MTH101: Lecture 25 – 26

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Bessel Functions J_ν with Half-Integer ν

In order to write down all the J_ν with $\nu = \pm\frac{1}{2}, \pm\frac{3}{2}\cdots$, we only need to check $J_{1/2}$, $J_{-1/2}$, and the rest can be found by the recurrence relation

$$J_{\nu+1}(x) = \frac{2\nu}{x} J_\nu(x) - J_{\nu-1}(x).$$

Claim

$$(a)\ J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x, \qquad (b)\ J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x.$$

Bessel Functions J_ν with Half-Integer ν

Proof.

$$J_{1/2}(x) = x^{\frac{1}{2}} \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+1/2} m! \Gamma(m+\frac{3}{2})}$$

$$\Rightarrow J_{1/2}(x) = \sqrt{\frac{2}{x}} \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{2^{2m+1} m! \Gamma(m+\frac{3}{2})}.$$

We can further express the denominator as $A \cdot B$, with $A = (2^m m!)$, $B = [2^{m+1}\Gamma(m+\frac{3}{2})]$.

Bessel Functions J_ν with Half-Integer ν

Proof.

Therefore,

$$
\begin{aligned}
A &= 2^m m! = 2m(2m-2)(2m-4)\cdots 4\cdot 2,\\
B &= 2^{m+1}\Gamma\left(m+\frac{3}{2}\right) = 2^{m+1}\left(m+\frac{1}{2}\right)\left(m-\frac{1}{2}\right)\cdots\frac{3}{2}\cdot\frac{1}{2}\cdot\Gamma\left(\frac{1}{2}\right)\\
&= (2m+1)(2m-1)\cdots 3\cdot 1\cdot\sqrt{\pi},
\end{aligned}
$$

where we have used the fact that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, which will be proved below.

Bessel Functions J_ν with Half-Integer ν

Proof.

Therefore,

$$\begin{aligned} AB &= 2m(2m-2)(2m-4)\cdots 2\cdot(2m+1)(2m-1)\cdots 1\cdot\sqrt{\pi} \\ &= (2m+1)!\sqrt{\pi}, \end{aligned}$$

and hence

$$\begin{aligned} J_{1/2}(x) &= \sqrt{\frac{2}{x}}\sum_{m=0}^{\infty}\frac{(-1)^m x^{2m+1}}{(2m+1)!\sqrt{\pi}} = \sqrt{\frac{2}{\pi x}}\sum_{m=0}^{\infty}\frac{(-1)^m x^{2m+1}}{(2m+1)!} \\ &= \sqrt{\frac{2}{\pi x}}\sin x. \end{aligned}$$

$\square$

Bessel Functions J_ν with Half-Integer ν

Proof.

On the other hand, $J_{-1/2}$ can be found by

$$[x^\nu J_\nu(x)]' = x^\nu J_{\nu-1}(x),$$

with $\nu = 1/2$.

$$[x^{\frac{1}{2}} J_{1/2}]' = \sqrt{\frac{2}{\pi}} \cos x = x^{\frac{1}{2}} J_{-1/2}$$

$$\Rightarrow J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x.$$

$\square$

Bessel Functions J_ν with Half-Integer ν

Proof.

Finally, we need to prove that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

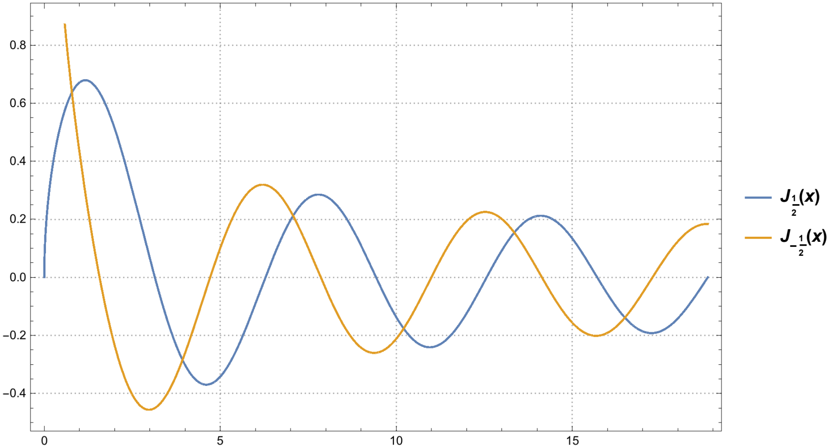
$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty e^{-t} t^{-1/2} dt = 2\int_0^\infty e^{-u^2} du,$$

where we use $t = u^2$ as the trick to perform this integral. Therefore,

$$\left[\Gamma\left(\frac{1}{2}\right)\right]^2 = 4\int_0^\infty \int_0^\infty e^{-(u^2+v^2)} dudv = 4\int_0^\infty \int_0^{\frac{\pi}{2}} e^{-r^2} rdrd\theta$$
$$= \pi \int_0^\infty e^{-r^2} dr^2 = \pi. \qquad \text{Thus } \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

$\square$

Bessel Functions J_ν with Half-Integer ν



To obtain the general solution of Bessel's equation for any ν, we now introduce the **Bessel's function of the second kind** $Y_\nu(x)$. We start with $\nu = n = 0$, and Bessel's equation can be written as

$$xy'' + y' + xy = 0,$$

with the indicial equation $r^2 = 0$. This is the Case 2 in Frobenius method, and we know the second solution to the equation is

$$y_2(x) = J_0 \ln x + \sum_{m=1}^{\infty} A_m x^m,$$

and we can find the coefficients A_m by substituting it into Bessel's equation.

We have

$$y_2(x) = J_0 \ln x + \sum_{m=1}^{\infty} A_m x^m,$$

$$y_2'(x) = J_0' \ln x + \frac{J_0}{x} + \sum_{m=1}^{\infty} m A_m x^{m-1},$$

$$y_2''(x) = J_0'' \ln x + \frac{2J_0'}{x} - \frac{J_0}{x^2} + \sum_{m=1}^{\infty} m(m-1) A_m x^{m-2}.$$

The Bessel's equation becomes

$$
\begin{aligned}
&(xJ_0'' + J_0' + xJ_0)\ln x + 2J_0' - \frac{J_0}{x} + \frac{J_0}{x} \\
&+ \sum_{m=1}^{\infty} m(m-1)A_m x^{m-1} + \sum_{m=1}^{\infty} mA_m x^{m-1} + \sum_{m=1}^{\infty} A_m x^{m+1} = 0 \\
\Rightarrow\ & 2J_0' + \sum_{m=1}^{\infty} m^2 A_m x^{m-1} + \sum_{m=1}^{\infty} A_m x^{m+1} = 0 \quad (J_0 \text{ is a solution.}) \\
\Rightarrow\ & 2\sum_{m=1}^{\infty} \frac{(-1)^m 2m x^{2m-1}}{2^{2m}(m!)^2} + \sum_{m=1}^{\infty} m^2 A_m x^{m-1} + \sum_{m=1}^{\infty} A_m x^{m+1} = 0 \\
\Rightarrow\ & \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m-1}}{2^{2m-2} m!(m-1)!} + \sum_{m=1}^{\infty} m^2 A_m x^{m-1} + \sum_{m=1}^{\infty} A_m x^{m+1} = 0.
\end{aligned}
$$

From the series, we first find that the power x^0 occurs only in the second series with coefficient A_1. Hence, $A_1 = 0$.
Next, we consider the even powers x^{2s}. The first series contains none. In the second series, we need $m - 1 = 2s$, with the coefficient $(2s+1)^2 A_{2s+1}$. In the third series, we need $m + 1 = 2s$, with the coefficient A_{2s-1}. Therefore, we require

$$(2s+1)^2 A_{2s+1} + A_{2s-1} = 0.$$

Since $A_1 = 0$, we thus obtain $A_3 = 0$, $A_5 = 0$, $\cdots$.

We then consider the odd powers x^{2s+1}. For $s = 0$, only the first and the second series contribute,

$$-\frac{1}{1 \cdot 1 \cdot 1} + 4A_2 = 0, \qquad \text{thus } A_2 = \frac{1}{4}.$$

For other values of s, in the first series we need $2m - 1 = 2s + 1$, in the second we need $m - 1 = 2s + 1$, and the third with $m + 1 = 2s + 1$. We thus obtain

$$\frac{(-1)^{s+1}}{2^{2s}(s+1)!s!} + (2s+2)^2 A_{2s+2} + A_{2s} = 0.$$

For $s = 1$.

$$\frac{1}{8} + 16A_4 + A_2 = 0, \qquad \textit{thus} \qquad A_4 = -\frac{3}{128}.$$

One can find that

$$A_{2m} = \frac{(-1)^{m-1}}{2^{2m}(m!)^2}\left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3}\cdots\frac{1}{m}\right), \qquad m = 1, 2, \cdots .$$

Using the notation $h_m = \left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3}\cdots\frac{1}{m}\right)$, and realizing that $A_1 = A_3 = \cdots = 0$, we have

$$y_2(x) = J_0 \ln x + \sum_{m=1}^{\infty} \frac{(-1)^{m-1} h_m}{2^{2m}(m!)^2} x^{2m}.$$

Since J_0 and y_2 are linearly independent functions, they form a basis for the Bessel's equation with $\nu = 0$; equivalently, we can replace y_2 by the linear combination of J_0 and y_2. We choose $a(y_2 + bJ_0)$ with $a = 2/\pi$, and $b = \gamma - \ln 2$, where γ is the **Euler constant**

$$\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \ln n\right) \approx 0.5772.$$

Therefore, we obtain the **Bessel's function of the second kind** of order zero

$$Y_0(x) = \frac{2}{\pi}\left[J_0(x)\left(\ln\frac{x}{2} + \gamma\right) + \sum_{m=1}^{\infty}\frac{(-1)^{m-1}h_m}{2^{2m}(m!)^2}x^{2m}\right].$$

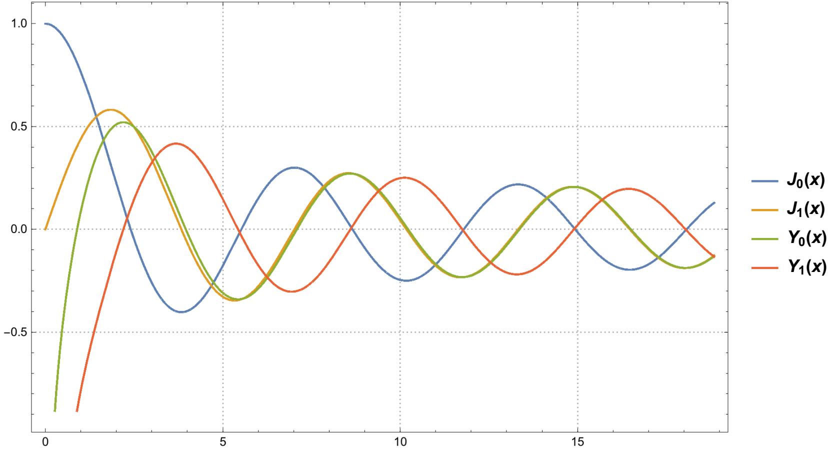
For small $x > 0$ the function behaves like $\ln x$.

For $\nu = n = 1, 2, \cdots$ can be obtained by the similar way, or we can write down the general form

$$Y_\nu(x) = \frac{1}{\sin(\nu\pi)} \left[J_\nu(x)\cos(\nu\pi) - J_{-\nu}(x)\right],$$
$$Y_n(x) = \lim_{\nu\to n} Y_\nu(x).$$

This is the **Bessel function of the second kind** of order ν. For any ν, J_ν and Y_ν are linearly independent, and the general solution to the Bessel's equation for all values of ν is

$$y(x) = C_1 J_\nu(x) + C_2 Y_\nu(x).$$


1.0
0.5
0.0
−0.5
0
5
10
15
$J_0(x)$
$J_1(x)$
$Y_0(x)$
$Y_1(x)$

Example

Consider

$$4xy'' + 4y' + y = 0,$$

this is not a Bessel's equation but with a coordinate transformation we can change it into a Bessel's equation. We let $y(x) = y(z^2)$, $x = z^2$, $\sqrt{x} = z$. *Therefore, we have*

$$y' = \frac{dz}{dx}\frac{dy}{dz} = \frac{1}{2\sqrt{x}}\frac{dy}{dz} = \frac{1}{2z}\frac{dy}{dz}$$

$$y'' = \frac{dz}{dx}\frac{d}{dz}\left(\frac{1}{2z}\frac{dy}{dz}\right) = \frac{1}{4z^2}\frac{d^2y}{dz^2} - \frac{1}{4z^3}\frac{dy}{dz}.$$

Example

Hence the differential equation becomes

$$4z^2\left(\frac{1}{4z^2}\frac{d^2y}{dz^2}-\frac{1}{4z^3}\frac{dy}{dz}\right)+4\left(\frac{1}{2z}\frac{dy}{dz}\right)+y=0$$
$$\Rightarrow \frac{d^2y}{dz^2}-\frac{1}{z}\frac{dy}{dz}+\frac{2}{z}\frac{dy}{dz}+y=0$$
$$\Rightarrow z\frac{d^2y}{dz^2}+\frac{dy}{dz}+zy=0,$$

which is a Bessel's equation with $\nu=0$ *and the general solution is*

$$y(z)=C_1J_0(z)+C_2Y_0(z),\Rightarrow y(x)=C_1J_0(\sqrt{x})+C_2Y_0(\sqrt{x}).$$

Example

Consider

$$xy'' + 11y' + xy = 0, \qquad (\textit{Hint: } y = x^{-5}u(x)).$$

$$y' = \frac{dy}{dx} = -5x^{-6}u + x^{-5}\frac{du}{dx}$$
$$y'' = \frac{d^2y}{dx^2} = 30x^{-7}u - 10x^{-6}\frac{du}{dx} + x^{-5}\frac{d^2u}{dx^2}$$

Example

Hence the differential equation becomes

$$x\left(30x^{-7}u - 10x^{-6}\frac{du}{dx} + x^{-5}\frac{d^2u}{dx^2}\right) + 11\left(-5x^{-6}u + x^{-5}\frac{du}{dx}\right) + x\left(x^{-5}u\right) = 0$$

$$\Rightarrow \frac{1}{x^4}\frac{d^2u}{dx^2} + \frac{1}{x^5}\frac{du}{dx} + \left(\frac{1}{x^4} - \frac{25}{x^6}\right)u = 0$$

$$\Rightarrow x^2\frac{d^2u}{dx^2} + x\frac{du}{dx} + \left(x^2 - 25\right)u = 0,$$

which is a Bessel's equation with $\nu = 5$.

Example

The general solution is thus

$$u(x) = C_1 J_5(x) + C_2 Y_5(x),$$
$$\Rightarrow y(x) = x^{-5} \left[C_1 J_5(x) + C_2 Y_5(x)\right].$$

Example

Consider

$$x^2y'' - 5xy' + 9\left(x^6 - 8\right)y = 0, \qquad (Hint:\ y = x^3u,\ z = x^3).$$

$$\begin{aligned}
\frac{du}{dx} &= \frac{dz}{dx}\frac{du}{dz} = 3x^2\frac{du}{dz},\\
y' &= \frac{dy}{dx} = 3x^2u + x^3\frac{du}{dx} = 3x^2u + 3x^5\frac{du}{dz},\\
y'' &= \frac{d^2y}{dx^2} = 6xu + 3x^2\frac{du}{dx} + 15x^4\frac{du}{dz} + 3x^5\frac{dz}{dx}\frac{d^2u}{dz^2}\\
&= 6xu + 24x^4\frac{du}{dz} + 9x^7\frac{d^2u}{dz^2}.
\end{aligned}$$

Example

Hence the differential equation becomes

$$x^2\left(6xu+24x^4\frac{du}{dz}+9x^7\frac{d^2u}{dz^2}\right)$$
$$-5x\left(3x^2u+3x^5\frac{du}{dz}\right)+9\left(x^6-8\right)x^3u=0$$
$$\Rightarrow 9x^9\frac{d^2u}{dz^2}+(24-15)x^6\frac{du}{dz}+\left(6-15+9x^6-72\right)x^3u=0$$
$$\Rightarrow 9z^3\frac{d^2u}{dz^2}+9z^2\frac{du}{dz}+\left(9z^2-81\right)zu=0,\quad (z=x^3)$$
$$\Rightarrow z^2\frac{d^2u}{dz^2}+z\frac{du}{dz}+(z^2-9)u=0,\quad \left(\times\frac{1}{9z}\right),$$

which is a Bessel's equation with $\nu=3$.

Example

The general solution is thus

$$u(z) = C_1 J_3(z) + C_2 Y_3(z),$$
$$\Rightarrow y(x) = x^3 \left[C_1 J_3(x^3) + C_2 Y_3(x^3) \right].$$

We consider the **Sturm-Liouville problems**, which consist of an ODE of the form

$$[p(x)y']' + [q(x) + \lambda r(x)]y = 0$$

on some closed interval $a \le x \le b$, satisfying conditions of the form

$$\begin{aligned} k_1 y + k_2 y' &= 0 \quad \text{at} \quad x = a \qquad (k_1^2 + k_2^2 > 0), \\ l_1 y + l_2 y' &= 0 \quad \text{at} \quad x = b \qquad (l_1^2 + l_2^2 > 0). \end{aligned}$$

where λ is an unknown parameter, k_1, k_2, l_1, l_2 are given real constants, $p(x)$, $q(x)$, $r(x)$ are functions of x, and $r(x)$ is called **weight** function. (We will discuss on this later.)

Eigenvalues, Eigenfunctions

It is easy to check that for any **Sturm-Liouville problems**, $y = 0$ is a solution–the "**trivial solution**", which is not interesting. The solution we want to find are the so-called **eigenfunctions** $y(x)$, which are non-zero solutions to the Sturm-Liouville problems, and we call the number λ for which an eigenfunction exists an **eigenvalue** of the Sturm-Liouville problem.

Eigenvalues, Eigenfunctions

Example

Find the eigenvalues and eigenfunctions of the Sturm-Liouville problem

$$y'' + \lambda y = 0, \qquad y(0) = 0, \quad y(\pi) = 0.$$

Example

Compare this problem to the most general Sturm-Liouville problem, we find in this case $p(x) = 1$, $q(x) = 0$, $r(x) = 1$, *and* $a = 0$, $b = \pi$, $k_1 = l_1 = 1$, $k_2 = l_2 = 0$. *There are three possiblilities:* $\lambda < 0$, $\lambda = 0$, *and* $\lambda > 0$.

Eigenvalues, Eigenfunctions

Example

For $\lambda < 0$, we can let $\lambda = -\nu^2$, and the differential equation is thus $y'' - \nu^2 y = 0$, which admits the general solution $y(x) = c_1 e^{\nu x} + c_2 e^{-\nu x}$. Using the boundary condition we get

$$\begin{aligned} y(0) &= c_1 + c_2 = 0, \\ y(\pi) &= c_1 e^{\nu\pi} + c_2 e^{-\nu\pi} = 0, \end{aligned}$$

therefore we obtain $c_1 = c_2 = 0$, which is a trivial solution.

Eigenvalues, Eigenfunctions

Example

For $\lambda = 0$, the differential equation is thus $y'' = 0$, which admits the general solution $y(x) = c_1 + c_2 x$. Using the boundary condition we get

$$y(0) = c_1 = 0,$$
$$y(\pi) = c_1 + c_2\pi = 0,$$

therefore we obtain $c_1 = c_2 = 0$, which is a trivial solution.

Eigenvalues, Eigenfunctions

Example

For $\lambda > 0$, *we can let* $\lambda = \nu^2$, *and the differential equation is thus* $y'' + \nu^2 y = 0$, *which admits the general solution* $y(x) = c_1 \cos \nu x + c_2 \sin \nu x$. *Using the boundary condition we get*

$$y(0) = c_1 = 0,$$

therefore we obtain $c_1 = 0$. *The other boundary condition gives us*

$$y(\pi) = c_2 \sin \nu\pi = 0,$$

and in order to have a non-trivial solution, we want $c_2 \neq 0$, *thus we need* $\nu = 0, \pm 1, \pm 2 \cdots$.

Eigenvalues, Eigenfunctions

Example

We then find that $\nu = 0$ *is still a trivial solution since* $\sin 0 = 0$. *Therefore the eigenfunctions to this Sturm-Liouville problem are*

$$y(x) = \sin \nu x, \qquad \textit{with } \nu = 1, 2, \cdots$$

and the eigenvalues corresponding to those eigenfunctions are $\lambda = \nu^2 = 1, 4, 9, \cdots$.

Eigenvalues, Eigenfunctions

Remark

1. *It can be shown that under general conditions on the functions p, q, r, the Sturm-Liouville problem has infinitely many eigenvalues.*
2. *If p, q, r, p' are real and continuous on the interval $a \leq x \leq b$ and $r(x)$ is positive for $a \leq x \leq b$, then all the eigenvalues of the Sturm-Liouville problem are* **real**. *These real eigenvalues usually correspond to the physical quantities such as energies, frequencies.*

Orthogonality

Functions $y_1(x)$, $y_2(x)$, $\cdots$ defined on some interval $a \leq x \leq b$ are called **orthogonal** on ths interval with respect to the **weight function** $r(x) > 0$ if for all $m \neq n$,

$$(y_m, y_n) = \int_a^b r(x) y_m(x) y_n(x) dx = 0 \qquad (m \neq n),$$

where (y_m, y_n) is a **standard notation** for the integral. The **norm** $||y_m||$ of y_m is defined by

$$||y_m|| = \sqrt{(y_m, y_m)} = \sqrt{\int_a^b r(x) y_m^2(x) dx}.$$

Orthogonality

The functions $\tilde{y}_1, \tilde{y}_2, \cdots$ are called **orthonormal** on the interval $[a, b]$ if they are orthogonal on the interval and all have norm 1. Then, we can write the above two equations jointly by the **Kronecker symbol** δ_{mn}

$$(\tilde{y}_m, \tilde{y}_n) = \int_a^b r(x)\tilde{y}_m(x)\tilde{y}_n(x)dx = 0 = \delta_{mn} = \begin{cases} 0 & if \quad m \neq n, \\ 1 & if \quad m = n. \end{cases}$$

Orthogonality

Theorem

If y_m and y_n are eigenfunctions of the Sturm-Liouville problem in the interval $[a, b]$ with continuous real-valued coefficients p, q, r, p' and $r(x) > 0$ within the interval $[a, b]$, then y_m, y_n are orthogonal on the interval with respect to the weight function $r(x)$, i.e.

$$(y_m, y_n) = \int_a^b r(x) y_m(x) y_n(x) dx = 0 \qquad (m \neq n).$$

Orthogonality

Theorem

Moreover, if $p(a) = 0$ and/or $p(b) = 0$, then the boundary condition on the point a and/or b can be dropped.
If $p(a) = p(b)$ then the boundary conditions on points a and b can be replaced by **periodic boundary conditions** *for the function y,*

$$y(a) = y(b) \quad \text{and} \quad y'(a) = y'(b).$$

Orthogonality

Proof.

By assumption, y_m and y_n satisfy the Sturm-Liouville equations

$$(py_m')' + (q + \lambda_m r)y_m = 0,$$
$$(py_n')' + (q + \lambda_n r)y_n = 0.$$

If we multiply the first one with y_n and the second one with $-y_m$ and then add, we find

$$(\lambda_m - \lambda_n)ry_my_m = y_m(py_n')' - y_n(py_m')' = [(py_n')y_m - (py_m')y_n]'.$$

Orthogonality

Proof.

Integrating the last equation over x from a to b, we obtain

$$(\lambda_m - \lambda_n) \int_a^b r y_m y_m dx = [(p y_n')y_m - (p y_m')y_n]\Big|_a^b .$$

If the RHS is zero, and the fact $\lambda_m - \lambda_n \neq 0$, this equation then implies the orthogonality. We thus need to use the boundary conditions to show that

$$p(b)[y_n'(b)y_m(b) - y_m'(b)y_n(b)] - p(a)[y_n'(a)y_m(a) - y_m'(a)y_n(a)]$$

is zero.

Orthogonality

Proof.

Case 1. $p(a) = p(b) = 0$. The equation is clearly zero, b.c. is not needed.

Orthogonality

Proof.

Case 2 $p(a) \neq 0$, $p(b) = 0$. The equation and the boundary condition are

$$\begin{aligned} &-p(a)[y_n'(a)y_m(a) - y_m'(a)y_n(a)], \\ &k_1 y_n(a) + k_2 y_n'(a) = 0, \quad k_1 y_m(a) + k_2 y_m'(a) = 0. \end{aligned}$$

If $k_2 \neq 0$, we multiply the first b.c. by $y_m(a)$ and the last by $-y_n(a)$ and add

$$k_2[y_n'(a)y_m(a) - y_m'(a)y_n(a)] = 0,$$

thus $[\cdots] = 0$ and the equation is zero even $p(a) \neq 0$. Similarly if $k_1 \neq 0$.

Orthogonality

Proof.

Case 3 $p(a) = 0$, $p(b) \neq 0$. The proof is basically same as **Case 2.**, with $a \leftrightarrow b$.

Case 4 $p(a) \neq 0$, $p(b) \neq 0$. The proof is same as **Case 2.**, but now we need to use the boundary condition for a and b at the same time.

Case 5 $p(a) = p(b)$. In this case, the equation becomes

$$p(b)[y_n'(b)y_m(b) - y_m'(b)y_n(b) - y_n'(a)y_m(a) + y_m'(a)y_n(a)].$$

By using the **periodic boundary conditions**, the first term in the bracket cancels the third term, while the second cancels the last term. □

Example

Show that the eigenfunctions $y_\nu(x) = \sin \nu x$ *with* $\nu = 1, 2 \cdots$ *in the last example are orthogonal on the interval* $0 \le x \le \pi$ *with the weight function* $r(x) = 1$.

Example

Solution:

$$(y_m, y_n) = \int_0^\pi \sin(mx)\sin(nx)dx \qquad (m \neq n)$$

$$\Rightarrow (y_m, y_n) = \frac{1}{2}\int_0^\pi \cos[(m-n)x]dx - \frac{1}{2}\int_0^\pi \cos[(m+n)x]dx$$

$$\Rightarrow (y_m, y_n) = \frac{\sin[(m-n)x]|_0^\pi}{2(m-n)} - \frac{\sin[(m+n)x]|_0^\pi}{2(m+n)} = 0.$$

Example

Show that the Legendre's equation can be written as a Sturm-Liouville problem within the interval $-1 \le x \le 1$, and the Legendre polynomials $P_n(x)$ are the eigenfunctions to the problem with eigenvalues $n(n+1)$.

Example

Solution:
Legendre's equation $(1-x^2)y'' - 2xy' + n(n+1)y = 0$ *can be written as*

$$[(1-x^2)y']' + \lambda y = 0,$$

which is a Sturm-Liouville equation with $p = 1 - x^2$, $q = 0$, $r = 1$, $\lambda = n(n+1)$.

Example

Solution:
Since $p(-1) = p(1) = 0$, we do not need any boundary conditions. From the analysis of the Legendre's equation, we know for $n = 0, 1 \cdots$, $P_n(x)$ are eigenfunctions to the equation with the eigenvalue $\lambda = n(n+1) = 0, 1 \cdot 2, 2 \cdot 3, \cdots$. The orthogonality of the $P_n(x)$ is

$$\int_{-1}^{1} P_m(x) P_n(x) dx = 0 \qquad (m \neq n).$$

We can check it, for example,

$$\int_{-1}^{1} P_0(x) P_2(x) dx = \frac{1}{2} \int_{-1}^{1} (3x^2 - 1) dx = \frac{1}{2} (x^3 - x)|_{-1}^{1} = 0.$$

Example

Show that the Bessel's equation with parameter n, $n \in \mathbb{N}$ can be written as a Sturm-Liouville equation, and the Bessel's function $J_n(x)$ are the eigenfunctions to problem.

Example

Solution:
Bessel's equation for fixed integer $n \geq 0$ is

$$\tilde{x}^2 \ddot{y}(\tilde{x}) + \tilde{x}\dot{y}(\tilde{x}) + (\tilde{x}^2 - n^2)y(\tilde{x}) = 0,$$

where $J_n(\tilde{x})$ is a solution to this equation. Now if we set $\tilde{x} = kx$, with some parameter k, we obtain

Example

Solution:

$$k^2x^2\frac{1}{k^2}y'' + kx\frac{1}{k}y' + (k^2x^2 - n^2)y = 0$$

$$\Rightarrow xy'' + y' + \frac{(k^2x^2 - n^2)}{x}y = 0$$

$$\Rightarrow [xy']' + \left(-\frac{n^2}{x} + k^2x\right) = 0,$$

which is a Sturm-Liouville equation with $p = x$, $q = \frac{-n^2}{x}$, $r = x$, $\lambda = k^2$.

Example

Solution:
Since $p(0) = 0$, *if we are given a boundary condition such that* $y(R) = 0$ *with fixed* R, *this is a Sturm-Liouville problem within the interval* $0 \leq x \leq R$. *The solution* $J_n(kx)$ *which satisfy* $J_n(kR) = 0$ *are the eigenfunctions with the eigenvalues* $\lambda = k^2$.
Since $J_n(kx)$ *has infinitely many zeroes, if those zeroes are at* $kx = \alpha_{n,m}$, $m = 1, 2 \cdots$, *the eigenfunctions are those with* $kR = \alpha_{n,m}$, *thus* $k_{n,m} = \frac{\alpha_{n,m}}{R}$.

Example

Solution:
It is a Sturm-Liouville problem within $0 \leq x \leq R$, with $r(x) = x$, and the eigenfunctions $J_n(k_{n,m}x)$. Therefore, the orthogonality of this problem is

$$\int_0^R x J_n(k_{n,m}x) J_n(k_{n,j}x) dx = 0 \qquad (j \neq m, \quad n \text{ fixed}).$$

Bibliography

1 *Kreyszig, E.* **Advanced Engineering Mathematics**. Wiley, 10th Edition.