MTH101: Lecture 21 – 22

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Example

Find a power series solution for the following ODE.

$$y'' - 4xy' + (4x^2 - 2)y = 0.$$

Example

We first write down the power series of y, y', and y''.

$$y = \sum_{m=0}^{\infty} a_m x^m, \ y' = \sum_{m=0}^{\infty} m a_m x^{m-1}, \ y'' = \sum_{m=0}^{\infty} m(m-1) a_m x^{m-2}.$$

Example

The ODE thus becomes

$$\sum_{m=0}^{\infty} m(m-1)a_m x^{m-2} - 4\sum_{m=0}^{\infty} m a_m x^m$$
$$+4\sum_{m=0}^{\infty} a_m x^{m+2} - 2\sum_{m=0}^{\infty} a_m x^m = 0$$
$$\Rightarrow \sum_{m=0}^{\infty}(m+2)(m+1)a_{m+2}x^m - 4\sum_{m=0}^{\infty} m a_m x^m$$
$$+4\sum_{m=2}^{\infty} a_{m-2}x^m - 2\sum_{m=0}^{\infty} a_m x^m = 0,$$

where for each term we choose the power of x to be m.

Example

$$\Rightarrow \sum_{m=2}^{\infty} \left[(m+2)(m+1)a_{m+2} - 4ma_m + 4a_{m-2} - 2a_m \right] x^m$$
$$+ (2a_2 - 2a_0) + (6a_3 - 4a_1 - 2a_1)\, x = 0.$$

Therefore, we know

$$a_2 = a_0, \quad a_3 = a_1,$$

$$(m+2)(m+1)a_{m+2}x^m - (4m+2)a_m + 4a_{m-2} = 0.$$

Example

$$\Rightarrow 12a_4 = 10a_2 - 4a_0 = 6a_0, \quad or, \quad a_4 = \frac{1}{2}a_0$$

$$\vdots$$

$$\Rightarrow a_{2k} = \frac{1}{k!}a_0 \quad for\ k \in \mathbb{Z}^+.$$

Similarly, we can find

$$a_{2k+1} = \frac{1}{k!}a_1 \quad for\ k \in \mathbb{Z}^+.$$

Therefore

$$y = (a_0 + a_1 x)\left(1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots\right) = (a_0 + a_1 x)e^{x^2}.$$

Legendre's differential equation

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0$$

with constant n is one of the most important ODE in physics. It arises when the Laplace equations for central force problems are separated in spherical coordinates.

This equation involves a **parameter** n, which is dependent on the physical problem we are considering, and the solution to it is called **Legendre function**, which is one of the most important **special functions**.

We can solve the Legendre's differential equation by **Power Series** technique. We first divide the whole equation by the factor $(1-x^2)$

$$y'' - \frac{2x}{(1-x^2)}y' + \frac{n(n+1)}{(1-x^2)}y = 0,$$

where we can see the coefficients of y' and y are analytic at $x = 0$ with radius of convergence $R = 1$. Hence we can use the power series $y = \sum_{m=0}^{\infty} a_m x^m$ to get

$$(1-x^2)\sum_{m=0}^{\infty} m(m-1)a_m x^{m-2} \; - 2x\sum_{m=0}^{\infty} m a_m x^{m-1} + n(n+1)\sum_{m=0}^{\infty} a_m x^m = 0.$$

$$\Rightarrow \sum_{m=0}^{\infty} m(m-1)a_m x^{m-2} - \sum_{m=0}^{\infty} m(m-1)a_m x^m - 2\sum_{m=0}^{\infty} m a_m x^m + n(n+1)\sum_{m=0}^{\infty} a_m x^m = 0.$$

Again, we can choose the power for x to be m for each term, and get

$$\Rightarrow \sum_{m=0}^{\infty}(m+2)(m+1)a_{m+2}x^m - \sum_{m=0}^{\infty} m(m-1)a_m x^m - 2\sum_{m=0}^{\infty} m a_m x^m + n(n+1)\sum_{m=0}^{\infty} a_m x^m = 0.$$

By collecting all the terms, we have

$$\sum_{m=0}^{\infty} \{(m+2)(m+1)a_{m+2} - [m(m-1) + 2m - n(n+1)]\, a_m\}\, x^m = 0,$$

or ,

$$a_{m+2} = \frac{m(m+1) - n(n+1)}{(m+2)(m+1)} a_m = -\frac{(n-m)(n+m+1)}{(m+2)(m+1)} a_m.$$

We can use this relation to find

$$a_2 = -\frac{n(n+1)}{2!}a_0, \qquad a_3 = -\frac{(n-1)(n+2)}{3!}a_1,$$

$$\begin{aligned} a_4 &= -\frac{(n-2)(n+3)}{4\cdot 3}a_2, \\ &= \frac{(n-2)n(n+1)(n+3)}{4!}a_0, \\ &\vdots \end{aligned} \qquad \begin{aligned} a_5 &= -\frac{(n-3)(n+4)}{5\cdot 4}a_3, \\ &= \frac{(n-3)(n-1)(n+2)(n+4)}{5!}a_1, \\ &\vdots \end{aligned}$$

and

$$y(x) = a_0 y_1(x) + a_1 y_2(x),$$

where

$$y_1(x) = 1 - \frac{n(n+1)}{2}x^2 + \frac{(n-2)n(n+1)(n+3)}{4!}x^4 - + \cdots$$
$$y_2(x) = x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-3)(n-1)(n+2)(n+4)}{5!}x^5 - + \cdots$$

Again, this series solution converges for $|x| < 1$, since for Legendre's differential equation, $x = \pm 1$ are the regular singular points.

Legendre Polynomials $P_n(x)$

Remark

The solution to the Legendre's differential equation with general parameter n is a linear combination of infinite series. However, when n is an integer, the recurrence relation will stop at some point, and the infinite series is terminated and become a n-th order polynomial, which we call it **Legendre polynomial** *and denoted as $P_n(x)$.*

Legendre Polynomials $P_n(x)$

Remark

We consider n is an integer. By the recurrence relation, we know $a_{n+2} = a_{n+4} = \cdots = 0$, and by convention the coefficient a_n of the highest power x^n is chosen as

$$a_n = \frac{(2n)!}{2^n(n!)^2} = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{n!},$$

and $a_n = 1$ if $n = 0$. Then we can inverse the recurrence relation to find

$$a_{n-2} = -\frac{n(n-1)}{2(2n-1)} a_n = -\frac{(2n-2)!}{2^n(n-1)!(n-2)!}$$

Legendre Polynomials $P_n(x)$

Remark

Similarly,

$$a_{n-4} = -\frac{(n-2)(n-3)}{4(2n-3)} a_{n-2} = \frac{(2n-4)!}{2^n 2!(n-2)!(n-4)!},$$

and so on. After all, one can find when $n - 2m \geq 0$,

$$a_{n-2m} = (-1)^m \frac{(2n-2m)!}{2^n m!(n-m)!(n-2m)!}.$$

Legendre Polynomials $P_n(x)$

Remark

The Legendre Polynomial of degree n is thus

$$\begin{aligned} P_n(x) &= \sum_{m=0}^{M} (-1)^m \frac{(2n-2m)!}{2^n m!(n-m)!(n-2m)!} x^{n-2m} \\ &= \frac{(2n)!}{2^n (n!)^2} x^n - \frac{(2n-2)!}{2^n 1!(n-1)!(n-2)!} x^{n-2} + - \cdots, \end{aligned}$$

where $M = \frac{n}{2}$ (if n even) and $\frac{(n-1)}{2}$ (if n odd).

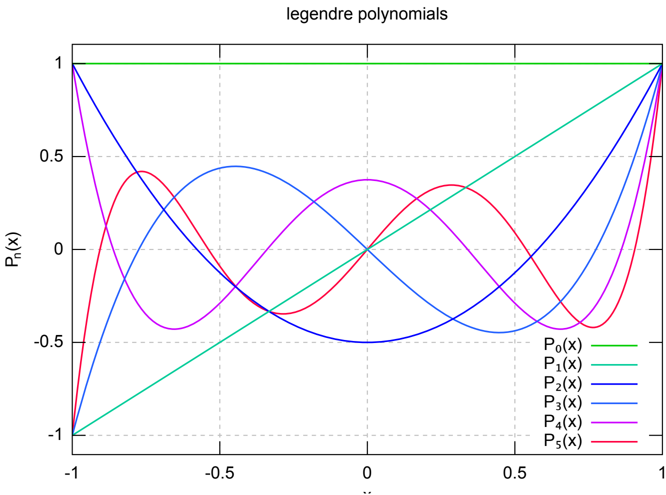
Legendre Polynomials $P_n(x)$

Remark

The first few Legendre polynomials are

$$P_0(x) = 1, \qquad P_1(x) = x, \qquad P_2(x) = \frac{1}{2}(3x^2 - 1),$$
$$P_3(x) = \frac{1}{2}\left(5x^3 - 3x\right), \qquad P_4(x) = \frac{1}{8}\left(35x^4 - 30x^2 + 3\right).$$

Legendre Polynomials $P_n(x)$



Rodrigues's formula

There are other ways to express the Legendre's polynomials, one of them is the Rodrigues's formula

Example

By applying the binomial theorem to $(x^2 - 1)^n$, *differentiating it* n *times term by term, show that*

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}[(x^2 - 1)^n].$$

Rodrigues's formula

Proof.

$$(x^2-1)^n = \sum_{m=0}^{n} \frac{(-1)^m n!}{m!(n-m)!} x^{2(n-m)}$$

$$\Rightarrow \frac{d^n}{dx^n}[(x^2-1)^n] = \sum_{m=0}^{M} \frac{(-1)^m n!}{m!(n-m)!} \frac{(2n-2m)!}{(n-2m)!} x^{n-2m},$$

where $M = n/2$ or $(n-1)/2$, since we need $(n-2m) \geq 0$.

$$\Rightarrow \frac{1}{2^n n!} \frac{d^n}{dx^n}[(x^2-1)^n] = \sum_{m=0}^{M} \frac{(-1)^m (2n-2m)!}{2^n m!(n-m)!(n-2m)!} x^{n-2m}$$

$\square$

We have seen how to use power series to solve some ODEs with analytic coefficients. In fact, there are some important ODEs with coefficients that are non-analytic (e.g. Bessel's equation), but they approach to infinities in a **controlled** way, it this case, we can use **Frobenius method** to solve them.

Theorem

Let $b(x)$ and $c(x)$ be any functions that are analytic at $x = 0$. Then the ODE

$$y'' + \frac{b(x)}{x} y' + \frac{c(x)}{x^2} y = 0,$$

admits at least one solution that can be represented in the form

$$y(x) = x^r \sum_{m=0}^{\infty} a_m x^m, \qquad (a_0 \neq 0),$$

where r may be any real or complex number and is chosen such that $a_0 \neq 0$.

Let's see how does the Frobenius method work. We first multiply the ODE by x^2

$$x^2 y'' + x b(x) y' + c(x) y = 0,$$

and since $b(x)$, $c(x)$ are analytic, we can expand them in power series

$$b(x) = \sum_{m=0}^{\infty} b_m x^m, \qquad c(x) = \sum_{m=0}^{\infty} c_m x^m,$$

and the ansatz

$$y(x) = x^r \sum_{m=0}^{\infty} a_m x^m.$$

The differentiation of $y(x)$ can be found by the termwise operation

$$y'(x) = \sum_{m=0}^{\infty}(r+m)a_m x^{r+m-1} = x^{r-1}\left[ra_0 + (r+1)a_1 x \cdots\right]$$

$$y''(x) = \sum_{m=0}^{\infty}(r+m)(r+m-1)a_m x^{r+m-2} = x^{r-2}\left[r(r-1)a_0 + \cdots\right].$$

Substituting them into the ODE, we find

$$x^r\left[r(r-1)a_0 + \cdots\right] + x^r\left[ra_0 + (r+1)a_1 x + \cdots\right]\left(b_0 + b_1 x + \cdots\right) + x^r\left(a_0 + a_1 x + \cdots\right)\left(c_0 + c_1 x + \cdots\right) = 0,$$

and we use the requirement that all the coefficients for x^r, x^{r+1}, x^{r+2}, $\cdots$ need to be zero to solve the unknown a_m.

Indicial Equation

We first consider the lowest order, the coefficient of x^r

$$[r(r-1)+b_0 r+c_0]\, a_0=0,$$

since by assumption $a_0 \neq 0$, we find the so called **indicial equation** of the ODE:

$$r(r-1)+b_0 r+c_0=0.$$

By the first theorem, one of the solution to this second order ODE takes the form $y(x)=x^r \sum_{m=0}^{\infty} a_m x^m$, and the other solution will be a form indicated by the indicial equation.

Theorem

Suppose that the ODE satisfies the assumption in last theorem. Let r_1 and r_2 be the roots of the indicial equation. There are three different cases for the other solution.

Case 1. Distinct roots not differing by an integer. *A basis is*

$$y_1(x) = x^{r_1}\left(a_0 + a_1 x + a_2 x^2 + \cdots\right),$$
$$y_2(x) = x^{r_2}\left(A_0 + A_1 x + A_2 x^2 + \cdots\right),$$

where the coefficients can be found by the requirement that coefficients of x^{r+1}, x^{r+2} $\cdots$ need to be zero for $r = r_1, r_2$.

Theorem

Case 2. Double roots $r_1 = r_2 = r$. *A basis is*

$$y_1(x) = x^r \left(a_0 + a_1 x + a_2 x^2 + \cdots\right),$$
$$y_2(x) = y_1(x) \ln x + x^r \left(A_1 x + A_2 x^2 + \cdots\right).$$

Case 3. Roots differing by an integer. *A basis is*

$$y_1(x) = x^{r_1} \left(a_0 + a_1 x + a_2 x^2 + \cdots\right),$$
$$y_2(x) = k y_1(x) \ln x + x^{r_2} \left(A_0 + A_1 x + A_2 x^2 + \cdots\right),$$

where the roots are so denoted that $r_1 - r_2 > 0$ and k may turn out to be zero.

Applications

Example

Solve the ODE

$$x(x-1)y'' + (3x-1)y' + y = 0.$$

Solution

$$x(x-1)y'' + (3x-1)y' + y = 0$$
$$\Rightarrow y'' + \frac{3x-1}{x(x-1)}y' + \frac{1}{x(x-1)}y = 0.$$

Applications

Solution Comparing it with the standard form of Frobenius method, we can find

$$b(x) = \frac{3x-1}{x-1}, \qquad c(x) = \frac{x}{x-1},$$

which are analytic at $x = 0$. We can Taylor expand $b(x)$ and $c(x)$ and get

$$b(x) = \frac{1-3x}{1-x} = (1-3x)\sum_{n=0}^{\infty} x^n, \quad \textit{with } b_0 = 1,$$

$$c(x) = -\frac{x}{1-x} = -x\sum_{n=0}^{\infty} x^n, \qquad \textit{with } c_0 = 0.$$

Applications

Solution The indicial equation is thus

$$r(r-1)+r=0 \Rightarrow r^2=0. \qquad \textbf{Case 2.}$$

The first solution can be found by substituting

$$y_1(x)=\sum_{m=0}^{\infty} a_m x^m, \quad y_1'(x)=\sum_{m=0}^{\infty} m a_m x^{m-1},$$
$$y_1''(x)=\sum_{m=0}^{\infty} m(m-1) a_m x^{m-2},$$

into the ODE.

Applications

Solution Therefore

$$x(x-1)\sum_{m=0}^{\infty} m(m-1)a_m x^{m-2}$$

$$+(3x-1)\sum_{m=0}^{\infty} m a_m x^{m-1} + \sum_{m=0}^{\infty} a_m x^m = 0$$

$$\Rightarrow \sum_{m=0}^{\infty} [m(m-1)+3m+1]\, a_m x^m - \sum_{m=0}^{\infty} [m(m-1)+m]\, a_m x^{m-1} = 0$$

$$\Rightarrow \sum_{m=0}^{\infty} \left[(m+1)^2 a_m - (m+1)^2 a_{m+1}\right] x^m \;\Rightarrow\; a_{m+1} = a_m$$

$$\Rightarrow y_1(x) = \sum_{m=0}^{\infty} x^m = \frac{1}{1-x}.$$

Applications

Solution The second solution to the ODE can be found by either using **reduction of order** or substituting $y_2(x) = y_1(x)\ln x + (A_1 x + \cdots)$ into the ODE and find the coefficients A_m. Here we demonstrate how to use the reduction of order to find the other solution.
We first guess $y_2(x) = u(x)y_1(x)$ with some arbitray function $u(x)$. By substituting $y_2(x)$ into the ODE, we can find the ODE for $u(x)$

$$
\begin{aligned}
& u'' + \left[\frac{2y_1'}{y_1} + p(x)\right] u' = 0, \quad p(x) = \frac{3x-1}{x(x-1)} \\
\Rightarrow\ & \frac{d\ln|u'|}{dx} = -2\frac{d\ln|y_1|}{dx} - p(x) \\
\Rightarrow\ & u' = \frac{1}{y_1^2} e^{-\int p(x)dx}.
\end{aligned}
$$

Applications

Solution Since

$$-\int p(x)dx = -\int \frac{3x-1}{x(x-1)}dx = -\int \left(\frac{-2}{1-x} + \frac{1}{x}\right) dx$$

$$\Rightarrow -\int p(x)dx = -2\ln(1-x) - \ln x.$$

Therefore,

$$u' = (1-x)^2 \frac{1}{x(1-x)^2} = \frac{1}{x}, \quad u(x) = \ln x$$

$$\Rightarrow y_2(x) = u(x)y_1(x) = \frac{\ln x}{1-x}.$$

Bibliography

1 *Kreyszig, E.* **Advanced Engineering Mathematics**. Wiley, 10th Edition.
2 *Wikipedia* **Legendre polynomials**.