MTH101: Lecture 15-16

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Second-Order Linear ODEs

You have already learned how to solve second-order linear **ODEs** (Ordinary Differential Equations) with constant coefficients,

$$y'' + ay' + by = r(t) \; ,$$

and corresponding **IVPs** (Initial Value Problems) like

$$y'' + ay' + by = r(t) \; , \qquad y(t_0) = K_0 \; , \quad y'(t_0) = K_1 \; .$$

Please brush up your knowledge, read the sections:

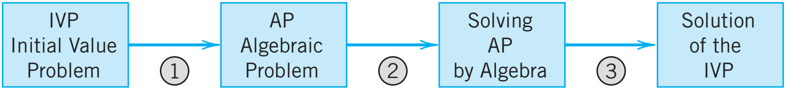
1.5 Linear ODEs of First Order

2.1 Homogeneous Linear ODEs of Second Order

2.2 Homogeneous Linear ODEs with Constant Coefficients

2.7 Nonhomogeneous ODEs

Strategy



1. Transform a given **IVP** into a **subsidiary equation** using Laplace transformation.
2. Solve the subsidiary equation algebraically.
3. Transform the algebraic solution back into a solution of the IVP using $\mathcal{L}^{-1}$.

Definition

For a function $f(t)$ defined for all $t \geq 0$, the **Laplace transform** $F = \mathcal{L}[f]$ *of $f(t)$ is defined by:*

$$F(s) = \mathcal{L}[f](s) = \int_0^\infty e^{-st} f(t)dt \ .$$

If $F(s)$ exists, then the original function $f(t)$ is **the** *(essentially unique)* **inverse transform** *of $F(s)$,*

$$f(t) = \mathcal{L}^{-1}[F](t) \quad \text{or short} \quad f = \mathcal{L}^{-1}[F] \ .$$

Example

Let $f(t) = 1$ *for* $t \geq 0$. *Find* $F(s)$.

Example

Solution

$$\mathcal{L}[f] = \mathcal{L}[1] = \int_0^{\infty} e^{-st} dt = -\frac{1}{s} e^{-st} \Big|_0^{\infty} = \frac{1}{s} \quad (\text{if } s > 0).$$

Example

Let $f(t) = e^{\alpha t}$ for $t \geq 0$, where α is a constant. Find $F(s)$.

Example

Solution

$$\mathcal{L}\left[f\right] = \mathcal{L}\left[e^{\alpha t}\right] = \int_0^{\infty} e^{-st} e^{\alpha t} dt = \frac{1}{\alpha - s} e^{-(s-\alpha)t}\Big|_0^{\infty},$$

when $s - \alpha > 0$,

$$\mathcal{L}\left[e^{\alpha t}\right] = \frac{1}{s-\alpha}.$$

Remark

$$f = \mathcal{L}^{-1}\big[\mathcal{L}[f]\big], \qquad F = \mathcal{L}\big[\mathcal{L}^{-1}[F]\big]$$

Remark

Linearity

$$\begin{aligned} \mathcal{L}\big[af + bg\big] &= a\mathcal{L}\big[f\big] + b\mathcal{L}\big[g\big], \\ or, \quad \mathcal{L}\big[af(t) + bg(t)\big](s) &= a\mathcal{L}\big[f(t)\big](s) + b\mathcal{L}\big[g(t)\big](s). \end{aligned}$$

Integration is linear operation!

Example

Let $f(t) = \cosh \alpha t$ *for* $t \geq 0$, *find* $F(s)$.

Example

Solution

$$\cosh \alpha t = \frac{e^{\alpha t} + e^{-\alpha t}}{2}$$

$$\mathcal{L}[f] = \mathcal{L}[\cosh \alpha t] = \frac{1}{2}\mathcal{L}[e^{\alpha t}] + \frac{1}{2}\mathcal{L}[e^{-\alpha t}]$$
$$= \frac{1}{2}\left(\frac{1}{s-\alpha} + \frac{1}{s+\alpha}\right) = \frac{s}{s^2-\alpha^2}.$$

Well-defined when $s - |\alpha| > 0$.

Example

Find $\mathcal{L}\left[\cos\omega t\right] = \frac{s}{s^2+\omega^2}$ *and* $\mathcal{L}\left[\sin\omega t\right] = \frac{\omega}{s^2+\omega^2}$.

Example

Solution

Let $\mathcal{L}_c \equiv \mathcal{L}\left[\cos\omega t\right]$, $\mathcal{L}_s \equiv \mathcal{L}\left[\sin\omega t\right]$.

$$\begin{aligned}
\mathcal{L}_c &= \int_0^\infty e^{-st}\cos\omega t dt \\
&= \left(-\frac{1}{s}e^{-st}\cos\omega t\right)\Big|_0^\infty - \frac{\omega}{s}\int_0^\infty e^{-st}\sin\omega t dt \\
&= \frac{1}{s} - \frac{\omega}{s}\mathcal{L}_s, \qquad (\text{when } s > 0).
\end{aligned}$$

Example

Find $\mathcal{L}\left[\cos\omega t\right] = \frac{s}{s^2+\omega^2}$ *and* $\mathcal{L}\left[\sin\omega t\right] = \frac{\omega}{s^2+\omega^2}$.

Example

Solution

Similarly,

$$\begin{aligned}\mathcal{L}_s &= \int_0^\infty e^{-st}\sin\omega t dt\\ &= \left(-\frac{1}{s}e^{-st}\sin\omega t\right)\Big|_0^\infty + \frac{\omega}{s}\int_0^\infty e^{-st}\cos\omega t dt\\ &= \frac{\omega}{s}\mathcal{L}_c, \qquad (\textit{when } s > 0).\end{aligned}$$

Example

Find $\mathcal{L}\left[\cos\omega t\right] = \frac{s}{s^2+\omega^2}$ *and* $\mathcal{L}\left[\sin\omega t\right] = \frac{\omega}{s^2+\omega^2}$.

Example

Solution

$$\begin{aligned} \mathcal{L}_c &= \frac{1}{s} - \frac{\omega}{s}\mathcal{L}_s, \qquad & \mathcal{L}_s &= \frac{\omega}{s}\mathcal{L}_c. \\ \Rightarrow \mathcal{L}_c &= \frac{s}{s^2+\omega^2}, \qquad & \mathcal{L}_s &= \frac{\omega}{s^2+\omega^2}. \end{aligned}$$

Example

Find $\mathcal{L}\left[t^{n+1}\right] = \frac{(n+1)!}{s^{n+2}}$, *for* $n = -1, 0, \cdots$.

Example

Solution

$$\begin{aligned}
\mathcal{L}\left[t^{n+1}\right] &= \int_0^\infty e^{-st} t^{n+1} dt \\
&= -\frac{1}{s} e^{-st} t^{n+1}\Big|_0^\infty + \frac{n+1}{s} \int_0^\infty e^{-st} t^n dt \\
&= \frac{n+1}{s} \mathcal{L}\left[t^n\right] = \frac{(n+1)!}{s^{n+1}} \mathcal{L}\left[1\right] = \frac{(n+1)!}{s^{n+2}}.
\end{aligned}$$

Definition

The **Gamma function** *is defined for* $\alpha > 0$ *by the integral*

$$\Gamma(\alpha) = \int_0^\infty e^{-x} x^{\alpha-1} dx \ .$$

Integration by parts yields

$$\Gamma(\alpha+1) = \alpha\Gamma(\alpha) \ .$$

Since $\Gamma(1) = 1$, *this implies for* $n \in \mathbb{N}$

$$\Gamma(n+1) = n! \ .$$

Example

Let $\alpha > 0$, then, using the substitution $st = x$,

$$\begin{aligned}\mathcal{L}\left[t^{\alpha}\right](s) &= \int_0^{\infty} e^{-st} t^{\alpha} dt = \int_0^{\infty} e^{-x} \left(\tfrac{x}{s}\right)^{\alpha} \tfrac{dx}{s} \\ &= \frac{1}{s^{\alpha+1}} \int_0^{\infty} e^{-x} x^{\alpha} dx = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}}.\end{aligned}$$

In particular,

$$\mathcal{L}\left[t^{n}\right](s) = \frac{n!}{s^{n+1}}.$$

Theorem

First shifting theorem, s-Shifting

$$\mathcal{L}\left[e^{\alpha t} f(t)\right](s) = F(s-\alpha)$$
$$or, \quad e^{\alpha t} f(t) = \mathcal{L}^{-1}\left[F(s-\alpha)\right].$$

Proof.

$$F(s-\alpha) = \int_0^\infty e^{-(s-\alpha)t} f(t) dt = \int_0^\infty e^{-st}\left[e^{\alpha t} f(t)\right] dt,$$
$$= \mathcal{L}\left[e^{\alpha t} f(t)\right].$$

$\square$

Example

$$\mathcal{L}\left[e^{\alpha t}\cos\omega t\right] = \mathcal{L}\left[\cos\omega t\right](s-\alpha) = \frac{(s-\alpha)}{(s-\alpha)^2+\omega^2},$$
$$\mathcal{L}\left[e^{\alpha t}\sin\omega t\right] = \mathcal{L}\left[\sin\omega t\right](s-\alpha) = \frac{\omega}{(s-\alpha)^2+\omega^2}.$$

	$f(t)$	$\mathcal{L}(f)$		$f(t)$	$\mathcal{L}(f)$
1	1	$1/s$	7	$\cos \omega t$	$\dfrac{s}{s^2 + \omega^2}$
2	t	$1/s^2$	8	$\sin \omega t$	$\dfrac{\omega}{s^2 + \omega^2}$
3	t^2	$2!/s^3$	9	$\cosh at$	$\dfrac{s}{s^2 - a^2}$
4	t^n $(n = 0, 1, \cdots)$	$\dfrac{n!}{s^{n+1}}$	10	$\sinh at$	$\dfrac{a}{s^2 - a^2}$
5	t^a (a positive)	$\dfrac{\Gamma(a+1)}{s^{a+1}}$	11	$e^{at} \cos \omega t$	$\dfrac{s-a}{(s-a)^2 + \omega^2}$
6	e^{at}	$\dfrac{1}{s-a}$	12	$e^{at} \sin \omega t$	$\dfrac{\omega}{(s-a)^2 + \omega^2}$

Completing the square

Example

Consider $F(s) = \frac{3s-137}{s^2+2s+401}$. *Find* $f(t) = \mathcal{L}^{-1}\big[F(s)\big]$.

Example

Solution

$$
\begin{aligned}
f(t) &= \mathcal{L}^{-1}\Big[\frac{3(s+1)-140}{(s+1)^2+400}\Big] \\
&= 3\mathcal{L}^{-1}\Big[\frac{s+1}{(s+1)^2+20^2}\Big] - 7\mathcal{L}^{-1}\Big[\frac{20}{(s+1)^2+20^2}\Big] \\
&= 3e^{-t}\cos 20t - 7e^{-t}\sin 20t.
\end{aligned}
$$

Bibliography

1 *Kreyszig, E.* **Advanced Engineering Mathematics**. Wiley, 10th Edition.