## 4.4 Heat equation

In this section, we will solve the one dimensional heat equation by the method of separation of variables. Heat equation governs the temperature $u$ in a body in space.

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}, \tag{4.13}$$

where $u(x,t)$ satisfies the boundary condtions

$$u(0,t) = 0, \quad u(L,t) = 0, \quad t \geq 0 \tag{4.14}$$

and the initial condition

$$u(x,0) = f(x). \tag{4.15}$$

**Step 1:** We assume the function $u(x,t)$ can be separated into the product of two functions

$$u(x,t) = X(x)T(t).$$

Differentiating $u(x,t)$ we obtain

$$\frac{\partial u}{\partial t} = XT' \quad \text{and} \quad \frac{\partial^2 u}{\partial x^2} = X''T.$$

By inserting this into the heat equation, we have

$$XT' = c^2 X''T.$$

Dividing by $c^2XT$ and simplifying gives

$$\frac{T'}{c^2T} = \frac{X''}{X}.$$

The variables are now separated, the left side depending only on $t$ and the right side only on $x$. Hence both sides must be constant because, if they were variable, then changing $t$ or $x$ would affect only one side, leaving the other unaltered. Thus, say,

$$\frac{T'}{c^2T} = \frac{X''}{X} = k.$$

Multiplying by the denominators gives immediately two ordinary differential equations

$$X'' - kX = 0$$

and

$$T' - c^2kT = 0.$$

Here, the separation constant $k$ is arbitrary.

Similarly with the wave equation, when the constant $k$ is zero or positive, only zero solutions exist. Therefore we assume $k < 0$ and write $k = -p^2$. Then the two ODEs becomes

$$X'' + p^2 X = 0$$

and

$$T' + c^2 p^2 T = 0.$$

**Step 2: Satisfying the boundary conditions**

The boundary conditions are

$$u(0,t) = X(0)T(t) = 0, u(L,t) = X(L)T(t) = 0, \quad \text{for all } t. \tag{4.16}$$

If $T \equiv 0$, then $u = XT \equiv 0$, which is of no interest. Hence $T \not\equiv 0$ and then,

$$X(0) = 0, \quad X(L) = 0.$$

The general solution of the first ODE $X'' + p^2 X = 0$ is

$$X(x) = A\cos px + B\sin px.$$

From this and the boundary condition $X(0) = 0, X(L) = 0$ we have

$$X(0) = A = 0, \text{ and } X(L) = B\sin pL = 0.$$

Here we must take $B \neq 0$ since otherwise $X \equiv 0$. Hence $\sin pL = 0$. Thus

$$pL = n\pi \Longrightarrow p = \frac{n\pi}{L}, \quad n \text{ is integrer.}$$

Setting $B = 1$, we thus obtain infinitely many solutions $X(x) = X_n(x)$, where

$$X_n(x) = B\sin px = \sin\frac{n\pi}{L}x,\ n = 1, 2, \cdots .$$

These solutions satisfy the boundary conditions $X(0) = 0, X(L) = 0$.

All this literally the same as the wave equation. From now on it differs since the second ODE $T' - c^2kT = 0$ differs from the one in wave equation. For $p = \frac{n\pi}{L}$, as just obtained, $T' - c^2kT = 0$ becomes

$$T' + \lambda_n^2 T = 0, \quad \text{where } \lambda_n = \frac{cn\pi}{L}.$$

It has general solution

$$T_n(t) = B_n \mathrm{e}^{-\lambda_n^2 t},$$

where $B_n$ is a constant.
Hence the functions

$$u_n(x,t) = X_n(t)T_n(t) = B_n \sin\frac{n\pi x}{L}\mathrm{e}^{-\lambda_n^2 t}, \quad n = 1, 2, \cdots \qquad (4.17)$$

are solutions of the heat equation, satisfying the boundary conditions.

**Step 3: Solution satisfying the initial conditions**

The solutions $u_n(x,t)$ we have obtained (4.17) satisfy the heat equation and the given boundary conditions. We now seeking solutions that satisfy the initial conditions. A single $u_n(x,t)$ will generally not satisfy the initial conditions (4.15). By the superposition theorem, the following function still satisfies the wave equation and the boundary conditions

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} \mathrm{e}^{-\lambda_n^2 t}. \tag{4.18}$$

The initial condition is $u(x,0) = f(x)$, therefore, we have

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{L} x = f(x).$$

Hence we must choose the $B_n$ such that $u(x,0)$ becomes the Fourier sine series of $f(x)$. Thus $B_n$ can be written into

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx, \quad n = 1, 2, \cdots .$$

Now we have obtained the solution of the heat equation satisfying the given boundary conditions and initial condition

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}\mathrm{e}^{-\lambda_n^2 t},$$

where

$$\lambda_n = \frac{cn\pi}{L},$$

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx, \quad n = 1, 2, \cdots$$

## 4.6 Laplace equation

The Laplace equation is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Using separation of variables, let $u(x, y) = X(x)Y(y)$ and substituting into the Laplace equation we get

$$X''Y + XY'' = 0$$

i.e.

$$\frac{X''}{X} = -\frac{Y''}{Y} = k,$$

where $k$ is any constant. Hence

$$X'' - kX = 0 \text{ and } Y'' + kY = 0.$$

If $k > 0$, i.e. $k = p^2$, then solutions are

$$X(x) = A\mathrm{e}^{px} + B\mathrm{e}^{-px},$$

$$Y(y) = C\cos py + D\sin py.$$

If $k < 0$, i.e. $k = -p^2$, then solutions are

$$X(x) = A\cos px + B\sin px,$$

$$Y(y) = C\mathrm{e}^{py} + D\mathrm{e}^{-py}.$$

Then the solutions can be determined by the boundary conditions.