Rayleigh Distributions

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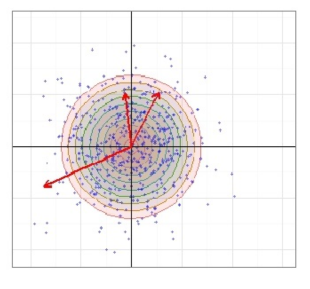
Rayleigh variables

The original derivation is to model the length of a vector defined as the sum of two uncorrelated (that is independent) Normal variables with equal variance and zero mean. If $x, y \sim N(0, \sigma^2)$ then



$$r = \sqrt{x^2 + y^2} \text{ is distributed like a Rayleigh variable.}$$

Example: shooting distance from the center.

Rayleigh distribution

Given a variable $x \geq 0$ and a parameter $\sigma \geq 0$, the density is given by

$$f(x) = \frac{x}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}},\ x \geq 0$$

The cdf is obtained by integrating the pdf:

$$F(x) = \int_0^x \frac{u}{\sigma^2} e^{-\frac{u^2}{2\sigma^2}} du = \left[-e^{-\frac{u^2}{2\sigma^2}} \right]_0^x = 1 - e^{-\frac{x^2}{2\sigma^2}},\ x \geq 0$$

because $\frac{d[e^{-u^2/2\sigma^2}]}{du} = -\frac{u}{\sigma^2} e^{-\frac{u^2}{2\sigma^2}}$

Distribution plots

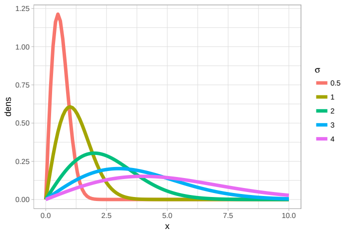


Figure 1: Probability density function

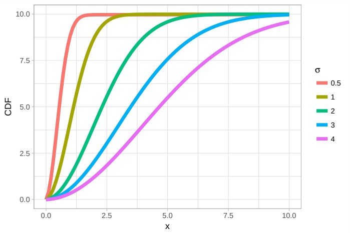


Figure 2: Cumulative density function

Mean

To compute the mean we need to use integration by parts

$[g(x)h(x)]_a^b = \int_a^b g'(x)h(x)dx + \int_a^b g(x)h'(x)dx$.

$$E(X) = \int_0^\infty \frac{x^2}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}} dx = \int_0^\infty \frac{x}{\sigma}\frac{x}{\sigma} e^{-\frac{1}{2}\frac{x^2}{\sigma^2}} dx = \int_0^\infty z\left(ze^{-\frac{1}{2}z^2}\right)\sigma dz$$

where $z = x/\sigma$, so that $dx = \sigma dz$.

Let $g(z) = z$ and $h(z) = e^{-z^2/2}$, with $g'(z) = 1$ and $h'(z) = -ze^{-z^2/2}$, we have

$$\begin{aligned} E(X) &= \sigma \int_0^\infty z(-(-ze^{-\frac{z^2}{2}}))dz = \left[-\sigma \int g\ h'\right] \\ &= \sigma \left\{[-ze^{-\frac{z^2}{2}}]_0^\infty + \int_0^\infty (e^{-\frac{z^2}{2}})dz\right\} = 0 + \sigma\sqrt{\frac{\pi}{2}} \end{aligned}$$

Variance

To compute the variance we first compute $E(X^2)$ and then obtain the variance as $V(x) = E(X^2) - [E(X)]^2$. Again we proceed by changing variable and integrating by parts. Consider that $x^3/\sigma^2 = \sigma z^3$ after the change of variable.

$$\begin{aligned} E(X^2) &= \int_0^\infty \frac{x^3}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}} dx = \sigma^2 \int_0^\infty z^2(-(-ze^{-\frac{z^2}{2}}))dz = \\ \left[-\sigma^2 \int g\ h'\right] &= \sigma^2 \left\{ [-z^2 e^{-\frac{z^2}{2}}]_0^\infty + 2\int_0^\infty ze^{-\frac{z^2}{2}} dz \right\} \\ &= 0 + 2\sigma^2 \lim_{x\to\infty} F(x) = 2\sigma^2 \end{aligned}$$

So, $V(x) = E(X^2) - [E(X)]^2 = 2\sigma^2 - \left(\sigma\sqrt{\frac{\pi}{2}}\right)^2 = \frac{4-\pi}{2}\sigma^2$

Quantiles and Median

The p-quantile is the value q for which $P(X \leq q) = F(q) = p$. For the Rayleigh distribution we have:

$$\begin{aligned} F(q) &= 1 - e^{-\frac{q^2}{2\sigma^2}} = p \Rightarrow \\ ln(1-p) &= -\frac{q^2}{2\sigma^2} \Rightarrow q = \sigma\sqrt{-2ln(1-p)} = \sigma\sqrt{ln(1/(1-p)^2)} \end{aligned}$$

The median is the value m for which $F(m) = p = 0.5$. So,

$$m = \sigma\sqrt{-2ln\left(1 - \frac{1}{2}\right)} = \sigma\sqrt{ln(4)}.$$

Example

A machine spreads seeds at random horizontal distance, x and random vertical distance, y. If x and y are independent and normally distributed with mean zero and variance $\sigma^2 = 2$, find:

1. the pdf and cdf of the distance of a seed from the machine;
2. the probability that a seed will fall between 1 and 2 meters from the machine;
3. the probability that a seed will fall more than 1.2 meters from the machine;

Example: solutions 1

Since both x and y are distributed according to a $N(0,2)$, the distance r is distributed as a rayleigh with parameter 2. Therefore,

The pdf is

$$f(r) = \frac{r}{2}e^{-\frac{r^2}{4}},\ r \geq 0$$

The pdf is obtained by integrating the pdf:

$$F(r) = 1 - e^{-\frac{r^2}{4}},\ x \geq 0.$$

Example: solutions 2

The probability that a seed will fall between 1 and 2 meters from the machine can be found as

$$P(1 < r \leq 2) = \int_1^2 \frac{r}{2} e^{-\frac{r^2}{4}} dr = \left[-e^{-\frac{r^2}{4}} \right]_1^2 = e^{-\frac{1}{4}} - e^{-1} \approx 0.4109$$

However, since we alread know the cdf, it is obviously more convenient to compute

$$P(1 < r \leq 2) = F(2) - F(1) = 1 - e^{-1} - \left[1 - e^{-\frac{1}{4}} \right] = e^{-\frac{1}{4}} - e^{-1}.$$

Example: solutions 3

The probability that a seed will fall more than 1.2 meters from the machine is

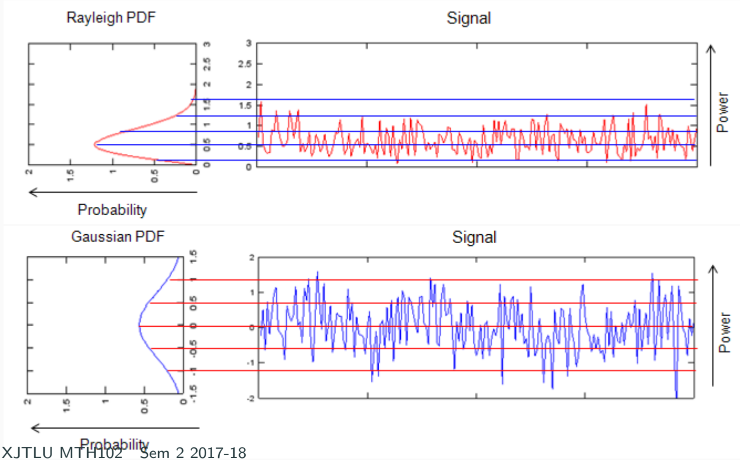
$$P(r > 1.2) = 1 - F(1.2) = e^{-1.2^2/4} = \approx 0.6977.$$

Uses of the Rayleigh distribution

- In communications theory, to model multiple paths of dense scattered signals reaching a receiver.
- In the physical sciences to model wind speed, wave heights and sound/light radiation.
- In engineering, to measure the lifetime of an object, where the lifetime depends on the objects age. For example: resistors, transformers, and capacitors in aircraft radar sets. This is a special case of the more general Weibull distribution.
- In medical imaging science, to model noise variance in magnetic resonance imaging.

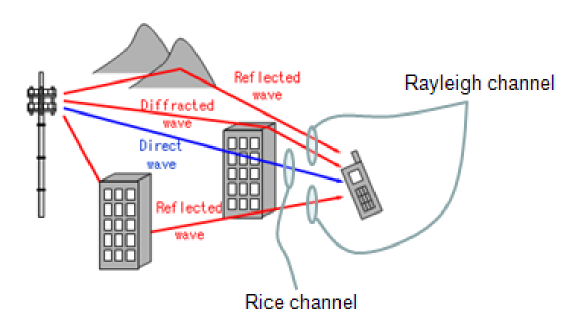
Rayleigh Channel Model 1

Simple definition of Rayleigh Channel is a channel which shows Rayleigh distribution of power profile as shown below.



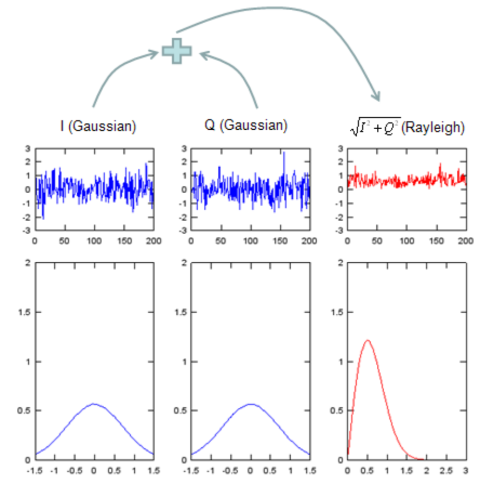
Rayleigh Channel Model 2

Why is this distribution important? In wireless communication, it is used for modelling faded channels. In most cases, the channels for reflected path is modelled with a Rayleigh distribution, as shown below.



Rayleigh Channel Model 2

Signal Generation for Rayleigh channel



Rayleigh Channel Model 4

Mathematical Presentation of Rayleigh Channel