EEE225 Advanced Electrical Circuits and Electromagnetics

Lecture 3 Materials in the static EM fields

Dr. Zhao Wang

zhao.wang@xjtlu.edu.cn

Room EE322

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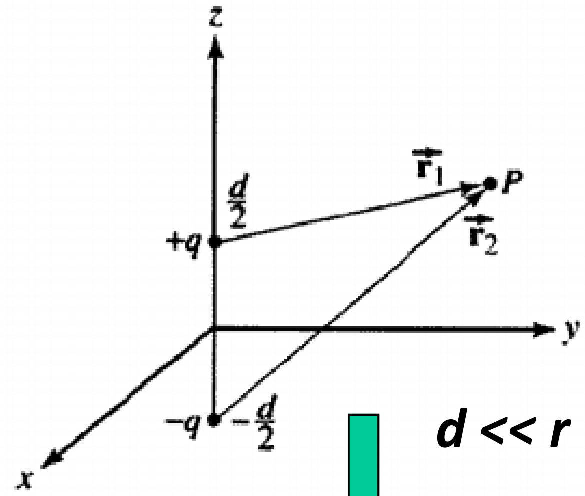
1.1 Electric Dipole

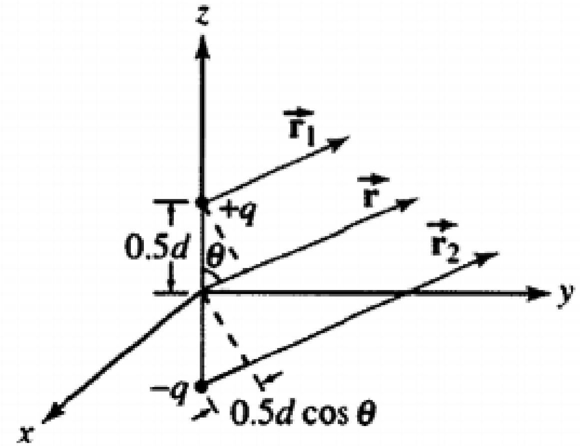
- An electric dipole is a pair of equal charges of opposite signs that are very close together.
  - Magnitude $+Q$ and $-Q$;
  - Distance d;
  - Observe at $P(r, \theta, \varphi)$ in spherical coordinates.
    - When the charges are symmetrically placed along the z axis, and
    - The point of observation is quite far away, so: $d << r$, d is much smaller compare with r
    - => we can approximate r_1 and r_2 as almost parallel

$$r_1 \approx r - 0.5d cos\theta$$

$$r_2 \approx r + 0.5d cos\theta$$



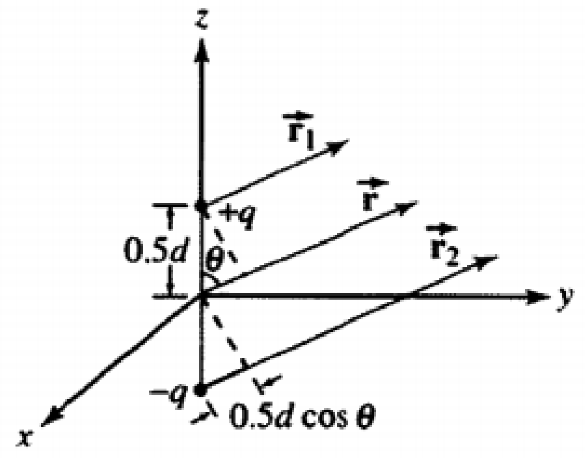
d << r



1.1 Electric Dipole

- The voltage at observation point P is:

$$V = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) = \frac{Q}{4\pi\varepsilon_0}\left(\frac{r_2 - r_1}{r_1 r_2}\right)$$



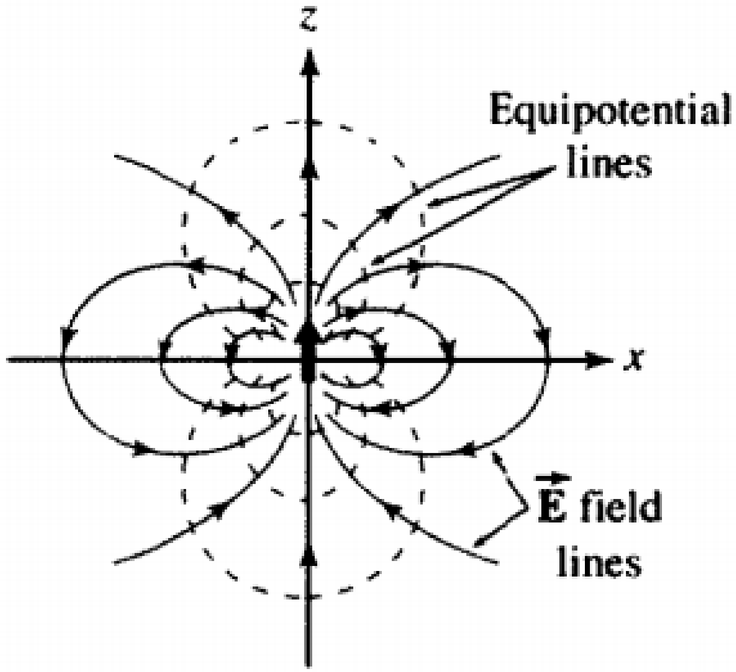
  – when P is far enough, consider $r_1 \parallel r \parallel r_2$
    - $r_1 r_2 \approx r^2$
    - $r_2 - r_1 = d cos\theta$

  – The final result of voltage is then

$$V = \frac{Q}{4\pi\varepsilon_0}\left(\frac{d cos\theta}{r^2}\right) = \frac{Q d cos\theta}{4\pi\varepsilon_0 r^2}$$

  – The equipotential surfaces are obtained by letting V be constant, so

$$\frac{cos\theta}{r^2} = constant$$



1.1 Electric Dipole

- Since $\vec{\boldsymbol{E}} = -\nabla V$, the electric field intensity is:

$$\vec{\boldsymbol{E}} = -\nabla V = -\left(-\frac{Qd\cos\theta}{2\pi\varepsilon_0 r^3}\hat{\boldsymbol{a}}_{\boldsymbol{r}} - \frac{Qd\sin\theta}{4\pi\varepsilon_0 r^3}\hat{\boldsymbol{a}}_{\boldsymbol{\theta}}\right) = \frac{Qd}{4\pi\varepsilon_0 r^3}(2\cos\theta\hat{\boldsymbol{a}}_{\boldsymbol{r}} + \sin\theta\hat{\boldsymbol{a}}_{\boldsymbol{\theta}})$$

- $\vec{d}$ is the vector length directed from $-Q$ to $+Q$. Define the ***dipole moment*** as $Q\vec{d}$ and assign it the symbol $\vec{p}$.

$$\vec{p} = Q\vec{d}$$

- Therefore, V and $\underline{\boldsymbol{E}}$ can be written as:

$$V = \frac{p\cos\theta}{4\pi\varepsilon_0 r^2} = \frac{\vec{\boldsymbol{p}}\cdot\hat{\boldsymbol{a}}_{\boldsymbol{r}}}{4\pi\varepsilon_0 r^2} = \frac{\vec{\boldsymbol{p}}\cdot\vec{\boldsymbol{r}}}{4\pi\varepsilon_0 r^3}$$

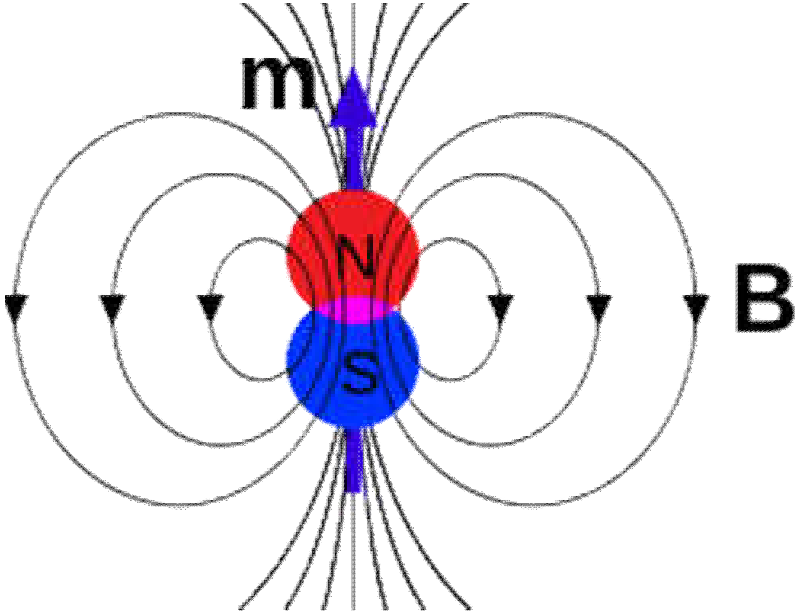
$$\vec{\boldsymbol{E}} = \frac{3(\vec{\boldsymbol{p}}\cdot\hat{\boldsymbol{a}}_r)\hat{\boldsymbol{a}}_r - \vec{\boldsymbol{p}}}{4\pi\varepsilon_0 r^3} = \frac{3(\vec{\boldsymbol{p}}\cdot\vec{\boldsymbol{r}})\vec{\boldsymbol{r}} - \vec{\boldsymbol{p}}}{4\pi\varepsilon_0 r^5}$$

1.1 Electric Dipole

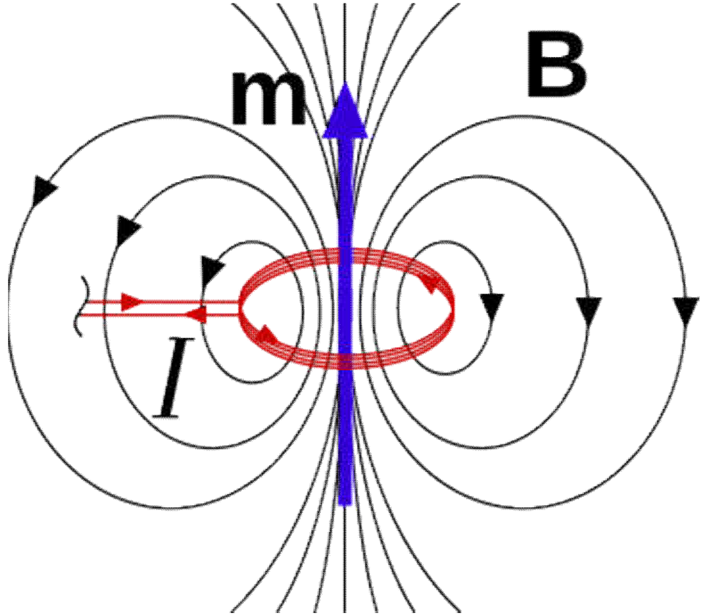
- An electron and a proton separated by a distance of 10^{-11} meter are symmetrically arranged along the z-axis with z=0 as its bisecting plane.
- Determine the potential V and $\underline{\mathbf{E}}$ field at $P(3, 4, 12)$ in Cartesian coordinates.

1.2 Magnetic Dipole

- The magnetic fields are generated by:



Natural magnet



Electric current

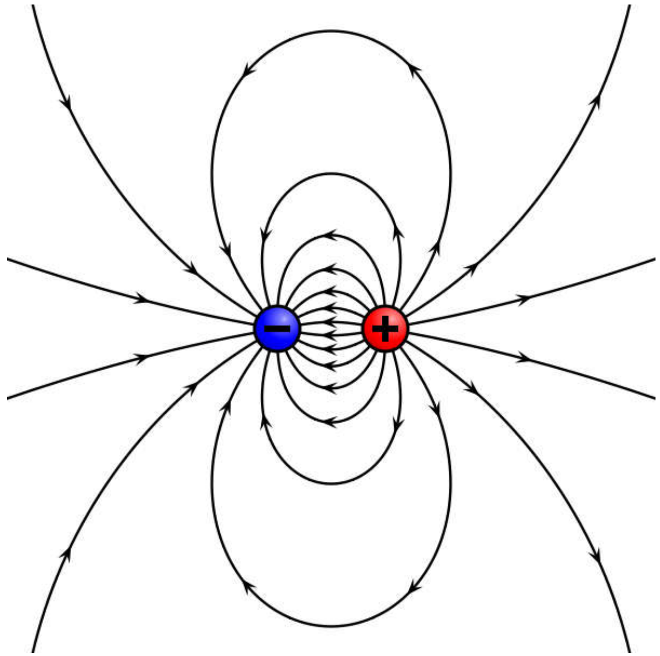
  - **B** is the magnetic field flux density, describing the strength of the magnetic intensity generated by the “dipole”;
  - **m** is the magnetic dipole moment, describing the property of the “dipole itself.”

1.2 Magnetic Dipole

- Pole model (a.k.a. Gilbert model):
  - In analogy to electrostatics;
  - A small magnet is modelled by a pair of magnetic poles of equal magnitude but opposite polarity;
  - Each pole is the source of magnetic force which weakens with distance;
  - The magnetic force produced by a bar magnet, at a given point in space, depends on two factors: the strength p of its poles (magnetic pole strength), and the vector $\vec{l}$ separating them:

$$\vec{m} = p\vec{l}$$

  - <u>Outside</u> the magnet, it points in the direction from ***North*** to ***South*** pole.
  - Notice: this model is only valid for the magnetic fields far from the source.



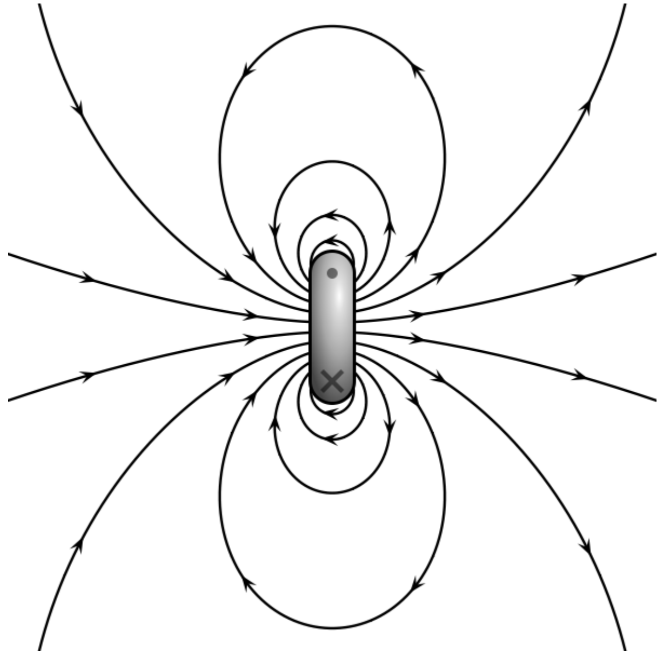
An electrostatic analogue for a magnetic moment: two opposing charges separated by a finite distance.

1.2 Magnetic Dipole

- Amperian loop model:
  - Since all electric currents attract and repel each other similar to magnets, it was natural to hypothesize that all magnetic fields are due to electric current loops.
  - In this model developed by Ampère, the elementary magnetic dipole that makes up all magnets is a sufficiently small amperian loop of current I.
  - The dipole moment of this loop is

$$\vec{m} = I\vec{S}$$

  - where S is the area of the loop. The direction of the magnetic moment is in a direction normal to the area enclosed by the current consistent with the direction of the current using the right hand rule.



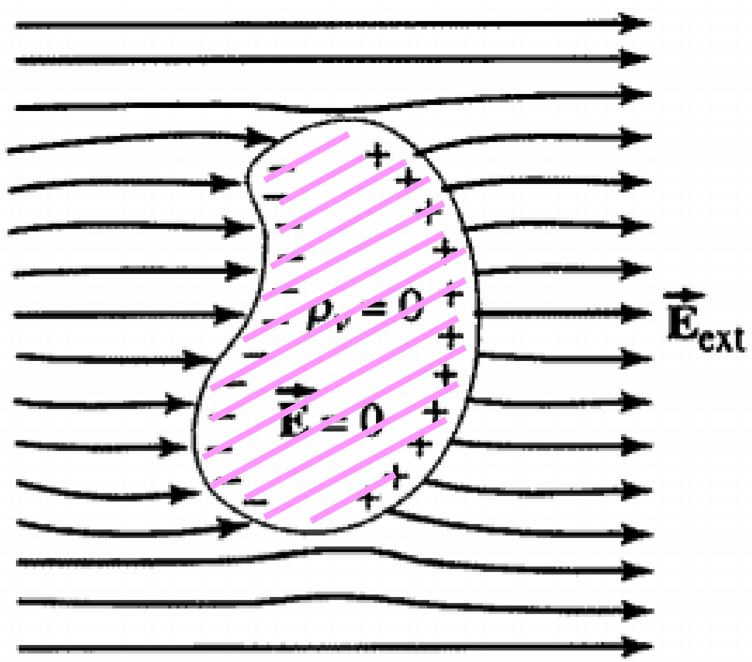
A current loop (ring) that goes into the page at the x and comes out at the dot produces a B-field (lines). The north pole is to the right and the south to the left.

2.1 Materials in E-field

- According to their electrical properties, classify materials into:
  - ***Conductors***: A conductor is a material, such as a metal, that possesses a relatively large number of free electrons. The electrons can drift freely in the conductor. Its ability of conducting electric current is described by the conductivity.
  - ***Dielectrics*** (Insulators): an ideal dielectric is a material with no free electrons in its lattice structure. In an ideal dielectric, positive and negative charges are so sternly bound that they are inseparable. It has zero conductivity.
  - ***Semiconductors***: In some special materials such as silicon and germanium, a small fraction of the total number of valence electrons are free to move about randomly with the space lattice.
    - NOT interested in this module.

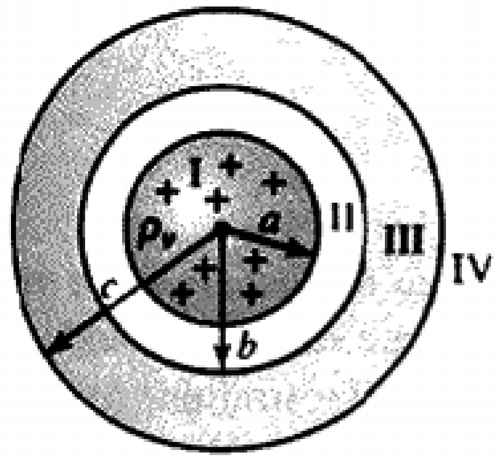
2.1 Materials in E-field

- Conductors: A conductor is a material, such as a metal, that possesses a relatively large number of free electrons.
  - The conductor has as many positive charges as it has electrons.
  - The excess charge cannot reside inside a conductor. It will redistribute itself on the surface of an isolated conductor.
    - Without external E-field
    - With external E-field
  - Under steady-state conditions, the net volume charge density within the conductor is zero, $\rho_v = 0$.
- Inside the conductor:
  - $\vec{\boldsymbol{E}} = 0$
  - V is constant



2.1 Materials in E-field

Example



- Charge is uniformly distributed within a spherical region of radius *a*. An isolated conducting spherical shell with inner radius *b* and outer radius *c* is placed concentrically.
- Determine E everywhere in the region.
- Solution:
  - Region I, $r < a$. Sphere enclosed charge is $Q = \frac{4\pi}{3} r^3 \rho_v$

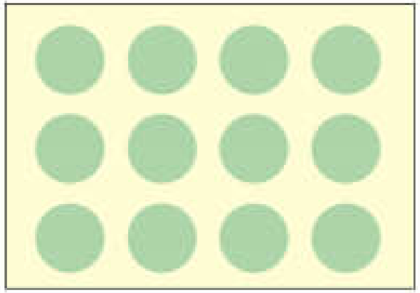
    According to Gauss's law $\oint_s \vec{\mathbf{E}} \cdot \vec{ds} = 4\pi r^2 E_r = Q/\varepsilon_0 \Rightarrow \vec{\mathbf{E}} = \frac{r}{3\epsilon_0} \rho_v \vec{\mathbf{a}}_r$ for $0 < r < a$
  - Region II, $a \leq r < b$. Sphere enclosed charge is $Q = \frac{4\pi}{3} a^3 \rho_v$

    And from Gauss's law: $\vec{\mathbf{E}} = \frac{a^3}{3\epsilon_0 r^2} \rho_v \vec{\mathbf{a}}_r$ for $a \leq r \leq b$
  - Region III, $b \leq r < c$. Inside the conductor shell, $\underline{\mathbf{E}} = 0$.
  - Region IV, $c \leq r$. Net charge equals the one for region II, so

$$\vec{\mathbf{E}} = \frac{a^3}{3\epsilon_0 r^2} \rho_v \vec{\mathbf{a}}_r \quad \text{for} \quad r \geq c$$

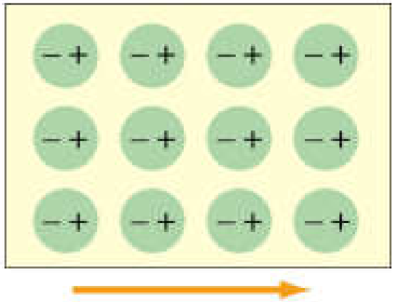
2.1 Materials in E-field

- Under the influence of an electric force, the molecules of a dielectric material experience distortion => being ***polarized***.
  - For non-polar molecules: the center of a positive charge of a molecule no longer coincides with the center of a negative charge



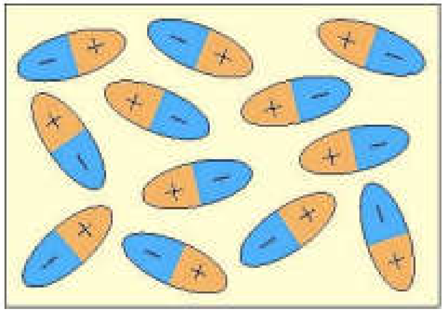
$\vec{E}_0 = 0$

$\vec{E}_0$



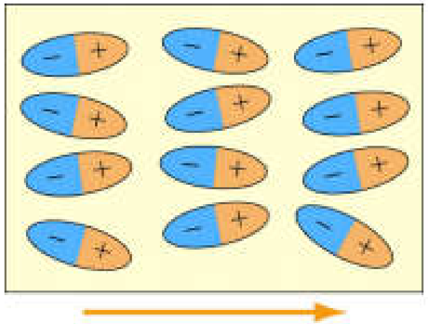
Dipoles

  - For polar molecules, the orientation of polar molecules is random in the absence of an external field. When an external electric field $\underline{\mathbf{E}_0}$ is present, a torque is set up and causes the molecules to align with $\underline{\mathbf{E}_0}$.



$\vec{E}_0 = 0$

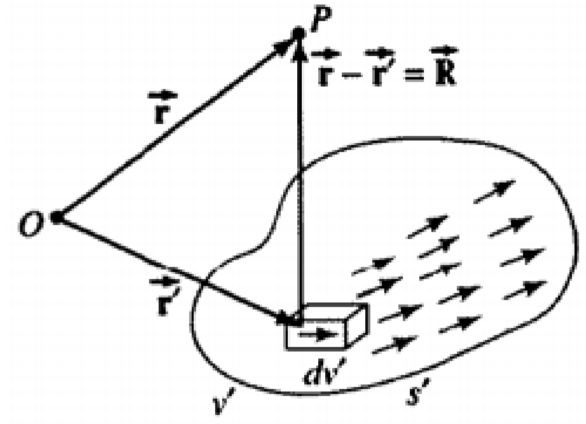
$\vec{E}_0$



Dipoles

2.1 Materials in E-field

Polarization vector P



- Polarization vector – the number of dipole moments per unit volume.

$$\vec{P} = \lim_{\Delta v \to 0} \frac{\Delta \vec{p}}{\Delta v}$$

  - where $\Delta \underline{p}$ is the dipole moment of volume Δv in the limit $\Delta v \to 0$. For another volume v', expressed $d\underline{p}$ as $d\vec{p} = \vec{P}\, dv'$
  - The potential at point P due to $d\underline{p}$ is $dV = \frac{\vec{P} \cdot \vec{a}_R}{4\pi\epsilon_0 R^2} dv'$

$$\nabla'\left(\frac{1}{R}\right) = \frac{1}{R^2}\vec{a}_R$$

$$dV = \frac{\vec{P} \cdot \nabla'(1/R)}{4\pi\epsilon_0} dv'$$

  - Because $\vec{P} \cdot \nabla'(1/R) = \nabla' \cdot (\vec{P}/R) - (\nabla' \cdot \vec{P})/R$
  - We get: $dV = \frac{1}{4\pi\epsilon_0}\left[\nabla' \cdot \left(\frac{\vec{P}}{R}\right) - \frac{\nabla' \cdot \vec{P}}{R}\right] dv'$

2.1 Materials in E-field

- Now integrating over volume *v'* of the polarized dielectric:

$$V = \frac{1}{4\pi\epsilon_0}\left[\boxed{\int_{v'} \nabla'\cdot\left(\frac{\vec{\mathbf{P}}}{R}\right)dv'} - \int_{v'} \frac{\nabla'\cdot\vec{\mathbf{P}}}{R}dv'\right]$$

- Applying the Gauss's theorem, get

$$V = \frac{1}{4\pi\epsilon_0}\oint_{s'} \frac{\vec{\mathbf{P}}\cdot\vec{\mathbf{a}}_n}{R}ds' - \frac{1}{4\pi\epsilon_0}\int_{v'} \frac{\nabla'\cdot\vec{\mathbf{P}}}{R}dv'$$

- Therefore, the potential *V* at point *P* due to a polarized dielectrics is the sum of two terms: a surface term and a volume term. Define:
  - Bound surface charge density

$$\rho_{sb} = \vec{\mathbf{P}}\cdot\vec{\mathbf{a}}_n$$

  - Bound volume charge density

$$\rho_{vb} = -\nabla\cdot\vec{\mathbf{P}}$$

- So the potential can be written as: $V = \dfrac{1}{4\pi\epsilon_0}\left[\oint_{s'} \dfrac{\rho_{sb}}{R}ds' + \int_{v'} \dfrac{\rho_{vb}}{R}dv'\right]$

2.1 Materials in E-field

- Thus, *the **polarization** of a dielectric material results in bound charge distributions*.
  - Different from free charges;
  - Created by separating the charge pairs.
- If a dielectric region contains the free charge density ρ_v in addition to the bound charge density ρ_{vb}
  - According to Gauss's law (differential form)

$$\nabla \cdot \vec{\mathbf{E}} = \frac{\rho_v + \rho_{vb}}{\epsilon_0} = \frac{\rho_v - \nabla \cdot \vec{\mathbf{P}}}{\epsilon_0} \quad \Longrightarrow \quad \nabla \cdot (\epsilon_0 \vec{\mathbf{E}} + \vec{\mathbf{P}}) = \rho_v = \nabla \cdot \vec{\boldsymbol{D}}$$

  - Therefore, for any medium

$$\vec{\mathbf{D}} = \epsilon_0 \vec{\mathbf{E}} + \vec{\mathbf{P}}$$

  - which includes the effect of polarization in a dielectric material.
  - Then $\nabla \cdot \vec{\boldsymbol{D}}$ always represent the free charge density in any medium.

2.1 Materials in E-field

- Express the polarization vector **P** in terms of **E** as:

$$\vec{\mathbf{P}} = \epsilon_0 \chi \vec{\mathbf{E}}$$

  - Where the constant χ is called the electric susceptibility.

- Therefore, the electric flux density **D** in a medium is:

$$\vec{\mathbf{D}} = \epsilon_0(1+\chi)\vec{\mathbf{E}} = \epsilon_0\epsilon_r\vec{\mathbf{E}} = \epsilon\vec{\mathbf{E}}$$

  - where ε_0 is the permittivity of free space (vacuum)
  - ε_r is the relative permittivity, also called dielectric constant, which is unitless
  - ε is the permittivity of the medium.

- Maxwell's equations: $\nabla \times \vec{\mathbf{E}} = 0$

$$\nabla \cdot \vec{\mathbf{D}} = \rho_v$$

$$\vec{\mathbf{D}} = \epsilon\vec{\mathbf{E}}$$

2.1 Materials in E-field

Example

- A point charge q is enclosed in a linear, isotropic and homogeneous dielectric medium of infinite extent.
- Calculate the **E** field, the **D** field, the polarization vector **P**, the bound surface charge density ρ_{sb}, and the bound volume charge density ρ_{vb}.
- Solution:
  - In a linear, isotropic and homogeneous medium, **E** // **D** // **P** in the radial direction ***r***
  - From Gauss's law, where q is the only free charge in the medium,

$$\oint_s \vec{\mathbf{D}} \cdot \vec{ds} = 4\pi r^2 D_r = q \Rightarrow \vec{\mathbf{D}} = \frac{q}{4\pi r^2}\vec{\mathbf{a}}_r \Rightarrow \vec{\mathbf{E}} = \frac{q}{4\pi\epsilon_0\epsilon_r r^2}\vec{\mathbf{a}}_r$$

$$\Rightarrow \vec{\mathbf{P}} = \vec{\mathbf{D}} - \epsilon_0\vec{\mathbf{E}} = \frac{q}{4\pi\epsilon_r r^2}(\epsilon_r - 1)\vec{\mathbf{a}}_r$$

2.1 Materials in E-field

Example (cont.)

- The dielectric medium is bounded by two surfaces: one at $r \to \infty$ and the other around the point charge.
  - The bound surface charge density on the surface at $r \to \infty$ does not contribute to the E field in the region $0 < r < \infty$.
  - However, the bound surface charge density on the surface at $r \to 0$ does contribute to the E field.
    - Based on the previously derived $\vec{\mathbf{P}} = \frac{q}{4\pi\epsilon_r r^2}(\epsilon_r - 1)\vec{\mathbf{a}}_r$, when $r \to 0$, $\underline{\boldsymbol{P}} \to \infty$.

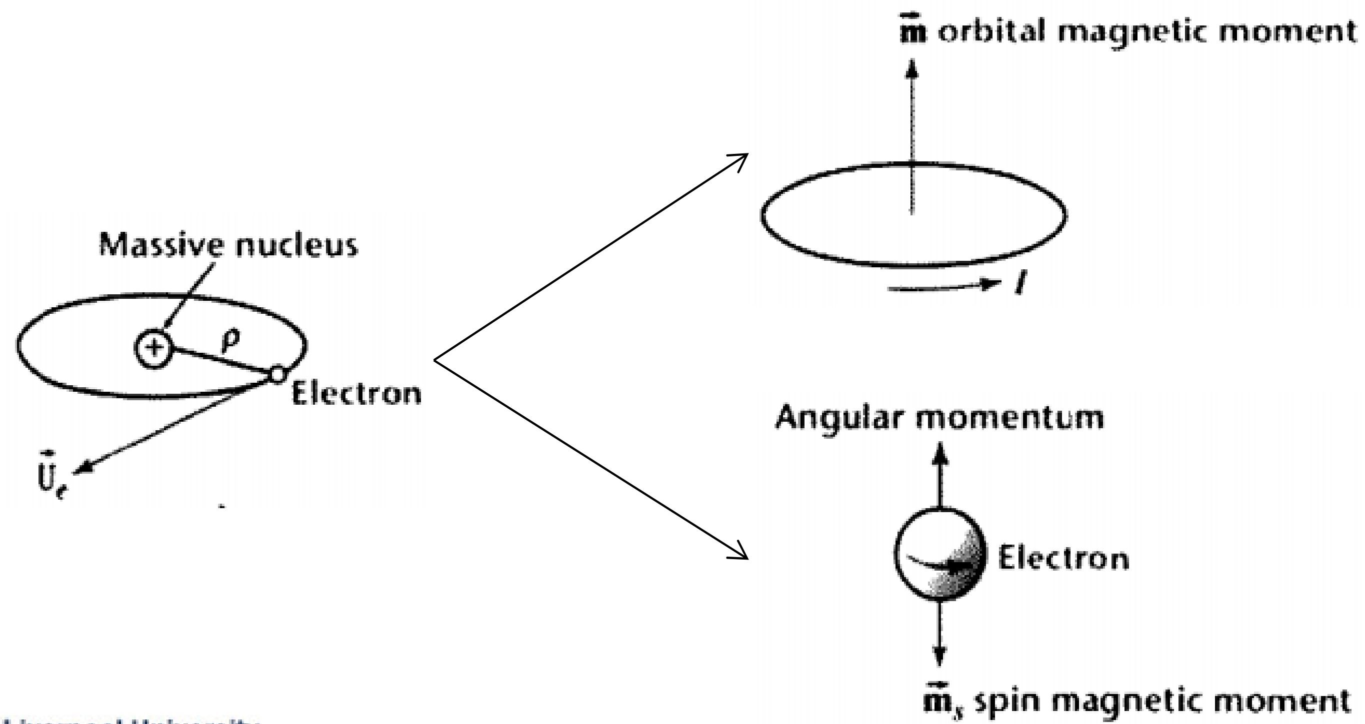
      This singularity exists because of the assumption of a point charge on a macroscopic scale.
    - Therefore, on a molecular scale, we assign a infinitesmall surface with a radius of b that in the limit approaches zero. Thus, the total bound charge on the surface of the dielectric next to the charge q is:

$$Q_{sb} = \lim_{b\to 0}[4\pi b^2 \rho_{sb}] = \lim_{b\to 0}[4\pi b^2 \vec{\mathbf{P}} \cdot (-\vec{\mathbf{a}}_r)]$$
$$= -q(\epsilon_r - 1)/\epsilon_r$$

    - The total charge responsible for $\underline{\mathbf{E}}$ field is $q_t = q + Q_{sb} = q/\epsilon_r$
    - Compare with ε which determines $\underline{\mathbf{D}}$, it reduced by a factor of ε_r.

2.2 Magnetic materials

- Although accurate quantitative results can only be predicted through the use of quantum theory, the simple atomic model yields reasonable qualitative results and provides a satisfactory theory
  - The simple atomic model assumes that there is a central positive nucleus surrounded by electrons in various circular orbits



$$\vec{\mathbf{m}} = \frac{eU_e\rho}{2}\,\vec{\mathbf{a}}_z$$

$$m_s = 9 \times 10^{-24}\ \mathrm{A \cdot m^2}$$

2.2 Magnetic materials

- Each atom contains many different component moments, and their combination determines the magnetic characteristics of the material and provides its general magnetic classification:
  - Diamagnetic material
    - produce a magnetization that opposes the magnetic field.
    - $B_{int} < B_{appl}$.
  - Paramagnetic material
    - produce a magnetization in the same direction as the applied magnetic field.
    - $B_{int} > B_{appl}$.
  - Ferromagnetic material
    - can have a magnetization independent of an applied B-field with a complex relationship between the two fields.
    - $B_{int} >> B_{appl}$.

2.2 Magnetic materials

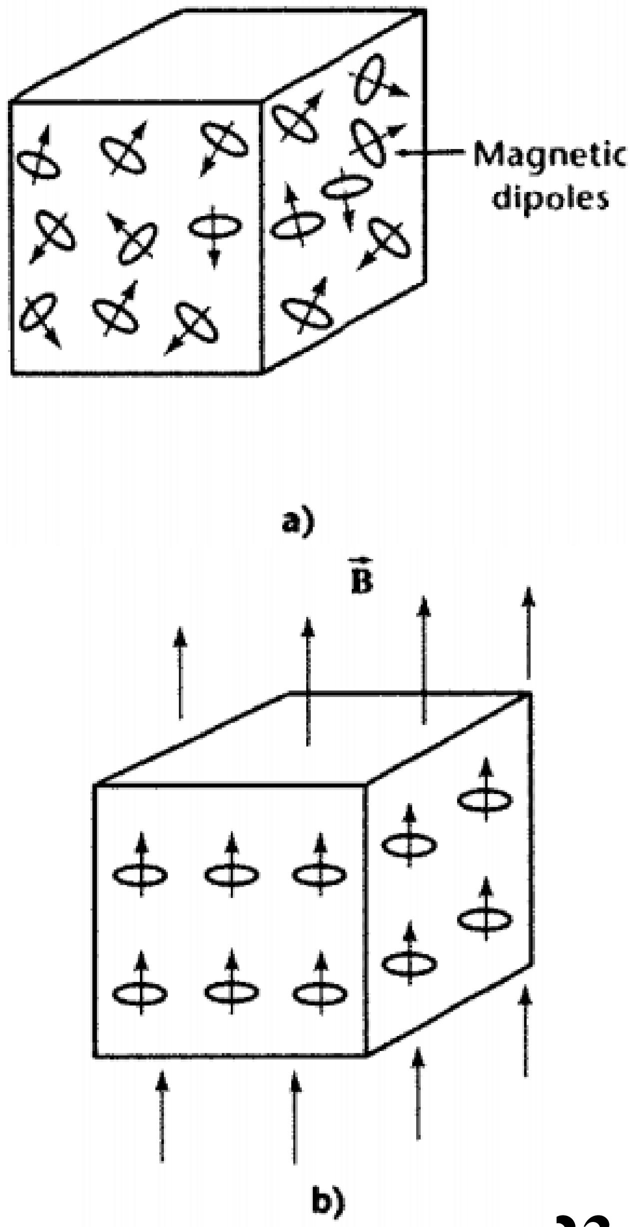
- For an orbiting electron whose moment **m** is in the same direction as the applied field $\mathbf{B_0}$.
  - The magnetic field produces an outward force on the orbiting electron. Since the orbital radius is quantized and cannot change, the inward Coulomb force of attraction is also unchanged. The force unbalance created by the outward magnetic force must therefore be compensated for by a reduced orbital velocity. Hence, the orbital moment decreases, and a smaller internal field results.
  - If we had selected an atom for which **m** and $\mathbf{B_0}$ were opposed, the magnetic force would be inward, the velocity would increase, the orbital moment would increase, and greater cancellation of $\mathbf{B_0}$ would occur. Again a smaller internal field would result.

- Diamagnetic effect is present in all materials, because it arises from an interaction of the external magnetic field with every orbiting electron. However, it is overshadowed by other effects in many materials.

2.2 Magnetic materials

Paramagentic

- When an external field is applied, there is a small torque on each atomic moment, and these moments tend to become aligned with the external field.
  - **m**, the orbital magnetic moment, is always aligned with the applied field, so it will increase **B** inside the material
  - the diamagnetic effect is still operating on the orbiting electrons and may counteract the increase.
    - If the net result is a decrease in B, the material is still called ***diamagnetic***. However, if there is an increase in B, the material is termed ***paramagnetic***.



2.2 Magnetic materials

- Define **Magnetization** as the *magnetic dipole moment per unit volume*

$$\vec{\mathbf{M}} = \lim_{\Delta v \to 0} \frac{\sum_{i=1}^{n} \vec{\mathbf{m}}_i}{\Delta v}$$

  - where $\mathbf{m_i}$ is the magnetic moment of the i^{th} atom in the volume Δv.

- After some derivation, the vector potential **A** can be written as

$$\vec{\mathbf{A}} = \frac{\mu_0}{4\pi}\int_{v'} \frac{\nabla' \times \vec{\mathbf{M}}}{R} dv' + \frac{\mu_0}{4\pi}\oint_{s'} \frac{\vec{\mathbf{M}} \times \vec{\mathbf{a}}_n}{R} ds' \quad \text{or} \quad \vec{\mathbf{A}} = \frac{\mu_0}{4\pi}\int_{v'} \frac{\vec{\mathbf{J}}_{vb}}{R} dv' + \frac{\mu_0}{4\pi}\int_{s'} \frac{\vec{\mathbf{J}}_{sb}}{R} ds'$$

  - where $\vec{\mathbf{J}}_{vb} = \nabla \times \vec{\mathbf{M}}$ is the bounded volume current density;
  - and $\vec{\mathbf{J}}_{sb} = \vec{\mathbf{M}} \times \vec{\mathbf{a}}_n$ is the bounded surface current density.

2.2 Magnetic materials

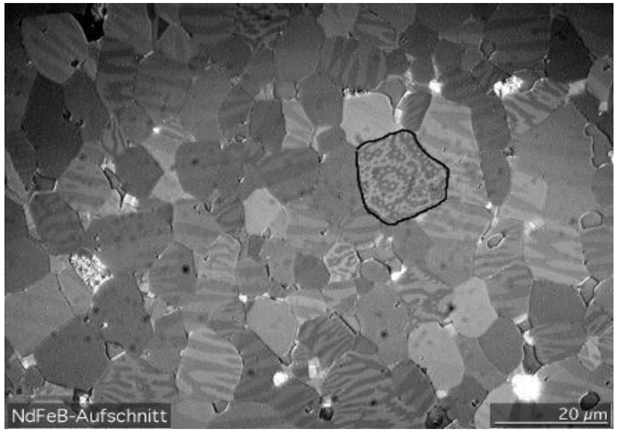
- Besides the bounded current densities, there must also be a free volume current density J_{vf} with $\vec{\mathbf{J}}_v = \vec{\mathbf{J}}_{vf} + \vec{\mathbf{J}}_{vb}$
- In free space, we have

$$\nabla \times \left[\frac{\vec{\mathbf{B}}}{\mu_0}\right] = \vec{\mathbf{J}}_v = \vec{\mathbf{J}}_{vf} + \vec{\mathbf{J}}_{vb} = \nabla \times \vec{\mathbf{H}} + \nabla \times \vec{\mathbf{M}}$$

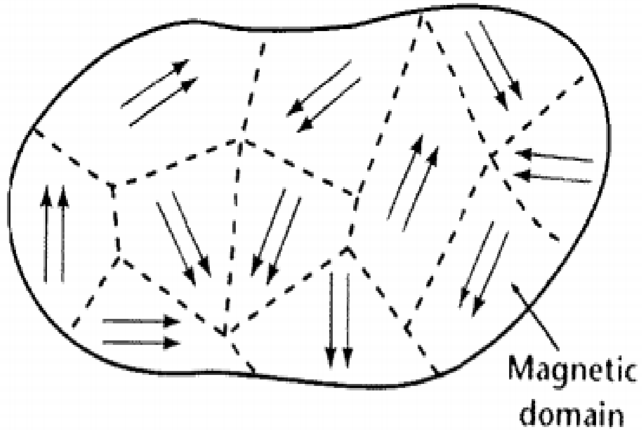
- That means: $\vec{\mathbf{B}} = \mu_0[\vec{\mathbf{H}} + \vec{\mathbf{M}}]$
  - For a linear, homogenous and isotropic medium, we can express **M** in terms of **H** as $\vec{\mathbf{M}} = \chi_m \vec{\mathbf{H}}$, where χ_m is the *susceptibility*.
- In this case: $\vec{\mathbf{B}} = \mu_0[1 + \chi_m]\vec{\mathbf{H}} = \mu_0\mu_r\vec{\mathbf{H}} = \mu\vec{\mathbf{H}}$
  - The quantity $\mu = \mu_0\mu_r$ is the ***permeability*** of the medium. μ_r is the *relative permeability*.
  - For diamagnetic and paramagnetic materials, $\mu_r \approx 1$.

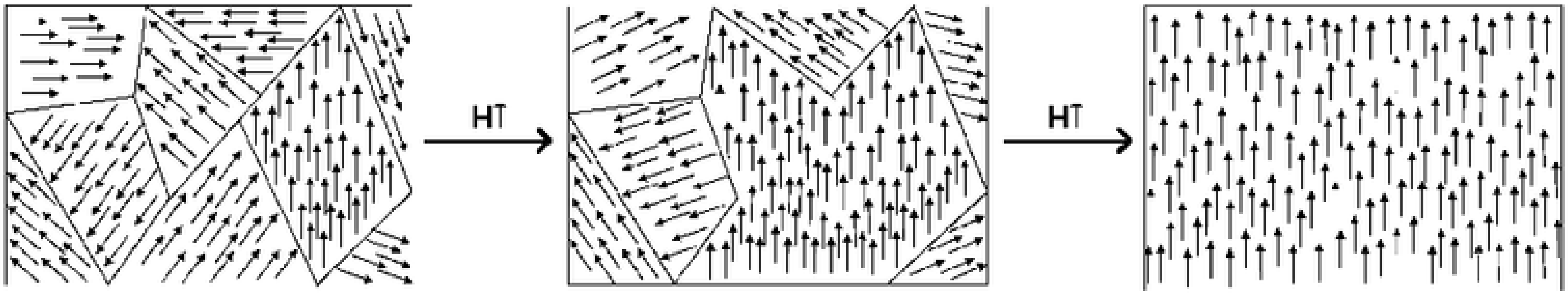
2.2 Magnetic materials

Ferromagnetic



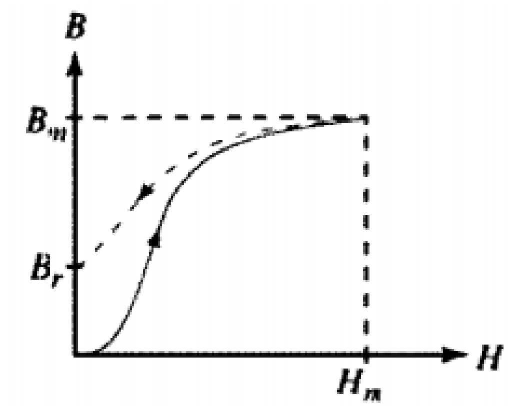
- In ferromagnetic materials, each atom has a relatively large dipole moment, Interatomic forces cause these moments to line up over regions containing a large number of atoms.
  - These regions are called **magnetic domains**.
  - With an external magnetic field applied, those domains which have moments in the direction of the applied field increase their size: $B_{int} >> B_{appl}$
  - Ferromagnetic materials: iron, cobalt, nickel.

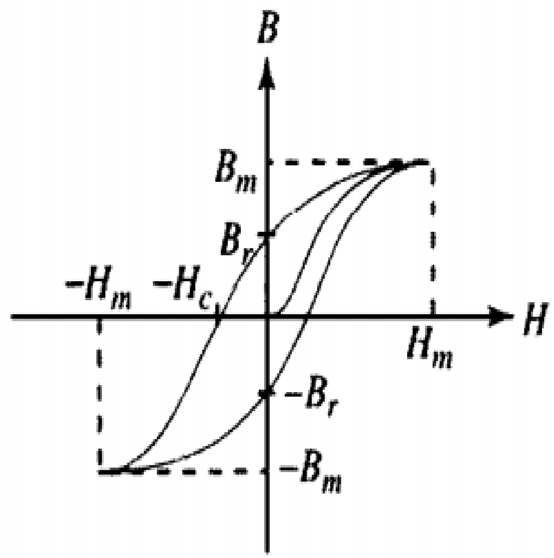




2.2 Magnetic materials

- Upon application of an external magnetic field, however, those domains which have moments in the direction of the applied field increase their size at the expense of their neighbours, and the internal magnetic field increases greatly over that of the external field alone.
- When the external field is removed, a completely random domain alignment is not usually attained, and a residual dipole field remains in the macroscopic structure. The fact that the magnetic moment of the material is different after the field has been removed, or that the magnetic state of the material is a function of its magnetic history, is called ***hysteresis***

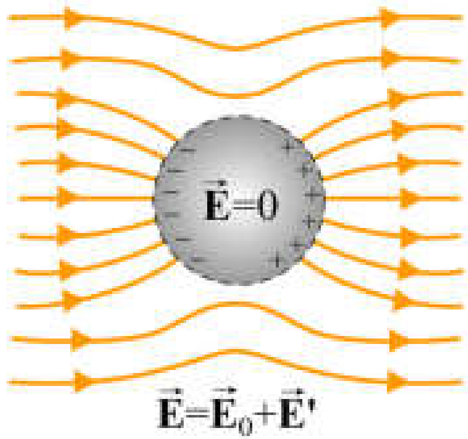




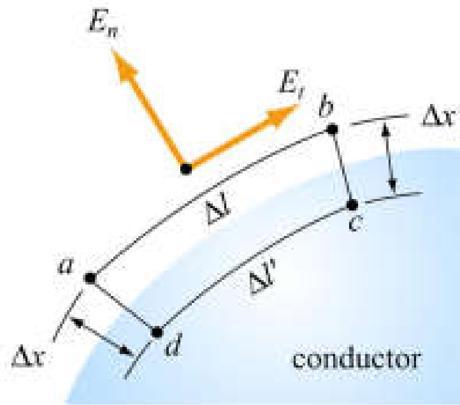
$\mu_r >> 1$

3.1 Boundary Conditions

- The basic properties of a conductor:
  - (1) The electric field inside a conductor is zero.
  - (2) Any net charge must reside on the surface of the conductor.
  - (3) The surface of a conductor is an equipotential surface.

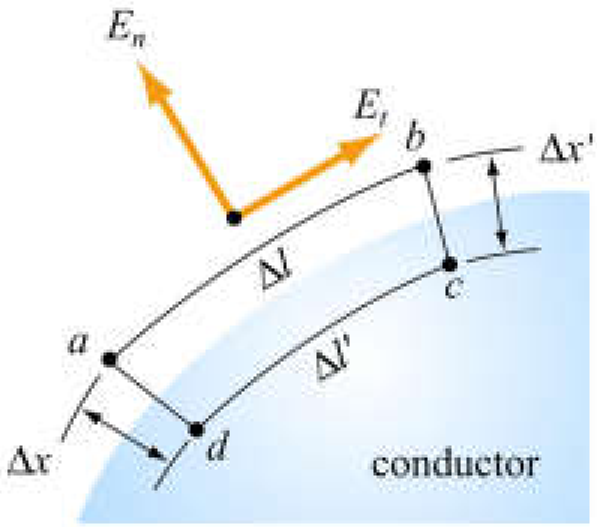


  - (4) The tangential component of the electric field on the surface is zero.
  - (5) Just outside the conductor, the electric field is normal to the surface.



3.1 Boundary Conditions

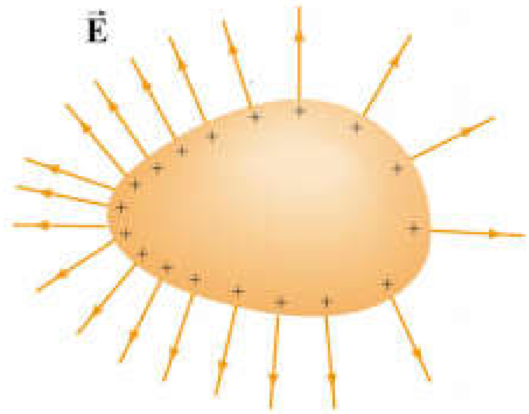
E_t and E_n



- Consider the line integral around a closed path *abcda* in the figure.
  - Since the electric field **E** is conservative, the line integral over the closed path should be 0:

$$E_t(\Delta l) - E_n(\Delta x') + 0(\Delta l') + E_n(\Delta x) = 0$$

  - where E_t and E_n are the tangential and the normal components of the electric field.
  - While approaching to the surface, Δx and $\Delta x'$ $\rightarrow$ 0. So we must have $E_t \Delta l = 0$. Δl does not need to be 0, so E_t must be 0 for anytime.
  - Since the **E** has no tangential components, $\vec{E} = \hat{n} E_n$, so **E** must be normal to the surface.

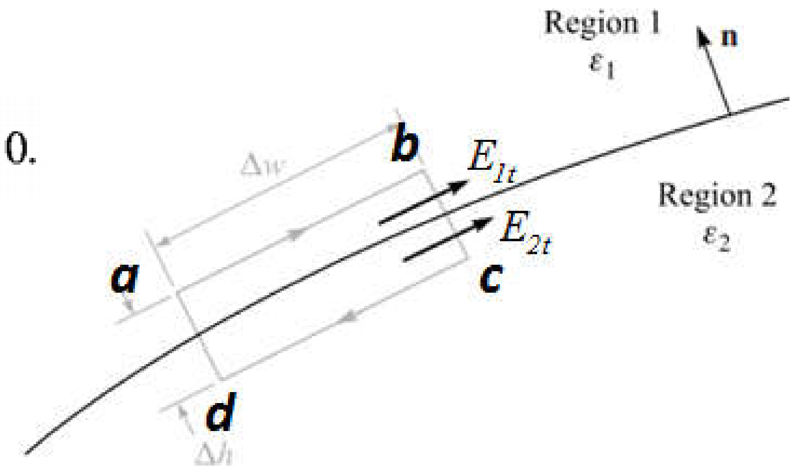


3.1 Boundary conditions

- To solve the ***tangential*** component
- Construct a small path *abcda*
  - Sides *ab* and *cd* are parallel to the interface and equal to Δw.
  - Sides *bc* and *da* are penetrating the interface, $bc = da = \Delta h \rightarrow 0$.
  - Therefore:

$$\oint_{abcda} \mathbf{E} \cdot d\ell = \mathbf{E}_1 \cdot \Delta\mathbf{w} + \mathbf{E}_2 \cdot (-\Delta\mathbf{w}) = E_{1t}\Delta w - E_{2t}\Delta w = 0.$$

$$\boxed{E_{1t} = E_{2t}}$$



- If material 2 is conductor

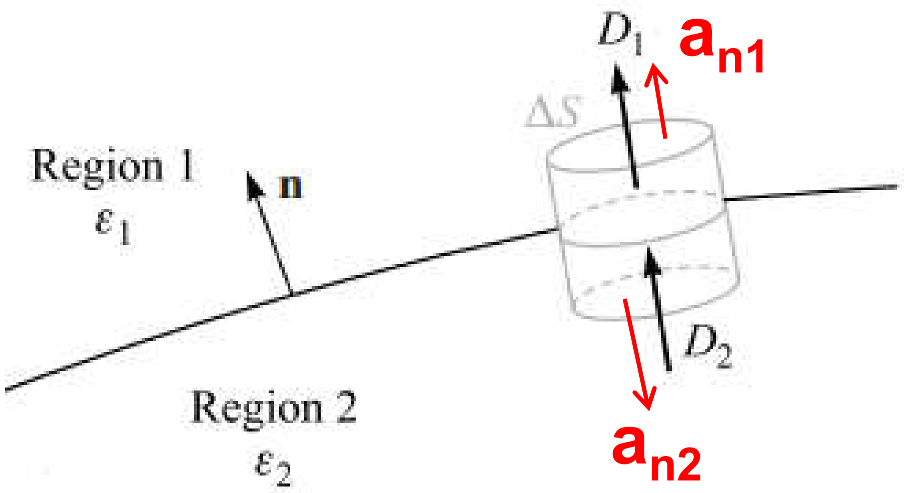
  Inside the conductor, E_{2t}=0, so outside the conductor, E_{1t} also equals to 0.
- If both materials are dielectrics:

$$\frac{D_{1t}}{\varepsilon_1} = \frac{D_{2t}}{\varepsilon_2}$$

3.1 Boundary conditions

In E-field: general materials

- To solve the ***normal*** component
- Construct a small pillbox surface
  - Top and bottom faces are parallel to the interface and equal to ΔS.
  - Height $\Delta h \rightarrow 0$.
  - Apply Gauss's Law, get:



$$\oint_S \mathbf{D} \cdot d\mathbf{s} = (\mathbf{D}_1 \cdot \mathbf{a}_{n1} + \mathbf{D}_2 \cdot \mathbf{a}_{n2})\Delta S$$
$$= \boldsymbol{n} \cdot (\mathbf{D}_1 - \mathbf{D}_2)\Delta S$$
$$= \rho_s \Delta S,$$

  - Therefore $\boxed{D_{1n} - D_{2n} = \rho_s}$
    - If material 2 is conductor

      Inside the conductor, $D_{2n} = 0$, so outside the conductor, $D_{1n} = \rho_s$.
    - If there is no free charges at the interface, i.e. $\rho_s = 0$, we have

$$D_{1n} = D_{2n} \quad \text{or} \quad \varepsilon_1 E_{1n} = \varepsilon_2 E_{2n}$$

3.1 Boundary conditions

In E-field: general materials

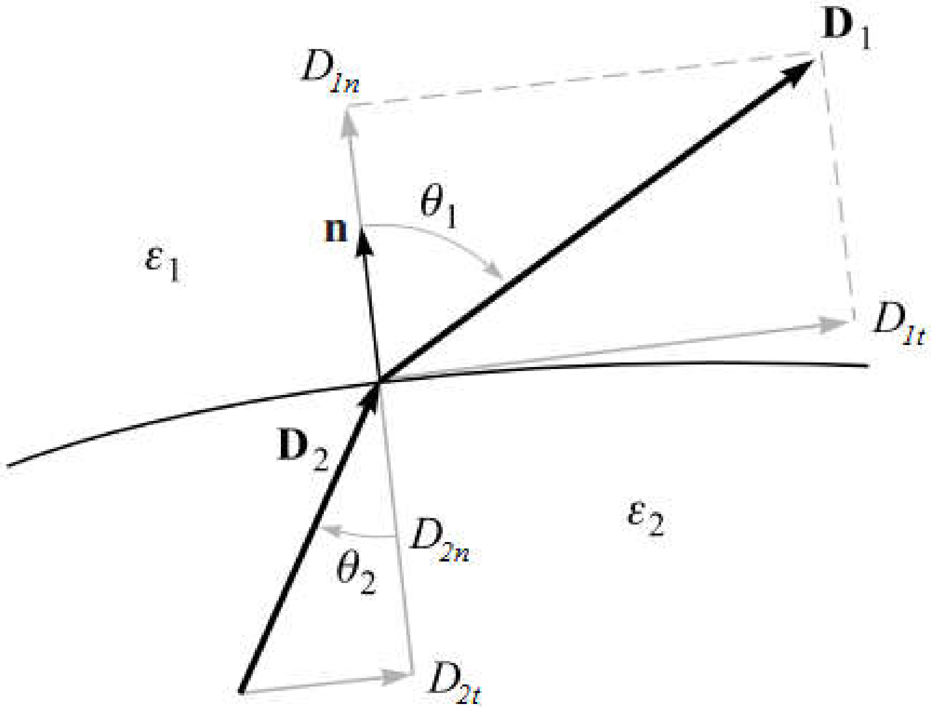
$$E_{1t} = E_{2t} \qquad \Rightarrow \qquad (\vec{E}_1 - \vec{E}_2) \times \hat{n} = 0$$

$$D_{1n} - D_{2n} = \rho_s \qquad \qquad (\vec{D}_1 - \vec{D}_2) \cdot \hat{n} = \rho_s$$

- The boundary conditions state that:
  - The tangential component of an **E** field is continuous across an interface;
  - The normal component of **D** field is discontinuous across an interface where a surface charge exists – the amount of discontinuity being equal to the surface charge density.

3.1 Boundary condition

- The refraction of the **E** (**D**) field at a dielectric interface (source free)



- Solution:
  - Normal components of **D** are continuous:
    $$D_{1n} = D_1 cos\theta_1 = D_{2n} = D_2 cos\theta_2$$
  - Tangential components of **E** are continuous:
    $$E_{1t} = \frac{D_{1t}}{\varepsilon_1} = \frac{D_1 sin\theta_1}{\varepsilon_1} = E_{2t} = \frac{D_{2t}}{\varepsilon_2} = \frac{D_2 sin\theta_2}{\varepsilon_2}$$
  - Combine the two equations together, remove D_1 and D_2, we get:
    $$\frac{tan\theta_1}{tan\theta_2} = \frac{\varepsilon_1}{\varepsilon_2}$$

3.1 Boundary Conditions

Example: normal incident

- Locate a slab of Teflon ($\varepsilon_r = 2.1$) in the region $0 \le x \le a$, and assume free space where $x < 0$ and $x > a$. Outside the Teflon there is a uniform field $\mathbf{E}_{\mathbf{out}} = E_0\mathbf{a}_\mathbf{x}$ V/m
- Find values for **D**, **E**, and **P**.
- Solution:
  - Outside the Teflon slab:

$$\vec{E} = E_0\hat{a}_x$$

$$\vec{D} = \varepsilon E_0\hat{a}_x = \varepsilon_0 E_0\hat{a}_x$$

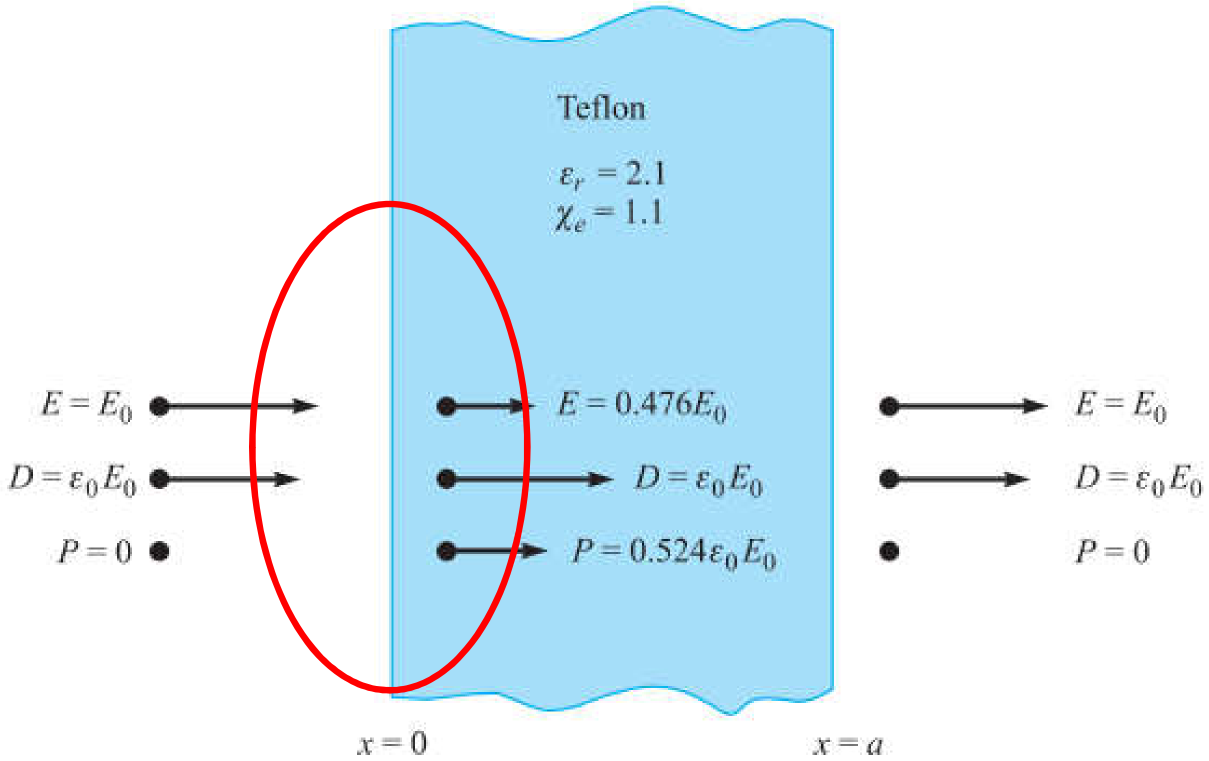
$$\vec{P} = \vec{D} - \varepsilon_0\vec{E} = 0$$

  - Inside the Teflon slab:

$$\vec{D}_2 = \vec{D}_1 = \varepsilon E_0\hat{a}_x = \varepsilon_0 E_0\hat{a}_x$$

$$\vec{E}_2 = \frac{\varepsilon_1}{\varepsilon_2}\vec{E}_1 = \frac{1}{2.1}E_0\hat{a}_x = 0.476E_0\hat{a}_x$$

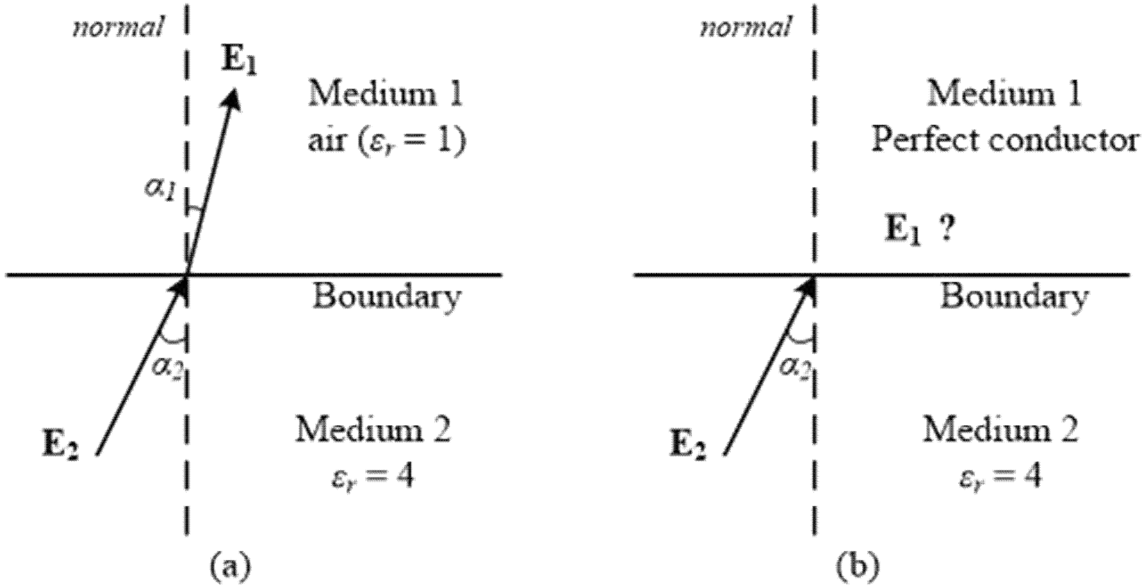
$$\vec{P}_2 = \vec{D}_2 - \varepsilon_0\vec{E}_2 = 0.524\varepsilon_0 E_0\hat{a}_x$$



Normal Incident: no tangential components, so:
D$_1$ = **D**$_2$ and ε_1**E**$_1$ = ε_2**E**$_2$

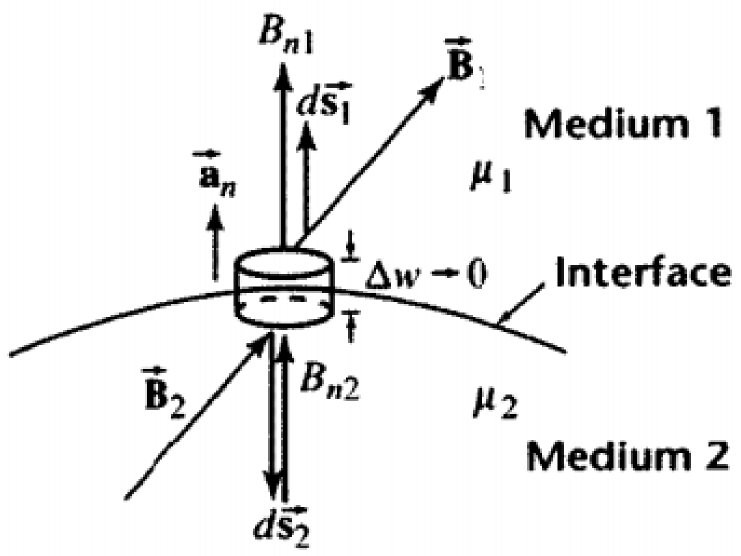
Quiz

- Two isotropic dielectric media are separated by a charge-free plane boundary as shown below. The relative permittivities of the two materials are $\varepsilon_{r1} = 1$ and $\varepsilon_{r2} = 4$.
  - If the angle between the E-field intensity $\mathbf{E_2}$ and normal direction is $\alpha_2 = 30^o$, find the angle between $\mathbf{E_1}$ and the normal direction;
  - If medium 1 is changed to be a perfect conductor as shown in the figure on the right, sketch the electric field lines in medium 1. Explain your answer



3.2 Boundary Conditions

In H-field: normal component



- Construct a pillbox shaped Gaussian surface with vanishing thickness
  - Since the magnetic flux lines are continuous, we have

$$\oint_s \vec{\mathbf{B}} \cdot \vec{ds} = 0$$

  - Neglecting the flux flowing through the vanishing side walls of the pillbox, the equation becomes

$$\int_{s_1} \vec{\mathbf{B}} \cdot \vec{ds} + \int_{s_2} \vec{\mathbf{B}} \cdot \vec{ds} = 0$$

  - Taking the direction of s_1 and s_2 into consideration, we get

$$\int_{s_1} B_{n1}\, ds_1 - \int_{s_2} B_{n2}\, ds_2 = 0 \quad \Rightarrow \quad \int_s (B_{n1} - B_{n2})\, ds = 0$$

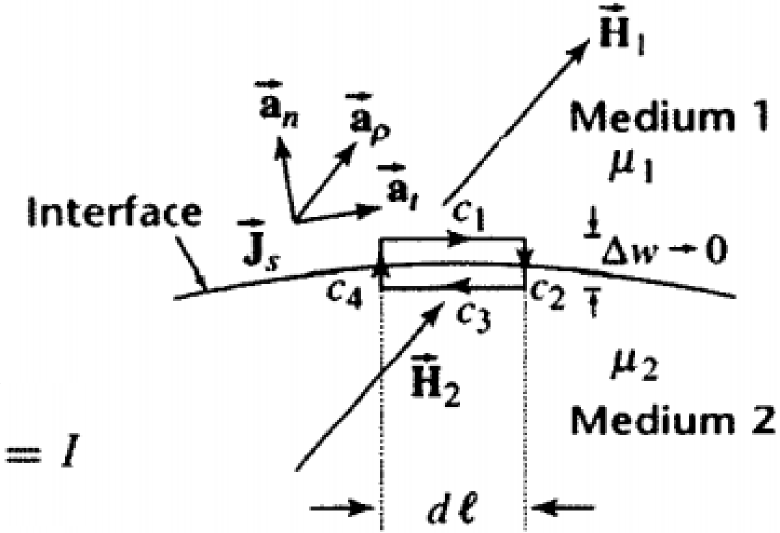
  - Therefore, $B_{n1} = B_{n2}$, or $\vec{\mathbf{a}}_n \cdot (\vec{\mathbf{B}}_1 - \vec{\mathbf{B}}_2) = 0$

Xi'an Jiaotong-Liverpool University
西交利物浦大學

3.2 Boundary Conditions

In H-field: tangential component

- Consider a close path $c_1c_2c_3c_4$, with vanishing sides c_2 and c_4
  - Applying Ampere's law to the close path, we get



$$\oint_c \vec{H}\cdot\vec{d\ell} = \int_{c_1} \vec{H}\cdot\vec{d\ell} + \cancel{\int_{c_2} \vec{H}\cdot\vec{d\ell}} + \int_{c_3} \vec{H}\cdot\vec{d\ell} + \cancel{\int_{c_4} \vec{H}\cdot\vec{d\ell}} = I$$

    - where I is the total current enclosed by the loop
  - Neglecting the terms for the vanishing sides c_2 and c_4, the equation becomes

$$\int_{c_1} \vec{H}\cdot\vec{d\ell} + \int_{c_3} \vec{H}\cdot\vec{d\ell} = \int_{c_1} (\vec{H}_1 - \vec{H}_2)\cdot\vec{a}_t\, d\ell = I = \int_s \vec{J}_v\cdot\vec{a}_\rho\, d\ell\, \Delta w = \int_{c_1} \vec{J}_s\cdot\vec{a}_\rho\, d\ell$$

  - So $H_{t1} - H_{t2} = J_s$, or $\vec{a}_n \times (\vec{H}_1 - \vec{H}_2) = \vec{J}_s$

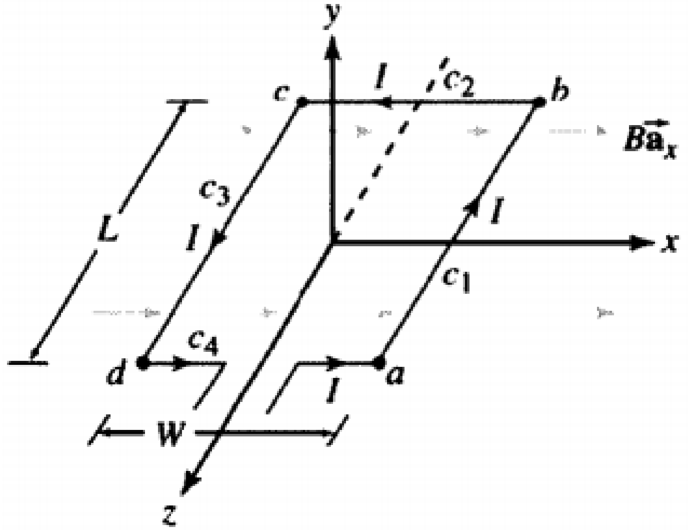
Quiz

- Assume that a plane located at $y = 0$ separates 2 mediums. Medium 1 is in $y > 0$ with relative permeability $\mu_{r1} = 2$ and medium 2 is in $y < 0$ with relative permeability $\mu_{r2} = 1$. The magnetic field intensity vector in medium 1 near the boundary is $\mathbf{H_1} = (4\mathbf{x}\text{-}2\mathbf{y}+8\mathbf{z})$ A/m.
- If no free current density exists on the boundary ($J_s = 0$), find the magnetic field intensity vector $\mathbf{H_2}$ in medium 2 near the boundary.

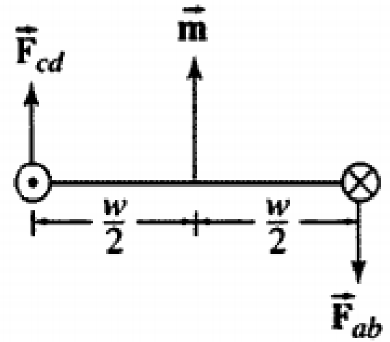
Next

- Steady current
  - Electric properties
  - As the source of magnetic field
- Resistor, capacitor and inductor
  - Structure
  - Calculation

Appendix A Magnetic Torque

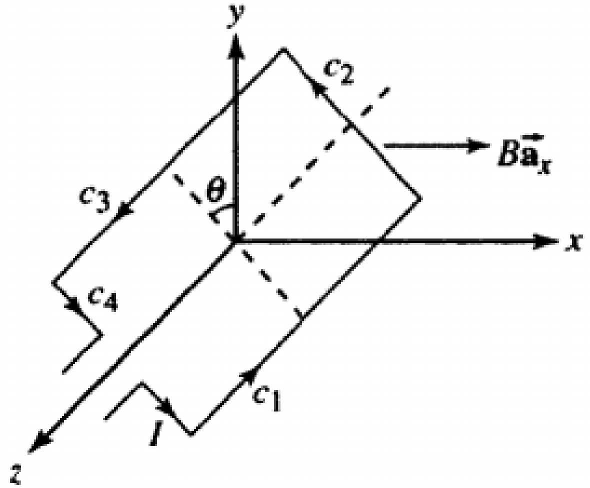


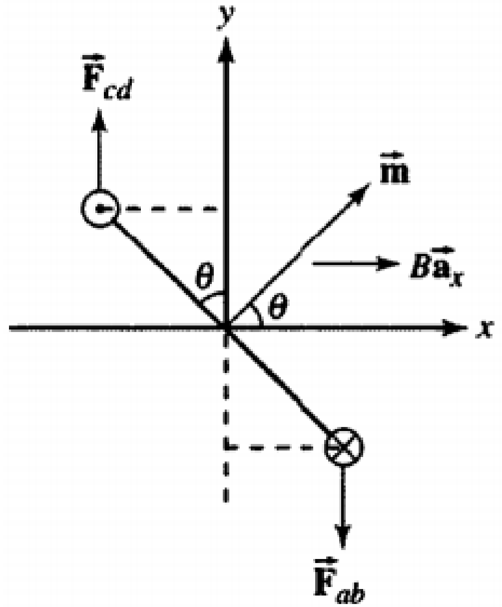
- If a current-carrying coil is placed in a magnetic field, the magnetic force acting on the coil may impart a rotation to the coil.
- For a coil as shown on the right,
  - The force exerted on side *ab* is: $\vec{\mathbf{F}}_{ab} = -BIL\vec{\mathbf{a}}_y$
  - And the force for side *cd* is: $\vec{\mathbf{F}}_{cd} = BIL\vec{\mathbf{a}}_y$
  - Forces for side *bc* and *da* should be zero.
  - The moment arm of side *ab* is W/2 $\mathbf{a}_x$ and the one for side *cd* is –W/2 $\mathbf{a}_x$
  - Therefore, the net torque of the coil is: $\vec{\mathbf{T}} = \vec{\mathbf{T}}_{ab} + \vec{\mathbf{T}}_{cd} = -BILW\vec{\mathbf{a}}_z = \vec{\mathbf{m}} \times \vec{\mathbf{B}}$
    - where **m** is the magnetic dipole moment with $\vec{\mathbf{m}} = ILW\vec{\mathbf{a}}_y = IA\vec{\mathbf{a}}_y$



Appendix A Magnetic Torque

- Then we assume that the coil has rotated under the influence of the torque and makes an angle θ with the y axis.
  - The force acting on side *bc* is
    $$\vec{\mathbf{F}}_{bc} = \int_{c_2} I(dx\vec{\mathbf{a}}_x + dy\vec{\mathbf{a}}_y) \times \vec{\mathbf{a}}_x B = -BIW\cos\theta\,\vec{\mathbf{a}}_z$$
  - Likewise, the force on side *da* is $\vec{\mathbf{F}}_{da} = BIW\cos\theta\,\vec{\mathbf{a}}_z$
  - Since the lines of actions of the forces $\mathbf{F}_{bc}$ and $\mathbf{F}_{da}$ are the same, the resultant force in the *z* direction is zero.
  - The torques acting on sides *ab* and *cd* are:
    $$\vec{\mathbf{T}}_{ab} = \frac{W}{2}[\vec{\mathbf{a}}_x\sin\theta + \vec{\mathbf{a}}_y\cos\theta] \times (-\vec{\mathbf{a}}_y BIL) = -\frac{1}{2}BILW\sin\theta\,\vec{\mathbf{a}}_z$$
    $$\vec{\mathbf{T}}_{cd} = \frac{W}{2}[-\vec{\mathbf{a}}_x\sin\theta + \vec{\mathbf{a}}_y\cos\theta] \times (\vec{\mathbf{a}}_y BIL) = -\frac{1}{2}BILW\sin\theta\,\vec{\mathbf{a}}_z$$
  - The resultant torque is: $\vec{\mathbf{T}} = \vec{\mathbf{T}}_{ab} + \vec{\mathbf{T}}_{cd} = -BILW\sin\theta\,\vec{\mathbf{a}}_z = \vec{\mathbf{m}} \times \vec{\mathbf{B}}$





Appendix B Magnetic Potentials

Recall:

$$\nabla \times \boldsymbol{E} = 0$$

$$\boldsymbol{E} = -\nabla V$$

Since the **E** field is irrotational, so it can be represented as the gradient of a scalar V.

Can we have the similar representation for magnetic field **B** or **H**?

- For magnetism:
  - The normal magnetic field is not irrotational, because:

    $$\nabla \times \boldsymbol{H} = \boldsymbol{J}$$

  - In the other word: the magnetic field is irrotational only when there is no current source;
  - In this case, define a scalar magnetic potential V_m:

    $$\boldsymbol{H} = -\nabla V_m \quad (\boldsymbol{J} = 0)$$

  - Laplace's equation of V_m:

$$0 = \nabla \cdot \boldsymbol{B} = \mu_0 \nabla \cdot \boldsymbol{H} = \mu_0 \nabla \cdot (-\nabla V_m) = -\nabla^2 V_m \quad (\boldsymbol{J} = 0)$$

Appendix B Magnetic Potentials

- For any vector **A**: $\nabla \cdot (\nabla \times \boldsymbol{A}) = 0$
  - The divergence of the curl of any vector field is zero.
  - Since $\nabla \cdot \boldsymbol{B} = 0$ is always true, we can express **B** as $\boldsymbol{B} = \nabla \times \boldsymbol{A}$ in any case, where **A** is the ***vector magnetic potential***.
  - The **H** field is $\boldsymbol{H} = \dfrac{1}{\mu_0} \nabla \times \boldsymbol{A}$
  - Express the magnetic flux Φ by in terms of **A** as:

$$\Phi = \int_s \vec{\mathbf{B}} \cdot \vec{ds} = \int_s (\nabla \times \vec{\mathbf{A}}) \cdot \vec{ds} \Longrightarrow \Phi = \oint_c \vec{\mathbf{A}} \cdot \vec{d\ell}$$

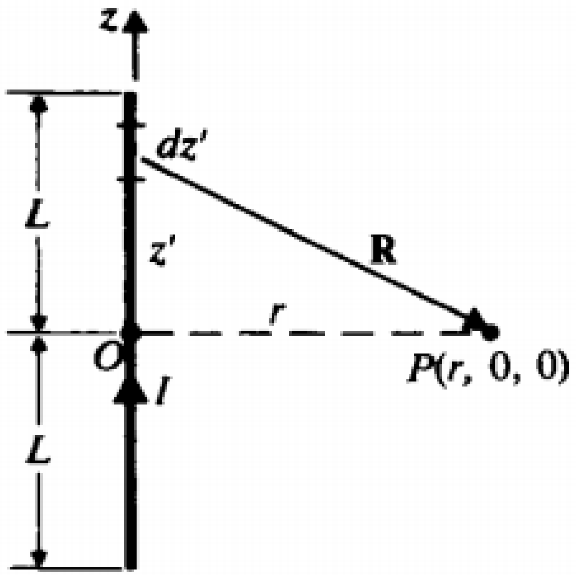
  - Given the Biot-Savart Law, **A** may be determined from the differential current elements by

$$\mathbf{A} = \oint \frac{\mu_0 I \, d\mathbf{L}}{4\pi R}$$

Appendix B Magnetic Potentials

Example

- A direct current I flows in a straight wire of length $2L$. Find the magnetic flux density **B** at a point located at a distance r from the wire in the bisecting plane.



- Solution:
  - The current-carrying line segment is aligned with the z-axis, so the segment is: $d\boldsymbol{\ell}' = \mathbf{a}_z\, dz'$.
  - With $R = \sqrt{z'^2 + r^2}$, get

$$\mathbf{A} = \mathbf{a}_z \frac{\mu_0 I}{4\pi} \int_{-L}^{L} \frac{dz'}{\sqrt{z'^2 + r^2}} = \mathbf{a}_z \frac{\mu_0 I}{4\pi} \left[\ln\left(z' + \sqrt{z'^2 + r^2}\right)\right]\Big|_{-L}^{L} = \mathbf{a}_z \frac{\mu_0 I}{4\pi} \ln \frac{\sqrt{L^2 + r^2} + L}{\sqrt{L^2 + r^2} - L}.$$

  - Therefore

$$\mathbf{B} = \nabla \times \mathbf{A} = \nabla \times (\mathbf{a}_z A_z) = \mathbf{a}_r \frac{1}{r}\frac{\partial A_z}{\partial \phi} - \mathbf{a}_\phi \frac{\partial A_z}{\partial r}$$

  - Cylindrical symmetry around the wire assures that $\partial A_z/\partial \phi = 0$. So

$$\mathbf{B} = -\mathbf{a}_\phi \frac{\partial}{\partial r}\left[\frac{\mu_0 I}{4\pi} \ln \frac{\sqrt{L^2 + r^2} + L}{\sqrt{L^2 + r^2} - L}\right] = \mathbf{a}_\phi \frac{\mu_0 I L}{2\pi r \sqrt{L^2 + r^2}} \xrightarrow{r << L} \mathbf{B}_\phi = \mathbf{a}_\phi \frac{\mu_0 I}{2\pi r}$$