EEE225 Advanced Electrical Circuits and Electromagnetics

Lecture 5 Electric and Magnetic Components

(Resistor, Capacitors and Inductors)

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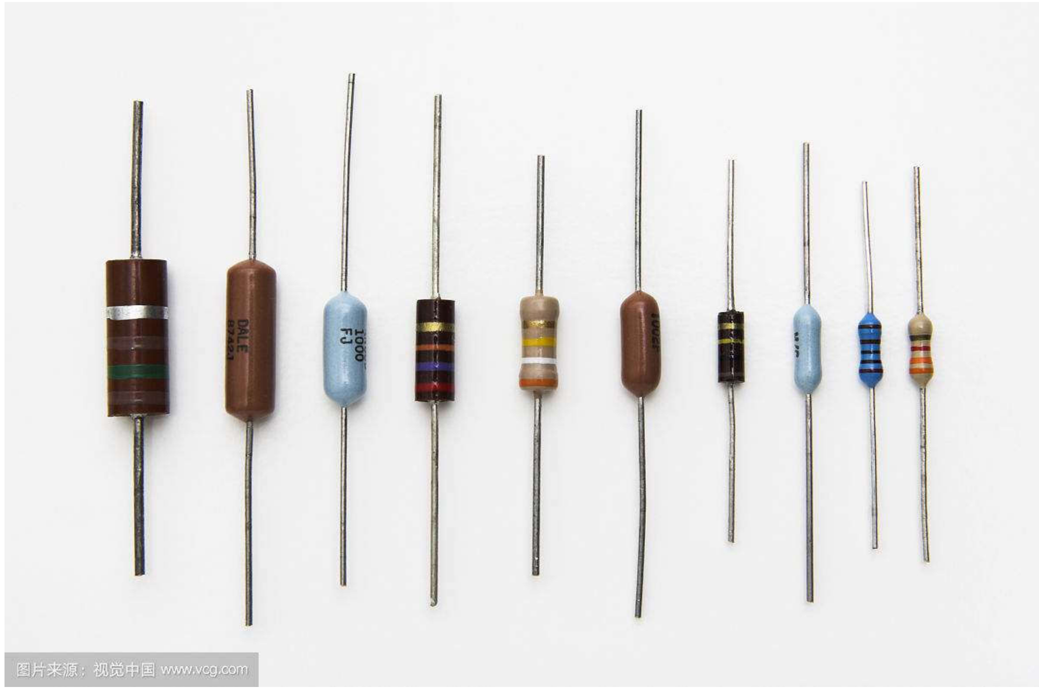
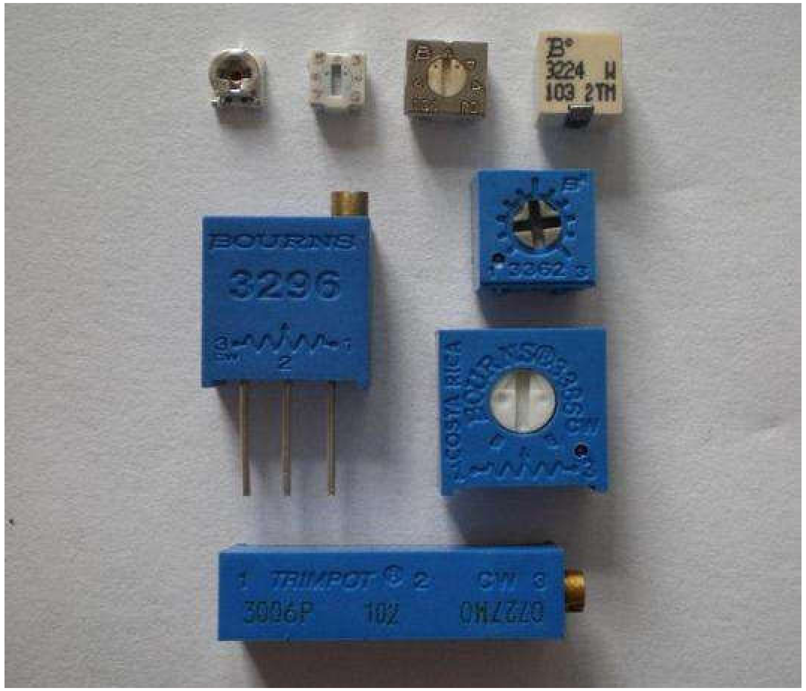
1. Resistor

- A resistor is a passive two-terminal electrical component that implements electrical resistance as a circuit element.



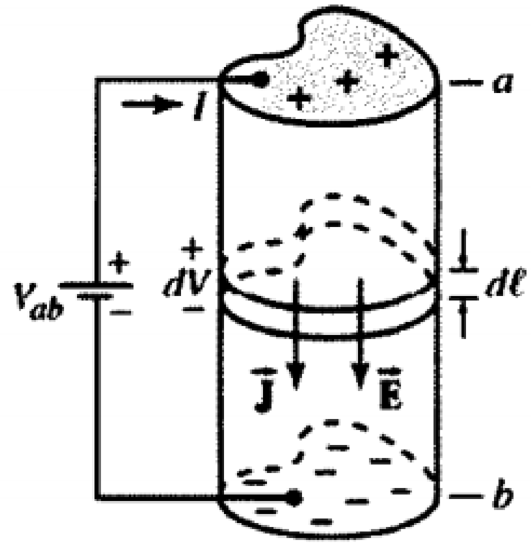
Fixed resistors



Variable resistors

1. Resistance

- The resistance of a conductor of length *dl* can be obtained by



$$dR = \frac{dV}{I} = \frac{-\vec{E}\cdot d\vec{l}}{\iint_S \vec{J}\cdot d\vec{s}}$$

  - If we assume that the potential at end *a* of the conductor is higher than that at end *b*.
  - The total resistance of the conductor is:

$$R = \int_b^a \frac{-\vec{E}\cdot d\vec{l}}{\iint_S \vec{J}\cdot d\vec{s}}$$

  - This is a general equation to determine the resistance of a conducting medium whose conductivity changes in the direction of the current. For homogeneous medium having constant σ, it reduces to:

$$R = \int_b^a \frac{-\vec{E}\cdot d\vec{l}}{\iint_S \vec{J}\cdot d\vec{s}} = \frac{-\int_b^a \vec{E}\cdot d\vec{l}}{\iint_S \vec{J}\cdot d\vec{s}} = \frac{V_{ab}}{I}$$

1. Resistance

How to calculate?

- Simplified model: A potential difference of V_0 is maintained across the two ends of a conducting wire of length l. If A is the cross-sectional area of the wire, obtain an expression for the resistance of the wire.
  - Assume the potential difference between the two ends of the conductor is V_0, the electric field holds:

$$V_0 = -\int_b^a \vec{\boldsymbol{E}} \cdot \boldsymbol{d}\vec{\boldsymbol{l}} = El \Rightarrow E = \frac{V_0}{l}$$

  - If σ is the conductivity of the conducting material, the current density at any cross section of the wire is:

$$J = \sigma E = \frac{\sigma V_0}{l}$$

  - The current through the wire is:

$$I = \iint_S \vec{J} \cdot d\vec{s} = JA = \frac{\sigma V_0 A}{l} = \frac{V_0}{R}$$

  - So the resistance of the piece of the conducting material is

$$R = \frac{V_0}{I} = \frac{l}{\sigma A}$$

1. Resistance

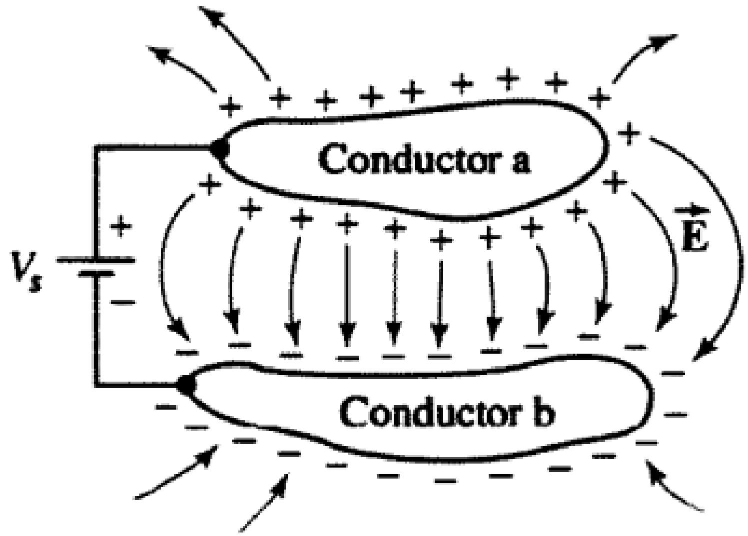
Example 1

- A long, round wire of radius a and conductivity σ is coated with a material of conductivity 0.1σ.
  - a) What must be the thickness of the coating so that the resistance per unit length of the uncoated wire is reduced by 50%?
  - b) Assuming a total current I in the coated wire, find **J** and **E** in both the core and the coating material.

2. Capacitor

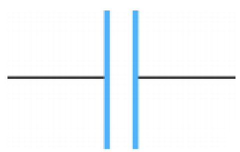
- A capacitor is also a two-terminal passive device which stores electric charge.

2. Capacitor

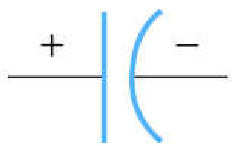


- Capacitor
  - A capacitor is a device which stores electric charge.
  - Its basic configuration is two conductors carrying equal but opposite charges
- Capacitance
  - measures the capability of energy storage in electrical devices.
  - the amount of charge Q stored in a capacitor is linearly proportional to the electric potential difference V between the two conductors:

$$\frac{Q}{V} = constant = C$$



  - Unit: 1 F (farad) = 1 C / V (coulomb/volt)

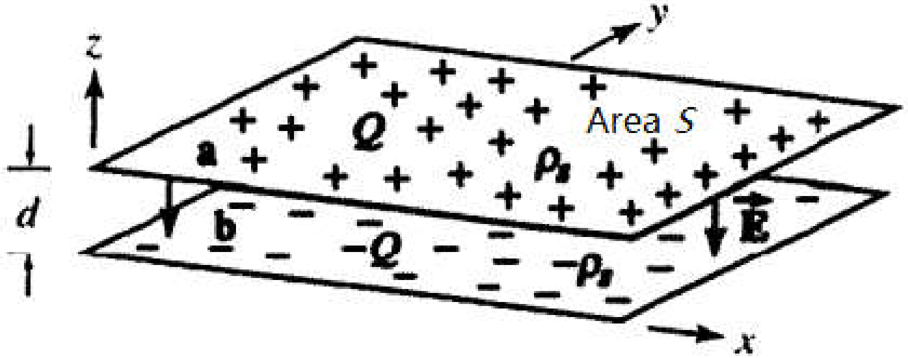


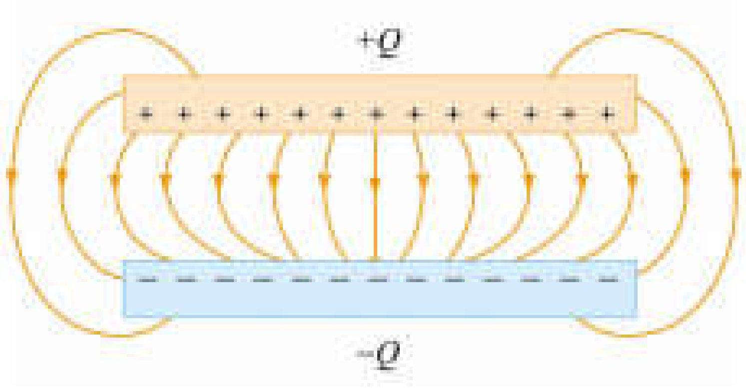
2.1 Capacitor Examples

Parallel Plates

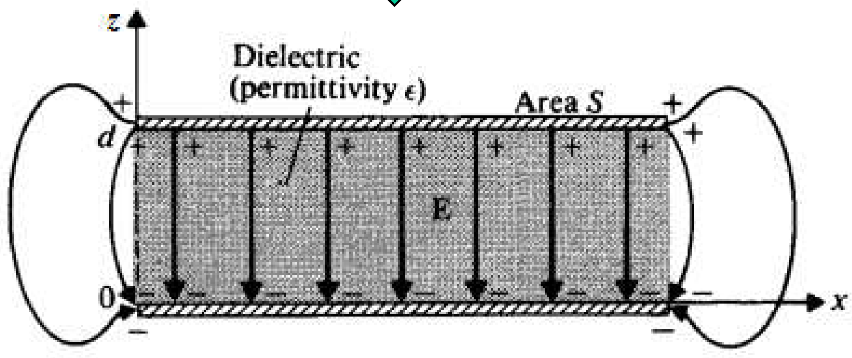
- Two parallel conducting plates, each of area *S*, and separated by a distance *d*, form a parallel-plate capacitor. The total charge on the top plate is $+Q$ and that on the other plate is $-Q$.
  - What is its capacitance?

- Solution:
  - Edge effects: The electric field lines at the edge of the plates are not straight lines, and the field is not contained entirely between the plates.
  - Fringing fields: The non-uniform fields near the edge.

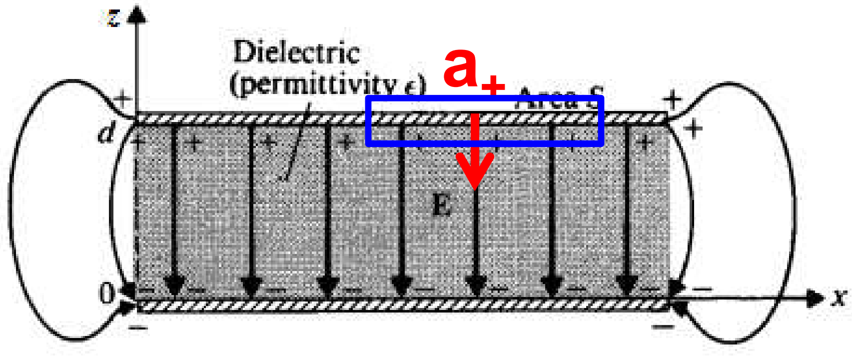




$d \ll \sqrt{S}$



2.1 Capacitor Examples



- Solution:
  - The surface charge density is:

    $$\rho_s = Q/S$$

  - Based on Gauss's Law:

    $$\oiint_{S'} \vec{E} \cdot d\vec{S} = \frac{q_{enc}}{\varepsilon_0} \xrightarrow[\textbf{S' is a unit area}]{} \vec{E} = -\hat{a}_z \frac{\rho_s}{\varepsilon_0} = -\hat{a}_z \frac{Q}{\varepsilon_0 S}$$

  - The potential *V* is:

    $$V = -\int_{z=0}^{z=d} \vec{E} \cdot d\vec{l} = -\int_0^d \left(-\hat{a}_z \frac{Q}{\varepsilon_0 S}\right) \cdot (\hat{a}_z dz) = \frac{Q}{\varepsilon_0 S} d$$

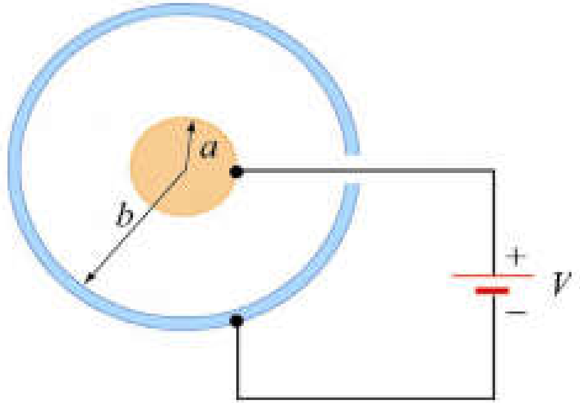
  - Therefore, the capacitance of a parallel – plate is:

    $$C = \frac{Q}{V} = \frac{\varepsilon_0 S}{d}$$

2.1 Capacitor Examples

Spherical capacitor

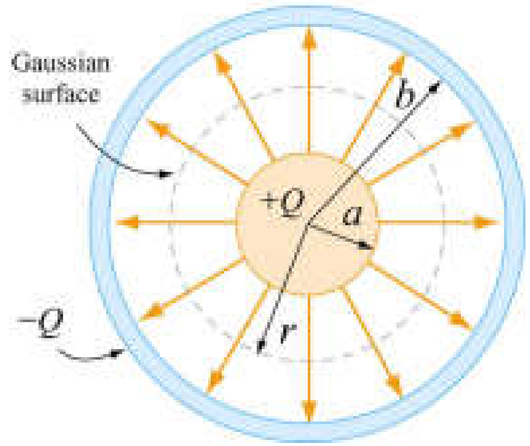
- A spherical capacitor:
  - two concentric spherical shells of radii *a* and *b*
  - The inner shell has a charge $+Q$ uniformly distributed over its surface, and the outer shell an equal but opposite charge $-Q$.



- Solution:
  - The electric field is non-vanishing only in the region $a < r < b$. Using Gauss's law, we obtain

$$E_r = \frac{1}{4\pi\varepsilon_o}\frac{Q}{r^2}$$

  - Therefore, the potential difference between the two conducting shells is

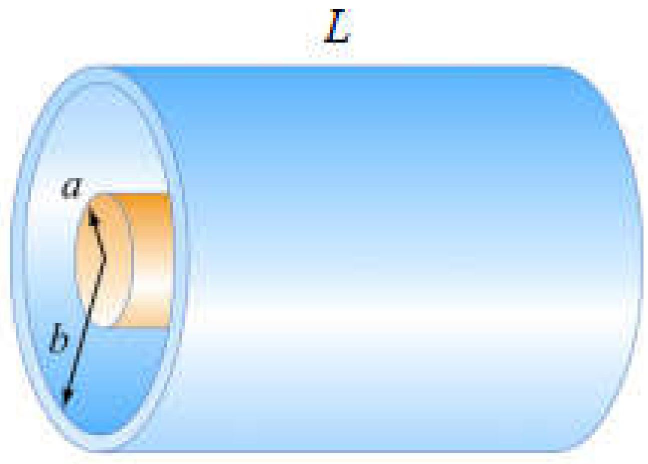


$$\Delta V = V_b - V_a = -\int_a^b E_r\,dr = -\frac{Q}{4\pi\varepsilon_0}\int_a^b \frac{dr}{r^2} = -\frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{a}-\frac{1}{b}\right) = -\frac{Q}{4\pi\varepsilon_0}\left(\frac{b-a}{ab}\right)$$

$$\Longrightarrow \quad C = \frac{Q}{|\Delta V|} = 4\pi\varepsilon_0\left(\frac{ab}{b-a}\right)$$

2.1 Capacitor Examples

- A cylindrical conductor
  - Inner radius *a* surrounded by a coaxial cylindrical shell of inner radius *b*. Filled with dielectrics with ε. The length of both cylinders is *L*.
  - The capacitor is charged so that the inner cylinder has charge +*Q* while the outer shell has a charge –*Q*.



- Solution:
  - Assume that *L* is much larger than *b*−*a*, the separation of the cylinders, so that edge effects can be neglected.
  - E can be calculated by:

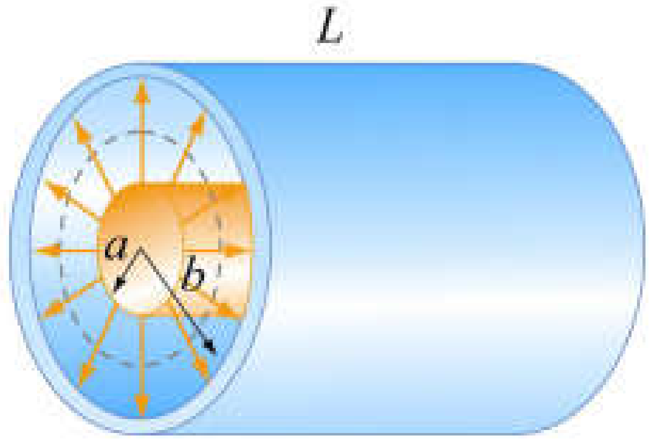
$$\mathbf{E} = \mathbf{a}_r E_r = \mathbf{a}_r \frac{Q}{2\pi\epsilon L r}.$$

  - So potential V is:

$$V_{ab} = -\int_{r=b}^{r=a} \mathbf{E} \cdot d\ell = -\int_b^a \left(\mathbf{a}_r \frac{Q}{2\pi\epsilon L r}\right) \cdot (\mathbf{a}_r\, dr)$$

$$= \frac{Q}{2\pi\epsilon L} \ln\left(\frac{b}{a}\right).$$

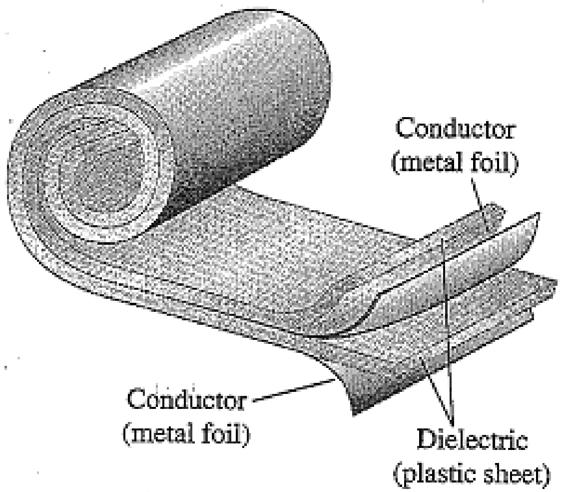
$$\Longrightarrow \quad C = \frac{Q}{V_{ab}} = \frac{2\pi\epsilon L}{\ln\left(\frac{b}{a}\right)}.$$



2.2 Capacitor with dielectrics

- Most capacitors have an insulating material, such as paper or plastic, between their conducting plates.
- Reasons:
  - To maintain a physical separation of the plates;
  - Increase the maximum possible potential difference between the conducting plates;
  - Capacitance increases when the space between the conductors is filled with dielectrics.

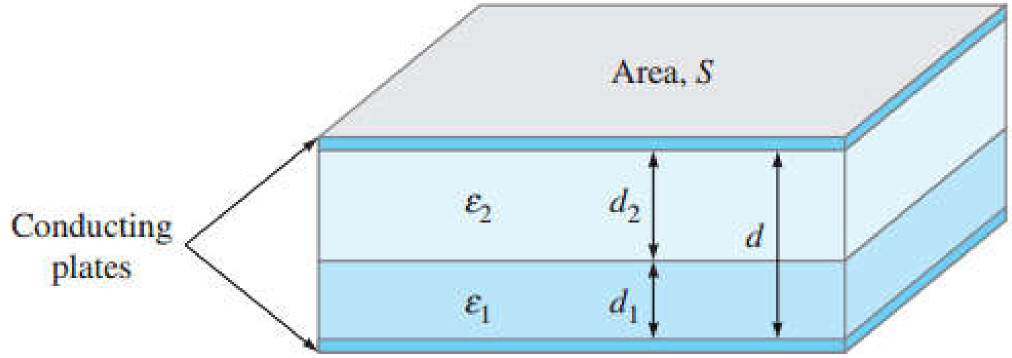
$$C = \frac{Q}{V} = \frac{\varepsilon S}{d}$$



2.2 Capacitor with dielectrics

Example 1

- A parallel-plate capacitor containing two dielectrics with the dielectric interface parallel to the plates.
- What is its capacitance?



- Solution 1:
  - It can be considered as two serially connected parallel-plate capacitors.
  - So the total capacitance is: $C = \dfrac{1}{\dfrac{1}{C_1} + \dfrac{1}{C_2}}$
  - where $C_1 = \epsilon_1 S/d_1$ $\quad$ $C_2 = \epsilon_2 S/d_2$

This is the correct result, but let's try to obtain it using less intuition and a more basic approach (from the definition).

2.2 Capacitor with dielectrics

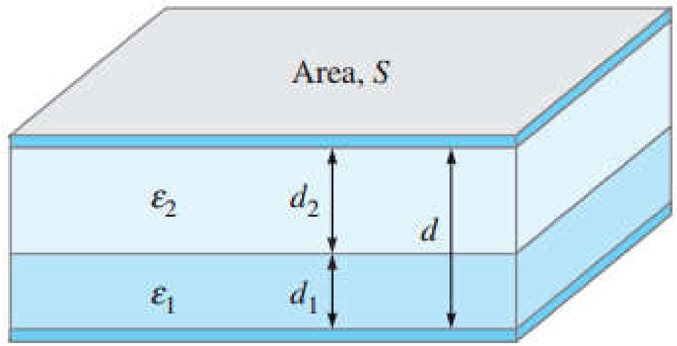
Example 1 cont.

- Solution 2:
  - Suppose we assume a potential difference V_0 between the plates. The electric field intensities in the two regions, E_2 and E_1, are both uniform, and $V = E_1 d_1 + E_2 d_2$
  - At the dielectric interface, ***E*** is normal to the interface, and our boundary condition tells us that $D_1 = D_2$, or $\varepsilon_1 E_1 = \varepsilon_2 E_2$
  - The surface charge density

$$\rho_{s1} = D_1 = \varepsilon_1 E_1 = \varepsilon_2 E_2 = \rho_{s2} = \rho_s$$

  - So we have:

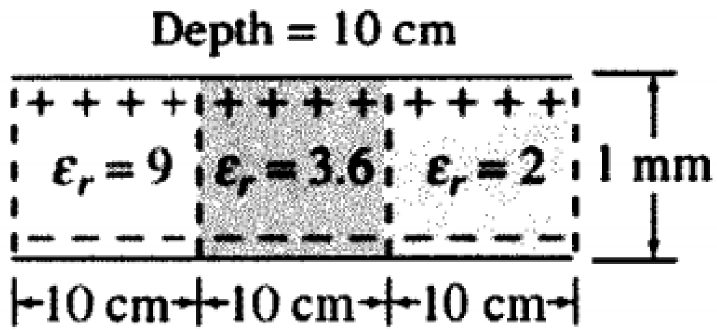
$$C = \frac{Q}{V} = \frac{\rho_s S}{V} = \frac{1}{\frac{d_1}{\varepsilon_1 S} + \frac{d_2}{\varepsilon_2 S}} = \frac{1}{\frac{1}{C_1} + \frac{1}{C_2}}$$



2.2 Capacitor with dielectrics

Example 2

- A parallel-plate capacitor with three dielectric media is shown below. What is the total capacitance of the system?



2.3 Energy stored in a capacitor

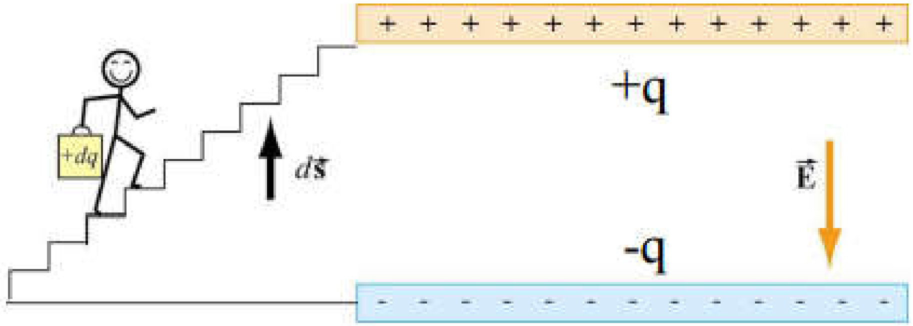
✓ 1. Capacitor starts uncharged.

✓ 2. Carry $+dq$ from bottom to top.

Now top has charge $q = +dq$, bottom $-dq$

✓ 3. Repeat

✓ 4. Finish when top has charge $q = +Q$, bottom $-Q$.



- At some point top plate has +q, potential difference is: $\Delta\varphi = q/C$
- Work done to lift dq from the bottom to top is: $dW = dq\Delta\varphi = qdq/C$
- So work done to move Q from bottom to top is:

$$W = \int dW = \frac{1}{C}\int_0^Q qdq = \frac{1}{C}\frac{Q^2}{2}$$

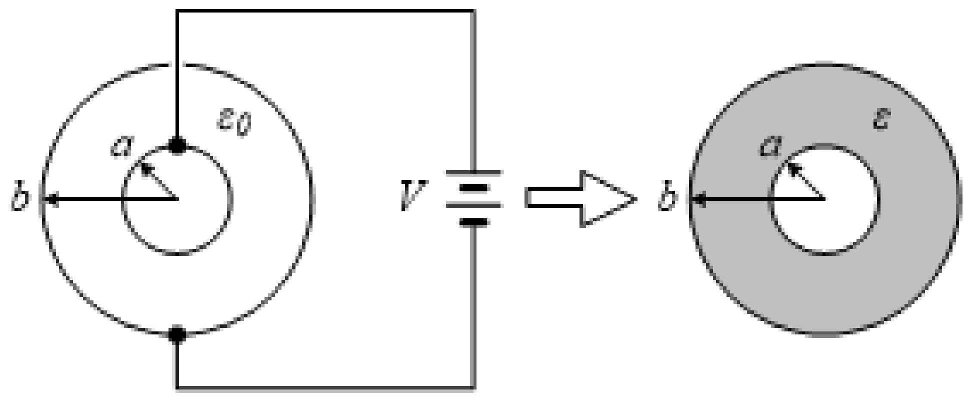
- After charging, in $\Delta\varphi = q/C$, $\Delta\varphi$ is V, and q is Q, the total energy stored is:

$$W = \frac{1}{C}\frac{Q^2}{2} = \frac{1}{C}\frac{(CV)^2}{2} = \frac{1}{2}CV^2$$

2.3 Energy stored in a capacitor

Example 3

- An air-filled spherical capacitor with conductor radii $a = 3$ cm and $b = 15$ cm is connected to a source of voltage $V = 15$ kV as shown in Figure (b).
- After an electrostatic state is established, the source is disconnected. The capacitor is then filled with a liquid dielectric of dielectric constant $\varepsilon_r = 2$ as shown in Figure (c).
- Determine the energy stored between the electrodes of the capacitor.



(b) (c)

2.4 Current – Voltage Relationship

- Start from the known relationship:

$$C = \frac{Q}{V} \Rightarrow V = \frac{Q}{C}$$

- In a time-dependent scenario:

$$V(t) = \frac{Q(t)}{C} = \frac{1}{C}\int_{t_0}^{t} I(\tau)d\tau + V(t_0)$$

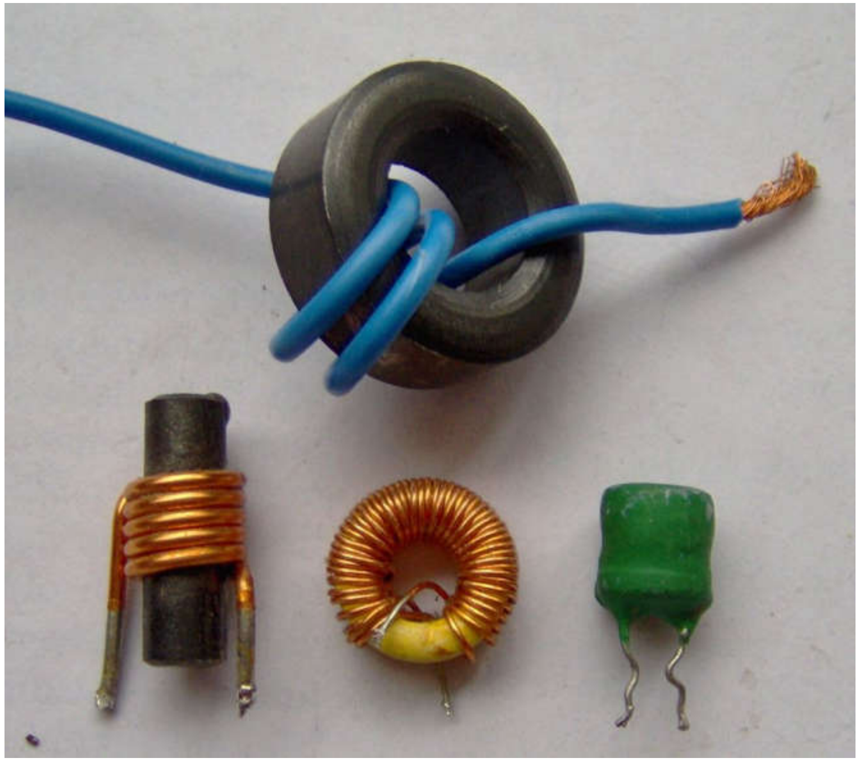
- Taking the derivative of this and multiplying by *C*, get:

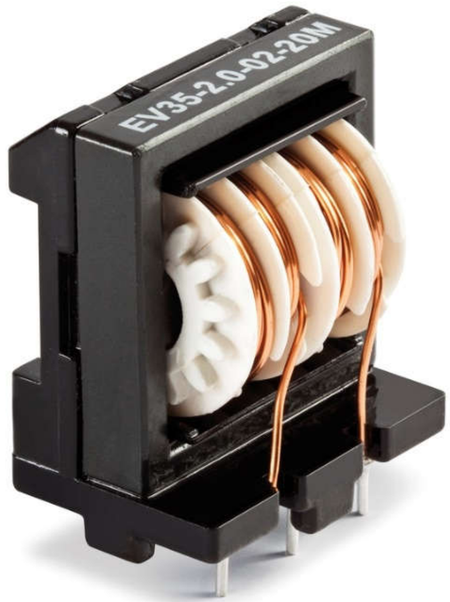
$$I(t) = \frac{dQ(t)}{dt} = C\frac{dV(t)}{dt}$$

  – which means "the voltage on the capacitor is always continuous";
  – Also points out that the current "flows" through the capacitor is proportional to the capacitance and the changing rate of the voltage on the capacitor.

3. Inductors

- An inductor, also called a coil, is a passive two-terminal electrical component that stores energy in a magnetic field when electric current flows through it.





3.1 Self-inductance

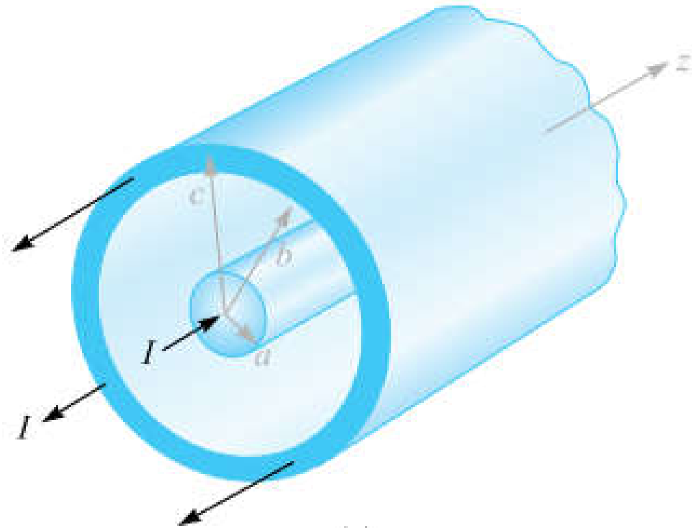
- Flux linkage:
  - Consider a toroid of N turns in which a current I produces a total flux Φ. We assume first that this flux links or encircles each of the N turns, and we also see that each of the N turns links the total flux Φ. The flux linkage $N\Phi$ is defined as the product of the number of turns N and the flux Φ linking each of them.
  - For a single turn coil, the flux linkage is equal to the total flux.
- Inductance(or self-inductance) is defined as the ratio of the total flux linkages to the current which they link

$$L = \frac{N\Phi}{I}$$

  - This definition is for linear materials (not ferromagnetic materials)
  - SI unit for inductance: H (Henry) = 1 Weber-turn per ampere.

3.1 Self-inductance

Example 1



- Calculate the inductance per meter length of a coaxial cable of inner radius *a* and outer radius *b*
- Solution:
  - The magnetic field intensity should be

$$H_\phi = \frac{I}{2\pi\rho} \quad (a < \rho < b) \quad \Longrightarrow \quad \mathbf{B} = \mu_0 \mathbf{H} = \frac{\mu_0 I}{2\pi\rho}\mathbf{a}_\phi$$

  - The magnetic flux contained between the conductors in a length *d* is

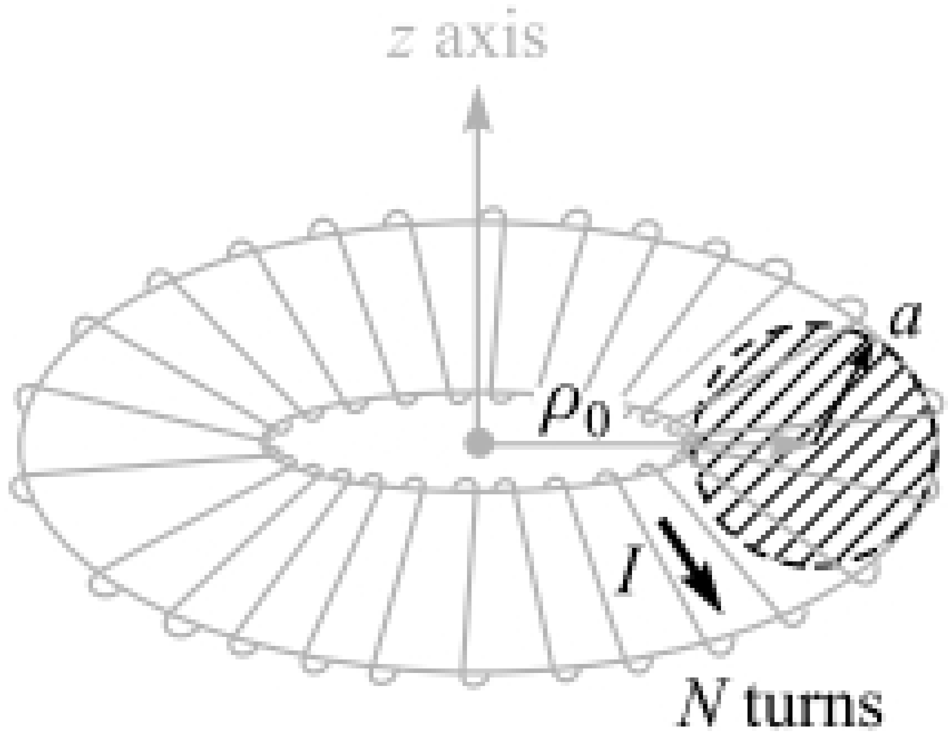
$$\Phi = \int_S \mathbf{B}\cdot d\mathbf{S} = \int_0^d \int_a^b \frac{\mu_0 I}{2\pi\rho}\mathbf{a}_\phi \cdot d\rho\, dz\, \mathbf{a}_\phi = \frac{\mu_0 I d}{2\pi}\ln\frac{b}{a}$$

  - So the inductance rapidly for a length *d* is $L = \frac{\mu_0 d}{2\pi}\ln\frac{b}{a}$ H
  - or, on a per-meter basis $L = \frac{\mu_0}{2\pi}\ln\frac{b}{a}$ H/m

3.1 Self-inductance

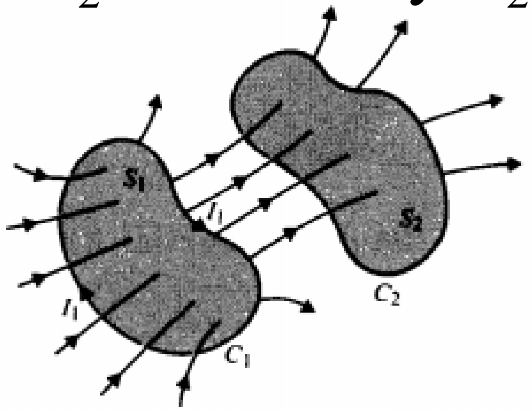
Example 2

- A toroid with a cross-section radius of a, consisting of N closely wound turns of a wire that carries a current I is shown below.
  - Determine the magnetic field intensity at points within the toroid;
  - Determine the inductance of the toroid.



3.2 Mutual inductance

– Consider two neighboring closed loops C_1 and C_2 bounding surfaces S_1 and S_2. If a current I_1 flows in C_1, a magnetic field B_1 will be created and its flux will pass through the surface S_2 bounded by C_2.



- The mutual flux Φ_{12} is defined as:

$$\Phi_{12} = \int_{S_2} \boldsymbol{B_1} \cdot d\boldsymbol{s_2} = L_{12} I_1$$

– For the Biot-Savart law, we know that B_1 is proportional to I_1, hence Φ_{12} is also proportional to I_1, the proportionality constant L_{12} is called the mutual inductance.

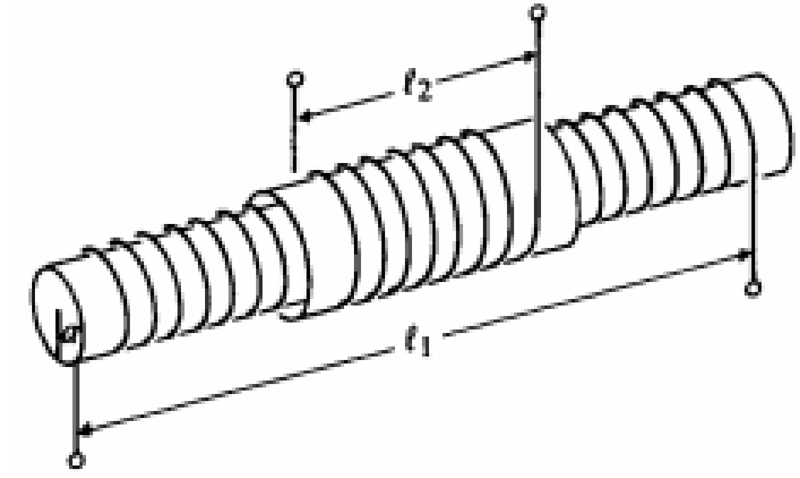
- Neumann formula: the mutual inductance can be calculated:

$$\boxed{L_{12} = \frac{\mu_0}{4\pi} \oint_{C_1} \oint_{C_2} \frac{dl_1 \cdot dl_2}{R}}$$

3.2 Mutual inductance

Example

- Two coils of N_1 and N_2 turns are wound concentrically on a straight cylindrical core of radius a and permeability μ. The windings have lengths l_1 and l_2, respectively.
- Find the mutual inductance between the coils.



3.3 Energy in a magnetic field

- In an electric field
  - The energy density:

$$w_e = \frac{1}{2}\vec{\mathbf{D}}\cdot\vec{\mathbf{E}}$$

  - The total electric energy stored in a medium:

$$W_e = \frac{1}{2}\int_v \vec{\mathbf{D}}\cdot\vec{\mathbf{E}}\,dv$$

  - In a capacitor

$$W_C = \frac{1}{2}CV^2$$

- In a magnetic field
  - The energy density:

$$w_m = \frac{1}{2}\vec{\mathbf{B}}\cdot\vec{\mathbf{H}}$$

$$= \frac{1}{2}\mu H^2 = \frac{1}{2\mu}B^2$$

  - The total electric energy stored in a medium:

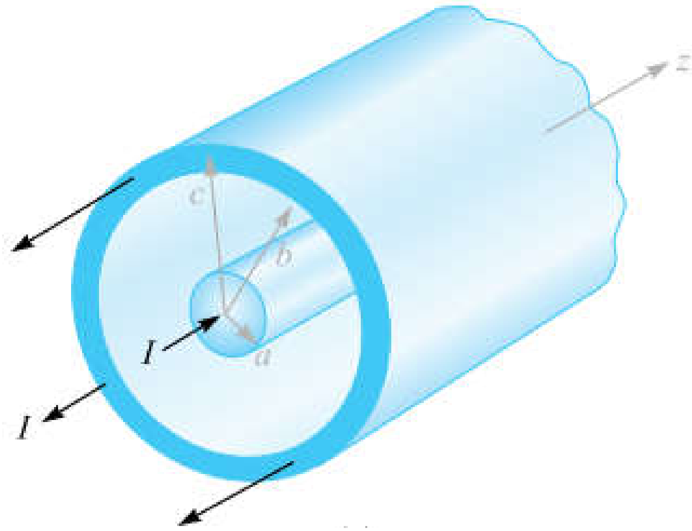
$$W_m = \int_v w_m\,dv$$

  - In an inductor:

$$W_L = \frac{1}{2}LI^2$$

3.3 Energy in a magnetic field

Example



- Calculate the energy stored in a unit-length coaxial cable of inner radius *a* and outer radius *b*
- Solution:
  - The magnetic field intensity should be

$$H_\phi = \frac{I}{2\pi\rho} \quad (a < \rho < b)$$

  - So the total energy stored in this coaxial cable is

$$W_m = \int_v \frac{1}{2}\mu_0 H^2 dv = \frac{\mu_0}{2}\int_0^1 dz \int_0^{2\pi}\int_a^b \left(\frac{I}{2\pi\rho}\right)^2 \rho d\varphi d\rho = \frac{\mu_0 I^2}{4\pi} ln\left(\frac{b}{a}\right) \ (W)$$

  - which agrees with the result calculated from $W_L = \frac{1}{2}LI^2$

3.4 Current – voltage relationship

- Any change in the current through an inductor creates a changing flux, inducing a voltage across the inductor. By Faraday's law of induction, the voltage induced by any change in magnetic flux through the circuit is given by:

$$\mathcal{E} = -\frac{d\Phi_B}{dt}$$

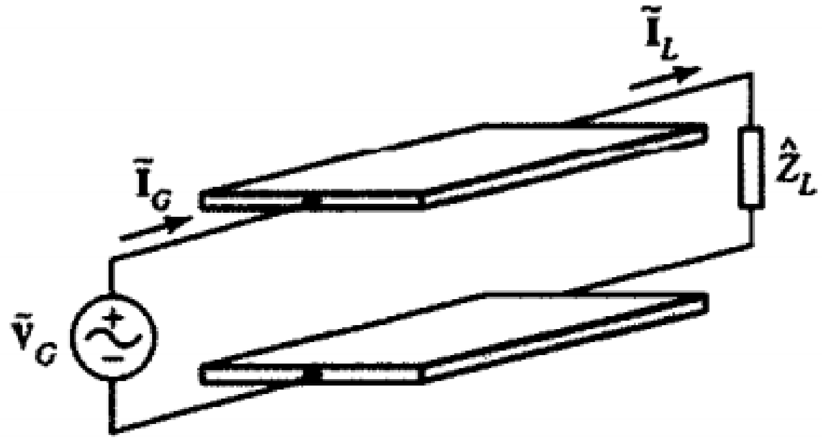
- And we know that $\Phi_B = LI$, substitute into the equation above, get:

$$\mathcal{E} = -\frac{d(LI)}{dt} = -L\frac{dI}{dt}$$

  – which means "the current on the inductor is always continuous";
  – Also points out that the voltage "induced" on the inductor is proportional to the inductance and the changing rate of the current flowing through the inductor.

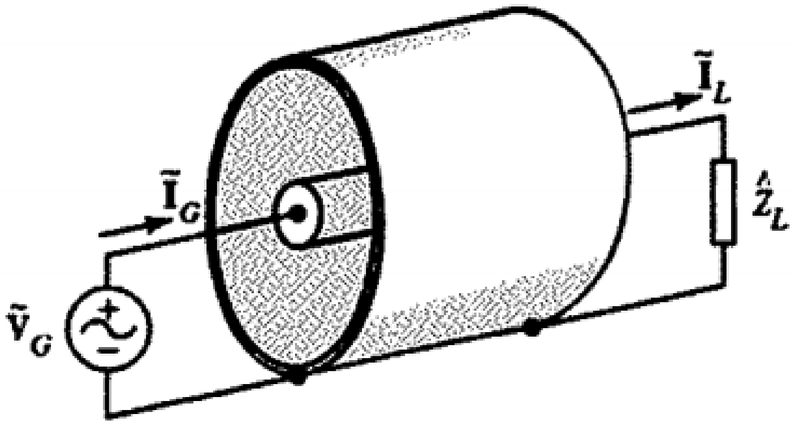
4.1 Case Study – Parallel plates

- For a pair of parallel plates, determine the resistance, capacitance, inductance and the leakage resistance (also called admittance).



4.2 Case Study – Coaxial cable

- For a piece of coaxial cable, determine the resistance, capacitance, inductance and the leakage resistance (also called admittance).



Next

- Electrical circuit
  - Review the electric circuital properties of R, L and C
  - Review the AC circuit evaluation
  - Introduce “frequency response”