MTH101: Lecture 11

Dr. Tai-Jun Chen, Dr. Xinyao Yang

Xi'an Jiaotong-Liverpool University, Suzhou

October 13, 2017

Laurent Series

Theorem

Let $f(z)$ be an analytic function in the **Annulus**

$$\{z \in \mathbb{C} : R_1 < |z - z_0| < R_2\}.$$

Then the function $f(z)$ can be represented by a Power series with both positive and negative powers of $(z - z_0)$:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} b_n (z - z_0)^{-n}, \quad \text{for } R_1 < |z - z_0| < R_2.$$

The above series is called **Laurent Series**.

The sum of negative powers $\sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$ *is called the* **Principal Part** *of the* **Laurent series**.

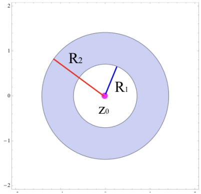
Theorem (Cont.)

Let γ be any Circle with center z_0, counterclockwise orientation and radius $r \in (R_1, R_2)$.
Then the coefficients a_n and b_n of the **Laurent Series** *are given by*

$$a_n = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{(z - z_0)^{n+1}}\, dz,$$

and

$$b_n = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{(z - z_0)^{1-n}}\, dz.$$



Remark

In general it is very difficult to write the Laurent Series of a function $f(z)$ using the definition, that is computing the coefficients a_n, b_n.
The Idea is to manipulate the function $f(z)$ in order to obtain something similar to a function $g(z)$ whose Power Series is known.

Example (Use of Maclaurin Series)

Expand the function

$$\frac{\cos z}{z^4}$$

in a Laurent series that converges for $0 < |z| < R$ and determine the precise region of convergence.

Solution: First, we notice that

$$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}, \qquad \text{for all } z \in \mathbb{C}.$$

Then,

$$\frac{\cos z}{z^4} = z^{-4} \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n-4}$$

for $0 < |z| < \infty$.

Example (Substitution)

Find the Laurent series of $z^2 e^{1/z}$ with center 0.

Solution: We note that

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \quad \text{for all } z \in \mathbb{C}.$$

With replacing z by $1/z$ we obtain the Laurent series representation

$$z^2 e^{1/z} = z^2 \sum_{n=0}^{\infty} \frac{z^{-n}}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!} z^{2-n}$$

for $0 < |z| < \infty$ whose principal part is an infinite series.

Example

We know that

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n, \quad \text{for } |z| < 1.$$

Can the function $f(z)$ be represented by a Power Series of center $z_0 = 0$ outside the Disk $|z| < 1$?

Solution

The function $f(z)$ is Analytic in $\mathbb{C} \setminus \{z^* = 1\}$ which means that it is **Analytic** in the following **Annulus** with center $z_0 = 0$:

$$1 < |z| < +\infty.$$

Then $f(z)$ can be Represented by a **Laurent Series** in that **Annulus**.

We observe that

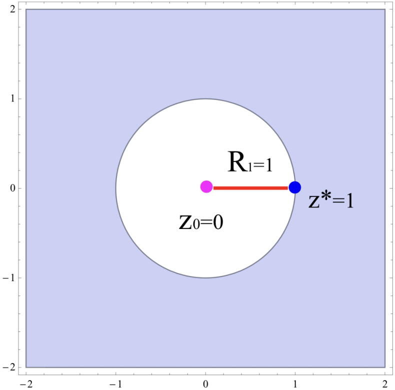
$$|z| > 1 \quad \Longleftrightarrow \quad \left|\frac{1}{z}\right| < 1.$$

Then we manipulate the function $f(z)$ in order to obtain a Geometric Series in powers of $\frac{1}{z}$:

$$\frac{1}{1-z} = \frac{1}{z(\frac{1}{z}-1)} = -\frac{1}{z}\left(\frac{1}{1-\frac{1}{z}}\right) = -\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n, \quad \text{for } \left|\frac{1}{z}\right| < 1.$$

Finally,

$$f(z) = \frac{1}{1-z} = \sum_{n=0}^{\infty}(-1)z^{-n-1}, \quad \text{for } |z| > 1.$$



Example

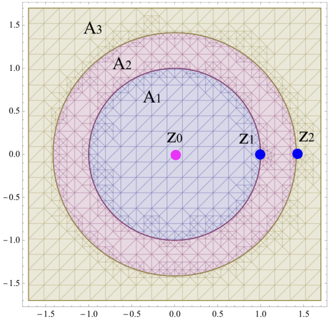
Find all the Power Series with Center $z_0 = 0$ of the function:

$$f(z) = \frac{1}{1-z} + \frac{2}{2-z}.$$

Solution

The function $f(z)$ is Analytic in $D = \mathbb{C} \setminus \{z_1 = 1,\ z_2 = 2\}$.
Then we have three different Power Series in three different Sets:

1. A_1: Taylor Series for $|z| < 1$,
2. A_2: Laurent Series for $1 < |z| < 2$,
3. A_3: Laurent Series for $2 < |z| < \infty$ (or $|z| > 2$).



1. In the set $|z| < 1$ both functions $\frac{1}{1-z}$ and $\frac{2}{2-z}$ are analytic and then they can be represented by a Taylor Series with center $z_0 = 0$:

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n, \quad \text{for } |z| < 1,$$

$$\frac{2}{2-z} = \frac{1}{1-\frac{z}{2}} = \sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n, \quad \text{for } \left|\frac{z}{2}\right| < 1.$$

Then

$$f(z) = \sum_{n=0}^{\infty} z^n + \sum_{n=0}^{\infty} \frac{1}{2^n} z^n = \sum_{n=0}^{\infty} \left(1 + \frac{1}{2^n}\right) z^n, \quad \text{for } |z| < 1.$$

2. The set $1 < |z| < 2$.
The function $\frac{1}{1-z}$ is Analytic in the Annulus $1 < |z| < 2$ and then it can be represented by a Laurent Series on it.
The function $\frac{2}{2-z}$ is Analytic in the set $|z| < 2$ which contains the Annulus $1 < |z| < 2$ and then admits a Taylor Series on it. Then:

$$\frac{1}{1-z} = -\frac{1}{z}\left(\frac{1}{1-\frac{1}{z}}\right) = -\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n, \quad \text{for } \left|\frac{1}{z}\right| < 1,$$

$$\frac{2}{2-z} = \frac{1}{1-\frac{z}{2}} = \sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n, \quad \text{for } |z| < 2,$$

Finally

$$f(z) = \sum_{n=0}^{\infty}(-1)z^{-n-1} + \sum_{n=0}^{\infty}\frac{1}{2^n}z^n, \quad \text{for } 1 < |z| < 2.$$

3. The set $|z| > 2$.
Both functions $\frac{1}{1-z}$ and $\frac{2}{2-z}$ are Analytic in the Annulus $|z| > 2$ and then they can be represented by a Laurent Series on it.

$$\frac{1}{1-z} = -\frac{1}{z}\left(\frac{1}{1-\frac{1}{z}}\right) = -\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n, \quad \text{for } \left|\frac{1}{z}\right| < 1,$$

$$\frac{2}{2-z} = -\frac{2}{z}\left(\frac{1}{1-\frac{2}{z}}\right) = -\frac{2}{z}\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n, \quad \text{for } \left|\frac{2}{z}\right| < 1.$$

Finally

$$f(z) = -\sum_{n=0}^{\infty} z^{-n-1} - \sum_{n=0}^{\infty} 2^{n+1} z^{-n-1} = \sum_{n=0}^{\infty}(-1-2^{n+1})z^{-n-1}, \text{ for } |z| > 2.$$

Bibliography

1 *Kreyszig, E.* **Advanced Engineering Mathematics**. Wiley, 10th Edition.