Xi'an Jiaotong-Liverpool University
西交利物浦大學

EEE204 Continuous and Discrete Time Signals and Systems II

2018–2019 Semester 2

Electrical and Electronic Engineering

Xi'an Jiaotong-Liverpool University

Week 3

Find N_0 if Existing

$x_1[k] = 5 \times (-1)^k$

$$\begin{aligned} x_1[k] &= 5 \times (-1)^k, \\ &= 5 \times \cos(k\pi), \\ &= 5 \times \cos\left(2\pi \frac{1}{2} k\right), \end{aligned}$$

Since $f = \frac{1}{2}$, the smallest N to make fN an integer is 2. Thus the signal is periodic with fundamental period N_0 of 2.

Find N_0 if Existing

$x_2[k] = \exp(j(7\pi k/4)) + \exp(j(3k/4))$

$$\begin{aligned}
x_2[k] &= \exp(j(7\pi k/4)) + \exp(j(3k/4)), \\
&= \exp\left(j\left(2\pi\frac{7}{8}k\right)\right) + \exp\left(j\left(2\pi\frac{3}{8\pi}k\right)\right), \\
&= \exp\left(j\left(2\pi f_1 k\right)\right) + \exp\left(j\left(2\pi f_2 k\right)\right).
\end{aligned}$$

For $f_2 = \dfrac{3}{8\pi}$, there does not exist any integer N to make $f_2 N$ an integer. Thus the signal is aperiodic.

Find N_0 if Existing

$x_3[k] = \exp(j(7\pi k/4)) + \exp(j(3\pi k/4))$

$$
\begin{aligned}
x_3[k] &= \exp(j(7\pi k/4)) + \exp(j(3\pi k/4)), \\
&= \exp\left(j\left(2\pi\frac{7}{8}k\right)\right) + \exp\left(j\left(2\pi\frac{3}{8}k\right)\right), \\
&= \exp\left(j\left(2\pi f_1 k\right)\right) + \exp\left(j\left(2\pi f_2 k\right)\right).
\end{aligned}
$$

For both $f_1 = \dfrac{7}{8}$ and $f_2 = \dfrac{3}{8}$, the smallest N to make $f_1 N$ and $f_2 N$ integers is 8. Thus the signal is periodic with fundamental period N_0 of 8.

Find N_0 if Existing

$x_4[k] = \sin(3\pi k/8) + \cos(63\pi k/64)$

$$\begin{aligned} x_4[k] &= \sin(3\pi k/8) + \cos(63\pi k/64), \\ &= \sin\left(2\pi\frac{3}{16}k\right) + \cos\left(2\pi\frac{63}{128}k\right), \\ &= \sin\left(2\pi f_1 k\right) + \cos\left(2\pi f_2 k\right). \end{aligned}$$

For both $f_1 = \dfrac{3}{16}$ and $f_2 = \dfrac{63}{128}$, the smallest N to make $f_1 N$ and $f_2 N$ integers is 128. Thus the signal is periodic with fundamental period N_0 of 128.

Find N_0 if Existing

$x_5[k] = \exp(j(7\pi k/4)) + \cos(4\pi k/7 + \pi)$

$$
\begin{aligned}
x_5[k] &= \exp(j(7\pi k/4)) + \cos(4\pi k/7 + \pi), \\
&= \exp\left(2\pi\frac{7}{8}k\right) + \cos\left(2\pi\frac{2}{7}k + \pi\right), \\
&= \exp\left(2\pi f_1 k\right) + \cos\left(2\pi f_2 k + \pi\right).
\end{aligned}
$$

For both $f_1 = \dfrac{7}{8}$ and $f_2 = \dfrac{2}{7}$, the smallest N to make $f_1 N$ and $f_2 N$ integers is 56. Thus the signal is periodic with fundamental period N_0 of 56.

Find N_0 if Existing

$x_6[k] = \sin(3\pi k/8)\cos(63\pi k/64)$

$$
\begin{aligned}
x_6[k] &= \sin(3\pi k/8)\cos(63\pi k/64), \\
&= \frac{1}{2}[\sin(87\pi k/64) - \sin(39\pi k/64)] \\
&= \frac{1}{2}\left[\sin\left(2\pi\frac{87}{128}k\right) - \sin\left(2\pi\frac{39}{128}k\right)\right], \\
&= \frac{1}{2}\left[\sin\left(2\pi f_1 k\right) - \sin\left(2\pi f_2 k\right)\right],
\end{aligned}
$$

For both $f_1 = \dfrac{87}{128}$ and $f_2 = \dfrac{39}{128}$, the smallest N to make $f_1 N$ and $f_2 N$ integers is 128. Thus the signal is periodic with fundamental period N_0 of 128.

Find the Energy of the Signal

$$x_7[k] = \begin{cases} \cos\left(\frac{3\pi k}{16}\right), & -10 \leqslant k \leqslant 0 \\ 0, & \text{otherwise} \end{cases}$$

$$E = \sum_{k=-10}^{0} \cos^2\left(\frac{3\pi k}{16}\right),$$

$$= \sum_{k=-10}^{0} \frac{1+\cos\left(\frac{3\pi k}{8}\right)}{2},$$

$$= \sum_{k=-10}^{0} \frac{1}{2} + \sum_{k=-10}^{0} \frac{\cos\left(\frac{3\pi k}{8}\right)}{2},$$

Find the Energy of the Signal

$$x_7[k] = \begin{cases} \cos\left(\frac{3\pi k}{16}\right), & -10 \leqslant k \leqslant 0 \\ 0, & \text{otherwise} \end{cases}$$

$$E = \sum_{k=-10}^{0} \frac{1}{2} + \sum_{k=-10}^{0} \frac{\cos\left(\frac{3\pi k}{8}\right)}{2},$$

$$= \frac{11}{2} + \frac{1}{4}\sum_{k=-10}^{0} e^{j\frac{3\pi k}{8}} + \frac{1}{4}\sum_{k=-10}^{0} e^{-j\frac{3\pi k}{8}},$$

$$= \frac{11}{2} + \frac{1}{4}\cdot\frac{e^{-j30\pi/8}(1-e^{j33\pi/8})}{1-e^{j3\pi/8}} + \frac{1}{4}\cdot\frac{e^{j30\pi/8}(1-e^{-j33\pi/8})}{1-e^{-j3\pi/8}},$$

$$= 5.5 + 0.1622 = 5.6622.$$

Find the Power of the Signal

$x_8[k] = \cos(\pi k/4)\sin(3\pi k/8)$

$$\begin{aligned} x_8[k] &= \cos(\pi k/4)\sin(3\pi k/8), \\ &= \frac{1}{2}[\sin(5\pi k/8) + \sin(\pi k/8)] \\ &= \frac{1}{2}\left[\sin\left(2\pi\frac{5}{16}k\right) + \sin\left(2\pi\frac{1}{16}k\right)\right] \end{aligned}$$

It is a period signal with fundamental period of 16. Since periodic signals are always power signals, $x_1[k]$ is a power signal. The average power is given by $(\frac{1}{2})^2/2 + (\frac{1}{2})^2/2 = \frac{1}{4}$. Is this always correct?

Consider the following DT sequence

$$x[n] = A_1 \sin(\omega_1 n + \phi_1) + A_2 \sin(\omega_2 n + \phi_2).$$

Assuming $\omega_1 = \frac{m_1}{N_1} \cdot 2\pi, \omega_2 = \frac{m_2}{N_2} \cdot 2\pi$ and determine the power of the signal $(0 \leqslant m_1 \leqslant N_1; 0 \leqslant m_2 \leqslant N_2; m_1, m_2 \in Z^+ \cup \{0\}; N_1, N_2 \in Z^+)$.

Power of Sum of Sinusoids

1. $m_1 = 0, m_2 = 0$, for $\forall n$

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2,$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} (A_1 \sin\phi_1 + A_2 \sin\phi_2)^2,$$

$$= \lim_{N\to\infty} \frac{2N+1}{2N+1} (A_1 \sin\phi_1 + A_2 \sin\phi_2)^2,$$

$$= A_1^2 \sin^2\phi_1 + A_2^2 \sin^2\phi_2 + 2A_1A_2 \sin\phi_1 \sin\phi_2.$$

Power of Sum of Sinusoids

2. $m_1 = 0, m_2 = 1, N_2 = 1$, for $\forall n$

$$
\begin{aligned}
P &= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2, \\
&= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} (A_1 \sin\phi_1 + A_2 \sin\phi_2)^2, \\
&= \lim_{N\to\infty} \frac{2N+1}{2N+1} (A_1 \sin\phi_1 + A_2 \sin\phi_2)^2, \\
&= A_1^2 \sin^2\phi_1 + A_2^2 \sin^2\phi_2 + 2A_1A_2 \sin\phi_1 \sin\phi_2.
\end{aligned}
$$

General Case

Power of Sum of Sinusoids

3. $m_1 = 1, N_1 = 1, m_2 = 0$, for $\forall n$

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2,$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} (A_1 \sin\phi_1 + A_2 \sin\phi_2)^2,$$

$$= \lim_{N\to\infty} \frac{2N+1}{2N+1} (A_1 \sin\phi_1 + A_2 \sin\phi_2)^2,$$

$$= A_1^2 \sin^2\phi_1 + A_2^2 \sin^2\phi_2 + 2A_1A_2 \sin\phi_1 \sin\phi_2.$$

Power of Sum of Sinusoids

4. $m_1 = 0, m_2 = 1, N_2 = 2$, for $\forall n$, the fundamental period is 2

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2 = \frac{1}{2}\sum_{n=0}^{1} |x[n]|^2,$$

$$= \frac{1}{2}\sum_{n=0}^{1}[A_1 \sin\phi_1 + (-1)^n A_2 \sin\phi_2]^2,$$

$$= \frac{1}{2}(2A_1^2 \sin^2\phi_1 + 2A_2^2 \sin^2\phi_2),$$

$$= A_1^2 \sin^2\phi_1 + A_2^2 \sin^2\phi_2.$$

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Power of Sum of Sinusoids

5. $m_1 = 1, N_1 = 2$, $m_2 = 0$, for $\forall n$, the fundamental period is 2

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2 = \frac{1}{2} \sum_{n=0}^{1} |x[n]|^2,$$

$$= \frac{1}{2} \sum_{n=0}^{1} [(-1)^n A_1 \sin\phi_1 + A_2 \sin\phi_2]^2,$$

$$= \frac{1}{2}(2A_1^2 \sin^2\phi_1 + 2A_2^2 \sin^2\phi_2),$$

$$= A_1^2 \sin^2\phi_1 + A_2^2 \sin^2\phi_2.$$

General Case

Power of Sum of Sinusoids

6. $m_1 = m_2 = 1, N_1 = N_2 = 2$, for $\forall n$

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2,$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} [(-1)^n A_1 \sin\phi_1 + (-1)^n A_2 \sin\phi_2]^2,$$

$$= \lim_{N\to\infty} \frac{2N+1}{2N+1} (A_1 \sin\phi_1 + A_2 \sin\phi_2)^2,$$

$$= A_1^2 \sin^2\phi_1 + A_2^2 \sin^2\phi_2 + 2A_1 A_2 \sin\phi_1 \sin\phi_2.$$

Power of Sum of Sinusoids

7. For $1 \leqslant m_1 < N_1, N_1 > 2, 1 \leqslant m_2 < N_2, N_2 > 2$, then N_1N_2 is the (fundamental) period.

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2 = \frac{1}{N_1N_2} \sum_{n=0}^{N_1N_2-1} |x[n]|^2,$$

$$= \frac{1}{N_1N_2} \sum_{n=0}^{N_1N_2-1} |A_1 \sin(\omega_1 n + \phi_1) + A_2 \sin(\omega_2 n + \phi_2)|^2,$$

$$= \frac{1}{N_1N_2} \sum_{n=0}^{N_1N_2-1} [A_1^2 \sin^2(\omega_1 n + \phi_1) + A_2^2 \sin^2(\omega_2 n + \phi_2)$$

$$+ 2A_1A_2 \sin(\omega_1 n + \phi_1) \sin(\omega_2 n + \phi_2)],$$

Power of Sum of Sinusoids

$$
\begin{aligned}
P &= \frac{1}{N_1N_2}\sum_{n=0}^{N_1N_2-1}[A_1^2\sin^2(\omega_1 n+\phi_1)+A_2^2\sin^2(\omega_2 n+\phi_2) \\
&\quad + 2A_1A_2\sin(\omega_1 n+\phi_1)\sin(\omega_2 n+\phi_2)], \\
&= \frac{1}{N_1N_2}\sum_{n=0}^{N_1N_2-1}A_1^2\sin^2(\omega_1 n+\phi_1) \qquad P_1 \\
&\quad + \frac{1}{N_1N_2}\sum_{n=0}^{N_1N_2-1}A_2^2\sin^2(\omega_2 n+\phi_2) \qquad P_2 \\
&\quad + \frac{1}{N_1N_2}\sum_{n=0}^{N_1N_2-1}2A_1A_2\sin(\omega_1 n+\phi_1)\sin(\omega_2 n+\phi_2), \quad P_3
\end{aligned}
$$

Power of Sum of Sinusoids

$$P = \frac{1}{N_1 N_2} \sum_{n=0}^{N_1 N_2 - 1} [A_1^2 \sin^2(\omega_1 n + \phi_1) + A_2^2 \sin^2(\omega_2 n + \phi_2)$$

$$+ 2A_1 A_2 \sin(\omega_1 n + \phi_1) \sin(\omega_2 n + \phi_2)],$$

$$= \frac{1}{N_1 N_2} \sum_{n=0}^{N_1 N_2 - 1} A_1^2 \sin^2(\omega_1 n + \phi_1) \qquad P_1 = \frac{A_1^2}{2}$$

$$+ \frac{1}{N_1 N_2} \sum_{n=0}^{N_1 N_2 - 1} A_2^2 \sin^2(\omega_2 n + \phi_2) \qquad P_2 = \frac{A_2^2}{2}$$

$$+ \frac{1}{N_1 N_2} \sum_{n=0}^{N_1 N_2 - 1} 2A_1 A_2 \sin(\omega_1 n + \phi_1) \sin(\omega_2 n + \phi_2), \quad P_3?$$

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Power of Sum of Sinusoids

$$P_3 = \frac{1}{N_1N_2}\sum_{n=0}^{N_1N_2-1} 2A_1A_2\sin(\omega_1 n+\phi_1)\sin(\omega_2 n+\phi_2),$$

$$= \frac{A_1A_2}{N_1N_2}\sum_{n=0}^{N_1N_2-1}\{\cos[(\omega_1-\omega_2)n+\phi_1-\phi_2]$$

$$-\cos[(\omega_1+\omega_2)n+\phi_1+\phi_2]\},$$

$$=\frac{A_1A_2}{2N_1N_2}\left[e^{j(\phi_1-\phi_2)}\sum_{n=0}^{N_1N_2-1}e^{j(\omega_1-\omega_2)n}+e^{-j(\phi_1-\phi_2)}\sum_{n=0}^{N_1N_2-1}e^{-j(\omega_1-\omega_2)n}\right.$$

$$\left.-e^{j(\phi_1+\phi_2)}\sum_{n=0}^{N_1N_2-1}e^{j(\omega_1+\omega_2)n}-e^{-j(\phi_1+\phi_2)}\sum_{n=0}^{N_1N_2-1}e^{-j(\omega_1+\omega_2)n}\right],$$

General Case

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Power of Sum of Sinusoids

$$
=\frac{A_1A_2}{2N_1N_2}\left[e^{j(\phi_1-\phi_2)}\sum_{n=0}^{N_1N_2-1}e^{j(\omega_1-\omega_2)n}+e^{-j(\phi_1-\phi_2)}\sum_{n=0}^{N_1N_2-1}e^{-j(\omega_1-\omega_2)n}\right.
$$

$$
\left.-e^{j(\phi_1+\phi_2)}\sum_{n=0}^{N_1N_2-1}e^{j(\omega_1+\omega_2)n}-e^{-j(\phi_1+\phi_2)}\sum_{n=0}^{N_1N_2-1}e^{-j(\omega_1+\omega_2)n}\right],
$$

if $\omega_1 = \omega_2$:

$$
P_3=\frac{A_1A_2}{2N_1N_2}\left[e^{j(\phi_1-\phi_2)}\sum_{n=0}^{N_1N_2-1}1+e^{-j(\phi_1-\phi_2)}\sum_{n=0}^{N_1N_2-1}1\right.
$$

$$
\left.-e^{j(\phi_1+\phi_2)}\sum_{n=0}^{N_1N_2-1}e^{j2\omega_1n}-e^{-j(\phi_1+\phi_2)}\sum_{n=0}^{N_1N_2-1}e^{-j2\omega_1n}\right],
$$

Power of Sum of Sinusoids

$$
\begin{aligned}
P_3 = \frac{A_1A_2}{2N_1N_2}\Bigg[& e^{j(\phi_1-\phi_2)}\sum_{n=0}^{N_1N_2-1} 1 + e^{-j(\phi_1-\phi_2)}\sum_{n=0}^{N_1N_2-1} 1 \\
& -e^{j(\phi_1+\phi_2)}\sum_{n=0}^{N_1N_2-1} e^{j2\omega_1 n} - e^{-j(\phi_1+\phi_2)}\sum_{n=0}^{N_1N_2-1} e^{-j2\omega_1 n}\Bigg], \\
= \frac{A_1A_2}{2N_1N_2}\Bigg\{ & N_1N_2[e^{j(\phi_1-\phi_2)} + e^{-j(\phi_1-\phi_2)}] \\
& -e^{j(\phi_1+\phi_2)}\frac{1-e^{j2\omega_1 N_1N_2}}{1-e^{j2\omega_1}} - e^{-j(\phi_1+\phi_2)}\frac{1-e^{-j2\omega_1 N_1N_2}}{1-e^{-j2\omega_1}}\Bigg\},
\end{aligned}
$$

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Power of Sum of Sinusoids

$$P_3 = \frac{A_1A_2}{2N_1N_2}\Big\{N_1N_2[e^{j(\phi_1-\phi_2)} + e^{-j(\phi_1-\phi_2)}]$$

$$-e^{j(\phi_1+\phi_2)}\frac{1-e^{j2\omega_1N_1N_2}}{1-e^{j2\omega_1}} - e^{-j(\phi_1+\phi_2)}\frac{1-e^{-j2\omega_1N_1N_2}}{1-e^{-j2\omega_1}}\Bigg],$$

$$= \frac{A_1A_2}{2N_1N_2}\Big[2N_1N_2\cos(\phi_1-\phi_2)$$

$$-e^{j(\phi_1+\phi_2)}\frac{1-e^{j2\omega_1N_1N_2}}{1-e^{j2\omega_1}} - e^{-j(\phi_1+\phi_2)}\frac{1-e^{-j2\omega_1N_1N_2}}{1-e^{-j2\omega_1}}\Bigg],$$

$$= \frac{A_1A_2}{2N_1N_2}\Big[2N_1N_2\cos(\phi_1-\phi_2)$$

$$-e^{j(\phi_1+\phi_2)}\frac{1-e^{j2m_1N_22\pi}}{1-e^{j2\omega_1}} - e^{-j(\phi_1+\phi_2)}\frac{1-e^{-j2m_1N_22\pi}}{1-e^{-j2\omega_1}}\Bigg],$$

Power of Sum of Sinusoids

$$
\begin{aligned}
P_3 = & \frac{A_1 A_2}{2N_1 N_2} \Big[2N_1 N_2 \cos(\phi_1 - \phi_2) \\
& - e^{j(\phi_1+\phi_2)} \frac{1 - e^{j2m_1 N_2 2\pi}}{1 - e^{j2\omega_1}} - e^{-j(\phi_1+\phi_2)} \frac{1 - e^{-j2m_1 N_2 2\pi}}{1 - e^{-j2\omega_1}} \Big], \\
= & \frac{A_1 A_2}{2N_1 N_2} \left[2N_1 N_2 \cos(\phi_1 - \phi_2) - 0 - 0 \right], \\
= & A_1 A_2 \cos(\phi_1 - \phi_2).
\end{aligned}
$$

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Power of Sum of Sinusoids

if $\omega_1 \neq \omega_2$:

$$P_3 = \frac{A_1 A_2}{2N_1 N_2}\left[e^{j(\phi_1-\phi_2)}\frac{1-e^{j(\omega_1-\omega_2)N_1N_2}}{1-e^{j(\omega_1-\omega_2)}} + e^{-j(\phi_1-\phi_2)}\frac{1-e^{-j(\omega_1-\omega_2)N_1N_2}}{1-e^{-j(\omega_1-\omega_2)}}\right.$$

$$\left.-e^{j(\phi_1+\phi_2)}\frac{1-e^{j(\omega_1+\omega_2)N_1N_2}}{1-e^{j(\omega_1+\omega_2)}} - e^{-j(\phi_1+\phi_2)}\frac{1-e^{-j(\omega_1+\omega_2)N_1N_2}}{1-e^{-j(\omega_1+\omega_2)}}\right],$$

$$= \frac{A_1 A_2}{2N_1 N_2}\left[e^{j(\phi_1-\phi_2)}\frac{1-e^{j(m_1N_2-m_2N_1)2\pi}}{1-e^{j(\omega_1-\omega_2)}} + e^{-j(\phi_1-\phi_2)}\frac{1-e^{-j(m_1N_2-m_2N_1)2\pi}}{1-e^{-j(\omega_1-\omega_2)}}\right.$$

$$\left.-e^{j(\phi_1+\phi_2)}\frac{1-e^{j(m_1N_2+m_2N_1)2\pi}}{1-e^{j(\omega_1+\omega_2)}} - e^{-j(\phi_1+\phi_2)}\frac{1-e^{-j(m_1N_2+m_2N_1)2\pi}}{1-e^{-j(\omega_1+\omega_2)}}\right],$$

$$= \frac{A_1 A_2}{2N_1 N_2}(0+0-0-0) = 0.$$

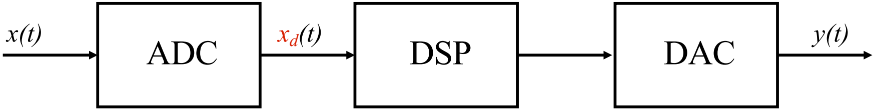
Example

$$x[n] = A_1 \sin(\omega_1 n + \phi_1) + A_2 \sin(\omega_2 n + \phi_2)$$

$$P = \begin{cases} A_1^2 \sin^2\phi_1 + A_2^2 \sin^2\phi_2 & m_1 = 0, m_2 = 0, \\ +2A_1A_2 \sin\phi_1 \sin\phi_2 & m_1 = m_2 = 1, N_1 = N_2 = 1, 2, \\ & \\ A_1^2 \sin^2\phi_1 & m_1 = 0, m_2 = 1, N_2 = 2, \\ +A_2^2 \sin^2\phi_2, & m_1 = 1, N_1 = 2, m_2 = 0, \\ & \omega_1 \neq \omega_2 \\ \dfrac{A_1^2}{2} + \dfrac{A_2^2}{2}, & 1 \leqslant m_1 < N_1, N_1 > 2, \\ & 1 \leqslant m_2 < N_2, N_2 > 2, \\ & \omega_1 = \omega_2 \\ \dfrac{A_1^2}{2} + \dfrac{A_2^2}{2} & 1 \leqslant m_1 < N_1, N_1 > 2, \\ +A_1A_2\cos(\phi_1 - \phi_2) & 1 \leqslant m_2 < N_2, N_2 > 2. \end{cases}$$

Sampling Theory

- Digital signal processing



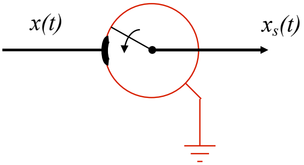
ADC: convert the continuous signal into digital signal

Sampling:

Is the operation that transforms a continuous-time/space signal into a discrete-time/space version.

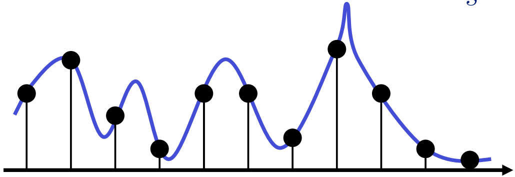
Sampler:

Is a system that measures the amplitude of the continuous signal at specific instances of time.



Uniform sampling:

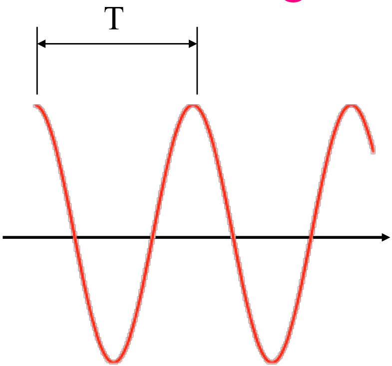
- Samples of signal are taken at equally spaced points along the signal waveform.
- Samples are taken at time instant $t_s = nT_s$ (n is integer).



- The distance between two consecutive samples (T_s) is called the sampling period/interval, the sampling frequency/rate f_s is the number of samples in one second, i.e. $f_s = \frac{1}{T_s}$ [Hz or samples per second].

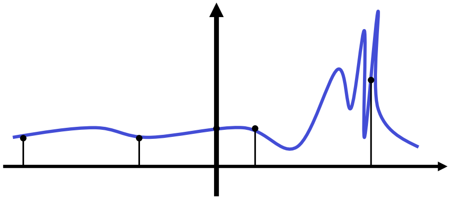
When representing a signal by its samples, we have to ask ourselves:

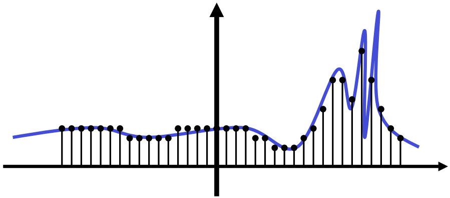
- Does the samples have the same shape / characteristics as the original signal?



Uniform Sampling

How fast should we sample





- Fewer samples are needed for a slowly-changing signal. More samples are required for fast-changing signals.
- What is the critical sampling rate?

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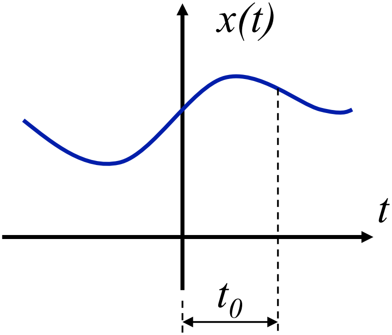
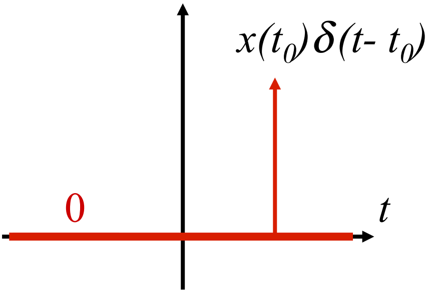
Ideal Impulse-train Sampling

Assuming that $x(t)$ is continuous function then

$$x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0).$$



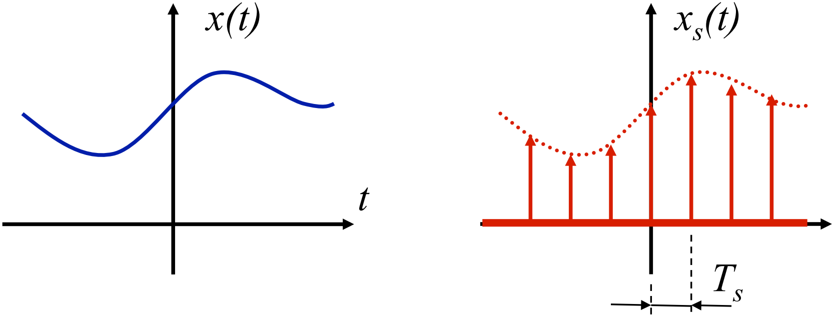


The weight of $\delta(t)$ is the value of $x(t)$ at t_0.

Ideal Impulse-train Sampling

The resulting sampled signal can be represented by multiplying $x(t)$ with an impulse train

$$x_s(t) = x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT_s) = \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s)$$



Frequency Domain Representation

We take continuous Fourier Transform on both sides of $x_s(t) = x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$

$$\mathcal{F}[x_s(t)] = \mathcal{F}\left[x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT_s)\right],$$

$$X_s(\omega) = \frac{1}{2\pi}\mathcal{F}[x(t)] * \mathcal{F}\left[\sum_{n=-\infty}^{\infty} \delta(t - nT_s)\right],$$

Frequency convolution

$$x_1(t) \times x_2(t) \rightarrow \frac{1}{2\pi}[X_1(\omega) * X_2(\omega)]$$

Frequency Domain Representation

$$\begin{aligned}
X_s(\omega) &= \frac{1}{2\pi}\mathcal{F}[x(t)] * \mathcal{F}\left[\sum_{n=-\infty}^{\infty} \delta(t - nT_s)\right], \\
&= \frac{1}{2\pi}\left[X(\omega) * \omega_s \sum_{m=-\infty}^{\infty} \delta(\omega - m\omega_s)\right], \\
&= \frac{\omega_s}{2\pi} \sum_{m=-\infty}^{\infty} X(\omega - m\omega_s), \\
&= \frac{1}{T_s} \sum_{m=-\infty}^{\infty} X\left(\omega - \frac{2m\pi}{T_s}\right).
\end{aligned}$$

The second step refers to Page 329 row 7 in Table 4.2.

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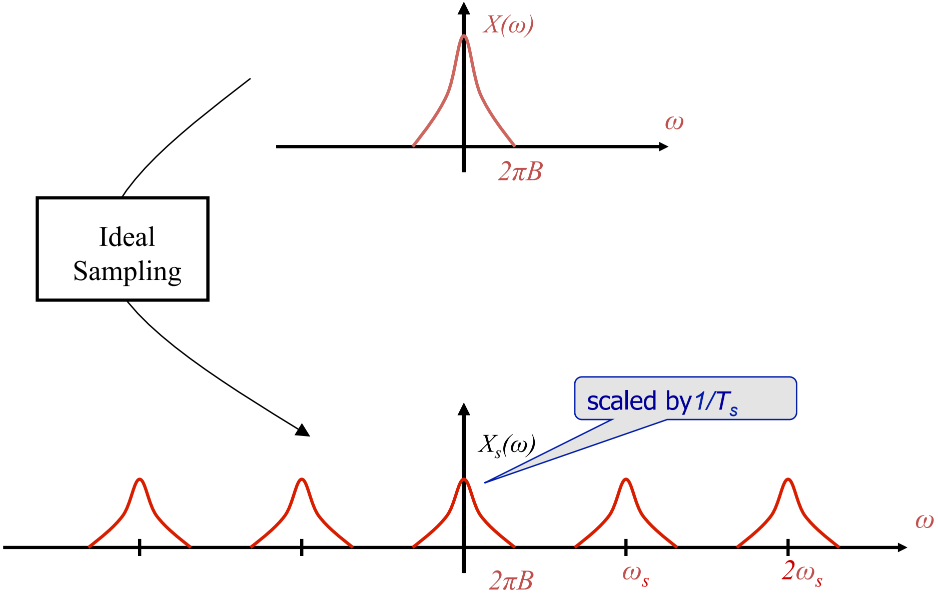
Frequency Domain Representation



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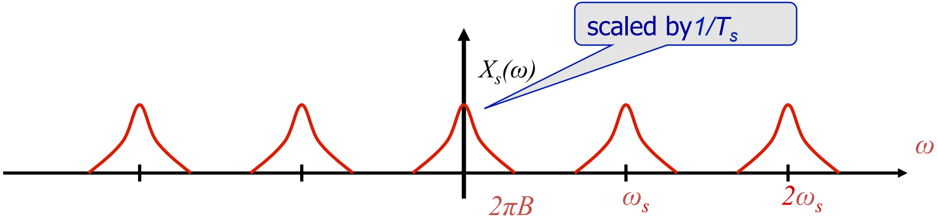
Theory of Ideal Impulse-train Sampling



A signal $x(t)$ can be reconstructed perfectly from its samples if the sampling rate $f_s = \dfrac{\omega_s}{2\pi}$ satisfies the condition:

$$\omega_s \geqslant 4\pi B,$$
$$2\pi f_s \geqslant 4\pi B,$$
$$\boxed{f_s \geqslant 2B}.$$

Theory of ideal impulse-train sampling

The minimum sampling rate required for perfect reconstruction of the original signal is called Nyquist rate,

$$\boxed{f_s = 2B}.$$

This theorem was first articulated by Nyquist in 1928 and was formally proven by Shannon in 1949.

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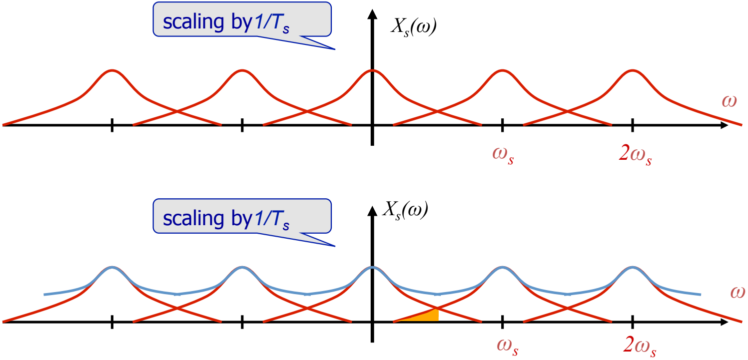
What happens when the sampling rate is below the Nyquist rate?



Frequency folding error (or aliasing) results from not having fast enough sampling. If the Nyquist sampling criterion is not met, perfect reconstruction of the original signal is not achieved.

- Page 61, Q1.26: (a)–(e);
- Page 514–545, read section 7.0–7.1.1, 7.2–7.4
- Page 556, Q7.1;
- Page 556, Q7.2: (a)–(c).
- Page 556, Q7.3: (a)–(c).
- Page 556, Q7.4: (a)–(d).

Thank you for your attention.