MTH101: Lecture 9

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Definition

A **Sequence** *of complex numbers* $\{z_n\}$ *is defined by associating to each positive integer* n *a complex number* z_n*.*

Definition

We say that the **Sequence** $\{z_n\}$ **Converges** *to* c *:*

$$\lim_{n\to\infty} z_n = c,$$

if for any $\epsilon > 0$ *there exists an integer* N *such that*

$$|z_n - c| < \epsilon, \quad \text{for any } n > N.$$

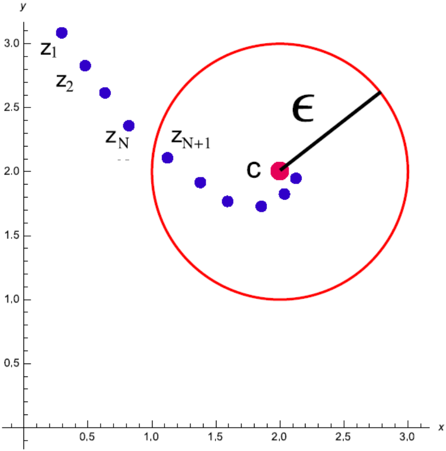

Figure : The Definition of Convergence of a Sequence: for $n > N$ the points z_n are inside the Disk with center c and radius ϵ.

Theorem

Let $z_n = x_n + iy_n$ *be a* **Sequence** *and* $c = a + ib$. *Then*

$$\lim_{n\to\infty} z_n = c \iff \begin{cases} \lim\limits_{n\to\infty} x_n = a, \\ \lim\limits_{n\to\infty} y_n = b. \end{cases}$$

Definition

A sequence defined as

$$s_n = \sum_{k=1}^{n} z_k,$$

(where $\{z_k\}$ are a sequence of Complex Numbers) is called the **n-th partial sum** *of the* **serie**

$$\sum_{k=1}^{\infty} z_k = z_1 + z_2 + \cdots .$$

Definition

We say that the **Series** $\{s_n\}$ **Converges** *to* s*, or that it is* **Convergent***, if*

$$\lim_{n\to\infty} s_n = s,$$

and we write

$$\sum_{k=1}^{\infty} z_k = s.$$

Definition

We regard s as the **Sum of the Series** *and s_n as the* n^{th} **Partial Sums** *of the Series.*

The series

$$R_n = \sum_{m=n+1}^{\infty} z_m = s - s_n,$$

is called the **Remainder.**

Remark

If $\lim_{n\to\infty} s_n = s$ *then*

$$\lim_{n\to\infty} R_n = \lim_{n\to\infty} (s - s_n) = 0.$$

Cauchys Convergence Principle for Series

Theorem

A series $\sum_{k=1}^{\infty} z_k$ *is convergent, if and only if for every* $\epsilon > 0$ *there exists an* N *such that*

$$\left|\sum_{k=n+1}^{n+p} z_k\right| = |z_{n+1} + z_{n+2} + \cdots + z_{n+p}| < \epsilon, \text{ for every } n > N \text{ and } p > 0.$$

Corollary

If the series $\sum_{k=1}^{\infty} z_k$ *is convergent then* $\lim_{k \to \infty} z_k = 0$.

Remark

The previous result gives a **Necessary Condition** *for Convergence: that is, if* $\lim_{k\to\infty} z_k \neq 0$ *then the series* $\sum_{k=1}^{\infty} z_k$ *does not converge.*

Absolute convergence

Definition

If the series $\sum_{k=1}^{\infty} |z_k|$ *is convergent, then the series* $\sum_{k=1}^{\infty} z_k$ *is called* **Absolutely Convergent**.

Theorem

Absolutely convergent series are convergent.

Remark

The converse of the previous theorem is false. For examples

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k},$$

is convergent but **not** *absolutely convergent. This kind of series are called* **Conditionally Convergent**.

Comparison Test

Theorem

If $|z_k| < a_k$ *for* $k \geq 1$ *and* $\sum_{k=1}^{\infty} a_k$ *is convergent, then the series*

$$\sum_{k=1}^{\infty} z_k$$

is **absolutely convergent**.

Theorem

The **Geometric Series**

$$\sum_{k=1}^{\infty} z^k = \frac{1}{1-z} \iff |z| < 1$$

and the convergence is absolute.
The Geometric series does not converge if $|z| \geq 1$.

Ratio Test

Theorem

If there exists $N > 0$, such that for all $n > N$,

$$\left|\frac{z_{n+1}}{z_n}\right| \leq q < 1,$$

then the series $\sum_{k=1}^{\infty} z_k$ is absolutely convergent.
If for every $n > N$

$$\left|\frac{z_{n+1}}{z_n}\right| \geq 1,$$

then the series $\sum_{k=1}^{\infty} z_k$ does not converge.

Corollary

If the limit

$$\lim_{n\to\infty}\left|\frac{z_{n+1}}{z_n}\right| = L,$$

then

(i) *if* $L < 1$*, the series* $\sum_{k=1}^{\infty} z_k$ *converges absolutely.*

(ii) *if* $L > 1$*, the series* $\sum_{k=1}^{\infty} z_k$ *does not converge.*

(iii) *if* $L = 1$*, the series* $\sum_{k=1}^{\infty} z_k$ *may converge or not.*

Example

Study the Convergence of the following Series:

$$\sum_{n=0}^{\infty} \frac{(-3+10i)^n}{n!}.$$

Solution:

We use the **Ratio Test**

$$\left|\frac{z_{n+1}}{z_n}\right| = \left|\frac{(-3+10i)^{n+1}}{(n+1)!}\,\frac{n!}{(-3+10i)^n}\right| = \left|\frac{(-3+10i)}{n+1}\right| = \frac{\sqrt{109}}{n+1}$$

from which

$$\lim_{n\to\infty}\left|\frac{z_{n+1}}{z_n}\right| = \lim_{n\to\infty}\frac{\sqrt{109}}{n+1} = 0.$$

Then $L = 0$ and from the **Ration Test** we have that the series is Convergent.

Root Test

Theorem

If there exists $N > 0$, such that for all $n > N$

$$\sqrt[n]{|z_n|} \leq q < 1,$$

then the series $\sum_{k=1}^{\infty} z_k$ *is* **absolutely convergent**. *If for infinitely many $k > N$*

$$\sqrt[n]{|z_n|} \geq 1,$$

then the series $\sum_{k=1}^{\infty} z_k$ *does not converge.*

Corollary

If the limit

$$\lim_{n\to\infty} \sqrt[n]{|z_n|} = L,$$

then

(i) *if $L < 1$, the series $\sum_{k=1}^{\infty} z_k$ converges absolutely.*

(ii) *if $L > 1$, the series $\sum_{k=1}^{\infty} z_k$ does not converge.*

(iii) *if $L = 1$, the series $\sum_{k=1}^{\infty} z_k$ may converge or not.*

Power Series

Definition

The series

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n$$

is called **Power Series** *in powers of* $z - z_0$ *with* **Coefficients** a_n.

Theorem

The **Power Series** $\sum_{n=0}^{\infty} a_n (z - z_0)^n$

(i) *converges for* $z = z_0$.

(ii) *If it converges for* $z = z_1$ *then it converges for all* z *such that* $|z - z_0| < |z_1 - z_0|$.

(iii) *if it does not converge for* $z = z_2$ *then it does not converge for any* z *such that* $|z - z_0| > |z_2 - z_0|$.

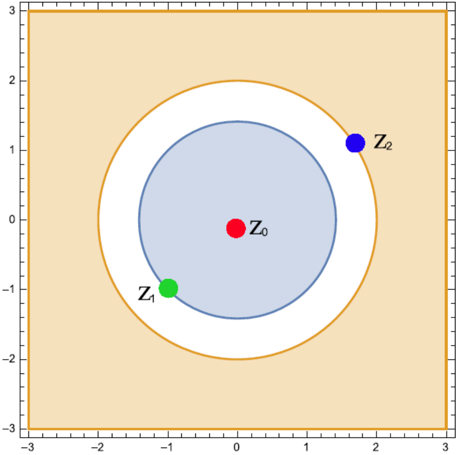

Figure : The Power Series Converges in the disk $|z - z_0| < |z_1 - z_0|$. The Power Series Does not Converge in the set $|z - z_0| > |z_2 - z_0|$.

Corollary

For the power series

$$\sum_{n=0}^{\infty} a_n(z-z_0)^n,$$

there exists a number $0 \leq R \leq \infty$, *called the* **Radius of Convergence**, *such that*

(i) *The series converges for all* z *such that* $|z-z_0| < R$ *(this is also called* **Disk of Convergence***).*

(ii) *The series does not converge all* $|z-z_0| > R$.

(iii) *On points of the set* $\{z \in \mathbb{C} : |z-z_0| = R\}$ *the power series may converge or not.*

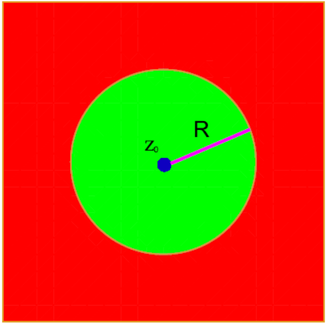

Figure : The Power Series Converges inside the Disk $|z - z_0| < R$ (Green Region, the **Disk of Convergence**).
The Power Series does not converge outside the Disk, that is for $|z - z_0| > R$ (Red Region).
On the circle $|z - z_0| = R$ the Power series may converge or not.

Theorem

Consider the **Power Series** $\sum_{n=0}^{\infty} a_n (z - z_0)^n$. *If the limit*

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = L^*,$$

then the **Radius of Convergence** *of the Power Series is given by*

$$R = \frac{1}{L^*},$$

where we set $R = \infty$ *if* $L^* = 0$ *and* $R = 0$ *if* $L^* = \infty$.

Example

Find the **Radius of convergence** of the following Series

$$\sum_{n=0}^{\infty} \frac{(3n)!}{(n!)^3}(z+2-i)^n.$$

Solution:

We have $a_n = \frac{(3n)!}{(n!)^3}$ and $z_0 = -2+i$. We use the previous theorem

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(3n+3)!}{[(n+1)!]^3}\frac{(n!)^3}{(3n)!}\right|$$
$$= \frac{(3n+3)(3n+2)(3n+1)}{(n+1)^3}$$

from which

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 27$$

and the **Radius of Convergence** is $R = \frac{1}{27}$, while the **Disk of Convergence** is:

$$|z+2-i| < \frac{1}{27}.$$

Theorem

Consider the **Power Series** $\sum_{n=0}^{\infty} a_n(z-z_0)^n$. *If the limit*

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \tilde{L},$$

then the **Radius of Convergence** *of the Power Series is given by*

$$R = \frac{1}{\tilde{L}},$$

where we set $R = \infty$ *if* $\tilde{L} = 0$ *and* $R = 0$ *if* $\tilde{L} = \infty$.

Example

Find the **Radius of convergence** of the following Series

$$\sum_{n=0}^{\infty} \frac{5^n}{(1+2i)^n}(z+i)^n.$$

Solution:

We have $a_n = \frac{5^n}{(1+2i)^n}$ and $z_0 = -i$. We use the previous theorem:

$$\sqrt[n]{|a_n|} = \sqrt[n]{\left|\frac{5^n}{(1+2i)^n}\right|} = \frac{5}{|1+2i|} = \sqrt{5},$$

from which

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \sqrt{5}$$

and the **Radius of Convergence** is $R = \frac{1}{\sqrt{5}}$, while the **Disk of Convergence** is

$$|z+i| < \frac{1}{\sqrt{5}}.$$

Bibliography

1 *Kreyszig, E.* **Advanced Engineering Mathematics**. Wiley, 10th Edition.