EEE336 Signal Processing and Digital Filtering

Revision

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Outline

- 1. Sampling and reconstruction
- 2. Quantization
- 3. Discrete-time signals and systems in time and frequency domain => DTFT
- 4. DFT (Discrete Fourier Transform)
- 5. FFT (Fast Fourier Transform)
- 6. Z-transform
- 7. Filter structures
- 8. FIR filters
- 9. IIR filters

Lecture 3 Sampling and Reconstruction

- 1. Sampling
  - In TD and FD
  - Derivation & Frequencies relationship (f vs Ω vs ω)
  - Graphical illustration
- 2. Anti-aliasing
  - What causes aliasing? – Nyquist theorem
  - Anti-aliasing filter
- 3. Reconstruction

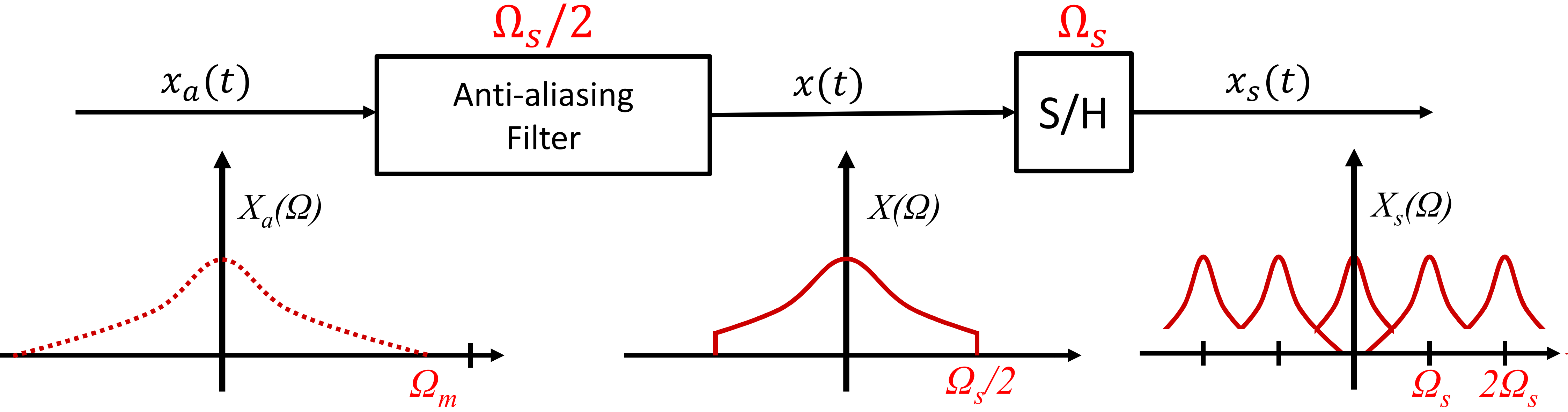
3.1 Sampling

(a) Ideal data flow for the digital processing of continuous-time signals

In Time Domain

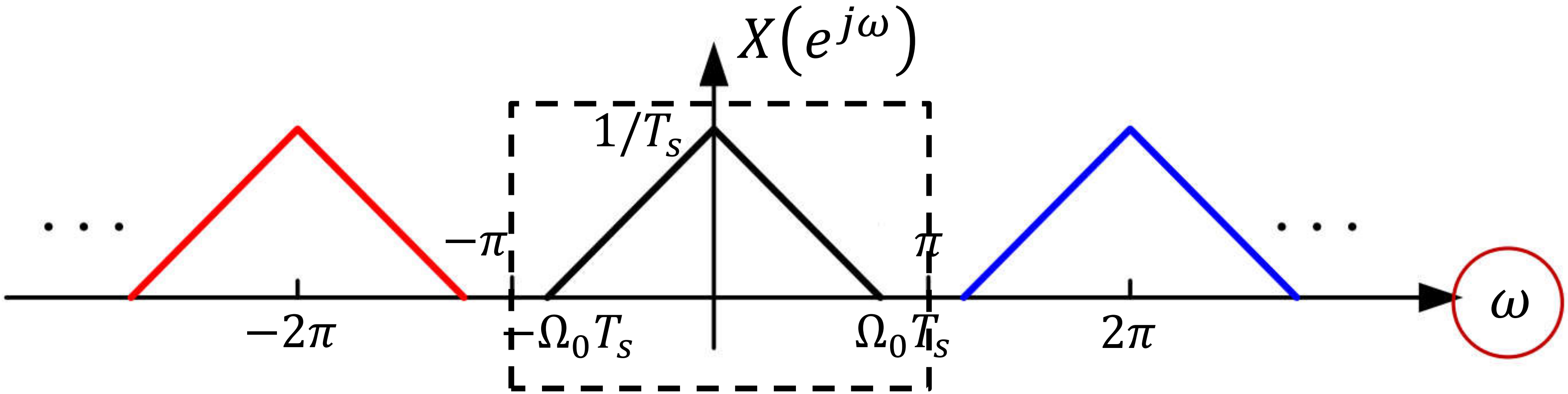
In Frequency Domain

3.2 Anti-aliasing



- Nyquist theorem: to avoid aliasing, the sampling frequency
- Three types of sampling: over sampling, critical sampling and under sampling
- Aliasing: If Nyquist theorem was not satisfied, aliasing happens
  - To reduce the aliasing error, passing the CT signal through an "Anti-aliasing filter" before sampling it.

3.3 Reconstruction



- The spectrum of the sampled signal contains all the information of the original CT signal
  - So the CT signal can be recovered without any loss;
  - But a condition needs to be satisfied:

$$\Omega_0 T_s \leq \pi \iff 2\Omega_0 \leq \Omega_s$$

  - The Nyquist theorem!

Lecture 4 Quantization

- 1. Quantization and error evaluation
  - Relationship among the key parameters R, Q and B
  - Error ***e*** and dynamic range SNR
  - Truncation and rounding
- 2. D/A Conversion: 3 types of codes
  - Natural binary
  - Offset binary
  - 2's combined binary
- 3. A/D Conversion
  - How to perform – the example

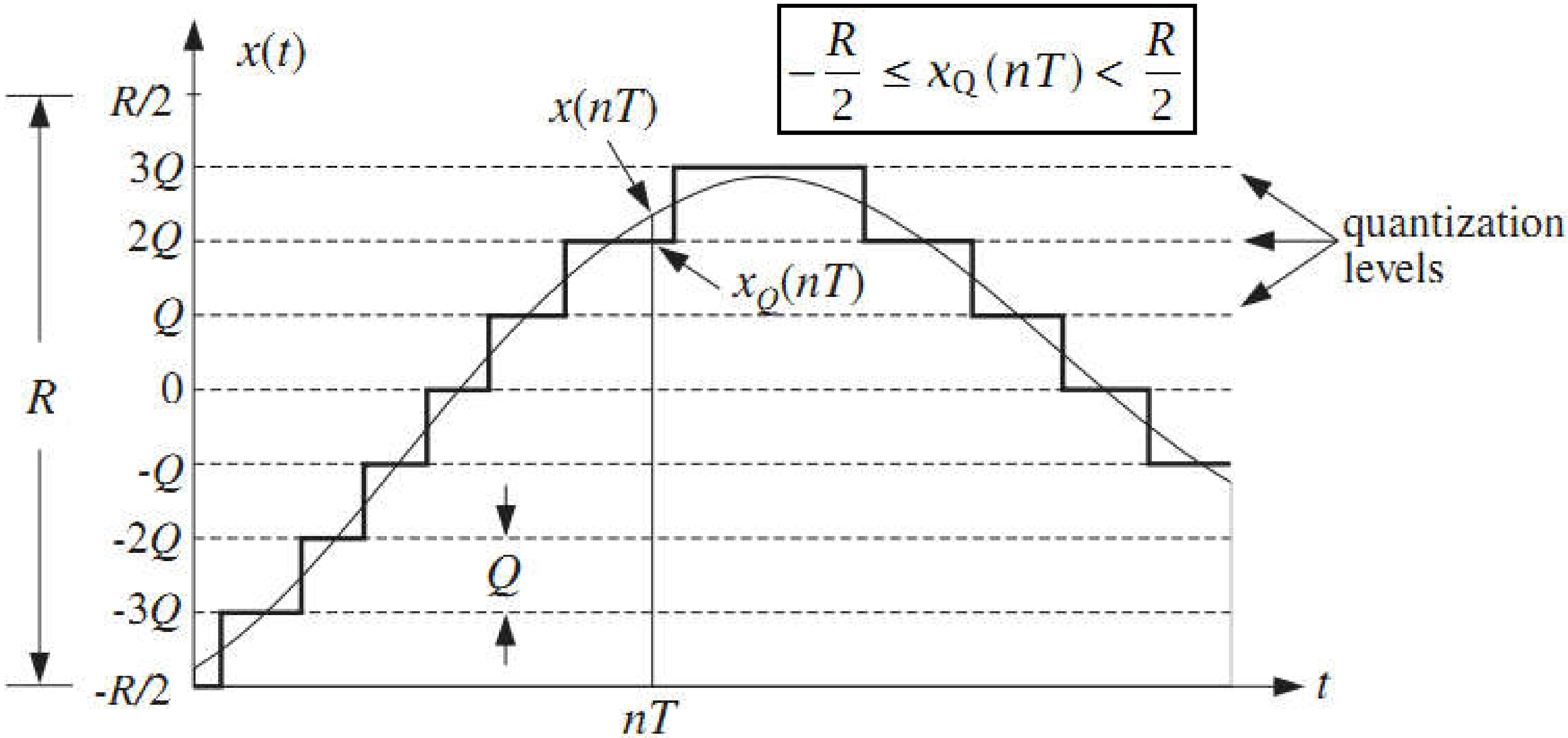
4.1 Quantisation Process

- R is the full-scale range which is divided equally (for a uniform quantizer) into 2^B *quantization levels*.
- The spacing between levels are called the *quantization width* / *quantization level* or *quantizer resolution* Q

$$\boxed{Q = \frac{R}{2^B}}$$

$$\Updownarrow$$

$$\boxed{\frac{R}{Q} = 2^B}$$



4.1 Quantisation error

- Root-mean-square of error ***e***:
  - Truncation: $e_{rms} = \sqrt{\bar{e}^2} = \frac{Q}{\sqrt{12}}$
  - Rounding: $e_{rms} = \sqrt{\bar{e}^2} = \frac{Q}{\sqrt{3}}$

- Signal-to-noise ratio (SNR)
  - R – range of signal
  - Q – range of noise

$$SNR = 20\log_{10}\left(\frac{R}{Q}\right) = 6B \text{ (dB)}$$

SNR is also called the *dynamic range* of the quantiser

4.2 D/A Converters

- Three types of converter and the coding conventions
  - Natural Binary: the unipolar natural binary

$$x_Q = R(b_1 2^{-1} + b_2 2^{-2} + \cdots + b_B 2^{-B}) \Longleftrightarrow x_Q = Qm \qquad (1)$$

where m is the integer whose binary representation is $(b_1 b_2 \cdots b_B)$

- LSB (Least Significant Bit): b_B
- MSB (Most Significant Bit): b_1

  - Offset Binary: the bipolar natural binary

$$x_Q = R(b_1 2^{-1} + b_2 2^{-2} + \cdots + b_B 2^{-B} - 0.5) \qquad (2)$$

  - 2's Combine: the two's complement

$$x_Q = R(\overline{b}_1 2^{-1} + b_2 2^{-2} + \cdots + b_B 2^{-B} - 0.5) \qquad (3)$$

4.3 A/D Converters

- Example: Convert the analog values x = 3.5 and x =−1.5 volts to their offset binary representation, assuming B = 4 bits and R = 10 volts

x = 3.5

test	$b_1 b_2 b_3 b_4$	x_Q	$C = u(x - x_Q)$
b_1	1 0 0 0	0.000	1
b_2	1 1 0 0	2.500	1
b_3	1 1 1 0	3.750	0
b_4	1 1 0 1	3.125	1
	1 1 0 1	3.125	

x =−1.5

test	$b_1 b_2 b_3 b_4$	x_Q	$C = u(x - x_Q)$
b_1	1 0 0 0	0.000	0
b_2	0 1 0 0	−2.500	1
b_3	0 1 1 0	−1.250	0
b_4	0 1 0 1	−1.875	1
	0 1 0 1	−1.875	

Lecture 5-8 Discrete Signals and Systems

- 1. Convolution
  - Linear convolution
  - Multiplication VS convolution

- 2. DTFT (Discrete-Time Fourier Transform)
  - Definition and relationship to time domain
  - Calculation
  - Properties & applications

6.1 Convolution

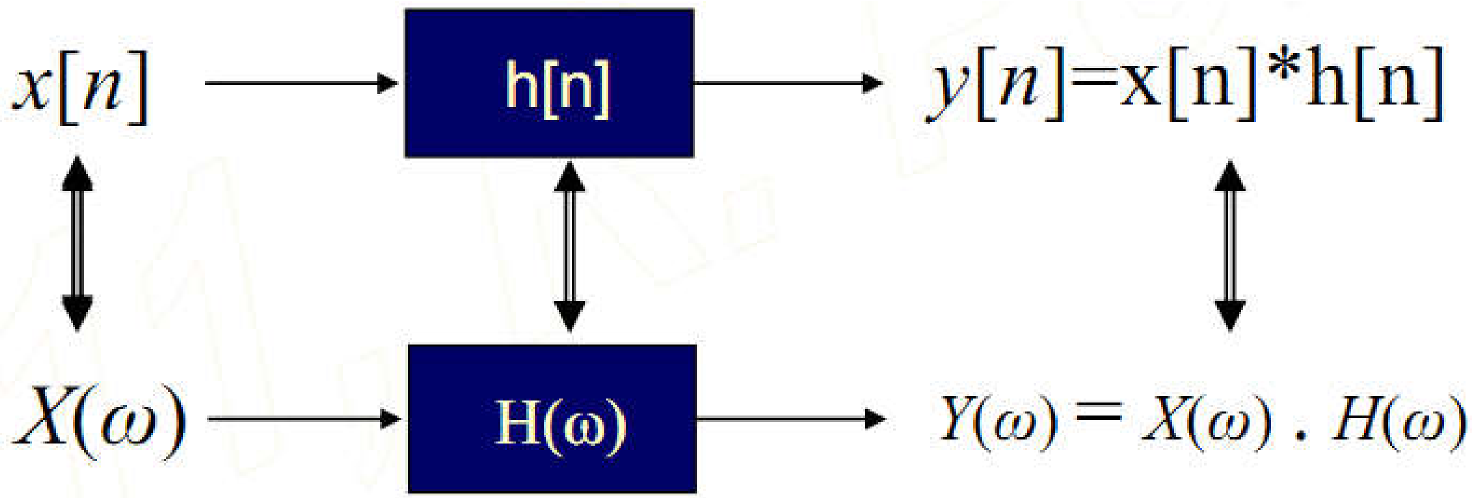
- Linear convolution:
  - Input x[n];
  - System impulse response h[n];
  - Output y[n]:

$$y[n_0] = \sum_{k=-\infty}^{\infty} x[k]h[n_0 - k]$$

- Calculation
  - Different methods …

8.1 Time-Frequency Domain Relationship

- If $x[n]$ is input to an LTI system with an impulse response of $h[n]$, then the DTFT of the output is the product of $X(\omega)$ and $H(\omega)$



7.1 DTFT Definition

- The discrete-time Fourier transform (DTFT) $X(e^{j\omega})$ of a sequence x[n] is defined by:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

  – DTFT $X(e^{j\omega})$ of a sequence x[n] is a continuous function of ω

- Inverse Discrete-Time Fourier Transform - the Fourier coefficients {x[n]} can be computed from $X(e^{j\omega})$ using

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

7.2 DTFT Properties

- 1. Linearity: $ax_1[n] + bx_2[n] \xleftrightarrow{\text{DTFT}} aX_1(\omega) + bX_2(\omega)$
- 2. Time-reversal: $x[-n] \xleftrightarrow{\text{DTFT}} X(-\omega)$
- 3. Symmetric:

  $x^*[n] \xleftrightarrow{\text{DTFT}} X^*(-\omega) \qquad x^*[-n] \xleftrightarrow{\text{DTFT}} X^*(\omega)$
- 4. Shifting

  $x[n-M] \xleftrightarrow{\text{DTFT}} X(\omega)e^{-j\omega M}$

  $e^{j\omega_0 n}x[n] \xleftrightarrow{\text{DTFT}} X(\omega-\omega_0)$
- 5. Parseval Theorem:

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(\omega)|^2 d\omega$$

Lecture 9 DFT

- 1. Definition and calculation
  - Analysis and synthesis equations
  - Twiddle factor
  - Calculation
- 2. Relationships among CTFT, DTFT and DFT
- 3. Properties
- 4. Circular convolution

9.1 DFT Definition

- The analysis equation

$$\boxed{X[k] = \sum_{n=0}^{N-1} x[n]\, W_N^{kn}, \quad k = 0, 1, \ldots, N-1}$$

- The synthesis equation

$$\boxed{x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-kn}, \quad k = 0, 1, \ldots, N-1}$$

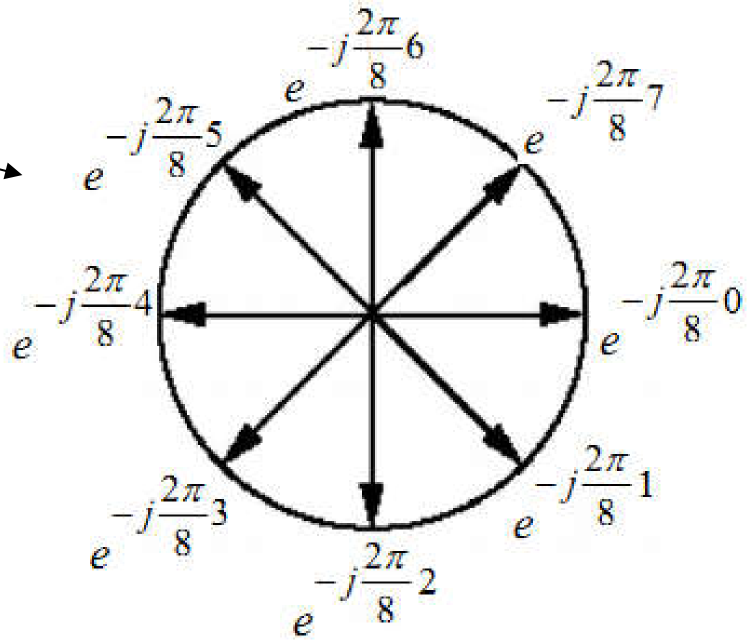
- The DFT pair is denoted as x[n] ←→ X[k]

9.1 Computing DFT

- For any given k, the DFT is computed by multiplying each x[n] with each of the complex exponentials $W_N^{nk} = e^{-j2\pi nk/N}$ and then adding up all these components
- If, for example, we wish to compute an 8-point DFT, the complex exponentials are 8 unit vectors placed at equal distances from each other on the unit circle

Complex exponential wheel

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}, \quad k = 0, 1, \dots, N-1$$



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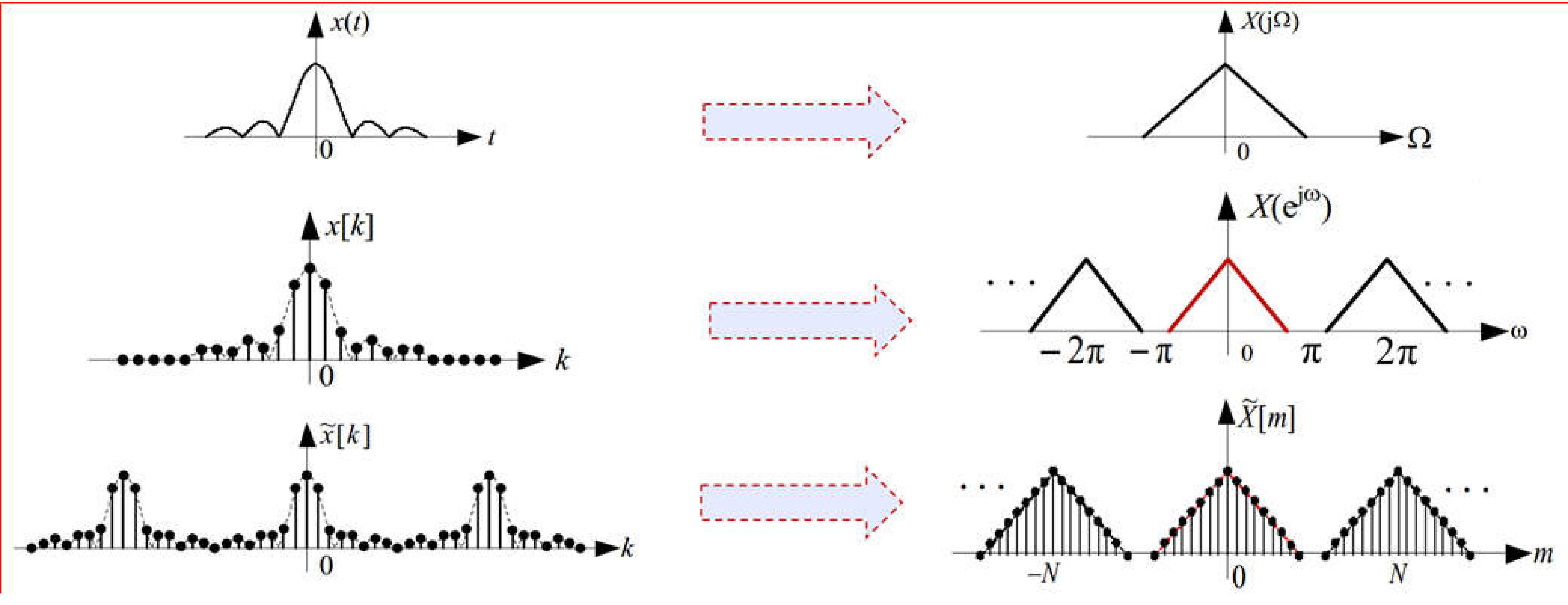
9.2 CTFT → DTFT → DFT

Time Domain	$x(t)$	$x[k]$	$\tilde{x}[k]$
	$\updownarrow$ CTFT	$\updownarrow$ DTFT	$\updownarrow$ DFT
Frequency Domain	$X(j\Omega)$	$X(e^{j\omega})$	$X[m]$



9.3 DFT Properties

- 1. Linearity: $ax_1[n] + bx_2[n] \xleftrightarrow{\text{DFT}} aX_1[k] + bX_2[k]$
- 2. Shifting: $x[\langle n - M \rangle_N] \xleftrightarrow{\text{DFT}} W_N^{kM} X[k]$

$$W_N^{-kM} x[n] \xleftrightarrow{\text{DTFT}} X[\langle n - M \rangle_N]$$

- 3. Symmetry:
  - Many different cases
  - Example: real sequence x[n], whose X[k] is conjugate symmetric
- 4. Parseval Theorem:

$$\sum_{n=0}^{N-1} |x[n]|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |X[k]|^2$$

9.4 Linear VS circular convolution

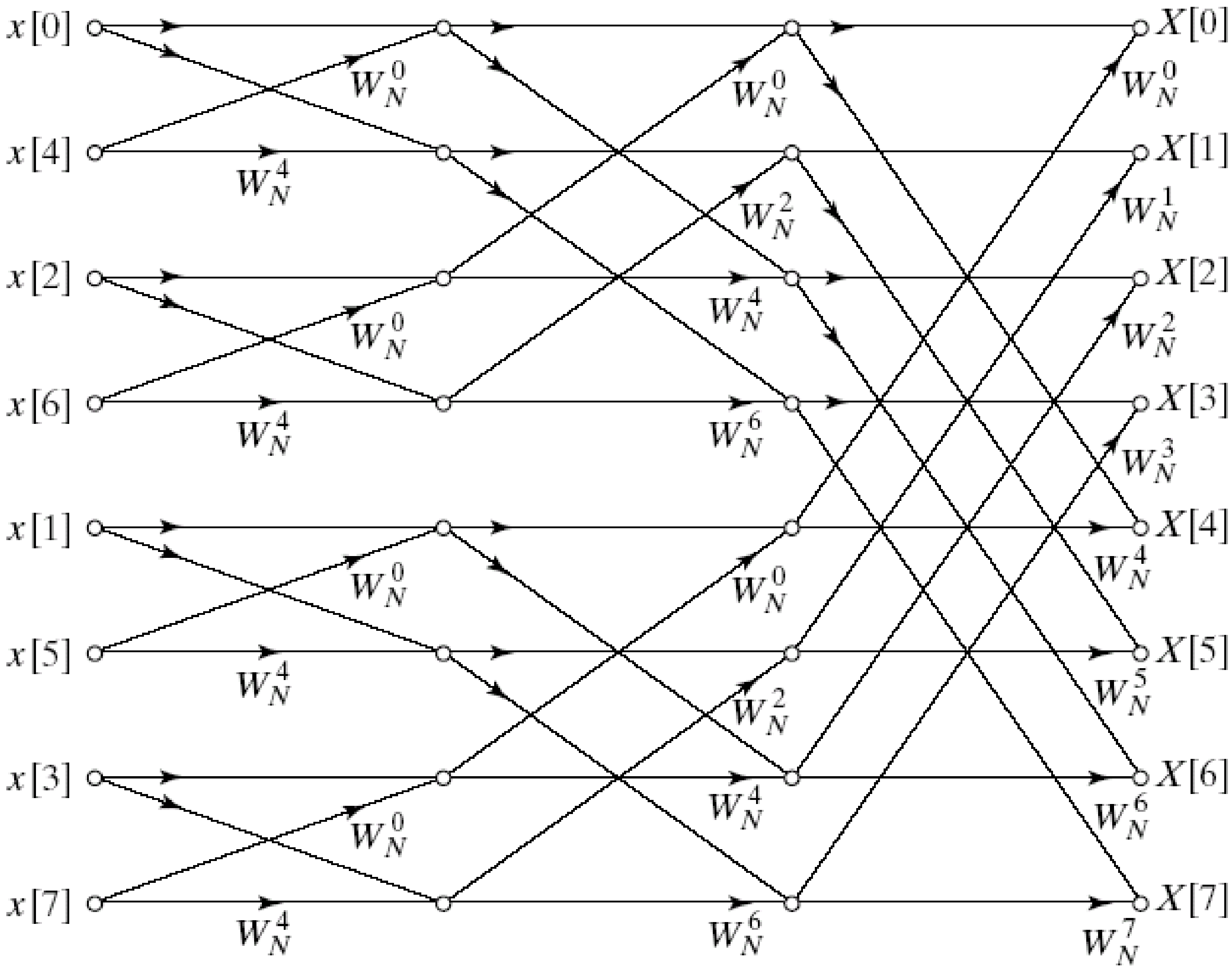
- Is there any relationship between the linear and circular convolutions? Can one be obtained from the other?
- YES!
  - FACT: If we zero pad both sequences x[n] and h[n], so that they are both of length N1+N2-1, then linear convolution and circular convolution result in identical sequences
  - Furthermore: If the respective DFTs of the zero padded sequences are X[k] and H[k], then the inverse DFT of X[k]·H[k] is equal to the linear convolution of x[n] and h[n]
  - Note that, normally, the inverse DFT of X[k].H[k] is the circular convolution of x[n] and h[n]. If they are zero padded, then the inverse DFT is also the linear convolution of the two.

Lecture 10 FFT

- 1. Computational complexity, i.e. number of additions and multiplications
  - DFT, convolution, and FFT
  - How to save?

- 2. FFT flow chart (DIT Radix-2)
  - How to draw?
  - How to calculate DFT based on the flow chart?
  - Bit reversal

Final flow-graph of DIT-2



- Number of stages:

$$p = log_2 N$$

- Number of butterflies per stage:

$$N/2$$

- Two computational complexity:
  - $N(p-1) = N(log_2N-1)$ complex multiplications;
  - $Np = Nlog_2N$ complex additions.

Lecture 11 Z-Transform

- 1. Definition and ROC
- 2. Relationship between DTFT and Z-transform
- 3. Some concepts: stable, causal, right-handed, left-handed, etc.
- 4. Properties
- 5. Zeroes and poles

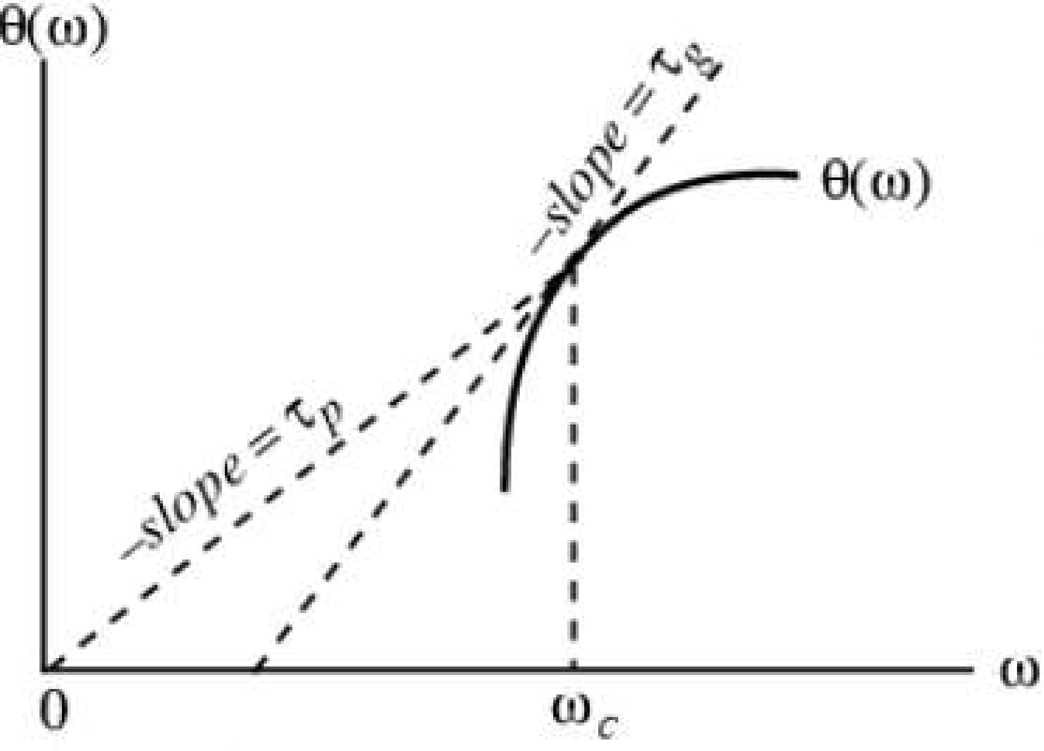
- Generally speaking, everything in this lecture is important!

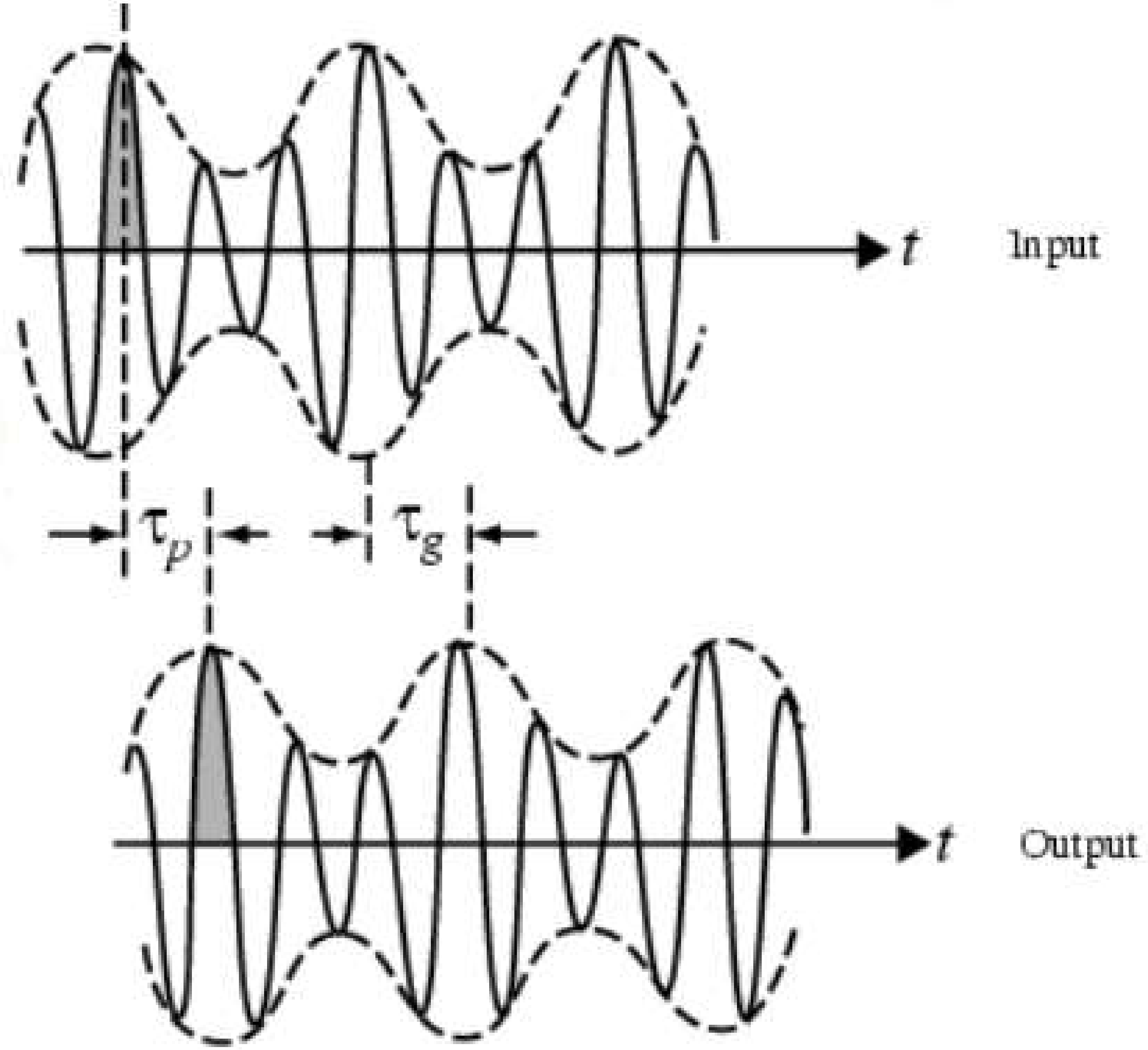
Lecture 12 Filters Classifications

- 1. Phase
  - Calculate phase response of a system
  - Classification according to phase response
- 2. Magnitude
  - Calculate magnitude response of a system
  - Classification according to magnitude response
- 3. Linear-phase FIR filters
  - Type I, II, III and IV
    - Zeroes locations
    - Magnitude responses

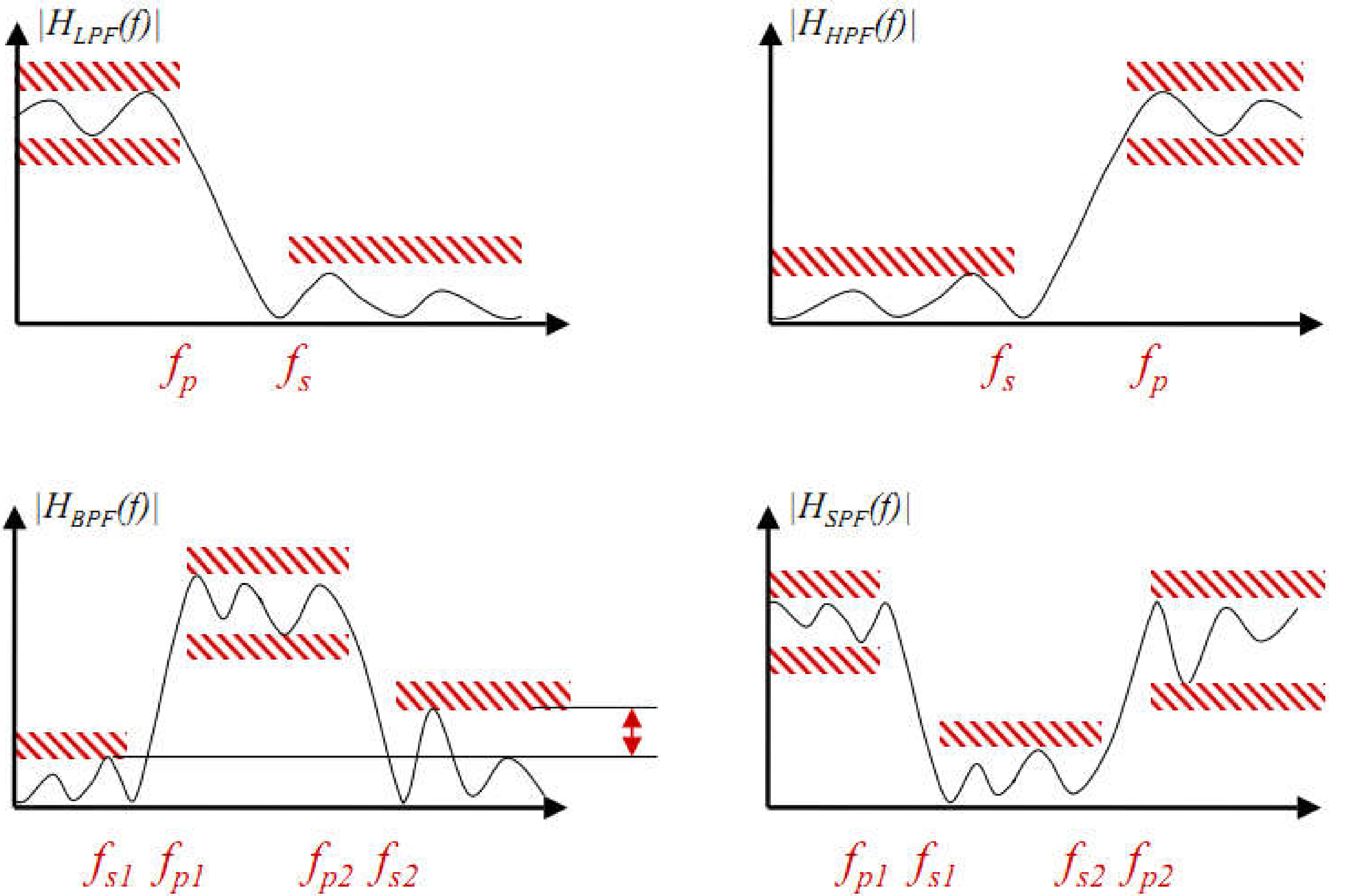
12.1 Phase delay and group delay

- Note that both phase delay and group delay are slopes of the phase function, just defined slightly differently



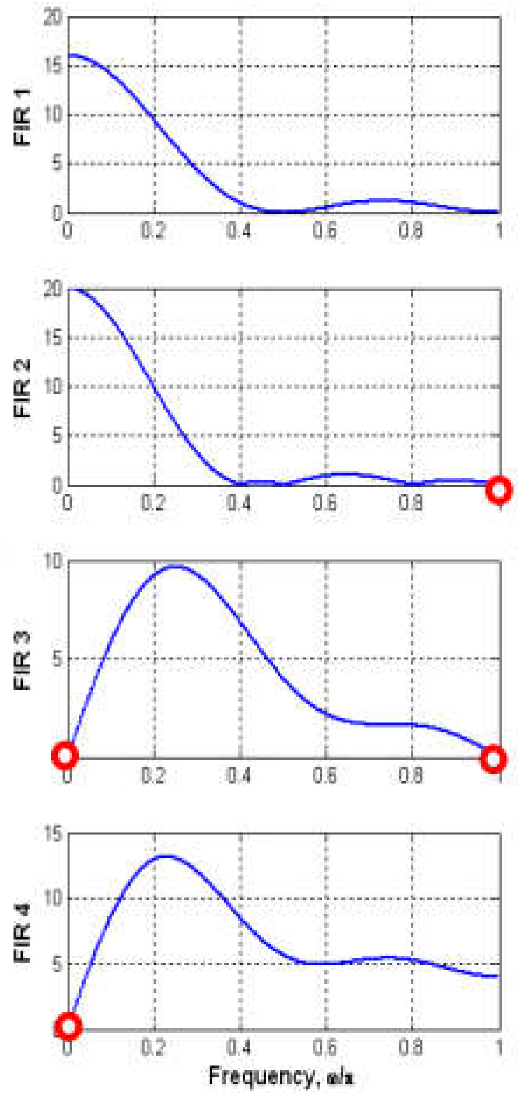


12.2 LPF, HPF, BPF and BSF



12.3 Zero locations of the linear-phase FIR filters

- The presence of zeroes at $z=\pm 1$ leads to some limitations on the use of these linear-phase transfer functions for designing frequency-selective filters
  - A Type 2 FIR filter cannot be used to design a highpass filter since it always has a zero at z=-1
  - A Type 3 FIR filter has zeroes at both z = 1 and z=-1, and hence cannot be used to design either a lowpass or a highpass or a bandstop filter
  - A Type 4 FIR filter is not appropriate to design a lowpass filter due to the presence of a zero at z = 1
  - Type 1 FIR filter has no such restrictions and can be used to design almost any type of filter



Lecture 13 Filter Structures

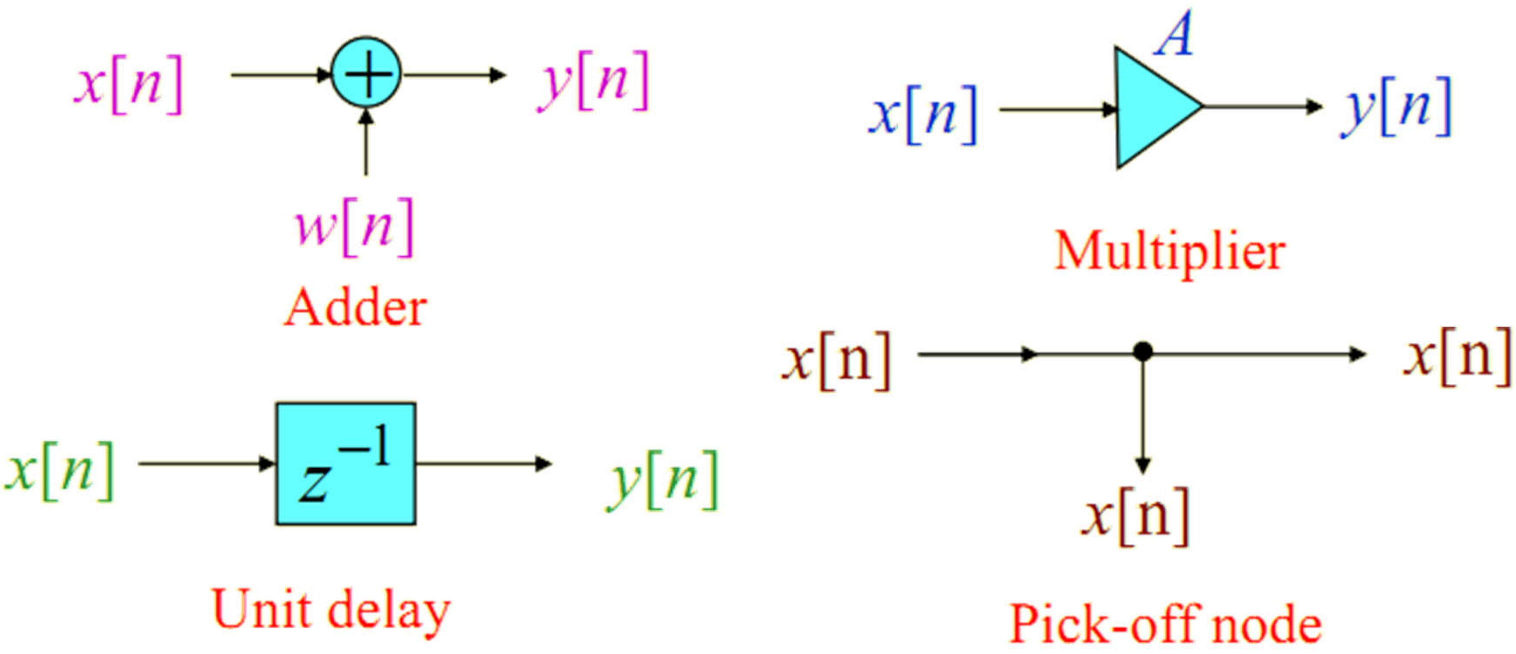
- 1. Two ways
  - Block diagram
  - Signal flow chart

- 2. From math expressions to graphs
  - From CCLDE
  - From transfer function

- 3. From graphs to math expressions

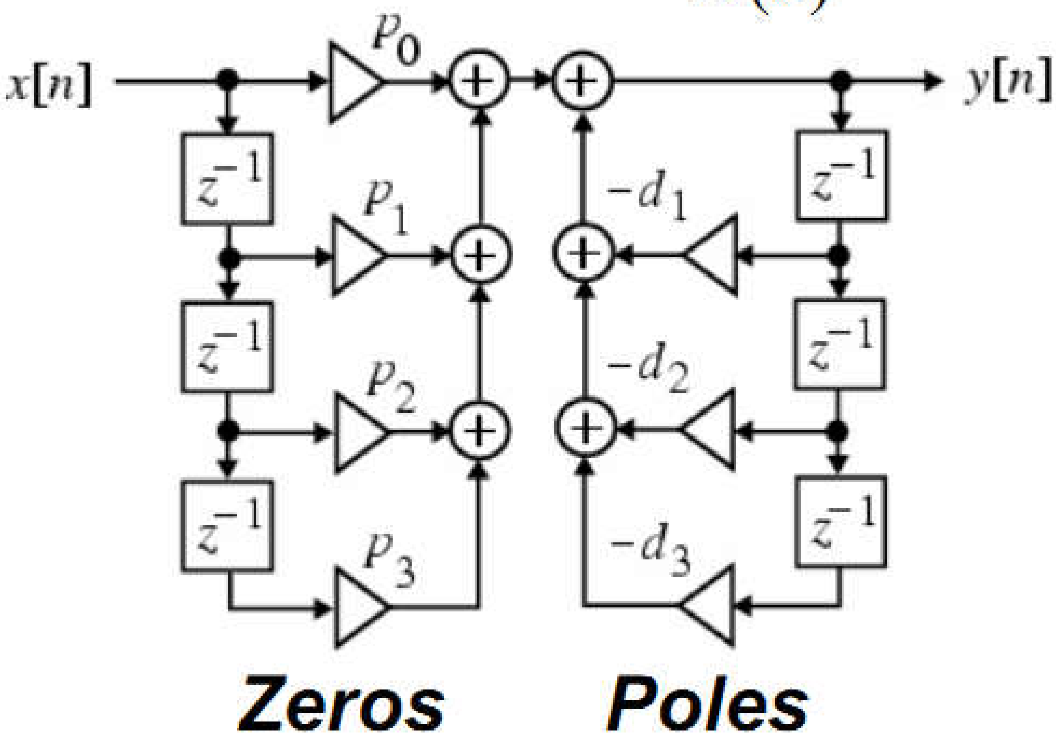
13.1 Basic building blocks

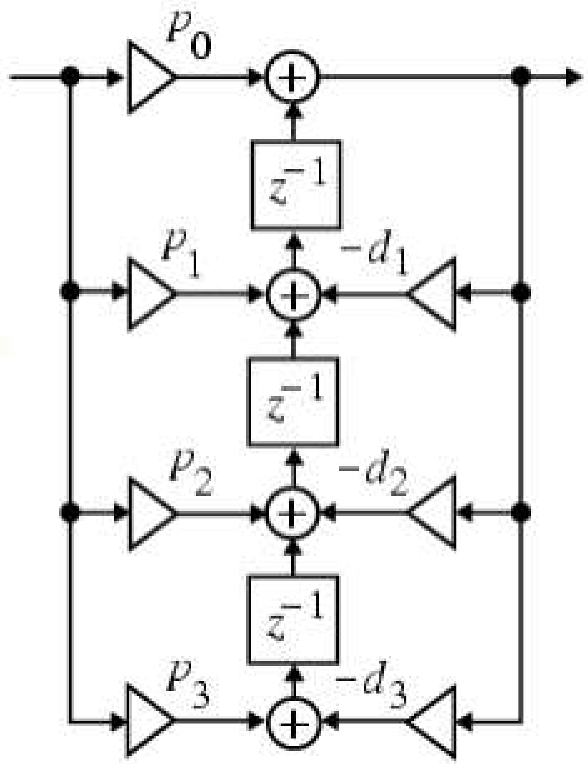
- The computational algorithm of an LTI digital filter can be conveniently represented in block diagram form using the basic building blocks:



13.2 Drawing

$$H(z) = P(z) \cdot \frac{1}{D(z)}$$

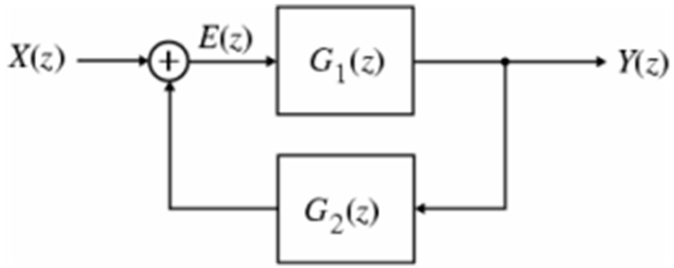




13.3 From graphs to math expressions

- Given a block diagram, the filter implemented by that diagram can be obtained by
  - writing down the input / output equations on key points of the block diagram;
  - eliminating the internal variables;
  - finally obtaining the main input / output expression.

- Example: consider the single-loop feedback structure:



Lecture 14 FIR Filters

- 1. Linear-phase FIR filters
  - In lecture 12

- 2. FIR filter design
  - Window method

14.2 FIR Filter Design

- This is the basic, straightforward approach to FIR filter design:
  - Step 1: Start with an ideal filter that meets the design criteria, say a filter $H_d(\omega)$
  - Step 2: Take the inverse DTFT of this $H_d(\omega)$ to obtain $h_d[n]$.
    - This $h_d[n]$ will be double infinitely long, and non-causal -> unrealizable
  - Step 3: Truncate using a window, say a rectangle, so that 2M+1 coefficients of $h_d[n]$ are retained, and all the others are discarded.
    - We now have a finite length (order 2M) filter, $h_t[n]$, however, it is still non-causal
  - Step 4: Shift the truncated $h_t[n]$ to the right (i.e., delay) by M samples, so that the first sample now occurs at n=0.
    - The resulting impulse response, $h_t[n-M]$ is a causal, stable, FIR filter, which has an almost identical magnitude response and a phase factor or $e^{-j\omega M}$ compared to the original filter, due to delay introduced.

Lecture 15 IIR Filters

- 1. Bilinear Transformation
  - Frequency warping
- 2. Analogue Filters
  - Butterworth filter
- 3. IIR Filter Design
  - Frequency prewar
  - Find H(s)
  - H(s) => H(z) by using bilinear transformation

15.1 Bilinear transformation

- Since, the frequency response is defined on the unit circle,

$$z = e^{j\omega} \Leftrightarrow s = j\Omega$$

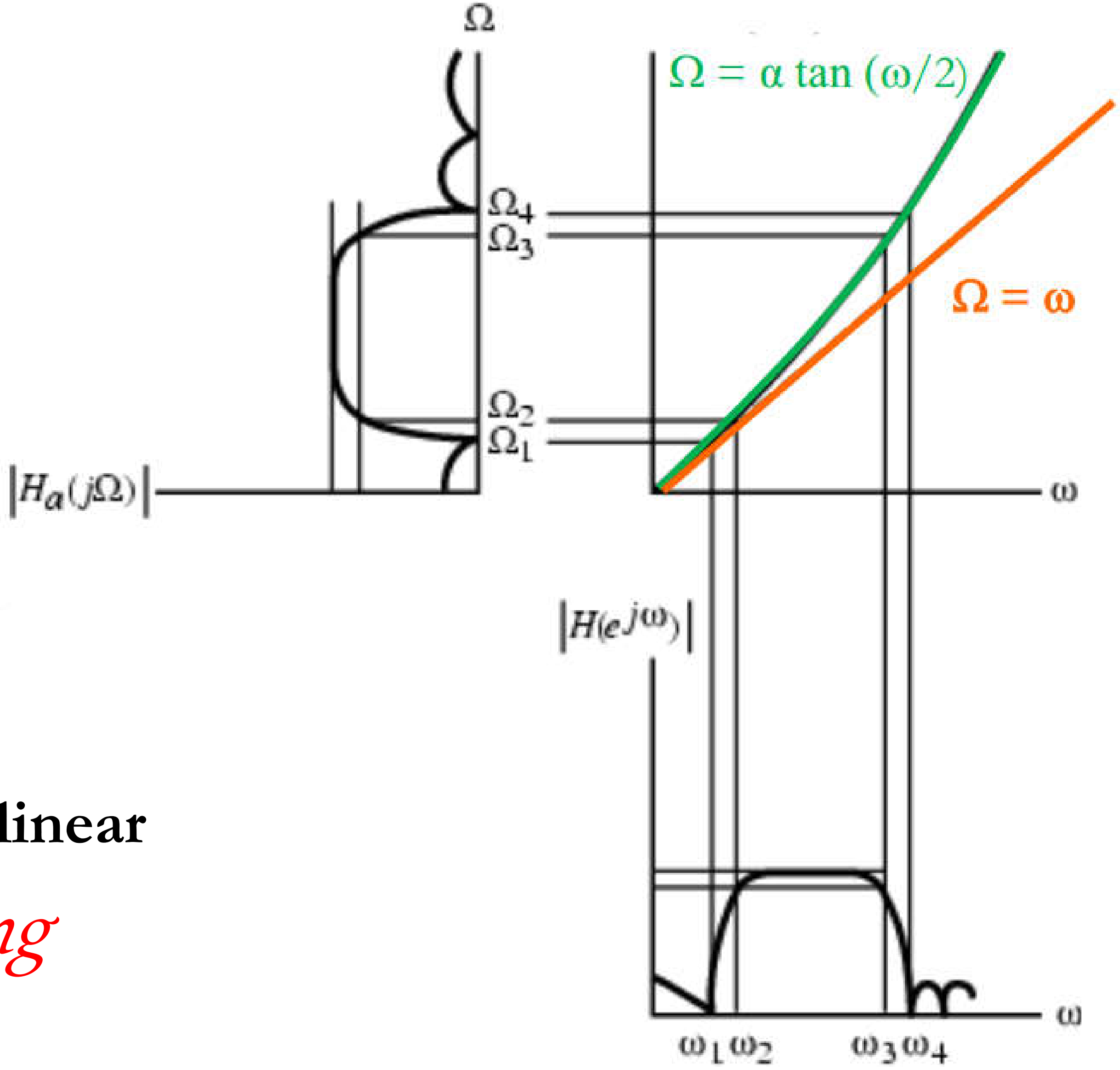
$$\boxed{s = \frac{2}{T_s}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

$$\Downarrow$$

$$\boxed{j\Omega = \frac{2}{T_s}\left(\frac{1-e^{-j\omega}}{1+e^{-j\omega}}\right)}$$

$$\Downarrow$$

$$\boxed{\Omega = \frac{2}{T_s}\tan\frac{\omega}{2} \Leftrightarrow \omega = 2\tan^{-1}\frac{\Omega T_s}{2}}$$



This mapping is (highly) nonlinear

=> Frequency warping

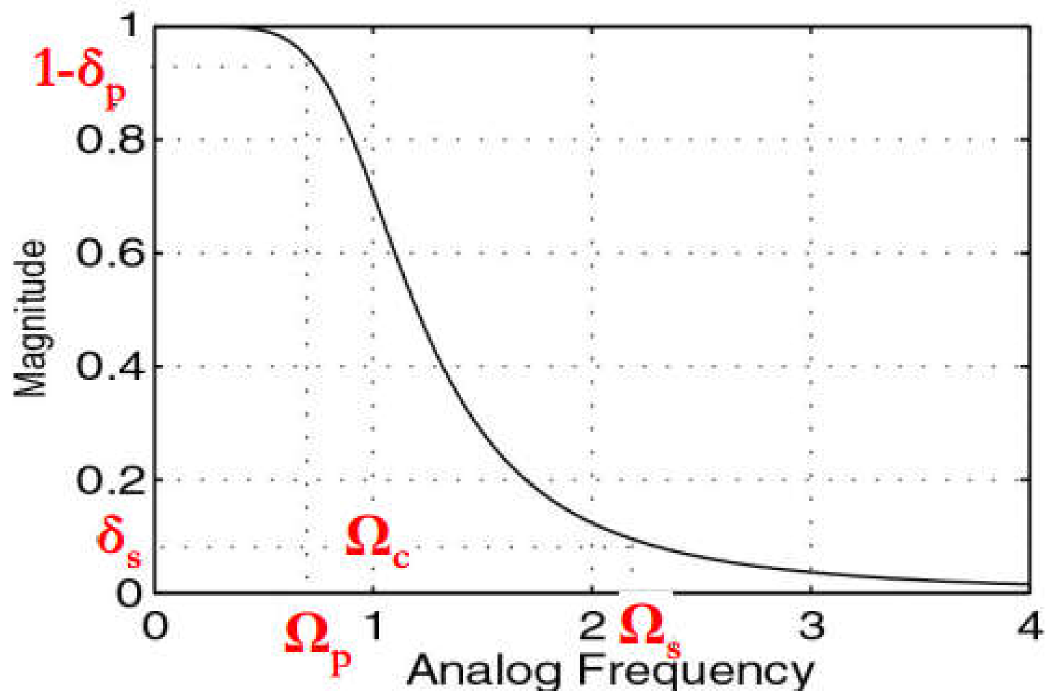
15.2 Butterworth Filter Design

- Two parameters completely characterizing a Butterworth lowpass filter are Ω_c and N

$$H_a(s) = \frac{\Omega_c^N}{\prod_{l=1}^{N}(s - p_l)}$$

where $p_l = \Omega_c e^{j\frac{\pi(2l+1+N)}{2N}}$

- To design a Butterworth filter, we thus need to find out Ω_c and N. They are determined from the specified band edges Ω_p and Ω_s, and minimum passband magnitude $1-\delta_p$, and maximum stopband ripple δ_s.



15.3 Designing Procedure

- 1. Prewarp ω_p, ω_s to find their analog equivalents Ω_p, Ω_s;
- 2. Design the analog filter: Determine N and Ω_c:
  - a) From δ_p, δ_s, Ω_p and Ω_s obtain the order of the filter N
    - Note that the order N must be integer, so the value obtained from this expression must be rounded up to exceed the specifications
  - Use N, δ_p, and Ω_p to calculate the 3dB cutoff frequency Ωc
  - Determine the corresponding H(s) and its poles
- 3. Apply bilinear transformation to obtain H(z)

Next lecture

- Tomorrow we will give some information of the final exam

- Q/A
  - Prepare some questions if you had any