MTH101: Lecture 6

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Parametrization of a Curve

Consider a **Curve** γ in the Complex Plane. It can be represented by a **Parametrization** in the following way:

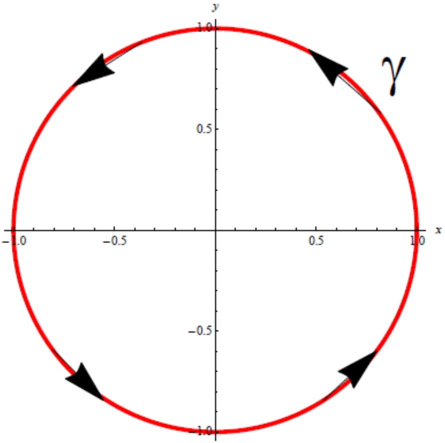
$$\gamma: \qquad z(t) = x(t) + iy(t), \qquad t \in [a, b]$$

Example

Consider the following path:

$$\gamma: \qquad z(t) = \cos t + i \sin t, \quad t \in [0, 2\pi].$$

It is the circle with radius 1 and center $z_0 = 0$. We observe that when t increases the point $z(t)$ moves counterclockwise through the circle. We say that the path γ is **Counterclockwise Oriented.**


y
1.0
0.5
−0.5
−1.0
x
−1.0
−0.5
0.5
1.0
γ

Remark

In general the equation

$$\gamma: \qquad z(t) = x(t) + iy(t), \quad t \in [a, b]$$

defines the **Orientation** *of the path.*
For instance, $z(t) = t + 3ti$ $(0 \leq t \leq 2)$ *describes a line segments on* $y = 3x$ *from* $(0, 0)$ *to* $(2, 6)$*, not the other way around.*

Definition

We say a curve γ *is* **smooth** *if it has continuous derivatives.*

Complex line integral

Definition

Given a partition $\{z_1, z_2, ..., z_n\}$ of the curve γ,the complex line integral is defined as

$$\int_\gamma f(z)\ dz = \lim_{n\to\infty} \sum_{m=1}^{n} f(\zeta_m)\Delta z_m$$

If the curve γ is closed, the integral is denoted by $\oint_\gamma f(z)\ dz$.

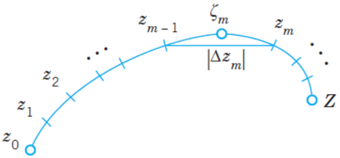


Fig. 340. Complex line integral

ML-Inequality - Darboux Inequality

Theorem

Suppose that the function f is bounded in the domain D, that is, there exists $M > 0$ such that

$$|f(z)| \leq M, \qquad \forall z \in D.$$

Then

$$\left| \int_\gamma f(z)\, dz \right| \leq ML$$

where L is the length of γ.

Example

Find an upper bound for $\int_\gamma z^2\, dz$, γ is the line segment from 0 to $1+i$.

Method 1: Integration by Parametrization

Theorem

Let γ be piecewise smooth path represented by $z(t)$, $t \in [a, b]$. Let $f(z)$ a continuous function on γ. Then

$$\int_\gamma f(z)\, dz = \int_a^b f(z(t))\dot{z}(t)\, dt,$$

where $\dot{z}(t) = \dot{x}(t) + i\dot{y}(t)$.

Basic Properties of Line Integrals

1. **Linearity.** If f_1 and f_2 are two continuous function on the path γ and K_1 and K_2 are two complex constants then

$$\int_\gamma [K_1 f_1(z) + K_2 f_2(z)]\, dz = K_1 \int_\gamma f_1(z)\, dz + K_2 \int_\gamma f_2(z)\, dz.$$

2. **Sense Reversal.** Let $-\gamma$ be the path γ with opposite orientation

$$\int_{-\gamma} f(z)\, dz = -\int_\gamma f(z)\, dz.$$

3. **Additivity.** Let $\gamma = \gamma_1 \cup \gamma_2$ be the union of two paths:

$$\int_\gamma f(z)\ dz = \int_{\gamma_1 \cup \gamma_2} f(z)\ dz = \int_{\gamma_1} f(z)\ dz + \int_{\gamma_2} f(z)\ dz.$$

Example

Consider the Counterclockwise oriented path $\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3$ where

$$\begin{aligned} &\gamma_1 \text{ is the segment joining } z_1 \text{ to } z_2, \\ &\gamma_2 \text{ is the segment joining } z_2 \text{ to } z_3, \\ &\gamma_3 \text{ is the segment joining } z_3 \text{ to } z_1, \end{aligned}$$

and

$$z_1 = 0, \qquad z_2 = 1 + i, \qquad z_3 = 2i.$$

Compute the integral

$$I = \oint_\gamma \bar{z} dz.$$

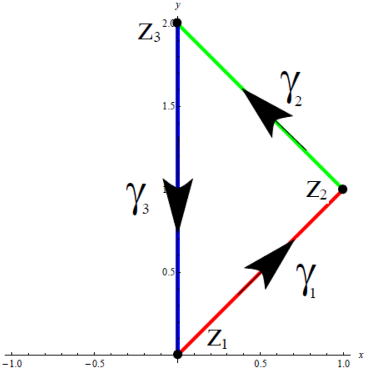


Figure: The path $\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3$

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Solution

We use the additivity so that

$$I = \oint_{\gamma} \bar{z}\ dz = \oint_{\gamma_1 \cup \gamma_2 \cup \gamma_3} \bar{z}\ dz = \oint_{\gamma_1} \bar{z}\ dz + \oint_{\gamma_2} \bar{z}\ dz + \oint_{\gamma_3} \bar{z}\ dz.$$

Then start by writing a Parametrization $z(t) = x(t) + iy(t)$ of the 3 paths:

$$\gamma_1 : \quad \begin{cases} x(t) = t, \\ y(t) = t, \end{cases} \quad t \in [0,1], \qquad \text{or} \quad z(t) = t + it, \quad t \in [0,1].$$

Along γ_1 we have that $\dot{z}(t) = 1 + i, t \in [0,1]$.

For γ_2 it is easier to consider the opposite orientation $-\gamma_2$:

$$-\gamma_2 : \begin{cases} x(t) = t, \\ y(t) = 2 - t, \end{cases} \quad t \in [0,1], \qquad \text{or} \quad z(t) = t + i(2-t),\ t \in [0,1]$$

Along $-\gamma_2$ we have that $\dot{z}(t) = 1 - i$.
Also for γ_3 we consider the opposite orientation $-\gamma_3$:

$$-\gamma_3 : \begin{cases} x(t) = 0, \\ y(t) = t, \end{cases} \quad t \in [0,2], \quad \text{or} \quad z(t) = it,\ t \in [0,2]$$

Along $-\gamma_3$ we have $\dot{z}(t) = i$.

Then we compute the 3 integrals separately using the formula

$$\oint_{\gamma} f(z)\,dz = \int_a^b f(z(t))\dot{z}(t)dt$$

where $z(t)$ is a parametrization of γ. In this case we have that $f(z(t)) = \bar{z}(t)$.Then:

$$\begin{aligned}
\oint_{\gamma_1} f(z)dz &= \int_0^1 (t - it)(1+i)dt = (1+i)(1-i)\int_0^1 tdt \\
&= 2\int_0^1 tdt = [t^2]_0^1 = 1, \\
\oint_{\gamma_2} f(z)dz &= -\oint_{-\gamma_2} f(z)dz = -\int_0^1 (t - i(2-t))(1-i)dt \\
&= -(1-i)\int_0^1 [t(1+i) - 2i]dt
\end{aligned}$$

$$
\begin{aligned}
&= -(1-i)\left[\frac{t^2}{2}(1+i) - 2it\right]_0^1 \\
&= -(1-i)\left[\frac{(1+i)}{2} - 2i\right] = 1+2i.
\end{aligned}
$$

$$
\begin{aligned}
\oint_{\gamma_3} f(z)dz &= -\oint_{-\gamma_3} f(z)dz = -\int_0^2 (-it)i\ dt \\
&= -\int_0^2 tdt = -[t^2/2]_0^2 = -2
\end{aligned}
$$

We conclude:

$$
\oint_{\gamma} f(z)dz = 1 + 1 + 2i - 2 = 2i
$$

Guideline of integrating by parametrization:

1. Represent the curve γ in the form $z = x(t) + iy(t),\ t \in [a, b]$;
2. Compute the derivative (with respect to t)

$$\dot{z}(t) = dz/dt = \dot{x}(t) + i\dot{y}(t);$$

3. Substitute $z(t)$ for every z in the integral;
4. Integrate $f(z(t))\dot{z}(t)$ from a to b.

Method 2: Indefinite integral and substitute the limits

A **simple closed path** is a closed path that does not intersect or touch it self as shown in figure below.

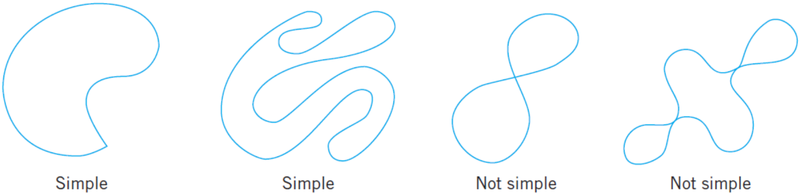


Fig. 345. Closed paths

Definition

A domain D is called **Simply Connected** if any simple closed curve in D encloses only points belonging to D.

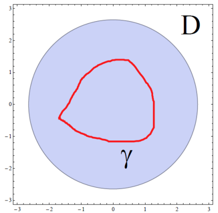


Figure:
The Domain D is an **Open Disk**: $|z - z_0| < R$,
it is **Simply Connected**:

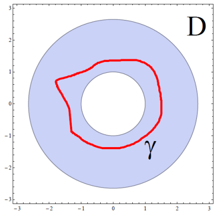


Figure:
The Domain D is an **Open Anulus**: $R_1 < |z - z_0| < R_2$,
it is **NOT Simply Connected**:
the closed path γ encloses points that do not belong to D.

Theorem

Let $f(z)$ be an Analytic function on a Simply Connected Domain D. Then there exists an Analytic function $F(z)$ such that

1. $F'(z) = f(z)$ *in* D
2. *For any path* γ *on* D *with initial point* z_1 *and final point* z_2 *we have:*
$$\int_{\gamma} f(z)\, dz = F(z_2) - F(z_1).$$

Remark

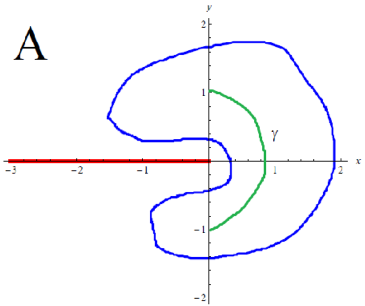
If $f(z)$ is Analytic on a Simply Connected Domain, then the line integral depends only on the initial point z_1 and on the final point z_2.

Example

Let A be the simply connect domain

$$A = \mathbb{C} \setminus \{z \in \mathbb{C} : \mathrm{Im}\ z = 0, \mathrm{Re}\ z \leq 0\}.$$

Integrate $\int_\gamma \frac{1}{z}\, dz$ where γ is a path in A joining $z_1 = -i$ and $z_2 = i$.



Solution: The function

$$f(z) = \frac{1}{z}$$

is Analytic in A. Moreover we have that

$$F(z) = \mathrm{Ln}\ z, \qquad F'(z) = f(z),$$

and $F(z)$ is Analytic in A.
We can use the previous theorem in the set A then, if γ is any path joining $z_1 = -i$ to $z_2 = i$ we have

$$\int_\gamma \frac{1}{z}\, dz = \mathrm{Ln}\ (i) - \mathrm{Ln}\ (-i) = i\mathrm{Arg}\ (i) - i\mathrm{Arg}\ (-i) = \pi i.$$

Guideline of method 2:

1. Check simple connectedness of domain D;
2. Check analyticity of $f(z)$ in D;
3. Find anti-derivative $F(z)$ and check its analyticity in D;
4. Substitute the limits.

Bibliography

1 *Kreyszig, E.* **Advanced Engineering Mathematics**. Wiley, 10th Edition.