MTH101: Lecture 13

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Residues integration method

Let $z_0 \in \mathbb{C}$ be an **Isolated Singularity** of $f(z)$ and consider the Laurent Series of $f(z)$:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n},$$

in the **Annulus** in which converges:

$$\{z \in \mathbb{C} : 0 < |z - z_0| < R\}.$$

Let γ be a simple, closed, counterclockwise oriented path which encloses z_0, then:

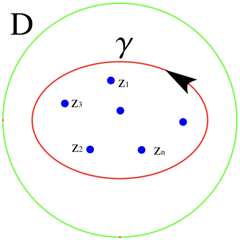
$$b_1 = \frac{1}{2\pi i} \oint_{\gamma} f(z)\, dz := \mathrm{Res}_{z_0}(f).$$

The coefficient b_1 is called the **Residue** of f at the point z_0.

Theorem

Let D be a simply connected Domain, and let $f(z)$ be analytic in D except for finitely many isolated singularities $z_1, z_2, ..., z_n$. Let γ be a simple closed path with counterclockwise orientation, contained in D which encloses all the isolated singularities $z_1, z_2, ..., z_n$. Then

$$\oint_\gamma f(z)\ dz = 2\pi i \sum_{k=1}^{n} Res_{z_k}(f).$$



Remark

In general it is very difficult to compute the Residue of a function at a point z_0 by writing the Laurent Series of $f(z)$ with center z_0. We will use several formulas.

Proposition

If z_0 is a **Simple Pole** *(Pole of order 1) for the function $f(z)$ then*

$$Res_{z_0} f(z) = \lim_{z \to z_0} (z - z_0) f(z).$$

Proposition

If z_0 is a **Pole of Order m** *for the function $f(z)$ then*

$$Res_{z_0} f(z) = \frac{1}{(m-1)!} \lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}} [(z - z_0)^m f(z)].$$

Example

Compute the Residues at the singular points of the function

$$f(z) = \frac{7z+1}{(z-2)(z+1)^3}.$$

Solution

The function $f(z)$ has two isolated singularities: $z_1 = -1$, $z_2 = 2$.
Step 1: $z_1 = -1$. The function $f(z)$ can be written as:

$$f(z) = \frac{h(z)}{g(z)}, \quad \text{with } h(z) = \frac{7z+1}{(z-2)}, \ g(z) = (z+1)^3.$$

The functions $h(z)$ and $g(z)$ are Analytic at z_1. Moreover $h(z_1) \neq 0$ while the function $g(z) = (z+1)^3$ has a zero of order 3 at z_1, then, from the previous theorem,the function $f(z)$ has a **Pole of Order 3** at z_1. We can use the formula, with $m = 3$, to compute the residue:

$$\begin{aligned}
\text{Res}_{z_1} f(z) &= \frac{1}{(3-1)!} \lim_{z \to z_1} \frac{d^{3-1}}{dz^{3-1}} [(z - z_1)^3 f(z)] \\
&= \frac{1}{2} \lim_{z \to -1} \frac{d^2}{dz^2} \left[\frac{7z+1}{z-2} \right] = \frac{1}{2} \lim_{z \to -1} \frac{30}{(z-2)^3} = -\frac{5}{9}.
\end{aligned}$$

Step 2: $z_2 = 2$. The function $f(z)$ can be written as:

$$f(z) = \frac{h(z)}{g(z)}, \quad \text{with } h(z) = \frac{7z+1}{(z+1)^3}, \; g(z) = z - 2.$$

The functions $h(z)$ and $g(z)$ are Analytic at $z_2 = 2$. Moreover, $h(z_2) = \frac{15}{27} \neq 0$ while the function $g(z) = (z-2)$ has a zero of 1 order 1 at $z_2 = 2$, then, from the previous theorem, the function $f(z)$ has a **Pole of Order 1** at $z_2 = 2$. We can use the formula, with $m = 1$, to compute the residue:

$$\begin{aligned} \operatorname{Res}_{z_2} f(z) &= \lim_{z \to z_2} [(z - z_2) f(z)] \\ &= \lim_{z \to 2} \left[\frac{7z+1}{(z+1)^3} \right] = \frac{15}{27}. \end{aligned}$$

Proposition

If $f(z)$ can be written as the quotient of two functions $p(z)$ and $q(z)$ which are analytic at z_0:

$$f(z) = \frac{p(z)}{q(z)},$$

such that $p(z_0) \neq 0$ and $q(z)$ has a simple zero at z_0, then

$$Res_{z_0} f(z) = \frac{p(z_0)}{q'(z_0)}.$$

Example

Compute the following integral

$$\oint_{\gamma} \frac{5-2z}{z(z+3)} dz,$$

where γ is the circle with center $z_0 = -1$ and radius $R = 3$ with counterclockwise orientation.

Solution

The function $f(z)$ has two **Isolated Singularities** at $z_1 = 0$ and $z_2 = -3$, they are both inside γ then we can use the Residue Theorem:

$$\oint_\gamma f(z)dz = 2\pi i[\mathrm{Res}_{z_1}(f) + \mathrm{Res}_{z_2}(f)].$$

We start by computing the Residue at $z_1 = 0$.
We observe that $f(z)$ can be written in the following form

$$f(z) = \frac{p(z)}{q(z)}, \quad \text{with } p(z) = \frac{5-2z}{z+3}, \ q(z) = z.$$

Both p(z) and q(z) are Analytic at z_1. Moreover, $p(0) \neq 0$ while $z_1 = 0$ is a **Zero of Order 1** of $q(z)$. Then we can use the formula:

$$\mathrm{Res}_{z_1}(f) = \frac{p(z_1)}{q'(z_1)} = \frac{p(0)}{q'(0)} = \frac{5}{3}.$$

We pass to compute the Residue at $z_2 = -3$.
We observe that $f(z)$ can be written in the following form

$$f(z) = \frac{p(z)}{q(z)}, \quad \text{with } p(z) = \frac{5-2z}{z}, \; q(z) = z+3.$$

Both $p(z)$ and $q(z)$ are Analytic at z_2. Moreover, $p(z_2) \neq 0$ while z_2 is a **Zero of Order 1** of $q(z)$. Then we can use the formula:

$$\mathrm{Res}_{z_2}(f) = \frac{p(z_2)}{q'(z_2)} = \frac{p(-3)}{q'(-3)} = -\frac{11}{3}.$$

Finally

$$\oint_\gamma f(z)dz = 2\pi i[\mathrm{Res}_{z_1}(f) + \mathrm{Res}_{z_2}(f)] = 2\pi i\left(\frac{5}{3} - \frac{11}{3}\right) = -\frac{12}{3}\pi i = -4\pi i$$

Remark

- *The Propositions provide us shortcut of computing residues for pole type singularities.*
- *For essential singularity, we will still use the Laurent series expanding functions at z_0.*
- *What about residues for removable singularities?*

Bibliography

1 *Kreyszig, E.* **Advanced Engineering Mathematics**. Wiley, 9th Edition.