Mathematical preliminaries

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Content i

- Sigma notation
- Derivatives
    - definition
    - rules

      chain rule; multiplication*; substitution*
- Integrals
    - definition
    - properties
    - (substitution) change of variable*
    - by parts*
- Double integrals
    - definition

Content ii

- properties
- over rectangular regions*
- over general regions*
- (substitution) change of variables*

Sigma (Σ) notation

Sigma (Σ) notation

We use the symbol $\Sigma_{i=1}^{n} x_i$ to indicate the sum of the elements

$x_1 + x_2 + \cdots + x_n$.

If $x = 1, 3, 5$, $\Sigma_{i=1}^{3} x_i = 1 + 3 + 5 = 9$.

Properties

$$(1)\ \Sigma_{i=1}^{n} c = nc; \quad (2)\ \Sigma_{i=1}^{n} c x_i = c \Sigma_{i=1}^{n} x_i;$$

$$(3)\ \Sigma_{i=1}^{n} (x_i + y_i) = \Sigma_{i=1}^{n} x_i + \Sigma_{i=1}^{n} y_i;$$

$$(4)\ \Sigma_{i=1}^{n} x_i = \Sigma_{i=1}^{m} x_i + \Sigma_{i=m}^{n} x_i, m \leq n.$$

Arithmetic and geometric series

- Arithmetic:

$$\sum_{i=1}^{n}(a_0 + (i-1)d) = \frac{n}{2}[2a_0 + (n-1)d]$$

- Geometric:

$$\sum_{i=0}^{n} ar^i = a\left(\frac{1-r^{n+1}}{1-r}\right), \; r \neq 1$$

When $|r| < 1$, $\lim_{n\to\infty} \sum_{1=0}^{n} ar^i = \frac{a}{1-r}$.

Derivatives

Derivatives

Given function $f(x)$ the first order derivative is defined as the limit of the difference quotient

$$\frac{df(x)}{dx} = f'(x) = \lim_{\Delta x \to \infty} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

The derivative has different interpretations. Geometrically, it is the slope of the tangent to the curve $y = f(x)$.

Differentiation rules

- power rule $\dfrac{dx^n}{dx} = nx^{n-1}$
- product-rule $\dfrac{df(x)g(x)}{dx} = f'(x)g(x) + f(x)g'(x)$
- chain-rule if $h(x) = g[f(x)]$ is a compound function of x, we define the variable $u = f(x)$, then

$$\frac{dh(x)}{dx} = \frac{dg(u)}{du}\frac{du(x)}{dx},$$

that is

$$h'(x) = g'(u)f'(x) = g'[f(x)]f'(x).$$

Note we simply write $\frac{dh}{dx} = \frac{dg}{du}\frac{du}{dx}$.

Example chain rule

If $h(x) = (x^2+1)^5$, then set $u = (x^2+1)$, so that

$$h(x) = g(u) = u^5 \text{ with } \frac{dg}{du} = 5u^4 \text{ and } \frac{du}{dx} = 2x.$$

$$h'(x) = g'[f(x)]f'(x) = 5(x^2+1)^4\,2x = 10x(x^2+1)^4$$

This gives us another "rule"

$$\frac{d[f(x)]^n}{dx} = nf'(x)[f(x)]^{n-1},$$

Important (for us) derivatives

We will use exponential functions quite a lot.

$$\frac{dln(x)}{dx} = \frac{1}{x}; \ \frac{de^x}{dx} = e^x;$$

Let $h(x) = ln(f(x))$. Setting $u = f(x)$, by the chain rule we have:

$$\frac{dln(f(x))}{dx} = \frac{dln(u)}{du}\frac{du}{dx} = \frac{1}{u}f'(x) = \frac{f'(x)}{f(x)}.$$

Similarly, for $h(x) = e^{f(x)}$ we have

$$\frac{de^{f(x)}}{dx} = \frac{de^u}{du}\frac{du}{dx} = f'(x)e^{f(x)}$$

Example chain rule with an exponential function

Consider $h(x) = \exp(-(x-\mu)^2/2\sigma^2)$. [$\exp(x)$ means e^x]

Set $u = -(x-\mu)^2/2\sigma^2$ and $g(u) = e^u$, so that [using $d[f^n]/dx = nf'[f]^{n-1}$] $\frac{du}{dx} = -2(x-\mu)/2\sigma^2 = -(x-\mu)/\sigma^2$ and $\frac{dg}{du} = e^u$.

Then,

$$\frac{dh}{dx} = \frac{du}{dx}\frac{dg}{du} = \frac{-(x-\mu)}{\sigma^2} exp(-(x-\mu)^2/2\sigma^2)$$

Exercise chain rule with an exponential function

Let $f(x) = xe^{\frac{-x^2}{2\sigma^2}}$. Find $\frac{df}{dx}$.

Solution

We need to use the product rule and the chain rule. Since we know already the rule for exponential functions $[\frac{de^{f(x)}}{dx} = f'(x)e^{f(x)}]$, we can use that

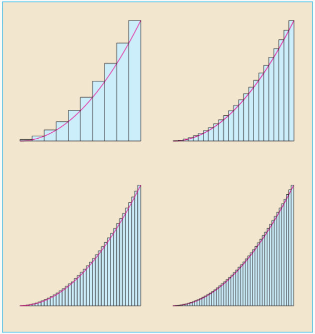
$$\begin{aligned}\frac{df}{dx} &= \frac{dx}{dx}e^{\frac{-x^2}{2\sigma^2}} + x\frac{d[e^{\frac{-x^2}{2\sigma^2}}]}{dx} = e^{\frac{-x^2}{2\sigma^2}} + x\left(-\frac{2x}{2\sigma^2}\right)e^{\frac{-x^2}{2\sigma^2}} \\ &= e^{\frac{-x^2}{2\sigma^2}} - \left(\frac{x^2}{\sigma^2}\right)e^{\frac{-x^2}{2\sigma^2}} = \left(1 - \frac{x^2}{\sigma^2}\right)e^{\frac{-x^2}{2\sigma^2}}\end{aligned}$$

Integration

Integration

To compute an area we approximate a region by rectangles and let the number of rectangles become large. The precise area is the limit of these sums of areas of rectangles.

$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \int_a^b f(x)dx$



Definite integral

Definition

If $f(x)$ is a function defined on an interval a, b, the definite integral of f from a to b is given by

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

provided the limit exists. If this limit exists, the function f(x) is said to be integrable on a, b, or is an integrable function.

Properties of the definite integral

1. If the limits of integration are the same, the integral is just a line and contains no area

$$\int_a^a f(x)dx = 0$$

2. If the limits are reversed, then place a negative sign in front of the integral

$$\int_b^a f(x)dx = -\int_a^b f(x)dx$$

3. The integral of a sum is the sum of the integrals

$$\int_a^b [f(x) + g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

Properties of the definite integral 2

4. for constant c, the integral of the product of a constant and a function is equal to the constant multiplied by the integral of the function
$$\int_a^b cf(x)dx = c\int_a^b f(x)dx$$
5. for constant c, the integral of a constant is equal to the constant multiplied by the interval of integration
$$\int_a^b cdx = c(b-a)$$
6. for three values a, b, c, an integral can be broken down as
$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$
Although this formula normally applies when $a \leq c \leq b$, it holds for all values of a, b, c, provided $f(x)$ is integrable on the largest interval.

Properties of the definite integral 3

7. If $f(x) \geq 0$ for $a \leq x \leq b$ then

$$\int_a^b f(x)dx \geq 0$$

8. If $f(x) \geq g(x)$ for $a \leq x \leq b$ then

$$\int_a^b f(x)dx \geq \int_a^b g(x)dx$$

9. If $m \leq f(x) \leq M$ for $a \leq x \leq b$ then

$$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$$

Fundamental theorem of calculus, Part 1

Definition

If $f(x)$ is continuous over an interval a, b, and the **function** $F(x)$ is defined by

$$F(x) = \int_a^x f(t)dt$$

then $F'(x) = f(x)$ over $[a, b]$.

this establishes the relationship between differentiation and integration. Basically it says $F(x) = \int F'(x)dx$ [note I didn't put the limits] or $\frac{dF}{dx} = f(x)$. F is the *antiderivative* of f.

Fundamental theorem of calculus, Part 2

Definition

If $f(x)$ is continuous over an interval a, b, and $F(x)$ is any antiderivative of $f(x)$, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

this gives us a method for evaluating integrals. We write $F(b) - F(a) = [F(x)]_a^b$.

Integration using the power rule

$$\int_a^b x^n dx = \left[\frac{x^{n+1}}{n+1}\right]_a^b$$

Example

$$\begin{aligned} \int_1^4 \sqrt{x}(1+1)dx &= \int_1^4 \left(x^{1/2} + x^{3/2}\right) dx = \left[\frac{2x^{3/2}}{3} + \frac{2x^{5/2}}{5}\right]_1^4 \\ & \left[\frac{2(4)^{3/2}}{3} + \frac{2(4)^{5/2}}{5}\right] - \left[\frac{2(1)^{3/2}}{3} + \frac{2(1)^{5/2}}{5}\right] = \frac{256}{15} \end{aligned}$$

Integration using the power rule, Exercise

Find $\int_{1}^{2} \left(\frac{x-1}{x^3}\right) dx$

Integration using the power rule, Exercise solutions

Find $\int_1^2 \left(\frac{x-1}{x^3}\right) dx$

Solution

$$\int_1^2 \left(\frac{x-1}{x^3}\right) dx = \int_1^2 \left(x^{-2} - x^{-3}\right) dx = \left[\frac{x^{-1}}{-1} + \frac{x^{-2}}{-2}\right]_1^2 = -\left[x^{-1} + \frac{x^{-2}}{2}\right]_1^2 = \left(1 + \frac{1}{2}\right) - \left(\frac{1}{2} + \frac{1}{8}\right) = \frac{7}{8}$$

Substitution

Finding integrals can be hard. We need to make the integrand look like a known derivative. One useful technique is to substitute (that is change) the variable.

Recall that $\frac{dg(f(x))}{dx} = f'(x)g'(f(x))$. If we can write the integrand as $f'(x)g'(f(x))$ the integral is equal to $g(f(x))$.

This means that, if $u = f(x)$ and we know how to integrate $g'(u)$, we can solve

$$\int f'(x)g'(u)dx = \int g'(u)(f'(x)dx) = \int g'(u)du = g(f(x))(+c)$$

because $du = f'(x)dx$.

Substitution, example

Example: $\int \left(x^2-3\right)^3 2x dx$

Let $u = x^2 - 3$, then $\frac{du}{dx} = 2x$, that is $du = 2xdx$. Let now $g(u) = u^4$, since $g'(u) = \frac{dg}{du} = 4u^3$, we can write

$$\int \left(x^2-3\right)^3 2xdx = \int \frac{4}{4} u^3 (2xdx) = \frac{1}{4} \int g'(u) du = \frac{1}{4} u^4 = \frac{1}{4}(x^2-3)^4$$

Integration of exponential functions

From the rules of differentiation we have:

$$\int e^x dx = e^x$$

However, in general we will have to integrate expressions like $e^{f(x)}$.

In this case we need to use substitution to exploit the derivative $\frac{de^{f(x)}}{dx} = f'(x)e^{f(x)}$

Example: $\int e^{-x} dx = -\int (-e^{-x}) dx = -e^{-x}$ because $\frac{de^{-x}}{dx} = -e^{-x}$

Integration of exponential functions, example

Find $\int 2xe^{-x^2}dx$.

Let $u = x^2$, so that $du = 2xdx$ and set $g(u) = e^{-u}$ so that $g'(u) = \frac{dg(u)}{du} = -e^{-u}$. Then

$$\int 2xe^{-x^2}dx = -\int(-e^{-u})(2xdx) = -\int(-e^{-u})du = -e^{-x^2}$$

Integration by parts

Recall the derivative of a product

$$\frac{d[f(x)g(x)]}{dx} = f'(x)g(x) + f(x)g'(x).$$

Then

$$\int f'(x)g(x)dx = f(x)g(x) - \int f(x)g'(x)dx.$$

For definite integrals

$$\int_a^b f'(x)g(x)dx = [f(x)g(x)]_a^b - \int_a^b f(x)g'(x)dx.$$

We usually write $\int_a^b u\ dv = [uv]_a^b - \int_a^b v\ du$.

Integration by parts, example

Find $\int_0^\infty \lambda x e^{-\lambda x} dx$.

Solution Let $u = x$ and $v = e^{-\lambda x}$ so that $u' = 1$ and $v' = -\lambda e^{-\lambda x}$. Then

$$\begin{aligned}
\int_0^\infty \lambda x e^{-\lambda x} dx &= \int_0^\infty u[-v'] dx = -\int_0^\infty u v' dx \\
&= -\left[[uv]_0^\infty - \int_0^\infty u' v dx \right] = -\left[[x e^{-\lambda x}]_0^\infty - \int_0^\infty e^{-\lambda x} dx \right] \\
&= -[x e^{-\lambda x}]_0^\infty - \left[\frac{1}{\lambda} e^{-\lambda x} \right]_0^\infty = 0 + \frac{1}{\lambda} = \frac{1}{\lambda}
\end{aligned}$$

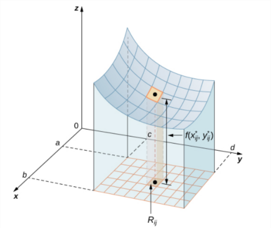
This is the expectation of an exponential variable.

Try on your own to compute the variance

$V(x) = \int_0^\infty (x - 1/\lambda)^2 \lambda e^{-\lambda x} dx$. Need to use integration by parts twice.

Double integrals

Double integrals compute the volume under curve defined by $z = f(x, y)$, over a rectangular region $R = [a, b] \times [c, d]$. To do so, we first subdivide R into mn small rectangles R_{ij}, each having area ΔA, where $i = 1, \ldots, n; i = j, \ldots, m$.



Then, for each pair of pints (x_{ij}, y_{ij}) we approximate the volume as the sum of all the volumes of the columns $f(x_{ij}, y_{ij})\Delta A$.

The double integral is defined as
$\int\int_R f(x, y)\Delta A = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}, y_{ij})\Delta A.$

Iterated Integrals - Fubini's theorem-

We can compute the double integrals first over the x-direction and then over the y-direction (or viceversa).

That is, if $f(x, y)$ is a continuous function on a rectangle $R = [a, b] \times [c, d]$, then

$$\int\int_R f(x, y)\Delta A = \int_c^d \int_a^b f(x, y)\, dx\, dy = \int_a^b \int_c^d f(x, y)\, dy\, dx$$

Example, for $R = [0, 1] \times [3, 4]$

$$\int\int_R f(x, y)\Delta A = \int_3^4 \left[\int_0^1 f(x, y)\, dx\right]\, dy = \int_0^1 \left[\int_3^4 f(x, y)\, dy\right]\, dx$$

Double integrals example

We compute first the integral with respect to one variable, keeping the other constant, and then with resect to the other.

Compute the integral of $f(x,y) = 3xy - x^2$ over the region $R = \{(x,y) | 0 \le x \le 2, 0 \le y \le 1\}$.
Solution

$$\begin{aligned}\int\int_R f(x,y)\Delta A &= \int_0^1 \left[\int_0^2 (3xy - x^2)\,dx\right] dy = \int_0^1 \left[\frac{3}{2}x^2 y - \frac{x^3}{3}\right]_0^2 dy \\ &= \int_0^1 \left(6y - \frac{8}{3}\right) dy = \left[3y^2 - \frac{8}{3}y\right]_0^1 = \frac{1}{3}\end{aligned}$$

Double integrals exercise

Exercise:

Compute the integral of $z = 16 - x^2 - 3y^2$ over

$R = \{(x, y)|0 \leq x \leq 2, 0 \leq y \leq 2\}$

Double integrals exercise solution

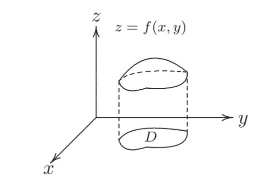
Compute the integral of $z = 16 - x^2 - 3y^2$ over

$R = \{(x, y) | 0 \leq x \leq 2, 0 \leq y \leq 2\}$

$$\begin{aligned} \int\int_R f(x,y)\Delta A &= \int_0^2 \int_0^2 (16 - x^2 - 3y^2)\, dy\, dx = \int_0^2 \left[16y - x^2y - y^3\right]_0^2 dx \\ &= \int_0^2 \left(24 - 2x^2\right) dx = \left[24x - \frac{2}{3}x^3\right]_0^2 = \frac{70}{3} \end{aligned}$$

Double Integral over General Regions

We can compute double integrals over a general closed and bounded region $D \in \mathbb{R}^2$ using some care. We distinguish two cases:



(Type 1) $D = \{(x, y)|a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$ and

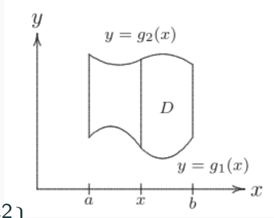
(Type 2) $D = \{(x, y)|h_1(y) \leq x \leq h_2(y), c \leq y \leq d\}$

Double Integral over General Regions, type 1

When y is bounded

by functions of x we set these functions

as the limits for the integration over y.



Example:
evaluate $\int\int_D (x+2y)dA$ where
$D = \{(x,y) | -1 \le x \le 1, 2x^2 \le y \le 1 + x^2\}$.

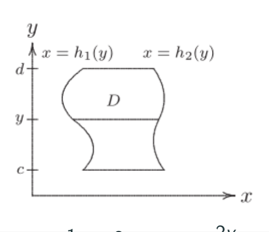
$$\begin{aligned}\int\int_D (x+2y)\Delta A &= \int_{-1}^{1}\int_{2x^2}^{1+x^2}(x+2y)\,dy\,dx = \int_{-1}^{1}\left[xy+y^2\right]_{2x^2}^{1+x^2}\,dx \\ &= \int_{-1}^{1}\left(1+2x^2+x^3-3x^4\right)\,dx = \frac{32}{15}\end{aligned}$$

Double Integral over General Regions, type 2

When x is bounded

by functions of y we set these functions

as the limits for the integration over x.



Example: evaluate $\int\int_D (x+2y)dA$ where $D = \{(x,y)|0 \le y \le 1, y^2 \le x \le 2y\}$.

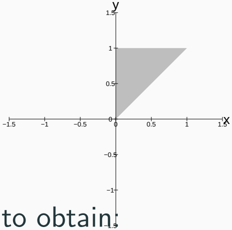
$$\begin{aligned}\int\int_D (x+2y)\Delta A &= \int_0^1 \int_{y^2}^{2y} (x+2y)\,dx\,dy = \int_0^1 \left[\frac{x^2}{2} + 2xy\right]_{y^2}^{2y} dy \\ &= \int_0^1 \left(6y^2 - \frac{y^4}{2} - 2y^3\right) dy = \frac{7}{5}\end{aligned}$$

Double Integral over a triangular region

We may want to integrate over

the region $R = \{(x, y)|0 \le x \le y \le 1\}$

this is the triangle shown.



Example: compute $\int\int_R (-12x^2 + 6y)dA$

We don't have two functions but simply the condition $x \le y \le 1$. We simply use this to obtain:

$$\begin{aligned}\int\int_R (-12x^2 + 6y)dA &= \int_0^1 \int_x^1 (-12x^2 + 6y)\, dy\, dx = \int_0^1 \left[-12x^2 y + 3y^2\right]_x^1 dx \\ &= \int_0^1 \left(-15x^2 + 12x^3 + 3\right) dx = \left[3x^4 - 5x^3 + 3x\right]_0^1 = 1\end{aligned}$$

Do on your own the same inverting the order of integration.

Change of variables in double integrals

In double integrals we can change one variable in the same way we do in simple integrals.

However, changing both variables is a more complex operation, explaining which is beyond the scope of this course.

So, do not attempt to simplify double integrals by changing variables, unless the changes do not involve the other variable.

For example, it is alright to change $x \to u = x/a$ or $y \to v = y^2$ (in this case check the limits) but not $x \to u = (x - y)$.

Change of variables in bivariate Normal

Consider

$$\int\int_{\mathbb{R}^2} \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} e^{\left\{-\frac{1}{2(1-\rho^2)}\left[\frac{(x-\mu_x)^2}{\sigma_x^2} - 2\rho\frac{(x-\mu_x)}{\sigma_x}\frac{(y-\mu_y)}{\sigma_y} + \frac{(y-\mu_y)^2}{\sigma_y^2}\right]\right\}} dx\, dy$$

We can set $u = \frac{(x-\mu_x)}{\sigma_x}$ and $v = \frac{(y-\mu_y)}{\sigma_y}$ with $du = dx/\sigma_x$ and $dv = dy/\sigma_y$, to obtain the equivalent

$$\int\int_{\mathbb{R}^2} \frac{1}{2\pi\sqrt{1-\rho^2}} e^{\left\{-\frac{1}{2(1-\rho^2)}\left(u^2 - 2\rho uv + v^2\right)\right\}} du\, dv$$