MTH101: Lecture 10

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Power series representation of Analytic functions

Theorem

Suppose that the function $f(z)$ is given by the sum of a power series

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n,$$

with radius of convergence $R > 0$. Then

- *$f(z)$ is* **analytic** *in the set $\{z \in \mathbb{C} : |z - z_0| < R\}$.*
- *Its derivative, $f'(z)$, can be represented by a power series*

$$f'(z) = \sum_{n=1}^{\infty} n a_n (z - z_0)^{n-1},$$

which converges in the set $\{z \in \mathbb{C} : |z - z_0| < R\}$.

Theorem (Cont.)

- *The anti-derivative F (that is, $F'(z) = f(z)$) can be represented by a power series*

$$F(z) = \sum_{n=0}^{\infty} \frac{a_n}{n+1}(z - z_0)^{n+1},$$

which converges in the set $\{z \in \mathbb{C} : |z - z_0| < R\}$.

Operations on Power series

Termwise addition or substraction: If we have two power series with radii of convergence R_1 and R_2, $R_1 < R_2$

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \ |z| < R_1 \quad \text{and} \quad g(z) = \sum_{n=0}^{\infty} b_n z^n, \ |z| < R_2,$$

then

$$f(z) \pm g(z) = \sum_{n=0}^{\infty} (a_n \pm b_n) z^n, \quad |z| < R_1 \ \text{(the smaller one)}$$

Termwise Multiplication: If we have two power series with radii of convergence R_1 and R_2, $R_1 < R_2$

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \ |z| < R_1 \quad \text{and} \quad g(z) = \sum_{n=0}^{\infty} b_n z^n, \ |z| < R_2,$$

Then the product $f(z)g(z)$ is convergent in the smaller disk $|z| < R_1$, but (!!)

$$f(z)g(z) \neq \sum_{n=0}^{\infty} (a_n b_n) z^n.$$

Taylor Series

Theorem

Let $f(z)$ be an analytic function, given by the power series

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n,$$

which is convergent in the set $\{z \in \mathbb{C} : |z - z_0| < R\}$, $R > 0$. Then

$$a_n = \frac{1}{n!} f^{(n)}(z_0) = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{(z - z_0)^{n+1}} dz,$$

where the function $f(z)$ is analytic in a Simply Connected Domain containing the closed, simple, counterclockwise oriented path γ which encloses the point z_0.
The power series is called **Taylor Series** *of $f(z)$, while if $z_0 = 0$ it is called* **McLaurin Series** *of $f(z)$.*

Taylors Theorem

Theorem

Let $f(z)$ be an **Analytic function** *in the domain D and let $z_0 \in D$. Then, the* **Taylor series** *of $f(z)$ converges to $f(z)$, that is*

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(z_0)(z - z_0)^n.$$

in the largest open disc with center in z_0

$$\{z \in \mathbb{C} : |z - z_0| < R\},$$

in which $f(z)$ is analytic.
Moreover, if we set $M(r) = \max_{|z-z_0|=r} |f(z)|$, then

$$|f^{(n)}(z_0)| \leq \frac{n!}{r^n} M(r), \quad \text{with } 0 < r < R.$$

Remark

If R is the largest number such that the function $f(z)$ is analytic in

$$\{z \in \mathbb{C} : |z - z_0| < R\},$$

then there exists at least one point, z^*, on

$$\{z \in \mathbb{C} : |z - z_0| = R\},$$

at which $f(z)$ is not Analytic.

Remark

Fix a point z_0 and consider the **Taylor Series with center** *z_0 of a function $f(z)$.*
Then its **Radius of Convergence** *is given by*

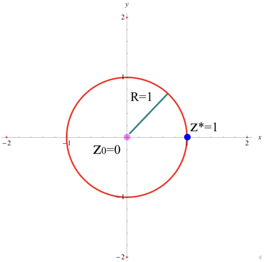
$$R = |z_0 - \tilde{z}|,$$

where $\tilde{z}$ is the nearest point to z_0, at which $f(z)$ is **not Analytic.**
We call $\tilde{z}$ a **Singular Point**.

The Geometric Series

$$f(z) = \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n, \quad \text{if } |z| < 1.$$

Then the **Radius of Convergence** is $R = 1$, $z_0 = 0$ and $a_n = 1$ for all n.
There is a Singular point $z^* = 1$, since $f(z)$ is Analytic in $\mathbb{C} \setminus \{1\}$, observe that $R = |z_0 - z^*| = 1.$



The Exponential Series

$$e^z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n, \quad \text{for all} \quad z \in \mathbb{C}.$$

The function $f(z)$ is Analytic in the whole Complex Plane (that is, $f(z)$ is **Entire**) then

$$R = +\infty.$$

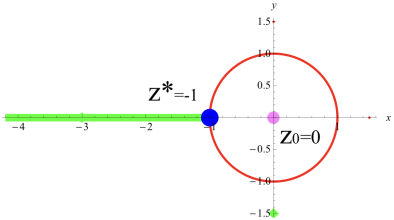
Here we have $z_0 = 0$ and $a_n = \frac{1}{n!}$ for all n.

The Logarithm (Principal Value)

The function $\mathrm{Ln}\,(1+z)$ is Analytic in the set

$$\mathbb{C} \setminus \{\mathrm{Im}(z+1) = 0, \mathrm{Re}(z+1) < 0\}.$$

We have infinitely many **Singular points** on the half-line $\{z = x + iy \in \mathbb{C} : y = 0, x \leq -1\}$.



Then if we chose $z_0 = 0$ as the center of the **Taylor Series** we have that $z^* = -1$ is the nearest point to $z_0 = 0$ at which the function is not Analytic.

Then the **Radius of Convergence** is

$$R = |z_0 - z^*| = |0 - 1| = 1,$$

and the **Taylor Series** of $f(z)$ converges to $f(z)$ for $|z - z_0| < R$, that is for $|z| < 1$:

$$\mathrm{Ln}\ (1 + z) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} z^n, \quad \text{for} \quad |z| < 1.$$

Trigonometric series

The functions $\cos z$ and $\sin z$ are **Entire functions** then their **Taylor Series** converges for all $z \in \mathbb{C}$, that is their **Radius of Convergence** is $R = +\infty$ for any fixed $z_0 \in \mathbb{C}$.
Then for $z_0 = 0$ we have

$$\cos z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n}, \quad \text{for all } z \in \mathbb{C},$$

and

$$\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1}, \quad \text{for all } z \in \mathbb{C}.$$

Hyperbolic Series

The functions $\cosh z$ and $\sinh z$ are **Entire functions** then their **Taylor Series** converges for all $z \in \mathbb{C}$, that is their **Radius of Convergence** is $R = +\infty$ for any fixed $z_0 \in \mathbb{C}$.
Then for $z_0 = 0$ we have

$$\cosh z = \sum_{n=0}^{\infty} \frac{1}{(2n)!} z^{2n}, \qquad \text{for all } z \in \mathbb{C},$$

and

$$\sinh z = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} z^{2n+1}, \qquad \text{for all } z \in \mathbb{C}.$$

Remark

In general it is very difficult to write the Taylor Series of a function $f(z)$ using the definition, that is computing the coefficients

$$a_n = \frac{f^{(n)}(z_0)}{n!}.$$

The Idea is to manipulate the well known series of a function $f(z)$ to obtain the Taylor Series of a different function.

Example (Substitution)

Write the Taylor Series with center $z_0 = 0$ of the function

$$f(z) = \frac{1}{2 + z^2},$$

and find its Radius of Convergence.

Solution

The function $f(z) = \frac{1}{2+z^2}$ is similar to the sum of the **Geometric Series**

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$$

which converges for $|z| < 1$.

The Idea is to manipulate the function $f(z)$:

$$\frac{1}{2+z^2} = \frac{1}{2}\left(\frac{1}{1+\frac{z^2}{2}}\right) = \frac{1}{2}\left(\frac{1}{1-\left(-\frac{z^2}{2}\right)}\right) = \frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{z^2}{2}\right)^n$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{2^n}z^{2n}$$

which converges for $\left|-\frac{z^2}{2}\right| < 1$.

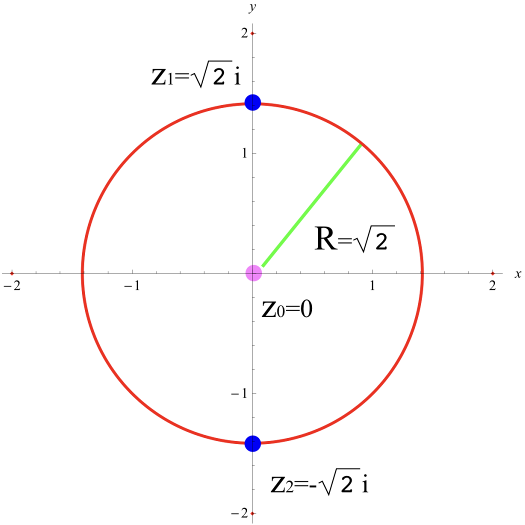
Then

$$f(z) = \frac{1}{2+z^2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} z^{2n}, \quad \text{for } |z| < \sqrt{2},$$

and its **Radius of Convergence** is $R = \sqrt{2}$.
We observe that $f(z)$ is not Analytic at the Singular Points $z_1 = \sqrt{2}i$ and at $z_2 = -\sqrt{2}i$, then both z_1 and z_2 are the nearest point to z_0 at which the function $f(z)$ is **not Analytic**.
Then the **Radius of Convergence** of the Taylor Series with center z_0 is

$$R = |z_0 - z_1| = |0 - \sqrt{2}i| = \sqrt{2},$$

and we get the same result obtained by the previous computation.


y
2
1
-1
-2
x
-2
-1
1
2
Z1=√2 i
R=√2
Z0=0
Z2=-√2 i

Example (Shift)

Write the Taylor Series with center $z_0 = 3$ of the Function

$$f(z) = e^z$$

and find its Radius of Convergence.

The function $f(z)$ is **Entire** then for any z_0 the **Radius of Convergence** of the Taylor Series with center z_0 is $R = +\infty$, that is the Taylor Series converges for all $z \in \mathbb{C}$.
We already know that if $z = 0$ then

$$e^{z} = \sum_{n=0}^{\infty} \frac{1}{n!} z^n, \quad \text{for all } z \in \mathbb{C}.$$

In this case $z_0 = 3$, so we need to manipulate the function $f(z)$:

$$\begin{aligned} e^z &= e^{z-3+3} = e^3 e^{z-3} = e^3 \sum_{n=0}^{\infty} \frac{1}{n!} (z-3)^n \\ &= \sum_{n=0}^{\infty} \frac{e^3}{n!} (z-3)^n, \quad \text{for all } z \in \mathbb{C}. \end{aligned}$$

Observe that the Taylor Series must be in powers of $(z - z_0)$.

Example (Integration)

Find the Maclaurin series (Taylor series centered at 0) of $f(z) = \arctan z$

Solution: We have $f'(z) = \frac{1}{1+z^2}$, thus by the geometric series

$$f'(z) = \frac{1}{1-(-z^2)} = \sum_{n=0}^{\infty}(-z^2)^n = \sum_{n=0}^{\infty}(-1)^n z^{2n}$$

where $|-z^2| < 1 \quad \Rightarrow \quad |z| < 1.$
By termwise integration,

$$f(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} z^{2n+1}, \text{ for all } |z| < 1.$$

Example (Differentiation)

Find the Maclaurin series of $f(z) = \frac{1}{(z+1)^2}$.

Solution: We note that

$$f(z) = \left[-\frac{1}{1+z}\right]' = \left[-\sum_{n=0}^{\infty}(-z)^n\right]' \quad \text{for all } |z| < 1.$$

Then by termwise differentiation,

$$f(z) = \sum_{n=1}^{\infty}(-1)^{n+1} n \cdot z^{n-1} \quad \text{for all } |z| < 1.$$

Bibliography

1 *Kreyszig, E.* **Advanced Engineering Mathematics**. Wiley, 9th Edition.