MTH101: Lecture 3

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Circles and Disks

Definition

The set

$$\{z \in \mathbb{C} : |z - z_0| = R\},$$

is the set of the points z whose distance from the point z_0 is R, that is the **Circle** with center z_0 and radius R. The set

$$\{z \in \mathbb{C} : |z - z_0| < R\},$$

is the interior of the Circle, called **Open Disk,**
while the set

$$\{z \in \mathbb{C} : |z - z_0| \leq R\},$$

is called **Closed Disk**.
The Circle is also called **Boundary of the Disk**.

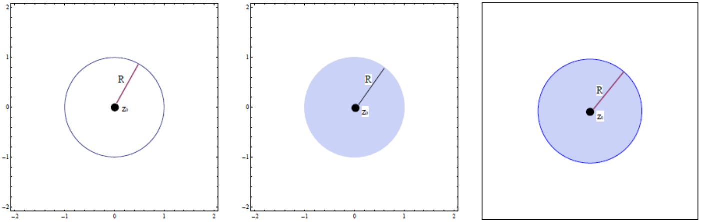


Figure: The circle (Boundary of the Disk), the Open Disk and the Closed Disk.

Definition

A **Neighborhood** of a point z_0 is an **Open Disk** with center z_0.

Definition

A set S is **Open** if for any z_0 in S there exists a **Neighborhood** of z_0 consisting entirely of points that belong to S.
Example: **Open Disk**.

Definition

A set S is **Connected** if any two points of S can be joined by a chain of finitely many line segments whose points are all in S.

Definition

A set D is a **Domain** if it is **Open** and **Connected**.

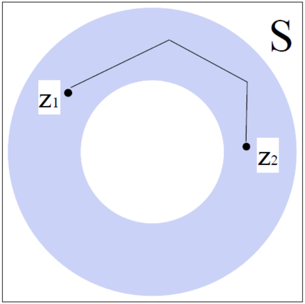


Figure: S is a **Connected Set**.
It is an Anulus: $S = \{z \in \mathbb{C} : \quad R_1 < |z - z_0| < R_2\}$.

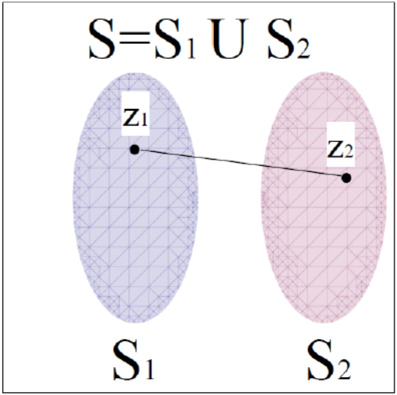


Figure: S is **not** a **Connected Set**.
It is the union of two open disks: $S = S_1 \cup S_2$.

Half-Planes

The (open) **upper half-plane** is the set of complex numbers with positive imaginary part:

$$\{x + iy \mid y > 0; x, y \in \mathbb{R}\}.$$

Similarly, we define

$$\begin{aligned} \textbf{lower half-plane}: &\quad \{x + iy \mid y < 0; x, y \in \mathbb{R}\} \\ \textbf{left half-plane}: &\quad \{x + iy \mid x < 0; x, y \in \mathbb{R}\} \\ \textbf{right half-plane}: &\quad \{x + iy \mid x > 0; x, y \in \mathbb{R}\} \end{aligned}$$

Definition

A function f, defined in a complex set S and taking values in $\mathbb{C}$ is called a **Complex Function** *(of one variable). We write:*

$$f : S \subseteq \mathbb{C} \to \mathbb{C}.$$

The **domain** *S is open and connect in most cases;*
Than **range** *$f(S)$ is the set of all possible output $\{f(z) \mid z \in S\}$*

A Complex Function $f(z)$ can be written as the sum of two real functions:

$$f(z) = u(x, y) + iv(x, y),$$

where

$$u(x, y), v(x, y) : \mathbb{R}^2 \to \mathbb{R}, \qquad z = x + iy,$$

and

$$u(x, y) = \text{Re } f(z), \qquad \textbf{Real part of } f(z),$$
$$v(x, y) = \text{Im } f(z), \qquad \textbf{Imaginary part of } f(z).$$

Example

Write $f(z) = z^2$ in the form $f = u + iv$

Solution

Using $z = x + iy$ we obtain:

$$f(z) = z^2 = (x + iy)^2 = x^2 - y^2 + 2xyi,$$

then

$$u(x, y) = x^2 - y^2,$$
$$v(x, y) = 2xy.$$

Exercise

Write in the form $f = u + iv$ the following Complex Functions:

$$f(z) = z^2 - 3z + 2, \qquad f(z) = |z|^2 + \bar{z} - 5z, \qquad f(z) = \frac{1}{\bar{z}}.$$

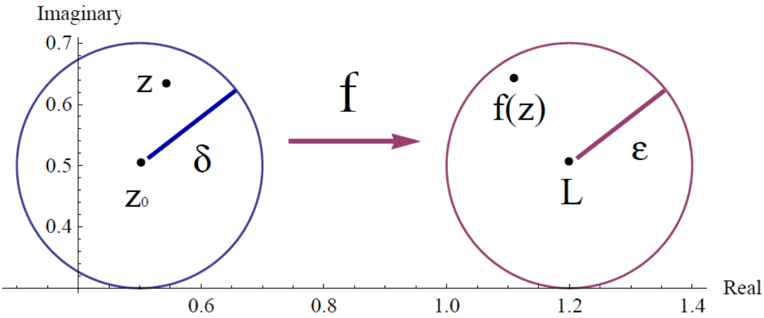
Definition

A complex function $f(z)$ is said to have the limit L as z approaches z_0, and we write

$$\lim_{z \to z_0} f(z) = L,$$

if for any $\epsilon > 0$ there exist a $\delta > 0$ such that

$$0 < |z - z_0| < \delta \quad \Rightarrow \quad |f(z) - L| < \epsilon.$$


Imaginary
0.7
0.6
0.5
0.4
z
z_0
δ
f
f(z)
L
ε
Real
0.6
0.8
1.0
1.2
1.4

Definition

The function $f(z)$ is Continuous at a point z_0 if

$$\lim_{z \to z_0} f(z) = f(z_0).$$

Definition

We say that the Complex Function $f(z)$ is **differentiable** at z_0 if the following limit exists:

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z},$$

$f'(z_0)$ is called the **derivative** of f at the point z_0

Proposition

(1) *If $c \in \mathbb{C}$ is a constant, then* $(cf)' = cf'$,

(2) $(f + g)' = f' + g'$,

(3) $(fg)' = f'g + fg'$,

(4) $(\frac{f}{g})' = \frac{f'g - fg'}{g^2}, \quad g \neq 0.$

Analytic Functions

Definition

The function $f(z)$ is **Analytic** at a **point** z_0 if it is **defined and differentiable** in a whole neighborhood of z_0.

Definition

The function $f(z)$ is **Analytic** in a **Domain** D if it is **defined and differentiable** at any points of the Domain D.

Definition

A function $f(z)$ is called **Entire** if it is **Analytic** on the whole Complex Plane $\mathbb{C}$.

Proposition

The Complex **Polynomial** *functions*

$$f(z) = c_n z^n + c_{n-1} z^{n-1} + ... c_1 z + c_0,$$

where n is non-negative integer and $c_k \in \mathbb{C}$ for $k = 0, 1, 2, ..., n$, are **Entire functions***.*

Example

Some examples of Complex Polynomial Functions:

$$f(z) = z + 1, f(z) = z^5 + i, f(z) = iz^{27} - (2 - i)z^{10} + 5z.$$

Proposition

The Complex **Rational** *functions, that is, the quotient of two Polynomial functions $P(z)$ and $Q(z)$:*

$$f(z) = \frac{P(z)}{Q(z)}$$

are **Analytic** *functions where they are defined.*
The function $f(z)$ is defined in the set

$$A = \{z \in \mathbb{C} : Q(z) \neq 0\},$$

and as a consequence $f(z)$ is **Analytic** *in A.*

Example

Consider the Complex **Rational** Functions:

$$f(z) = \frac{z^2 + 3z}{z - 1}$$

it is the quotient of the two Polynomials $P(z) = z^2 + 3z$ and $Q(z) = z - 1$.
The function $f(z)$ is defined in the set

$$\begin{aligned} A &= \{z \in \mathbb{C} : Q(z) \neq 0\} \\ &= \{z \in \mathbb{C} : z - 1 \neq 0\} \\ &= \{z \in \mathbb{C} : z \neq 1\}. \end{aligned}$$

Then $f(z)$ is Analytic in the set $A = \{z \in \mathbb{C} : z \neq 1\}$.

Bibliography

1 *Kreyszig, E.* **Advanced Engineering Mathematics**. Wiley, 9th Edition.