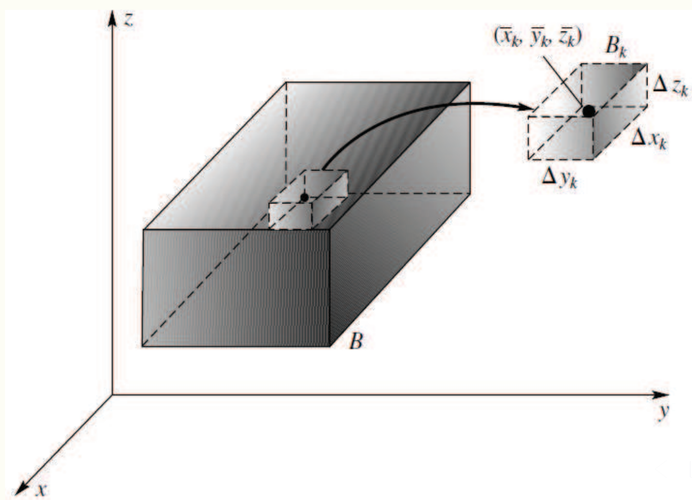
2.8 Triple integral and divergence theorem of Gauss
(page 452)

In this section we discuss the divergence theorem, which transforms surface integrals into triple integrals. So let us begin with a review of the latter.

A **triple integral** is an integral of a function $f(x, y, z)$ taken over a closed bounded, three-dimensional region in space.

We first consider the triple integrals over rectangular boxes. Let $f(x, y, z)$ be defined over a box-shaped region B (figure)

$$B = \{(x, y, z) : a \leq x \leq b, c \leq y \leq d, r \leq z \leq s\}.$$



1. Form a partition P of B using planes parallel to the coordinate planes. This divides B into n small subboxes B_k with the lengths of sides Δx_k, Δy_k, and Δz_k, $k = 1, 2, \ldots, n$. Then the volume of B_k is $\Delta V_k = \Delta x_k \Delta y_k \Delta z_k$
2. Pick a sample point $(\bar{x}_k, \bar{y}_k, \bar{z}_k)$ in each B_k and form the Riemann sum

$$R_p = \sum_{k=1}^{n} f(\bar{x}_k, \bar{y}_k, \bar{z}_k)\Delta V_k.$$

3. Take the limit as the partition get finer and finer by $\|P\| \to 0$ ($\|P\|$ is the length of the longest diagonal of the subboxes). Then we define the **triple integral** of f over B by

$$\iiint_B f(x, y, z)dV = \lim_{\|P\|\to 0} f(\bar{x}_k, \bar{y}_k, \bar{z}_k)\Delta V_k$$

if this limit exists.

Remark: The triple integrals can be also written as triple iterated integrals, for example

$$\iiint\limits_B f(x,y,z)dV = \int_r^s \int_c^d \int_a^b f(x,y,z)dxdydz.$$

There are five other possible orders of integration, all of which give the same answer.

Example 1: Evaluate $\iiint\limits_B x^2yzdV$, where B is the box

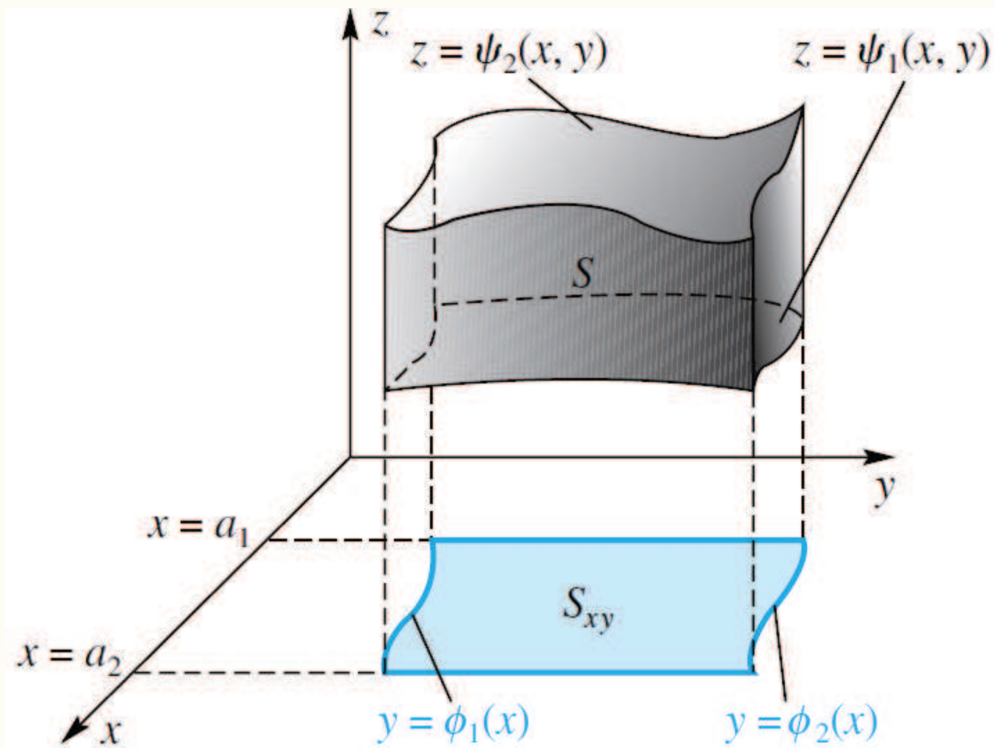
$$B = \{(x,y,z) : 1 \le x \le 2, 0 \le y \le 1, 0 \le z \le 2\}.$$

Solution:

$$\begin{aligned}
\iiint\limits_B x^2yz\,dV &= \int_0^2\int_0^1\int_1^2 x^2yz\,dx\,dy\,dz \\
&= \int_0^2\int_0^1\left[\frac{1}{3}x^3yz\right]_1^2 dy\,dz = \int_0^2\int_0^1 \frac{7}{3}yz\,dy\,dz \\
&= \frac{7}{3}\int_0^2\left[\frac{1}{2}y^2z\right]_0^1 dz = \frac{7}{3}\int_0^2 \frac{1}{2}z\,dz \\
&= \frac{7}{6}\left[\frac{z^2}{2}\right]_0^2 = \frac{7}{3}
\end{aligned}$$

Triple integrals over general regions

1. Let S be a $z-$**simple set**: vertical lines intersect S in a single line segment. Let S_{xy} be the projection of S onto the $xy-$ plane. Notice that S lies between the graphs of two functions. The upper boundary is the surface $z = \psi_2(x, y)$, the lower boundary is the surface $z = \psi_1(x, y)$. Thus
$S = \{(x, y, z) : (x, y) \in D, \psi_1(x, y) \leq z \leq \psi_2(x, y)\}$.



2. If S be a $z-$simple set, then the triple integral can be computed by the following integral (first definite integral with respect to z, then the double integral on the $xy-$plane)

$$\iiint\limits_S f(x,y,z)dV = \iint\limits_{S_{xy}} \left[\int_{\psi_1(x,y)}^{\psi_2(x,y)} f(x,y,z)dz\right] dA.$$

3. If in addition S_{xy} is a $y-$simple set,

$$S_{xy} = (x,y) : \phi_1(x) \leq y \leq \phi_2(x), a_1 \leq y \leq a_2,$$

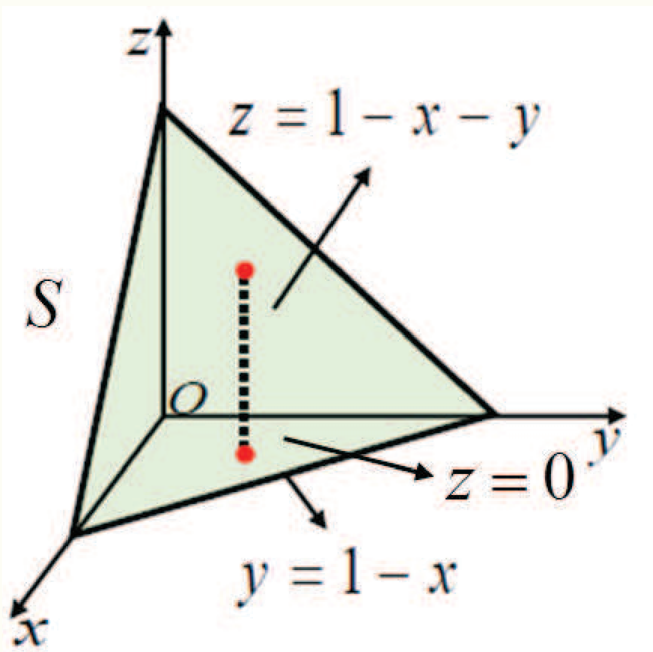
then we can further rewrite the outer double integral as an iterated integral.

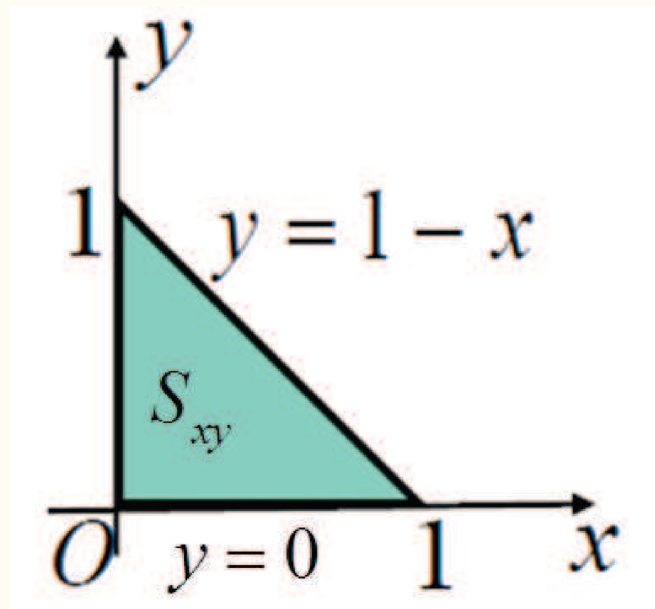
$$\iiint\limits_S f(x,y,z)dV = \int_{a_1}^{a_2} \int_{\phi_1(x)}^{\phi_2(x)} \int_{\psi_1(x,y)}^{\psi_2(x,y)} f(x,y,z)dzdydx.$$

The integral on the right is a triple iterated integral.

Example 2: Evaluate $\iiint\limits_S x\,dV$, where S is the solid bounded by the plane $x+y+z=1$ and the three coordinate planes in the first octant.

Solutions: Step 1: Sketch the solid region in three space and its projection in the xy-plane.
$S=\{(x,y,z):(x,y)\in S_{xy}, 0\le z\le 1-x-y\},\ z-$simple set
$S_{xy}=\{(x,y):0\le y\le 1-x, 0\le x\le 1\}$





Step 2: Write the triple integral as an triple iterated integral

$$
\begin{aligned}
\iiint\limits_S x dV &= \iint\limits_{S_{xy}} \left[\int_0^{1-x-y} x dz\right] dA \\
&= \int_0^1 \int_0^{1-x} \int_0^{-x-y} x dz dy dx
\end{aligned}
$$

Step 3: Compute the triple iterated integral by N-L formula.

$$
\begin{aligned}
\iiint\limits_S x dV &= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} x dz dy dx \\
&= \int_0^1 \int_0^{1-x} x(1-x-y) dy dx \\
&= \int_0^1 \left[xy - x^2 y - x\frac{y^2}{2}\right]_{y=0}^{y=1-x} dx \\
&= \frac{1}{2}\int_0^1 (x - 2x^2 + x^3) dx = \frac{1}{24}
\end{aligned}
$$

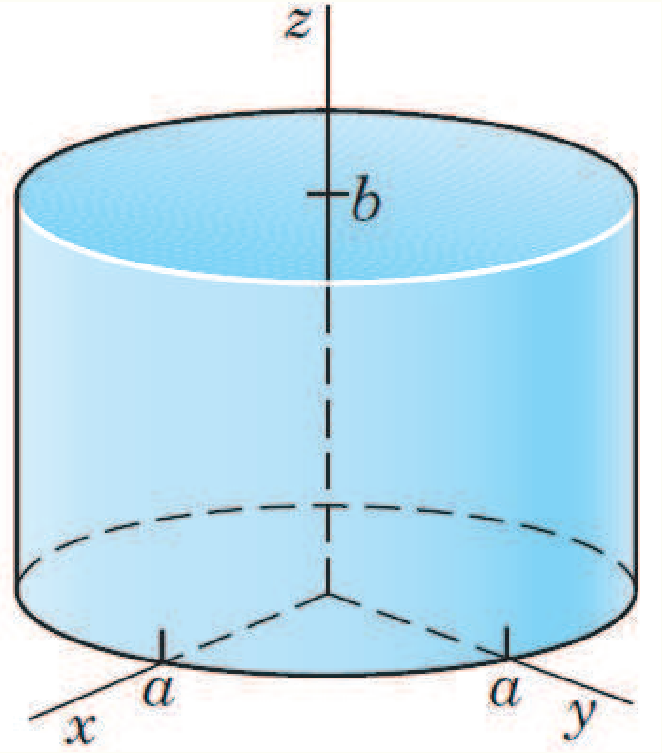
$\boxed{\textbf{Divergence theorem of Gauss:}}$ Let T be a closed bounded region in space whose boundary is a piecewise smooth orientable surface S. Let $\boldsymbol{F}(x, y, z)$ be a vector function that is continuous and has continuous first partial derivatives in some domain containing T. Then

$$\iiint_T \mathrm{div}\boldsymbol{F} dV = \iint_S \boldsymbol{F} \cdot \boldsymbol{n} dA. \tag{2.20}$$

In components of $\boldsymbol{F} =< F_1, F_2, F_3 >$ and of the outer unit normal vector $\boldsymbol{n} =< \cos\alpha, \cos\beta, \cos\gamma >$ of S, formula (2.20) becomes

$$\begin{aligned} & \iiint_T \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right) dxdydz \\ = & \iint_S (F_1 \cos\alpha + F_2 \cos\beta + F_3 \cos\gamma) dA \\ = & \iint_S (F_1 dydz + F_2 dzdx + F_3 dxdy) \end{aligned} \tag{2.21}$$

Example 3: Evaluate $I = \iint\limits_S x^3 dydz + x^2 ydzdx + x^2 zdxdy$ by using the divergence theorem, where S is the closed surface in the following figure.



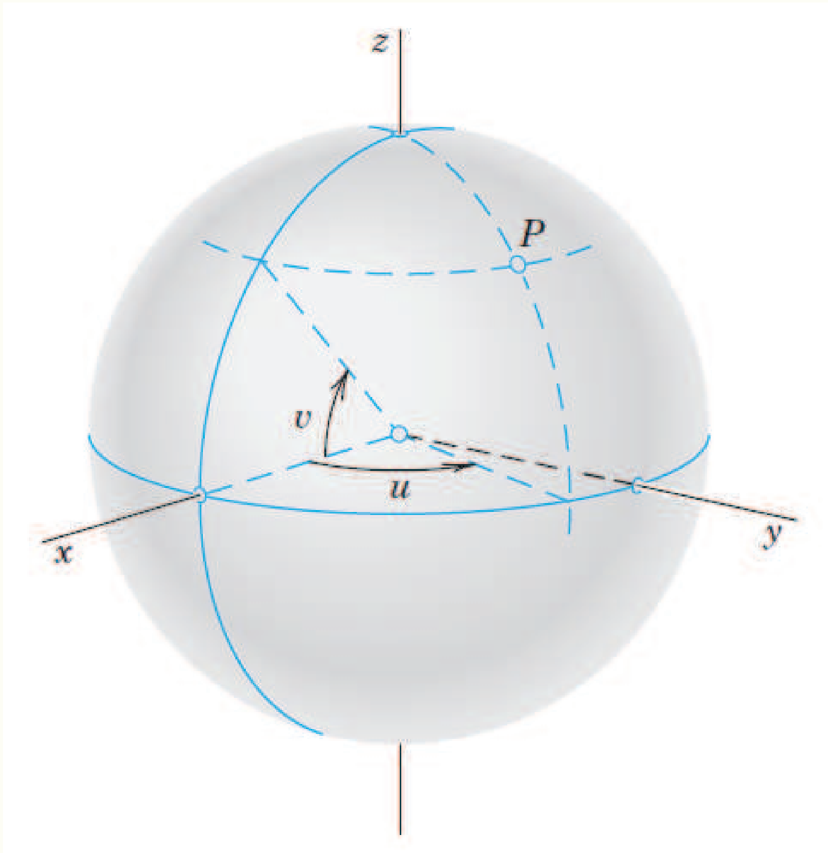
Solution: We know $F_1 = x^3, F_2 = x^2 y, F_3 = x^2 z$. Hence $\mathrm{div}\boldsymbol{F} = 3x^2 + x^2 + x^2 = 5x^2$. The form of the surface suggests that we introduce polar coordinates r, θ defined by $x = r\cos\theta, y = r\sin\theta$ (thus cylindrical coordinates r, θ, z). Then the volume element is $dxdydz = rdrd\theta dz$, and we obtain

$$
\begin{aligned}
I &= \iiint\limits_T \mathrm{div}\boldsymbol{F} dV = \iiint\limits_T 5x^2 dxdydz \\
&= \iiint\limits_T 5(r\cos\theta)^2 rdrd\theta dz = \int_0^b \int_0^{2\pi} \int_0^a 5r^3 \cos^2\theta drd\theta dz \\
&= \int_0^b \int_0^{2\pi} \frac{5a^4}{4} \cos^2\theta d\theta dz = \frac{5a^4}{4} \int_0^b \int_0^{2\pi} \frac{1+\cos 2\theta}{2} d\theta dz \\
&= \frac{5a^4}{4} \int_0^b \pi dz = \frac{5\pi a^4 b}{4}
\end{aligned}
$$

Example 4: Verification of divergence theorem. Evaluate $\iint_S (7x\mathbf{i} - z\mathbf{k} \cdot dA)$ over the sphere $S : x^2 + y^2 + z^2 = 4$

(a) by the divergence theorem;

(b) by definition of surface integral.



Solution: (a)

$$
\begin{aligned}
\iiint\limits_{T} \operatorname{div} \boldsymbol{F} dV &= \iiint\limits_{T} \left[\frac{\partial(7x)}{\partial x} + \frac{\partial(0)}{\partial y} + \frac{\partial(-z)}{\partial z} \right] \\
&= \iiint\limits_{T} (7 + 0 - 1) dV = \iiint\limits_{T} 6 dV \\
&= 6 \iiint\limits_{T} dV = 6 \left(\frac{4}{3} \pi 2^3 \right) = 64\pi.
\end{aligned}
$$

(b) The sphere can be represented by

$$\boldsymbol{r}(u,v) = 2\cos v\cos u\mathbf{i} + 2\cos v\sin u\mathbf{j} + 2\sin v\mathbf{k},$$

where $0 \le u \le 2\pi$, $-\frac{\pi}{2} \le v \le \frac{\pi}{2}$.
So the normal vector of the sphere is

$$\begin{aligned}
\boldsymbol{N} &= \boldsymbol{r}_u \times \boldsymbol{r}_v \\
&= < -2\cos v\sin u, 2\cos v\cos u, 0 > \times \\
&\quad < -2\sin v\cos u, -2\sin v\sin u, 2\cos v > \\
&= < 2^2\cos^2 v\cos u, 2^2\cos^2 v\sin u, 2^2\cos v\sin v >
\end{aligned}$$

We have
$\boldsymbol{F}(\boldsymbol{r}(u,v)) =< 7x, 0, -z >=< 2(2\cos v\cos u), 0, -(2\sin v) >$, so
by the definition of the surface integral,

$$
\begin{aligned}
& \iint_S \boldsymbol{F} \cdot \boldsymbol{n} dA = \iint_R \boldsymbol{F}(\boldsymbol{r}(u,v)) \cdot \boldsymbol{N}(u,v) du dv \\
= \quad & \iint_R (56 \cos^3 v \cos^2 u - 8 \sin^2 v \cos u) du dv \\
= \quad & \iint_R (56 \cos^3 v \cos^2 u) du dv - \iint_R (8 \sin^2 v \cos u) du dv \\
= \quad & 56 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^3 v dv \int_0^{2\pi} \cos^2 u du - 8 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 v \cos v dv \int_0^{2\pi} 1 du \\
= \quad & 56 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1 - \sin^2 v) d \sin v \int_0^{2\pi} \frac{1 + \cos 2u}{2} du \\
& -16\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 v d \sin v \\
= \quad & 64\pi
\end{aligned}
$$

Compare the amount of work!

Example 5: (optional) Modeling of heat flow, heat or diffusion equation (page 459)
Physical experiments show that in a body, heat flows in the direction of decreasing temperature, and the rate of flow is proportional to the gradient of the temperature. This means that the velocity $\boldsymbol{v}$ of the heat flow in a body is of the form

$$\boldsymbol{v} = -\kappa \text{grad} U,$$

where $U(x, y, z)$ is temperature, t is time, and κ is called the thermal conductivity of the body, in ordinary physical circumstances κ is a constant. Using this information, set up the mathematical model of heat flow, the so-called heat equation or diffusion equation.

Solution: Let T be a region in the body bounded by a surface S with outer unit normal vector $\boldsymbol{n}$ such that the divergence theorem applies. Then $\boldsymbol{v} \cdot \boldsymbol{n}$ is the component of $\boldsymbol{v}$ in the direction of $\boldsymbol{n}$, and the amount of heat leaving T per unit time is

$$\begin{aligned}
\iint_S \boldsymbol{v} \cdot \boldsymbol{n} dA &= \iint_s (-\kappa \text{grad} U) \cdot \boldsymbol{n} dA \\
&= -\kappa \iint_S (\text{grad} U) \cdot \boldsymbol{n} dA \\
&= -\kappa \iiint_T \text{div}(\text{grad} U) \cdot dV \, (\text{By divergence theorem}) \\
&= -\kappa \iiint_T (U_{xx} + U_{yy} + U_{zz}) dx dy dz \\
&= -\kappa \iiint_T \nabla^2 U dx dy dz,
\end{aligned}$$

where $\nabla^2 U = U_{xx} + u_{yy} + U_{zz}$.

On the other hand, the total amount of heat H in T is

$$H = \iiint\limits_T \sigma\rho U(x,y,z,t)dxdydz,$$

where the constant σ is the specific heat of the material of the body and ρ is the density (=mass per unit volume) of the material. Hence the time rate of decrease of H is

$$-\frac{\partial H}{\partial t} = -\iiint\limits_T \sigma\rho\frac{\partial U}{\partial t}dxdydz$$

and this must be equal to the above amount of heat leaving T. From the amount of heat leaving T per unit time,
$-\kappa \iiint\limits_T \nabla^2 U dxdydz$ we thus have

$$-\iiint\limits_T \sigma\rho\frac{\partial U}{\partial t}dxdydz = -\kappa\iiint\limits_T \nabla^2 U dxdydz.$$

Therefore

$$\iiint\limits_T \left(\sigma\rho\frac{\partial U}{\partial t} - \kappa\nabla^2 U\right)dxdydz = 0.$$

Since this holds for any region T in the body, the integrand (if continuous) must be zero everywhere, that is

$$\sigma\rho\frac{\partial U}{\partial t} - \kappa\nabla^2 U = 0,$$

$$\sigma\rho\frac{\partial U}{\partial t} = \kappa\nabla^2 U,$$

$$\frac{\partial U}{\partial t} = \frac{\kappa}{\sigma\rho}\nabla^2 U,$$

If we let $c^2 = \frac{\kappa}{\sigma\rho}$,

$$\frac{\partial U}{\partial t} = c^2\nabla^2 U.$$

We call c^2 the thermal diffusivity of the material.
This partial differential equation is called the **heat equation**.
It is the fundamental equation for heat conduction.
It is also called the **diffusion equation**.
If heat flow does not depend on time, it is called **steady-state heat flow**. Then $\frac{\partial U}{\partial t} = 0$, so $\nabla^2 U = 0$ which is Laplace's equation.

Potential theory. Harmonic functions The theory of solution of Laplace's equation:

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0 \tag{2.22}$$

is called **potential theory**.
A solution of (2.22) with continuous second-order partial derivatives is called a **harmonic function**.

Green's theorem (page 461)
Let f and g be scalar functions such that $\boldsymbol{F} = f\,\mathrm{grad}\,g$ satisfies the assumptions of the divergence theorem in some region T, bounded by a piecewise smooth closed orientable surface S. Then we have the following two formulae:

1. Green's first formula

$$\iiint_T (f\nabla^2 g + \mathrm{grad}\, f \cdot \mathrm{grad}\, g)\, dV = \iint_S f \frac{\partial g}{\partial n}\, dA;$$

2. Green's second formula
   If $g\,\mathrm{grad}\,f$ also satisfies the assumptions of the divergence theorem, then we have

$$\iiint_T (f\nabla^2 g - g\nabla^2 f)\, dV = \iint_S \left(f \frac{\partial g}{\partial n} - g \frac{\partial f}{\partial n} \right) dA.$$