MTH101: Tutorial 11

Dr. Tai-Jun Chen, Dr. Xinyao Yang

Xi'an Jiaotong-Liverpool University, Suzhou

November 27 – December 2, 2017

Exercise 1.1

Find the power series solution to the following questions.

1. $xy' - 4y = k, \quad k$ *constant.*
2. $(1 - x^2)y'' - 2xy' + 2y = 0.$

Solution

1. We first write down the power series expansion of $y(x)$

$$y(x) = \sum_{m=0}^{\infty} a_m x^m, \qquad y'(x) = \sum_{m=0}^{\infty} m a_m x^{m-1}.$$

Substituting it into the ODE, we find

$$\begin{aligned}
& x\sum_{m=0}^{\infty} m a_m x^{m-1} - 4\sum_{m=0}^{\infty} a_m x^m = k \\
\Rightarrow\ & \sum_{m=0}^{\infty} (m-4) a_m x^m = k \\
\Rightarrow\ & -4a_0 - 3a_1 x - 2a_2 x^2 - a_3 x^3 + a_5 x^5 \cdots = k.
\end{aligned}$$

Solution

Therefore, we require

$$a_0 = -\frac{k}{4}, \quad a_1 = a_2 = a_3 = a_5 = \cdots = 0, \textit{ and } a_4 \text{ undetermined.}$$

We thus conclude that

$$y(x) = -\frac{k}{4} + a_4 x^4.$$

Solution

2. Similarly, we first write down the power series expansion of $y(x)$

$$y(x) = \sum_{m=0}^{\infty} a_m x^m, \qquad y'(x) = \sum_{m=0}^{\infty} m a_m x^{m-1},$$

$$y''(x) = \sum_{m=0}^{\infty} m(m-1) a_m x^{m-2}.$$

Substituting it into the ODE, we find

$$\sum_{m=0}^{\infty} m(m-1) a_m x^{m-2} - \sum_{m=0}^{\infty} m(m-1) a_m x^m$$

$$-2 \sum_{m=0}^{\infty} m a_m x^m + 2 \sum_{m=0}^{\infty} a_m x^m = 0$$

$$\Rightarrow \sum_{m=0}^{\infty} m(m-1) a_m x^{m-2} + \sum_{m=0}^{\infty} \left[-m(m-1) - 2m + 2\right] a_m x^m = 0.$$

Solution

By choosing $n = m - 2$ for the first term and factorize the second term, we can find

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{m=0}^{\infty}\left[-(m-1)(m+2)\right]a_m x^m = 0$$

$$\Rightarrow \sum_{m=0}^{\infty}\left[(m+2)(m+1)a_{m+2} - (m-1)(m+2)a_m\right]x^m = 0,$$

where we change the dummy variable n to be m for the first term to simplify the calculation. Therefore, we have

$$\begin{cases} a_{m+2} = \frac{m-1}{m+1}a_m, & for \;\; m \neq 1, \\ a_3 = 0 & for \;\; m = 1. \end{cases}$$

Solution

We can thus conclude that

$$a_{2k+2} = \frac{2k-1}{2k+1}a_{2k} = \frac{2k-3}{2k+1}a_{2k-2} = \cdots = \frac{1}{2k+1}a_2 = -\frac{1}{2k+1}a_0,$$
$$a_{2k+1} = \frac{2k-2}{2k}a_{2k-1} = \frac{2k-4}{2k}a_{2k-3} = \cdots = \frac{2}{2k}a_3 = 0.$$

Therefore,

$$y(x) = a_1 x + a_0\left(1 - \sum_{k=0}^{\infty}\frac{x^{2k+2}}{2k+1}\right).$$

Exercise 2.1

We know that the Legendre's differential equation admit the general solution

$$y(x) = a_0 y_1(x) + a_1 y_2(x),$$

where

$$y_1(x) = 1 - \frac{n(n+1)}{2}x^2 + \frac{(n-2)n(n+1)(n+3)}{4!}x^4 - + \cdots$$
$$y_2(x) = x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-3)(n-1)(n+2)(n+4)}{5!}x^5 \cdots .$$

For $n = 1$, *show that* $y_2(x) = P_1(x) = x$ *and*

$$y_1(x) = 1 - \frac{1}{2}x \ln\left(\frac{1+x}{1-x}\right).$$

Solution

For $n = 1$, only the first term in $y_2(x)$ survives, and thus $y_2(x) = x = P_1(x)$.
For $n = 1$, we can write down $y_1(x)$ as

$$
\begin{aligned}
y_1(x) &= 1 - x^2 - \frac{1}{3}x^4 - \frac{1}{5}x^6 + \cdots \\
&= 1 - x\left(x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \cdots\right).
\end{aligned}
$$

Since we know

$$
\begin{aligned}
\ln(1+x) &= 0 + x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \\
\ln(1-x) &= 0 - x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots
\end{aligned}
$$

Solution

$$\ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5}\cdots\right)$$
$$\Rightarrow y_1(x) = 1 - \frac{x}{2}\ln\left(\frac{1+x}{1-x}\right).$$

You can check that this is actually the solution to Legendre's differential equation with $n = 1$.

Exercise 3.1

Find a basis of solutions by the Frobenius method for the following question.

1. $x^2y'' + 2x^3y' + (x^2 - 2)y = 0.$
2. $x^2y'' + 6xy' + (4x^2 + 6)y = 0.$

Solution

1. We first write the ODE into the standard form

$$y'' + \frac{2x^2}{x}y' + \frac{(x^2-2)}{x^2}y = 0.$$

Since $b(x) = 2x^2$, $c(x) = x^2 - 2$ are analytic at $x = 0$, we can use the Frobenius theorem which states that there is at least one solution

$$y(x) = x^r \sum_{m=0}^{\infty} a_m x^m,$$

and

$$y'(x) = \sum_{m=0}^{\infty} (m+r) a_m x^{m+r-1},$$

$$y''(x) = \sum_{m=0}^{\infty} (m+r)(m+r-1) a_m x^{m+r-2}.$$

Solution

Therefore, the ODE becomes

$$\sum_{m=0}^{\infty}(m+r)(m+r-1)a_m x^{m+r} + 2\sum_{m=0}^{\infty}(m+r)a_m x^{m+r+2} + \sum_{m=0}^{\infty} a_m x^{m+r+2} - 2\sum_{m=0}^{\infty} a_m x^{m+r} = 0.$$

The coefficient of x^r is

$$(r)(r-1)a_0 - 2a_0 = (r^2 - r - 2)a_0 = (r-2)(r+1)a_0,$$

and by assumption $a_0 \neq 0$, the roots to the indicial equation is $r_1 = 2$, $r_2 = -1$, which is **Case 3**.

Solution

The two solutions take the form

$$y_1(x) = x^2 \sum_{m=0}^{\infty} a_m x^m = \sum_{m=0}^{\infty} a_m x^{m+2},$$

$$y_2(x) = k y_1(x) \ln x + \sum_{m=0}^{\infty} A_m x^{m-1}.$$

Substitute y_1 in to the ODE, we obtain

$$\sum_{m=0}^{\infty} (m+2)(m+1) a_m x^{m+2} + 2 \sum_{m=0}^{\infty} (m+2) a_m x^{m+4} + \sum_{m=0}^{\infty} a_m x^{m+4} - 2 \sum_{m=0}^{\infty} a_m x^{m+2} = 0.$$

Solution

Then

$$\sum_{m=0}^{\infty} \{[(m+4)(m+3)-2]\, a_{m+2} + [2(m+2)+1]\, a_m\}\, x^{m+4}$$
$$+(2a_0-2a_0)x^2 + (6a_1-2a_1)x^3 = 0.$$

We thus require $a_1 = 0$ and

$$a_{m+2} = -\frac{(2m+5)}{m^2+7m+10} a_m = -\frac{(2m+5)}{(m+2)(m+5)} a_m$$
$$\Rightarrow y_1(x) = a_0 x^2 \left(1 - \frac{x^2}{2} + \frac{9x^4}{56} - + \cdots\right).$$

Solution

For $y_2(x)$, since y_1 is linearly independent of $\sum_{m=0}^{\infty} A_m x^{m-1}$, and our goal is to find a y_2 which is linearly independent of y_1, we can simply choose $k = 0$. Substitute y_2 into the ODE, we obtain

$$\sum_{m=0}^{\infty}(m-1)(m-2)A_m x^{m-1} + 2\sum_{m=0}^{\infty}(m-1)A_m x^{m+1} + \sum_{m=0}^{\infty} A_m x^{m+1} - 2\sum_{m=0}^{\infty} A_m x^{m-1} = 0.$$

Solution

Then

$$\sum_{m=0}^{\infty} \left\{\left[(m+1)m-2\right]A_{m+2}+\left[2(m-1)+1\right]A_m\right\}x^{m+1}$$
$$+(2A_0-2A_0)x^{-1}+(0-2A_1)=0.$$

We thus require $A_1 = 0$ and

$$A_{m+2}=-\frac{(2m-1)}{m^2+m-2}A_m=-\frac{(2m-1)}{(m+2)(m-1)}A_m$$
$$\Rightarrow y_2(x)=A_0x^{-1}\left(1-\frac{x^2}{2}+\frac{3x^4}{8}-+\cdots\right).$$

Solution

2. We first write the ODE into the standard form

$$y'' + \frac{6}{x}y' + \frac{(4x^2+6)}{x^2}y = 0.$$

Since $b(x) = 6$, $c(x) = 4x^2 + 6$ are analytic at $x = 0$, we can use the Frobenius theorem, and since $b_0 = 6$, $c_0 = 6$, the indicial equation is

$$r(r-1) + 6r + 6 = 0 \Rightarrow (r+2)(r+3) = 0,$$

therefore $r_1 = -2$, $r_2 = -3$, which is **Case 3**.

Solution

The two solutions take the form

$$y_1(x) = x^{-2}\sum_{m=0}^{\infty} a_m x^m = \sum_{m=0}^{\infty} a_m x^{m-2},$$

$$y_2(x) = k y_1(x)\ln x + \sum_{m=0}^{\infty} A_m x^{m-3},$$

and since y_1 is independent of $\sum_{m=0}^{\infty} A_m x^{m-3}$, we can choose $k = 0$.

Solution

Substitute y_1 into the ODE, we obtain

$$\sum_{m=0}^{\infty}(m-2)(m-3)a_m x^{m-2} + 6\sum_{m=0}^{\infty}(m-2)a_m x^{m-2}$$
$$+4\sum_{m=0}^{\infty} a_m x^m + 6\sum_{m=0}^{\infty} a_m x^{m-2} = 0$$
$$\Rightarrow \sum_{m=0}^{\infty}\left\{\left[m(m-1)+6m+6\right]a_{m+2}+4a_m\right\}x^m$$
$$+(6a_0-12a_0+6a_0)x^{-2}+(2a_1-6a_1+6a_1)x^{-1}=0$$
$$\Rightarrow a_1 = 0 \quad a_{m+2} = -\frac{4}{(m+2)(m+3)}a_m.$$

Solution

Therefore

$$y_1(x) = x^{-2}\left(1 - \frac{2}{3}x^2 + \frac{2}{15}x^4 - + \cdots\right)$$
$$\Rightarrow y_1(x) = \frac{\sin(2x)}{2x^3}.$$

Solution

Similarly, substitute y_2 into the ODE, we obtain

$$\sum_{m=0}^{\infty}(m-3)(m-4)A_m x^{m-3} + 6\sum_{m=0}^{\infty}(m-3)A_m x^{m-3}$$

$$+4\sum_{m=0}^{\infty}A_m x^{m-1} + 6\sum_{m=0}^{\infty}A_m x^{m-3} = 0$$

$$\Rightarrow \sum_{m=0}^{\infty}\left\{\left[(m-1)(m-2)+6(m-1)+6\right]A_{m+2}+4A_m\right\}x^{m-1}$$

$$+(12A_0-18A_0+6A_0)x^{-3}+(6A_1-12A_1+6A_1)x^{-2}=0$$

$$\Rightarrow A_0, A_1 \neq 0 \quad A_{m+2} = -\frac{4}{(m+1)(m+2)}A_m.$$

Solution

Therefore

$$y_2(x) = A_0 x^{-3}(1 - 2x^2 + \frac{2}{3}x^4 \cdots) + A_1 x^{-2}(1 - \frac{2}{3}x^2 + \frac{2}{15}x^4 \cdots).$$

$$\Rightarrow y_2(x) = A_0 x^{-3}(1 - 2x^2 + \frac{2}{3}x^4 \cdots) + A_1 y_1(x)$$

$$\Rightarrow y_2(x) = x^{-3}(1 - 2x^2 + \frac{2}{3}x^4 \cdots) = \frac{\cos(2x)}{x^3}.$$

Exercise 4.1

Find a general solution to the following Bessel's equation in terms of J_ν, $J_{-\nu}$, or indicate when this is not possible.

$$x^2y'' + xy' + \left(x^2 - \frac{1}{9}\right)y = 0.$$

Solution

This is Bessel's equation with $\nu = \frac{1}{3}$, and since ν is not a integer, the most general solution is the linear combination of $J_{1/3}$ and $J_{-1/3}$

$$y(x) = c_1 J_{\frac{1}{3}}(x) + c_2 J_{-\frac{1}{3}}(x).$$