# EEE336 Signal Processing and Digital Filtering

## Lecture 3 Sampling and Reconstruction

## 3_1 Sampling Process

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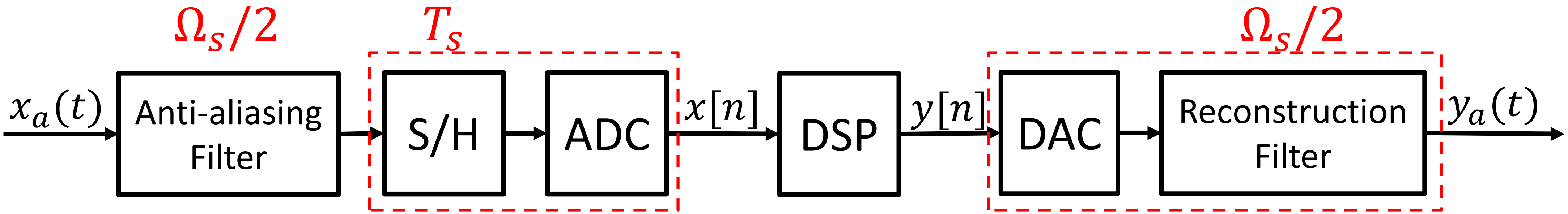
Room EE322

# *Digital Processing of CT Signals*

- Most signals in nature are continuous in time

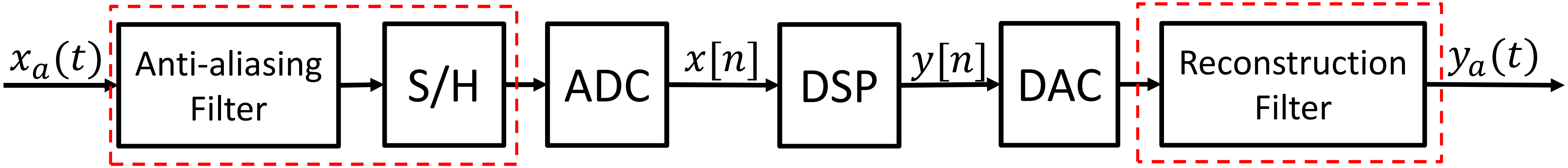
  => Need a way for "digital processing of continuous-time signals" => ***SAMPLING***!


(a) Ideal data flow for the digital processing of continuous-time signals




(b) Practical data flow for the digital processing of continuous-time signals

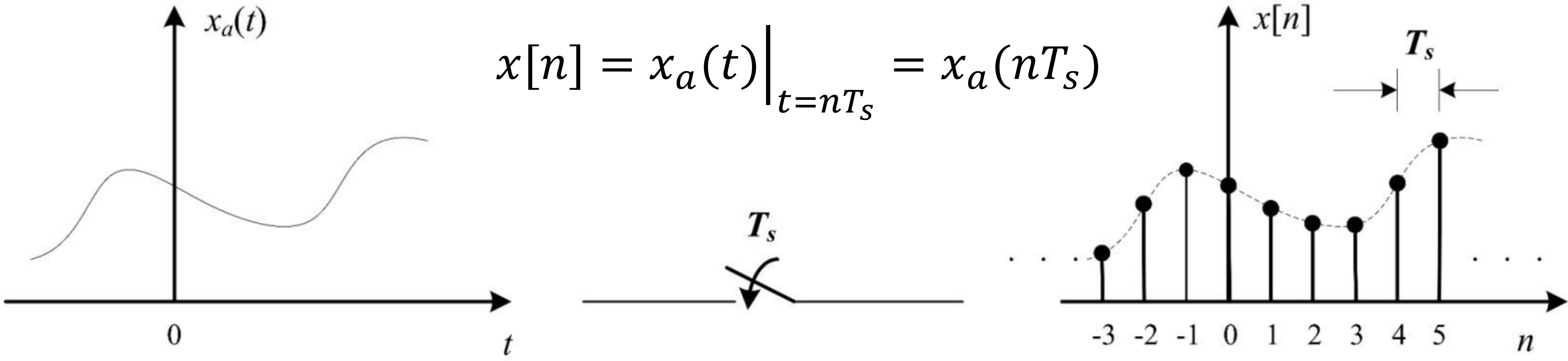
# *Analog -> Digital ->Analog*




- Conversion of the continuous-time signal into a discrete-time signal
  - Anti-aliasing filter – to prevent potentially detrimental effects of sampling
  - Sample & Hold – discrete in time and keep the sampling values for a while to allow the A/D converter to do its job
  - ***Analog to Digital Converter (A/D) – conversion in amplitude***
- Processing of the discrete-time signal
  - Digital Signal Processing –Filter, digital processor
- Conversion of the processed discrete-time signal back into a cont.-time signal
  - ***Digital to analog converter (D/A) –to obtain the continuous signal***
  - Reconstruction / smoothing filter -smooth out the signal from the D/A

# Sampling in Time domain (TD)

- A discrete-time sequence is developed by uniformly sampling the continuous-time signal $x_a(t)$




$$x[n] = x_a(t)\Big|_{t=nT_s} = x_a(nT_s)$$

- The time variable - time $t$ is related to the discrete time variable $n$ only at discrete-time instants $t_n$

$$t_n = nT_s = \frac{n}{F_s} = \frac{2\pi n}{\Omega_s} \begin{cases} T_s = 1/F_s \text{ (Sampling period, second, second/sample)} \\ F_s = 1/T_s \text{ (Sampling frequency, Hz, cycles/second)} \\ \Omega_s = 2\pi/T_s \text{ (Sampling angular frequency, radian/second)} \end{cases}$$

# Sampling in Time domain (TD)

- Consider $x_a(t) = A\cos(\Omega_0 t + \phi)$
- Now $x[n] = A\cos(\Omega_0 n T_s + \phi)$

$$= A\cos(\frac{2\pi\Omega_0}{\Omega_s}n + \phi) = A\cos(\omega_0 n + \phi) = x_a(nT)$$

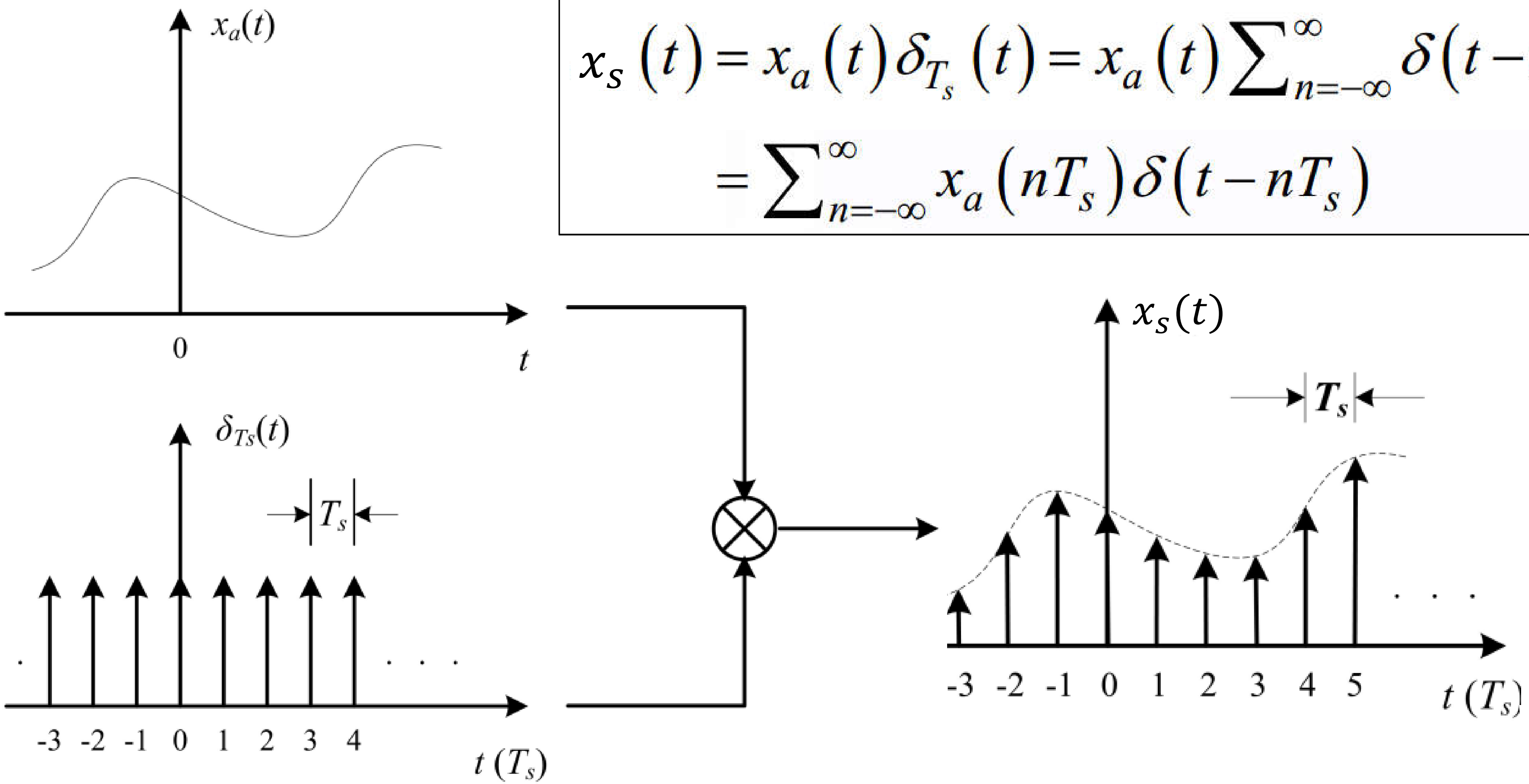
  – Where $\omega_0 = \frac{2\pi\Omega_0}{\Omega_s} = \Omega_0 T_s$

  – $\omega_0$ is the (normalized) digital angular frequency of the signal
    - Unit: radians/sample
  – $\Omega_0$ is the analog angular frequency of signal
    - Unit: radians/second
  – $\Omega_s$ is the sampling analog angular frequency
    - Unit: radians/second

# Sampling in Time domain (TD)

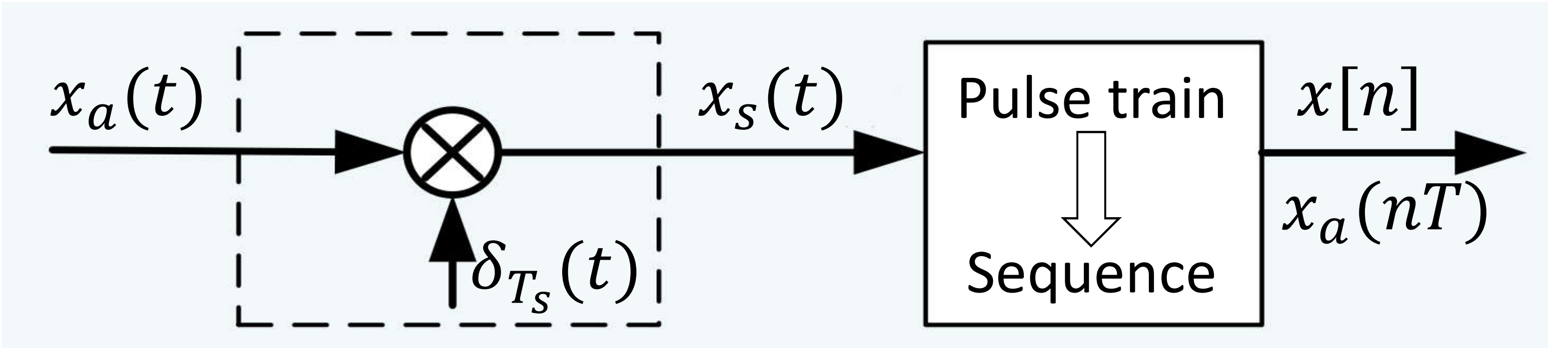
- In mathematics, the periodic sampling is modelled as the multiplication of continuous signal and impulse train

$$x_s(t) = x_a(t)\,\delta_{T_s}(t) = x_a(t)\sum_{n=-\infty}^{\infty}\delta(t-nT_s)$$
$$= \sum_{n=-\infty}^{\infty} x_a(nT_s)\,\delta(t-nT_s)$$

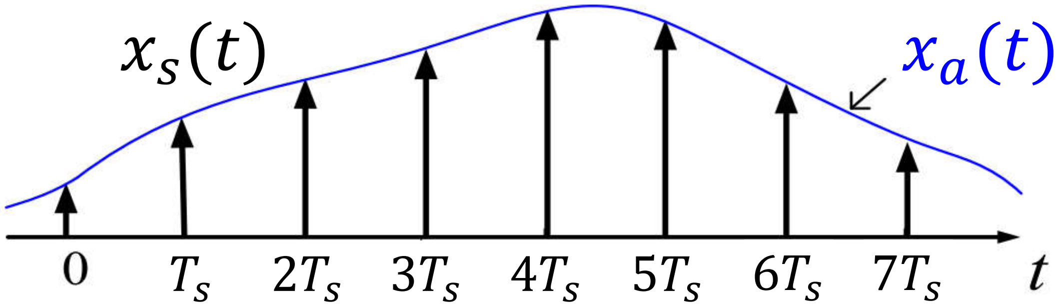


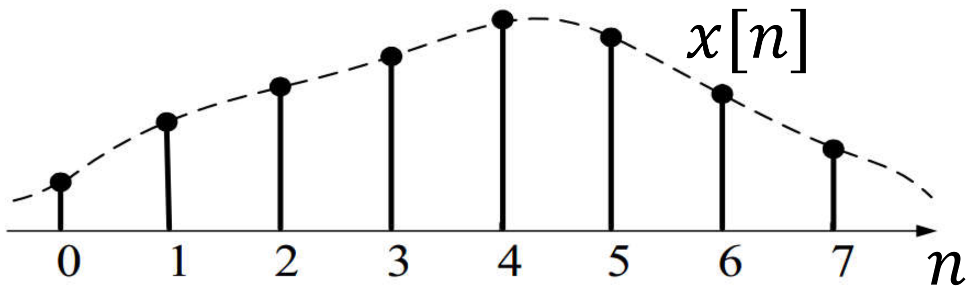
# Sampling in Time domain (TD)

- The system to convert the continuous-time (CT) signal $x_a(t)$ to a discrete-time (DT) signal $x[n]$ is shown:



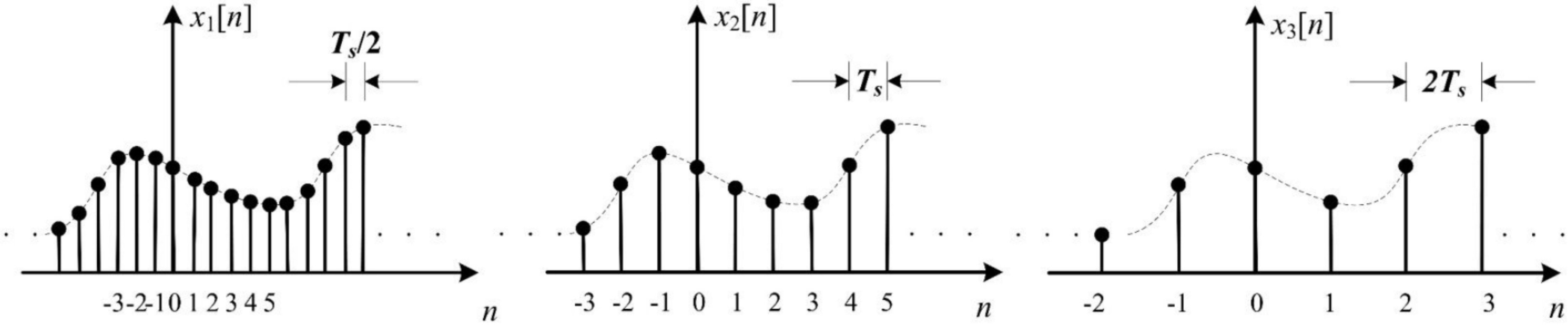

$$x_s(t) = x_a(t)\delta_{T_s}(t) = \sum_{n=-\infty}^{\infty} x_a(nT_s)\delta(t - nT_s)$$



# How signal changed after sampling

- In time domain, continuous -> discrete




- Different sampling rates, different details
  - More samples = higher sampling rate/frequency = more detail = more information kept = more resource occupation
  - Less samples = lower sampling rate/frequency = less details = more information loss = less resource occupation
- How to choose the sampling period/rate?

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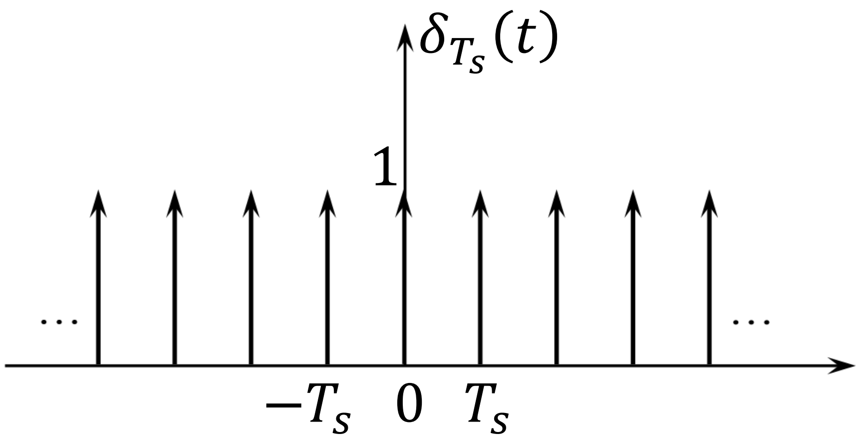
# Frequency domain analyses

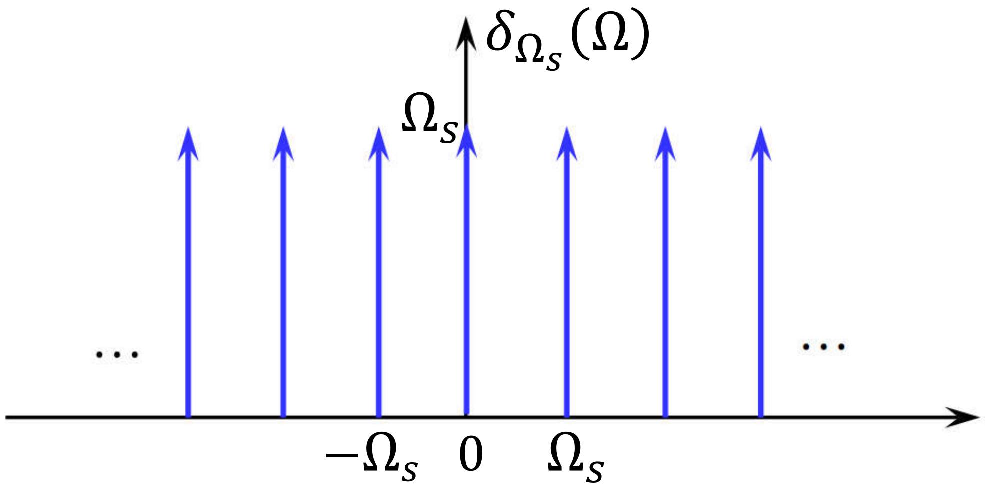
- Review: CTFT of a pulse train $\delta_{T_s}(t)$

$$\delta_{T_s}(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s) \quad \xLeftrightarrow{\text{CTFT}} \quad \delta_{\Omega_s}(\Omega) = \Omega_s \sum_{k=-\infty}^{\infty} \delta(\Omega - k\Omega_s)$$

$$\Omega_s = 2\pi f_s = \frac{2\pi}{T_s}$$




TD: Time domain




FD: Frequency domain

# Sampling in Frequency domain (FD)

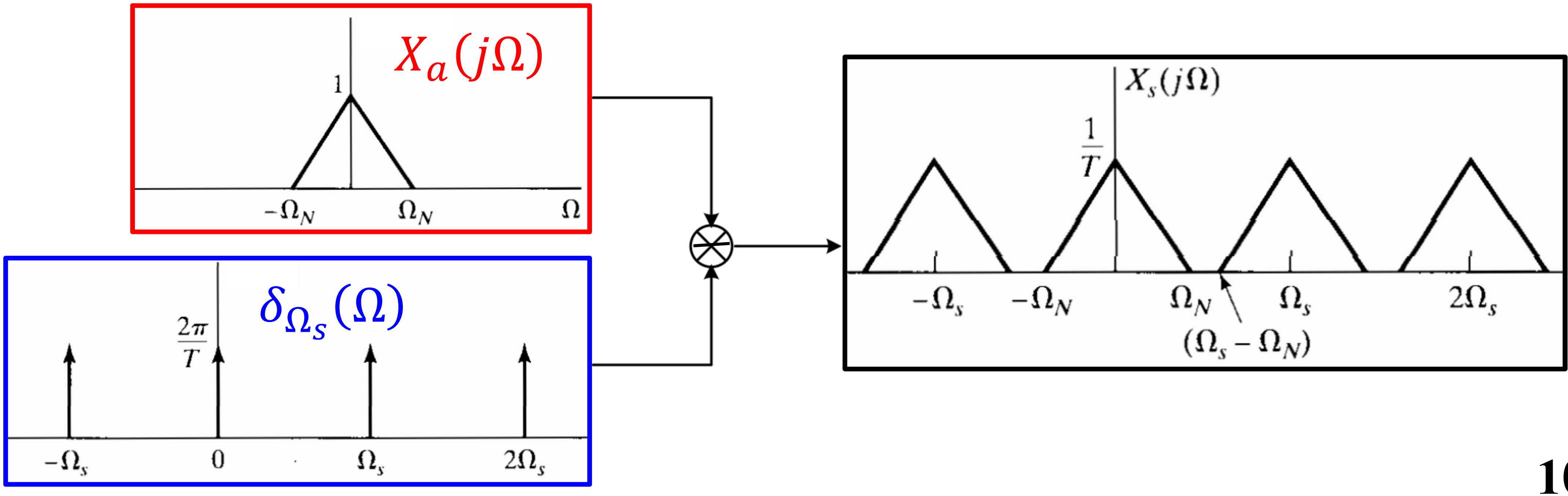
- In TD: multiplication between $x_a(t)$ and $\delta_{T_s}(t)$

$$x_s(t) = x_a(t) \cdot \delta_{T_s}(t)$$

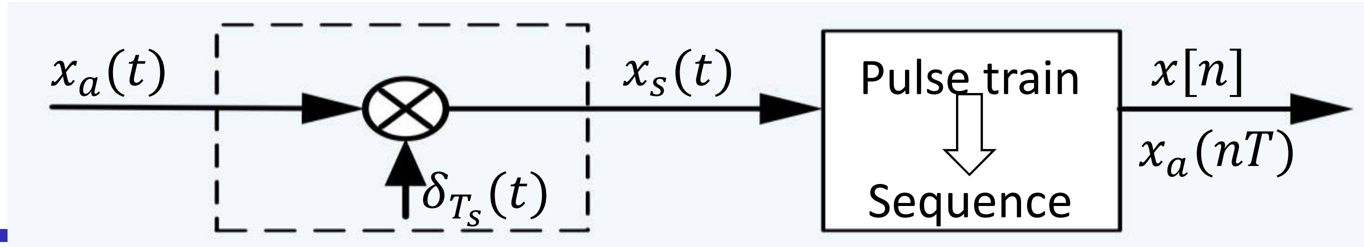
CTFT

- In FD: convolution between $X_a(j\Omega)$ and $\delta_{\Omega_s}(\Omega)$

$$X_s(j\Omega) = \frac{1}{2\pi} X_a(j\Omega) * \Omega_s \sum_{k=-\infty}^{\infty} \delta(\Omega - k\Omega_s) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X_a[j(\Omega - k\Omega_s)]$$



# Sampling in FD




- An alternative expression of $X_s(j\Omega)$ is:

$$x_s(t) = \sum_{n=-\infty}^{\infty} x_a(nT_s)\delta(t - nT_s)$$

$$\Big\Updownarrow \text{CTFT}$$

$$X_s(j\Omega) = \sum_{n=-\infty}^{\infty} x_a(nT_s)e^{-j\Omega T_s n}$$

Since: $$x[n] = x_a(nT_s)$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

$$X_s(j\Omega) = X(e^{j\omega})\Big|_{\omega=\Omega T_s} = X(e^{j\Omega T_s})$$

Recall

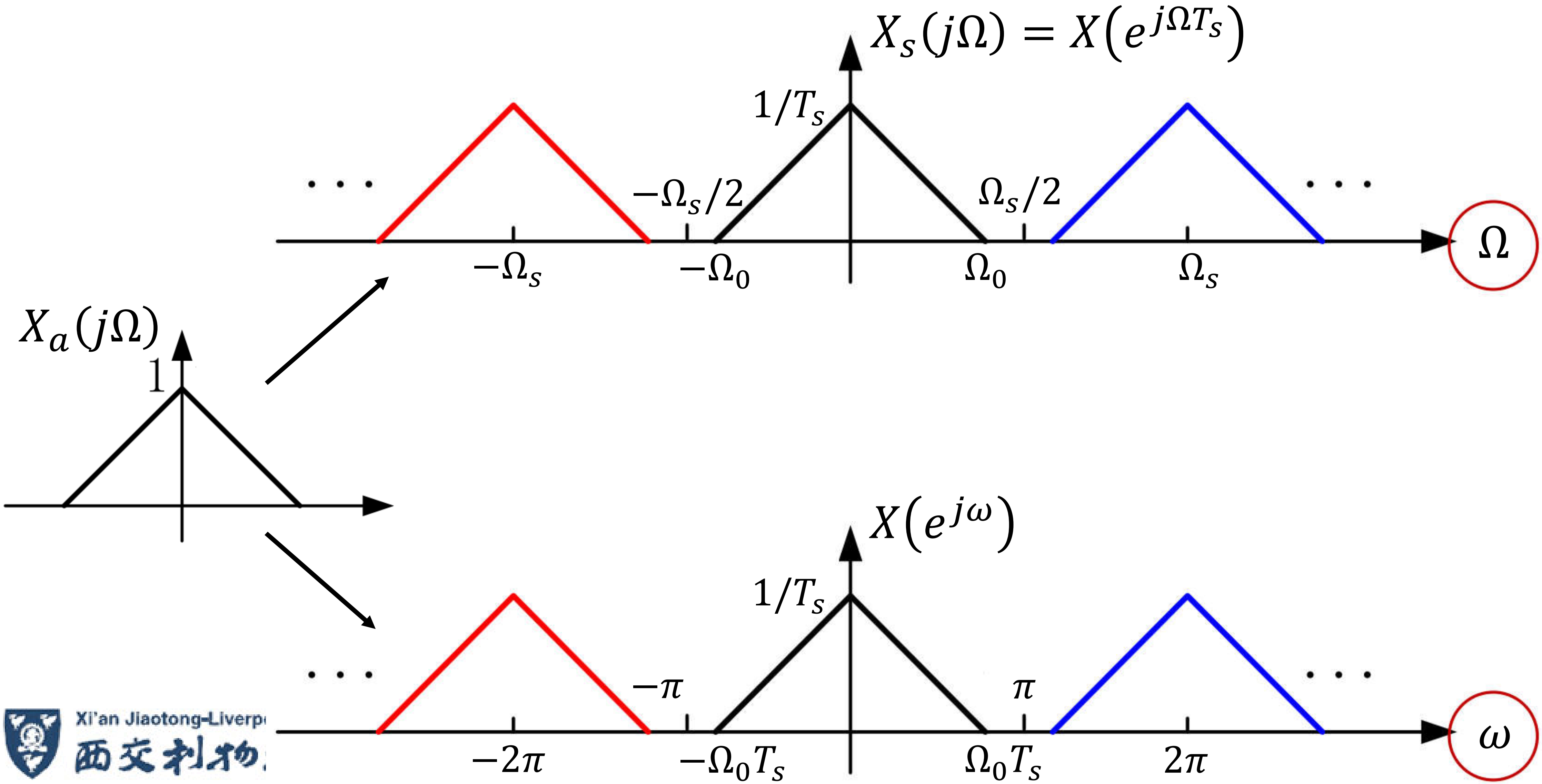
$$X_s(j\Omega) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X_a[j(\Omega - k\Omega_s)]$$

$$X(e^{j\Omega T_s}) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X_a[j(\Omega - k\Omega_s)]$$

$$X(e^{j\omega}) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X_a\left[j\left(\frac{\omega}{T_s} - \frac{2\pi k}{T_s}\right)\right]$$

# Sampling in Frequency domain (FD)

- Discretization in TD => Periodicity in FD

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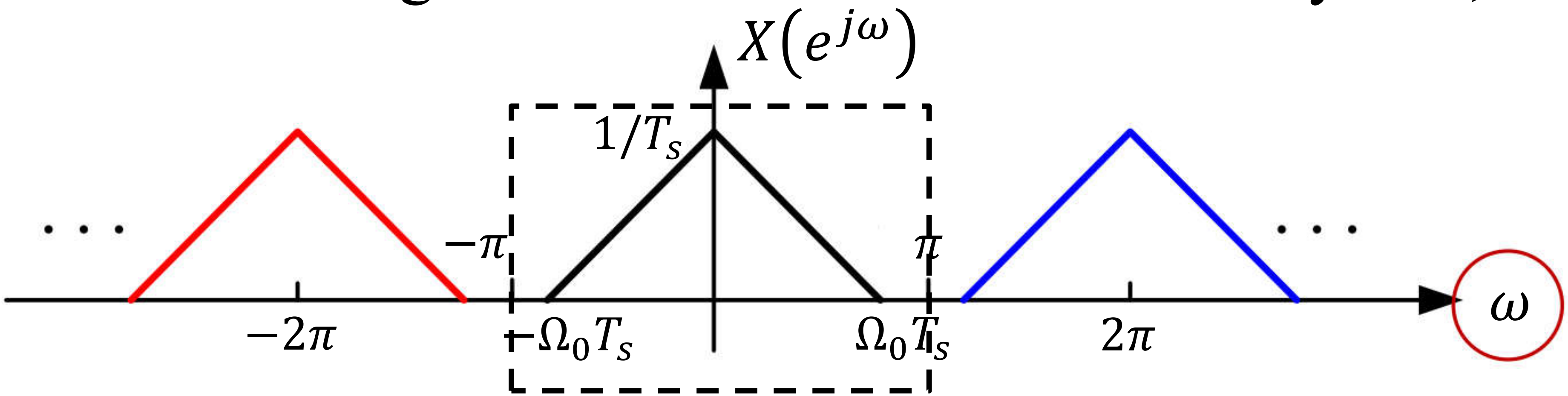
# Recovery of the CT signal

- The spectrum of the sampled signal contains all the information of the original CT signal
  - So the CT signal can be recovered without any loss;




  - But a condition needs to be satisfied:

$$\Omega_0 T_s \leq \pi \quad \Longleftrightarrow \quad 2\Omega_0 \leq \Omega_s$$

  - The Nyquist theorem!

# 3_1 Wrap up

- What is sampling process?
  – The first step to convert a continuous-time signal to a discrete-time signal;

- In time domain: multiplication the CT signal to a pulse train, then convert the modulated pulse train to sequence;
- In frequency domain: copy and shift (create infinite replica) the spectrum of the CT signal;
  – The CT signal can be recovered from the sampled signal if Nyquist theorem is satisfied, i.e., $2\Omega_0 \leq \Omega_s$

# EEE336 Signal Processing and Digital Filtering

## Lecture 3 Sampling and Reconstruction

## 3_2 Aliasing

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# An Example of Sampling

*In Time Domain*

- Three continuous-time signals are sampled at $T_s = 0.1\ s$

$$g_1(t) = \cos(6\pi t)$$

$$g_2(t) = \cos(14\pi t)$$

$$g_3(t) = \cos(26\pi t)$$

$$F_s = 10\ Hz$$

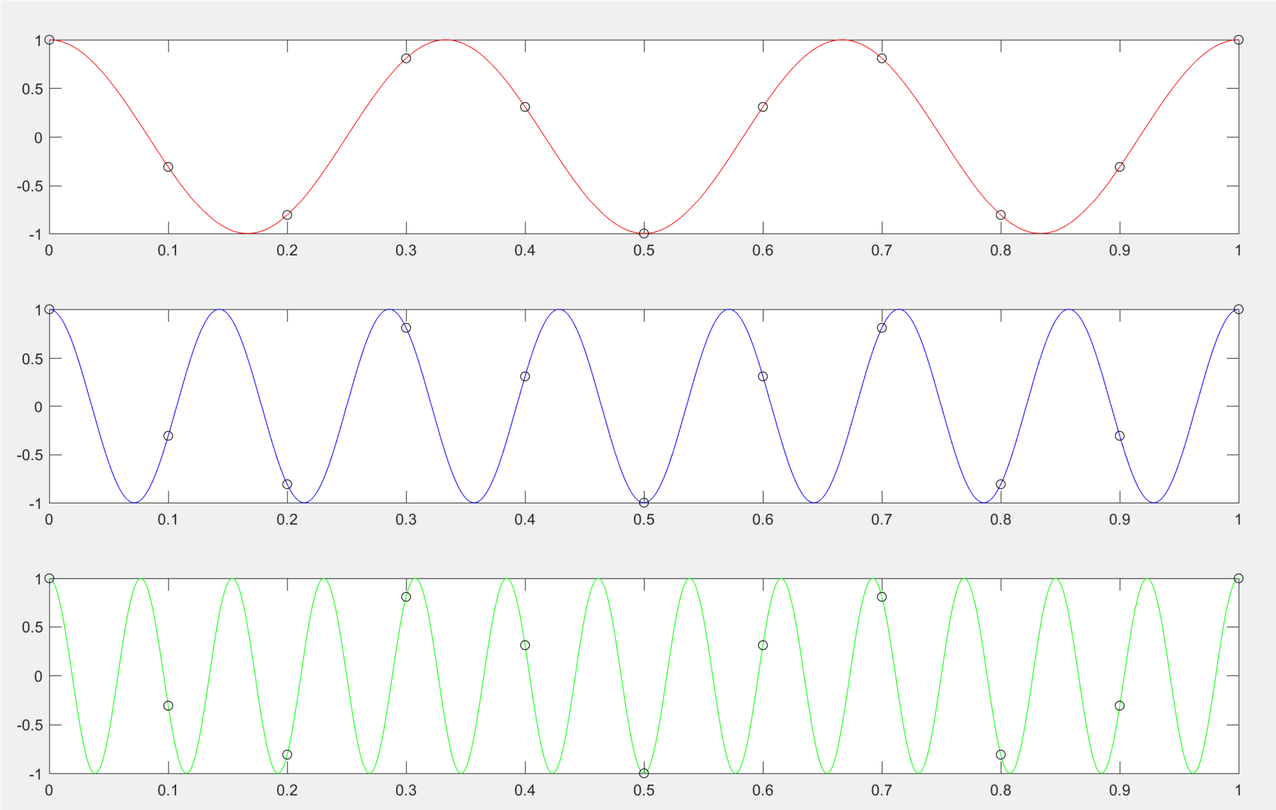
$$\Omega_s = 20\pi$$

- Generating the following sequence as shown in figure

$g_1(t) = \cos(6\pi t)$

$g_2(t) = \cos(14\pi t)$

$g_3(t) = \cos(26\pi t)$



# An Example of Sampling

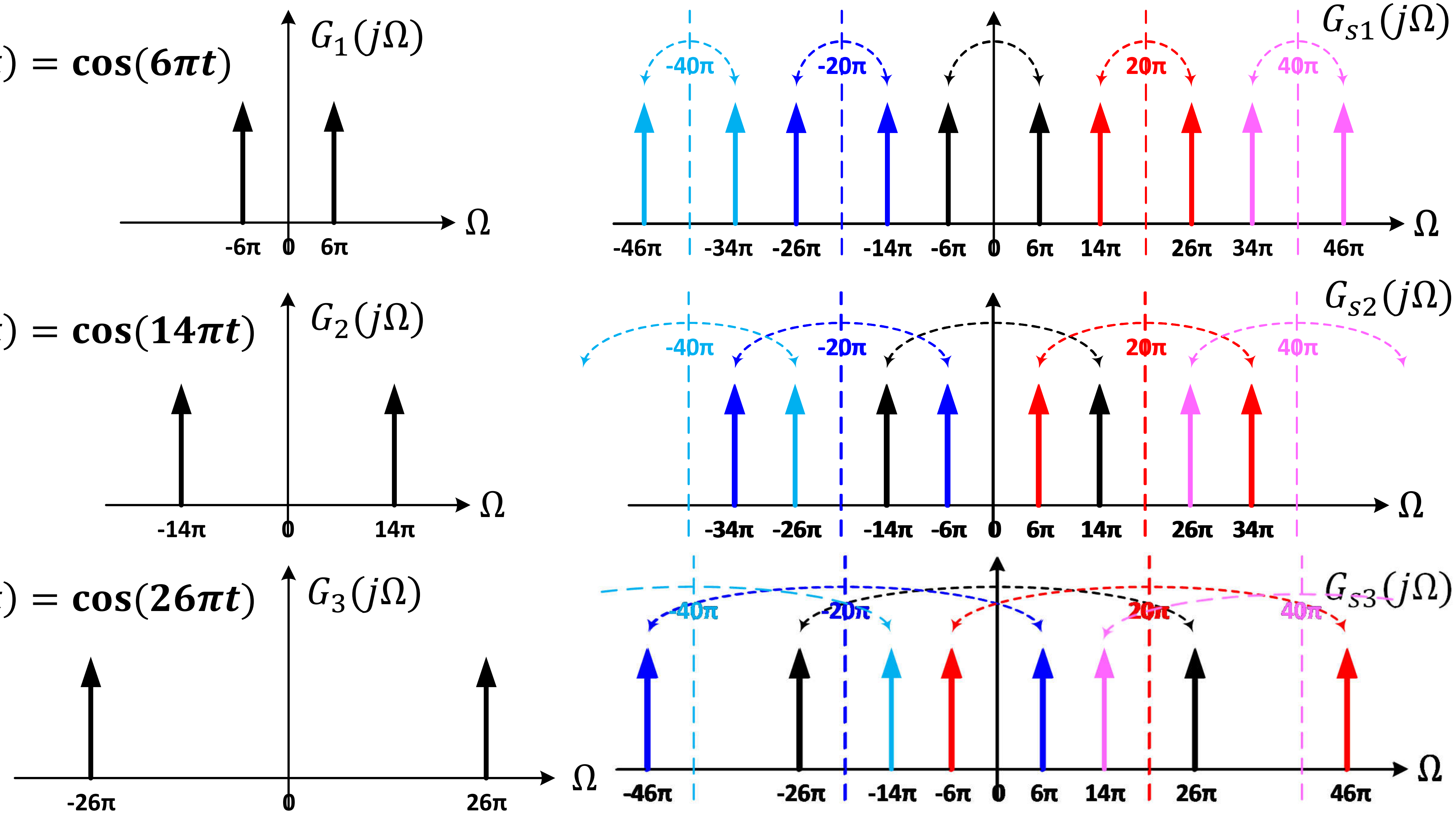
## *In Frequency Domain*

- The angular sampling frequency is $\Omega_s = 20\pi$

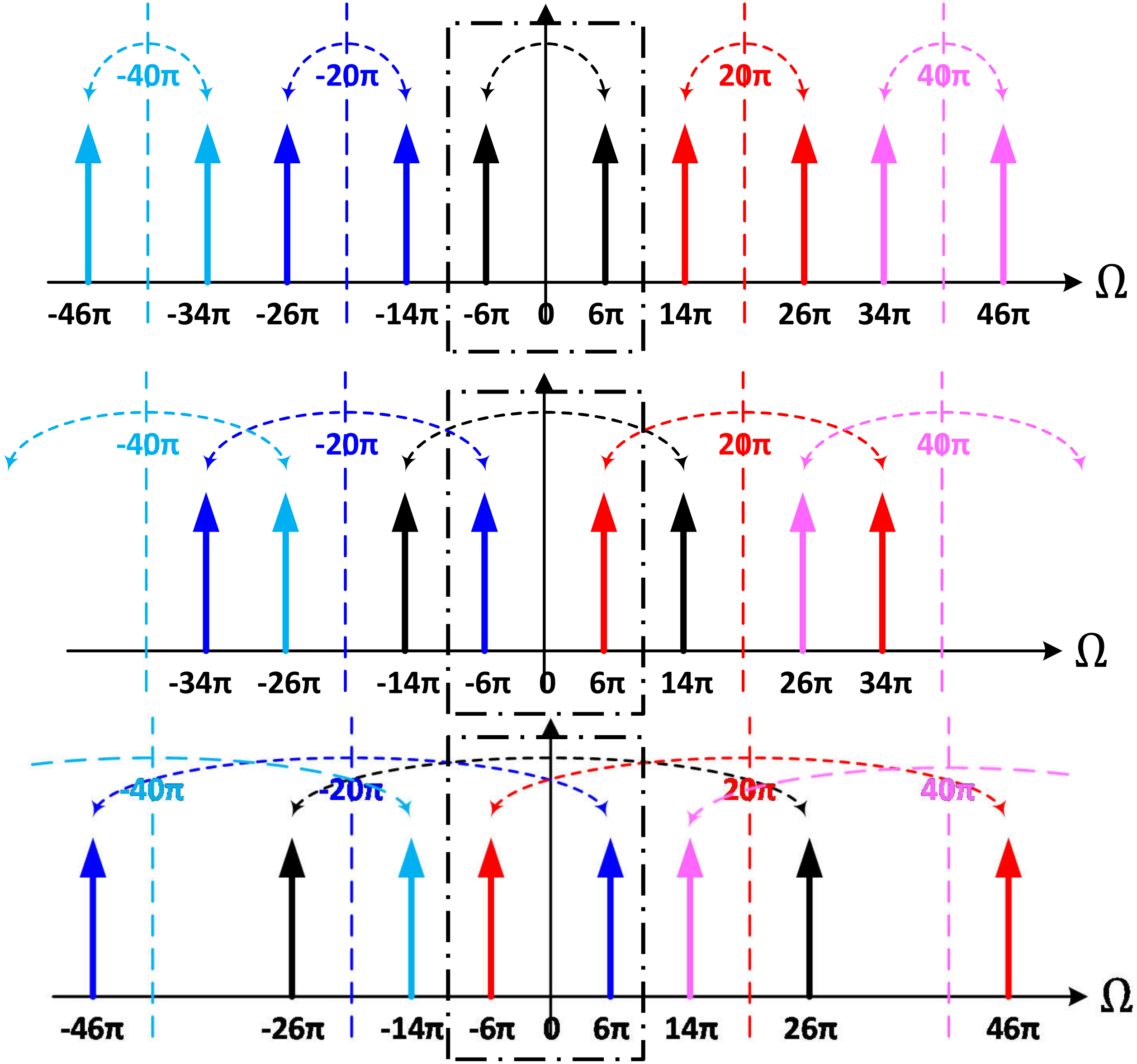
$$g_1(t) = \cos(6\pi t)$$

$$g_2(t) = \cos(14\pi t)$$

$$g_3(t) = \cos(26\pi t)$$



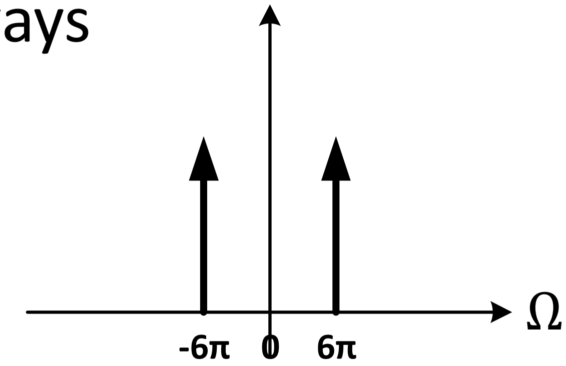
# To recover the CT signal




Use a lowpass filter with cutoff frequency at

$$\Omega_c = \Omega_s/2 = 10\pi$$

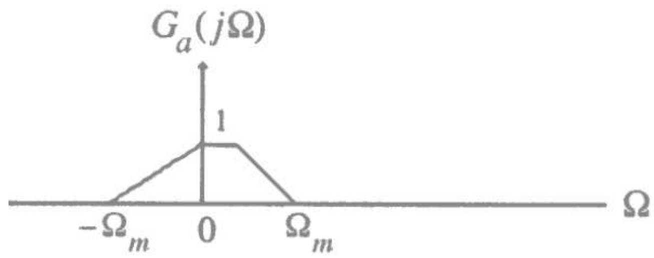
The signal filtered out is always




That is $g_1(t) = \cos(6\pi t)$ in time domain

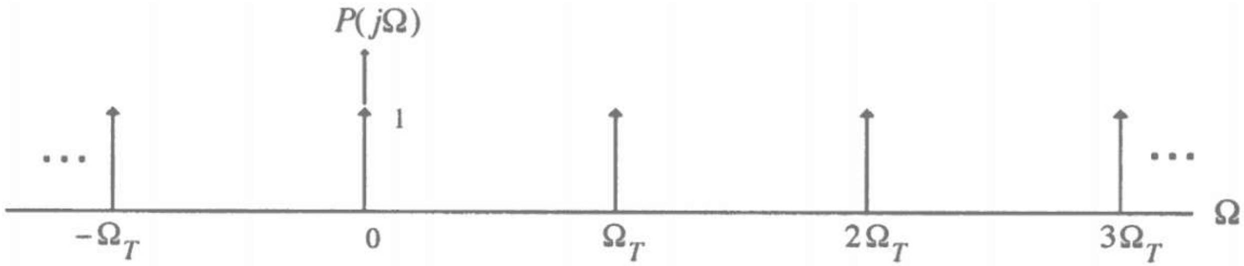
# Another Example of Sampling

- Assume $g_a(t)$ is a band-limited signal with a CTFT $G_a(j\Omega)$:

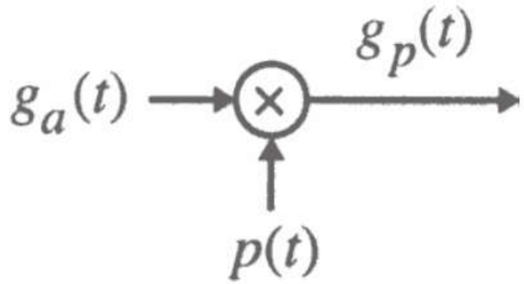



The highest frequency of $g_a(t)$ is $\Omega_m$

- The spectrum $P(j\Omega)$ of $p(t)$ having a sampling period $T = \frac{2\pi}{\Omega_T}$

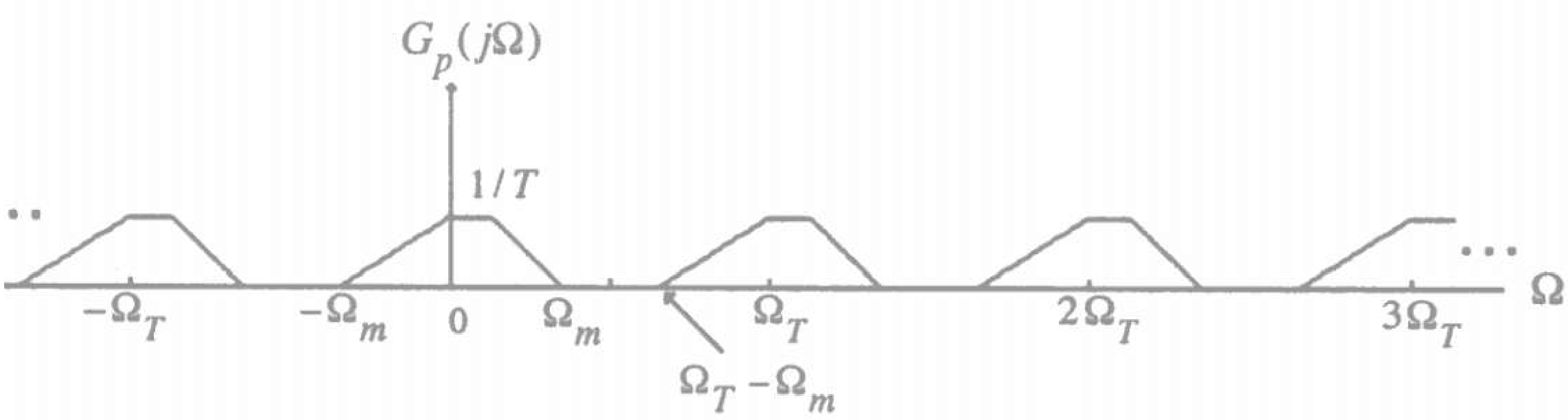



- Perform the sampling is to copy and paste $G_a(j\Omega)$ at every $\Omega_T$

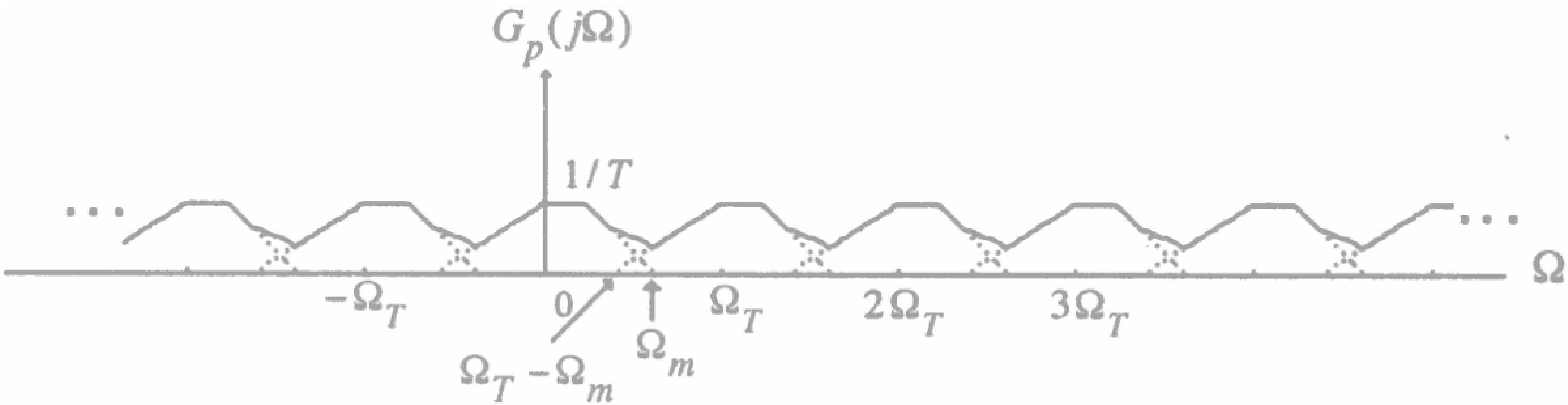


# Another Example of Sampling

- If $\Omega_m \leq \Omega_T - \Omega_m$




- If $\Omega_m \geq \Omega_T - \Omega_m$



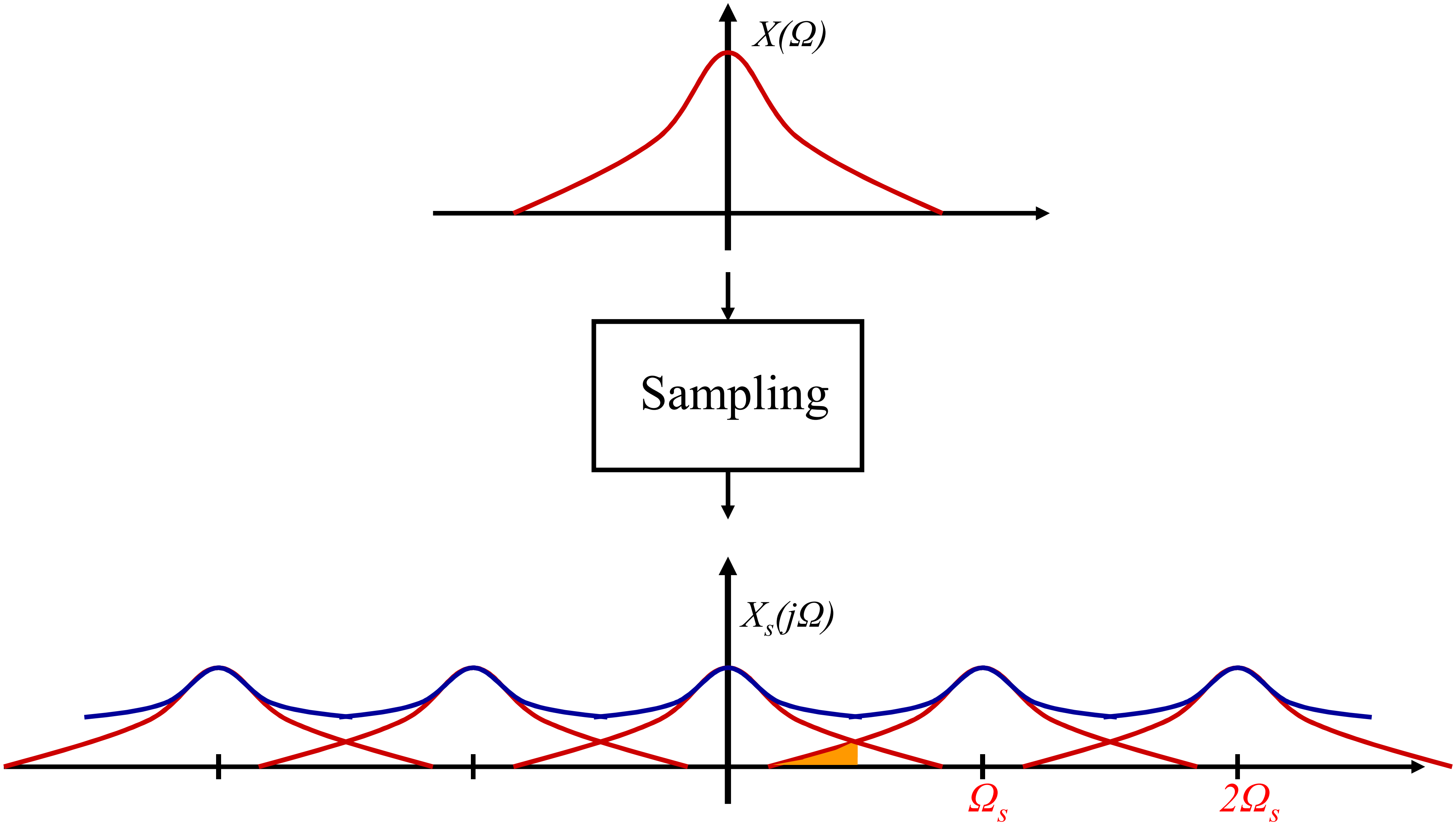
# Sampling Theorem

- The highest frequency $\Omega_m$ contained in the signal is called the Nyquist frequency since it determines the minimum sampling frequency $\Omega_T = 2\Omega_m$

- The frequency $\Omega_T /2$ is referred to as the folding frequency
  - Critical sampling corresponds to $\Omega_T = 2\Omega_m$
  - Oversampling corresponds to $\Omega_T >> 2\Omega_m$
  - Undersampling corresponds to $\Omega_T < 2\Omega_m$

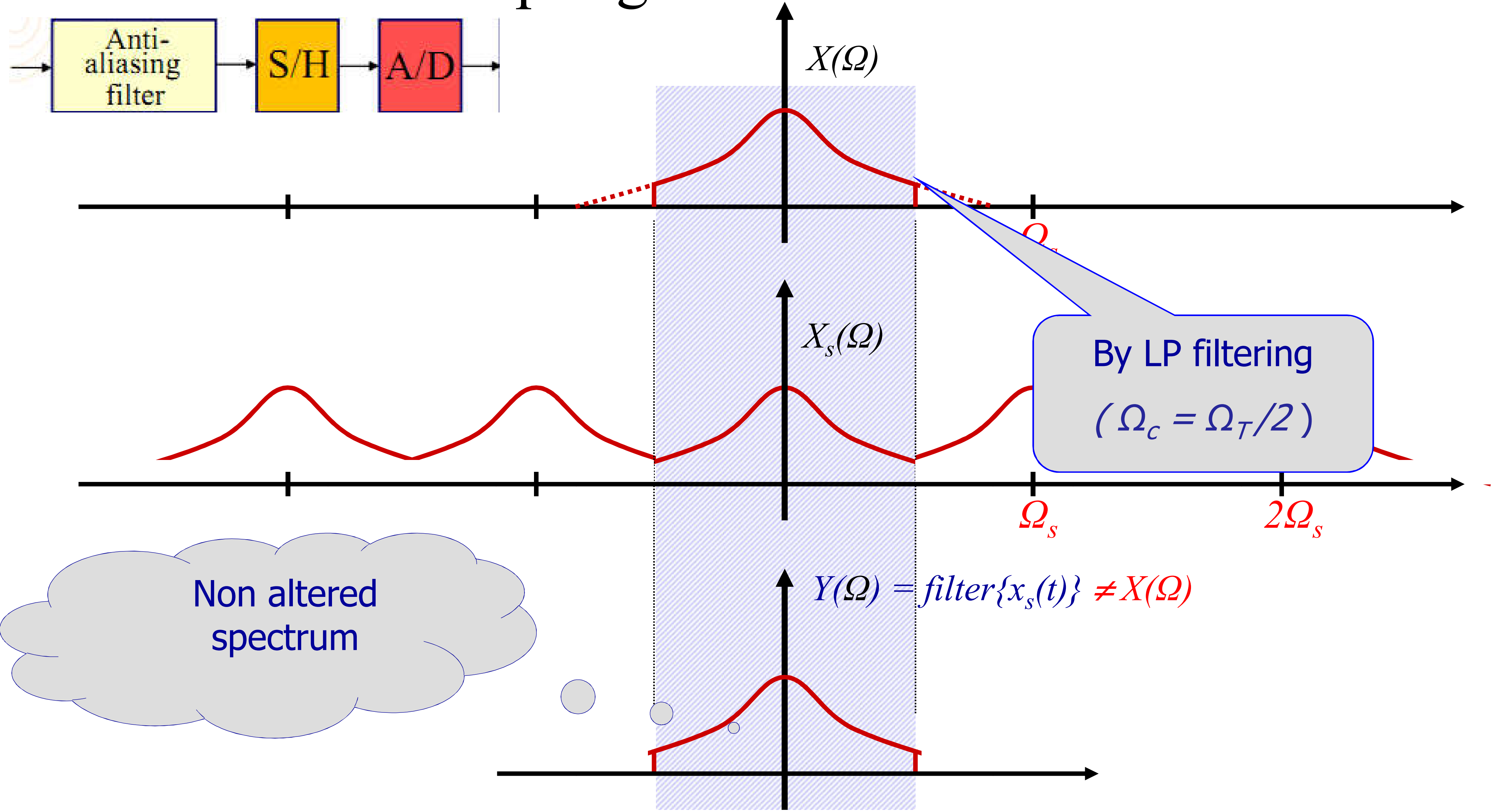
# Under-sampling

- Frequency folding error (aliasing)



# Anti-Aliasing Filter

- When the sampling rate is below the 2B

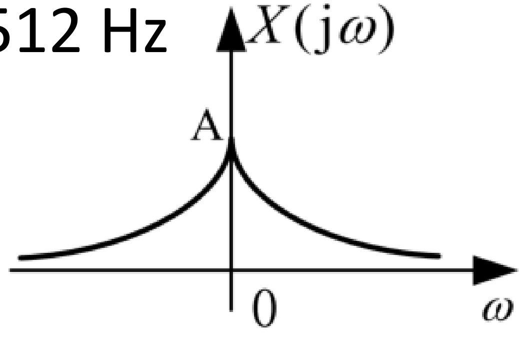


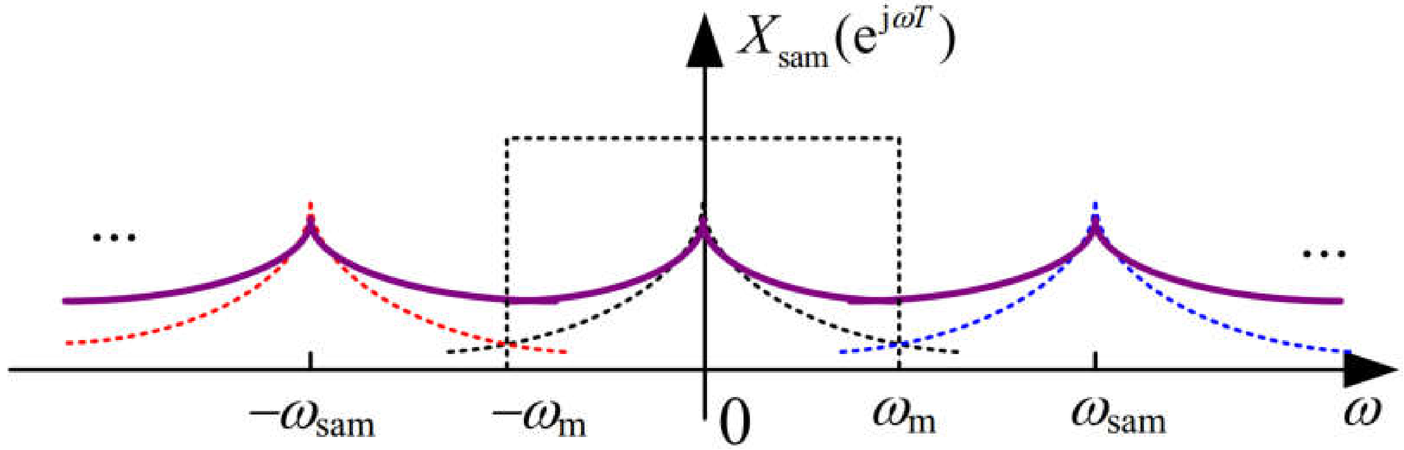
# *Anti-Aliasing Filter*

Original, Fs = 44100 Hz

Folding Error

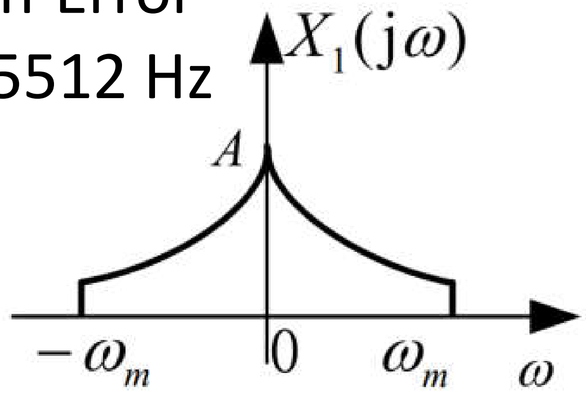
Fs = 5512 Hz

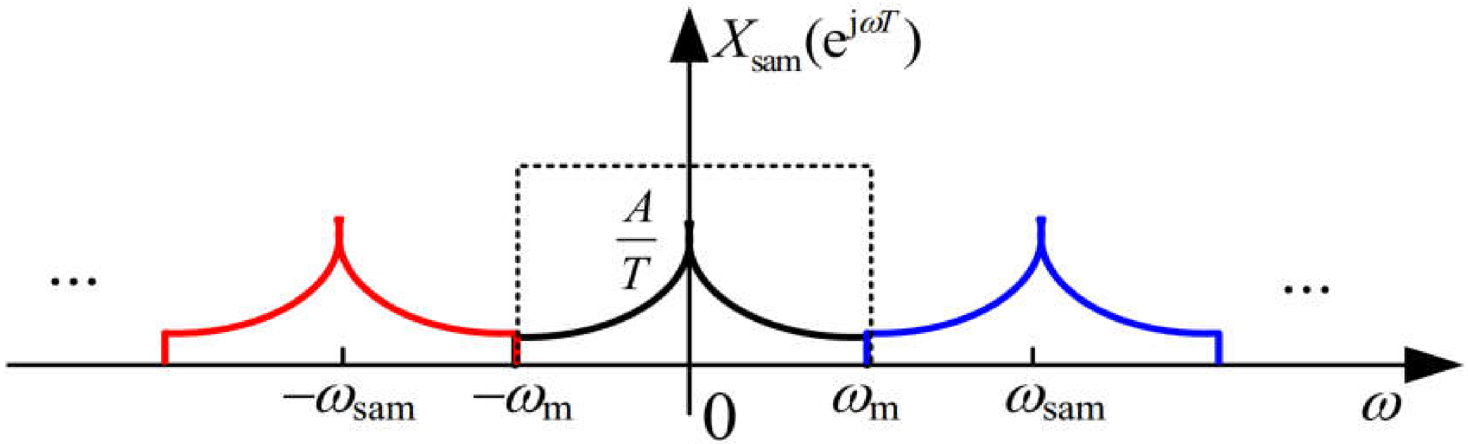



Cutoff Error

Fs = 5512 Hz



# Aliasing in Digital Images

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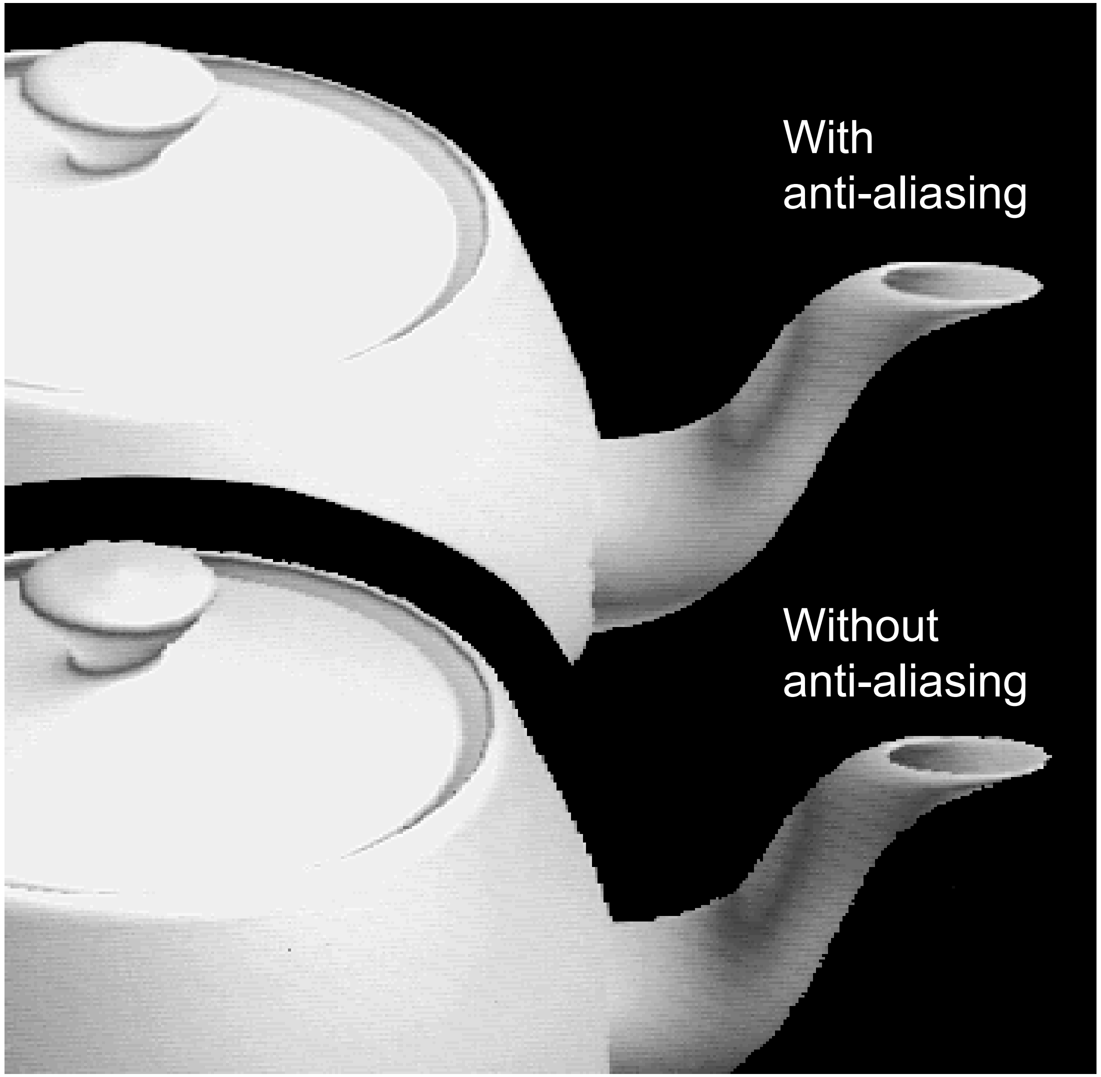
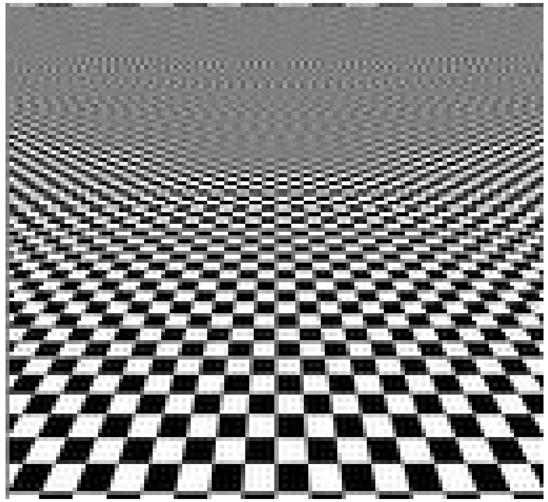
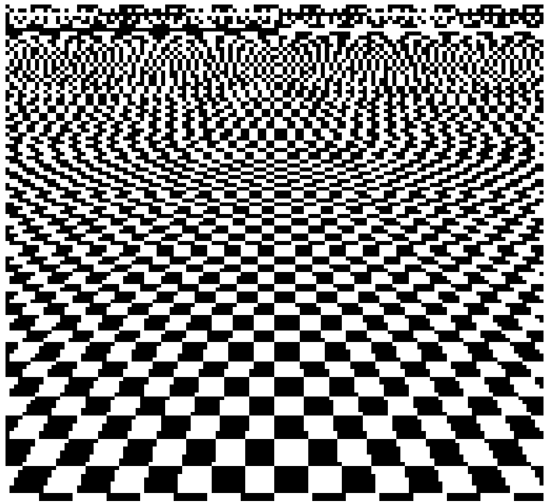


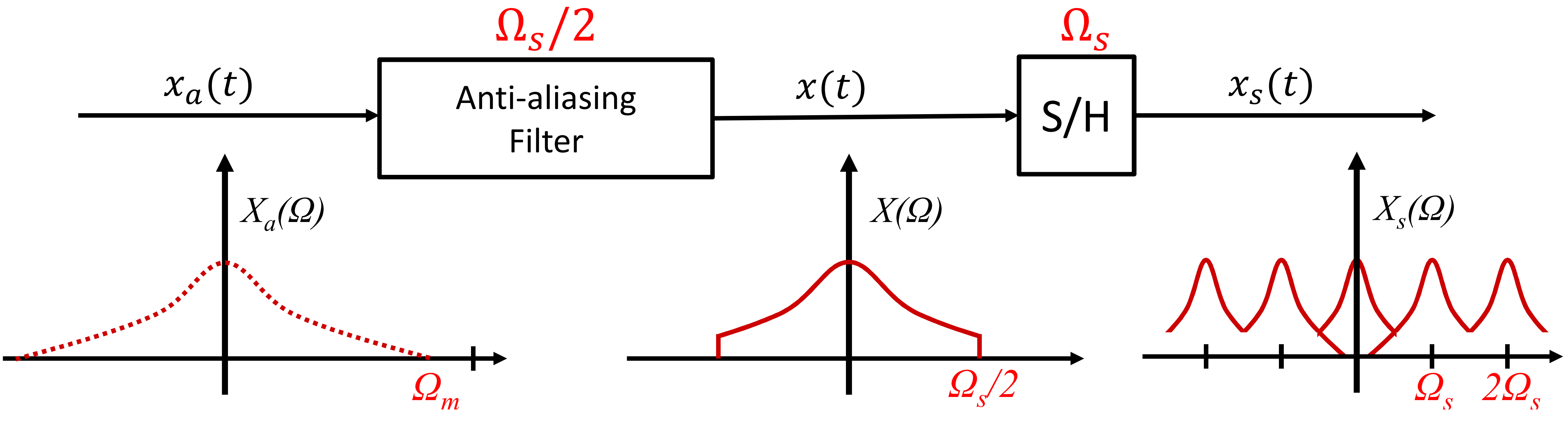


# 3_2 Wrap up

- Nyquist theorem: to avoid aliasing, the sampling frequency

$$\Omega_s \geq 2\Omega_{max}$$

- Three types of sampling: over sampling, critical sampling and under sampling
- Aliasing: If Nyquist theorem was not satisfied, aliasing happens
  – To reduce the aliasing error, passing the CT signal through an "Anti-aliasing filter" before sampling it.



# EEE336 Signal Processing and Digital Filtering

## Lecture 3 Sampling and Reconstruction

## 3_3 Interpolation / Reconstruction

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# Interpolation

- What is interpolation?
  - In the mathematical field of numerical analysis, interpolation is a method of ***constructing new data points*** within the range of a discrete set of known data points.
  - In this module, interpolation is a procedure whereby we convert a discrete-time (DT) sequence $x[n]$ to a continuous-time (CT) function $x(t)$.
  - Requirement: for the CT function x(t), its values at multiples of $T_s$ should be equal to the corresponding points of the DT sequence x[n]:

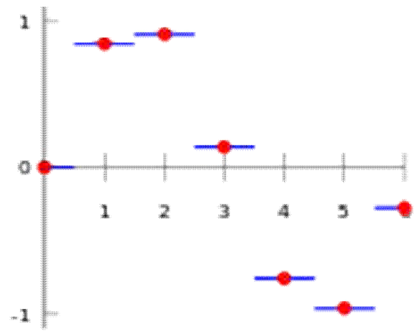
$$x(t)\Big|_{t=nT_s} = x[n]$$

The interpolation problem now reduces to "filling the gap" between these instants.

# Interpolation methods

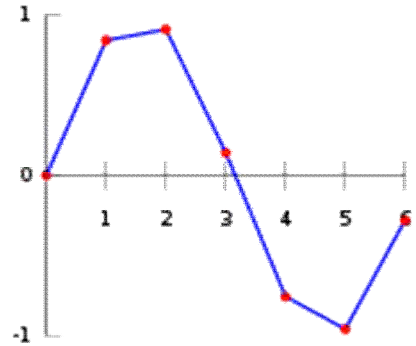
- 1. Zero-order/Local Interpolation

$$I_0(t) = \mathrm{rect}(t)$$

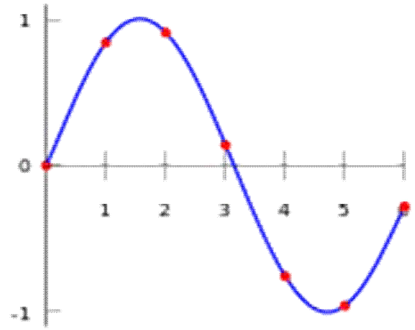



- 2. First-order/Linear Interpolation

$$I_1(t) = \begin{cases} 1 - |t| & \text{if } |t| < 1 \\ 0 & \text{otherwise} \end{cases}$$



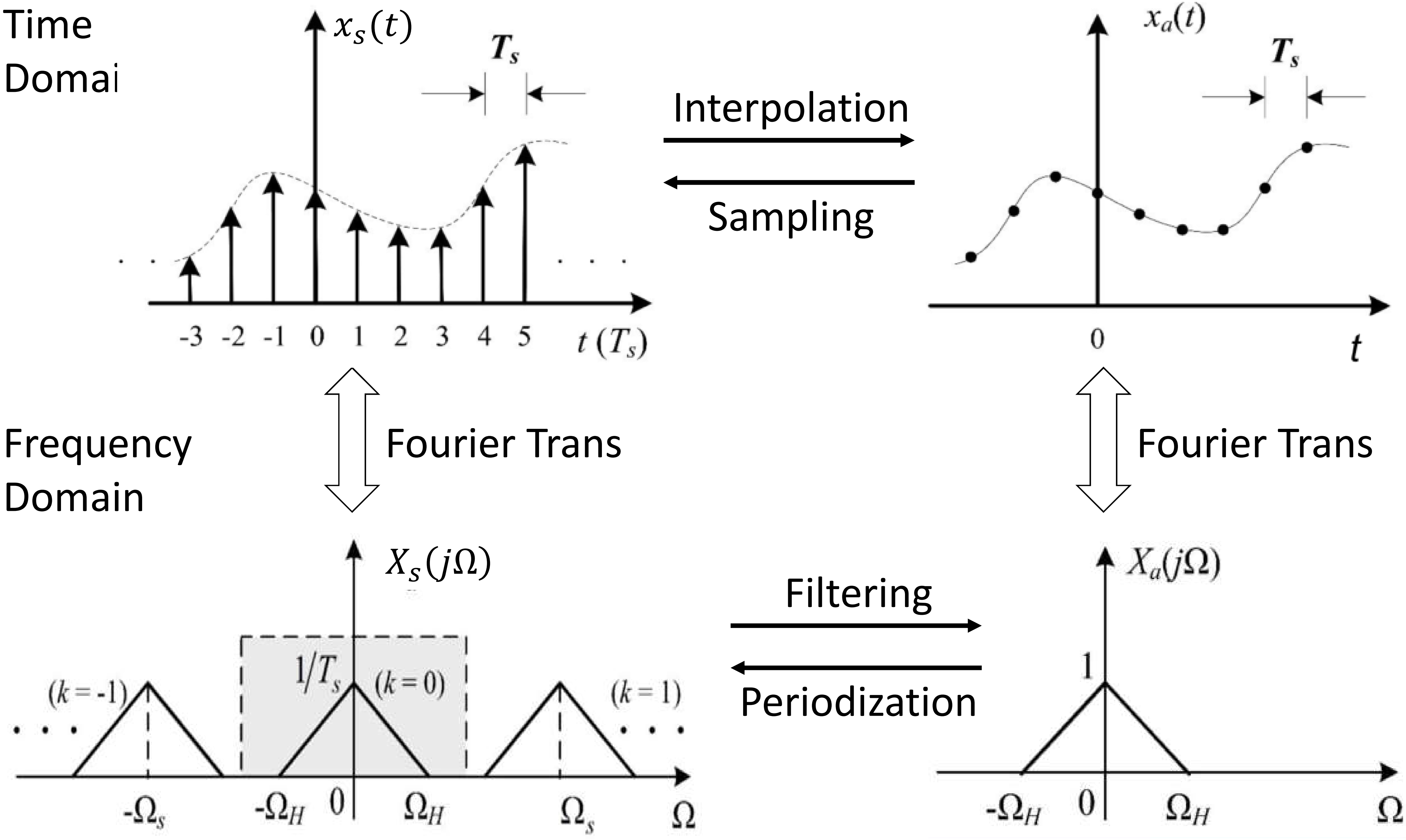

- 3. Higher-order/Polynomial Interpolation






Which one should we use in signal reconstruction?

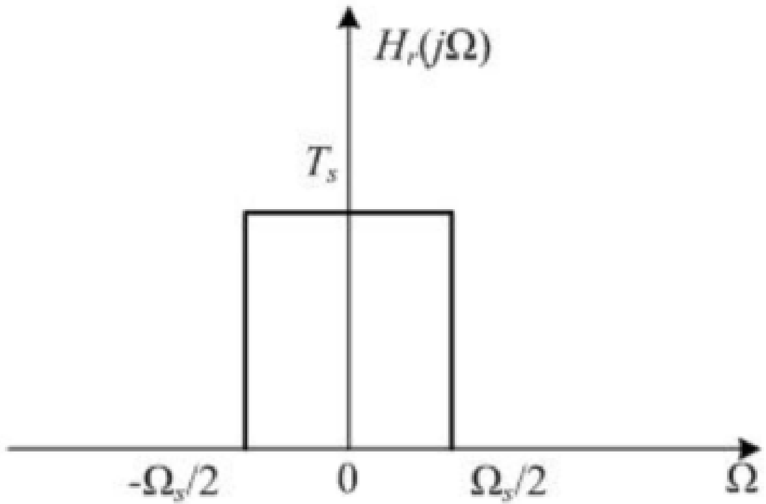
# Reconstruction Theory



# Reconstruction filter in Frequency domain

- Reconstruction or smoothing filter is used to eliminate all the replicas of the spectrum outside the baseband

- Ideal lowpass filter
  – Frequency domain $H_r(j\Omega) = \begin{cases} T_s, & |\Omega| < \Omega_c \\ 0, & |\Omega| > \Omega_c \end{cases}$

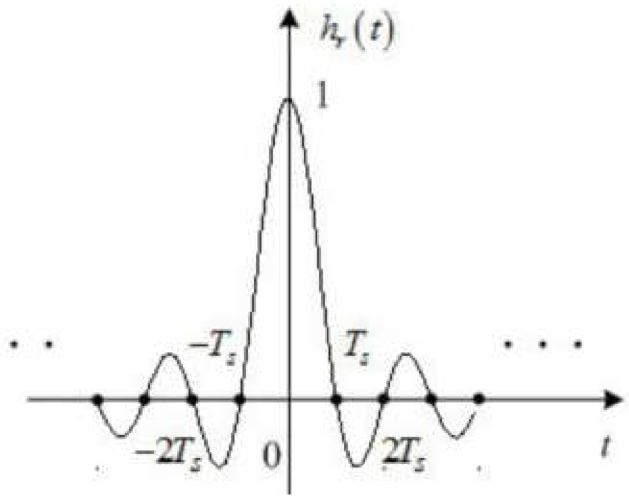


# Reconstruction filter in time domain

- This lowpass filter in time domain is a "sinc" function:
  - Time domain

$$h_r(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H_r(j\Omega)e^{j\Omega t}d\Omega = \frac{T_s}{2\pi}\int_{-\Omega_c}^{\Omega_c} e^{j\Omega t}d\Omega$$

$$= \frac{\sin(\Omega_s t/2)}{\Omega_s t/2} = \frac{\sin(\pi t/T_s)}{\pi t/T_s} = \mathrm{sinc}(\frac{t}{T_s})$$



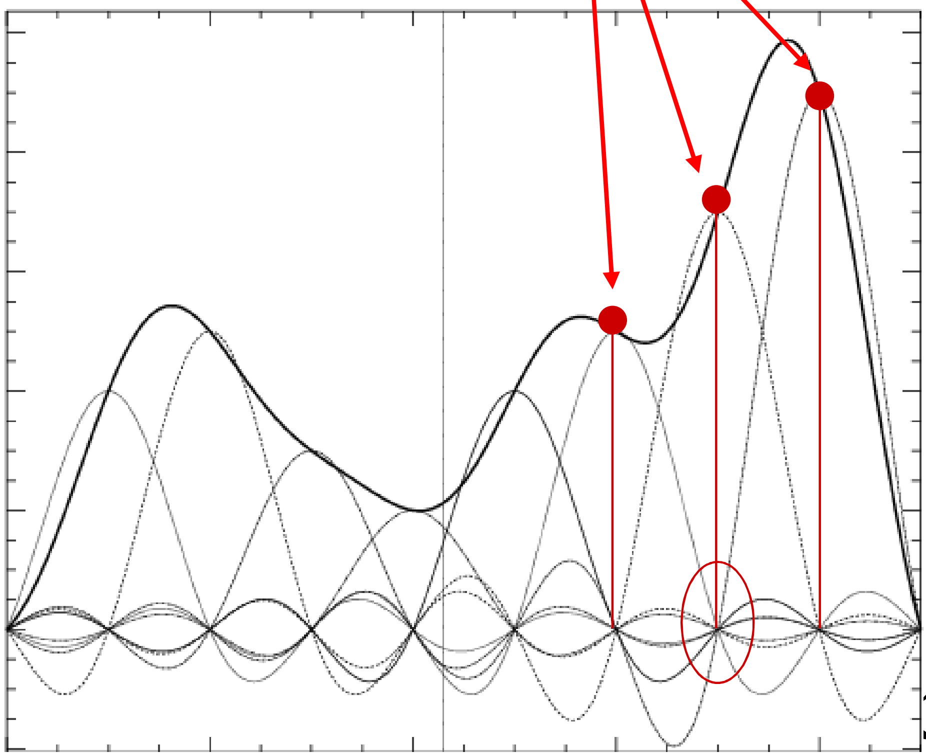

  - Multiply with $H_r(j\Omega)$ (in FD) is equivalent to convolve with $h_r(t)$ (in TD), the recovered signal $x_r(t) = x_s(t) * h_r(t)$
  - Impulse train $x_s(t)$: $x_s(t) = \sum_{n=-\infty}^{\infty} x_a(nT_s)\delta(t - nT_s)$

# Reconstruction filter in time domain

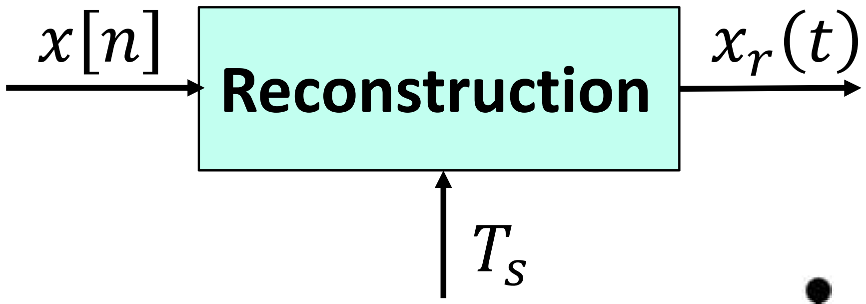
- Convolution between the discretized signal $x_s(t)$ and the reconstruction lowpass filter $h_r(t)$:

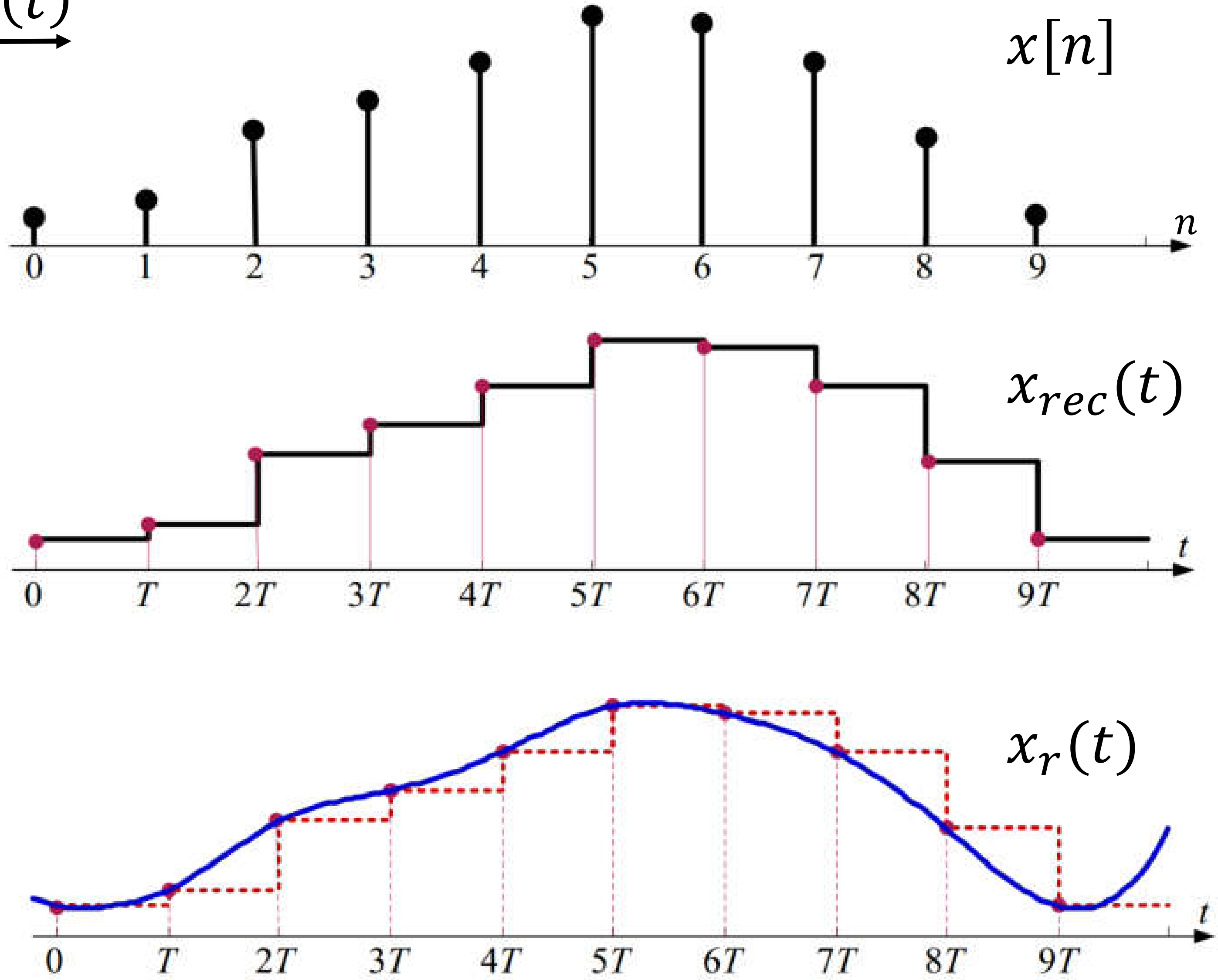
$$x_r(t) = x_s(t) * h_r(t) = \sum_{n=-\infty}^{\infty} x_a(nT_s)h_r(t - nT_s) = \sum_{n=-\infty}^{\infty} x[n]\mathrm{sinc}(t - nT_s)$$

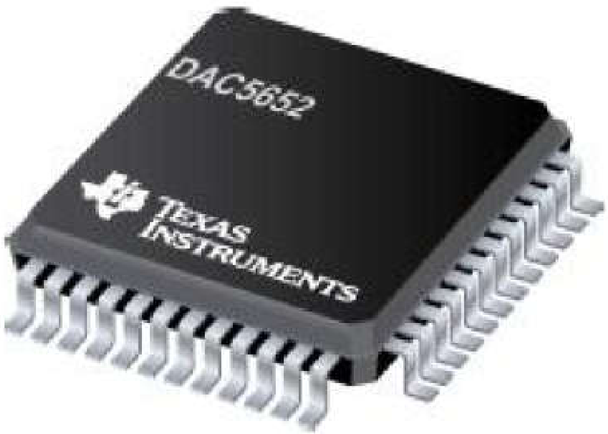
- The values are interpolated as a linear combination of the time-shifted sinc functions
- The amplitudes are scaled according to the sample values at the center locations of the sinc (the interpolation functions)



# Realization



# 3_3 Wrap up

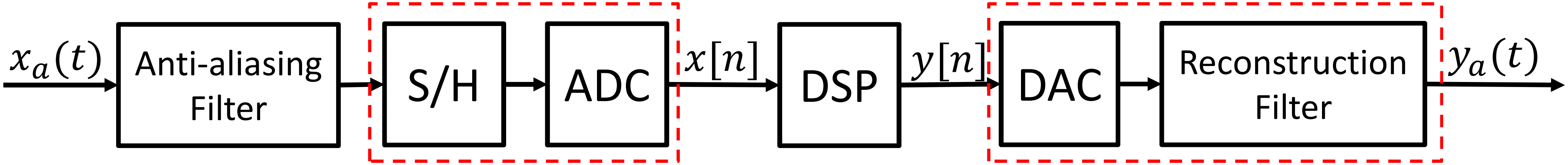
- Continuous-time signal can be reconstructed from the discrete-time sequence;
- Reconstruction can be realized as
  – In time domain: interpolation;
  – In frequency domain: filtering.
- Ideal reconstruction:

Filter with ideal lowpass filter $\Longleftrightarrow$ Interpolate with sinc function

Frequency domain | Time domain

# Chapter 3 Summary

- The whole process of A-D-A




- What is sampling? The time domain and frequency domain representation, including the equations and drawing;
- What is aliasing? What is the effect from aliasing and how to reduce it? Explain how the anti-aliasing filter works.
- What are interpolation and reconstruction? How do they work in time domain and frequency domain?

# Acronym

- CT – Continuous time
- DT – Discrete time

- TD – Time domain
- FD – Frequency domain
- TransD – Transform domain