Xi'an Jiaotong-Liverpool University

EEE108 Electromagnetism and Electromechanics

Lecture 3

Gauss's Law

Dr. Jinling Zhang

Dept. of Electrical and Electronic Engineering
University of Xi'an Jiaotong-Liverpool
Email: jinling.zhang@xjtlu.edu.cn

Last Week

- Vector:
  - Calculus
  - Line Integral
  - Surface Integral
  - Fields

- Coulomb's law : $\mathbf{F}_{12} = k_e \frac{Q_1 Q_2}{r^2} \mathbf{a}_r$

- The electric field intensity : $\mathbf{E} = \frac{\mathbf{F}}{q}$

- Principle of superposition : $\mathbf{E}_{Total} = \sum_{i=1}^{N} \mathbf{E}_i$

Today

- Electric Fields Produced by Continuous Charge Distributions: Examples
- Electric Flux– Review
- Gauss's Law:
  - The statement
  - The mathematical expression
  - Applications (1)

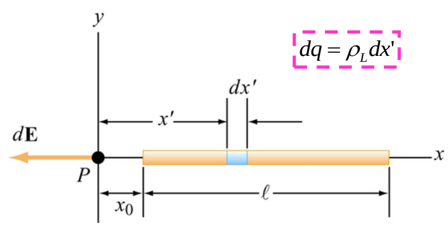
Electric Field due to Continuous Charge Distributions

Example 1: Line of Charge

A rod of length l with a uniform positive charge density ρ_L and a total charge Q is lying along the x-axis. Calculate the electric field at a point P located along the axis of the rod and a distance x_0 from one end.

$$d\mathbf{E} = \frac{1}{4\pi\varepsilon_0} \frac{dQ}{R^2} \mathbf{a}_R$$

$$dq = \rho_L dx'$$

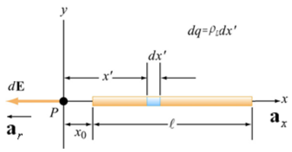


Example 1: Line of Charge Solution

The uniform linear charge density $\rho_L = Q/\ell$

The amount of charge contained in a small segment of length dx': $dq = \rho_L dx'$

The unit vector that points from the source to P $\quad \mathbf{a}_r = -\mathbf{a}_x$



$$d\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r^2}\mathbf{a}_r = \frac{1}{4\pi\varepsilon_0}\frac{\rho_L dx'}{x'^2}(-\mathbf{a}_x) = -\frac{1}{4\pi\varepsilon_0}\frac{Qdx'}{\ell\, x'^2}\mathbf{a}_x$$

Integrating over the entire length

$$\mathbf{E} = \int d\mathbf{E} = -\frac{1}{4\pi\varepsilon_0}\frac{Q}{\ell}\boxed{\int_{x_0}^{x_0+\ell}\frac{dx'}{x'^2}}\mathbf{a}_x = -\frac{1}{4\pi\varepsilon_0}\frac{Q}{\ell}\left(\frac{1}{x_0} - \frac{1}{x_0+\ell}\right)\mathbf{a}_x = -\frac{1}{4\pi\varepsilon_0}\frac{Q}{x_0(\ell + x_0)}\mathbf{a}_x$$

When P is very far away from the rod, $x_0 >> l$

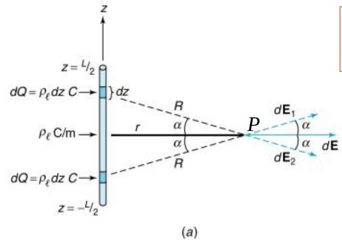
$$\mathbf{E} \approx -\frac{1}{4\pi\varepsilon_0}\frac{Q}{x_0^2}\mathbf{a}_x \quad \Rightarrow \quad \text{Point charge}$$

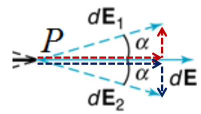
$$\boxed{\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{R^2}\mathbf{a}_R}$$

Example 2: Line of Charge

A rod of length L with a uniform positive charge density ρ_L and a total charge Q is lying along the z-axis. Calculate the electric field at point P.

$$\boxed{d\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\frac{dQ}{R^2}\mathbf{a}_R}$$





Example 2: Line of Charge Solution

$$d\mathbf{E} = 2\frac{1}{4\pi\varepsilon_0}\frac{dQ}{R^2}\cos(\alpha)\mathbf{a}_r$$

where $\cos(\alpha) = \dfrac{r}{R}$, $\; R = \sqrt{r^2 + z^2}$

Then

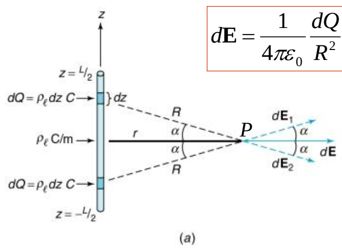
$$\mathbf{E} = 2\int_{z=0}^{L/2}\frac{1}{4\pi\varepsilon_0}\frac{\rho_l r}{(r^2+z^2)^{3/2}}dz\mathbf{a}_r$$

with $\displaystyle\int\frac{1}{(r^2+z^2)^{3/2}}dz = \frac{z}{r^2\sqrt{r^2+z^2}}$

we have

$$\mathbf{E} = \frac{\rho_l}{2\pi\varepsilon_0 r}\frac{L}{\sqrt{4r^2+L^2}}\mathbf{a}_r \quad \text{V/m}$$

$$\boxed{d\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\frac{dQ}{R^2}\mathbf{a}_R}$$



If the rod is infinite long, i.e. $L \to \infty$, we have

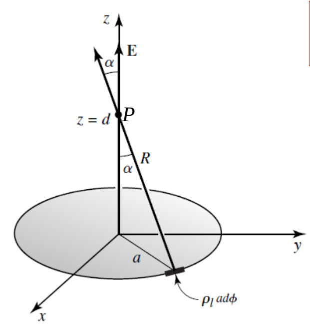
$$\mathbf{E} = \frac{\rho_l}{2\pi\varepsilon_0 r}\mathbf{a}_r \quad \text{V/m}$$

Example 3: Line of Charge

A ring of charge of radius a has a linear charge density of ρ_l C/m that is uniformly distributed around the ring.

Find the electric field at point P.

$$\boxed{d\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\frac{dQ}{R^2}\mathbf{a}_R}$$



Example 3: Line of Charge Solution

Divide the charge into chunks of charge,

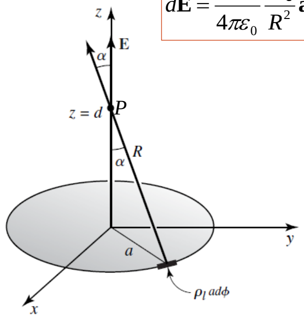
$$dQ = \rho_l \underbrace{a d\phi}_{dl}$$

$$\boxed{d\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\frac{dQ}{R^2}\mathbf{a}_R}$$

At a distance d from the center and on a line perpendicular to the ring, the horizontal components cancel out leaving only the vertical components so

$$\mathbf{E} = \int_{\phi=0}^{2\pi} \frac{1}{4\pi\varepsilon_o}\frac{\rho_l a d\phi}{R^2}\cos(\alpha)\,\mathbf{a}_z$$

where $R = \sqrt{d^2 + a^2}$ and $\cos(\alpha) = \dfrac{d}{R}$

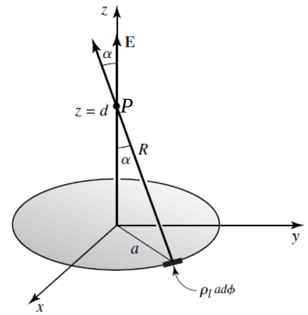


Example 3: Line of Charge Solution

$$\mathbf{E} = \int_{\phi=0}^{2\pi} \frac{1}{4\pi\varepsilon_o}\frac{\rho_l a d}{\left(d^2 + a^2\right)^{3/2}}d\phi\;\mathbf{a}_z$$

$$= \frac{1}{2\varepsilon_o}\frac{\rho_l a d}{\left(d^2 + a^2\right)^{3/2}}\,\mathbf{a}_z \quad z > 0$$

V/m



At a large distance from the center, $d >> a$, this result reduces to

$$\mathbf{E} = \frac{1}{2\varepsilon_o}\frac{\rho_l a}{d^2}\,\mathbf{a}_z \quad d >> a$$

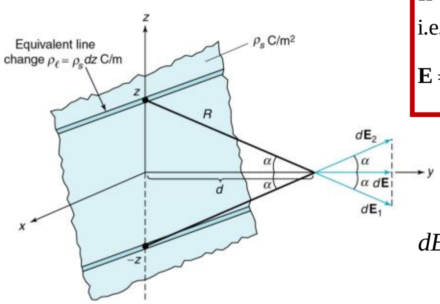
$$= \frac{\overset{\textbf{Q}}{2\pi a \rho_l}}{4\pi\varepsilon_o d^2}\,\mathbf{a}_z \quad d >> a$$

V/m

Point charge

Example 4: Surface of Charge

An infinite sheet of charge with uniform charge distribution ρ_s C/m^2 is located in the *xz* plane. Determine the electric field at some distance *d* away.



If the rod is infinite long, i.e. $L \to \infty$, we have

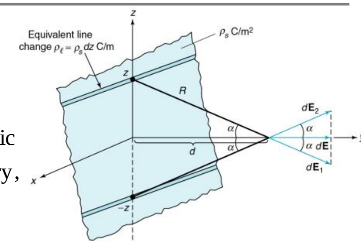
$$\mathbf{E} = \frac{\rho_l}{2\pi\varepsilon_0 r}\mathbf{a}_r \quad \text{V/m}$$

$$dE = \frac{\rho_l}{2\pi\varepsilon_0 R} = \frac{\rho_s dz}{2\pi\varepsilon_0 R}$$

Example 4: Surface of Charge Solution

For an infinite line charge :

$$dE = \frac{\rho_l}{2\pi\varepsilon_0 R} = \frac{\rho_s dz}{2\pi\varepsilon_0 R}$$



The vertical components of the electric field will be canceled due to symmetry, only the horizontal components left.

$$d\mathbf{E} = 2\frac{\rho_l}{2\pi\varepsilon_0 R}\cos(\alpha)\mathbf{a}_y$$

where $R = \sqrt{d^2 + z^2}$, $\cos(\alpha) = \dfrac{d}{R}$

$$\text{Then } \mathbf{E} = \int_{z=0}^{\infty} \frac{\rho_s d}{\pi\varepsilon_0 R^2} dz\mathbf{a}_y = \frac{\rho_s d}{\pi\varepsilon_0}\int_{z=0}^{\infty}\frac{1}{d^2 + z^2}dz\mathbf{a}_y$$

Example 4: Surface of Charge Solution

$$\mathbf{E}=\frac{\rho_s d}{\pi\varepsilon_0}\int_{z=0}^{\infty}\frac{1}{d^2+z^2}dz\,\mathbf{a}_y$$

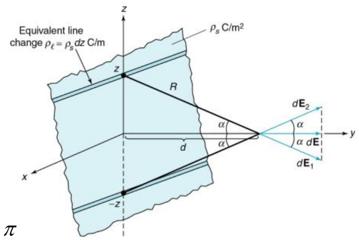
Since $\displaystyle\int_{z=0}^{\infty}\frac{1}{d^2+z^2}dz=\frac{1}{d}\tan^{-1}\left(\frac{z}{d}\right)\Bigg|_{z=0}^{z=\infty}$

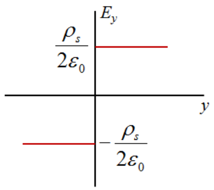
and $\tan^{-1}\left(\frac{0}{d}\right)=0,\quad \tan^{-1}\left(\frac{\infty}{d}\right)=\frac{\pi}{2}$

so: $\displaystyle\int_{z=0}^{\infty}\frac{1}{d^2+z^2}dz=\frac{1}{d}\tan^{-1}\left(\frac{z}{d}\right)\Bigg|_{z=0}^{z=\infty}=\frac{1}{d}\frac{\pi}{2}$

The result:

$$\mathbf{E}=\begin{cases}\dfrac{\rho_s}{2\varepsilon_0}\mathbf{a}_y & y>0\\[2mm] -\dfrac{\rho_s}{2\varepsilon_0}\mathbf{a}_y & y<0\end{cases}$$

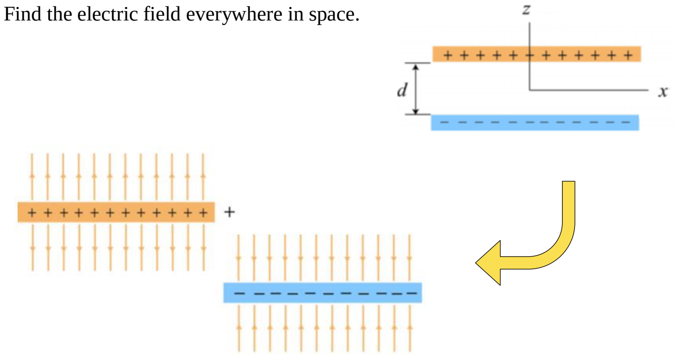




Example 5: Surface of Charge

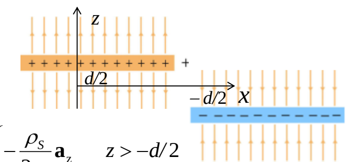
Two parallel infinite non-conducting planes lying in the *xy*-plane are separated by a distance . Each plane is uniformly charged with equal but opposite surface charge densities.

Find the electric field everywhere in space.



Example 5: Surface of Charge Solution

Since the planes carry equal but opposite surface charge densities, both fields have equal magnitude: $E_+=E_-=\dfrac{\rho_S}{2\varepsilon_0}$

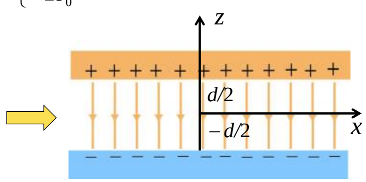


The electric field of the positive and the negative planes are given by:

$$\mathbf{E}_+=\begin{cases}+\dfrac{\rho_S}{2\varepsilon_0}\mathbf{a}_z & z>d/2\\[2mm] -\dfrac{\rho_S}{2\varepsilon_0}\mathbf{a}_z & z<d/2\end{cases}\qquad \mathbf{E}_-=\begin{cases}-\dfrac{\rho_S}{2\varepsilon_0}\mathbf{a}_z & z>-d/2\\[2mm] +\dfrac{\rho_S}{2\varepsilon_0}\mathbf{a}_z & z<-d/2\end{cases}$$

Adding these two fields together:

$$\mathbf{E}=\begin{cases}0 & z>d/2\\[2mm] -\dfrac{\rho_S}{\varepsilon_0}\mathbf{a}_z & -d/2<z<d/2\\[2mm] 0 & z<-d/2\end{cases}$$



Summary

1. Start with $d\mathbf{E}=\dfrac{1}{4\pi\varepsilon_0}\dfrac{dQ}{r^2}\mathbf{a}_r$
2. Rewrite the charge element dQ
3. Substitute dQ into the expression for $d\mathbf{E}$
4. Specify an appropriate coordinate system and express the differential element and r in terms of the coordinates
5. Rewrite $d\mathbf{E}$ in terms of the integration variable(s), and apply symmetry argument to identify non-vanishing component(s) of the electric field
6. Complete the integration to obtain the electric field.

Electric Flux

•Field lines visualize the electric field.

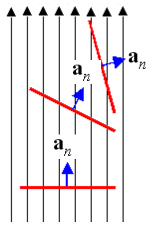
•The strength of an electric field is proportional to the number of field lines per area – called density of the field lines

•The number of electric field lines that penetrates a given surface is called an "**electric flux**".

The number of field lines (electric flux) passing through a geometrical surface of given area depends on three factors:

- the strength of the field
- the surface area
- the orientation of the surface

$\left.\begin{array}{l}\text{the surface area}\\\text{the orientation of the surface}\end{array}\right\}$ Vector area

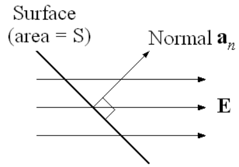


Electric Flux

Vector Area and Surface Normal

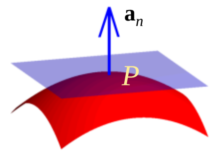
In geometry, for a finite planar surface of scalar area S, the vector area $\mathbf{S}$ is defined as a vector whose magnitude is S and whose direction $\mathbf{a}_n$ is perpendicular to the plane: $\mathbf{S} = \mathbf{a}_n S$

- A surface normal, or simply normal, to a flat surface is a vector that is perpendicular to that surface.



A flat surface

- A normal to a non-flat surface at a point P on the surface is a vector perpendicular to the tangent plane to that surface at P.

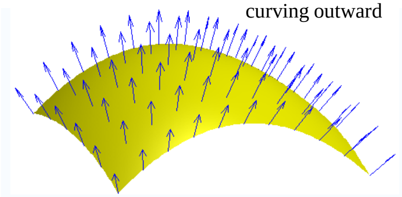


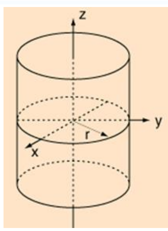
A non-flat surface

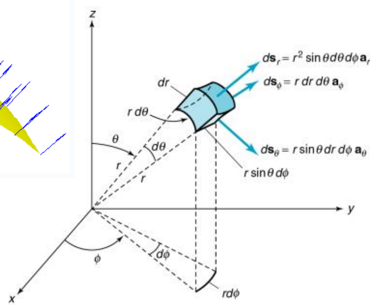
Electric Flux

Vector Area and Surface Normal

curving outward







Electric Flux

Pass through a Planar Surface

$\mathbf{E}$ is a constant vector field directed at angle θ to a planar surface $\mathbf{S}$ of area A, the flux:

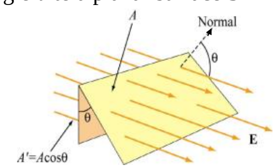
$$\Phi_E = \iint \mathbf{E} \bullet d\mathbf{s}$$

$$\Phi_E = EA' = EA\cos\theta$$

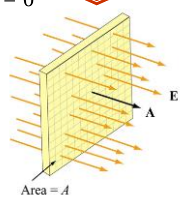
SI unit: **Nm^2/C**

The number of the flux can be larger than zero, smaller than zero, and zero.

When angle $\theta = 0$, the number reaches its maximum value, EA.



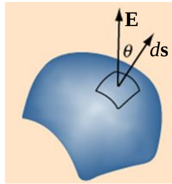
$\theta = 0$



Electric Flux

General Case

In general, a surface can be curved and the electric field may vary over the surface.



The caculation of the flux generally requires an area integral :

The area elements : $\Delta\mathbf{s}_i = \Delta s_i \mathbf{a}_{ni}$

The electric flux through $\Delta\mathbf{s}_i$: $\Delta\Phi_E = \mathbf{E}_i \bullet \Delta\mathbf{s}_i = E_i \Delta s_i \cos\theta$

The total flux through the entire surface is

$$\boxed{\Phi_E = \lim_{\Delta s_i \to 0} \sum \mathbf{E}_i \bullet \Delta\mathbf{s}_i = \iint \mathbf{E} \bullet d\mathbf{s}}$$

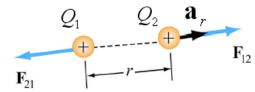
Electric Flux Density Vector

Electric force

$$\mathbf{F}_{12} = k\frac{Q_1 Q_2}{r^2}\mathbf{a}_r$$

Electric field

$$\mathbf{E} = \frac{\mathbf{F}}{q}$$



Electric flux density vector/Flux density

$\mathbf{D} = \varepsilon_0 \mathbf{E}$ in free space

C/m^2

$\mathbf{D} = \varepsilon\, \mathbf{E}$ in dielectric material

Gauss's Law

- **Gauss**'s law, also known as Gauss's flux theorem
- The relationship between the electric charge and the resulting electric field -- A convenient tool for evaluating electric field
- A very useful computational technique and gives people a great deal of insight into the problem
- The first Maxwell's Equation

A set of four equations:
Gauss's law,
Gauss's law for magnetism,
Ampère's law, and
Faraday's law of induction.

Relate the electric and magnetic fields to their sources, charge density and current density.

The set of equations is named after **James Clerk Maxwell**.

Carl Friedrich Gauss

Gauss was a German mathematician and scientist who contributed significantly to many fields, including:

number theory, statistics, analysis, differential geometry, geodesy, geophysics, electrostatics, astronomy and optics.

•Gauss's law

•Gauss's law for magnetism

Carl Friedrich Gauss

Johann Carl Friedrich Gauss (1777– 1855)

Nationality German

Fields Mathematician and physicist

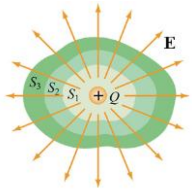
Gauss's Law

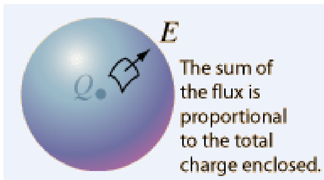
Gauss's law states:

The electric flux passing through any closed surface is proportional to the enclosed electric charge.

The law was formulated in 1835, and published in1867.

The total "flux" of field lines penetrating any of these surfaces is the same and depends only on the amount of charge inside.





Gauss's Law

Stated Mathematically

In free space:

$$\Phi_E = \oint\!\!\!\oint_S \mathbf{E}\bullet d\mathbf{s} = \frac{Q_{enc}}{\varepsilon_0} \Rightarrow \oint\!\!\!\oint_S \varepsilon_0 \mathbf{E}\bullet d\mathbf{s} = Q_{enc} \Rightarrow \oint\!\!\!\oint_S \mathbf{D}\bullet d\mathbf{s} = Q_{enc}$$

In a dielectric material with the permittivity of $\varepsilon = \varepsilon_r \varepsilon_0$:

$$\Phi_E = \oint\!\!\!\oint_S \mathbf{E}\bullet d\mathbf{s} = \frac{Q_{enc}}{\varepsilon_r \varepsilon_0} \Rightarrow \oint\!\!\!\oint_S \varepsilon_r \varepsilon_0 \mathbf{E}\bullet d\mathbf{s} = Q_{enc} \Rightarrow \oint\!\!\!\oint_S \mathbf{D}\bullet d\mathbf{s} = Q_{enc}$$

Then

$$\oint\!\!\!\oint_S \mathbf{D}\bullet d\mathbf{s} = Q_{enc}$$

We take ε as a scalar constant in module EEE108 unless it is indicated.

Under some conditions, the surface integral can be work out by volume integral:

$$\oint\!\!\!\oint_S \mathbf{D}\bullet d\mathbf{s} = \oint_V \rho_v dv = Q_{enc}$$

where V is the volume enclosed by the surface and

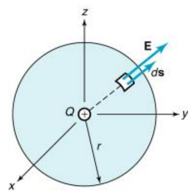
ρ_v is the volume charge density.

Gauss's Law

Point Charge

Consider a positive point charge Q located at the center of a sphere of radius r. Find the electric field on the sphere and electric flux through the sphere.

The dot product in this case is reduced to the simple form:



$$\oint\!\!\!\oint_S \mathbf{D}\bullet d\mathbf{s} = \oint\!\!\!\oint_S D ds = Q \Rightarrow \quad \mathbf{D} = \frac{Q}{4\pi r^2}\mathbf{a}_r$$

by $\mathbf{D} = \varepsilon \mathbf{E}$, we have:

$$\mathbf{E} = \frac{Q}{4\pi\varepsilon r^2}\mathbf{a}_r$$

And by $\Phi_E = \oint\!\!\!\oint_S \mathbf{E}\bullet d\mathbf{s} = \frac{Q_{enc}}{\varepsilon}$:

$$\Phi_E = \frac{Q}{\varepsilon}$$

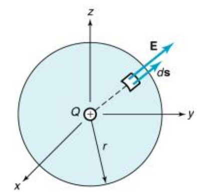
The sphere of radius r called the "**Gaussian surface**".

The total of the electric flux out of a closed surface is equal to the charge enclosed divided by the permittivity.

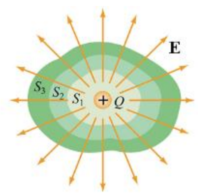
Gauss's Law

Gaussian Surface

From Gauss's Law: $\Phi_E = \oint\!\!\!\oint_S \mathbf{E}\bullet d\mathbf{s} = \dfrac{Q_{enc}}{\varepsilon}$



The flux is independent from the surface. The total "flux" through any of the **enclosed surfaces**, such as $\mathbf{S}_1$, $\mathbf{S}_2$, and $\mathbf{S}_3$, is the same and depends only on the amount of charge inside.



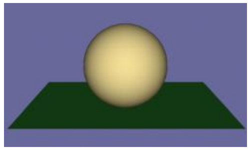
Gaussian Surface is imaginary, there needs not be any material object at the position of the surface.

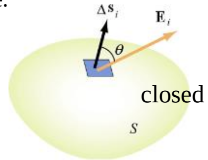
Choose Gaussian Surface wisely.

Gauss's Law

Open and Closed Surfaces

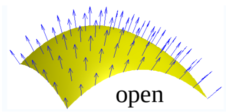
We shall be specially interested in the closed surface.





A rectangle is an open surface — it does NOT contain a volume

A sphere is a closed surface — it DOES contain a volume



For a closed surface the unit vector is chosen to point in the outward normal direction.

The total flux through the entire closed surface is

$$\Phi_E = \lim_{\Delta s_i \to 0} \sum \mathbf{E}_i \bullet \Delta \mathbf{s}_i = \oiint_S \mathbf{E} \bullet d\mathbf{s}$$

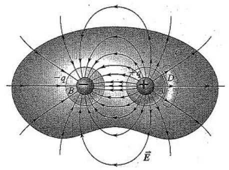
$\Phi_E > 0$ if $\mathbf{E}$ points out

$\Phi_E < 0$ if $\mathbf{E}$ points in

Gauss's Law

Example

The picture shows the field produced by two point charges $+q$ and $-q$ of equal magnitude but opposite sign (an electric dipole) in free space.



Find the electric flux through each of the closed surface A, B, C, and D.

Solution

Surface A: Total net charge enclosed: $Q_{encl} = +q$

so the total flux through surface A: $\Phi_{E,A} = +q/\varepsilon_0$

Surface B: Total net charge enclosed: $Q_{encl} = -q$

so the total flux through surface B: $\Phi_{E,B} = -q/\varepsilon_0$

Surface C: Encloses both charges: $Q_{encl} = +q + (-q) = 0$

so the total flux through surface C: $\Phi_{E,C} = 0$

Surface D: No charges enclosed: $Q_{encl} = 0$

so the total flux through surface D: $\Phi_{E,D} = 0$

Gauss's Law

Applications

Gauss's law is valid for any distribution of charges and for any closed surface.

Gauss's law can be used in two ways:

1. If the charge distribution is known and if it has **enough symmetry** to evaluate the integral in Gauss's law, we can find the field.
2. If the field is known, we can find the charge distribution.

Steps:

1. Identify regions in which to calculate electric field.
2. Choose Gaussian surfaces S: symmetry
3. Calculate Q_{in} -- Charge enclosed by Gaussian surface
4. Apply Gauss's law to get electric field

$$\oiint_S \varepsilon \mathbf{E} \bullet d\mathbf{s} = \oiint_S \mathbf{D} \bullet d\mathbf{s} = Q_{in}$$

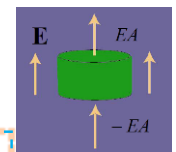
Gauss's Law Applications

Choosing Gaussian Surface

Choose surface where $\mathbf{E}/\mathbf{D}$ is perpendicular and constant:

then the flux can be written as EA or $-EA$, OR

Choose surface where $\mathbf{E}/\mathbf{D}$ is parallel: then the flux is zero



Cylinder Surface with a uniform field:
Flux is EA on top
Flux is $-EA$ on bottom
Flux is zero on sides

1. If you want to find the field at a particular point, then that point should lie on your Gaussian surface.
2. Again: The Gaussian surface does not have to be a real physical surface. It is an imaginary geometric surface, such as: empty surface, embedded in a solid body, or both.
3. If the charge distribution has cylindrical or spherical symmetry, chose the Gaussian surface to be a coaxial cylinder or a concentric sphere, respectively.

Gauss's Law Applications

Symmetry

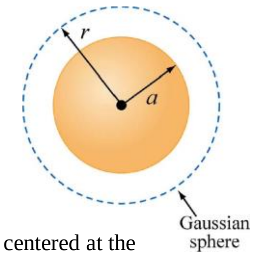
Use Gauss's Law to calculate **E** field (**D** field) from symmetric sources

Symmetry	Gaussian Surface
Spherical	Concentric Sphere
Cylindrical	Coaxial Cylinder
Planar	Gaussian "Pillbox"

Gauss's Law Applications

Example 1: Spherical Symmetry

+Q uniformly distributed throughout **non-conducting solid sphere** of radius a. Find electric field every where.



Solution

In the case, Symmetry is Spherical

Gaussian Surface: Gaussian Spheres

Draw a spherical Gaussian surface of radius r centered at the center of the spherical charge distribution. r is arbitrary but is the radius for which you will calculate the electric field!

The electric field on the Gaussian surface:

$$\mathbf{E} = E\mathbf{a}_r$$

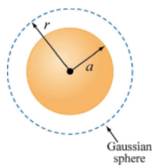
Gauss's Law Applications

Example 1: Spherical Symmetry

Solution Cont.

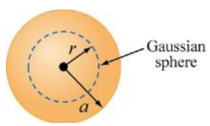
Region 1: $r \geq a$

Total charge enclosed $= +Q$, so, $\Phi_E = \oiint_S \mathbf{E} \bullet d\mathbf{s} = E \oiint_S ds = E(4\pi r^2)$



By using Gauss's Law: $\Phi_E = E(4\pi r^2) = \dfrac{Q}{\varepsilon_0}$

Then: $E = \dfrac{Q}{4\pi\varepsilon_0 r^2} \Rightarrow \boxed{\mathbf{E} = \dfrac{Q}{4\pi\varepsilon_0 r^2}\mathbf{a}_r}$



Region 2: $r < a$

$$\text{Total charge enclosed} = \left(\frac{\frac{4}{3}\pi r^3}{\frac{4}{3}\pi a^3}\right)Q = \left(\frac{r^3}{a^3}\right)Q = \rho_V V$$

By using Gauss's Law: $\Phi_E = E(4\pi r^2) = \dfrac{q_{in}}{\varepsilon_0} = \left(\dfrac{r^3}{a^3}\right)\dfrac{Q}{\varepsilon_0}$

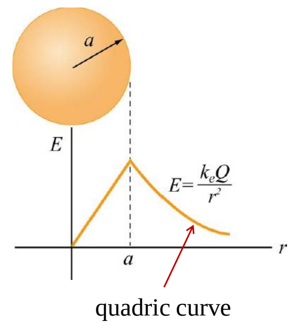
Then: $E = \dfrac{Q}{4\pi\varepsilon_0}\dfrac{r}{a^3} \Rightarrow \boxed{\mathbf{E} = \dfrac{Q}{4\pi\varepsilon_0}\dfrac{r}{a^3}\mathbf{a}_r}$

Gauss's Law Applications

Example 1: Spherical Symmetry

Solution Cont.

$$\begin{cases} \mathbf{E} = \dfrac{Q}{4\pi\varepsilon_0}\dfrac{r}{a^3}\mathbf{a}_r & \text{when } r < a \\ \mathbf{E} = \dfrac{Q}{4\pi\varepsilon_0 r^2}\mathbf{a}_r & \text{when } r \geq a \end{cases}$$



The direction is outward for a positive distribution, and inward for a negative distribution.

Gauss's Law Applications

Example 2: Spherical Symmetry

A thin spherical shell of radius a has a charge $+Q$ evenly distributed over its surface. Find the electric field both inside and outside the shell in free space.

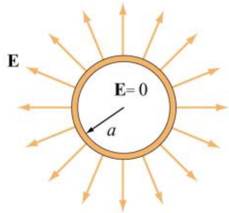
Solution

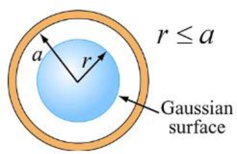
Charge distribution : spherically symmetric

Surface charge density : $\rho_S = Q / A_S = Q / 4\pi a^2$

where $A_S = 4\pi a^2$ is the surface area of the sphere.

Treat the regions $r \le a$ and $r > a$ sepearately.





Case 1 : $r \le a$

Charge enclosed by Gaussian surface :

$q_{enc} = 0$

So, from Gauss's law : $\Phi_E = \dfrac{q_{enc}}{\varepsilon_0}$

Then $\mathbf{E} = 0$, when $r \le a$

Gauss's Law Applications

Example 2: Spherical Symmetry

Solution Cont.

Case 2 : $r > a$

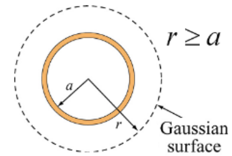
Charge enclosed by Gaussian surface : $q_{enc} = Q$

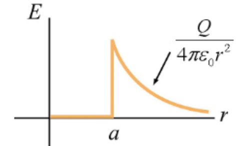
So, the flux through the Gaussian surface :

$$\Phi_E = \oiint_S \mathbf{E} \bullet d\mathbf{s} = EA = E(4\pi r^2)$$

Then from Gauss's law :

$$E = \frac{Q}{4\pi\varepsilon_0 r^2} \Rightarrow \mathbf{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\mathbf{a}_r \quad \text{when } r > a$$





$$\boxed{\mathbf{E} = \begin{cases} 0 & r \le a \\ \dfrac{Q}{4\pi\varepsilon_0 r^2}\mathbf{a}_r & r \ge a \end{cases}}$$

E field is discontinued at $r = a$.

The change from outer to the inner surface

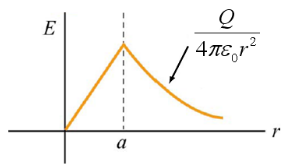
$$\Delta E = \frac{Q}{4\pi\varepsilon_0 a^2} - 0 = \frac{\rho_S}{\varepsilon_0}$$

Gauss's Law Applications

Example 2: Spherical Symmetry

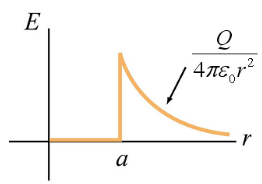
Non-conducting solid sphere

$$\begin{cases} \mathbf{E} = \dfrac{Q}{4\pi\varepsilon_0}\dfrac{r}{a^3}\mathbf{a}_r & r \le a \\ \mathbf{E} = \dfrac{Q}{4\pi\varepsilon_0 r^2}\mathbf{a}_r & r \ge a \end{cases}$$



Thin spherical shell

$$\mathbf{E} = \begin{cases} 0 & r \le a \\ \dfrac{Q}{4\pi\varepsilon_0 r^2}\mathbf{a}_r & r \ge a \end{cases}$$



Gauss's Law Applications

Example 3: Cylindrical Symmetry

An infinitely long rod of negligible radius has a uniform charge density ρ_l.

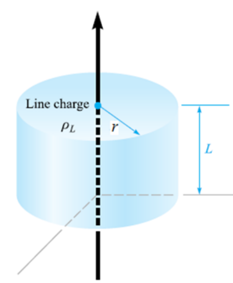
Calculate the **D**(**E**) field outside the wire.

Solution

In the case, Symmetry is Cylindrical

Gaussian Surface: Coaxial Cylinder

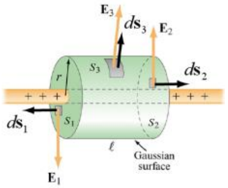
Draw a cylinder of radius r centered at the center of the rod. r is arbitrary but is the radius for which you will calculate the **D**(**E**) field!
l is arbitrary and should divide out!



$$\mathbf{E} = E\mathbf{a}_r \qquad \mathbf{D} = D\mathbf{a}_r$$

Example 3: Cylindrical Symmetry

Solution Cont.



Gaussian surface : $S_1 + S_2 + S_3$

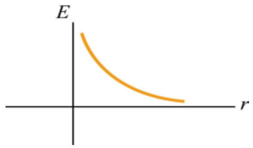
Total charge enclosed $Q_{in} = \rho_L l$

The flux through the Gaussian surface :

$$\Phi_E = \oiint_S \mathbf{E} \bullet d\mathbf{s}$$

$$= \iint_{S_1} \mathbf{E}_1 \bullet d\mathbf{s}_1 + \iint_{S_2} \mathbf{E}_2 \bullet d\mathbf{s}_2 + \iint_{S_3} \mathbf{E}_3 \bullet d\mathbf{s}_3$$

$$= 0 + 0 + E_3 A_3 \ = E(2\pi r l)$$

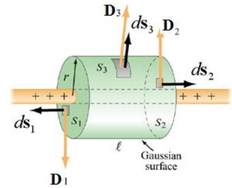


By using Gauss's Law :

$$\Phi_E = E(2\pi r l) = \frac{Q_{in}}{\varepsilon_0} = \frac{\rho_L l}{\varepsilon_0} \qquad \text{Then} : E = \frac{\rho_L}{2\pi\varepsilon_0 r} \Rightarrow \boxed{\mathbf{E} = \frac{\rho_L}{2\pi\varepsilon_0 r}\mathbf{a}_r}$$

Example 3: Cylindrical Symmetry

Solution Cont.



Gaussian surface : $S_1 + S_2 + S_3$

Total charge enclosed $Q_{in} = \rho_L l$

By using Gauss's Law :

$$Q_{in} = \oiint_S \mathbf{D} \bullet d\mathbf{s}$$

$$= \iint_{S_1} \mathbf{D}_1 \bullet d\mathbf{s}_1 + \iint_{S_2} \mathbf{D}_2 \bullet d\mathbf{s}_2 + \iint_{S_3} \mathbf{D}_3 \bullet d\mathbf{s}_3$$

$$= 0 + 0 + D_3 A_3 \ = D(2\pi r l)$$

$$Q_{in} = D(2\pi r l) = \rho_L l \qquad \text{Then} : D = \frac{\rho_L}{2\pi r} \Rightarrow \boxed{\mathbf{D} = \frac{\rho_L}{2\pi r}\mathbf{a}_r}$$

Today

- Electric Fields Produced by Continuous Charge Distributions : Examples
- Electric Flux– Review
- Gauss's Law:
  - The statement
  - The mathematical expression
  - Applications (1)

Next

- Gauss's Law Applications (2)
- Divergence Theorem
- Energy and Electric Potential

Thanks for your attendance