EEE108 Electromagnetism and Electromechanics

Lecture 6

Conductors and Dielectrics
Capacitors and Capacitance (1)

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Today

- **Conductors**
  Basic properties in electrostatics
- **Dielectrics**
- **Capacitors and Capacitance (1)**

Conductors and Insulators/Dielectrics

$*$ Conductor : a material with free electrons /charges

- Excellent conductors : metals such as Au, Ag, Cu, Al, ...
- OK conductors : ionic solutions such as NaCl in H_2O, ...
- A conductor is assumed to have an infinite supply of electric charges

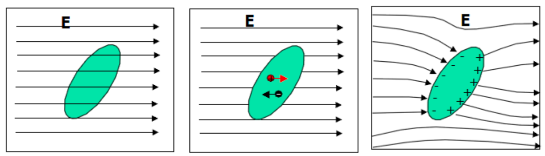
$*$ Insulator : a material without free electrons /charges

- Organic material : rubber, plastic, ...
- Inorganic material : quartz, glass, ...

Electric Fields and Potential in Conductors

◆ Inside a conductor, electric field is zero.

- Why? if $\mathbf{E}$ is not zero, then charges will move from where the potential is higher to where the potential is lower. The moving will stop only when $\mathbf{E} = 0$.
- How long does it take? $10^{-17} \sim 10^{-16}$ second for typical resistivity of metals.



◆ Electric potential inside a conductor is constant

If there are 2 points inside the conductor P_1 and P_2, the $\Delta\varphi$ would be :

$$\Delta\varphi = \int_{P_1}^{P_2} \mathbf{E} \bullet d\mathbf{L} = 0$$

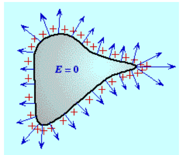
since electric field $\mathbf{E}$ is zero inside the conductor.

Electric Fields and Potential in Conductors

◆Net charge can only reside on the surface

If there is net charge inside the conductor, then $\mathbf{E} \neq 0$

From Gauss's law : $\Phi_E = \oint\!\!\oint_S \mathbf{E} \bullet d\mathbf{s} = \dfrac{Q_{enc}}{\varepsilon_r \varepsilon_0}$

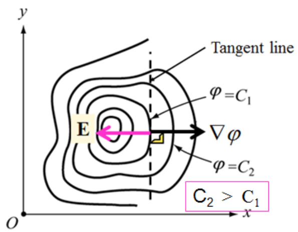


◆External field lines must be perpendicular to surface

If any field line did NOT come it at a 90º angle, then a component of the field line would be parallel to the conductor’s surface and electrons would respond to its presence and be accelerated within the conductor

- Conductor’s surface is an equipotential

  Because it’s perpendicular to field lines.

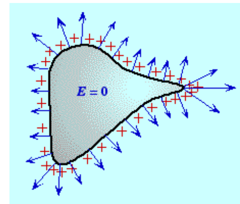


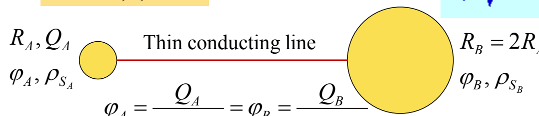
Electric Fields and Potential in Conductors

◆Net charge can only reside on the surface

How the charges distribute on the surface?

1. Uniformly?
2. Smaller curvature, denser charges?
3. Bigger curvature, denser charges? √
4. None 1, 2, and 3.





$$\varphi_A = \frac{Q_A}{4\pi\varepsilon_0 R_A} = \varphi_B = \frac{Q_B}{4\pi\varepsilon_0 R_B}$$

$$\frac{Q_A}{R_A} = \frac{Q_B}{R_B}$$

$$\boxed{\rho_{S_A} = \frac{Q_A}{4\pi R_A^2}; \rho_{S_B} = \frac{Q_B}{4\pi R_B^2}}$$

$$\frac{\rho_{S_A}}{\rho_{S_B}} = \frac{Q_A / R_A^2}{Q_B / R_B^2} = 2$$

Electric Fields and Potential in Conductors

Corollary

A charge $+q$ in the cavity will induce a charge $+q$ on the outside of the conductor.

Apply Gauss's law to the surface inside the conductor

$$\begin{cases} \oint \mathbf{E} \bullet d\mathbf{s} = 0, \text{ Because } E = 0 \text{ inside a conductor} \\ \oint \mathbf{E} \bullet d\mathbf{s} = \dfrac{1}{\varepsilon_0}(q + q_{inside}) \end{cases}$$

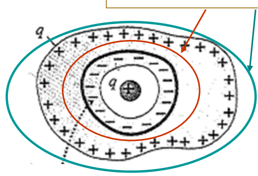
$$\Downarrow$$

$$(q + q_{inside}) = 0$$

Then, we have $q_{inside} = -q$

$$\Rightarrow \quad q_{outside} = -q_{inside} = q$$

Gaussian surface



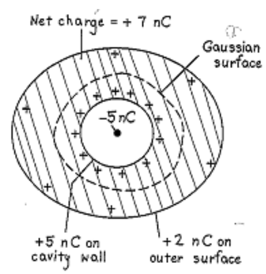
For **E** to be zero at all points on the Gaussian surface, the surface of the cavity must have a total charge $-q$.

Electric Fields and Potential in Conductors

Example

A solid conductor with a cavity carries a total charge of +7 nC. Within the cavity, insulated from the conductor, is a point charge of -5 nC. How much charge is on each surface (inner and outer) of the conductor?

Solution



The net charge enclosed by the Gaussian surface must be zero.

A charge $q = -(-5\text{nC})$ must have been induced on the cavity surface.

The conductor itself has a charge +7 nC, the amount of charge on the outer surface of the conductor must be $Q - q = 7 - 5\text{nC} = +2$ nC.

The basic properties of a conductor:

(1) The electric field inside a conductor is zero.

(2) The surface of a conductor is an equipotential surface.

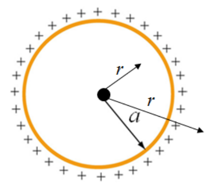
(3) Just outside the conductor, the electric field is normal to the surface, the tangential component of the electric field on the surface is zero.

(4) Any net charge must reside on the surface of the conductor.

Electric Potential Due to a Spherical Shell

Consider a metallic spherical shell of radius a and charge Q

(a) Find the electric potential everywhere.
(b) Calculate the potential energy of the system.



Electric Potential Due to a Spherical Shell

(a) Find the electric potential everywhere.

Solution

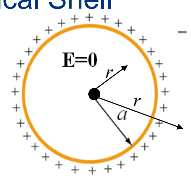
$$\mathbf{E}=\begin{cases}0 & r\le a\\ \dfrac{Q}{4\pi\varepsilon_0 r^2}\mathbf{a}_r & r\ge a\end{cases}$$

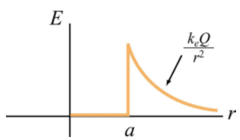
The electric potential can be calculated by:

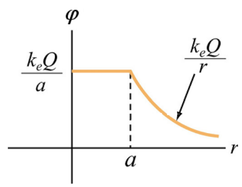
$$\varphi_r-\varphi_\infty=-\int_\infty^r \frac{Q}{4\pi\varepsilon_0 r'^2}dr'=\frac{1}{4\pi\varepsilon_0}\frac{Q}{r}=k_e\frac{Q}{r}$$

If $r<a$, the potential becomes:

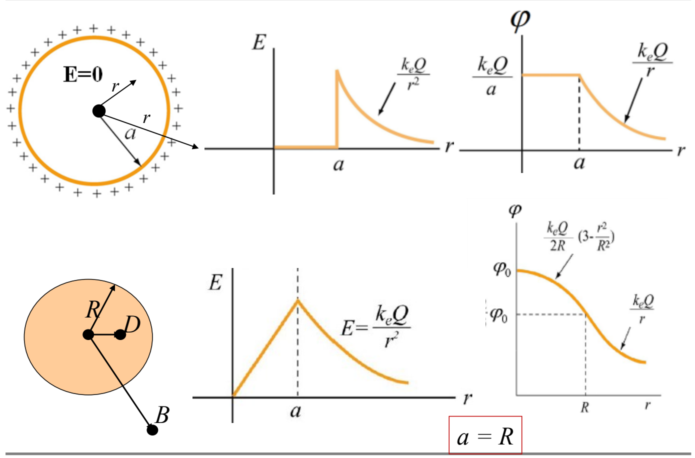
$$\varphi_r-\varphi_\infty=-\underbrace{\int_\infty^a \frac{Q}{4\pi\varepsilon_0 r'^2}dr'}_{r>a}-\underbrace{\int_a^r E\,dr}_{r<a,\,E=0}=\frac{1}{4\pi\varepsilon_0}\frac{Q}{a}=k_e\frac{Q}{a}$$







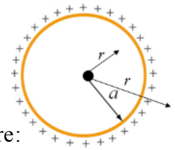
Electric Potential Due to a Spherical Shell



Electric Potential Due to a Spherical Shell

Potential Energy

(b) Calculate the potential energy of the system.



The potential energy U can be thought of as the work that needs to be done to build up the system.

To bring charge dq from infinity and deposit it on the sphere:

$$dW_{ext} = \varphi dq = \frac{1}{4\pi\varepsilon_0}\frac{q}{a}dq$$

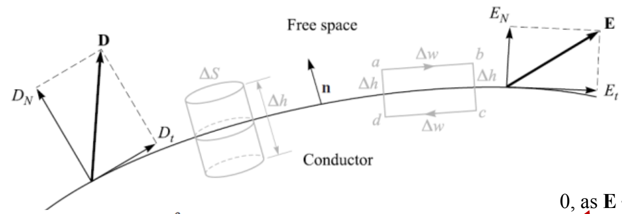
The total amount of work needed to charge the sphere to Q:

$$W_{ext} = \int_0^Q \frac{1}{4\pi\varepsilon_0}\frac{q}{a}dq = \frac{1}{8\pi\varepsilon_0}\frac{Q^2}{a}$$

Since $\varphi = \frac{1}{4\pi\varepsilon_0}\frac{Q}{a}$ and $W_{ext} = U$, then the potential energy U is:

$$\mathrm{U} = \frac{1}{2}Q\varphi$$

Boundary Condition of Conductors



Using Gauss's law, $\oint_S \mathbf{D}\cdot d\mathbf{S} = Q$

0, as $\mathbf{E} \perp \mathbf{n}$

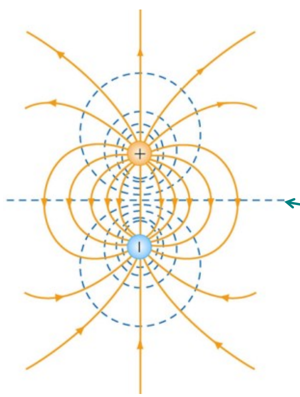
0, as $\mathbf{E} = 0$

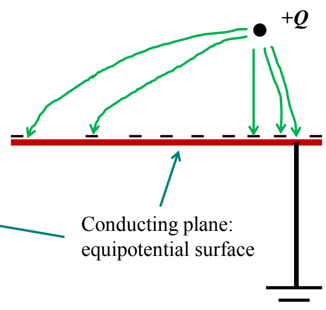
we integrate over the three distinct surfaces $\int_{top} + \int_{bottom} + \int_{sides} = Q$

and find that the last two are zero. Then $D_N \Delta S = Q = \rho_S \Delta S$

$$\boxed{D_N = \epsilon_0 E_N = \rho_S}$$

Method of Images





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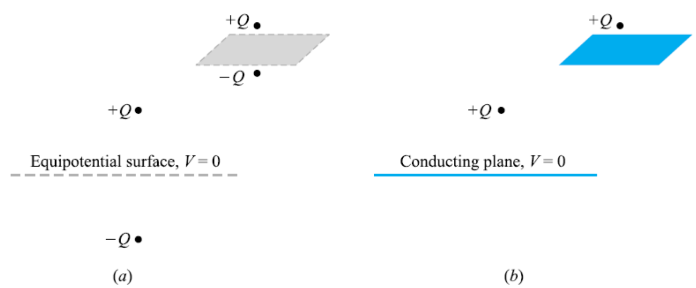

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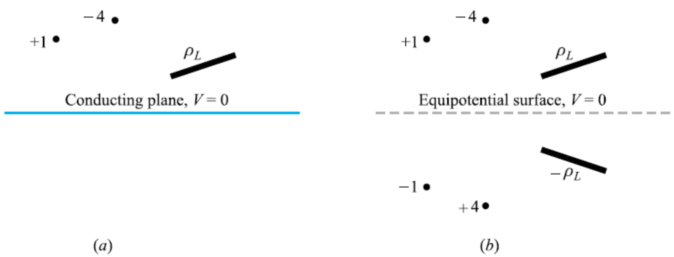
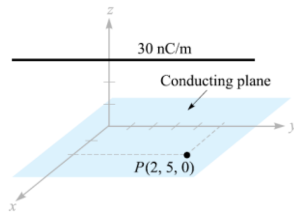


Method of Images

Example

find the surface charge density at $P(2, 5, 0)$ on the conducting plane $z = 0$ if there is a line charge of 30 nC/m located at $x = 0$, $z = 3$.

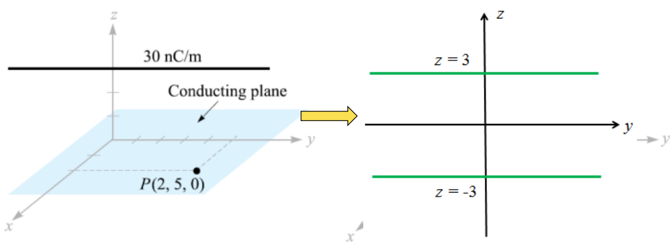


Method of Images

Example

Solution

We remove the plane and install an image line charge of -30 nC/m at $x = 0$, $z = -3$



Method of Images

Example

Solution Cont.

The radial vector from the positive line charge to P is $\mathbf{R}_+ = 2\mathbf{a}_x - 3\mathbf{a}_z$, while $\mathbf{R}_- = 2\mathbf{a}_x + 3\mathbf{a}_z$. Thus, the individual fields are

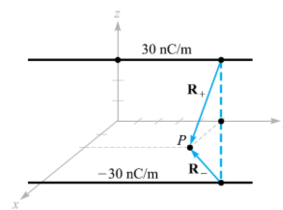
$$\mathbf{E}_+ = \frac{\rho_L}{2\pi\epsilon_0 R_+}\mathbf{a}_{R+} = \frac{30 \times 10^{-9}}{2\pi\epsilon_0\sqrt{13}}\,\frac{2\mathbf{a}_x - 3\mathbf{a}_z}{\sqrt{13}}$$

and

$$\mathbf{E}_- = \frac{30 \times 10^{-9}}{2\pi\epsilon_0\sqrt{13}}\,\frac{2\mathbf{a}_x + 3\mathbf{a}_z}{\sqrt{13}}$$

Adding these results, we have

$$\mathbf{E} = \frac{-180 \times 10^{-9}\mathbf{a}_z}{2\pi\epsilon_0(13)} = -249\mathbf{a}_z \text{ V/m}$$



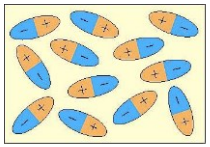
Thus, $\mathbf{D} = \epsilon_0\mathbf{E} = -2.20\mathbf{a}_z$ nC/m^2, and because this is directed *toward* the conducting plane, ρ_S is negative and has a value of -2.20 nC/m^2 at P.

Dielectric Materials

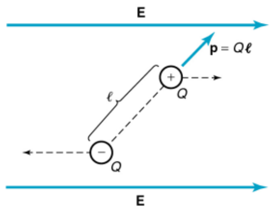
Dielectrics/Insulators: there are not significant numbers of free charges present within them, quartz, Teflon, rubber, glass, …

Bound charges/Polarization charges: electric charges confined to atoms or molecules.

Electric dipoles consisting of a pair of charges of opposite polarity that are bound together in that they are not free to move at the atomic level.

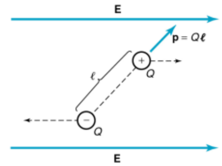


Without external field: Random, unpolarized



With external field: Oriented

Electric Fields and Dielectric Materials



- The dipoles rotate to try to align with the field.
- Not all the dipoles completely align with the field.
- The net alignment is given by the ***polarization vector***: indicate the net polarization of the dielectric.

Polarization vector: defined as the dipole moment per unit volume

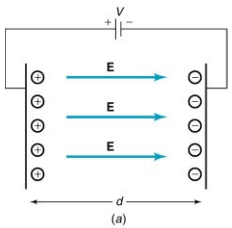
$$\mathbf{P} = \lim_{\Delta v \to 0} \frac{\sum_i \mathbf{p}_i}{\Delta v}$$

Charge × displacement: C × m

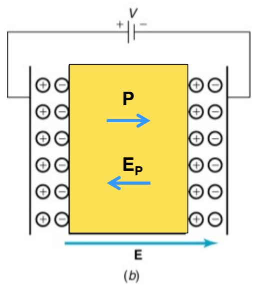
The unit of $\mathbf{P}$ is $\mathrm{C/m^2}$ or *charge per unit of area*.

The strength of the polarization vector depends on the strength of the applied electric field and material.

Electric Fields and Dielectric Materials

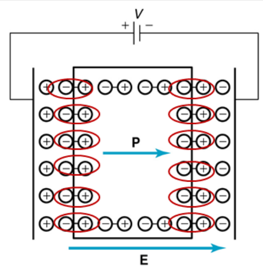


- A parallel-plate capacitor without a dielectrics.
- The charge creates an electric field between the plates separated.



- A dielectric material is inserted between the plates.
- The bound charge dipoles attempt to align with the field, creating a polarization vector in the dielectric.
- A surface charge is created on the two surfaces of the dielectric. *More free charge is drawn from the battery.*

Electric Fields and Dielectric Materials



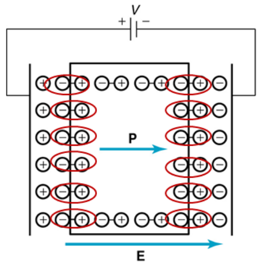
- There are two vector quantities between the plates: the applied electric field, **E**, and the polarization vector due to the polarization of the dielectric, **P**.
- **E** and **P** are in the same direction.

The strength of the polarization vector depends on the strength of the applied electric field and material.

This related by the electric susceptibility of the material, χ_e :

$$\mathbf{P} = \varepsilon_0 \chi_e \mathbf{E}$$

Electric Flux Density Vector and Dielectric Materials



Electric flux density vector/Flux density:
defined as:

$$\boxed{\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} \quad \mathrm{C/m^2}}$$

Relative permittivity/ dielectric constant

$$\mathbf{D} = \varepsilon_0 (1+\chi_e)\mathbf{E} \qquad \varepsilon_r = (1+\chi_e)$$

$$\mathbf{D} = \varepsilon\, \mathbf{E} \qquad \varepsilon = \varepsilon_r \varepsilon_0$$

permittivity

$$[\mathbf{E}] = \frac{\text{Newton}}{\text{Coulomb}} = \frac{\text{Volt}}{\text{meter}}$$

$$[\varepsilon] = \frac{\text{Farad}}{\text{meter}} = \frac{\text{Coulomb/Volt}}{\text{meter}}$$

$$[\varepsilon\, \mathbf{E}] = \frac{\text{Coulomb/Volt}}{\text{meter}} \times \frac{\text{Volt}}{\text{meter}} = \mathrm{C/m^2}$$

Polarization vector:

$$\mathbf{P} = \varepsilon_0 \chi_e \mathbf{E} \quad \mathrm{C/m^2}$$

Boundary Conditions for Dielectric Materials

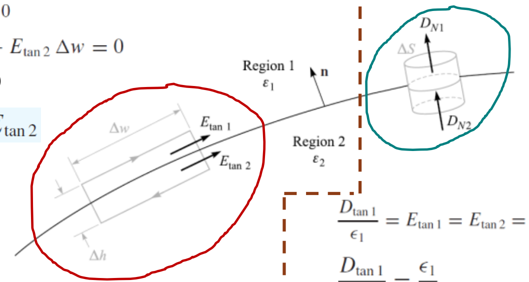
Dielectric – dielectric

$$\oint \mathbf{E} \cdot d\mathbf{L} = 0$$

$$E_{\tan 1}\, \Delta w - E_{\tan 2}\, \Delta w = 0$$

$$\mathbf{E} \bullet \Delta \mathbf{h} \approx 0$$

$$E_{\tan 1} = E_{\tan 2}$$



$$\frac{D_{\tan 1}}{\epsilon_1} = E_{\tan 1} = E_{\tan 2} = \frac{D_{\tan 2}}{\epsilon_2}$$

$$\frac{D_{\tan 1}}{D_{\tan 2}} = \frac{\epsilon_1}{\epsilon_2}$$

For perfect dielectrics, no free charge is on the interface:

$$D_{N1} = D_{N2}$$

$$\epsilon_1 E_{N1} = \epsilon_2 E_{N2}$$

On the normal direction:

$$D_{N1}\Delta S - D_{N2}\Delta S = \Delta Q = \rho_S \Delta S$$

$$D_{N1} - D_{N2} = \rho_S$$

Boundary Conditions for Dielectric Materials

Dielectric – dielectric

Because the normal components of **D** are continuous,

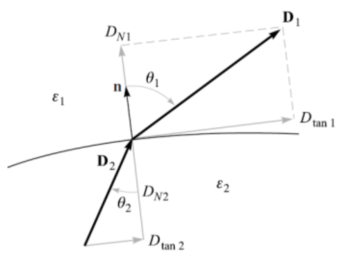
$$D_{N1} = D_1 \cos\theta_1 = D_2 \cos\theta_2 = D_{N2}$$

$$\frac{D_{\tan 1}}{D_{\tan 2}} = \frac{D_1 \sin\theta_1}{D_2 \sin\theta_2} = \frac{\epsilon_1}{\epsilon_2}$$

$$\frac{\tan\theta_1}{\tan\theta_2} = \frac{\epsilon_1}{\epsilon_2}$$

$$D_2 = D_1 \sqrt{\cos^2\theta_1 + \left(\frac{\epsilon_2}{\epsilon_1}\right)^2 \sin^2\theta_1}$$

$$E_2 = E_1 \sqrt{\sin^2\theta_1 + \left(\frac{\epsilon_1}{\epsilon_2}\right)^2 \cos^2\theta_1}$$



D is larger in the region of larger permittivity
E is larger in the region of smaller permittivity

Capacitors and Capacitance

•A capacitor is a device which stores electric charge.

•Capacitors vary in shape and size, but the basic configuration is two conductors carrying equal but opposite charges.

•Capacitors have many important applications in electronics:

- ➢ Storing electric potential energy,
- ➢Filtering out unwanted frequency signals,
- ➢…

•Physically, capacitance is a measure of the capacity of storing electric charge for a given potential difference.

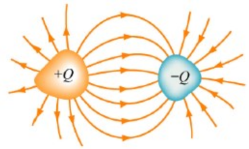
Basic Configuration of a Capacitor

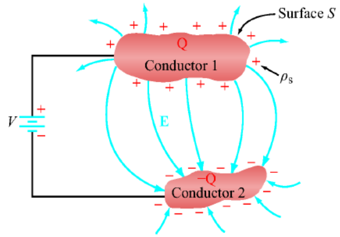
✓Any two conductors separated by an insulator (or a vacuum) form a Capacitor.

✓In most practical applications, each conductor initially has zero net charge.

✓Electrons can be transferred from one conductor to the other, called charging the capacitor.

✓Whether charged or uncharged, the net charge on the capacitor as a whole is zero.





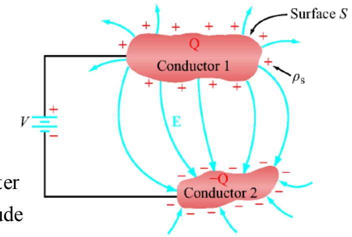
A DC voltage source connected to a capacitor composed of two conducting bodies.

Capacitance

When charging a capacitor, the amount of charge Q stored in the capacitor is linearly proportional to the electric potential difference between the two conductors:

$$Q = C|\Delta\varphi| = CV$$

where C is a positive proportionality constant called **capacitance.**



Capacitance is a measure of the ability of a capacitor to store energy. The greater the capacitance, the greater the magnitude of charge on either conductor for a given potential difference.

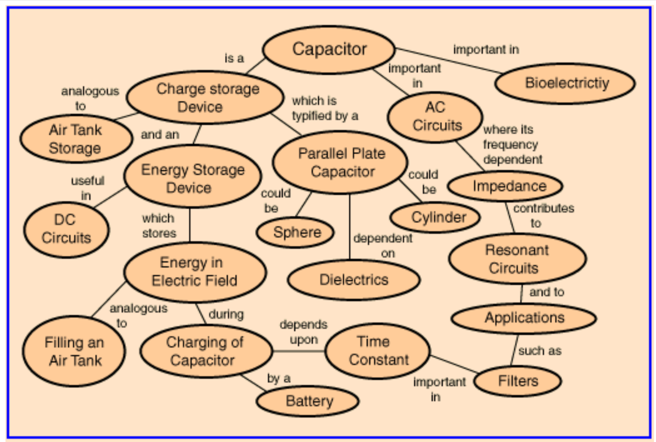
Units of Capacitance

* Definition of capacitance: $C = \dfrac{Q}{V}$

* SI units: Farad (F) = Coulomb/Volt

* Remember:
  - ∘ 1 Coulomb is a BIG charge ⇒ 1F is a BIG capacitance
  - ∘ The most common subunits of capacitance in use today are: millifarad (mF), microfarad (μF), nanofarad (nF) and picofarad (pF)
    $1\,\text{F} = 10^3\,\text{mF} = 10^6\,\mu\text{F} = 10^9\,\text{nF} = 10^{12}\,\text{pF}$
  - ∘ Don't confuse the symbol C for capacitance (in italics) and abbreviation C for coulombs.

Capacitors Are Everywhere



The Prototypical Capacitor

Physical Configuration

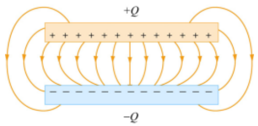
Physical configuration :

* Two parallel plates, each of area A, separated by a distance d
* Deposit $+Q$ on top plate and $-Q$ on bottom plate

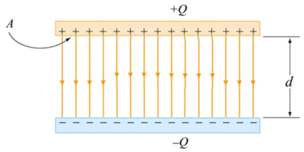
•A real capacitor is finite in size.

•Edge effects: The electric field lines at the edge of the plates are not straight lines, and the field is not contained entirely between the plates.

•Fringing fields: The non-uniform fields near the edge.



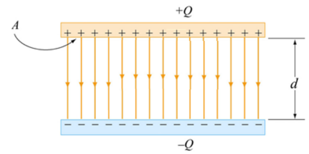
If $d^2 << A \Rightarrow$ infinite parallel planes.
The electric field in the region between the plates: $E = \dfrac{\rho_S}{\varepsilon}$



The Prototypical Capacitor

Capacitance Calculation

* The potential difference between the plates:



$$\Delta\varphi = -\int_{+}^{-} \mathbf{E} \bullet d\mathbf{l} = -Ed$$

The path of integration : straight line from the positive plate to the negative plate.
The sign is immaterial. The magnitude of the potential difference between the two plates: $|\Delta\varphi| = Ed$

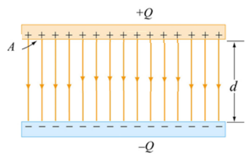
* From the definition : $C = \dfrac{Q}{|\Delta\varphi|} = \dfrac{\rho_S A}{Ed} = \dfrac{\rho_S A}{\dfrac{\rho_S}{\varepsilon} d} = \dfrac{\varepsilon A}{d}$

- C depends only on the geometric factors A and d.
  - C increases linearly with the area A.
  - C is inversely proportional to d.

The Prototypical Capacitor

Size of a 1-F Capacitor

A parallel-plate capacitor in free space has a capacitance of 1.0 F. If the plates are 1.0 mm apart, what is the area of the plate?



Solution

From $C = \dfrac{\varepsilon_0 A}{d} \Rightarrow A = \dfrac{Cd}{\varepsilon_0}$ where $\varepsilon_0 = 8.85\times10^{-12}$ F/m

Then $A = \dfrac{Cd}{\varepsilon_0} = \dfrac{1.0\times(1.0\times10^{-3})}{8.85\times10^{-12}} = 1.1\times10^{8}\ \text{m}^2$

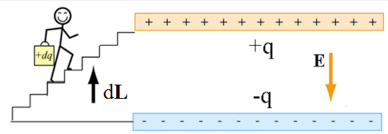
** This corresponds to a square about 10 km on a side!

➢The permittivity in free space is too small!

➢1-F capacitor is a relatively big capacitor!

Energy Stored in a Capacitor

1. Capacitor starts uncharged.
2. Carry $+dq$ from bottom to top. Now top has charge $q = +dq$, bottom $-dq$
3. Repeat
4. Finish when top has charge $q = +Q$, bottom $-Q$



At some point top plate has $+q$, bottom has $-q$ with the potential difference : $\Delta\varphi = \dfrac{q}{C}$

Work done to lift dq from the bottom to top: $dW = dq\Delta\varphi = \dfrac{q}{C}dq$

So work done to move Q from bottom to top: $W = \int dW = \dfrac{1}{C}\int_0^Q q\,dq = \dfrac{1}{C}\dfrac{Q^2}{2}$

The total energy stored: $U = W = \dfrac{1}{C}\dfrac{Q^2}{2} = \dfrac{|\Delta\varphi|}{Q}\dfrac{Q^2}{2} = \dfrac{1}{2}Q|\Delta\varphi| = \dfrac{1}{2}C|\Delta\varphi|^2 = \dfrac{1}{2}CV^2$

$\boxed{\mathrm{U} = \dfrac{1}{2}CV^2}$ ← Energy Stored in a Capacitor

Energy Stored in a Capacitor

Where is energy stored?

In a parallel plate capacitor: $C = \frac{\varepsilon A}{d}$ and $|\Delta\varphi| = Ed$

Then: $U_E = \frac{1}{2}C|\Delta\varphi|^2 = \frac{1}{2}\left(\frac{\varepsilon A}{d}\right)(Ed)^2 = \frac{1}{2}\varepsilon E^2(Ad)$

Energy is stored in the electric field.

Since the quantity Ad represents the volume between the plates, we can define the ***electric energy density*** as:

$$u_E = \frac{U_E}{\text{Volume}} = \frac{1}{2}\underline{\underline{\varepsilon E^2}} \quad \text{J/m}^3$$

The electric energy density depends on the two factors.

Even though this expression was derived for a parallel-plate capacitor, it is valid for any dielectric medium containing an electric field **E**, include vacuum.

Today

- **Conductors**
  Basic properties in electrostatics
- **Dielectrics**

Book: Engineering Electromagnetics
Chapters 4&5

- **Capacitors and Capacitance (1)**

Book: Engineering Electromagnetics
Chapter 6

Next

- Capacitors and Capacitance
- Lab 1 Tutorial

Thanks for your attendance