MTH101: Lecture 1

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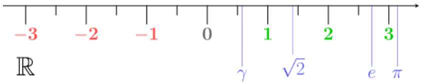
Review of Real Numbers

Integers : $\ldots, -4, -3, -2, -1, 0, 1, 2, 3, \ldots$
Rational Numbers: $\frac{3}{4}, -\frac{2}{1}, \frac{1}{9}, \ldots.$
Irrational Numbers: $\pi, e, \sqrt{2}, \ldots$



Real numbers can be thought of as points on an infinitely long line called the Real number line.

Imaginary Unit (An unreal number)

Question:

$$\sqrt{-1} = ?$$

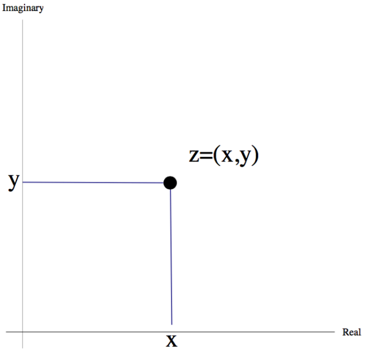
Answer:

$$\sqrt{-1} = i \;\Rightarrow\; i^2 = -1$$

Remark

The **imaginary unit** *which solves $x^2 = -1$ is denoted by i.*

We call **Real Axis** the Horizontal Axis (whose unit is 1) of a **Cartesian coordinate system** and **Imaginary Axis** the Vertical one (whose unit is i).



Geometric Representation of Complex Numbers

Definition

A complex number $z = (x, y)$ is a point in the plane.
The first coordinate is called **Real Part**

$$\mathrm{Re}\ z = x,$$

the second coordinate is called **Imaginary Part**

$$\mathrm{Im}\ z = y.$$

Remark

The **Imaginary unit** *using the Geometric Representation is*

$$i = (0, 1).$$

Remark

The set of all complex numbers is denoted by

$$\mathbb{C} =: \{z = (x, y) |\ x, y \in \mathbb{R}\}.$$

Note that $\mathbb{R} \subsetneq \mathbb{C}$.

Algebraic Representation of Complex Numbers

Recall that the **Imaginary unit** i such that $i^2 = -1$. Then a **Complex Number** $z \in \mathbb{C}$ can also be written as

$$z = \underbrace{(x, y)}_{\textbf{Geometric}} = \underbrace{x + iy}_{\textbf{Algebraic}}$$

where $x, y \in \mathbb{R}$ are **Real and Imaginary Part** respectively.

Example

$3 + 2i = (3, 2)$;

$\sqrt{3}i = (0, \sqrt{3})$; Note: $z = iy$ is called pure imaginary

$e = (e, 0); \ldots$

Operations

Consider the **Complex Numbers** $z_1 = (x_1, y_1) = x_1 + iy_1$ and $z_2 = (x_2, y_2) = x_2 + iy_2$.

Sum

$$z_1+z_2 = (x_1+iy_1)+(x_2+iy_2) = (x_1+x_2)+i(y_1+y_2) = (x_1+x_2, y_1+y_2).$$

Product

$$\begin{aligned} z_1 \cdot z_2 &= (x_1 + iy_1) \cdot (x_2 + iy_2) = x_1x_2 + ix_1y_2 + ix_2y_1 + i^2 y_1y_2 \\ &= x_1x_2 + (-1)y_1y_2 + i(x_1y_2 + x_2y_1) \\ &= (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1) \\ &= (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1). \end{aligned}$$

Substraction If $z = x + iy$, then its **Additive Inverse** is

$$-z = -x - iy.$$

Using the **Additive Inverse** we can define the **Subtraction**, the **difference** is

$$z_1 - z_2 := z_1 + (-z_2) = (x_1 - x_2) + i(y_1 - y_2).$$

Remark

Two **Complex Numbers** $z_1 = x_1 + iy_1$ *and* $z_2 = x_2 + iy_2$ *are equal if and only if*

$$Re\ z_1 = Re\ z_2, \qquad x_1 = x_2,$$

and

$$Im\ z_1 = Im\ z_2, \qquad y_1 = y_2.$$

Division

Exercise

Find the **Multiplicative Inverse** z^{-1} *of* $z = x + iy$*, and use the inverse to define* **Division***, the* **quotient** *is*

$$\frac{z_1}{z_2} = z_1 \cdot z_2^{-1}, \qquad z_2 \neq 0$$

Solution

Write $z = x + iy$, then

$$\frac{1}{z} = \frac{1}{x+iy} = \frac{1}{x+iy} \cdot \frac{x-iy}{x-iy} = \frac{x-iy}{x^2+y^2} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$$

Continued

Therefore, the **quotient**

$$\frac{z_1}{z_2} = z_1 \cdot z_2^{-1} = (x_1 + iy_1) \cdot \left(\frac{x_2}{x_2^2 + y_2^2} - i \frac{y_2}{x_2^2 + y_2^2} \right)$$
$$= \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + i \frac{x_2 y_1 - x_1 y_2}{x_2^2 + y_2^2}$$

The set $\mathbb{C}$ of complex numbers is a field. (With respect to the sum and product defined above it obeys the following rules of operations)

Commutative Law

$$z_1 + z_2 = z_2 + z_1,$$

$$z_1 \cdot z_2 = z_2 \cdot z_1.$$

Associative Law

$$z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3,$$
$$z_1 \cdot (z_2 \cdot z_3) = (z_1 \cdot z_2) \cdot z_3.$$

Existence of Identities

$$z + 0 = z, \qquad \text{Additive Identity},$$

$$z \cdot 1 = z, \qquad \text{Multiplicative Identity}.$$

Existence of Inverses

$$z + (-z) = 0, \qquad \text{Additive Inverse,}$$

$$z \cdot z^{-1} = 1, \qquad \text{Multiplicative Inverse if } \; z \neq 0$$

Distributivity of multiplication over addition

$$z_1 \cdot (z_2 + z_3) = (z_1 \cdot z_2) + (z_1 \cdot z_3).$$

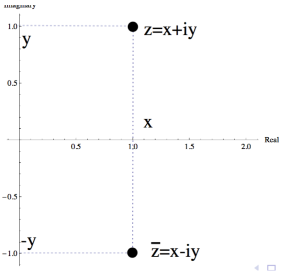
Complex Conjugate of a Complex Number

Definition

If $z = x + iy$ then we define the **Complex Conjugate** of z as

$$\bar{z} = x - iy.$$

The point $\bar{z}$ is the reflection of z with respect to the **Real Axis**.



Remark

If $z = x + iy$, *then*

$$Re\ z = x = \frac{1}{2}(z + \bar{z}); \qquad Im\ z = y = \frac{1}{2i}(z - \bar{z}).$$

Some Properties of the Conjugate.

$$\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2,$$
$$\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2,$$
$$\overline{z_1 \cdot z_2} = \bar{z}_1 \cdot \bar{z}_2$$
$$\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$$

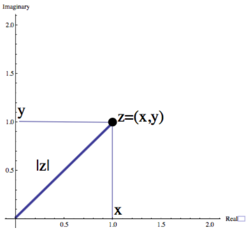
Modulus of a Complex Number

Definition

If $z = x + iy = (x, y)$ we define the **Modulus** of z as

$$|z| = \sqrt{x^2 + y^2}.$$

This quantity represents the distance of the point z from the origin $O = (0, 0)$.



Some Properties of the Modulus

$$
\begin{aligned}
&|z_1 \pm z_2| \leq |z_1| + |z_2|, && \textbf{Triangle Inequality}, \\
&|z_1 \cdot z_2| = |z_1| \cdot |z_2|, \\
&|z^n| = |z|^n, \\
&|z|^2 = z \cdot \bar{z}, \\
&z^{-1} = \frac{\bar{z}}{|z|^2}.
\end{aligned}
$$

Exercise

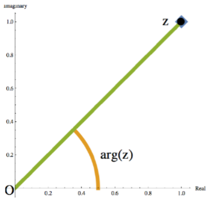
Verify that for any $z_1, z_2 \in \mathbb{C}$ *we have*

$$|z_1 \cdot z_2| = |z_1| \cdot |z_2|.$$

Argument of a Complex Number

Definition

An **argument** of the Complex Number $z = x + iy \neq 0$, denoted by $\arg(z)$, is the angle from the Real Axis to the line Oz (see figure below):
Remark: All angles are measured in radius and positive in the counterclockwise direction!!



Remark

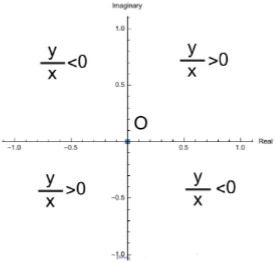
If θ is an argument of z then also $\theta + 2n\pi$, with $n = \pm 1, \pm 2, \ldots$, is also an argument of z.
That is, we can associate to any $z \in \mathbb{C}$ infinite many values of $\arg(z)$.

Definition

The Principal Argument of z, denoted by $\mathrm{Arg}(z)$, is the unique value of $\arg(z)$ which is in the interval $(-\pi, \pi]$. We can write

$$\arg(z) = \mathrm{Arg}(z) + 2n\pi, \quad \text{with} \quad n = 0, \pm 1, \pm 2, \ldots$$

$$\theta = \mathrm{Arg}(z) = \begin{cases} \arctan(\frac{y}{x}), & \text{if } \; x > 0, \\ \arctan(\frac{y}{x}) + \pi, & \text{if } \; x < 0 \; \text{ and } \; y \geq 0, \\ \arctan(\frac{y}{x}) - \pi, & \text{if } \; x < 0 \; \text{ and } \; y < 0, \\ \frac{\pi}{2}, & \text{if } \; x = 0 \; \text{ and } \; y > 0, \\ -\frac{\pi}{2}, & \text{if } \; x = 0 \; \text{ and } \; y < 0. \end{cases}$$



Exercise

Find the expression of $\arg(z)$ *and* $Arg(z)$ *for the following Complex Numbers:*

$$z_1 = 1 + i, \quad z_2 = -1 + i, \quad z_3 = \sqrt{3} - i, \quad z_4 = -\sqrt{3} - i.$$

Bibliography

1 *Kreyszig, E.* **Advanced Engineering Mathematics**. Wiley, 10th Edition.