MTH101: Lecture 4

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Cauchy-Riemann Equations

Theorem

Let $D \subseteq \mathbb{C}$ be a domain and $f(z) = u(x, y) + iv(x, y)$ be a complex function defined on D.
Then the following statements are equivalent

- *f is* **Analytic** *on D.*
- *u and v have continuous first partial derivatives that satisfy the* **Cauchy-Riemann equations**

$$u_x = v_y, \qquad u_y = -v_x.$$

Moreover we have a formula for the Complex Derivative of f :

$$f'(z) = u_x + iv_x = v_y - iu_y.$$

Example

Consider the function

$$f(z) = \bar{z}$$

Check if it is **Analytic** in $\mathbb{C}$.

Solution:
We write the function in the form of $f = u + iv$:

$$f(z) = \bar{z} = x - iy,$$

then

$$u(x, y) = x, \quad v(x, y) = -y.$$

We compute the partial derivatives

$$\begin{aligned} u_x &= 1, \quad u_y = 0, \\ v_x &= 0, \quad v_y = -1. \end{aligned}$$

The partial u_x, u_y, v_x, v_y derivatives are continuous functions. It remains to check the Cauchy-Riemann equations:

$$u_x = v_y \quad \Rightarrow \qquad\qquad 1 = -1 \text{ (wrong!)},$$

Thus $f(z)$ is **NOT** Analytic on $\mathbb{C}$.

Exercise

Check if the following function is Analytic in $\mathbb{C}$:

$$f(z) = e^x(\sin \; y + i \cos \; y).$$

Cauchy-Riemann equations in polar form

Given $z = r(\cos\theta + i\sin\theta)$, and $f(z) = u(r,\theta) + iv(r,\theta)$, then the Cauchy-Riemann euqations are

$$u_r = \frac{1}{r} v_\theta$$
$$v_r = -\frac{1}{r} u_\theta$$

Harmonic Functions

Theorem

If the function $f(z) = u(x,y) + iv(x,y)$ is **Analytic** *in a* **Domain** *D then both $u(x,y)$ and $v(x,y)$ satisfy the* **Laplace Equation***:*

$$\nabla^2 u = u_{xx} + u_{yy} = 0,$$
$$\nabla^2 v = v_{xx} + v_{yy} = 0,$$

in the Domain D. Moreover, u and v have continuous second order partial derivatives in D.
(We say that u and v are **Harmonic Functions***.)*

Harmonic Conjugate function

Definition

If two **Harmonic Functions** u *and* v *satisfy the Cauchy-Riemann equations in a* **Domain** D, *then the function* $f := u + iv$ *is an* **Analytic** *function on* D, *and* v *is said to be the* **Harmonic Conjugate Function** *of* u *in* D.

Example

Verify that the function $u(x,y) = x^4 - 6x^2y^2 + y^4 + 7$ is **Harmonic**, and find its **Harmonic Conjugate**.

Solution:
We have

$$u_x = 4x^3 - 12xy^2, \quad u_{xx} = 12x^2 - 12y^2,$$

and

$$u_y = -12x^2y + 4y^3, \quad u_{yy} = -12x^2 + 12y^2,$$

from which we get that u is **Harmonic**:

$$u_{xx} + u_{yy} = 12x^2 - 12y^2 + (-12x^2 + 12y^2) = 0.$$

The **Harmonic Conjugate** v of u satisfies the Cauchy-Riemann equations:

$$v_x = -u_y = 12x^2y - 4y^3,$$
$$v_y = u_x = 4x^3 - 12xy^2.$$

Integrating the first Cauchy-Riemann equation with respect to x we find:

$$\begin{aligned} v(x,y) &= \int v_x(x,y)\ dx + g(y) + C \\ &= \int (12x^2y - 4y^3)dx + g(y) + C \\ &= 4x^3y - 4y^3x + g(y) + C, \end{aligned}$$

where $C \in \mathbb{C}$ is a constant and g is an unknown function of y.

Now we use the second Cauchy-Riemann equation:

$$\begin{aligned} v_y(x,y) = u_x(x,y) &= 4x^3 - 12xy^2 \\ &= (4x^3y - 4y^3x + g(y) + C)_y \\ &= 4x^3 - 12y^2x + g'(y) \end{aligned}$$

from which we obtain that $g'(y) = 0$, that is, $g(y)$ is a constant. Finally, the expression of $v(x,y)$ is: $v(x,y) = 4x^3y - 4y^3x + K$, where $K \in \mathbb{C}$ is a constant.

Bibliography

1 *Kreyszig, E.* **Advanced Engineering Mathematics**. Wiley, 10th Edition.