MTH101: Lecture 19 – 20

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Example

When using Laplace transform to solve nonhomogeneous linear ODEs, we frequently have to deal with some **multiplication of transforms***.*

E.g.

$$y'' + ay' + by = r(t) \qquad y(0) = K_0,\ y'(0) = K_1,$$
$$\Rightarrow y(t) = \mathcal{L}^{-1}\big[[(s+a)K_0 + K_1]\,Q(s)\big] + \mathcal{L}^{-1}\big[Q(s)R(s)\big],$$

where the first is easy to solve, and we need **convolution theorem** *to deal with the second term.*

Theorem

If two functions f and g satisfy the existence theorem in Sec. 6.1, and their transforms are denoted as F and G, the product $H = FG$ is the Laplace transform of function h:

$$h(t) \equiv (f * g) \equiv \int_0^t f(\tau)g(t-\tau)d\tau.$$

Proof.

By definition, F, G can be written as follows

$$F(s) = \int_0^\infty e^{-s\tau} f(\tau) d\tau \quad and \quad G(s) = \int_0^\infty e^{-sp} g(p) dp.$$

Therefore

$$F(s)G(s) = \int_0^\infty e^{-s\tau} f(\tau) d\tau \int_0^\infty e^{-sp} g(p) dp.$$

Proof.

Now, if we introduce a variable $t = \tau + p$ and integrate over variables t and τ instead of p and τ, the limits of integration become $(0 \leq t \leq \infty)$, $(0 \leq \tau \leq t)$, the integral becomes

$$F(s)G(s) = \int_{t=0}^{\infty} \left[\int_{\tau=0}^{t} e^{-s(t-\tau)} g(t-\tau) e^{-s\tau} f(\tau) d\tau \right] dt$$

$$\Rightarrow \; F(s)G(s) = \int_{t=0}^{\infty} e^{-st} \left[\int_{\tau=0}^{t} g(t-\tau) f(\tau) d\tau \right] dt$$

$$\Rightarrow \; F(s)G(s) = \int_{0}^{\infty} e^{-st} h(t) dt = \mathcal{L}\left[h(t) \right].$$

$\square$

Remark

$$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)d\tau$$

$$\mathcal{L}\left[(f * g)\right] = \mathcal{L}\left[f\right]\mathcal{L}\left[g\right]$$

$$f * g = g * f \qquad \textit{(commutative law)},$$

$$f * (g_1 + g_2) = f * g_1 + f * g_2 \qquad \textit{(distributive law)},$$

$$(f * g) * v = f * (g * v) \qquad \textit{(associative law)},$$

$$f * 0 = 0 * f = 0.$$

Example

Using convolution to solve the damped mass-spring system modeled by

$$y'' + 3y' + 2y = r(t) \qquad y(0) = y'(0) = 0,$$
$$r(t) = \begin{cases} 1 & \text{if } 1 < t < 2, \\ 0 & \text{otherwise.} \end{cases}$$

Example

Solution:
We know the solution to the nonhomogeneous ODE is

$$y(t) = \mathcal{L}^{-1}\big[\left[(s+a)K_0 + K_1\right] Q(s)\big] + \mathcal{L}^{-1}\big[Q(s)R(s)\big],$$

and with the initial conditions, we can express the solution as

$$y(t) = \mathcal{L}^{-1}\big[Q(s)R(s)\big] = q(t) * r(t) = \int_0^t q(t-\tau)r(\tau)d\tau,$$

where $Q(s)$, $R(s)$ are Laplace transform of $q(t)$ and $r(t)$.

Example

Solution:
Since

$$Q(s) = \frac{1}{s^2 + 3s + 2} = \frac{1}{s+1} - \frac{1}{s+2},$$

we know $q(t) = e^{-t} - e^{-2t}$, *and the solution is thus*

$$y(t) = \int_0^t \left[e^{-(t-\tau)} - e^{-2(t-\tau)} \right] r(\tau) d\tau.$$

There are **three** *different intervals we need to consider.*

Example

Solution:

1. *For* $0 < t < 1$, *the integration is zero since* $r(\tau) = 0$ *for* $0 < \tau < t < 1$.

$$y_1(t) = 0, \quad \text{if} \quad 0 < t < 1.$$

2. *For* $1 < t < 2$, *we have to integrate from* $\tau = 1$ *to* $\tau = t$, *with* $r(\tau) = 1$ *in this interval.*

$$\begin{aligned} y_2(t) &= \left[e^{-(t-\tau)} - \frac{1}{2} e^{-2(t-\tau)} \right] \Big|_1^t \\ &= \frac{1}{2} - e^{-(t-1)} + \frac{1}{2} e^{-2(t-1)}, \quad \text{if} \quad 1 < t < 2. \end{aligned}$$

Example

Solution:

3. *For $t > 2$, we have to integrate from $\tau = 1$ to $\tau = 2$, with $r(\tau) = 1$ in this interval.*

$$
\begin{aligned}
y_3(t) &= \left[e^{-(t-\tau)} - \frac{1}{2} e^{-2(t-\tau)} \right] \Big|_1^2 \\
&= e^{-(t-2)} - \frac{1}{2} e^{-2(t-2)} - e^{-(t-1)} + \frac{1}{2} e^{-2(t-1)},
\end{aligned}
$$

which is for $t > 2$. There for $y(t) = y_i(t)$, for $i = 1, 2, 3$, depending on the interval we are interested in. One can check that $y(t)$ is continuous in $0 < t < \infty$.

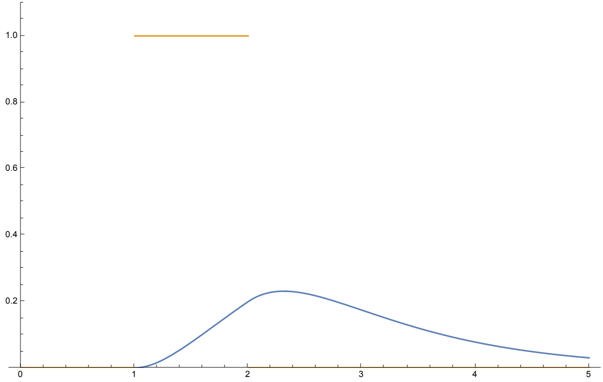

Figure: $r(t)$ and $y(t)$.

Example

Using Laplace transform to solve the following ODE

$$y''(t) + y(t) = r(t), \quad y(0) = 0, \quad y'(0) = 1,$$

where

$$r(t) = \begin{cases} 10 \sin 2t, & \text{if} \quad 0 < t < \pi, \\ 0, & \text{if} \quad t > \pi. \end{cases}$$

Example

Solution:
We first use the Laplace transform for the ODE

$$\mathcal{L}\left[y''\right] + \mathcal{L}\left[y\right] = \mathcal{L}\left[r(t)\right],$$

where

$$\mathcal{L}\left[y''\right] = s^2\mathcal{L}\left[y\right] - sy(0) - y'(0) = s^2\mathcal{L}\left[y\right] - 1,$$

and the the equation becomes

$$\left(s^2+1\right)\mathcal{L}\left[y\right] - 1 = \mathcal{L}\left[r(t)\right].$$

Example

Solution:
Therefore,

$$\mathcal{L}\left[y\right] = \frac{\mathcal{L}\left[r(t)\right]}{s^2+1} + \frac{1}{s^2+1},$$

and we can perform inverse transform to obtain

$$y = \mathcal{L}^{-1}\left[\mathcal{L}\left[r(t)\right]\mathcal{L}\left[\sin t\right]\right] + \sin t$$
$$\Rightarrow y = r(t) * \sin t + \sin t.$$

Example

Solution:
The first term can be expressed as follows

$$I \equiv r(t) * \sin t = \int_0^t \sin(t-\tau)\left\{10\sin(2\tau)\left[1-u(\tau-\pi)\right]\right\} d\tau,$$

if $t < \pi$

$$\begin{aligned} I &= 10\int_0^t (\sin t\cos\tau - \cos t\sin\tau)\sin(2\tau)d\tau \\ &= 10\left(\sin t\int_0^t 2\sin\tau\cos^2\tau\, d\tau - \cos t\int_0^t 2\sin^2\tau\cos\tau\, d\tau\right) \end{aligned}$$

Example

Solution:

if $t < \pi$

$$
\begin{aligned}
I &= 10\left[-\sin t\int_{\tau=0}^{t} 2\cos^2\tau\, d(\cos\tau) - \cos t\int_{\tau=0}^{t} 2\sin^2\tau\, d(\sin\tau)\right] \\
&= 10\left[-\sin t\left(\frac{2}{3}\cos^3\tau\right)\Big|_0^t - \cos t\left(\frac{2}{3}\sin^3\tau\right)\Big|_0^t\right] \\
&= \frac{20}{3}\left(\sin t - \sin t\cos^3 t - \sin^3 t\cos t\right) \\
&= \frac{20}{3}\sin t - \frac{20}{3}\sin t\cos t = \frac{20}{3}\sin t - \frac{10}{3}\sin 2t.
\end{aligned}
$$

Example

Solution:
if $t > \pi$

$$I = 10\int_0^{\pi} (\sin t\cos\tau - \cos t\sin\tau)\sin(2\tau)d\tau$$
$$= 10\left(\sin t\int_0^{\pi} 2\sin\tau\cos^2\tau d\tau - \cos t\int_0^{\pi} 2\sin^2\tau\cos\tau d\tau\right)$$
$$= 10\left[-\sin t\left(\frac{2}{3}\cos^3\tau\right)\Big|_0^{\pi} - \cos t\left(\frac{2}{3}\sin^3\tau\right)\Big|_0^{\pi}\right]$$
$$= 10\left(\frac{4}{3}\sin t\right) = \frac{40}{3}\sin t$$

Example

Solution:

Or,

$$I = \frac{20}{3}\sin t - \frac{10}{3}\sin(2t) + \left[\frac{20}{3}\sin t + \frac{10}{3}\sin(2t)\right] u(t-\pi)$$

$$y = I + \sin t.$$

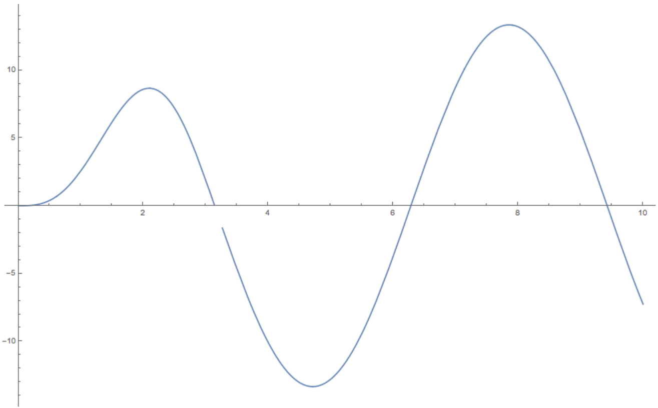

Figure: $I(t)$.

Differentiation of Transforms

Theorem

Suppose $f(t)$ satisfies the existence theorem in Sec. 6.1, and its Laplace transform is denoted as $F(s)$, then

$$\mathcal{L}\left[tf(t)\right] = -F'(s).$$

Proof.

By definition, $F(s) = \int_0^\infty e^{-st} f(t)dt$ and the derivative of it with respect to s is

$$F'(s) = \frac{d}{ds}\int_0^\infty e^{-st} f(t)dt = \int_0^\infty -te^{-st} f(t)dt = -\mathcal{L}\left[tf(t)\right].$$

□

Differentiation of Transforms

Example

Using differentiation of transform to calculate the Laplace transform of the following functions.

$$1. t\sin(\omega t). \qquad 2. t\cos(\omega t). \qquad 3. \sin(\omega t) + \omega t\cos(\omega t).$$

Example

Solution:

1.

$$\mathcal{L}\left[t\sin(\omega t)\right] = -\frac{d}{ds}\mathcal{L}\left[\sin(\omega t)\right] = -\left(\frac{\omega}{s^2+\omega^2}\right)' = \frac{2s\omega}{(s^2+\omega^2)^2}.$$

Differentiation of Transforms

Example

Solution:

2.

$$\mathcal{L}\left[t\cos(\omega t)\right] = -\left(\frac{s}{s^2+\omega^2}\right)' = -\frac{(s^2+\omega^2)-2s^2}{(s^2+\omega^2)^2} = \frac{(s^2-\omega^2)}{(s^2+\omega^2)^2}.$$

3.

$$\mathcal{L}\left[\sin(\omega t)+\omega t\cos(\omega t)\right] = \mathcal{L}\left[(t\sin\omega t)'\right]$$
$$= s\mathcal{L}\left[t\sin(\omega t)\right] - \left[t\sin(\omega t)\right]\big|_{t=0} = \frac{2s^2\omega}{(s^2+\omega^2)^2}.$$

Integration of Transforms

Theorem

Suppose $f(t)$ satisfies the existence theorem in Sec. 6.1, and its Laplace transform is denoted as $F(s)$, then if $\lim_{t\to 0^+}(f(t)/t)$ exists, then

$$\mathcal{L}\left[\frac{f(t)}{t}\right] = \int_s^\infty F(\tilde{s})d\tilde{s}.$$

Proof.

By definition, $F(s) = \int_0^\infty e^{-st} f(t)dt$. Therefore,

$$\int_s^\infty F(\tilde{s})d\tilde{s} = \int_s^\infty \int_0^\infty e^{-\tilde{s}t} f(t)dtd\tilde{s}.$$

Integration of Transforms

Proof.

$$\int_s^\infty F(\tilde{s})d\tilde{s} = \int_s^\infty \int_0^\infty e^{-\tilde{s}t} f(t)dtd\tilde{s}$$

$$\Rightarrow \int_s^\infty F(\tilde{s})d\tilde{s} = \int_0^\infty f(t)\left[\int_s^\infty e^{-\tilde{s}t}d\tilde{s}\right]dt$$

$$\Rightarrow \int_s^\infty F(\tilde{s})d\tilde{s} = \int_0^\infty f(t)\left[-\frac{1}{t}e^{-\tilde{s}t}\Big|_s^\infty\right]dt = \int_0^\infty e^{-st}\left[\frac{f(t)}{t}\right]dt$$

$$\Rightarrow \int_s^\infty F(\tilde{s})d\tilde{s} = \mathcal{L}\left[\frac{f(t)}{t}\right].$$

$\square$

Integration of Transforms

Example

Find the inverse transform of $F(s) = \ln\left(1 + \frac{\omega^2}{s^2}\right)$.

Example

Solution: We are using two different way to solve this problem.
1. We first calculate $F'(s)$

$$F'(s) = \frac{d}{ds}\left[\ln\left(s^2+\omega^2\right) - \ln(s^2)\right] = \frac{2s}{(s^2+\omega^2)} - \frac{2s}{s^2}$$
$$\Rightarrow \mathcal{L}^{-1}\left[F'(s)\right] = 2\cos\omega t - 2 = -tf(t)$$
$$\Rightarrow f(t) = \frac{2\,(1-\cos\omega t)}{t}.$$

Integration of Transforms

Example

Solution:

2. Alternatively, we can let

$$G(s) = \frac{2s}{(s^2+\omega^2)} - \frac{2}{s}, \qquad g(t) = \mathcal{L}^{-1}\left[G\right] = 2\left(\cos\omega t - 1\right).$$

$$\Rightarrow \mathcal{L}\left[\frac{g(t)}{t}\right] = \int_s^\infty G(\tilde{s})d\tilde{s} = -F(s)$$

$$\Rightarrow f(t) = \mathcal{L}^{-1}\left[F(s)\right] = -\frac{g(t)}{t} = \frac{2\left(1-\cos\omega t\right)}{t}.$$

Claim

$$\mathcal{L}\left[f(\alpha t)\right](s) = \frac{1}{\alpha}\mathcal{L}\left[f(t)\right]\left(\frac{s}{\alpha}\right), \qquad \alpha \textit{ constant.}$$

Proof.

By definition,

$$\begin{aligned}
\mathcal{L}\left[f(\alpha t)\right] &= \int_0^\infty e^{-st} f(\alpha t) dt \\
(\text{let } \tau = \alpha t) \qquad &= \int_0^\infty e^{-\frac{s}{\alpha}\tau} f(\tau)\frac{d\tau}{\alpha} = \frac{1}{\alpha}\mathcal{L}\left[f(t)\right]\left(\frac{s}{\alpha}\right).
\end{aligned}$$

$\square$

Example

We know $\mathcal{L}\left[\cos t\right] = \frac{s}{s^2+1}$, *we can use this trick to show* $\mathcal{L}\left[\cos \omega t\right] = \frac{s}{s^2+\omega^2}$.

Example

Using the equation $\mathcal{L}\left[f(\alpha t)\right](s) = \frac{1}{\alpha}\mathcal{L}\left[f(t)\right]\left(\frac{s}{\alpha}\right)$, *with* $f(t) = \cos t$, $\alpha = \omega$, *we can express*

$$\begin{aligned}\mathcal{L}\left[\cos \omega t\right] &= \frac{1}{\omega}\mathcal{L}\left[\cos t\right]\left(\frac{s}{\omega}\right) \\ &= \frac{1}{\omega}\frac{\left(\frac{s}{\omega}\right)}{\left(\frac{s}{\omega}\right)^2+1} = \frac{s}{s^2+\omega^2}.\end{aligned}$$

Power Series

The power series is a very useful tool for solving linear ODEs, with variable coefficients. In the last part of this module, we are going to discuss how to use the power series and introduce some standard special functions which play important roles in engineering, modeling.

Power Series

A real power series is an infinite series of the form

$$\sum_{m=0}^{\infty} a_m (x - x_0)^m = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots .$$

Here, x is our variable, and a_0, a_1, a_2, $\cdots$ are constants coefficients of the series. x_0 is a constant, called **center** or **reference point** of the series. In particular, if we choose $x_0 = 0$, we obtain a **power series in powers of** x

$$\sum_{m=0}^{\infty} a_m x^m = a_0 + a_1 x + a_2 x^2 + \cdots .$$

Analytic Function

Definition

An function $f(x)$ is real analytic on an open set D if for any x_0 in D one can write

$$f(x) = \sum_{m=0}^{\infty} a_m (x - x_0)^m = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots$$

in which $a_0, a_1, a_2, \cdots$ are real constants coefficients and the series is convergent to $f(x)$ for x in a neighborhood of x_0. Or, we can consider $x_0 = x_0 + 0 \times i$ is the **center**, *and $f(x)$ is convergent with radius of convergence $R > 0$. I.e. $D \equiv (x_0 - R, x_0 + R)$.*

Analytic Function

Remark

Similar to the analytic function in complex analysis ($\mathbb{R} \subset \mathbb{C}$), we can find the radius of convergence R by the ratio or root test

$$(a) \quad R = \frac{1}{\lim\limits_{m\to\infty} \left| \frac{a_{m+1}}{a+m} \right|}, \qquad (b) \quad R = \frac{1}{\lim\limits_{m\to\infty} \sqrt[m]{|a_m|}}.$$

Analytic Function

Example

Find the Taylor's expansion and the radius of convergence of the following function with the center at $x = 0$.

$$(a) \quad \frac{1}{1-x}, \qquad (b) \quad e^x.$$

Analytic Function

Example

Solutions:

$$(a)\quad \frac{1}{1-x} = \sum_{m=0}^{\infty} x^m, \quad \left|\frac{a_{m+1}}{a_m}\right| = \frac{1}{1} = 1, \quad R = 1.$$

$$(b)\quad e^x = \sum_{m=0}^{\infty} \frac{x^m}{m!}, \quad \left|\frac{a_{m+1}}{a_m}\right| = \frac{1}{(m+1)} \to 0, R = \infty.$$

Operations on Power Series

Remark

Consider two analytic functions $f(x) = \sum_{m=0}^{\infty} a_m(x - x_0)^m$ *and* $g(x) = \sum_{m=0}^{\infty} b_m(x - x_0)^m$ *are convergent in open set* D*, then the following operation holds in* D*.*

1. **Termwise differentiation**

$$f'(x) = \sum_{m=0}^{\infty} m a_m (x - x_0)^{m-1}$$

$$f''(x) = \sum_{m=0}^{\infty} m(m-1) a_m (x - x_0)^{m-2}$$

$$\vdots$$

Operations on Power Series

Remark

2. **Termwise addition**

$$f(x) + g(x) = \sum_{m=0}^{\infty} a_m (x - x_0)^m + \sum_{m=0}^{\infty} b_m (x - x_0)^m$$
$$= \sum_{m=0}^{\infty} (a_m + b_m)(x - x_0)^m.$$

Operations on Power Series

Remark

3. **Termwise multiplication**

$$
\begin{aligned}
f(x)g(x) &= \left[\sum_{m=0}^{\infty} a_m (x-x_0)^m\right]\left[\sum_{m=0}^{\infty} b_m (x-x_0)^m\right] \\
&= a_0 b_0 + (a_0 b_1 + a_1 b_0)(x-x_0) + \cdots \\
&= \sum_{m=0}^{\infty} (a_0 b_m + a_1 b_{m-1} \cdots a_m b_0)(x-x_0)^m.
\end{aligned}
$$

Operations on Power Series

Remark

4. **Identity theorem for power series** *The power series representation for function $f(x)$ is unique within D. More precisely, if*

$$f(x) = \sum_{m=0}^{\infty} a_m (x - x_0)^m = \sum_{m=0}^{\infty} b_m (x - x_0)^m,$$

we must have $a_0 = b_0$, $a_1 = b_1$, and so on. In particular, if $f(x) = 0$, this theorem guarantees that all the coefficients $a_m = 0$.

Our goal is to use **Power Series** to solve ODEs. The following example can give you some idea of it.

Example

Find power series solution to $y' - y = 0$.

Example

We first let

$$y = a_0 + a_1 x + a_2 x^2 + \cdots = \sum_{m=0}^{\infty} a_m x^m,$$

$$\text{and} \quad y' = a_1 + 2a_2 x + 3a_3 x^2 \cdots = \sum_{m=0}^{\infty} m a_m x^{m-1}.$$

Example

Substituting it into the ODE, we have

$$\begin{aligned}
&(a_1 + 2a_2x + 3a_3x^2 \cdots) - (a_0 + a_1x + a_2x^2 + \cdots) = 0, \\
\Rightarrow\ &(a_1 - a_0) + (2a_2 - a_1)x + (3a_3 - a_2)x^2 \cdots = 0, \\
\Rightarrow\ &a_1 = a_0, \quad a_2 = \frac{a_1}{2} = \frac{a_0}{2}, \quad a_3 = \frac{a_2}{3} = \frac{a_0}{3!}, \cdots .
\end{aligned}$$

Therefore,

$$y = a_0\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\right) = a_0e^x.$$

Theorem

Consider the following ODE

$$y'' + p(x)y' + q(x)y = r(x),$$

if $p(x)$, $q(x)$, $r(x)$ *are analytic at* $x = x_0$, *then every solution of it is analytic at* $x = x_0$ *and can be represented by a power series in powers of* $x - x_0$ *with radius of convergence* $R > 0$.

Following the theorem,we can use power series to solve second order nonhomogeneous ODE.

Example

Solve the following ODE.

$$y'' + p(x)y' + q(x)y = r(x), \quad y(x_0) = k_0,\ y'(x_0) = k_1.$$

Example

Since p, q, r are analytic at $x = x_0$, we can express p, q, r, and y by the power series.

$$y(x) = \sum_{m=0}^{\infty} a_m (x - x_0)^m, \quad p(x) = \sum_{m=0}^{\infty} p_m (x - x_0)^m,$$

$$q(x) = \sum_{m=0}^{\infty} q_m (x - x_0)^m, \quad r(x) = \sum_{m=0}^{\infty} r_m (x - x_0)^m,$$

Example

where a_m are unknown and p_m, q_m, r_m can be found by

$$p_m = \frac{p^{(m)}(x_0)}{m!} \cdots .$$

We can also find y', y'' by the operation of power series

$$y' = \sum_{m=0}^{\infty} m a_m (x - x_0)^{m-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} (x - x_0)^n,$$

$$y'' = \sum_{m=0}^{\infty} m(m-1) a_m (x - x_0)^{m-2}$$

$$= \sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} (x - x_0)^n.$$

Example

With the operation of power series we know

$$q(x)y(x) = \left[\sum_{m=0}^{\infty} q_m(x-x_0)^m\right]\left[\sum_{l=0}^{\infty} a_l(x-x_0)^l\right]$$
$$= \sum_{m=0}^{\infty}\sum_{l=0}^{\infty} q_m a_l (x-x_0)^{m+l} = \sum_{n=0}^{\infty} c_n(x-x_0)^n,$$

where we set $n = m + l$ and

$$c_n = \sum_{m=0}^{n} q_m a_{n-m}.$$

Example

Similarly, we have

$$
\begin{aligned}
p(x)y'(x) &= \left[\sum_{m=0}^{\infty} p_m(x-x_0)^m\right]\left[\sum_{l=0}^{\infty}(l+1)a_{l+1}(x-x_0)^l\right] \\
&= \sum_{m=0}^{\infty}\sum_{l=0}^{\infty}(l+1)p_m a_{l+1}(x-x_0)^{m+l} = \sum_{n=0}^{\infty} d_n(x-x_0)^n,
\end{aligned}
$$

where we set $n = m + l$ *and*

$$d_n = \sum_{m=0}^{n}(n-m+1)p_m a_{n-m+1}.$$

Example

Therefore, the ODE becomes

$$\sum_{n=0}^{\infty}\left[(n+1)(n+2)a_{n+2}+\sum_{m=0}^{n}(n-m+1)p_m a_{n-m+1} + \sum_{m=0}^{n} q_m a_{n-m} - r_n\right](x-x_0)^n = 0.$$

The coefficients in the bracket is called **recurrence relation**, *which tells us how to find* a_{n+2} *with lower order* a_k. *From the initial conditions, we know*

$$y(x_0) = k_0 = a_0, \qquad y'(x_0) = k_1 = a_1,$$

we can thus solve this IVP.

Example

For $n = 0$, *we have*

$$2a_2 + p_0 a_1 + q_0 a_0 = r_0$$
$$\Rightarrow a_2 = \frac{r_0 - p_0 a_1 - q_0 a_0}{2}.$$

Similarly, we can choose $n = 1, 2, 3 \cdots$ *to find* a_3, a_4, a_5, $\cdots$.

Bibliography

1 *Kreyszig, E.* **Advanced Engineering Mathematics**. Wiley, 10th Edition.