Xi'an Jiaotong-Liverpool University
西交利物浦大学

EEE204 Continuous and Discrete Time Signals and Systems II

2018–2019 Semester 2

Electrical and Electronic Engineering

Xi'an Jiaotong-Liverpool University

Week 5

Causality, memory, stability and LTI?

$y[k] = ax[k] + b$

Linearity: For $x_1[k]$ applied as the input, the output $y_1[k]$ is given by

$$y_1[k] = ax_1[k] + b.$$

For $x_2[k]$ applied as the input, the output $y_2[k]$ is given by

$$y_2[k] = ax_2[k] + b.$$

For $x_3[k] = \alpha x_1[k] + \beta x_2[k]$ applied as the input, the output $y_3[k]$ is given by

$$y_3[k] = ax_3[k] + b = a(\alpha x_1[k] + \beta x_2[k]) + b.$$

$$y[k] = ax[k] + b$$

$$\begin{aligned} y_3[k] &= a(\alpha x_1[k] + \beta x_2[k]) + b, \\ &= \alpha(ax_1[k] + b) + \beta(ax_2[k] + b) + (1 - \alpha - \beta)b, \\ &\neq \alpha y_1[k] + \beta y_2[k]. \end{aligned}$$

Therefore, the system is NOT linear.

The invariance: For inputs $x_1[k]$ and $x_2[k] = x_1[k - k_0]$, the outputs are given by

$$\begin{aligned} x_1[k] \rightarrow y_1[k] = ax_1[k] + b, \\ x_2[k] \rightarrow y_2[k] = ax_2[k] + b. \end{aligned}$$

The second equation implies that

$$y_2[k] = ax_1[k - k_0] + b.$$

Causality, memory, stability and LTI?

$y[k] = ax[k] + b$

We also notice that

$$y_1[k-k_0] = ax_1[k-k_0] + b = y_2[k].$$

The system is time-invariant.

Stability: Assuming that the input is bounded $|x[k]| \leqslant B_x$, the output

$$\begin{aligned} |y[k]| &= |ax[k] + b|, \\ &\leqslant |a||x[k]| + |b|, \\ &\leqslant |a|B_x + |b|, \end{aligned}$$

is also bounded. Therefore, the system is BIBO stable.

Causality, memory, stability and LTI?

$y[k] = ax[k] + b$

Causality: Since the output does not require future values of the input, the system is causal.

Memory: Since the output requires only current values of the input, the system is memoryless.

Causality, memory, stability and LTI?

$y[k] = 5x[3k-2]$

Linearity: For $x_1[k]$ applied as the input, the output $y_1[k]$ is given by

$$y_1[k] = 5x_1[3k-2].$$

For $x_2[k]$ applied as the input, the output $y_2[k]$ is given by

$$y_2[k] = 5x_2[3k-2].$$

For $x_3[k] = ax_1[k] + bx_2[k]$ applied as the input, the output $y_3[k]$ is given by

$$y_3[k] = 5x_3[3k-2] = 5(ax_1[3k-2] + bx_2[3k-2]).$$

Causality, memory, stability and LTI?

$y[k] = 5x[3k-2]$

$$\begin{aligned} y_3[k] &= 5(ax_1[3k-2] + bx_2[3k-2]), \\ &= a(5x_1[3k-2]) + b(5x_2[3k-2]), \\ &= ay_1[k] + by_2[k]. \end{aligned}$$

Therefore, the system is linear.

The invariance: For inputs $x_1[k]$ and $x_2[k] = x_1[k - k_0]$, the outputs are given by

$$\begin{aligned} x_1[k] \rightarrow y_1[k] = 5x_1[3k-2], \\ x_2[k] \rightarrow y_2[k] = 5x_2[3k-2]. \end{aligned}$$

The second equation implies that

$$y_2[k] = 5x_1[3k - 2 - k_0].$$

Causality, memory, stability and LTI?

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$y[k] = 5x[3k-2]$

We also notice that

$$y_1[k-k_0] = 5x_1[3(k-k_0)-2] \neq y_2[k].$$

The system is NOT time-invariant.

Stability: Assuming that the input is bounded $|x[k]| \leqslant B_x$, the output

$$\begin{aligned} |y[k]| &= |5x[3k-2]|, \\ &= 5|x[3k-2]|, \\ &\leqslant 5B_x, \end{aligned}$$

is also bounded. Therefore, the system is BIBO stable.

$y[k] = 5x[3k-2]$

Causality: Since the output requires future values of the input when $k \geqslant 2$, the system is NOT causal.

Memory: Since the output requires both previous and future values of the input, the system is NOT memory-less.

$y[k] = 2^{x[k]}$

Linearity: For $x_1[k]$ applied as the input, the output $y_1[k]$ is given by

$$y_1[k] = 2^{x_1[k]}.$$

For $x_2[k]$ applied as the input, the output $y_2[k]$ is given by

$$y_2[k] = 2^{x_2[k]}.$$

For $x_3[k] = ax_1[k] + bx_2[k]$ applied as the input, the output $y_3[k]$ is given by

$$y_3[k] = 2^{x_3[k]} = 2^{ax_1[k]+bx_2[k]}.$$

Causality, memory, stability and LTI?

$y[k] = 2^{x[k]}$

$$\begin{aligned} y_3[k] &= 2^{ax_1[k]+bx_2[k]}, \\ &= \left(2^{x_1[k]}\right)^a \cdot \left(2^{x_2[k]}\right)^b, \\ &\neq ay_1[k] + by_2[k]. \end{aligned}$$

Therefore, the system is NOT linear.

The invariance: For inputs $x_1[k]$ and $x_2[k] = x_1[k-k_0]$, the outputs are given by

$$\begin{aligned} x_1[k] &\to y_1[k] = 2^{x_1[k]}, \\ x_2[k] &\to y_2[k] = 2^{x_2[k]}. \end{aligned}$$

The second equation implies that

$$y_2[k] = 2^{x_1[k-k_0]}.$$

Causality, memory, stability and LTI?

$y[k] = 2^{x[k]}$

We also notice that

$$y_1[k - k_0] = 2^{x_1[k-k_0]} = y_2[k].$$

The system is time-invariant.

Stability: Assuming that the input is bounded $|x[k]| \leqslant B_x$, the output

$$\begin{aligned} |y[k]| &= |2^{x[k]}|, \\ &\leqslant 2^{|x[k]|}, \\ &\leqslant 2^{B_x}, \end{aligned}$$

is also bounded. Therefore, the system is BIBO stable.

Causality, memory, stability and LTI?

$y[k] = 2^{x[k]}$

Causality: Since the output requires only the current values of the input, the system is causal.

Memory: Since the output requires only the current values of the input, the system is memoryless.

Causality, memory, stability and LTI?

$y[k] = \sum_{m=-\infty}^{k} x[m]$

Linearity: For $x_1[k]$ applied as the input, the output $y_1[k]$ is given by

$$y_1[k] = \sum_{m=-\infty}^{k} x_1[m].$$

For $x_2[k]$ applied as the input, the output $y_2[k]$ is given by

$$y_2[k] = \sum_{m=-\infty}^{k} x_2[m].$$

For $x_3[k] = ax_1[k] + bx_2[k]$ applied as the input, the output $y_3[k]$ is given by

$$y_3[k] = \sum_{m=-\infty}^{k} x_3[m] = \sum_{m=-\infty}^{k} (ax_1[m] + bx_2[m]).$$

Causality, memory, stability and LTI?

$y[k] = \sum_{m=-\infty}^{k} x[m]$

$$
\begin{aligned}
y_3[k] &= \sum_{m=-\infty}^{k} (ax_1[m] + bx_2[m]), \\
&= a \sum_{m=-\infty}^{k} x_1[m] + b \sum_{m=-\infty}^{k} x_2[m], \\
&= ay_1[k] + by_2[k].
\end{aligned}
$$

Therefore, the system is linear.

Causality, memory, stability and LTI?

$y[k] = \sum_{m=-\infty}^{k} x[m]$

The invariance: For inputs $x_1[k]$ and $x_2[k] = x_1[k - k_0]$, the outputs are given by

$$x_1[k] \to y_1[k] = \sum_{m=-\infty}^{k} x_1[m],$$

$$x_2[k] \to y_2[k] = \sum_{m=-\infty}^{k} x_2[m].$$

The second equation implies that

$$y_2[k] = \sum_{m=-\infty}^{k} x_1[m - k_0].$$

Causality, memory, stability and LTI?

$y[k] = \sum_{m=-\infty}^{k} x[m]$

We also notice that

$$y_1[k-k_0] = \sum_{m=-\infty}^{k-k_0} x_1[m] = y_2[k].$$

The system is time-invariant. (Refer to Slide 16 in Week 4.)

Stability: Assuming that the input is bounded $|x[k]| \leqslant B_x$, the output

$$|y[k]| = \left| \sum_{m=-\infty}^{k} x[m] \right|,$$

$$\leqslant \sum_{m=-\infty}^{k} |x[m]|,$$

may become unbounded. Therefore, the system is NOT BIBO stable.

Causality, memory, stability and LTI?

$y[k] = \sum_{m=-\infty}^{k} x[m]$

Causality: Since the output does not require future the current values of the input, the system is causal.

Memory: Since the output requires the previous values of the input, the system is NOT memoryless.

Discrete-time(DT) LTI Systems

We know

$$
\begin{aligned}
y[n] &= T\{x[n]\}, \\
&= T\left\{\sum_{k=-\infty}^{+\infty} x[k]\delta[n-k]\right\}, \text{Slide 28 of Week 1} \\
&= \sum_{k=-\infty}^{+\infty} x[k] \cdot T\left\{\delta[n-k]\right\}, \\
&= \sum_{k=-\infty}^{+\infty} x[k]h[n-k], \text{Assume } T\{\cdot\} \text{ is LTI} \\
&= x[n] * h[n]. \ h[n] \text{ is called the impulse response}
\end{aligned}
$$

- $y[n] = x[n] * h[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$ is referred to as the convolution sum.
- Given $h[n]$, it is possible to calculate the output $y[n]$ due to any input $x[n]$ using the convolution sum
- An LTI system is completely characterised by its impulse response $h[n]$.

Convolution Sum

$y[n] = x[n] * h[n]$

What is the expression for $y[n - n_0]$?

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k],$$

$$\begin{aligned} y[n-n_0] &= \sum_{k=-\infty}^{+\infty} x[k]h[n-n_0-k], \\ &= x[n] * h[n-n_0]. \end{aligned}$$

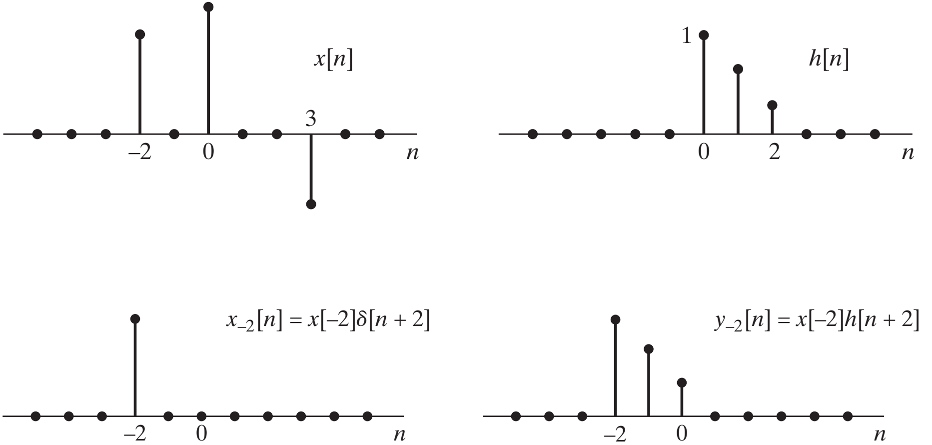
Convolution Sum

$y[n] = x[n] * h[n]$

- $y[n] = x[n] * h[n]$ is shorthand notation for $\sum_{k=-\infty}^{+\infty} x[k]h[n-k]$ and any use of the shorthand form should be referred back to the full expression of convolution sum.
- Blindly trying the substitution or any other transformation may lead to wrong answers.

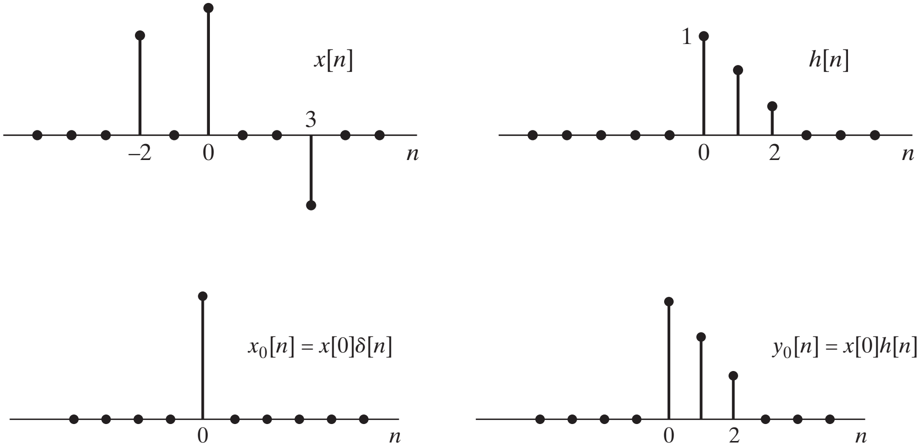
Example

Graphical Approach I



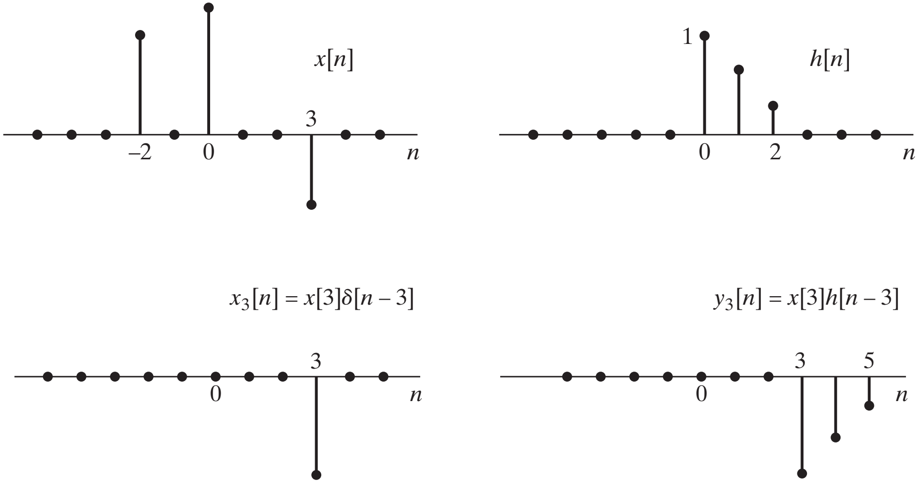
Example

Graphical Approach I



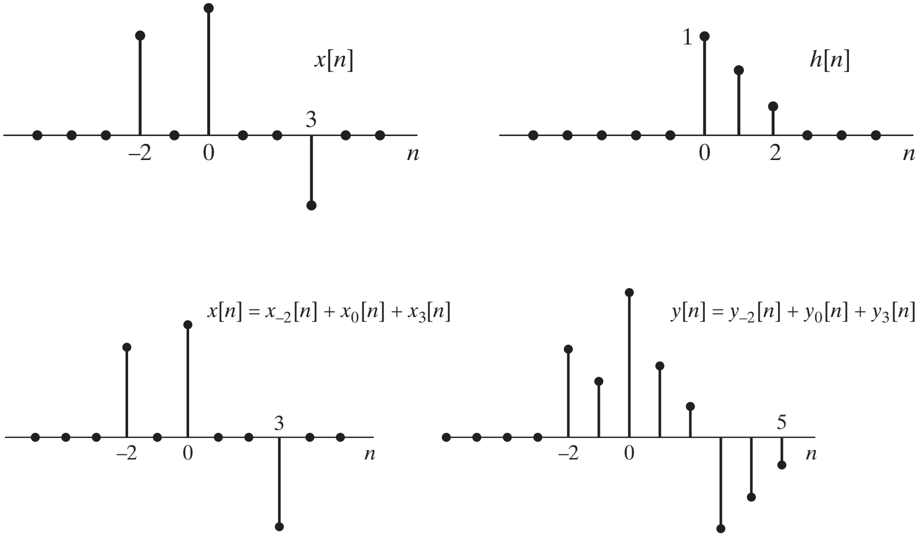
Example

Graphical Approach I



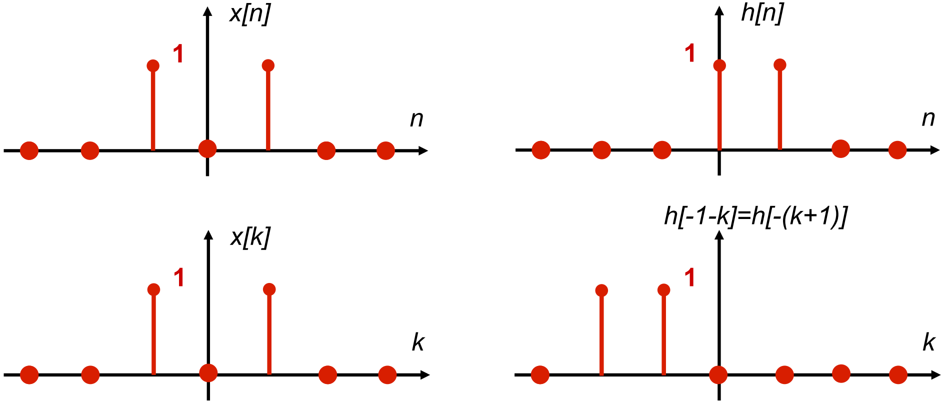
Example

Graphical Approach I



Example

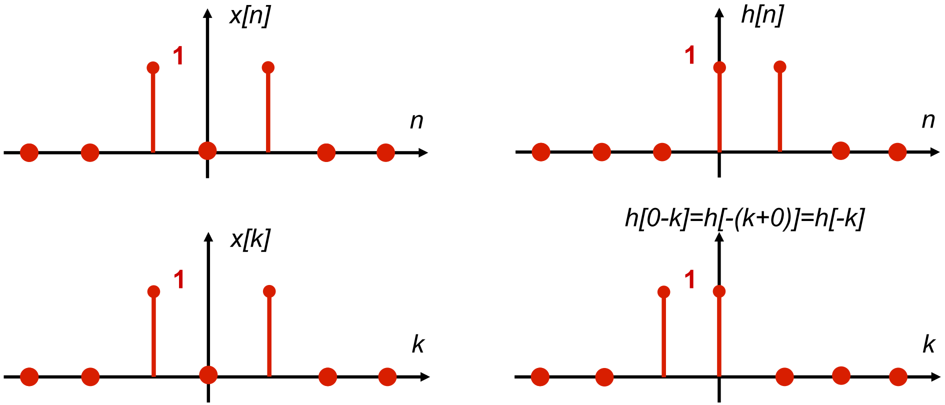
Graphical Approach II



$$y[-1] = 1$$

Example

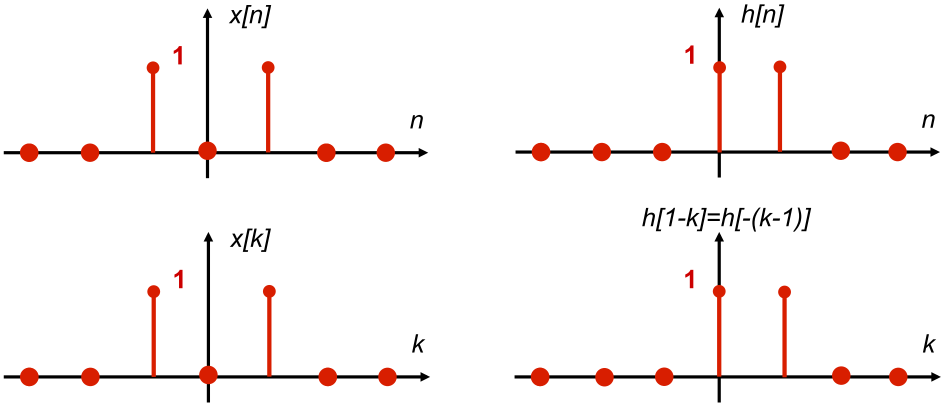
Graphical Approach II



$$y[0] = 1$$

Example

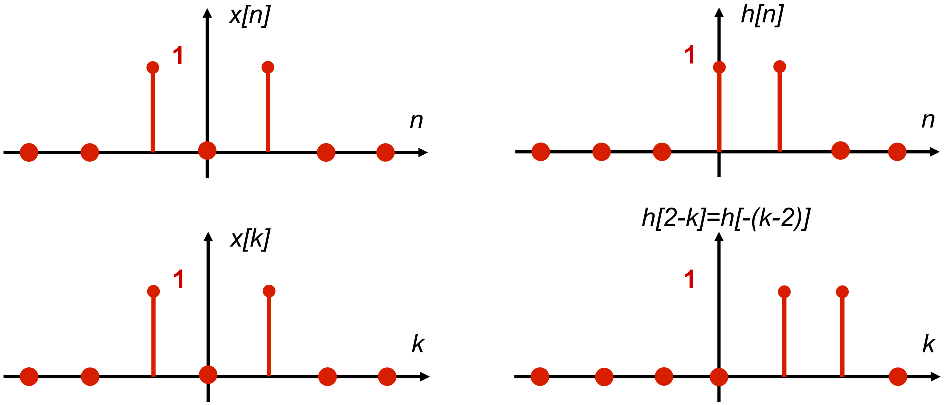
Graphical Approach II



$$y[1] = 1$$

Example

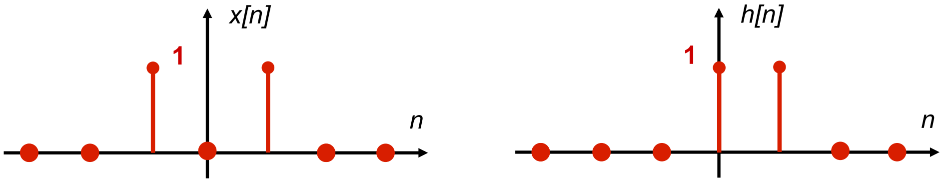
Graphical Approach II

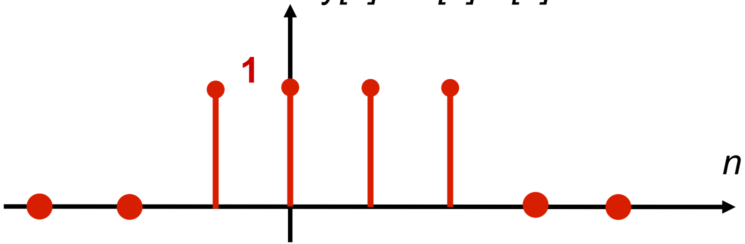


$$y[2] = 1$$

Example

Graphical Approach II





$$y[n] = u[n+1] - u[n-3]$$

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Analytical Approach

Consider a system with impulse response

$$h[n] = u[n] - u[n-N].$$

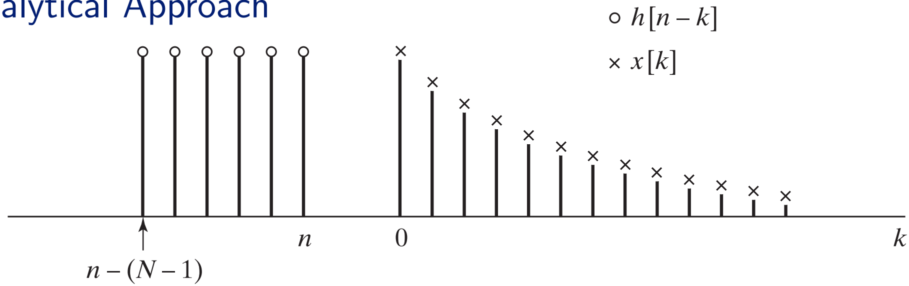
The input is

$$x[n] = a^n u[n].$$

Find the output $y[n]$ at a particular index n.

Example

Analytical Approach



All negative values of n give a similar picture; i.e., the nonzero portions of the sequences $x[k]$ and $h[n-k]$ do not overlap, so

$$y[n] = 0, \text{ for } n < 0.$$

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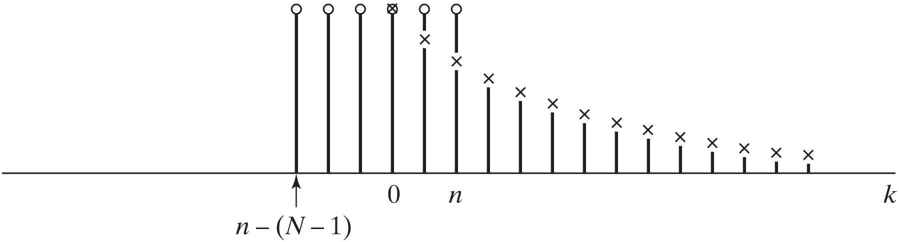
Analytical Approach

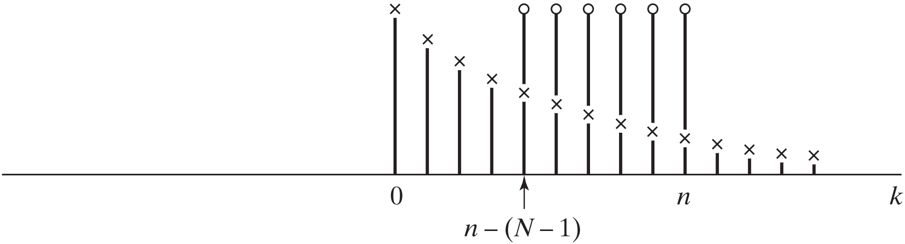


$$y[n] = \sum_{k=0}^{n} x[k]h[n-k],$$

$$= \sum_{k=0}^{n} a^k = \frac{1-a^{n+1}}{1-a}, \text{ for } 0 \leqslant n \leqslant N-1.$$

Example

Analytical Approach

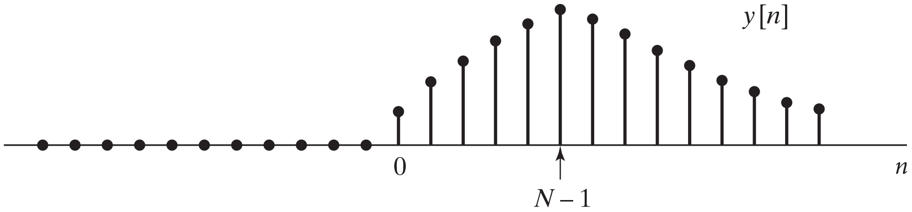


$$y[n] = \sum_{k=n-N+1}^{n} x[k]h[n-k],$$

$$= \sum_{k=n-N+1}^{n} a^k = \frac{a^{n-N+1}(1-a^N)}{1-a}, \text{ for } n > N-1.$$

Example

Analytical Approach



$$y[n] = \begin{cases} 0, & n < 0, \\ \dfrac{1-a^{n+1}}{1-a}, & 0 \leqslant n \leqslant N-1, \\ \dfrac{a^{n-N+1}(1-a^N)}{1-a}, & n > N-1. \end{cases}$$

Homework

- Page 74–90, 103–116 read section 2.0–2.1, 2.3;
- Page 137, Q2.1: (a)–(c);
- Page 138, Q2.2;
- Page 138, Q2.3;
- Page 138, Q2.4;
- Page 138, Q2.5;
- Page 138, Q2.6;
- Page 138–139, Q2.7: (a)–(d).

Thank you for your attention.