MTH101: Tutorial 2

Dr. Tai-Jun Chen, Dr. Xinyao Yang

Xi'an Jiaotong-Liverpool University, Suzhou

September 23, 2017

Exercise 0.1

Find out, and give reasons, whether $f(z)$ is continuous at $z = 0$ if $f(0) = 0$ and for $z \neq 0$ the function f is equal to

1. $(\mathrm{Im}\ z^2)/|z|^2$
2. $(\mathrm{Re}\ z^2)/|z|$

Solution:

1. This function is not continuous at 0 since given $z = r(\cos\theta + i\sin\theta)$, we have

$$(\mathrm{Im}\ z^2)/|z|^2 = r^2\sin(2\theta)/r^2 = \sin 2\theta$$

which depends on θ, on the direction of approach to 0, so that it has not limit, by definition.

2. Yes, because

$$(\mathrm{Re}\ z^2)/|z| = r^2\cos(2\theta)/r = r\cos(2\theta) \to 0 = f(0), \qquad \text{as } r \to 0.$$

Exercise 0.2

Determine whether the following $f(z)$ is differentiable at 0, if yes, find the derivative $f'(0)$; if not, state the reason.

1. $f(z) = i(1-z)^n$;
2. $f(z) = \mathrm{Re}\ z$

Solution:

1. The function is differentiable at 0 because it is a polynomial function (entire), the derivative by using the chain rule:

$$f'(0) = i \cdot n(1-z)^{n-1} \cdot (-1)|_{z=0} = -ni.$$

2. The function is not differentiable at 0 since

$$\frac{f(z+\Delta z) - f(z)}{(z + \Delta z - z)} = \frac{\Delta x}{\Delta z},$$

which is 0 if $\Delta x = 0$ but 1 if $\Delta y = 0$, so that it has no limit as $\Delta z \to 0$.

Exercise 0.3

Are the following functions analytic?

1. $f(z) = iz\bar{z}$;
2. $f(z) = e^y(\sin x + i\cos x)$;
3. $f(z) = 1/(z - z^5)$

Solution:

1. No, $u = 0$, $v = |z|^2 = x^2 + y^2$, the C-R equation is not satisfied.
2. Yes, $u(x, y) = e^y \sin x$, $v(x, y) = e^y \cos x$. We compute the partial derivatives:

$$
\begin{aligned}
u_x &= e^y \cos x, \quad u_y = e^y \sin x, \\
v_x &= -e^y \sin x, \quad v_y = e^y \cos x.
\end{aligned}
$$

The partial derivatives u_x, u_y, v_x, v_y are continuous for all $x, y \in \mathbb{R}$. Moreover the Cauchy-Riemann equations:

$$
\begin{aligned}
u_x &= v_y = e^y \cos x, \\
u_y &= -v_x = e^y \sin x,
\end{aligned}
$$

are satisfied for all $x, y \in \mathbb{R}$. Then we conclude that f is analytic for any $z = x + iy \in \mathbb{C}$.

3. The function is rational, thus it is analytic iff $z - z^5 \neq 0$, that is, it is analytic when $z \neq 0, \pm 1, \pm i$.

Exercise 0.4

Verify that the function $v(x, y) = xy$ is Harmonic, and find its Harmonic Conjugate u so that $f = u + iv$ is analytic.

Solution

We have

$$v_x = y, \quad v_{xx} = 0$$

and

$$v_y = x, \quad v_{yy} = 0$$

then v is Harmonic since:

$$v_{xx} + v_{yy} = 0.$$

The Harmonic Conjugate u of v satisfies the Cauchy-Riemann equations:

$$u_x = v_y = x$$
$$u_y = -v_x = -y$$

Integrating the first Cauchy-Riemann equation with respect to x we find:

$$u(x, y) = \int u_x dx + g(y) + C = \frac{x^2}{2} + g(y) + C$$

where $C \in \mathbb{C}$ is a constant and g is an unknown function of y.

Now we use the second Cauchy-Riemann equation:

$$
\begin{aligned}
u_y &= -v_x = -y \\
&= \left[\frac{x^2}{2} + g(y) + C\right]_y = g'(y)
\end{aligned}
$$

from which we obtain that $g'(y) = -y$, that is, $g(y) = -y^2/2 + C_1$. Finally, the expression of $u(x, y)$ is:

$$
u(x, y) = \frac{x^2 - y^2}{2} + K
$$

where $K \in \mathbb{C}$ is a constant.
Moreover, the analytic function

$$
f = u + iv = \frac{x^2 - y^2}{2} + i\ xy + K = \frac{z^2}{2} + K.
$$

Exercise 0.5

Write in the form $u + iv$ the following functions

1. $e^{-\pi z}$
2. $\exp(z^2)$

Solution:

1. We have

$$e^{-\pi z} = e^{-\pi(x+iy)} = e^{-\pi x} e^{-\pi y \ i}$$
$$= e^{-\pi x}(\cos(-\pi y) + i \sin(-\pi y),$$

thus,

$$u = \mathrm{Re}\ f = e^{-\pi x} \cdot \cos(-\pi y) = e^{-\pi x} \cos(\pi y)$$
$$v = \mathrm{Im}\ f = e^{-\pi x} \cdot \sin(-\pi y) = -e^{-\pi x} \sin(\pi y).$$

2. We have

$$\exp(z^2) = \exp(x^2 - y^2 + 2xy\,i) = e^{x^2-y^2} e^{2xyi}$$
$$= e^{x^2-y^2}(\cos(2xy) + i\sin(2xy)),$$

therefore,

$$u = \mathrm{Re}\ f = e^{x^2-y^2} \cdot \cos(2xy)$$
$$v = \mathrm{Im}\ f = e^{x^2-y^2} \cdot \sin(2xy).$$

Exercise 0.6

Compute all the values of

1 $\ln(-1)$;

2 $(-1)^{2-i}$.

Solution:

1. We can use the formula

$$\ln z = \mathrm{Ln}\ z + i2n\pi = \ln|z| + i\mathrm{Arg}\ z + i2n\pi, \quad n = 0, \pm 1, \pm 2, \ldots.$$

Then we compute

$$|z| = 1, \quad \mathrm{Arg}\ z = \pi,$$

from which

$$\ln(-1) = \ln(1) + i\pi + i2n\pi, \quad n = 0, \pm 1, \pm 2, \ldots.$$

that is

$$\ln(-1) = i(2n+1)\pi, \quad n = 0, \pm 1, \pm 2, \ldots.$$

and

$$\mathrm{Ln}\ (-1) = i\pi.$$

2. We use the formula $z^c = e^{c \ln z}$ to obtain:

$$(-1)^{2-i} = e^{(2-i)\ln(-1)},$$

from the previous execise, we know that

$$\begin{aligned} &\ln(-1) = i(2n+1)\pi, \quad n = 0, \pm 1, \pm 2, ... \\ &\mathrm{Ln}\,(-1) = i\pi, \end{aligned}$$

Thus

$$(-1)^{2-i} = e^{(2-i)(2n+1)\pi i},$$

and the **principle value** of $(-1)^{2-i}$ is

$$e^{(2-i)\cdot i\pi} = e^{\pi + i2\pi} = e^{\pi} \cdot e^{2\pi i} = e^{\pi} \cdot 1 = e^{\pi}.$$