MTH101: Lecture 5

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Exponential Function

Consider the function

$$f(z) = e^z = e^{x+iy} = e^x e^{iy},$$

then using the **Eulers Formula** we get

$$e^z = e^x(\cos\ y + i \sin\ y),$$

and in the form of $f = u + iv$ we obtain

$$u(x,y) = e^x \cos\ y, \quad v(x,y) = e^x \sin\ y.$$

The **Complex Exponential function** is defined for any $z \in \mathbb{C}$.

Analyticity

- e^z is an **entire function**. (Proved in Example 2 of Sec.13.4)
- The derivative of e^z is e^z: $(e^z)' = e^z$

Further Properties

- $e^{z_1+z_2} = e^{z_1}e^{z_2}$
- $e^{2\pi i} = 1 \qquad (|e^{iy}| = 1$ for any $y)$
- $e^z \neq 0$ for all z.
- $|e^z| = e^x, \quad \arg(z) = y + 2n\pi \; (n = 0, \pm 1, \pm 2, ...)$
- $e^{z+2\pi i} = e^z$ for all z (**Periodicity with period** $2\pi i$)

Remark

We assume that $-\pi < y \leq \pi$, *this is called a* **fundamental region** *of* e^z.

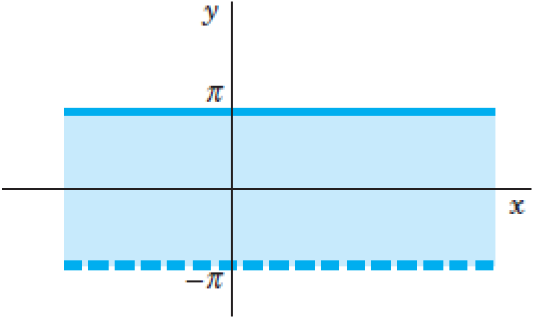

Fig. 336. Fundamental region of the exponential function e^z in the z-plane

Example

Solve the equation $e^z = 3 + 4i$ for z.

Solution: Let $z = x + iy$, we set up the equation

$$e^x e^{iy} = 5e^{i\,\arctan(\frac{4}{3})} \approx 5e^{i0.927}$$

$$\downarrow$$

$$z = ln5 + i(0.927 + 2n\pi), \;\; n = 0, \pm 1, \dots$$

Natural Logarithm is the inverse function of exponential function: given $z \in \mathbb{C}$

$$e^{\omega} = z \qquad \Leftrightarrow \qquad \ln z = \omega$$

We set $\omega = x + iy$ and write z in exponential form:

$$e^{\omega} = e^{x+iy} = e^{x} e^{iy} = re^{i\theta}.$$

Then we have

$$e^{x} = r \quad \Rightarrow \quad x = \ln r = \ln |z|,$$

and

$$y = \theta + 2n\pi, \quad n = 0, \pm 1, \pm 2, \ldots \quad \text{that is} \quad y = \arg z.$$

Definition

The **Complex Logarithm function** *is defined by*

$$\ln z = \ln |z| + i \arg z,$$

it takes infinitely many values, thus is a **Multivalued function**. *We define the* **Principal Value of the Logarithm functions** *:*

$$Ln\ z = \ln |z| + i Arg\ z.$$

The function Ln z takes only one value. Moreover we have:

$$\ln z = Ln\ z + i2n\pi, \quad n = 0, \pm 1, \pm 2, \ldots.$$

Example

Compute all the complex values of $\ln 1$.

Solution:
We can use the formula

$$\ln z = Ln\ z + i2n\pi = \ln|z| + i\ Arg\ z + i2n\pi, \quad n = 0, \pm 1, \pm 2, \ldots.$$

Then we need to compute

$$|z| = 1, \quad Arg\ z = 0,$$

from which

$$\ln(1) = \ln(1) + i \cdot 0 + i2n\pi, \quad n = 0, \pm 1, \pm 2, \ldots.$$

that is

$$\ln(1) = i2n\pi, \quad n = 0, \pm 1, \pm 2, \ldots.$$

Exercise

Compute all the values of $\ln(-1)$.

Theorem

For any $n = 0, \pm 1, \pm 2, \ldots,$ *the function*

$$\ln z = Ln\ z + i2n\pi,$$

is **Analytic** *except at zero and at the points of the negative real axis.*
Moreover, when $\ln z$ *is Analytic its derivative is*

$$(\ln z)' = \frac{1}{z}$$

General Powers

Definition

A **General Power function** *is defined by*

$$f(z) = z^c,$$

where $z \neq 0$ *and* $c \in \mathbb{C}$.
It can be defined using $\ln z$*:*

$$f(z) = e^{c \ln z} = e^{c(Ln\ z + 2n\pi i)}, \qquad n = 0, \pm 1, \pm 2, \ldots$$

It takes infinite many values, thus is a **Multivalued Function**.
We define its **Principal Value** *as:*

$$z^c = e^{c\ Ln\ z}.$$

Example

Coumpute all the values of i^i.

Solution:

By definition we have

$$i^i = e^{i \ln i} = e^{i(Ln\ i + 2n\pi i)}, \qquad n = 0, \pm 1, \pm 2, \ldots$$

We need to compute

$$Ln\ i = \ln |i| + i\ Arg\ (i),$$

then since

$$|i| = 1 \Rightarrow \ln |i| = 0, \quad \text{and} \quad Arg\ (i) = \frac{\pi}{2}$$

we have

$$Ln\ i = \ln |i| + i\ Arg\ (i) = 0 + i\frac{\pi}{2} = i\frac{\pi}{2}.$$

Then

$$i^i = e^{i \ln i} = e^{i(i\frac{\pi}{2}+2n\pi i)}, \qquad n = 0, \pm 1, \pm 2, \ldots$$

from which

$$i^i = e^{i \ln i} = e^{-(\frac{\pi}{2}+2n\pi)}, \qquad n = 0, \pm 1, \pm 2, \ldots$$

while its **Principal Value** is

$$i^i = e^{-\frac{\pi}{2}}.$$

Remark

We consider a general power: z^c, *where* $z \neq 0$ *and* $c \in \mathbb{C}$

- *If* $c = 0, 1, 2, ..,$ *then* z^n *is single-valued and simply the* n^{th} *power of z;*
- *If* $c = 1/n$, *where* $n = 2, 3, ...,$ *then* z^c *is the* n^{th} *root of* z, *thus is n-vlued.*
- *If c is irrational or genuinely complex, then* z^c *is infinitely many-valued.*

Trigonometric Functions

$$\cos z = \frac{1}{2}(e^{iz} + e^{-iz}), \quad \sin z = \frac{1}{2i}(e^{iz} - e^{-iz}).$$

They satisfy the fundamental formula:

$$\cos^2 z + \sin^2 z = 1.$$

Remark

They are **Entire functions** *since e^z is an entire function.*

The derivatives of the Trigonometric functions:

$$(\cos z)' = -\sin z, \qquad (\sin z)' = \cos z.$$

Exercise

Write $\cos z$, $\sin z$ *in the form* $f = u + iv$.

Exercise

Find the expression of $|\cos z|^2$ *and of* $|\sin z|^2$.

Remark

$\sin z$ *and* $\cos z$ *are periodic with period* 2π*, but unlike real trigonometric functions, they are unbounded. That is,* $|\sin z|, |\cos z| \to \infty$ *as* $y \to \infty$.

Other Trigonometric functions

$$\tan z = \frac{\sin z}{\cos z}, \qquad \cot z = \frac{\cos z}{\sin z},$$
$$\sec z = \frac{1}{\cos z}, \qquad \csc z = \frac{1}{\sin z}$$

They are **NOT!** Entire functions.

Exercise

Determine the set in which $\sec z$ *is an Analytic function.*

Hyperbolic functions

$$\cosh z = \frac{1}{2}(e^z + e^{-z}), \qquad \sinh z = \frac{1}{2}(e^z - e^{-z}).$$

They satisfy the fundamental formula:

$$\cosh^2 z - \sinh^2 z = 1.$$

Remark

They are **Entire functions** *since e^z is an entire function.*

The derivatives of the Hyperbolic functions:

$$(\cosh z)' = \sinh z, \qquad (\sinh z)' = \cosh z.$$

Other Hyperbolic functions

$$\tanh z = \frac{\sinh z}{\cosh z}, \qquad \coth z = \frac{\cosh z}{\sinh z}$$
$$sech\ z = \frac{1}{\cosh z}, \qquad csch\ z = \frac{1}{\sinh z}$$

They are **NOT!** Entire functions.

Exercise

Determine the set in which $\tanh z$ *is an Analytic function.*

Some important formulas

$$\cosh(iz) = \cos(z), \qquad \sinh(iz) = i \cdot \sin(z),$$
$$\cos(iz) = \cosh(z), \qquad \sin(iz) = i \cdot \sinh(z),$$

directly from the definitions of trigonometric and hyperbolic functions.

Bibliography

1 *Kreyszig, E.* **Advanced Engineering Mathematics**. Wiley, 10th Edition.