2.7 Surface integrals (page 443)

To define a surface integral, we take a piecewise smooth surface S, given by a parametric representation

$$\boldsymbol{r}(u,v) =< x(u,v), y(u,v), z(u,v) >= x(u,v)\mathbf{i}+y(u,v)\mathbf{j}+z(u,v)\mathbf{k},$$

where (u,v) varies over a region R in the uv-plane.
Then S has a normal vector

$$\boldsymbol{N} = \boldsymbol{r}_u \times \boldsymbol{r}_v = \frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v} \quad \text{and unit normal vector} \quad \boldsymbol{n} = \frac{1}{|\boldsymbol{N}|}\boldsymbol{N}$$

at every point (except perhaps for some edges or cusps).

For a given vector function $\boldsymbol{F}$ we can now define the **surface integral** over S by

$$\iint\limits_S \boldsymbol{F} \cdot \boldsymbol{n} dA, \tag{2.12}$$

where $\boldsymbol{n} dA = \boldsymbol{n}|\boldsymbol{N}|dudv = \boldsymbol{N}dudv$. Here $dA = |\boldsymbol{N}|dudv$ is the element of area of S. Therefore the surface integral (2.12) can be written into

$$\iint\limits_S \boldsymbol{F} \cdot \boldsymbol{n} dA = \iint\limits_R \boldsymbol{F}(\boldsymbol{r}(u,v)) \cdot \boldsymbol{N}(u,v)dudv \tag{2.13}$$

Also $\boldsymbol{F} \cdot \boldsymbol{n}$ is the normal component of $\boldsymbol{F}$. This integral arises naturally in flow problems, where it gives the **flux** across S when $\boldsymbol{F} = \rho \boldsymbol{v}$. Recall, from Sec. 9.8, that the flux across S is the mass of fluid crossing S per unit time. Furthermore, ρ is the density of the fluid and $\boldsymbol{v}$ is the velocity vector of the flow, as illustrated by example 1 below. We may thus call the surface integral (2.13) the **flux integral**.

we can write (2.13) in components, using $\boldsymbol{F} =< F_1, F_2, F_3 >$, $\boldsymbol{N} =< N_1, N_2, N_3 >$ and $\boldsymbol{n} =< \cos\alpha, \cos\beta, \cos\gamma >$. Here, α, β, γ are the angles between $\boldsymbol{n}$ and the coordinate axes; indeed, for the angle between $\boldsymbol{n}$ and $\mathbf{i}$, we have $\cos\alpha = \frac{\boldsymbol{n}\cdot\mathbf{i}}{|\boldsymbol{n}||\mathbf{i}|} = \boldsymbol{n}\cdot\mathbf{i}$ and so on. We thus obtain from (2.13)

$$\begin{aligned} \iint\limits_S \boldsymbol{F}\cdot\boldsymbol{n}dA &= \iint\limits_R (F_1\cos\alpha + F_2\cos\beta + F_3\cos\gamma)dA \\ &= \iint\limits_R (F_1N_1 + F_2N_2 + F_3N_3)dudv. \qquad (2.14) \end{aligned}$$

In (2.14) we can write $\cos\alpha dA = dydz$, $\cos\beta dA = dzdx$, $\cos\gamma dA = dxdy$. Then (2.14) becomes the following integral for the flux:

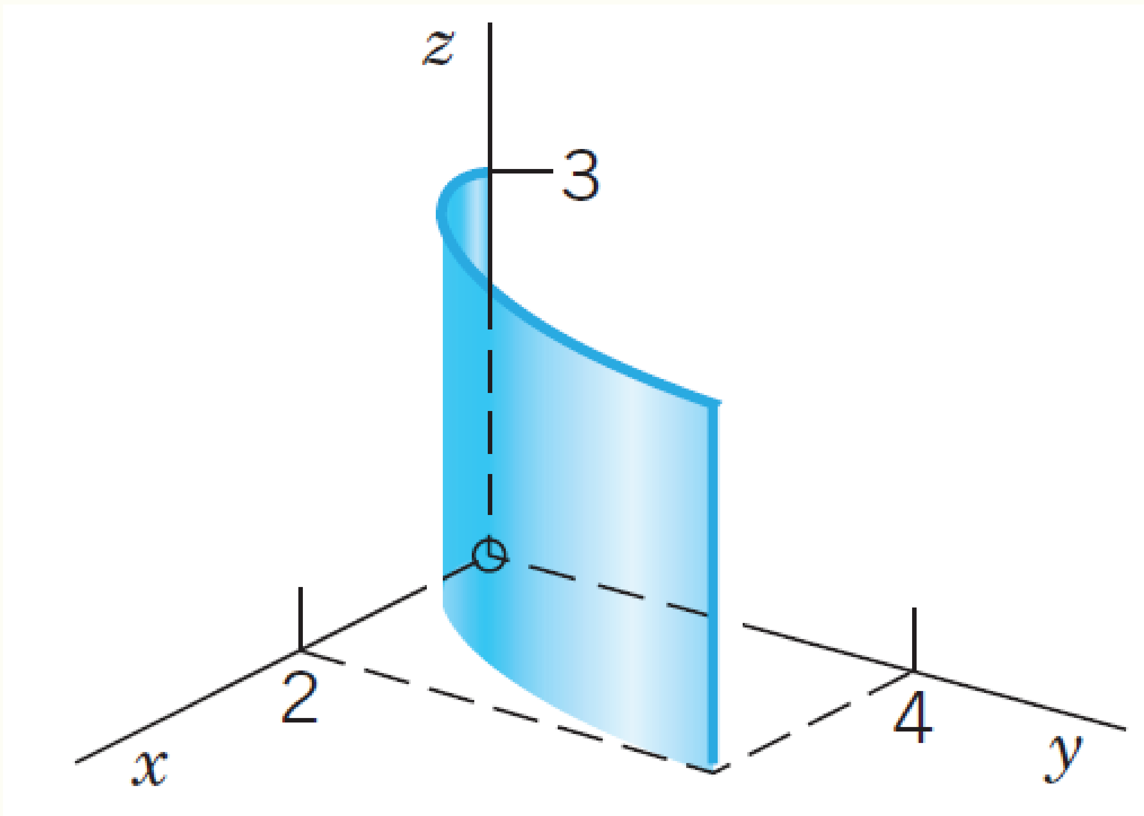
$$\iint\limits_S \boldsymbol{F}\cdot\boldsymbol{n}dA = \iint\limits_S (F_1dydz + F_2dzdx + F_3dxdy). \qquad (2.15)$$

We can use this formula to evaluate surface integrals by converting them to double integrals over regions in the coordinate planes of the xyz-coordinate system. But we must carefully take into account the orientation of S (the choice of $\boldsymbol{n}$).

Notes:

Here we define a surface integral by a double integral over a region R in the uv-plane, like a line integral by a definite integral over an interval $[a, b]$ in the real number set.

Example 1: Flux through a surface.
Compute the flux of water through the parabolic cylinder $S : y = x^2, 0 \leq x \leq 2,\ 0 \leq z \leq 3$ (see the figure below) if the velocity vector is $\boldsymbol{v} = \boldsymbol{F} =< 3z^2, 6, 6xz >$, speed being measured in meters/sec. (Generally, $\boldsymbol{F} = \rho\boldsymbol{v}$, but water has the density $\rho = 1\text{g/cm}^3 = 1\text{ton/m}^3$.)



Solution: Writing $x = u$ and $z = v$, we have $y = x^2 = u^2$. Hence a representation of S is

$$\boldsymbol{r}(u, v) =< u, u^2, v > .$$

So the normal vector of the surface is

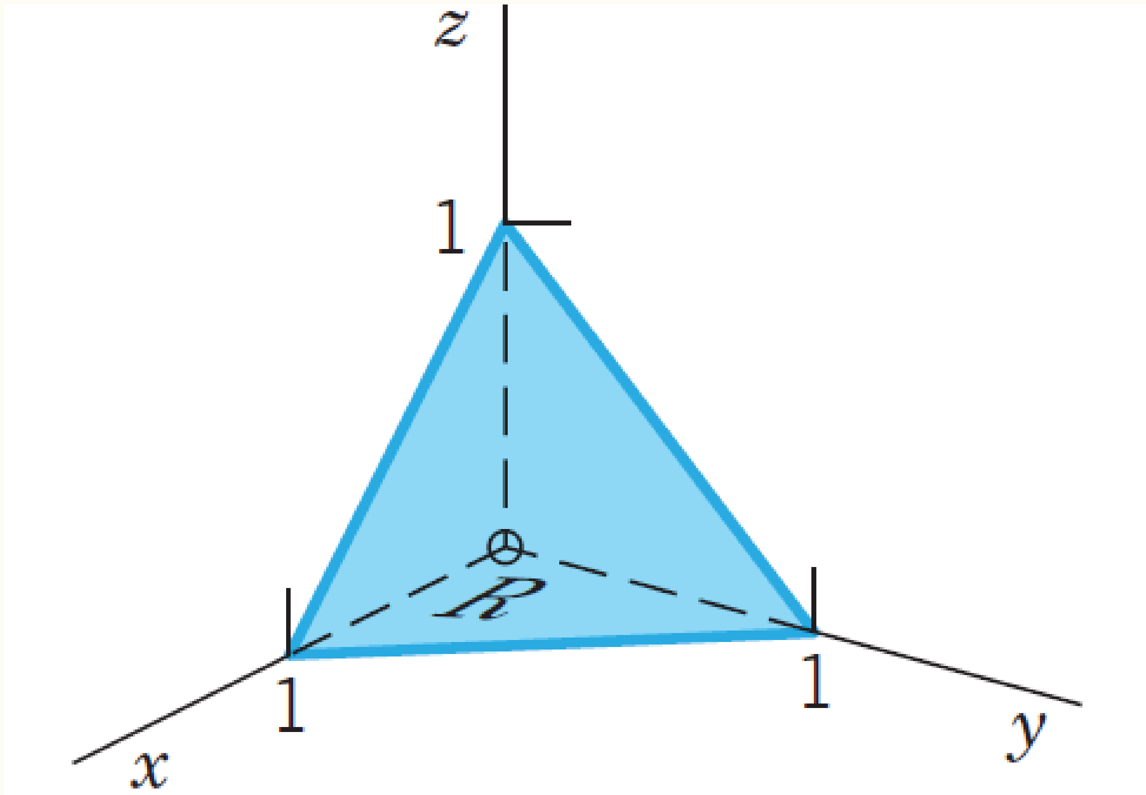
$$\begin{aligned} \boldsymbol{N} &= \boldsymbol{r}_u \times \boldsymbol{r}_v =< 1, 2u, 0 > \times < 0, 0, 1 > \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2u & 0 \\ 0 & 0 & 1 \end{vmatrix} \\ &= 2u\mathbf{i} - \mathbf{j} =< 2u, -1, 0 > \end{aligned} \tag{2.16}$$

Another way to find the normal vector of the surface is by (2.11). The surface is $y = x^2$, so we can write its equation into $g(x, y, z) = x^2 - y = 0$. Therefor the normal vector is given by $\boldsymbol{N} = \nabla g =< 2x, -1, 0 >$, which is the same with (2.16).

Thus from (2.13) the flux through the cylinder is

$$\begin{aligned}
\iint\limits_S \boldsymbol{F}\cdot\boldsymbol{n}dA &= \iint\limits_R \boldsymbol{F}(\boldsymbol{r}(u,v))\cdot\boldsymbol{N}(u,v)dudv \\
&= \iint\limits_R < 3v^2, 6, 6uv > \cdot < 2u, -1, 0 > dudv \\
&= \iint\limits_R (6uv^2 - 6)dudv \\
&= \int_0^3 \left[\int_0^2 (6uv^2 - 6)du\right] dv \\
&= \int_0^3 (12v^2 - 12)dv \\
&= 72.
\end{aligned}$$

Example 2: (page 445) Evaluate $\iint\limits_S \boldsymbol{F}\cdot\boldsymbol{n}dA$ when $\boldsymbol{F} =< x^2, 0, 3y^2 >$ and S is the portion of the plane $x+y+z=1$ in the first octant.



Solution: Writing $x = u$ and $y = v$, we have $z = 1 - x - y = 1 - u - v$. Hence we can represent the plane in the form $\boldsymbol{r}(u, v) =< u, v, 1 - u - v >$. The projection of the plane in the first octant R is the triangle bounded by the two coordinate axes and the straight line $x + y = 1$, obtained from $x + y + z = 1$ by setting $z = 0$. Thus R can be represented by $0 \leq x \leq 1 - y$, $0 \leq y \leq 1$. The normal vector of the plane is

$$\boldsymbol{N} = \boldsymbol{r}_u \times \boldsymbol{r}_v =< 1, 0, -1 > \times < 0, 1, -1 >=< 1, 1, 1 >$$

We can also obtain the normal vector by writing the equation of the plane into $g(x, y, z) = x + y + z - 1 = 0$ and by (2.11),

$$\boldsymbol{N} = \text{grad} g(x, y, z) =< 1, 1, 1 > .$$

By (2.13)

$$\begin{aligned}
\iint\limits_S \boldsymbol{F}\cdot\boldsymbol{n}\,dA &= \iint\limits_R \boldsymbol{F}(\boldsymbol{r}(u,v))\cdot\boldsymbol{N}(u,v)\,dudv \\
&= \iint\limits_R < u^2, 0, 3v^2 > \cdot < 1,1,1 > dudv \\
&= \iint\limits_R (u^2+3v^2)\,dudv \\
&= \int_0^1 \int_0^{1-v} (u^2+3v^2)\,dudv \\
&= \int_0^1 \left[\frac{1}{3}(1-v)^3 + 3v^2(1-v)\right] dv \\
&= \frac{1}{3}.
\end{aligned}$$

Orientation of a surface

From (2.13) we see that the value of the integral depends on the choice of the unit normal vector $\boldsymbol{n}$. Instead of $\boldsymbol{n}$ we could choose $-\boldsymbol{n}$ as the unit normal vector. In this case, corresponding surface integral will become $-\iint\limits_S \boldsymbol{F}\cdot\boldsymbol{n}dA$. So such a surface integral is an integral over an oriental surface S.

Theorem: The replacement of $\boldsymbol{n}$ by $-\boldsymbol{n}$ (hence of $\boldsymbol{N}$ by $-\boldsymbol{N}$) corresponds to the multiplication of the integral $\iint\limits_S \boldsymbol{F}\cdot\boldsymbol{n}dA$ by -1.

In practice, how do we make such a change of $\boldsymbol{N}$ happen, if S is given in the form $\boldsymbol{r}(u,v) =< x(u,v), y(u,v), z(u,v) >$? The easiest way is to interchange u and v, because then $\boldsymbol{r}_u$ becomes $\boldsymbol{r}_v$ and conversely, so that $\boldsymbol{N} = \boldsymbol{r}_u \times \boldsymbol{r}_v$ becomes

$$\boldsymbol{r}_v \times \boldsymbol{r}_u = -\boldsymbol{r}_u \times \boldsymbol{r}_v = -\boldsymbol{N}$$

as wanted.

Example 3: Use example 1 and now we represent S by $\boldsymbol{r} =< v, v^2, u >$, $0 \le v \le 2$, $0 \le u \le 3$. Then

$$\boldsymbol{N} = \boldsymbol{r}_u \times \boldsymbol{r}_v =< 0, 0, 1 > \times < 1, 2v, 0 >=< -2v, 1, 0 > .$$

For $\boldsymbol{F} =< 3z^2, 6, 6xz >$ we now get $\boldsymbol{F}(\boldsymbol{r}) =< 3u^2, 6, 6uv >$. Hence

$$\begin{aligned}
\iint\limits_S \boldsymbol{F} \cdot \boldsymbol{n} dA &= \iint\limits_R \boldsymbol{F}(\boldsymbol{r}(u,v)) \cdot \boldsymbol{N}(u,v) du dv \\
&= \iint\limits_R < 3u^2, 6, 6vu > \cdot < -2v, 1, 0 > du dv \\
&= \iint\limits_R (-6u^2 v + 6) du dv \\
&= \int_0^3 \left[\int_0^2 (-6u^2 v + 6) dv \right] du \\
&= \int_0^3 (-12u^2 + 12) dv = -72.
\end{aligned}$$

Surface integrals without regard to orientation

Another type of surface integral is

$$\iint_S G(\boldsymbol{r}) dA = \iint_R G(\boldsymbol{r}(u,v))|\boldsymbol{N}(u,v)| du dv. \tag{2.17}$$

Here $dA = |\boldsymbol{N}| du dv = |\boldsymbol{r}_u \times \boldsymbol{r}_v|$ is the element of area of the surface S represented by $\boldsymbol{r}(u,v) =< x(u,v), y(u,v), z(u,v) >$ and we disregard the orientation.

As for application, if $G(\boldsymbol{r})$ is the mass density of S, then (2.17) is the total mass of S. If $G = 1$, then (2.17) gives the **area** $A(S)$ of S,

$$A(S) = \iint\limits_S dA = \iint\limits_R |\boldsymbol{r}_u \times \boldsymbol{r}_v| du dv. \qquad (2.18)$$

Next we will have two examples to show how to apply (2.18) to find the area of a sphere and a torus.

Example 4: Find the area of a sphere

$$\boldsymbol{r}(u, v) = < a \cos v \cos u, a \cos v \sin u, a \sin v >,$$

where $0 \le u \le 2\pi$, $-\pi/2 \le v \le \pi/2$.

Solution: The normal vector of the sphere is

$$
\begin{aligned}
\boldsymbol{r}_u \times \boldsymbol{r}_v &= < -a\cos v \sin u, a\cos v \cos u, 0 > \times \\
& \quad < -a\sin v\cos u, -a\sin v \sin u, a\cos v > \\
&= < a^2\cos^2 v\cos u, a^2\cos^2 v \sin u, a^2\cos v\sin v >
\end{aligned}
$$

Therefore

$$
\begin{aligned}
|\boldsymbol{r}_u \times \boldsymbol{r}_v| &= \sqrt{(a^2\cos^2 v\cos u)^2 + (a^2\cos^2 v\sin u)^2 + (a^2\cos v\sin v)^2} \\
&= a^2|\cos v|.
\end{aligned}
$$

Therefore by (2.18) the area of the sphere is

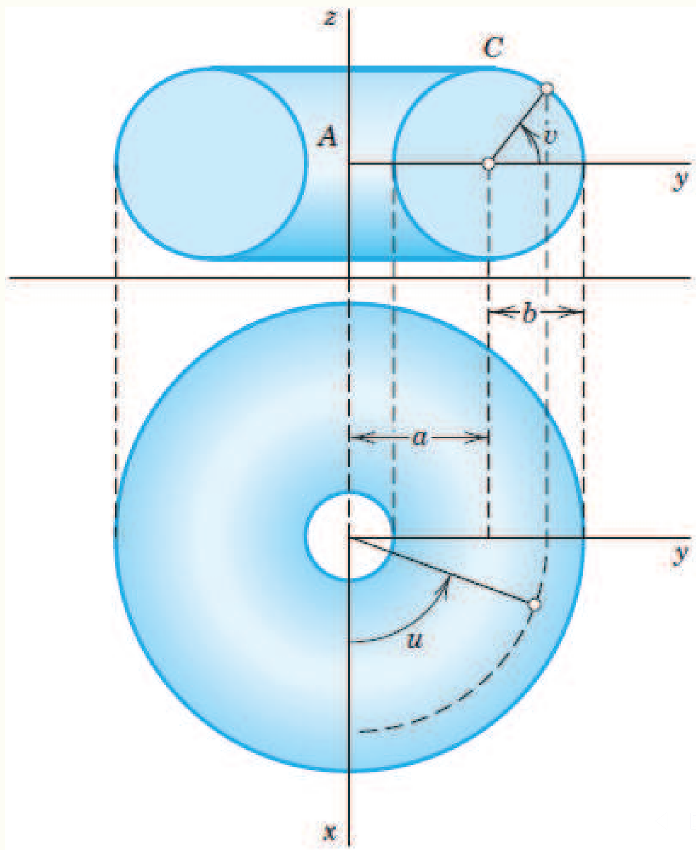
$$
\begin{aligned}
A(S) &= \iint\limits_S dA = \iint\limits_R |\boldsymbol{r}_u \times \boldsymbol{r}_v| du dv \\
&= \iint\limits_R a^2|\cos v| du dv = a^2 \int_{-\pi/2}^{\pi/2}\int_0^{2\pi} \cos v du dv \\
&= 2\pi a^2 \int_{-\pi/2}^{\pi/2} \cos v dv = 4\pi a^2.
\end{aligned}
$$

Example 2: Torus surface (doughnut surface)

A torus surface S is obtained by rotating a circle C about a straight line L in space so that C does not intersect or touch L but its plane always passes through L. We assume L is the z-axis and C has radius b and its center has distance $a(> b)$ from L, as in the figure below. Then S can be represented by

$$\boldsymbol{r}(u, v) = (a + b\cos v)\cos u\mathbf{i} + (a + b\cos v)\sin u\mathbf{j} + b\sin v\mathbf{k},$$

where $0 \leq u \leq 2\pi$, $0 \leq v \leq 2\pi$. Find the area of S.



Solution: The normal vector of the surface is

$$
\begin{aligned}
\boldsymbol{N} &= \boldsymbol{r}_u \times \boldsymbol{r}_v \\
&= < -(a + b\cos v)\sin u, (a + b\cos v)\cos u, 0 > \times \\
&\quad < -b\sin v\cos u, -b\sin v\sin u, b\cos v > \\
&= b(a + b\cos v) < \cos u\cos v, \sin u\cos v, \sin v >,
\end{aligned}
$$

so $|\boldsymbol{r}_u \times \boldsymbol{r}_v| = b(a + b\cos v)$. By (2.18), the area of the torus surface is

$$
\begin{aligned}
A(S) &= \iint\limits_S dA = \iint\limits_R |\boldsymbol{r}_u \times \boldsymbol{r}_v| du dv \\
&= \iint\limits_R b(a + b\cos v) du dv = \int_0^{2\pi}\int_0^{2\pi} b(a + b\cos v) du dv \\
&= \int_0^{2\pi} 2\pi b(a + b\cos v) dv = 4\pi^2 ab.
\end{aligned}
$$

Another representation) If a surface is given by $z = f(x, y)$, then setting $u = x, v = y$, $\boldsymbol{r} =< u, v, f >$ gives

$$\begin{aligned} |\boldsymbol{N}| &= |\boldsymbol{r}_u \times \boldsymbol{r}_v| = | < 1, 0, f_u > \times < 0, 1, f_v > | \\ &= | < -f_u, -f_v, 1 > | = \sqrt{1 + f_u^2 + f_v^2}, \end{aligned}$$

and, since $f_u = f_x$, $f_v = f_y$, so

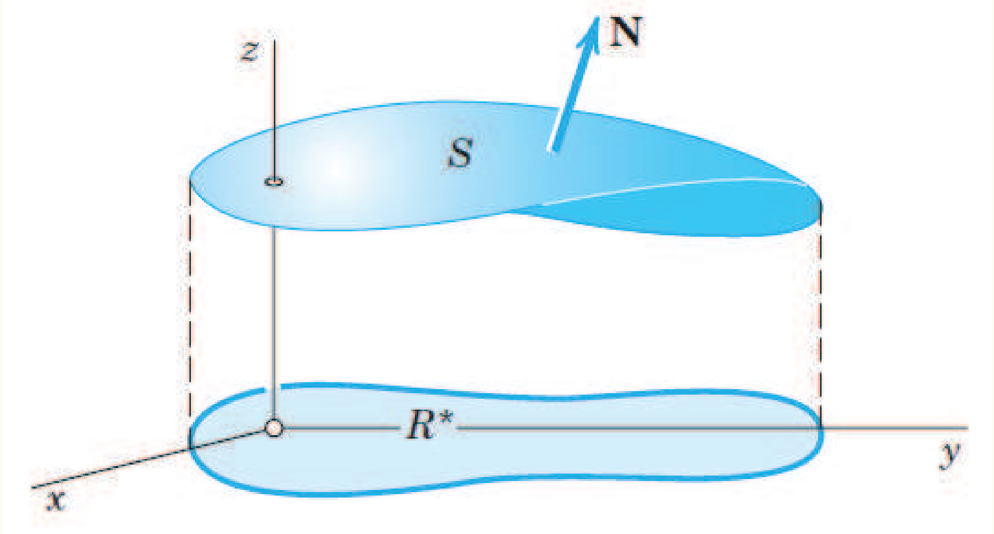
$$\iint\limits_S G(\boldsymbol{r})dA = \iint\limits_R G(\boldsymbol{r}(u, v))|\boldsymbol{N}(u, v)|dudv$$

becomes

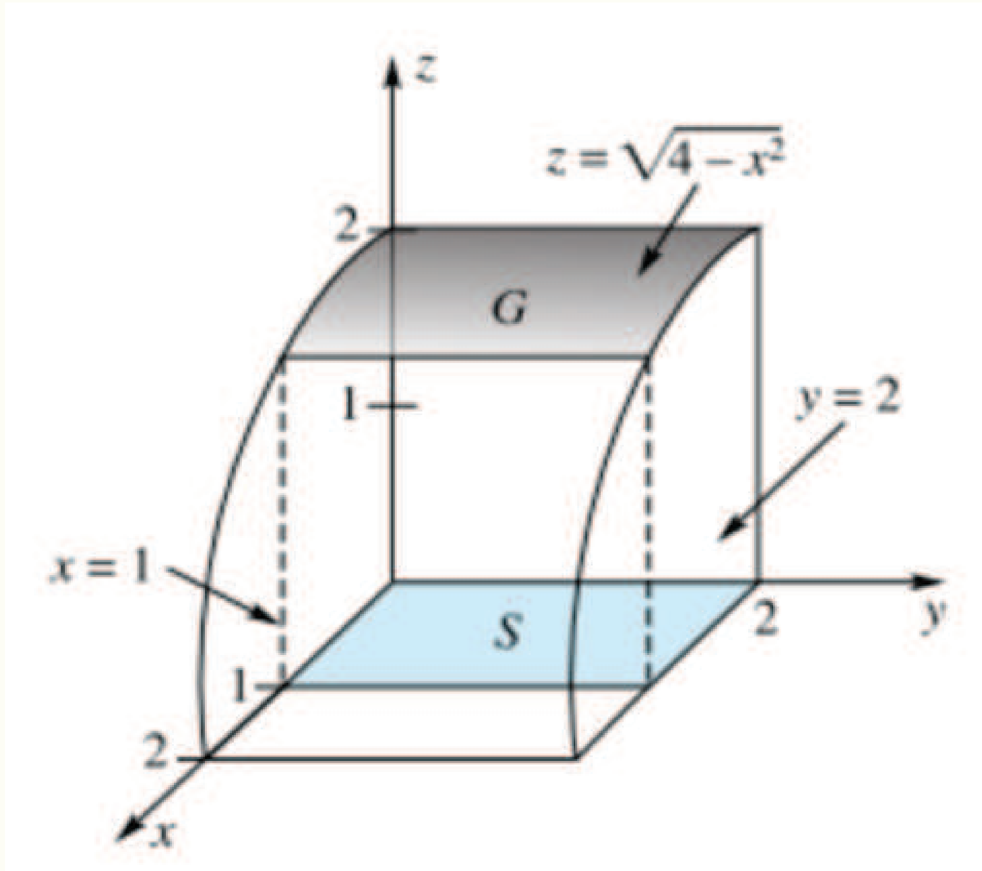
$$\iint\limits_S G(\boldsymbol{r})dA = \iint\limits_{R^*} G(x, y, f(x, y))\sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}dxdy$$

Here R^* is the projection of S into the $xy-$plane and the normal vector $\boldsymbol{N}$ on S points up. If its points down, the integral on the right is preceded by a minus sign. For $G(\boldsymbol{r}) = 1$ we obtain for the area $A(S)$ of $S : z = f(x, y)$ the formula

$$A(S) = \iint_{R^*} \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2} dxdy. \tag{2.19}$$



Example 3: If S is the region in the xy-plane that is bounded by the lines $x = 0$, $x = 1$, $y = 0$ and $y = 2$, find the area of the part of the cylindrical surface $z = \sqrt{4 - x^2}$ that is projects onto S.



Solution: Let $f(x,y)=\sqrt{4-x^2}$, then $f_x=-\frac{x}{\sqrt{4-x^2}}$, $f_y=0$,

$$\begin{aligned}
A(G) &= \iint\limits_S \sqrt{f_x^2+f_y^2+1}dA = \iint\limits_S \sqrt{\frac{x^2}{4-x^2}+1}dA \\
&= \iint\limits_S \sqrt{\frac{4}{4-x^2}}dA = \int_0^1\int_0^2 \sqrt{\frac{4}{4-x^2}}dydx \\
&= 4\int_0^1 \frac{1}{\sqrt{4-x^2}}dx = 4[\arcsin\frac{x}{2}]_0^1 = \frac{2\pi}{3}.
\end{aligned}$$

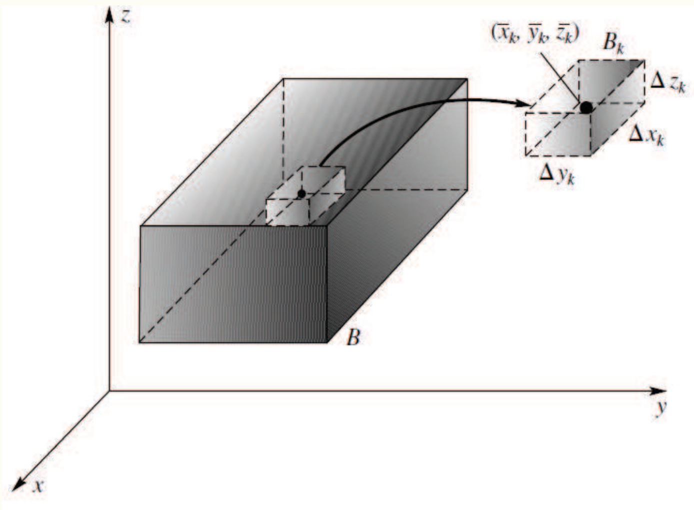
2.8 Divergence theorem of Gauss (page 452)

In this section we discuss the divergence theorem, which transforms surface integrals into triple integrals. So let us begin with a review of the latter.

A **triple integral** is an integral of a function $f(x, y, z)$ taken over a closed bounded, three-dimensional region in space.

We first consider the triple integrals over rectangular boxes. Let $f(x, y, z)$ be defined over a box-shaped region B (figure)

$$B = \{(x, y, z) : a \leq x \leq b, c \leq y \leq d, r \leq z \leq s\}.$$



1. Form a partition P of B using planes parallel to the coordinate planes. This divides B into n small subboxes B_k with the lengths of sides Δx_k, Δy_k, and Δz_k, $k = 1, 2, \ldots, n$. Then the volume of B_k is $\Delta V_k = \Delta x_k \Delta y_k \Delta z_k$
2. Pick a sample point $(\bar{x}_k, \bar{y}_k, \bar{z}_k)$ in each B_k and form the Riemann sum

$$R_p = \sum_{k=1}^{n} f(\bar{x}_k, \bar{y}_k, \bar{z}_k)\Delta V_k.$$

3. Take the limit as the partition get finer and finer by $\|P\| \to 0$ ($\|P\|$ is the length of the longest diagonal of the subboxes). Then we define the **triple integral** of f over B by

$$\iiint\limits_B f(x, y, z)dV = \lim_{\|P\| \to 0} f(\bar{x}_k, \bar{y}_k, \bar{z}_k)\Delta V_k$$

if this limit exists.

Remark: The triple integrals can be also written as triple iterated integrals, for example

$$\iiint\limits_B f(x,y,z)dV = \int_r^s \int_c^d \int_a^b f(x,y,z)dxdydz.$$

There are five other possible orders of integration, all of which give the same answer.

Example 1: Evaluate $\iiint\limits_B x^2yzdV$, where B is the box

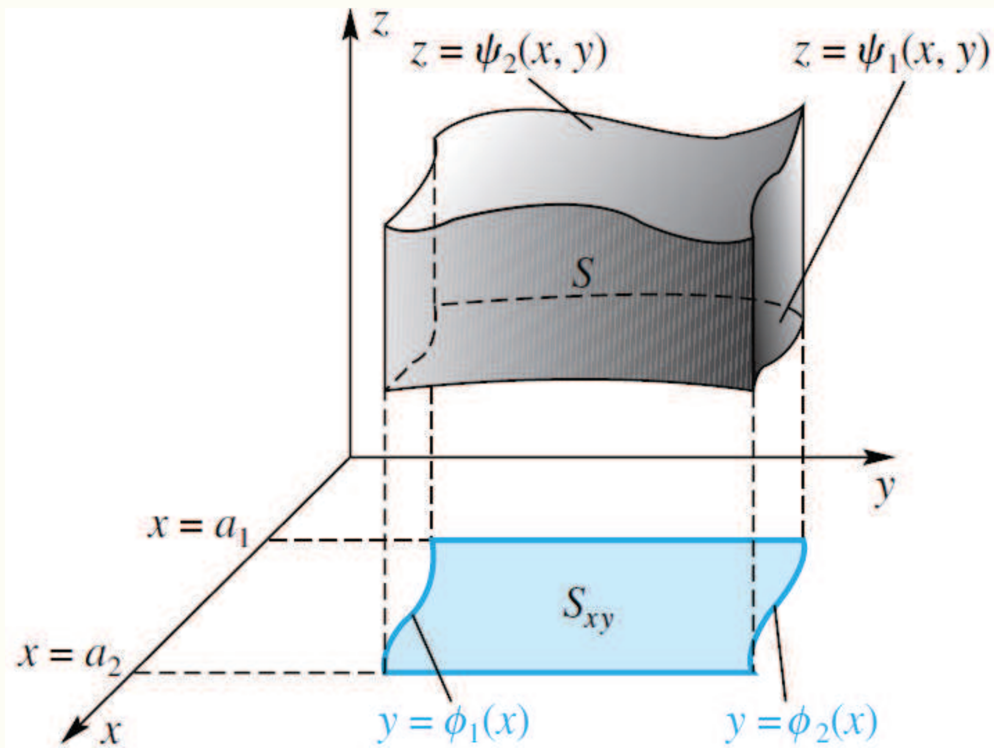
$$B = \{(x,y,z) : 1 \le x \le 2, 0 \le y \le 1, 0 \le z \le 2\}.$$

Solution:

$$\begin{aligned}
\iiint\limits_B x^2yz\,dV &= \int_0^2\int_0^1\int_1^2 x^2yz\,dx\,dy\,dz \\
&= \int_0^2\int_0^1 \left[\frac{1}{3}x^3yz\right]_1^2 dy\,dz = \int_0^2\int_0^1 \frac{7}{3}yz\,dy\,dz \\
&= \frac{7}{3}\int_0^2 \left[\frac{1}{2}y^2z\right]_0^1 dz = \frac{7}{3}\int_0^2 \frac{1}{2}z\,dz \\
&= \frac{7}{6}\left[\frac{z^2}{2}\right]_0^2 = \frac{7}{3}
\end{aligned}$$

Triple integrals over general regions

1. Let S be a $z-$**simple set**: vertical lines intersect S in a single line segment. Let S_{xy} be the projection of S onto the $xy-$ plane. Notice that s lies between the graphs of two functions. The upper boundary is the surface $z = \psi_2(x, y)$, the lower boundary is the surface $z = \psi_1(x, y)$. Thus
$S = \{(x, y, z) : (x, y) \in D, \psi_1(x, y) \leq z \leq \psi_2(x, y)\}$.



2. If S be a $z-$simple set, then the triple integral can be computed by the following integral (first definite integral with respect to z, then the double integral on the $xy-$plane)

$$\iiint\limits_{S} f(x,y,z)dV = \iint\limits_{S_{xy}} \left[\int_{\psi_1(x,y)}^{\psi_2(x,y)} f(x,y,z)dz\right] dA.$$

3. If in addition S_{xy} is a $y-$simple set,

$$S_{xy} = (x,y) : \phi_1(x) \leq y \leq \phi_2(x), a_1 \leq y \leq a_2,$$

then we can further rewrite the outer double integral as an iterated integral.

$$\iiint\limits_{S} f(x,y,z)dV = \int_{a_1}^{a_2} \int_{\phi_1(x)}^{\phi_2(x)} \int_{\psi_1(x,y)}^{\psi_2(x,y)} f(x,y,z)dzdydx.$$

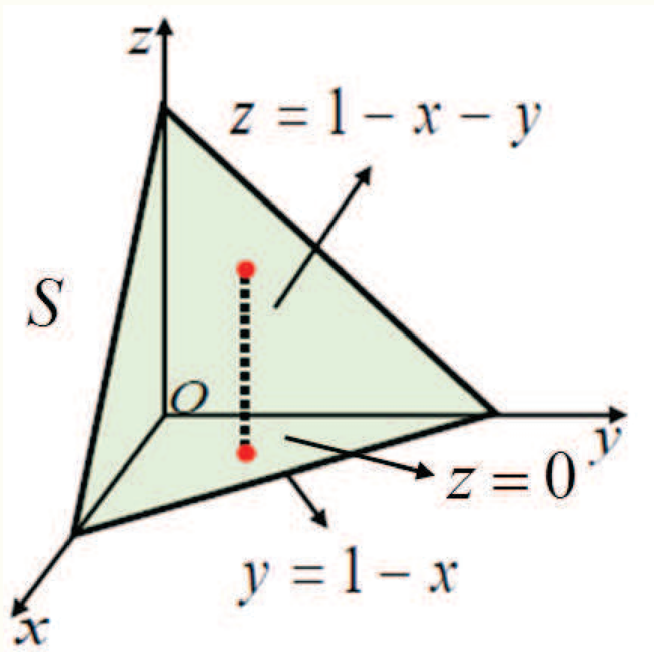
The integral on the right is a triple iterated integral.

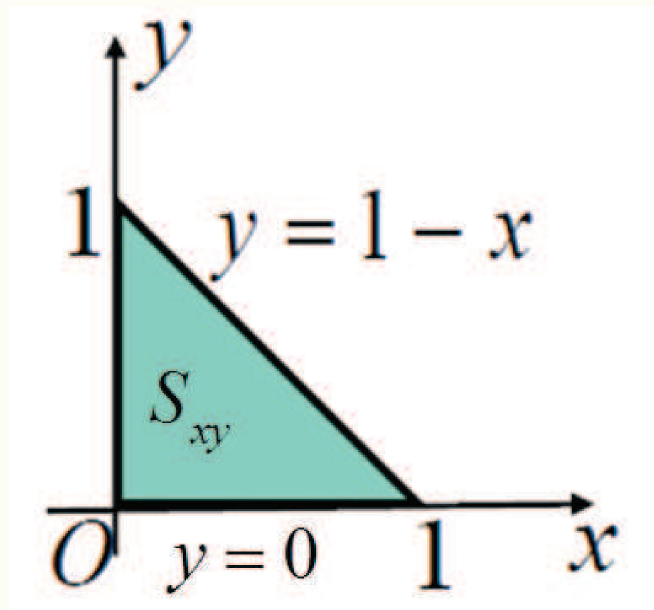
Example 2: Evaluate $\iiint\limits_S x\,dV$, where S is the solid bounded by the plane $x+y+z=1$ and the three coordinate planes in the first octant.

Solutions: Step 1: Sketch the solid region in three space and its projection in the xy-plane.

$S=\{(x,y,z):(x,y)\in S_{xy}, 0\leq z\leq 1-x-y\},\ \ z-$ simple set

$S_{xy}=\{(x,y):0\leq y\leq 1-x, 0\leq x\leq 1\}$





Step 2: Write the triple integral as an triple iterated integral

$$\begin{aligned}\iiint\limits_{S} x dV &= \iint\limits_{S_{xy}} \left[\int_0^{1-x-y} x dz\right] dA \\ &= \int_0^1 \int_0^{1-x} \int_0^{-x-y} x dz dy dx\end{aligned}$$

Step 3: Compute the triple iterated integral by N-L formula.

$$\begin{aligned}\iiint\limits_{S} x dV &= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} x dz dy dx \\ &= \int_0^1 \int_0^{1-x} x(1-x-y) dy dx \\ &= \int_0^1 \left[xy - x^2 y - x\frac{y^2}{2}\right]_{y=0}^{y=1-x} dx \\ &= \frac{1}{2}\int_0^1 (x - 2x^2 + x^3) dx = \frac{1}{24}\end{aligned}$$