Xi'an Jiaotong-Liverpool University
西交利物浦大学

EEE204 Continuous and Discrete Time Signals and Systems II

2018–2019 Semester 2

Electrical and Electronic Engineering

Xi'an Jiaotong-Liverpool University

Week 7

Find z-transform of the DT Signal

$$x_1[k] = \begin{cases} 1, & k = 10, 11 \\ 2, & k = 12, 15 \\ 0, & \text{otherwise.} \end{cases}$$

$$= \delta[k-10] + \delta[k-11] + 2\delta[k-12] + 2\delta[k-15].$$

$$X_1(z) = \sum_{k=-\infty}^{+\infty} x_1[k] z^{-k} = \sum_{k=-\infty}^{+\infty} (\delta[k-10] + \delta[k-11]$$

$$+ 2\delta[k-12] + 2\delta[k-15]) z^{-k},$$

$$= z^{-10} + z^{-11} + 2z^{-12} + 2z^{-15}.$$

The ROC is $\mathbb{C} - \{0\}$.

Find z-transform of the DT Signal

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$x_2[k] = 3^{-k+2}u[k] + \sum_{m=1}^{4} m\delta[k-m]$

$$X_2(z) = \sum_{k=-\infty}^{+\infty} \left(3^{-k+2}u[k] + \sum_{m=1}^{4} m\delta[k-m] \right) z^{-k},$$

$$= \sum_{k=0}^{+\infty} 3^{-k+2} z^{-k} + \sum_{k=-\infty}^{+\infty} (\delta[k-1] + 2\delta[k-2]$$

$$+ 3\delta[k-3] + 4\delta[k-4])z^{-k},$$

$$= 9\sum_{k=0}^{+\infty} (3z)^{-k} + z^{-1} + 2z^{-2} + 3z^{-3} + 4z^{-4},$$

Find z-transform of the DT Signal

$x_2[k] = 3^{-k+2}u[k] + \sum_{m=1}^{4} m\delta[k-m]$

$$= 9\sum_{k=0}^{+\infty}(3z)^{-k} + z^{-1} + 2z^{-2} + 3z^{-3} + 4z^{-4},$$

$$= \frac{9}{1-(3z)^{-1}} + z^{-1} + 2z^{-2} + 3z^{-3} + 4z^{-4}.$$

The ROC is $|(3z)^{-1}| < 1 \cap |z| \neq 0$, which can be simplified to $|z| > \frac{1}{3}$.

More on ROC

$Z(a^n u[n]) = \dfrac{z}{z-a}$ and the ROC is $|z| > |a|$, the exterior of circle.

$Z(-a^n u[-n-1]) = \dfrac{z}{z-a}$ and the ROC is $|z| < |a|$, the interior of circle.

What is the general shape?

The ROC is always an annulus, i.e., $\{r_2 < |z| < r_1\}$.

Note that r_2 can be zero (possibly with $\leqslant$) and r_1 can be ∞ (possibly with $\leqslant$).

More on ROC

Explanation. Let $z = re^{j\theta}$ be polar form

$$
\begin{aligned}
|X(z)| &= \left| \sum_{n=-\infty}^{+\infty} x[n] z^{-n} \right|, \\
&\leqslant \sum_{n=-\infty}^{+\infty} |x[n]||z|^{-n} = \sum_{n=-\infty}^{+\infty} |x[n]| r^{-n}, \\
&= \sum_{n=-\infty}^{-1} |x[n]| r^{-n} + \sum_{n=0}^{+\infty} |x[n]| r^{-n}, \\
&= \sum_{n=1}^{+\infty} |x[-n]| r^{n} + \sum_{n=0}^{+\infty} \frac{|x[n]|}{r^n}.
\end{aligned}
$$

More on ROC

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$$|X(z)| = \sum_{n=1}^{+\infty} |x[-n]| r^n + \sum_{n=0}^{+\infty} \frac{|x[n]|}{r^n}$$

The ROC is the subset of $\mathbb{C}$ where both of the above sums are finite.

If the right sum (the "causal part") is finite for some z_2 with magnitude $r_2 = |z_2|$, then that sum will also be finite for any z with magnitude $r \geqslant r_2$, since for such an r each term in the sum is smaller.

So the ROC for the right sum is the subset of $\mathbb{C}$ for which $|z| > r_2$, for some r_2, which is the exterior of some circle.

More on ROC

$$|X(z)| = \sum_{n=1}^{\infty} |x[-n]|r^n + \sum_{n=0}^{\infty} \frac{|x[n]|}{r^n}$$

The ROC is the subset of $\mathbb{C}$ where both of the above sums are finite.

Likewise if the left sum (the "anti-causal part") is finite for some z_1 with magnitude $r_1 = |z_1|$, then that sum will also be finite for any z with magnitude $r \leqslant r_1$, since for such an r each term in the sum is smaller.

So the ROC for the left sum is the subset of $\mathbb{C}$ for which $|z| < r_1$, for some r_1, which is the interior of some circle.

More on ROC

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$|X(z)| = \sum_{n=1}^{\infty} |x[-n]| r^n + \sum_{n=0}^{\infty} \frac{|x[n]|}{r^n}$

The ROC of a causal signal is the exterior of a circle of some radius r_2.

The ROC of an anti-causal signal is the interior of a circle of some radius r_1.

For a general signal $x[n]$, the ROC will be the intersection of the ROC of its causal and non-causal parts, which is an annulus.

If $r_2 < r_1$, then that intersection is an annulus (nonempty). Otherwise the z-transform is undefined (does not exist).

More on ROC

Example of a signal with empty ROC

$$x[n] = 1 = u[n] + u[-n-1].$$

Recall $u[n] \overset{z}{\leftrightarrow} U(z) = \frac{z}{z-1}$, for $\{|z| > 1\}$,

$$u[-n-1] \overset{z}{\leftrightarrow} U'(z) = \frac{-z}{z-1}, \text{ for } \{|z| < 1\},$$

ROC for the causal part is $\{|z| > 1\}$, ROC for the anti-causal part is $\{|z| < 1\}$. The z-transform does not exist.

Properties of z-transform

Linearity

If $x_1[n] \overset{z}{\leftrightarrow} X_1(z)$ and $x_2[n] \overset{z}{\leftrightarrow} X_2(z)$ then

$$\begin{aligned} Z\{x[n]\} &= Z\{a_1x_1[n] + a_2x_2[n]\}, \\ &= a_1X_1(z) + a_2X_2(z). \end{aligned}$$

The ROC of the sum contains at least as much of the z-plane as the intersection of the two ROC's.

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Linearity

Proof:

$$x[n] = a_1 x_1[n] + a_2 x_2[n] \overset{z}{\leftrightarrow} a_1 X_1(z) + a_2 X_2(z).$$

$$\begin{aligned}
X(z) &= \sum_{n=-\infty}^{+\infty} x[n] z^{-n}, \\
&= \sum_{n=-\infty}^{+\infty} (a_1 x_1[n] + a_2 x_2[n]) z^{-n}, \\
&= a_1 \sum_{n=-\infty}^{+\infty} x_1[n] z^{-n} + a_2 \sum_{n=-\infty}^{+\infty} x_2[n] z^{-n}, \\
&= a_1 X_1(z) + a_2 X_2(z).
\end{aligned}$$

Example

$x[n] = \cos(\omega_0 n + \phi)u[n]$

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n]z^{-n} = \sum_{n=-\infty}^{+\infty} [\cos(\omega_0 n + \phi)u[n]]z^{-n},$$

$$= \sum_{n=0}^{+\infty} [\cos(\omega_0 n + \phi)]z^{-n},$$

$$= \frac{1}{2}e^{j\phi}\sum_{n=0}^{+\infty}(e^{j\omega_0})^n z^{-n} + \frac{1}{2}e^{-j\phi}\sum_{n=0}^{+\infty}(e^{-j\omega_0})^n z^{-n},$$

$$= \frac{\frac{1}{2}e^{j\phi}}{1 - e^{j\omega_0}z^{-1}} + \frac{\frac{1}{2}e^{-j\phi}}{1 - e^{-j\omega_0}z^{-1}},$$

Example

$x[n] = \cos(\omega_0 n + \phi)u[n]$

$$
\begin{aligned}
X(z) &= \frac{\frac{1}{2}e^{j\phi}}{1 - e^{j\omega_0}z^{-1}} + \frac{\frac{1}{2}e^{-j\phi}}{1 - e^{-j\omega_0}z^{-1}}, \\
&= \frac{\frac{1}{2}e^{j\phi}(1 - e^{-j\omega_0}z^{-1}) + \frac{1}{2}e^{-j\phi}(1 - e^{j\omega_0}z^{-1})}{(1 - e^{j\omega_0}z^{-1})(1 - e^{-j\omega_0}z^{-1})}, \\
&= \frac{\frac{1}{2}e^{j\phi} + \frac{1}{2}e^{-j\phi} - \frac{1}{2}e^{j\phi}e^{-j\omega_0}z^{-1} - \frac{1}{2}e^{-j\phi}e^{j\omega_0}z^{-1}}{1 - e^{j\omega_0}z^{-1} - e^{-j\omega_0}z^{-1} + z^{-2}}, \\
&= \frac{\cos\phi - z^{-1}\cos(\omega_0 - \phi)}{1 - 2z^{-1}\cos\omega_0 + z^{-2}}.
\end{aligned}
$$

The ROC is $|e^{j\omega_0}z^{-1}| < 1$ and $|e^{-j\omega_0}z^{-1}| < 1$, which yields $|z| > 1$.

Time Shifting

If $x[n] \overset{z}{\leftrightarrow} X(z)$,then

$$x[n-k] \overset{z}{\leftrightarrow} z^{-k}X(z).$$

The ROC is unchanged, except for adding or deleting $|z| = 0$ or $|z| = \infty$.

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Time Shifting

Proof: $x[n-k] \overset{z}{\leftrightarrow} z^{-k}X(z)$.

$$
\begin{aligned}
Z(x[n-k]) &= \sum_{n=-\infty}^{+\infty} x[n-k]z^{-n}, \\
&\xlongequal{m=n-k} \sum_{m=-\infty}^{+\infty} x[m]z^{-k-m}, \\
&= z^{-k}\sum_{m=-\infty}^{+\infty} x[m]z^{-m}, \\
&= z^{-k}X(z).
\end{aligned}
$$

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Scaling the z-domain

If $x[n] \overset{z}{\leftrightarrow} X(z)$ with ROC$= \{r_1 < |z| < r_2\}$, then

$$a^n x[n] \overset{z}{\leftrightarrow} X(a^{-1}z),$$

with ROC$= \{|a|r_1 < |z| < |a|r_2\}$.

Scaling the z-domain

Proof: $a^n x[n] \overset{z}{\leftrightarrow} X(a^{-1}z)$.

$$\begin{aligned} Z(a^n x[n]) &= \sum_{n=-\infty}^{\infty} a^n x[n] z^{-n}, \\ &= \sum_{n=-\infty}^{\infty} x[n](a^{-1}z)^{-n}, \\ &= X(a^{-1}z). \end{aligned}$$

Example

$x[n] = \frac{1}{3^n}\cos(\omega_0 n)u[n]$

$$Z(\cos(\omega_0 n + \phi)u[n]) = \frac{\cos\phi - z^{-1}\cos(\omega_0 - \phi)}{1 - 2z^{-1}\cos\omega_0 + z^{-2}},$$

$$\phi = 0, Z(\cos(\omega_0 n)u[n]) = \frac{1 - z^{-1}\cos\omega_0}{1 - 2z^{-1}\cos\omega_0 + z^{-2}},$$

$$\because a = \frac{1}{3}, \therefore X(z) = \frac{1 - \frac{1}{3}z^{-1}\cos\omega_0}{1 - \frac{2}{3}z^{-1}\cos\omega_0 + \frac{1}{9}z^{-2}}.$$

The ROC is $|z| > \frac{1}{3}$.

If $x[n] \overset{z}{\leftrightarrow} X(z)$ with ROC$= \{r_1 < |z| < r_2\}$, then

$$x[-n] \overset{z}{\leftrightarrow} X(z^{-1}),$$

with ROC$= \{1/r_2 < |z| < 1/r_1\}$.

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Time Reversal

Proof: $x[-n] \overset{z}{\leftrightarrow} X(z^{-1})$.

$$
\begin{aligned}
Z(x[-n]) &= \sum_{n=-\infty}^{\infty} x[-n] z^{-n}, \\
&\xlongequal{m=-n} \sum_{m=-\infty}^{\infty} x[m] z^{m}, \\
&= \sum_{m=-\infty}^{\infty} x[m] (z^{-1})^{-m}, \\
&= X(z^{-1}).
\end{aligned}
$$

Differentiation in z-domain

If $x[n] \overset{z}{\leftrightarrow} X(z)$, then

$$nx[n] \overset{z}{\leftrightarrow} -z\frac{\mathrm{d}}{\mathrm{d}z}X(z),$$

The ROC is unchanged.

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Differentiation in z-domain

Proof: $nx[n] \overset{z}{\leftrightarrow} -z\dfrac{\mathrm{d}}{\mathrm{d}z}X(z)$.

$$Z(nx[n]) = \sum_{n=-\infty}^{\infty} nx[n]z^{-n} = z\sum_{n=-\infty}^{\infty} nx[n]z^{-n-1},$$

$$= z\frac{\mathrm{d}}{\mathrm{d}z}\sum_{n=-\infty}^{\infty}\int nx[n]z^{-n-1}\,\mathrm{d}z,$$

$$= z\frac{\mathrm{d}}{\mathrm{d}z}\sum_{n=-\infty}^{\infty} -x[n]z^{-n},$$

$$= -z\frac{\mathrm{d}}{\mathrm{d}z}X(z).$$

Example

$x[n] = nu[n]$(unit ramp signal)

$$\begin{aligned} X(z) &= -z \frac{\mathrm{d}}{\mathrm{d}z} U(z), \\ &= -z \left(\frac{z}{z-1} \right)', \\ &= -z \frac{-1}{(z-1)^2}, \\ &= \frac{z}{(z-1)^2}, \end{aligned}$$

The ROC is $|z| > 1$.

Convolution

If $x_1[n] \stackrel{z}{\leftrightarrow} X_1(z)$ and $x_2[n] \stackrel{z}{\leftrightarrow} X_2(z)$ then

$$x[n] = x_1[n] * x_2[n] \stackrel{z}{\leftrightarrow} X_1(z)X_2(z).$$

The ROC of the convolution contains at least as much of the z-plane as the intersection of the ROC of $X_1(z)$ and the ROC of $X_2(z)$.

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Convolution

Proof: $x[n] = x_1[n] * x_2[n] \overset{z}{\leftrightarrow} X_1(z)X_2(z)$.

$$
\begin{aligned}
X(z) &= \sum_{n=-\infty}^{\infty} x[n]z^{-n}, \\
&= \sum_{n=-\infty}^{\infty} \left[\sum_{k=-\infty}^{\infty} x_1[k]x_2[n-k] \right] z^{-n}, \\
&= \sum_{k=-\infty}^{\infty} x_1[k] \left[\sum_{n=-\infty}^{\infty} x_2[n-k]z^{-n} \right], \\
&= \sum_{k=-\infty}^{\infty} x_1[k]z^{-k} X_2(z) = X_1(z)X_2(z).
\end{aligned}
$$

Example

$x[n] = u[n] * u[n-1]$

$$
\begin{aligned}
X(z) &= U(z)Z(u[n-1]), \\
&= \frac{z}{z-1} \cdot z^{-1} \cdot \frac{z}{z-1}, \\
&= \frac{z}{(z-1)^2}.
\end{aligned}
$$

The ROC is $|z| > 1$.

This means $x[n] = u[n] * u[n-1] = nu[n]$.

Example

DT LTI System $y[n] = x[n] * h[n]$

$$h[n] = \delta[n] - \delta[n-1],$$
$$H(z) = 1 - z^{-1},\ \text{ROC} = \mathbb{C} - \{0\}$$
$$x[n] = u[n-2],$$
$$X[z] = z^{-2} \cdot U(z) = \frac{z^{-1}}{z-1},\ \text{ROC} = |z| > 1$$
$$Y[z] = X(z)H(z) = \frac{z^{-1}}{z-1}(1 - z^{-1}) = z^{-2},$$
$$\text{ROC} = \mathbb{C} - \{0\},\ y[n] = \delta[n-2].$$

The ROC of $y[n]$ is bigger than intersection of ROC of $x[n]$ and ROC of $h[n]$.

Understand all the properties in Table 10.1 (P.775)

- Page 748–757, 767–776 read section 10.2 and 10.5 – 10.7 ;
- Page 799, Q10.13;
- Page 799, Q10.14;
- Page 804, Q10.30;
- Page 804, Q10.32;
- Page 804, Q10.33: (b).

Thank you for your attention.