MTH101: Lecture 8

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Cauchy's Integral Formula

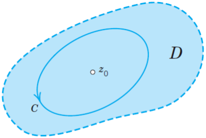
Theorem

If $f(z)$ is **Analytic** *in a* **Simply Connected Domain** *D, then for any point $z_0 \in D$ and any conterclockwise orinted simple closed path γ that encloses the point z_0 we have*

$$\oint_\gamma \frac{f(z)}{z - z_0}\, dz = 2\pi i f(z_0),$$

or

$$f(z_0) = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{z - z_0}\, dz.$$

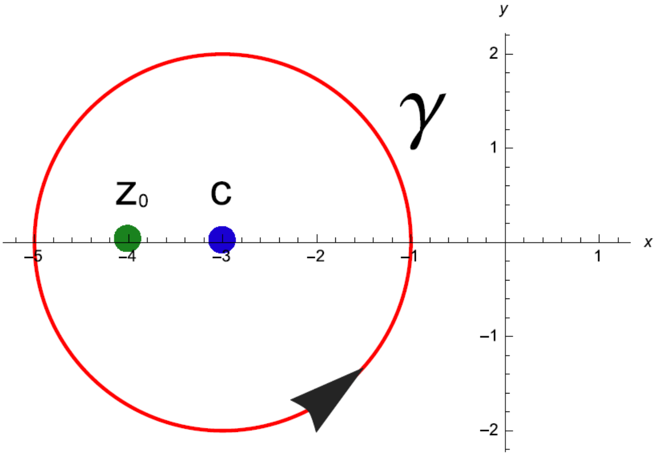


Example

Compute the following integral:

$$\oint_{\gamma} \frac{e^{2z}}{z+4}\, dz,$$

where γ is the circle with center $c = -3$ and radius $R = 2$ with counterclockwise orientation.


y
2
1
−1
−2
x
−5
−4
−3
−2
−1
1
γ
z_0
c

Solution:

The function $f(z) = e^{2z}$ is **Entire**, thus is **Analytic** in $\mathbb{C}$ (**Simply Connected**), while γ is a **counterclockwise simple closed path** in $\mathbb{C}$.

The point $z_0 = -4$ is in the **interior** of γ (that is γ encloses z_0).

Then we can use **Cauchy's Integral Formula**

$$\oint_\gamma \frac{f(z)}{z - z_0}\, dz = 2\pi i f(z_0),$$

where $f(z) = e^{2z}$, $z_0 = -4$.

Therefore,

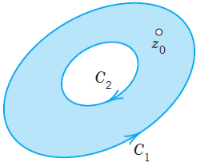
$$\oint_\gamma \frac{e^{2z}}{z + 4}\, dz = 2\pi i f(-4) = 2\pi i\ e^{-8}.$$

Multiply Connected Domains - Application 1

Theorem

If $f(z)$ is analytic on C_1 and C_2 and on the ring shaped domain bounded by C_1 and C_2 (see figure below) and z_0 is any point in that domain, then

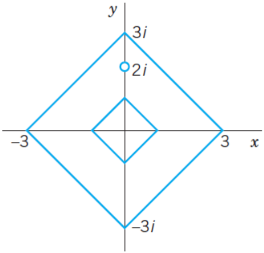
$$\oint_{C_1} \frac{f(z)}{z - z_0}\, dz + \oint_{C_2} \frac{f(z)}{z - z_0}\, dz = 2\pi i\ f(z_0).$$



Example

$$\oint_C \frac{\sin z}{4z^2 - 8iz}\, dz,$$

C consists of the boundaries of the squares with vertices ± 3, $\pm 3i$ counterclockwise and ± 1, $\pm i$ clockwise. (See figure below)



Solution:
Let $4z^2 - 8iz = 0$ we have two "bad" points $z = 0$ and $z = 2i$, only $z = 2i$ is in the "ring" shaped doubly connected domain.
We let

$$f(z) = \frac{\sin z}{4z}, \quad z_0 = 2i,$$

and observe that $f(z)$ is **Analytic** both in the interior and on the boundary of the ring. The outer boundary is counterclockwise and the inner is clockwise.
Therefore,

$$\oint_C \frac{\sin z}{4z^2 - 8iz}\, dz = \oint_C \frac{\frac{\sin z}{4z}}{z - 2i}\, dz = 2\pi i \left[\frac{\sin z}{4z}\right]_{z_0 = 2i}$$

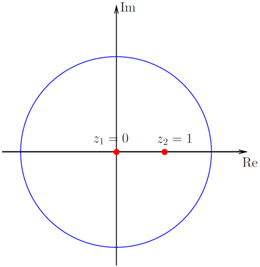
$$= 2\pi i\, \frac{\sin(2i)}{8i} = \frac{\pi i}{4} \sinh 2.$$

Multiply Connected Domains - Application 2

Example

$$\oint_C \frac{z-2}{z^2-z}\, dz,$$

where C is a circle with center $c = 0$ and radius $R = 2$ counterclockwise.



Solution:

Let $z^2 - z = 0$ we obtain the two "bad" points $z_1 = 0$ and $z_2 = 1$, both are enclosed by C.

Let C_1 be circle centered at z_1 counterclockwise that only encloses z_1, and C_2 be a circle centered at z_2 counterclockwise that only encloses z_2, then

$$\begin{aligned}\oint_C \frac{z-2}{z^2-z}\,dz &= \oint_{C_1} \frac{\frac{z-2}{z-1}}{z}\,dz + \oint_{C_2} \frac{\frac{z-2}{z}}{z-1}\,dz \\ &= 2\pi i \left[\frac{z-2}{z-1}\right]_{z=0} + 2\pi i \left[\frac{z-2}{z}\right]_{z=1} \\ &= 2\pi i.\end{aligned}$$

Alternatively, we write

$$\frac{z-2}{z^2-z} = \frac{2}{z} - \frac{1}{z-1}.$$

Then,

$$\begin{aligned}
\oint_C \frac{z-1}{z^2-z}\, dz &= \oint_C \frac{2}{z}\, dz - \oint_C \frac{1}{z-1}\, dz \\
&= 2\pi i \cdot 2 - 2\pi i \cdot 1 \\
&= 2\pi i.
\end{aligned}$$

Theorem

If $f(z)$ is analytic in a domain D, then for any point $z_0 \in D$, any counterclockwise oriented simple closed path γ that encloses z_0 and whose interior belongs to D and any positive integer n, we have

$$\oint_\gamma \frac{f(z)}{(z-z_0)^{n+1}}\, dz = \frac{2\pi i}{n!} f^{(n)}(z_0),$$

or

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_\gamma \frac{f(z)}{(z-z_0)^{n+1}}\, dz.$$

Moreover, $f(z)$ has derivatives of all order in D, which are also analytic functions in D.

Application 1 - evaluate the line integrals

Example

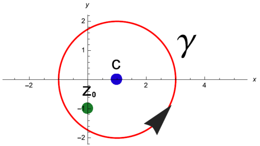
Compute the following integral:

$$\oint_{\gamma} \frac{\sin z + \cos z}{(z+i)^3}\, dz$$

where γ is the circle with center $c = 1$ and radius $R = 2$ with counterclockwise orientation.

Solution:

Let $(z+i)^3 = 0$ we have $z = -i$, which is in the interior of γ.



We observe that $f(z) = \sin z + \cos z$ is **Entire**, thus **Analytic** in the interior and on the boundary of γ which is **simple closed path**. Then we use **Cauchy's integral Formula for derivatives**:

$$\oint_\gamma \frac{f(z)}{(z-z_0)^{n+1}}\, dz = \frac{2\pi i}{n!} f^{(n)}(z_0),$$

where $f(z) = \sin z + \cos z$, $z_0 = -i$ and $n = 2$.
Therefore,

$$\begin{aligned}\oint_\gamma \frac{\sin z + \cos z}{(z+i)^3}\, dz &= \frac{2\pi i}{2!}\left[\sin z + \cos z\right]'' |_{z_0=-i} \\ &= \pi i \left[-\sin z - \cos z\right]_{z_0=-i} \\ &= -\pi i[\sin(-i) + \cos(-i)] \\ &= -\pi i[\cosh 1 - i \ \sinh 1]\end{aligned}$$

Example

Compute the integral

$$\oint_{\gamma} \frac{dz}{z^2(z-2)}$$

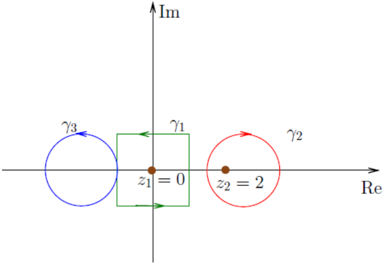
along the following paths:

γ_1: square with vertices $1 \pm i$ and $-1 \pm i$ counterclockwise;

γ_2: circle with center $c_2 = 2.5$ and radius $R_2 = 1$ clockwise;

γ_3: circle $|z+2| = 1$ counterclockwise.

Solution:



Let $z^2(z-2) = 0$ we have $z_1 = 0$ and $z_2 = 2$.

1 The counterclockwise simple closed path γ_1 encloses $z_1 = 0$ and the function $f_1(z) = \frac{1}{z-2}$ is analytic both in the interior and on the boundary of γ_1.
We use Cauchy's integral formula for derivatives:

$$\oint_{\gamma_1} \frac{dz}{z^2(z-2)} = \oint_{\gamma_1} \frac{\frac{1}{z-2}}{z^2}\, dz = \frac{2\pi i}{1!}\left[\frac{1}{z-2}\right]'_{z_1=0}$$

$$= 2\pi i \left[-\frac{1}{(z-2)^2}\right]_{z_1=0}$$

$$= -\frac{\pi i}{2}$$

2 The clockwise(!) simple closed path γ_2 encloses $z_2 = 2$ and the function $f_2(z) = \frac{1}{z^2}$ is analytic both in the interior and one the boundary of γ_2.
Thus we use Cauchy's integral formula (according to counterclockwise path $-\gamma_2$):

$$\oint_{\gamma_2} \frac{dz}{z^2(z-2)} = -\oint_{-\gamma_2} \frac{\frac{1}{z^2}}{z-2}\,dz = -2\pi i\left[\frac{1}{z^2}\right]_{z_2=2}$$
$$= -\frac{\pi i}{2}$$

3 The counterclockwise simple closed path γ_3 encloses no "bad" points and the function $f(z) = \frac{1}{z^2(z-2)}$ is analytic both in the interior and one the boundary of γ_3.
Thus we use Cauchy's integral theorem

$$\oint_{\gamma_3} \frac{dz}{z^2(z-2)} = 0.$$

Application 2-Cauchy's Inequality

Remark

Let $f(z)$ be analytic function in a domain D, and C be a circle if radius r and center z_0 whose full interior belongs to D, with $|f(z)| \leq M$ on C
Using **ML-inequality** *we have*

$$|f^{(n)}(z_0)| = \left| \frac{n!}{2\pi i} \oint_{C_r} \frac{f(z)}{(z-z_0)^{n+1}}\, dz \right| \leq \frac{n!}{2\pi} \cdot \frac{M}{r^{n+1}} \cdot 2\pi r = \frac{n! M}{r^n}.$$

Bibliography

1 *Kreyszig, E.* **Advanced Engineering Mathematics**. Wiley, 9th Edition.