Xi'an Jiaotong-Liverpool University
西交利物浦大学

EEE204 Continuous and Discrete Time Signals and Systems II

2018–2019 Semester 2

Electrical and Electronic Engineering

Xi'an Jiaotong-Liverpool University

Week 3

Find N_0 if Existing

$x_1[k] = 5 \times (-1)^k$

Find N_0 if Existing

Xi'an Jiaotong-Liverpool University 西交利物浦大学

$$x_2[k] = \exp[j(7\pi k/4)] + \exp[j(3k/4)]$$

Find N_0 if Existing

$$x_3[k] = \exp[j(7\pi k/4)] + \exp[j(3\pi k/4)]$$

Find N_0 if Existing

Xi'an Jiaotong-Liverpool University 西交利物浦大学

$$x_4[k] = \sin(3\pi k/8) + \cos(63\pi k/64)$$

Find N_0 if Existing

Xi'an Jiaotong-Liverpool University 西交利物浦大学

$$x_5[k] = \exp[j(7\pi k/4)] + \cos(4\pi k/7 + \pi)$$

Find N_0 if Existing

Xi'an Jiaotong-Liverpool University 西交利物浦大学

$x_6[k] = \sin(3\pi k/8)\cos(63\pi k/64)$

Find the Energy of the Signal

$$x_7[k] = \begin{cases} \cos\left(\frac{3\pi k}{16}\right), & -10 \leqslant k \leqslant 0 \\ 0, & \text{otherwise} \end{cases}$$

Find the Energy of the Signal

$$x_7[k] = \begin{cases} \cos\left(\frac{3\pi k}{16}\right), & -10 \leqslant k \leqslant 0 \\ 0, & \text{otherwise} \end{cases}$$

Find the Power of the Signal

$x_8[k] = \cos(\pi k/4)\sin(3\pi k/8)$

Power of Sum of Sinusoids

Consider the following DT sequence

$$x[n] = A_1 \sin(\omega_1 n + \phi_1) + A_2 \sin(\omega_2 n + \phi_2).$$

Assuming $\omega_1 = \frac{m_1}{N_1} \cdot 2\pi, \omega_2 = \frac{m_2}{N_2} \cdot 2\pi$ and determine the power of the signal $(0 \leqslant m_1 \leqslant N_1; 0 \leqslant m_2 \leqslant N_2; m_1, m_2 \in Z^+ \cup \{0\}; N_1, N_2 \in Z^+)$.

General Case

Power of Sum of Sinusoids

1. $m_1 = 0, m_2 = 0$, for $\forall n$

$$
\begin{aligned}
P &= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2, \\
&= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} (A_1 \sin\phi_1 + A_2 \sin\phi_2)^2, \\
&= \lim_{N\to\infty} \frac{2N+1}{2N+1} (A_1 \sin\phi_1 + A_2 \sin\phi_2)^2, \\
&= A_1^2 \sin^2\phi_1 + A_2^2 \sin^2\phi_2 + 2A_1A_2 \sin\phi_1 \sin\phi_2.
\end{aligned}
$$

Xi'an Jiaotong-Liverpool University 西交利物浦大学

Power of Sum of Sinusoids

2. $m_1 = 0, m_2 = 1, N_2 = 1$, for $\forall n$

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2,$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} (A_1 \sin\phi_1 + A_2 \sin\phi_2)^2,$$

$$= \lim_{N\to\infty} \frac{2N+1}{2N+1} (A_1 \sin\phi_1 + A_2 \sin\phi_2)^2,$$

$$= A_1^2 \sin^2\phi_1 + A_2^2 \sin^2\phi_2 + 2A_1A_2 \sin\phi_1 \sin\phi_2.$$

Power of Sum of Sinusoids

3. $m_1 = 1, N_1 = 1, m_2 = 0$, for $\forall n$

$$
\begin{aligned}
P &= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2, \\
&= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} (A_1 \sin\phi_1 + A_2 \sin\phi_2)^2, \\
&= \lim_{N\to\infty} \frac{2N+1}{2N+1} (A_1 \sin\phi_1 + A_2 \sin\phi_2)^2, \\
&= A_1^2 \sin^2\phi_1 + A_2^2 \sin^2\phi_2 + 2A_1A_2 \sin\phi_1 \sin\phi_2.
\end{aligned}
$$

Xi'an Jiaotong-Liverpool University 西交利物浦大学

Power of Sum of Sinusoids

4. $m_1 = 0, m_2 = 1, N_2 = 2$, for $\forall n$, the fundamental period is 2

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2 = \frac{1}{2}\sum_{n=0}^{1} |x[n]|^2,$$

$$= \frac{1}{2}\sum_{n=0}^{1}[A_1 \sin\phi_1 + (-1)^n A_2 \sin\phi_2]^2,$$

$$= \frac{1}{2}(2A_1^2 \sin^2\phi_1 + 2A_2^2 \sin^2\phi_2),$$

$$= A_1^2 \sin^2\phi_1 + A_2^2 \sin^2\phi_2.$$

Xi'an Jiaotong-Liverpool University 西交利物浦大学

Power of Sum of Sinusoids

5. $m_1 = 1, N_1 = 2$, $m_2 = 0$, for $\forall n$, the fundamental period is 2

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2 = \frac{1}{2}\sum_{n=0}^{1} |x[n]|^2,$$

$$= \frac{1}{2}\sum_{n=0}^{1} [(-1)^n A_1 \sin\phi_1 + A_2 \sin\phi_2]^2,$$

$$= \frac{1}{2}(2A_1^2 \sin^2\phi_1 + 2A_2^2 \sin^2\phi_2),$$

$$= A_1^2 \sin^2\phi_1 + A_2^2 \sin^2\phi_2.$$

General Case

Power of Sum of Sinusoids

6. $m_1 = m_2 = 1, N_1 = N_2 = 2$, for $\forall n$

$$
\begin{aligned}
P &= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2, \\
&= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} [(-1)^n A_1 \sin\phi_1 + (-1)^n A_2 \sin\phi_2]^2, \\
&= \lim_{N\to\infty} \frac{2N+1}{2N+1} (A_1 \sin\phi_1 + A_2 \sin\phi_2)^2, \\
&= A_1^2 \sin^2\phi_1 + A_2^2 \sin^2\phi_2 + 2A_1 A_2 \sin\phi_1 \sin\phi_2.
\end{aligned}
$$

Xi'an Jiaotong-Liverpool University 西交利物浦大学

Power of Sum of Sinusoids

7. For $1 \leqslant m_1 < N_1, N_1 > 2, 1 \leqslant m_2 < N_2, N_2 > 2$, then $N_1 N_2$ is the (fundamental) period.

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2 = \frac{1}{N_1 N_2} \sum_{n=0}^{N_1 N_2 - 1} |x[n]|^2,$$

$$= \frac{1}{N_1 N_2} \sum_{n=0}^{N_1 N_2 - 1} |A_1 \sin(\omega_1 n + \phi_1) + A_2 \sin(\omega_2 n + \phi_2)|^2,$$

$$= \frac{1}{N_1 N_2} \sum_{n=0}^{N_1 N_2 - 1} [A_1^2 \sin^2(\omega_1 n + \phi_1) + A_2^2 \sin^2(\omega_2 n + \phi_2)$$

$$+ 2A_1 A_2 \sin(\omega_1 n + \phi_1) \sin(\omega_2 n + \phi_2)],$$

General Case

Power of Sum of Sinusoids

$$
\begin{aligned}
P = \frac{1}{N_1N_2} \sum_{n=0}^{N_1N_2-1} & [A_1^2 \sin^2(\omega_1 n + \phi_1) + A_2^2 \sin^2(\omega_2 n + \phi_2) \\
& + 2A_1A_2 \sin(\omega_1 n + \phi_1)\sin(\omega_2 n + \phi_2)], \\
= \frac{1}{N_1N_2} \sum_{n=0}^{N_1N_2-1} & A_1^2 \sin^2(\omega_1 n + \phi_1) \qquad P_1 \\
+ \frac{1}{N_1N_2} \sum_{n=0}^{N_1N_2-1} & A_2^2 \sin^2(\omega_2 n + \phi_2) \qquad P_2 \\
+ \frac{1}{N_1N_2} \sum_{n=0}^{N_1N_2-1} & 2A_1A_2 \sin(\omega_1 n + \phi_1)\sin(\omega_2 n + \phi_2), \quad P_3
\end{aligned}
$$

General Case

Power of Sum of Sinusoids

$$P = \frac{1}{N_1 N_2} \sum_{n=0}^{N_1 N_2 - 1} [A_1^2 \sin^2(\omega_1 n + \phi_1) + A_2^2 \sin^2(\omega_2 n + \phi_2)$$

$$+ 2A_1 A_2 \sin(\omega_1 n + \phi_1) \sin(\omega_2 n + \phi_2)],$$

$$= \frac{1}{N_1 N_2} \sum_{n=0}^{N_1 N_2 - 1} A_1^2 \sin^2(\omega_1 n + \phi_1) \qquad P_1 = \frac{A_1^2}{2}$$

$$+ \frac{1}{N_1 N_2} \sum_{n=0}^{N_1 N_2 - 1} A_2^2 \sin^2(\omega_2 n + \phi_2) \qquad P_2 = \frac{A_2^2}{2}$$

$$+ \frac{1}{N_1 N_2} \sum_{n=0}^{N_1 N_2 - 1} 2A_1 A_2 \sin(\omega_1 n + \phi_1) \sin(\omega_2 n + \phi_2), \quad P_3?$$

Power of Sum of Sinusoids

$$
\begin{aligned}
P_3 &= \frac{1}{N_1N_2}\sum_{n=0}^{N_1N_2-1} 2A_1A_2\sin(\omega_1 n+\phi_1)\sin(\omega_2 n+\phi_2),\\
&= \frac{A_1A_2}{N_1N_2}\sum_{n=0}^{N_1N_2-1}\{\cos[(\omega_1-\omega_2)n+\phi_1-\phi_2]\\
&\quad -\cos[(\omega_1+\omega_2)n+\phi_1+\phi_2]\},\\
&= \frac{A_1A_2}{2N_1N_2}\left[e^{j(\phi_1-\phi_2)}\sum_{n=0}^{N_1N_2-1}e^{j(\omega_1-\omega_2)n}+e^{-j(\phi_1-\phi_2)}\sum_{n=0}^{N_1N_2-1}e^{-j(\omega_1-\omega_2)n}\right.\\
&\quad \left.-e^{j(\phi_1+\phi_2)}\sum_{n=0}^{N_1N_2-1}e^{j(\omega_1+\omega_2)n}-e^{-j(\phi_1+\phi_2)}\sum_{n=0}^{N_1N_2-1}e^{-j(\omega_1+\omega_2)n}\right],
\end{aligned}
$$

General Case

Power of Sum of Sinusoids

$$
=\frac{A_1A_2}{2N_1N_2}\Bigg[e^{j(\phi_1-\phi_2)}\sum_{n=0}^{N_1N_2-1}e^{j(\omega_1-\omega_2)n}+e^{-j(\phi_1-\phi_2)}\sum_{n=0}^{N_1N_2-1}e^{-j(\omega_1-\omega_2)n}
$$

$$
-e^{j(\phi_1+\phi_2)}\sum_{n=0}^{N_1N_2-1}e^{j(\omega_1+\omega_2)n}-e^{-j(\phi_1+\phi_2)}\sum_{n=0}^{N_1N_2-1}e^{-j(\omega_1+\omega_2)n}\Bigg],
$$

if $\omega_1=\omega_2$:

$$
P_3=\frac{A_1A_2}{2N_1N_2}\Bigg[e^{j(\phi_1-\phi_2)}\sum_{n=0}^{N_1N_2-1}1+e^{-j(\phi_1-\phi_2)}\sum_{n=0}^{N_1N_2-1}1
$$

$$
-e^{j(\phi_1+\phi_2)}\sum_{n=0}^{N_1N_2-1}e^{j2\omega_1n}-e^{-j(\phi_1+\phi_2)}\sum_{n=0}^{N_1N_2-1}e^{-j2\omega_1n}\Bigg],
$$

Power of Sum of Sinusoids

$$
\begin{aligned}
P_3 = \frac{A_1A_2}{2N_1N_2}\Bigg[& e^{j(\phi_1-\phi_2)}\sum_{n=0}^{N_1N_2-1}1 + e^{-j(\phi_1-\phi_2)}\sum_{n=0}^{N_1N_2-1}1 \\
& -e^{j(\phi_1+\phi_2)}\sum_{n=0}^{N_1N_2-1}e^{j2\omega_1 n} - e^{-j(\phi_1+\phi_2)}\sum_{n=0}^{N_1N_2-1}e^{-j2\omega_1 n}\Bigg], \\
= \frac{A_1A_2}{2N_1N_2}\Bigg\{ & N_1N_2[e^{j(\phi_1-\phi_2)} + e^{-j(\phi_1-\phi_2)}] \\
& -e^{j(\phi_1+\phi_2)}\frac{1-e^{j2\omega_1N_1N_2}}{1-e^{j2\omega_1}} - e^{-j(\phi_1+\phi_2)}\frac{1-e^{-j2\omega_1N_1N_2}}{1-e^{-j2\omega_1}}\Bigg\},
\end{aligned}
$$

General Case

Xi'an Jiaotong-Liverpool University 西交利物浦大学

Power of Sum of Sinusoids

$$P_3 = \frac{A_1A_2}{2N_1N_2}\left\{N_1N_2[e^{j(\phi_1-\phi_2)} + e^{-j(\phi_1-\phi_2)}]\right.$$

$$\left. -e^{j(\phi_1+\phi_2)}\frac{1-e^{j2\omega_1N_1N_2}}{1-e^{j2\omega_1}} - e^{-j(\phi_1+\phi_2)}\frac{1-e^{-j2\omega_1N_1N_2}}{1-e^{-j2\omega_1}}\right],$$

$$= \frac{A_1A_2}{2N_1N_2}\left[2N_1N_2\cos(\phi_1-\phi_2)\right.$$

$$\left. -e^{j(\phi_1+\phi_2)}\frac{1-e^{j2\omega_1N_1N_2}}{1-e^{j2\omega_1}} - e^{-j(\phi_1+\phi_2)}\frac{1-e^{-j2\omega_1N_1N_2}}{1-e^{-j2\omega_1}}\right],$$

$$= \frac{A_1A_2}{2N_1N_2}\left[2N_1N_2\cos(\phi_1-\phi_2)\right.$$

$$\left. -e^{j(\phi_1+\phi_2)}\frac{1-e^{j2m_1N_22\pi}}{1-e^{j2\omega_1}} - e^{-j(\phi_1+\phi_2)}\frac{1-e^{-j2m_1N_22\pi}}{1-e^{-j2\omega_1}}\right],$$

General Case

Power of Sum of Sinusoids

$$
\begin{aligned}
P_3 = \frac{A_1 A_2}{2N_1N_2} & \bigg[2N_1N_2 \cos(\phi_1 - \phi_2) \\
& - e^{j(\phi_1+\phi_2)} \frac{1 - e^{j2m_1N_22\pi}}{1 - e^{j2\omega_1}} - e^{-j(\phi_1+\phi_2)} \frac{1 - e^{-j2m_1N_22\pi}}{1 - e^{-j2\omega_1}} \bigg], \\
= \frac{A_1 A_2}{2N_1N_2} & \left[2N_1N_2 \cos(\phi_1 - \phi_2) - 0 - 0 \right], \\
= A_1 A_2 & \cos(\phi_1 - \phi_2).
\end{aligned}
$$

Xi'an Jiaotong-Liverpool University 西交利物浦大学

Power of Sum of Sinusoids

if $\omega_1 \neq \omega_2$:

$$P_3 = \frac{A_1 A_2}{2N_1 N_2} \left[e^{j(\phi_1-\phi_2)} \frac{1 - e^{j(\omega_1-\omega_2)N_1 N_2}}{1 - e^{j(\omega_1-\omega_2)}} + e^{-j(\phi_1-\phi_2)} \frac{1 - e^{-j(\omega_1-\omega_2)N_1 N_2}}{1 - e^{-j(\omega_1-\omega_2)}} \right.$$

$$\left. - e^{j(\phi_1+\phi_2)} \frac{1 - e^{j(\omega_1+\omega_2)N_1 N_2}}{1 - e^{j(\omega_1+\omega_2)}} - e^{-j(\phi_1+\phi_2)} \frac{1 - e^{-j(\omega_1+\omega_2)N_1 N_2}}{1 - e^{-j(\omega_1+\omega_2)}} \right],$$

$$= \frac{A_1 A_2}{2N_1 N_2} \left[e^{j(\phi_1-\phi_2)} \frac{1 - e^{j(m_1 N_2 - m_2 N_1)2\pi}}{1 - e^{j(\omega_1-\omega_2)}} + e^{-j(\phi_1-\phi_2)} \frac{1 - e^{-j(m_1 N_2 - m_2 N_1)2\pi}}{1 - e^{-j(\omega_1-\omega_2)}} \right.$$

$$\left. - e^{j(\phi_1+\phi_2)} \frac{1 - e^{j(m_1 N_2 + m_2 N_1)2\pi}}{1 - e^{j(\omega_1+\omega_2)}} - e^{-j(\phi_1+\phi_2)} \frac{1 - e^{-j(m_1 N_2 + m_2 N_1)2\pi}}{1 - e^{-j(\omega_1+\omega_2)}} \right],$$

$$= \frac{A_1 A_2}{2N_1 N_2} (0 + 0 - 0 - 0) = 0.$$

Example

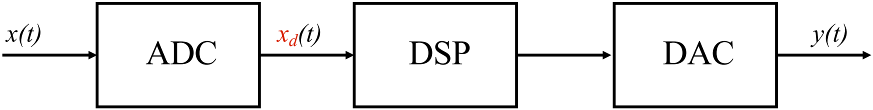
Xi'an Jiaotong-Liverpool University 西交利物浦大学

$$x[n] = A_1 \sin(\omega_1 n + \phi_1) + A_2 \sin(\omega_2 n + \phi_2)$$

$$P = \begin{cases} A_1^2 \sin^2\phi_1 + A_2^2 \sin^2\phi_2 & m_1 = 0, m_2 = 0, \\ +2A_1 A_2 \sin\phi_1 \sin\phi_2 & m_1 = m_2 = 1, N_1 = N_2 = 1, 2, \\ & \\ A_1^2 \sin^2\phi_1 & m_1 = 0, m_2 = 1, N_2 = 2, \\ +A_2^2 \sin^2\phi_2, & m_1 = 1, N_1 = 2, m_2 = 0, \\ & \\ & \omega_1 \neq \omega_2 \\ \dfrac{A_1^2}{2} + \dfrac{A_2^2}{2}, & 1 \leqslant m_1 < N_1, N_1 > 2, \\ & 1 \leqslant m_2 < N_2, N_2 > 2, \\ & \\ & \omega_1 = \omega_2 \\ \dfrac{A_1^2}{2} + \dfrac{A_2^2}{2} & 1 \leqslant m_1 < N_1, N_1 > 2, \\ +A_1 A_2 \cos(\phi_1 - \phi_2) & 1 \leqslant m_2 < N_2, N_2 > 2. \end{cases}$$

Sampling Theory

- Digital signal processing



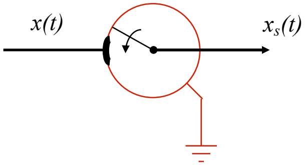
ADC: convert the continuous signal into digital signal

Sampling:

Is the operation that transforms a continuous-time/space signal into a discrete-time/space version.

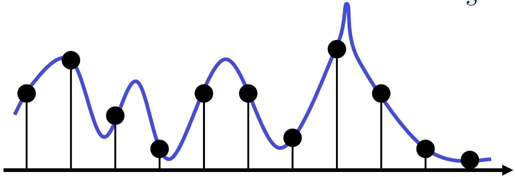
Sampler:

Is a system that measures the amplitude of the continuous signal at specific instances of time.



Uniform sampling:

- Samples of signal are taken at equally spaced points along the signal waveform.
- Samples are taken at time instant $t_s = nT_s$ (n is integer).

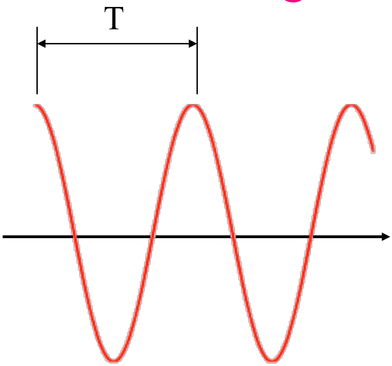


- The distance between two consecutive samples (T_s) is called the sampling period/interval, the sampling frequency/rate f_s is the number of samples in one second, i.e. $f_s = \frac{1}{T_s}$ [Hz or samples per second].

Sampling of Sinusoid Signal

When representing a signal by its samples, we have to ask ourselves:

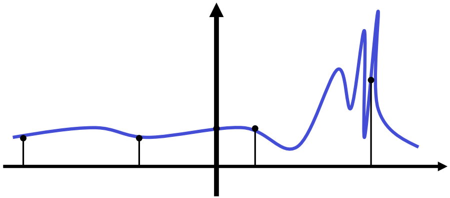
- Does the samples have the same shape / characteristics as the original signal?

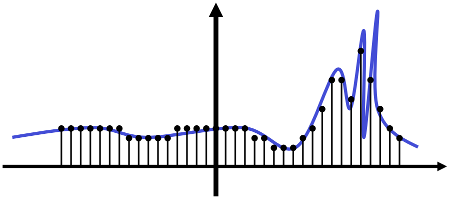


Uniform Sampling

Xi'an Jiaotong-Liverpool University 西交利物浦大學

How fast should we sample





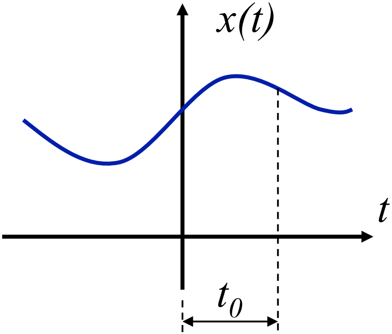
- Fewer samples are needed for a slowly-changing signal. More samples are required for fast-changing signals.
- What is the critical sampling rate?

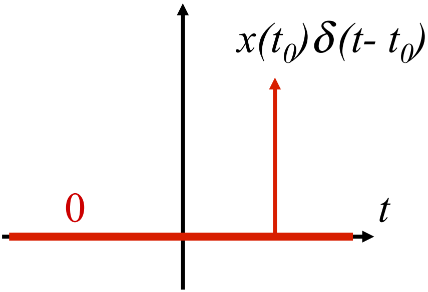
Xi'an Jiaotong-Liverpool University 西交利物浦大学

Ideal Impulse-train Sampling

Assuming that $x(t)$ is continuous function then

$$x(t)\delta(t - t_0) = x(t_0)\delta(t - t_0).$$



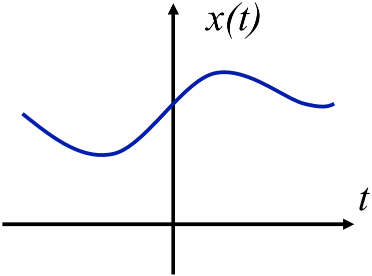


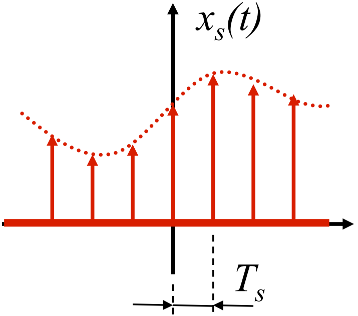
The weight of $\delta(t)$ is the value of $x(t)$ at t_0.

Ideal Impulse-train Sampling

The resulting sampled signal can be represented by multiplying $x(t)$ with an impulse train

$$x_s(t) = x(t)\sum_{n=-\infty}^{\infty}\delta(t-nT_s) = \sum_{n=-\infty}^{\infty}x(nT_s)\delta(t-nT_s)$$





Frequency Domain Representation

We take continuous Fourier Transform on both sides of $x_s(t) = x(t)\sum_{n=-\infty}^{\infty}\delta(t-nT_s)$

$$\mathcal{F}[x_s(t)] = \mathcal{F}\left[x(t)\sum_{n=-\infty}^{\infty}\delta(t-nT_s)\right],$$

$$X_s(\omega) = \frac{1}{2\pi}\mathcal{F}[x(t)] * \mathcal{F}\left[\sum_{n=-\infty}^{\infty}\delta(t-nT_s)\right],$$

Frequency convolution

$$x_1(t)\times x_2(t) \rightarrow \frac{1}{2\pi}[X_1(\omega) * X_2(\omega)]$$

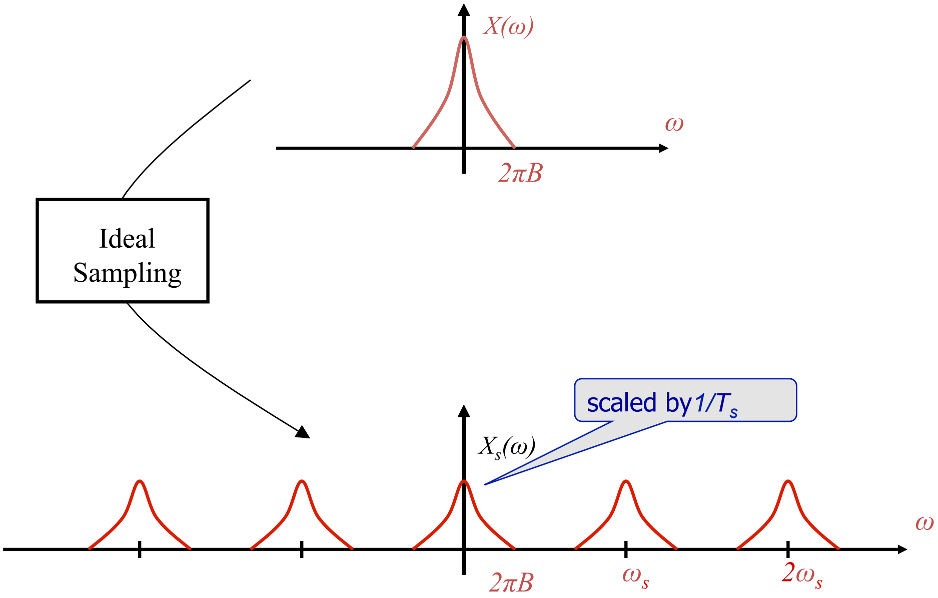
Frequency Domain Representation

$$
\begin{aligned}
X_s(\omega) &= \frac{1}{2\pi}\mathcal{F}[x(t)] * \mathcal{F}\left[\sum_{n=-\infty}^{\infty}\delta(t-nT_s)\right], \\
&= \frac{1}{2\pi}\left[X(\omega) * \omega_s \sum_{m=-\infty}^{\infty}\delta(\omega - m\omega_s)\right], \\
&= \frac{\omega_s}{2\pi}\sum_{m=-\infty}^{\infty} X(\omega - m\omega_s), \\
&= \frac{1}{T_s}\sum_{m=-\infty}^{\infty} X\left(\omega - \frac{2m\pi}{T_s}\right).
\end{aligned}
$$

The second step refers to Page 329 row 7 in Table 4.2.

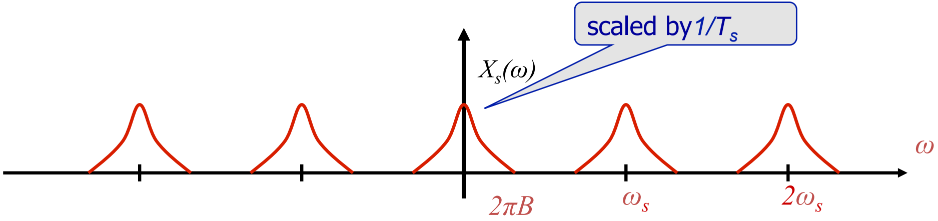
Xi'an Jiaotong-Liverpool University 西交利物浦大学

Frequency Domain Representation



Xi'an Jiaotong-Liverpool University 西交利物浦大学

Theory of Ideal Impulse-train Sampling



A signal $x(t)$ can be reconstructed perfectly from its samples if the sampling rate $f_s = \dfrac{\omega_s}{2\pi}$ satisfies the condition:

$$\omega_s \geqslant 4\pi B,$$

$$2\pi f_s \geqslant 4\pi B,$$

$$\boxed{f_s \geqslant 2B}.$$

B is the highest frequency of the signal.

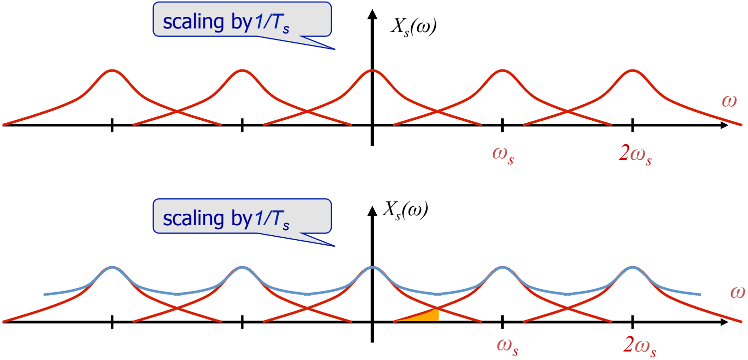
Theory of ideal impulse-train sampling

The minimum sampling rate required for perfect reconstruction of the original signal is called Nyquist rate,

$$\boxed{f_s = 2B}.$$

This theorem was first articulated by Nyquist in 1928 and was formally proven by Shannon in 1949.

What happens when the sampling rate is below the Nyquist rate?



Frequency folding error (or aliasing) results from not having fast enough sampling. If the Nyquist sampling criterion is not met, perfect reconstruction of the original signal is not achieved.

Homework

- Page 61, Q1.26: (a)–(e);
- Page 514–545, read section 7.0–7.1.1, 7.2–7.4
- Page 556, Q7.1;
- Page 556, Q7.2: (a)–(c).
- Page 556, Q7.3: (a)–(c).
- Page 556, Q7.4: (a)–(d).

Thank you for your attention.