MTH101: Review for Laplace Transform and Special Functions

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Laplace Transform

Definition

For a function $f(t)$ *defined for all* $t \geq 0$, *the* **Laplace transform** $F = \mathcal{L}[f]$ *of* $f(t)$ *is defined by:*

$$F(s) = \mathcal{L}[f](s) = \int_0^\infty e^{-st} f(t)dt \ .$$

If $F(s)$ *exists, then the original function* $f(t)$ *is* **the** *(essentially unique)* **inverse transform** *of* $F(s)$,

$$f(t) = \mathcal{L}^{-1}[F](t) \quad \text{or short} \quad f = \mathcal{L}^{-1}[F] \ .$$

Remark

Remember that the function $f(t)$ needs to statisfy the **existence theorem!**

$$f = \mathcal{L}^{-1}\left[\mathcal{L}\left[f\right]\right], \qquad F = \mathcal{L}\left[\mathcal{L}^{-1}\left[F\right]\right]$$

Linearity

$$\begin{aligned} & \mathcal{L}\left[af + bg\right] = a\mathcal{L}\left[f\right] + b\mathcal{L}\left[g\right], \\ or, \quad & \mathcal{L}\left[af(t) + bg(t)\right](s) = a\mathcal{L}\left[f(t)\right](s) + b\mathcal{L}\left[g(t)\right](s). \end{aligned}$$

Integration is linear operation!

Laplace Transform of $u(t-\alpha)$

Definition

Heaviside function (or unit step function) $u(t-\alpha)$ is defined as follows:

$$u(t-\alpha) = \begin{cases} 0 & \textit{if} \quad t < \alpha, \\ 1 & \textit{if} \quad t > \alpha. \end{cases}$$

Remark

Laplace transform for the Heaviside function is as follows:

$$\mathcal{L}\left[u(t-\alpha)\right] = \int_0^\infty e^{-st} u(t-\alpha) dt = \int_\alpha^\infty e^{-st} dt$$
$$= -\frac{e^{-st}}{s}\bigg|_{t=\alpha}^{\infty} = \frac{e^{-s\alpha}}{s}.$$

Laplace Transform of $\delta(t-\alpha)$

Definition

Dirac's delta function $\delta(t-\alpha)$ is defined as follows

$$\delta(t-\alpha) = \begin{cases} \infty, & \text{if } t = \alpha, \\ 0, & \text{otherwise} \end{cases}$$

$$f(\alpha) = \int_a^b f(x)\delta(t-\alpha)dt, \quad \text{for } \alpha \in (a,b).$$

Remark

In practical, we approximate $\delta(t-\alpha)$ as

$$\delta(t-\alpha) = \lim_{k\to 0} f_k(t-\alpha), \quad f_k(t-\alpha) = \begin{cases} \frac{1}{k} & \text{if } \alpha \le t \le \alpha + k \\ 0 & \text{otherwise.} \end{cases}$$

Laplace Transform of $\delta(t-\alpha)$

Since

$$f_k(t-\alpha) = \frac{1}{k}\left[u(t-\alpha) - u(t-(\alpha+k))\right],$$

$$\begin{aligned}
\mathcal{L}\left[\delta(t-\alpha)\right] &= \mathcal{L}\left[\lim_{k\to 0} f_k(t-\alpha)\right] = \lim_{k\to 0} \mathcal{L}\left[f_k(t-\alpha)\right] \\
&= \lim_{k\to 0} \frac{1}{k}\left[\frac{e^{-\alpha s}}{s} - \frac{e^{-(\alpha+k)s}}{s}\right] = \lim_{k\to 0} \frac{e^{-\alpha s} - e^{-(\alpha+k)s}}{ks} \\
&= e^{-\alpha s}.
\end{aligned}$$

Shifting Theorems

Theorem

First shifting theorem, s-Shifting

$$\mathcal{L}\left[e^{\alpha t} f(t)\right](s) = F(s-\alpha) \quad or, \quad e^{\alpha t} f(t) = \mathcal{L}^{-1}\left[F(s-\alpha)\right].$$

Second shifting theorem, t-Shifting

$$\begin{aligned} \mathcal{L}\left[f(t-\alpha)u(t-\alpha)\right] &= e^{-\alpha s}F(s), \\ or, \quad f(t-\alpha)u(t-\alpha) &= \mathcal{L}^{-1}\left[e^{-\alpha s}F(s)\right]. \end{aligned}$$

	$f(t)$	$\mathcal{L}(f)$
1	1	$1/s$
2	t	$1/s^2$
3	t^2	$2!/s^3$
4	t^n $(n = 0, 1, \cdots)$	$\dfrac{n!}{s^{n+1}}$
5	t^a (a positive)	$\dfrac{\Gamma(a+1)}{s^{a+1}}$
6	e^{at}	$\dfrac{1}{s-a}$

	$f(t)$	$\mathcal{L}(f)$
7	$\cos \omega t$	$\dfrac{s}{s^2+\omega^2}$
8	$\sin \omega t$	$\dfrac{\omega}{s^2+\omega^2}$
9	$\cosh at$	$\dfrac{s}{s^2-a^2}$
10	$\sinh at$	$\dfrac{a}{s^2-a^2}$
11	$e^{at} \cos \omega t$	$\dfrac{s-a}{(s-a)^2+\omega^2}$
12	$e^{at} \sin \omega t$	$\dfrac{\omega}{(s-a)^2+\omega^2}$

Laplace Transform of Derivative and Integral

Laplace Transform of Derivatives

Theorem

The Laplace transform of n-th derivative of $f(t)$ is

$$\mathcal{L}\left[f^{(n)}\right](s) = s^n \mathcal{L}\left[f\right] - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0)\ .$$

In particular, for $n = 1$ and $n = 2$,

$$\begin{aligned} \mathcal{L}\left[f'\right] &= s\mathcal{L}\left[f\right] - f(0), \\ \mathcal{L}\left[f''\right] &= s^2\mathcal{L}\left[f\right] - sf(0) - f'(0). \end{aligned}$$

Laplace Transform of Integrals

Theorem

Let $F(s)$ denotes the Laplace transform of a function $f(t)$. By definition, $f(t)$ has to be piecewise continuous for $t \geq 0$ and satisfies growth restriction $|f(t)| \leq Me^{kt}$. Then, for $s > k > 0$ and $t > 0$,

$$\mathcal{L}\Big[\int_0^t f(\tau)d\tau\Big] = \frac{1}{s}F(s), \quad or, \quad \int_0^t f(\tau)d\tau = \mathcal{L}^{-1}\Big[\frac{1}{s}F(s)\Big].$$

Convolution

Theorem

If two functions f and g satisfy the existence theorem in Sec. 6.1, and their transforms are denoted as F and G, the product $H = FG$ is the Laplace transform of function h:

$$h(t) \equiv (f * g) \equiv \int_0^t f(\tau)g(t-\tau)d\tau.$$

Remark

$$\begin{aligned} \mathcal{L}\left[(f*g)\right] &= \mathcal{L}\left[f\right]\mathcal{L}\left[g\right] \\ f*g &= g*f && \textit{(commutative law)}, \\ f*(g_1+g_2) &= f*g_1+f*g_2 && \textit{(distributive law)}, \\ (f*g)*v &= f*(g*v) && \textit{(associative law)}, \\ f*0 &= 0*f = 0. \end{aligned}$$

Laplace Transform and IVP

The idea of using Laplace transform to solve nonhomogeneous linear ODEs is as follows. Consider the following initial value problem

$$y'' + ay' + by = r(t) \qquad y(0) = K_0,\ y'(0) = K_1,$$

where a, b, K_0, K_1 are constants.
To solve this problem with Laplace transform, there are three steps.
Step 1: We first transform the ODE into functions of s:

$$\begin{aligned} & [s^2 Y - sy(0) - y'(0)] + a[sY - y(0)] + bY = R(s), \\ \Rightarrow\ & (s^2 + as + b)Y = (sK_0 + K_1 + aK_0) + R(s). \end{aligned}$$

Step 2: Then we solve $Y(s)$ by

$$Y(s) = \left[(s+a)K_0 + K_1\right] Q(s) + Q(s)R(s),$$

where $R(s) = \mathcal{L}\left[r(t)\right]$, and

$$Q(s) \equiv \frac{1}{s^2 + as + b} = \frac{1}{\left(s + \frac{1}{2}a\right)^2 + b - \frac{1}{4}a^2}.$$

Step 3: After finding $Y(s)$, the solution can be found by inverse transform

$$y(t) = \mathcal{L}^{-1}\left[\left[(s+a)K_0 + K_1\right] Q(s)\right] + \mathcal{L}^{-1}\left[Q(s)R(s)\right].$$

Therefore,

$$y(t) = \mathcal{L}^{-1}\left[\left[(s+a)K_0 + K_1\right] Q(s)\right] + q * r(t).$$

Differentiation and Integration of Transforms

Theorem

Suppose $f(t)$ satisfies the existence theorem in Sec. 6.1, and its Laplace transform is denoted as $F(s)$, then

$$\mathcal{L}\left[tf(t)\right] = -F'(s).$$

Theorem

Suppose $f(t)$ satisfies the existence theorem in Sec. 6.1, and its Laplace transform is denoted as $F(s)$, then if $\lim_{t\to 0^+}(f(t)/t)$ exists, then

$$\mathcal{L}\left[\frac{f(t)}{t}\right] = \int_s^\infty F(\tilde{s})d\tilde{s}.$$

Summary for Laplace Transform

1. Practice how to do Laplace transform and inverse transform for the functions we mentioned in this part, with the basic transformation, the shifting theorem and other techniques.
2. Practice how to do Laplace transform to solve initial value problem or ordinary differential equation, with Laplace transform of derivatives and convolution theorem.

Power Series and ODE

Operations on Power Series

Remark

Consider two analytic functions $f(x) = \sum_{m=0}^{\infty} a_m(x - x_0)^m$ *and* $g(x) = \sum_{m=0}^{\infty} b_m(x - x_0)^m$ *are convergent in open set* D*, then the following operation holds in* D*.*

1. **Termwise differentiation**

$$f'(x) = \sum_{m=0}^{\infty} m a_m (x - x_0)^{m-1}$$

$$f''(x) = \sum_{m=0}^{\infty} m(m-1) a_m (x - x_0)^{m-2}$$

$$\vdots$$

Operations on Power Series

Remark

2. **Termwise addition**

$$f(x)+g(x)=\sum_{m=0}^{\infty}a_m(x-x_0)^m+\sum_{m=0}^{\infty}b_m(x-x_0)^m$$
$$=\sum_{m=0}^{\infty}(a_m+b_m)(x-x_0)^m.$$

Operations on Power Series

Remark

3. **Termwise multiplication**

$$
\begin{aligned}
f(x)g(x) &= \left[\sum_{m=0}^{\infty} a_m (x-x_0)^m\right]\left[\sum_{m=0}^{\infty} b_m (x-x_0)^m\right] \\
&= a_0 b_0 + (a_0 b_1 + a_1 b_0)(x-x_0) + \cdots \\
&= \sum_{m=0}^{\infty} (a_0 b_m + a_1 b_{m-1} \cdots a_m b_0)(x-x_0)^m.
\end{aligned}
$$

Operations on Power Series

Remark

4. **Identity theorem for power series** *The power series representation for function $f(x)$ is unique within D. More precisely, if*

$$f(x) = \sum_{m=0}^{\infty} a_m (x - x_0)^m = \sum_{m=0}^{\infty} b_m (x - x_0)^m,$$

we must have $a_0 = b_0$, $a_1 = b_1$, and so on. In particular, if $f(x) = 0$, this theorem guarantees that all the coefficients $a_m = 0$.

Theorem

Consider the following ODE

$$y'' + p(x)y' + q(x)y = r(x),$$

if $p(x)$, $q(x)$, $r(x)$ are analytic at $x = x_0$, then every solution of it is analytic at $x = x_0$ and can be represented by a power series in powers of $x - x_0$ with radius of convergence $R > 0$. I.e. we can write down an ansatz

$$y = \sum_{m=0}^{\infty} a_m (x - x_0)^m.$$

Following the theorem,we can use power series to solve second order nonhomogeneous ODE.

Example

Solve the following ODE.

$$y'' + p(x)y' + q(x)y = r(x), \quad y(x_0) = k_0, \; y'(x_0) = k_1.$$

Example

Since p, q, r are analytic at $x = x_0$, we can express p, q, r, and y by the power series.

$$y(x) = \sum_{m=0}^{\infty} a_m (x - x_0)^m, \quad p(x) = \sum_{m=0}^{\infty} p_m (x - x_0)^m,$$
$$q(x) = \sum_{m=0}^{\infty} q_m (x - x_0)^m, \quad r(x) = \sum_{m=0}^{\infty} r_m (x - x_0)^m,$$

Example

where a_m are unknown and p_m, q_m, r_m can be found by

$$p_m = \frac{p^{(m)}(x_0)}{m!} \cdots .$$

We can also find y', y'' by the operation of power series

$$y' = \sum_{m=0}^{\infty} m a_m (x - x_0)^{m-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} (x - x_0)^n,$$

$$y'' = \sum_{m=0}^{\infty} m(m-1) a_m (x - x_0)^{m-2}$$

$$= \sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} (x - x_0)^n.$$

Example

With the operation of power series we know

$$\begin{aligned} q(x)y(x) &= \left[\sum_{m=0}^{\infty} q_m (x-x_0)^m\right]\left[\sum_{l=0}^{\infty} a_l (x-x_0)^l\right] \\ &= \sum_{m=0}^{\infty}\sum_{l=0}^{\infty} q_m a_l (x-x_0)^{m+l} = \sum_{n=0}^{\infty} c_n (x-x_0)^n, \end{aligned}$$

where we set $n = m + l$ *and*

$$c_n = \sum_{m=0}^{n} q_m a_{n-m}.$$

Example

Similarly, we have

$$p(x)y'(x) = \left[\sum_{m=0}^{\infty} p_m (x-x_0)^m\right]\left[\sum_{l=0}^{\infty}(l+1)a_{l+1}(x-x_0)^l\right]$$
$$= \sum_{m=0}^{\infty}\sum_{l=0}^{\infty}(l+1)p_m a_{l+1}(x-x_0)^{m+l} = \sum_{n=0}^{\infty} d_n (x-x_0)^n,$$

where we set $n = m + l$ *and*

$$d_n = \sum_{m=0}^{n}(n-m+1)p_m a_{n-m+1}.$$

Example

Therefore, the ODE becomes

$$\sum_{n=0}^{\infty}\left[(n+1)(n+2)a_{n+2}+\sum_{m=0}^{n}(n-m+1)p_m a_{n-m+1} + \sum_{m=0}^{n} q_m a_{n-m} - r_n\right](x-x_0)^n = 0.$$

The coefficients in the bracket is called **recurrence relation**, *which tells us how to find* a_{n+2} *with lower order* a_k. *From the initial conditions, we know*

$$y(x_0) = k_0 = a_0, \qquad y'(x_0) = k_1 = a_1,$$

we can thus solve this IVP.

Example

For $n = 0$*, we have*

$$2a_2 + p_0 a_1 + q_0 a_0 = r_0$$
$$\Rightarrow a_2 = \frac{r_0 - p_0 a_1 - q_0 a_0}{2}.$$

Similarly, we can choose $n = 1, 2, 3 \cdots$ *to find* a_3*,* a_4*,* a_5*,* $\cdots$.

Gamma function

Definition

The **Gamma function** *is defined for* $\alpha > 0$ *by the integral*

$$\Gamma(\alpha) = \int_0^\infty e^{-x} x^{\alpha-1} dx \ .$$

Integration by parts yields

$$\Gamma(\alpha+1) = \alpha\Gamma(\alpha) \ .$$

Since $\Gamma(1) = 1$, *this implies for* $n \in \mathbb{N}$

$$\Gamma(n+1) = n! \ .$$

Legendre's Equation and Legendre's Funcation

Legendre's differential equation is

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0,$$

where n is a real constant. This equation involves a given **parameter** n, which is dependent on the problem we are considering, and the solution to it is called **Legendre function**, which is one of the most important **special functions**.

By using the **Power Series** technique, we find the recurrence relation

$$a_{m+2} = \frac{m(m+1) - n(n+1)}{(m+2)(m+1)} a_m = -\frac{(n-m)(n+m+1)}{(m+2)(m+1)} a_m,$$

and the general solution is

$$y(x) = a_0 y_1(x) + a_1 y_2(x),$$

where

$$y_1(x) = 1 - \frac{n(n+1)}{2} x^2 + \frac{(n-2)n(n+1)(n+3)}{4!} x^4 - + \cdots$$
$$y_2(x) = x - \frac{(n-1)(n+2)}{3!} x^3 + \frac{(n-3)(n-1)(n+2)(n+4)}{5!} x^5 - + \cdots$$

This series solution converges for $|x| < 1$, since for Legendre's differential equation, $x = \pm 1$ are the regular singular points.

Legendre's Polynomial

Remark

When n is an integer. By the recurrence relation, we know $a_{n+2} = a_{n+4} = \cdots = 0$, and by convention the coefficient a_n of the highest power x^n is chosen as

$$a_n = \frac{(2n)!}{2^n (n!)^2} = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{n!},$$

and $a_n = 1$ if $n = 0$. Then we can inverse the recurrence relation to find

$$a_{n-2m} = (-1)^m \frac{(2n-2m)!}{2^n m!(n-m)!(n-2m)!}.$$

Remark

The Legendre Polynomial of degree n is thus

$$P_n(x) = \sum_{m=0}^{M} (-1)^m \frac{(2n-2m)!}{2^n m!(n-m)!(n-2m)!} x^{n-2m},$$

where $M = \frac{n}{2}$ *(if n even) and* $\frac{(n-1)}{2}$ *(if n odd) and the first few Legendre polynomials are*

$$P_0(x) = 1, \qquad P_1(x) = x, \qquad P_2(x) = \frac{1}{2}(3x^2 - 1),$$

$$P_3(x) = \frac{1}{2}\left(5x^3 - 3x\right), \qquad P_4(x) = \frac{1}{8}\left(35x^4 - 30x^2 + 3\right).$$

Frobenius Method

Theorem

Let $b(x)$ and $c(x)$ be any functions that are analytic at $x = 0$. Then the ODE

$$y'' + \frac{b(x)}{x}y' + \frac{c(x)}{x^2}y = 0,$$

admits at least one solution that can be represented in the form

$$y_1(x) = x^r \sum_{m=0}^{\infty} a_m x^m, \qquad (a_0 \neq 0),$$

where r may be any real or complex number and is chosen such that $a_0 \neq 0$.

Let's see how does the Frobenius method work. We first multiply the ODE by x^2

$$x^2 y'' + xb(x)y' + c(x)y = 0,$$

and since $b(x)$, $c(x)$ are analytic, we can expand them in power series

$$b(x) = \sum_{m=0}^{\infty} b_m x^m, \qquad c(x) = \sum_{m=0}^{\infty} c_m x^m,$$

and the ansatz

$$y_1(x) = x^r \sum_{m=0}^{\infty} a_m x^m.$$

Substituting them into the ODE, we find

$$x^r\left[r(r-1)a_0+\cdots\right]+x^r\left[ra_0+(r+1)a_1x+\cdots\right](b_0+b_1x+\cdots)$$
$$+\,x^r(a_0+a_1x+\cdots)(c_0+c_1x+\cdots)=0,$$

and we use the requirement that all the coefficients for x^r, x^{r+1}, x^{r+2}, $\cdots$ need to be zero to solve the unknown a_m. The coefficient of x^r gives us the indicial equation ($a_0\neq 0$),

$$[r(r-1)+b_0r+c_0]=0,$$

and we can find two roots r_1 and r_2, and there are three cases for the other solution y_2.

Theorem

Case 1. $r_1 \neq r_2$, $r_1 - r_2 \notin \mathbb{Z}$. *A basis is*

$$y_1(x) = x^{r_1}\left(a_0 + a_1 x + a_2 x^2 + \cdots\right),$$
$$y_2(x) = x^{r_2}\left(A_0 + A_1 x + A_2 x^2 + \cdots\right),$$

Case 2. $r_1 = r_2 = r$. *A basis is*

$$y_1(x) = x^{r}\left(a_0 + a_1 x + a_2 x^2 + \cdots\right),$$
$$y_2(x) = y_1(x)\ln x + x^{r}\left(A_1 x + A_2 x^2 + \cdots\right).$$

Case 3. $r_1 \neq r_2$, $r_1 - r_2 \in \mathbb{Z}^+$. *A basis is*

$$y_1(x) = x^{r_1}\left(a_0 + a_1 x + a_2 x^2 + \cdots\right),$$
$$y_2(x) = k y_1(x)\ln x + x^{r_2}\left(A_0 + A_1 x + A_2 x^2 + \cdots\right),$$

where k may turn out to be zero, and $\ln x$ is defined for $x > 0$.

In pratical, we can first find out y_1 by substituting it into the ODE, fix all the coefficients. After find out y_1, we can use the reduction of order to find y_2. I.e. we assume $y_2(x) = y_1(x)u(x)$, and by substituting it into the ODE, the second order ODE becomes the first order ODE of $u'(x)$, we can thus solve $u'(x)$ and then integrate it out to find $u(x)$ and $y_2(x)$.

Check the example in the lecture slides!!

Bessel's Equation and Bessel Function J_ν

One of the most important ODEs in applied mathematics is the Bessel's equation

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0,$$

where ν is a given real number which is positive or zero. This equation corresponding to the ODE in Frobenius method with $b(x) = 1$ and $c(x) = x^2 - \nu^2$, we thus know that it has a solution of the form

$$y(x) = \sum_{m=0}^{\infty} a_m x^{m+r}.$$

By substituting it into the Bessel's equation, we find the indicial equation is

$$r^2 - \nu^2 = 0 \quad \Rightarrow \quad (r-\nu)(r+\nu) = 0.$$

and the roots are $r_1 = \nu$ (≥ 0) and $r_2 = -\nu$.
On the other hand, if we choose $r = \nu$, the coefficient of x^{r+1} shows that $a_1 = 0$ and the general recurrence relation is

$$a_s = -\frac{1}{s(s+2\nu)} a_{s-2}.$$

Therefore the nonvanishing terms are

$$a_{2m} = \frac{(-1)^m a_0}{2^{2m} m! (\nu+1)(\nu+2)\cdots(\nu+m)}, \quad m = 1, 2, \cdots .$$

If we choose $a_0 = \frac{1}{2^\nu \Gamma(\nu+1)}$, we find **Bessel function of the first kind of order** ν is thus

$$J_\nu(x) = x^\nu \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+\nu} m! \Gamma(\nu + m + 1)}.$$

In particular, for ν an integer n, we have **Bessel function of the first kind** of order n

$$J_n(x) = x^n \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+n} m! (n + m)!}.$$

Properties of the Bessel Equations

Remark

There are four useful properties of the Bessel's equation

$$
\begin{aligned}
(a)\ & [x^{\nu} J_{\nu}(x)]' = x^{\nu} J_{\nu-1}(x), \\
(b)\ & [x^{-\nu} J_{\nu}(x)]' = -x^{-\nu} J_{\nu+1}(x), \\
(c)\ & J_{\nu-1}(x) + J_{\nu+1}(x) = \frac{2\nu}{x} J_{\nu}(x), \\
(d)\ & J_{\nu-1}(x) - J_{\nu+1}(x) = 2J_{\nu}'(x).
\end{aligned}
$$

General Solution, Bessel Function Y_ν

For ν not an integer, we can easily find out the other solution which is linearly independent of J_ν, which is

$$J_{-\nu}(x) = x^{-\nu} \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m-\nu} m! \Gamma(-\nu + m + 1)},$$

if ν not an integer, $\Gamma(-\nu + m + 1)$ is finite and the first terms of J_ν and $J_{-\nu}$ are finite nonzero multiples of x^ν and $x^{-\nu}$, and thus are independent of each other and the general solution is thus

$$y(x) = c_1 J_\nu(x) + c_2 J_{-\nu}(x).$$

For ν an integer n, one can show that $J_{-n}(x) = (-1)^n J_n(x)$, and J_n, J_{-n} are linearly dependent thus cannot form a basis of the general solution.

The **Bessel function of the second kind** of order ν is defined by

$$Y_\nu(x) = \frac{1}{\sin(\nu\pi)}\left[J_\nu(x)\cos(\nu\pi) - J_{-\nu}(x)\right],$$
$$Y_n(x) = \lim_{\nu\to n} Y_\nu(x),$$

where $n = 0, 1, 2, 3\cdots$, and for any ν, J_ν and Y_ν are linearly independent, and the general solution to the Bessel's equation for all values of ν is

$$y(x) = C_1 J_\nu(x) + C_2 Y_\nu(x).$$

I.e., for ν not an integer, we have

$$y(x) = C_1 J_\nu(x) + C_2 Y_\nu(x) = C_3 J_\nu(x) + C_4 J_{-\nu}(x).$$

One can use one of the two basis to write down the general solution.

Summary for Special Functions and Series Solution to ODEs

1. Practice how to use power series method and Frobenius method to solve the corresponding ordinary differential equation or initial value problems.
2. Practice how to use power series to solve Legendre's equation, understands the properties of Legendre's function and Legendre's polynomials.
3. Practice how to use Frobenius method to solve Bessel's equation, understands the properties of first and second kind of Bessel's function and in which case J_ν and $J_{-\nu}$ are linearly independent.
4. Some equation, although doesn't look like a Legendre or Bessel equation, under some coordinate transformation, it can be a Legendre or Bessel equation, practice how to do it.

Sturm-Liouville Problem

A **Sturm-Liouville problem** consists of an ODE of the form

$$[p(x)y']' + [q(x) + \lambda r(x)]y = 0$$

on some closed interval $a \leq x \leq b$, satisfying conditions of the form

$$\begin{aligned} k_1 y + k_2 y' &= 0 \quad \text{at} \quad x = a \qquad (k_1^2 + k_2^2 > 0), \\ l_1 y + l_2 y' &= 0 \quad \text{at} \quad x = b \qquad (l_1^2 + l_2^2 > 0). \end{aligned}$$

where λ is an unknown parameter, k_1, k_2, l_1, l_2 are given real constants, $p(x)$, $q(x)$, $r(x)$ are functions of x, and $r(x)$ is called **weight** function. (We will discuss on this later.)

Eigenvalues, Eigenfunctions

It is easy to check that for any **Sturm-Liouville problems**, $y = 0$ is a solution–the "**trivial solution**", which is not interesting. The solution we want to find are the so-called **eigenfunctions** $y(x)$, which are non-zero solutions to the Sturm-Liouville problems, and we call the number λ for which an eigenfunction exists an **eigenvalue** of the Sturm-Liouville problem.

Remark

1. *It can be shown that under general conditions on the functions p, q, r, the Sturm-Liouville problem has infinitely many eigenvalues.*
2. *If p, q, r, p' are real and continuous on the interval $a \leq x \leq b$ and $r(x)$ is positive for $a \leq x \leq b$, then all the eigenvalues of the Sturm-Liouville problem are* **real**. *These real eigenvalues usually correspond to the physical quantities such as energies, frequencies.*

Check the examples for Sturm-Liouville problem in the lecture slides!

Orthogonality

Orthogonality

Functions $y_1(x)$, $y_2(x)$, $\cdots$ defined on some interval $a \leq x \leq b$ are called **orthogonal** on ths interval with respect to the **weight function** $r(x) > 0$ if for all $m \neq n$,

$$(y_m, y_n) = \int_a^b r(x)y_m(x)y_n(x)dx = 0 \qquad (m \neq n),$$

where (y_m, y_n) is a **standard notation** for the integral. The **norm** $||y_m||$ of y_m is defined by

$$||y_m|| = \sqrt{(y_m, y_m)} = \sqrt{\int_a^b r(x)y_m^2(x)dx}.$$

Orthogonality

The functions $\tilde{y}_1, \tilde{y}_2, \cdots$ are called **orthonormal** on the interval $[a, b]$ if they are orthogonal on the interval and all have norm 1. Then, we can write the above two equations jointly by the **Kronecker symbol** δ_{mn}

$$(\tilde{y}_m, \tilde{y}_n) = \int_a^b r(x)\tilde{y}_m(x)\tilde{y}_n(x)dx = 0 = \delta_{mn} = \begin{cases} 0 & if \quad m \neq n, \\ 1 & if \quad m = n. \end{cases}$$

Orthogonality

Theorem

If y_m and y_n are eigenfunctions of the Sturm-Liouville problem in the interval $[a, b]$ with continuous real-valued coefficients p, q, r, p' and $r(x) > 0$ within the interval $[a, b]$, then y_m, y_n are orthogonal on the interval with respect to the weight function $r(x)$, i.e.

$$(y_m, y_n) = \int_a^b r(x) y_m(x) y_n(x) dx = 0 \qquad (m \neq n).$$

Orthogonality

Theorem

Moreover, if $p(a) = 0$ and/or $p(b) = 0$, then the boundary condition on the point a and/or b can be dropped.
If $p(a) = p(b)$ then the boundary conditions on points a and b can be replaced by **periodic boundary conditions** *for the function y,*

$$y(a) = y(b) \quad \text{and} \quad y'(a) = y'(b).$$

Summary for Sturm-Liouville Problem and Orthogonality

1. Practice how to solve a Sturm-Liouville problem, remember that we need to consider all possible λ, i.e. ($\lambda > 0$, $\lambda = 0$, $\lambda < 0$.)
2. Understand the definition of orthogonality and the weight function. Remember that all the eigenfunctions to a Sturm-Liouville problem are orthogonal to each other.
3. Legendre's and Bessel's equations can be rewritten as Sturm-Liouville problems, and thus the solutions to the two equations are eigenfunctions to the problems, with associated eigenvalue. The eigenfunctions also satisfy the orthogonality theorem of the Sturm-Liouville problem.