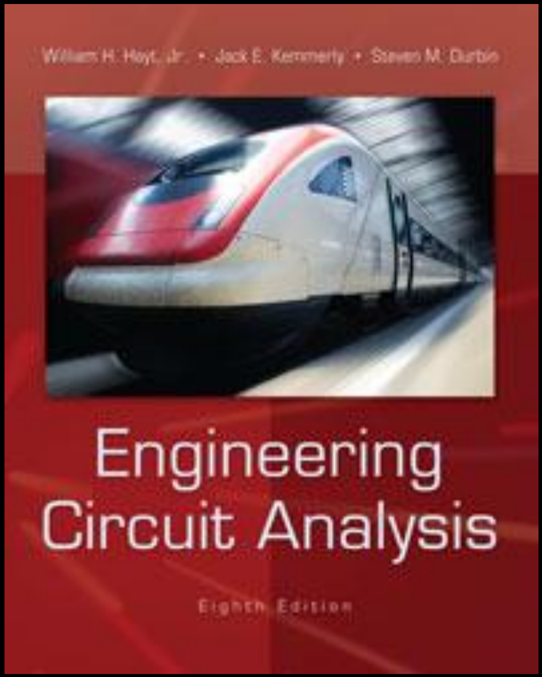
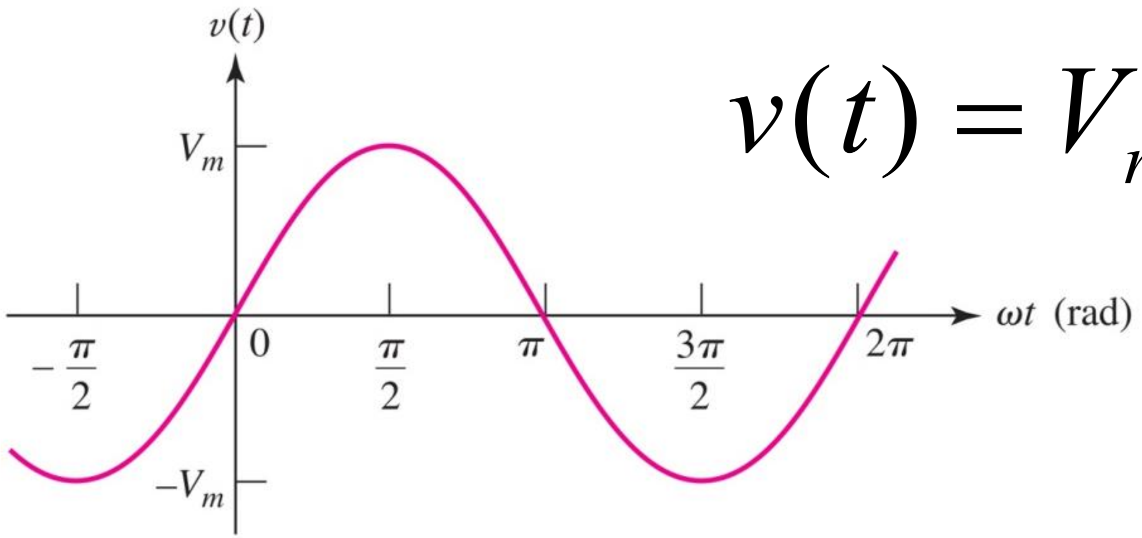
Chapter 10
Sinusoidal Steady-State Analysis



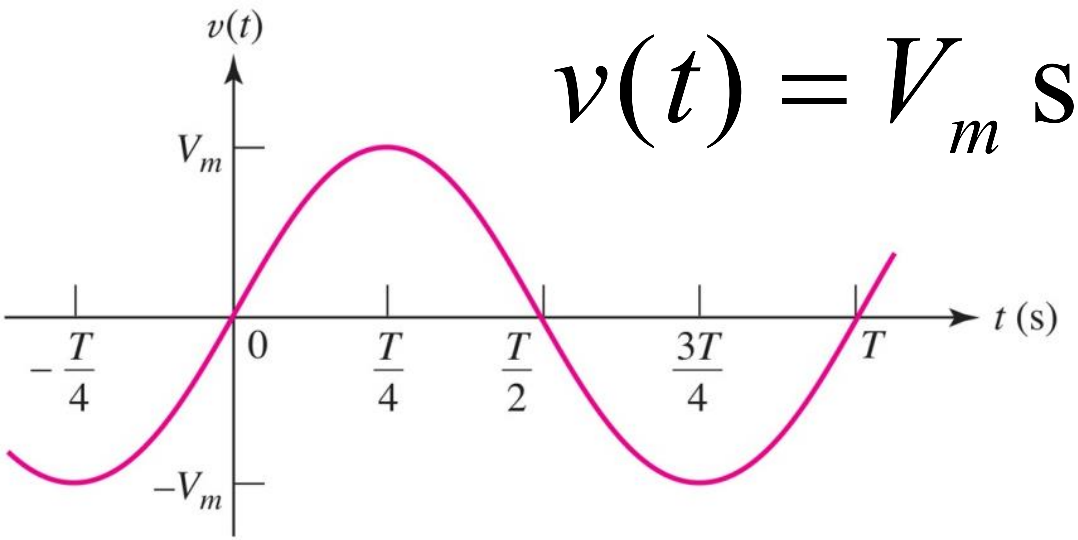
Sinusoids: Defining Terms



$$v(t) = V_m \sin(\omega t)$$

- the *amplitude* of the wave is V_m
- the *argument* is ωt
- the *radian or angular frequency* is ω
- note that sin() is *periodic*

Period of Sine Wave



$$v(t) = V_m \sin(\omega t)$$

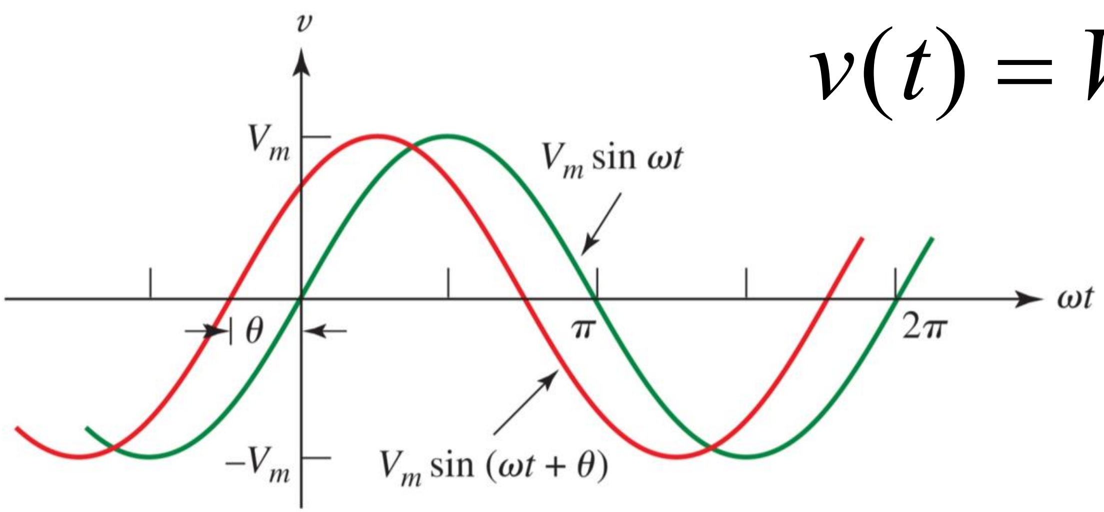
- the period of the wave is T
- the frequency f is 1/T: units Hertz (Hz)

$$f = \frac{1}{T} = \frac{\omega}{2\pi} \qquad \omega = 2\pi f$$

Sine Wave Phase

A more general form of a sine wave includes a *phase* θ

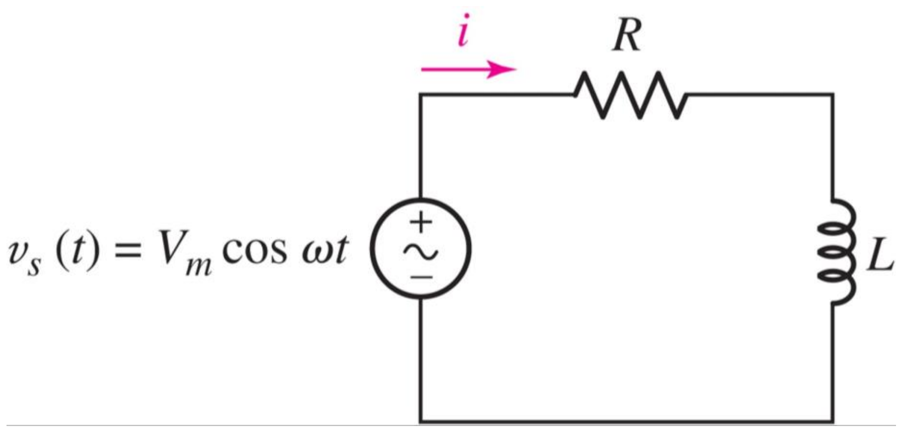
$$v(t) = V_m \sin(\omega t + \theta)$$



- The new wave (in red) is said to *lead* the original (in green) by θ.
- The original *sin(ωt)* is said to *lag* the new wave by θ.
- θ can be in degrees or radians, but the argument of sin() is always *radians*.

Forced Response to Sine Sources

When the source is sinusoidal, we often ignore the transient/natural response and consider only the forced or "steady-state" response.

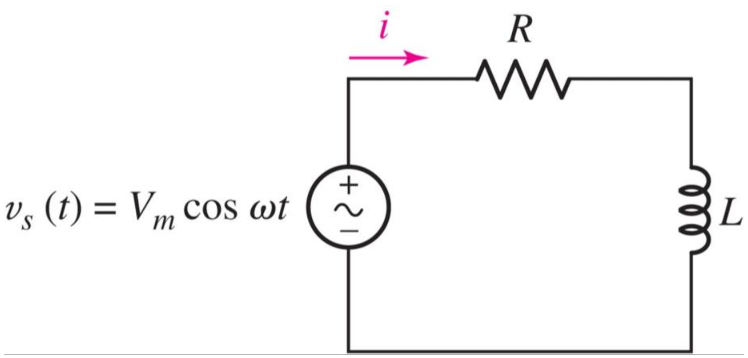


The source is assumed to exist forever: $-\infty < t < \infty$

Finding the Steady-State Response

1. Apply KVL:

$$L\frac{di}{dt} + Ri = V_m \cos(\omega t)$$



2. Make a good guess:

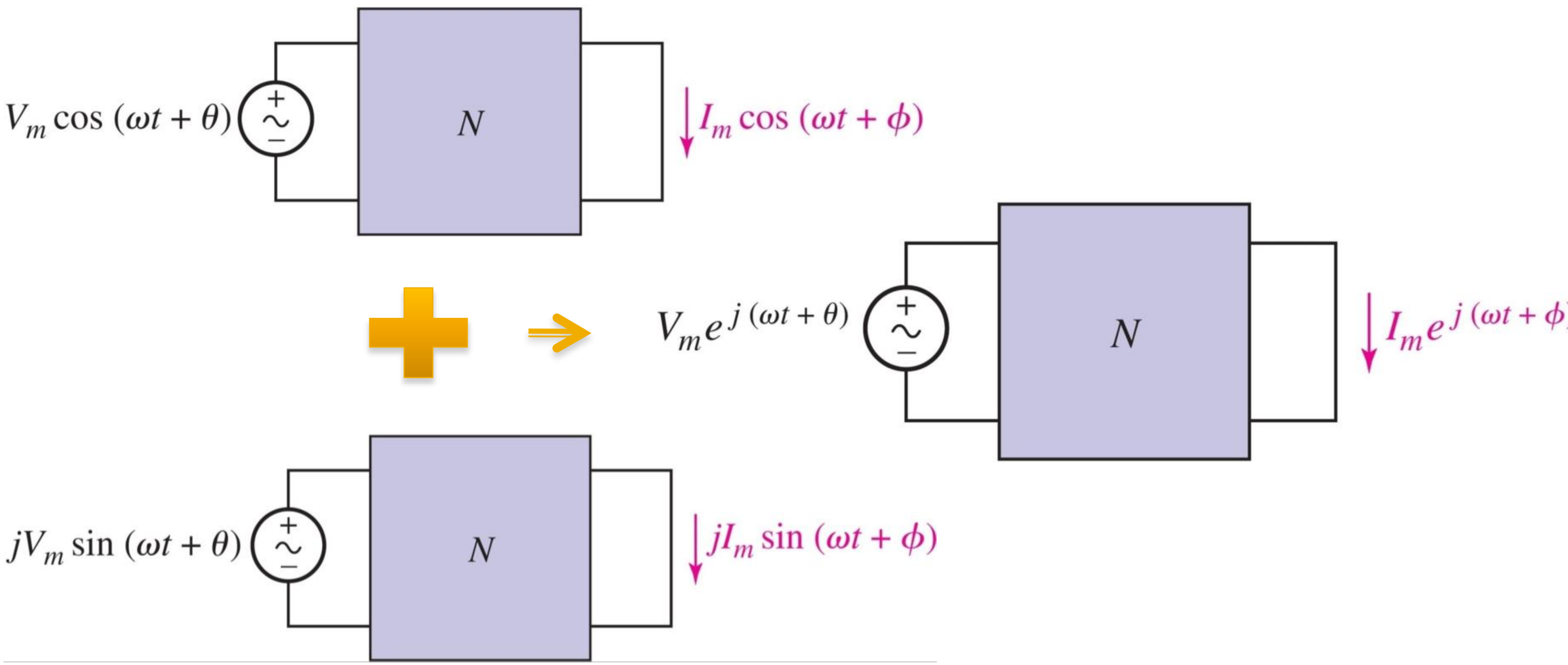
$$i(t) = I_1 \cos \omega t + I_2 \sin \omega t$$

3. Solve for the constants:

$$i(t) = \frac{RV_m}{R^2 + \omega^2 L^2} \cos \omega t + \frac{\omega L V_m}{R^2 + \omega^2 L^2} \sin \omega t$$

The Complex Forcing Function

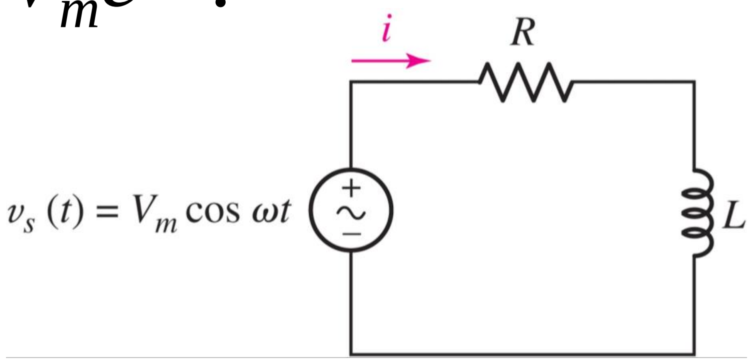
Apply superposition and use $e^{j\theta} = \cos(\theta) + j\sin(\theta)$



The Steady-State Response via Complex Forcing Function

1. Apply KVL, assume $v_s = V_m e^{j\omega t}$.

$$L\frac{di}{dt} + Ri = v_s$$



2. Find the complex response

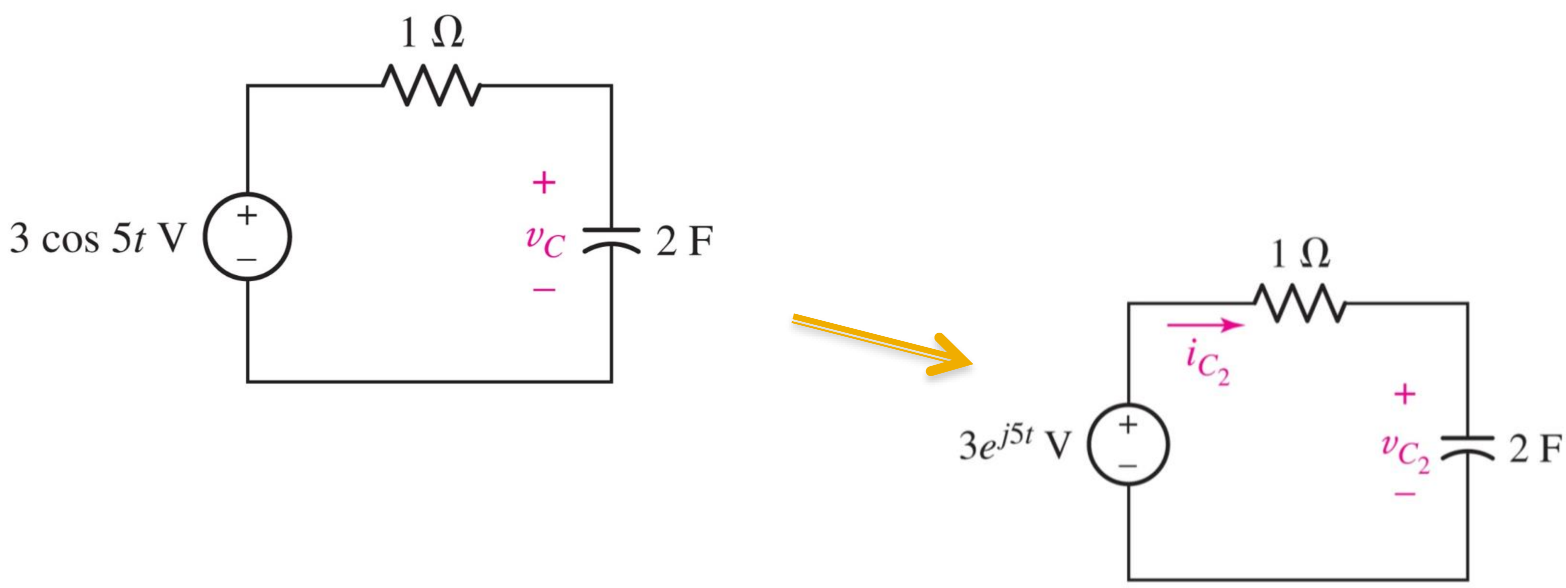
$$\mathrm{i(t)} = I_m e^{j\omega t+\theta}$$

3. Find I_m and θ, *(discard the imaginary part)*

$$i(t) = I_m \cos(\omega t + \phi) = \frac{V_m}{\sqrt{R^2 + \omega^2 L^2}} \cos\left(\omega t - \tan^{-1}\frac{\omega L}{R}\right)$$

Example: Sine Wave Analysis

Find the voltage on the capacitor.



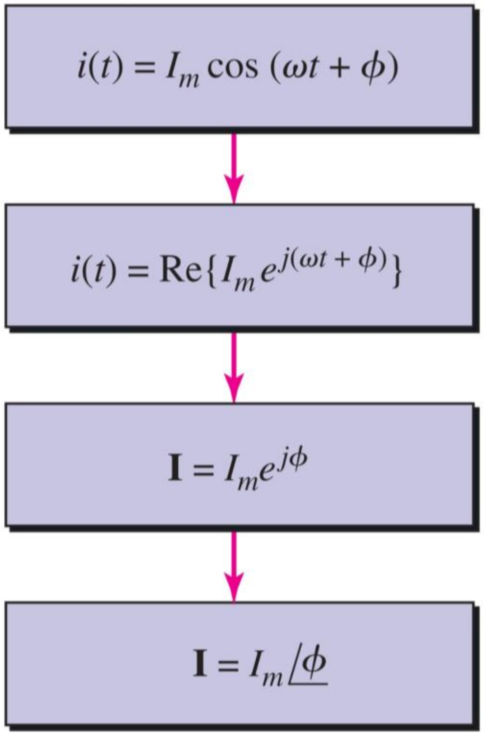
Answer: $v_c(t) = 298.5 \cos(5t - 84.3^\circ)$ *mV*

The Phasor

The term $e^{j\omega t}$ is common to all voltages and currents and can be ignored in all intermediate steps, leading to the phasor:

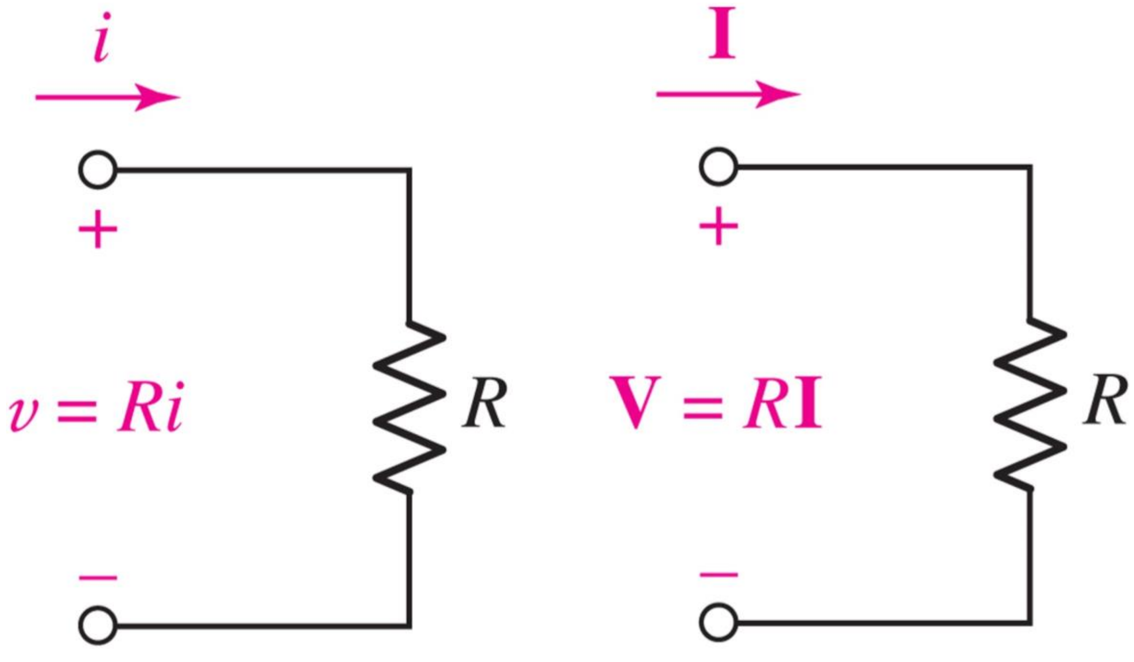
$$\mathbf{I} = I_\mathrm{m} e^{j\phi} = I_\mathrm{m} \angle \phi$$

The phasor representation of a current (or voltage) is in the *frequency domain*



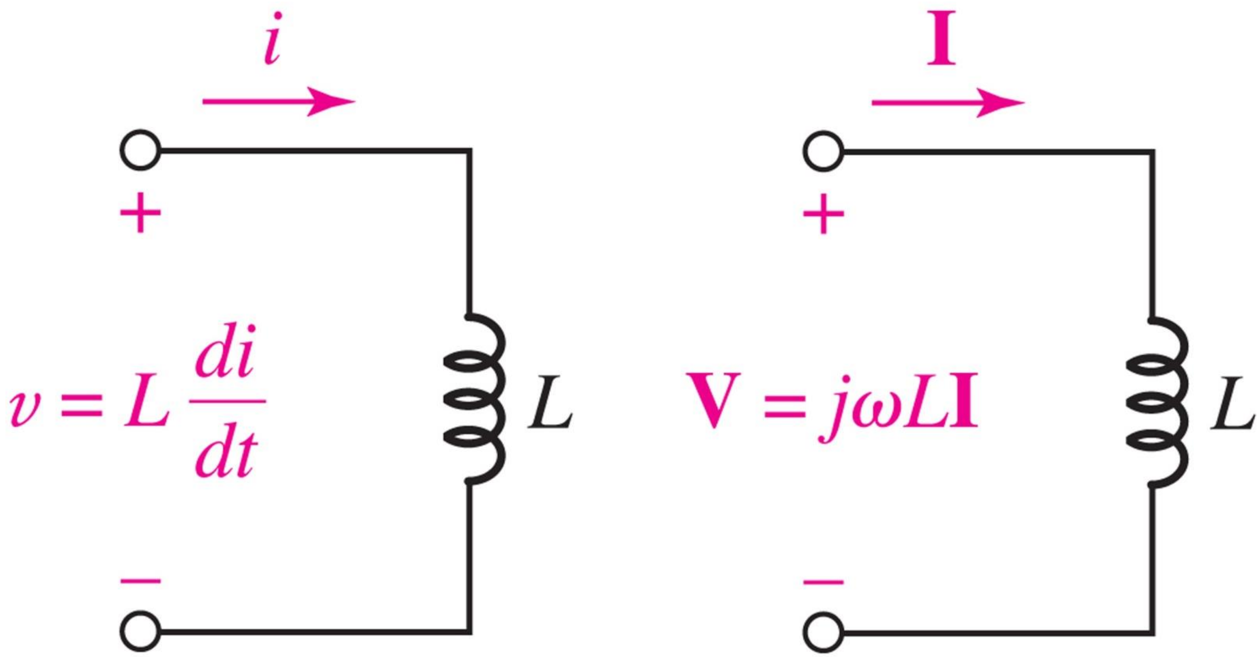
Phasors: The Resistor

In the frequency domain, Ohm's Law takes the same form:



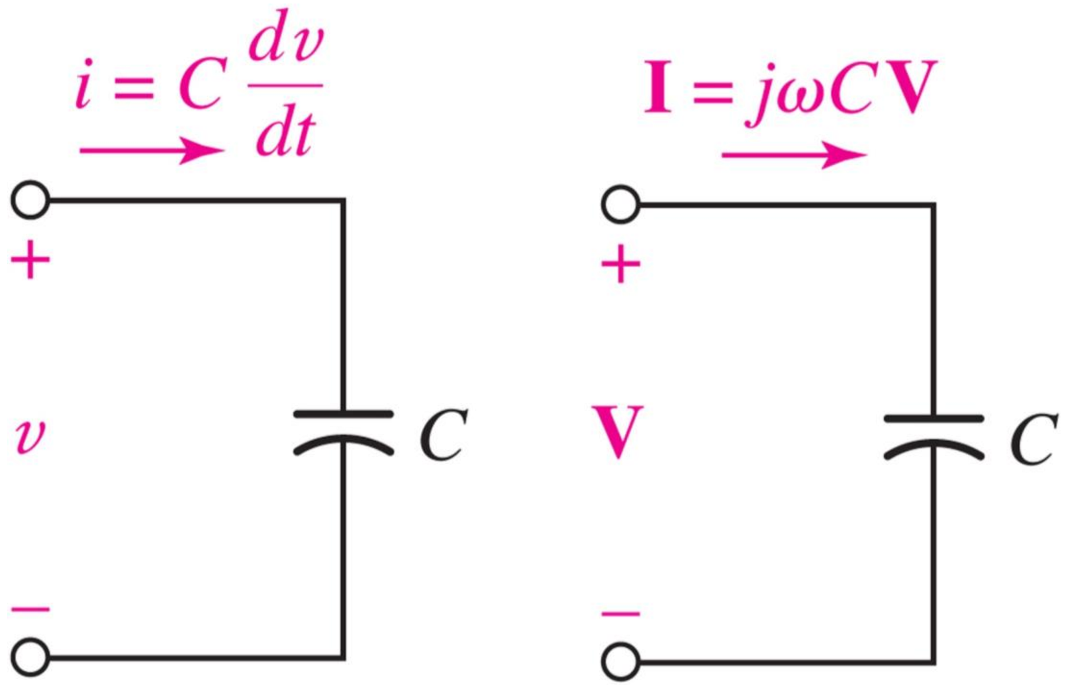
Phasors: The Inductor

Differentiation in time becomes multiplication in phasor form: (calculus becomes algebra!)



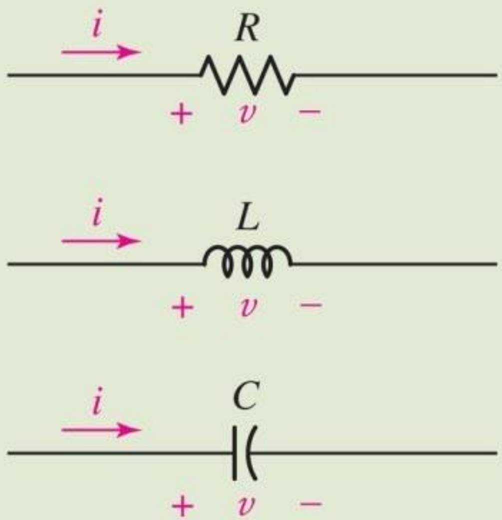
Phasors: The Capacitor

Differentiation in time becomes multiplication in phasor form: (calculus becomes algebra!)



Summary: Phasor Voltage/Current Relationships

Time Domain



$$v = Ri$$

$$v = L\frac{di}{dt}$$

$$v = \frac{1}{C}\int i\,dt$$

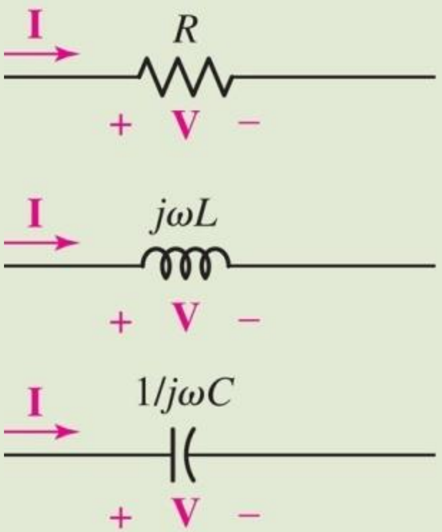
Calculus (hard but real)

Frequency Domain

$$\mathbf{V} = R\mathbf{I}$$

$$\mathbf{V} = j\omega L\mathbf{I}$$

$$\mathbf{V} = \frac{1}{j\omega C}\mathbf{I}$$



Algebra (easy but complex)

Kirchhoff's Laws for Phasors

Applying KVL in time implies KVL for phasors:

$$V_1 + V_2 + \ldots + V_N = 0$$

Applying KCL in time implies KCL for phasors:

$$I_1 + I_2 + \ldots + I_N = 0$$

Impedance

- Define impedance as $Z = \boldsymbol{V}/\boldsymbol{I}$, i.e. $\boldsymbol{V} = \boldsymbol{I}Z$

$$Z_R = R \qquad Z_L = j\omega L \qquad Z_C = 1/j\omega C$$

- Impedance is the equivalent of resistance in the frequency domain.
- Impedance is a complex number (unit ohm).
- Impedances in series or parallel can be combined using "resistor rules."

Impedance Relationships

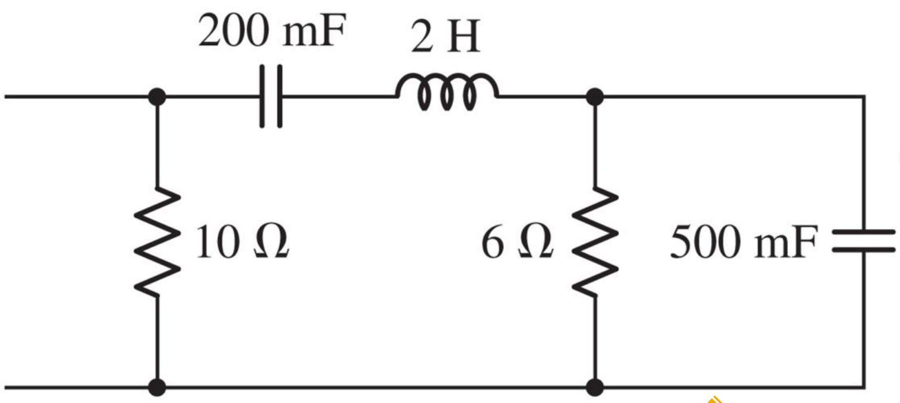
- the admittance is $Y=1/Z$

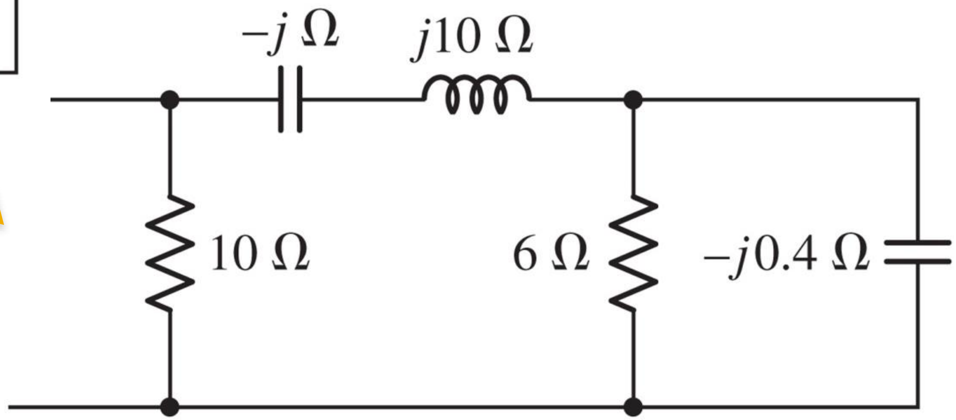
$$Y_R=1/R \qquad Y_L=1/j\omega L \qquad Y_C=j\omega C$$

- if $Z=R+jX$; R is the *resistance*, X is the *reactance* (unit ohm Ω)
- if $Y=G+jB$; G is the *conductance*, B is the *susceptance*: (unit siemen S)

Example: Equivalent Impedance

Find the impedance of the network at 5 rad/s.

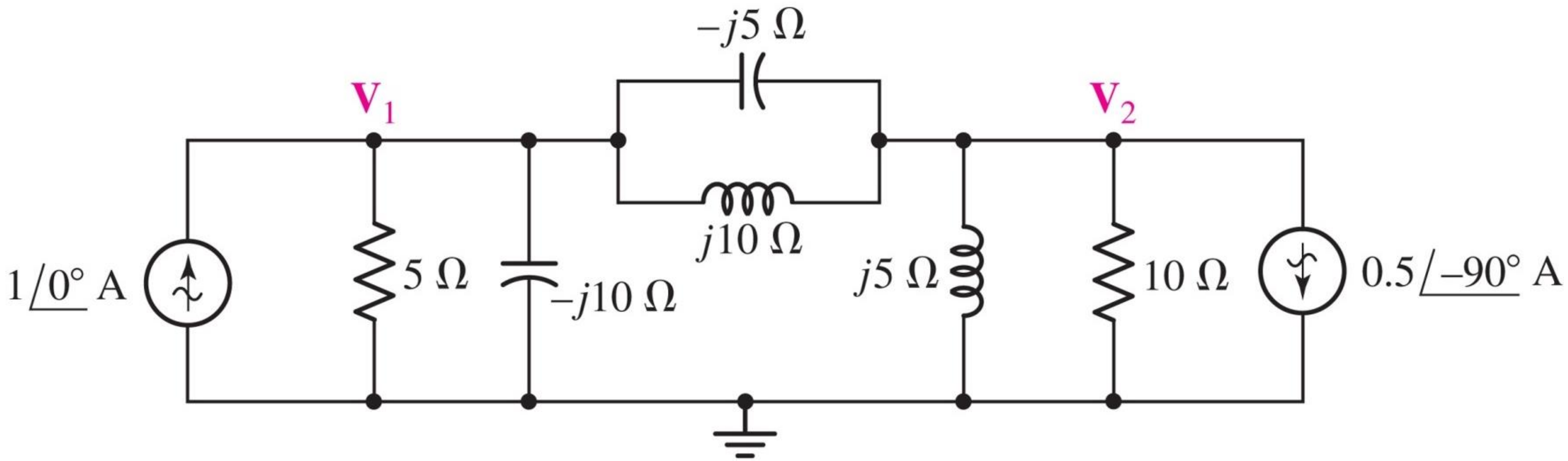




Answer: $4.255 + j4.929\ \Omega$

Nodal and Mesh Analysis

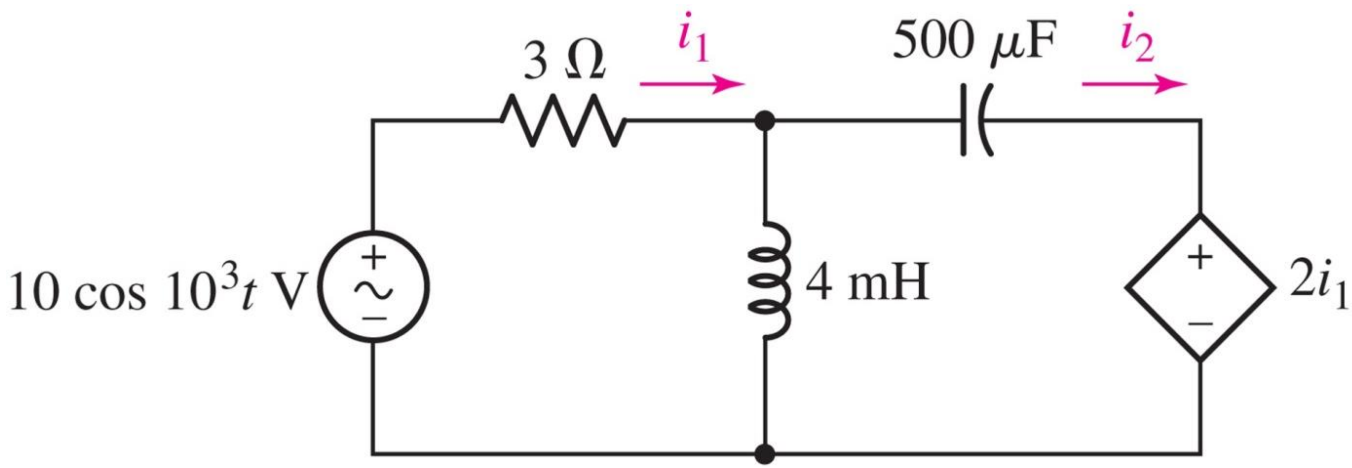
Find the phasor voltages $\boldsymbol{V_1}$ and $\boldsymbol{V_2}$.



Answer: $\boldsymbol{V_1} = 1\text{-}j2$ V *and* $\boldsymbol{V_2} = \text{-}2+j4$ V

Nodal and Mesh Analysis

Find the currents $i_1(t)$ and $i_2(t)$.



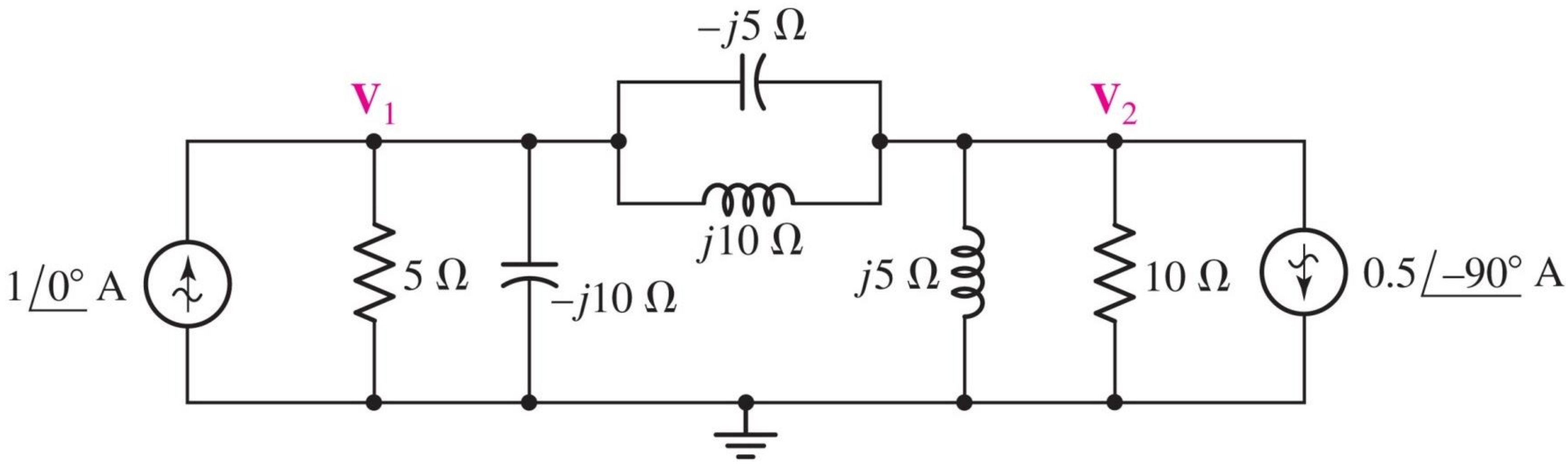
Answer:

$i_1(t) = 1.24 \cos(10^3 t + 29.7^\circ)$ A

$i_2(t) = 2.77 \cos(10^3 t + 56.3^\circ)$ A

Superposition Example

The superposition principle applies to phasors; use it to find $\boldsymbol{V}_1$.

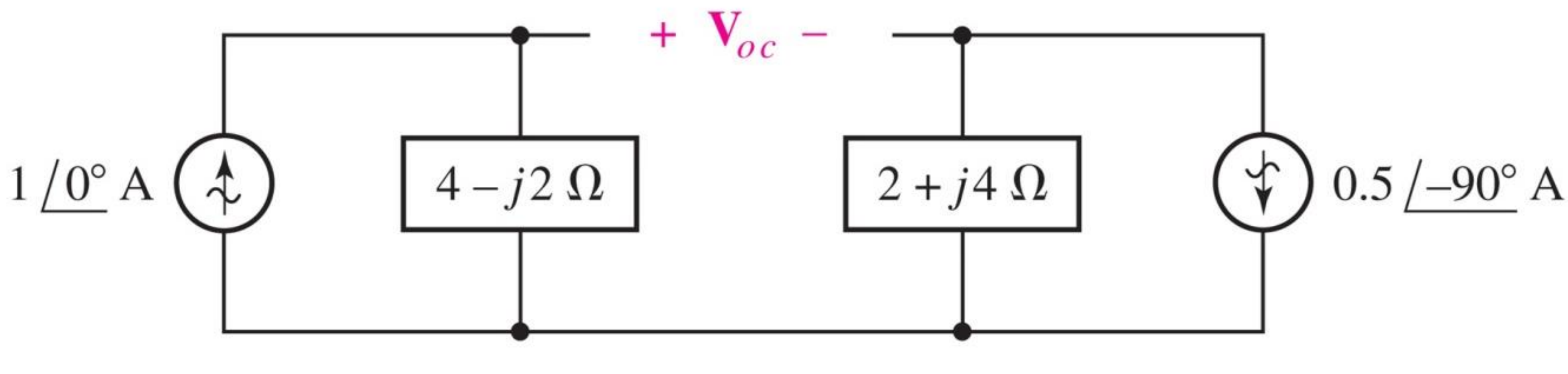


Answer: $\boldsymbol{V}_1 = \boldsymbol{V}_{1L} + \boldsymbol{V}_{1R} = (2-j2)+(-1) = 1-j2$ V

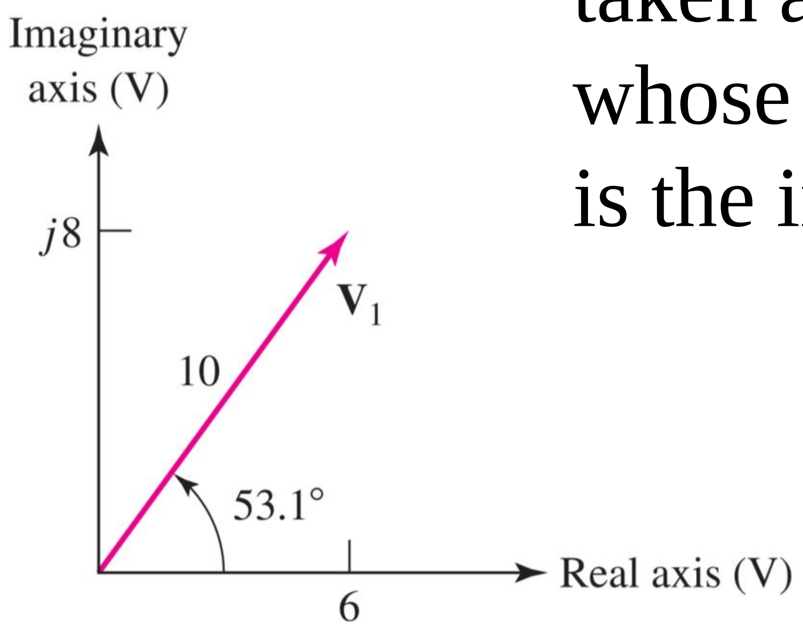
Thévenin Example

Thévenin's theorem also applies to phasors; we can use it to find $\boldsymbol{V_1}$.

The setup is shown below:



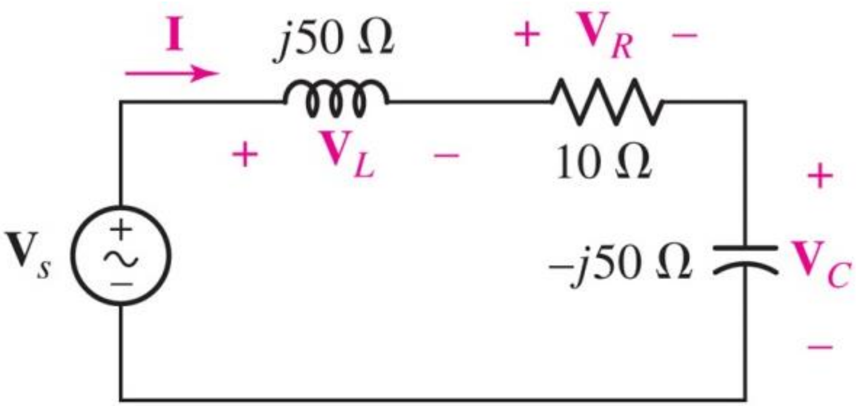
Phasor Diagrams

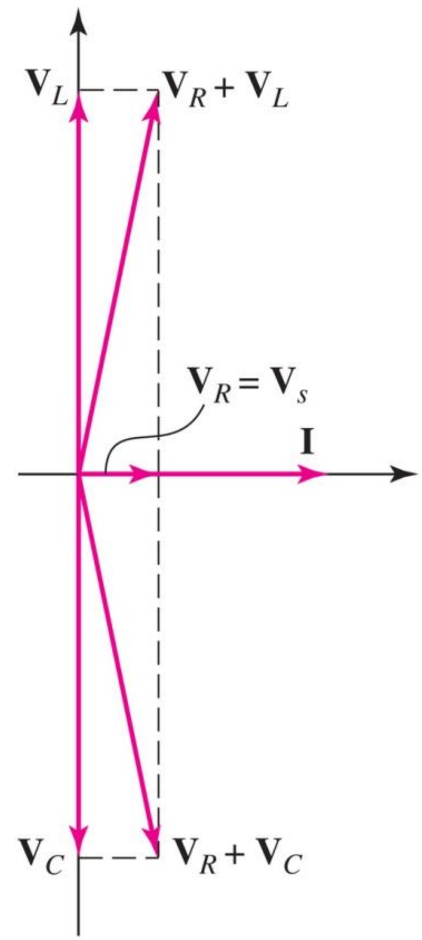


The arrow for the phasor **V** on the phasor diagram is a photograph, taken at $\omega t = 0$, of a rotating arrow whose projection on the real axis is the instantaneous voltage $v(t)$.

Example Phasor Diagram

If we assume $\mathbf{I}=1\angle 0^\circ$ A





Phasor Diagram: Parallel RLC

Assume $\mathbf{V} = 1\,\underline{/0^\circ}\ V$