EEE225 Advanced Electrical Circuits and Electromagnetics

Lecture 8 Transient Analysis – 2nd order circuits

Dr. Zhao Wang

zhao.wang@xjtlu.edu.cn

Room EE322

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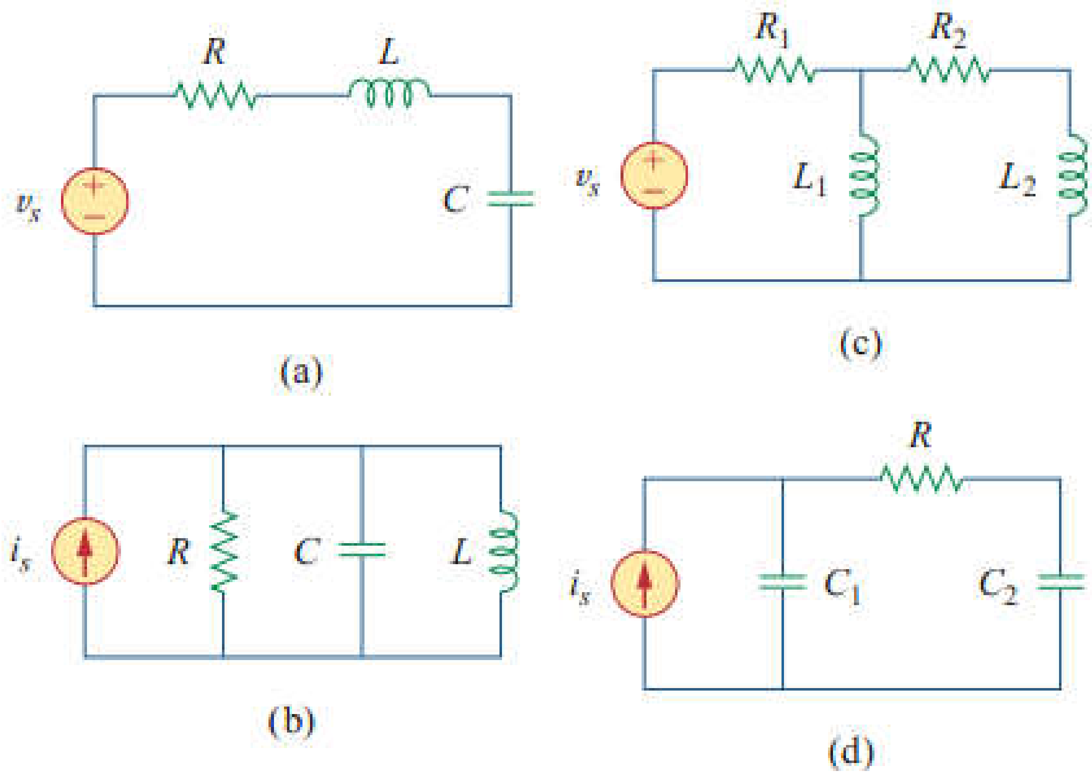
Second-order Circuits

- A second order circuit is characterized by a second order differential equation. It consists of resistors and the equivalent of **TWO** energy storage elements.

- Typical examples of second order circuits:
  - Series RLC circuit
  - Parallel RLC circuit
  - RLL circuit
  - RCC circuit

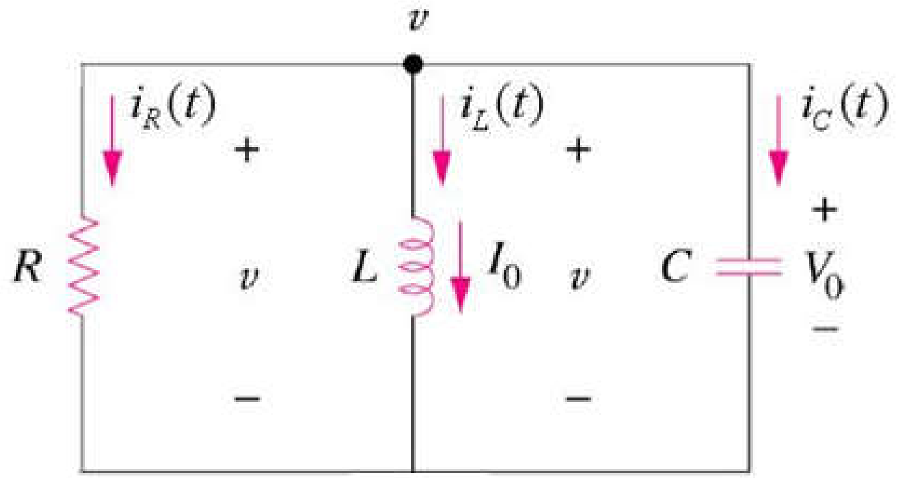


Parallel RLC Circuit - Obtaining the SODE

SODE - Second Order Differential Equation

- By applying the KCL at the node:

$$i_R(t)+i_L(t)+i_C(t)=0$$



- Where we have

$$\left.\begin{aligned} i_R(t)&=\frac{v}{R}\\ i_C(t)&=C\frac{dv}{dt}\\ i_L(t)&=\frac{1}{L}\int_{t_0}^{t} v\,dt'-i(t_0)\end{aligned}\right\}\quad \frac{v}{R}+C\frac{dv}{dt}+\frac{1}{L}\int_{t_0}^{t} v\,dt'-i(t_0)=0$$

- Take derivative of both sides, then divide by C, we get

$$\frac{d^2v}{dt^2}+\frac{1}{RC}\frac{dv}{dt}+\frac{1}{LC}v=0$$

Parallel RLC Circuit - Solving the SODE

- By assuming the solution of the form Ae^{st},
  - A is a constant determined by initial conditions
  - s is a constant determined by the coefficients of the differential equation (by circuit components)
- The ***characteristic equation*** is

$$As^2e^{st} + \frac{1}{RC}Ase^{st} + \frac{A}{LC}e^{st} = 0 \;\Longrightarrow\; s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0$$

- The two solutions are:

$$s_1 = -\frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}} \quad \text{and} \quad s_2 = -\frac{1}{2RC} - \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

- The general form of the natural response

$$v_n(t) = A_1e^{s_1t} + A_2e^{s_2t}$$

Where A_1 and A_2 must be found by applying the given initial condition

Parallel RLC Circuit - *Determining A_1 and A_2*

- Since $v_n(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$
  - 1) $v_n(0^+) = A_1 + A_2$
  - 2) $\dfrac{dv_n(0^+)}{dt} = A_1 s_1 e^{s_1 t} + A_2 s_2 e^{s_2 t}\Big|_{t=0^+} = A_1 s_1 + A_2 s_2$

- Therefore, with the knowledge of s_1 and s_2, we need two initial conditions to determine the value of A_1 and A_2:

$$v_n(0^+) \quad \text{and} \quad \frac{dv_n(0^+)}{dt}$$

  - Where 0^+ is the time just after the changes in circuit, i.e. switch movement.

Parallel RLC Circuit – *SODE of $i_L(t)$*

- Similarly, we can solve for the inductor current $i_L(t)$, and get the SODE like:

$$\frac{d^2 i_L}{dt^2} + \frac{1}{RC}\frac{di_L}{dt} + \frac{1}{LC} i_L = 0$$

- By assuming the exponential solution Ae^{st}, it can also be written as

$$s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0$$

- Whose solution also has the form

$$i_{L,n}(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

where A_1 and A_2 are determined by the given initial conditions.

Parallel RLC Circuit - Finding initial values

- Two key points:
  - Polarity of voltage across the capacitor, and the direction of the current through the inductor.
  - The capacitor voltage is always continuous, and the inductor current is always continuous

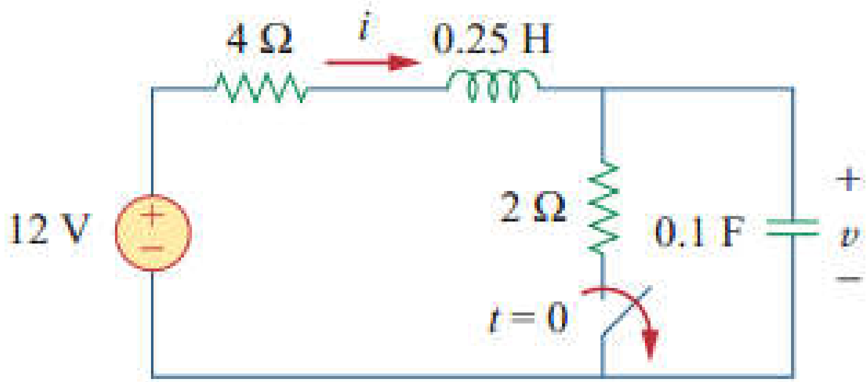
$$v_C(t=0^+) = v_C(t=0^-)$$

$$i_L(t=0^+) = i_L(t=0^-)$$

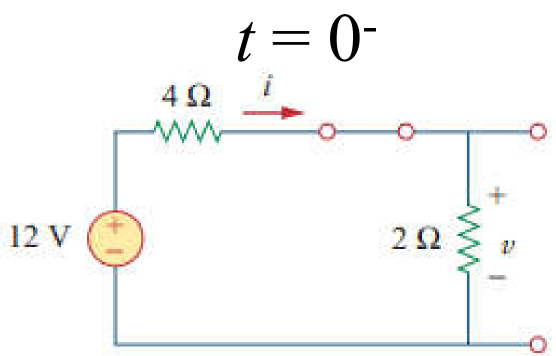
Normally start from finding variables that cannot change abruptly.

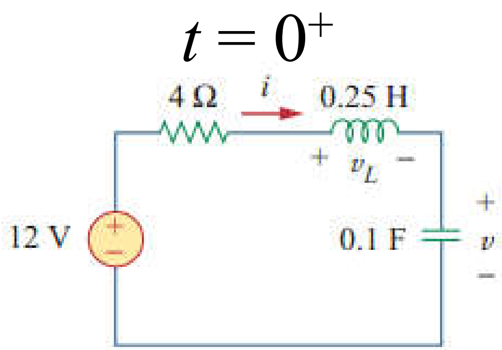
Parallel RLC Circuit - Finding initial values

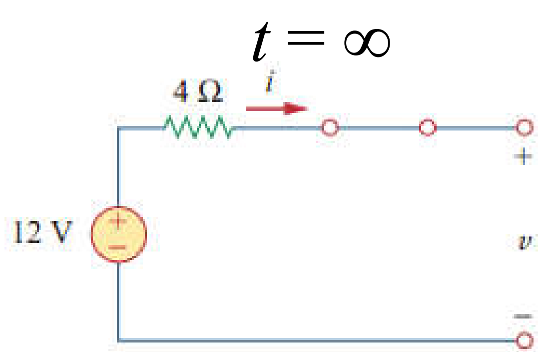
- Example:
- The switch has been closed for a long time. It is open at $t = 0$.
  - Find: $i(0^+), v_C(0^+), \dfrac{di(0^+)}{dt}, \dfrac{dv_C(0^+)}{dt}, i(\infty)$ and $v_C(\infty)$



Ref. 3, example 8.1



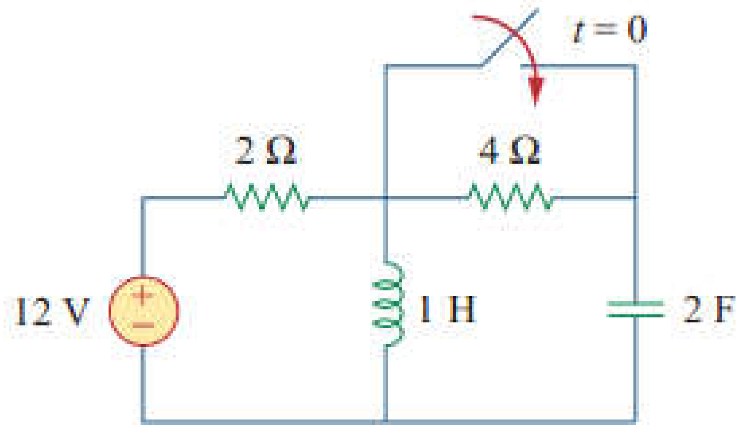




Quiz

- For the circuit as shown, the capacitor voltage at $t = 0^+$ (just after the switch is closed) is:
  - (a) 0 V; (b) 4 V; (c) 8 V; (d) 12 V.

- For the same circuit as shown, the inductor voltage at $t = 0^+$ (just after the switch is closed) is:
  - (a) 0 V; (b) 4 V;
  - (c) 8 V; (d) 12 V.



Parallel RLC Circuit - Definition of Frequency Terms

- Define ω_0 as the ***resonant frequency***, and α as the ***neper frequency***, or the ***exponential damping coefficient***:

$$\omega_0 = \frac{1}{\sqrt{LC}} \qquad \alpha = \frac{1}{2RC}$$

- The characteristic equation can also be written as

$$s^2 + 2\alpha s + \omega_0^2 = 0$$

- Then the s_1 and s_2 are called ***complex frequencies***

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}$$

$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$$

- The natural response is still

$$v_n(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

Parallel RLC Circuit - Damping factors

$$\boxed{s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}, \quad s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}}$$

- The value of the term $\sqrt{\alpha^2 - \omega_0^2}$ determines the behavior of the response.
  - 1. Over Damped System: $\alpha > \omega_0$, s_1 and s_2 are two unequal real numbers;
  - 2. Critical Damped System: $\alpha = \omega_0$, s_1 and s_2 are two equal real numbers;
  - 3. Under Damped System: $\alpha < \omega_0$, s_1 and s_2 are two complex numbers,
    $s_1 = -\alpha + j\sqrt{\omega_0^2 - \alpha^2}, s_2 = -\alpha - j\sqrt{\omega_0^2 - \alpha^2}$

    The system exhibits oscillatory behavior.

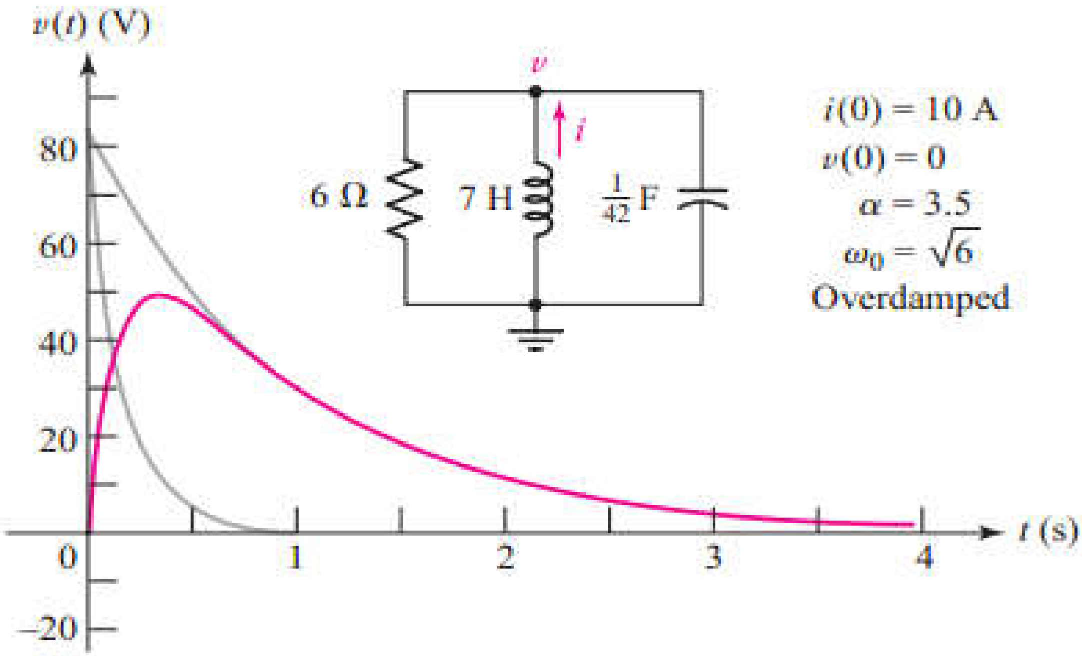
Parallel RLC Circuit - 1. Over Damped System

$$\omega_0 = \frac{1}{\sqrt{LC}} \quad \text{and} \quad \alpha = \frac{1}{2RC}$$

- Over Damped System: $\alpha > \omega_0$, implies $L > 4R^2C$.
- Both s_1 and s_2 are two negative real numbers, the response is

$$v_n(t) = A_1 e^{S_1 t} + A_2 e^{S_2 t}$$

- Thus, $v_n(t)$ decays and approach zero as time increases $t \rightarrow \infty$.

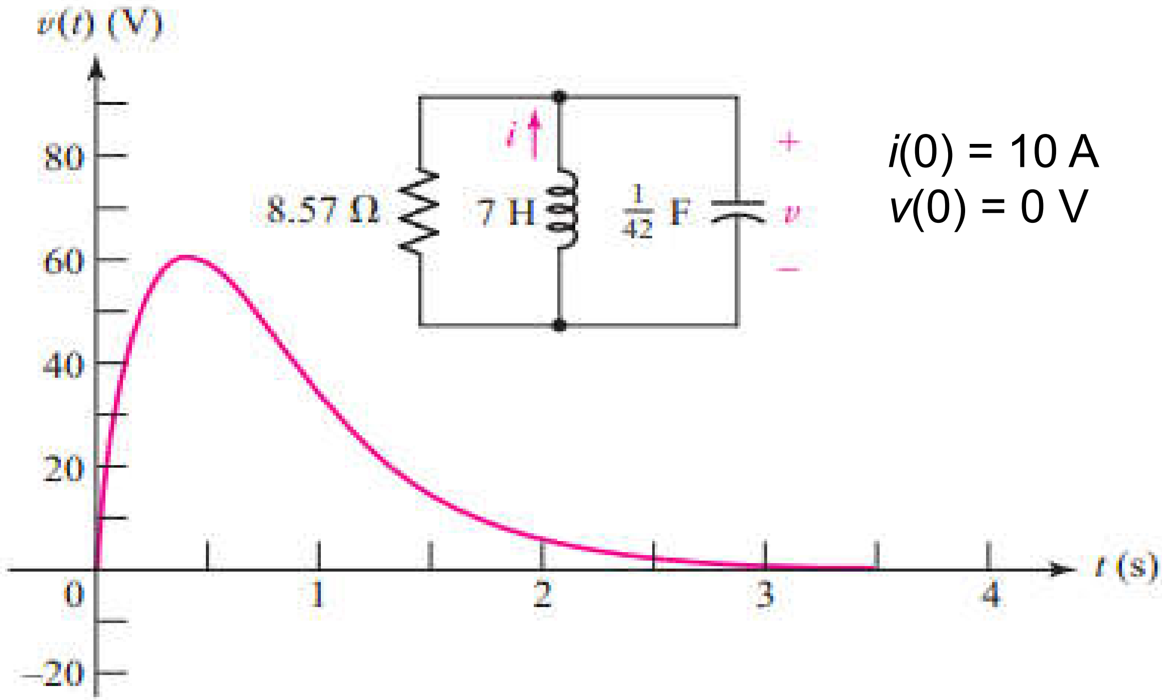


Parallel RLC Circuit - 2. Critical Damped System

- Critical Damped System: $\alpha = \omega_0$, implies $L = 4R^2C$.
- Since s_1 and s_2 are two equal real numbers, s_1=s_2=-α, the response is

$$v_n(t) = e^{-\alpha t}\left(A_1 t + A_2\right)$$

- $v_n(t)$ also decays and approach zero as time increases $t \to \infty$.



$i(0)$ = 10 A
$v(0)$ = 0 V

Parallel RLC Circuit - 3. Under Damped System

- Under Damped System: $\alpha < \omega_0$, implies $L < 4R^2C$.
- Since s_1 and s_2 are two complex numbers,

$$s_1 = -\alpha + j\sqrt{\omega_0^2 - \alpha^2},\; s_2 = -\alpha - j\sqrt{\omega_0^2 - \alpha^2}$$

- Define $\omega_d = \sqrt{\omega_0^2 - \alpha^2}$, the response now be written as

$$\begin{aligned} v_n(t) &= e^{-\alpha t}\left(A_1 e^{j\omega_d t} + A_2 e^{-j\omega_d t}\right) \\ &= e^{-\alpha t}\left[\underbrace{(A_1 + A_2)}_{B_1}\cos \omega_d t + \underbrace{j(A_1 - A_2)}_{B_2}\sin \omega_d t\right] \end{aligned}$$

- Therefore, $v_n(t) = e^{-\alpha t}\left(B_1 \cos \omega_d t + B_2 \sin \omega_d t\right)$

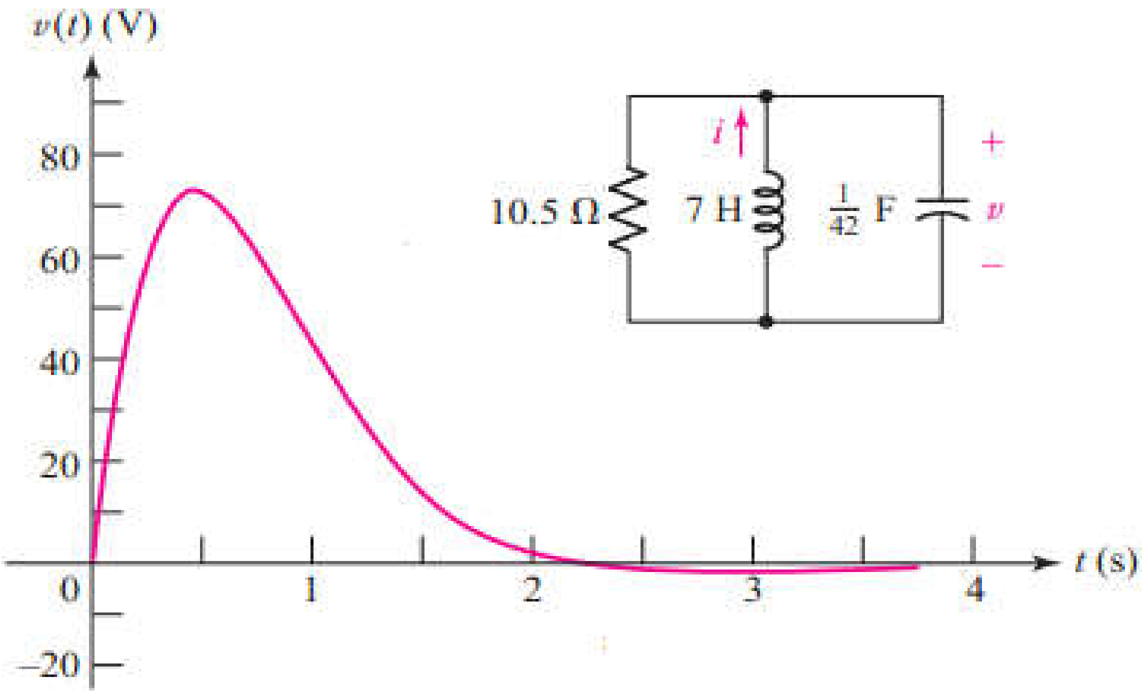
  Where B_1 and B_2 are determined by the initial conditions

Parallel RLC Circuit - 3. Under Damped System

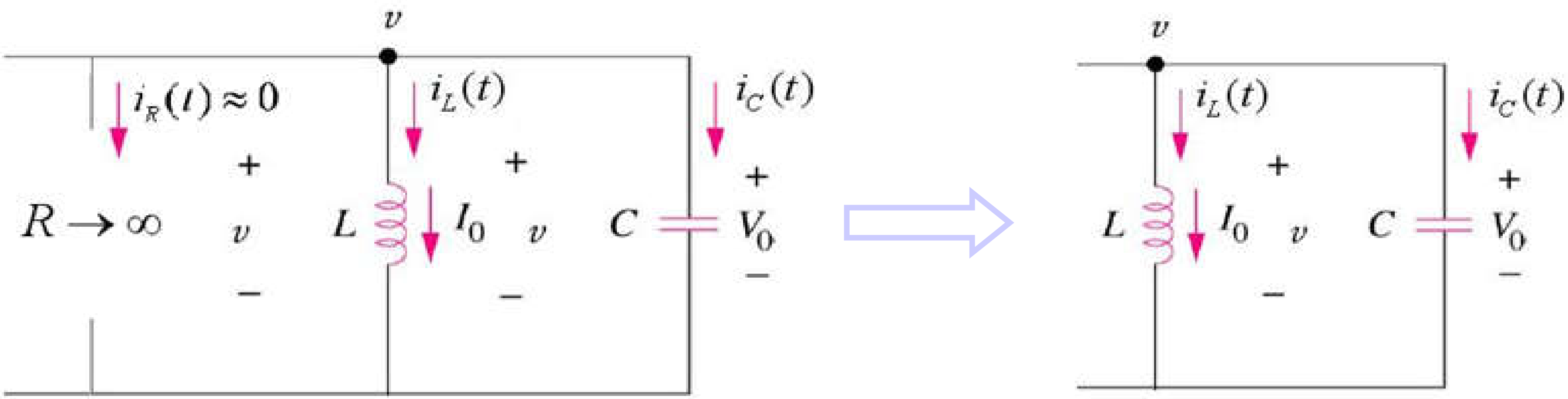
$$v_n(t) = e^{-\alpha t}\left(B_1 \cos \omega_d t + B_2 \sin \omega_d t\right)$$

Where B_1 and B_2 are determined by the initial conditions

$i(0) = 10$ A
$v(0) = 0$ V



Parallel RLC Circuit - Role of the Resistor



$$\alpha = \frac{1}{2RC}, \quad \omega_0 = \frac{1}{\sqrt{LC}}$$

when $R \to \infty,\ \alpha \to 0,\ \alpha << \omega_0,$

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2} \approx \omega_0$$

$\Longrightarrow$

$$v_n(t) = B_1 \cos \omega_0 t + B_2 \sin \omega_0 t$$

$$i_L(t) = K_1 \sin \omega_0 t + K_2 \cos \omega_0 t$$

Actual parallel RLC circuits can be made to have effective values of R so large that a natural ***undamped*** sinusoidal response can be maintained for years without supplying any additional energy.

Parallel RLC Circuit – Summary of solving procedure

- 1. Obtain the SODE: $\frac{d^2v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{1}{LC}v = 0$
- 2. With $\omega_0 = \frac{1}{\sqrt{LC}}$ and $\alpha = \frac{1}{2RC}$, evaluate the damped condition:
  - 1) Over Damped System: $\alpha > \omega_0$, s_1 and s_2 are two unequal real numbers, $v_n(t) = A_1 e^{S_1 t} + A_2 e^{S_2 t}$;
  - 2) Critical Damped System: $\alpha = \omega_0$, s_1 and s_2 are two equal real numbers α, $v_n(t) = e^{-\alpha t}\left(A_1 t + A_2\right)$
  - 3) Under Damped System: $\alpha < \omega_0$, s_1 and s_2 are two complex numbers $s_1 = -\alpha + j\sqrt{\omega_0^2 - \alpha^2}$, $s_2 = -\alpha - j\sqrt{\omega_0^2 - \alpha^2}$ and

$$v_n(t) = e^{-\alpha t}\left(B_1 \cos \omega_d t + B_2 \sin \omega_d t\right)$$

Parallel RLC Circuit – Summary of solving procedure

- 3. Determine coefficients A_1 and A_2 (or B_1 and B_2) according to the initial conditions $v_n(0^+)$ and $\frac{dv_n(0^+)}{dt}$:
  - 1) Over damped system:

$$v_n(0^+) = A_1 + A_2 \text{ , and } \frac{dv_n(0^+)}{dt} = A_1 s_1 + A_2 s_2$$

  - 2) Critical damped system:

$$v_n(0^+) = A_2 \text{ , and } \frac{dv_n(0^+)}{dt} = A_1 - A_2 \alpha$$

  - 3) Under damped system:

$$v_n(0^+) = B_1 \text{ , and } \frac{dv_n(0^+)}{dt} = -\alpha B_1 + \omega_d B_2$$

Parallel RLC Circuit - Example 1

In a parallel RLC circuit: $R = 500\ \Omega$, $C = 1\ \mu$F, $L = 0.2$ H. The initial conditions are $i_L(0) = 50$ mA and $v(0) = 0$ V.

Find i_L(t), i_R(t) and v_c(t).

Solution

$$\alpha = \frac{1}{2RC} = \frac{1}{2 \times 500 \times 1 \times 10^{-6}} = 10^3, \quad \omega_0^2 = \frac{1}{LC} = \frac{1}{0.2 \times 1 \times 10^{-6}} = 5 \times 10^6$$

$$\alpha^2 < \omega_0^2, underdamped$$

From the characteristic equation: $s^2 + 2\alpha s + \omega_0^2 = 0$, we have

$$s^2 + 2 \times 10^3 s + 5 \times 10^6 = 0$$

$$s_1 = -1000 + j2000, \quad s_2 = -1000 - j2000$$

The response of $i_L(t)$: $i_L(t) = e^{-\alpha t}\left(B_1 \cos \omega_d t + B_2 \sin \omega_d t\right)$

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2} = \sqrt{5 \times 10^6 - 10^6} = 2000$$

Parallel RLC Circuit - Example 1 (cont.)

$$i_L(0) = 1 \times \left(B_1 \cos 0 + B_2 \sin 0\right) = 50 \times 10^{-3} \text{ A} \;\Rightarrow\; B_1 = 50 \times 10^{-3}$$

$$v(0) = L \left.\frac{di_L(t)}{dt}\right|_{t=0} \;\Rightarrow\; -1000\,B_1 + 2000\,B_2 = 0$$

$$\Downarrow$$

$$B_2 = 25 \times 10^{-3}$$

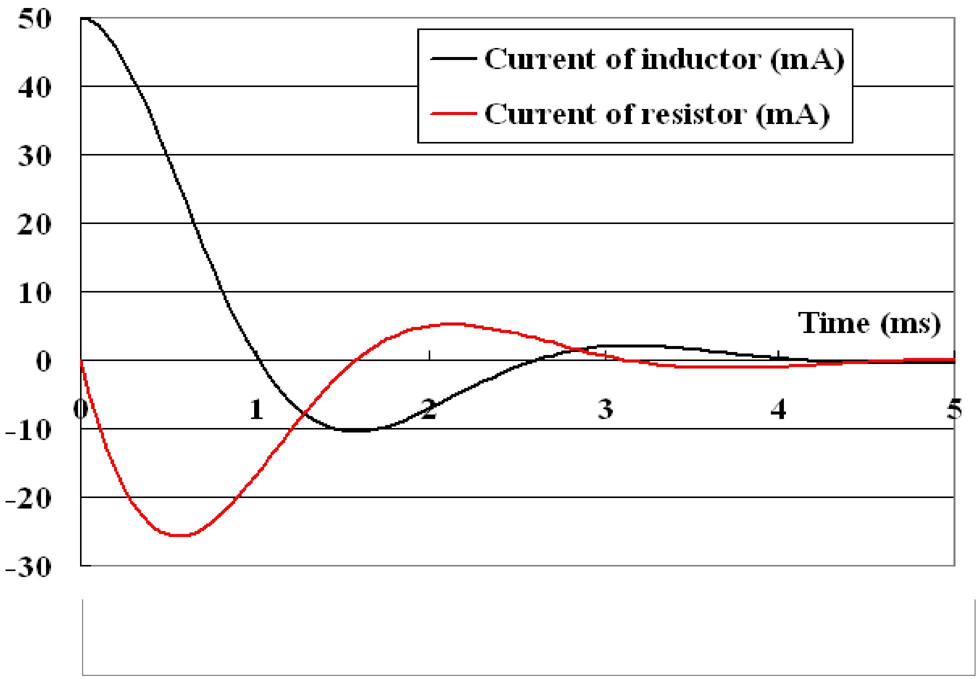
So $i_L(t) = e^{-1000t}\left(50 \cos 2000t + 25 \sin 2000t\right)$ mA $\quad t \geq 0$

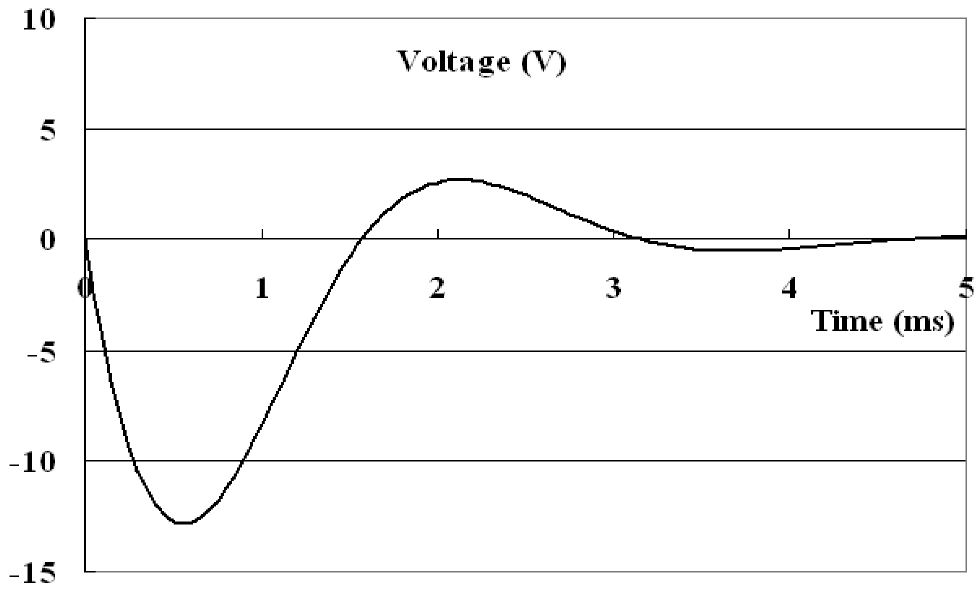
$$v_L(t) = L\frac{di_L}{dt} = -25e^{-1000t} \sin 2000t \text{ V} \qquad t \geq 0$$

$$v_C(t) = v_L(t) \qquad t \geq 0$$

$$i_R = \frac{v_R(t)}{R} = \frac{v_L(t)}{R} = -50e^{-1000t} \sin 2000t \text{ mA} \qquad t \geq 0$$

Parallel RLC Circuit - Example 1 (cont.)





Quiz

- If the roots of the characteristic equation of an RLC circuit are -2 and -3, the response is:
  - (a) $(A\cos 2t + B\sin 2t)e^{-3t}$
  - (b) $(A + 2Bt)e^{-3t}$
  - (c) $Ae^{-2t} + Bte^{-3t}$
  - (d) $Ae^{-2t} + Be^{-3t}$

- A parallel RLC circuit has $L = 4$ H and $C = 0.25$ F. The value of R that will produce under damping factor is:
  - (a) 0.5 Ω; (b) 1 Ω;
  - (c) 2 Ω; (d) 4 Ω.

Series RLC Circuit – Obtaining the SODE

By using KVL : $v_R + v_C + v_L = v_s$

The current flowing in the circuit :

$$i = C\frac{dv_C}{dt}$$

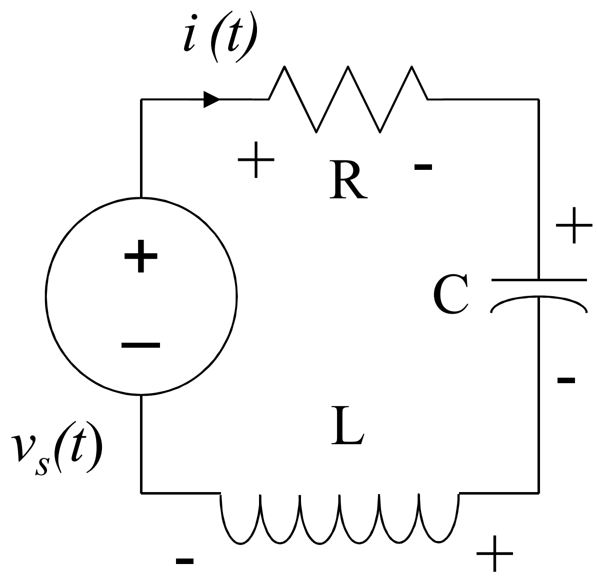
The voltage v_R and v_L are given by

$$v_R = iR = RC\frac{dv_C}{dt}$$

$$v_L = L\frac{di}{dt} = LC\frac{d^2 v_C}{dt^2}$$

Then we have :

$$\boxed{\frac{d^2 v_C}{dt^2} + \frac{R}{L}\frac{dv_C}{dt} + \frac{1}{LC}v_c = \frac{1}{LC}v_s \qquad (1)}$$



Simpler we have:

$$v_C = v_{C,p} + v_{C,h}$$

where

v_C : the complete solution

$v_{C,p}$: the particular solution

$v_{C,h}$: the homogeneous solution

Series RLC Circuit – Solving the SODE

$$\frac{d^2 v_{C,h}}{dt^2} + \frac{R}{L}\frac{dv_{C,h}}{dt} + \frac{1}{LC}v_{C,h} = 0 \qquad (2)$$

Assuming a homogeneous solution has the following form :

$v_{C,h} = Ae^{st} \qquad (A \neq 0)$

A is a constant determined by initial conditions.

s is a constant determined by the coefficients of the differential equation.

Substituting $v_{c,h} = Ae^{st}$ into Eq. (2) :

$$As^2e^{st} + \frac{R}{L}Ase^{st} + \frac{A}{LC}e^{st} = 0 \quad or$$

$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0 \qquad (3)$ usually called the *characteristic equation*

Series RLC Circuit – Solving the SODE

Two solutions obtained for

the *characteristic equation* :

$$s_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

$$s_2 = -\frac{R}{2L} - \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

Replace s by s_1, and s_2 :

$$v_{C,h1} = A_1 e^{s_1 t} \qquad v_{C,h2} = A_2 e^{s_2 t}$$

Both $v_{C,h1}$, and $v_{C,h2}$ satisfy the Eq. (2),

so we have $\quad v_{C,h} = A_1 e^{s_1 t} + A_2 e^{s_2 t} \quad (4)$

The two constants A_1 and A_2 can be obtained by two initial conditions :

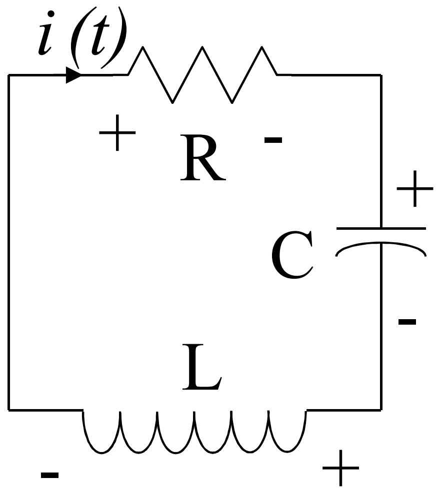
$$v_{C,h}(0) = V_0 \text{ and } \frac{dv_{C,h}(0)}{dt} = \frac{1}{C} i(0) = \frac{I_0}{C}$$

From $v_{C,h} = A_1 e^{s_1 t} + A_2 e^{s_2 t} \quad (4)$

with $v_{C,h}(0) = V_0$,

and by $\frac{dv_{C,h}(0)}{dt} = \frac{1}{C} i(0) = \frac{I_0}{C}$,
$$\begin{cases} A_1 + A_2 = V_0 \\ A_1 s_1 + A_2 s_2 = \dfrac{I_0}{C} \end{cases}$$

$$A_1 = \frac{V_0 s_2 - \dfrac{I_0}{C}}{s_2 - s_1} \qquad A_2 = \frac{\dfrac{I_0}{C} - V_0 s_1}{s_2 - s_1}$$



Series RLC Circuit – SODE of $i_L(t)$

By using KVL $v_R + v_C + v_L = 0$,

instead of $i(t) = C\dfrac{dv_c(t)}{dt}$

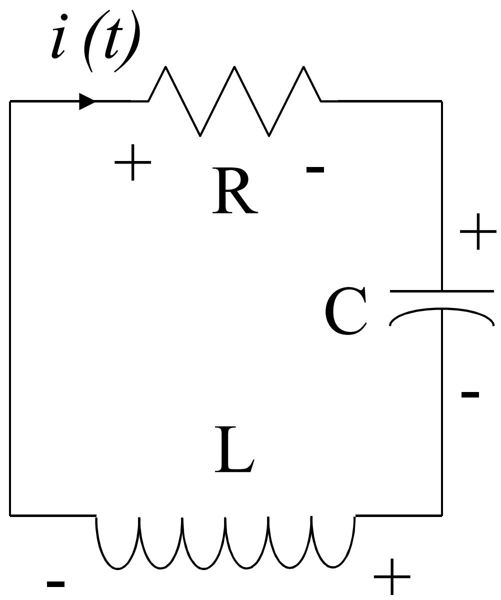
we use $v_C(t) = \dfrac{q(t)}{C} = \dfrac{1}{C}\int_{-\infty}^{t} i(t)dt$

we can have:

$$iR + L\frac{di}{dt} + \frac{1}{C}\int_{-\infty}^{t} idt = 0$$

$$\Downarrow$$

$$\boxed{\frac{d^2i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{1}{LC}i = 0} \Leftrightarrow \boxed{\frac{d^2v_C}{dt^2} + \frac{R}{L}\frac{dv_C}{dt} + \frac{1}{LC}v_c = 0 \quad (1)}$$



Series RLC Circuit – Damping factor

$$s_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} \qquad s_2 = -\frac{R}{2L} - \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

Definition of α and ω_0 :

$\alpha = \frac{\mathrm{R}}{2\mathrm{L}}$, called damping factor

$\omega_0 = \frac{1}{\sqrt{LC}}$ $(rad/\sec)$: Called the undamped natural frequency,

or the resonant frequency

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2} \qquad s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$$

Series RLC Circuit – Damping factor

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2} \quad s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$$

The value of the term $\sqrt{\alpha^2 - \omega_0^2}$ determines the behavior of the response:

1. Over Damped System: $\alpha > \omega_0$, s_1 and s_2 are two unequal real numbers

2. Critically Damped System: $\alpha = \omega_0$, s_1 and s_2 are two equal real numbers

3. Under Damped System: $\alpha < \omega_0$, s_1 and s_2 are complex numbers:

$$s_1 = -\alpha + j\sqrt{\omega_0^2 - \alpha^2}, s_2 = -\alpha - j\sqrt{\omega_0^2 - \alpha^2}$$

System exhibits oscillatory behavior

Special case: $R = 0 \Rightarrow \alpha = 0$

$$s_1 = j\omega_0, \quad s_2 = -j\omega_0$$

Series RLC Circuit – Over damped

$$\alpha = \frac{R}{2L}, \quad \omega_0 = \frac{1}{\sqrt{LC}}$$

1. Over Damped System:

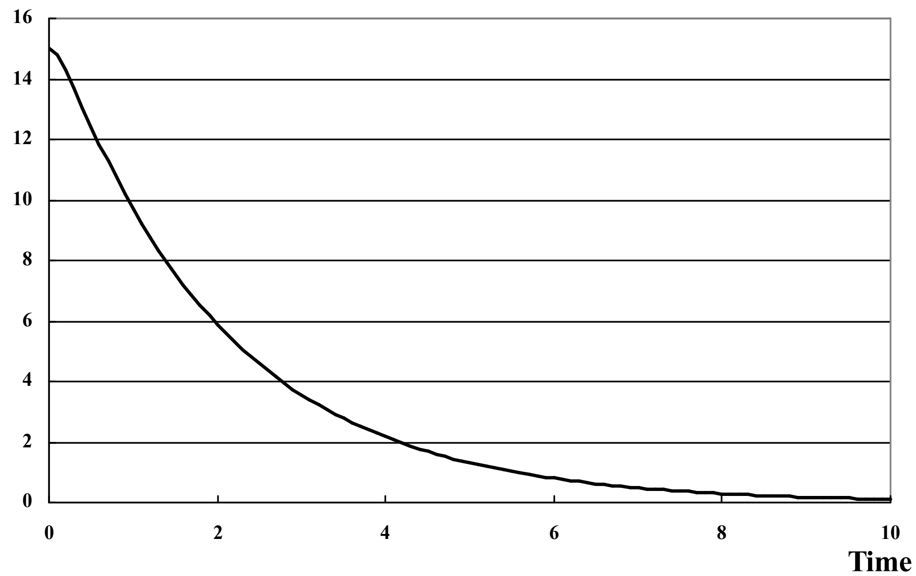
$\alpha > \omega_0$, implies $C > 4L/R^2$

s_1 and s_2 are two unequal real numbers

The response is:

$$\boxed{v_C(t) = A_1 e^{S_1 t} + A_2 e^{S_2 t}}$$

$v_C(t)$ decays and approaches zero as time increases.



Series RLC Circuit – Critical damped

When $\alpha = \omega_0,\ R = 2\sqrt{\frac{L}{C}},\ \Rightarrow\ C = 4L/R^2$ and $s_1 = s_2 = -\alpha = -\frac{R}{2L}$, the differential equation

$$\frac{d^2 v_{C,h}}{dt^2} + \frac{R}{L}\frac{dv_{C,h}}{dt} + \frac{1}{LC}v_{C,h} = 0 \quad (2) \quad \text{becomes}: \frac{d^2 v_{C,h}}{dt^2} + 2\alpha\frac{dv_{C,h}}{dt} + \alpha^2 v_{C,h} = 0 \quad (2a)$$

Then we have $$\frac{d}{dt}\left(\frac{dv_{C,h}}{dt} + \alpha v_{C,h}\right) + \alpha\left(\frac{dv_{C,h}}{dt} + \alpha v_{C,h}\right) = 0 \quad (3)$$

Let $f = \frac{dv_{C,h}}{dt} + \alpha v_{C,h}$ (4), the equation (3) becomes: $\frac{df}{dt} + \alpha f = 0$ (5)

For the first-order equation (5), we have $f = A_1 e^{-\alpha t}$

Then: $\frac{dv_{C,h}}{dt} + \alpha v_{C,h} = A_1 e^{-\alpha t} \quad \Rightarrow \quad e^{\alpha t}\frac{dv_{C,h}}{dt} + e^{\alpha t}\alpha v_{C,h} = A_1$

$$\Rightarrow \frac{d(e^{\alpha t} v_{C,h})}{dt} = A_1 \Rightarrow e^{\alpha t} v_{C,h} = A_1 t + A_2$$

The solution finally is: $\boxed{v_{C,h}(t) = e^{-\alpha t}\left(A_1 t + A_2\right)}$

Series RLC Circuit – Under damped

$$\alpha = \frac{R}{2L}, \quad \omega_0 = \frac{1}{\sqrt{LC}}$$

When $\alpha < \omega_0, \;\; C < 4L/R^2, \;\;$ we define

$\omega_d = \sqrt{\omega_0^2 - \alpha^2}$ called *natural resonant frequency*

$$s_1 = -\alpha + \sqrt{-\left(\omega_0^2 - \alpha^2\right)} = -\alpha + j\omega_d$$

$$s_2 = -\alpha - \sqrt{-\left(\omega_0^2 - \alpha^2\right)} = -\alpha - j\omega_d$$

The solution may be:

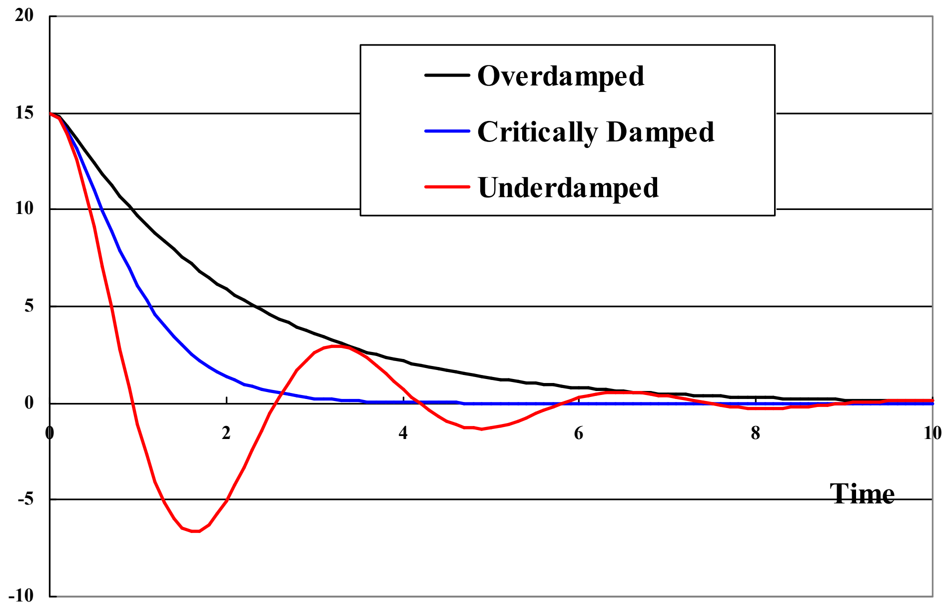
$$v_{C,h}(t) = A_1 e^{-(\alpha - j\omega_d)t} + A_2 e^{-(\alpha + j\omega_d)t}$$

$$= e^{-\alpha t}\left(A_1 e^{j\omega_d t} + A_2 e^{-j\omega_d t}\right)$$

The final form is:

$$\boxed{v_{C,h}(t) = e^{-\alpha t}\left(B_1 \cos\omega_d t + B_2 \sin\omega_d t\right)}$$

The constants of B_1 and B_2 are also determined by initial conditions



Series RLC Circuit – Example

The circuit below has $C = 0.25\ \mu\text{F}$ and $L = 1\ \text{H}$.

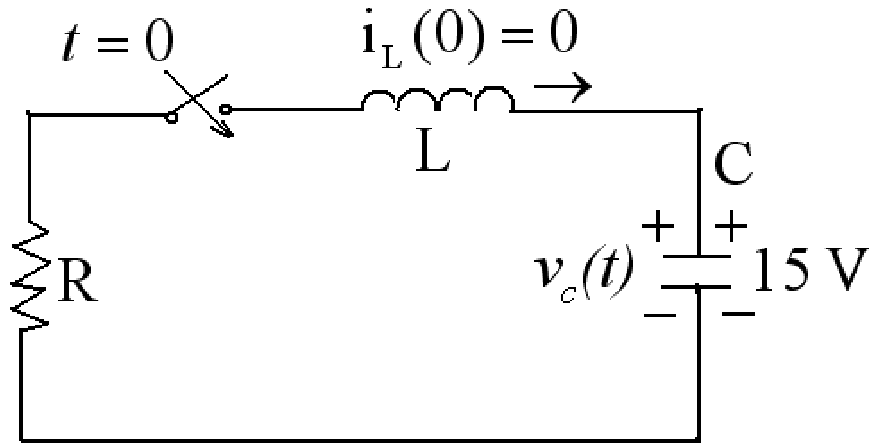
The switch has been open for a long time and is closed at $t = 0$.

Find the capacitor voltage for $t \geq 0$ for

1. $R = 8.5\ \text{k}\Omega$
2. $R = 4\ \text{k}\Omega$ and
3. $R = 1\ \text{k}\Omega$.

The initial conditions are

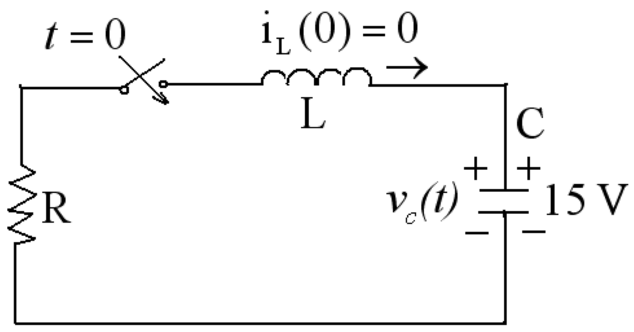
$I_0 = 0$ and $V_0 = 15\ \text{V}$.



Series RLC Circuit – Step of solving 2^{nd} problems

- 0. Get the SODE of the circuit;
- 1. Find out the initial conditions $i_L(0^+)$, $v_C(0^+)$, $v_L(0^+)$, $i_C(0^+)$;
- 2. Get the characteristic equation from SODE (SODE -> s);
- 3. Evaluate α and ω_0;
  - Compare $\alpha^2 - \omega_0^2$ with 0, determine the damping case, which determines the general form of the solution.
- 4. Solve for s_1 and s_2, substitute into the general form of the solution;
- 5. using initial conditions to solve for coefficients A_1, A_2 or B_1, B_2;
- 6. Get final expression of $v_C(t)$ or $i_L(t)$.

Series RLC Circuit – Example solution



$C = 0.25\mu F\ ,\ L = 1H$

1. $R = 8.5\ \text{k}\Omega$

The characteristic equation

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0 \qquad (3)$$

$$s^2 + \frac{8.5\times 10^3}{1}s + \frac{1}{1\times 0.25\times 10^{-6}} = 0$$

$$0.25\times 10^{-6}s^2 + 2.125\times 10^{-3}s + 1 = 0$$

$$s_1 = -500, \qquad s_2 = -8000$$

s_1 and s_2 are two unequal real numbers

$$v_{C,h}(t) = A_1e^{-500t} + A_2e^{-8000t} \qquad t \geq 0$$

By initial conditions :

$v_{c,h}(0) = 15$ V and $i_L(0) = 0$

$15 = A_1 + A_2$

$$\frac{dv_C(0)}{dt} = \frac{i_L(0)}{C} = 0$$

$$\begin{cases} A_1 + A_2 = 15 \\ -500A_1 - 8000A_2 = 0 \end{cases}$$

$A_1 = 16$ and $A_2 = -1$, so we have

$$v_{C,h}(t) = 16e^{-500t} - e^{-8000t} \quad \text{V} \qquad t \geq 0$$

s_1 and s_2 are two unequal real numbers

-- Over Damped System

Series RLC Circuit – Example solution (cont.)

$$s^2+\frac{R}{L}s+\frac{1}{LC}=0 \qquad (3)$$

$$s^2+\frac{4\times10^3}{1}s+\frac{1}{1\times0.25\times10^{-6}}=0$$

$$0.25\times10^{-6}s^2+10^{-3}s+1=0$$

$$s_1=s_2=-2000$$

$$v_{c,h}(t)=A_1e^{-2000t}+A_2te^{-2000t} \qquad t\ge0$$

By initial conditions $:v_{c,h}(0)=15\ V$ and $i_L(0)=0$

$$15=A_1$$

$$\frac{dv_c(0)}{dt}=\frac{i_L(0)}{C}=0$$

$$\begin{cases} A_1=15 \\ -2000A_1-A_2=0 \end{cases}$$

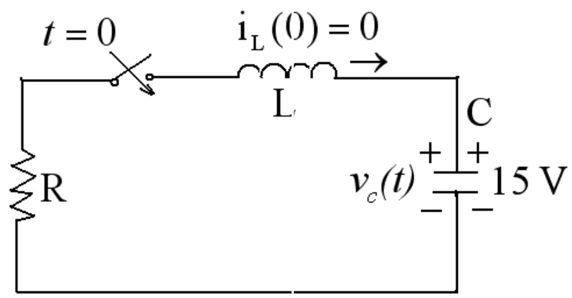
$A_1=15$ and $A_2=15\times2000$, so we have

$$v_{c,h}(t)=15e^{-2000t}+15(2000t)e^{-2000t}\ \text{V} \qquad t\ge0$$

s_1 and s_2 are two equal real numbers -- Critically Damped System

$C=0.25\mu F$, $L=1H$

2. $R=4\ \text{k}\Omega$



Series RLC Circuit – Example solution (cont.)

The characteristic equation

$$s^2 + \frac{1\times10^3}{1}s + \frac{1}{1\times0.25\times10^{-6}} = 0$$

$$0.25\times10^{-6}s^2 + 0.25\times10^{-3}s + 1 = 0$$

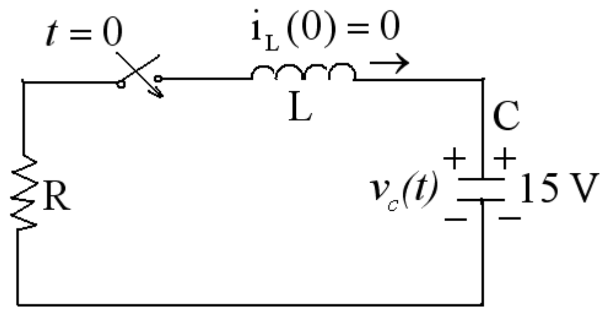
$$s_1 = -500 + j500\sqrt{15}, \quad s_2 = -500 - j500\sqrt{15}$$

$$v_{C,h}(t) = e^{-\alpha t}(B_1\cos\omega_d t + B_2\sin\omega_d t) \qquad t \ge 0$$

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2} = \sqrt{(\frac{1}{\sqrt{LC}})^2 - (\frac{R}{2L})^2} = 500\sqrt{15}$$

$C = 0.25\mu F$, $L = 1H$

3. $R = 1\ \text{k}\Omega$



By initial conditions:

$$v_{C,h}(0) = 15\ \text{V and } i_L(0) = 0$$

$$15 = B_1$$

$$\frac{dv_{C,h}(0)}{dt} = \frac{i_L(0)}{C} = 0$$

$$\begin{cases} B_1 = 15 \\ -500B_1 + 500\sqrt{15}B_2 = 0 \Rightarrow B_2 = \sqrt{15} \end{cases}$$

so we have $v_{C,h}(t) = e^{-500t}(15\cos 500\sqrt{15}t + \sqrt{15}\sin 500\sqrt{15}t)$ V $\qquad t \ge 0$

s_1 and s_2 are complex numbers -- Under Damped System

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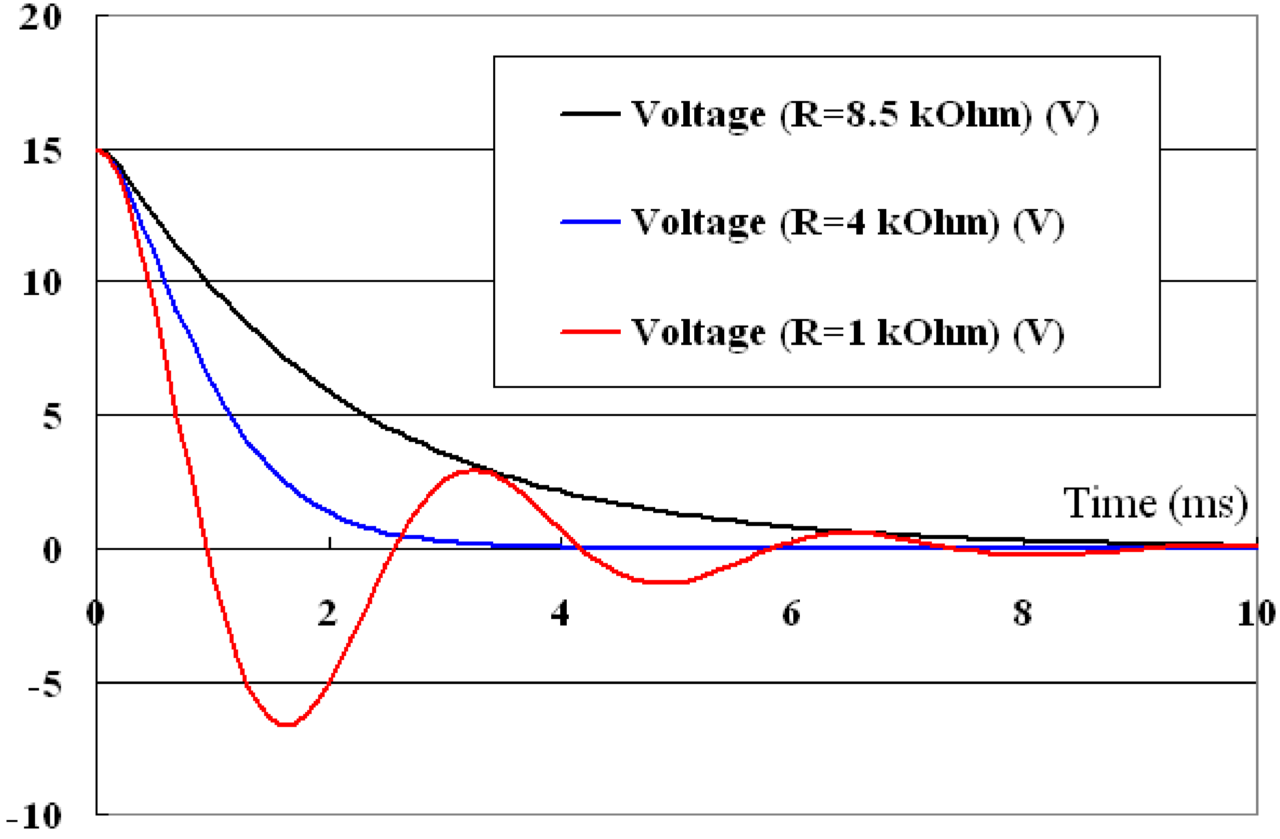
Series RLC Circuit – Example solution (cont.)



Series RLC Circuit – Example solution (cont.)

$C = 0.25\mu F$, $L = 1H$

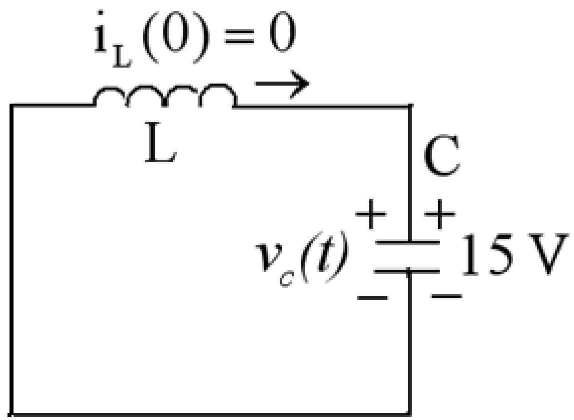
R=0

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0 \quad (3) \rightarrow \quad s^2 + \frac{1}{LC} = 0 \quad (3a)$$

$$s_{1,2} = \pm\sqrt{-\frac{1}{LC}} = \pm j\omega_0$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{1\times 0.25\times 10^{-6}}} = 20$$

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2} = \sqrt{\left(\frac{1}{\sqrt{LC}}\right)^2 - \left(\frac{R}{2L}\right)^2} = \omega_0$$



$$v_{C,h}(t) = e^{-\alpha t}(B_1 \cos\omega_d t + B_2 \sin\omega_d t) \qquad t \geq 0$$

$$\rightarrow v_{C,h}(t) = B_1 \cos\omega_0 t + B_2 \sin\omega_0 t \qquad t \geq 0$$

$$\boxed{\omega_d = \omega_0}$$

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Series RLC Circuit – Example solution (cont.)

By initial conditions: $v_{C,h}(0) = 15\ V$ and $i_L(0) = 0$

$C = 0.25\mu F$, $L = 1H$

R=0

$15 = B_1 \qquad\qquad B_1 = 15$

$$\frac{dv_{C,h}(0)}{dt} = \frac{i_L(0)}{C} = 0 \qquad B_2 = 0$$

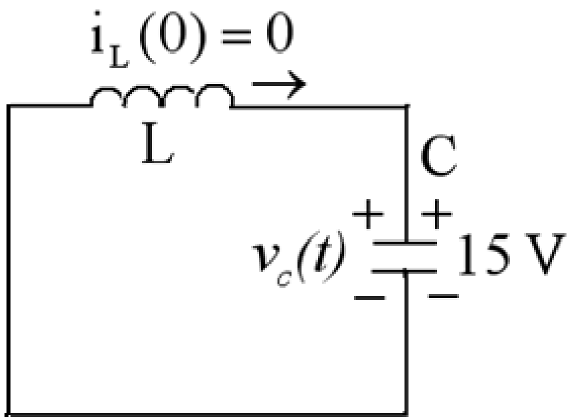
so we have

$$v_{C,h}(t) = 15\cos 20t \ \text{ V} \qquad t \geq 0$$

$$= V_0 \cos\omega_0 t \quad \text{V}$$

$$i_{L,h} = C\frac{dv_{C,h}}{dt} = -V_0 C\omega_0 \sin(\omega_0 t) \quad \text{A} \quad t \geq 0$$

$$v_{L,h} = L\frac{di_{L,h}}{dt} = -V_0 \cos\omega_0 t \quad \text{V} \qquad t \geq 0$$

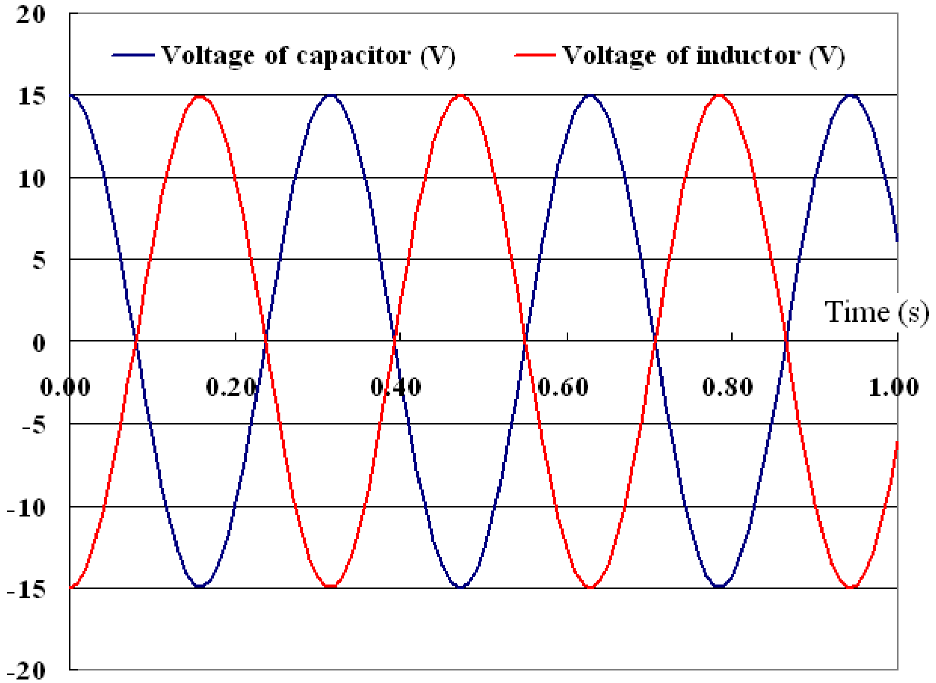


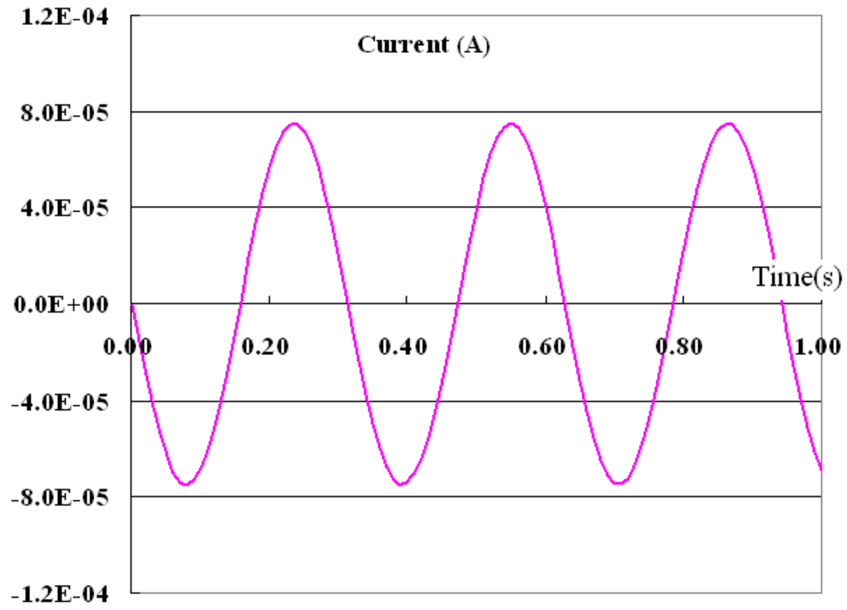
Series RLC Circuit – Example solution (cont.)

$C = 0.25\mu\text{F} \quad L = 1\text{H}$

$I_0 = 0 \quad V_0 = 15\text{ V}$

R=0





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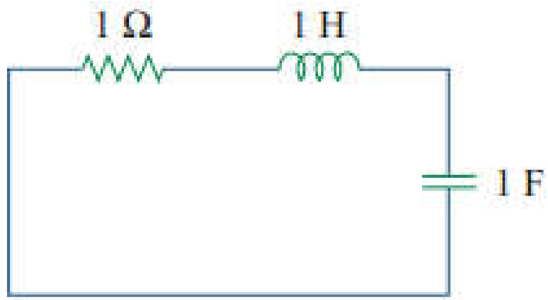
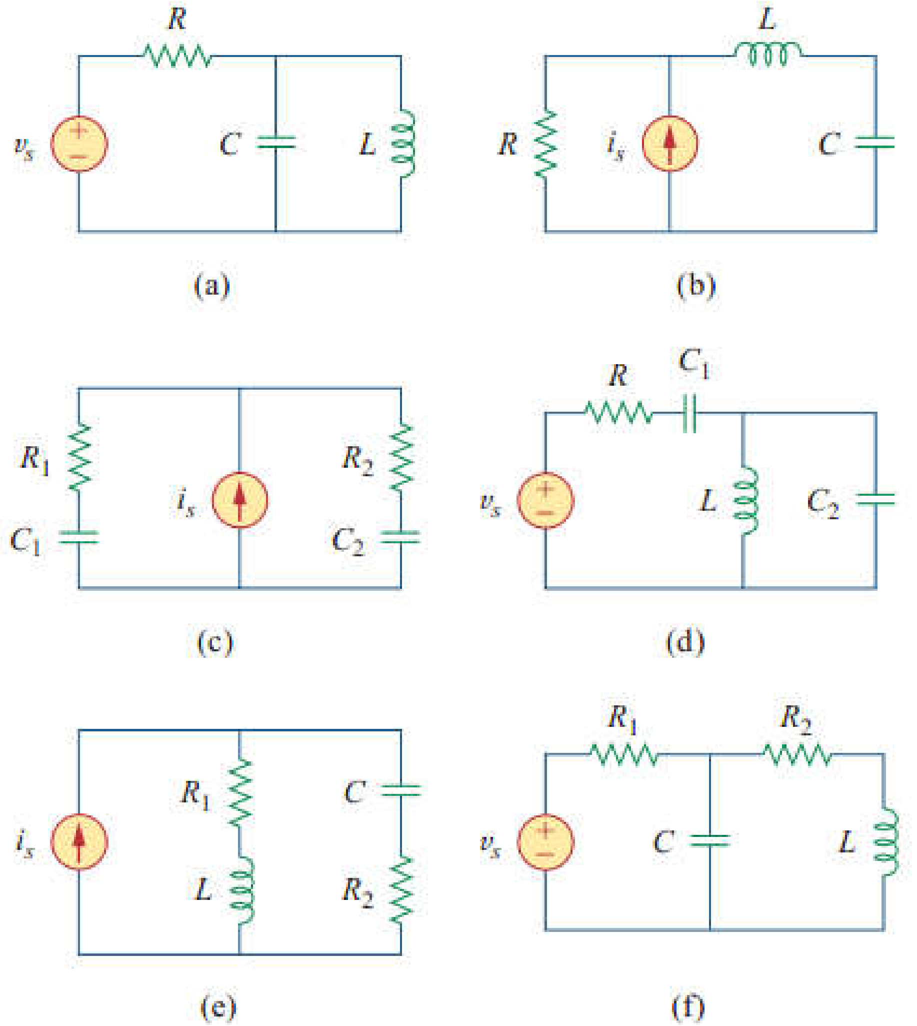
Quiz



- Refer to the series RLC circuit, what kind of response will it produce?
  - (a) over damped;
  - (b) under damped;
  - (c) critically damped;
  - (d) none of the above.



- Match the circuits shown on the right with the following items:
  - (i) first-order circuit
  - (ii) second-order series circuit
  - (iii) second-order parallel circuit
  - (iv) none of the above

Second Order Circuit – Source free summary

	Series	Parallel
α	$\alpha = \dfrac{R}{2L}$	$\alpha = \dfrac{1}{2RC}$
ω_0	$\dfrac{1}{\sqrt{LC}}$	$\dfrac{1}{\sqrt{LC}}$
Over Damped	$\alpha > \omega_0$ Response: $A_1 e^{s_1 t} + A_2 e^{s_2 t}$	
Critically Damped	$\alpha = \omega_0$ Response: $e^{-\alpha t}(A_1 t + A_2)$	
Under Damped	$\alpha < \omega_0$ Response: $e^{-\alpha t}(B_1 \cos \omega_d t + B_2 \sin \omega_d t)$	
Undamped	R=0	$R \rightarrow \infty$
	Response: $B_1 \cos \omega_0 t + B_2 \sin \omega_0 t$	

Complete response

- The complete response of a second order system consists of a forced response $x_f(t)$ and a natural response $x_n(t)$ which is the same form as source-free natural response.)
- If the input is not zero, we need to find the complete solution $x(t)$

$$\underbrace{x(t) = x_f(t)}_{\text{Forced}} + \underbrace{x_n(t)}_{\text{natural}}$$

- Remember the output follows the form of the input

input function	Constant	Exponential	Sinusoid
particular solution	A	$Ae^{-\alpha t} + Bte^{-\alpha t}$	$A\cos(\omega t) + B\sin(\omega t)$

Step-function ⇨ Step-response

Step response

- Now consider those RLC circuits in which dc sources are switched into the network and produce forced responses that do not necessarily vanish as time becomes infinite. – ***Step response***.
- The basic steps are (not necessarily in this order) as follows:
  - 1. Determine the initial conditions.
  - 2. Obtain a numerical value for the forced response.
  - 3. Write the appropriate form of the natural response with the necessary number of arbitrary constants.
  - 4. Add the forced response and natural response to form the complete response.
  - 5. Evaluate the response and its derivative at $t = 0$, and employ the initial conditions to solve for the values of the unknown constants.

Step response – parallel RLC circuit

By applying KCL at the indicated node:

$$i_s(t) = i_R(t) + i_L(t) + i_C(t)$$

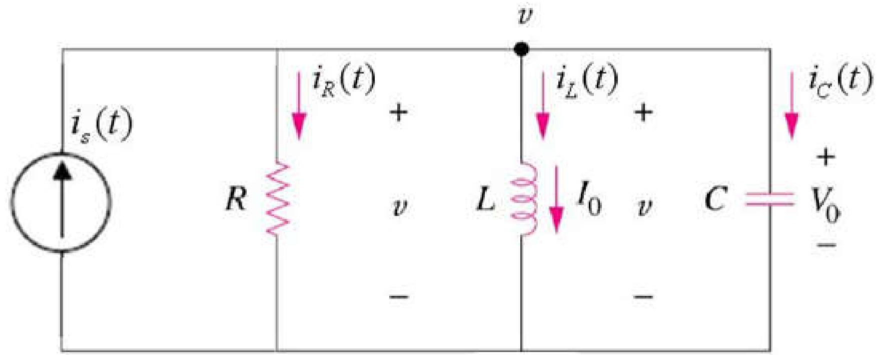
The voltage across the inductor: $v = L\dfrac{di_L}{dt}$

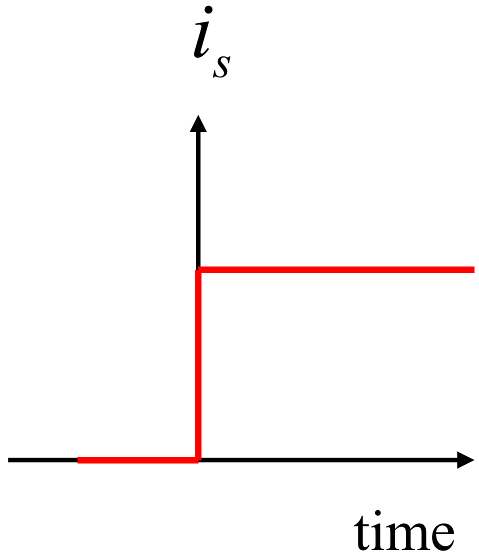
The currents $i_R(t)$ and $i_C(t)$:

$$i_R(t) = \frac{v}{R} = \frac{L}{R}\frac{di_L}{dt}, \qquad i_C(t) = C\frac{dv}{dt} = LC\frac{d^2 i_L}{dt^2}$$

Then we can have:

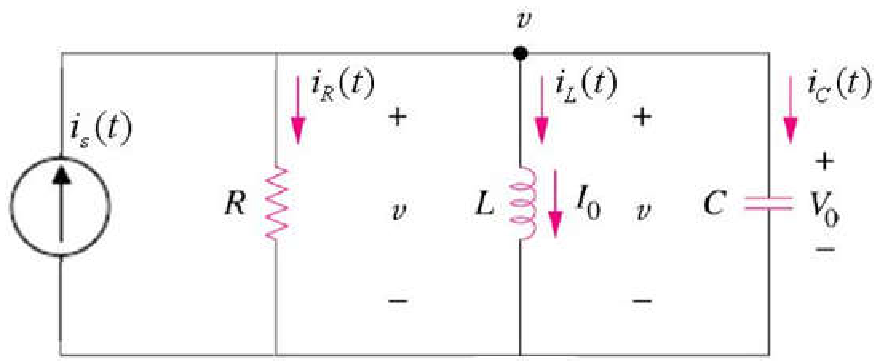
$$\boxed{\frac{d^2 i_L}{dt^2} + \frac{1}{RC}\frac{di_L}{dt} + \frac{1}{LC} i_L = \frac{1}{LC} i_s \qquad (1)}$$





Step response – *parallel RLC circuit*

$$\frac{d^2 i_L}{dt^2}+\frac{1}{RC}\frac{di_L}{dt}+\frac{1}{LC}i_L=\frac{1}{LC}i_s \qquad (1)$$



The solution is :

$$i_L(t)=i_{L,f}(t)+i_{L,n}(t)$$

The natural response is the same as the source free case, the forced response of the current should be the value at t = ∞, in this case, i_s

$$i_L(t)=i_s+A_1e^{s_1t}+A_2e^{s_2t} \longrightarrow \text{Over damped}$$

$$i_L(t)=i_s+(A_1+A_2t)e^{-\alpha t} \longrightarrow \text{Critical damped}$$

$$i_L(t)=i_s+(A_1\cos\omega_d t+A_2\sin\omega_d t)e^{-\alpha t} \longrightarrow \text{Under damped}$$

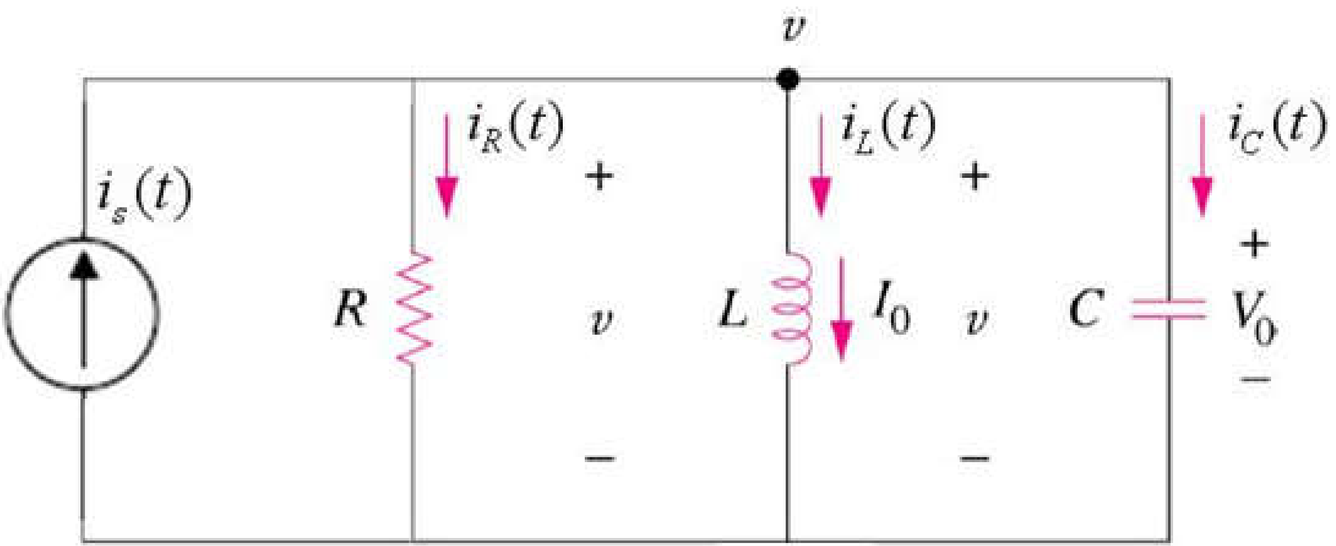
Step response – Example 1 (parallel RLC circuit)

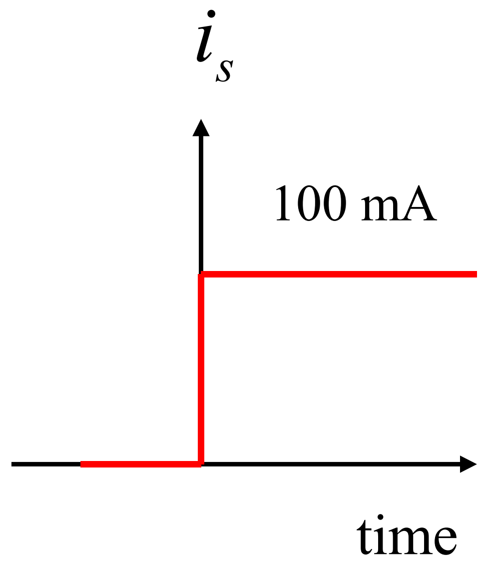
In a parallel RLC circuit $R = 500\Omega,\ C = 1\mu\text{F},\ L = 0.2\text{H}$

The initial conditions are $i_L(0) = 50$ mA and $v(0) = 0$

Input current $i_s = 100$ mA

Find the response of $i_L(t), i_R(t)$ and $v_C(t)$





Recall – parallel RLC circuit (source free) – Example 1

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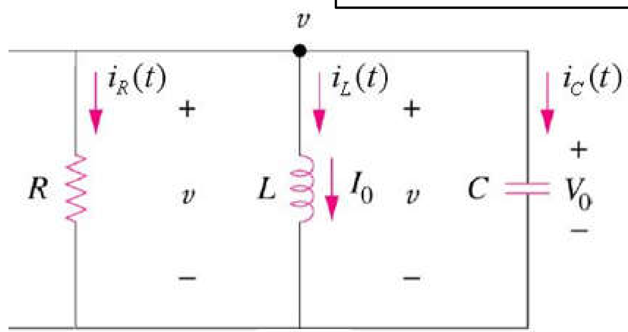
In a parallel RLC circuit: $R = 500\ \Omega$, $C = 1\mu F$, $L = 0.2H$. The initial conditions are $i_L(0) = 50mA$ and $v(0)=0$.

Find the zero-input response of $i_L(t)$, $i_R(t)$ and $v_c(t)$.



Solution

$$\alpha = \frac{1}{2RC} = \frac{1}{2\times 500\times 1\times 10^{-6}} = 10^3\ ,\ \omega_0^2 = \frac{1}{LC} = \frac{1}{0.2\times 1\times 10^{-6}} = 5\times 10^6$$

$$\alpha^2 < \omega_0^2, underdamped$$

From the characteristic equation : $s^2 + 2\alpha s + \omega_0^2 = 0$, we have

$$s^2 + 2\times 10^3 s + 5\times 10^6 = 0$$

$$s_1 = -1000 + j2000,\ \ s_2 = -1000 - j2000$$

The response of $i_L(t) : i_L(t) = e^{-\alpha t}\left(B_1 \cos \omega_d t + B_2 \sin \omega_d t\right)$

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2} = \sqrt{5\times 10^6 - 10^6} = 2000$$

$$i_L(0) = 1\times\left(B_1 \cos 0 + B_2 \sin 0\right) = 50\times 10^{-3}\ \text{A} \Rightarrow \quad B_1 = 50\times 10^{-3}$$

?

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Step response – Example 1 solution

The response of $i_{L,n}(t)$: $i_{L,n}(t) = e^{-\alpha t}\left(B_1 \cos\omega_d t + B_2 \sin\omega_d t\right)$

The complete response : $i_L(t) = i_{L,f}(t) + i_{L,n}(t)$

$i_L(t) = i_s + e^{-\alpha t}\left(B_1 \cos\omega_d t + B_2 \sin\omega_d t\right)$

Now let's determine the constants B_1 and B_2

$i_L(0) = 50\text{ mA} \quad i_s + B_1 = 50\text{ mA} \quad \Rightarrow \underline{B_1} = 50\times10^{-3} - 100\times10^{-3} = \underline{-50\times10^{-3}}$

$$\frac{di_L(t)}{dt} = -\alpha e^{-\alpha t}\left(B_1 \cos\omega_d t + B_2 \sin\omega_d t\right) + e^{-\alpha t}\left(-B_1\omega_d \sin\omega_d t + B_2\omega_d \cos\omega_d t\right)$$

$$v(0) = L\left.\frac{di_L(t)}{dt}\right|_{t=0} = 0 \qquad \Rightarrow \qquad -\alpha B_1 + B_2\omega_d = 0$$

$$\Downarrow$$

$$\underline{B_2} = \frac{\alpha}{\omega_d}B_1 = \frac{1000}{2000}(-50\times10^{-3}) = \underline{-25\times10^{-3}}$$

Step response – Example 1 solution (cont.)

Step Response

$$i_L(t) = 100 + e^{-1000t}\left(-50\cos 2000t - 25\sin 2000t\right) \quad \text{mA} \quad t \geq 0$$

$$v_L(t) = L\frac{di_L}{dt} = 25e^{-1000t}\sin 2000t \quad \text{V} \quad t \geq 0$$

$$v_C(t) = v_L(t) \quad t \geq 0$$

$$i_R = \frac{v_R(t)}{R} = \frac{v_L(t)}{R} = 50e^{-1000t}\sin 2000t \quad \text{mA} \quad \text{t} \geq 0$$

Source Free

$$i_L(t) = e^{-1000t}\left(50\cos 2000t + 25\sin 2000t\right) \quad \text{mA} \quad t \geq 0$$

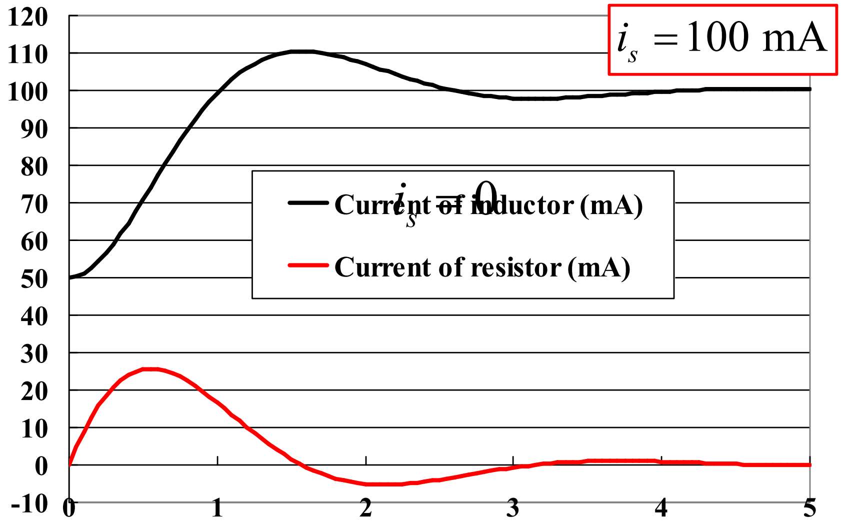
$$v_L(t) = L\frac{di_L}{dt} = -25e^{-1000t}\sin 2000t \quad \text{V} \quad t \geq 0$$

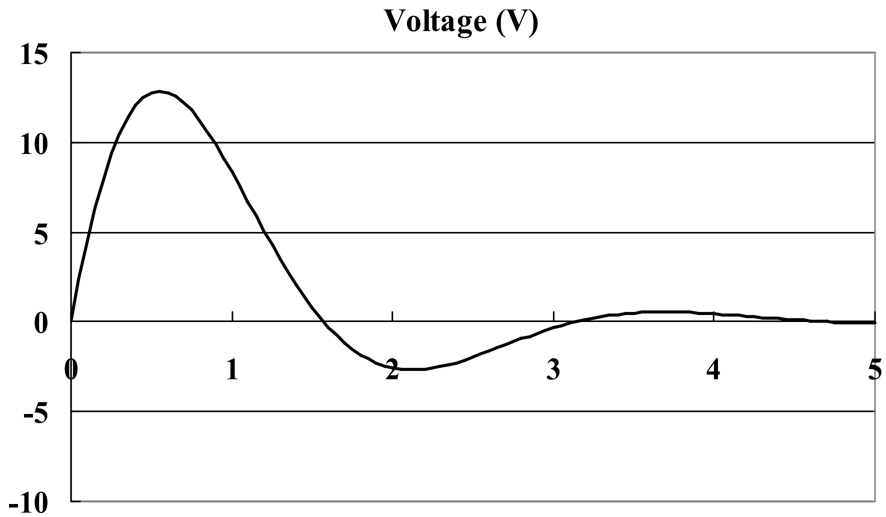
$$v_C(t) = v_L(t) \quad t \geq 0$$

$$i_R = \frac{v_R(t)}{R} = \frac{v_L(t)}{R} = -50e^{-1000t}\sin 2000t \quad \text{mA} \quad \text{t} \geq 0$$

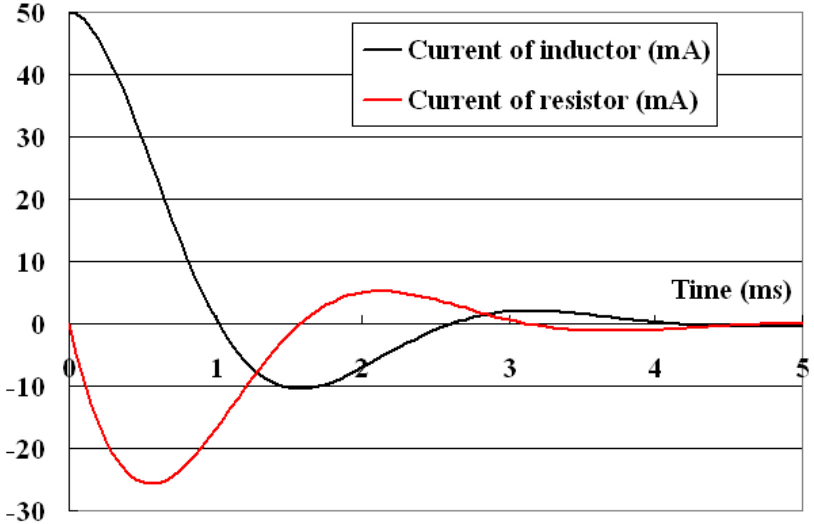
Step response – Example 1 solution (cont.)

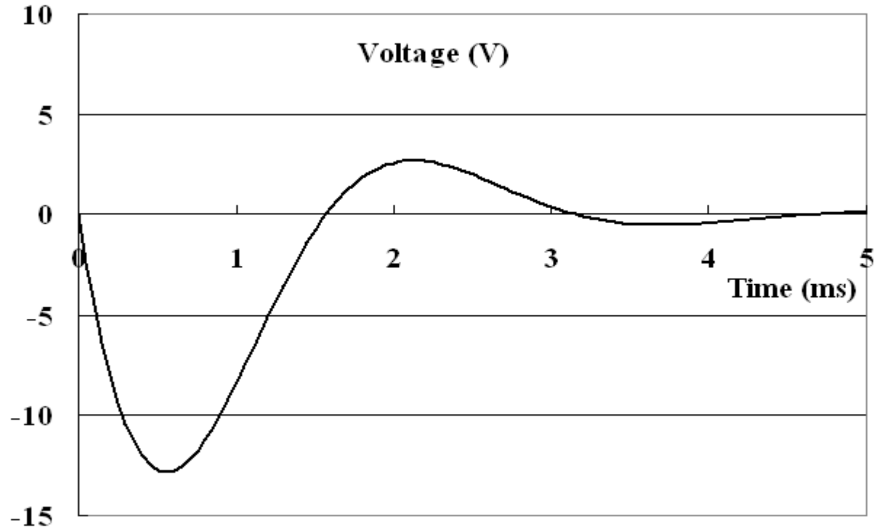
Step Response



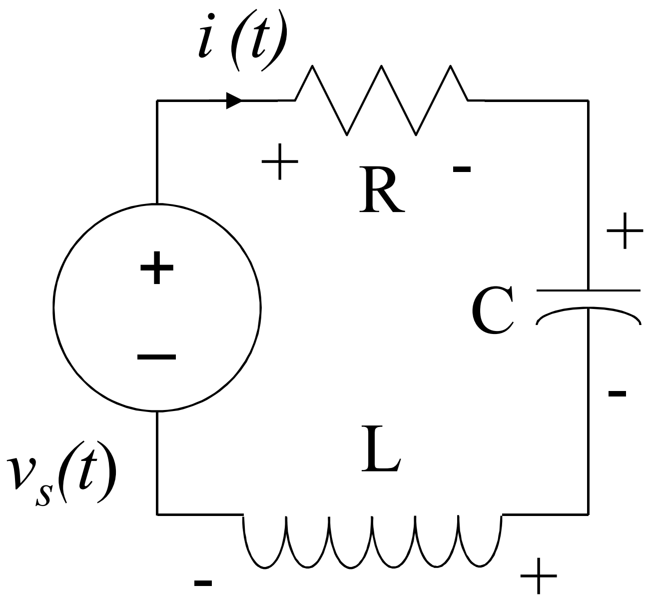


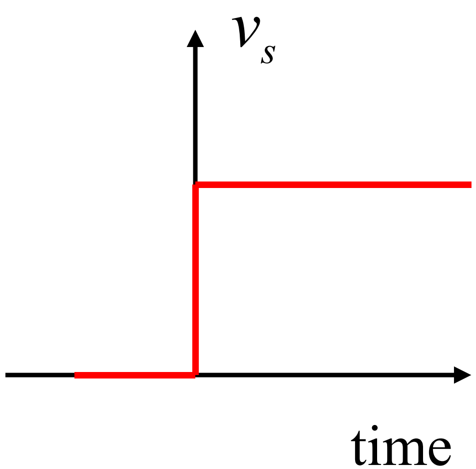
Source Free





Step response – *series RLC circuit*





By using KVL : $v_R + v_C + v_L = v_s$

The current flowing in the circuit :

$$i = C\frac{dv_C}{dt}$$

The voltage v_R and v_L are given by

$$v_R = iR = RC\frac{dv_C}{dt}$$

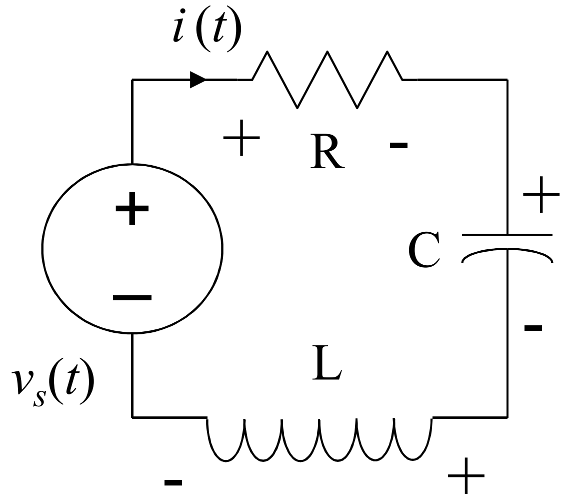
$$v_L = L\frac{di}{dt} = LC\frac{d^2v_C}{dt^2}$$

Then we have :

$$\boxed{\frac{d^2v_C}{dt^2} + \frac{R}{L}\frac{dv_C}{dt} + \frac{1}{LC}v_c = \frac{1}{LC}v_s \qquad (1)}$$

Step response – series RLC circuit

$$\frac{d^2 v_C}{dt^2} + \frac{R}{L}\frac{dv_C}{dt} + \frac{1}{LC}v_c = \frac{1}{LC}v_s \qquad (1)$$



The solution is :

$$v_C(t) = v_{C,f}(t) + v_{C,n}(t)$$

The natural response is the same as the source free case, the forced value of the voltage across the capacitor should be the voltage at t = ∞, in this circuit, the same as the source voltage v_s

$$v_C(t) = v_s + A_1 e^{s_1 t} + A_2 e^{s_2 t} \qquad \text{(Overdamped)}$$

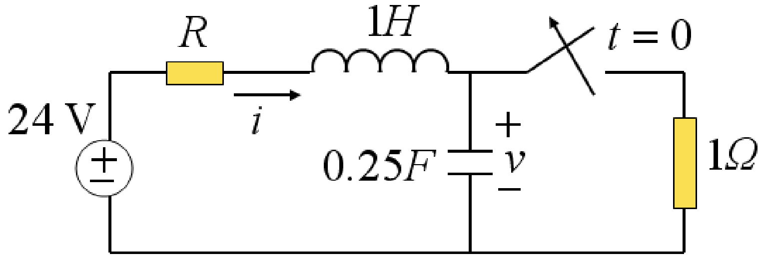
$$v_C(t) = v_s + (A_1 + A_2 t)e^{-\alpha t} \qquad \text{(Critically damped)}$$

$$v_C(t) = v_s + (A_1 \cos \omega_d t + A_2 \sin \omega_d t)e^{-\alpha t} \qquad \text{(Underdamped)}$$

Step response – Example 2 (series RLC circuit)

For the circuit below, the switch is closed for a long time before it is opened. Find $v(t)$ and $i(t)$ for $t > 0$. Consider these cases: $R = 5\Omega$, $R = 4\Omega$, $R = 1\Omega$

Solution Case 1: R = 5Ω



When $t < 0$: $\quad i(0) = \dfrac{24}{5+1} = 4 \text{ A}$

$$v(0) = 1 \times i(0) = 4 \text{ V}$$

$t > 0$: $\quad \alpha = \dfrac{R}{2L} = \dfrac{5}{2\times 1} = 2.5$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{1\times 0.25}} = 2$$

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} = -1, \quad -4 \Rightarrow \text{Overdamped response}$$

$$v(t) = 24 + A_1 e^{-t} + A_2 e^{-4t}$$

$$v(0) = 4 = 24 + A_1 + A_2 \quad \Rightarrow \quad -20 = A_1 + A_2$$

Step response – Example 2 solution

The current through the inductor cannot change abruptly,

and is the same current through the capacitor at $t = 0_+$

$$i(0) = C\frac{dv(0)}{dt} = 4 \quad \Rightarrow \quad \frac{dv(0)}{dt} = \frac{4}{C} = 16$$

$$\frac{dv(t)}{dt} = -A_1 e^{-t} - 4A_2 e^{-4t} \qquad \text{At } t = 0, \quad \frac{dv(0)}{dt} = 16 = -A_1 - 4A_2$$

$$\boxed{\begin{cases} A_1 + A_2 = -20 \\ -A_1 - 4A_2 = 16 \end{cases}}$$

$$A_1 = -64/3, \quad A_2 = 4/3$$

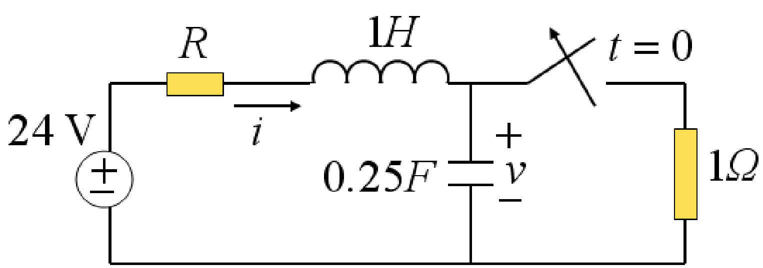
$$v(t) = 24 + \frac{4}{3}(-16e^{-t} + e^{-4t}) \quad \text{V}$$

As the inductor and the capacitor are in series when $t > 0$,

The current through the inductor is the same as one through the capacitor :

$$i(t) = C\frac{dv}{dt} = 0.25 \times \frac{4}{3}(16e^{-t} - 4e^{-4t})$$

$$i(t) = \frac{4}{3}(4e^{-t} - e^{-4t}) \quad \text{A}$$



Step response – Example 2 solution (cont.)

Solution Case 2: R = 4Ω

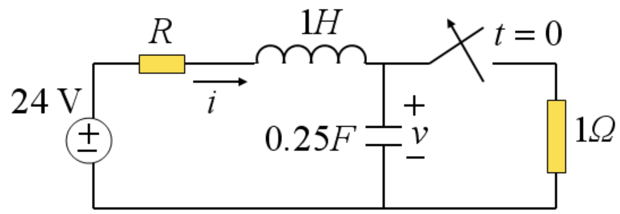
$$t<0:\quad i_L(0)=24/(4+1)=4.8\ \ A,\qquad v(0)=1\times i(0)=4.8\quad V$$

$$\alpha=R/2L=4/(4\times1)=2,\qquad \omega_0=2\quad\Rightarrow\quad s_1=s_2=-\alpha=-2$$

As $s_1=s_2$, we have the critically damped response.

And $v(t)=24+(A_1+A_2t)e^{-2t}\quad V$

$$v(0)=4.8=24+A_1\qquad\Rightarrow\qquad A_1=-19.2$$



$$\frac{dv(0)}{dt}=\frac{4.8}{C}=19.2$$

$$\frac{dv(t)}{dt}=(-2A_1-2tA_2+A_2)e^{-2t}\quad\Rightarrow\quad\frac{dv(0)}{dt}=-2A_1+A_2=19.2\quad\Rightarrow\quad A_2=-19.2$$

Thus $\qquad v(t)=24-19.2(1+t)e^{-2t}\quad$ V

The current through the inductor is the same as one through the capacitor :

$$i(t)=C\frac{dv}{dt}=(4.8+9.6t)e^{-2t}\quad \text{A}$$

Step response – Example 2 solution (cont.)

$t < 0$: $\quad i_L(0) = 24/(1+1) = 12 \;\; A, \quad v(0) = 1 \times i(0) = 12 \quad \text{V}$

$\alpha = R/2L = 1/(2\times 1) = 0.5, \quad \omega_0 = 2 \quad \Rightarrow \quad s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} = -0.5 \pm j1.936$

$\alpha < \omega_0$: the underdamped response

then : $v(t) = 24 + (A_1 \cos 1.936t + A_2 \sin 1.936t)e^{-0.5t} \quad V$

$$v(0) = 12 = 24 + A_1 \quad \Rightarrow \quad A_1 = -12 \quad \text{and} \quad \frac{dv(0)}{dt} = \frac{12}{C} = 48$$

$$\frac{dv(t)}{dt} = e^{-0.5t}(-1.936 A_1 \sin 1.936t + 1.936 A_2 \cos 1.936t) - 0.5e^{-0.5t}(A_1 \cos 1.936t + A_2 \sin 1.936t)$$

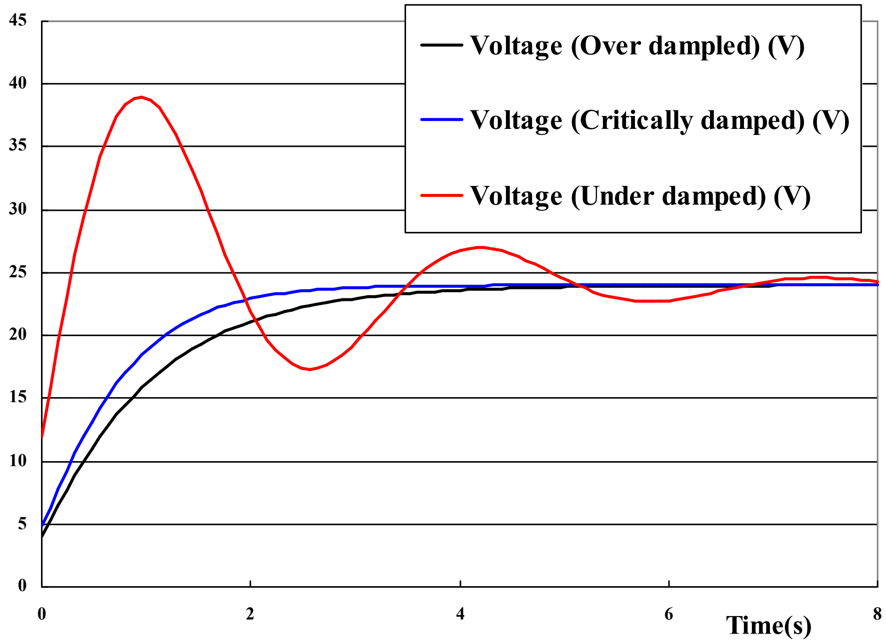
$$\frac{dv(0)}{dt} = (-0 + 1.936 A_2) - 0.5(A_1 + 0) = 48 \qquad \Rightarrow \qquad A_2 = 21.694$$

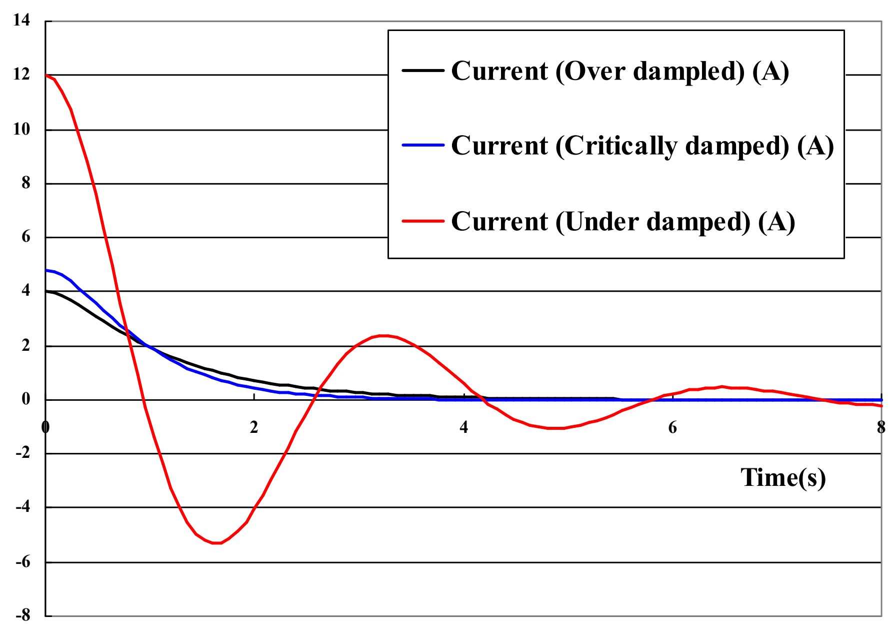
Thus $\quad v(t) = 24 + (21.694 \sin 1.936t - 12 \cos 1.936t)e^{-0.5t} \quad \text{V}$

The current through the inductor is the same as one through the capacitor :

$$i(t) = C\frac{dv}{dt} = (3.1 \sin 1.936t + 12 \cos 1.936t)e^{-0.5t} \quad \text{A}$$

Step response – Example 2 solution (cont.)





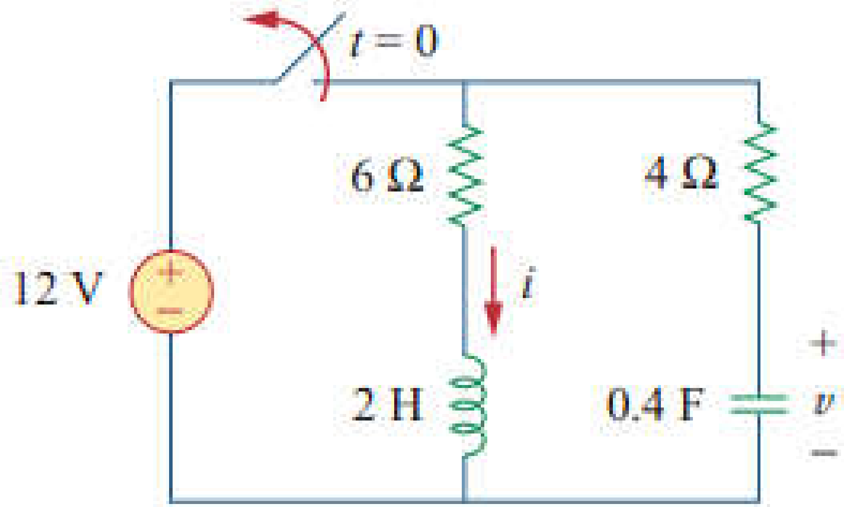
Second Order Circuit – Step response summary

	Series	Parallel
α	$\alpha = \dfrac{R}{2L}$	$\alpha = \dfrac{1}{2RC}$
ω_0	$\dfrac{1}{\sqrt{LC}}$	$\dfrac{1}{\sqrt{LC}}$
Over Damped	$\alpha > \omega_0$ Response: $A_1 e^{s_1 t} + A_2 e^{s_2 t}$	
Critically Damped	$\alpha = \omega_0$ Response: $e^{-\alpha t}(A_1 t + A_2)$	
Under Damped	$\alpha < \omega_0$ Response: $e^{-\alpha t}(B_1 \cos \omega_d t + B_2 \sin \omega_d t)$	
Undamped	R=0	$R \to \infty$
	Response: $B_1 \cos \omega_0 t + B_2 \sin \omega_0 t$	

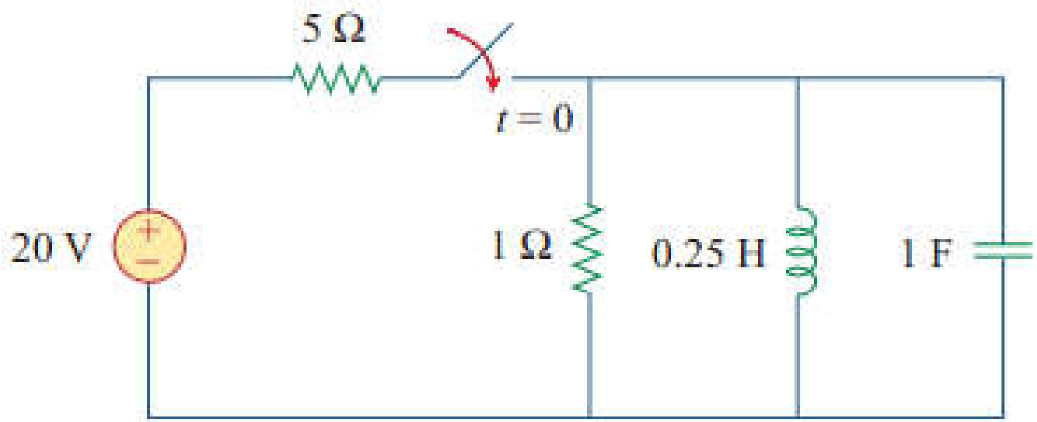
Xi'an Jiaotong-Liverpool University 西交利物浦大學

Quiz

- For the circuit in the figure on the right, the initial value of $i(0^+)$ and $di(0^+)/dt$ are:
  - (a) $i(0^+) = 2$ A, $di(0^+)/dt = -4$ A/s;
  - (b) $i(0^+) = 2$ A, $di(0^+)/dt = 0$ A/s;
  - (c) $i(0^+) = 1.2$ A, $di(0^+)/dt = 2$ A/s;
  - (d) $i(0^+) = 1.2$ A, $di(0^+)/dt = 1.2$ A/s;



- For the circuit in the figure on the right, determine its damping case:
  - (a) over damped;
  - (b) critical damped;
  - (c) under damped;
  - (d) un-damped.

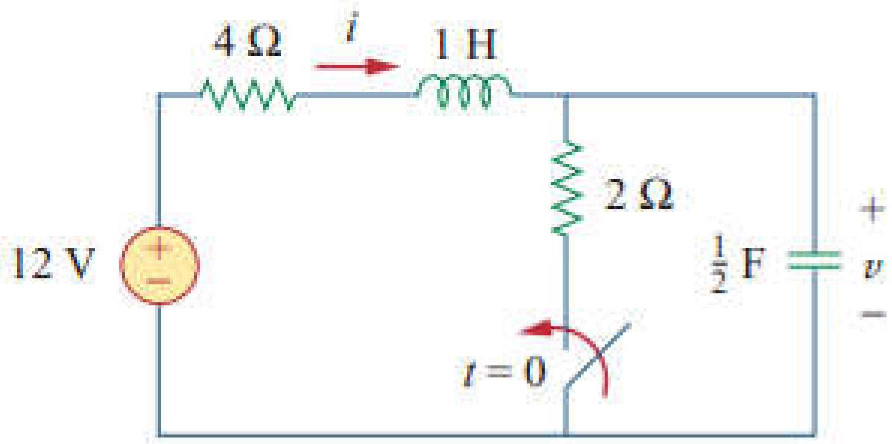


General Second Order Circuits

- The idea of parallel/series RLC circuits can be extended to any second order circuits.
- Find the step response of a 2nd order system takes five steps:
  - 1. First determine the initial conditions $x(0)$ and $dx(0)/dt$, and the final value $x(\infty)$.
  - 2. Turn off the independent sources and find the form of the natural response $x_n(t)$ by applying KCL and KVL.
  - 3. We obtain the forced response as $x_f(t) = x(\infty)$
  - 4. The total response is now found as the sum of the natural response and forced response $x(t) = x_n(t) + x_f(t)$
  - 5. We finally determine the constants associated with the natural response by imposing the initial conditions $x(0)$ and $dx(0)/dt$ determined in step 1.

General Second Order Circuits – Example 3

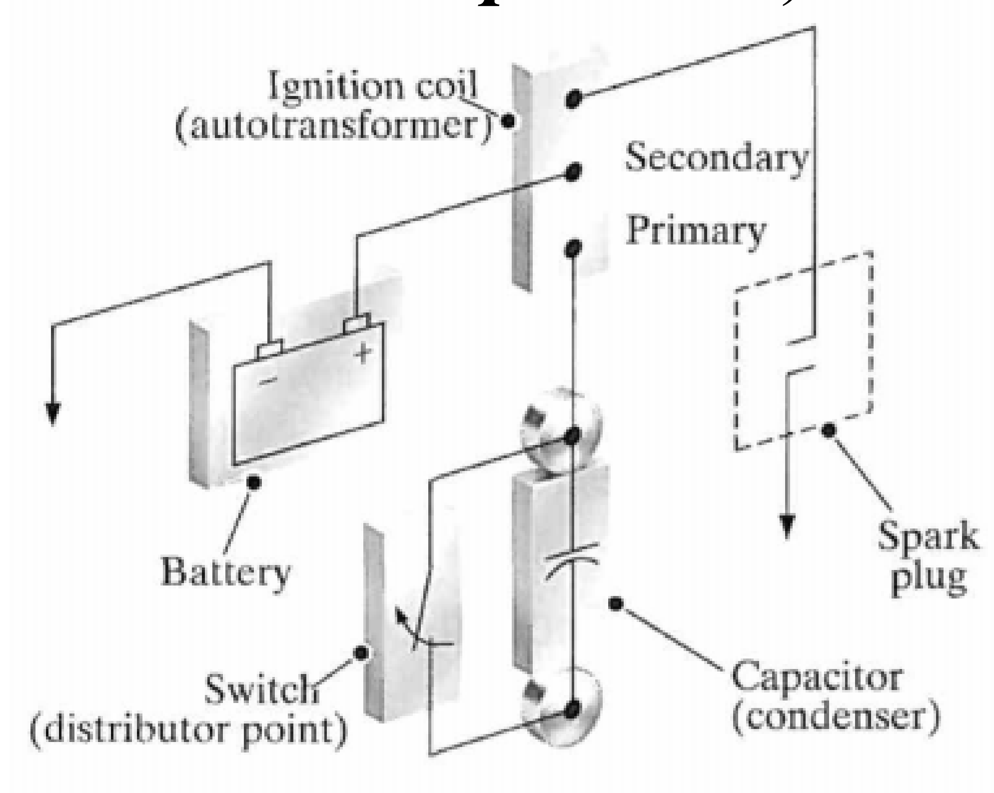
- Find the complete response v and then i for $t > 0$ in the following circuit.



Ref.3, example 8.9

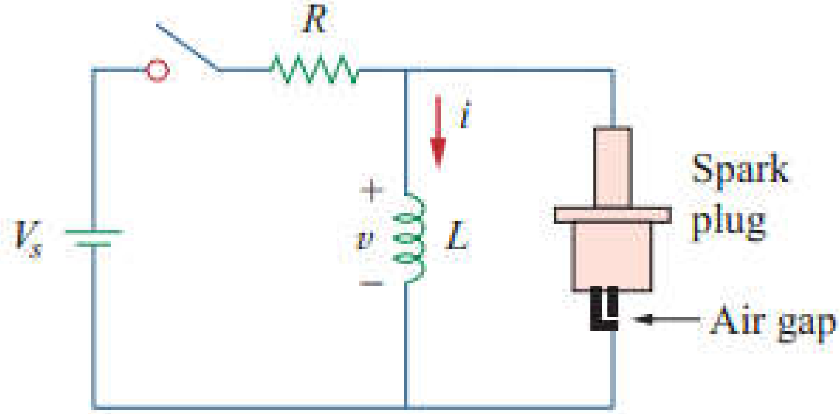
Practical Application – *Automobile Ignition Circuit*

- An automobile ignition circuit is based on the transient response of an RLC circuit.
- In such a circuit, a switching operation causes a rapid change in the current in an inductive winding known as an ***ignition coil*** (also called the ***spark coil***).



Automobile Ignition Circuit – 1st order model

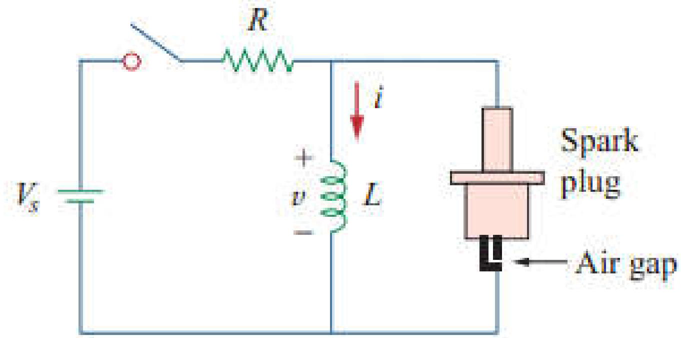
- In gasoline engine, the ignition of the fuel-air mixture in each cylinder is achieved by means of a spark plug.
  - By creating a large voltage (thousands of volts) between the electrodes, a spark is formed across the air gap, thereby igniting the fuel.



  - Question: how can such a large voltage be obtained from the car battery, which supplies only 12 V?
- This is achieved by means of an inductor (the spark coil) L.
  - Voltage across the inductor is $v = L di/dt$
  - So rapidly changing current $i(t)$ can generate large voltage v.
  - After the switching, the current increases and reaches the final value of $i = V_s/R$, where $V_s = 12$V.

Automobile Ignition Circuit – 1st order model

- A solenoid with resistance 4Ω and inductance 6 mH is used in an auto-mobile ignition circuit.



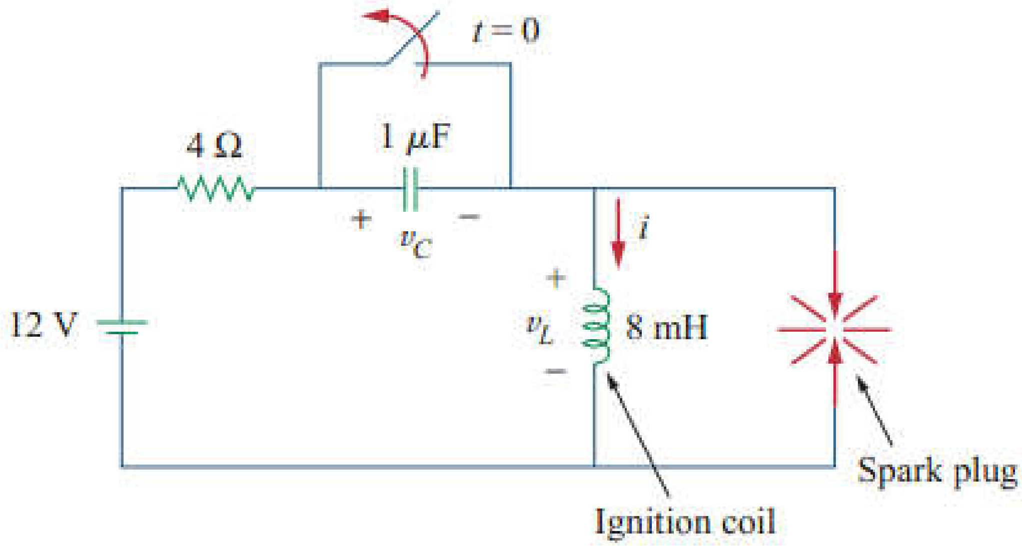
- If the battery supplies 12 V, determine:
  - the final current through the solenoid when the switch is closed;
  - the energy stored in the coil, and the voltage across the air gap assuming that the switch takes 1 μs to open.

- Solution:
  - The final current through the coil is $I = \frac{V_s}{R} = \frac{12}{4} = 3\text{ A}$
  - The energy stored in the coil is $W = \frac{1}{2}L\,I^2 = \frac{1}{2} \times 6 \times 10^{-3} \times 3^2 = 27\text{ mJ}$
  - The voltage across the gap is

$$V = L\frac{\Delta I}{\Delta t} = 6 \times 10^{-3} \times \frac{3}{1 \times 10^{-6}} = 18\text{ kV}$$

Automobile Ignition Circuit – 2nd order model

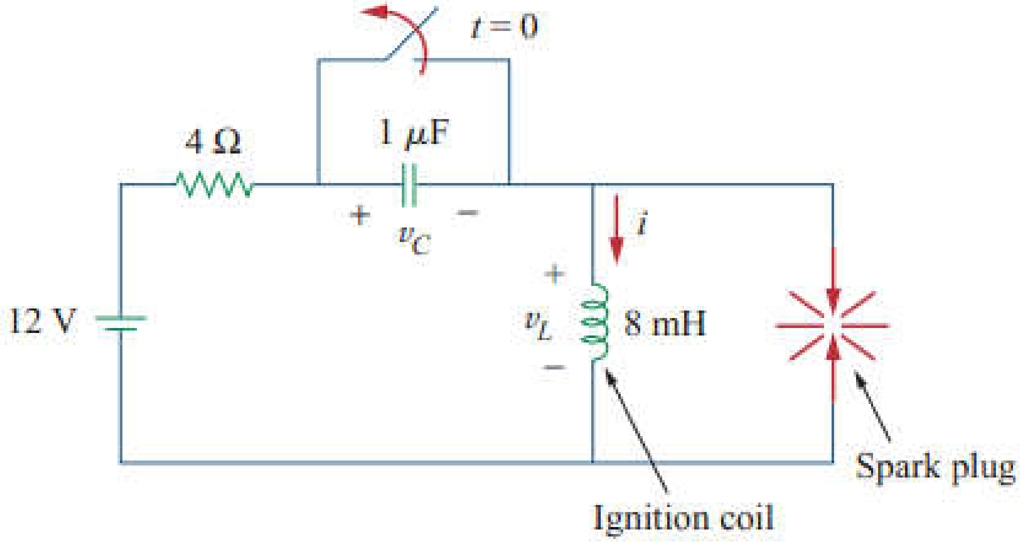
- In 1st order model, we considered the automobile ignition system as a charging system.



- Here, we consider another part: the voltage generating system.
  - The 12-V source is due to the battery and alternator.
  - The resistor represents the resistance of the wiring.
  - The ignition coil is modelled by the 8-mH inductor.
  - The capacitor (known as the condenser to automechanics) is in parallel with the switch (known as the breaking points or electronic ignition).

Automobile Ignition Circuit – 2nd order model

- Assuming that the switch in the figure is closed prior to $t = 0^-$. Find the inductor voltage v_L for $t > 0$.

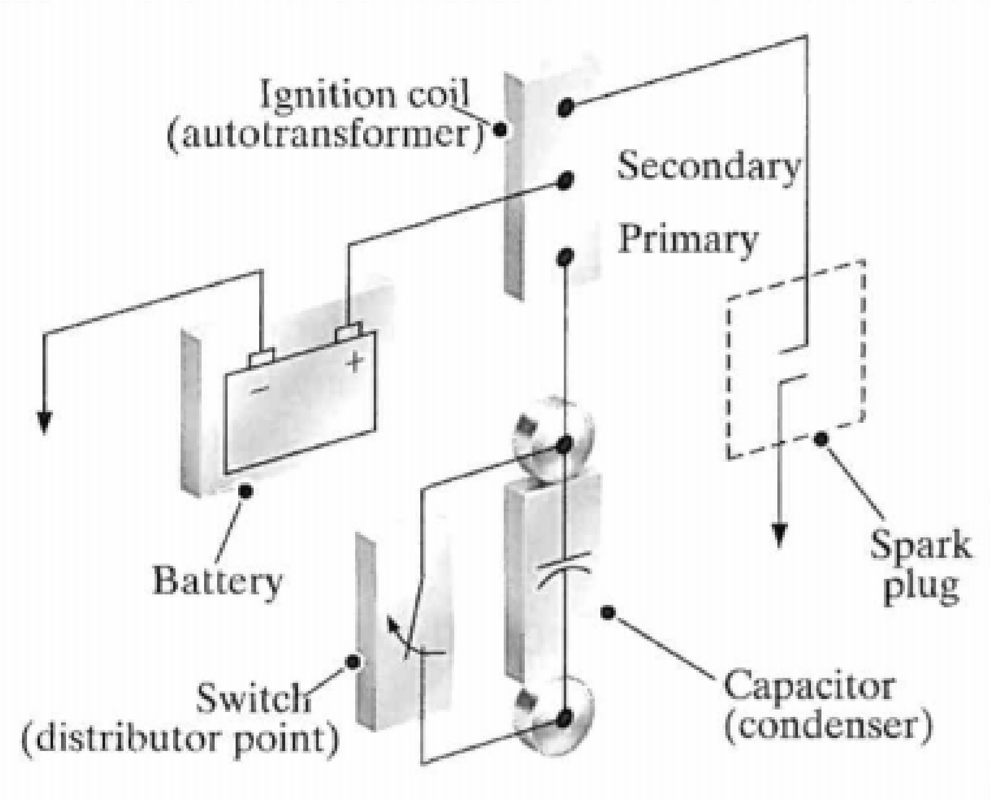


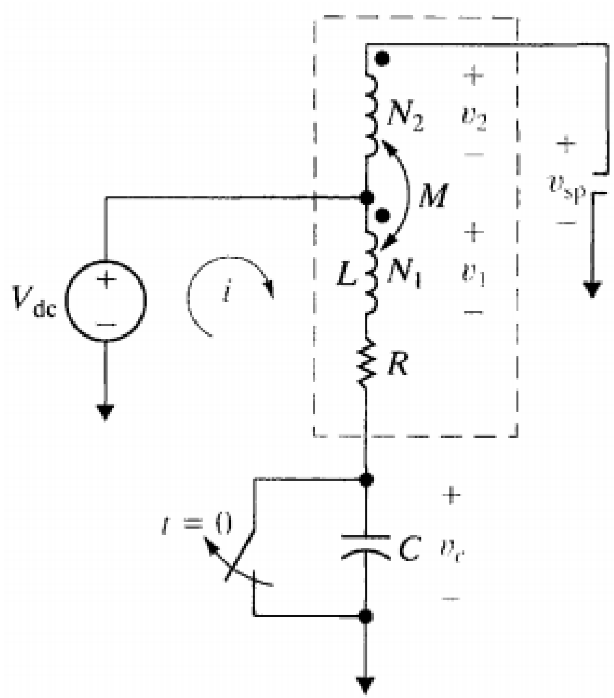
Ref.3, example 8.16

- After calculation, the inductor voltage's peak value is –259V.
  - Still far less than the voltage range of 6000 to 10,000 V required to fire the spark plug in a typical automobile
  - A device known as a transformer is used to step up the inductor voltage to the required level.

Automobile Ignition Circuit – 2^{nd} order model

- By using a transformer (autotransformer in this case), the voltage on the spark plug can be boosted up to 40 kV to ignite the fuel-air mixture in the cylinder.





Practice

- The switch in the circuit shown in Figure Q2 has been closed for a long time before it is opened at $t = 0$.
  - Draw the equivalent circuit at $t = 0^-$ and $t = \infty$;
  - Find $i_L(t)$ for $t \geq 0$;
  - How many microseconds after the switch opens is the inductor voltage $v_L(t)$ maximum?
  - If resistor is removed from the circuit at t = 0 s, draw $i_L(t)$.