Xi'an Jiaotong-Liverpool University
西交利物浦大学

EEE204 Continuous and Discrete Time Signals and Systems II

2018–2019 Semester 2

Electrical and Electronic Engineering

Xi'an Jiaotong-Liverpool University

Week 12

Causality and Stability

Causality

A system is causal if, for every choice of n_0, the output sequence value at the index $n = n_0$ depends only on the input sequence values for $n \leqslant n_0$.

The impulse response of such a system does not have values for $n < 0$.

Consider a causal system described by difference equation

$$y[n] = x[n-1] + x[n-2],$$

its impulse response is given by

$$h[n] = \delta[n-1] + \delta[n-2].$$

Causality

The z-transform (transfer function) of the impulse response is defined by

$$H(z) = \sum_{n=0}^{\infty} h[n]z^{-n} = z^{-1} + z^{-2},$$

which does not include any positive powers of z. Consequently, the ROC includes infinity.

We then have the following principle: A DT LTI system is causal iff the ROC of its transfer function is the exterior of a circle, including infinity.

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Causality

For $H(z)$ is rational, then: A DT LTI system with rational system function $H(z)$ is causal iff: (a) the ROC is the exterior of a circle outside the outermost pole; and (b) with $H(z)$ expressed as a ratio of polynomials in z, the order of the numerator cannot be greater than the order of the denominator.

Example 1: $H(z) = \dfrac{1 - 2z^{-1} + z^{-2}}{z^{-1} + z^{-2} - z^{-3}}$, non-causal.

Example 2: $H(z) = \dfrac{1}{1 - 2z^{-1}} - \dfrac{1}{1 - 3z^{-1}}, |z| > 3$, causal.

Stability

The stability of a DT LTI system is equivalent to its impulse response being absolutely summable.

$$\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=-\infty}^{\infty} |h[n]||e^{-j\omega n}| < \infty,$$

which means the DTFT of $h[n]$ must converge and hence the ROC of $H(z)$ must include the unit circle.

We have the following principle: An LTI system is stable iff the ROC of its transfer function $H(z)$ includes the unit circle, $|z| = 1$.

Stability

It is perfectly possible for a system to be stable but not causal.

However, if we combine the principles of causality and the stability check, we have:

A causal LTI system with rational transfer function $H(z)$ is stable iff all of the poles of $H(z)$ lie inside the unit circle.

Example: $H(z) = \dfrac{1}{1 - \frac{1}{3}z^{-1}} + \dfrac{1}{1 - \frac{1}{4}z^{-1}}, |z| > \dfrac{1}{3},$

causal and stable.

Digital Filter Implementations

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LTI systems can be described by general linear constant-coefficient difference equations(LCCDE) (or just **dif-feq** systems):

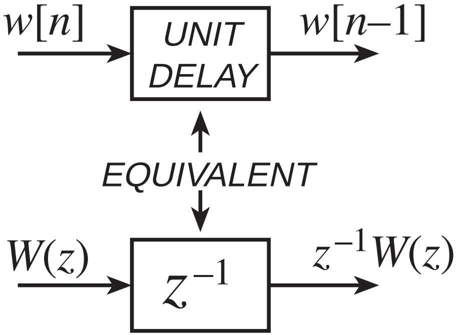
$$y[n] = -\sum_{k=1}^{N} a_k y[n-k] + \sum_{k=0}^{M} b_k x[n-k].$$

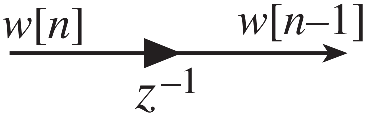
That is to say, LTI systems can be built up from delays elements, adders, and multipliers. Below is the z-transform of the above LCCDE,

$$Y(z) = -\sum_{k=1}^{N} a_k z^{-k} Y(z) + \sum_{k=0}^{M} b_k z^{-k} X(z).$$

LTI Building Blocks

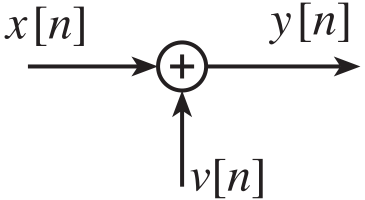
Delay

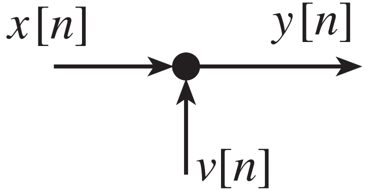




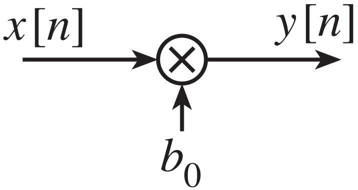
Adder and multiplier

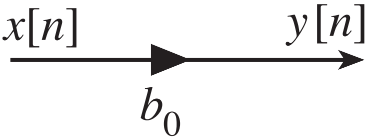
Adder:





Multiplier:





Direct Form

2nd order signal flow graph

The implementation of using adders, multipliers and delay elements is called the direct form.

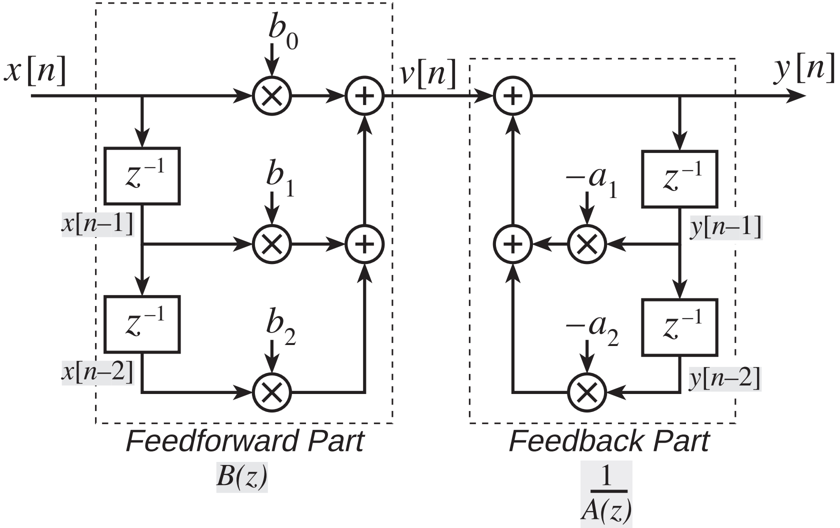
If we take $N = M = 2$,

$$
\begin{aligned}
y[n] &= -\sum_{k=1}^{2} a_k y[n-k] + \sum_{k=0}^{2} b_k x[n-k], \\
&= -a_1 y[n-1] - a_2 y[n-2] \\
&\quad + b_0 x[n] + b_1 x[n-1] + b_2 x[n-2].
\end{aligned}
$$

Direct Form

2^{nd} order signal flow graph

$$y[n] = -a_1y[n-1]-a_2y[n-2]+b_0x[n]+b_1x[n-1]+b_2x[n-2].$$



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2^{nd} order signal flow graph

Note the intermediate signal $v[n]$ is:

$$v[n] = b_0 x[n] + b_1 x[n-1] + b_2 x[n-2],$$

and the output can be written in terms of $v[n]$,

$$y[n] = -a_1 y[n-1] - a_2 y[n-2] + v[n].$$

In the z-domain we have,

$$V(z) = B(z)X(z) \quad \text{and} \quad Y(z) = \frac{1}{A(z)} \cdot V(z).$$

2nd order signal flow graph

We can switch the order of the two LTI sub-system in:

$$V(z) = B(z)X(z) \quad \text{and} \quad Y(z) = \frac{1}{A(z)} \cdot V(z),$$

and we get,

$$W(z) = \frac{1}{A(z)} \cdot X(z) \quad \text{and} \quad Y(z) = B(z)W(z).$$

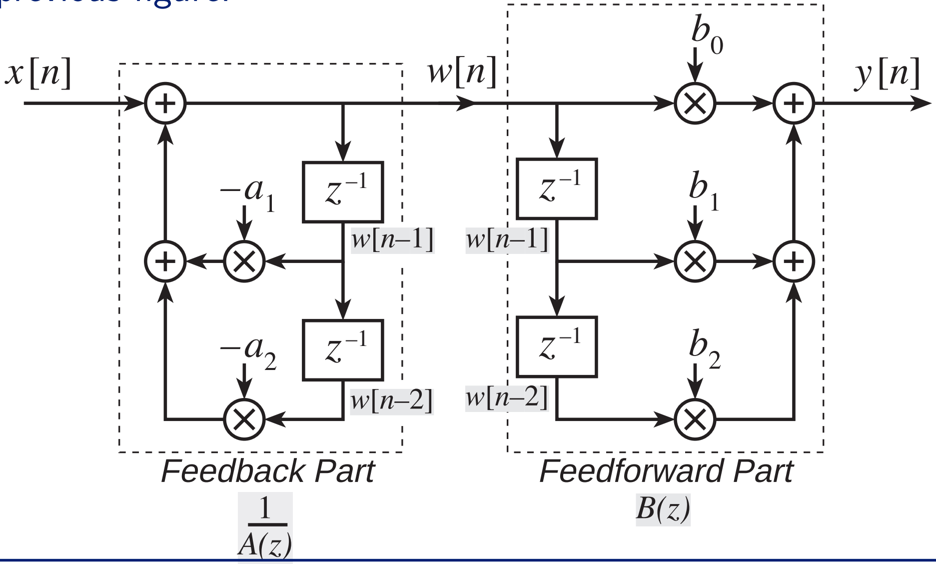
The overall transfer function is still the same,

$$Y(z) = B(z)W(z) = B(z)\left[\frac{1}{A(z)}X(z)\right] = \frac{B(z)}{A(z)}X(z).$$

Direct Form

2^{nd} order signal flow graph

For the signal flow graph, we just switch the order of the previous figure:



Direct Form

2^{nd} order signal flow graph

Now the intermediate signal $w[n]$ is:

$$w[n] = -a_1 w[n-1] - a_2 w[n-2] + x[n],$$

and the output can be written in terms of $w[n]$,

$$y[n] = b_0 w[n] + b_1 w[n-1] + b_2 w[n-2].$$

In the z-domain we have,

$$W(z) = \frac{1}{A(z)} \cdot X(z) \quad \text{and} \quad Y(z) = B(z)W(z).$$

Direct Form II

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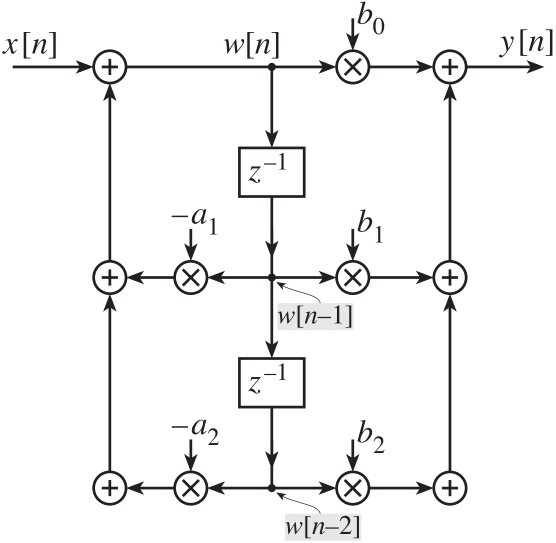
2nd order signal flow graph

If we merge the delay elements in the signal flow graph, then we get the standard structure called direct form II,



Transposed Form

Transposed Structures

From direct form to transposed structure

The Transposition Theorem will create a new structure with exactly the same transfer function. Here are the steps:

- Reverse the direction of all branches (Multipliers and Delays).
- Change summing nodes into branch points.
- Change branch points into summing nodes.
- Relabel the input and output.

Note: a summing node in the original structure has only one output, but multiple inputs. When the branch directions are reversed, it will now have one input and multiple outputs, so it must become a branch point.

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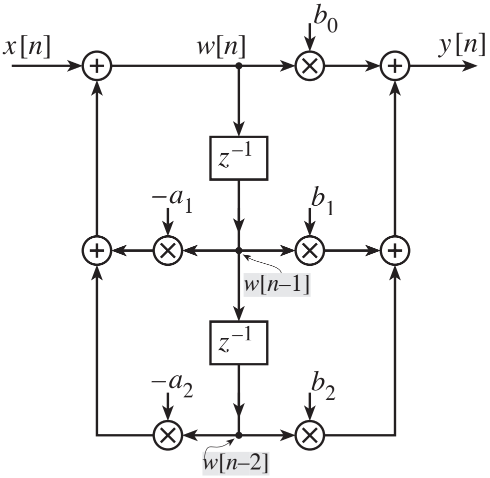
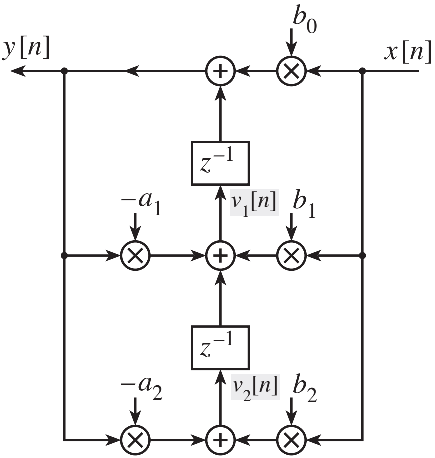
Direct form II v.s. transposed form

Here is a second-order filter shown in direct form II (left) and transposed form (right).





Transposed Direct Form II

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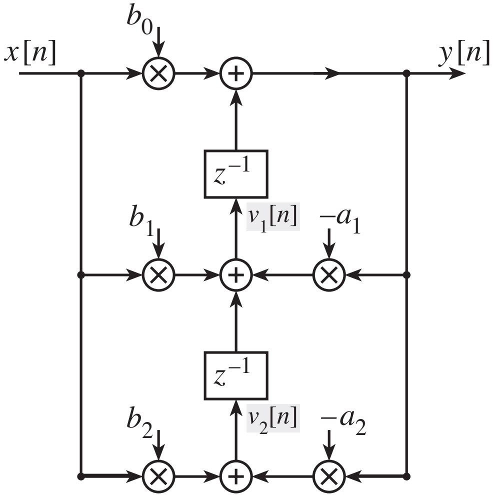
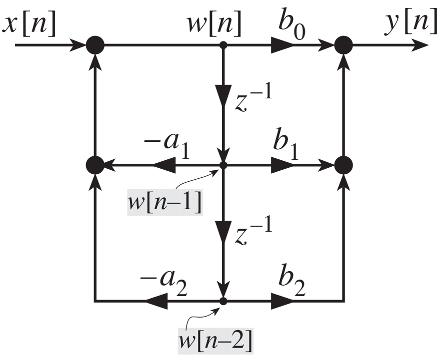
Transposed direct form II

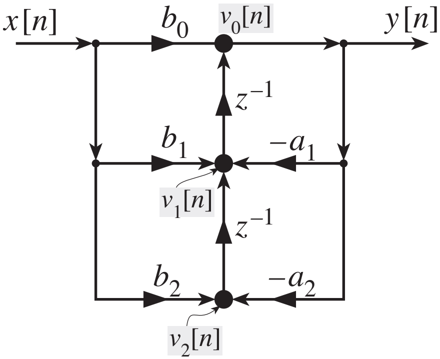
If we put the input signal on the left, then we get the standard structure called transposed direct form II.



Direct form II v.s. transposed direct form II

Here is a second-order filter shown in direct form II (left) and transposed direct form II (right).





The signal flow graphs are in place of explicit block diagrams for elements.

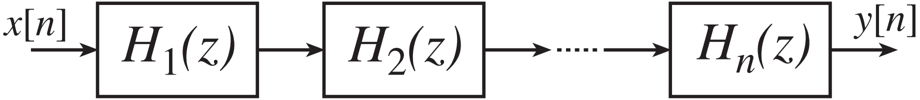
Cascade Form

Cascade Form

If we factor $H(z)$ into low-order factors (i.e., second-order sections):

$$H(z) = H_1(z)H_2(z)H_3(z)\cdots H_n(z),$$
$$\frac{B(z)}{A(z)} = \left[\frac{B_1(z)}{A_1(z)}\right]\left[\frac{B_2(z)}{A_2(z)}\right]\left[\frac{B_3(z)}{A_3(z)}\right]\cdots\left[\frac{B_n(z)}{A_n(z)}\right].$$

In block diagram, the cascade is,



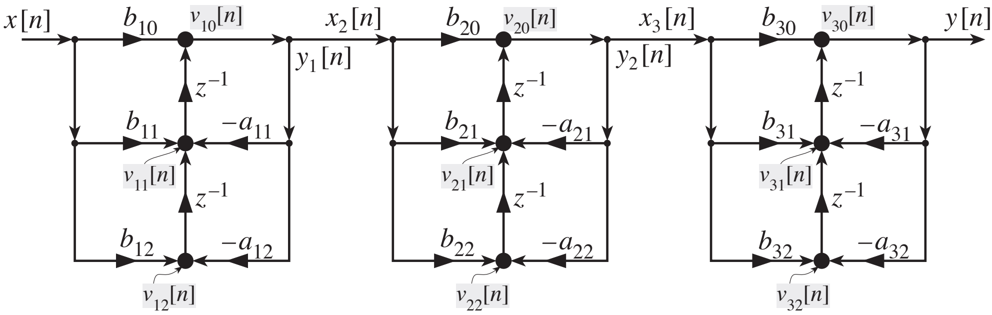
Cascade Form

If each of the low-order system is a second-order section, we can then factorise to get the coefficients and write (for an order of 6):

$$H(z) = \left(\frac{b_{10} + b_{11}z^{-1} + b_{12}z^{-2}}{1 + a_{11}z^{-1} + a_{12}z^{-2}}\right)\left(\frac{b_{20} + b_{21}z^{-1} + b_{22}z^{-2}}{1 + a_{21}z^{-1} + a_{22}z^{-2}}\right) \times \left(\frac{b_{30} + b_{31}z^{-1} + b_{32}z^{-2}}{1 + a_{31}z^{-1} + a_{32}z^{-2}}\right)$$

We can always group the poles and zeros into complex conjugate pairs, so that the coefficients of the second-order sections will be real when the coefficients of $B(z)$ and $A(z)$ are real.

If we use the transposed direct form II in the cascade structure, here is the result for a sixth-order filer,



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Parallel Form

Parallel Form

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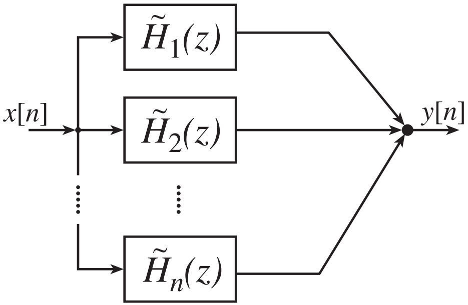
If we factor $H(z)$ into a partial fraction form with low-order terms (i.e., second-order sections):

$$H(z) = \tilde{H}_1(z) + \tilde{H}_2(z) + \tilde{H}_3(z) + \cdots + \tilde{H}_n(z),$$

$$\frac{B(z)}{A(z)} = \left[\frac{C_1(z)}{A_1(z)}\right] + \left[\frac{C_2(z)}{A_2(z)}\right] + \left[\frac{C_3(z)}{A_3(z)}\right] + \cdots + \left[\frac{C_n(z)}{A_n(z)}\right].$$

In block diagram, the parallel form is,



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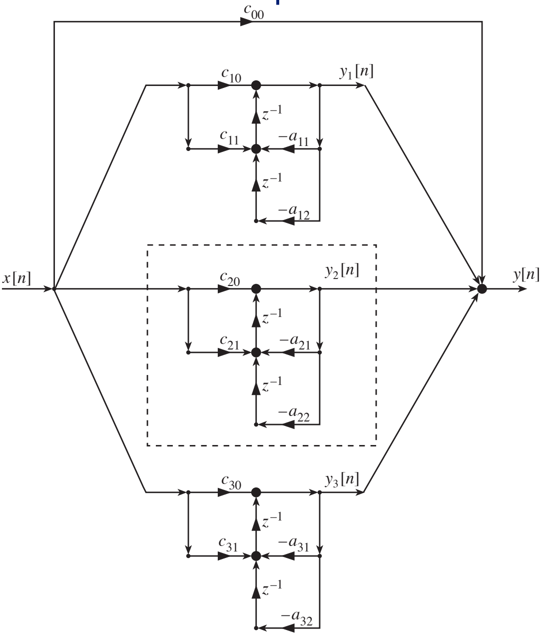
If each of the low-order system is a second-order section, we can then expand out the coefficients and get (for three section):

$$H(z) = c_{00} + \left(\frac{c_{10} + c_{11} z^{-1}}{1 + a_{11} z^{-1} + a_{12} z^{-2}} \right) + \left(\frac{c_{20} + c_{21} z^{-1}}{1 + a_{21} z^{-1} + a_{22} z^{-2}} \right) + \left(\frac{c_{30} + c_{31} z^{-1}}{1 + a_{31} z^{-1} + a_{32} z^{-2}} \right)$$

Parallel Form

Transposed direct form II in the parallel structure



Flow Graph to LCCDE

Flow Graph to LCCDE

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General rules

Starting with the signal flow graph, it is possible to specify a procedure for writing down the difference equations described by the flow graph, and also for deriving the transfer function of the system.

- Give an unique signal name to the input of every delay branch.
- Label the output of each delay branch.
- Write the summation equations at each summing node.
- Combine the equations to get $H(z)$.

Flow Graph to LCCDE

General rules

Not all signal flow graphs are computable, but if there are no delay-free loops, then a recursive computation is possible. Furthermore, the flow graph will specify the order of computation by virtue of its graphical representation of the algorithm.

- After step #2 all branches in the signal flow graph should be labelled, so when the summation equations are written every adder input is known.
- The result of step #3 is the computational algorithm described by the signal flow graph. Each signal flow graph specifies an unique implementation.
- In the last step, a z-domain approach is needed when there are coupled equations.

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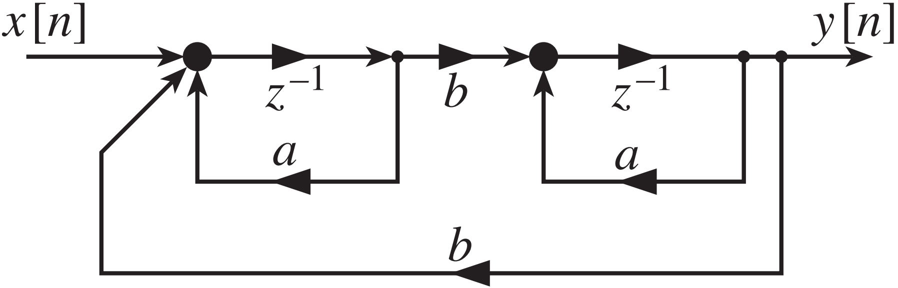
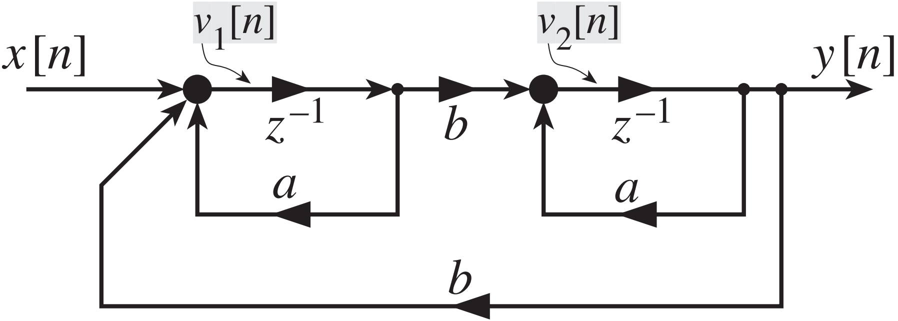
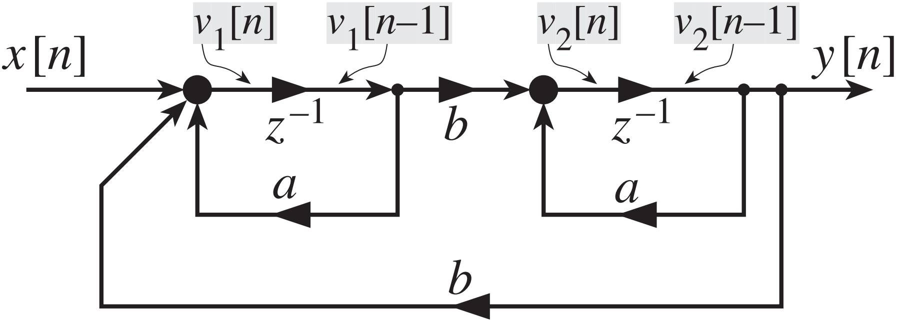
Label the flow graph and write out the corresponding difference equation.



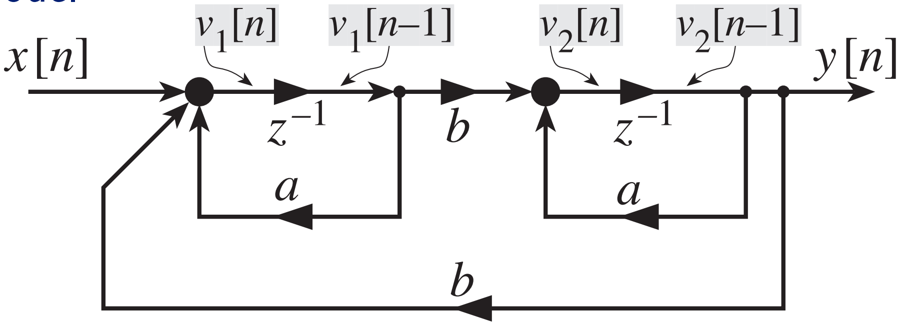
Give an unique signal name to the input of every delay branch.



Label the output of each delay branch.



Write the summation equations at each summing node.



$$y[n] = v_2[n-1],$$
$$v_2[n] = bv_1[n-1] + av_2[n-1],$$
$$v_1[n] = x[n] + by[n] + av_1[n-1].$$

Example

Combine the equations to get $H(z)$

$$
\begin{aligned}
y[n] &= v_2[n-1], \\
v_2[n] &= bv_1[n-1] + av_2[n-1], \\
v_1[n] &= x[n] + by[n] + av_1[n-1],
\end{aligned}
$$

Take z-transform for all the above equations:

$$
\begin{aligned}
Y(z) &= z^{-1}V_2(z), \\
V_2(z) &= bz^{-1}V_1(z) + az^{-1}V_2(z), \\
V_1(z) &= X(z) + bY(z) + az^{-1}V_1(z),
\end{aligned}
$$

Example

Combine the equations to get $H(z)$

$$
\begin{aligned}
Y(z) &= z^{-1}V_2(z), \\
V_2(z) &= bz^{-1}V_1(z) + az^{-1}V_2(z), \\
V_1(z) &= X(z) + bY(z) + az^{-1}V_1(z),
\end{aligned}
$$

$$
\begin{aligned}
Y(z) &= z^{-1}V_2(z), \\
V_2(z) &= \frac{bz^{-1}}{1 - az^{-1}}V_1(z), \\
V_1(z) &= \frac{1}{1 - az^{-1}}X(z) + \frac{b}{1 - az^{-1}}Y(z).
\end{aligned}
$$

Example

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Combine the equations to get $H(z)$

$$
\begin{aligned}
Y(z) &= z^{-1}V_2(z), \\
V_2(z) &= \frac{bz^{-1}}{1-az^{-1}}V_1(z), \\
V_1(z) &= \frac{1}{1-az^{-1}}X(z) + \frac{b}{1-az^{-1}}Y(z),
\end{aligned}
$$

$$
Y(z) = z^{-1}\left[\frac{bz^{-1}}{1-az^{-1}}\right]\left[\frac{1}{1-az^{-1}}X(z) + \frac{b}{1-az^{-1}}Y(z)\right],
$$

$$
\left[1 - \frac{b^2z^{-2}}{(1-az^{-1})^2}\right]Y(z) = \frac{bz^{-2}}{(1-az^{-1})^2}X(z),
$$

Example

Combine the equations to get $H(z)$

$$\left[1-\frac{b^2z^{-2}}{(1-az^{-1})^2}\right]Y(z)=\frac{bz^{-2}}{(1-az^{-1})^2}X(z),$$

$$Y(z)=\frac{bz^{-2}}{1-2az^{-1}+(a^2-b^2)z^{-2}}X(z),$$

The difference equation is given by

$$y[n]-2ay[n-1]+(a^2-b^2)y[n-2]=bx[n-2].$$

Homework

- Page 776–777, read section 10.7.1–10.7.2;
- Page 783–789, read section 10.8;
- Page 800, Q10.16: (a)–(c);
- Page 800, Q10.17: (a)–(b);
- Page 800, Q10.18: (a)–(b);
- Page 804, Q10.31;
- Page 805, Q10.34: (a)–(c);
- Page 805, Q10.35–10.36;
- Page 805, Q10.37: (a)–(b);
- Page 805–806, Q10.38: (a)–(e).

Thank you for your attention.