Week 10

Laplace Transform

Laplace Transform

- Introduction to the Laplace Transform
- Laplace Transform Definition
- Region of Convergence
- Inverse Laplace Transform
- Properties of the Laplace Transform

Introduction

The **Laplace Transform** is a tool used to **convert** an operation of a real **time domain** variable (t) into an operation of a **complex domain** variable (s)

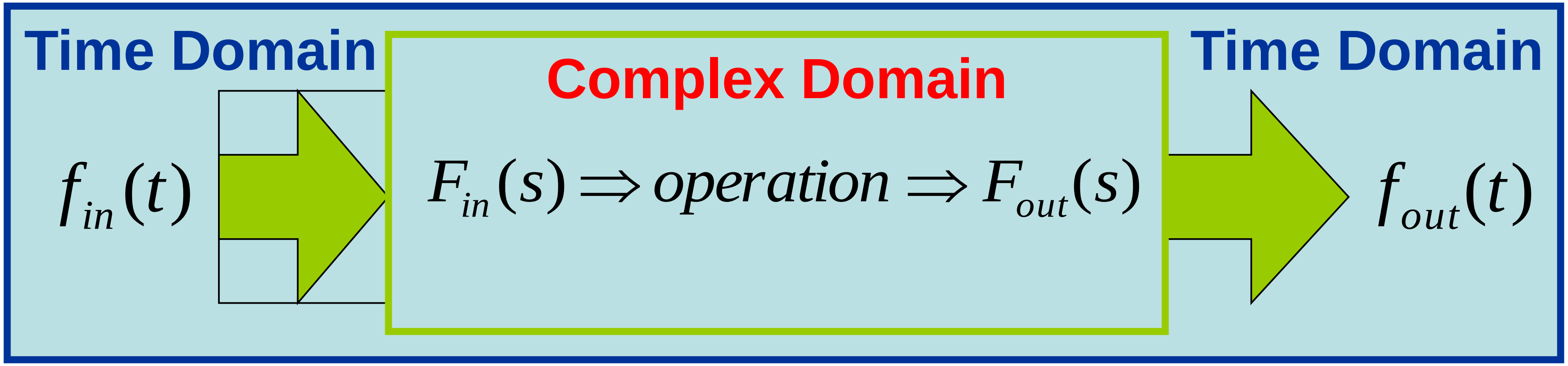
By operating on the transformed complex signal rather than the original real signal it is often possible to **Substantially Simplify** a problem involving:

- **Linear Differential Equations**
- **Convolutions**
- **Systems with Memory**

Signal Analysis

Operations on signals involving linear differential equations may be **difficult to perform** strictly in the **time domain**

- These operations may be **Simplified** by:
  - **Converting** the signal to the **Complex Domain**
  - **Performing Simpler Equivalent Operations**
  - **Transforming** back to the **Time Domain**



Laplace Transform Definition

The **Laplace Transform** of a continuous-time signal is given by:

$$X(s) = LT\{x(t)\} = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

x(t) **= Continuous Time Signal**

X(s) **= Laplace Transform** of x(t)

s **= Complex Variable** of the form **σ+jω**

Laplace Transform Definition

Unilateral

$$X(s) = \int_{0}^{\infty} x(t)e^{-st}dt$$

Bilateral

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

Unilateral:

Analyzing causal systems, systems specified by linear constant-coefficient differential equations with nonzero initial conditions.

Relation between FT and LT

Laplace transform:

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

Fourier transform:

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = X(s)|_{s=j\omega}$$

•The difference between the CTFT and the LT lies in the choice of the basis functions used in the two representations.

•The LT is a generalization of the CTFT, since the independent variable s can take any value in the complex s-plane and is not simply restricted to the imaginary $j\omega$-axis.

Relation between FT and LT

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The LT also bears a straightforward relationship to the FT when the complex variable s is not purely imaginary.

- By replacing **s** with **σ+jω**

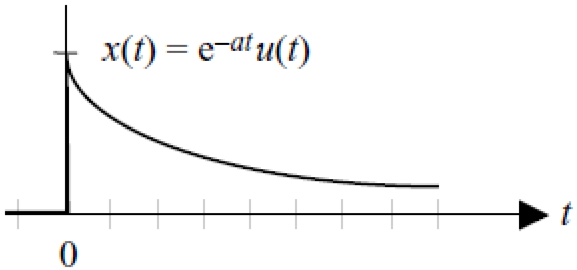
$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-(\sigma + j\omega)t}dt = \int_{-\infty}^{\infty} [x(t)e^{-\sigma t}]e^{-j\omega t}dt$$

Right-hand side of above equation is the FT of $x(t)e^{-\sigma t}$

The real exponential $e^{-\sigma t}$ may be decaying or growing…

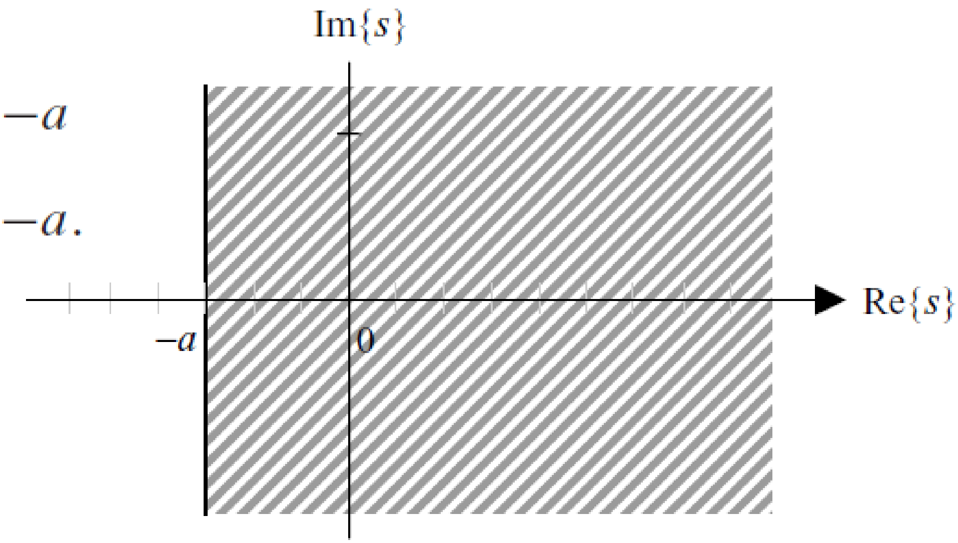
Example

Find the LT for $x(t) = e^{-at}u(t)$



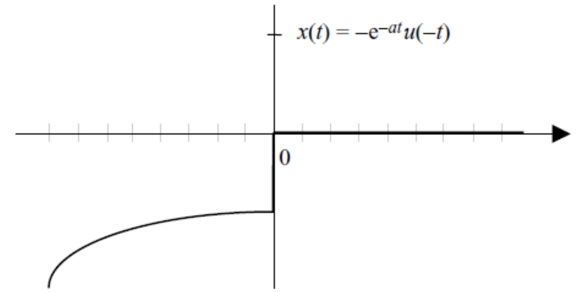
$$X(s) = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-st}\,dt = \int_{0}^{\infty} e^{-(s+a)t}\,dt = -\frac{1}{(s+a)}e^{-(s+a)t}\Big|_{0}^{\infty}$$

$$X(s) = \begin{cases} \dfrac{1}{(s+a)} & \text{for } \mathrm{Re}\{s\} > -a \\ \text{undefined} & \text{for } \mathrm{Re}\{s\} \le -a. \end{cases}$$

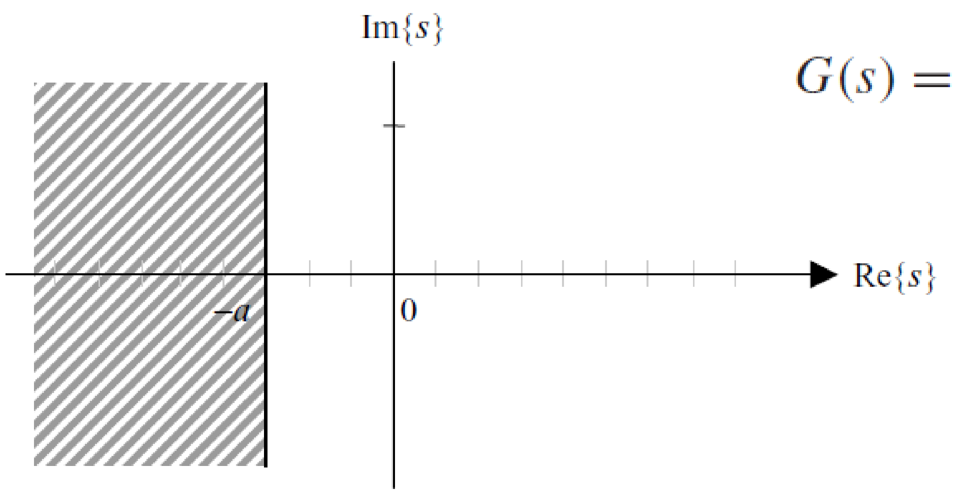


Example

Find the LT for $g(t) = -e^{-at}u(-t)$



$$G(s) = \int_{-\infty}^{\infty} -\mathrm{e}^{-at}u(-t)\mathrm{e}^{-st}\,\mathrm{d}t = -\int_{-\infty}^{0} \mathrm{e}^{-(s+a)t}\,\mathrm{d}t = \frac{1}{(s+a)}\mathrm{e}^{-(s+a)t}\Big|_{-\infty}^{0}$$



$$G(s) = \begin{cases} \dfrac{1}{(s+a)} & \text{for } \mathrm{Re}\{s\} < -a \\ \text{undefined} & \text{for } \mathrm{Re}\{s\} \geq -a. \end{cases}$$

Convergence

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Finding the Laplace Transform requires **integration** of the function from minus infinity to infinity

$$X(s) = LT\{x(t)\} = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

- For X(s) to exist, the integral must **converge**
- Convergence means that the **area** under the integral is **finite**
- Laplace Transform, X(s), **exists only for a** set of points in the s domain called the **Region of Convergence (ROC)**

Magnitude of X(s)

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For a complex X(s) to exist, it's magnitude must **converge**

$$\boxed{|X(s)| < \infty}$$

- By replacing **s** with **σ+jω**, |X(s)| can be rewritten as:

$$|X(s)| = \left| \int_{-\infty}^{\infty} x(t) e^{-st} dt \right| = \left| \int_{-\infty}^{\infty} x(t) e^{-(\sigma + j\omega)t} dt \right| < \infty$$

$$|X(s)| = \left| \int_{-\infty}^{\infty} x(t) e^{-\sigma t} e^{-j\omega t} dt \right| < \infty$$

|X(s)| Depends on σ

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The Magnitude of X(s) is **bounded** by the integral of the multiplied magnitudes of x(t), $e^{-\sigma t}$, and $e^{-j\omega t}$

$$\left|X(s)\right| = \left|\int_{-\infty}^{\infty} x(t)e^{-\sigma t}e^{-j\omega t}dt\right| \le \int_{-\infty}^{\infty}\left|x(t)e^{-\sigma t}e^{-j\omega t}\right|dt \le \int_{-\infty}^{\infty}\left|x(t)\right|\left|e^{-\sigma t}\right|\left|e^{-j\omega t}\right|dt$$

- **$e^{-\sigma t}$** is a **Real** number, therefore $|e^{-\sigma t}| = e^{-\sigma t}$
- **$e^{-j\omega t}$** is a **Complex** number with a magnitude of 1

- Therefore the **Magnitude Bound** of X(s) is dependent only upon the magnitude of x(t) and the **Real Part of s**

$$\boxed{\left|X(s)\right| \le \int_{-\infty}^{\infty}\left|x(t)\right|e^{-\sigma t}dt}$$

Region of Convergence

Laplace Transform X(s) exists only for a set of points in the **Region of Convergence (ROC)**

- The Region of Convergence is defined as the region where the **Real Portion of s (σ) meets the following criteria:**

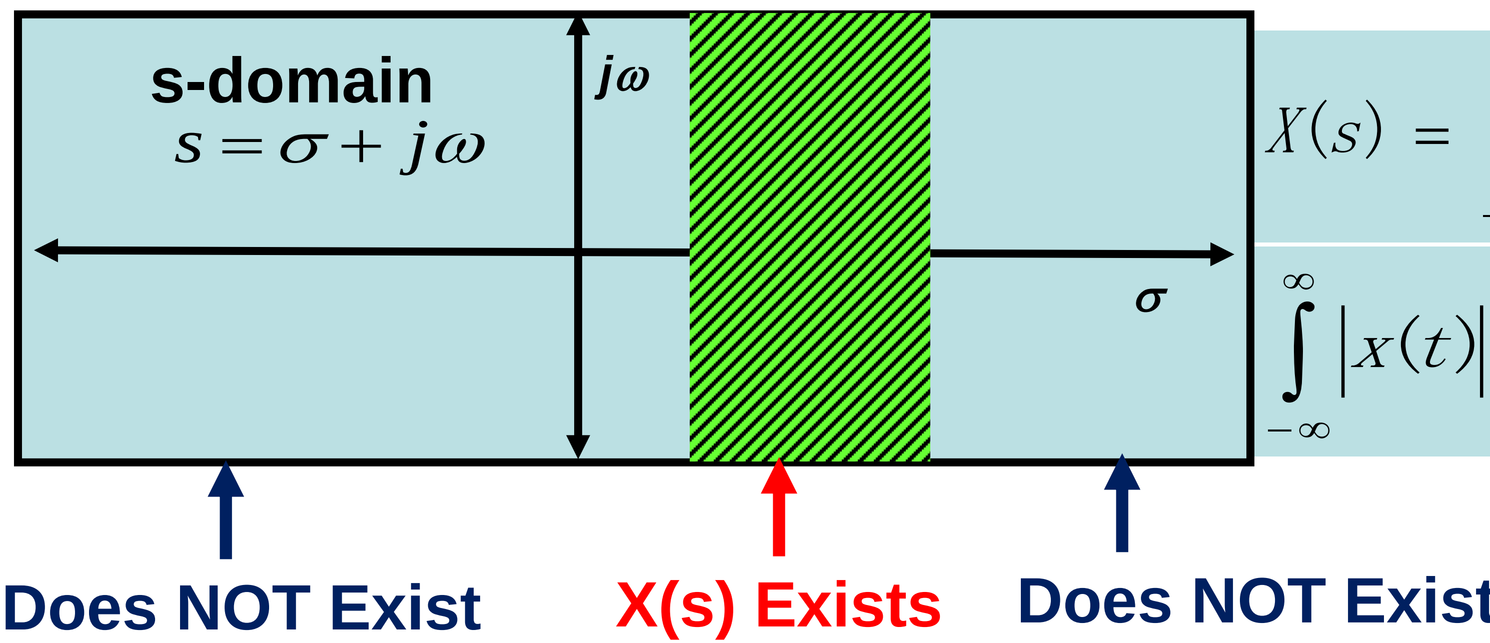
$$\boxed{\int_{-\infty}^{\infty} |x(t)| e^{-\sigma t} dt < \infty}$$

- X(s) only exists when the above integral is **finite**

ROC Graphical Depiction

- The s-domain can be graphically depicted as a 2D plot of the real and imaginary portions of s

In general the ROC is a strip in the complex s-domain



$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

$$\int_{-\infty}^{\infty} |x(t)| e^{-\sigma t} dt < \infty$$

Left- and right-sided ROCs

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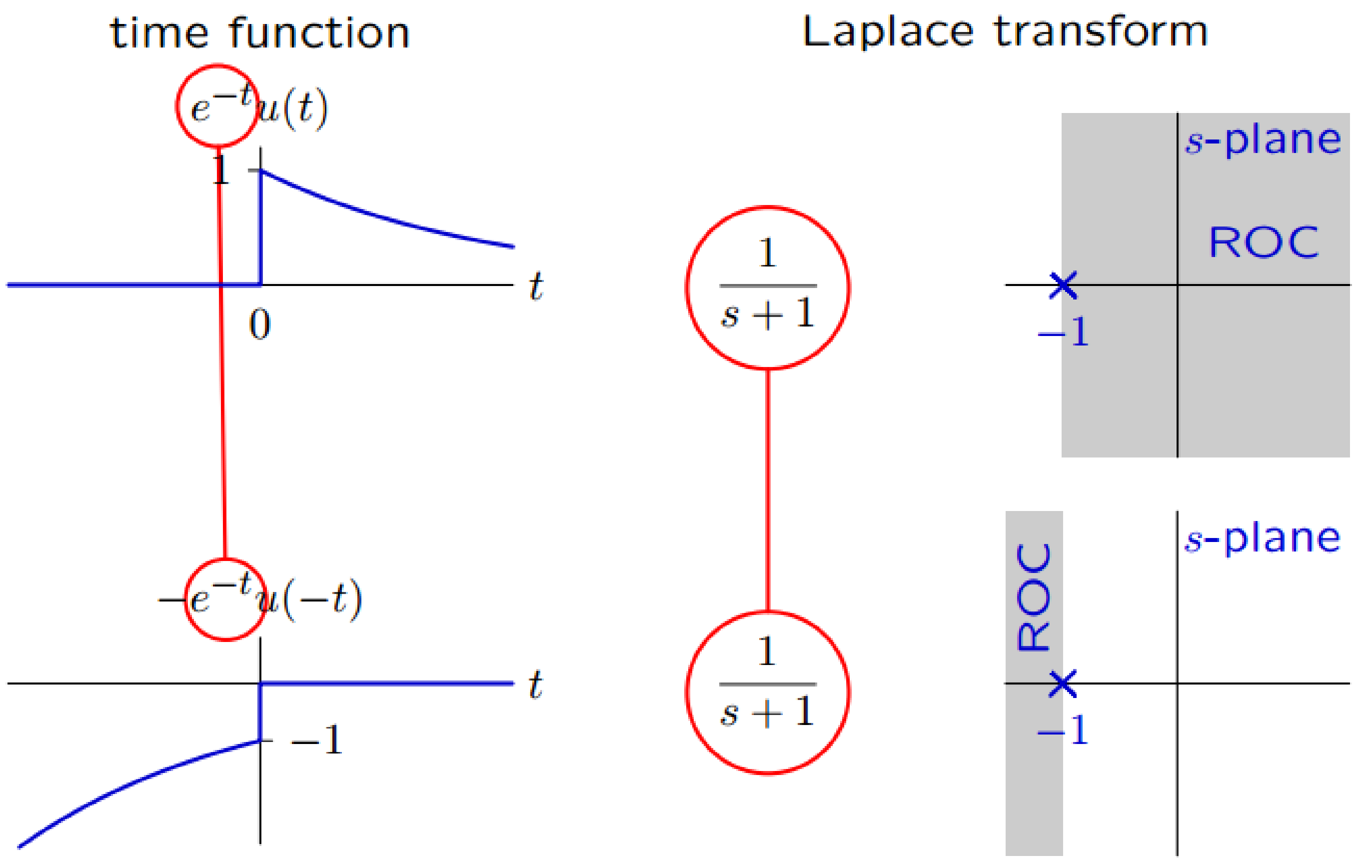
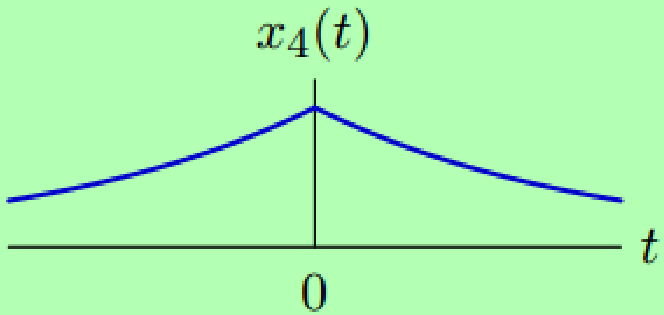
Laplace transforms of left- and right-sided exponentials have the same form (except −); with left- and right-sided ROCs, respectively.



Check yourself

Find the Laplace transform of $x_4(t)$.

$$x_4(t) = e^{-|t|}$$



1. $X_4(s) = \frac{2}{1-s^2}$; $-\infty < \mathrm{Re}(s) < \infty$
2. $X_4(s) = \frac{2}{1-s^2}$; $-1 < \mathrm{Re}(s) < 1$
3. $X_4(s) = \frac{2}{1+s^2}$; $-\infty < \mathrm{Re}(s) < \infty$
4. $X_4(s) = \frac{2}{1+s^2}$; $-1 < \mathrm{Re}(s) < 1$
5. none of the above

Check yourself

$$X_4(s) = \int_{-\infty}^{\infty} e^{-|t|} e^{-st} dt$$

$$= \int_{-\infty}^{0} e^{(1-s)t} dt + \int_{0}^{\infty} e^{-(1+s)t} dt$$

$$= \left. \frac{e^{(1-s)t}}{(1-s)} \right|_{-\infty}^{0} + \left. \frac{e^{-(1+s)t}}{-(1+s)} \right|_{0}^{\infty}$$

$$= \underbrace{\frac{1}{1-s}}_{\mathrm{Re}(s)<1} + \underbrace{\frac{1}{1+s}}_{\mathrm{Re}(s)>-1}$$

$$= \frac{1+s+1-s}{(1-s)(1+s)} = \frac{2}{1-s^2} \; ; \quad -1 < \mathrm{Re}(s) < 1$$

Inverse Laplace Transform

Inverse Laplace Transform is used to **compute x(t) from X(s)**

- The Inverse Laplace Transform is **strictly defined** as:

$$x(t) = LT^{-1}\{X(s)\} = \frac{1}{j2\pi}\int_{c-j\infty}^{c+j\infty} X(s)e^{st}ds$$

- Strict computation is **complicated** and rarely used in engineering

- Practically, the Inverse Laplace Transform of a **rational function** is calculated using a method of **table look-up**

Partial-Fraction Expansion 1

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To obtain an inverse transform of X(s), we form a sum by resorting to a partial fraction expansion. This requires x(s) to be strictly proper rational fraction.

$$X(S) = \frac{P(s)}{(S+P_1)(S+P_2)\cdots(S+P_N)} = \frac{K_1}{S+P_1} + \frac{K_2}{S+P_2} + \cdots + \frac{K_N}{S+P_N}$$

To find K_m, we multiply both side by $(S+P_m)$ and then, with both sides evaluated at $S=-P_m$, we have

$$K_m = (S+P_m)X(S)|_{S=-P_m}$$

For Repeated Factors:

$$X(S) = \frac{P(s)}{(S+P_1)(S+r)^k} = \frac{K_1}{S+P_1} + \frac{A_0}{(S+r)^k} + \frac{A_1}{(S+r)^{k-1}} \cdots + \frac{A_{k-1}}{S+r}$$

$$A_0 = (S+r)^k X(S)|_{s=-r} \qquad A_2 = \frac{1}{2!}\frac{d^2}{dS^2}(S+r)^k X(S)|_{s=-r}$$

$$A_1 = \frac{d}{dS}(S+r)^k X(S)|_{s=-r} \qquad A_n = \frac{1}{n!}\frac{d^n}{dS^n}(S+r)^k X(S)|_{s=-r}$$

Partial-Fraction Expansion 2

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$$\mathrm{H}(s)=\frac{10s}{(s+4)(s+9)}=\frac{K_1}{s+4}+\frac{K_2}{s+9}\quad,\ s>-4$$

$$K_1=\left[\cancel{(s+4)}\frac{10s}{\cancel{(s+4)}(s+9)}\right]_{s=-4}=\left[\frac{10s}{s+9}\right]_{s=-4}=\frac{-40}{5}=-8$$

$$K_2=\left[\cancel{(s+9)}\frac{10s}{(s+4)\cancel{(s+9)}}\right]_{s=-9}=\left[\frac{10s}{s+4}\right]_{s=-9}=\frac{-90}{-5}=18$$

$$\mathrm{H}(s)=\frac{-8}{s+4}+\frac{18}{s+9}=\frac{-8s-72+18s+72}{(s+4)(s+9)}=\frac{10s}{(s+4)(s+9)}\quad.\ \text{Check.}$$

$$\downarrow\downarrow\downarrow\downarrow\downarrow$$

$$\mathrm{h}(t)=\left(-8e^{-4t}+18e^{-9t}\right)\mathrm{u}(t)$$

Properties of Laplace Transform

1. Linearity
2. Time Scaling
3. Right Time Shift
4. Shifted in the s-domain
5. Convolution
6. Differentiation in Time Domain
7. Integration in Time Domain
8. Initial Value Theorem
9. Final Value Theorem
10. Differentiation in s-domain

1. Linearity

The Laplace Transform is a Linear Operation

Superposition Principle can be applied

$$x(t) \xleftrightarrow{LT} X(s)$$

$$y(t) \xleftrightarrow{LT} Y(s)$$

$$ax(t) + by(t) \xleftrightarrow{LT} aX(s) + bY(s)$$

2. Time Scaling

- x(t) **Compressed** in the time domain

$$x(t)\ \ compressed\ \ to\ \ x(at)\ \ if\ \ |a|>1$$

x(t) **Stretched** in the time domain

$$x(t)\ \ stretched\ \ to\ \ x(at)\ \ if\ \ |a|<1$$

- Laplace Transform of **compressed or stretched** version of x(t)

$$x(at) \xleftrightarrow{LT} \frac{1}{|a|} X\left(\frac{s}{a}\right)$$

3. Right Time Shift

Given a time domain signal **delayed by t_0 seconds**

The Laplace Transform of the delayed signal is **$e^{-t_o s}$**

multiplied by the Laplace Transform of the original signal

$$x(t) \xleftrightarrow{LT} X(s)$$

$$x(t - t_o) \xleftrightarrow{LT} e^{-t_o s} X(s)$$

4. Shifted in the s-domain

A time domain signal x(t) **multiplied** by an **exponential** function of t, results in the Laplace Transform of x(t) being a **shifted in the s-domain**

$$x(t) \xleftrightarrow{LT} X(s)$$

$$e^{at}x(t) \xleftrightarrow{LT} X(s-a)$$

5. Convolution

The **convolution** of two signals in the **time domain** is equivalent to a **multiplication** of their **Laplace Transforms** in the s-domain

$$LT\{x(t) * y(t)\} = X(s)Y(s)$$

- * is the sign for convolution

$$x(t) * y(t) = \int_{0}^{\infty} x(\tau)y(t-\tau)d\tau$$

6. Differentiation in the Time Domain

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In general, the Laplace Transform of the nth derivative of a continuous function x(t) is given by:

$$LT\left\{\frac{d^n x(t)}{dt^n}\right\} = s^n X(s) - s^{n-1}x(0^-) - s^{n-2}\frac{dx(0^-)}{dt} - \ldots - s\frac{d^{(n-2)}x(0^-)}{dt^{(n-2)}} - \frac{d^{(n-1)}x(0^-)}{dt^{(n-1)}}$$

- 1st Derivative Example:

$$LT\left\{\frac{dx(t)}{dt}\right\} = sX(s) - x(0^-)$$

- 2nd Derivative Example:

$$LT\left\{\frac{d^2 x(t)}{dt^2}\right\} = s^2 X(s) - sx(0^-) - \frac{dx(0^-)}{dt}$$

7. Integration in Time Domain

The Laplace Transform of the **integral** of a time domain function is the functions **Laplace Transform divided by s**

$$LT\left\{\int_0^t x(\tau)d\tau\right\} = \frac{1}{s}X(s)$$

8. Initial Value Theorem

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The initial value of x(t) can be found using the Laplace Transform as follows:

$$x(0) = \lim_{s \to \infty} \left\{ sX(s) \right\}$$

- Assume:
  - **$x(t) = 0$ for $t < 0$**
  - **x(t) does not contain impulses or higher order singularities**

Example:

Given;

$$F(s) = \frac{(s+2)}{(s+1)^2 + 5^2}$$

Find f(0)

$$f(0) = \lim_{s \to \infty} sF(s) = \lim_{s \to \infty} s\frac{(s+2)}{(s+1)^2 + 5^2} = \lim_{s \to \infty}\left[\frac{s^2 + 2s}{s^2 + 2s + 1 + 25}\right]$$

$$= \lim_{s \to \infty} \frac{s^2/s^2 + 2s/s^2}{s^2/s^2 + 2s/s^2 + (26/s^2)} = 1$$

9. Final Value Theorem

The **steady-state** value of the signal x(t) can also be determined using the Laplace Transform

$$\lim_{t \to \infty} x(t) = \lim_{s \to 0} \left\{ sX(s) \right\}$$

10. Differentiation in *s*-domain

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

$$\frac{dX(s)}{ds} = \int_{-\infty}^{\infty} (-t)x(t)e^{-st}dt$$

$$LT\{-\ tx(t)\} = \frac{dX(s)}{ds}$$

Examples

Solve the following initial-value differential equations using the Laplace transform method:

$$\frac{d^2 y}{dt^2} + 6\frac{dy}{dt} + 8y(t) = te^{-3t}u(t)\,; \quad y(0^-) = y'(0^-) = 1\,;$$

Summary

- **Laplace Transform** Definition
- **Region of Convergence** where Laplace Transform is valid
- **Inverse Laplace Transform** Definition
- **Properties** of the Laplace Transform that can be used to simplify difficult time domain operations such as differentiation and convolution

Exercises

9.2. Consider the signal

$$x(t) = e^{-5t}u(t-1),$$

and denote its Laplace transform by $X(s)$.

(a) Using eq. (9.3), evaluate $X(s)$ and specify its region of convergence.

(b) Determine the values of the finite numbers A and t_0 such that the Laplace transform $G(s)$ of

$$g(t) = Ae^{-5t}u(-t-t_0)$$

has the same algebraic form as $X(s)$. What is the region of convergence corresponding to $G(s)$?

Exercises

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9.9. Given that

$$e^{-at}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+a}, \qquad \mathcal{Re}\{s\} > \mathcal{Re}\{-a\},$$

determine the inverse Laplace transform of

$$X(s) = \frac{2(s+2)}{s^2+7s+12}, \qquad \mathcal{Re}\{s\} > -3.$$