MTH101: Tutorial 13

Dr. Tai-Jun Chen, Dr. Xinyao Yang

Xi'an Jiaotong-Liverpool University, Suzhou

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Exercise 1.1

Show that $y'' + fy' + (g + \lambda h)y = 0$ *takes the form*

$$[p(x)y']' + [q(x) + \lambda r(x)]y = 0$$

if you set $p = exp(\int fdx)$, $q = pg$, $r = hp$.
Why would you do such a transformation?

Solution

$$
\begin{aligned}
&[p(x)y']' + [q(x) + \lambda r(x)]y = 0 \\
\Rightarrow\ & py'' + p'y' + (q + \lambda r)\, y = 0 \\
\Rightarrow\ & y'' + \frac{p'}{p}y' + \left(\frac{q}{p} + \lambda \frac{r}{p}\right) y = 0.
\end{aligned}
$$

If we compare this equation with $y'' + fy' + (g + \lambda h)y = 0$, we find that

$$
f = \frac{p'}{p}, \quad g = \frac{q}{p}, \quad h = \frac{r}{p} \Rightarrow \frac{d \ln p}{dx} = f, \quad q = pg, \quad r = hp,
$$

$$
\Rightarrow \ln p = \int f dx \Rightarrow p = e^{\int f dx}.
$$

This transformation helps us to find the weight function for the orthogonality, $r(x) = h(x)p(x) = h(x)e^{\int^x f(\tilde{x})d\tilde{x}}$.

Exercise 2.1

Find the eigenvalues and eigenfunctions of the following questions. Verify orthogonality.

1. $y'' + \lambda y = 0, \qquad y(0) = y(1), \quad y'(0) = y'(1).$
2. $y'' + \lambda y = 0, \qquad y(0) = 0, \quad y'(L) = 0.$

Solution

1. It is a Sturm-Liouville problem with pediodic boundary condition. For negative $\lambda = -\nu^2$, the general solution to the ODE is $C_1 e^{\nu x} + C_2 e^{-\nu x}$ and $y' = \nu \left(C_1 e^{\nu x} - C_2 e^{-\nu x} \right)$. Therefore, the boundary condition gives us

$$\begin{aligned} C_1 + C_2 &= C_1 e^{\nu} + C_2 e^{-\nu}, \\ \nu \left(C_1 - C_2 \right) &= \nu \left(C_1 e^{\nu} - C_2 e^{-\nu} \right), \\ \Rightarrow C_1 = C_2 = 0, &\quad \Rightarrow \quad \text{trivial solution.} \end{aligned}$$

On the other hand, for $\lambda = 0$, we have $y = C_1 x + C_2$, $y' = C_1$, and the boundary condition gives us

$$\begin{aligned} C_2 &= C_1 + C_2, \\ C_1 &= C_1, \\ \Rightarrow C_1 &= 0, \quad \Rightarrow \quad y = C_2, \quad (constant). \end{aligned}$$

Solution

For $\lambda = \nu^2$, the general solution is $y = C_1 \cos \nu x + C_2 \sin \nu x$, and $y' = \nu \left(-C_1 \sin \nu x + C_2 \cos \nu x\right)$, and the boundary condition gives us

$$\begin{aligned} C_1 &= C_1 \cos \nu + C_2 \sin \nu, \\ \nu \left(C_2\right) &= \nu \left(-C_1 \sin \nu + C_2 \cos \nu\right), \\ \Rightarrow &\begin{cases} (1 - \cos \nu) C_1 - \sin \nu \, C_2 = 0 \\ (\sin \nu) C_1 + (1 - \cos \nu) C_2 = 0. \end{cases} \end{aligned}$$

For non-trivial solution, we need to ask the determinant of the following matrix to be zero

$$\begin{bmatrix} (1 - \cos \nu) & -\sin \nu \\ \sin \nu & (1 - \cos \nu) \end{bmatrix}.$$

Solution

Therefore we need

$$(1 - \cos\nu)^2 + \sin^2\nu = 0$$
$$\Rightarrow 2 - 2\cos\nu = 0, \quad \textit{or} \quad \nu = 2m\pi, \quad m = 0, 1, 2\cdots .$$

One can see if we choose $m = 0$, this cover the non-trivial solution in the $\lambda = 0$ case, and the eigenfunctions to the equation are

$$y_m = C_1\cos(2m\pi x) + C_2\sin(2m\pi x), \quad m = 0, 1, 2\cdots ,$$

with the eigenvalues $\lambda = (2m\pi)^2$.

Solution

Orthogonality: For $m \neq n$

$$\int_0^1 [C_1 \cos(2m\pi x) + C_2 \sin(2m\pi x)][C_1^* \cos(2n\pi x) + C_2^* \sin(2n\pi x)]dx$$

$$= A \int_0^1 \cos(2m\pi x)\cos(2n\pi x)dx + B \int_0^1 \sin(2m\pi x)\cos(2n\pi x)dx$$

$$+ C \int_0^1 \cos(2m\pi x)\sin(2n\pi x)dx + D \int_0^1 \sin(2m\pi x)\sin(2n\pi x)dx,$$

with some constants A, B, C, D.

Solution

Since

$$
\begin{aligned}
&\int_0^1 \cos(2m\pi x)\cos(2n\pi x)dx \\
=& \int_0^1 \frac{\cos[2\pi(m-n)x] + \cos[2\pi(m+n)x]}{2}dx \\
=& \frac{1}{4\pi(m-n)}\sin[2\pi(m-n)x]\Big|_0^1 + \frac{1}{4\pi(m+n)}\sin[2\pi(m+n)x]\Big|_0^1 = 0.
\end{aligned}
$$

Similarly, one can find the other three terms vanish, and we thus verify the orthogonality.

Solution

2. For negative $\lambda = -\nu^2$, the general solution to the ODE is $C_1 e^{\nu x} + C_2 e^{-\nu x}$ and $y' = \nu \left(C_1 e^{\nu x} - C_2 e^{-\nu x} \right)$. Therefore, the boundary condition gives us

$$
\begin{aligned}
& C_1 + C_2 = 0, \\
& \nu \left(C_1 e^{\nu L} - C_2 e^{-\nu L} \right) = 0 \Rightarrow \nu C_1 \left(e^{\nu L} + e^{-\nu L} \right) = 0, \\
& C_1 = C_2 = 0, \quad \Rightarrow \quad \text{trivial solution.}
\end{aligned}
$$

On the other hand, for $\lambda = 0$, we have $y = C_1 x + C_2$, $y' = C_1$, and the boundary condition gives us

$$
\begin{aligned}
& C_2 = 0, \\
& C_1 = 0, \Rightarrow \qquad \text{trivial solution.}
\end{aligned}
$$

Solution

For $\lambda = \nu^2$, the general solution is $y = C_1 \cos \nu x + C_2 \sin \nu x$, and $y' = \nu \left(-C_1 \sin \nu x + C_2 \cos \nu x\right)$, and the boundary condition gives us

$$
\begin{aligned}
& C_1 = 0, \\
& \nu \left[-C_1 \sin(\nu L) + C_2 \cos(\nu L)\right] = 0, \\
\Rightarrow\ & \nu C_2 \cos(\nu L) = 0, \\
\Rightarrow\ & \nu L = \frac{(1+2m)\,\pi}{2}, \quad or, \quad \nu = \frac{(1+2m)\,\pi}{2L}, \quad (m = 0, 1, 2 \cdots)
\end{aligned}
$$

Therefore, the eigenfunctions to the equation are

$$
y_m = \sin\left[\frac{(1+2m)\,\pi}{2L} x\right], \quad m = 0, 1, 2 \cdots,
$$

with the eigenvalues $\lambda = \left[\frac{(1+2m)\pi}{2L}\right]^2$.

Solution

Orthogonality: For $m \neq n$

$$\int_0^L \sin\left[\frac{(1+2m)\,\pi}{2L}x\right]\sin\left[\frac{(1+2n)\,\pi}{2L}x\right]dx$$

$$= \int_0^L \frac{\cos\left[\frac{(m-n)\pi}{L}x\right] - \cos\left[\frac{(m+n+1)\pi}{L}x\right]}{2}dx$$

$$= \frac{L}{2(m-n)\pi}\sin\left[\frac{(m-n)\pi}{L}x\right]\Big|_0^L - \frac{L}{2(m+n+1)\pi}\sin\left[\frac{(m+n+1)\pi}{L}x\right]\Big|_0^L$$

$$= \frac{L}{2(m-n)\pi}[0-0] - \frac{L}{2(m+n+1)\pi}[0-0] = 0.$$

We thus verify the orthogonality.