Chapter 11 Independence and Bivariate Normal Distribution

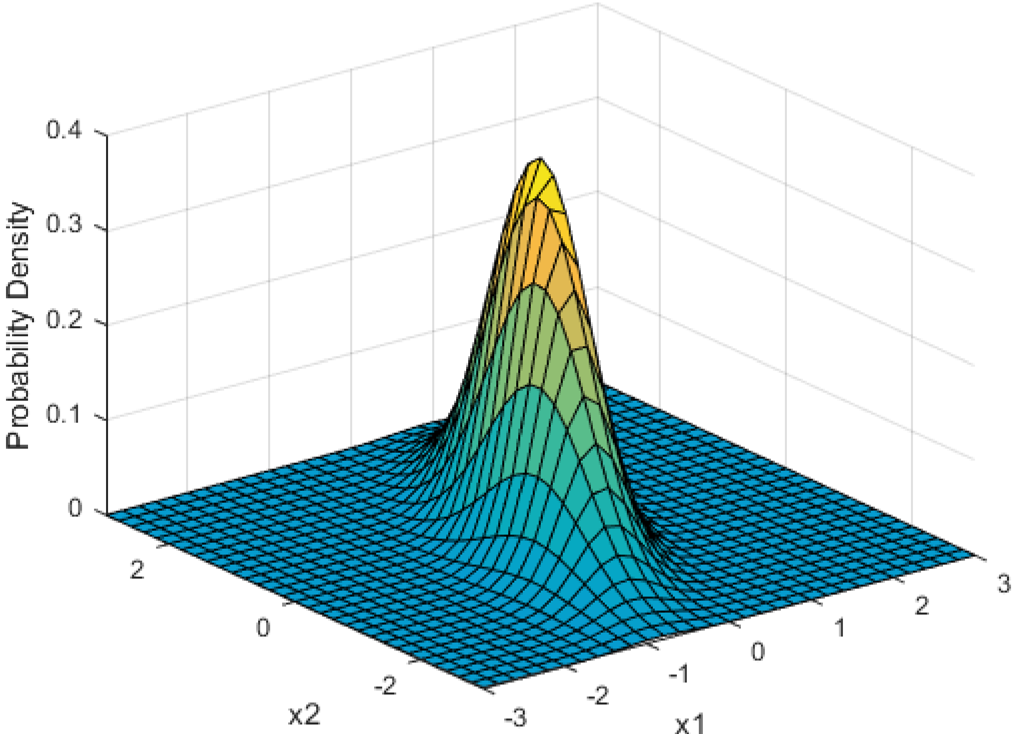
11.1 Independent variables

11.2 Conditional distributions

11.3 Jointly Normal Random Variables

11.4 Summary

05 April, 2018



11.1 Independent variables (1)

Recall that for two events, A and B, are independent if $P(A \cap B) = P(A)P(B)$.

Similarly, Two discrete variables are independent when

$$P\left(X = x_i, Y = y_j\right) = P\left(X = x_i\right)P\left(Y \leq y_j\right), for\ all\ x_i, y_j$$

11.1 Independent variables (2)

Two variables X and Y are independent when

$$P(X \leq x, Y \leq y) = P(X \leq x)P(Y \leq y)$$
$$\Leftrightarrow F_{XY}(x,y) = F_X(x)F_Y(y)$$

for all x and y. In terms of the pdf, this is

$$f_{XY}(x,y) = f_X(x)f_Y(y), for\ all\ x, y$$

11.1 Independent variables, example 1

Example 1

Let Y_1, Y_2 be discrete random variables with joint probability mass function $f(y_1, y_2)$ given by:

		Y_1		
		0	1	2
Y_2	0	0	0.1	0.2
	1	0.1	0.2	0
	2	0.4	0	0

Are Y_1 and Y_2 independent? Explain.

11.1 Independent variables, example 1

For independence, we check if $f(x_i, y_j) = f(x_i)f(y_j)$ for **all** x_i's and y_j's.
We compute $f(x_i)f(y_j)$ for all cells and write their values in brackets below:

		Y_1			
		0	1	2	$f(Y_2)$
Y_2	0	0 $(0.5 * 0.3 = 0.15)$	0.1 $(0.3 * 0.3 = 0.09)$	0.2 (0.06)	**0.3**
	1	0.1 (0.15)	0.2 (0.09)	0 (0.06)	**0.3**
	2	0.4 (0.2)	0 (0.12)	0 (0.08)	**0.4**
	$f(Y_1)$	**0.5**	**0.3**	**0.2**	**1**

Since $f(x_i, y_j) \neq f(x_i)f(y_j)$ for **all** x_i's and y_j's, $\boldsymbol{Y_1}$ **and** $\boldsymbol{Y_2}$ **are not independent**. ■

11.1 Independent variables, example 2

Example 2

Are the variables X and Y with joint pdf $f(x,y) = e^{-y-x};\ x, y > 0$ independent?

Solution

$f(x,y) = e^{-y-x} = e^{-x}e^{-y} = f(x)f(y)$

Hence X and Y are independent. Also,

$$F_{XY}(x,y) = \int_0^x \int_0^y e^{(-u-v)} du dv = \int_0^x e^{-v}(1-e^{-y}) dv = (1-e^{-x})(1-e^{-y}) = F_X(x)F_Y(y)$$

Note, $X \sim \mathrm{Exp}(1), Y \sim \mathrm{Exp}(1)$.

11.1 Conditional distributions (1)

Recall that for two events, A and B, the conditional probability $P(A|B)$ is equal to $\frac{P(A \cap B)}{P(B)}$.

Similarly, we define the conditional probabilities as

$$P(X \leq x | Y \leq y) = \frac{P(X \leq x \cap Y \leq y)}{P(Y \leq y)} = \frac{F_{XY}(x, y)}{F_Y(y)} \; for \; all \; y{:}\, F(y) > 0.$$

11.1 Conditional distributions (2)

We can also define the pmf and pdf conditional on a single value as

$$p\left(X = x_i \middle| Y = y_j\right) = \frac{P\left(X = x_i \cap Y = y_j\right)}{P\left(Y = y_j\right)}, for\ discrete\ variables$$

$$f_{X|Y}(x, y) = \frac{f_{XY}(x,y)}{f_Y(y)} \Rightarrow F_{X|Y}(x|Y = y) = \int_{-\infty}^{x} f_{X|Y}(u, y)du$$

for continuous variables

11.1 Conditional distribution, discrete example

- Assume X and Y have the distribution

X	Y 1	2	3	
1	0.33	0.04	0.03	0.40
2	**0.15**	**0.20**	**0.25**	**0.60**
	0.48	0.24	0.28	1

$P(Y = k|X = 2) = \frac{P(Y=k \cap X=2)}{P(X=2)} = \frac{P(Y=k \cap X=2)}{0.6}$, so

Y	1	2	3	Tot
P(Y|X=2)	0.15/0.6=1/4	0.2/0.6=1/3	0.25/0.6 = 5/12	1

11.1 Conditional distribution, continuous example

Consider two variables, X, Y, with joint pdf

$$f_{XY}(x,y) = \begin{cases} \frac{4}{3}x(1+y), if\ 0 < x \leq 1, 0 < y \leq 1 \\ 0, otherwise \end{cases}$$

The marginal pdf $f_Y(y) = \int_0^1 f_{XY}(x,y)dx = \frac{2}{3}(1+y)$

The conditional pdf

$$f_{X|Y}(x|y = 0.5) = \frac{f_{XY}(x, 0.5)}{f_Y(0.5)} = \frac{\frac{4}{3}x\left(\frac{3}{2}\right)}{\frac{2}{3}\frac{3}{2}} = 2x$$

11.1 Conditional distribution, continuous example2

The conditional pdf

$$f_{X|Y}(x|y = 0.5) = 2x, 0 < x \leq 1$$

is a proper pdf:

$\mathrm{F}_{\mathrm{X|Y}}(\mathrm{x}|y = 0.5) = \int_0^x 2u\, du = \left[\frac{2}{2}u^2\right]_0^x = x^2$, with $F_{X|Y}(1|0.5) = 1$.

We can take expectations:

$$E(X|Y = 0.5) = \int_0^1 2x^2 dx = \left[\frac{2}{3}x^3\right]_0^1 = \frac{2}{3}$$

$$Var(X|Y = 0.5) = \int_0^1 2x^3 dx - \left[\frac{2}{3}\right]^2 = \frac{1}{2} - \frac{4}{9} = \frac{1}{18}$$

11.1 Conditions for independence

<u>Two</u> random variables X and Y

1) Are **independent** if $f(x, y) = f(x)f(y)$,

   or equivalently, if both

   $f(x|y) = f(x)$ and $f(y|x) = f(y)$ or

$P(X \leq x|Y = y) = P(X \leq x)$ and $P(Y \leq y|X = x) = P(Y \leq y)$

3) Independence $\Rightarrow$ Uncorrelated but

   Uncorrelated $\neq>$ Independence

   two dependent variables can be uncorrelated (in some cases).

11.1 Independence and uncorrelatedness

The *covariance* between two variables is defined as

$Cov(X,Y) = E\big((X-\mu_x)(Y-\mu_Y)\big) = E(XY) - E(X)E(Y).$

The *correlation* is $\rho = \frac{Cov(X,Y)}{\sigma_X \sigma_Y}$.

Theorem 1

1) X and Y are uncorrelated if $\rho = 0$, which means if $Cov(X,Y) = 0$, if and only if $E(XY) = E(X)E(Y)$.

2) If X and Y are independent, $Cov(X,Y) = Cor(X,Y) = 0$.
   If $Cov(X,Y) = Cor(X,Y) \neq 0, X$ and Y are not independent

11.1 independent discrete variables

If X and Y are independent, then

$\mathrm{p}(x_i|y_j) = p(x_i)$ and $\mathrm{p}(y_i|x_j) = p(y_i)$ or
$\mathrm{p}(x_i, y_j) = p(x_i)p(y_j)$ for **all** x_i's and y_j's.

Then we have to check this property for all pairs (x_i, y_j). If we find one pair for which this is not true, we can stop: the variables are not independent.
Otherwise we keep checking until all pairs have been considered.

We could use also $f(x|y) = f(x)$ but then we have to check also $f(y|x) = f(y)$ as in some cases this is not true for only one variable.

11.1 Independent discrete variables, example

Example 3

Let Y_1, Y_2 be discrete random variables with joint probability mass function $f(y_1, y_2)$ given by:

		Y_1			
		0	1	2	Tot
Y_2	0	0	0.1	0.2	0.3
	1	0.1	0.2	0	0.3
	2	0.4	0	0	0.4
	Tot	0.5	0.3	0.2	1

i. Check if $\mathrm{P}_{\mathrm{Y}_1\mathrm{Y}_2}(y_1, y_2) = \mathrm{P}_{\mathrm{Y}_1}(y_1)\mathrm{P}_{\mathrm{Y}_2}(y_2)$;

ii. Find $P(Y_2 = 2|Y_1 = 0)$;

iii. Find $cov(X, Y)$;

iv. Are the variables independent? Explain for each case above;

v. Find the conditional probability $P(Y_1 \geq 1|Y_2 = 1)$.

11.1 independent discrete variables, example

i. For independence, we check if

$f(x_i, y_j) = f(x_i)f(y_j)$ for **all** x_i's and y_j's. We compute $f(x_i)f(y_j)$ for all cells and write their values in brackets below:

		Y_1			
		0	1	2	
Y_2	0	0 $(0.5 * 0.3 = 0.15)$	1/10 $(0.3 * 0.3 = 0.09)$	2/10 (0.06)	0.3
	1	1/10 (0.15)	2/10 (0.09)	0 (0.05)	0.3
	2	4/10 (0.2)	0 (0.12)	0 (0.08)	0.4
		0.5	0.3	0.2	1

Since $f(x_i, y_j) \neq f(x_i)f(y_j)$ for **all** x_i's and y_j's, Y_1 and Y_2 are not independent. ■

11.1 independent discrete variables, example

Solution

		Y_1			
		0	1	2	Tot
Y_2	0	0	0.1	0.2	0.3
	1	**0.1**	0.2	0	0.3
	2	0.4	0	0	0.4
	Tot	0.5	0.3	0.2	1

ii. $P(Y_2 = 1 | Y_1 = 0) = \frac{P(Y_1=0, Y_2=1)}{P(Y_1=0)}$

$= \frac{0.1}{0.5} = 0.2$

$P(Y_2 = 1 | Y_1 = 0) \neq P(Y_2 = 1)$. This is enough to say that the variables are not independent.

11.1 independent discrete variables, example

iii. $\mathrm{E}(Y_1 Y_2) = \sum_{i=0}^{2} \sum_{j=0}^{2} ijP(Y_1 = i, Y_2 = j) =$

$$0 + 0 + \sum_{i=1}^{2} \sum_{j=1}^{2} ijP(Y_1 = i, Y_2 = j) =$$

$1 * P(Y_1 = 1, Y_2 = 1) + 2 * P(Y_1 = 2, Y_2 = 1) + 2 * P(Y_1 = 1, Y_2 = 2) + 4 * P(Y_1 = 2, Y_2 = 2) = 0.2 + 0 + 0 + 0 + 0 = 0.2$

	$p(x_i, y_j)$	Y_1			
		0	1	2	Tot
Y_2	0	0	0.1	0.2	0.3
	1	**0.1**	0.2	0	0.3
	2	0.4	0	0	0.4
	Tot	0.5	0.3	0.2	1

	$x_i y_j$	Y_1		
		0	1	2
Y_2	0	0	0	0
	1	0	1	2
	2	0	2	4

11.1 independent discrete variables, example

iii. $\mathrm{E}(Y_1) = \sum_{i=0}^{2} iP(Y_1 = i) = 0 * 0.5 + 1 * 0.3 + 2 * 0.2 = 0.7$

$\mathrm{E}(Y_2) = \sum_{j=0}^{2} jP(Y_2 = j) = 0 * 0.3 + 1 * 0.3 + 2 * 0.4 = 1.1$

$$Cov(X, Y) = E(XY) - E(X)E(Y) = 0.2 - 0.77 = -0.57$$

Since $Cov(X, Y) \neq 0$ [or $E(XY) = 0.2 \neq E(X)E(Y) = 0.77$], the variables are correlated. Then X and Y are not independent.

Note if X and Y are independent,

Cov must be zero.

Hence, if $Cov \neq 0$ the variables are not independent

		Y_1			
	$p(x_i, y_j)$	0	1	2	Tot
Y_2	0	0	0.1	0.2	0.3
	1	**0.1**	0.2	0	0.3
	2	0.4	0	0	0.4
	Tot	0.5	0.3	0.2	1

11.1 independent discrete variables, example

Solution

		Y_1			
		0	1	2	Tot
Y_2	0	0	0.1	0.2	0.3
	1	0.1	0.2	0	0.3
	2	0.4	0	0	0.4
	Tot	0.5	0.3	0.2	1

v. $P(Y_1 \geq 1 | Y_2 = 1) = \frac{P(Y_1 \geq 1, Y_2 = 1)}{P(Y_2 = 1)}$

$$= \frac{P(Y_1 = 1, Y_2 = 1) + P(Y_1 = 2, Y_2 = 1)}{P(Y_2 = 1)}$$

So, $P(Y_1 \geq 1 | Y_2 = 1) = \frac{0.2 + 0}{0.3} = \frac{0.2+0}{0.1+0.2+0} = \frac{2}{3}$.

11.1 Independence and uncorrelatedness, example

Example 3

Suppose random variable X has the pmf:

$\boldsymbol{x}$	$\mathbf{-1}$	$\mathbf{0}$	$\mathbf{1}$
$\boldsymbol{P(X = x)}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

Let $Y = X^2$.

i. Are X and Y independent?

ii. Are X and Y are uncorrelated?

11.1 Independence and uncorrelatedness , example

Solution The joint pmf is given as

		Y		
		0	1	
X	-1	0	1/4	$P(X=-1)=\frac{1}{4}$
	0	1/2	0	$P(X=0)=\frac{1}{2}$
	1	0	1/4	$P(X=1)=\frac{1}{4}$
		$P(Y=0)=\frac{1}{2}$	$P(Y=1)=\frac{1}{2}$	

i. Since $P(X=-1, Y=0)=0 \neq P(X=-1)P(Y=0)=\frac{1}{8}$, X and Y are not independent.

11.1 Independence and Uncorrelatedness, example

ii. We find the expectations:

$$E(X) = (-1)\left(\tfrac{1}{4}\right) + (0)\left(\tfrac{1}{2}\right) + (1)\left(\tfrac{1}{4}\right) = 0; \qquad E(Y) = (0)\left(\tfrac{1}{2}\right) + (1)\left(\tfrac{1}{2}\right) = \tfrac{1}{2}$$

$$E(XY) = (-1)(0)(0) + (-1)(1)\left(\tfrac{1}{4}\right) + (0)(0)\left(\tfrac{1}{2}\right) + (0)(1)(0) + (1)(0)(0) + (1)(1)\left(\tfrac{1}{4}\right) = 0$$

		Y		
		0	1	
X	-1	0	$\frac{1}{4}$	$P(X=-1)=\frac{1}{4}$
	0	$\frac{1}{2}$	0	$P(X=0)=\frac{1}{2}$
	1	0	$\frac{1}{4}$	$P(X=1)=\frac{1}{4}$
		$P(Y=0)=\frac{1}{2}$	$P(Y=1)=\frac{1}{2}$	

Since $E(XY) = E(X)E(Y)$, $Cov(X,Y) = E(XY) - E(X)E(Y) = 0$, X and Y are uncorrelated. ■

The example illustrates that you can have a pair of dependent random variables that is uncorrelated.

11.1 Independence continuous variables, example

Example 4

Consider X and Y with joint pdf

$$f_{XY}(x, y) = \frac{4}{3}(1 - xy), 0 < x, y \leq 1$$

i. Find $f_X(x)$ and $f_Y(y)$;

ii. Find $f_{X|Y}(x|y)$ and $F_{X|Y}(x|y)$

iii. Are the variables independent? comment the above cases

11.1 Independence continuous variables, example

i. $f_X(x) = \frac{4}{3}\int_0^1 (1 - xy)dy = \frac{4}{3}\left[y - \frac{xy^2}{2}\right]_0^1 = \frac{4}{3} - \frac{2}{3}x = \frac{2}{3}(2 - x);$

$$f_Y(y) = \frac{4}{3}\int_0^1 (1 - xy)dx = \frac{4}{3}\left[x - \frac{yx^2}{2}\right]_0^1 = \frac{4}{3} - \frac{2}{3}y = \frac{2}{3}(2 - y);$$

ii. $f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)} = \frac{2(1-xy)}{2-y};$

$$F_{X|y}(X|Y) = \frac{2}{2-y}\int_0^x (1 - uy)du = \frac{2}{2-y}\left(x - \frac{x^2y}{2}\right) = \frac{2x - x^2y}{2-y}$$

iii. Since $f_X(x)\, f_Y(y) = \frac{4}{9}(2 - x)(2 - y) \neq f_{XY}(x, y)$, X and Y are not indep.t.

since $f_{X|Y}(x|y) \neq f_X(x)$ $[or\ F_X(x) = \frac{1}{3}(4x - x^2) \neq F_{X|Y}(x)]$, X and Y are not indep.t.

11.2 Mean of product of independent variables

Theorem 2

The mean of the product of **independent** random variables equals the product of means, i.e.

$$E(X_1X_2\cdots X_n)=E(X_1)E(X_2)\cdots E(X_n).$$

The result is true for continuous/discrete random variables. ■

11.3 Bivariate Normal Random Variables

Theorem 1

Two random variables X and Y are said to have a **bivariate normal distribution** with parameters μ_X, σ_X^2, μ_Y, σ_Y^2 and ρ, if their joint pdf is given by

$$f(x, y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2 - 2\rho\frac{(x-\mu_X)(y-\mu_Y)}{\sigma_X\sigma_Y} + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right]}$$

where ρ is the correlation between X and Y. ■

11.3 Bivariate Normal Random Variables

The correlation ρ is a measure of *closeness* or *agreement* between two variables.

$$-1 \leq \rho \leq 1$$

The closer it is to 1 and the more the variables are positively correlated, the closer to -1 the more the variables are negatively correlated.

$\rho = 0$ means that the variables are uncorrelated.

11.3 Bivariate Normal Random Variables

When the variables are perfectly correlated ($\rho = \pm 1$) the distribution is improper ($1 - \rho^2 = 0$). This is because the variables are linearly dependent.

We can think of perfectly correlated variables linked by

$$X = a + bY$$

Hence it is enough to define the probability distribution of one of the two variables to have that of the other because

$$P(X \leq x) = P\left(Y \leq \frac{x - a}{b}\right)$$

11.3 Bivariate Normal Random Variables

We get an insight by looking at the contour plots

where $\left(\frac{x-\mu_X}{\sigma_X}\right)^2 - 2\rho\frac{(x-\mu_X)(y-\mu_Y)}{\sigma_X\sigma_Y} + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2 = \text{constant.}$

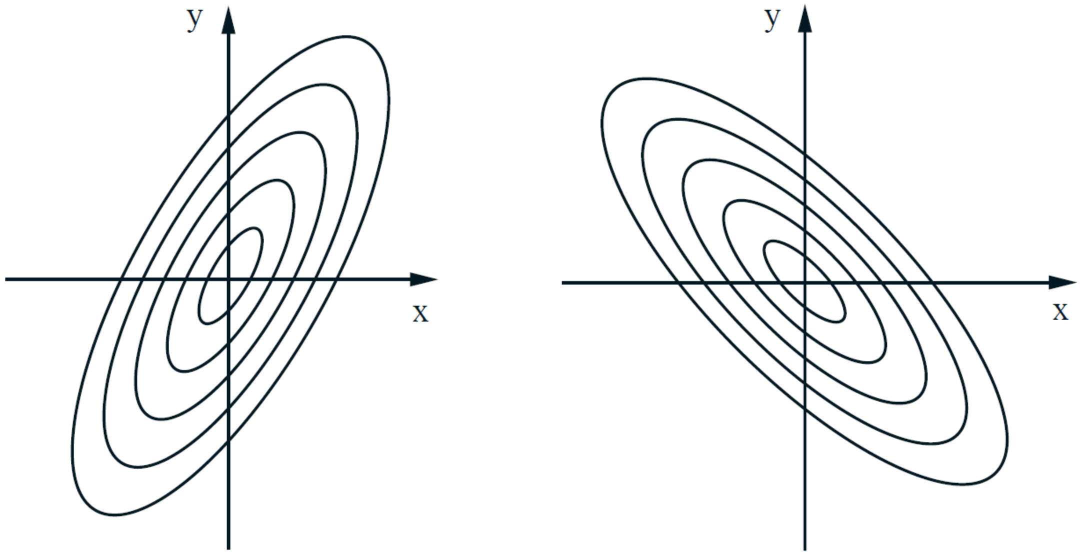
$u = \frac{x-\mu_X}{\sigma_X}$ and $v = \frac{y-\mu_Y}{\sigma_Y}$

Are the standardized values of X and Y.

These are very useful in proofs. You are familiar with them as they are the same as the z we use for univariate normal

11.3 Bivariate Normal Random Variables

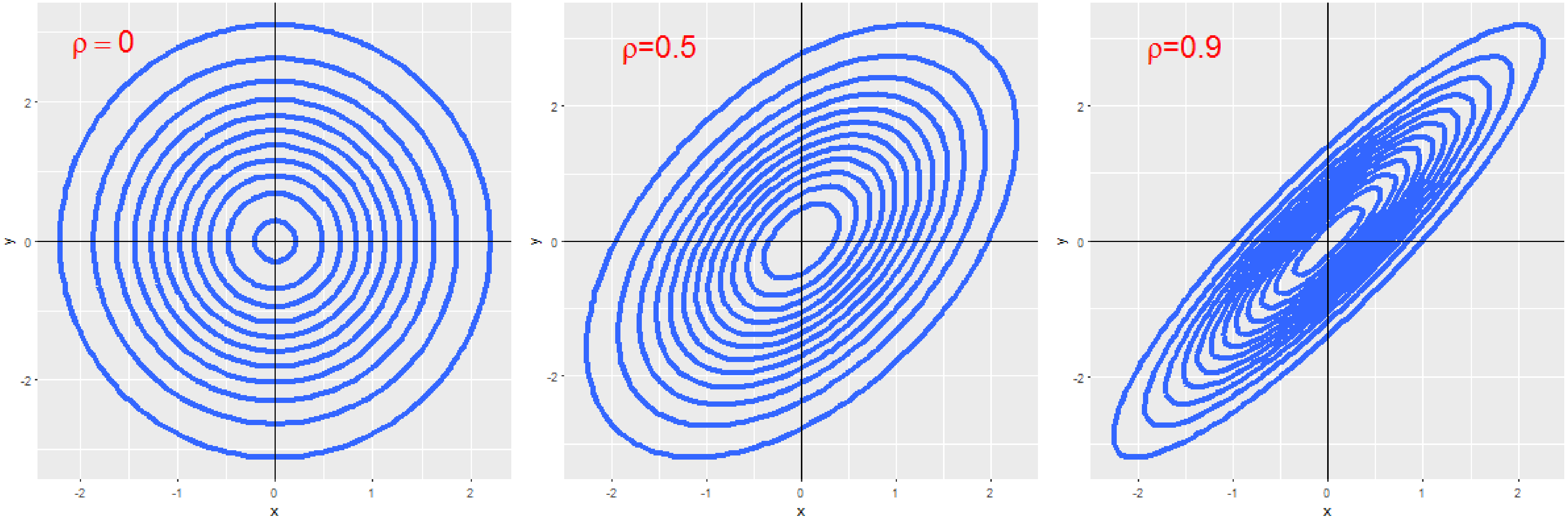
The contours are the values of X and Y for which the pdf is constant



The contour plots of the bivariate normal (X, Y) pdf. The diagram on the left corresponds to a case of positive value of ρ, on the right of negative.

11.3 Bivariate Normal Random Variables

The contour plots of the bivariate normal (X, Y) pdf. The diagram on shows the contours of equal density for increasing positive correlation (ρ). The means are both zero, $\mu_x = \mu_y = 0$, the contours are elliptical because $\sigma_X^2 = 1, \sigma_y^2 = 2$. When $\sigma_X^2 = \sigma_y^2$ the contours are circular.



11.3 Bivariate Normal Random Variables

Theorem 2

Let X and Y be bivariate normal random variables and uncorrelated, $(\rho = 0)$. Then they are independent.

Proof

$$f(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2 - 2\rho\frac{(x-\mu_X)(y-\mu_Y)}{\sigma_X\sigma_Y} + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right]} =$$

$$[substitute\ \rho = 0]\ \frac{1}{2\pi\sigma_X\sigma_Y} e^{-\frac{1}{2}\left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2 + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right]} =$$

$$\left[\frac{1}{\sqrt{2\pi}\sigma_X} e^{-\frac{1}{2}\left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2\right]}\right]\left[\frac{1}{\sqrt{2\pi}\sigma_Y} e^{-\frac{1}{2}\left[\left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right]}\right] = f(x)f(y) \blacksquare$$

NB. This is true for Normal variables, and not for all variables!!! In general uncorrelated variables may be dependent.

11.3 Bivariate Normal Random Variables

Theorem 3

Let X and Y be bivariate normal random variables then X and Y are normal. Enough to prove for U, V standard Normal

Proof

$$f_{U,V}(u,v) = \frac{1}{2\pi\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}[u^2+v^2-2\rho uv]}$$

$$= \frac{1}{2\pi\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}[(u^2-\rho^2u^2)+\rho^2u^2+v^2-2\rho uv]}$$

$$\boxed{\begin{aligned}&(u^2-\rho u^2)\\&=(1-\rho)u^2\end{aligned}}$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}[u^2]} \frac{1}{\sqrt{2\pi}\sqrt{1-\rho^2}} e^{-\frac{(v-\rho u)^2}{2(1-\rho^2)}} = f_U(u) f_{V|U}(v|u) \blacksquare$$

So $u \sim N(0,1)$ and $v|u \sim N(\rho u, (1-\rho^2))$

NB. Analogously, $v \sim N(0,1)$ and $u|v \sim N(\rho v, (1-\rho^2))$

11.3 Bivariate Normal Random Variables

Corollary to Theorem 3

Let X and Y be bivariate normal random variables then $Y|X$ has a normal distribution with mean and variance equal to

$$E(Y|X) = E(Y) + \rho\,\frac{\sigma_Y}{\sigma_X}\big(X - E(X)\big) = \mu_Y + \rho\,\frac{\sigma_Y}{\sigma_X}(X - \mu_X)$$

$$Var(Y|X) = \sigma_Y^2(1 - \rho^2)$$

To see this consider that

$$\mathrm{E_{U|V}}(v|u) = E\left(\frac{y - \mu_Y}{\sigma_Y}\right) = \frac{1}{\sigma_Y}(E(Y|u) - \mu_Y) = \rho u.\,\mathrm{Hence}$$

$$E(Y|u) = \mu_Y + \rho\frac{\sigma_Y}{\sigma_X}(x - \mu_X)$$

$$V(v|u) = V\left(\frac{y-\mu_Y}{\sigma_Y}|u\right) = \frac{1}{\sigma_Y^2}V(Y|u) = (1 - \rho^2) \Rightarrow \sigma_{Y|X}^2 = \sigma_Y^2(1 - \rho^2)$$

11.3 Bivariate Normal Random Variables

Theorem 4 **important**

Two random variables X and Y are said to be *bivariate (jointly) normal*, if $\mathrm{Z} = aX + bY$ has a normal distribution for all $a, b \in \mathbb{R}$, with

$E(Z) = a\mu_X + b\mu_Y$ and

$var(Z) = a^2\sigma_X^2 + b^2\sigma_Y^2 + 2ab\ cov(XY) = a^2\sigma_X^2 + b^2\sigma_Y^2 + 2ab\ \rho\sigma_X\sigma_Y$

Note

1) If X and Y are independent normal random variables, then they are bivariate normal.
2) If X and Y are dependent normal random variables, then they <u>may not</u> be bivariate normal.

11.3 Bivariate Normal Random Variables

Example 1

Suppose that X and Y are zero-mean bivariate normal random variables, such that $\sigma_X^2 = 4, \sigma_Y^2 = 9$ and $E[XY] = 3$. We define a new random variable $\mathrm{Z} = 2X - Y$. Determine the

i. Pdf $f_Z(z)$

ii. Pdf $f_{X,Z}(x, z)$

iii. Factorize the joint pdf $f(x, y) = \mathrm{f_X(x)f_{y|X}(y|x)}$.

11.3 Bivariate Normal Random Variables, sol 1

i. Since X and Y are bivariate Normal, also Z is with $E(Z) = 2E(X) - E(Y) = 0, \sigma_Z^2 = 4\sigma_X^2 + \sigma_Y^2 - 4cov(XY)$

- $cov(XY) = E(XY) = 3$, so $\sigma_Z^2 = 16 + 9 - 12 = 13$, and
- $f_Z(z) = \frac{1}{\sqrt{26\pi}} e^{-\frac{1}{2}\frac{z^2}{13}}$

ii. $Cov(X, Z) = E(XZ) - E(X)E(Z) = E\big(X(2X - Y)\big) = 2E(X^2) - E(XY) = 2\sigma_X^2 - 3 = 8 - 3 = 5$

So, $\rho_{XY}^2 = \frac{25}{13*4} = \frac{25}{52}$

11.3 Bivariate Normal Random Variables, sol 2

And $f_{X,Z}(x,z) = \frac{1}{2\pi\sqrt{4*13\left(1-\frac{25}{52}\right)}} e^{-\frac{1}{2\left(1-\frac{25}{52}\right)}\left[\frac{x^2}{4} - \frac{25}{26}\frac{xz}{\sqrt{4*13}} + \frac{y^2}{13}\right]}$

iii. $f(x,z) = \frac{1}{\sigma_X\sigma_Z 2\pi\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left(\left(\frac{x-\mu_X}{\sigma_X}\right)^2 - 2\rho\frac{(x-\mu_X)(z-\mu_Z)}{\sigma_X\sigma_Z} + \left(\frac{z-\mu_Z}{\sigma_Z}\right)^2\right)}$.

Call $u = (x-\mu_X)/\sigma_X$. Remember the trick $u^2 = (u^2 - \rho^2 u^2) + \rho^2 u^2$? That is still valid. So,

$$f(x,z) = \frac{1}{\sigma_X\sqrt{2\pi}} e^{-\frac{1}{2}\left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2\right]} \frac{1}{\sigma_Z\sqrt{2\pi}\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left[\left(\left(\frac{z-\mu_Z}{\sigma_Z}\right) - \rho\left(\frac{x-\mu_X}{\sigma_X}\right)\right)^2\right]}$$

11.3 Bivariate Normal Random Variables, ex 2

Example

Consider two jointly standard normal variables U and V with $cor(u, v) = \rho = \frac{1}{2}$. Find the pdf of V$/U$ and $P(V > 0|U = u)$.

Solution

We have $\mu_U = \mu_V = 0, \sigma_U = \sigma_V = 1$

$$f_{UV}(u, v) = \frac{1}{2\pi\sqrt{3/4}} e^{-\frac{1}{2\left(1-\frac{1}{4}\right)}\left[u^2-2\frac{1}{2}uv+v^2\right]} =$$

11.3 Bivariate Normal Random Variables, ex 2

$$f_{UV}(u,v) = \frac{1}{2\pi\sqrt{3/4}} e^{-\frac{1}{2\frac{3}{4}}\left[u^2 - \frac{1}{4}u^2 + \frac{1}{4}u^2 - 2\frac{1}{2}uv + v^2\right]} =$$

$$\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} \frac{1}{\sqrt{2\pi}\sqrt{3/4}} e^{-\frac{1}{2\frac{3}{4}}\left(v - \frac{1}{2}u\right)^2} = N(0,1)N\left(\frac{1}{2}u, \frac{3}{4}\right)$$

$$= f_U(u) f_{V|U}(v|u)$$

So $V|U \sim N\left(\frac{u}{2}, \frac{3}{4}\right)$.

$E(V|U=u) = E(V) + \rho \frac{\sigma_V}{\sigma_U}\left(u - E(U)\right) = \frac{1}{2}u,\ Var(V|U) = \sigma_V(1-\rho^2) = \frac{3}{4}.$

11.3 Bivariate Normal Random Variables, sol 2

Therefore,

$$P(V > 0|U = u) = P\left(Z > \frac{-{}^{u}/_{2}}{{}^{\sqrt{3}}/_{2}}\right) = P\left(Z > -\frac{u}{\sqrt{3}}\right),$$

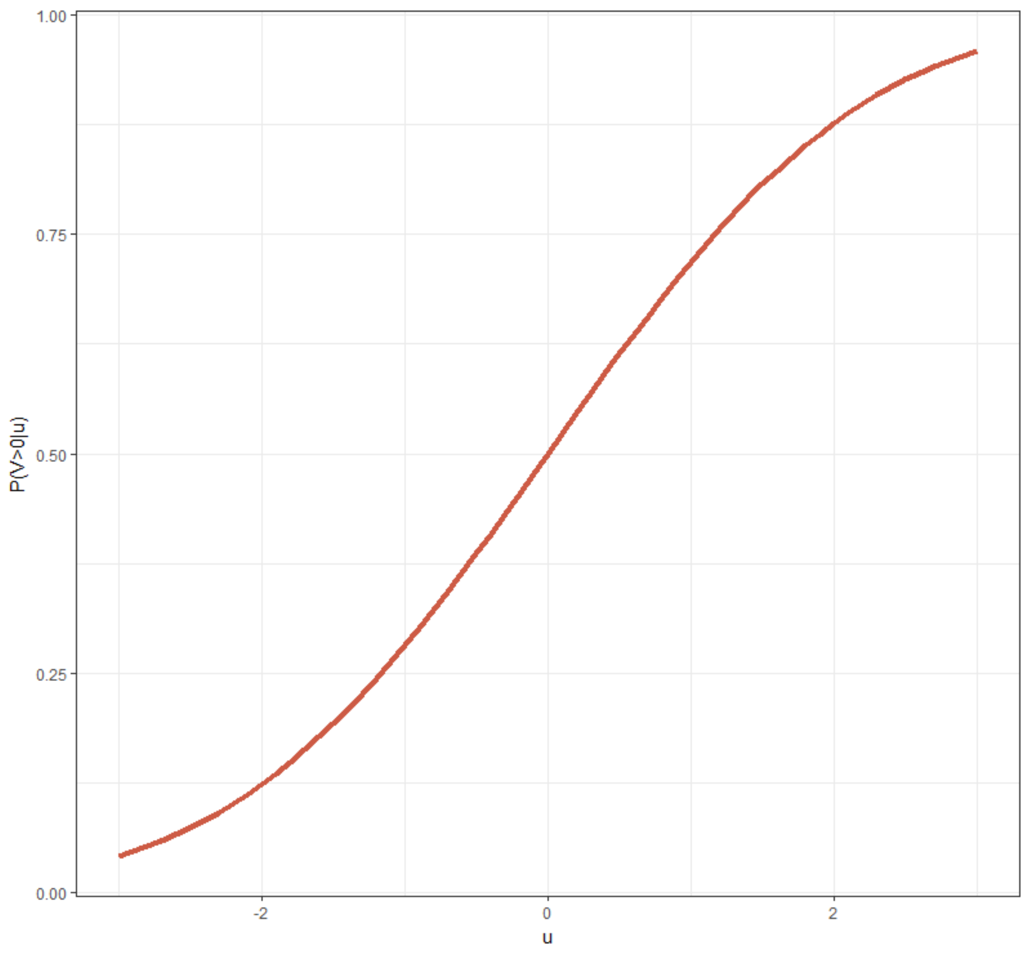
where $Z \sim N(0,1)$

So $P(V > 0|U = 0) = 0.5$ (as usual),

but $P(V > 0|U = 1) = 0.7182$

11.3 Bivariate Normal Random Variables,

Probability $P(V > 0|U = u)$ as u increases. Since U and V are positively correlated, the probability of V being positive increases the larger is u.



11.2 Summary

- Variables are independent if $f_{XY}(x,y) = f_X(x)f_Y(y)$ or $f_{X|Y}(x|y) = f_X(x)$ and $f_{Y|X}(y|x) = f_Y(y)$
- If two variables are independent $cov(X,Y) = cor(X,Y) = 0$ but $cov(X,Y) = cor(X,Y) = 0$ is not enough for independence
- Know what is meant by bivariate (jointly) normal
- Know that bivariate normal + uncorrelated $(\rho = 0)$ ⇒ are independent
- Know that normal independent variables are always bivariate (jointly) normal
  - If $(X,Y) \sim MN_2\left((\mu_X,\mu_Y),(\sigma_X^2,\sigma_Y^2)\right)$, then $X \sim N(\mu_X,\sigma_X^2)$ and $Y \sim N(\mu_Y,\sigma_Y^2)$
- Two Normal variables are bivariate if a linear combination of them $(Z = aX + bY)$ also is Normal
- If $(X,Y) \sim MN_2$ then

$$Y|X \sim MN\left(\mu_Y - \rho\,\frac{\sigma_Y}{\sigma_X}(X - \mu_x),\ \sigma_Y^2(1-\rho^2)\right)$$