EEE336 Signal Processing and Digital Filtering

Lecture 11 Z-Transform

11_1 What is z-transform?

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Why do we need another transform?

- Think about all the transforms you have seen so far
  - Laplace transform, Fourier series, CTFT, DTFT and DFT
- Why do we need another one?
  - Convergence issues with the Fourier transforms:

    The DTFT of a sequence exists if and only if the sequence x[n] is absolutely summable, that is, if

$$\boxed{\sum_{n=-\infty}^{\infty} |x[n]| < \infty}$$

  - DTFT may not exist for certain signals of practical interest or some analytical signals, whose frequency analysis can therefore not be obtained through DTFT

Z-Transform

- A generalization of the DTFT leads to the z-transform that may exist for many signals for which the DTFT does not.
  - DTFT is in fact a special case of the z-transform
  - …just like the CTFT is a special case of Laplace's transform.
- Importance of z-transform
  - The use of z-transform techniques permits simple algebraic manipulations
  - The z-transform has become an important tool in the analysis and design of digital filters
  - The representation of an LTI discrete-time system in the z-domain is given by its transfer function which is the z-transform of the impulse response of the system

Z-Transform

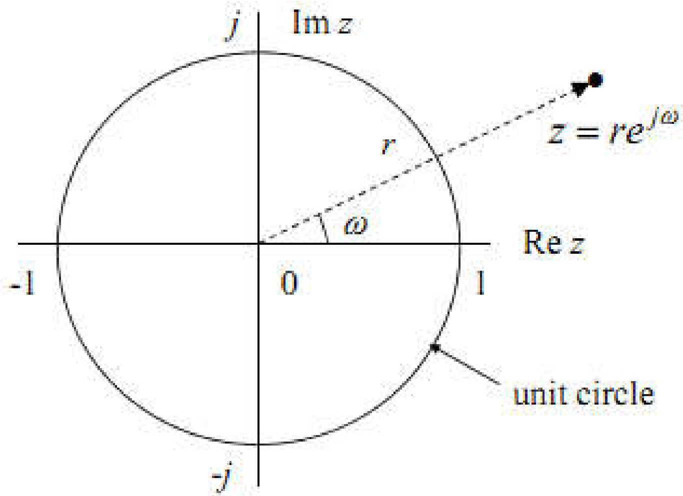
- For a given sequence x[n], its z-transform X(z) is defined as

$$X[z] = \sum_{n=-\infty}^{\infty} x[n] z^{-n} = \sum_{n=-\infty}^{\infty} x[n] \left(re^{j\omega}\right)^{-n} = \sum_{n=-\infty}^{\infty} x[n] r^{-n} e^{-j\omega n}$$

  where z lies in the complex space, that is : $z=a+jb=re^{j\omega}$

  - It follows that the DTFT is indeed a special case of the z-transform, specifically, z-transform reduces to DTFT for the special case of r=1, that is, $|z|$=1.
  - The contour $|z|$=1 is a circle in the z-plane of unit radius -> the unit circle
  - Hence, the DTFT is really the z-transform evaluated on the unit circle.

$$X(\omega) = X[z]\big|_{e^{j\omega}}$$



Convergence

- Just like the DTFT, z-transform also has its own convergence requirements: $x[n]r^{-n}$ must be absolutely summable, that is, $\boxed{\sum_{n=-\infty}^{\infty} \left| x[n] r^{-n} \right| < \infty}$
- For a given sequence, the set R of values of z for which its z-transform converges is called the region of convergence (ROC).
  - The area where the above condition is satisfied defines the ROC, which in general is an annular region of the z-plane

    $$R^{-} < \left| z \right| < R^{+} \quad where\ 0 \le R^{-} < R^{+} \le \infty$$

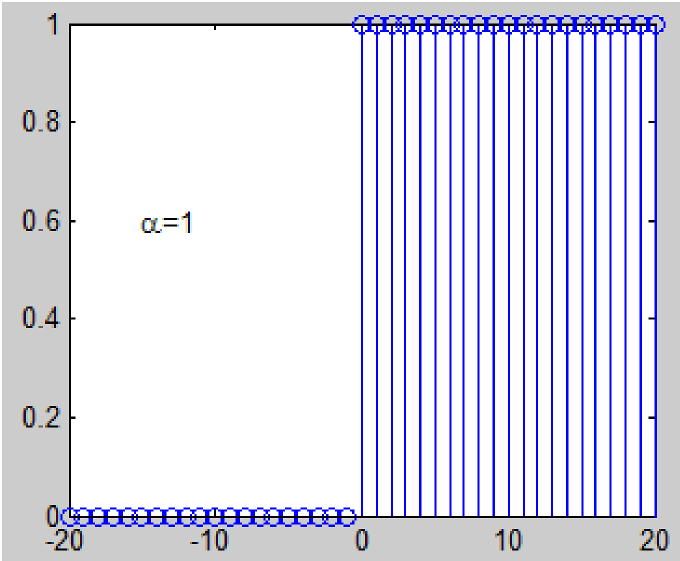
  - The z-transform must always be specified with its ROC !

DTFT exists only when ROC include |z|=1, the unit circle!

Example 1

- Determine the z-transform and the corresponding ROC of the unit step sequence $u[n]$

$$U[z] = \sum_{n=-\infty}^{\infty} u[n] z^{-n} = \sum_{n=0}^{\infty} z^{-n} = 1 + z^{-1} + z^{-2} + \cdots + z^{-n} + \cdots$$



which converges to

$$U[z] = \frac{1}{1 - z^{-1}}, \quad \mathit{for}\ \left|z^{-1}\right| < 1$$

$$= \frac{z}{z-1}, \quad \mathit{for}\ \left|z\right| > 1$$

- The region of convergence is the annular region in the z-plane

$$1 < |z| < \infty$$

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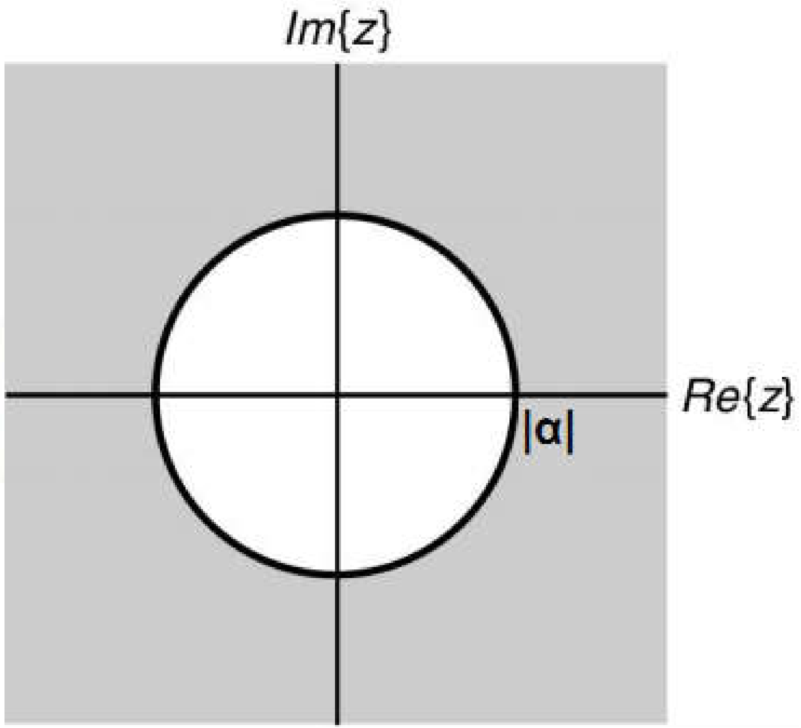
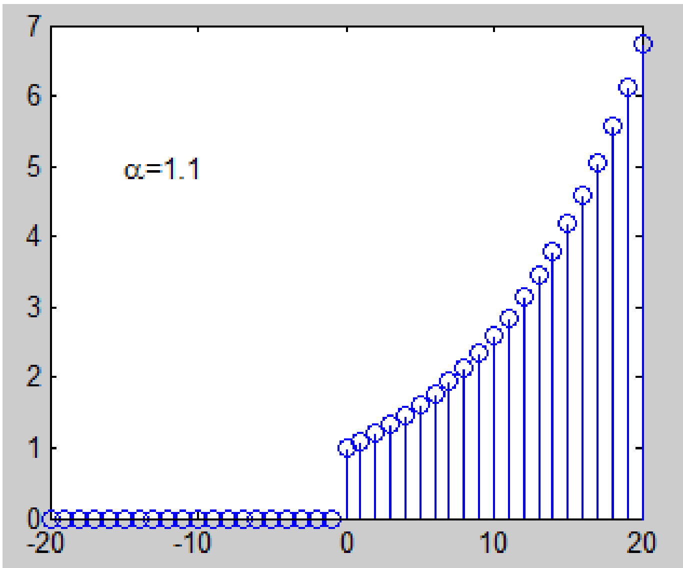
Example 2

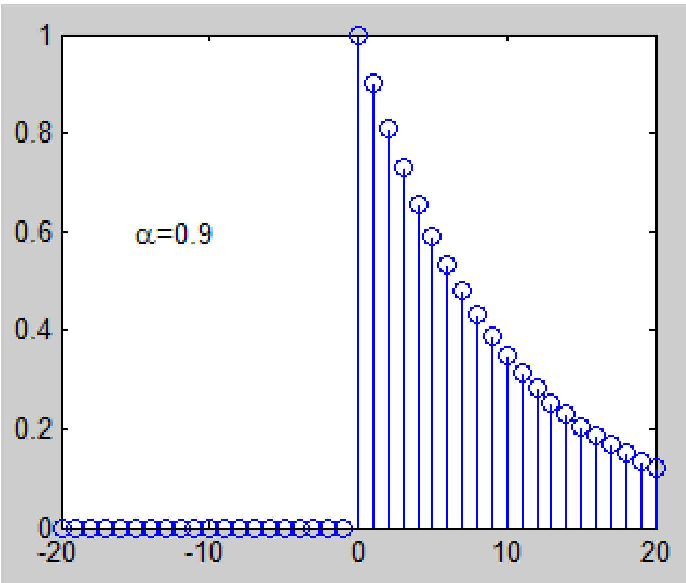
- Determine the z-transform and the corresponding ROC of the causal sequence x[n]=$\alpha^n u[n]$ (right-sided)

$$X[z]=\sum_{n=-\infty}^{\infty}\alpha^n u[n] z^{-n}=\sum_{n=0}^{\infty}\left(\alpha z^{-1}\right)^n \Longrightarrow X[z]=\frac{1}{1-\alpha z^{-1}},\quad for\ \left|\alpha z^{-1}\right|<1$$

$$=\frac{z}{z-\alpha},\qquad for\ |\alpha|<z$$







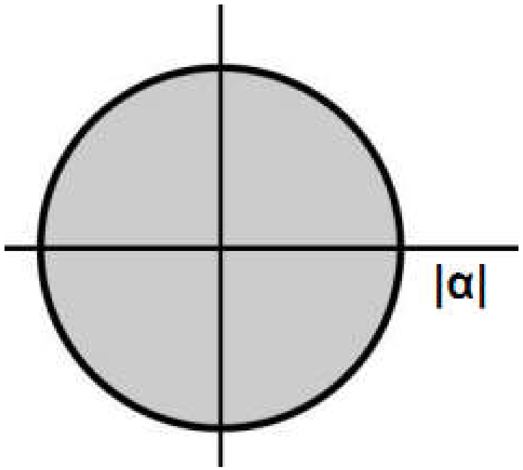
Example 3

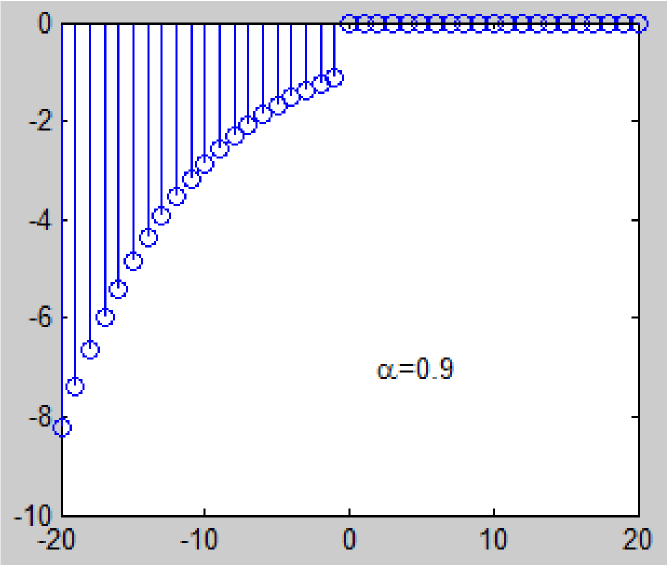
Is the same with that in previous slide, but with different ROC

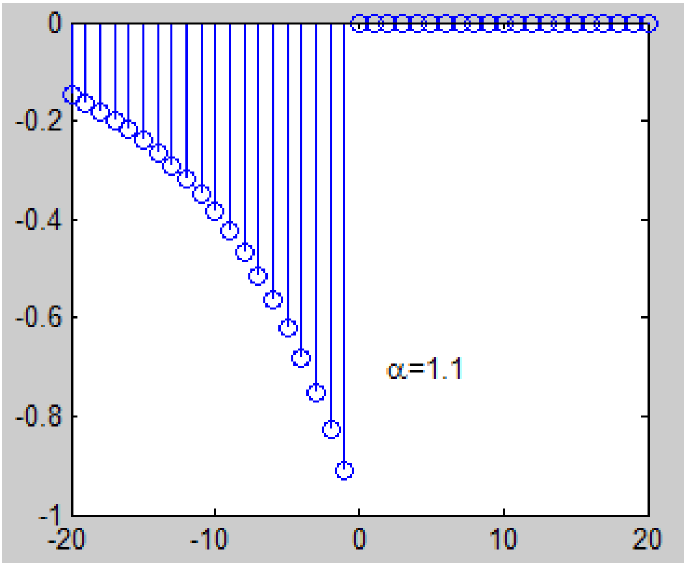
- Now consider the anti-causal y[n]=-$\alpha^n u[-n-1]$ (<u>left-sided</u>)

$$Y[z]=\sum_{n=-\infty}^{\infty}-\alpha^n u[-n-1]z^{-n}=-\sum_{n=-\infty}^{-1}\alpha^n z^{-n}=-\sum_{m=1}^{\infty}\alpha^{-m}z^{m}=-\alpha^{-1}z\sum_{m=0}^{\infty}\alpha^{-m}z^{m}$$

$$=-\frac{\alpha^{-1}z}{1-\alpha^{-1}z}=\frac{1}{1-\alpha z^{-1}}=\frac{z}{z-\alpha},\qquad for\ z<|\alpha|$$







Impulse response and transfer function

- Impulse responses: $x[n]=\alpha^n u[n]$ and $y[n]=-\alpha^n u[-n-1]$
- Transfer functions: $$X[z]=\frac{z}{z-\alpha}, \qquad for\ z>|\alpha|$$ $$Y[z]=\frac{z}{z-\alpha}, \qquad for\ z<|\alpha|$$
  - The z-transforms of the two sequences x[n] and y[n] are identical even though the two parent sequences are different
  - Only way a unique sequence can be associated with a z-transform is by specifying its ROC
  - Both transfer functions have a pole at z=α, which make the transfer function asymptotically approach to infinity at this value. Therefore, z=α is not included in either of the ROCs.

11_1 Wrap up

- Relationships between DTFT and z-transform
  - The reason of introducing z-transform
- Definition of z-transform + ROC
  - Importance of ROC
- Examples and their calculations

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Lecture 11 Z-Transform

11_2 ROC of z-transform

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ROC of the z-transform

$$X[z] = \frac{N[z]}{D[z]} = \frac{z}{z-\alpha}, \qquad for\ |z| > |\alpha|$$

- In the X[z] given above, z = 0 is its ***zero***, and z = α is its ***pole***.
- The circle with the radius of ***α*** is called the ***pole circle***. A system may have many poles, and hence many pole circles.
- For right sided sequences, the ROCs extend outside of the outermost pole circle, whereas for left sided sequences, the ROCs are the inside of the innermost pole circle.
- For two-sided sequences, the ROC will be the intersection of the two ROC areas corresponding to the left and right sides of the sequence.

ROC of the z-Transform

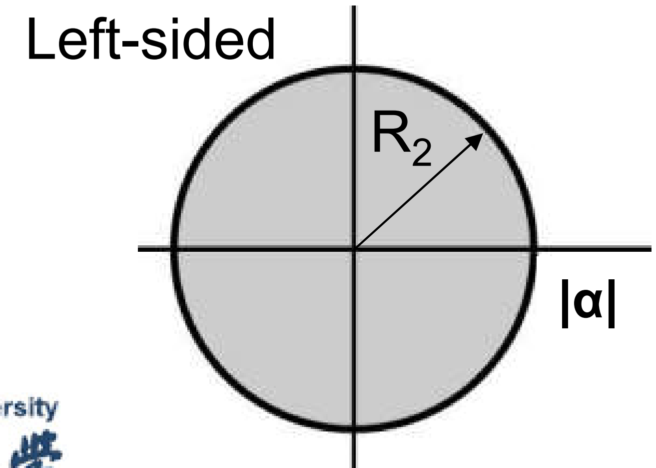
- For double sided sequence:

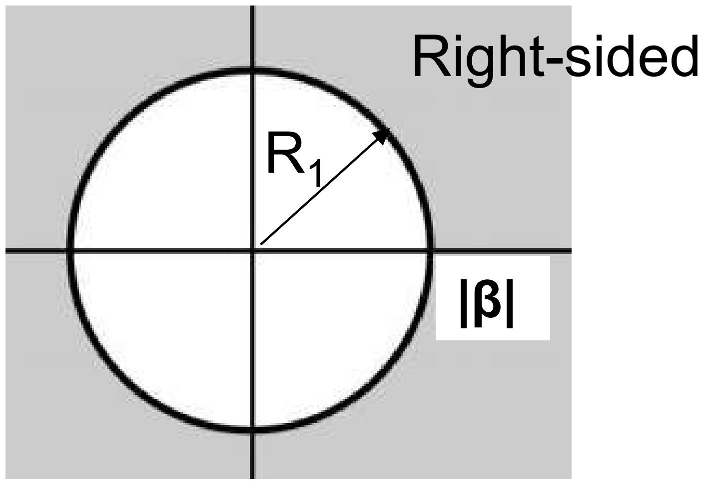
$$x[n] = \beta^n u[n] - \alpha^n u[-n-1]$$

- Its z-transform is:

$$X(z) = \frac{1}{1-\alpha z^{-1}} + \frac{1}{1-\beta z^{-1}}$$

  – Two poles of the transfer function: $|z| = |\alpha|$and $|z| = |\beta|$
  – ROC: $|z| > |\beta|$ and $|z| < |\alpha|$

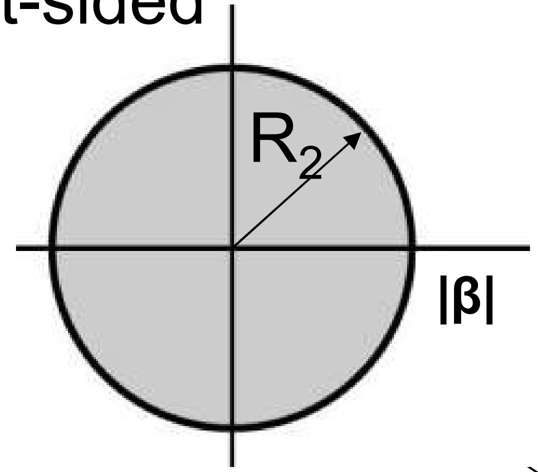




ROC of the z-Transform

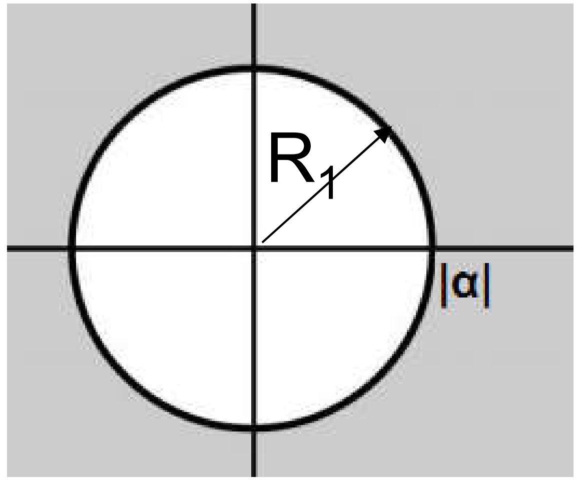
- When $R_1 < R_2$

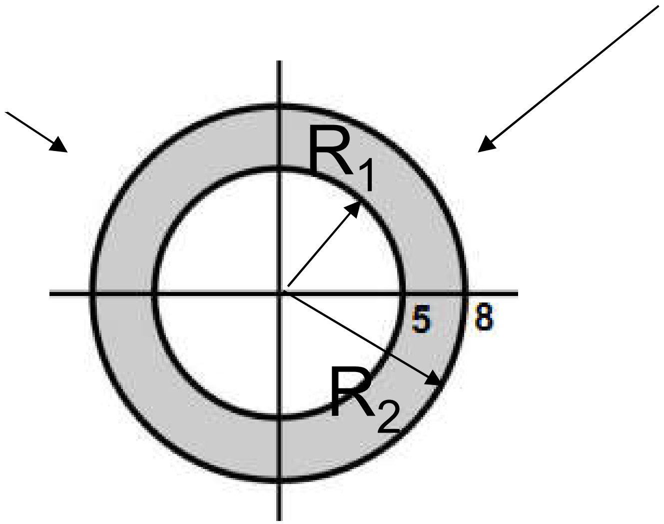
Left-sided



$$R_1 < |z| < R_2$$
$$if\ 0 \le R_1 < R_2 \le \infty$$

Right-sided





ROC of a left-sided sequence is inside of a circular area

ROC of a right-sided sequence is outside of a circular area

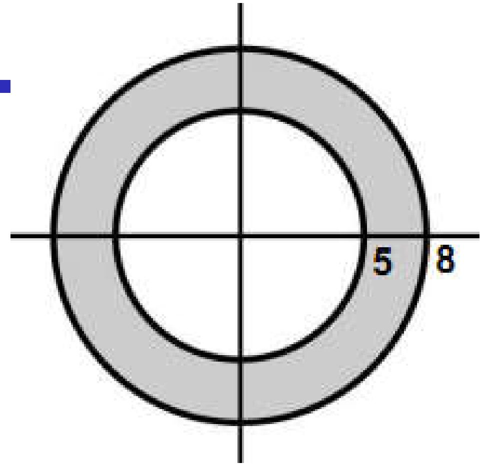
- When $R_1 > R_2$
  - No valid ROC => z-transform doesn't exist.

Example 4

- Consider x[n]=5^nu[n]-8^nu[-n-1]

$$X[z] = \frac{z}{z-5} + \frac{z}{z-8}$$

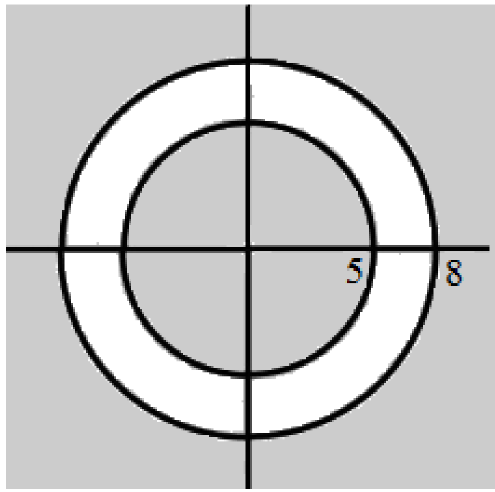
  - Corresponding ROCs are |z|>5 and |z|<8
  - Therefore the ROC for this signal is the annular region 5<|z|<8



- Consider x[n]=8^nu[n]-5^nu[-n-1]

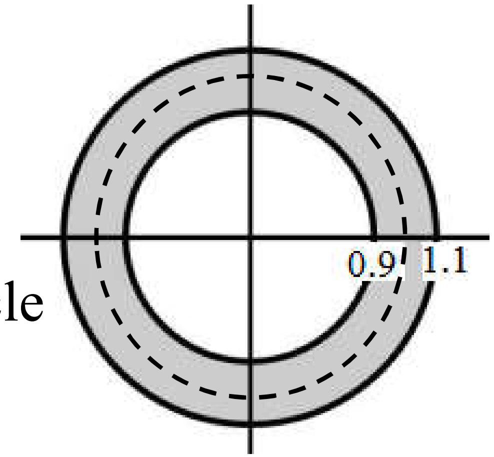
$$X[z] = \frac{z}{z-5} + \frac{z}{z-8}$$

  - Corresponding ROCs are |z|<5 and |z|>8
  - Therefore, the z-transform of this sequence does not exist!



Existence of DTFT and z-transform

- Since DTFT is the z-transform evaluated on the unit circle, that is for $z=e^{j\omega}$, DTFT of a sequence exists if and only if the ROC includes the unit circle!
  - The DTFT for $x[n]=5^n u[n]-8^n u[-n-1]$ clearly does not exist, since the ROC does not include the unit circle!
  - Consider the sequence $x[n]=0.9^n u[n]-1.1^n u[-n-1]$
    - Its transfer function is: $X[z]=\dfrac{z}{z-0.9}+\dfrac{z}{z-1.1}$
    - with the ROC as $0.9<|z|<1.1$, which includes the unit circle
    - Therefore, the DTFT of x[n] exists



The existence of DTFT is not a guarantee for the existence of the z-transform either!

10_2 Wrap up

- Zeroes and poles
- ROC: intersection of all ROCs of the constitutes
  - All right sided;
  - All left sided;
  - Double sided.
- Existence of DTFT: ROC includes $|z|=1$

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Lecture 11 Z-Transform

11_3 Inverse z-transform

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Inverse z-transform

- The inverse z-transform is defined as

$$\boxed{x[n] = \frac{1}{2\pi j} \oint_C X(z) \cdot z^{n-1} dz}$$

where C is a counter-clockwise contour encircling the origin in the ROC of X(z) gives the contour integral

- There are three methods for the evaluation of the inverse z-transform in practice
  - ~~0. Observe the X[z] and directly get x[n] from the commonly used z-transform pair;~~
  - ~~1. Direct evaluation by the contour integration using the Cauchy Residue theorem~~
  - 2. Long division of the numerator by the denominator
  - 3. Partial-fraction expansion and table lookup

Commonly used z-transform pairs

	Signal, $x(n)$	z-Transform, $X(z)$	ROC
1	$\delta(n)$	1	All z
2	$u(n)$	$\dfrac{1}{1-z^{-1}}$	$\|z\| > 1$
3	$a^n u(n)$	$\dfrac{1}{1-az^{-1}}$	$\|z\| > \|a\|$
4	$na^n u(n)$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| > \|a\|$
5	$-a^n u(-n-1)$	$\dfrac{1}{1-az^{-1}}$	$\|z\| < \|a\|$
6	$-na^n u(-n-1)$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| < \|a\|$
7	$(\cos \omega_0 n)u(n)$	$\dfrac{1-z^{-1}\cos\omega_0}{1-2z^{-1}\cos\omega_0+z^{-2}}$	$\|z\| > 1$
8	$(\sin \omega_0 n)u(n)$	$\dfrac{z^{-1}\sin\omega_0}{1-2z^{-1}\cos\omega_0+z^{-2}}$	$\|z\| > 1$
9	$(a^n \cos \omega_0 n)u(n)$	$\dfrac{1-az^{-1}\cos\omega_0}{1-2az^{-1}\cos\omega_0+a^2z^{-2}}$	$\|z\| > \|a\|$
10	$(a^n \sin \omega_0 n)u(n)$	$\dfrac{az^{-1}\sin\omega_0}{1-2az^{-1}\cos\omega_0+a^2z^{-2}}$	$\|z\| > \|a\|$

Inverse z-Transform by long division

- The z-transform of a causal sequence can be expanded in a power series in z^{-1}.
- For a rational z-transform expressed as a ratio of polynomials in z^{-1}, the power series expansion can be obtained by long division.

- Example – Evaluate the inverse z-transform of

$$H(z) = \frac{1 + 2z^{-1}}{1 + 0.4z^{-1} - 0.12z^{-2}}$$

Using the long division

Not close-form expression, not good enough!

Inverse z-Transform by Partial Fraction Expansion

- A rational H(z) can be expressed as

$$H(z)=\frac{Y(z)}{X(z)}=\frac{P(z)}{D(z)}=\frac{b_0+b_1z^{-1}+b_2z^{-2}+\cdots+b_Mz^{-M}}{1+a_1z^{-1}+a_2z^{-2}+\cdots+a_Nz^{-N}}=\frac{\sum_{i=0}^{M}b_iz^{-i}}{\sum_{i=0}^{N}a_iz^{-i}}$$

- If M ≥ N then H(z) can be re-expressed through long division

$$\boxed{H(z)=\sum_{\ell=0}^{M-N}\eta_\ell z^{-\ell}+\frac{P_1(z)}{D(z)}}$$

$$H(z)=\frac{2+0.8z^{-1}+0.5z^{-2}+0.3z^{-3}}{1+0.8z^{-1}+0.2z^{-2}} \Rightarrow H(z)=-3.5+1.5z^{-1}+\frac{5.5+2.1z^{-1}}{1+0.8z^{-1}+0.2z^{-2}}$$

where the degree of $P_1(z)$ is less than N. The rational fraction $P_1(z)/D(z)$ is then called a *proper polynomial* .

Inverse z-Transform by Partial Fraction Expansion

- ***Simple Poles***: In most practical cases, the rational z-transform of interest H(z) is a proper fraction with simple poles, then it can be written in the following form

$$H(z)=\frac{A_1}{z-p_1}+\frac{A_2}{z-p_2}+\cdots+\frac{A_N}{z-p_N}$$

$$H(z)\cdot z=\frac{z\cdot A_1}{z-p_1}+\frac{z\cdot A_2}{z-p_2}+\cdots+\frac{z\cdot A_N}{z-p_N} \Longleftrightarrow A_1(p_1)^n u[n]+A_2(p_2)^n u[n]+\cdots+A_N(p_N)^n u[n]$$

is not the inverse transform of the original H(z) we are interested in

- So, we simply compute the partial fraction of H(z)/z, which will then give us the inverse z-transform of H(z)

$$\frac{H(z)}{z}=\frac{b_0+b_1z^{-1}+b_2z^{-2}+\cdots+b_Mz^{-M}}{z(z-p_1)(z-p_2)\cdots(z-p_N)}=\frac{A_0}{z}+\frac{A_1}{z-p_1}+\frac{A_2}{z-p_2}+\cdots+\frac{A_N}{z-p_N}$$

Inverse z-Transform by Partial Fraction Expansion

- The constants A_i, which are the residues at the poles of H(z)/z, can be computed as follows:

$$\boxed{A_i = (z - p_i)\frac{H(z)}{z}\bigg|_{z=p_i} \qquad i = 0,1,2,\cdots N}$$

- Find the inverse z-transform of H(z) given the ROC
  - i) $0.2<|z|<0.6$
  - ii) $|z|>0.6$

$$H(z) = \frac{z^2 + 2z + 1}{z^2 + 0.4z - 0.12}$$

Inverse z-Transform by Partial Fraction Expansion

- ***Multiple Poles***: If the z-domain function contains an m-multiple pole, that is, a term as the following is included

$$\frac{H(z)}{z} = \frac{P(z)}{(z-p)^m}$$

- this term is expanded as follows:

$$\boxed{\frac{H(z)}{z} = \frac{A_1}{z-p} + \frac{A_2}{(z-p)^2} + \cdots + \frac{A_{m-1}}{(z-p)^{m-1}} + \frac{A_m}{(z-p)^m}}$$

where each coefficient can be computed by taking consecutive derivatives and evaluating the function at the pole

$$\begin{aligned} A_{m-i} &= \frac{1}{(i)!} \left. \frac{d^i\left((z-p)^m \frac{H(z)}{z}\right)}{dz^i} \right|_{z=p} \qquad i = 0,1,2,\ldots,m-1 \\ &= \frac{1}{(i)!} \left. \frac{d^i P(z)}{dz^i} \right|_{z=p} \qquad i = 0,1,2,\ldots,m-1 \end{aligned}$$

Inverse z-Transform in Matlab

- **`[r,p,k]=residuez(num,den)`** develops the partial-fraction expansion of a rational z-transform with numerator and denominator coefficients given by vectors **`num`** and **`den`**.
  - Vector r contains the residues, vector p contains the poles, vector k contains the direct term constants
- **`[num,den]=residuez(r,p,k)`** converts a z-transform expressed in a partial-fraction expansion form to its rational form.
- Example: Consider the first question in the exercise, and write it in terms of z^{-1}: $X(z)=\frac{z}{2z^2-3z+1}=\frac{z}{z^2\left(2-3z^{-1}+z^{-2}\right)}=\frac{z^{-1}}{2-3z^{-1}+z^{-2}}$

```
>> b=[0 1];
>> a=[2 -3 1];
>>[r p k]=residuez (b,a);

r=[1, -1], p=[1 1/2], k=[]
```

$$\Rightarrow X(z)=\frac{1}{1-(1)z^{-1}}+\frac{-1}{1-(0.5)z^{-1}}$$

$$\Rightarrow x[n]=-u[-n-1]-(-1)(0.5)^n u[-n-1],\quad |z|<\frac{1}{2}$$

10_3 Wrap up

- Familiar with the methods for inverse-z transform
  - Directly using the common z-transform pairs
  - Long division
  - Partial Fraction Expansion

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Lecture 11 Z-Transform

11_4 Properties of z-transform

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Z-transform properties

- Linearity

$$x_1(n) \xleftrightarrow{z} X_1(z)$$

$$x_2(n) \xleftrightarrow{z} X_2(z)$$

$$x(n) = a_1 x_1(n) + a_2 x_2(n) \xleftrightarrow{z} X(z) = a_1 X_1(z) + a_2 X_2(z)$$

  – ROC of X(z) is the intersection of ROCs of $X_1(z)$ and $X_2(z)$

- Example
  – 1. Determine the z-Transform and the ROC of $x(n) = \left[3(2^n) - 4(3^n)\right]u(n)$
  – 2. Determine the z-Transform and the ROC of $x[n] = a^n u[n] - b^n u[-n-1]$
  – 3. Determine the z-Transform and the ROC of $x[n] = (\cos\omega_0 n)u(n)$

Z-transform properties

- Time-shifting $x(n) \xleftrightarrow{z} X(z)$

$$x(n-k) \xleftrightarrow{z} z^{-k} X(z)$$

And the ROC remains unchanged except for z = 0 if k > 0 and z = ∞ if k < 0

- Example
  - Determine the z-Transform of the signal

$$x(n) = \begin{cases} 1, & 0 \le n \le N-1 \\ 0, & \text{elsewhere} \end{cases}$$

Z-transform properties

- <u>Scaling in the z-domain</u>

$$x(n) \overset{z}{\longleftrightarrow} X(z), \quad \text{ROC: } r_1 < |z| < r_2$$

$$a^n x(n) \overset{z}{\longleftrightarrow} X(a^{-1}z), \quad \text{ROC: } |a| r_1 < |z| < |a| r_2$$

For any constant *a*

- Example
  - Determine the z-transform and its ROC of the causal sequence

$$x(n) = r^n (\cos \omega_0 n) u(n)$$

$$x(n) = r^n (\sin \omega_0 n) u(n)$$

Z-transform properties

- Time Reversal

$$x(n) \xleftrightarrow{z} X(z), \quad \text{ROC}: r_1 < |z| < r_2$$

$$x(-n) \xleftrightarrow{z} X(z^{-1}), \quad \text{ROC}: \frac{1}{r_2} < |z| < \frac{1}{r_1}$$

- Example
  - Determine the z-transform and its ROC of $x(n) = u(-n)$

Z-transform properties

- <u>Differentiation in the z-Domain</u>

$$x(n) \xleftrightarrow{z} X(z)$$

$$nx(n) \xleftrightarrow{z} -z\frac{dX(z)}{dz}$$

ROC remains unchanged

- Example
  - Find the z-Transform of $x(n) = na^n u(n)$

Z-transform properties

- <u>Convolution of Two Sequences</u>

$$x_1(n) \overset{z}{\longleftrightarrow} X_1(z) \quad \text{and} \quad x_2(n) \overset{z}{\longleftrightarrow} X_2(z)$$

$$x(n) = x_1(n) * x_2(n) \overset{z}{\longleftrightarrow} X(z) = X_1(z)X_2(z)$$

The ROC is the intersection of that for $X_1(z)$ and $X_2(z)$

Z-transform properties

- Parseval's relation

$$x_1(n) \xleftrightarrow{z} X_1(z) \quad \text{and} \quad x_2(n) \xleftrightarrow{z} X_2(z)$$

$$\sum_{n=-\infty}^{\infty} x_1(n) x_2^*(n) = \frac{1}{2\pi j} \oint_C X_1(v) X_2^*\left(\frac{1}{v^*}\right) v^{-1} dv$$

11_4 Wrap up

Property	Time Domain	z-Domain	ROC
Notation	$x(n)$	$X(z)$	ROC: $r_2 < \|z\| < r_1$
	$x_1(n)$	$X_1(z)$	ROC_1
	$x_2(n)$	$X_2(z)$	ROC_2
Linearity	$a_1x_1(n) + a_2x_2(n)$	$a_1X_1(z) + a_2X_2(z)$	At least the intersection of ROC_1 and ROC_2
Time shifting	$x(n-k)$	$z^{-k}X(z)$	That of $X(z)$, except $z = 0$ if $k > 0$ and $z = \infty$ if $k < 0$
Scaling in the z-domain	$a^n x(n)$	$X(a^{-1}z)$	$\|a\|r_2 < \|z\| < \|a\|r_1$
Time reversal	$x(-n)$	$X(z^{-1})$	$\frac{1}{r_1} < \|z\| < \frac{1}{r_2}$
Conjugation	$x^*(n)$	$X^*(z^*)$	ROC
Real part	$\text{Re}\{x(n)\}$	$\frac{1}{2}[X(z) + X^*(z^*)]$	Includes ROC
Imaginary part	$\text{Im}\{x(n)\}$	$\frac{1}{2}j[X(z) - X^*(z^*)]$	Includes ROC
Differentiation in the z-domain	$nx(n)$	$-z\frac{dX(z)}{dz}$	$r_2 < \|z\| < r_1$
Convolution	$x_1(n) * x_2(n)$	$X_1(z)X_2(z)$	At least, the intersection of ROC_1 and ROC_2
Correlation	$r_{x_1x_2}(l) = x_1(l) * x_2(-l)$	$R_{x_1x_2}(z) = X_1(z)X_2(z^{-1})$	At least, the intersection of ROC of $X_1(z)$ and $X_2(z^{-1})$
Initial value theorem	If $x(n)$ causal	$x(0) = \lim_{z\to\infty} X(z)$	
Multiplication	$x_1(n)x_2(n)$	$\frac{1}{2\pi j}\oint_C X_1(v)X_2\left(\frac{z}{v}\right)v^{-1}\,dv$	At least, $r_{1l}r_{2l} < \|z\| < r_{1u}r_{2u}$
Parseval's relation	$\sum_{n=-\infty}^{\infty} x_1(n)x_2^*(n) = \frac{1}{2\pi j}\oint_C X_1(v)X_2^*(1/v^*)v^{-1}dv$		

EEE336 Signal Processing and Digital Filtering

Lecture 11 Z-Transform

11_5 Zeroes and Poles

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Rational z-transform

- The z-transforms of LTI systems can be expressed as a ratio of two polynomials in z^{-1}, hence they are rational transforms.
  - Starting with the constant coefficient linear difference equation (CCLDE) representation of an LTI system:

$$\sum_{i=0}^{N} a_i y[n-i] = \sum_{j=0}^{M} b_j x[n-j], \quad a_0 = 1$$

$$y[n] + a_1 y[n-1] + a_2 y[n-2] + \cdots a_N y[n-N] = b_0 x[n] + b_1 x[n-1] + \cdots + b_M x[n-M]$$

$$\Big\Updownarrow \mathcal{Z} \qquad\qquad\qquad\qquad \Big\Updownarrow \mathcal{Z}$$

$$Y[z] + a_1 z^{-1} Y[z] + a_2 z^{-2} Y[z] + \cdots + a_N z^{-N} Y[z] = X[z] + b_1 z^{-1} X[z] + b_2 z^{-2} X[z] + \cdots + b_M z^{-M} X[z]$$

$$\Rightarrow H[z] = \frac{Y[z]}{X[z]} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_M z^{-M}}{a_0 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{-N}}$$

A ratio of two polynomials

Degree of H(z) is max{M,N}

Rational z-transform

- A rational z-transform can be alternately written in factored form as

$$H(z)=\frac{b_0\prod_{\ell=1}^{M}\left(1-\zeta_\ell z^{-1}\right)}{a_0\prod_{\ell=1}^{N}\left(1-p_\ell z^{-1}\right)}=z^{(N-M)}\frac{p_0\prod_{\ell=1}^{M}\left(z-\zeta_\ell\right)}{d_0\prod_{\ell=1}^{N}\left(z-p_\ell\right)}$$

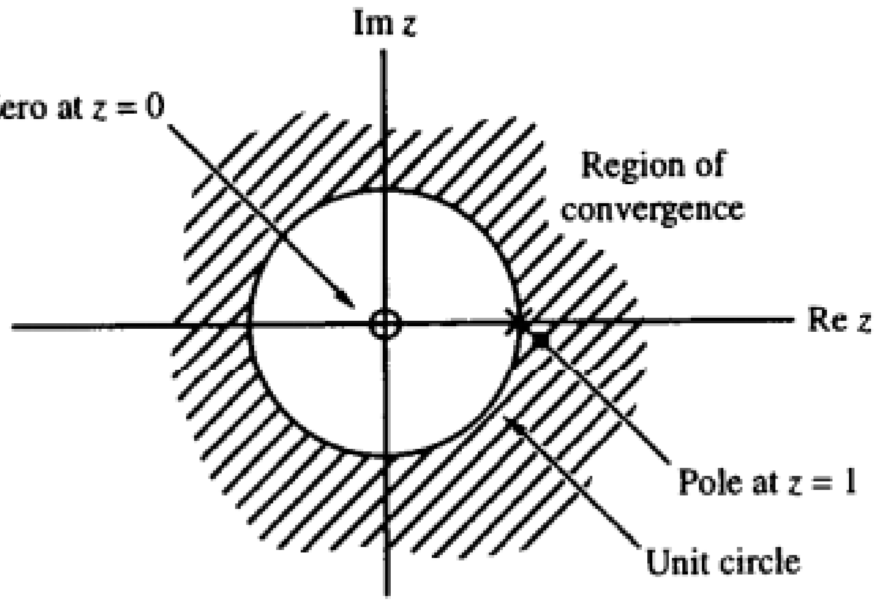
  - At a root $z=\xi_l$ of the numerator polynomial, $H(\xi_l)=0$ and these values of z are called the ***zeroes*** of H(z)
  - At a root $z=p_\ell$ of the denominator polynomial $H(p_\ell)\to\infty$, and as a result, these values of z are known as the ***poles*** of H(z)
  - There are M finite zeroes and N finite poles of H(z)
  - There are additional (N-M) zeros at the origin if N>M or (N-M) poles at z = 0 if N<M

Rational z-transform

- Example: z-Transform of the Unit Step

$$\mu(z) = \frac{1}{1-z^{-1}} = \frac{z}{z-1} \qquad \begin{cases} \text{zero}: & z=0 \\ \text{pole}: & z=1 \end{cases}$$

  – The region of convergence in the z-plane

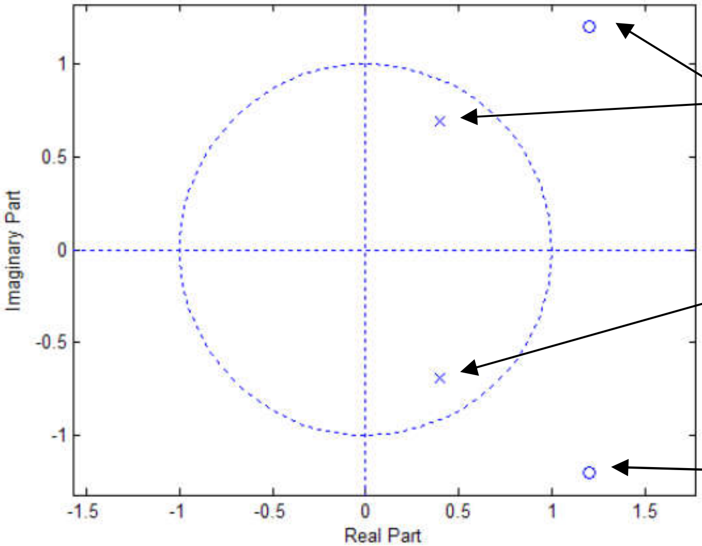


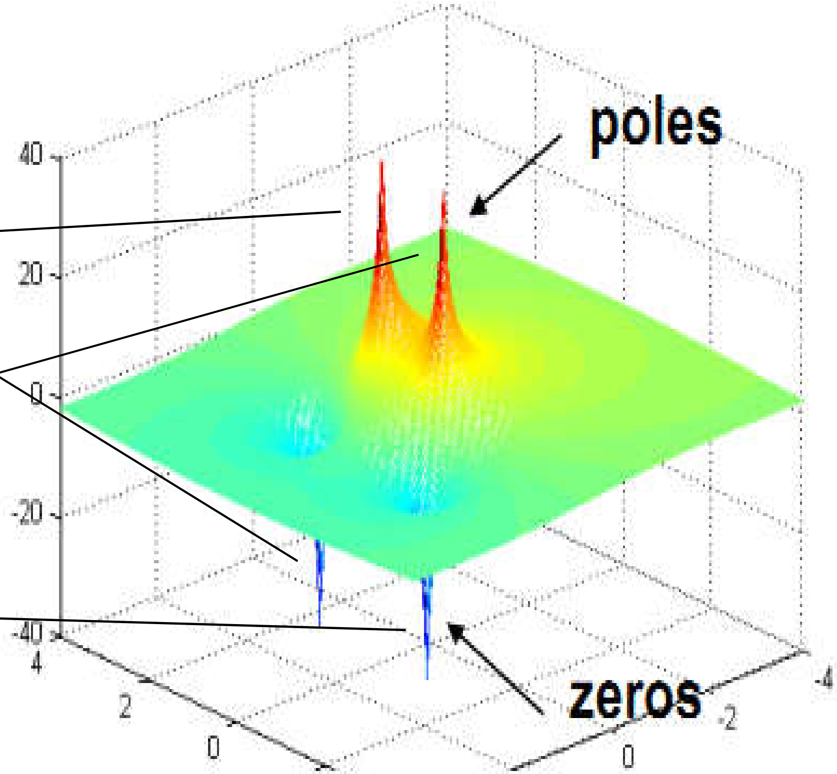
The ROC of a rational z-transform is bounded by the locations of its poles

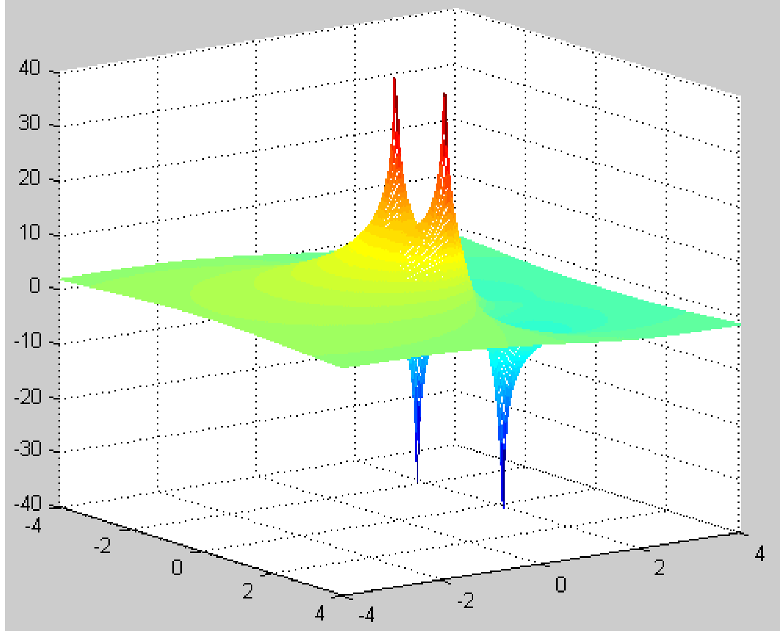
- Example: A physical interpretation of the concepts of poles and zeros can be given by plotting the log-magnitude 20log10|G(z)| of G(z)

$$G(z)=\frac{1-2.4z^{-1}+2.88z^{-2}}{1-0.8z^{-1}+0.64z^{-2}}$$

The poles are at z=0.4±j0.6928
The zeroes are at z=1.2±j1.2








```
clear;
close all;
N=256;
rez=linspace(-4,4,N);
imz=linspace(-4,4,N);
%create a uniform z-plane
for n=1:N
  z(n,:)=ones(1,N).*rez(n)+j*ones(1,N).*imz(1:N);
end
%Compute the H function on the z-plane
for n=1:N
  for m=1:N
    Hz(n,m)=(1-2.4*z(n,m)^(-1)+2.88*z(n,m)^(-
2))/(1-0.8*z(n,m)^(-1)+0.64*z(n,m)^(-2));
  end
end
%Logarithmic mesh plot of the H function
mesh(rez, imz, 20*log10(abs(Hz)))
```

- Matlab has simple functions to determine and plot the poles and zeros of a function in the z-plane
  - **`tf2zpk():[Z,P,K]=tf2zpk(NUM,DEN)`** finds the zeros, poles, and gain.

```
b=[1 -2.4 2.88];
a=[1 -0.8 0.64];
[z,p,k] = tf2zpk(b,a)
```

⇨

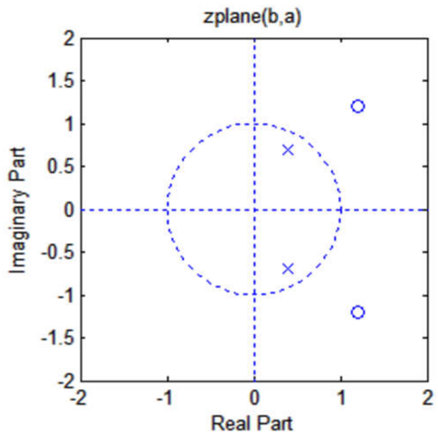
```
z =   1.2000 + 1.2000i
      1.2000 - 1.2000i
p =   0.4000 + 0.6928i
      0.4000 - 0.6928i
k=1
```

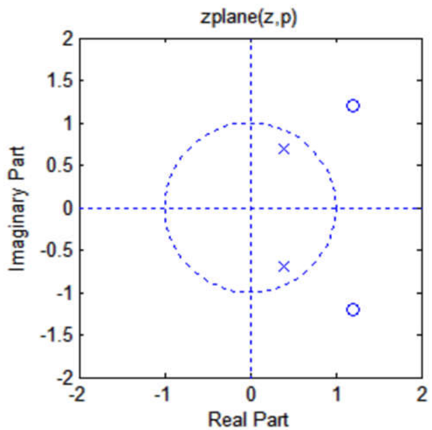
  - **`[num,den] = zp2tf(z,p,k)`** implements the reverse process
  - **`zplane(): zplane(Z,P)`** plots the zeros Z and poles P (in column vectors) with the unit circle for reference.

    `zplane(B,A)` plots the poles and zeros of B(z)/A(z) where B and A are row vectors containing transfer function polynomial coefficients

```
zplane(b,a);
zplane(z,p);
```

⇨





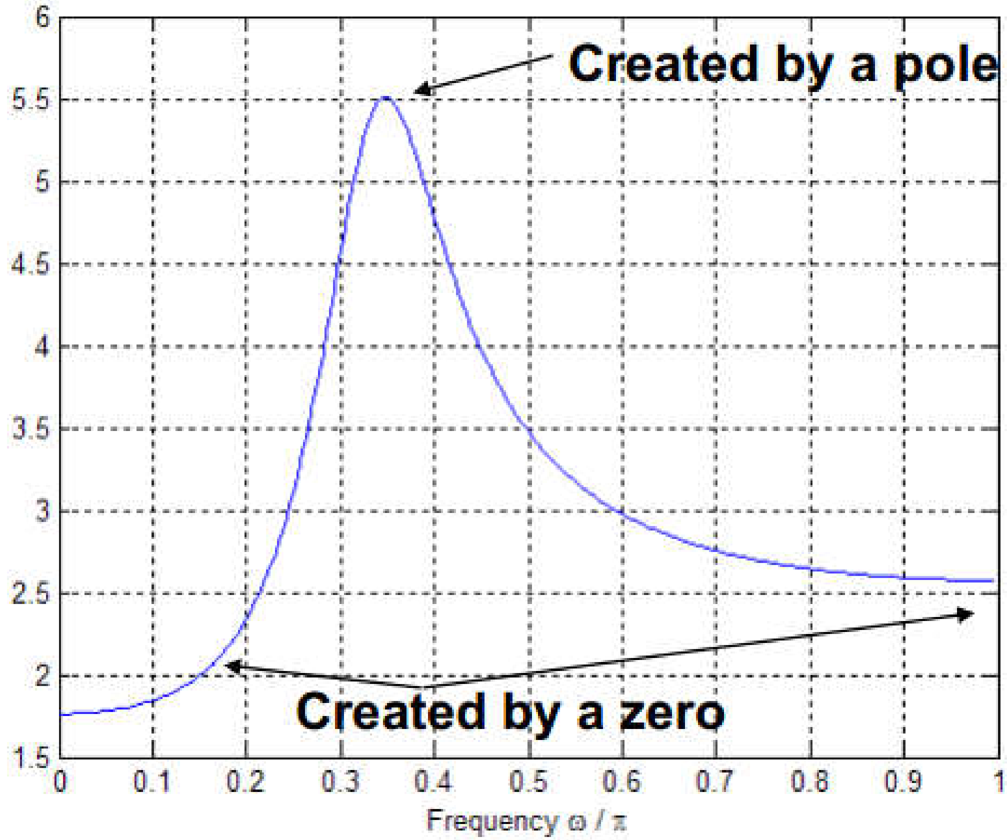
Frequency Response

$$G(z) = \frac{\boxed{1 - 2.4z^{-1} + 2.88z^{-2}}}{\boxed{1 - 0.8z^{-1} + 0.64z^{-2}}}$$

(Numerator) b- coefficients

(Denominator) a- coefficients

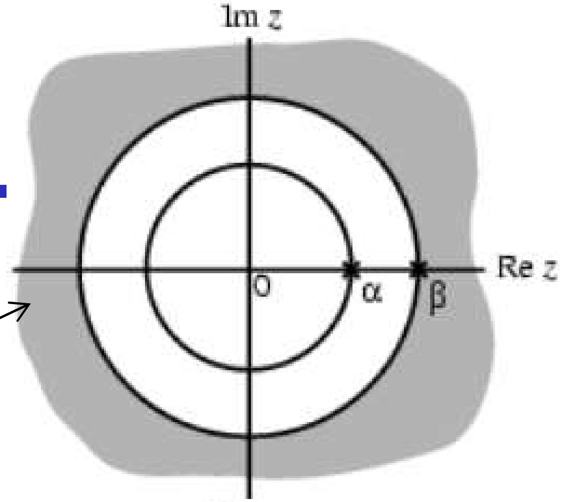
```
[H w]=freqz([1 -2.4 2.88],[1 -0.8 0.64],256);
figure
plot(w/pi, abs(H))
grid
title('Transfer function')
xlabel('Frequency \omega / \pi')
```

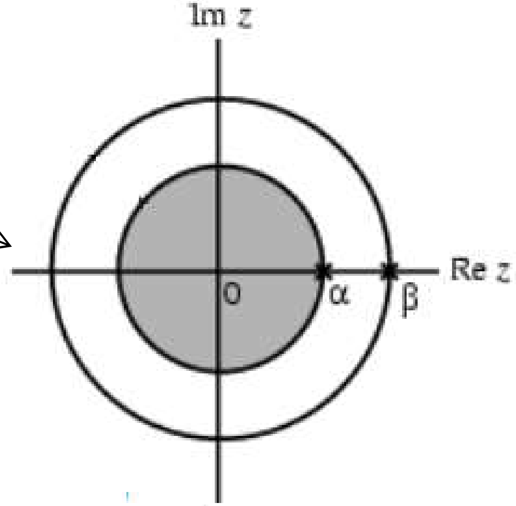


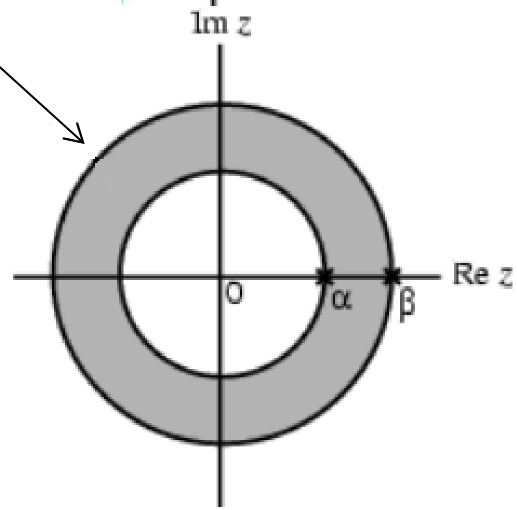
This system has two zeros at $z=1.2 \pm j1.2$ and two poles at $z=0.4 \pm j0.6928$

Poles and Zeros

- The ROC of a rational z-transform cannot contain any poles and is bounded by the poles
  - For a right sided sequence, the ROC is outside of the largest pole
  - For a left sided sequence, the ROC is inside of the smallest pole
  - For a two sided sequence, some of the poles contribute to terms in the parent sequence for $n<0$ and other to terms for $n>0$. Therefore, the ROC is between two circular regions: outside of the largest pole coming from the $n>0$ sequence and inside of the smallest pole coming from the $n<0$ sequence.
  - If the sequence is of finite length, then the ROC includes the entire z-plane, except possibly $z=0$ and/or $z=\infty$.







Stability & ROC

- Now, for an LTI system to be stable it must be absolutely summable, or in other words, it must have a DTFT. But for a system to have a DTFT, its ROC must include the unit circle.

⇒ An LTI system is stable, if and only if the ROC of its transfer function H(z) includes the unit circle!

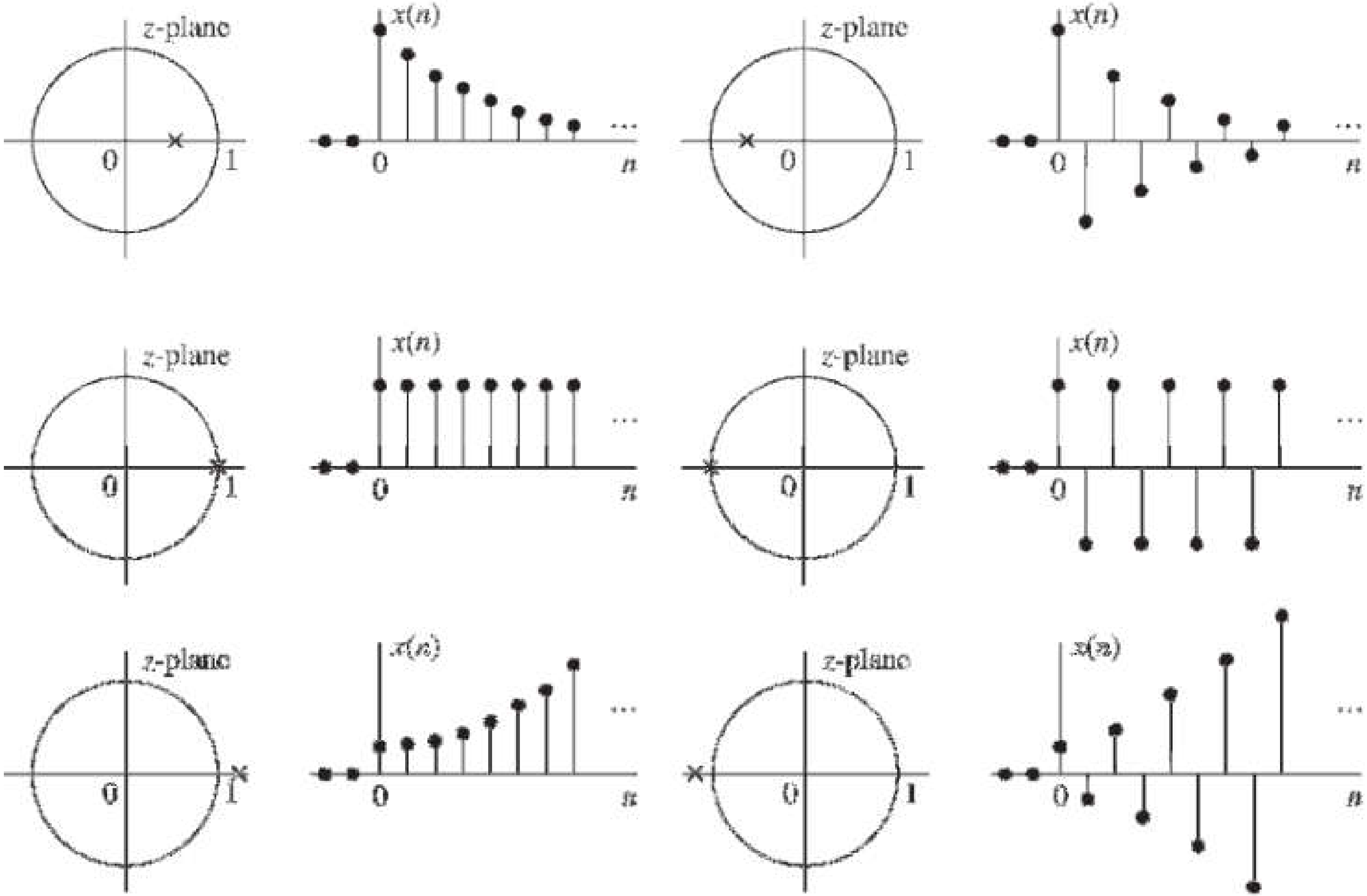
- Furthermore, a causal system's ROC lies outside of a pole circle. If that system is also stable, its ROC must include unit circle

⇒ Then a causal system is stable, if and only if, all poles are inside the unit circle!

- Similarly, an anti-causal system is stable, if and only if its poles lie outside the unit circle.

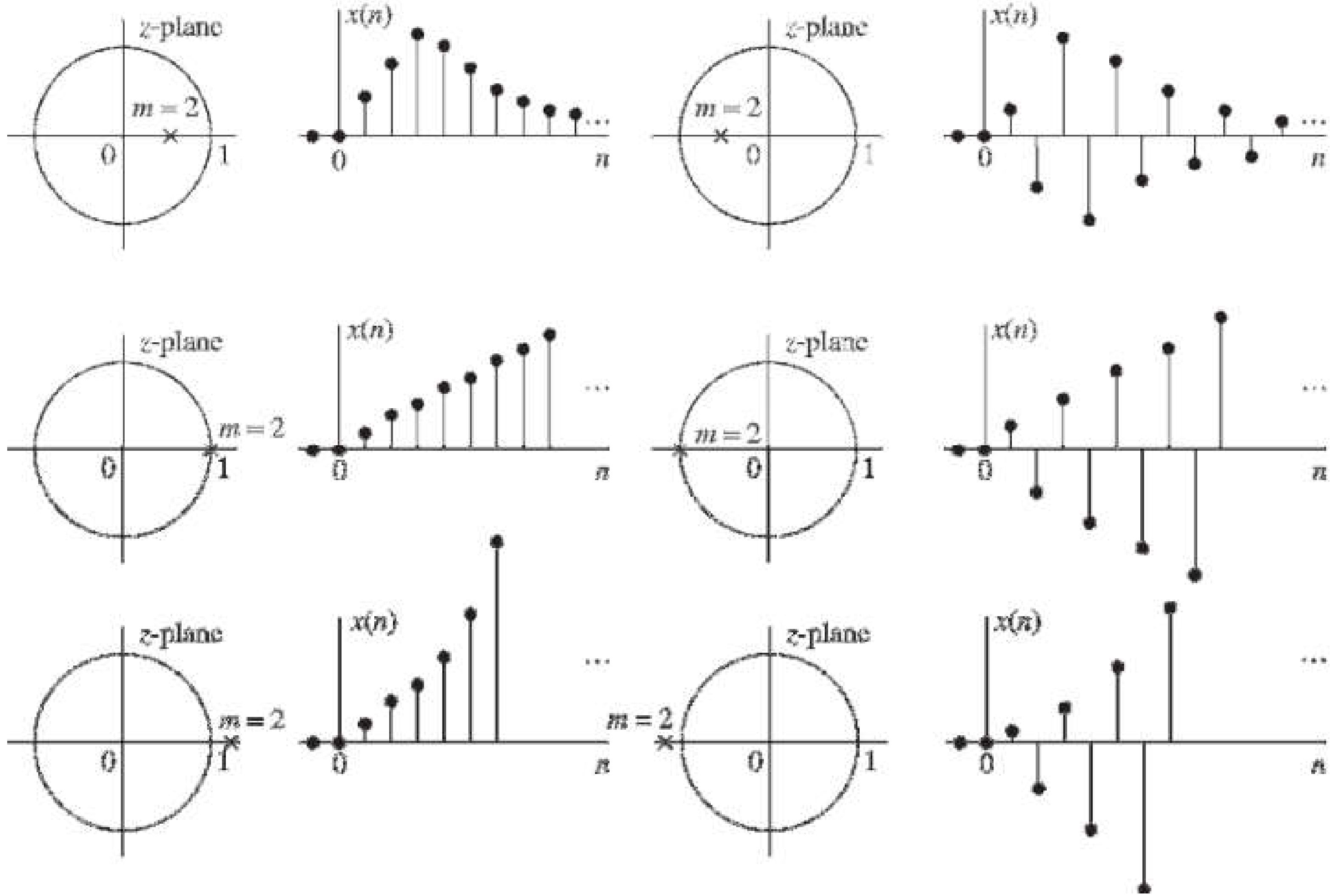
Behavior of a Single Real-Pole Causal Signal

$$x(n) = a^n u(n) \overset{z}{\longleftrightarrow} X(z) = \frac{1}{1 - az^{-1}}, \quad \text{ROC: } |z| > |a|$$



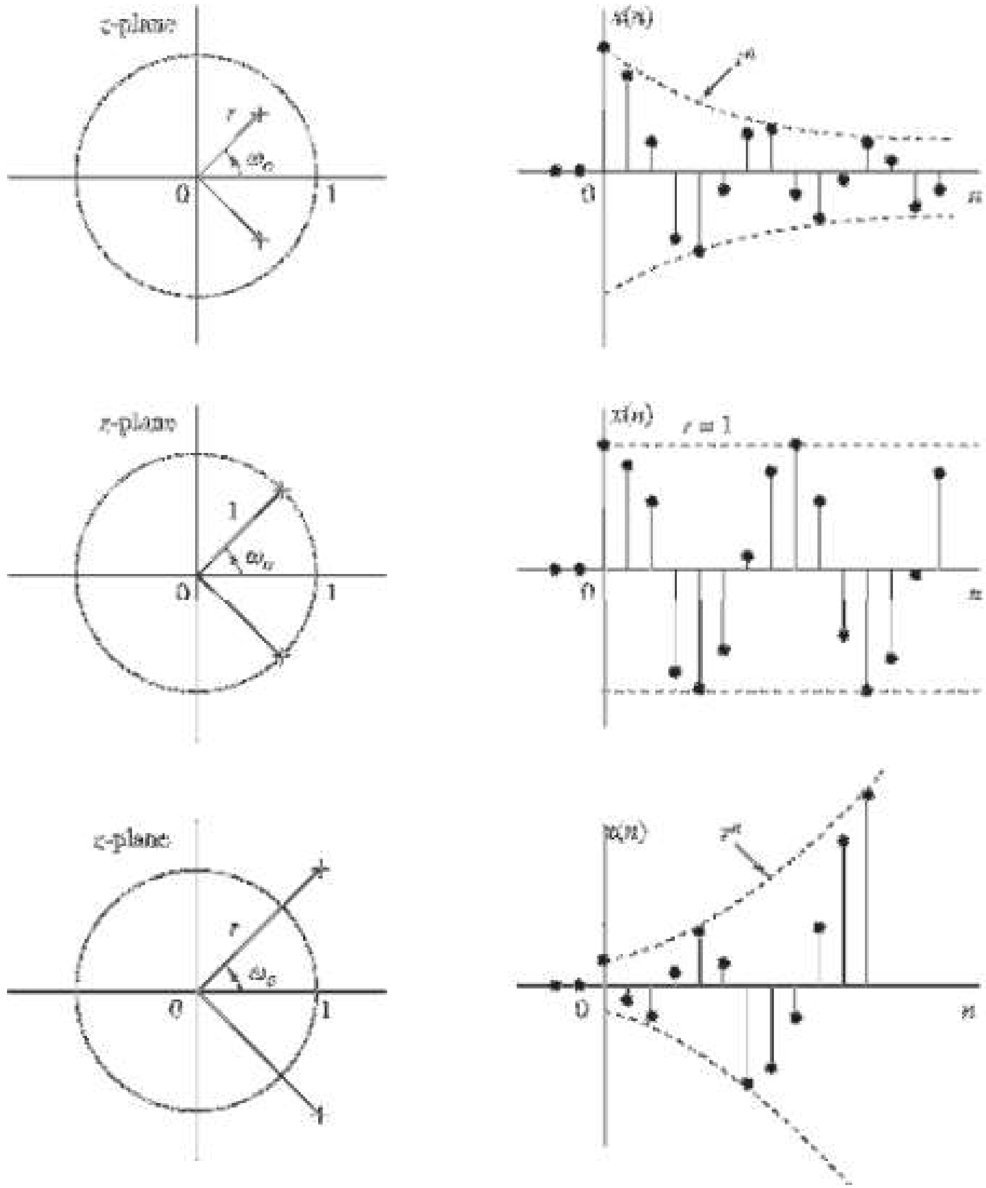
Behavior of a Double Real-Pole Causal Signal

$$x(n) = na^n u(n) \overset{z}{\longleftrightarrow} X(z) = \frac{1}{\left(1-az^{-1}\right)^2}, \quad \text{ROC: } |z| > |a|$$



Behavior of a Causal Signal with a Pair of Complex-Conjugate Poles

$$\boxed{x(n) = \left(r^n \cos \omega_0 n\right) u(n)}$$



11_5 Wrap up

- Analysing the transfer function:
  - Representation: polynomial ratio VS factor form
  - Roots of numerator ➔ zeroes ➔ H(zero) = 0
  - Roots of denominator ➔ poles ➔ H(pole) $\rightarrow \infty$
    - Pole circles: bounds the ROC

- For an LTI system to be causal and stable
  - ROC includes $|z|=1$
  - Right sided

  $\Big\}$ All poles inside the unit circle

EEE336 Signal Processing and Digital Filtering

Lecture 11 Z-Transform

11_6 Transfer Functions and Frequency Responses

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CCLDE coefficients

- All discrete systems can be represented using **<u>Constant Coefficient, Linear Difference Equations</u>** (CCLDE), of the form

$$\boxed{y[n]+a_1y[n-1]+a_2y[n-2]+\cdots a_N y[n-N]=b_0x[n]+b_1x[n-1]+\cdots+b_M x[n-M]}$$

$$\boxed{\sum_{i=0}^{N} a_i y[n-i]=\sum_{j=0}^{M} b_j x[n-j],\ \ a_0=1}$$

- The function H(z), which is the z-transform of the impulse response h[n] of the LTI system, is called the <u>transfer function</u>
  - Using the CCLDE coefficients

$$\boxed{H(z)=\frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}=z^{(N-M)}\frac{\sum_{k=0}^{M} b_k z^{M-k}}{\sum_{k=0}^{N} a_k z^{N-k}}=\frac{b_0}{a_0}\cdot\frac{\prod_{k=1}^{M}(1-\zeta_k z^{-1})}{\prod_{k=1}^{N}(1-p_k z^{-1})}=\frac{b_0}{a_0}z^{(N-M)}\frac{\prod_{k=1}^{M}(z-\zeta_k)}{\prod_{k=1}^{N}(z-p_k)}}$$

CCLDE coefficients

Zeros & poles

Zero & pole factors

Frequency response and the transfer function

- If the ROC of the transfer function H(z) includes the unit circle, then the frequency response H(ω) of the LTI digital filter can be obtained simply as follows:

$$H(\omega) = H\left(e^{j\omega}\right) = H(z)\Big|_{z=e^{j\omega}}$$

- So the frequency response of a typical LTI system is

$$\boxed{H(z) = \frac{b_0}{a_0} z^{(N-M)} \frac{\prod_{k=1}^{M}(z-\zeta_k)}{\prod_{k=1}^{N}(z-p_k)}} \Rightarrow \boxed{H\left(e^{j\omega}\right) = \frac{b_0}{a_0} e^{j\omega(N-M)} \frac{\prod_{k=1}^{M}\left(e^{j\omega}-\zeta_k\right)}{\prod_{k=1}^{N}\left(e^{j\omega}-p_k\right)}}$$

- From which we can obtain the magnitude and phase response

$$\left|H\left(e^{j\omega}\right)\right| = \left|\frac{b_0}{a_0}\right| \frac{\prod_{k=1}^{M}\left|e^{j\omega}-\zeta_k\right|}{\prod_{k=1}^{N}\left|e^{j\omega}-p_k\right|}$$

$$\arg H\left(e^{j\omega}\right) = \arg\left(b_0 / a_0\right) + \omega(N-M) + \sum_{k=1}^{M}\arg\left(e^{j\omega}-\zeta_k\right) - \sum_{k=1}^{N}\arg\left(e^{j\omega}-p_k\right)$$

Frequency response and the transfer function

$$\left|H\left(e^{j\omega}\right)\right| = \left|\frac{b_0}{a_0}\right| \frac{\prod_{k=1}^{M}\left|e^{j\omega} - \zeta_k\right|}{\prod_{k=1}^{N}\left|e^{j\omega} - p_k\right|}$$

– The magnitude response |H(ω)| at a specific value of ω is given by the product of the distances to all zeros divided by the product of the distances to all poles!

$$\arg H\left(e^{j\omega}\right) = \arg\left(b_0 / a_0\right) + \omega\left(N - M\right) + \sum_{k=1}^{M} \arg\left(e^{j\omega} - \zeta_k\right) - \sum_{k=1}^{N} \arg\left(e^{j\omega} - p_k\right)$$

– The phase response at a specific value of ω is obtained by adding the phase of the term b_0/a_0 and the linear-phase term ω(N-M) to the sum of the angles of the zero vectors minus the angles of the pole vectors

Frequency response by pole and zero distances

- Example:
  - The transfer function of a filter has zeros at $z_o = r_o e^{\pm j\theta_o}$ and poles at $z_p = r_p e^{\pm j\omega_p}$, thus

$$H(z) = \frac{(z - r_o e^{j\theta_o})(z - r_o e^{-j\theta_o})}{(z - r_p e^{j\theta_p})(z - r_p e^{-j\theta_p})} = \frac{z^2 - 2r_o \cos\theta_o z + r_o^2}{z^2 - 2r_p \cos\theta_p z + r_p^2}$$

  - Choosing $r_o = 1.2$, $\theta_o = 30^0$, $r_{\mathrm{p}} = 0.9$, $\theta_{\mathrm{p}} = 60^0$

$$H(z) = \frac{z^2 - 2.078z + 1.440}{z^2 - 0.900z + 0.810}$$

  - The frequency response at ω is given by

$$H(e^{j\omega}) = \frac{(e^{j\omega} - r_o e^{j\theta_0})(e^{j\omega} - r_o e^{-j\theta_0})}{(e^{j\omega} - r_p e^{j\theta_p})(e^{j\omega} - r_p e^{-j\theta_p})} = \frac{\vec{u}_1 \vec{u}_2}{\vec{u}_3 \vec{u}_4}$$

    - Where the $\vec{u}_k$ are complex phasors pointing from a zero or pole to the point $e^{j\omega}$ on the unit circle.

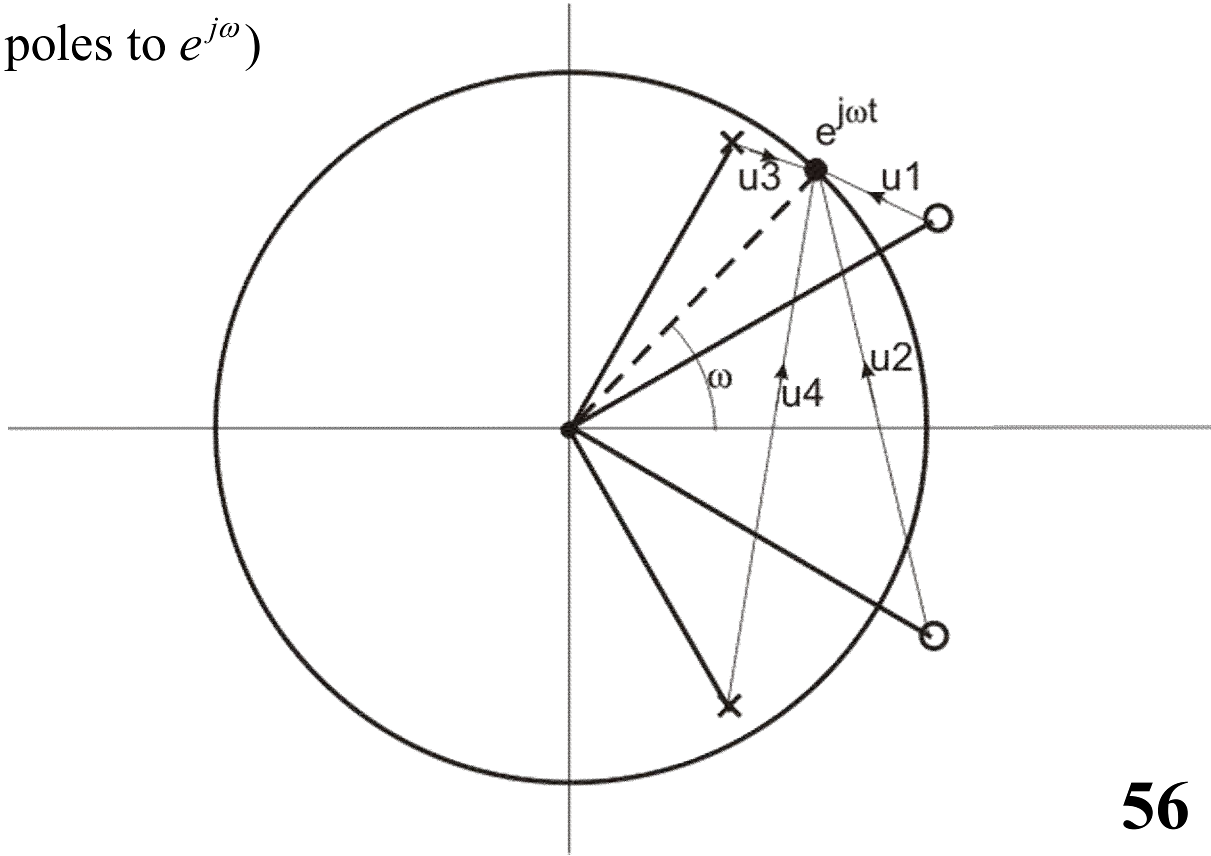
Frequency response by pole and zero distances

- Thus we may evaluate the frequency response at a given frequency in terms of the magnitudes and angles of the phasors $\vec{u}_k$

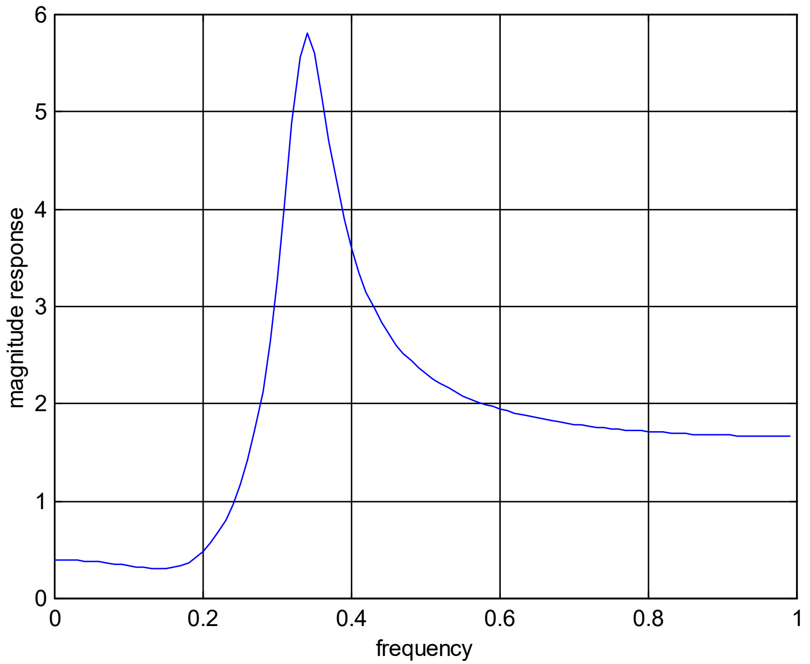
$$\left|H(e^{j\omega})\right| = \frac{\text{product of distances to zeros}}{\text{product of distances to poles}}$$

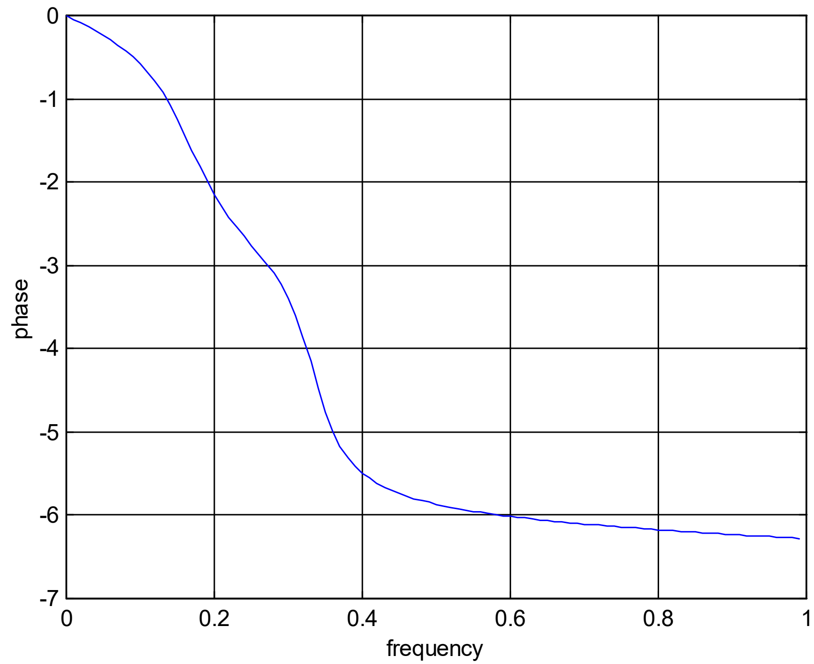
$$Arg\{H(e^{j\omega})\} = (\text{sum of angles from zeros to } e^{j\omega})$$
$$-(\text{sum of angles from poles to } e^{j\omega})$$

This pole – zero diagram shows u_1 u_2 u_3 and u_4 for our example



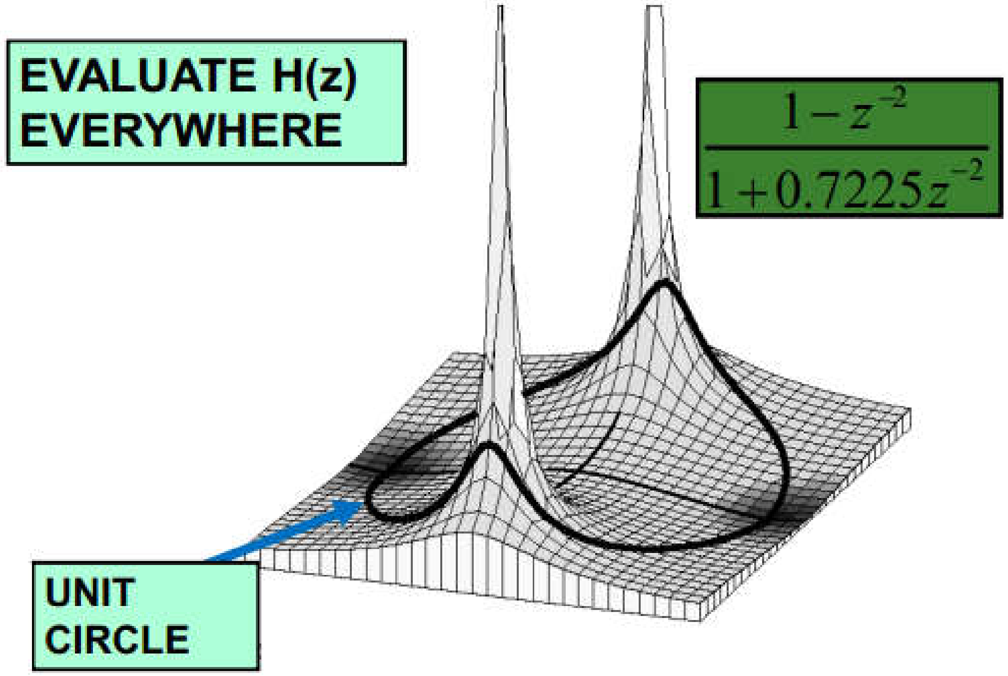
- An approximate plot of the magnitude and phase responses of the transfer function of an LTI digital filter can be developed by examining the pole and zero locations
- Now, the frequency response has the smallest magnitude around ω=ζ, and the largest magnitude around ω=p.
- Of course, at ω=p, the response is infinitely large, and at ω=ζ, the response is zero





Frequency response by pole and zero distances

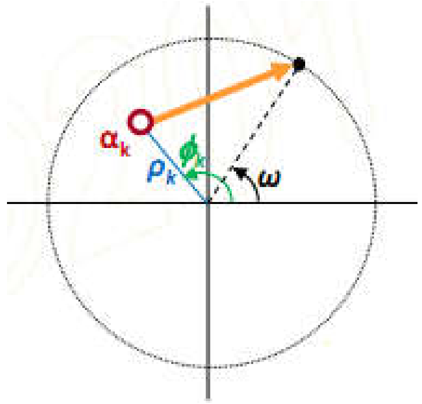
- Therefore:
  - To highly attenuate signal components in a specified frequency range, we need to place zeros very close to or on the unit circle in this range.
  - Likewise, to highly emphasize signal components in a specified frequency range, we need to place poles very close to or on the unit circle in this range.



Graphical interpretation

$$\left|H\left(e^{j\omega}\right)\right| = \left|\frac{b_0}{a_0}\right| \frac{\prod_{k=1}^{M}\left|e^{j\omega} - \zeta_k\right|}{\prod_{k=1}^{N}\left|e^{j\omega} - p_k\right|}$$

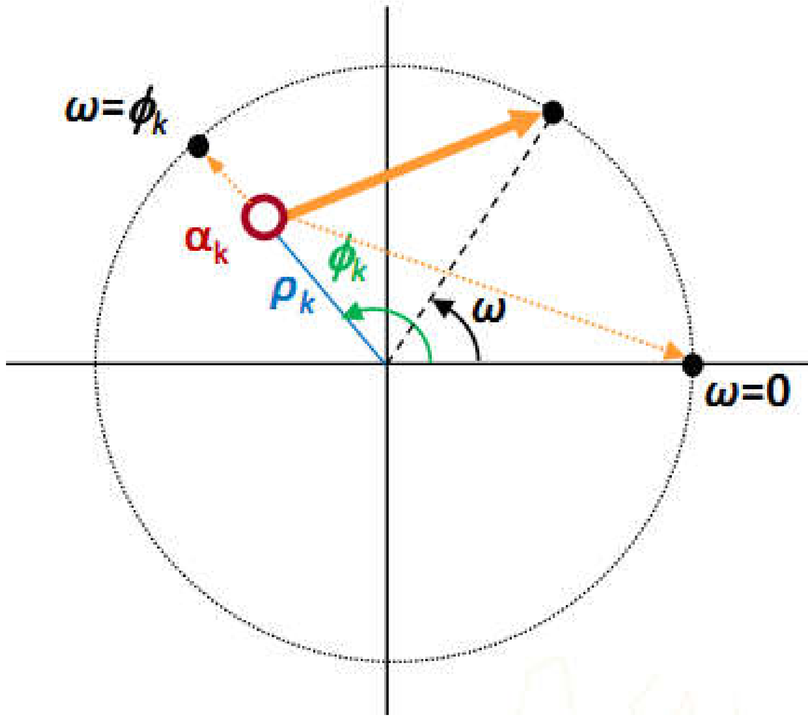
- Complex vector
  - α_k is in general a complex quantity, let's write that as $\alpha_k = \rho_k e^{j\phi_k}$
  - Then we have $e^{j\omega} - \alpha_k = e^{j\omega} - \rho_k e^{j\phi_k}$
  - the term $e^{j\omega} - \rho_k e^{j\phi_k}$ represents a vector in the z-plane, that starts at the point $z = \rho_k e^{j\phi_k}$ and ends at the point $z = e^{j\omega}$, which is on the unit circle
  - As ω varies from 0 to 2π, the tip of this vector moves counter-clockwise tracing the unit circle.



Zero vectors: α_k is ξ_k, in numerator;

Pole vectors: α_k is p_k, in denominator.

- The magnitude response $|H(\omega)|$, at a given frequency ω, is the product of the magnitude (length of orange vector) of all zeros, divided by the magnitude of all poles, as evaluated at that ω.
  - If α_k is a zero (i.e., a numerator factor), the overall magnitude vector of $H(\omega)$ will be small at frequencies around φ_k, and will be exactly zero if α_k is on the unit circle, causing $H(\omega_k) = 0$.
  - Conversely, if α_k is a pole (i.e., a denominator factor), the overall magnitude vector of $H(\omega)$ will be large at frequencies around φ_k, and will go to infinity if α_k is on the unit circle.
  - This is why the zeros and the poles that are at or close to the unit circle have a larger impact on the overall frequency response than those that are further away from the unit circle.



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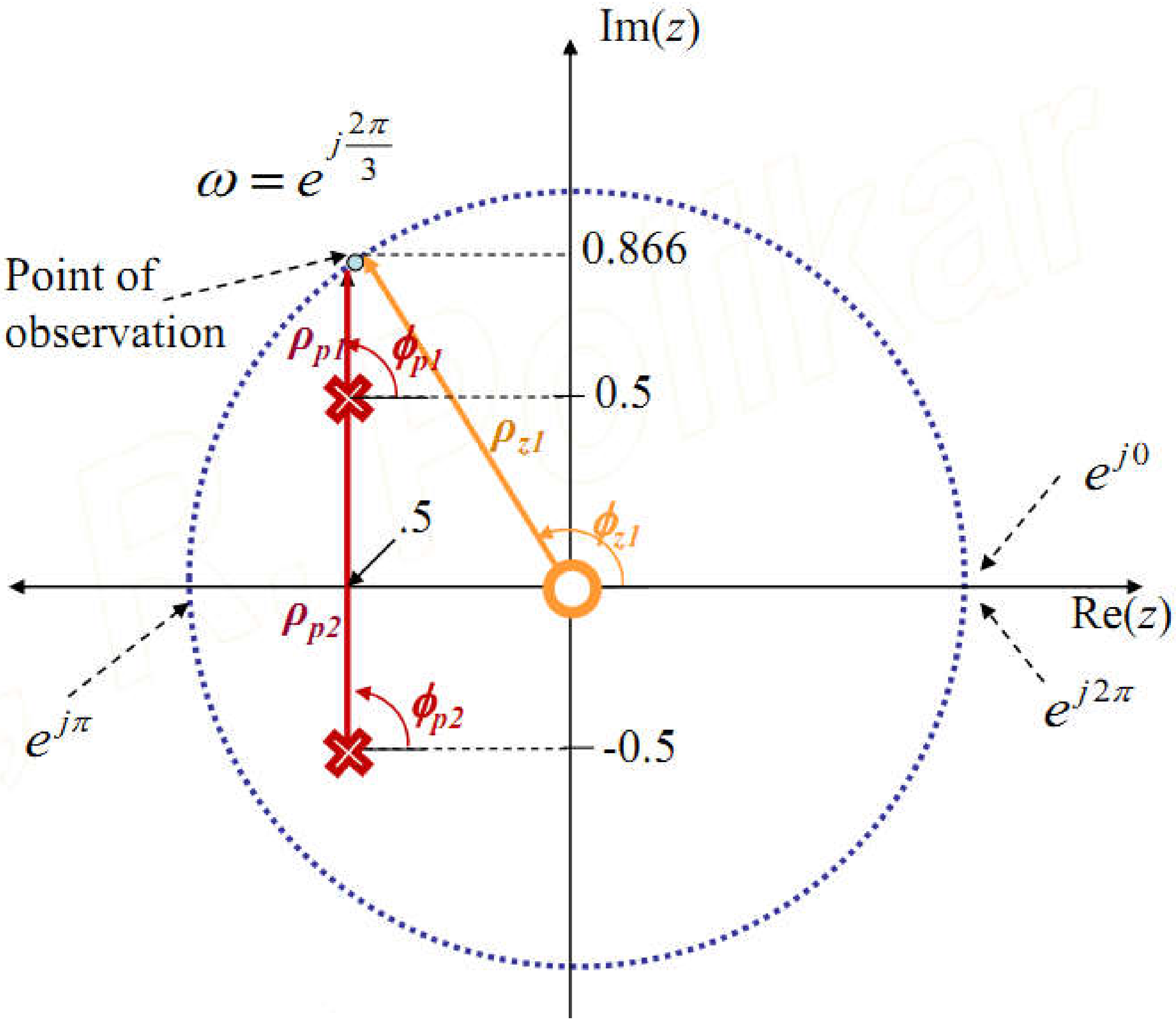
Graphical interpretation – An example

$$H(z) = \frac{z^{-1}}{1 + z^{-1} + 0.5z^{-2}} = \frac{z}{z^2 + z + 0.5}$$



One zero at z = 0

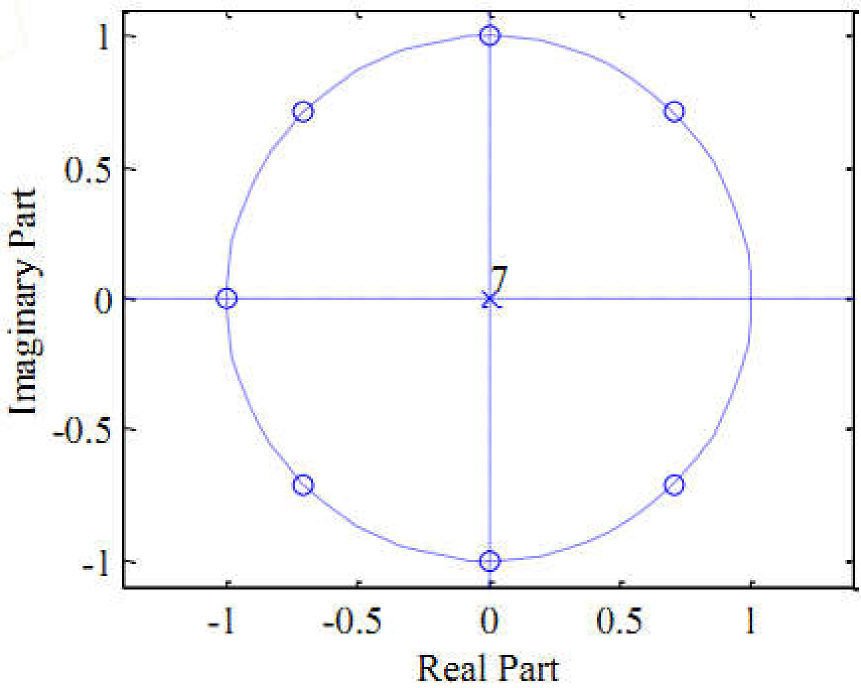
One pair of conjugate poles at z = -0.5000 ± j0.5000

Graphical interpretation – Another example

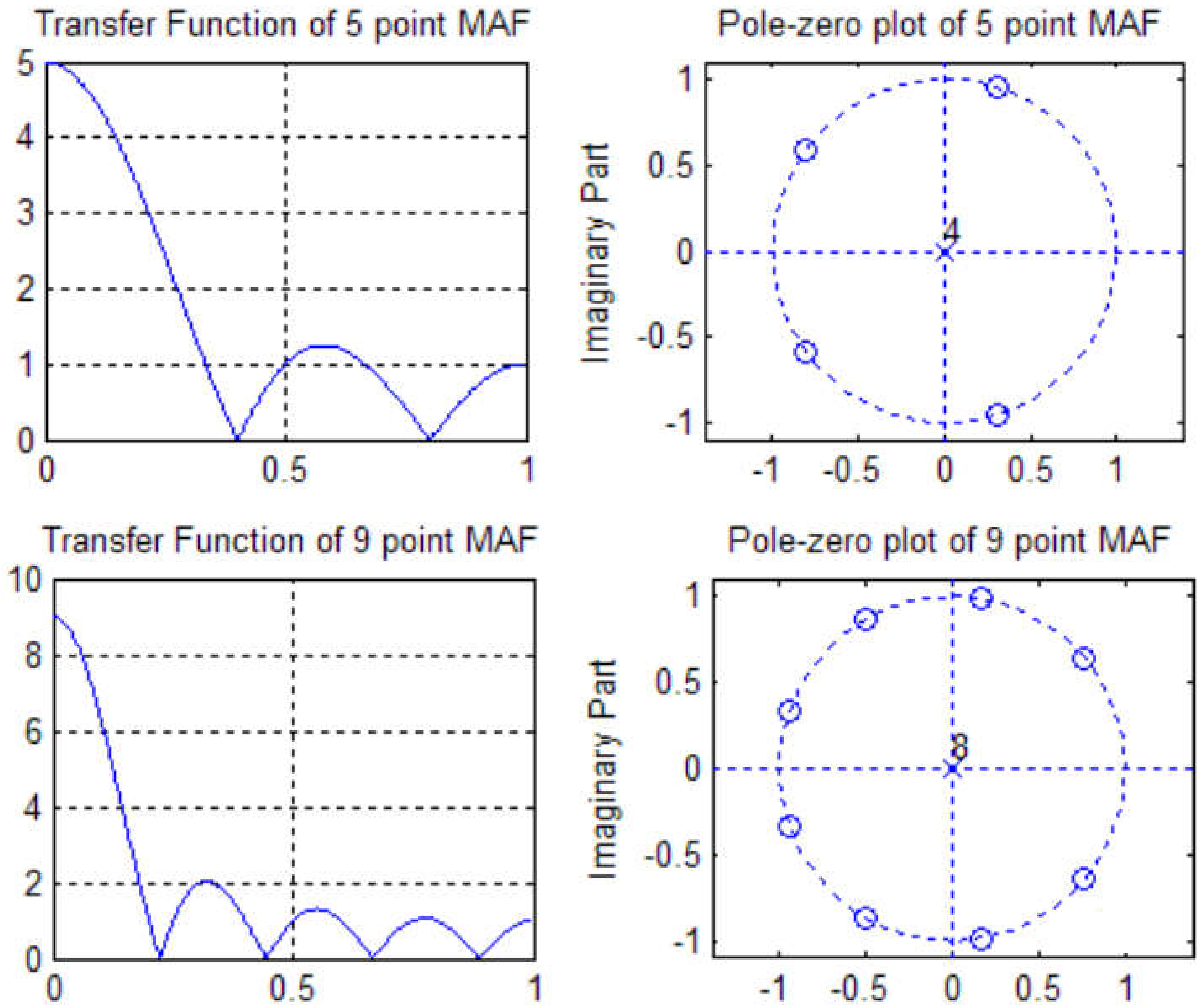
- Consider the M-point moving-average FIR filter with an impulse response $h[n]=\begin{cases}1/M, & 0\le n\le M-1\\ 0, & \text{otherwise}\end{cases}$

$$H(z)=\frac{1}{M}\sum_{n=0}^{M-1}z^{-n}=\frac{1-z^{-M}}{M\left(1-z^{-1}\right)}=\frac{z^{M}-1}{M\left[z^{M}(z-1)\right]}$$

  - The transfer function has M zeros on the unit circle at $z=e^{j2\pi k/M}$,$0\le k\le M-1$
  - There are M-1 poles at z = 0 and a single pole at z = 1
  - The pole at z = 1 exactly cancels the zero at z = 1
  - The ROC is the entire z-plane except z = 0



Moving Average Filter



11_6 Wrap up

- Frequency response
  - Magnitude response
  - Phase response

Z-transform
(Zero-pole positions)

- Graphic explanation
- Filter design based on zero position arrangement

Chapter 11 Summary

- DTFT ←→ Z-transform
- Z-transform
  - Definition
  - ROC
  - Properties
  - Inverse
  - Zeroes and poles
- Frequency response (DTFT)