EEE336 Signal Processing and Digital Filtering

Lecture 6 Discrete-Time Systems in Time Domain

6_1 Introduction

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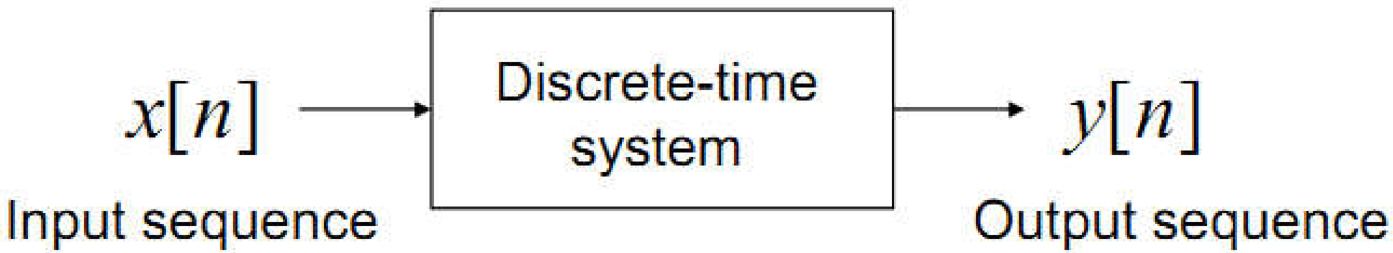
Discrete-Time System

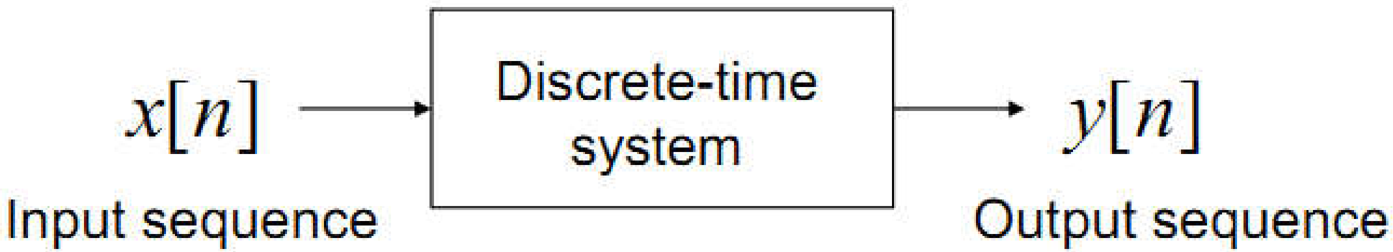
- Discrete-time system: A device or an algorithm that performs some prescribed operation on a discrete-time signal (input or excitation) to produce another discrete-time signal (output or response)
- In most applications, the discrete-time system is a single-input, single-output system:



$$y[n]=\mathcal{H}\{x[n]\}$$

DT System in Time and Frequency domain

- Discrete systems can be characterized in several ways:
  - Constant Coefficient Linear Difference Equations (time)
  - Impulse response (time)
  - Frequency response (frequency)
  - Transfer function (frequency)



$$y[n]=\mathcal{H}\{x[n]\}$$

Input-output description

- The input-output description of a discrete-time system consists of a mathematical expression or a rule, which explicitly defines the relation between the input and output signals:

$$x[n] \xrightarrow{\mathcal{H}} y[n] \quad \Longleftrightarrow \quad y[n] = \mathcal{H}\{x[n]\}$$

- Example: Determine the response of the following systems to the input signal x[n]

$$x[n] = \begin{cases} |n|, & -2 \leq n \leq 2 \\ 0, & otherwise \end{cases}$$

$$y_1[n] = x[n-1]$$

$$y_2[n] = \frac{1}{2}\{x[n-1] + x[n]\}$$

Constant Coefficient Linear Difference Equations

- All discrete systems can be represented using CCLDE (Constant Coefficient Linear Difference Equations), of the form:

$$\boxed{y[n] + a_1 y[n-1] + \cdots + a_N y[n-N] = b_0 x[n] + b_1 x[n-1] + \cdots + b_M x[n-M]}$$

$$\Downarrow$$

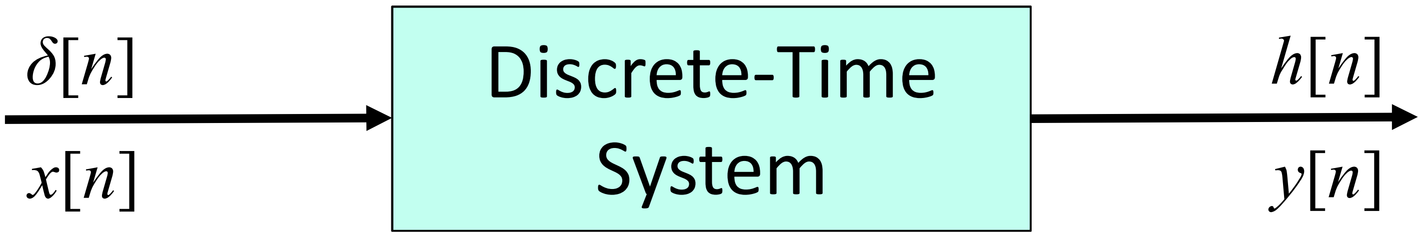
$$\boxed{y[n] + \sum_{i=1}^{N} a_i y[n-i] = \sum_{j=0}^{M} b_j x[n-j]}$$

Constant coefficients

- Constant coefficients a_i and b_j are called **filter coefficients**
- Integers M and N represent the maximum **delay** in the input and output, respectively. The larger of the two numbers is known as the **order of the filter**.
- **Any** LTI system can be represented as two finite sum of products!

Impulse and Step Responses

- Impulse response: the response of a discrete-time system to a unit impulse sequence $\delta[n]$ is called the *unit sample response*, or simply, the *impulse response*
  - Denoted as $h[n]$



$$\mathrm{x[n]} = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

$$\mathrm{y[n]} = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

$$y[n] = x[n] * h[n]$$

Impulse and Step Responses

- Eg1: Consider the CCLDE of a discrete-time system

$$y[n] = a_1 x[n] + a_2 x[n-1] + a_3 x[n-2] + a_4 x[n-3]$$

  – Its impulse response $\{h[n]\}$ is obtained by setting $x[n] = \delta[n]$:

$$h[n] = a_1 \delta[n] + a_2 \delta[n-1] + a_3 \delta[n-2] + a_4 \delta[n-3]$$

  – The impulse response is thus a finite-length sequence of length 4 given by:

$$\{h[n]\} = \{a_1, a_2, a_3, a_4\}, 0 \le n \le 3$$

Impulse and Step Responses

- Step response: the response of a discrete-time system to a unit step sequence $u[n]$ is its *unit step response*, or simply, the *step response*
  - Denoted as $s[n]$

- An LTI system is completely characterized in the time domain by its impulse response (or step response)

6_1 Wrap up

- What is a DT system?
- Approaches to characterize systems in:
  - Time domain
  - Frequency domain
- Input-output relationship = CCLDE
- Impulse / Step Responses

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Lecture 6 Discrete-Time Systems in Time Domain

6_2 Classifications

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Classification of Discrete-Time Systems

- Linearity
- Time-invariance
- Causality
- Stability
- Recursiveness

Linear Time-invariant System
(LTI System)

Consider the examples:

- *the Accumulator*
- *the Moving Average System*
- *the Median Filter*
- *the linear interpolation*
- *the down sampler*

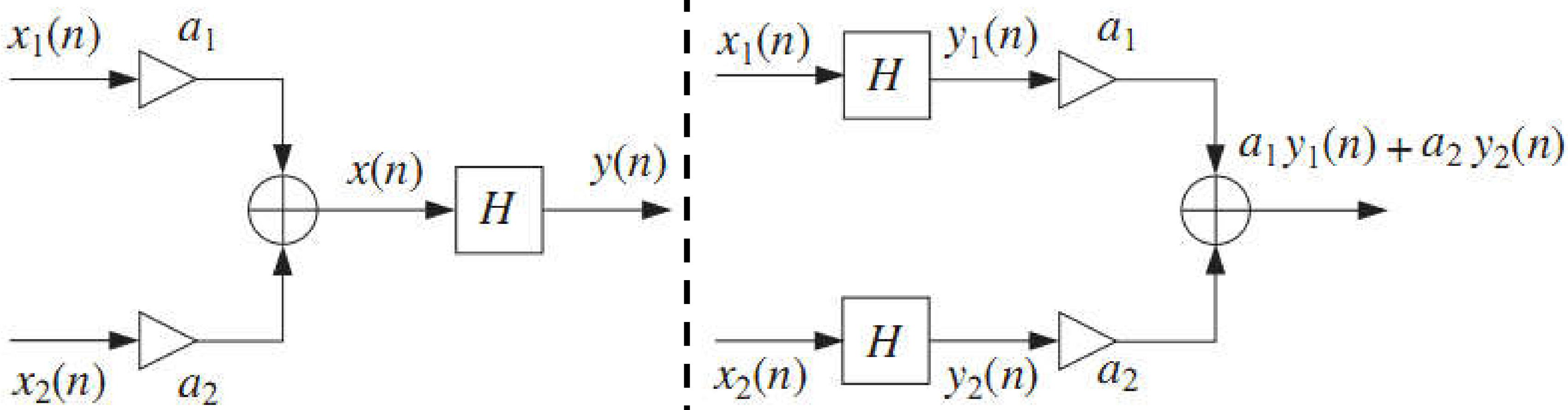
Are they linear, time-invariant, causal, stable and recursive?

Linearity

- Linear vs. non-linear systems
  - A **linear** system is one that satisfies the superposition and homogeneity principle

$$x[n]=\alpha x_1[n]+\beta x_2[n] \Leftrightarrow y[n]=\alpha y_1[n]+\beta y_2[n]$$

$$\mathcal{H}\{ax_1[n]+bx_2[n]\} = a\mathcal{H}\{x_1[n]\}+b\mathcal{H}\{x_2[n]\}$$

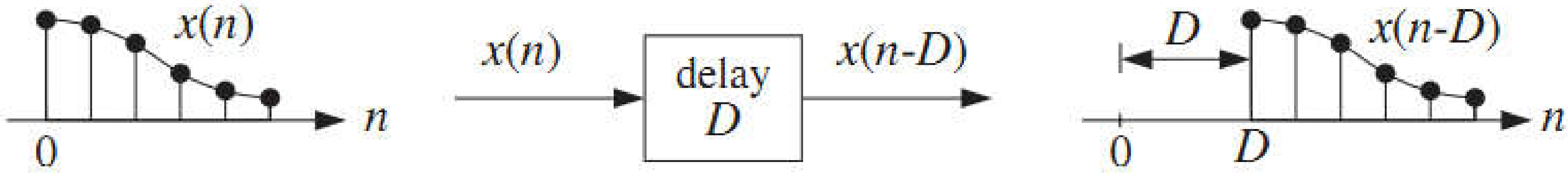


Linearity (cont.)

- Examples:
  - (1) $y[n]=2x[n]+3$
  - (2) $y[n]=x^2[n]$
  - (3) $y[n]=x[n^2]$

Time-invariance

- Time-invariant vs. time-variant systems
  - A system is called time-invariant if its input-output characteristics do not change with time



  - A system $\mathcal{H}$ is time-invariant or shift-invariant if and only if
  - Time-invariance property ensures that for a specified input, the output is independent of the time the input applied

$$x[n] \xrightarrow{\mathcal{H}} y[n]$$

$$x[n-n_0] \xrightarrow{\mathcal{H}} y[n-n_0]$$

Time-invariance (cont.)

- Examples:
  - (1) $y[n]=\sin(x[n])$
  - (2) $y[n]=nx[n]$
  - (3) $y[n]=x^2[n]$

Causality

- Causal vs. non-causal systems
  - A system is said to be **causal** if the output of the system at any time n_0 depends only on present and past inputs (x[n] for $n \le n_0$), but does not depend on future inputs ($n \ge n_0$)

$$y[n] = \mathcal{H}\{x[n], x[n-1], x[n-2], \ldots\}$$

$$y(n) = \sum_{m=-\infty}^{\infty} h(m)x(n-m) = \boxed{\cdots + h_{-2}x_{n+2} + h_{-1}x_{n+1}} + h_0x_n + h_1x_{n-1} + h_2x_{n-2} + \cdots$$

To be ZERO!

$$h[k] = 0, n < 0$$

Causality (cont.)

- Examples:
  - (1) $y[n] = x[n] - x[n-1]$
  - (2) $y[n] = (x[n-1]+x[n]+x[n+1])/3$
  - (3) $y[n] = x[-n]$

Stability

- Stable vs. unstable systems
  – A relaxed system is said to be **bounded input-bounded output (BIBO) stable** if and only if every bounded input produces a bounded output

$$|x(n)| \le M_x < \infty \Rightarrow |y(n)| \le M_y < \infty \quad \text{for all } n$$

- Examples:
  – (1) $y[n] = x[n] - x[n-1]$
  – (2) $y[n] = \tan(x[n])$

Linear Time-Invariant (LTI) System

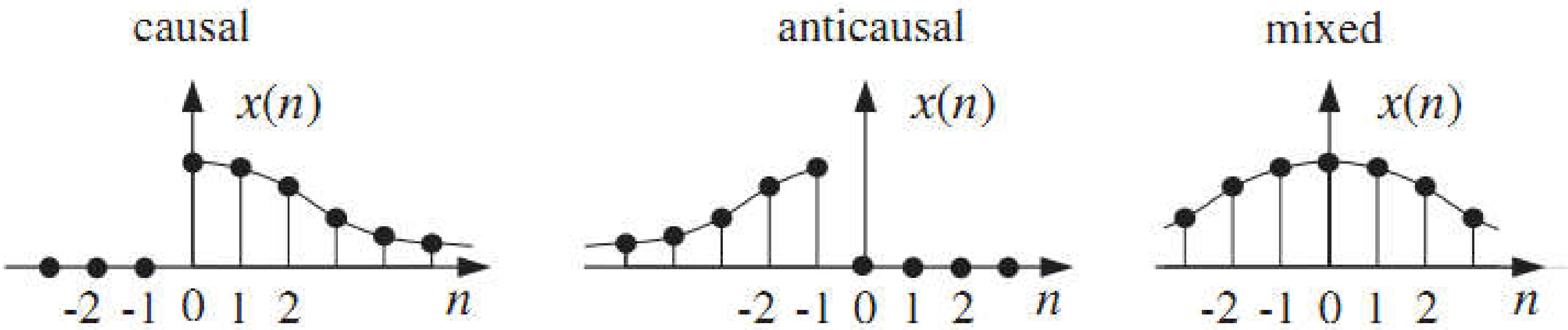
- Linear Time-Invariant (LTI) System: A system that satisfies both the linearity and the time (shift) invariance properties
  - LTI systems are mathematically easy to analyse and characterise, and consequently, easy to design
  - For a given input, a certain output will be obtained
  - The systems do not change with time, which is a reasonably good approximation of most systems
  - Many useful signal processing algorithms have been developed utilizing LTI systems

Properties of LTI System

- An LTI system is BIBO stable, if its impulse response is absolutely summable:

$$\sum_{k=-\infty}^{\infty} |h[k]| < \infty$$

- An LTI system is causal, if its impulse response is a causal sequence: $h[k] = 0, n < 0$



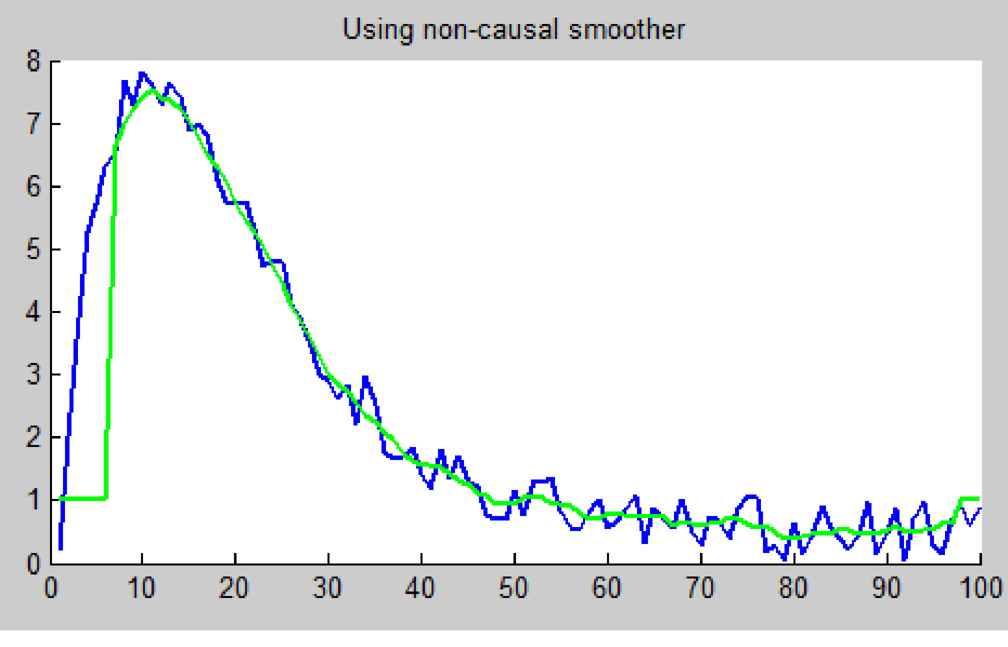
- Example: Consider the typical 5-tap smoothing filter with filter coefficients h[n]=1/5 for −2≤n≤2. The convolution equation becomes

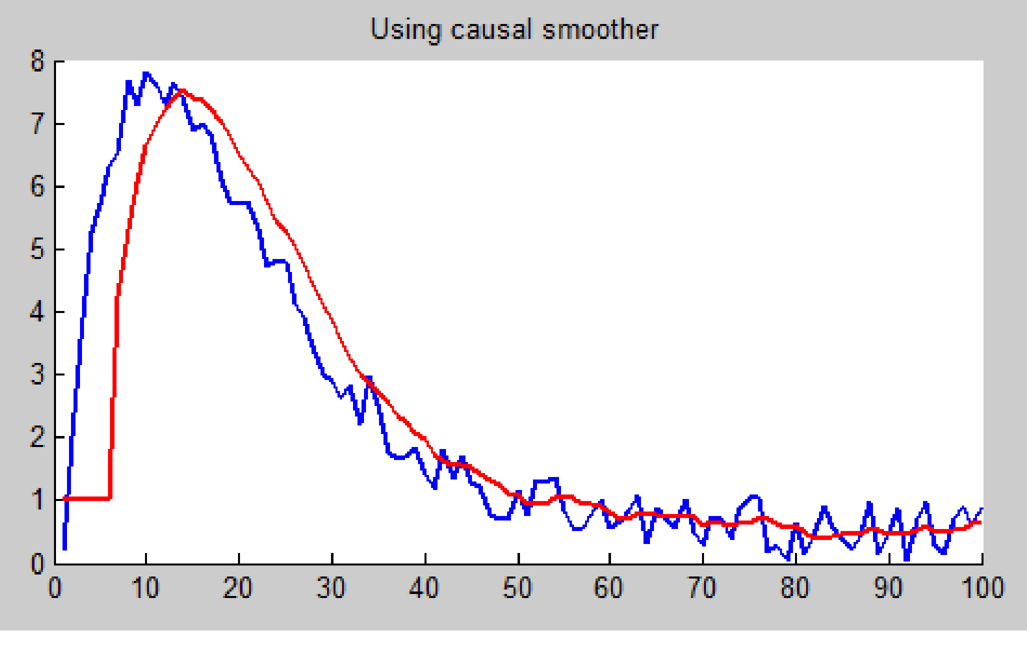
$$y(n) = \sum_{m=-2}^{2} h(m)x(n-m) = \frac{1}{5}\sum_{m=-2}^{2} x(n-m)$$

$$= \frac{1}{5}[x(n+2)+x(n+1)+x(n)+x(n-1)+x(n-2)]$$

$$\Downarrow$$ *Non-causal, delay of 2 units:*

$$y_2(n) = y(n-2) = \frac{1}{5}[x(n)+x(n-1)+x(n-2)+x(n-3)+x(n-4)]$$





Finite-dimensional LTI system

- An important subclass of LTI discrete-time system is the ***finite-dimensional LTI system***, characterized by a constant coefficient difference linear equation (CCLDE) of the form:

$$\boxed{\sum_{k=0}^{N} d_k y[n-k] = \sum_{k=0}^{M} p_k x[n-k]}$$

  - d_k and p_k are constants
  - The order of the system is given by $max(N, M)$

Finite-dimensional LTI system

- Assuming the system to be causal and $d_0 \neq 0$, then:

$$y[n] = -\sum_{k=1}^{N} \frac{d_k}{d_0} y[n-k] + \sum_{k=0}^{M} \frac{p_k}{d_0} x[n-k]$$

  - The output $y[n]$ can be computed for all $n \geq n_0$, knowing $x[n]$ and the initial conditions $y[n_0-1]$, $y[n_0-2]$, … , $y[n_0-N]$.
  - Eg.1: $y[n] = \frac{1}{M} \sum_{k=0}^{M-1} x[n-k]$
  - Eg.2: $y[n] - y[n-1] = x[n], with\ y[n] = 0\ for\ n < 0$

Recursiveness

- Classified according to the method of calculation employed to determine the output samples:
  - If the output can be calculated by knowing only the present and past input samples, the system is said to be a ***non-recursive*** system;
    - Eg: FIR system

$$y[n] = \sum_{k=N_1}^{N_2} h[k]x[n-k]$$

  - If the computation of output involves past output samples in addition to the input samples, it is known as a ***recursive*** system.
    - Eg: IIR system

$$y[n] = -\sum_{k=1}^{N} \frac{d_k}{d_0} y[n-k] + \sum_{k=0}^{M} \frac{p_k}{d_0} x[n-k]$$

  - However, it is also possible to implement an FIR system using a recursive computational scheme.
    - Eg: FIR system

$$y[n] = \frac{1}{M} \sum_{l=0}^{M-1} x[n-l] \Rightarrow y[n] = y[n-1] + \frac{1}{M}(x[n] - x[n-M])$$

6_2 Wrap up

- Introduced the properties of DT systems;
- You should be able to classify the systems according to these properties;
- LTI system is very important

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6_3 Examples

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Accumulator

- Accumulator: $$y(n)=\sum_{k=-\infty}^{n} x(k)=\sum_{k=-\infty}^{n-1} x(k)+x(n)$$ $$=y(n-1)+x(n)$$
  - The output $y[n]$ is the sum of the input sample $x[n]$ and the previous output $y[n-1]$
- Input-output relation can also be written in the form
$$y(n)=\sum_{k=-\infty}^{n_0} x(k)+\sum_{k=n_0+1}^{n} x(k)=y(n_0)+\sum_{k=n_0+1}^{n} x(k),\quad n\ge n_0$$
  - The initial condition $y[n_0]$ “summarizes” the effect on the system from all the inputs which had been applied to the system before n_0
  - When n_0=-1, y[-1] is called the ***initial condition*** for the casual system.
  - If $y[n_0]=0$, this system is said to be initially relaxed

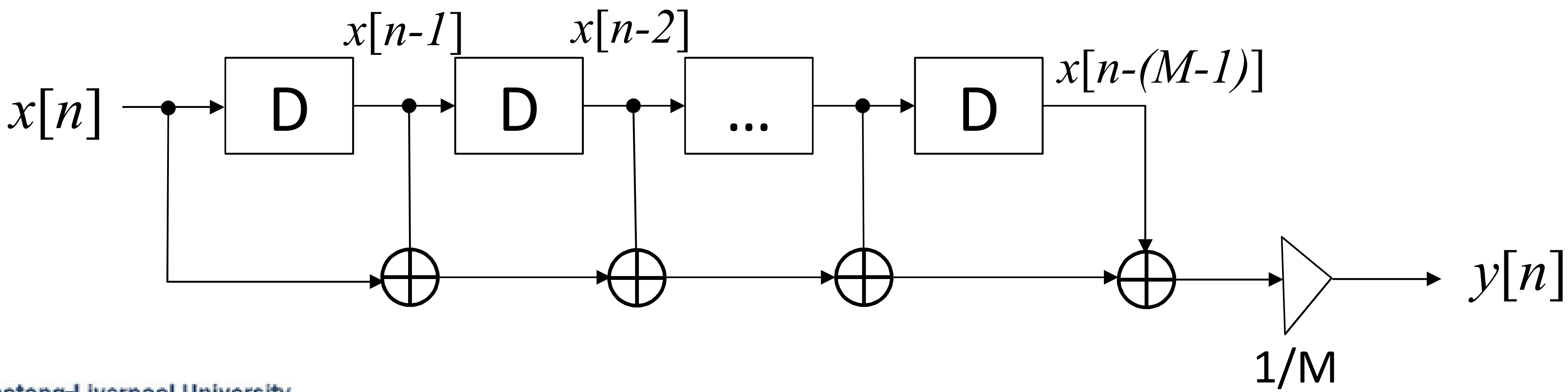
Moving Average Filter

- M-point moving-average system:

$$y[n] = \frac{1}{M} \sum_{k=0}^{M-1} x(n-k)$$

  – Used in smoothing random variations in data

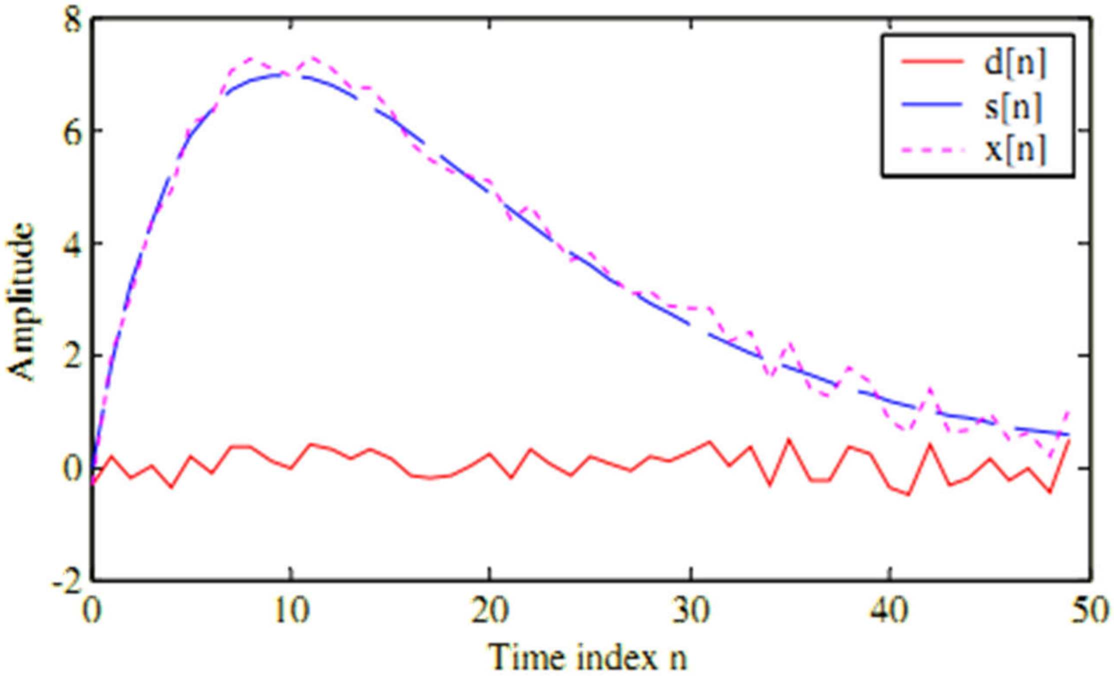
- A direct implementation of the M-point moving average system requires M−1 delay units and adders, and 1 multiplier.

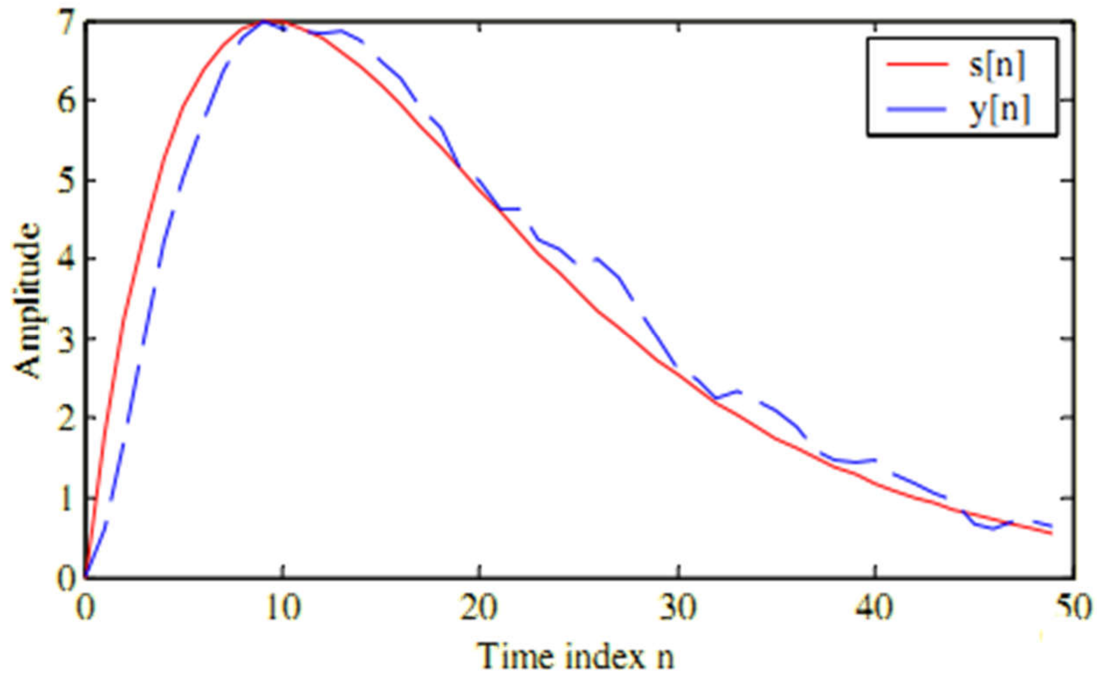


Moving Average Filter (cont.)

- An application: Consider x(n) = s(n) + d(n)
  - Where s(n) is the signal corrupted by a random noise d(n)

$$s(n) = 2[n(0.9)^n]$$





Median Filter

- The median of a set of (2K+1) numbers is the number such that K numbers from the set have values greater than this number and the other K numbers have values smaller
- Median can be determined by rank-ordering the numbers in the set by their values and choosing the number at the middle
- Example: Consider the set of numbers

$$\{2, \quad -3, \quad 10, \quad 5, \quad -1\}$$

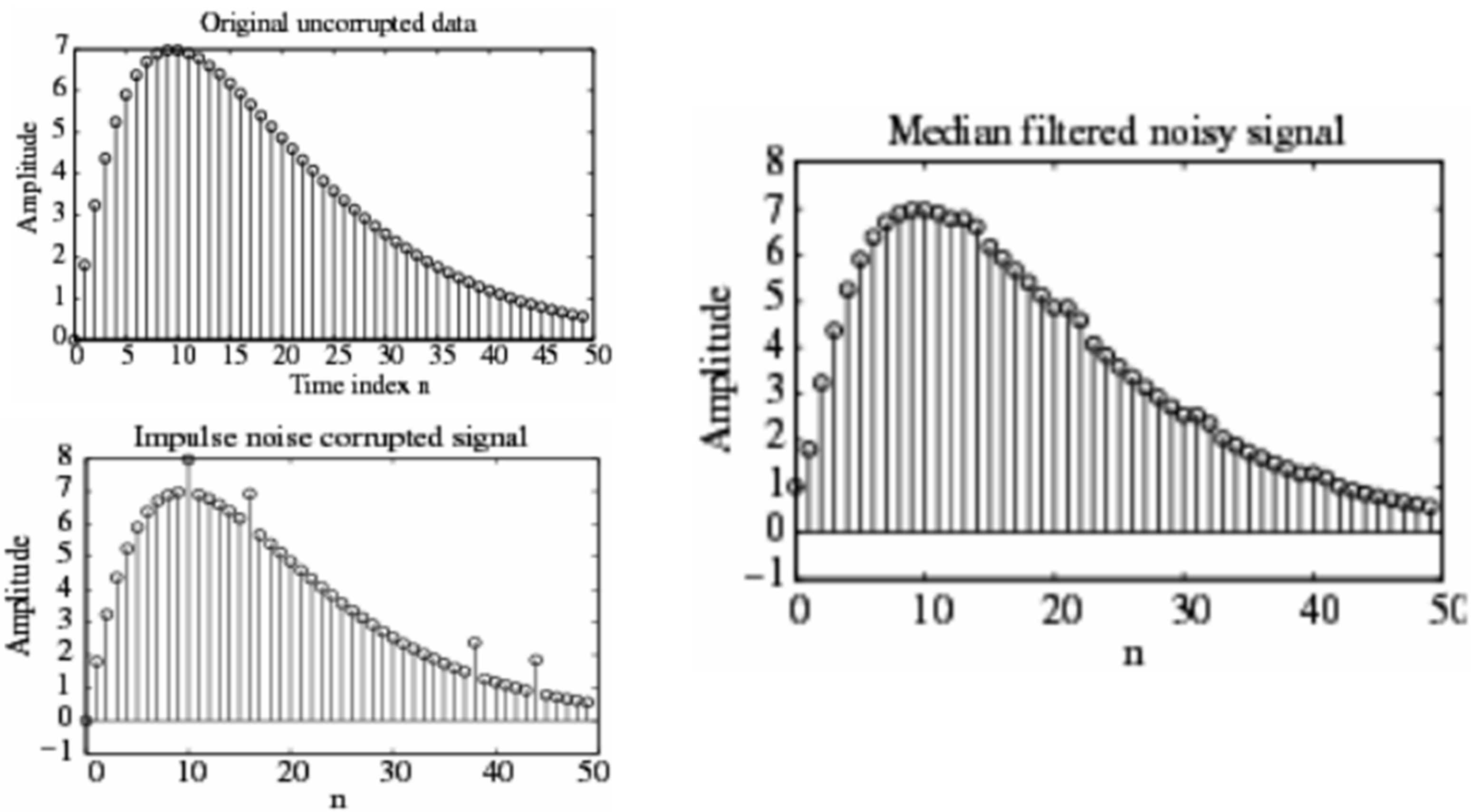
- Rank-order set is given by

$$\{-3, \quad -1, \quad 2, \quad 5, \quad 10\}$$

$$\Longrightarrow \quad \operatorname{med}\{2, \quad -3, \quad 10, \quad 5, \quad -1\} = 2$$

Median Filter (cont.)

- Finds applications in removing additive random noise, which shows up as sudden large errors in the corrupted signal
- Usually used for the smoothing of signals corrupted by impulse noise



Median Filter (cont.)

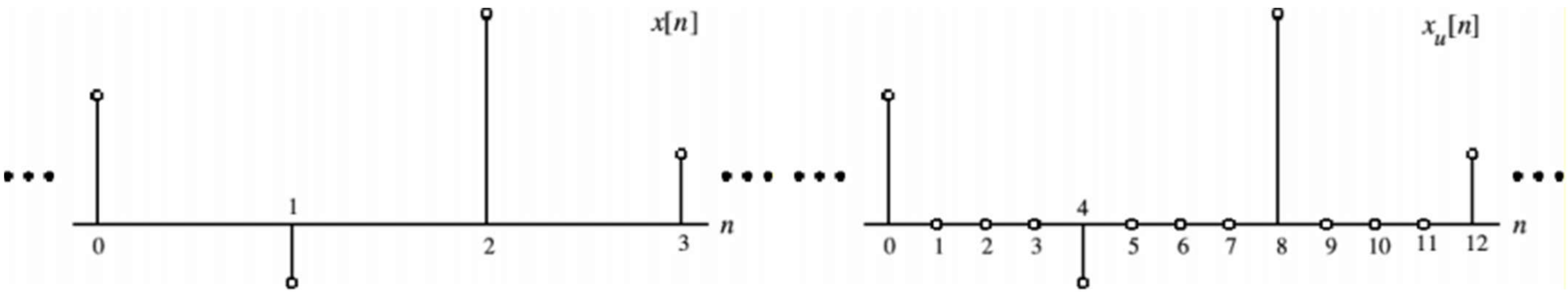
Original Image

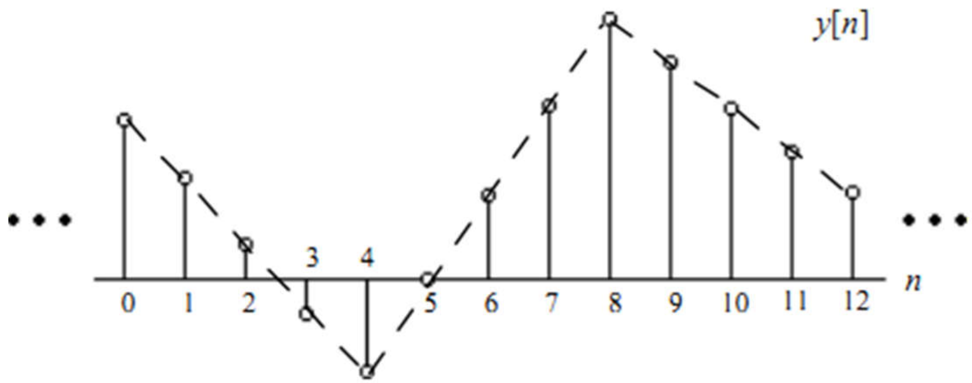
Noisy Image
(pepper-and-salt noise)

Filtered Image

Up-sampler (Linear Interpolator)

- Linear interpolation – Employed to estimate sample values between pairs of adjacent samples of a discrete-time sequence
- Example: Factor-of-4 interpolation





Examples – Linear Interpolation (cont.)

- Factor-of-2 interpolator –

$$y[n] = x_u[n] + \frac{1}{2}\left(x_u[n-1] + x_u[n+1]\right)$$

Original (512×512)

Down–sampled (256×256)

Interpolated (512 × 512)

6_3 Wrap up

- Introduced the examples of DT systems
- Analyze their properties and determine whether they are linear, time-invariant, causal and stable.

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Lecture 6 Discrete-Time Systems in Time Domain

6_4 Convolution

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Input-output relationship: Convolution

- An LTI system is completely characterised by its impulse response
  - If you know the impulse response of a discrete LTI system, then you know the response of the system to any arbitrary input by performing the *convolution*
- The summation

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{\infty} x[n-k]h[k]$$

is called the convolution of the sequences x[n] and h[n], and represented as $y[n] = x[n] * h[n]$

Properties of convolution

- 1. the commutative property:

$$\boldsymbol{x[n] * h[n] = h[n] * x[n]}$$

- 2. the associative property:

$$\boldsymbol{x[n] * (h_1[n] * h_2[n]) = (x[n] * h_1[n]) * h_2[n]}$$

- 3. the distributive property:

$$\boldsymbol{x[n] * (h_1[n] + h_2[n]) = (x[n] * h_1[n]) + (x[n] * h_2[n])}$$

Computing convolution

- The output of an LTI system at $n = n_0$ is given by

$$y[n_0] = \sum_{k=-\infty}^{\infty} x[k]h[n_0 - k]$$

- To compute $y[n_0]$
  - Folding. Time reverse $h[k]$ about $k = 0$ to obtain $h[-k]$
  - Shifting. Shift $h[-k]$ by n_0 to the right (left) if is positive (negative), to obtain $h[n_0-k]$
  - Multiplication. Multiply $x[k]$ by $h[n_0-k]$ for every k to obtain the product sequence

$$v_{n_0}[k] = x[k]h[n_0 - k]$$

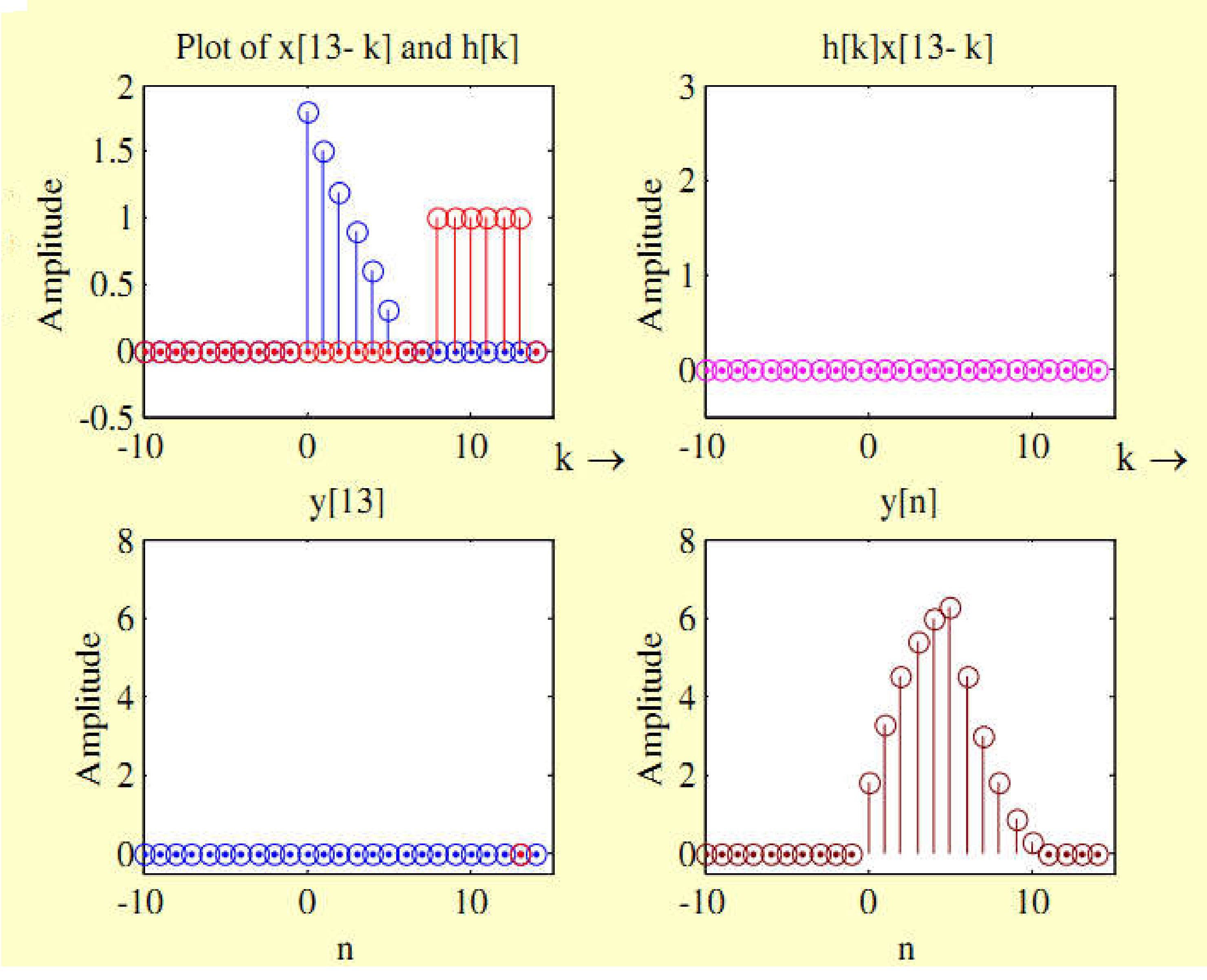
  - Summation. Sum all the values of $v_{n_0}[k]$ to obtain $y[n_0]$

Computing convolution (cont.)

- Calculate the convolution of x[n] and h[n]:

$$x[n] = \begin{cases} 1, & 0 \le n \le 5 \\ 0, & \text{otherwise} \end{cases}$$

$$h[n] = \begin{cases} 1.8 - 0.3n, & 0 \le n \le 5 \\ 0, & \text{otherwise} \end{cases}$$



Computing convolution (cont.)

- If both the input sequence and the impulse response sequence are of infinite length, convolution cannot be used to compute the output
- If one of them is of finite length, the convolution can be used to compute the output, since it involves a finite sum of products. (but the output is infinite length)
- If both of them are of finite length, the output sequence can be calculated by convolution, and it is also of finite length
  - If the lengths of the two sequences being convolved are M and N, then the sequence generated by the convolution is of length M+N-1

Computation: Vector method

$$\{x[n]\} = \{1,2,\mathbf{0},3,2\}$$
$$\{h[n]\} = \{1,\mathbf{4},2,3\}$$

Computation: Diagonal

$$\{x[n]\} = \{1,2,\mathbf{0},3,2\}$$
$$\{h[n]\} = \{1,\mathbf{4},2,3\}$$

Computation: Multiplication

$$\{x[n]\} = \{1,2,\mathbf{0},3,2\}$$
$$\{h[n]\} = \{1,\mathbf{4},2,3\}$$

6_4 Wrap up

- Convolution: with the knowledge of input x[n] and system impulse response h[n], the output y[n] is obtained by performing the convolution: x[n]*h[n]

- Calculate convolution: graphical method, vector method, diagonal method and the long multiplication method

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6_5 Interconnection of Systems

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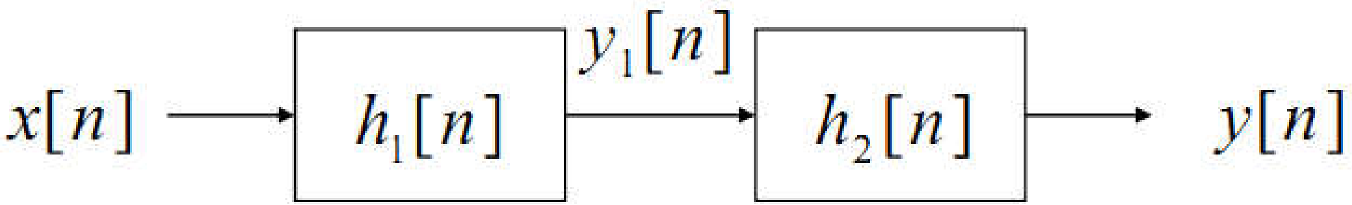
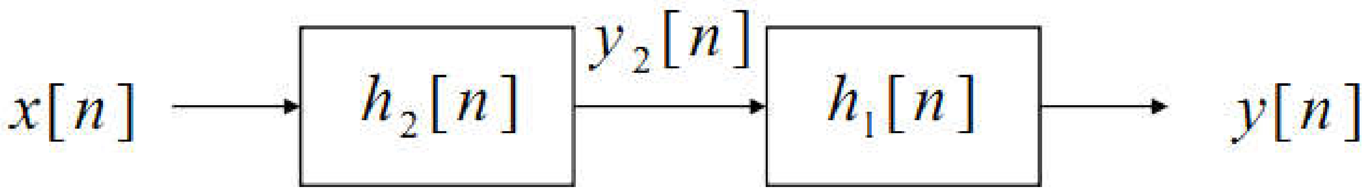
Interconnection of LTI Systems (cont.)

- Series (cascade) interconnection – Commutative Law

$$y[n] = \left(x[n] * h_1[n]\right) * h_2[n] = y_1[n] * h_2[n]$$



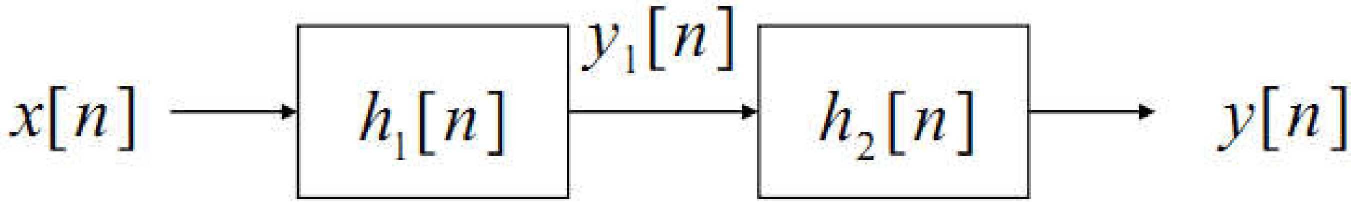
$$y[n] = \left(x[n] * h_2[n]\right) * h_1[n] = y_2[n] * h_1[n]$$



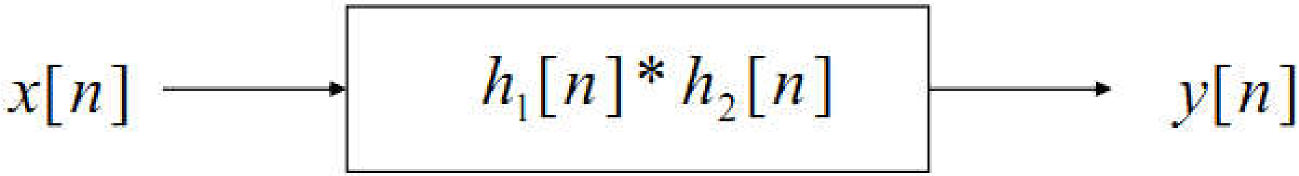
Interconnection of LTI Systems

- Series (cascade) interconnection – Associative Law

$$y[n] = \left(x[n] * h_1[n]\right) * h_2[n] = y_1[n] * h_2[n]$$



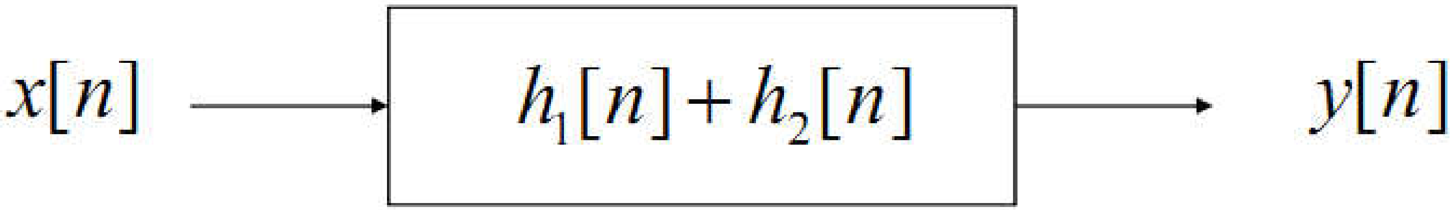
$$y[n] = x[n] * \left(h_1[n] * h_2[n]\right)$$



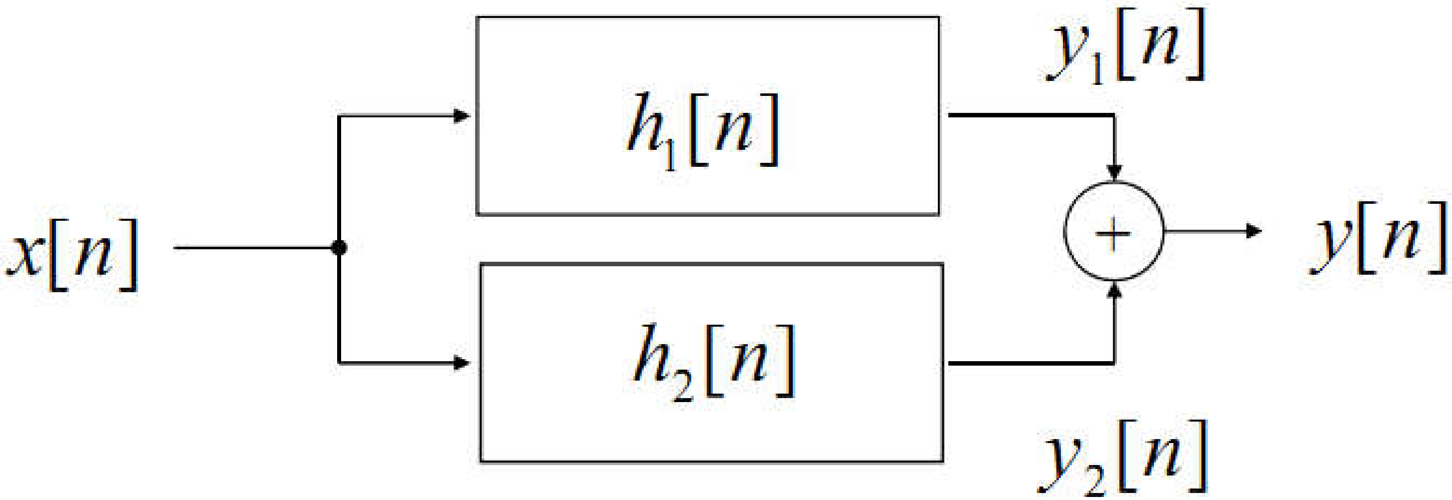
Interconnection of LTI Systems (cont.)

- Parallel interconnection – Distributive Law

$$x[n] * \left(h_1[n] + h_2[n]\right)$$

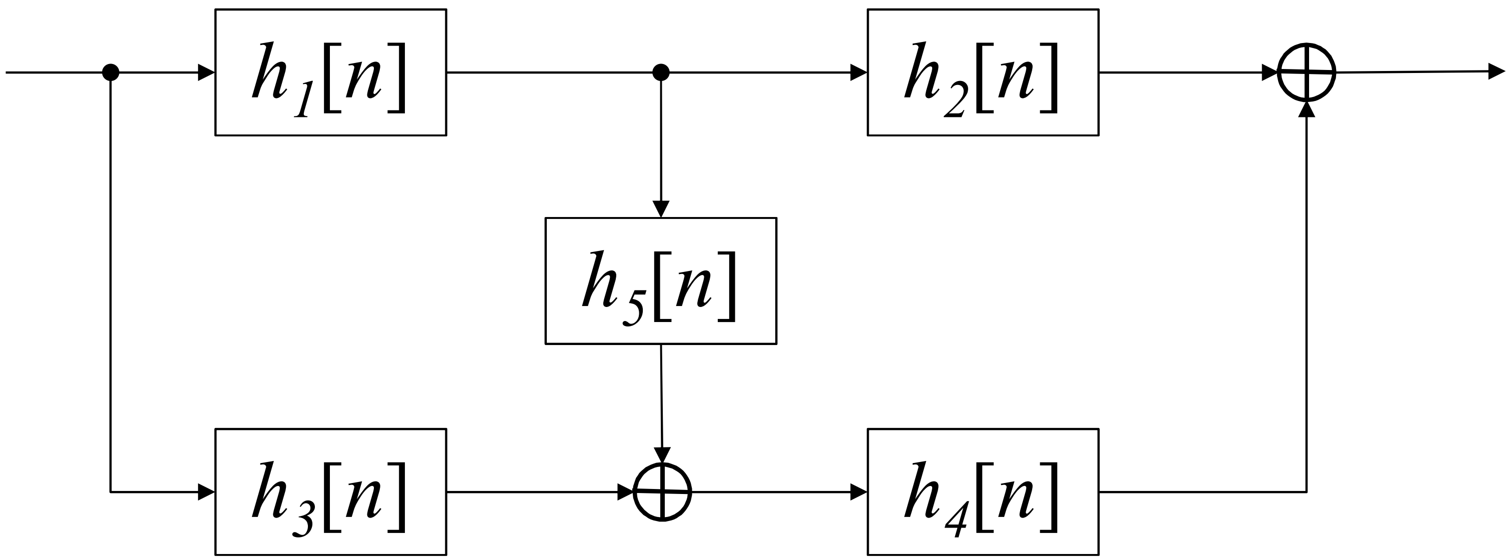


$$x[n] * h_1[n] + x[n] * h_2[n]$$



Example

- Determine the expression for the impulse response of the LTI system shown below:



6_5 Wrap up

- Be able to write the system total impulse response based on given interconnected system block diagram;
- Reversely, be able to draw the block diagram with given combined impulse response.

Chapter 6 Summary

Systems	Properties (Classification)	System IN-OUT
• Accumulator	• Linearity	Convolution
• MAF	• Time-invariance	
• Median Filter	• Causality	**Interconnection**
• Interpolator	• BIBO Stability	• Series (cascade)
• Decimator	• Recursiveness	• Parallel
		• (Feedback loop)