Xi'an Jiaotong-Liverpool University
西交利物浦大学

EEE220 Instrumentation and Control System

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Lecture 19

Outline

The Root Locus Method

- ❑ The Root Locus Concept
- ❑ The Root Locus Procedure
- ❑ **The Root Locus Using Matlab**
- ❑ **Parameter Design by the Root Locus Method**
- ❑ PID Controllers
  - Concept
  - PID Tuning
- ❑ Design Examples

A Complete Example 19.1

Obtain the root locus for the following characteristic equation of a system as K varies for $0 \leq K < \infty$.

$$1 + \frac{K}{s^4 + 12s^3 + 64s^2 + 128s} = 0.$$

Step 1. prepare the sketch.

$$1 + \frac{K}{s(s+4)(s+4+j4)(s+4-j4)} = 0$$

- The system has no finite zeros. There are four poles.
- Because $n = 4, M = 0$, we have four separate loci.
- The root loci are symmetrical with respect to the real axis.

Step 2. determine the segments on the real axis that are root loci.

- A segment of the root locus exists on the real axis between $s = 0$ and $s = -4$.

Step 3. determine the asymptotes.

Angles:

$$\phi_A = \frac{(2k+1)}{4}180°, \qquad k = 0, 1, 2, 3;$$

$$\phi_A = +45°, 135°, 225°, 315°.$$

Centroid:

$$\sigma_A = \frac{-4-4-4j-4+4j}{4} = -3$$

Step 4. determine the point where the locus crosses the imaginary axis (if any).

The characteristic equation is rewritten as

$$s(s + 4)(s^2 + 8s + 32) + K = s^4 + 12s^3 + 64s^2 + 128s + K = 0.$$

Therefore, the Routh array is

$$\begin{array}{c|ccc} s^4 & 1 & 64 & K \\ s^3 & 12 & 128 & \\ s^2 & b_1 & K & \\ s^1 & c_1 & & \\ s^0 & K & & \end{array},$$

where $$b_1 = \frac{12(64) - 128}{12} = 53.33 \quad \text{and} \quad c_1 = \frac{53.33(128) - 12K}{53.33}.$$

The gain K for marginally stability is $K = 568.89$, and the roots for the auxiliary equation are

$$53.33s^2 + 568.89 = 53.33(s^2 + 10.67) = 53.33(s + j3.266)(s - j3.266).$$

Therefore, the root locus crosses the $j\omega$-axis at $s = \pm j3.266$ when $K = 568.89$.

Step 5. determine the breakaway point (if any).

The breakaway point is estimated by evaluating

$$K = p(s) = -s(s + 4)(s + 4 + j4)(s + 4 - j4)$$

between $s = -4$ and $s = 0$.

We set $\frac{dK}{ds} = \frac{dp(s)}{ds} = 0$, and find $s = -1.577$.

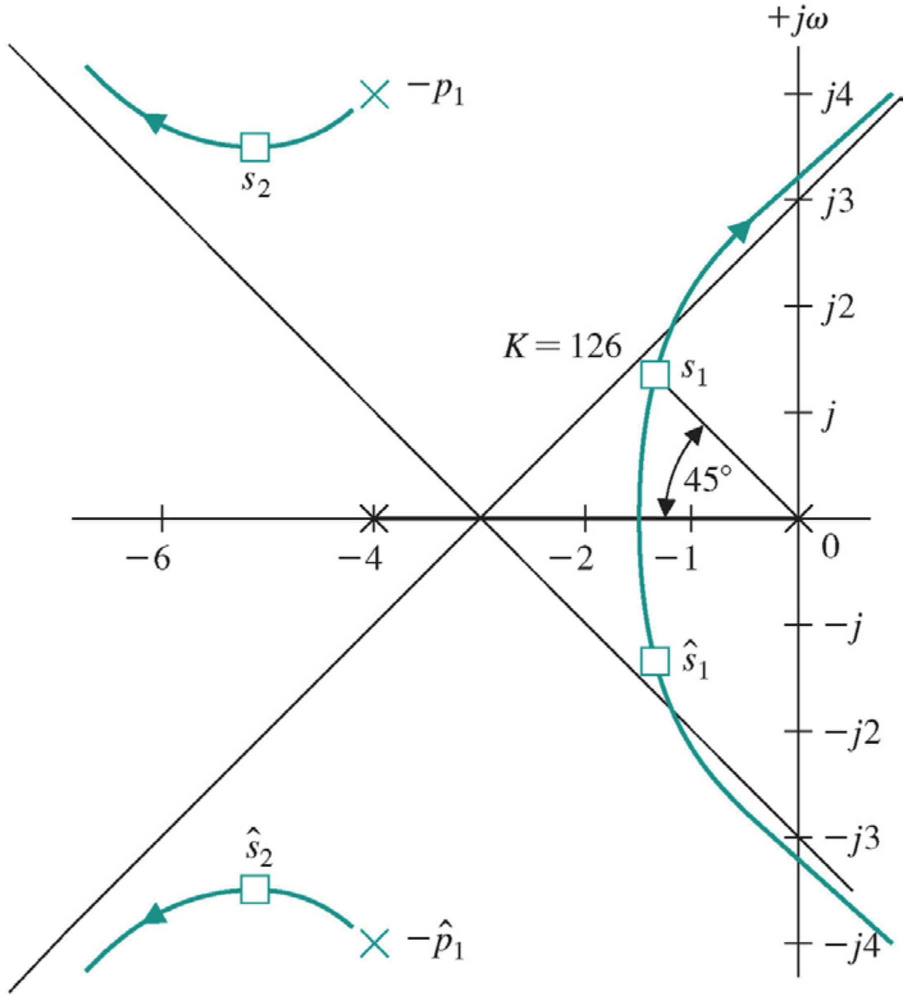
Step 6. determine the departure angles.

For angle of departure at complex pole $-p_1$, utilize angle criterion as follows

$$\theta_1 + 90^\circ + 90^\circ + \theta_3 = 180^\circ + k360^\circ.$$

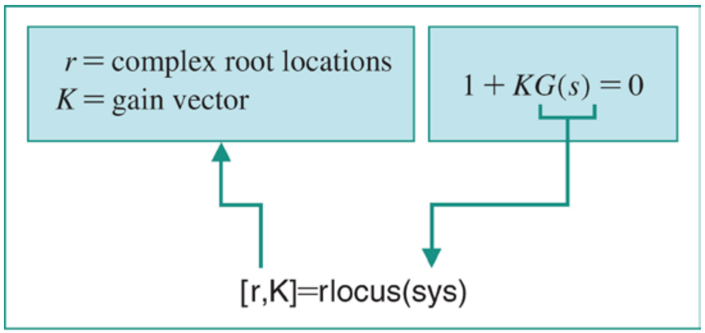
Since $\theta_3 = 135^o$ → $\theta_1 = -135^o \equiv 225^o$

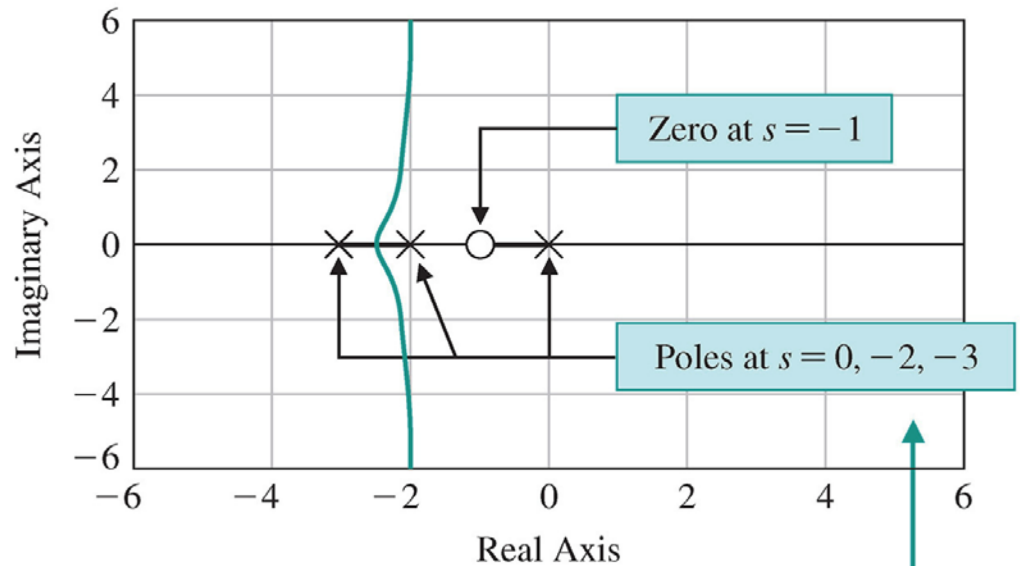
Step 7. complete the sketch.



Root Locus Using Matlab

The **rlocus** function.





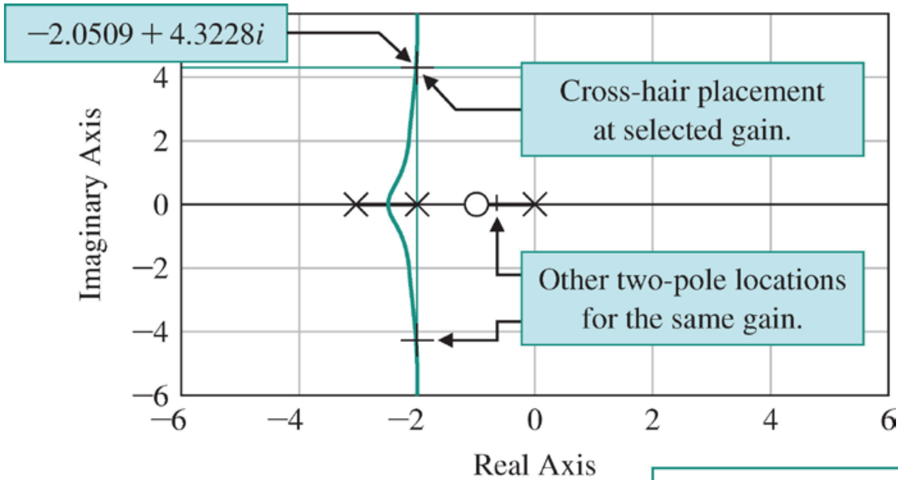

```
>>p=[1 1]; q=[1 5 6 0]; sys=tf(p,q); rlocus(sys)
```

Generating a root locus plot.

```
>>p=[1 1]; q=[1 5 6 0]; sys=tf(p,q); [r,K]=rlocus(sys);
```

Obtaining root locations r associated with various values of the gain K.

Using the **rlocfind** function.




```
>>p=[1 1]; q=[1 5 6 0]; sys=tf(p,q); rlocus(sys)
>>rlocfind(sys)

   Select a point in the graphics window

    selected_point =
      −2.0509 + 4.3228i
    ans =
       20.5775
```

rlocfind follows the rlocus function.

Value of K at selected point

Parameter Design by the Root Locus Method

- Originally, the root locus method was developed to determine the locus of roots of the characteristic equation as the system gain K is varied from zero to infinity; while the effect of other system parameters can be readily investigated by rearranging the characteristic equation;

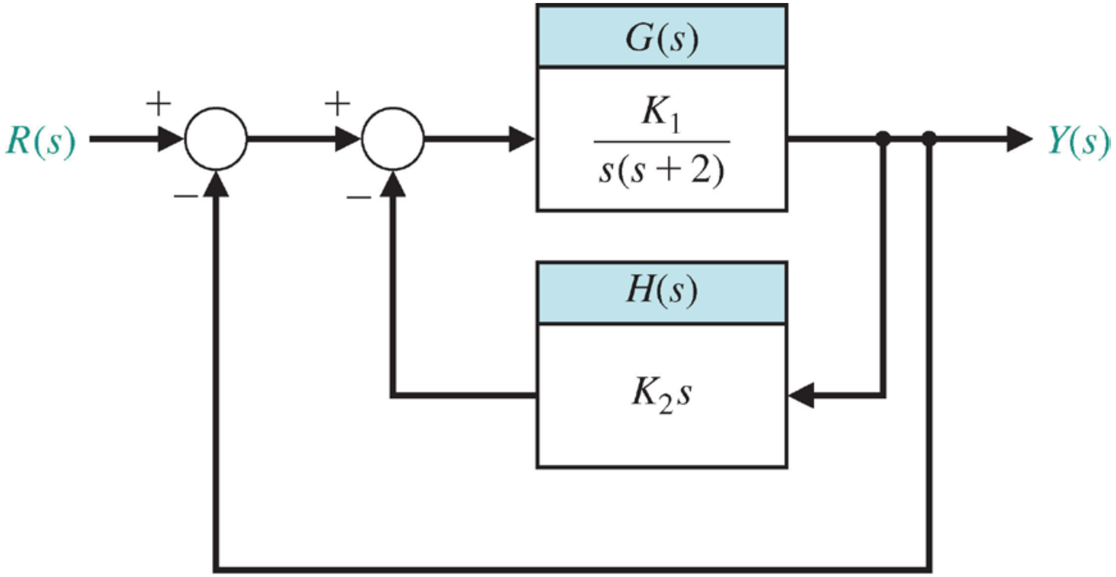
- In appears that the root locus method is a single-parameters. The question arises: How do we investigate the effect of two or more parameters?

--- Fortunately, this method can be extended to more than one parameters, based on which **parameter design** for the system is enabled.

Example 19.2: Welding Head Control

A welding head for an auto body requires an accurate control system for positioning the welding head. The feedback control system is to be designed (i.e., values of K_1 and K_2 are to be determined) to satisfy the following specifications:

1. Steady-state error for a ramp input is $e_{ss} \leq 35\%$ of the input slope
2. Damping ratio of dominant roots is $\zeta \geq 0.707$
3. Settling time to within 2% of the final value is $T_s \leq 3s$



Solutions:

Step 1. determine root locations in the s-plane to satisfy the design specifications.

- For steady-state error requirement:

$$E(s) = R(s) - Y(s) = \frac{s^2 + (K_1K_2 + 2)s}{s^2 + (K_1K_2 + 2)s + K_1} R(s)$$

Ramp input -> $R(s) = \frac{A}{s^2}$

$$e_{ss} = \lim_{s \to 0} sE(s) = \lim_{s \to 0} s \frac{s^2 + (K_1K_2 + 2)s}{s^2 + (K_1K_2 + 2)s + K_1} \frac{A}{s^2} = \frac{K_1K_2 + 2}{K_1} A \leq 0.35A$$

$$K_2 + \frac{2}{K_1} \leq 0.35$$ --> we need small value of K_2.

- For damping ratio requirement:

$$\zeta \geq 0.707$$

as $\theta = \cos^{-1}\zeta$

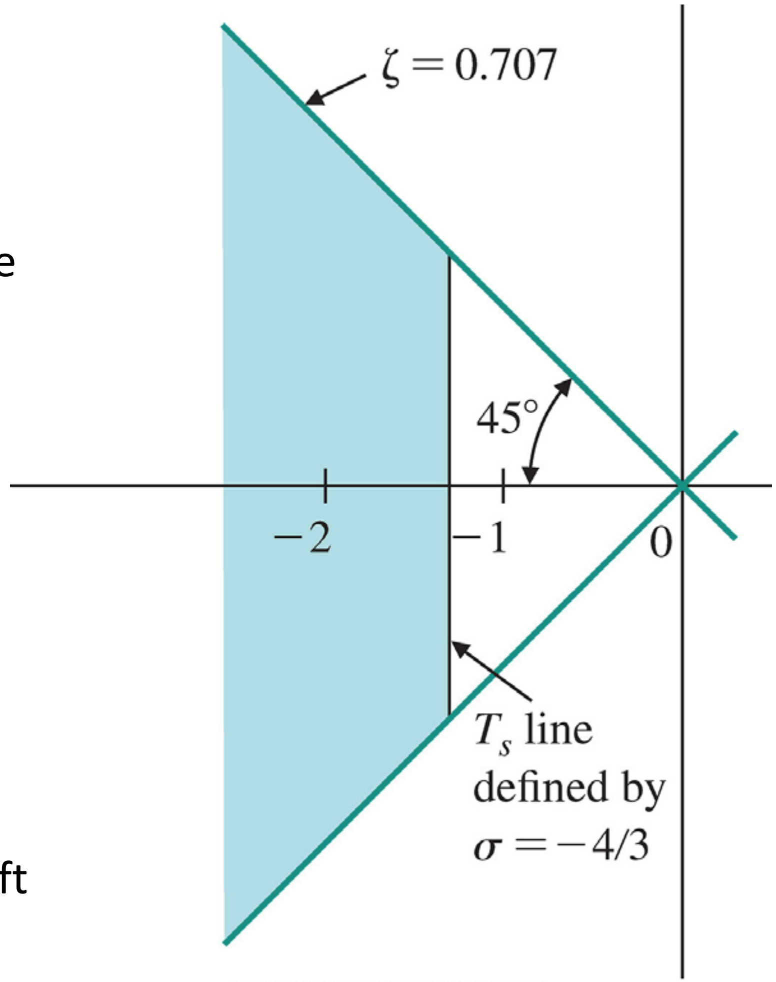
--> The roots of the closed-loop system must be below the line at 45° in the left-hand s-plane.

- For settling time requirement:

$$T_s = \frac{4}{\zeta\omega_n} = \frac{4}{-\sigma} \leq 3$$

$$-\sigma \geq \frac{4}{3}$$

--> We want the dominant roots to lie to the left of the line defined by $\sigma = -\frac{4}{3}$.



Step 2. Look into the root locus with one varying parameter, while setting the other parameter to be zero.

Characteristic equation for the closed-loop system:

$$\Delta(s) = s^2 + (K_1 K_2 + 2)s + K_1$$

Assume $K_1 = \alpha$, $K_1 K_2 = \beta$, then

$$\Delta(s) = s^2 + \beta s + 2s + \alpha$$

Set $\beta = 0$, sketch the root locus with varying α from zero to infinity

$$1 + \alpha \frac{1}{s(s+2)} = 0$$

Step 3. Select a fixed value of α, investigate the effect of another parameter by sketching the corresponding root locus.

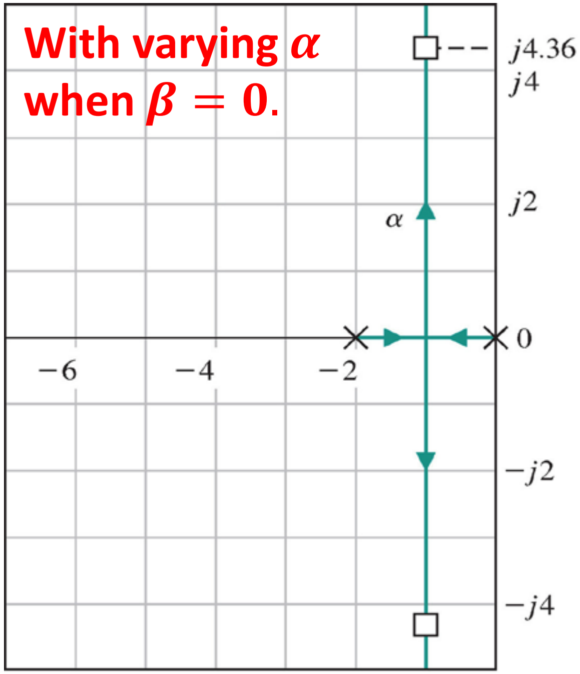
For example, choose a gain of $K_1 = \alpha = 20$, the roots are

$$s = -1 \pm j4.36$$

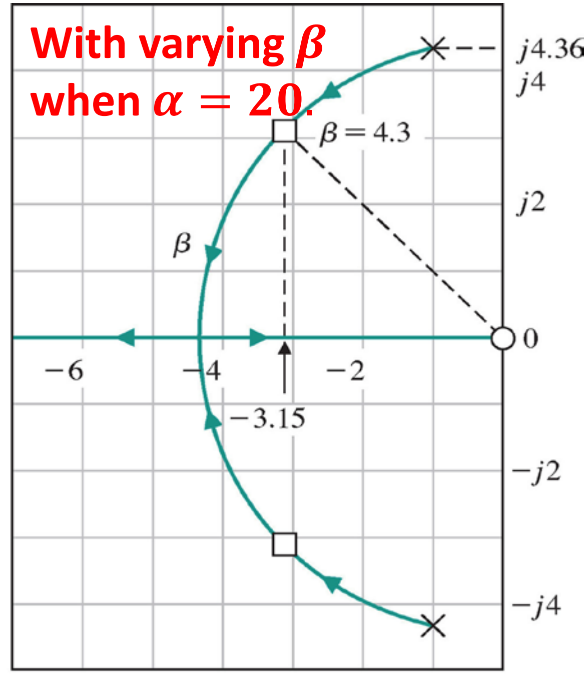
Then the effect of varying $\beta = 20K_2$ is determined from the locus equation

$$1 + \beta \frac{s}{s^2 + 2s + 20} = 0$$

The root locus for $\alpha = 20$ and with varying β can be then obtained.



(a)



(b)

Step 4. determine the parameter values.

The root with $\zeta = 0.707$ are obtained when $\beta = 4.3$, the real part of these roots is $\sigma = -3.15$, then $T_s = 1.27s$. Therefore, when $K_1 = 20$, $K_2 = 0.215$, the design specifications can be met.

The root locus method can be extended to more than two parameters by extending the number of steps in the method.

Root Contours

Actually, a family of root loci can be generated for two parameters in order to determine the total effect of varying two parameters. For example, let us determine the effect of varying α and β of the following characteristic equation:

$$s^3 + 3s^2 + 2s + \beta s + \alpha = 0$$

The root locus equation as a function of α is (set $\beta = 0$)

$$1 + \frac{\alpha}{s(s+1)(s+2)} = 0 \qquad (1)$$

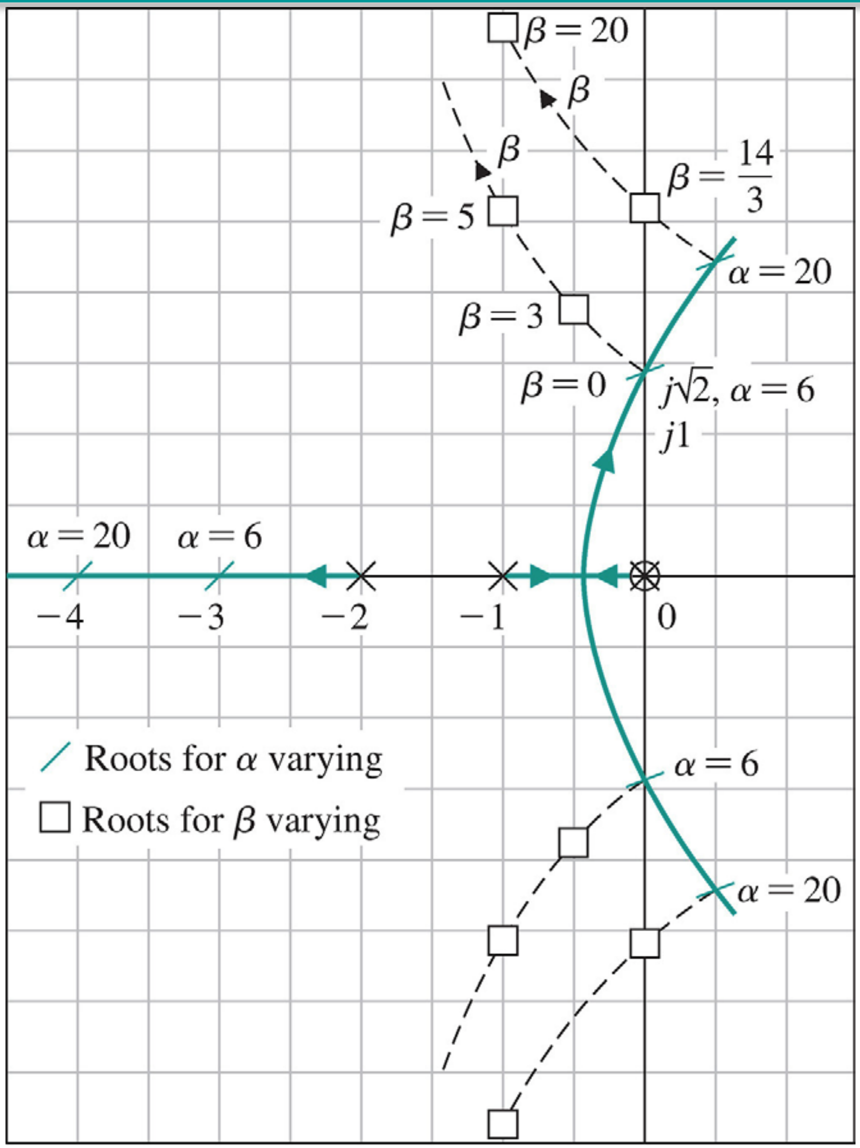
The root locus as a function of β is

$$1 + \frac{\beta s}{s^3 + 3s^2 + 2s + \alpha} = 0 \qquad (2)$$

Note: the roots of eq.(1) become poles of eq.(2).

A family of loci, often called root contours can be sketched, which illustrates the effect of varying both α and β on the roots of the system's characteristic equation.

Two-parameter root locus. The loci for α varying are solid; the loci for β varying are dashed.



Quiz 19.1

Two unity feedback control systems have the loop transfer functions:

(a) $$L(s) = G_c(s)G(s) = \frac{K}{s(s+2)(s^2+4s+5)}.$$

(b) $$L(s) = G_c(s)G(s) = \frac{K(s^2+4s+8)}{s^2(s+4)}.$$

Thank You !