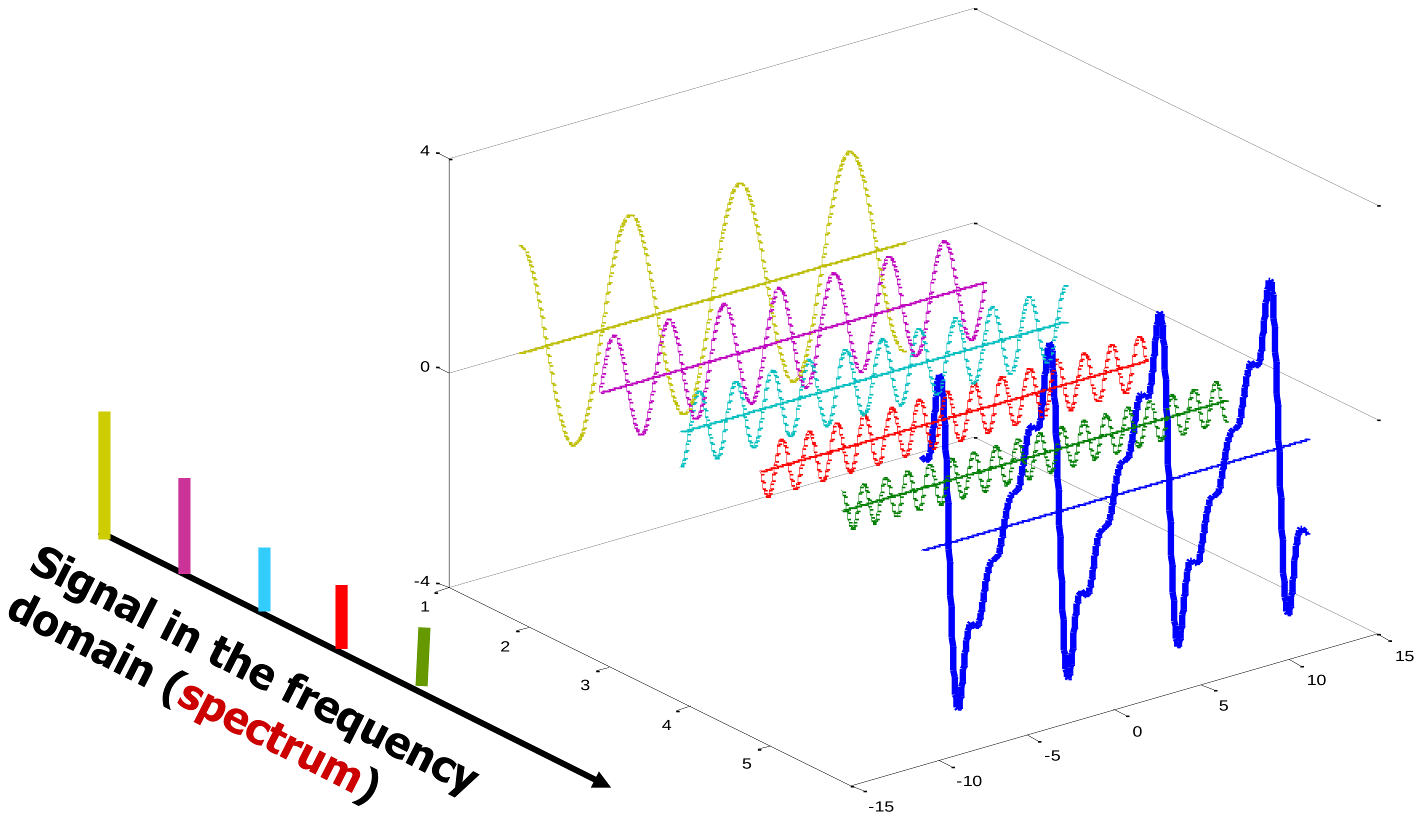
- **Relation between time and frequency !**



- **Relation between time and frequency !**



- **Relation between time and frequency !**



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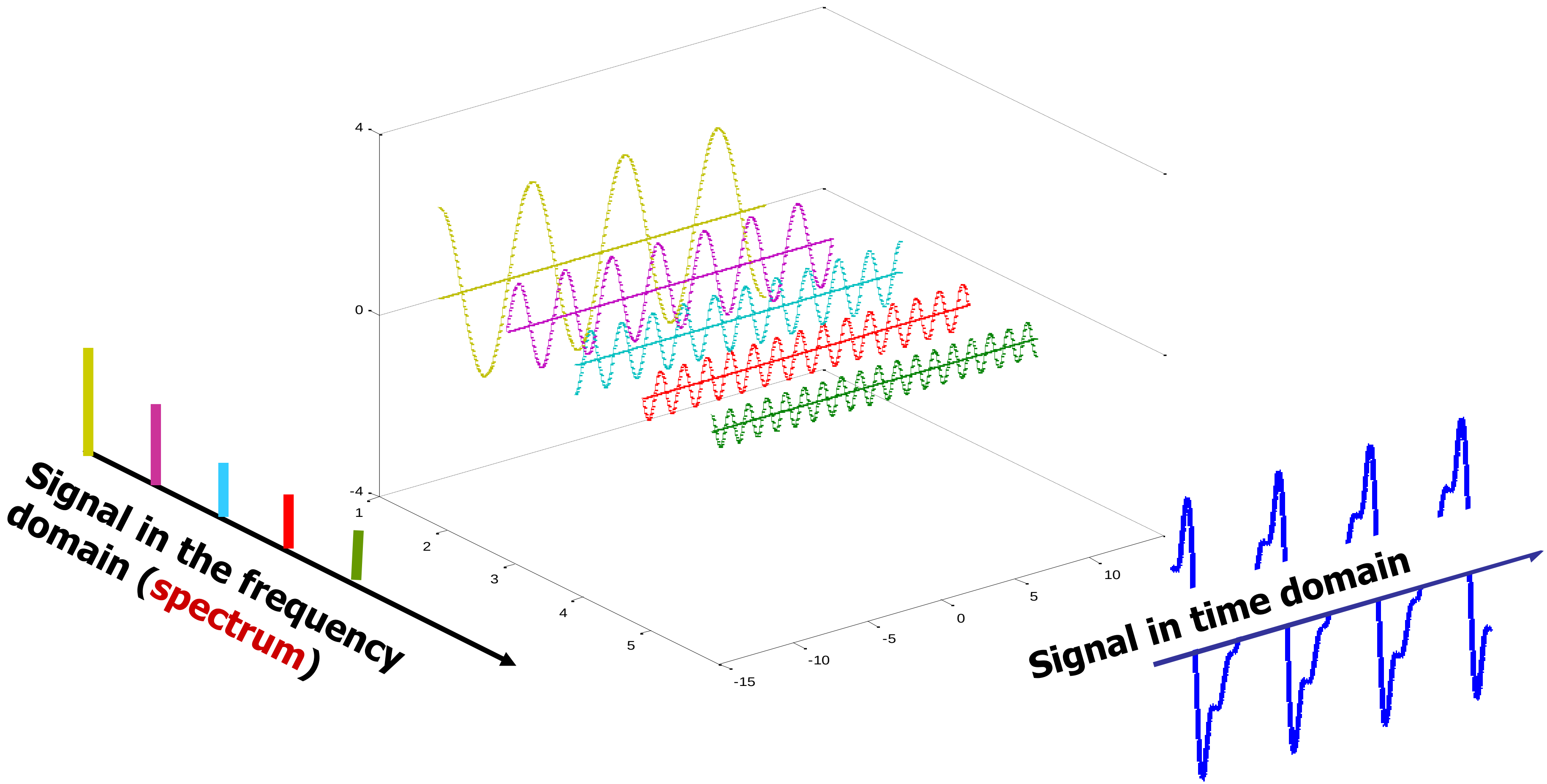
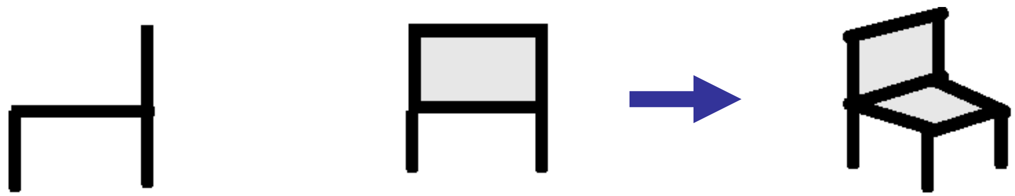
- **Relation between time and frequency !**

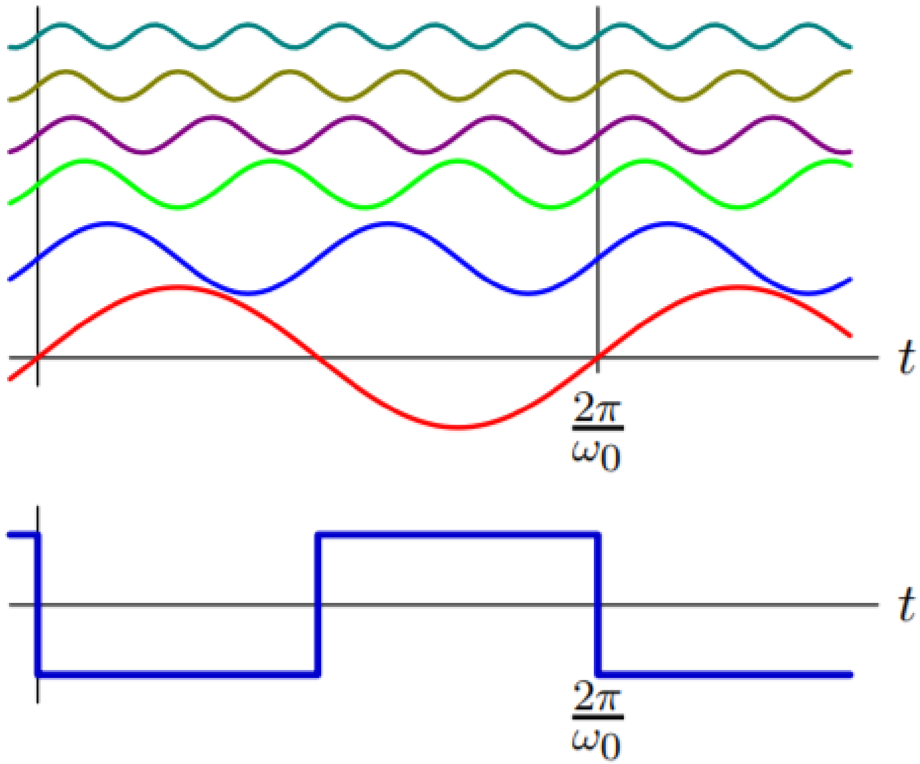


- **Relation between time and frequency !**
  - The Fourier series may be thought of as a tool for looking at a signal from a different perspective



# Harmonic Representations

Is it possible to represent ALL periodic signals with harmonics? What about discontinuous signals?




Fourier claimed YES — even though all harmonics are continuous!

Lagrange ridiculed the idea that a discontinuous signal could be written as a sum of continuous signals.

We will assume the answer is YES and see if the answer makes sense.

# Periodic signals

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$x(t)$ is periodic if, for some positive constant $T_0$

- For all values of $t$, $x(t) = x(t + T_0)$
- Smallest value of $T_0$ is the period of $x(t)$.
- For example:

$$x(t) = \cos(\omega_0 t) \text{ and } x(t) = e^{j\omega_0 t}$$

Fundamental frequency are all $\omega_0$
Fundamental periods are $\frac{2\pi}{\omega_0}$

# Fourier series of periodic signals

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Harmonically related complex exponential:

$$\varphi_k(t) = e^{jk\omega_0 t}, \qquad k = 0, \pm 1, \pm 2 \ldots$$

Linear combination of harmonically related complex exponentials:

$$\boxed{x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}}$$

Fourier Series

For $k = +N \; and \; k = -N$ are referred as the Nth harmonic components.

# Fourier series coefficients



How to determine coefficients $a_k$ ?    (textbook pg. 190)

$$\boxed{a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt}$$

Underlying properties.

1. Multiplying two harmonics produces a new harmonic with the same fundamental frequency:

$$e^{jk\omega_0 t} \times e^{jl\omega_0 t} = e^{j(k+l)\omega_0 t}.$$

2. The integral of a harmonic over any time interval with length equal to a period $T$ is zero unless the harmonic is at DC:

$$\int_{t_0}^{t_0+T} e^{jk\omega_0 t}dt \equiv \int_T e^{jk\omega_0 t}dt = \begin{cases} 0, & k \neq 0 \\ T, & k = 0 \end{cases}$$

# Separating harmonic components

Assume that $x(t)$ is periodic in $T$ and is composed of a weighted sum of harmonics of $\omega_0 = 2\pi/T$.

$$x(t) = x(t+T) = \sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 kt}$$

Then

$$\begin{aligned}\int_T x(t)e^{-jl\omega_0 t}dt &= \int_T \sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 kt} e^{-j\omega_0 lt} dt \\ &= \sum_{k=-\infty}^{\infty} a_k \int_T e^{j\omega_0 (k-l)t} dt \\ &= \sum_{k=-\infty}^{\infty} a_k T\delta[k-l] = Ta_l\end{aligned}$$

Therefore

$$a_k = \frac{1}{T}\int_T x(t)e^{-j\omega_0 kt}dt \quad = \frac{1}{T}\int_T x(t)e^{-j\frac{2\pi}{T}kt}dt$$

# Existence of the Fourier Series

Dirichlet conditions of existence :

Convergence for all $t$ in one period $\int_0^T |f(t)|\,dt < \infty$

Finite number of maxima and minima in one period of $f(t)$.

Finite discontinuities.

Example of signals that violate the Dirichlet condition is shown in textbook pg. 199.

# Example



- **finite-power periodic signal x(t) could be expressed using the complex exponentials Fourier Series as:**

$$\boxed{x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi k \frac{t}{T}}}$$

"synthesis" equation

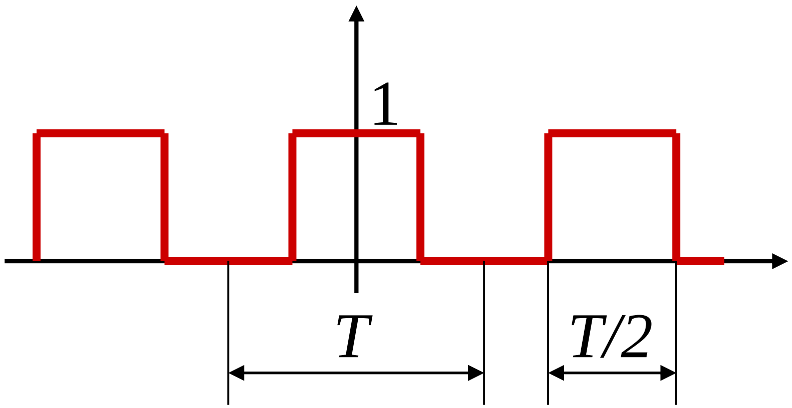
$$\boxed{a_k = \frac{1}{T} \int_{-T/2}^{T/2} x(t) e^{-j2\pi k \frac{t}{T}} dt}$$

"analysis" equation

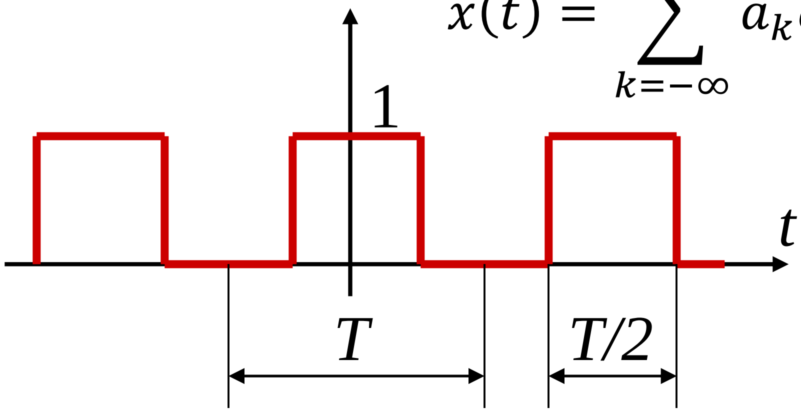
# Example

- **Find the complext exponential (C. E.) Fourier series of *x(t)***

$$x(t) = \sum_{n=-\infty}^{\infty} rect(\frac{t-nT}{T/2})$$



# C.E. Fourier series of the square wave




$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi k\frac{t}{T}}$$

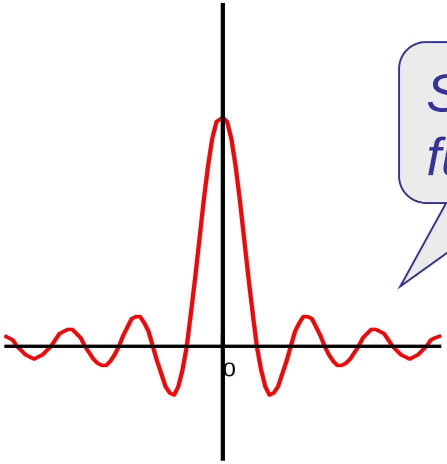
$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-j2\pi k\frac{t}{T}}dt$$

$$a_k = \frac{1}{T}\int_{-T/4}^{T/4} e^{-j2\pi k\frac{t}{T}}dt$$

$$a_k = \left.\frac{e^{-j2\pi k\frac{t}{T}}}{-j2\pi k}\right|_{-T/4}^{T/4}$$

$$a_k = \frac{e^{-j2\pi k\frac{1}{4}} - e^{j2\pi k\frac{1}{4}}}{-j2\pi k}$$

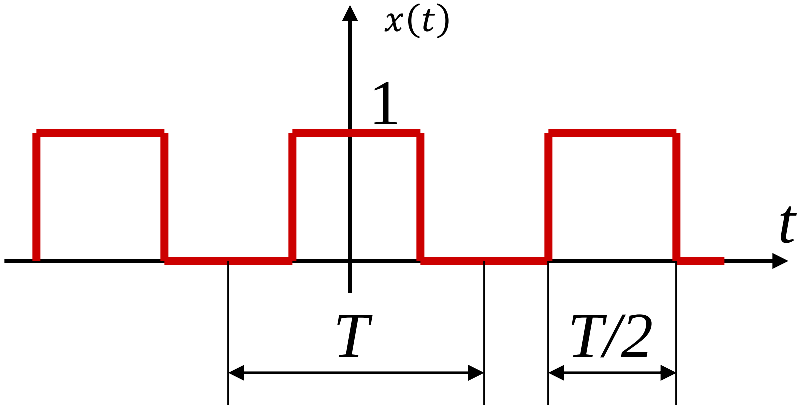
$$a_k = \frac{1}{2}\frac{\sin(\frac{\pi k}{2})}{\frac{\pi k}{2}} = \frac{1}{2}sinc(\frac{k}{2})$$




Sinc function

# Example

- **C.E. Fourier series of the square wave**




$$x(t) = \sum_{n=-\infty}^{\infty} rect\left(\frac{t-nT}{T/2}\right)$$

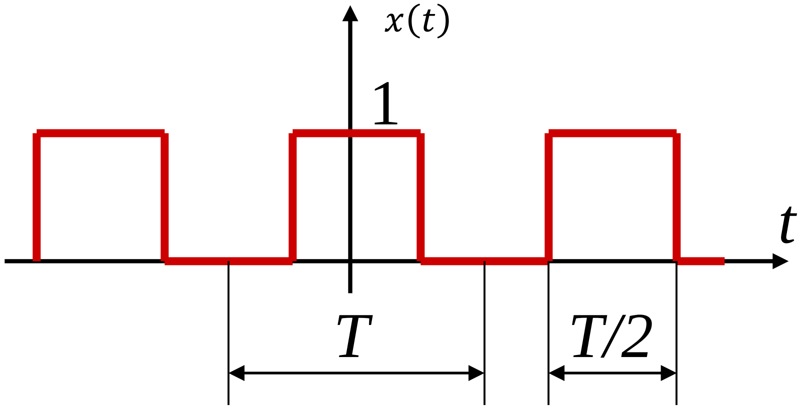
Using Fourier Series :

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi k\frac{t}{T}} \qquad a_k = \frac{1}{2} sinc\left(\frac{k}{2}\right)$$

$$x(t) = \sum_{n=-\infty}^{\infty} rect\left(\frac{t-nT}{T/2}\right) = \sum_{k=-\infty}^{\infty} \frac{1}{2} sinc\left(\frac{k}{2}\right) e^{j2\pi k\frac{t}{T}}$$

# Example

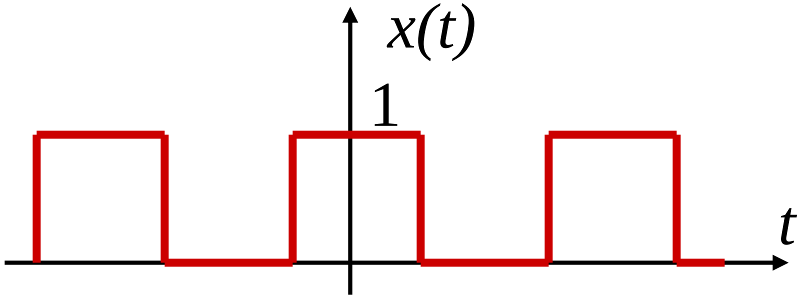
- **C.E. Fourier series of the square wave**

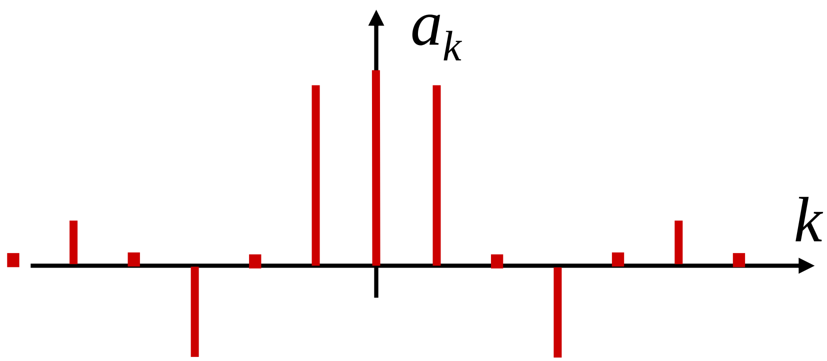



$$x(t) = \sum_{n=-\infty}^{\infty} rect\left(\frac{t-nT}{T/2}\right) = \sum_{k=-\infty}^{\infty} \boldsymbol{\frac{1}{2} sinc\left(\frac{k}{2}\right)} e^{j2\pi k\frac{t}{T}}$$

– Once we know $a_k$ we could completely describe the function *x(t)* ➔ knowing $a_k$ is equivalent to knowing x(t)

- **C.E. Fourier series of the square wave**




$$a_k = \frac{1}{2} sinc(\frac{k}{2})$$

$$a_0 = \frac{1}{2}$$

$$a_1 = a_{-1} = \frac{1}{\pi}$$

$$a_2 = a_{-2} = 0$$

$$a_3 = a_{-3} = -\frac{1}{3\pi}$$

$$\ldots$$

$$\ldots\ldots$$

$$\lim_{k\to\infty} a_k = 0$$

# Signal spectrum

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- **Signal spectrum is the graphical representation of** $a_k$

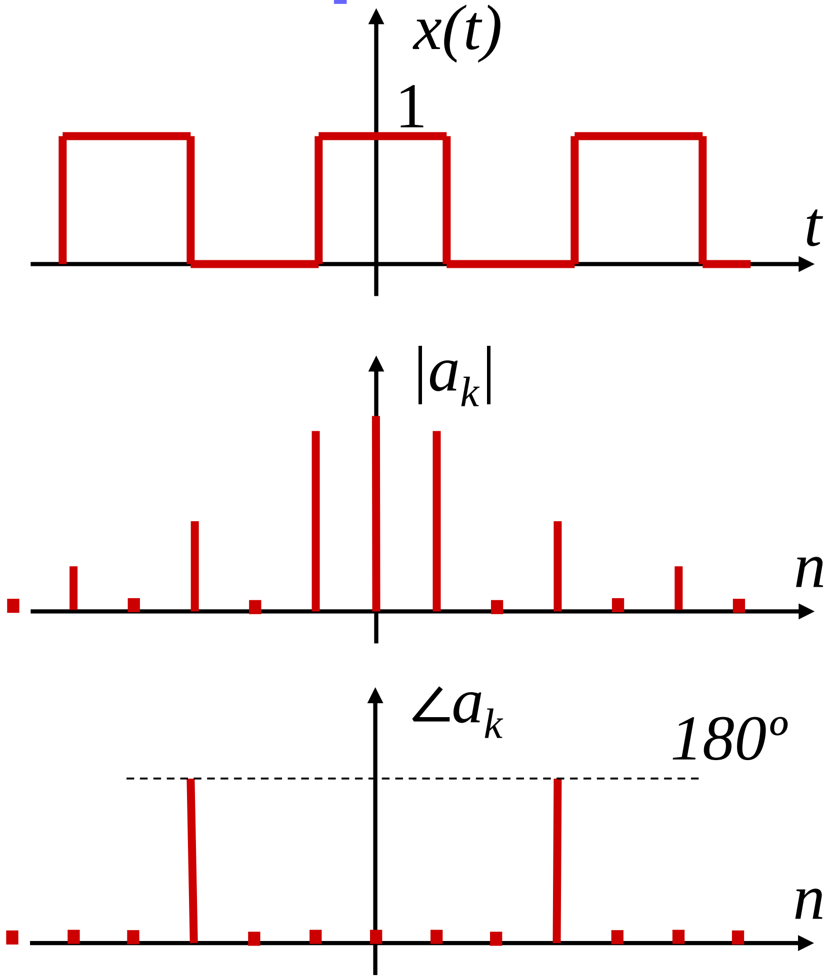
$$\boxed{a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-j2\pi k\frac{t}{T}}dt}$$

  - $a_k$ is in general a complex function of the variable $k$ ➔ how could we draw it (complex function)?

$$a_k = |a_k|e^{j\ phase(a_k)} = |a_k|e^{j\angle a_k}$$

# Signal spectrum (an example)

- **For the square wave**




$$x(t) = \sum_{k=-\infty}^{\infty} rect\left(\frac{t-kT}{T/2}\right)$$

$$a_k = \frac{1}{2} sinc\left(\frac{k}{2}\right)$$

$$a_k = |a_k| e^{j\angle a_k}$$

# Signal spectrum (an example)

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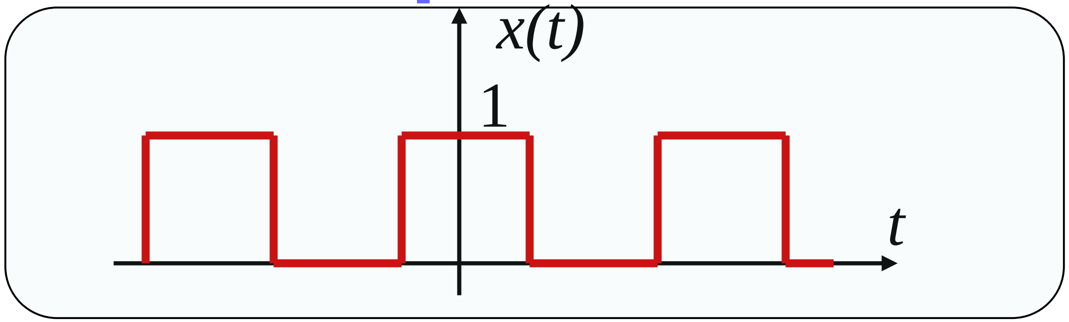
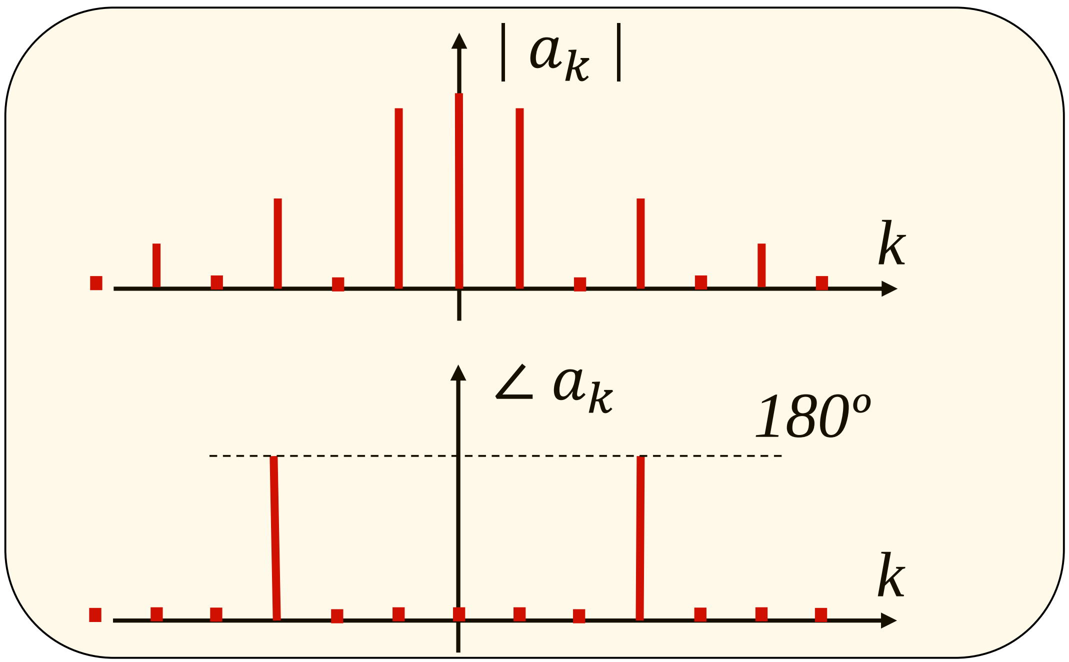
- **For the square wave**



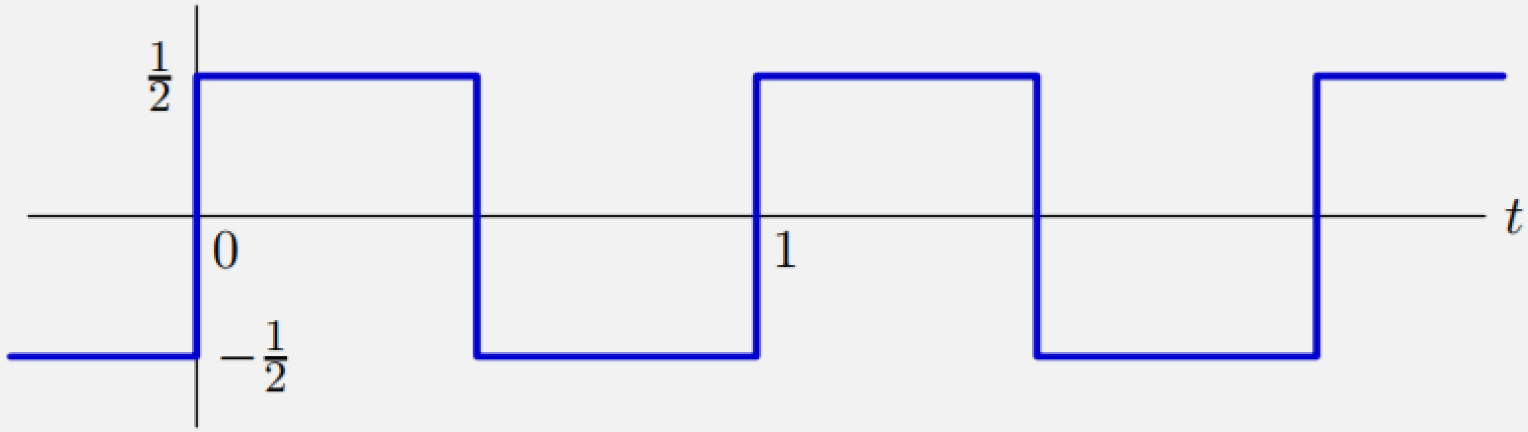

Time domain




Frequency domain

# Check Yourself

Let $a_k$ represent the Fourier series coefficients of the following square wave.

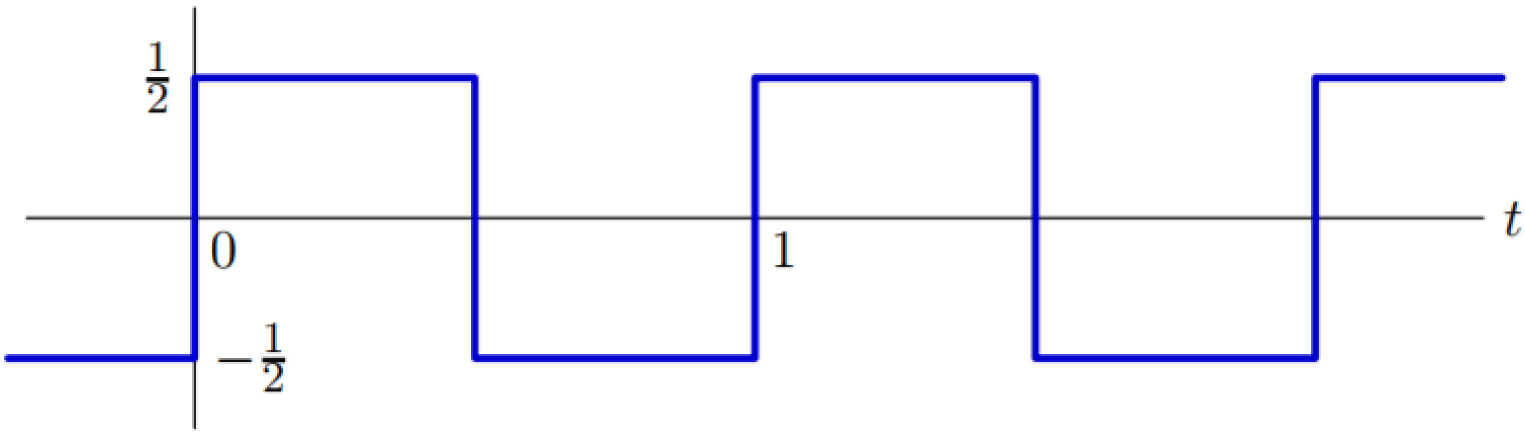



How many of the following statements are true?

1. $a_k = 0$ if $k$ is even
2. $a_k$ is real-valued
3. $|a_k|$ decreases with $k^2$
4. there are an infinite number of non-zero $a_k$
5. all of the above

# Check Yourself

Let $a_k$ represent the Fourier series coefficients of the following square wave.




$$a_k = \int_T x(t)e^{-j\frac{2\pi}{T}kt}dt = -\frac{1}{2}\int_{-\frac{1}{2}}^{0} e^{-j2\pi kt}dt + \frac{1}{2}\int_{0}^{\frac{1}{2}} e^{-j2\pi kt}dt$$

$$= \frac{1}{j4\pi k}\left(2 - e^{j\pi k} - e^{-j\pi k}\right)$$

$$= \begin{cases} \dfrac{1}{j\pi k}\ ; & \text{if } k \text{ is odd} \\ 0\ ; & \text{otherwise} \end{cases}$$

## Check Yourself

Let $a_k$ represent the Fourier series coefficients of the following square wave.

$$a_k = \begin{cases} \dfrac{1}{j\pi k} \ ; & \text{if } k \text{ is odd} \\ 0 \ ; & \text{otherwise} \end{cases}$$

How many of the following statements are true?

1. $a_k = 0$ if $k$ is even $\surd$
2. $a_k$ is real-valued $\times$
3. $|a_k|$ decreases with $k^2$ $\times$
4. there are an infinite number of non-zero $a_k$ $\surd$
5. all of the above $\times$

## Fourier Series Properties

If a signal is differentiated in time, its Fourier coefficients are multiplied by $j\frac{2\pi}{T}k$.

Proof: Let

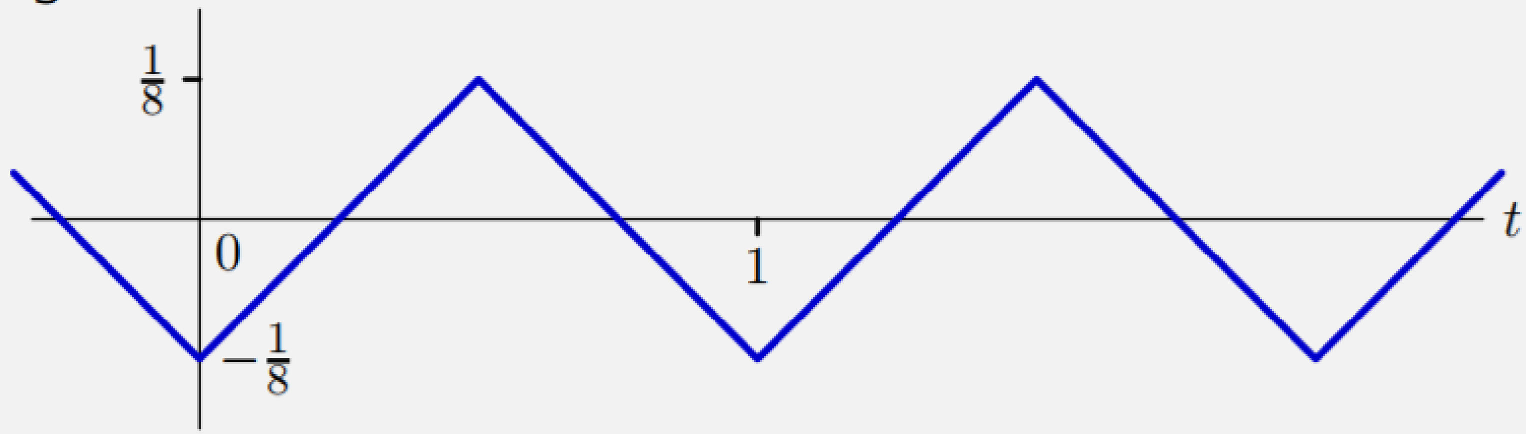
$$x(t) = x(t+T) = \sum_{k=-\infty}^{\infty} a_k e^{j\frac{2\pi}{T}kt}$$

then

$$\dot{x}(t) = \dot{x}(t+T) = \sum_{k=-\infty}^{\infty} \left(j\frac{2\pi}{T}k\, a_k\right) e^{j\frac{2\pi}{T}kt}$$

# Check Yourself

Let $b_k$ represent the Fourier series coefficients of the following triangle wave.

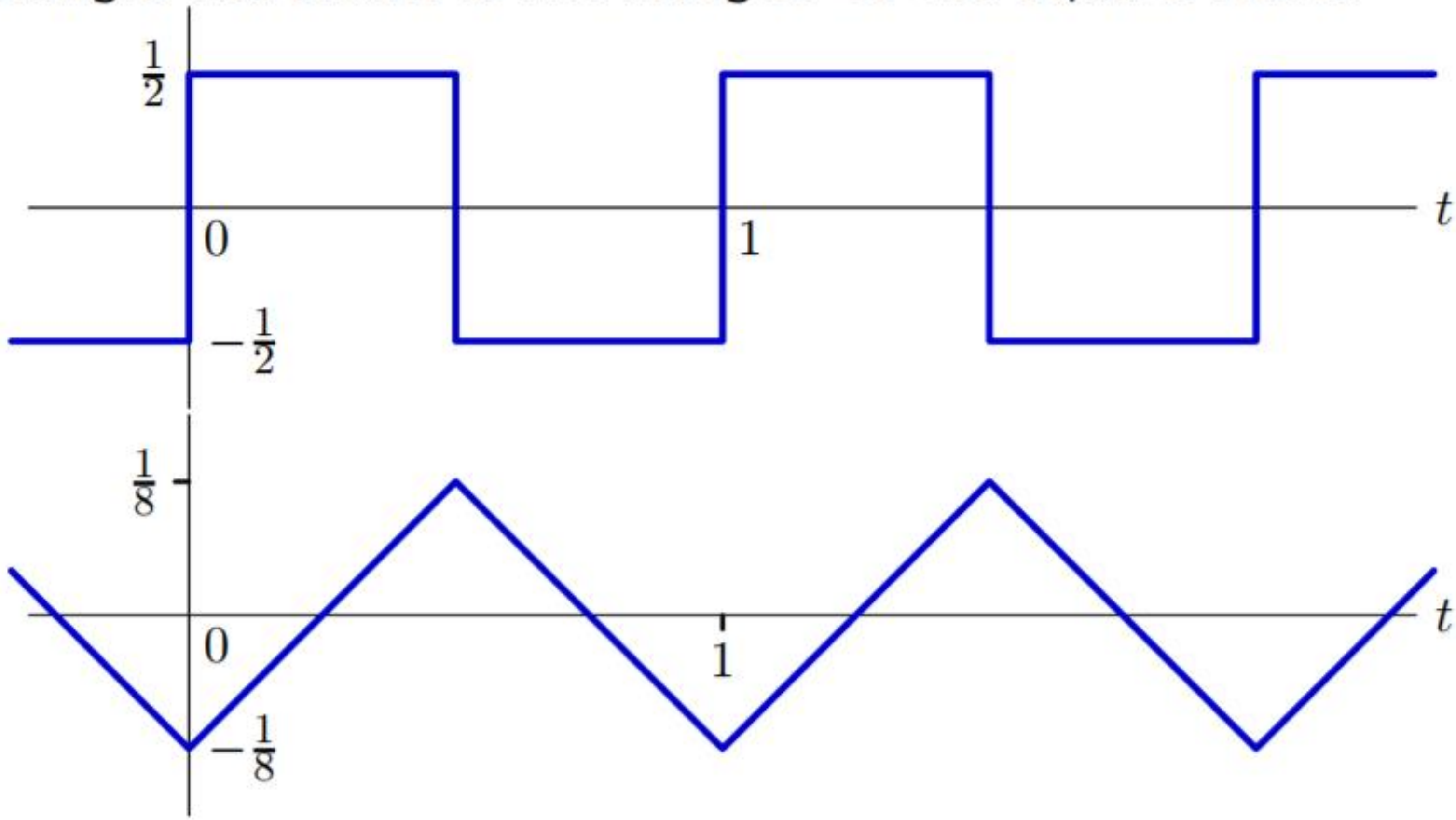



How many of the following statements are true?

1. $b_k = 0$ if $k$ is even
2. $b_k$ is real-valued
3. $|b_k|$ decreases with $k^2$
4. there are an infinite number of non-zero $b_k$
5. all of the above

## Check Yourself

The triangle waveform is the integral of the square wave.




Therefore the Fourier coefficients of the triangle waveform are $\frac{1}{j2\pi k}$ times those of the square wave.

$$b_k = \frac{1}{jk\pi} \times \frac{1}{j2\pi k} = \frac{-1}{2k^2\pi^2} \ ; \ k \text{ odd}$$

# Check Yourself

Let $b_k$ represent the Fourier series coefficients of the following triangle wave.

$$b_k = \frac{-1}{2k^2\pi^2} \; ; \; k \text{ odd}$$

How many of the following statements are true?

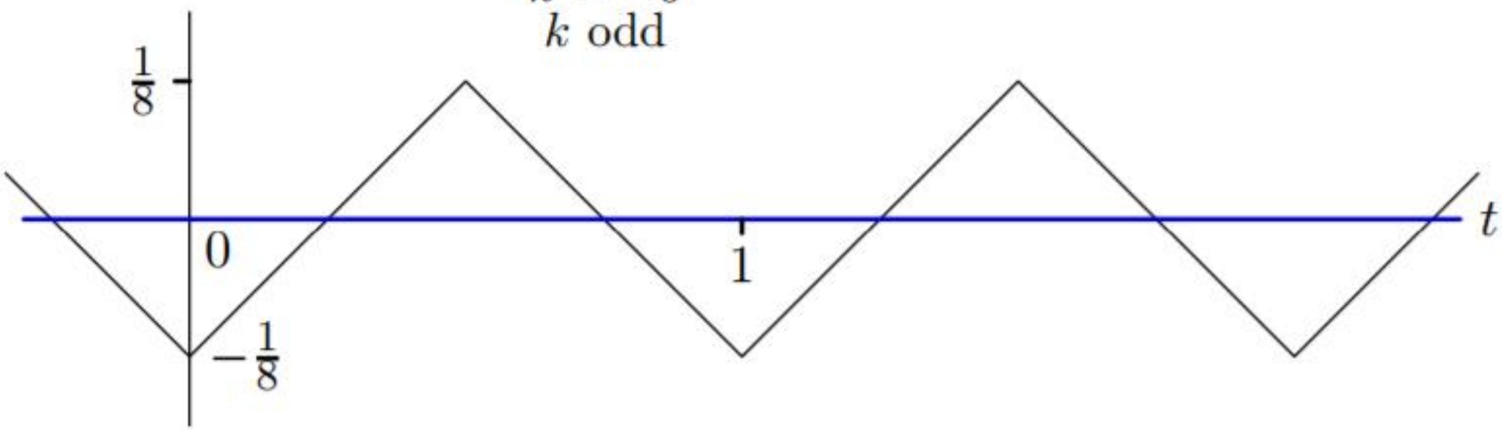
1. $b_k = 0$ if $k$ is even $\surd$
2. $b_k$ is real-valued $\surd$
3. $|b_k|$ decreases with $k^2$ $\surd$
4. there are an infinite number of non-zero $b_k$ $\surd$
5. all of the above $\surd$

# Fourier Series

One can visualize convergence of the Fourier Series by incrementally adding terms.

Example: triangle waveform

$$\sum_{\substack{k=-0\\ k\text{ odd}}}^{0} \frac{-1}{2k^2\pi^2} e^{j2\pi kt}$$

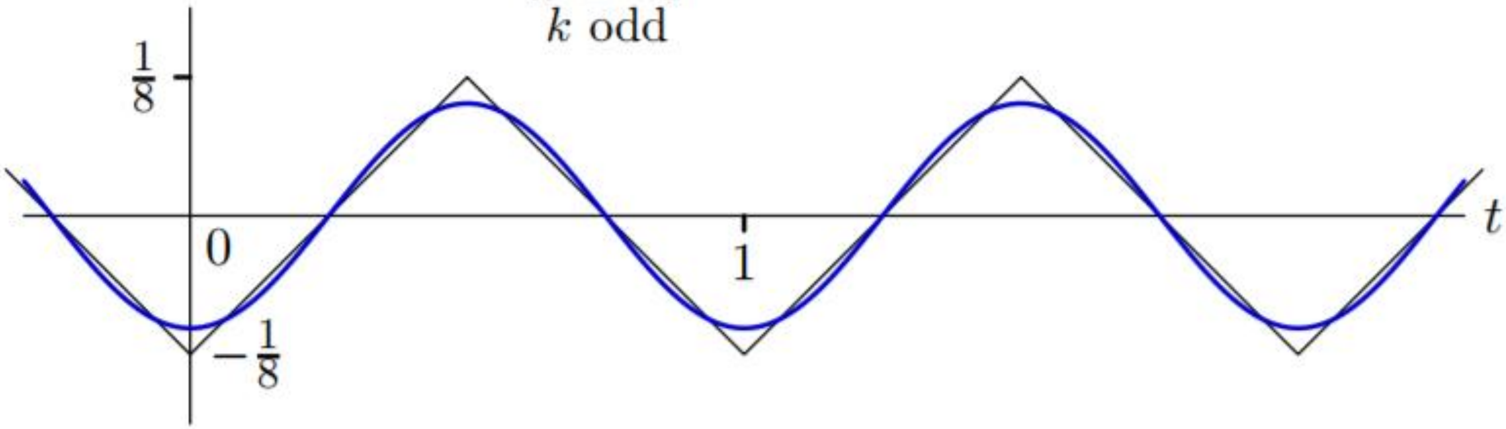


# Fourier Series

One can visualize convergence of the Fourier Series by incrementally adding terms.

Example: triangle waveform

$$\sum_{\substack{k=-1 \\ k \text{ odd}}}^{1} \frac{-1}{2k^2\pi^2} e^{j2\pi kt}$$

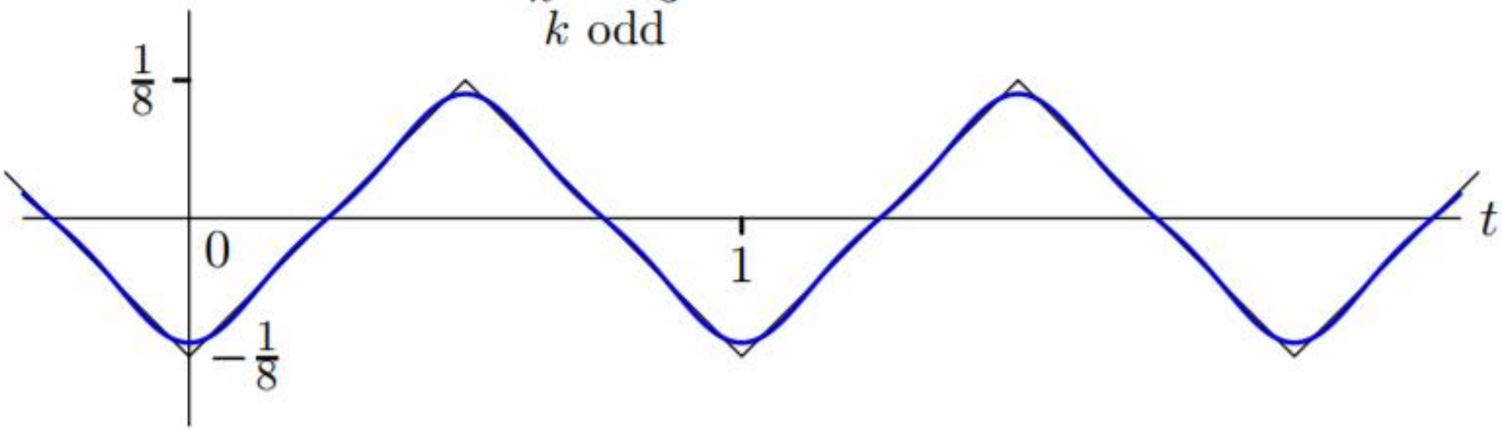


## Fourier Series

One can visualize convergence of the Fourier Series by incrementally adding terms.

Example: triangle waveform

$$\sum_{\substack{k=-3 \\ k \text{ odd}}}^{3} \frac{-1}{2k^2\pi^2} e^{j2\pi kt}$$

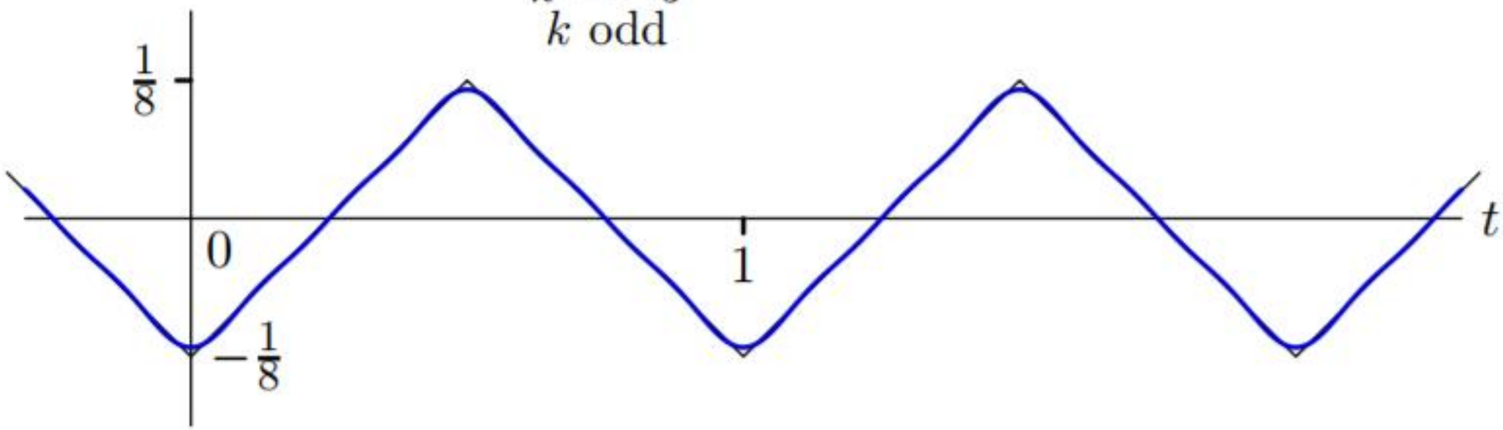


# Fourier Series

One can visualize convergence of the Fourier Series by incrementally adding terms.

Example: triangle waveform

$$\sum_{\substack{k=-5 \\ k \text{ odd}}}^{5} \frac{-1}{2k^2\pi^2} e^{j2\pi kt}$$

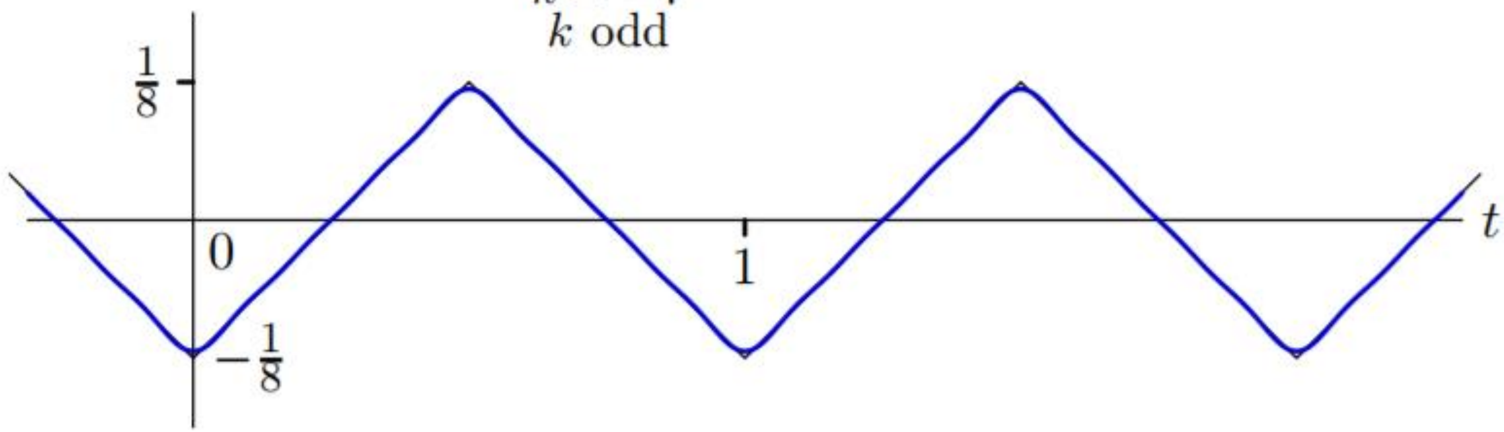


# Fourier Series

One can visualize convergence of the Fourier Series by incrementally adding terms.

Example: triangle waveform

$$\sum_{\substack{k=-7 \\ k \text{ odd}}}^{7} \frac{-1}{2k^2\pi^2} e^{j2\pi kt}$$

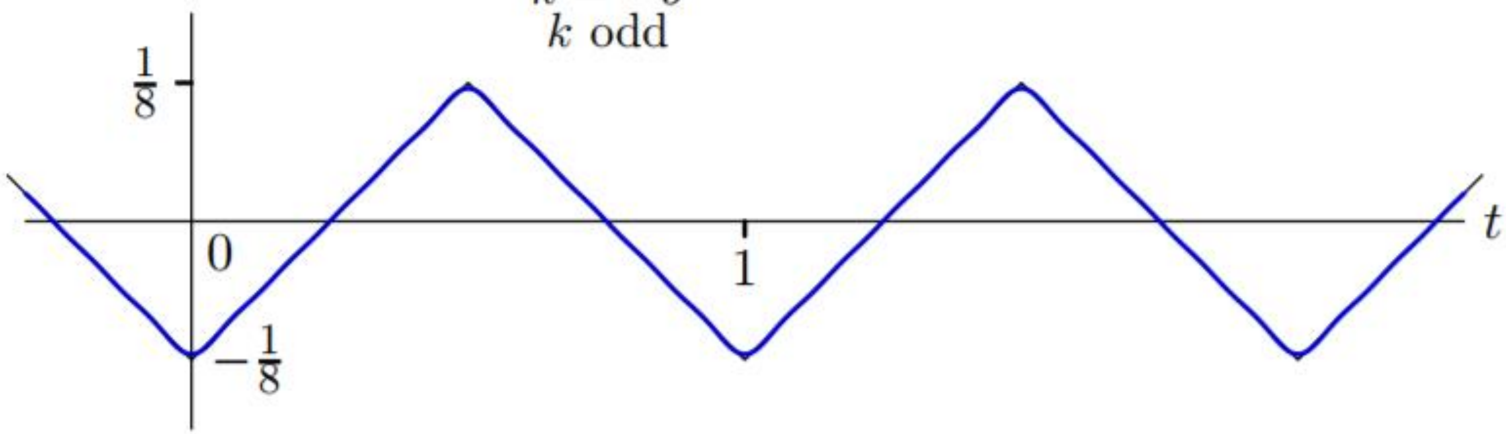


# Fourier Series

One can visualize convergence of the Fourier Series by incrementally adding terms.

Example: triangle waveform

$$\sum_{\substack{k=-9 \\ k \text{ odd}}}^{9} \frac{-1}{2k^2\pi^2} e^{j2\pi kt}$$

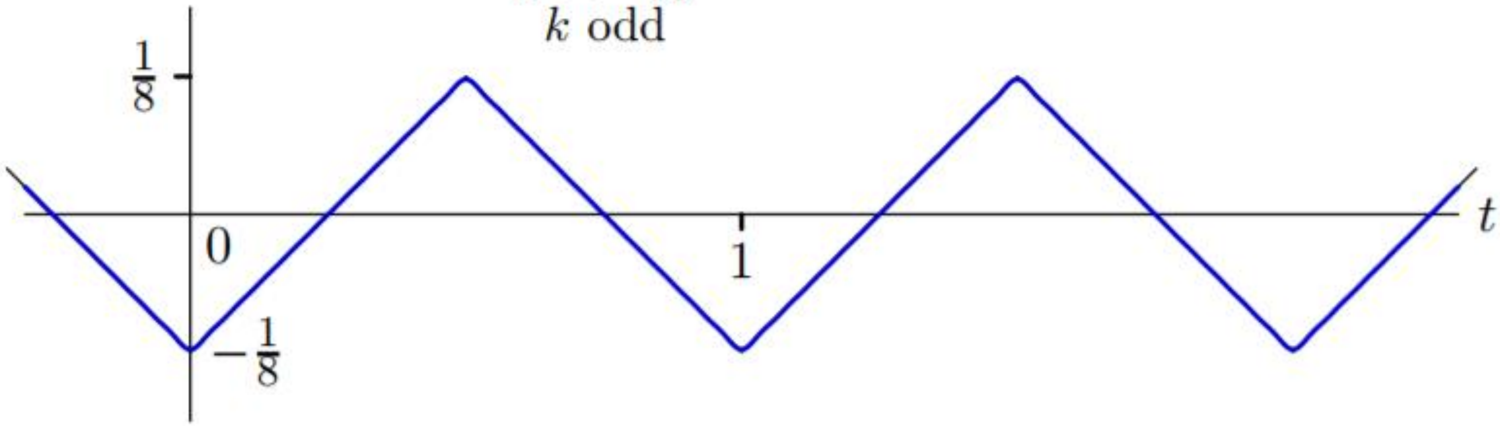


# Fourier Series

One can visualize convergence of the Fourier Series by incrementally adding terms.

Example: triangle waveform

$$\sum_{\substack{k=-19 \\ k \text{ odd}}}^{19} \frac{-1}{2k^2\pi^2} e^{j2\pi kt}$$

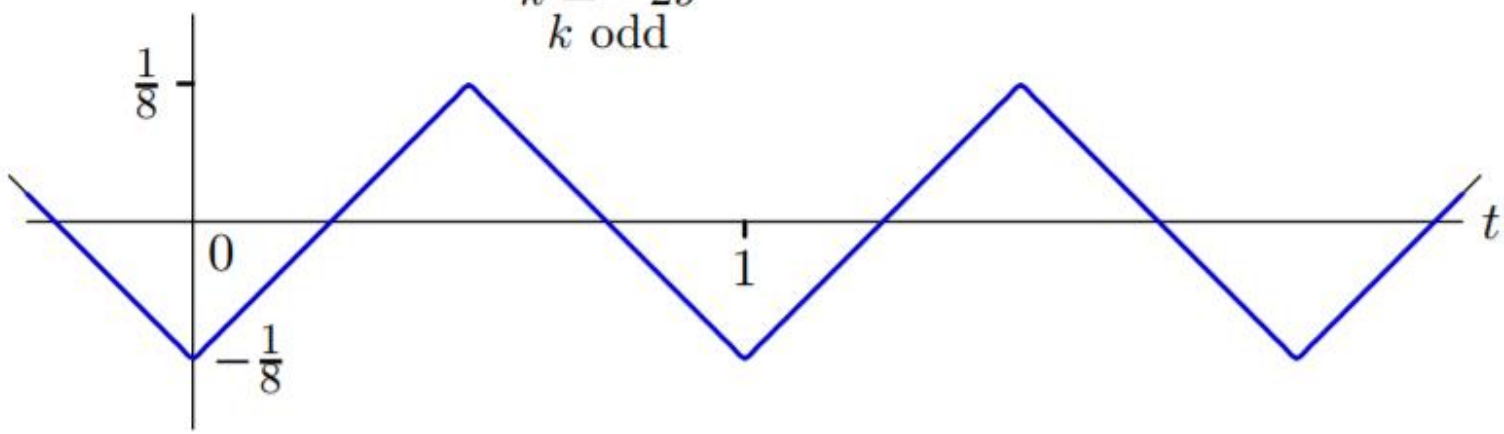


# Fourier Series

One can visualize convergence of the Fourier Series by incrementally adding terms.

Example: triangle waveform

$$\sum_{\substack{k=-29 \\ k \text{ odd}}}^{29} \frac{-1}{2k^2\pi^2} e^{j2\pi kt}$$

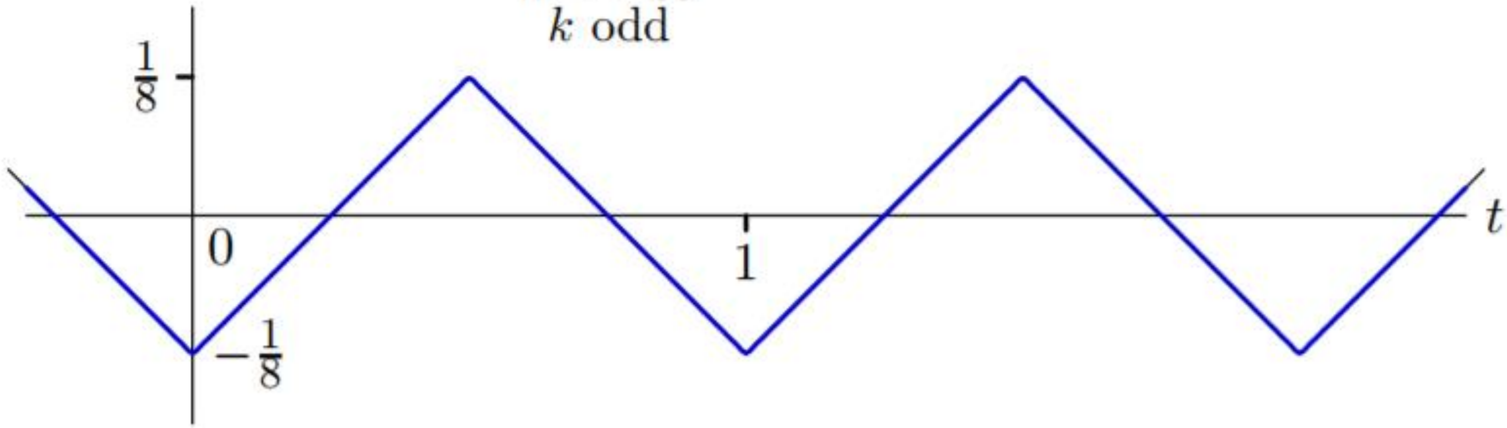


# Fourier Series

One can visualize convergence of the Fourier Series by incrementally adding terms.

Example: triangle waveform

$$\sum_{\substack{k=-39 \\ k \text{ odd}}}^{39} \frac{-1}{2k^2\pi^2} e^{j2\pi kt}$$



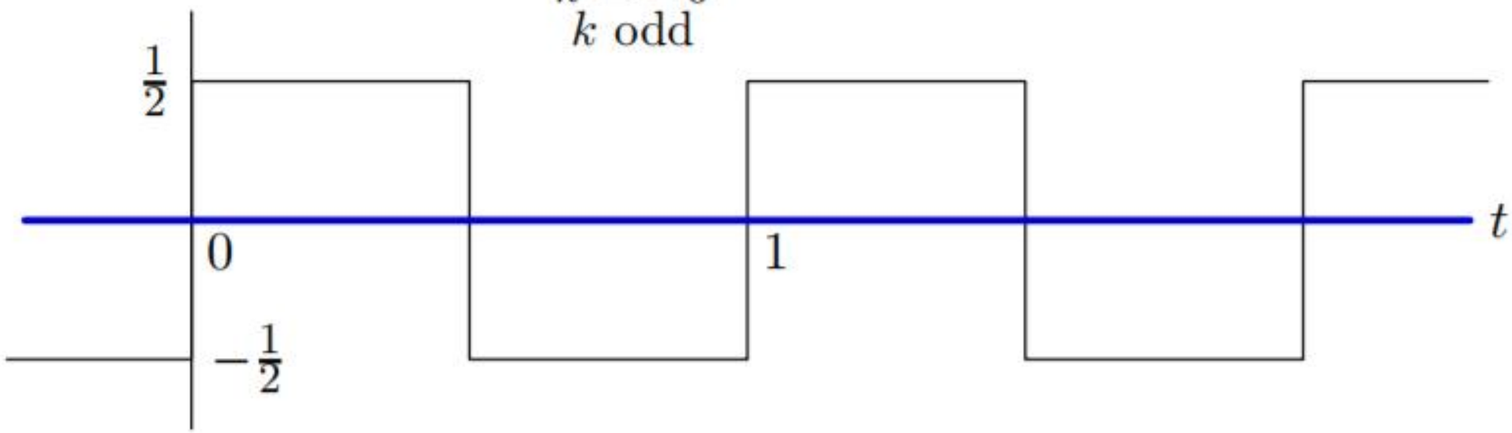

Fourier series representations of functions with discontinuous slopes converge toward functions with discontinuous slopes.

# Fourier Series

One can visualize convergence of the Fourier Series by incrementally adding terms.

Example: square wave

$$\sum_{\substack{k=-0 \\ k \text{ odd}}}^{0} \frac{1}{jk\pi} e^{j2\pi kt}$$

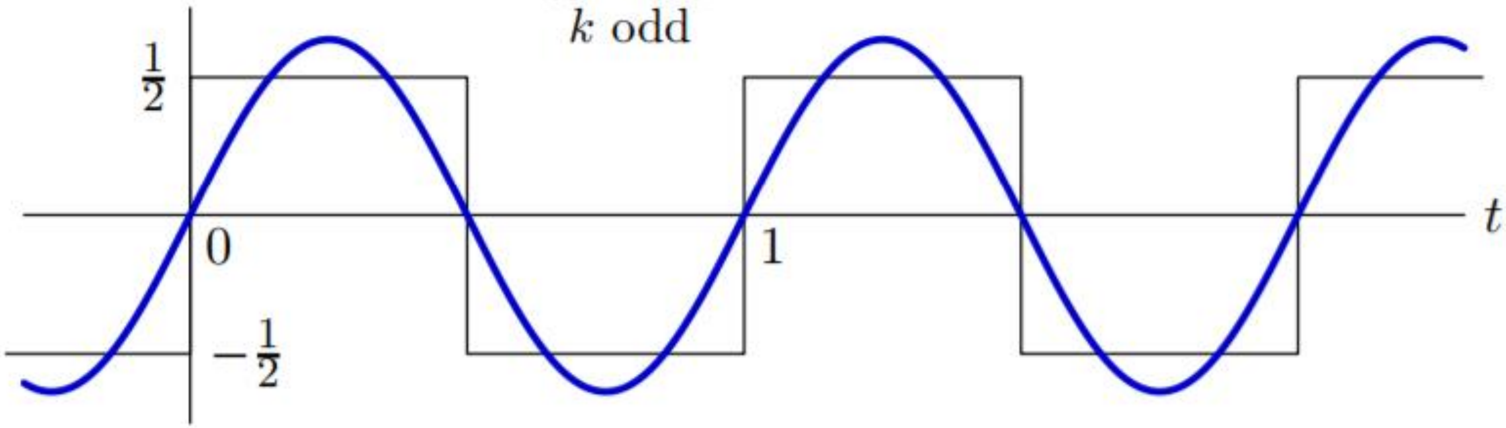


# Fourier Series

One can visualize convergence of the Fourier Series by incrementally adding terms.

Example: square wave

$$\sum_{\substack{k=-1 \\ k \text{ odd}}}^{1} \frac{1}{jk\pi} e^{j2\pi kt}$$

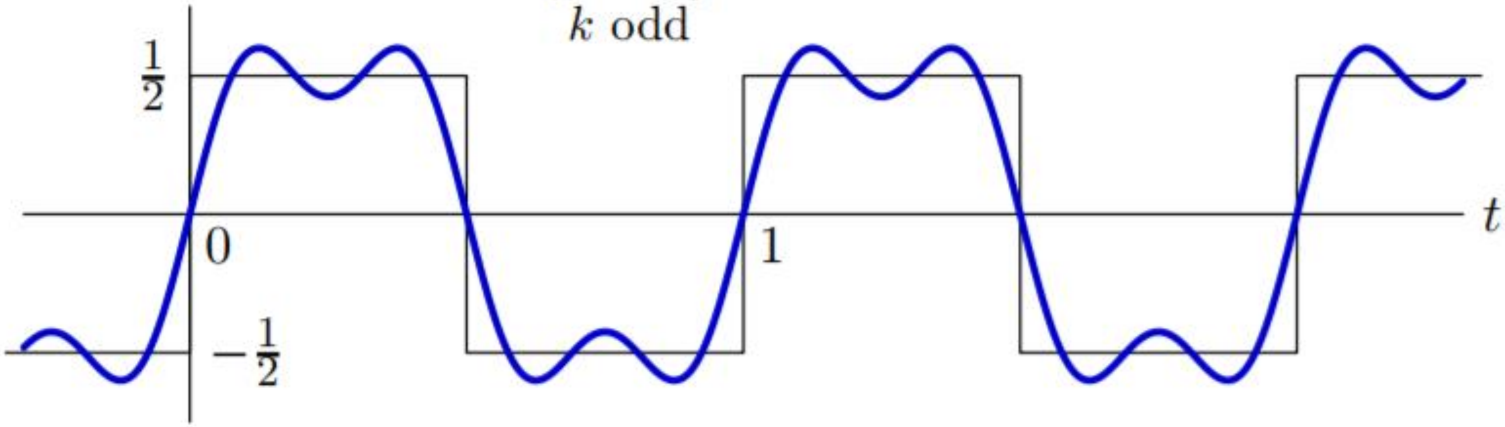


## Fourier Series

One can visualize convergence of the Fourier Series by incrementally adding terms.

Example: square wave

$$\sum_{\substack{k=-3 \\ k \text{ odd}}}^{3} \frac{1}{jk\pi} e^{j2\pi kt}$$

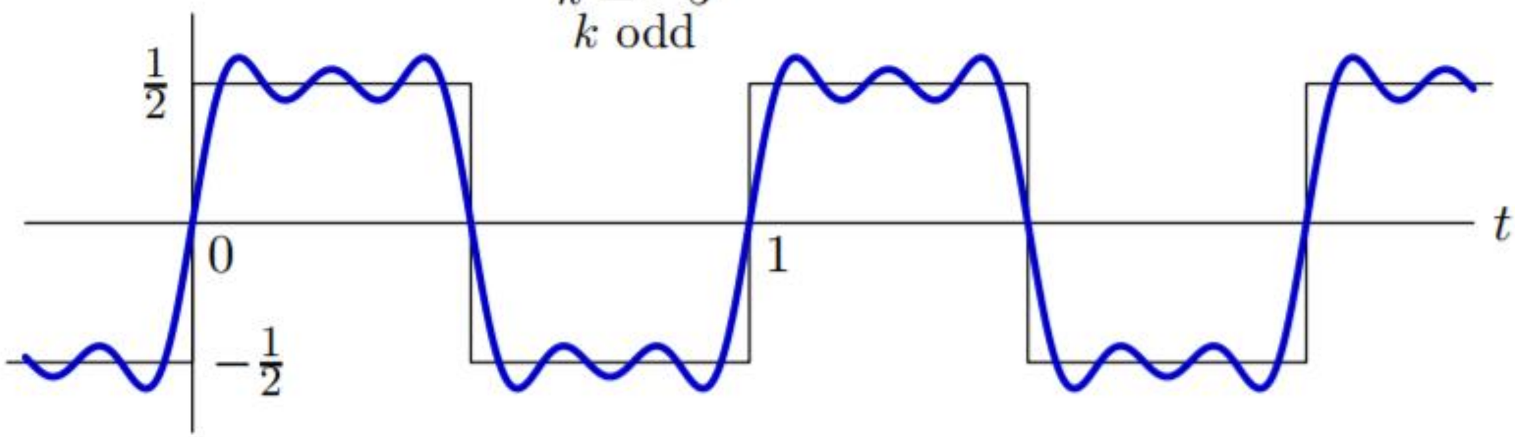


# Fourier Series

One can visualize convergence of the Fourier Series by incrementally adding terms.

Example: square wave

$$\sum_{\substack{k=-5 \\ k \text{ odd}}}^{5} \frac{1}{jk\pi} e^{j2\pi kt}$$

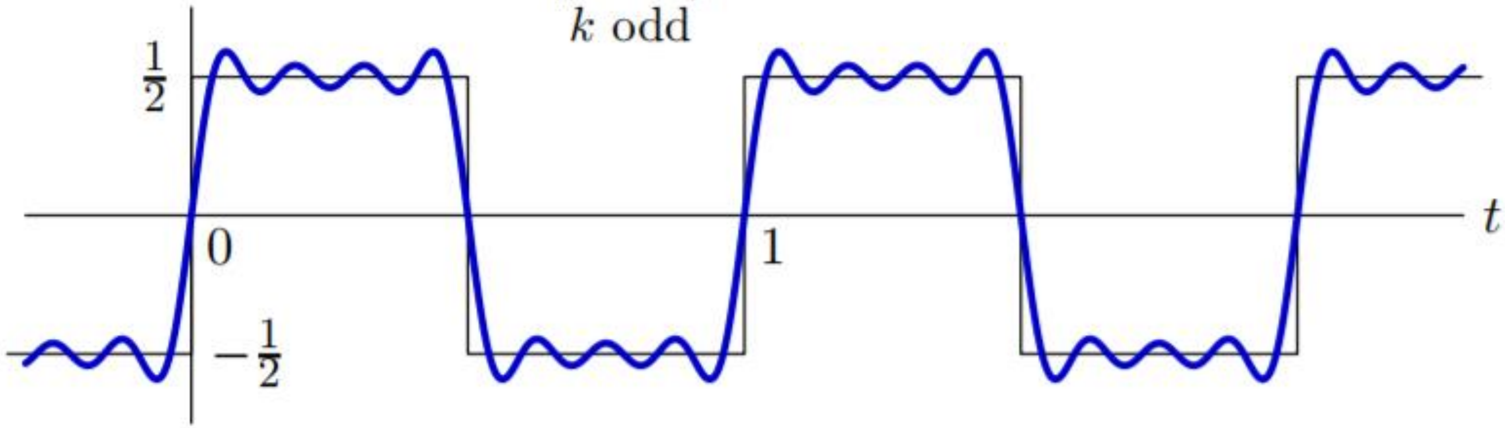


# Fourier Series

One can visualize convergence of the Fourier Series by incrementally adding terms.

Example: square wave

$$\sum_{\substack{k=-7 \\ k \text{ odd}}}^{7} \frac{1}{jk\pi} e^{j2\pi kt}$$

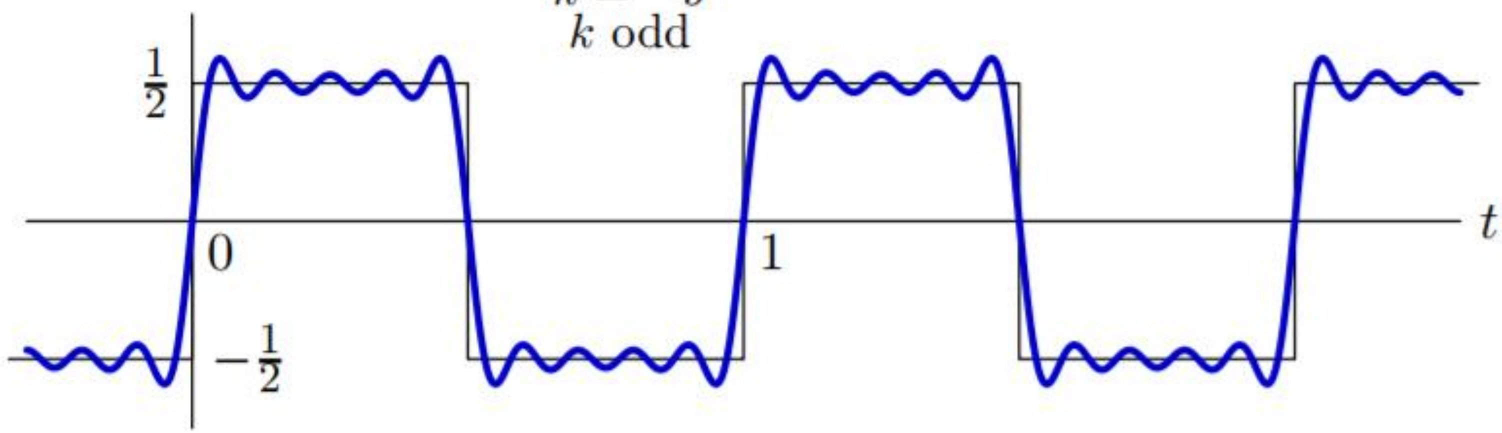


# Fourier Series

One can visualize convergence of the Fourier Series by incrementally adding terms.

Example: square wave

$$\sum_{\substack{k=-9\\k \text{ odd}}}^{9} \frac{1}{jk\pi} e^{j2\pi kt}$$

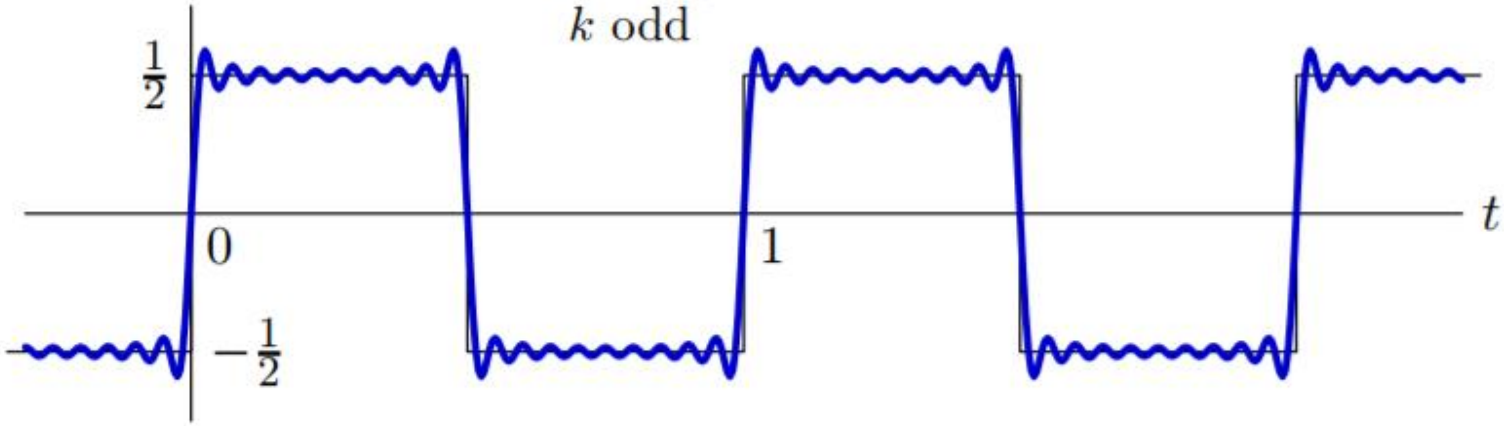


# Fourier Series

One can visualize convergence of the Fourier Series by incrementally adding terms.

Example: square wave

$$\sum_{\substack{k=-19 \\ k \text{ odd}}}^{19} \frac{1}{jk\pi} e^{j2\pi kt}$$

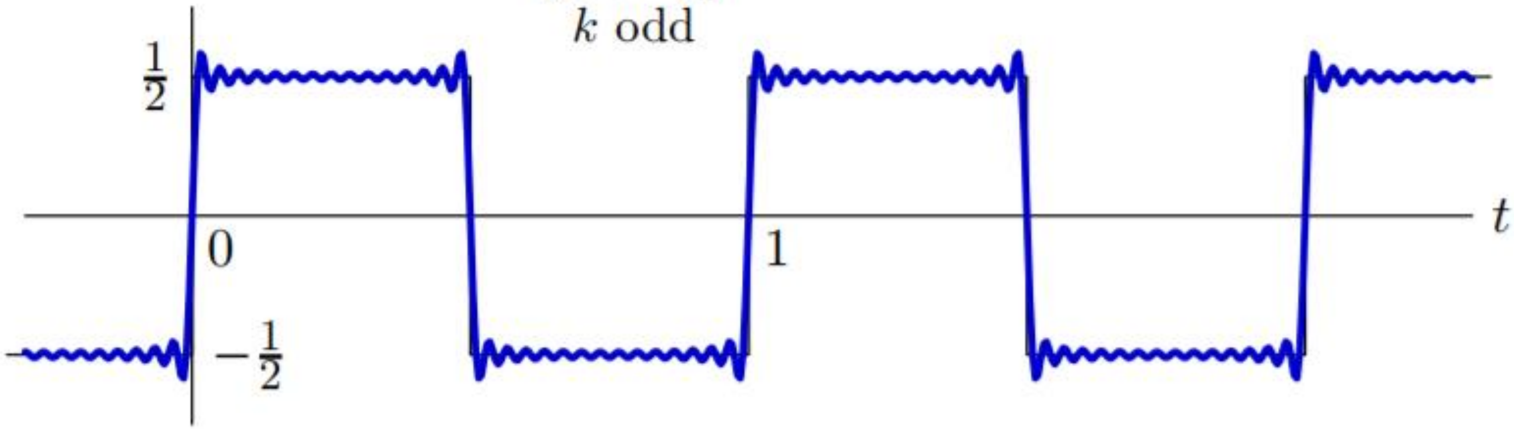


# Fourier Series

One can visualize convergence of the Fourier Series by incrementally adding terms.

Example: square wave

$$\sum_{\substack{k=-29 \\ k \text{ odd}}}^{29} \frac{1}{jk\pi} e^{j2\pi kt}$$

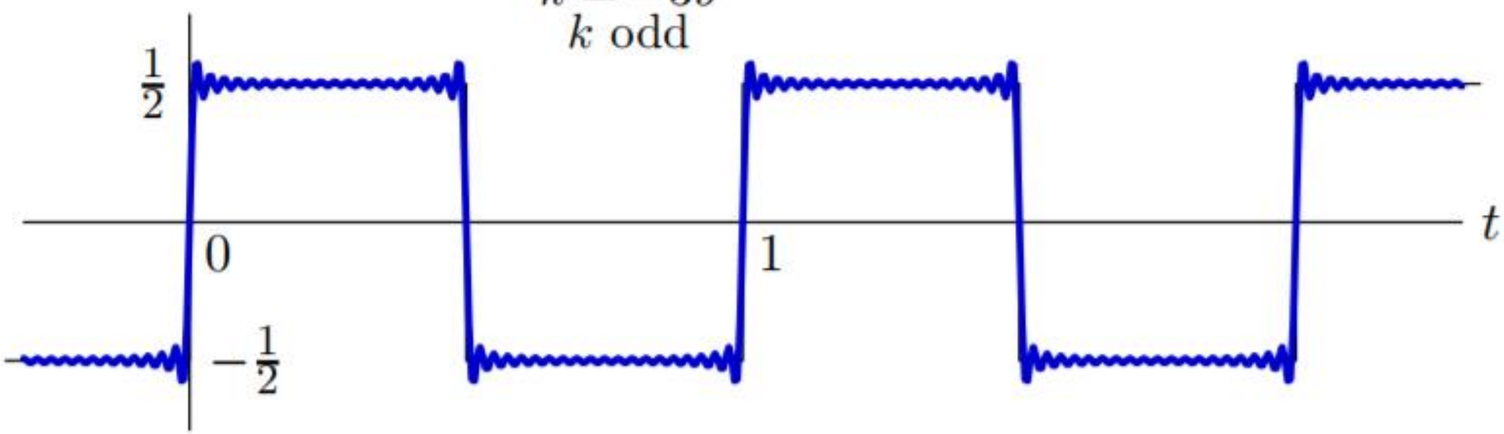


# Fourier Series

One can visualize convergence of the Fourier Series by incrementally adding terms.

Example: square wave

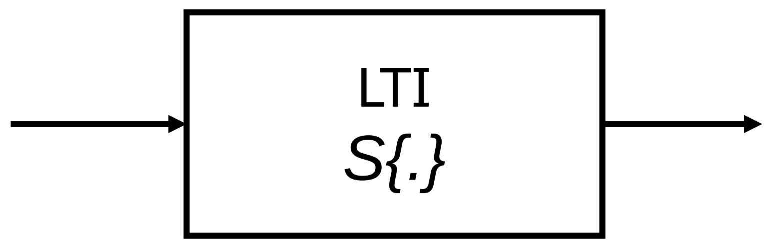
$$\sum_{\substack{k=-39 \\ k \text{ odd}}}^{39} \frac{1}{jk\pi} e^{j2\pi kt}$$



# Response of complex exponentials

Theorem: If a complex exponential function is applied to an LTI system with a real-valued impulse response function, the output response of the system is identical to the complex exponential function except for changes in amplitude and phase.

$$k_1 e^{j\omega_1 t}$$




$$A_1 k_1 e^{j(\omega_1 t + \varphi_1)}$$

$$k_1 e^{j\omega_1 t} H(\omega_1)$$

# Response of complex exponentials

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Proof by using convolution:

$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau \quad = k_1 e^{j\omega_1 t} \int_{-\infty}^{\infty} h(\tau)e^{-j\omega_1 \tau}d\tau$$

$$H(\omega_1) = \int_{-\infty}^{\infty} h(\tau)e^{-j\omega_1 \tau}d\tau$$

$$y(t) = k_1 e^{j\omega_1 t} H(\omega_1)$$

## Filtering

The output of an LTI system is a "filtered" version of the input.

Input: Fourier series $\rightarrow$ sum of complex exponentials.

$$x(t) = x(t+T) = \sum_{k=-\infty}^{\infty} a_k e^{j\frac{2\pi}{T}kt}$$

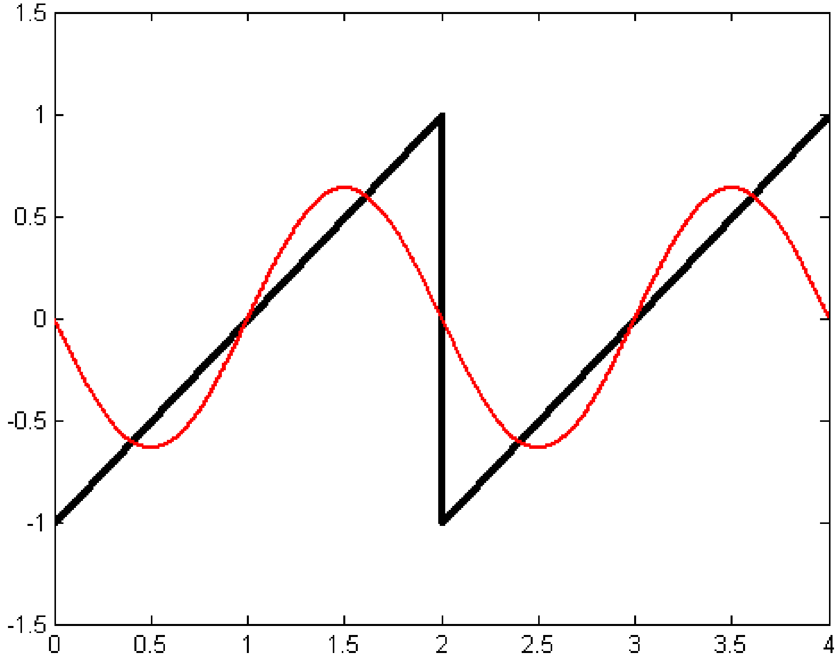
Complex exponentials: eigenfunctions of LTI systems.

$$e^{j\frac{2\pi}{T}kt} \rightarrow H(j\frac{2\pi}{T}k)e^{j\frac{2\pi}{T}kt}$$

Output: same eigenfunctions, amplitudes/phases set by system.

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\frac{2\pi}{T}kt} \rightarrow y(t) = \sum_{k=-\infty}^{\infty} a_k H(j\frac{2\pi}{T}k)e^{j\frac{2\pi}{T}kt}$$

# Trigonometric Fourier Series




Proposed by Joseph Fourier (1768-1830) in the French Academy of Sciences

- **To obtain the Trigonometric Fourier Series from the Complex exponentials Fourier series we need to use Euler formula:**

$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

$$\cos(\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

$$e^{-j\theta} = \cos(\theta) - j\sin(\theta)$$

$$\sin(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

- **Trigonometric fourier Series:**
  - General representation of a real periodic signal is:

$$x(t) = a_0 + \sum_{n=1}^{\infty}[a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)]$$

  - The coefficients of the Fourier series are:

$$a_0 = \frac{1}{T_0}\int_{T_0} x(t)dt$$

$$a_n = \frac{2}{T_0}\int_{T_0} x(t)\cos(n\omega_0 t)dt$$

$$b_n = \frac{2}{T_0}\int_{T_0} x(t)\sin(n\omega_0 t)dt$$

- **Compact Fourier series :**
  - General representation of a real periodic signal with the Compact Fourier series is:

$$x(t) = c_0 + \sum_{n=1}^{\infty} c_n \cos(n\omega_0 t + \theta_n)$$

  - In this case the coefficients of the Fourier series are:

$$c_0 = a_0$$

The DC offset

$$c_n = \sqrt{a_n^2 + b_n^2}$$

The amplitude

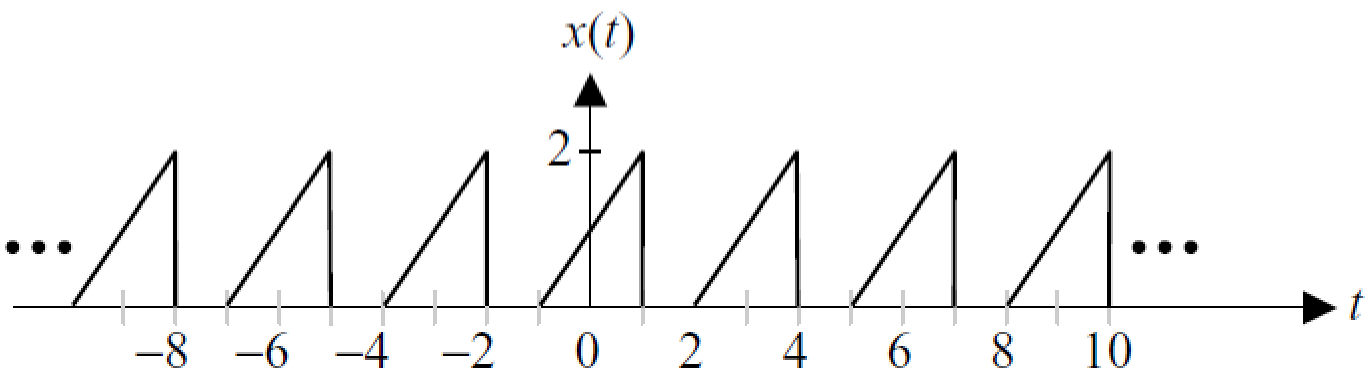
$$\theta_n = \tan^{-1}\left(\frac{-b_n}{a_n}\right)$$

The phase

# Example

Calculate the trigonometric CTFS coefficients of the periodic signal $x(t)$ defined over one period $T_0 = 3$ as follows:

$$x(t) = \begin{cases} t+1 & -1 \le t \le 1 \\ 0 & 1 < t < 2. \end{cases}$$



# Answer

Answer will be provided during the lecture

# Answer

Answer will be provided during the lecture

# Different forms of Fourier Series



- **Trigonometric Fourier Series:**

$$x(t) = a_0 + \sum_{n=1}^{\infty} [a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)]$$

- **Exponential Fourier Series:**

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

# Different forms of Fourier Series

Xi'an Jiaotong-Liverpool University 西交利物浦大学

**Trigonometric Fourier Series:**

$$x(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t))$$

$$= a_0 + \sum_{n=1}^{\infty} \frac{a_n}{2}(\mathrm{e}^{\mathrm{j}n\omega_0 t} + \mathrm{e}^{-\mathrm{j}n\omega_0 t}) + \sum_{n=1}^{\infty} \frac{b_n}{2\mathrm{j}}(\mathrm{e}^{\mathrm{j}n\omega_0 t} - \mathrm{e}^{-\mathrm{j}n\omega_0 t}).$$

**Complex Exponential coefficient:**

$$a_k = \begin{cases} a_0 & n = 0 \\ \frac{1}{2}(a_n - jb_n) & n > 0 \\ \frac{1}{2}(a_{-n} + jb_{-n}) & n < 0 \end{cases}$$

# Coefficients for symmetric signals



- Fourier series of any even periodic function x(t) consists of cosine terms only.
- Fourier series of any odd periodic function x(t) consists of sine terms only.
- In these cases, Fourier coefficients can be determined by integrating over only half of the period.

If x(t) is a real function, then the trigonometric CTFS coefficients $a_0$, $a_n$ and $b_n$ are also real-valued for all n.

# Summary

1. A signal x(t) can be represented by a Linear combination of harmonically related complex exponentials.

2. A complex exponential function is applied to an LTI system, the output response of the system is identical to the complex exponential function except for changes in amplitude and phase.

3. Fourier Series is a bridge between time domain and frequency domain.

# Problem

**3.3.** For the continuous-time periodic signal

$$x(t) = 2 + \cos\left(\frac{2\pi}{3}t\right) + 4\sin\left(\frac{5\pi}{3}t\right),$$

determine the fundamental frequency $\omega_0$ and the Fourier series coefficients $a_k$ such that

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}.$$

**3.4.** Use the Fourier series analysis equation (3.39) to calculate the coefficients $a_k$ for the continuous-time periodic signal

$$x(t) = \begin{cases} 1.5, & 0 \le t < 1 \\ -1.5, & 1 \le t < 2 \end{cases}$$

with fundamental frequency $\omega_0 = \pi$.

# Problem (cont.)



**3.13.** Consider a continuous-time LTI system whose frequency response is

$$H(j\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}\,dt = \frac{\sin(4\omega)}{\omega}.$$

If the input to this system is a periodic signal

$$x(t) = \begin{cases} 1, & 0 \le t < 4 \\ -1, & 4 \le t < 8 \end{cases}$$

with period $T = 8$, determine the corresponding system output $y(t)$.

**3.25.** Consider the following three continuous-time signals with a fundamental period of $T = 1/2$:

$$x(t) = \cos(4\pi t),$$
$$y(t) = \sin(4\pi t),$$
$$z(t) = x(t)y(t).$$

**(a)** Determine the Fourier series coefficients of $x(t)$.
**(b)** Determine the Fourier series coefficients of $y(t)$.
**(c)** Use the results of parts (a) and (b), along with the multiplication property of the continuous-time Fourier series, to determine the Fourier series coefficients of $z(t) = x(t)y(t)$.
**(d)** Determine the Fourier series coefficients of $z(t)$ through direct expansion of $z(t)$ in trigonometric form, and compare your result with that of part (c).