西交利物浦大學 Xi'an Jiaotong-Liverpool University

EEE108 Electromagnetism and Electromechanics

Lecture 2

Electrical Fields

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Last

Vector calculus:

Addition and subtraction
Dot product: $\mathbf{A} \bullet \mathbf{B} = AB\cos\theta \quad \leftarrow$ a scalar number
Cross product: $\mathbf{A} \times \mathbf{B} = \mathbf{a}_n AB\sin\theta \quad \leftarrow$ a vector

Coordinate systems:

Cartesian/Rectangular (x, y, z)
Cylindrical (r, φ, z)
Spherical (R, θ, φ)

Today

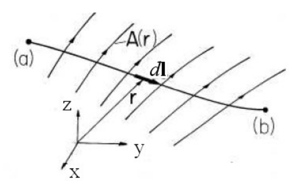
- Vector:
  - Line Integral
  - Surface Integral
  - Fields
- Electric Force -- Coulomb's Law
- Electric Field Intensity
- Field Produced by Continuous Charge Distributions (1)

Line Integral

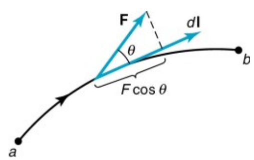
A path connecting points (a) and (b)
Assume that a vector field $\mathbf{A}(\mathrm{r})$ exists in the space in which the path is situated. Then the line integral of $\mathbf{A}(\mathrm{r})$ is defined by

$$\int_a^b \mathbf{A} \bullet d\mathbf{l}$$

Dot Product



Example



The work required to move the object from a to b is:

$$W = \int_a^b \mathbf{F} \bullet d\mathbf{l} = \int_a^b F\cos\theta dl$$

The line integral sums the components that are tangent to the path.

Line integral

$$\mathbf{A}\bullet\mathbf{dl} = A_x dx + A_y dy + A_z dz \quad \text{Cartesian}$$
$$\mathbf{A}\bullet\mathbf{dl} = A_r dr + A_\phi r d\phi + A_z dz \quad \text{Cylindrical}$$
$$\mathbf{A}\bullet\mathbf{dl} = A_R dR + A_\theta R d\theta + A_\phi R\sin\theta d\phi \quad \text{Spherical}$$

Then

$$\int_a^b \mathbf{A}\bullet\mathbf{dl} = \int_{x_a}^{x_b} A_x dx + \int_{y_a}^{y_b} A_y dy + \int_{z_a}^{z_b} A_z dz \quad \text{Cartesian}$$
$$\int_a^b \mathbf{A}\bullet\mathbf{dl} = \int_{r_a}^{r_b} A_r dr + \int_{\phi_a}^{\phi_b} A_\phi r d\phi + \int_{z_a}^{z_b} A_z dz \quad \text{Cylindrical}$$
$$\int_a^b \mathbf{A}\bullet\mathbf{dl} = \int_{R_a}^{R_b} A_R dR + \int_{\theta_a}^{\theta_b} A_\theta R d\theta + \int_{\phi_a}^{\phi_b} A_\phi R\sin\theta d\phi \quad \text{Spherical}$$

Also

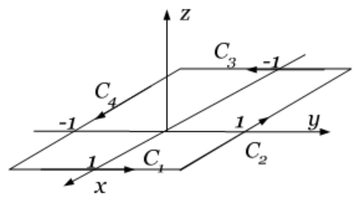
$$\int_a^b \mathbf{A}\bullet\mathbf{dl} = -\int_b^a \mathbf{A}\bullet\mathbf{dl}$$

Line integral

Example 1

Consider the line integral of $\oint_C \mathbf{B}\bullet d\mathbf{l}$, where $\mathbf{B} = y\mathbf{a}_x + z\mathbf{a}_y$ and C is a square path in the $z = 0$ plane with sides $x = -1, x = 1$, $y = -1$ and $y = 1$. The direction of the path is counterclockwise when looking downward from the $+z$ axis.

Calculate the line integral.



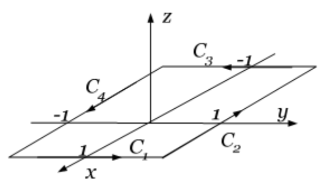
Line integral

Example 1 Solution

The closed line integral is split into 4 parts summed up after

$$\oint_C \mathbf{B}\bullet d\mathbf{l} = \int_{C_1}\mathbf{B}\bullet d\mathbf{l} + \int_{C_2}\mathbf{B}\bullet d\mathbf{l} + \int_{C_3}\mathbf{B}\bullet d\mathbf{l} + \int_{C_4}\mathbf{B}\bullet d\mathbf{l}$$
$$= \int_{-1}^{1} (y\mathbf{a}_x + z\mathbf{a}_y)\Big|_{x=1,z=0} \bullet \mathbf{a}_y dy + \int_{1}^{-1} (y\mathbf{a}_x + z\mathbf{a}_y)\Big|_{y=1,z=0} \bullet \mathbf{a}_x dx$$
$$+ \int_{1}^{-1} (y\mathbf{a}_x + z\mathbf{a}_y)\Big|_{x=-1,z=0} \bullet \mathbf{a}_y dy + \int_{-1}^{1} (y\mathbf{a}_x + z\mathbf{a}_y)\Big|_{y=-1,z=0} \bullet \mathbf{a}_x dx$$
$$= 0 + \int_{1}^{-1} dx + 0 + \int_{-1}^{1} (-1) dx = -4$$

$$\boxed{\begin{aligned} \mathbf{a}_x\bullet\mathbf{a}_x &= \mathbf{a}_y\bullet\mathbf{a}_y = \mathbf{a}_z\bullet\mathbf{a}_z = 1 \\ \mathbf{a}_x\bullet\mathbf{a}_y &= \mathbf{a}_y\bullet\mathbf{a}_z = \mathbf{a}_z\bullet\mathbf{a}_x = 0 \end{aligned}}$$



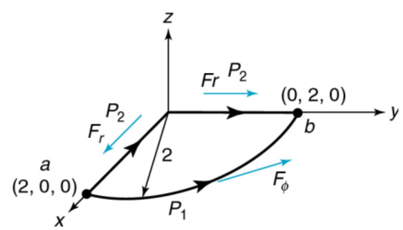
Line integral

Example 2

Evaluate the line integral of $\mathbf{F} = 3\mathbf{a}_r + 2r\mathbf{a}_\phi + \mathbf{a}_z$ along :

1. a circular path P_1 : radius 2 from point a (2, 0, 0) to point b (0, 2, 0);
2. path P_2

Both pathes are shown in the Figure.



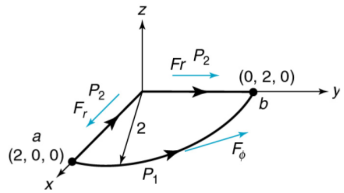
Line integral

Example 2 Solution

Along path P_1

The path fits a circular coordinate system where $r = 2$.

$\boxed{\mathbf{F} = 3\mathbf{a}_r + 2r\mathbf{a}_\phi + \mathbf{a}_z}$



We choose a cylindrical coordinate system:

$\mathbf{F} \bullet d\mathbf{l} = 3(dr) + 2r(rd\phi) + dz = 3dr + 2r^2 d\phi + dz$

along a circular path of radius 2 from point a (2, 0, 0) to point b (0, 2, 0):

$$\int_{P_1} \mathbf{F} \bullet d\mathbf{l} = \underbrace{\int_{r=2}^{2} 3dr}_{=0} + \underbrace{\int_{\phi=0}^{\pi/2} 2r^2 d\phi}_{r=2} + \underbrace{\int_{z=0}^{0} dz}_{=0} = 4\pi$$

Line integral

Example 2 Solution

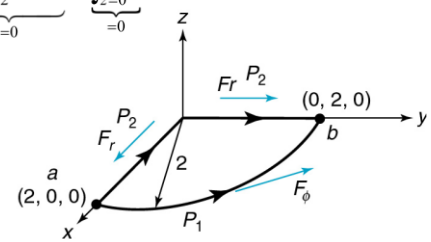
Along path P_2

$\boxed{\mathbf{F} \bullet d\mathbf{l} = 3(dr) + 2r(rd\phi) + dz = 3dr + 2r^2 d\phi + dz}$

$$\int_{P_2} \mathbf{F} \bullet d\mathbf{l} = \int_{r=2}^{0} 3dr + \underbrace{\int_{\phi=0}^{0} 2r^2 d\phi}_{=0} + \underbrace{\int_{z=0}^{0} dz}_{=0}$$

$$+ \int_{r=0}^{2} 3dr + \underbrace{\int_{\phi=\pi/2}^{\pi/2} 2r^2 d\phi}_{=0} + \underbrace{\int_{z=0}^{0} dz}_{=0}$$

= ?

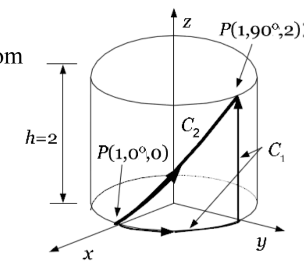


Line integral

Example 3

Vector $\mathbf{F} = r\mathbf{a}_r + z^2\mathbf{a}_\phi$

(a) Calculate the line integral $\int_c \mathbf{F} \bullet d\mathbf{l}$ from point $P(1,0^\circ,0)$ to point $P(1,90^\circ,2)$ along the path C_1, which consists of the arc $r = 1, 0 < \phi < \pi/2$, and $z = 0$, followd by the straight line $r = 1$, $\phi = \pi/2$, and $0 < z < 2$.



(b) Calculate the line integral $\int_c \mathbf{F} \bullet d\mathbf{l}$ from point $P(1,0^\circ,0)$ to point $P(1,90^\circ,2)$ along the path C_2, which is defined by the arc $r = 1,\ 0 < \phi < \pi/2$, and $z = (4\phi)/\pi$.

Line integral

Example 3 Solution

(a) The path consists of two parts:

1. $r = 1,\ 0 < \phi < \pi/2$, and $z = 0$

   $z = 0$: In the x-y plane

   The differential length:

   $d\mathbf{l} = rd\phi\mathbf{a}_\phi\big|_{r=1} = d\phi\mathbf{a}_\phi$

2. $r = 1,\ \phi = \pi/2$, and $0 < z < 2$

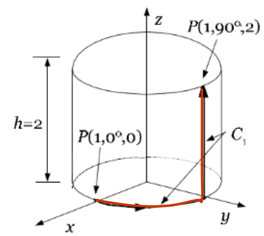
   $\phi = \pi/2$: In the $y-z$ plane

   The differential length:

   $d\mathbf{l} = dz\mathbf{a}_z$

So the integral along path C_1 split into two portions:

$$\int_{C_1} \mathbf{F} \bullet d\mathbf{l} = \int_0^{\pi/2} \mathbf{F} \bullet d\mathbf{l} + \int_0^2 \mathbf{F} \bullet d\mathbf{l}$$



$$\int_{C_1} \mathbf{F} \bullet d\mathbf{l} = \int_0^{\pi/2} \mathbf{F} \bullet d\mathbf{l} + \int_0^2 \mathbf{F} \bullet d\mathbf{l}$$

$$= \int_0^{\pi/2} (r\mathbf{a}_r + z^2\mathbf{a}_\phi)\big|_{r=1,z=0} \bullet d\phi\mathbf{a}_\phi$$

$$+ \int_0^2 (r\mathbf{a}_r + z^2\mathbf{a}_\phi)\big|_{r=1,\phi=\pi/2} \bullet dz\mathbf{a}_z$$

= ?

Line integral

Example 3 Solution

$$\mathbf{a}_r \bullet \mathbf{a}_\phi = \mathbf{a}_\phi \bullet \mathbf{a}_z = \mathbf{a}_z \bullet \mathbf{a}_r = 0$$
$$\mathbf{a}_r \bullet \mathbf{a}_r = \mathbf{a}_\phi \bullet \mathbf{a}_\phi = \mathbf{a}_z \bullet \mathbf{a}_z = 1$$

Vector $\mathbf{F} = r\mathbf{a}_r + z^2\mathbf{a}_\phi$

(b) $\int_c \mathbf{F} \bullet d\mathbf{l}$ along $C_2 : r = 1,\ 0 < \phi < \pi/2,$ and $z = (4\phi)/\pi$.

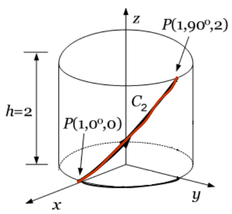
The differential length :

$d\mathbf{l} = (rd\phi)_{r=1}\mathbf{a}_\phi + dz\mathbf{a}_z = d\phi\mathbf{a}_\phi + dz\mathbf{a}_z$

The line integral :

$$\int_{C_2} \mathbf{F} \bullet d\mathbf{l} = \int_{C_2} (r\mathbf{a}_r + z^2\mathbf{a}_\phi) \bullet (d\phi\mathbf{a}_\phi + dz\mathbf{a}_z)$$

$$= \int_0^{\pi/2} z^2 d\phi = \int_0^{\pi/2} \left(\frac{4\phi}{\pi}\right)^2 d\phi = \frac{2\pi}{3}$$



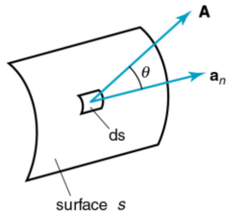
Surface Integral

Given a vector field $\mathbf{A}(r)$ in a region of space containing a specified (open or closed) surface $\mathbf{S}$, an important form of the surface integral of $\mathbf{A}$ over $\mathbf{S}$ is

$$\int_S \mathbf{A} \bullet d\mathbf{s} = \int_S \mathbf{A} \bullet \mathbf{a}_n ds = \int_S A\cos\theta ds$$

where $\mathbf{a}_n$ is a unit vector normal to the differential surface ds.

The surface integral $\int_S \mathbf{A} \bullet \mathbf{a}_n ds$ is called the "flux" of the vector $\mathbf{A}$ through the surface.



$\mathbf{A} \bullet \mathbf{ds} = A_x dydz + A_y dxdz + A_z dxdy$

Cartesian

$\mathbf{A} \bullet \mathbf{ds} = A_r rd\phi dz + A_\phi drdz + A_z rd\phi dr$

Cylindrical

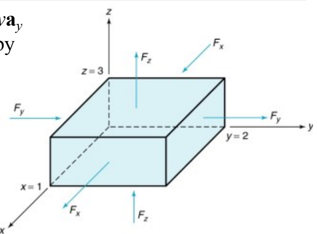
$\mathbf{A} \bullet \mathbf{ds} = A_r R^2 \sin\theta d\phi d\theta + A_\theta R\sin\theta dr d\phi + A_\phi R dR d\theta$

Spherical

Surface Integral

Example

Determine the flux of the vector $\mathbf{F} = 4x\mathbf{a}_x + 5y\mathbf{a}_y + 6\mathbf{a}_z$ out of the rectangular surface bounded by $x = 1$, $y = 2$, and $z = 3$.



Solution $\boxed{\mathbf{A} \bullet \mathbf{ds} = A_x dydz + A_y dxdz + A_z dxdy}$

The flux out of the surface over the front side :

$$\psi_{front} = \int_{y=0}^{2}\int_{z=0}^{3} 4xdydz = \int_{y=0}^{2}\int_{z=0}^{3} 4dydz = 24$$

The flux over the back side of $x = 0$:

$$\psi_{back} = -\int_{y=0}^{2}\int_{z=0}^{3} 4xdydz = 0$$

The flux over the right side of $y = 2$:

$$\psi_{right} = \int_{x=0}^{1}\int_{z=0}^{3} 5ydxdz = \int_{x=0}^{1}\int_{z=0}^{3} 5 \times 2dxdz = 30$$

The flux over the left side of $y = 0$:

$$\psi_{left} = -\int_{x=0}^{1}\int_{z=0}^{3} 5ydxdz = 0$$

The flux over the top of $z = 3$:

$$\psi_{top} = \int_{x=0}^{1}\int_{y=0}^{2} 6dxdy = 12$$

The flux over the bottom of $z = 0$:

$$\psi_{bottom} = -\int_{x=0}^{1}\int_{y=0}^{2} 6dxdy = -12$$

Then $\psi = 24 + 0 + 30 + 0 + 12 - 12 = 54$

Fields

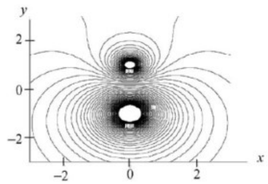
Scalar Fields

A scalar field is a function that gives us a single value of some variable for every point in space (two-dimension or three-dimension).

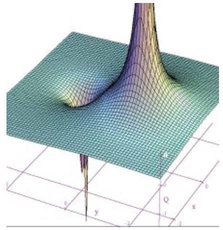
Normally there are three ways to represent a scalar field:

- Contour Map
- Color Coding
- Relief Map /Block Diagram

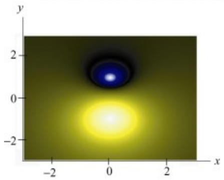
Scalar Fields



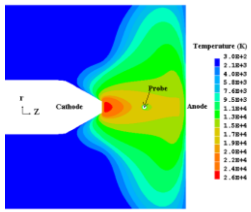
Contour lines show constant values



A relief map of the scalar field



A color-coded map in the *xy*-plane of the scalar field



Temperature distribution in a free burning arc

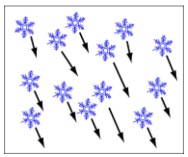
Cylindrical Coordinate

Vector Fields

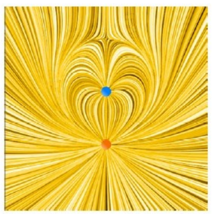
A vector is a quantity which has both a magnitude and a direction in space. Vectors are used to describe physical quantities such as velocity, momentum, acceleration and force, associated with an object.

How do we represent vector fields? Since there is much more information (magnitude and direction) in a vector field, our visualizations are correspondingly more complex when compared to the representations of scalar fields.

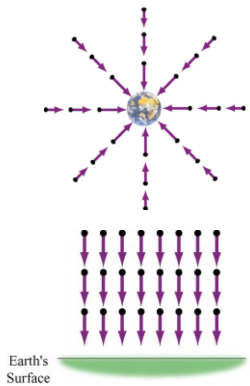
Vector Fields



Falling snow

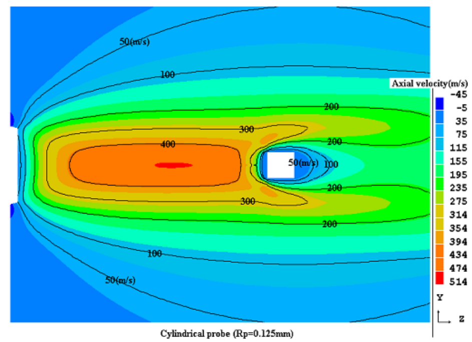


A “grass seeds” representation of the electric field



Gravitational field of the Earth

Vector Fields



Axial velocity distribution at 5ms after inserting the cylindrical probe into the 200A free-burning arc.

Electric Charges and Electric Force

- Two types of observed electric charge: positive and negative.
- The unit of charge is called Coulomb (C).
- The smallest unit of free charge in nature is the charge of an electron or proton:

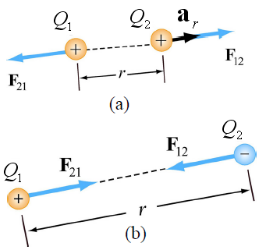
$$e = 1.602\text{x}10^{-19}\ \text{(C)}$$

- Charge is *conserved*.

The ***electric force*** between charges Q_1 and Q_2 :

(a) repulsive if charges have the same signs

(b) attractive if charges have opposite signs



interaction between two charges

Electric Force

Coulomb's Law

The electric force between charges Q_1 and Q_2 :

Q_1 on Q_2 :

$$\mathbf{F}_{12} = k\frac{Q_1 Q_2}{r^2}\mathbf{a}_r$$

where $k = \dfrac{1}{4\pi\varepsilon_0} = 8.9875\times10^9\ \ \text{N}\cdot\text{m}^2/\text{C}^2$

k : called Coulomb constant

ε_0 : permittivity of free space,

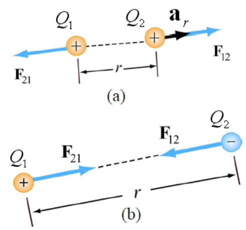
$$\varepsilon_0 = 8.85\times10^{-12}$$

$$\approx \frac{1}{36\pi}10^{-9}\quad C^2/\text{N}\cdot\text{m}^2 \qquad \text{Or F/m (farads per meter)}$$

$\mathbf{a}_r$: unit vector from Q_1 to Q_2

Q_2 on Q_1 : acting force and reacting force

$$\mathbf{F}_{21} = -\mathbf{F}_{12}$$



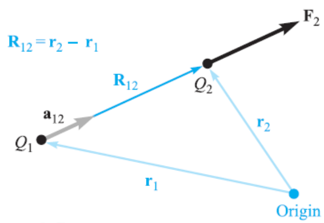
interaction between two charges

Electric Force

Coulomb's Law

The vector form of Coulomb's law is

$$\mathbf{F}_2 = \frac{Q_1 Q_2}{4\pi\epsilon_0 R_{12}^2}\mathbf{a}_{12}$$



$\mathbf{a}_{12}$ a unit vector in the direction of R_{12}

$$\mathbf{a}_{12} = \frac{\mathbf{R}_{12}}{|\mathbf{R}_{12}|} = \frac{\mathbf{R}_{12}}{R_{12}} = \frac{\mathbf{r}_2 - \mathbf{r}_1}{|\mathbf{r}_2 - \mathbf{r}_1|}$$

Electric Force

Example 1

We illustrate the use of the vector form of Coulomb's law by locating a charge of $Q_1 = 3\times10^{-4}$ C at $M(1,2,3\,\text{m})$ and a charge of $Q_2 = -10^{-4}$ C at $N(2,0,5\,\text{m})$ in a vacuum. We desire the force exerted on Q_2 by Q_1.

The vector form of Coulomb's law is

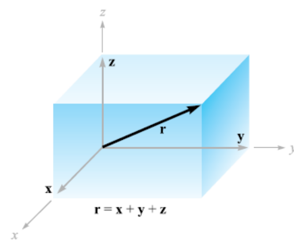
$$\boxed{\mathbf{F}_2 = \frac{Q_1 Q_2}{4\pi\epsilon_0 R_{12}^2}\mathbf{a}_{12}}$$

$$\mathbf{a}_{12} = \frac{\mathbf{r}_2 - \mathbf{r}_1}{|\mathbf{r}_2 - \mathbf{r}_1|}$$

$$\mathbf{r} = \mathbf{x} + \mathbf{y} + \mathbf{z}$$
$$= x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z$$

$$\mathbf{r}_2 = 2\mathbf{a}_x + 5\mathbf{a}_z$$
$$\mathbf{r}_1 = \mathbf{a}_x + 2\mathbf{a}_y + 3\mathbf{a}_z$$

$$\mathbf{r}_2 - \mathbf{r}_1 = (2-1)\mathbf{a}_x + (0-2)\mathbf{a}_y + (5-3)\mathbf{a}_z$$
$$= \mathbf{a}_x - 2\mathbf{a}_y + 2\mathbf{a}_z$$



Electric Force

Example 1

We illustrate the use of the vector form of Coulomb's law by locating a charge of $Q_1 = 3 \times 10^{-4}$ C at $M(1,2,3\text{m})$ and a charge of $Q_2 = -10^{-4}$ C at $N(2, 0, 5\text{m})$ in a vacuum. We desire the force exerted on Q_2 by Q_1.

The vector form of Coulomb's law is

$$\mathbf{F}_2 = \frac{Q_1 Q_2}{4\pi\epsilon_0 R_{12}^2}\mathbf{a}_{12}$$

$$\mathbf{a}_{12} = \frac{\mathbf{r}_2 - \mathbf{r}_1}{|\mathbf{r}_2 - \mathbf{r}_1|} \qquad R_{12} = |\mathbf{r}_2 - \mathbf{r}_1|$$

$|\mathbf{R}_{12}| = 3$, and the unit vector, $\mathbf{a}_{12} = \frac{1}{3}(\mathbf{a}_x - 2\mathbf{a}_y + 2\mathbf{a}_z)$. Thus,

$$\mathbf{F}_2 = \frac{3 \times 10^{-4}(-10^{-4})}{4\pi(1/36\pi)10^{-9} \times 3^2}\left(\frac{\mathbf{a}_x - 2\mathbf{a}_y + 2\mathbf{a}_z}{3}\right)$$

$$= -30\left(\frac{\mathbf{a}_x - 2\mathbf{a}_y + 2\mathbf{a}_z}{3}\right) \text{ N}$$

$$\mathbf{F}_2 = -10\mathbf{a}_x + 20\mathbf{a}_y - 20\mathbf{a}_z \text{ N}$$

Electric Force

Example 2

Two identical small metallic spheres are placed 10 cm apart. The spheres have charges of 1.7×10^{-9} C and -3.3×10^{-9} C, respectively. Find the force between the two spheres if there is no accumulation of charges on the needle.

Solution

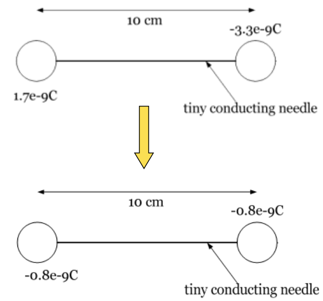
The charges with opposite signs will cancelled, the net charge remained is :

$(1.7-3.3)\times10^{-9} = -1.6\times10^{-9}$ C

And the net charge will evenly distributed between the two spheres.

The force :

$$F = \frac{(-0.8\times10^{-9})^2}{4\pi\varepsilon_0(0.1)^2} = 5.8\times10^{-7} \quad \text{N}$$



Fundamental SI Units

Dimension	Unit	Symbol
Length	meter	m
Mass	kilogram	kg
Time	second	s
Electric Current	ampere	A
Temperature	kelvin	K
Amount of Substance	mole	mol

Fundamental Physical Constants

Constant	Symbol	Value
Speed of light in vacuum	c	$2.998\text{x}10^{8}$ m/s
Gravitational constant	G	$6.67\text{x}10^{-11}$ N.m^2/kg^2
Boltzmann's constant	K	$1.38\text{x}10^{-23}$ J/K
Elementary charge	e	$1.6\text{x}10^{-19}$ C
Permittivity of free space	ε_0	$8.85\text{x}10^{-12}$ F/m
Permeability of free space	μ_0	$4\pi\text{x}10^{-7}$ H/m
Electron mass	m_e	$9.11\text{x}10^{-31}$ kg
Proton mass	m_p	$1.67\text{x}10^{-27}$ kg
Planck's constant	h	$6.63\text{x}10^{-34}$ J.s
Intrinsic impedance of free space	η_0	120π Ω

Some Derived Units Used in Electrostatics

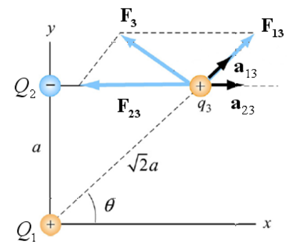
Quantity	Symbol	Units	Equivalent Units
Conductance	G	siemens(S) = ampere/volt	$m^{-2}\cdot kg^{-1}\cdot s^{3}\cdot A^{2}$
Capacitance	C	farad(F) = coulomb/volt	$m^{-2}\cdot kg^{-1}\cdot s^{4}\cdot A^{2}$
Charge	Q,q	coulomb(C)	$s\cdot A$
Conductivity	σ	siemens / meter	$m^{-3}\cdot kg^{-1}\cdot s^{3}\cdot A^{2}$
Energy	W,U	joule(J) = newton · meter	$m^{2}\cdot kg\cdot s^{-2}$
Electric dipole moment	$\boldsymbol{p}$	coulomb - meter(C - m)	$m\cdot s\cdot A$
Electric flux density	$\boldsymbol{D}$	coulomb/meter^2(C/m^2)	$m^{-2}\cdot s\cdot A$
Electric field intensity	$\boldsymbol{E}$	volt/meter(V/m) = newton/coulomb	$m\cdot kg\cdot s^{-3}\cdot A^{-1}$
Electric potential	φ, V	volt(V)	$m^{2}\cdot kg\cdot s^{-3}\cdot A^{-1}$
Force	$\boldsymbol{F}$	newton(N)	$m\cdot kg\cdot s^{-2}$
permittivity	ε	farad/meter(F/m)	$m^{-3}\cdot kg^{-1}\cdot s^{4}\cdot A^{2}$

Electric Force

Principle of Superposition

Coulomb's law applies to any pair of point charges.
When more than two charges are present, the net force on any one charge is the ***vector sum*** of the forces from other charges.

Example:



A system of three charges

$$\mathbf{F}_3 = \mathbf{F}_{13} + \mathbf{F}_{23}$$

In general :

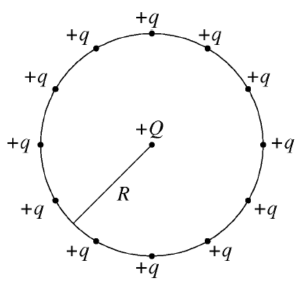
$$\boxed{\mathbf{F}_j = \sum_{\substack{i=1 \\ j\neq i}}^{N} \mathbf{F}_{ij}}$$

Electric Force

Twelve equal charges $+q$ are situated in a circle with radius R, and they are equally spaced.

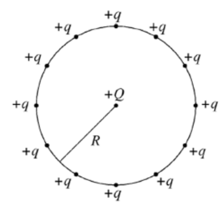
1. What is the net force (magnitude and direction) on a charge $+Q$ at the centre of the circle?

2. What is the net force (magnitude and direction) on the charge $+Q$ at the centre of the circle if we remove only the $+q$ charge which is located at "3-o'clock"?



Reference Answer

(a) The force on $+Q$ due to any particular charge $+q$ on the ring is exactly balanced by the force due to the charge $+q$ diametrically opposite. So the net force on $+Q$ is zero.



(b) With the charge on 3-o'clock removed, the 9-o'clock charge is now unbalanced, and $+Q$ thus experiences a force obtained by Coulomb's law:

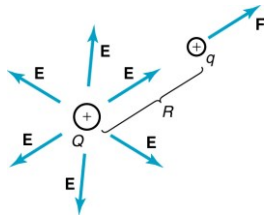
$$\frac{1}{4\pi\varepsilon_0}\frac{qQ}{R^2}$$ The direction points to the right.

Electric Field Intensity

The electric field at a point is the force acting on a test charge q:

$$\mathbf{E} = \frac{\mathbf{F}}{q} \qquad \text{SI unit}: \frac{\text{N}}{\text{C}} = \frac{\text{J/m}}{\text{C}} = \text{V/m}$$

For a point charge Q: $$\mathbf{E} = \frac{\overbrace{k_e \frac{Qq}{R^2}\mathbf{a}_R}^{\mathbf{F}}}{q} = k_e \frac{Q}{R^2}\mathbf{a}_R = \frac{1}{4\pi\varepsilon_0}\frac{Q}{R^2}\mathbf{a}_R$$



Superposition Principle

The total electric field due to a group of charges is equal to the ***vector sum*** of the electric fields of individual charges

$$\mathbf{E}_{total} = \mathbf{E}_1 + \mathbf{E}_2 + \ldots\ldots + \mathbf{E}_N = \sum_{i=1}^{N}\mathbf{E}_i$$

Electric Field Intensity

Expression of superposition in the coordinates

consider a charge that is *not* at the origin of coordinate system

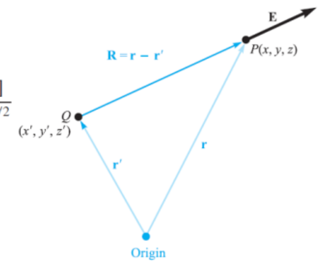
charge Q located at the source point $\mathbf{r}' = x'\mathbf{a}_x + y'\mathbf{a}_y + z'\mathbf{a}_z$

find the field at a general field point $\mathbf{r} = x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z$

expressing $\mathbf{R}$ as $\mathbf{r} - \mathbf{r}'$

$$\mathbf{E}(\mathbf{r}) = \frac{Q}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}'|^2}\overbrace{\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}}^{\mathbf{a}_R} = \frac{Q(\mathbf{r}-\mathbf{r}')}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}'|^3}$$

$$= \frac{Q[(x-x')\mathbf{a}_x + (y-y')\mathbf{a}_y + (z-z')\mathbf{a}_z]}{4\pi\epsilon_0[(x-x')^2 + (y-y')^2 + (z-z')^2]^{3/2}}$$



Electric Field Intensity

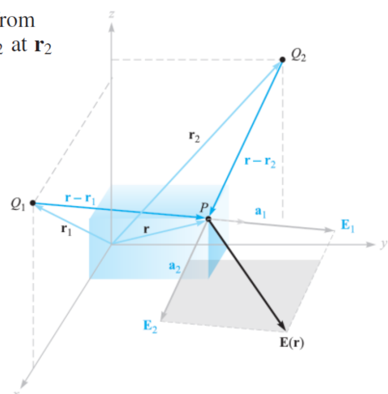
Expression of superposition in the coordinates

the electric field intensity arising from two point charges, Q_1 at $\mathbf{r}_1$ and Q_2 at $\mathbf{r}_2$

$$\mathbf{E}(\mathbf{r}) = \frac{Q_1}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}_1|^2}\mathbf{a}_1 + \frac{Q_2}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}_2|^2}\mathbf{a}_2$$

$$\mathbf{a}_1 = \frac{\mathbf{r}-\mathbf{r}_1}{|\mathbf{r}-\mathbf{r}_1|}$$

$$\mathbf{a}_2 = \frac{\mathbf{r}-\mathbf{r}_2}{|\mathbf{r}-\mathbf{r}_2|}$$



Electric Field Intensity

Expression of superposition in the coordinates

The field due to n point charges:

$$\mathbf{E}(\mathbf{r}) = \sum_{m=1}^{n}\frac{Q_m}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}_m|^2}\mathbf{a}_m$$

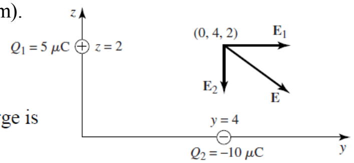
$$\mathbf{a}_m = \frac{\mathbf{r}-\mathbf{r}_m}{|\mathbf{r}-\mathbf{r}_m|}$$

Electric Field Intensity

Example 1

A positive point charge, $Q_1 = 5\mu C$, is located at (0,0,2 m) in a rectangular coordinate system, and a negative point charge, $Q_2 = -10\ \mu C$, is located at (0,4 m, 0).

Determine the electric field at (0,4 m,2 m).



Solution

The electric field due to the positive charge is

$$\mathbf{E}_1 = 9\times10^9\frac{Q_1}{(4)^2}\mathbf{a}_y = 2{,}812.5\mathbf{a}_y$$

The electric field due to the negative charge is

$$\mathbf{E}_2 = -9\times10^9\frac{Q_2}{(2)^2}\mathbf{a}_z = -22{,}500\mathbf{a}_z$$

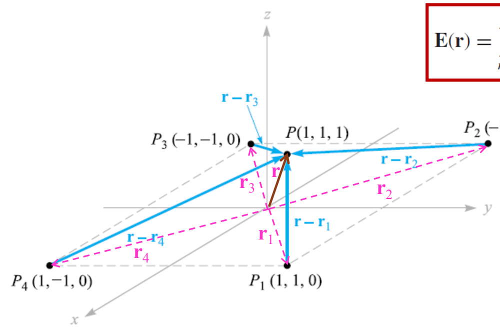
Hence the total electric field is $\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 = 2{,}812.5\mathbf{a}_y - 22{,}500\mathbf{a}_z \quad \dfrac{\text{V}}{\text{m}}$

Electric Field Intensity

Example 2

we find $\mathbf{E}$ at $P(1, 1, 1)$ caused by four identical 3-nC (nanocoulomb) charges located at $P_1(1, 1, 0)$, $P_2(-1, 1, 0)$, $P_3(-1, -1, 0)$, and $P_4(1, -1, 0)$, as shown in Figure 2.4.

$$\boxed{\mathbf{E}(\mathbf{r}) = \sum_{m=1}^{n}\frac{Q_m}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}_m|^2}\mathbf{a}_m}$$



$$\mathbf{r} = \mathbf{a}_x + \mathbf{a}_y + \mathbf{a}_z$$
$$\mathbf{r}_1 = \mathbf{a}_x + \mathbf{a}_y$$
$$\mathbf{r}_2 = -\mathbf{a}_x + \mathbf{a}_y$$
$$\mathbf{r}_3 = -\mathbf{a}_x - \mathbf{a}_y$$
$$\mathbf{r}_4 = \mathbf{a}_x - \mathbf{a}_y$$

Electric Field Intensity

Example 2 Solution

$$\boxed{\mathbf{E}(\mathbf{r}) = \sum_{m=1}^{n}\frac{Q_m}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}_m|^2}\mathbf{a}_m}$$

Solution. We find that $\mathbf{r} = \mathbf{a}_x + \mathbf{a}_y + \mathbf{a}_z$, $\mathbf{r}_1 = \mathbf{a}_x + \mathbf{a}_y$, and thus $\mathbf{r} - \mathbf{r}_1 = \mathbf{a}_z$. The magnitudes are: $|\mathbf{r}-\mathbf{r}_1| = 1$, $|\mathbf{r}-\mathbf{r}_2| = \sqrt{5}$, $|\mathbf{r}-\mathbf{r}_3| = 3$, and $|\mathbf{r}-\mathbf{r}_4| = \sqrt{5}$. Because $Q/4\pi\epsilon_0 = 3\times10^{-9}/(4\pi\times8.854\times10^{-12}) = 26.96\ \text{V}\cdot\text{m}$, we obtain

$$\mathbf{E} = 26.96\left[\frac{\mathbf{a}_z}{1}\frac{1}{1^2} + \frac{2\mathbf{a}_x + \mathbf{a}_z}{\sqrt{5}}\frac{1}{(\sqrt{5})^2} + \frac{2\mathbf{a}_x + 2\mathbf{a}_y + \mathbf{a}_z}{3}\frac{1}{3^2} + \frac{2\mathbf{a}_y + \mathbf{a}_z}{\sqrt{5}}\frac{1}{(\sqrt{5})^2}\right]$$

or

$$\mathbf{E} = 6.82\mathbf{a}_x + 6.82\mathbf{a}_y + 32.8\mathbf{a}_z\ \text{V/m}$$

Electric Field Intensity

Example 2 Solution

$$\mathbf{E} = 6.82\mathbf{a}_x + 6.82\mathbf{a}_y + 32.8\mathbf{a}_z \quad \text{V/m}$$

Show the electric field both in magnitude and direction:

Magnitude:

$$E = \sqrt{6.82^2 + 6.82^2 + 32.8^2} = 34.20 \quad \text{V/m}$$

Direction:

$$\mathbf{a} = \frac{\mathbf{E}}{|\mathbf{E}|} = \frac{6.82\mathbf{a}_x + 6.82\mathbf{a}_y + 32.8\mathbf{a}_z}{34.20} = 0.20\mathbf{a}_x + 0.20\mathbf{a}_y + 0.96\mathbf{a}_z$$

Magnitude and direction:

$$\mathbf{E} = 34.20(0.20\mathbf{a}_x + 0.20\mathbf{a}_y + 0.96\mathbf{a}_z) \quad \text{V/m}$$

Gravitational Field vs Electric Field

Gravitational Force: $\mathbf{F}_g = -G\dfrac{Mm}{r^2}\mathbf{a}_R$

Gravitational Field: $\mathbf{g} = \dfrac{\mathbf{F}_g}{m} = -G\dfrac{M}{r^2}\mathbf{a}_R$

where

$\mathbf{F}_g$: gravitational force between the two point masses,

G : gravitational constant,

M : mass of the second point mass

$\mathbf{a}_R$: unit vector from M to m

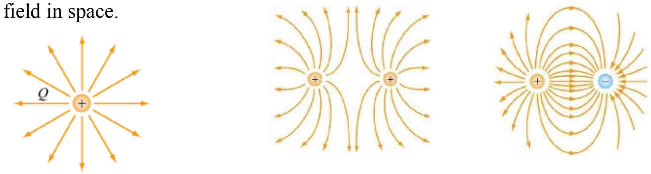
m : mass of the first point mass

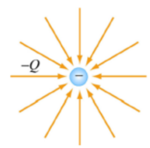
r : distance between the two points

Gravitation	Electrostatics
Mass m	Charge q
Gravitational Force: $\mathbf{F}_g = -G\dfrac{Mm}{r^2}\mathbf{a}_R$	Coulomb Force: $\mathbf{F}_e = k_e\dfrac{Qq}{r^2}\mathbf{a}_R$
Gravitational Field: $\mathbf{g} = \dfrac{\mathbf{F}_g}{m}$	Electric Field: $\mathbf{E} = \dfrac{\mathbf{F}_e}{q}$

Electric Field Lines

Electric field lines provide a convenient graphical representation of the electric field in space.





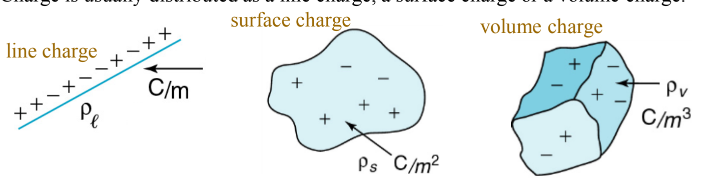
The properties of electric field lines:

- The direction of the electric field at any point is tangent to the field lines at that point.
- The number of lines per unit area through a surface perpendicular to the line is devised to be proportional to the magnitude of the electric field in a given region.
- The field lines must begin on positive charges (or at infinity) and then terminate on negative charges (or at infinity).
- No two field lines can cross each other.

Electric Field due to Continuous Charge Distributions

Charge Density

The charge of a point charge is considered to reside at an infinitesimally small point. Charge is usually distributed as a line charge, a surface charge or a volume charge.



When a large number of charges are tightly packed within a volume (or surface/ line), we can take the **continuum limit**: (Difference to differential)

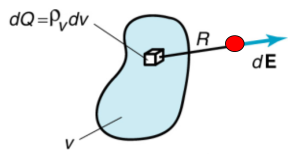
	Unit	Charge Density	Total Amount of Charge (C)
Volume:	C/m^3	$\rho_V = \dfrac{dQ}{dV}$	$Q = \int_V \rho_V dV$
Surface:	C/m^2	$\rho_S = \dfrac{dQ}{dA}$	$Q = \iint_S \rho_S dA$
Line:	C/m	$\rho_L = \dfrac{dQ}{dl}$	$Q = \int_l \rho_L dl$

Electric Field due to Continuous Charge Distributions

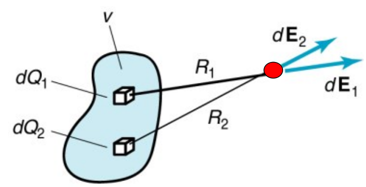
If we can take the continuum limit: (Difference to differential)

1. Divide the charge distribution into differential dQ
2. Use superposition to add up the contributions from all these dQ at the point where we are interested in computing **E**
3. Utilize symmetry to simplify the resulting integral

Superposition continuum: $\mathbf{E} = \int_V d\mathbf{E}$



$$d\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\frac{dQ}{R^2}\mathbf{a}_R$$



Summary

- Vector:
  - Line Integral
  - Surface Integral
  - Fields

- Coulomb's law : $\mathbf{F}_{12} = k_e \frac{Q_1 Q_2}{r^2} \mathbf{a}_r$
- The electric field intensity : $\mathbf{E} = \frac{\mathbf{F}}{q}$
- Principle of superposition : $\mathbf{E}_{Total} = \sum_{i=1}^{N} \mathbf{E}_i$

Next

- Field Produced by Continuous Charge Distributions: Examples
- Electric Flux -- Review
- Gauss's Law

Thanks for your attendance