Xi'an Jiaotong-Liverpool University
西交利物浦大学

EEE204 Continuous and Discrete Time Signals and Systems II

2018–2019 Semester 2

Electrical and Electronic Engineering

Xi'an Jiaotong-Liverpool University

Week 11

The Discrete Fourier Transform (DFT)

DFT

Why yet another transform? We have FT tools for periodic and aperiodic signals in both CT and DT! What is left?

- What if we want to automate DTFT using a computer?
- There is a problem since ω is a continuous variable that runs from $-\pi$ to π.
- We need an (uncountably) infinite number of ω's which cannot be done on a computer.
- For example, we cannot implement the ideal low-pass filter digitally.
- What happens if we do not use all the ω's, but rather just a finite set (which can be stored digitally).
- In general this will entail irrecoverable information loss.

DFT

- Any signal that is stored in a computer must be a finite length sequence, say $x[0], x[1], \cdots, x[L-1]$.
- Since there are only L signal time samples, we should not need an infinite number of frequencies to adequately represent the signal.
- In fact, exactly $N \geqslant L$ frequencies should be enough information.
- Using the DFT, we can convolve two generic sampled signals stored in a computer.

The N-point DFT of any signal $x[n]$ is defined as follows:

$$X[k] \triangleq \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn}, k = 0, \cdots, N-1,$$

or

$$X[\cdot] = \text{DFT}\{x[\cdot]\},$$

or shorthand:

$$x[n] \overset{\text{DFT}}{\leftrightarrow} X[k].$$

DFT

$X[k] \triangleq \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi}{N}kn}$

What are the choices for $X[k]$, when $k \notin \{0, 1, \cdots, N-1\}$?

- Treat $X[k]$ as an N-periodic function that is defined for all integer arguments $k \in \mathbb{Z}$.

$$X[k+N] \triangleq \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi}{N}(k+N)n} = X[k].$$

- Treat $X[k]$ as undefined for $k \notin \{0, 1, \cdots, N-1\}$.
- Treat $X[k]$ as being zero for $k \notin \{0, 1, \cdots, N-1\}$.

Example

Find the DFT of $x[n] = \delta[n] + 0.9\delta[n-3]$

What is L? $L = 4$, let us use $N = 4$

$$
\begin{aligned}
X[k] &= \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn}, \\
&= \sum_{n=0}^{3} x[n] e^{-j\frac{2\pi}{N}kn}, \\
&= \sum_{n=0}^{3} (\delta[n] + 0.9\delta[n-3]) e^{-j\frac{2\pi}{N}kn}, \\
&= 1 + 0.9 e^{-j\frac{2\pi}{4}k\cdot 3}.
\end{aligned}
$$

Example

$X[k] = 1 + 0.9e^{-j\frac{2\pi}{4}k3}$

$$X[0] = 1 + 0.9e^{-j\frac{2\pi}{4}\cdot 0\cdot 3} = 1.9.$$

$$X[1] = 1 + 0.9e^{-j\frac{2\pi}{4}\cdot 1\cdot 3} = 1 + 0.9j.$$

$$X[2] = 1 + 0.9e^{-j\frac{2\pi}{4}\cdot 2\cdot 3} = 1 - 0.9 = 0.1.$$

$$X[3] = 1 + 0.9e^{-j\frac{2\pi}{4}\cdot 3\cdot 3} = 1 - 0.9j.$$

Example

Find the N-point DFT of $x[n] = e^{j\frac{2\pi}{N}k_0 n}$

$$\begin{aligned} X[k] &= \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn}, \\ &= \sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}k_0 n} e^{-j\frac{2\pi}{N}kn}, \\ &= \sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}(k-k_0)n}, \end{aligned}$$

For $k = k_0 + lN, l \in \mathbb{Z}$,

$$X[k] = N.$$

Example

Find the N-point DFT of $x[n] = e^{j\frac{2\pi}{N}k_0 n}$

$$X[k] = \sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}(k-k_0)n},$$

For $k \neq k_0 + lN, l \in \mathbb{Z}$,

$$X[k] = \frac{1 - e^{-j\frac{2\pi}{N}(k-k_0)N}}{1 - e^{-j\frac{2\pi}{N}(k-k_0)}} = 0.$$

$$\therefore X[k] = N \sum_{l=-\infty}^{\infty} \delta[k - k_0 - lN], l \in \mathbb{Z}.$$

Example

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Find the N-point DFT of $x[n] = e^{j\omega_0 n}, \omega_0 \neq \frac{2\pi}{N}k_0$ for any k_0

$$
\begin{aligned}
X[k] &= \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn}, \\
&= \sum_{n=0}^{N-1} e^{j\omega_0 n} e^{-j\frac{2\pi}{N}kn}, \\
&= \sum_{n=0}^{N-1} \left[e^{j\left(\omega_0 - \frac{2\pi}{N}k\right)} \right]^n, \\
&= \frac{1 - \left[e^{j\left(\omega_0 - \frac{2\pi}{N}k\right)} \right]^N}{1 - e^{j\left(\omega_0 - \frac{2\pi}{N}k\right)}} = \frac{1 - e^{j\omega_0 N}}{1 - e^{j\left(\omega_0 - \frac{2\pi}{N}k\right)}}.
\end{aligned}
$$

The Modulo Function

The Modulo Function

If $m = m_0 + lN$ with $m_0 \in \{0, 1, \cdots, N-1\}$ and $l \in \mathbb{Z}$ then $m \bmod N = m_0$.

You can also think of $m \bmod N$ as the remainder when dividing m by N.

Example: $1 \bmod 4 = 1$; $7 \bmod 4 = 3$; $-1 \bmod 4 = 3$; $-8 \bmod 4 = 0$.

Periodic Superposition and Circular Extension

N-point Periodic Superposition

For any signal $x[n]$, be it time-limited or not, we define the N-point periodic superposition of $x[n]$ as follows:

$$x_{\text{ps}}[n] = \sum_{l=-\infty}^{\infty} x[n - lN].$$

Note that all the values of $x[n]$ affect $x_{\text{ps}}[n]$.

$$x_{\text{ps}}[n] \overset{\text{DFT}}{\leftrightarrow} X_{\text{ps}}[k] = X(e^{j\omega})\Big|_{\omega=\frac{2\pi}{N}k}.$$

N-point Circular Extension

For any signal $x[n]$, be it time-limited or not, we define the N-point circular extension of $x[n]$ as follows:

$$x([n])_N = x[n \text{ mod } N].$$

Note that only the values of $x[n]$ for $n \in \{0, 1, \cdots, N-1\}$ affect $x[n \text{ mod } N]$.

Difference between Two Extensions

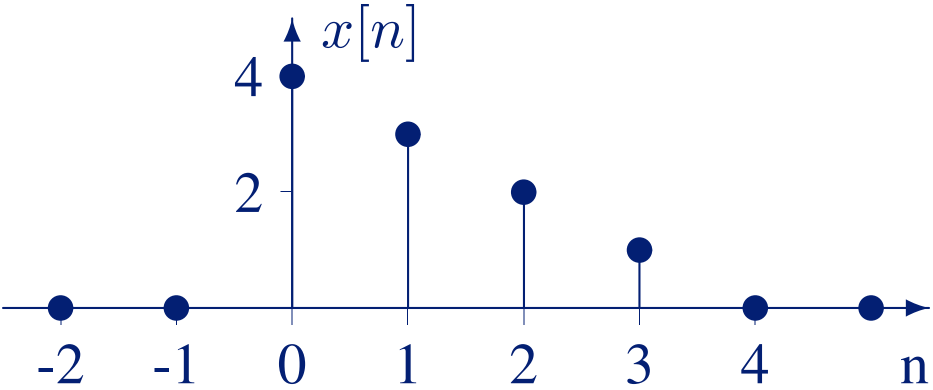
$$x_{\mathsf{ps}}[n] = \sum_{l=-\infty}^{\infty} x[n - lN], x([n])_N = x[n \bmod N]$$

- Both $x_{\mathsf{ps}}[n]$ and $x[n \bmod N]$ are N-periodic signals. They are both defined for all values of $n \in \mathbb{Z}$.
- In general, $x_{\mathsf{ps}}[n]$ and $x[n \bmod N]$ are different signals.
- If $x[n]$ is a time-limited signal over $0, \cdots, L-1$, also called a finite-length sequence, with $L \leqslant N$, then $x_{\mathsf{ps}}[n] = x[n \bmod N]$, and they consist of of shifted replicates of $x[n]$. Otherwise $x_{\mathsf{ps}}[n]$ and $x[n \bmod N]$ differ!

Example 1

$x[n] = \{\underline{4}, 3, 2, 1\}$

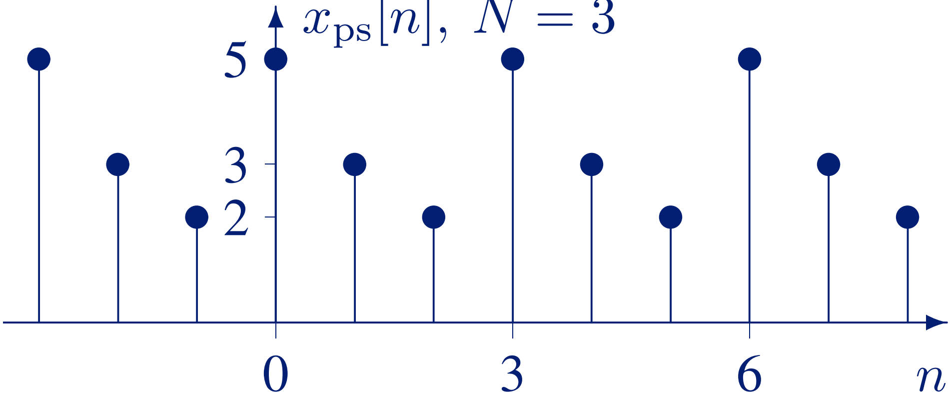
Find the 3-point periodic superposition and 3-point circular extension.



Example 1

$x[n] = \{\underline{4}, 3, 2, 1\}$

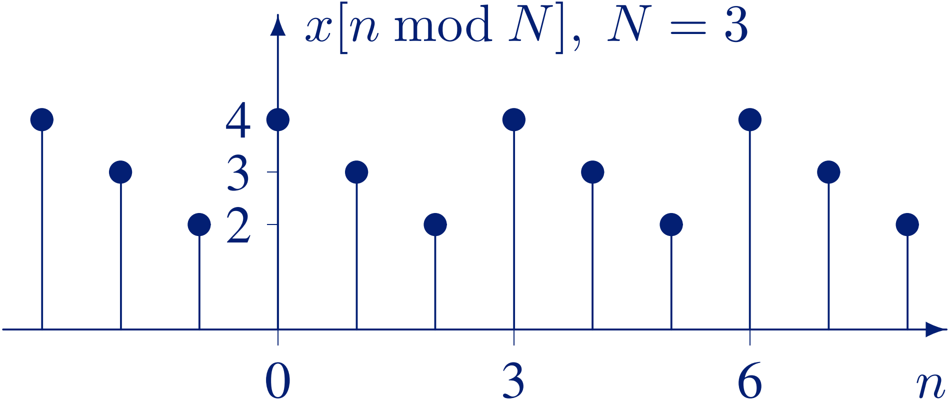
$$L = 4 > N = 3, x_{\text{ps}}[n] = \sum_{l=-\infty}^{\infty} x[n - lN].$$



Example 1

$x[n] = \{\underline{4}, 3, 2, 1\}$

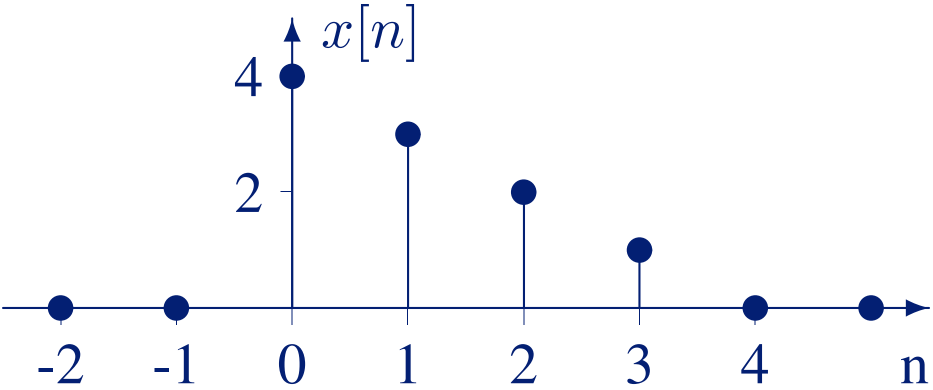
$$L = 4 > N = 3, x([n])_N = x[n \bmod N].$$



Example 2

$x[n] = \{\underline{4}, 3, 2, 1\}$

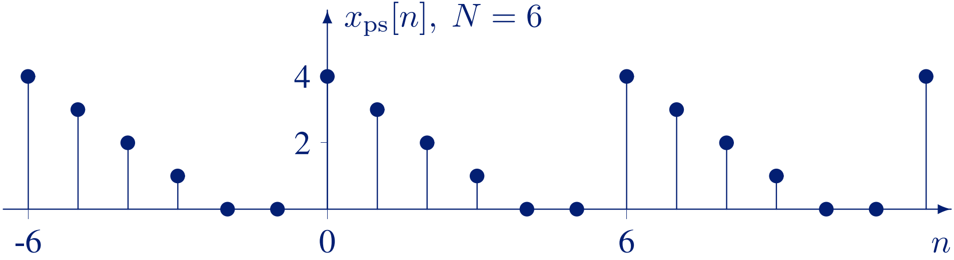
Find the 6-point periodic superposition and 6-point circular extension.



Example 2

$x[n] = \{\underline{4}, 3, 2, 1\}$

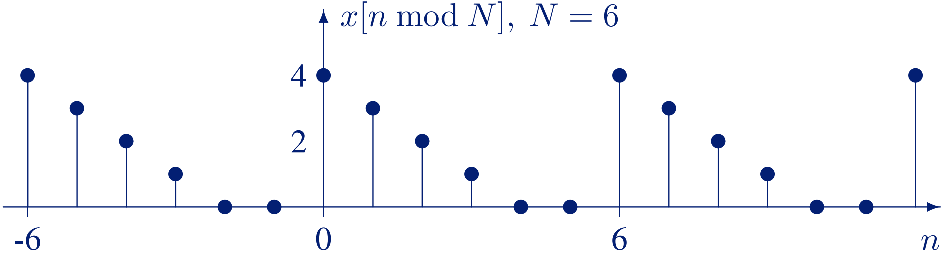
$$L = 4 < N = 6, x_{\mathsf{ps}}[n] = \sum_{l=-\infty}^{\infty} x[n - lN].$$



Example 2

$x[n] = \{\underline{4}, 3, 2, 1\}$

$$L = 4 < N = 6, x([n])_N = x[n \text{ mod } N].$$



Properties of the DFT

Linearity

If $x_1[n] \overset{\text{DFT}}{\leftrightarrow} X_1[k]$ and $x_2[n] \overset{\text{DFT}}{\leftrightarrow} X_2[k]$ then

$$x[n] = a_1 x_1[n] + a_2 x_2[n]$$

$$\overset{\text{DFT}}{\leftrightarrow} a_1 X_1[k] + a_2 X_2[k].$$

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Linearity

Proof:

$$x[n] = a_1 x_1[n] + a_2 x_2[n] \overset{\text{DFT}}{\leftrightarrow} a_1 X_1[k] + a_2 X_2[k].$$

$$\begin{aligned}
X[k] &= \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn}, \\
&= \sum_{n=0}^{N-1} (a_1 x_1[n] + a_2 x_2[n]) e^{-j\frac{2\pi}{N}kn}, \\
&= a_1 \sum_{n=0}^{N-1} x_1[n] e^{-j\frac{2\pi}{N}kn} + a_2 \sum_{n=0}^{N-1} x_2[n] e^{-j\omega n}, \\
&= a_1 X_1[k] + a_2 X_2[k].
\end{aligned}$$

Symmetries

The next set of properties of the DFT describes what happens when the signal $x[n]$ has certain symmetries. Hence we first need to appropriately define "symmetries" in the context of periodic sequences.

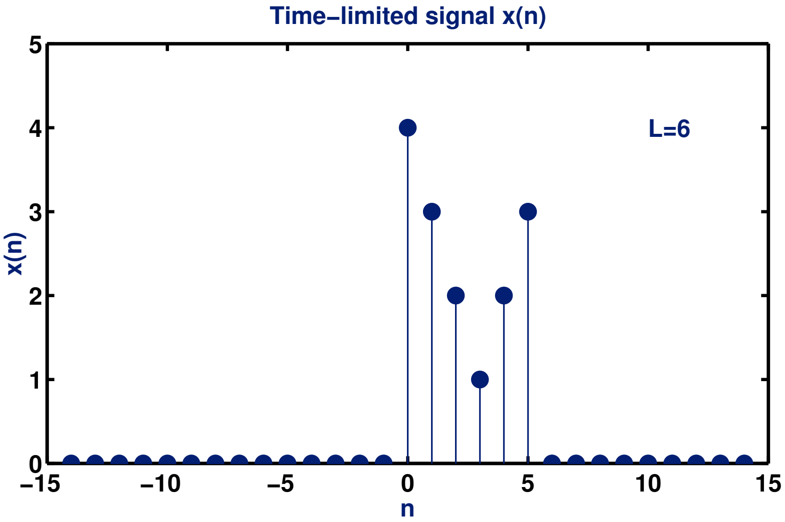
A signal $x[n]$ is called N-point circularly even iff its N-point circular extension, $x[n \bmod N]$, is even.

Equivalently, a signal $x[n]$ is N-point circularly even iff $x[n \bmod N] = x[-n \bmod N]$.

Example 1

$x[n] = \{\underline{4}, 3, 2, 1, 2, 3\}$

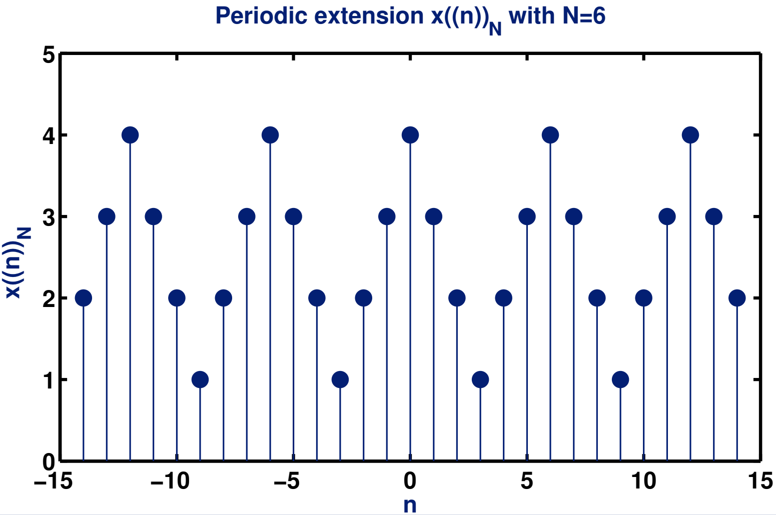
Is this signal 6-point circularly even?



Example 1

$x[n] = \{\underline{4}, 3, 2, 1, 2, 3\}$

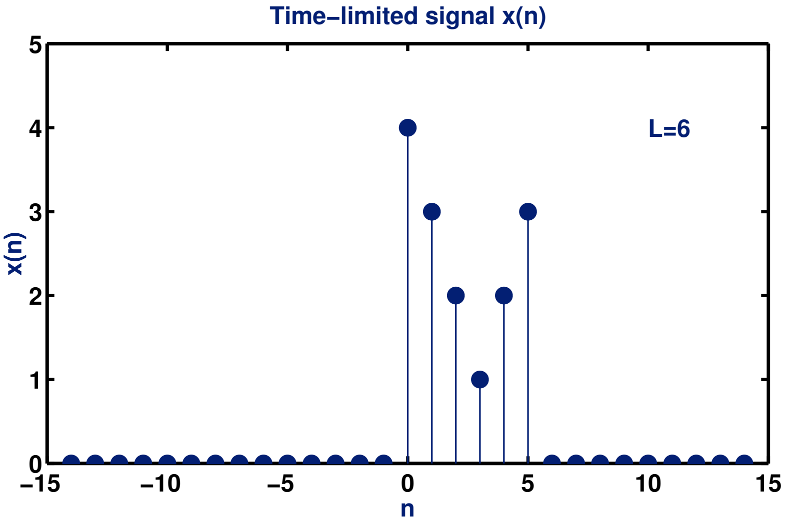
Yes, as the following figure illustrates.



Example 2

$x[n] = \{\underline{4}, 3, 2, 1, 2, 3\}$

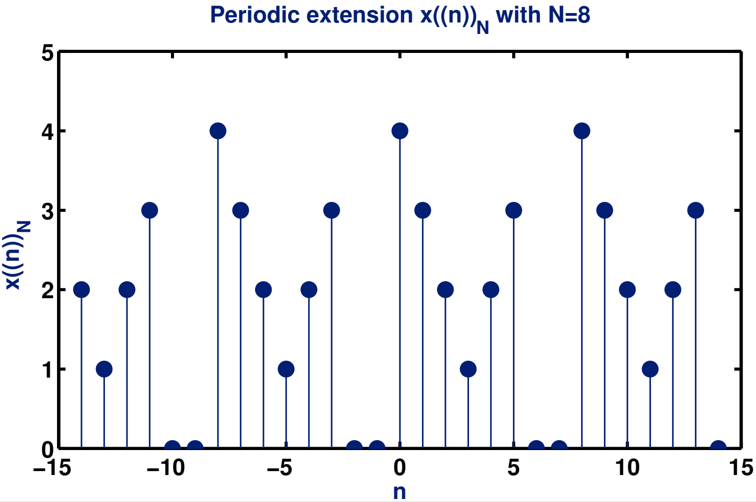
Is this signal 8-point circularly even?



Example 2

$x[n] = \{\underline{4}, 3, 2, 1, 2, 3\}$

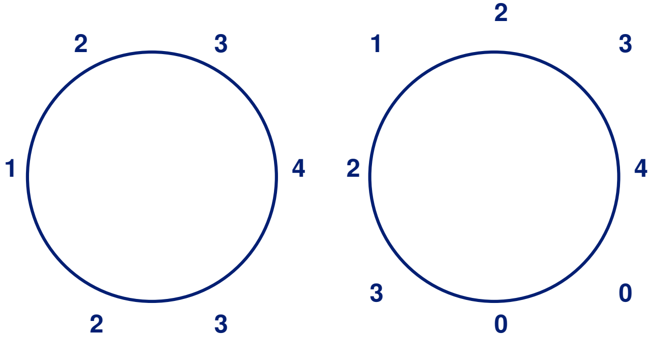
No, for $N = 8$ it is not circularly even, since $x[n \bmod N]$ is not even for $N = 8$.



Simple Test

$x[n] = \{\underline{4}, 3, 2, 1, 2, 3\}$

A simple test for whether a sequence is N-point circularly even is to draw it around a circle (in N evenly spaced points).



If the sequence is the same whether you read it out CW or CCW, then it is circularly even.

By considering the points around the circle, we conclude the following.

An signal is N-point circularly even iff $x[N-n] = x[n], n = 1, \cdots N-1$, where $x[0]$ is arbitrary.

An signal is N-point circularly odd iff $x[N-n] = -x[n], n = 1, \cdots N-1$, where $x[0] = 0$.

Example

$x[n] = \{\underline{0}, -3, 2, 0, -2, 3\}$

Whether the above sequence is circularly even or circularly odd?

$$x[0] = 0,$$

$$x[1] = -x[6-1] = -x[5] = -3,$$

$$x[2] = -x[6-2] = -x[4] = 2,$$

$$x[3] = -x[6-3] = -x[3] = 0,$$

Therefore, it is 6-point circularly odd signal.

Decomposition

We can always decompose an N-point sequence into circularly even and circularly odd components:

$$x[n] = x_{\text{ce}}[n] + x_{\text{co}}[n],$$

where

$$x_{\text{ce}}[n] = \begin{cases} \frac{1}{2}(x[n] + x[N-n]), & n = 1, \cdots, N-1 \\ x[0], & n = 0, \end{cases}$$

$$x_{\text{co}}[n] = \begin{cases} \frac{1}{2}(x[n] - x[N-n]), & n = 1, \cdots, N-1 \\ 0, & n = 0, \end{cases}$$

Example

$x[n] = \{\underline{4}, 3, 2, 1\}$

Decompose the above sequence into one circularly even and one circularly odd sequence.

$$x_{\text{ce}}[n] = \begin{cases} \frac{1}{2}(x[n] + x[N-n]), & n = 1, \cdots, N-1 \\ x[0], & n = 0, \end{cases}$$

$$x_{\text{co}}[n] = \begin{cases} \frac{1}{2}(x[n] - x[N-n]), & n = 1, \cdots, N-1 \\ 0, & n = 0, \end{cases}$$

$$x_{\text{ce}}[n] = \{\underline{4}, 2, 2, 2\},$$

$$x_{\text{co}}[n] = \{\underline{0}, 1, 0, -1\}.$$

Symmetry properties

If $x[n]$ is real,then its DFT has circular Hermitian symmetry:

$$X[k] = X^*[-k \text{ mod } N].$$

If $x[n]$ is circularly even,then $X[k]$ is circularly even.

Combing the above, if $x[n]$ is real and circularly even, then $X[k]$ is also real and circularly even.

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Circular time-reversal

Ordinary time-reversal of time-limited sequence would yield a sequence that is not limited to 0 to $N-1$. Instead, we first take the N-point circular extension of the signal, time-reverse that, and then pick out the values from 0 to $N-1$. This is called circular time-reversal. It is equivalent to writing the sequence CCW around a circle, and then reading the values CW.

If $x[n] = \{\underline{10}, 11, 12, 13, 14\}$ and $N = 6$,

then $x[-n \bmod N] = \{\underline{10}, 0, 14, 13, 12, 11\}$.

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Circular time-shift

Since time-shifting a time-limited sequence would yield a sequence that is not limited to 0 to $N-1$. Instead, we first take the N-point circular extension of the signal, time-shift that, and then pick out the values from 0 to $N-1$. This is called N-point circular time-shift. It is equivalent to writing the sequence CCW around a circle, and then reading the values CCW from point $-l$.

If $x[n] = \{\underline{10}, 11, 12, 13, 14\}$ and $N = 6$ and $l = 2$,

then $x[n - l \text{ mod } N] = \{\underline{14}, 0, 10, 11, 12, 13\}$.

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Circular time-reversal and time-shift

The DFT property for circular time-reversal is

$$x[-n \bmod N] \overset{\text{DFT}}{\leftrightarrow} X[-k \bmod N].$$

The DFT property for circular time-shifting is

$$x[n - n_0 \bmod N] \overset{\text{DFT}}{\leftrightarrow} e^{-j\frac{2\pi}{N}kn_0} X[k].$$

DFT v.s. DTFT v.s. the z-Transform

If $x[n]$ is an L-point signal with $L \leqslant N$, then

$$X[k] = X(e^{j\omega})\Big|_{\omega = \frac{2\pi}{N}k},$$

If $x[n]$ is an L-point signal with $L \leqslant N$, then

$$X[k] = X(z)\Big|_{z = e^{j\frac{2\pi}{N}k}},$$

If $x[n]$ is an L-point signal with $L \leqslant N$, then

$$X(z) = \frac{1 - z^{-N}}{N} \sum_{k=0}^{N-1} \frac{X[k]}{1 - e^{j\frac{2\pi}{N}k} z^{-1}}.$$

Inverse DFT

Inverse DFT

The N-point inverse DFT of given $X[k]$ is defined as follows:

$$x[n] \triangleq \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j\frac{2\pi}{N}kn}, n = 0, \cdots, N-1,$$

or

$$x[\cdot] = \text{IDFT}\{X[\cdot]\},$$

or shorthand:

$$X[k] \overset{\text{IDFT}}{\leftrightarrow} x[n].$$

Example

Find the IDFT of $X[k] = \delta[k-k_0], k_0 \in \{0, 1, \cdots, N-1\}$

$$
\begin{aligned}
x[n] &= \frac{1}{N}\sum_{k=0}^{N-1} X[k] e^{j\frac{2\pi}{N}kn}, \\
&= \frac{1}{N}\sum_{k=0}^{N-1} \delta[k-k_0] e^{j\frac{2\pi}{N}kn}, \\
&= \frac{1}{N} e^{j\frac{2\pi}{N}k_0 n},
\end{aligned}
$$

$$
\frac{1}{N} e^{j\frac{2\pi}{N}k_0 n} \overset{\text{DFT}}{\longleftrightarrow} \delta[k-k_0], k_0 \in \{0, 1, \cdots, N-1\}.
$$

Example

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Find the 8-point DFT of $x[n] = 6\cos^2(\frac{\pi}{4}n)$

$$\begin{aligned} x[n] &= 3 + 3\cos\left(\frac{\pi}{2}n\right), \\ &= 3 + \frac{3}{2}e^{j\frac{\pi}{2}n} + \frac{3}{2}e^{-j\frac{\pi}{2}n}, \\ &= 3 + \frac{3}{2}e^{j\frac{2\pi}{8}2n} + \frac{3}{2}e^{-j\frac{2\pi}{8}2n}, \\ &= 3 + \frac{3}{2}e^{j\frac{2\pi}{8}2n} + \frac{3}{2}e^{-j\frac{2\pi}{8}2n}e^{j2\pi n}, \\ &= \frac{1}{8}[24 + 12e^{j\frac{2\pi}{8}2n} + 12e^{j\frac{2\pi}{8}6n}], \end{aligned}$$

$$X[k] = \{\underline{24}, 0, 12, 0, 0, 0, 12, 0\}.$$

Homework

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Find the DFT of the following sequences

1. $x[n] = \begin{cases} 1 & n = 0, 3 \\ 0 & n = 1, 2 \end{cases}$ with length $N = 4$;
2. $x[n] = \begin{cases} 1 & n \text{ even} \\ -1 & n \text{ odd} \end{cases}$ with length $N = 8$;
3. $x[n] = 0.6^n$ with length $N = 8$;
4. $x[n] = u[n] - u[n-8]$ with length $N = 8$;
5. $x[n] = \cos(\omega_0 n)$ with $\omega_0 \neq \frac{2\pi m}{N}, m \in \mathbb{Z}$.

Thank you for your attention.