Chapter 4: Partial differential equations

PDEs are differential equations in which there is more than one independent variable. They arise in the modelling of a wide-range of physical phenomena including electromagnetism, fluid flow, elasticity, and heat conduction and as such the ability to solve them is of considerable importance. In this section we will concentrate on use of an important technique for the solution to PDEs.

4.1 Basic concepts of PDEs

Definitions:

1. **Order**: The order of the highest derivative is called the order of the PDE. For example,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^3 u}{\partial y^3} + u^4(x, y) = 0$$

   is a third-order PDE.

2. **Linear**: A PDE is linear if the dependent variable and its derivatives appear in linear fashion, i.e. it is of the first degree in the unknown function and its derivatives. For example,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

   is a linear PDE. But

$$u \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

   is not linear.

3. **Homogeneous:** We call a linear PDE homogeneous if each of its terms contains either the dependent variable u or one of its partial derivatives. For example

$$u + u_x + u_{yy} = 0$$

is homogeneous.
But

$$u + u_x + u_{yy} + x = 0$$

is inhomogeneous, because x is the independent variable.

$$u^2 + u_x + u_{yy} = 0$$

is also inhomogeneous because it is not linear.

Example 1: The following three are very important linear and homogeneous PDEs.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad \text{Laplace's equation} \tag{4.1}$$

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad \text{Wave equation} \tag{4.2}$$

$$\frac{\partial u}{\partial t} = k^2 \frac{\partial^2 u}{\partial x^2}, \quad \text{Hear or diffusion equation} \tag{4.3}$$

Here c is a constant, t is time, x and y are Cartesian coordinates.

Example 2: Verify that the following functions are solutions to

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

1. $u(x, y) = x^2 - y^2$
2. $u(x, y) = \mathrm{e}^{\mathrm{x}} \cos \mathrm{y}$
3. $u(x, y) = \sin x \cosh y$
4. $u(x, y) = \ln(x^2 + y^2)$

Solutions:

1.

$$u(x,y) = x^2 - y^2 \Rightarrow \begin{cases} \dfrac{\partial u}{\partial x} = 2x, & \dfrac{\partial^2 u}{\partial x^2} = 2 \\ \dfrac{\partial u}{\partial y} = -2y, & \dfrac{\partial^2 u}{\partial y^2} = -2 \end{cases}$$

Therefore $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, so $u(x,y) = x^2 - y^2$ is a solution of the Laplace equation.

2.

$$u(x,y) = \mathrm{e}^x \cos y \Rightarrow \begin{cases} \dfrac{\partial u}{\partial x} = \mathrm{e}^x \cos y, & \dfrac{\partial^2 u}{\partial x^2} = \mathrm{e}^x \cos y \\ \dfrac{\partial u}{\partial y} = -\mathrm{e}^x \sin y, & \dfrac{\partial^2 u}{\partial y^2} = -\mathrm{e}^x \sin y \end{cases}$$

Therefore $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, so $u(x,y) = \mathrm{e}^x \cos y$ is a solution of the Laplace equation.

3.

$$u(x,y) = \sin x \cosh y \Rightarrow \begin{cases} \dfrac{\partial u}{\partial x} = \cos x \cosh y, & \dfrac{\partial^2 u}{\partial x^2} = -\sin x \cosh y \\ \dfrac{\partial u}{\partial y} = \sin x \sinh y, & \dfrac{\partial^2 u}{\partial y^2} = \sin x \sinh y \end{cases}$$

Therefore $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, so $u(x,y) = \sin x \cosh y$ is a solution of the Laplace equation.

4.

$$u(x,y) = \ln(x^2+y^2) \Rightarrow \begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{2x}{x^2+y^2}, & \dfrac{\partial^2 u}{\partial x^2} = \dfrac{2y^2-2x^2}{(x^2+y^2)^2} \\ \dfrac{\partial u}{\partial y} = \dfrac{2y}{x^2+y^2}, & \dfrac{\partial^2 u}{\partial y^2} = \dfrac{-2y^2+2x^2}{(x^2+y^2)^2} \end{cases}$$

Therefore $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, so $u(x,y) = \ln(x^2+y^2)$ is a solution of the Laplace equation.

Example 3:

Verify that the functions $u(x, y) = \mathrm{e}^x \cos y + \mathrm{e}^x$ is a solution to

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \mathrm{e}^x.$$

Solution:

Because $u(x, y) = \mathrm{e}^x \cos y + \mathrm{e}^x$, we have

$$\begin{aligned} \frac{\partial u}{\partial x} &= \mathrm{e}^x \cos y + \mathrm{e}^x, & \frac{\partial^2 u}{\partial x^2} &= \mathrm{e}^x \cos y + \mathrm{e}^x \\ \frac{\partial u}{\partial y} &= -\mathrm{e}^x \sin y, & \frac{\partial^2 u}{\partial y^2} &= -\mathrm{e}^x \cos y \end{aligned}$$

Therefore we have $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \mathrm{e}^x$ so $u(x, y)$ is a solution of the given PDE.

Theorem: Fundamental theorem on superposition
If $u_1(x, y)$ and $u_2(x, y)$ are solutions of a homogeneous PDE in some region R, then for any constants c_1 and c_2, the function

$$u(x, y) = c_1 u(x, y) + c_2 u(x, y)$$

is also a solution of PDE in the region R.

4.2 Some basic methods for solving simple PDEs

Example 1: Solve $u_{xx} - u = 0$, where u is a function of x and y.

Solution: Since there is no y-derivative in the given PDE, we can solve it like an ODE: $u'' - u = 0$. The characteristic equation (or auxiliary equation) of this ODE is

$$\lambda^2 - 1 = 0$$

so we get $\lambda = \pm 1$ and the general solution is

$$u = A\mathrm{e}^{x} + B\mathrm{e}^{-x}.$$

Here A and B may be functions of y, so that the answer is

$$u(x, y) = A(y)\mathrm{e}^{x} + B(y)\mathrm{e}^{-x}.$$

Check the result by differentiation.

Example 2: Solving $u_{xy} = -u_x$, where u is a function of x and y.

Solution: Setting $\frac{\partial u}{\partial x} = h$, we have $\frac{\partial h}{\partial y} = -h$, therefore $\frac{h_y}{h} = -1$. Solving this ODE we get

$$\ln |h| = -y + c_1(x),$$

so $h = c(x)\mathrm{e}^{-y}$, where $c(x) = \mathrm{e}^{c_1(x)}$. Then by integration with respect to x,

$$u(x, y) = f(x)\mathrm{e}^{-y} + g(y), \text{ where } f(x) = \int c(x)dx,$$

here, $f(x)$ and $g(y)$ are arbitrary.

Example 3: Method of separating variables Solve the boundary value problem

$$\begin{cases} \dfrac{\partial u}{\partial x} &= 4\dfrac{\partial u}{\partial y} \\ u(0,y) &= 8\mathrm{e}^{-3y} + 4\mathrm{e}^{-5y} \end{cases}.$$

Solution:

Step 1. Suppose that $u(x,y)$ can be separated into the product of two functions: $u(x,y) = X(x)Y(y)$ and substituting into the PDE.

Since $X(x)$ is a function of single variable x and $Y(y)$ is a function of single variable y, by substituting $u = X(x)Y(y)$ into the PDE, we have

$$\frac{X'}{4X} = \frac{Y'}{Y} = c,$$

where c is a constant. We obtain

$$X' - 4cX = 0 \text{ and } Y' - cY = 0.$$

The solutions for these two ODE are

$$X(x) = A\mathrm{e}^{-\int -4cdx} = A\mathrm{e}^{4cx},\ Y(y) = B\mathrm{e}^{-\int -cdx} = A\mathrm{e}^{cx},$$

where c, A, B are arbitrary constants.
Thus $u(x, y) = X(x)Y(y) = AB\mathrm{e}^{c(4x+y)}$ is a solution of the PDE, where $k = AB$ and c are arbitrary constants.
Step 2. Find the solution which satisfies the boundary condition.

Firstly, find $u_1(x, y) = k_1\mathrm{e}^{c_1(4x+y)}$ which satisfies $u(0, y) = 8\mathrm{e}^{-3y}$, i.e. $k_1\mathrm{e}^{c_1 y} = 8\mathrm{e}^{-3y}$.
We get $k_1 = 8,\ c_1 = -3$.
So

$$u_1(x, y) = 8\mathrm{e}^{-3(4x+y)}.$$

Secondly, find $u_2(x, y) = k_2 \mathrm{e}^{c_2(4x+y)}$ which satisfies $u(0, y) = 4\mathrm{e}^{-5y}$, i.e. $k_2 \mathrm{e}^{c_2 y} = 4\mathrm{e}^{-5y}$.
We get $k_2 = 4,\ c_2 = -5$.
So

$$u_2(x, y) = 4\mathrm{e}^{-5(4x+y)}.$$

According to the theorem on superposition, $u(x, y) = u_1(x, y) + u_2(x, y) = 8\mathrm{e}^{-3(4x+y)} + 4\mathrm{e}^{-5(4x+y)}$ satisfies the boundary condition and the given PDE.

So $u(x, y) = 8\mathrm{e}^{-3(4x+y)} + 4\mathrm{e}^{-5(4x+y)}$ is the solution of the given boundary value problem.

4.3 Wave equation

In this section, we will solve the one dimensional wave equation by the method of separation of variables,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \tag{4.4}$$

which satisfies the boundary conditions

$$u(0,t) = 0, \quad u(L,t) = 0, \quad \text{for all } t \geq 0 \tag{4.5}$$

and the initial conditions

$$u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad 0 \leq x \leq L. \tag{4.6}$$

Review on solutions of second order ODE

When we apply the separation of variables, the PDE will be separated into two ODEs. Therefore, let's have a short review about the solution of the second order homogeneous linear ODE. We consider an ODE

$$y'' + ay' + by = 0,$$

where a, b are constants. The characteristic equation (or auxiliary equation) of this ODE is

$$\lambda^2 + a\lambda + b = 0.$$

The forms of the solution depends on the sign of the discriminant $a^2 - 4b$, namely,
Case I: Two distinct real roots if $a^2 - 4b > 0$,
Case II: A real double root if $a^2 - 4b = 0$,
Case III: Complex conjugate roots if $a^2 - 4b < 0$.

Case I: Two distinct real roots λ_1 and λ_2
In this case $a^2 - 4b > 0$ and the general solution to the given ODE is

$$y = c_1 \mathrm{e}^{\lambda_1 x} + c_2 \mathrm{e}^{\lambda_2 x},$$

where c_1 and c_2 are arbitrary constants.
Case II: A real double root $\lambda = -a/2$
In this case $a^2 - 4b = 0$ and then $\lambda = -\frac{a}{2}$ and the general solution is

$$y = (c_1 + c_2 x)\mathrm{e}^{-\frac{ax}{2}}.$$

Case III: Two conjugate complex roots
In this case $a^2 - 4b < 0$ and the characteristic equation has two complex roots $\lambda_1 = -\frac{a}{2} + \mathrm{i}\omega$ and $\lambda_2 = -\frac{a}{2} - \mathrm{i}\omega$. The general solution is

$$y = \mathrm{e}^{-\frac{ax}{2}}(A\cos\omega x + B\sin\omega x).$$

Now we start to solve the wave equation (4.4) that satisfies the boundary conditions (4.5) and the initial conditions (4.6).

Step 1: We assume the function $u(x,t)$ can be separated into the product of two functions

$$u(x,t) = X(x)T(t).$$

Differentiating $u(x,t)$ we obtain

$$\frac{\partial^2 u}{\partial t^2} = XT'' \quad \text{and} \quad \frac{\partial^2 u}{\partial x^2} = X''T.$$

By inserting this into the wave equation, we have

$$XT'' = c^2 X''T.$$

Dividing by c^2XT and simplifying gives

$$\frac{T''}{c^2T} = \frac{X''}{X}.$$

The variables are now separated, the left side depending only on t and the right side only on x. Hence both sides must be constant because, if they were variable, then changing t or x would affect only one side, leaving the other unaltered. Thus, say,

$$\frac{T''}{c^2T} = \frac{X''}{X} = k.$$

Multiplying by the denominators gives immediately two ordinary differential equations

$$X'' - kX = 0$$

and

$$T'' - c^2kT = 0.$$

Here, the separation constant k is arbitrary.

Step 2: Satisfying the boundary conditions

The boundary conditions are

$$u(0,t) = X(0)T(t) = 0, u(L,t) = X(L)T(t) = 0, \quad \text{for all } t. \quad (4.7)$$

Now let first solve $X'' - kX = 0$. If $T \equiv 0$, then $u = XT \equiv 0$, which is of no interest. Hence $T \not\equiv 0$ and then by (4.7),

$$X(0) = 0, \quad X(L) = 0.$$

We now show that k must be negative.

At first, if $k = 0$, the general solution of $X'' - kX = 0$ is $X(x) = ax + b$, and because $X(0) = 0, X(L) = 0$ we have $a = b = 0$. So that $X \equiv 0$ and $u = XT \equiv 0$, which is of no interest.

Secondly, if $k > 0$ we can assume $k = \mu^2$ and a general solution of $X'' - kX = 0$ is

$$X(x) = A\mathrm{e}^{\mu x} + B\mathrm{e}^{-\mu x}$$

and from the boundary conditions $X(0) = 0, X(L) = 0$ we obtain $X \equiv 0$ as before (verify it by yourself!).

Hence we are left with the possibility of choosing k negative, say, $k = -p^2$. Then $X'' - kX = 0$ becomes $X'' + p^2 X = 0$ and its general solution is

$$X(x) = A \cos px + B \sin px.$$

From this and the boundary condition $X(0) = 0, X(L) = 0$ we have

$$X(0) = A = 0, \text{ and } X(L) = B \sin pL = 0.$$

Here we must take $B \neq 0$ since otherwise $X \equiv 0$. Hence $\sin pL = 0$. Thus

$$pL = n\pi \Longrightarrow p = \frac{n\pi}{L}, \quad n \text{ is integrer.}$$

Setting $B = 1$, we thus obtain infinitely many solutions $X(x) = X_n(x)$, where

$$X_n(x) = B \sin px = \sin \frac{n\pi}{L} x, \; n = 1, 2, \cdots .$$

These solutions satisfy the boundary conditions $X(0) = 0, X(L) = 0$.

We now solve $T'' - c^2 kT = 0$ with $k = -p^2 = \left(\frac{n\pi}{L}\right)^2$, that is

$$T'' + \lambda_n^2 T = 0, \text{ where } \lambda_n = cp = \frac{cn\pi}{L}.$$

A general solution is

$$T_n(t) = B_n \cos \lambda_n t + B_n^* \sin \lambda_n t.$$

Hence solutions of the wave equation (4.4) satisfying the boundary condition (4.5) are

$$u_n(x,t) = X_n(x)T_n(t) = (B_n \cos \lambda_n t + B_n^* \sin \lambda_n t) \sin \frac{n\pi}{L} x, \; n = 1, 2, \cdots \tag{4.8}$$

Step 3: Solution satisfying the initial conditions

The solutions $u_n(x,t)$ we have obtained (4.8) satisfy the wave equation and the given boundary conditions. We now seeking solutions that satisfy the initial conditions. A single $u_n(x,t)$ will generally not satisfy the initial conditions (4.6). By the superposition theorem, the following function still satisfies the wave equation and the boundary conditions

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} (B_n \cos \lambda_n t + B_n^* \sin \lambda_n t) \sin \frac{n\pi}{L} x. \tag{4.9}$$

The first initial condition is $u(x,0) = f(x)$, therefore, we have

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{L} x = f(x).$$

Hence we must choose the B_n such that $u(x,0)$ becomes the Fourier sine series of $f(x)$. Thus B_n can be written into

$$B_n = \frac{2}{L}\int f(x)\sin\frac{n\pi x}{L}dx, \quad n = 1, 2, \cdots .$$

Similarly, by differentiating (4.9) with respect to t and using the second initial condition $u_t(x,0) = g(x)$, we obtain

$$\begin{aligned}
\left.\frac{\partial u}{\partial t}\right|_{t=0} &= \left[\sum_{n=1}^{\infty}(-B_n\lambda_n \sin\lambda_n t + B_n^*\lambda_n\cos\lambda_n t)\sin\frac{n\pi}{L}x\right]_{t=0} \\
&= \sum_{n=1}^{\infty} B_n^*\lambda_n \sin\frac{n\pi}{L}x = g(x).
\end{aligned}$$

Hence we must choose the B_n^* so that for $t = 0$ the derivative $\frac{\partial u}{\partial t}$ becomes the Fourier sine series of $g(x)$. Thus,

$$B_n^* \lambda_n = \frac{2}{L} \int_0^L g(x) \sin \frac{n\pi}{L} x.$$

Since $\lambda_n = \frac{cn\pi}{L}$, we obtain by division

$$B_n^* = \frac{2}{cn\pi} \int_0^L g(x) \sin \frac{n\pi}{L} x, \quad n = 1, 2, \cdots .$$

Now we have obtained the general solution of the wave equation satisfying the given boundary conditions and initial conditions

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} (B_n \cos \lambda_n t + B_n^* \sin \lambda_n t) \sin \frac{n\pi}{L} x,$$

where

$$B_n = \frac{2}{L} \int f(x) \sin \frac{n\pi x}{L} dx, \quad n = 1, 2, \cdots$$

and

$$B_n^* = \frac{2}{cn\pi} \int_0^L g(x) \sin \frac{n\pi}{L} x, \quad n = 1, 2, \cdots .$$