Assignment Regulations

- ✓ This is an individual assignment accounting for 10% of the overall assessment.

- ✓ For this module, you MUST submit a soft copy of the report in SINGLE PDF format must be submitted on the ICE.

- ✓ Write a short report which should contain a concise description of your results and observations to each question.

- ✓ You may refer to textbooks and lecture notes to discover approaches to problems, however the report should be your own work.

- ✓ Where you do make use of other references, please cite them in your work and provide a list of references in IEEE style.

- ✓ Reports may be accepted up to 5 days after the deadline has passed; a late penalty of 5% will apply for each day late without an extension being granted. Submissions over 5 days late will not be marked. Emailed submissions will NOT be accepted without exceptional circumstances.

Objects

The objects of this experiment are:

1. To study quantitatively the Fourier components of some simple periodic waveforms;
2. To demonstrate that periodic waveforms can be synthesised by the superposition of properly selected sinusoids.
3. To investigate the concept of signal filtering.
4. To understand the concept of amplitude modulation.

Fourier Analysis

Periodic Signals

Perhaps one of the most familiar type of waveform is the sinusoid, shown in Fig. 1. Here the signal $x(t)$ at any t is given by the equation:

$$x(t) = A\sin(2\pi f t + \phi),$$

where A is the "amplitude" of the sinusoid, f is the "frequency" and ϕ is a "phase" angle measured in radians.

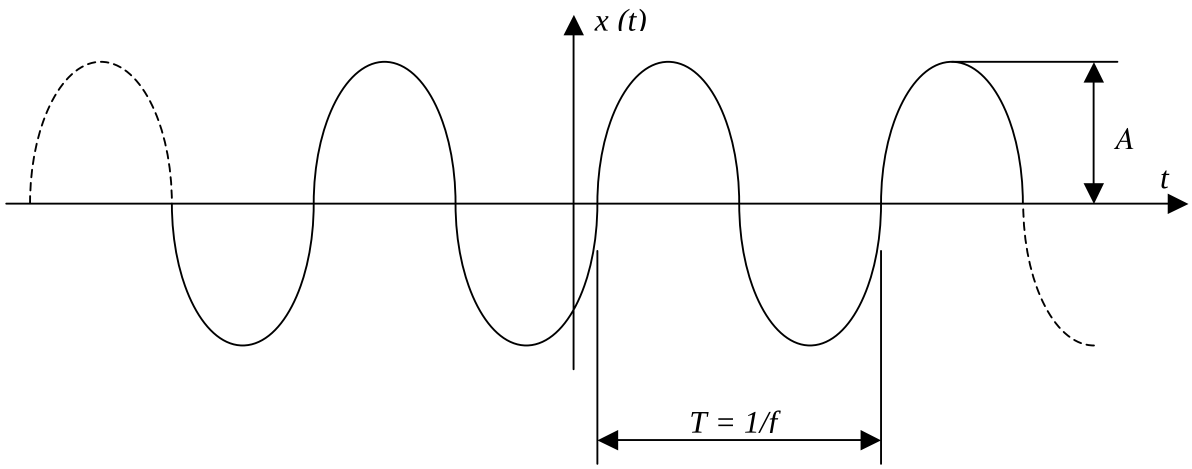

Figure 1: Sinusoid waveform with period $T = 1/f$, and amplitude A.

It may be seen from the graph that the waveform is "periodic" with period $T = \frac{1}{f}$, i.e. the waveform repeats itself after every T seconds. The frequency f, measured in Hertz (1/second), and the period T are related by:

$$T = \frac{1}{f}.$$

The phase angle determines the starting point of the sine wave at $t = 0$.
Although periodicity requires that the signal starts at $t \to -\infty$ and last until $t \to +\infty$, which is infeasible, many encountered signals can be regarded as periodic signals.

Fourier Series

Fourier series is based on the Harmonic analysis theorem, which is the branch of mathematics that studies the representation of functions or signals as the superposition of basic waves.

A Fourier series decomposes a periodic function into a sum of simple oscillating functions, namely sines and cosines. If T is the period of a signal $x(t)$ which satisfies Dirichlet conditions (sufficient conditions for a real-valued periodic function, that allows to determine the behaviour of the Fourier series at points of discontinuity), this signal may be expressed as follows (the trigonometric Fourier series):

$$x(t) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{2\pi n}{T} t\right) + b_n \sin\left(\frac{2\pi n}{T} t\right) \right], \tag{1}$$

where $\frac{1}{T}$ is termed the "fundamental frequency". a_0, a_n and b_n $(n = 1, 2, 3, \ldots)$ are the 'Fourier coefficients' and can be calculated from the following expressions:

$$a_0 = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\ \mathrm{d}t,$$

$$a_n = \frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) \cos\left(\frac{2\pi n}{T} t\right)\ \mathrm{d}t,$$

$$b_n = \frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) \sin\left(\frac{2\pi n}{T} t\right)\ \mathrm{d}t.$$

Equation (1) may also be written in the following alternative compact form:

$$x(t) = c_0 + \sum_{n=1}^{\infty} c_n \cos\left(\frac{2\pi n}{T} t + \theta_n\right), \tag{2}$$

where:

$$c_0 = a_0,$$

$$c_n = \sqrt{a_n^2 + b_n^2},$$

$$\theta_n = -\tan^{-1}\left(\frac{b_n}{a_n}\right).$$

It is worth noticing that the function $x(t)$ has been expressed as the sum of a constant value c_0, and sinusoids functions of frequency $\frac{1}{T}, \frac{2}{T}, \frac{3}{T}$, and so on. The sinusoid at frequency $\frac{1}{T}$, i.e. is called the fundamental frequency of $x(t)$.

Another convenient way to represent the Fourier series of a signal is by using the complex exponentials:

$$x(t) = \sum_{n=-\infty}^{\infty} h_n(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi n}{T} t}, \tag{3}$$

where

$$c_n = \frac{1}{T} \int_T x(t) e^{-j\frac{2\pi n}{T} t}\ \mathrm{d}t,$$

$h_n(t)$ are called the harmonic components of $x(t)$. The component at frequency $\pm\frac{1}{T}$ is called the fundamental harmonic of $x(t)$, and the component at frequency $\pm\frac{2}{T}$ being the second harmonic, the component at $\pm\frac{3}{T}$ being the third harmonic and so on.

Comments:

1. It is normal to assume that $x(t)$ exists as a periodic function for all values of t between $-\infty$ and ∞.

2. If $x(t) = x(-t)$, the function is even and only even terms appear in the series (1), i.e. cosine terms plus a_0.

3. If $x(t) = -x(-t)$ the function is odd and only odd terms appear in the series (1), i.e. sine terms.

4. By Fourier Analysis, the signal $x(t)$ has been broken down into its frequency components. These frequency components can be represented by two graphs, the amplitude spectrum that plots the harmonic amplitude versus the harmonic index $(n, |h_n|)$, (or equivalently $(\frac{n}{T}, |h_n|)$, with $\dfrac{n}{T}$ being the harmonic frequency), and the phase spectrum which plots $(n, \angle h_n)$, or $(\frac{n}{T}, \angle h_n)$.

5. The range of frequencies that "sufficiently" represent the signal $x(t)$ is called the bandwidth of $x(t)$ (i.e. the bandwidth of a periodic signal is $\frac{n_{\max}}{T}$, with $h_n > \epsilon$, $\forall |n| > n_{\max}$) is the bandwidth of $x(t)$). All electronic systems, in particular communication systems, operate only with restricted bandwidths so that frequency components outside a given frequency range are severely attenuated or eliminated. Further, the phases of the remaining components may be altered. A knowledge of the frequency spectrum of a signal provides the information necessary for predicting the type of distortion that will be caused by such systems.

Example: Fourier Analysis of a Triangular Wave

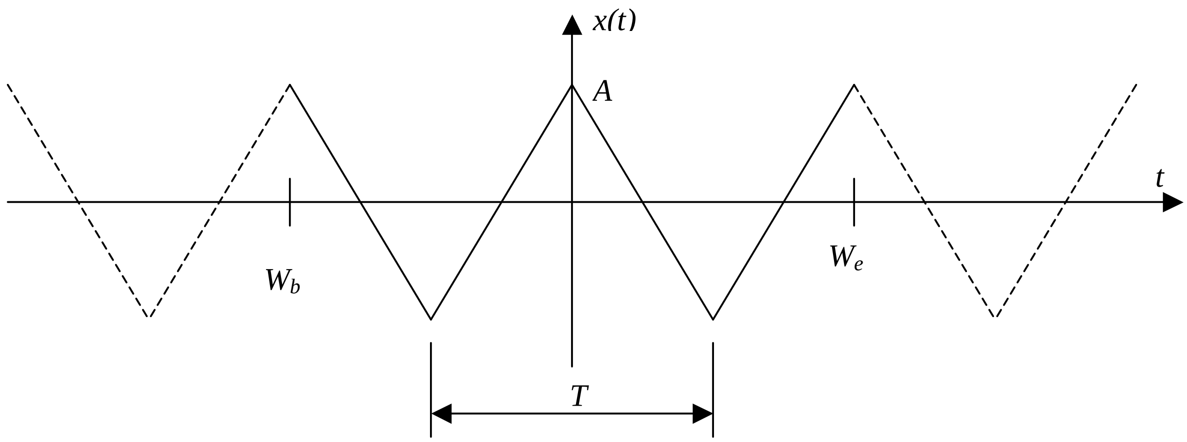

Figure 2: Triangular waveform.

One period of the triangular waveform[1] represented in Fig. 2 can be described by the following expression ($A = 1$):

$$x_T(t) = \begin{cases} 4\dfrac{t}{T} + 1, & \frac{-T}{2} \leqslant t \leq 0 \\ 1 - 4\dfrac{t}{T}, & 0 \leqslant t \leqslant \frac{T}{2} \end{cases}$$

By using equation (3), we get that:

$$c_n = \begin{cases} \dfrac{4}{(n\pi)^2} & , \quad \forall n \text{ odd;} \\ \\ 0 & , \quad \forall n \text{ even.} \end{cases}$$

[1]The signal $x_T(t)$represents one period of $x(t)$, i.e. $x_T(t) = x(t); \forall t \in (0, T)$.

Hence the Fourier series for $x(t)$ is:

$$x(t) \quad = \quad \frac{4}{\pi^2}\left[\ldots + \frac{1}{9}e^{-j2\pi 3\frac{t}{T}} + 0 + e^{-j2\pi\frac{t}{T}} + 0 + e^{j2\pi\frac{t}{T}} + 0 + \frac{1}{9}e^{j2\pi 3\frac{t}{T}} + \ldots\right]. \tag{4}$$

The spectrum of $x(t)$ contains an infinite number of frequency components. Hence the bandwidth of $x(t)$ is infinite. A practical system would restrict $x(t)$ to a finite bandwidth by removing all frequency components above some upper limit. This would distort the shape of the triangular waveform. Fortunately, the first few terms in the Fourier series of a triangular wave are the most important since the amplitudes.

Example: Fourier Analysis of a Square Wave

Similarly the square waveform shown in Fig. 3 has a Fourier series described by:

$$c_n = \begin{cases} \frac{2}{n\pi}(-1)^{\frac{n-1}{2}} & , \quad \forall n \text{ odd;} \\ 0 & , \quad \forall n \text{ even.} \end{cases}$$

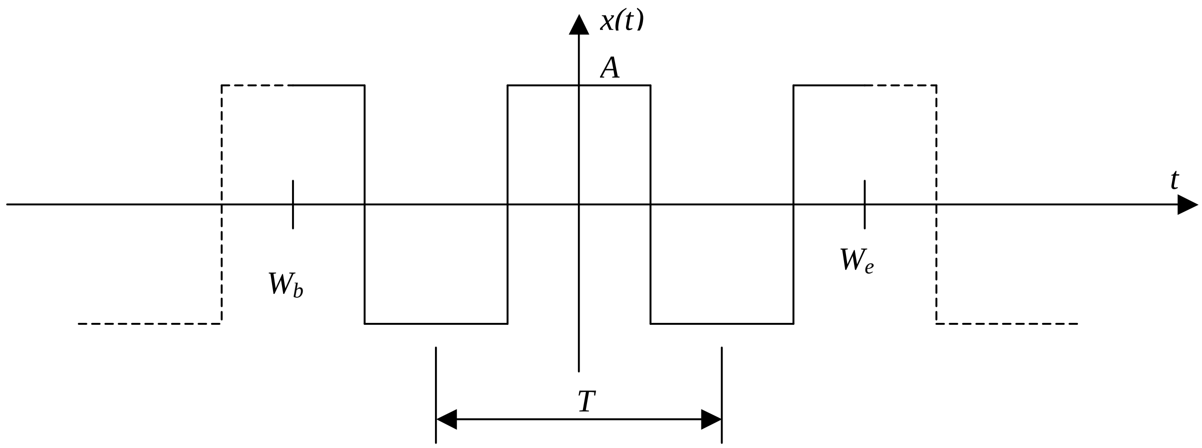

Figure 3: Square waveform.

Working with Discrete Signals

To process the continuous time signals in computers, we need to convert the signals to digital form. While an analogue signal is continuous in both time and amplitude, a digital signal is discrete in both time and amplitude. The first step to convert a signal from continuous to discrete time form is to sample it. The value of the signal $x(t)$ is measured at certain intervals in time $\frac{k}{T_s}$, with $k \in \mathbb{Z}$, and T_s is the sampling period. Each measurement $x(t)|_{t=\frac{k}{T_s}} = x(\frac{k}{T_s})$ is referred to as a sample, and it is simply represented as $x[k]$.

A sampled periodic signal $x(t)$ can be written using (3) as:

$$x(t)\big|_{t=\frac{k}{T_s}} = \sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi n}{T}t}\Bigg|_{t=\frac{k}{T_s}}, \tag{5}$$

$$x\left(\frac{k}{T_s}\right) = \sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi nk}{TT_s}}, \tag{6}$$

with

$$c_n = \frac{1}{T}\int_T x(t)e^{-j\frac{2\pi n}{T}t}\,\mathrm{d}t.$$

Fourier Synthesiser

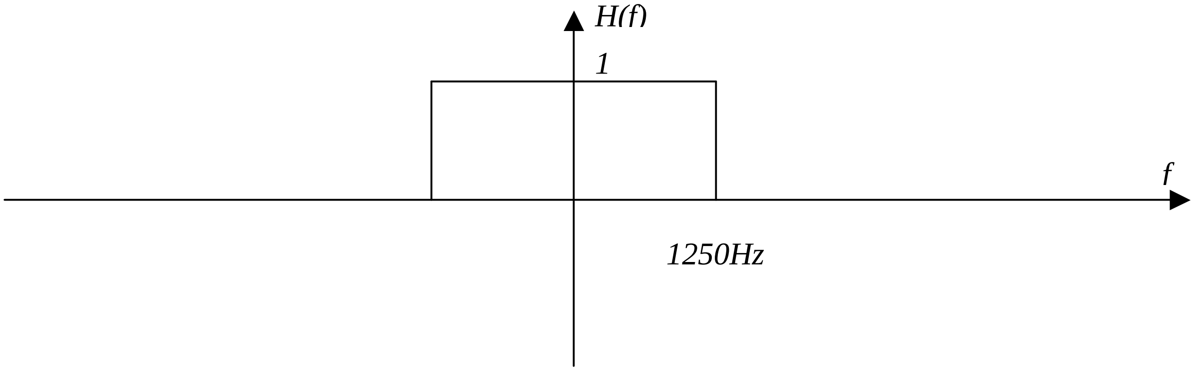

Figure 4: Low pass filter with cut-off frequency $f_c = 1250$ Hz.

Experimental Procedure

Unless otherwise noted, repeat the following steps for the triangular and square waveform (both of them will be denoted by $x(t)$ for brevity)

1. Plot a graph of the signal $x(t)$ with $T = 100$ ms, $A = 1$, sampling interval of 0.1 ms, and the observation window is $(W_b, W_e) = (-0.2, 0.2)$ s, using the functions `"square_wave_fun.m"` and `"triangular_wave_fun.m"`.

2. Plot a graph of the N-th partial sum $\tilde{x}(t)$ with $T = 100$ ms, $A = 1$, and for $N = [3, 4, 5, 11]$, using the functions `"square_wave_PFS_fun.m"` and `"triangular_wave_PFS_fun.m"`. Print the resulting waveforms and compare with the original signal $x(t)$, what do you observe about the behaviour of the partial sum when N gets increased ? what do you observe about the behaviour of the partial sum on the discontinuities of $x(t)$? (compare the behaviour of the triangular and square waveform).

3. Plot the signal $y(t)$ obtained by filtering the signal $x(t)$ ($T = 100$ ms and $A = 1$) using the low-pass filter represented in Fig. 4. Print the resulting waveform and observe the large oscillations near the jump discontinuity (what is the name of this phenomena?). Evaluate the overshoot for the square waveform as $a = \tilde{x}(t_0^+) - \tilde{x}(t_0^-)$, with $\tilde{x}(t_0^+)$ $(\tilde{x}(t_0^-))$ is the first maximum (minimum) value of $\tilde{x}$ to occur near the jump discontinuity point. Compare the evaluated overshoot value with the theoretical one.

4. Plot the amplitude and phase of the harmonics of $x(t)$, in the frequency range $(-1250, 1250)$ Hz, with $T = 100$ ms $A = 1$. Print the resulting spectrum. How many harmonics are there in the plotted range of frequency ? Note the higher frequency content of the square waveform.

Discrete Fourier Transform

The discrete Fourier transform (DFT) is one of the specific forms of Fourier analysis. It transforms one function into another one in the "discrete" frequency domain.
DFT is powerful tool to study signals and systems in the discrete domain. The main requirement for the DFT is to have an input function that is discrete and whose non-zero values have a limited (finite) duration. Such inputs are often created by sampling a continuous function. The DFT only evaluates enough frequency components to reconstruct the finite segment that was analysed. Its inverse transform cannot reproduce the entire time domain (i.e. $-\infty < t < \infty$), unless the input happens to be periodic (forever).
The DFT is ideal for processing information stored in computers. In particular, the DFT is widely employed in signal processing and related fields to analyse the frequencies contained in a sampled signal, to solve partial differential equations, and to perform other operations such as convolutions. In practice, the DFT can be computed efficiently using a fast Fourier transform (FFT) algorithm.
The forward DFT of the discrete function $x[n]$, $n \in [0\ ,\ N-1]$ is given by:

$$X[k] \triangleq \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn}, k = 0, \cdots, N-1. \tag{7}$$

The inverse discrete Fourier transform (IDFT) is given by:

$$x[n] \triangleq \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j\frac{2\pi}{N}kn}, n = 0, \cdots, N-1. \tag{8}$$

Note the analogy with the Fourier transform which is used in the continuous domain:

$$X(f) \triangleq \int_{-\infty}^{\infty} x(t) e^{-j2\pi tf}\ \mathrm{d}t,$$

$$x(t) \triangleq \int_{-\infty}^{\infty} X(f) e^{j2\pi tf} df.$$

Experimental Procedure

Load the audio signal `"music.wav"` into the MATLAB environment using the following command `audioread`.[2]. Generate a variable $x_{20}[n]$ that contains the first 20 s of the loaded sound data.

1. Use the command `audioplayer` and `play` to play the variable $x_{20}[n]$.

2. Use the command `fft` to evaluate the DFT of the signal $x_{20}[n]$, plot the amplitude and phase spectrum using the command `plotSpectrum(estimator)` [3] or the provided function `getSpectrum(f, FS)`. Print the amplitude spectrum of $x_{20}[n]$ and note that the low frequency components (up to 4000 Hz) carry most of the signal energy.

3. Scaling property (contracting and stretching signal): Generate a vector $x_d[n]$ by down-sampling the vector $x_{20}[n]$ by a factor of four. Using the original value of the sampling frequency, play the content

[2]Example: `[y, Fs]= audioread(filename)` reads the file `filename` into the variable `y`, and the sampling frequency into `Fs` (see the MATLAB help for further information)
[3]See the command help for further information about this command.

of x_d by the command `play`, what did you notice? Use the command `fft` to evaluate the DFT of $x_d[n]$, plot the amplitude and phase spectrum using the command `plotSpectrum(estimator)` or `getSpectrum(f, FS)`. Print the amplitude spectrum of $x_d[n]$ and compare it with the amplitude spectrum of the original signal. What will happen to the signal spectrum when it is played faster?