4.3 Wave equation

In this section, we will solve the one dimensional wave equation by the method of separation of variables,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \tag{4.4}$$

which satisfies the boundary conditions

$$u(0,t) = 0, \quad u(L,t) = 0, \quad \text{for all } t \geq 0 \tag{4.5}$$

and the initial conditions

$$u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad 0 \leq x \leq L. \tag{4.6}$$

Review on solutions of second order ODE

When we apply the separation of variables, the PDE will be separated into two ODEs. Therefore, let's have a short review about the solution of the second order homogeneous linear ODE. We consider an ODE

$$y'' + ay' + by = 0,$$

where a, b are constants. The characteristic equation (or auxiliary equation) of this ODE is

$$\lambda^2 + a\lambda + b = 0.$$

The forms of the solution depends on the sign of the discriminant $a^2 - 4b$, namely,

Case I: Two distinct real roots if $a^2 - 4b > 0$,

Case II: A real double root if $a^2 - 4b = 0$,

Case III: Complex conjugate roots if $a^2 - 4b < 0$.

Case I: Two distinct real roots λ_1 and λ_2
In this case $a^2 - 4b > 0$ and the general solution to the given ODE is

$$y = c_1 \mathrm{e}^{\lambda_1 x} + c_2 \mathrm{e}^{\lambda_2 x},$$

where c_1 and c_2 are arbitrary constants.
Case II: A real double root $\lambda = -a/2$
In this case $a^2 - 4b = 0$ and then $\lambda = -\frac{a}{2}$ and the general solution is

$$y = (c_1 + c_2 x)\mathrm{e}^{-\frac{ax}{2}}.$$

Case III: Two conjugate complex roots
In this case $a^2 - 4b < 0$ and the characteristic equation has two complex roots $\lambda_1 = -\frac{a}{2} + \mathrm{i}\omega$ and $\lambda_2 = -\frac{a}{2} - \mathrm{i}\omega$. The general solution is

$$y = \mathrm{e}^{-\frac{ax}{2}}(A\cos\omega x + B\sin\omega x).$$

Now we start to solve the wave equation (4.4) that satisfies the boundary conditions (4.5) and the initial conditions (4.6).

Step 1: We assume the function $u(x,t)$ can be separated into the product of two functions

$$u(x,t) = X(x)T(t).$$

Differentiating $u(x,t)$ we obtain

$$\frac{\partial^2 u}{\partial t^2} = XT'' \quad \text{and} \quad \frac{\partial^2 u}{\partial x^2} = X''T.$$

By inserting this into the wave equation, we have

$$XT'' = c^2 X''T.$$

Dividing by c^2XT and simplifying gives

$$\frac{T''}{c^2T} = \frac{X''}{X}.$$

The variables are now separated, the left side depending only on t and the right side only on x. Hence both sides must be constant because, if they were variable, then changing t or x would affect only one side, leaving the other unaltered. Thus, say,

$$\frac{T''}{c^2T} = \frac{X''}{X} = k.$$

Multiplying by the denominators gives immediately two ordinary differential equations

$$X'' - kX = 0$$

and

$$T'' - c^2kT = 0.$$

Here, the separation constant k is arbitrary.

Step 2: Satisfying the boundary conditions

The boundary conditions are

$$u(0,t) = X(0)T(t) = 0, u(L,t) = X(L)T(t) = 0, \quad \text{for all } t. \quad (4.7)$$

Now let first solve $X'' - kX = 0$. If $T \equiv 0$, then $u = XT \equiv 0$, which is of no interest. Hence $T \not\equiv 0$ and then by (4.7),

$$X(0) = 0, \quad X(L) = 0.$$

We now show that k must be negative.
At first, if $k = 0$, the general solution of $X'' - kX = 0$ is $X(x) = ax + b$, and because $X(0) = 0, X(L) = 0$ we have $a = b = 0$. So that $X \equiv 0$ and $u = XT \equiv 0$, which is of no interest.
Secondly, if $k > 0$ we can assume $k = \mu^2$ and a general solution of $X'' - kX = 0$ is

$$X(x) = A\mathrm{e}^{\mu x} + B\mathrm{e}^{-\mu x}$$

and from the boundary conditions $X(0) = 0, X(L) = 0$ we obtain $X \equiv 0$ as before (verify it by yourself!).

Hence we are left with the possibility of choosing k negative, say, $k = -p^2$. Then $X'' - kX = 0$ becomes $X'' + p^2 X = 0$ and its general solution is

$$X(x) = A\cos px + B\sin px.$$

From this and the boundary condition $X(0) = 0, X(L) = 0$ we have

$$X(0) = A = 0, \text{ and } X(L) = B\sin pL = 0.$$

Here we must take $B \neq 0$ since otherwise $X \equiv 0$. Hence $\sin pL = 0$. Thus

$$pL = n\pi \Longrightarrow p = \frac{n\pi}{L}, \quad n \text{ is integrer.}$$

Setting $B = 1$, we thus obtain infinitely many solutions $X(x) = X_n(x)$, where

$$X_n(x) = B\sin px = \sin\frac{n\pi}{L}x, \; n = 1, 2, \cdots .$$

These solutions satisfy the boundary conditions $X(0) = 0, X(L) = 0$.

We now solve $T'' - c^2kT = 0$ with $k = -p^2 = -\left(\frac{n\pi}{L}\right)^2$, that is

$$T'' + \lambda_n^2 T = 0, \text{ where } \lambda_n = cp = \frac{cn\pi}{L}.$$

A general solution is

$$T_n(t) = B_n \cos \lambda_n t + B_n^* \sin \lambda_n t.$$

Hence solutions of the wave equation (4.4) satisfying the boundary condition (4.5) are

$$u_n(x,t) = X_n(x)T_n(t) = (B_n \cos \lambda_n t + B_n^* \sin \lambda_n t) \sin \frac{n\pi}{L} x, \; n = 1, 2, \cdots \tag{4.8}$$

Step 3: Solution satisfying the initial conditions
The solutions $u_n(x,t)$ we have obtained (4.8) satisfy the wave equation and the given boundary conditions. We now seeking solutions that satisfy the initial conditions. A single $u_n(x,t)$ will generally not satisfy the initial conditions (4.6). By the superposition theorem, the following function still satisfies the wave equation and the boundary conditions

$$u(x,t)=\sum_{n=1}^{\infty}u_n(x,t)=\sum_{n=1}^{\infty}(B_n\cos\lambda_n t+B_n^*\sin\lambda_n t)\sin\frac{n\pi}{L}x. \tag{4.9}$$

The first initial condition is $u(x,0)=f(x)$, therefore, we have

$$u(x,0)=\sum_{n=1}^{\infty}B_n\sin\frac{n\pi}{L}x=f(x).$$

Hence we must choose the B_n such that $u(x, 0)$ becomes the Fourier sine series of $f(x)$. Thus B_n can be written into

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx, \quad n = 1, 2, \cdots .$$

Similarly, by differentiating (4.9) with respect to t and using the second initial condition $u_t(x, 0) = g(x)$, we obtain

$$\begin{aligned}
\left.\frac{\partial u}{\partial t}\right|_{t=0} &= \left[\sum_{n=1}^{\infty}(-B_n\lambda_n\sin\lambda_n t + B_n^*\lambda_n\cos\lambda_n t)\sin\frac{n\pi}{L}x\right]_{t=0} \\
&= \sum_{n=1}^{\infty} B_n^*\lambda_n\sin\frac{n\pi}{L}x = g(x).
\end{aligned}$$

Hence we must choose the B_n^* so that for $t = 0$ the derivative $\frac{\partial u}{\partial t}$ becomes the Fourier sine series of $g(x)$. Thus,

$$B_n^* \lambda_n = \frac{2}{L} \int_0^L g(x) \sin \frac{n\pi}{L} x.$$

Since $\lambda_n = \frac{cn\pi}{L}$, we obtain by division

$$B_n^* = \frac{2}{cn\pi} \int_0^L g(x) \sin \frac{n\pi}{L} x dx, \quad n = 1, 2, \cdots .$$

Now we have obtained the general solution of the wave equation satisfying the given boundary conditions and initial conditions

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} (B_n \cos \lambda_n t + B_n^* \sin \lambda_n t) \sin \frac{n\pi}{L} x,$$

where

$$\lambda_n = \frac{cn\pi}{L},$$

$$B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx, \quad n = 1, 2, \cdots$$

and

$$B_n^* = \frac{2}{cn\pi} \int_0^L g(x) \sin \frac{n\pi x}{L} dx, \quad n = 1, 2, \cdots .$$

In particular, for the sake of simplicity we consider only the case when the initial velocity $g(x)$ is identically zero. Then the B_n^* are zero, and the solution reduces to

$$
\begin{aligned}
u(x,t) &= \sum_{n=1}^{\infty} \sin\frac{n\pi x}{L} B_n \cos\lambda_n t = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{L} B_n \cos\frac{cn\pi}{L} t \\
&= \sum_{n=1}^{\infty} B_n \left[\sin\frac{n\pi x}{L}\cos\frac{cn\pi t}{L}\right] \\
&= \frac{1}{2}\sum_{n=1}^{\infty} B_n \left[\sin\frac{n\pi}{L}(x+ct) + \sin\frac{n\pi}{L}(x-ct)\right] \\
&= \frac{1}{2}\left[\sum_{n=1}^{\infty} B_n \sin\frac{n\pi}{L}(x+ct) + \sum_{n=1}^{\infty} B_n \sin\frac{n\pi}{L}(x-ct)\right] \\
&= \frac{1}{2}\left[f^*(x+ct) + f^*(x-ct)\right], \qquad (4.10)
\end{aligned}
$$

where f^* is the odd periodic extension of f with the period $2L$, see the figure below.

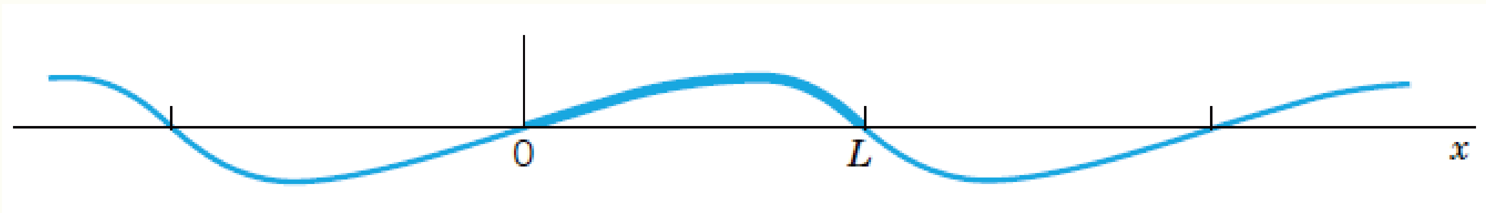

Figure: Odd periodic extension of $f(x)$

Example: Vibrating string if the initial deflection is triangular

Find the solution of the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

satisfying $u(0,t) = 0$, $u(L,t) = 0$, for all $t \geq 0$ and corresponding to the triangular initial deflection

$$f(x) = \begin{cases} \frac{2k}{L} x, & 0 < x < \frac{L}{2} \\ \frac{2k}{L}(L - x), & \frac{L}{2} < x < L \end{cases}$$

and initial velocity zero.

Since the initial velocity is zero, we have $g(x) = 0$, therefore $B_n^* = 0$. From the example 3 in chapter 3 (Fourier series), we know

$$\begin{aligned}
B_n &= \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{2}x\right)dx \\
&= \frac{2}{L}\left[\int_0^{\frac{L}{2}} \frac{2kx}{L}\sin\left(\frac{n\pi}{L}x\right)dx + \int_{\frac{L}{2}}^{L} \frac{2k(L-x)}{L}\sin\left(\frac{n\pi}{L}x\right)dx\right] \\
&= \frac{4k}{L^2}\left[\int_0^{\frac{L}{2}} x\sin\left(\frac{n\pi}{L}x\right)dx + \int_{\frac{L}{2}}^{L}(L-x)\sin\left(\frac{n\pi}{L}x\right)dx\right] \\
&= \frac{8k}{n^2\pi^2}\sin\frac{n\pi}{2},\ n = 1, 2, \cdots.
\end{aligned}$$

Thus the solution is

$$\begin{aligned}
u(x,t) &= \sum_{n=1}^{\infty}(B_n \cos\lambda_n t + B_n^* \sin\lambda_n t)\sin\frac{n\pi}{L}x \\
&= \sum_{n=1}^{\infty} B_n \cos\lambda_n t \sin\frac{n\pi}{L}x \\
&= \sum_{n=1}^{\infty}\left[\frac{8k}{n^2\pi^2}\sin\frac{n\pi}{2}\cos\frac{cn\pi}{L}t\right]\sin\frac{n\pi}{L}x \\
&= \frac{8k}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\cos\frac{cn\pi}{L}t\sin\frac{n\pi}{L}x.
\end{aligned}$$

4.4 D'Alembert's solution of the wave equation. Characteristics

In the last section, we have solved the wave equation by the method of separating variables and obtained

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} B_n \left[\sin \frac{n\pi x}{L} \cos \frac{cn\pi t}{L} \right] \\ &= \frac{1}{2} \left[\sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{L}(x+ct) + \sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{L}(x-ct) \right] \\ &= \frac{1}{2} \left[f^*(x+ct) + f^*(x-ct) \right], \end{aligned}$$

when the initial velocity is zero i.e. $u_t(x,0) = g(x) = 0$.

In this section we will show that this solution can be immediately obtained by transforming the wave equation in a suitable way.

We introduce two new independent variables

$$v = x + ct, \quad w = x - ct.$$

Then u becomes a function of v and w: $u = u(v, w)$ and

$$\frac{\partial v}{\partial x} = 1, \ \frac{\partial v}{\partial t} = c, \ \frac{\partial w}{\partial x} = 1, \ \frac{\partial w}{\partial t} = -c.$$

Therefore

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial w}\frac{\partial w}{\partial x}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial v} + \frac{\partial u}{\partial w}\right) = \frac{\partial}{\partial x}(u_v + u_w) \\
&= \frac{\partial(u_v + u_w)}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial(u_v + u_w)}{\partial w}\frac{\partial w}{\partial x} \\
&= \frac{\partial(u_v + u_w)}{\partial v} + \frac{\partial(u_v + u_w)}{\partial w} \\
&= u_{vv} + u_{wv} + u_{vw} + u_{ww} = u_{vv} + 2u_{vw} + u_{ww}.
\end{aligned}$$

Similarly,

$$
\begin{aligned}
\frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right) = \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial v}\frac{\partial v}{\partial t} + \frac{\partial u}{\partial w}\frac{\partial w}{\partial t}\right) \\
&= \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial v}c + \frac{\partial u}{\partial w}(-c)\right) = \frac{\partial}{\partial t}(cu_v - cu_w) \\
&= \frac{\partial(cu_v - cu_w)}{\partial v}\frac{\partial v}{\partial t} + \frac{\partial(cu_v - cu_w)}{\partial w}\frac{\partial w}{\partial t} \\
&= \frac{\partial(cu_v - cu_w)}{\partial v}c + \frac{\partial(cu_v - cu_w)}{\partial w}(-c) \\
&= c^2 u_{vv} - c^2 u_{wv} - c^2 u_{vw} + c^2 u_{ww} \\
&= c^2 u_{vv} - 2c^2 u_{vw} + c^2 u_{ww}.
\end{aligned}
$$

Hence

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

becomes

$$c^2 u_{vv} - 2c^2 u_{vw} + c^2 u_{ww} = c^2(u_{vv} + 2u_{vw} + u_{ww})$$

Thus we obtain

$$u_{vw} = 0$$

i.e.

$$\frac{\partial^2 u}{\partial w \partial v} = 0.$$

By two successive integrations, first with respect to w and then with respect to v, we have

$$\frac{\partial u}{\partial v} = h(v) \text{ and } u(v, w) = \int h(v)dv + \psi(w) = \phi(v) + \psi(w),$$

where $h(v)$ and $\psi(w)$ are arbitrary functions and $\phi(v) = \int h(v)dv$. By $v = x + ct$, $w = x - ct$, we thus have

$$u(x, t) = \phi(v) + \psi(w) = \phi(x + ct) + \psi(x - ct).$$

This is known as **d'Alembert's solution** of the wave equation.

D'Alembert's Solutions satisfying the initial conditions

We assume the initial conditions of the wave equation is

$$u(x,0) = f(x), \quad u_t(x,0) = g(x).$$

By differentiating $u(x,t) = \phi(x+ct) + \psi(x-ct)$ with respect to t we have

$$u_t(x,t) = c\phi'(x+ct) - c\psi'(x-ct)$$

where primes denote derivatives with respect to the entire arguments $x+ct$ and $x-ct$, respectively. Therefore we have

$$u(x,0) = \phi(x) + \psi(x) = f(x),$$

$$u_t(x,0) = c\phi'(x) - c\psi'(x) = g(x). \tag{4.11}$$

Dividing (4.11) by c and integrating with respect to x from x_0 to x, we obtain

$$\phi'(x) - \psi'(x) = \frac{1}{c} g(x)$$

$$\phi(x) - \psi(x) = \phi(x_0) - \psi(x_0) + \frac{1}{c}\int_{x_0}^{x} g(s)ds \quad (4.12)$$

Add this to

$$\phi(x) + \psi(x) = f(x)$$

we have

$$2\phi(x) = f(x) + \phi(x_0) - \psi(x_0) + \frac{1}{c}\int_{x_0}^{x} g(s)ds$$

$$\phi(x) = \frac{1}{2}\left[f(x) + \phi(x_0) - \psi(x_0) + \frac{1}{c}\int_{x_0}^{x} g(s)ds\right]$$

$$\phi(x) = \frac{1}{2}f(x) + \frac{1}{2}[\phi(x_0) - \psi(x_0)] + \frac{1}{2c}\int_{x_0}^{x} g(s)ds$$

Similarly, subtraction of (4.12) from

$$\phi(x) + \psi(x) = f(x)$$

we obtain

$$2\psi(x) = f(x) - \phi(x_0) + \psi(x_0) - \frac{1}{c}\int_{x_0}^{x} g(s)ds$$

$$\psi(x) = \frac{1}{2}\left[f(x) - \phi(x_0) + \psi(x_0) - \frac{1}{c}\int_{x_0}^{x} g(s)ds\right]$$

$$\psi(x) = \frac{1}{2}f(x) - \frac{1}{2}[\phi(x_0) - \psi(x_0)] - \frac{1}{2c}\int_{x_0}^{x} g(s)ds$$

Now we have obtained

$$\phi(x) = \frac{1}{2}f(x) + \frac{1}{2}[\phi(x_0) - \psi(x_0)] + \frac{1}{2c}\int_{x_0}^{x} g(s)ds$$
$$\psi(x) = \frac{1}{2}f(x) - \frac{1}{2}[\phi(x_0) - \psi(x_0)] - \frac{1}{2c}\int_{x_0}^{x} g(s)ds$$

Now replace x by $x + ct$ for $\phi(x)$ and x by $x - ct$ for $\psi(x)$, then

$$\phi(x+ct) = \frac{1}{2}f(x+ct) + \frac{1}{2}[\phi(x_0) - \psi(x_0)] + \frac{1}{2c}\int_{x_0}^{x+ct} g(s)ds$$
$$\psi(x-ct) = \frac{1}{2}f(x-ct) - \frac{1}{2}[\phi(x_0) - \psi(x_0)] - \frac{1}{2c}\int_{x_0}^{x-ct} g(s)ds$$

Therefore the solution $u(x,t)$ is

$$u(x,t) = \phi(x+ct)+\psi(x-ct) = \frac{1}{2}[f(x+ct)+f(x-ct)]+\frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds.$$

If the initial velocity is zero, that is, $u_t(x,0) = g(x) = 0$, then

$$u(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)],$$

which agrees with that obtained in previous section.

Characteristics: types and normal forms of PDEs

The idea of d'Alembert's solution is just a special instance of the method of characteristics. This concerns PDEs of the form

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} = F(x, y, u, u_x, u_y).$$

There are three types of PDEs, depending on the discriminant $AC - B^2$, as follows.

Type	Defining condition	Example
Hyperbolic	$AC - B^2 < 0$	Wave equation
Parabolic	$AC - B^2 = 0$	Heat equation
Elliptic	$AC - B^2 > 0$	Laplace equation