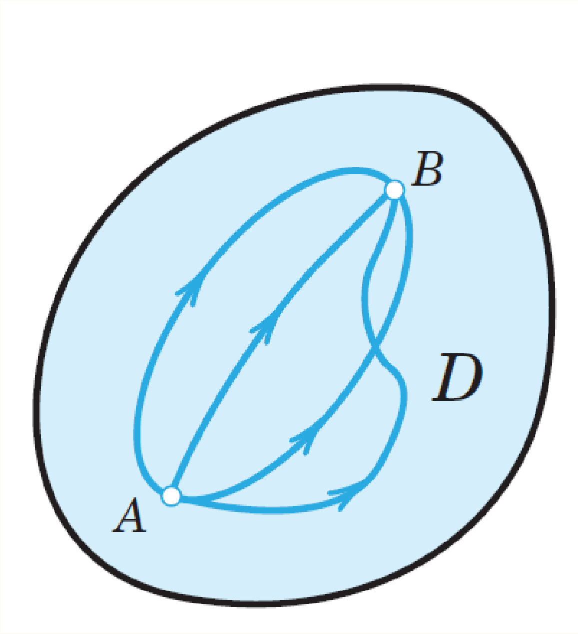
2.4 Path independence of line integrals (page419)

The line integral $\int_C \boldsymbol{F}(\boldsymbol{r}) \cdot dr = \int_C (F_1 dx + F_2 dy + F_3 dz)$, where $dr = (dx, dy, dz)$ is said to be path independent in a domain D in space if for every pair of endpoints A, B in domain D, $\int_C \boldsymbol{F}(\boldsymbol{r}) \cdot dr = \int_C (F_1 dx + F_2 dy + F_3 dz)$ has the same value for all paths in D that begin at A and end B.



Now let's discuss what conditions should be satisfied if the line integral is path independence.

Theorem: Assume domain D is **simply connected** and F_1, F_2, F_3 are continuous and have continuous first partial derivatives in D, then the following four conditions are equivalent to each other.

1. $\int_C \boldsymbol{F}(\boldsymbol{r}) \cdot d\boldsymbol{r} = \int_C (F_1 dx + F_2 dy + F_3 dz)$ is path independence in domain D.
2. $\boldsymbol{F}$=grad$f$, where grad$f$ is the gradient of f.
3. Integration around closed curves C in D always gives 0.
4. curl$\boldsymbol{F} = 0$.

A domain D is **simply connected** if every closed curve in D can be continuously shrunk to any point in D without leaving D.

Next we will have three theorems to justify this.

Theorem 1: A line integral $\int_C \boldsymbol{F}(\boldsymbol{r}) \cdot d\boldsymbol{r} = \int_C (F_1 dx + F_2 dy + F_3 dz)$ with continuous F_1, F_2, F_3 in a domain D in space is path independent in D if and only if $\boldsymbol{F} = [F_1, F_2, F_3]$ is the gradient of some function f in D,

$$\boldsymbol{F} = \text{grad} f, \text{ thus } F_1 = \frac{\partial f}{\partial x}, F_2 = \frac{\partial f}{\partial y}, F_3 = \frac{\partial f}{\partial z}.$$

By this theorem we could calculate the path independent line integral simply by the following formula:

$$\int_A^B (F_1 dx + F_2 dy + F_3 dz) = f(B) - f(A), \tag{2.2}$$

where $\boldsymbol{F} = \text{grad} f$ and A and B are two points in domain D.

(2.2) is the analog of the usual formula for definite integrals in calculus

$$\int_a^b g(x)dx = [G(x)]_a^b = G(b) - G(a), \quad \text{where } G'(x) = g(x).$$

Formula (2.2)should be applied whenever a line integral is independent of path. f is called a **potential** of $\boldsymbol{F}$.

Example 1 (page 421)

Show that the integral $\int_C \boldsymbol{F}(\boldsymbol{r}) \cdot d\boldsymbol{r} = \int_C (2xdx + 2ydy + 4zdz)$ is path independent in any domain in space and find its value in the integration from $A : (0,0,0)$ to $B : (2,2,2)$.

Solution:

If $\boldsymbol{F} = [2x, 2y, 4z]$ has a potential f, then

$$\boldsymbol{F} = \text{grad} f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle =< 2x, 2y, 4z > .$$

By integration of $\frac{\partial f}{\partial x} = 2x$, we obtain $f = x^2 + g(y,z)$.
By differentiation of $f = x^2 + g(y,z)$ with respect to y we get

$$\frac{\partial f}{\partial y} = \frac{\partial g(y,z)}{\partial y}$$

$$2y = \frac{\partial g(y,z)}{\partial y}$$

$$y^2 + h(z) = g(y,z)$$

Therefore

$$f = x^2 + g(y, z) = x^2 + y^2 + h(z).$$

Then

$$\frac{\partial f}{\partial z} = \frac{\partial[x^2 + y^2 + h(z)]}{\partial z} = h'(z)$$

$$4z = h'(z), \text{ so } 2z^2 = h(z)$$

Therefore,

$$f = x^2 + y^2 + 2z^2.$$

Hence the integral is independent of path by the path independence theorem.

By $\int_C \boldsymbol{F}(\boldsymbol{r}) \cdot d\boldsymbol{r} = \int_C (F_1 dx + F_2 dy + F_3 dz) = f(B) - f(A)$ we have

$$\int_C (2xdx + 2ydy + 4zdz) = f(2, 2, 2) - f(0, 0, 0) = 16.$$

Example 2 (page 421)

Evaluate the integral

$$I = \int_C (3x^2 dx + 2yzdy + y^2 dz)$$

from $A : (0, 1, 2)$ to $B : (1, -1, 7)$ by showing that $\boldsymbol{F} = (F_1, F_2, F_3)$ has potential and applying

$$\int_A^B (F_1 dx + F_2 dy + F_3 dz) = f(B) - f(A).$$

Solution: If $\boldsymbol{F}$ has a potential f, we should have

$$F_1 = f_x = 3x^2, \ F_2 = f_y = 2yz, \ F_3 = f_z = y^2.$$

We show that we can satisfy these conditions. By integrating of f_x and differentiation,

$$
\begin{array}{lll}
f = x^3 + g(y,z), & f_y = g_y = 2yz, & g = y^2 z + h(z), \\
f = x^3 + y^2 z + h(z), & f_z = h'(z) + y^2 = y^2, & \\
h'(z) = y^2 - y^2 = 0, & h = \text{constant}. &
\end{array}
$$

This gives $f(x,y,z) = x^2 + y^2 z$ (we take the constant be 0) and by (2.2)

$$
I = f(B) - f(A) = f(1,-1,7) - f(0,1,2) = 8 - 2 = 6.
$$

Theorem 2: The integral $\int_C \boldsymbol{F}(\boldsymbol{r}) \cdot d\boldsymbol{r} = \int_C (F_1 dx + F_2 dy + F_3 dz)$ is path independent in a domain D if and only if its value around every closed path in D is zero.

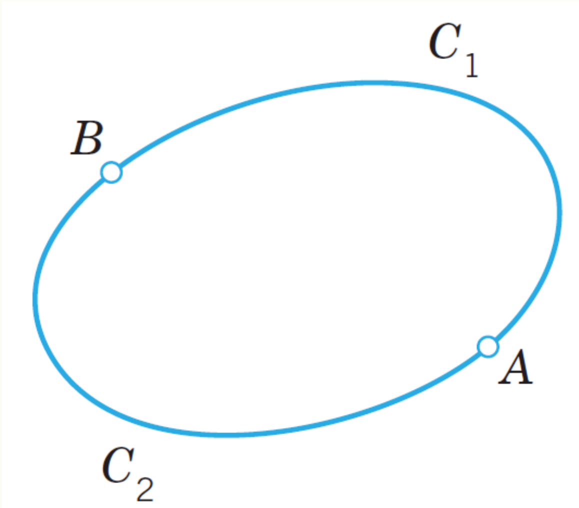


Figure: 2.2: Integral round a closed curve

Proof: $\Longrightarrow$ if the integral is path independent, by the definition of path independence, then (see figure 2.2)

$$\int_{C_1} \boldsymbol{F}(\boldsymbol{r}) \cdot d\boldsymbol{r} = \int_{C_2} \boldsymbol{F}(\boldsymbol{r}) \cdot d\boldsymbol{r} \tag{2.3}$$

Now let C be the closed curve given by $C = C_1 - C_2$, then by (2.3) we have

$$\begin{aligned}
\int_C \boldsymbol{F}(\boldsymbol{r}) \cdot dr &= \int_{C_1} \boldsymbol{F}(\boldsymbol{r}) \cdot d\boldsymbol{r} + \int_{-C_2} \boldsymbol{F}(\boldsymbol{r}) \cdot d\boldsymbol{r} \\
&= \int_{C_1} \boldsymbol{F}(\boldsymbol{r}) \cdot d\boldsymbol{r} - \int_{C_2} \boldsymbol{F}(\boldsymbol{r}) \cdot d\boldsymbol{r} \\
&= \int_{C_1} \boldsymbol{F}(\boldsymbol{r}) \cdot d\boldsymbol{r} - \int_{C_1} \boldsymbol{F}(\boldsymbol{r}) \cdot d\boldsymbol{r} \\
&= 0.
\end{aligned}$$

$\Longleftarrow$: Assume that the integral around any closed path C in domain D is 0. Given that $C = C_1 - C_2$, we get

$$0 = \int_C \boldsymbol{F}(\boldsymbol{r}) \cdot d\boldsymbol{r}$$
$$0 = \int_{C_1} \boldsymbol{F}(\boldsymbol{r}) \cdot d\boldsymbol{r} + \int_{-C_2} \boldsymbol{F}(\boldsymbol{r}) \cdot d\boldsymbol{r}$$
$$0 = \int_{C_1} \boldsymbol{F}(\boldsymbol{r}) \cdot d\boldsymbol{r} - \int_{C_2} \boldsymbol{F}(\boldsymbol{r}) \cdot d\boldsymbol{r}$$
$$\int_{C_1} \boldsymbol{F}(\boldsymbol{r}) \cdot d\boldsymbol{r} = \int_{C_2} \boldsymbol{F}(\boldsymbol{r}) \cdot d\boldsymbol{r}$$

Since for any closed path C, we have the last equality above, this implies the integral is path independent.

Path independence and exactness of differential forms

$\boldsymbol{F}(\boldsymbol{r}) \cdot dr = F_1 dx + F_2 dy + F_3 dz$ is called **exact** in a domain D in space if it is the differential

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz = \text{grad} f \cdot d\boldsymbol{r}$$

of a differential function $f(x, y, z)$ everywhere in D, that is if

$$\boldsymbol{F} \cdot d\boldsymbol{r} = df.$$

Theorem 1*: Path independence The integral $\int_C \boldsymbol{F}(\boldsymbol{r}) \cdot d\boldsymbol{r} = \int_C (F_1 dx + F_2 dy + F_3 dz)$ is path independent in a domain D in space if and only if the differential form $\boldsymbol{F}(\boldsymbol{r}) \cdot d\boldsymbol{r} = F_1 dx + F_2 dy + F_3 dz$ has continuous coefficient functions F_1, F_2, F_3 and is exact in D.

Theorem 3: Criterion for exactness and path independence
Let F_1, F_2, F_3 in the integral $\int_C \boldsymbol{F}(\boldsymbol{r}) \cdot d\boldsymbol{r} = \int_C (F_1 dx + F_2 dy + F_3 dz)$ be continuous and have continuous first partial derivative in a domain D in space. Then

1. If the differential form $\boldsymbol{F}(\boldsymbol{r}) \cdot d\boldsymbol{r} = F_1 dx + F_2 dy + F_3 dz$ is exact in D and thus $\int_C \boldsymbol{F}(\boldsymbol{r}) \cdot d\boldsymbol{r} = \int_C (F_1 dx + F_2 dy + F_3 dz)$ is path independent (by theorem above), then in D, $curl F = 0$, in components

$$\frac{\partial F_3}{\partial y} = \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}. \tag{2.4}$$

2. If curl$F = 0$ holds in D and D is simply connected, then $\boldsymbol{F}(\boldsymbol{r}) \cdot d\boldsymbol{r} = F_1 dx + F_2 dy + F_3 dz$ is exact in D and thus $\int_C \boldsymbol{F}(\boldsymbol{r}) \cdot d\boldsymbol{r} = \int_C (F_1 dx + F_2 dy + F_3 dz)$ is path independent.

For $\int_C \boldsymbol{F}(\boldsymbol{r}) \cdot d\boldsymbol{r} = \int_C F_1 dx + F_2 dy$ the curl has only one component (the z-component), so that (2.4) reduces to the single relation

$$\frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$$

Note:

$$
\begin{aligned}
& \operatorname{curl}\boldsymbol{F} \\
= \; & \operatorname{curl}(\operatorname{grad} f) \\
= \; & \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} \\
= \; & \left(\frac{\partial^2 f}{\partial z \partial y} - \frac{\partial^2 f}{\partial y \partial z}\right)\mathbf{i} + \left(\frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z}\right)\mathbf{j} + \left(\frac{\partial^2 f}{\partial y \partial x} - \frac{\partial^2 f}{\partial x \partial y}\right)\mathbf{k} \\
= \; & 0.
\end{aligned}
$$

Example (page 424)

Using

$$\frac{\partial F_3}{\partial y} = \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y} \tag{2.5}$$

show that the differential form under the integral sign of

$$I = \int_C [2xyz^2 dx + (x^2 z^2 + z \cos yz) dy + (2x^2 yz + y \cos yz) dz]$$

is exact, so that we have independence of path in any domain, and find the value of I from $A : (0, 0, 1)$ to $B : (1, \frac{\pi}{4}, 2)$.

Solution:

In this example,

$$F_1 = 2xyz^2, F_2 = x^2z^2 + z\cos yz, F_3 = 2x^2yz + y\cos yz.$$

Exactness follows from (2.5), which gives:

$$(F_3)_y = 2x^2z + \cos(yz) - yz\sin(yz) = (F_2)_z$$

$$(F_1)_z = 4xyz = (F_3)_x, \quad (F_2)_x = 2xz^2 = (F_1)_y.$$

Since (2.5) is satisfied in any domain (including simply connected domain), by Theorem 3, $\boldsymbol{F} = (F_1, F_2, F_3)$ is exact in any simple connected domain.

To find f,

$$\frac{\partial f}{\partial x} = F_1 = 2xyz^2 \rightarrow f = x^2yz^2 + g(y,z),$$

then

$$\frac{\partial f}{\partial y} = x^2z^2 + \frac{\partial g}{\partial y} = F_2 = x^2z^2 + z\cos(yz),$$

therefore

$$\frac{\partial g}{\partial y} = z\cos(yz).$$

From $\frac{\partial g}{\partial y} = z\cos(yz)$, we know $g = \sin(yz) + h(z)$.
Hence

$$f = x^2yz^2 + g(y,z) = x^2yz^2 + \sin(yz) + h(z)$$

So

$$\frac{\partial f}{\partial z} = 2x^2yz + y\cos(yz) + h'(z) = F_3 = 2x^2yz + y\cos yz.$$

Now we get $h'(z) = 0 \Longrightarrow h$ is a constant. If we take $h = 0$, then $f = x^2yz^2 + \sin(yz)$. This gives

$$\begin{aligned} I &= f(B) - f(A) \\ &= f(1, \frac{\pi}{4}, 2) - f(0,0,1) \\ &= \pi + 1. \end{aligned}$$