MTH101: Lecture 7

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Cauchys integral theorem

Theorem

If the function $f(z)$ is **Analytic** *in a* **Simply Connected** *domain D, then for every* **simple closed** *path γ in D we have*

$$\oint_{\gamma} f(z)\, dz = 0.$$

Remark

From the previous theorem we get that the integral of an Entire function over any closed path is zero:

$$\oint_{\gamma} e^{z} dz = 0, \ \oint_{\gamma} \cos z\, dz = 0, \ \oint_{\gamma} \sinh z\, dz = 0, \oint_{\gamma} (z^{n}+1)\, dz = 0 \ldots$$

Example

Compute the integral

$$\oint \frac{1}{z}\, dz,$$

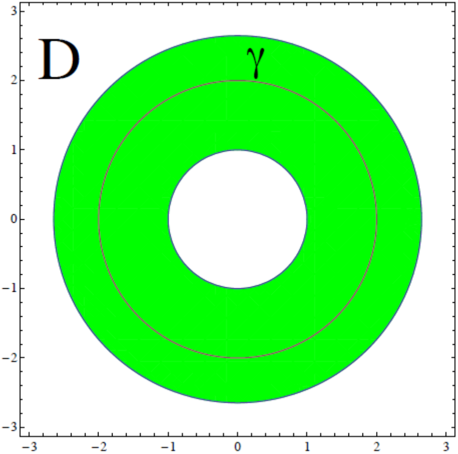
where γ is the circle with radius 1 and center 0 with counterclockwise orientation.

Solution:

Consider the set

$$D = \{z \in \mathbb{C} : \quad 0 < |z| < R\}, \qquad \text{with } R > 1.$$

we note that γ is in D and the function $f(z) = \frac{1}{z}$ is analytic in D.


D
γ
3
2
1
0
−1
−2
−3
−3
−2
−1
0
1
2
3

The set D is **not simply connect**!

$$\Downarrow$$

We **Cannot (!)** use the Cauchy's Integral Theorem

We need to use integration by parametrization:

$$z(t) = \cos t + i \sin t = e^{it}, \qquad t \in [0, 2\pi]$$

$$\dot{z}(t) = ie^{it}, \qquad t \in [0, 2\pi]$$

thus

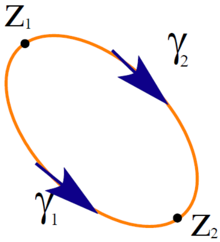
$$\oint_\gamma \frac{1}{z}\, dz = \int_0^{2\pi} \frac{1}{e^{it}} ie^{it}\, dt = [it]_0^{2\pi} = 2\pi i.$$

Independence of Path

Theorem

If $f(z)$ is an Analytic function in a Simply Connected Domain D, then the integral of $f(z)$ is independent of the path in D.

Brief Idea of the proof: Apply Cauchy's integral theorem according to following figure:



Multiply connected Domain

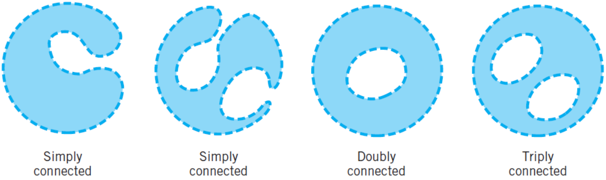


Fig. 346. Simply and multiply connected domains

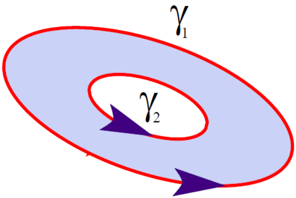
In general, a p-fold connected domain has p disjoint boundaries and $p-1$ "holes", e.g., an annulus, which is doubly connected, has 2 disjoint boundaries, and 1 "hole".

Cauchy's Integral Theorem for Multiply connected Domains

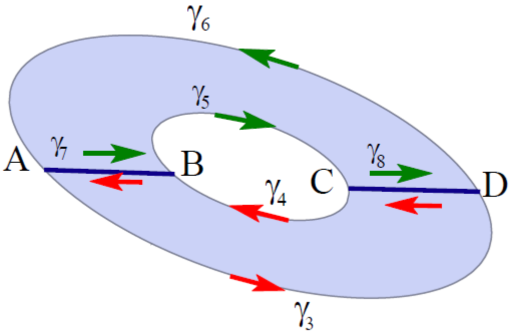
Theorem

Let D be a Doubly Connected Set with boundaries γ_1 and γ_2 with counterclockwise orientation. Let $f(z)$ be analytic in D^* Containing D, then

$$\oint_{\gamma_1} f(z)\,dz = \oint_{\gamma_2} f(z)\,dz$$



Brief Idea of the proof:
Consider the points A, D on γ_1 and B, C on γ_2 and the following paths:



Bibliography

1 *Kreyszig, E.* **Advanced Engineering Mathematics**. Wiley, 10th Edition.