Department Of Electrical and Electronics Engineering

Year 3 - Linear Filters

Object

To revise the basic electrical circuit theory needed to understand filters, and to gain experience of how this theory may be applied in practise to design and build active and passive low-pass, high-pass and band-pass filters.

Structure of the Assignment

Before attempting the laboratory exercises you should read carefully the tutorial notes presented in Part A, below, and complete the questions given in Part B. This should be followed by the three laboratory exercises described in Part C.

Equipment
Oscilloscope
Power supply providing –15,0,+15 volts
Signal generator and Frequency counter
AC mV meter
Patch-board
Three '741' operational amplifiers & data sheet
Passive 5th order low-pass filter

References

1) J.Millman & A.Grabel, "Microelectronics", (CH.16), McGraw-Hill, 1988
2) P.Horowitz & W.Hill "The Art Of Electronics", (CH.3,4), Cambridge UP, 1980
3) M.E. Van Valkenburg, "Analogue filter design", Holt Rinehart & Winston, 1982

Part A Tutorial Notes

1) **Introduction**

An electrical filter is often described as a circuit for removing or "filtering out" unwanted frequency components from a signal. A low-pass filter, for example, would be designed to allow low frequency components of an input signal to pass through without attenuation and to eliminate (as much as practically possible) the high frequency components. A "cut-off frequency", $f_c Hz$, would be specified to define the boundary between low and high frequencies; i.e. all frequencies below f_c would be considered low, all frequencies above f_c would be high. The "pass-band" of this low-pass filter would be the frequency range 0 to $f_c Hz$ and the "stop-band" would be $f_c Hz$ to infinity. A high-pass filter with cut off frequency f_c would have a pass-band $f_c Hz$ to infinity and a stop-band 0 Hz to f_c. "Band-pass" and "band-stop" filters are also commonly seen, with pass-bands and stop-bands defined between lower and upper cut-off frequencies.

Although the concept of a filter outlined above is sufficient for the purposes of this experiment, it does not tell the whole story and many other very interesting types of filter exist. For example an “all-pass” filter does not attenuate any frequency components at all and instead just changes their phase relationships. Tone control circuits are filters which emphasise certain frequency ranges, and some filters are adaptive and actually design themselves by adjusting to their input signals.

2) **<u>Frequency response</u>**

The behaviour of a filter is conveniently described by its “frequency response” which tells us how the filter affects sinusoidal input signals. Remembering that periodic signals can be expressed as Fourier series, i.e. sums of sine waves of different frequencies, knowledge of the frequency response is very useful, for it allows us to analyse the filter’s response to say square waves and other more complicated signals such as music or speech. The frequency response of a filter is a complex valued function of the frequency in Hz.

We shall adopt the notation $H(j\omega)$ for frequency response, in these experiments, although this is not universally used in textbooks. (N.B. $j=\sqrt{(-1)}$). $H(j\omega)$ is a complex number for any given value of ω. A formula can normally be found for $H(j\omega)$ for a given analogue electrical circuit (i.e. one using electrical components such as resistors, capacitors etc.) For the resistor-capacitor circuit shown in Figure 1, it may be shown that:

$$H(j\omega)=\frac{1}{1+j\omega\ CR}$$

In most cases we are not more interested in the modulus and argument of $H(j\omega)$ than in its real and imaginary parts. We define

$$G(\omega)=|H(j\omega)|$$

and $\phi\ (\omega)=\arg(H(j\omega))=\arctan\left(\frac{\mathrm{Im}(H(j\omega))}{\mathrm{Re}(H(j\omega))}\right)+(\pi\,, if\ \mathrm{Re}(j\omega))<0)$

where $G(\omega)$ is the “gain response” of the filter and $-\phi\ (\omega)$ against (ω), as shown in Figure 2 which represents $H(j\omega)$ for the resistor-capacitor circuit shown in Figure 1.We prefer to plot $-\phi\ (\omega)$ rather than $\phi\ (\omega)$ because it represents phase leg rather than phase lead.

An alternative way of plotting the gain response is shown in Figure 3 where $G(\omega)$ has been converted to decibels (dB) and a logarithmic scale is used for the horizontal axis with $f = \omega / 2\pi$. The gain in dB at fHz is 20 log 10$[G(2\pi f)]$. If the input to a filter with frequency response $H(j\omega)$ is the sine wave:

$$x(t) = A\cos(\omega t)$$

the output signal will be

$$y(t)AG(\omega)\cos(\omega t + \phi(\omega))$$

This is a sinusoid of the same frequency as the input but with amplitude scaled by $G(\omega)$ and with a phase lead of $\phi(\omega)$. If we re-express (t) as:

$$y(t) = AG(\omega)\cos(\omega[t - (-\phi(\omega)/\omega)])$$

we can deduce that the effect of the phase lead $\phi(\omega)$ is the same as a time delay of $-\phi(\omega)/\omega$ seconds and this is called the "phase delay". For an ideal filter, we would like to have the same phase delay for all frequencies, and this would be true if $-\phi(\omega)/\omega$ were a constant, k seconds, say, for all ω. In this case we should have $\phi(\omega) = -k\omega$, i.e. the phase response graph would be a straight line through the origin and the filter would be termed "linear phase".

3) Butterworth low-pass approximation

Ideally, we would like to be able to design filters with "brick wall" gain responses and linear phase responses as illustrated in Figure 4 for an ideal low pass filter. Unfortunately, this is not possible and we must accept approximations to these ideal responses that can actually be realised by practical circuits.

The gain and phase responses for an ideal-pass filter (as illustrated in Figure 4) with cut-off frequency ω_c radians per second and time delay k seconds are:

$$G(\omega) = 1 \text{ and } \phi(\omega) = -kw \quad 0 \le \omega \le \omega_c$$
$$G(\omega) = 0 \qquad \omega > \omega_c$$

Considering $G(\omega)$ first, since the ideal expression above cannot be realised exactly, we must find an approximation that can be realised by a practical circuit. There are many solutions to this problem, perhaps the most famous, and simplest, being the Butterworth low-pass approximation. Other well-known solutions are known as Chebychev, Ellipitical and Bessel low-pass approximations; these are a little more complicated.

The Butterworth low-pass gain approximation of order n has the following formula:

$$G(\omega)=\frac{1}{\sqrt{\left(1+(\omega/\omega_c)^{2n}\right)}}$$

The order n determines the accuracy of the approximation, and also the complexity of the required circuit. Figure 5 shows a graph of $G(\omega)$ against ω when $\omega_c = 1$, in the three cases $n = 1,3\, and\, 7$. Practical circuits can be designed to have Butterworth low-pass gain responses. Traditionally, the phase responses of these circuits are disregarded at this stage of a design.

Additional circuits can be used to bring the phase responses closer to the ideal, although this is beyond the scope of our experiment.

4) High-pass, band-pass and band-stop approximations

A low-pass approximation can easily be transformed to make it high-pass, as will be seen later. Formulae for band-pass and band-stop approximations can also be derived from low-pass formulae, for example by multiplying a low-pass by a high-pass. Hence it is useful to concentrate for the moment on low-pass filters as they are useful in their own right and also as building blocks or "prototypes" for other filter types.

5) The transfer function

To design practical filter circuits, we use the concept of a "transfer function". For analogue circuits this is a function of the Laplace transform $V_{out}(s)$ of the filter's output signal is equal to $V_{out}(s)=H(s)V_{in}(s)$ where $V_{in}(s)$ is the Laplace transform of the input signal and $H(s)$ is the transfer function. $H(s)$ is normally a ratio of polynomials in s. For example, the RC circuit in Figure 1has the transfer function:

$$H(s)=\frac{1}{1+RC\mathbf{s}}$$

The transfer function of a circuit and its frequency response are closely related. Replacing s by $j\omega$ in the transfer function $H(s)$ gives us the frequency response $H(\omega)$.

6) Filter design

To design practical filter circuits there are two stages:

a) Find a transfer function $H(s)$ such that replacing s by $j\omega$ and taking the modulus gives us the required expression for $G(\omega)$.
b) Design a practical circuit whose transfer function is $H(s)$

Fortunately both these problems have been solved already for the Butterworth low-pass approximation, and many others. It may be shown that for an nth order Butterworth low-pass gain approximation with cut-off frequency ω_C :

$$H(s)=\frac{1}{B_n(s/\omega_c)}$$

where

$$B_n(s)=\prod_{k=1}^{n/2}\left(1+2s\sin\frac{(2k-1)\pi}{2n}+s^2\right) \qquad \text{n even}$$

$$B_n(s)=(1+s)\prod_{k=1}^{(n-1/2)}\left(1+2s\sin\frac{(2k-1)\pi}{2n}+s^2\right) \qquad \text{n odd}$$

Replacing s by s/ω_c gives us $B_n(s/\omega_c)$. Table 1 presents expressions derived from this formula when $n=1,2,3$ and 4. When the factors are multiplied out as shown on the right hand side of the table, the resulting are polynomials in s termed "Butterworth polynomials".

n	$B_n(s)$	
1	$(s+1)$	$s+1$
2	$(s^2+1.414s+1)$	$s^2+\sqrt{2s}+1$
3	$(s+1)(s^2+s+1)$	s^3+2s^2+2s+1
4	$(s^2+0.765s+1)(s^2+1.848s+1)$	$s^4+2.613s^3+3.414s^2+1$

This completes stage (a). For stage (b), four different types of filter circuit are commonly seen

- (i) Active filters which use operational amplifiers.
- (ii) Passive filters which use only resistors, capacitors and inductors.
- (iii) Piezoelectric (ceramic or quartz-crystal) filters using crystals which mechanically vibrate when subject to changing electric fields and resonate over a narrow range of frequencies.
- (iv) Digital filters which receive their input signals in digitised form and are implemented essentially by computer or microprocessor programs.

7) Active filters

Active filters are suitable for lower frequency signals, below about 1 MHz, because of limitations of operational amplifiers. The circuit shown in Figure 9 is a form of "Sallen and Key" active low-pass filter section. It uses an amplifier (normally an operational amplifier with feedback) of gain K, two resistors of equal value R and two capacitors of equal value C. Several other forms of this circuit are often seen, but this is probably the simplest. It may be shown that its transfer function is:

$$H(s)=\frac{K}{1+(3-K)RCs+(RCs)^2}$$

Assume we wish to realise a second order Butterworth low-pass transfer function with cut-off frequency ω_c. From Table 1, the required function is:

$$H(s)=\frac{1}{1+1.414(s/\omega_c)+(s/\omega_c)^2}$$

The denominators of these two expressions can be made identical if R and C are chosen such that $RC=1/\omega_c$ and the amplifier gain K is set such that $(3-K)=1.414$, i.e. $K=1.59$. Taking R to be convenient value of $10k\Omega$, this requires C to be $100/\omega_c\mu F$. The numerators cannot now be made equal, but this simply means that the output of the Sallen and Key low-pass section will be scaled by a constant factor K which is usually greater than one. In practise, this scaling is not too much of a problem, and can be compensated for, if necessary, by a simple potentiometer. Higher order Butterworth low-pass transfer functions may be expressed as the product of second order (or "biquadratic") transfer functions with one first order transfer function when the order is odd.
Each first order transfer function may be realised by the circuit shown in Figure 10 whose transfer function is:

$$H(s)=\frac{K}{1+(1-K)RCs}$$

The separate sections may be cascaded by connecting the output of one to the input of the next, with scaling by potentiometers where necessary.

8) <u>High pass active filter sections</u>

Given the transfer function $H(s)$ for a low-pass filter with cut-off frequency 1 rad/sec replacing s by ω_c/s produces the transfer function of a high-pass filter with cut-off ω_c. Hence the transfer function for a second order high-pass filter derived from the Butterworth approximation is:

$$H(s)=\frac{s^2}{s^2+1.414\omega_c s+\omega_c^2}$$

If the two resistors of value R are interchanged with the two capacitors of value C in the Sallen Key low-pass section in Figure 9, we obtain a high-pass section whose transfer function is:

$$H(s) = \frac{Ks^2}{s^2 + (3-K)s/RC + 1/(RC)^2}$$

Hence high-pass active filters can be realised as well as low-pass. Also band-pass filters can be designed by cascading high-pass and low-pass sections as long as the cut-off frequencies are not too close.

9) **<u>Piezoelectric filters</u>**

Piezoelectric filters can have very narrow passbands (e.g. 10kHz) with centre frequencies between 400kHz and 50MHz and are commonly used in radio receivers. These are available from manufactures for a variety of applications.

10) **<u>Passive filters</u>**

Passive filters are used mainly for high frequency filtering operations, but are also seen in crossover circuits for bi-fi loudspeakers. The theory behind this type of filter is a little complicated. Fortunately, design tables for Butterworth and other analogue filter approximations are published in many textbooks. These tables usually give the component values needed to achieve a "normalised"(1 rad/sec) cut-off frequency when the load is a resistor R_L of value 1Ω and the source is an ideal voltage source in series with a resistor of value R_s (which may be selected). The transfer function of this circuit when correctly terminated by R_L and R_s will be $H(s)$ as obtained for a Butterworth low-pass approximation multiplied by a constant $R_L/(R_L + R_s)$. To scale the cut-off frequency from 1 rad/sec to some more practical value, ω_c, say, we must replace s in the transfer function expression by s/ω_c. It may be shown that the same effect can be achieved very simply by dividing the value of each inductor and each capacitor by ω_c.

Since the characteristics of passive filters are affected by load and source resistances, and since the load is unlikely to be 1Ω in practise, we need also to be to scale R_L and R_s. This again is straightforward: to scale the resistance of the load from 1Ω to some other value $R_L\Omega$, multiply all L's and any resistors (including R_s) by R_L and divide all C's by R_L. The effect of this scaling on R_S must be anticipated when selecting a suitable normalised filter.

High-pass and band-pass passive filters can be designed by applying the following component transformations (either before or after resistance scaling) to the capacitor C by an inductor of value $1/(C\omega_c)$.

Band-pass ω_1 to ω_u: replace each inductance L/d in series with capacitance $d/(pL)$ and replace each capacitance C by capacitance C/d in parallel with inductance $1/(pC)$, where $d=(\omega_u-\omega_1)$ and $p=\omega_u\omega_1$.

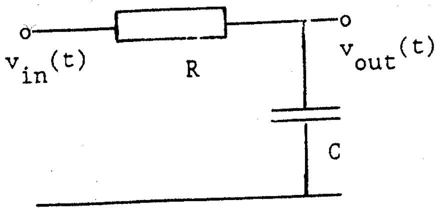


Figure 1

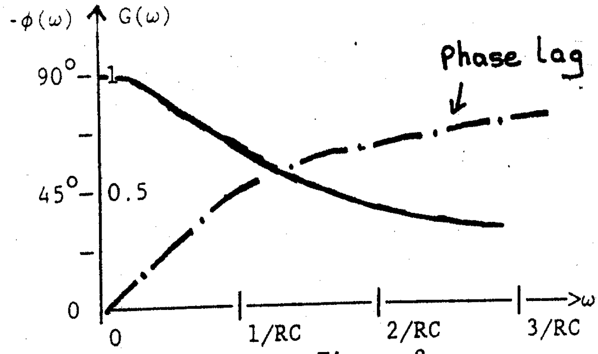


Figure 2

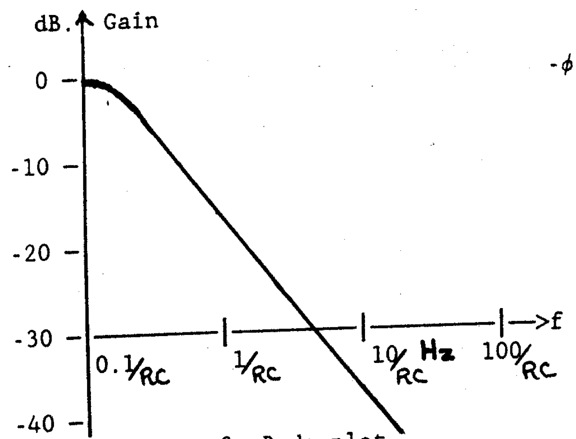


Figure 3: Bode plot

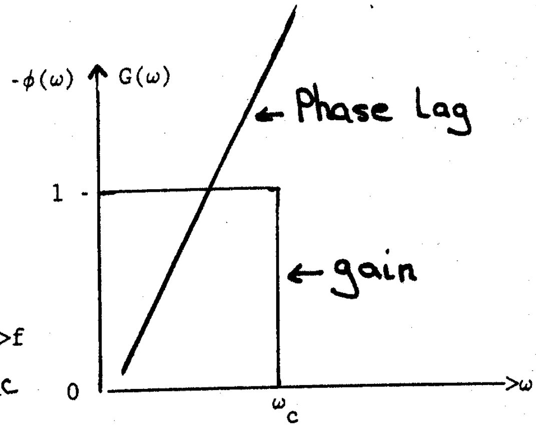


Figure 4: Ideal lowpass response

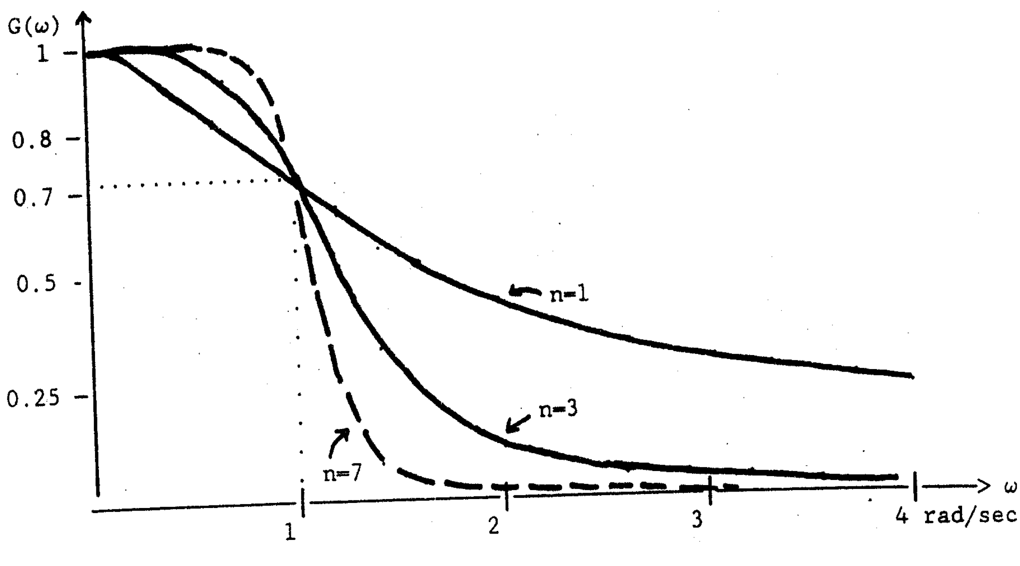


Figure 5: Butterworth lowpass gain responses

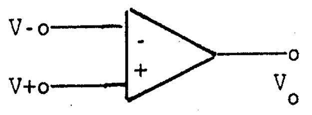


Figure 6(a): Op-amp.

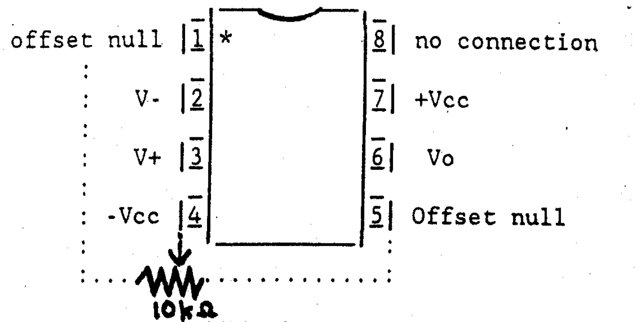
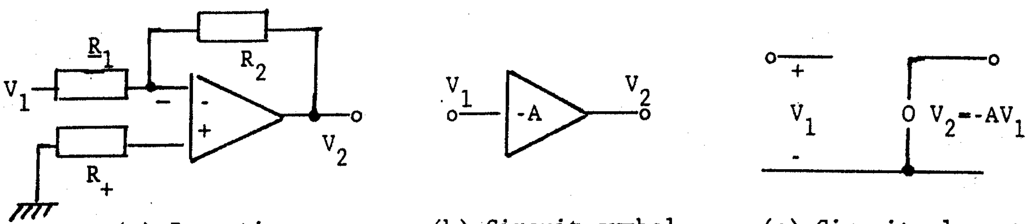
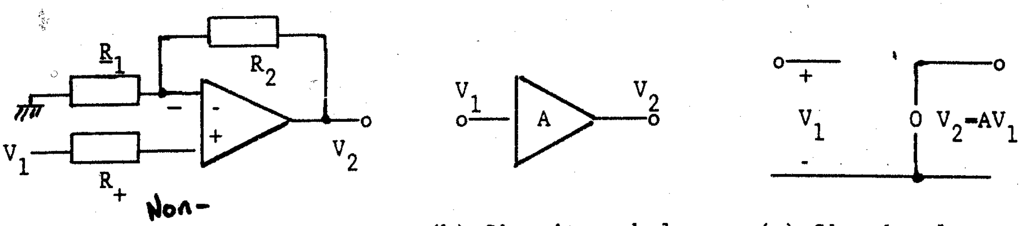


Figure 6(b): The "741" with null offset



(a) Inverting amp (b) Circuit symbol (c) Circuit element

Figure 7



(a) Non-Inverting amp (b) Circuit symbol (c) Circuit element

Figure 8

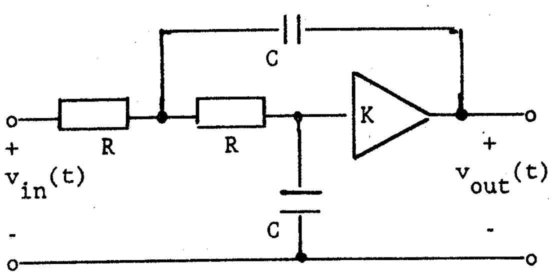


Figure 9: Active lowpass biquad section

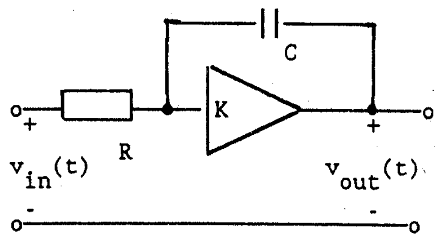


Figure 10: First order lowpass

PART B EXERCISES ON THE THEORY

1) Express the following values of the voltage gain in dB:

GAIN	1	1/2	$1/\sqrt{2}$	$\sqrt{2}$	2	10	100
dB							

2) Give formulae for (a) the transfer function, (b) the frequency response, (c) the gain response and (d) the phase response of a second order Butterworth low-pass approximation with cut-off frequency 4 rad/sec.

Transfer Function:

Frequency Response:

Gain Response:

Phase Response:

3) Complete the missing entries in the following table for the ~~2 rad/sec~~ 4 rad/sec cut-off low-pass filter referred to above:

ω (rad/sec)	0	0.5	1.0	1.5	4.0	10.0	20.0
Voltage gain	1	0.999		0.990			
Gain in dB		-0.001		-0.085			
Phase lead (deg)	0	-10.18	-20.66	-31.68	-90		
Phase lag (rad)	0	0.178	0.361	0.553			
Phase delay (sec)		0.355	0.361				

4) Referring to the table above, at frequencies greater than 4 radians/sec what is the reduction in gain in dB per octave (i.e. per doubling of frequency)? ANS……. dB/octave.

5) At frequencies greater than 4 radians/sec what is the reduction in gain in dB per decade (i.e. increasing the frequency 10 times)? ANS……. dB/ decade.

6) A 1 radian/second sine wave will be delayed by about 0.8 seconds. TRUE/FALSE?

7) Is the frequency response linear phase? YES/NO?

8) Why is a linear phase response desirable?

9) Butterworth low-pass filters of all orders have –3dB gain (relative to their gain at 0 Hz) at $\omega = \omega_C$. TRUE or FALSE?

10) Cascading three identical 2nd order Butterworth high-pass filters produces a 6th order Butterworth high-pass filter. TRUE or FALSE?

11) If a medium wave radio receiver were to be designed using a band-pass ceramic filter filter whose pass-band is centered on 455kHz, it could be turned to different stations by multiplying the incoming signal by a sine wave of correctly chosen frequency. The effect of this multiplication is to change the frequency of the input signal to make it pass through the filter. The filter will then remove other radio stations, leaving only the one passed by the filter. We can understand this by remembering the formula:

$$2\cos(\omega_1 t)\cos(\omega_2 t) = \cos(\omega_1 + \omega_2)t + \cos(\omega_1 - \omega_2)t$$

Assuming for simplicity that the incoming signal is a 750kHz sine-wave, what must be the frequency of the multiplying sine wave to make the frequency-changed signal pass through the ceramic filter?
ANS…………….kHz

PART C – LABORATORY EXERCISES

Exercise 2. Active filters

Build a circuit using an op-amp to provide a non-inverting amplifier of gain 1.59. Test it with sinusoidal inputs up to about 1MHz. What is the voltage gain at (a) 100Hz, (b) 1kHz, (c) 10kHz, (d) 100kHz, (e) 1MHz. (At the higher frequencies, be careful to keep the input voltage small enough to avoid slew rate limiting).

1) Complete the missing entries in the following table:

Frequency	100Hz	1kHz	10kHz	100kHz	1MHz
Voltage gain					
Gain in dB					

Use the method in section 7 to design an active 2nd order Butterworth filter whose cut-off frequency is 1592Hz. Build this circuit, measure its gain and phase response from 100Hz to 100kHz. For the phase response you can use the measurement technique described in Appendix 2. Plot separate graphs for (i) gain (dB), using a log scale for frequency and (ii) phase, using linear scales.

2) Complete the missing entries in the following table:

Frequency	100Hz	1kHz	1600Hz	5kHz	10kHz	100kHz	
Voltage gain							
Gain in dB							
Phase lead (deg)							

Design an active 2th order band-pass filter for telephone application. The pass band should be 318.31Hz to 3.183kHz, which is approximately the bandwidth for "toll quality" telephone speech. Construct this 2th order band-pass filter. Plot gain and phase response as before.

3) Complete the missing entries in the following table:

Frequency	100Hz	300Hz	1600Hz	3kHz	10kHz	100kHz	
Voltage gain							
Gain in dB							
Phase lead (deg)							

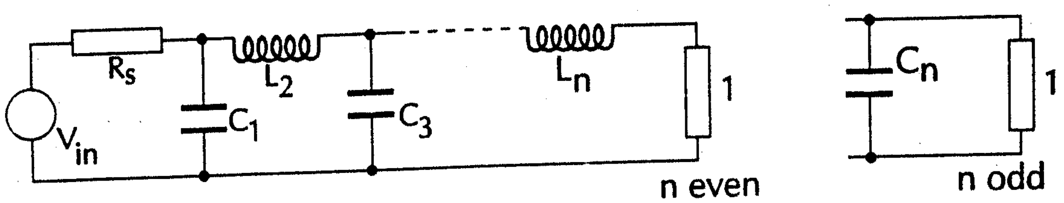
Exercise 3. Passive filters (Optional)

Use the method in section 10 and the tables in Appendix 1 to design a passive 5th order Butterworth filter using two inductors and three capacitors. The cut off frequency is to be 9.5kHz and the source and load resistances are to be 1800Ω. Compare the component values for your design with those shown on the low-pass filter board provided. Test the performance of the filter provided experimentally by gain and phase measurement. What is the phase delay at low frequencies? What is the attenuation (dB per octave) at high frequencies?

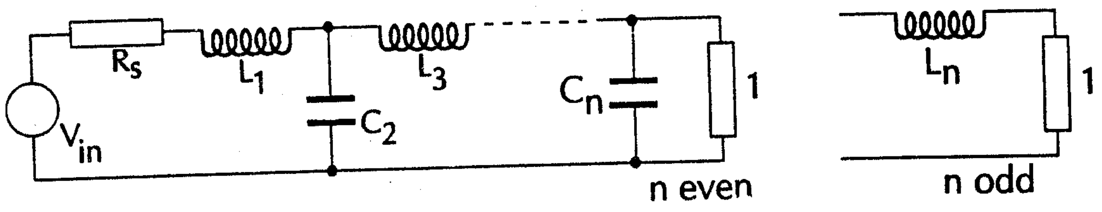
Appendix 1

Normalised Butterworth low-pass filter tables

Read top labels for upper circuit and bottom labels $(1/R_s\ etc.)$ for lower.

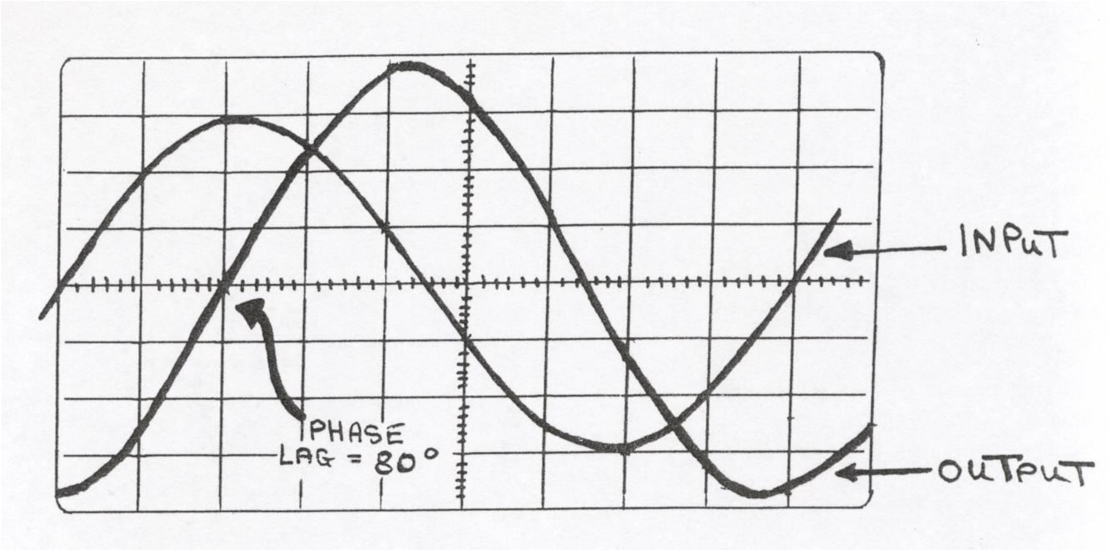


n	R_s	C_1	L_2	C_3	L_4	C_5	etc
2	1.0	1.41	1.41				
	2.0	0.448	3.35				
	2.5	0.34	4.10				
	5.0	0.16	7.71				
	10.0	0.07	14.81				
	∞	1.41	0.71				
3	1.0	1.0	2.0	1.0			
	0.5	1.18	0.78	3.26			
	0.4	1.43	0.60	4.06			
	0.2	2.67	0.28	7.91			
	0.1	5.17	0.14	15.46			
	∞	1.50	1.33	0.50			
4	1.0	0.77	1.85	0.77			
	2.0	0.22	2.45	3.19			
	2.5	0.17	2.99	4.01			
	5.0	0.08	5.68	7.94			
	10.0	0.04	11.09	15.64			
	∞	1.53	1.58	0.38			
5	1.0	0.62	1.62	2.00	1.62	0.62	
	0.5	0.69	0.50	305	0.92	3.13	
	0.4	0.84	0.39	3.74	0.73	3.96	
	0.2	1.61	0.19	7.18	0.35	7.93	
	0.1	3.15	0.09	14.09	0.17	15.71	
	∞	1.55	1.69	1.38	0.89	0.31	
n	$1/R_s$	L_1	C_2	L_3	C_4	L_5	



Appendix 2

Measuring phase difference using an oscilloscope



This is a straightforward way of measuring the phase difference, in degrees, between two sinusoidal waveforms of the same frequency. Here the method is used to measure the phase lag of the sinusoidal output of a circuit relative to the input sinusoid.

Connect the input signal to channel 1 and adjust the Y-sensitivity (volt/cm) and the Y-position controls so that the waveform almost fills the screen with equal excursions above and below the central X-axis. Adjust the time-base (time/cm), trigger-level and X-position controls so that a single cycle of the input waveform is seen with positive-going zero crossing at divisions zero and nine (yes, nine) cm on the horizontal axis. Next adjust the Y-sensitivity and Y-position controls for channel 2 to make the trace for the output waveform almost fill the screen symmetrically about the horizontal centre line. The diagram shows an example of the kind of display you should now see.

The phase lag in degrees of the output with respect to the input is now 40P, where P is the number of divisions (cm) along the horizontal centre-line from 0 (where you set the start of the input cycle) to the first upward-going crossing of the centre line by the output waveform. This works because you chose 9 divisions for a whole cycle at 40 times 9 is 360^0.

This method does not distinguish 360^0 lag from 0^0 lag, so you may have to add 360^0 , or multiples of 360^0, when plotting phase-response graphs, in order to get curves without vertical jumps. However, make sure you record enough points for you to be sure where this has to be done.