Xi'an Jiaotong-Liverpool University
西交利物浦大学

EEE220 Instrumentation and Control System

2018-19 Semester 2

Dr. Qing Liu

Email: qing.liu@xjtlu.edu.cn

Office: EE516

Department of Electrical and Electronic Engineering

18 April, 2019

Lecture 16

Outline

The Stability of Linear Feedback Systems

- ❑ The Concept of Stability
- ❑ The Routh-Hurwitz Stability Criterion
- ❑ The Relative Stability of Feedback Control Systems
- ❑ The Stability of State Variable Systems
- ❑ System Stability Using Matlab

Stability

A stable system is a dynamic system with a bounded (limited) response to a bounded input.

- When considering the design and analysis of feedback control systems, **stability** is of the utmost important. From a practical point of view, a closed-loop feedback system that is unstable is of minimal value.
- Many physical systems are inherently open-loop unstable. Using feedback, we can stabilize unstable systems and then with a judicious selection of controller parameters, we can adjust the transient performance.
- For open-loop stable systems, we still use feedback to adjust the closed-loop performance to meet the design specifications (i.e. percent overshoot, settling time, steady-state error etc.).
- We can say that a closed-loop feedback system is stable or it is not stable, this type of stable/not stable characterization is referred to as **absolute stability**.
- Given that a closed-loop system is stable, we can further characterize the degree of stability. This is referred to as **relative stability**.

Conditions for A Feedback System To Be Stable

In terms of linear systems, stability requirement may be defined in terms of the location of the poles of the closed-loop transfer function which can be written as

$$T(s) = \frac{p(s)}{q(s)} = \frac{K\prod_{i=1}^{M}(s+z_i)}{s^N\prod_{k=1}^{Q}(s+\sigma_k)\prod_{m=1}^{R}[s^2+2\alpha_m s+(\alpha_m^2+\omega_m^2)]}$$

where $\Delta(s) = q(s) = 0$ is the characteristic equation whose roots are the poles of the closed-loop system. The output response for an impulse function input (when $N = 0$) is then

$$y(t) = \sum_{k=1}^{Q} A_k e^{-\sigma_k t} + \sum_{m=1}^{R} B_m\left(\frac{1}{\omega_m}\right)e^{-\alpha_m t}\sin(\omega_m t + \theta_m)$$

Where A_k, B_m are constants that depend on K, z_i, σ_k, α_m and ω_m.

A necessary and sufficient condition for a feedback system to be stable is that all the poles of the system transfer function have negative real parts. --- A system is stable if all the poles of the transfer function are in the left-hand s-plane.

Stability and Root Location

- **Stable:**
  - All the roots of characteristic equation are in the left-hand s-plane;
  - Output is bounded to bounded input.

- **Marginally Stable:**
  - The characteristic equation has simple roots on the imaginary axis ($j\omega$-axis) with all other roots in the left hand-plane;
  - The steady-state output will be sustained oscillations for a bounded input, unless a sinusoid (which is bounded) whose frequency is equal to the magnitude of the $j\omega$-axis roots. For this case, the system becomes unbounded.

- **Unstable:**
  - The characteristic equation has at least one root in the right half of the s-plane or repeated $j\omega$ roots;
  - The output will become unbounded for any input.

Routh Array

In the late 1800s, A. Hurwitz and E. J. Routh independently published a method of investigating the stability of a linear system. The Routh-Hurwitz stability method provides an answer to the question of stability by considering the **characteristic equation** of the system. The characteristic equation can be written as

$$\Delta(s) = q(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0 = 0$$

❖ Routh Array

$$\begin{array}{c|ccc}
s^n & a_n & a_{n-2} & a_{n-4} \\
s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} \\
s^{n-2} & b_{n-1} & b_{n-3} & b_{n-5} \\
s^{n-3} & c_{n-1} & c_{n-3} & n_{n-5} \\
\vdots & \vdots & \vdots & \vdots \\
s^0 & h_{n-1} & &
\end{array}$$

where

$$b_{n-1} = \frac{a_{n-1}a_{n-2} - a_n a_{n-3}}{a_{n-1}} = \frac{-1}{a_{n-1}} \begin{vmatrix} a_n & a_{n-2} \\ a_{n-1} & a_{n-3} \end{vmatrix}$$

$$b_{n-3} = -\frac{1}{a_{n-1}} \begin{vmatrix} a_n & a_{n-4} \\ a_{n-1} & a_{n-5} \end{vmatrix},$$

$$c_{n-1} = \frac{-1}{b_{n-1}} \begin{vmatrix} a_{n-1} & a_{n-3} \\ b_{n-1} & b_{n-3} \end{vmatrix},$$

and so on.

The Routh-Hurwitz Stability Criterion

The Routh-Hurwitz criterion states that the number of roots of $q(s)$ with positive real parts is equal to the number of changes in sign of the first column of the Routh array.

--- This criterion requires that there be **no changes in sign of in the first column** for a stable system. This requirement is **both necessary and sufficient**.

Four distinct cases or configurations of the first column array must be considered, and each must be treated separately and requires suitable modifications of the array calculation procedure:

1) No element in the first column is zero;
2) There is a zero in the first column, but some other elements of the row containing the zero are nonzero;
3) There is a zero in the first column, and the other elements of the row containing the zero are also zero;
4) As in the third case, but with repeated roots on the $j\omega$-axis.

Case 1: Second-order System

- **Case 1. No element in the first column is zero.**

The characteristic polynomial of a second-order system is

$$q(s) = a_2 s^2 + a_1 s + a_0$$

The Routh array is written as

$$\begin{array}{c|cc} s^2 & a_2 & a_0 \\ s^1 & a_1 & 0 \\ s^0 & b_1 & 0 \end{array}$$

where

$$b_1 = \frac{a_1 a_0 - (0)a_2}{a_1} = \frac{-1}{a_1}\begin{vmatrix} a_2 & a_0 \\ a_1 & 0 \end{vmatrix} = a_0.$$

Therefore, the requirement for a stable second-order system is simply that all the coefficients be positive or all the coefficients be negative.

Case 1: Third-order System

The characteristic polynomial of a third-order system is

$$q(s) = a_3 s^3 + a_2 s^2 + a_1 s + a_0$$

The Routh array is

$$\begin{array}{c|cc} s^3 & a_3 & a_1 \\ s^2 & a_2 & a_0 \\ s^1 & b_1 & 0 \\ s^0 & c_1 & 0 \end{array}$$

where

$$b_1 = \frac{a_2 a_1 - a_0 a_3}{a_2} \quad \text{and} \quad c_1 = \frac{b_1 a_0}{b_1} = a_0.$$

Therefore, for the third-order system to be stable, it is necessary and sufficient that coefficients be positive and $a_2 a_1 > a_0 a_3$. The condition when $a_2 a_1 = a_0 a_3$ results in a marginally stability case, and one pair of roots lies on the imaginary axis in the s-plane.

Case 1. An Unstable Example

The characteristic polynomial of a third-order system is

$$q(s) = s^3 + s^2 + 2s + 24$$

The Routh array is

$$\begin{array}{c|cc} s^3 & 1 & 2 \\ s^2 & 1 & 24 \\ s^1 & -22 & 0 \\ s^0 & 24 & 0 \end{array}$$

Because two changes in sign appear in the first column, there are two roots of $q(s)$ lie in the right-hand plane, the system therefore is unstable.

Case 2.

- **Case 2. There is a zero in the first column, but some of other elements of the row containing the zero in the first column are nonzero.**

In this case, if only one zero element in the array is zero, it may be replaced with a small positive number, ε, that is allowed to approach zero after completing the array.

For example, consider the following characteristic polynomial:

$$q(s) = s^5 + 2s^4 + 2s^3 + 4s^2 + 11s + 10$$

The Routh array is

$$\begin{array}{c|ccc} s^5 & 1 & 2 & 11 \\ s^4 & 2 & 4 & 10 \\ s^3 & \epsilon & 6 & 0 \\ s^2 & c_1 & 10 & 0 \\ s^1 & d_1 & 0 & 0 \\ s^0 & 10 & 0 & 0 \end{array}$$

where

$$c_1 = \frac{4\epsilon - 12}{\epsilon} = \frac{-12}{\epsilon}$$

and $d_1 = \dfrac{6c_1 - 10\epsilon}{c_1} \to 6.$

There are two sign changes due to the large negative number in the first column, $c_1 = -12/\varepsilon$. Therefore, the system is unstable, and two roots lie in the right half of the plane.

Case 2. Example

For example, consider the following characteristic polynomial. Obtain the gain K that results in marginally stability.

$$q(s) = s^4 + s^3 + s^2 + s + K$$

The Routh array is

$$\begin{array}{c|ccc} s^4 & 1 & 1 & K \\ s^3 & 1 & 1 & 0 \\ s^2 & \epsilon & K & 0, \\ s^1 & c_1 & 0 & 0 \\ s^0 & K & 0 & 0 \end{array}$$

where

$$c_1 = \frac{\epsilon - K}{\epsilon} \rightarrow \frac{-K}{\epsilon}$$

Therefore, for any value of K greater than zero, the system is unstable. Also, because the last term in the first column is equal to K, a negative value of K will result in an unstable system. Consequently, the system is unstable for all values of gain K.

Case 3.

- **Case 3. There is a zero in the first column, and the other elements of the row containing the zero are also zero.**

- Case 3 occurs when all the elements in one row are zero or when the row consists of a single element that is zero.

- This condition occurs when the polynomial contains singularities that are symmetrically located about the origin of the s-plane. Therefore, case 3 occurs when factors such as $(s+\sigma)(s-\sigma)$ or $(s+j\omega)(s-j\omega)$ occur.

- This problem is circumvented by utilizing the **auxiliary polynomial**, $\boldsymbol{U(s)}$, which immediately precedes the zero entry in the Routh array. The order of the auxiliary polynomial is always even and indicates the number of symmetrical root pairs.

Case 3. Example

Let us consider a third-order system with the characteristic polynomial where K is an adjustable gain

$$q(s) = s^3 + 2s^2 + 4s + K$$

The Routh array is

$$\begin{array}{c|cc} s^3 & 1 & 4 \\ s^2 & 2 & K \\ s^1 & \dfrac{8-K}{2} & 0 \\ s^0 & K & 0 \end{array}$$

For a stable system, we require: $0 < K < 8$

When $K = 8$, we obtain a row of zeros. In this case we have two roots on the $j\omega$-axis and a marginally stability case.

The auxiliary polynomial is the equation of the row preceding the row of zeros, therefore, in this case, obtained obtain from the s^2-row

$$U(s) = 2s^2 + Ks^0 = 2s^2 + 8 = 2(s^2 + 4) = 2(s + j2)(s - j2).$$

The factors of the characteristic polynomial can be obtained by dividing $q(s)$ by $U(s)$.

$$q(s) = (s + 2)(s + j2)(s - j2).$$

Case 4.

- **Case 4. Repeated roots of the characteristic equation on the $j\omega$-axis.**

- If the $j\omega$-axis roots of the characteristic equation are simple, the system is neither stable nor unstable, it is instead marginally stable;

- If the $j\omega$-axis roots are repeated, they system response will be unstable. The Routh-Hurwitz criteria will not reveal this form of instability.

Consider the system with a characteristic polynomial

$$q(s) = (s+1)(s+j)(s-j)(s+j)(s-j) = s^5 + s^4 + 2s^3 + 2s^2 + s + 1.$$

The Routh array is

$$\begin{array}{c|ccc} s^5 & 1 & 2 & 1 \\ s^4 & 1 & 2 & 1 \\ s^3 & \epsilon & \epsilon & 0 \\ s^2 & 1 & 1 & \\ s^1 & \epsilon & 0 & \\ s^0 & 1 & & \end{array}$$

where $\varepsilon \to 0$

NO sign changes in the first column, but there are two zero rows, indicating the repeated roots on the $j\omega$-axis. It is an unstable system.

There are two auxiliary polynomials at the s^2-line ($s^2 + 1$) and s^4-line ($s^4 + 2s^2 + 1 = (s^2+1)^2$) respectively.

Example 16.1

Consider the characteristic polynomial

$$q(s) = s^5 + s^4 + 4s^3 + 24s^2 + 3s + 63.$$

The Routh array is

$$\begin{array}{c|ccc}
s^5 & 1 & 4 & 3 \\
s^4 & 1 & 24 & 63 \\
s^3 & -20 & -60 & 0. \\
s^2 & 21 & 63 & 0 \\
s^1 & 0 & 0 & 0 \\
s^0 & 63 & &
\end{array}$$

Therefore, the auxiliary polynomial is

$$U(s) = 21s^2 + 63 = 21(s^2 + 3) = 21\left(s + j\sqrt{3}\right)\left(s - j\sqrt{3}\right)$$

which indicates that two roots are on the imaginary axis. To examine the remaining roots, we divide by the auxiliary polynomial to obtain

$$\frac{q(s)}{s^2 + 3} = s^3 + s^2 + s + 21$$

$$\frac{q(s)}{s^2 + 3} = s^3 + s^2 + s + 21$$

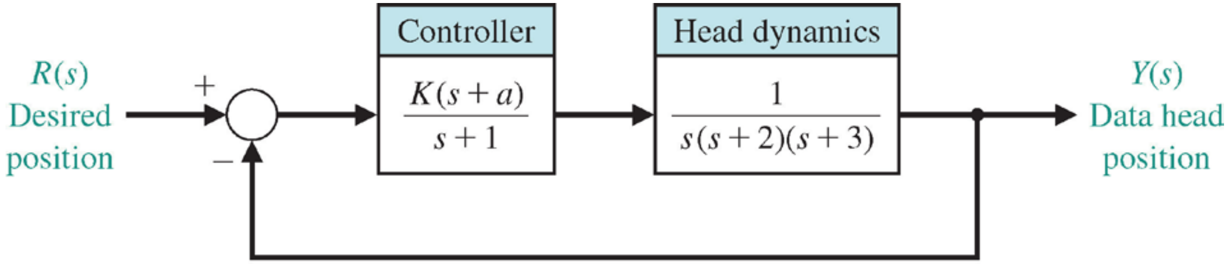
Establishing a Routh array for this equation, we have

$$\begin{array}{c|cc} s^3 & 1 & 1 \\ s^2 & 1 & 21 \\ s^1 & -20 & 0 \\ s^0 & 21 & 0 \end{array}$$

The two changes in sign in the first column indicate the presence of two roots in the right-hand plane, and the system is unstable.

Design Example 16.2: Welding Control

Large welding robots are used in today's auto plants. The welding head is moved to different positions on the auto body, and a rapid, accurate response is required. The diagram is shown as follows. Determine K and a to make the system stable.



Step 1. Obtain characteristic equation.

$$1 + G(s) = 1 + \frac{K(s + a)}{s(s + 1)(s + 2)(s + 3)} = 0.$$

Therefore, $q(s) = s^4 + 6s^3 + 11s^2 + (K + 6)s + Ka = 0.$

Step 2. Establish the Routh array.

$$\begin{array}{c|ccc} s^4 & 1 & 11 & Ka \\ s^3 & 6 & K+6 & \\ s^2 & b_3 & Ka & \\ s^1 & c_3 & & \\ s^0 & Ka & & \end{array}$$

where $b_3 = \dfrac{60 - K}{6}$ and $c_3 = \dfrac{b_3(K+6) - 6Ka}{b_3}$.

Step 3. Examine the first column, to determine K and a.

$b_3 > 0 \longrightarrow K < 60$

$c_3 > 0 \longrightarrow a \le \dfrac{(60-K)(K+6)}{36K}$ when a is positive.

Therefore, if $K = 40$, we require $a \le 0.639$.

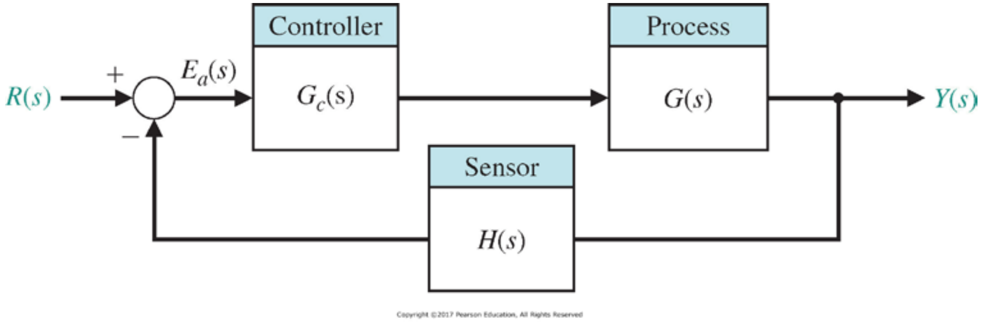
Quiz 16.1

Determine stability for a system with the following characteristic equation

$$q(s) = s^3 + Ks^2 + (1 + K)s + 6 = 0$$

Quiz 16.2

Consider the following system, determine K when it is marginally stable, and obtain the roots lying on the $j\omega$-axis.



$$G_c(s) = K \text{ and } G(s) = \frac{s + 40}{s(s + 10)} \qquad H(s) = 1/(s + 20).$$

Thank You！