EEE225 Advanced Electrical Circuits and Electromagnetics

Lecture 9 Two-port Networks

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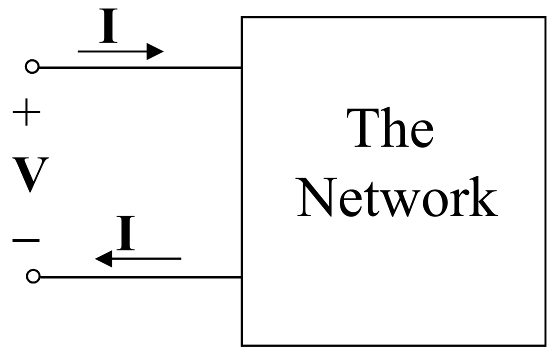
Room EE322

Content

- (Pre-lecture reading) Review of matrix operation
- 1-port and 2-port networks
- Four sets of parameters
  - Z-parameter
  - Y-parameter
  - T-parameter
  - ~~H-parameter~~
- Relationship between them
  - z and y parameters
- Interconnections
  - Series, parallel and cascade

One-port Network

- A port : A pair of terminals through which a signal (voltage or current) may enter or leave.
- Port condition: The same current must enter and leave a port.

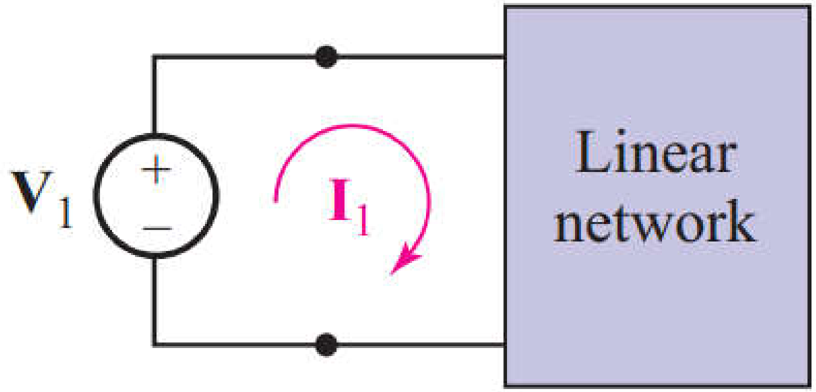


- One port network: a network has only <u>one pair of terminals</u>: two terminals
- For one-port network:
  - Current entering the port = current leaving the port
  - May be modeled by Thevenin or Norton equivalents

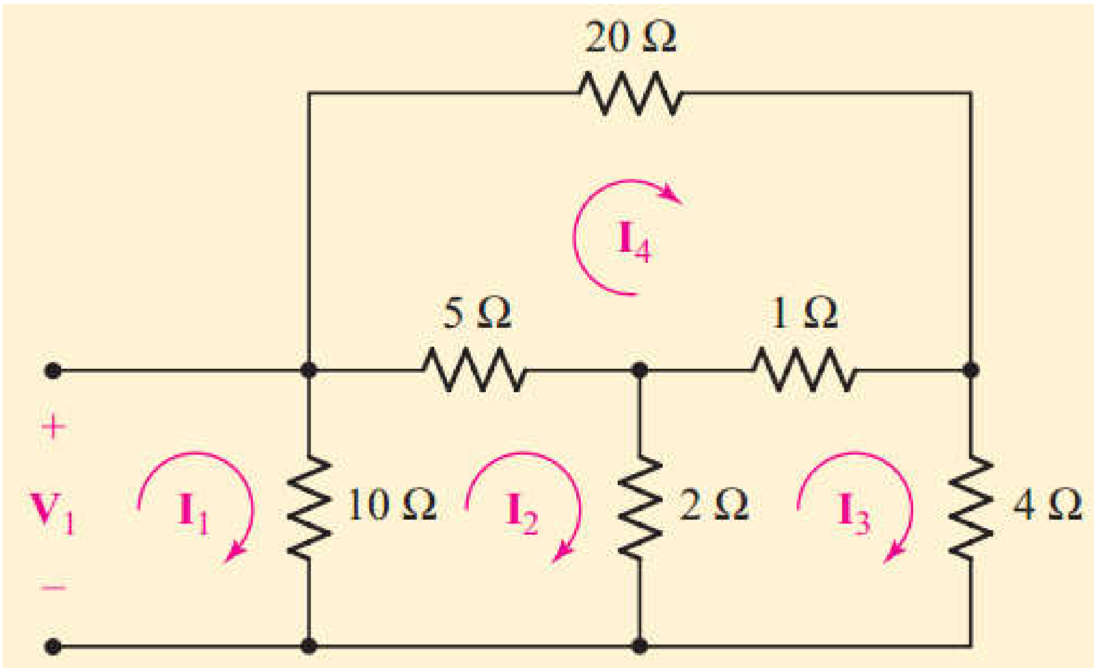
One-port Network – Example

- Input impedance of this one-port network is:

$$Z_{in} = \frac{V_1}{I_1}$$



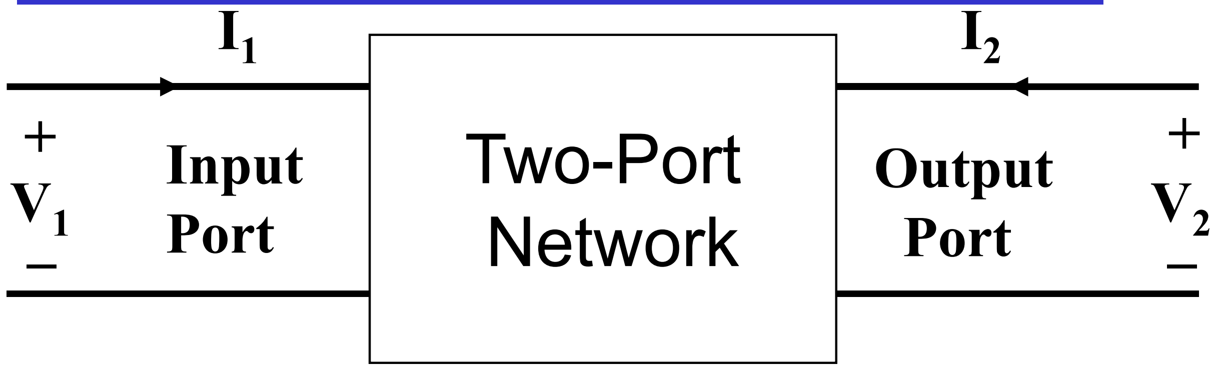
- Example:
  – Get Z_{in}



Two-port Network

- A two-port network is an electrical network with two separate ports for input and output.

The standard configuration of a two port:



- Any linear circuit with four terminals can be transformed into a two-port network provided that it does not contain an independent source and satisfies the port conditions.

- The voltages and currents at one port may be expressed as linear combinations of the voltages and currents at the other port, i.e. V_1, V_2, I_1, and I_2, are related by using two-port network *parameters.*

Two-port Network – Why?

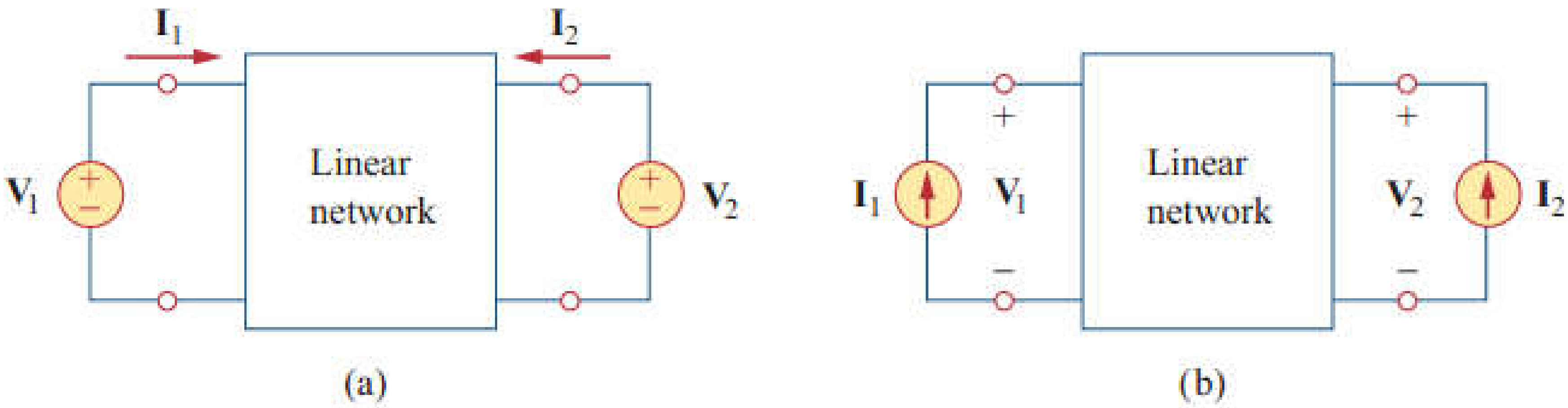
- 1. The network are useful, typically in communications, control systems, power system and electronics
- 2. Know how to model two-port network will help in the analysis of larger network (two-port network can be treated as 'black box'.)

Two-port Network – Four Sets of Network Parameters

In EEE207 we will study on four sets of network parameters

Impedance Z parameters **Admittance Y parameters**	→	Commonly used in the synthesis of filters, and are useful in the design and analysis of impedance-matching networks and power distribution networks
Transmission A, B, C, D parameters	→	The A,B,C,D parameters are best used when we want to cascade two networks together
Hybrid H parameters	→	H parameters are used almost solely in electronics -- in the equivalent circuit of a transistor.

Z Parameters – Definition



Voltage source driven 2-port network

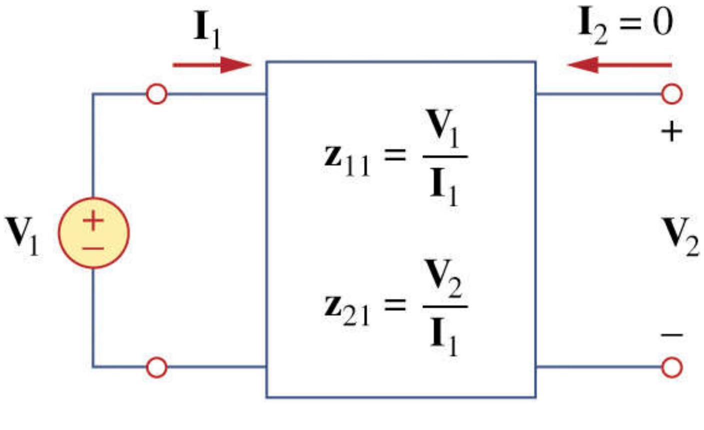
Current source driven 2-port network

The terminal voltages can be related to the terminal currents as

$$\mathbf{V}_1 = \mathbf{z}_{11}\mathbf{I}_1 + \mathbf{z}_{12}\mathbf{I}_2$$
$$\mathbf{V}_2 = \mathbf{z}_{21}\mathbf{I}_1 + \mathbf{z}_{22}\mathbf{I}_2$$

Notice: Only two of the four variables (V_1, V_2, I_1, and I_2) are independent. The other two can be found using above eq.s

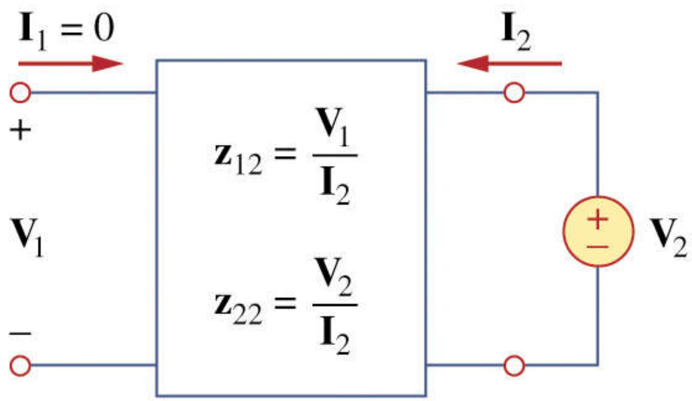
Z Parameters – Expression



(a)

$$\mathbf{z}_{11} = \frac{\mathbf{V}_1}{\mathbf{I}_1}, \quad \mathbf{z}_{21} = \frac{\mathbf{V}_2}{\mathbf{I}_1}$$

$$\mathbf{z}_{12} = \frac{\mathbf{V}_1}{\mathbf{I}_2}, \quad \mathbf{z}_{22} = \frac{\mathbf{V}_2}{\mathbf{I}_2}$$



(b)

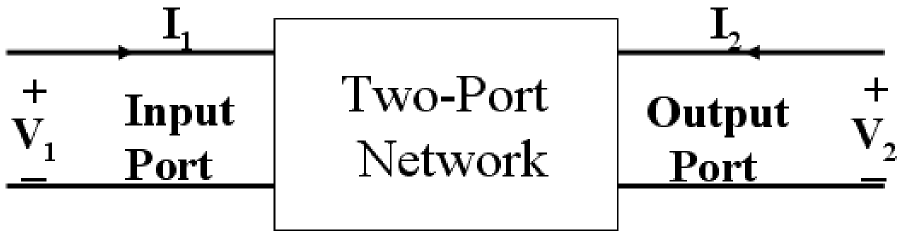
$$\mathbf{V}_1 = \mathbf{z}_{11}\mathbf{I}_1 + \mathbf{z}_{12}\mathbf{I}_2$$
$$\mathbf{V}_2 = \mathbf{z}_{21}\mathbf{I}_1 + \mathbf{z}_{22}\mathbf{I}_2$$

$$\begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{z}_{11} & \mathbf{z}_{12} \\ \mathbf{z}_{21} & \mathbf{z}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix} = [\mathbf{z}] \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix}$$

$\mathbf{z}_{11}$ and $\mathbf{z}_{22}$: called driving-point impedances

$\mathbf{z}_{12}$ and $\mathbf{z}_{21}$: called transfer impedances

Z Parameters – Expression



$$z_{11} = \frac{V_1}{I_1}\bigg|_{I_2=0}$$

z_{11} is the impedance seen looking into port 1 when port 2 is open.

$$z_{12} = \frac{V_1}{I_2}\bigg|_{I_1=0}$$

z_{12} is a transfer impedance. It is the ratio of the voltage at port 1 to the current at port 2 when port 1 is open.

$$z_{21} = \frac{V_2}{I_1}\bigg|_{I_2=0}$$

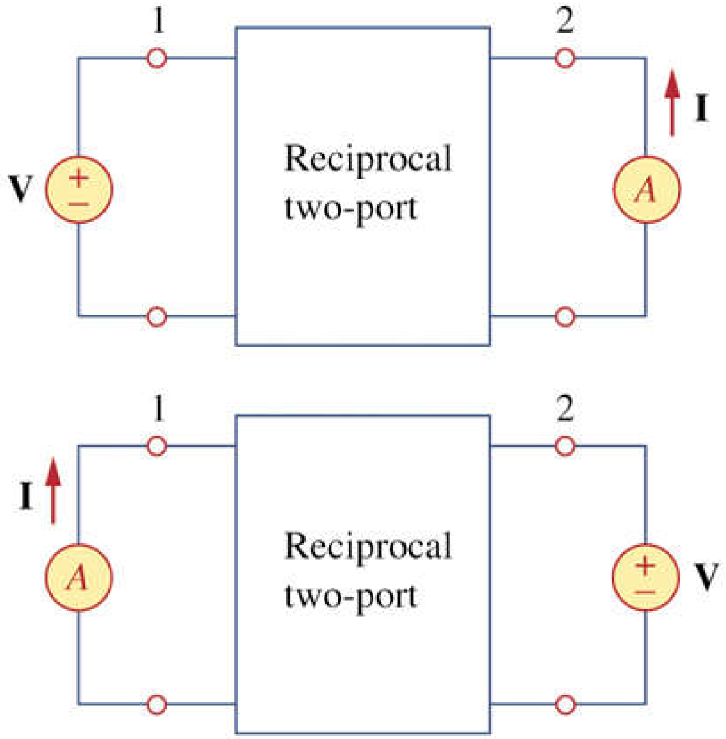
z_{21} is a transfer impedance. It is the ratio of the voltage at port 2 to the current at port 1 when port 2 is open.

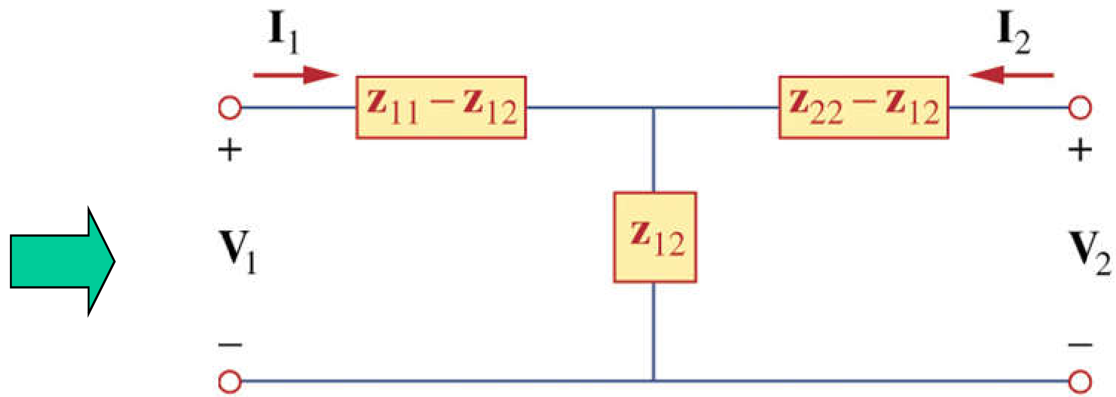
$$z_{22} = \frac{V_2}{I_2}\bigg|_{I_1=0}$$

z_{22} is the impedance seen looking into port 2 when port 1 is open.

Z Parameters – Reciprocal

When the two-port network is linear and has no dependent sources: $\mathbf{z}_{12} = \mathbf{z}_{21}$, and the two-port is said to be reciprocal – Reciprocal Two-port Network





T-equivalent circuit for reciprocal two-port network

Any two-port made entirely of resistors, capacitors and inductors must be reciprocal.

Z Parameters – *Symmetrical*

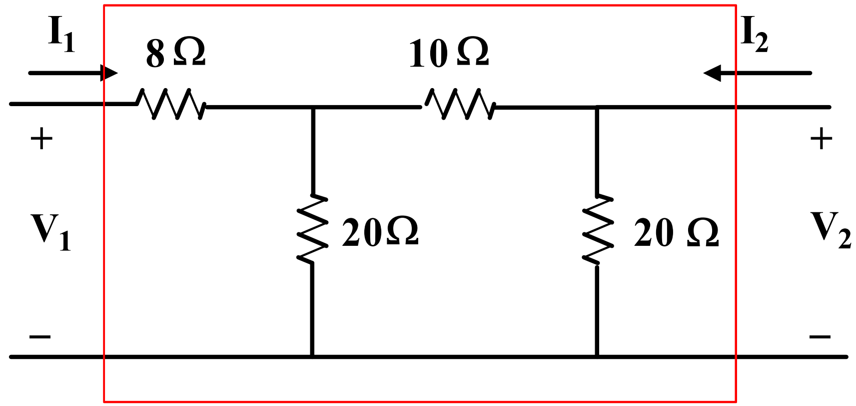
$$\mathbf{V}_1 = \mathbf{z}_{11}\mathbf{I}_1 + \mathbf{z}_{12}\mathbf{I}_2$$
$$\mathbf{V}_2 = \mathbf{z}_{21}\mathbf{I}_1 + \mathbf{z}_{22}\mathbf{I}_2$$

When $\mathbf{z}_{11} = \mathbf{z}_{22}$, the two-port network is said to be symmetrical – mirror-like.

If the two-port network is reciprocal and symmetrical, only 2 parameters need to be determined.

Z Parameters – Example 1

Given the following circuit. Determine the **Z** parameters.



Solution

$$z_{11} = \left.\frac{V_1}{I_1}\right|_{I_2=0} \qquad z_{22} = \left.\frac{V_2}{I_2}\right|_{I_1=0}$$

$$\mathbf{z}_{11} = 8 + 20\|30 = 20\ \Omega$$

$$\mathbf{z}_{22} = 20\|30 = 12\ \Omega$$

$$\mathbf{V}_1 = \frac{20\times \mathbf{I}_2 \times 20}{20+30} = 8\times \mathbf{I}_2$$

$$\mathbf{z}_{12} = \frac{8\times \mathbf{I}_2}{\mathbf{I}_2} = 8\ \ \Omega$$

Reciprocal Two-Port Network

$$\mathbf{z}_{21} = \mathbf{z}_{12} = 8\ \ \Omega$$

The **Z** parameter equations can be expressed in matrix form as follows:

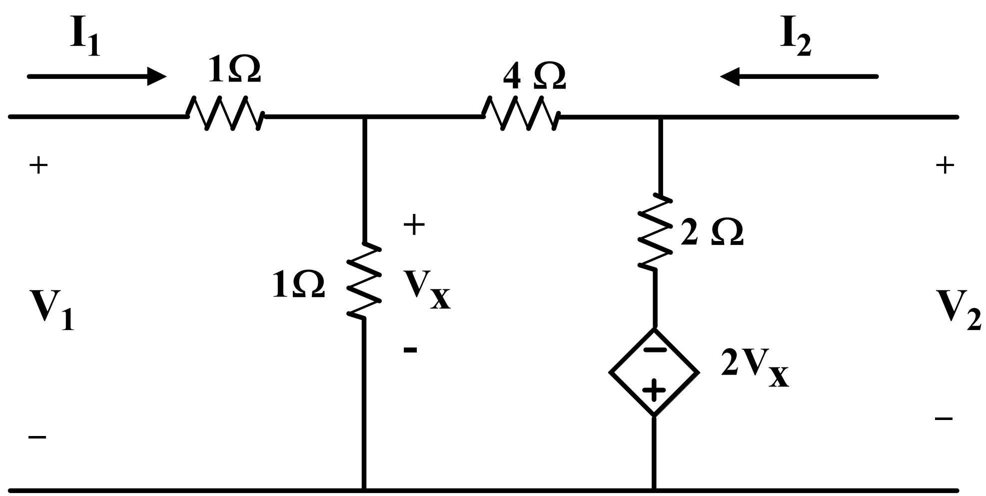
$$\begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{z}_{11} & \mathbf{z}_{12} \\ \mathbf{z}_{21} & \mathbf{z}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{bmatrix} = \begin{bmatrix} 20 & 8 \\ 8 & 12 \end{bmatrix} \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix}$$

Z Parameters – Example 2

Find the **Z** parameters the following circuit.

Solution



$$z_{11} = \left.\frac{V_1}{I_1}\right|_{I_2=0}$$

$$\mathbf{V}_x = (\mathbf{I}_1 - \frac{\mathbf{V}_x}{1}) \times (4+2) - 2\mathbf{V}_x$$

$$3\mathbf{V}_x = 6\mathbf{I}_1 - 6\mathbf{V}_x \Rightarrow \mathbf{I}_1 = \frac{3\mathbf{V}_x}{2}$$

$$\mathbf{V}_x = \mathbf{V}_1 - \mathbf{I}_1 \times 1$$

Substituting gives

$$\mathbf{I}_1 = \frac{3(\mathbf{V}_1 - \mathbf{I}_1)}{2} \quad \text{or} \quad \frac{\mathbf{V}_1}{\mathbf{I}_1} = \mathbf{z}_{11} = 1.667 \quad \Omega$$

$\mathbf{z_{21} = -0.667\ \Omega}$

$\mathbf{z_{12} = 0.222\ \Omega}$

$\mathbf{z_{22} = 1.111\ \Omega}$

Admittance Parameters – Definition

$$\mathbf{y}_{11} = \frac{\mathbf{I}_1}{\mathbf{V}_1}\bigg|_{\mathbf{V}_2=0}$$

$\mathbf{y}_{11}$ is the admittance seen looking into port 1 when port 2 is shorted.

$$\mathbf{y}_{12} = \frac{\mathbf{I}_1}{\mathbf{V}_2}\bigg|_{\mathbf{V}_1=0}$$

$\mathbf{y}_{12}$ is a transfer admittance. It is the ratio of the current at port 1 to the voltage at port 2 when port 1 is shorted.

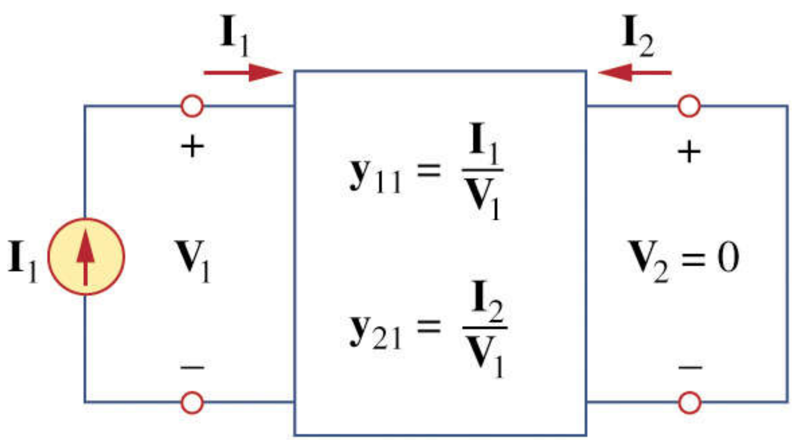
$$\mathbf{y}_{21} = \frac{\mathbf{I}_2}{\mathbf{V}_1}\bigg|_{\mathbf{V}_2=0}$$

$\mathbf{y}_{21}$ is a transfer admittance. It is the ratio of the current at port 2 to the voltage at port 1 when port 2 is shorted.

$$\mathbf{y}_{22} = \frac{\mathbf{I}_2}{\mathbf{V}_2}\bigg|_{\mathbf{V}_1=0}$$

$\mathbf{y}_{22}$ is the admittance seen looking into port 2 when port 1 is shorted.

Admittance Parameters – Expression



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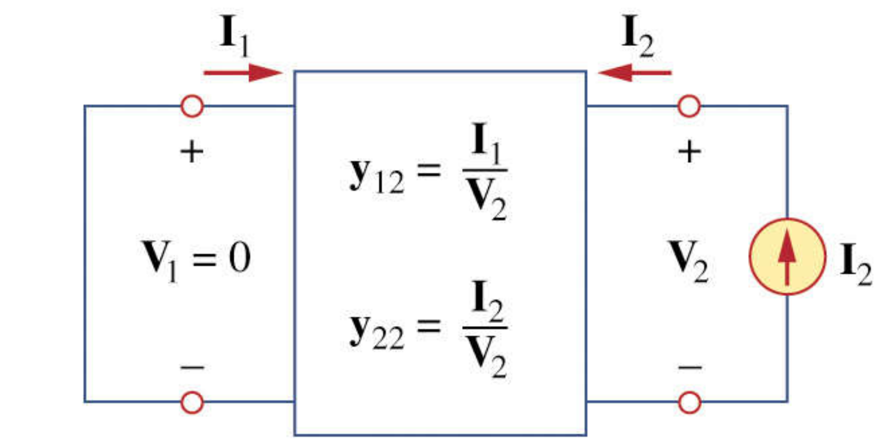


(b)

$$\mathbf{y}_{11} = \frac{\mathbf{I}_1}{\mathbf{V}_1}, \quad \mathbf{y}_{12} = \frac{\mathbf{I}_1}{\mathbf{V}_2}$$

$$\mathbf{y}_{21} = \frac{\mathbf{I}_2}{\mathbf{V}_1}, \quad \mathbf{y}_{22} = \frac{\mathbf{I}_2}{\mathbf{V}_2}$$

$$\mathbf{I}_1 = \mathbf{y}_{11}\mathbf{V}_1 + \mathbf{y}_{12}\mathbf{V}_2$$

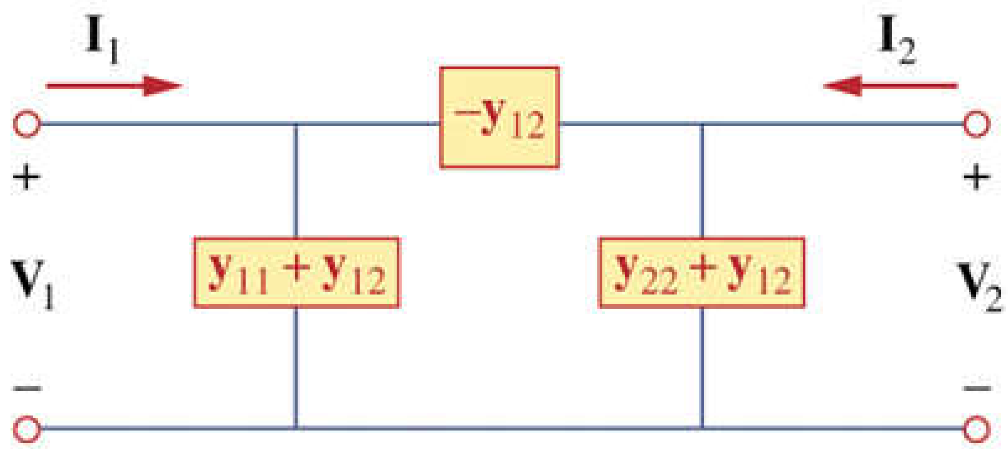
$$\mathbf{I}_2 = \mathbf{y}_{21}\mathbf{V}_1 + \mathbf{y}_{22}\mathbf{V}_2$$

$$\begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{y}_{11} & \mathbf{y}_{12} \\ \mathbf{y}_{21} & \mathbf{y}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{bmatrix} = [\mathbf{y}] \begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{bmatrix}$$

Admittance Parameters – *Reciprocal & Symmetrical*

When the two-port network is linear and has no dependent sources: $\mathbf{y}_{12} = \mathbf{y}_{21}$, Reciprocal Two-Port Network

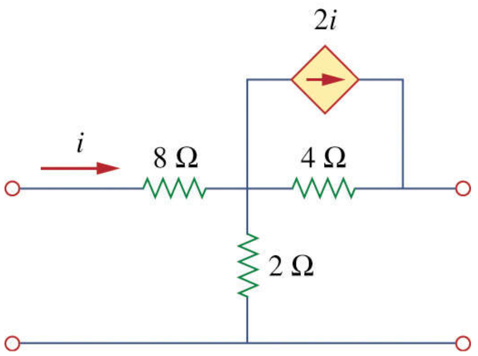
Π-equivalent circuit for reciprocal two-port network



When $\mathbf{y}_{11} = \mathbf{y}_{22}$, the two-port network is said to be symmetrical – mirror-like

Admittance Parameters – Example

Determine the **y** parameters in the circuit.

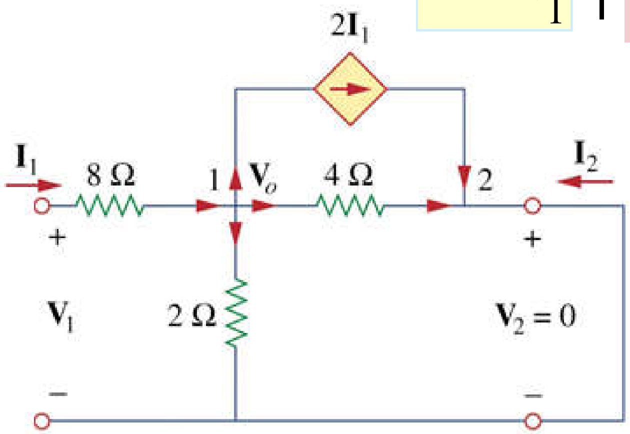


Solution

Find $\mathbf{y}_{11}$ and $\mathbf{y}_{21}$

$$\mathbf{y}_{11} = \frac{\mathbf{I}_1}{\mathbf{V}_1}\bigg|_{\mathbf{V}_2=0}$$

$$\mathbf{y}_{21} = \frac{\mathbf{I}_2}{\mathbf{V}_1}\bigg|_{\mathbf{V}_2=0}$$



At node 1, $\dfrac{\mathbf{V}_1 - \mathbf{V}_o}{8} = 2\mathbf{I}_1 + \dfrac{\mathbf{V}_o}{2} + \dfrac{\mathbf{V}_o}{4}$

But $\mathbf{I}_1 = \dfrac{\mathbf{V}_1 - \mathbf{V}_o}{8}$, therefore, $0 = \dfrac{\mathbf{V}_1 - \mathbf{V}_o}{8} + \dfrac{3\mathbf{V}_o}{4}$

$0 = \mathbf{V}_1 - \mathbf{V}_o + 6\mathbf{V}_o \Rightarrow \mathbf{V}_1 = -5\mathbf{V}_o$

Hence, $\mathbf{I}_1 = \dfrac{-5\mathbf{V}_o - \mathbf{V}_o}{8} = -0.75\,\mathbf{V}_o$

and $\mathbf{y}_{11} = \dfrac{\mathbf{I}_1}{\mathbf{V}_1} = \dfrac{-0.75\mathbf{V}_o}{-5\mathbf{V}_o} = 0.15\,\mathrm{S}$

$\mathbf{V}_1 = -5\mathbf{V}_o,\ \ \mathbf{I}_1 = 0.75\mathbf{V}_o$

At node 2, $\dfrac{\mathbf{V}_o - 0}{4} + 2\mathbf{I}_1 + \mathbf{I}_2 = 0$

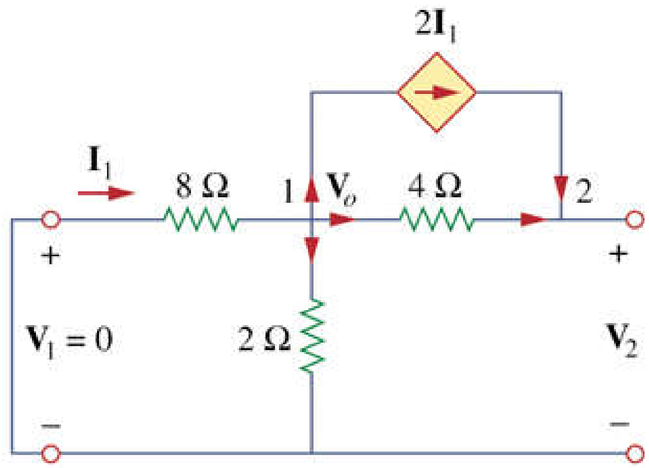
or $-\mathbf{I}_2 = 0.25\mathbf{V}_o - 1.5\mathbf{V}_o = -1.25\mathbf{V}_o$

Hence, $\mathbf{y}_{21} = \dfrac{\mathbf{I}_2}{\mathbf{V}_1} = \dfrac{1.25\mathbf{V}_o}{-5\mathbf{V}_o} = -0.25\,\mathrm{S}$

Admittance Parameters – Example cont.

Find $\mathbf{y}_{22}$ and $\mathbf{y}_{12}$

$y_{22}=\frac{I_2}{V_2}\Big|_{V_1=0}$ $\quad$ $y_{12}=\frac{I_1}{V_2}\Big|_{V_1=0}$



Similarly, at node 1: $\dfrac{0-\mathbf{V}_o}{8}=2\mathbf{I}_1+\dfrac{\mathbf{V}_o}{2}+\dfrac{\mathbf{V}_o-\mathbf{V}_2}{4}$

And $\mathbf{I}_1=\dfrac{0-\mathbf{V}_o}{8}, \Rightarrow 0=-\dfrac{\mathbf{V}_o}{8}+\dfrac{\mathbf{V}_o}{2}+\dfrac{\mathbf{V}_o-\mathbf{V}_2}{4}$

$\Rightarrow 0=-\mathbf{V}_o+4\mathbf{V}_o+2\mathbf{V}_o-2\mathbf{V}_2 \Rightarrow \mathbf{V}_2=2.5\mathbf{V}_o$

Hence: $\mathbf{y}_{12}=\dfrac{\mathbf{I}_1}{\mathbf{V}_2}=\dfrac{-\mathbf{V}_o/8}{2.5\mathbf{V}_o}=-0.05\,\mathrm{S}$

At node 2: $\dfrac{\mathbf{V}_o-\mathbf{V}_2}{4}+2\mathbf{I}_1+\mathbf{I}_2=0$

$\Rightarrow -\mathbf{I}_2=0.25\mathbf{V}_o-\dfrac{1}{4}(2.5)\mathbf{V}_o-\dfrac{2\mathbf{V}_o}{8}=-0.625\mathbf{V}_o$

Thus: $\mathbf{y}_{22}=\dfrac{\mathbf{I}_2}{\mathbf{V}_2}=\dfrac{0.625\mathbf{V}_o}{2.5\mathbf{V}_o}=0.25\,\mathrm{S}$

The **Y** parameter equations can be expressed in matrix form as follows:

$$\mathbf{Y}=\begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix}$$

$$\mathbf{Y}=\begin{bmatrix} 0.15 & -0.05 \\ -0.25 & 0.25 \end{bmatrix}$$

Transmission Parameters (A,B,C,D)

The defining equations are:

$$\mathbf{A} = \frac{\mathbf{V}_1}{\mathbf{V}_2}\bigg|_{\mathbf{I_2 = 0}}$$

$$\mathbf{B} = \frac{\mathbf{V}_1}{-\mathbf{I}_2}\bigg|_{\mathbf{V_2 = 0}}$$

$$\mathbf{C} = \frac{\mathbf{I}_1}{\mathbf{V}_2}\bigg|_{\mathbf{I_2 = 0}}$$

$$\mathbf{D} = \frac{\mathbf{I}_1}{-\mathbf{I}_2}\bigg|_{\mathbf{V_2 = 0}}$$

$$\mathbf{V}_1 = \mathbf{A}\mathbf{V}_2 - \mathbf{B}\mathbf{I}_2$$

$$\mathbf{I}_1 = \mathbf{C}\mathbf{V}_2 - \mathbf{D}\mathbf{I}_2$$

$$\begin{bmatrix} \mathbf{V}_1 \\ \mathbf{I}_1 \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \begin{bmatrix} \mathbf{V}_2 \\ -\mathbf{I}_2 \end{bmatrix} = [\mathbf{T}] \begin{bmatrix} \mathbf{V}_2 \\ -\mathbf{I}_2 \end{bmatrix}$$

A and **D** are dimensionless, **B** is in Ohms, and **C** is in siemens.

When **AD-BC** = 1, the two-port is reciprocal and **A** = **D**, symmetrical.

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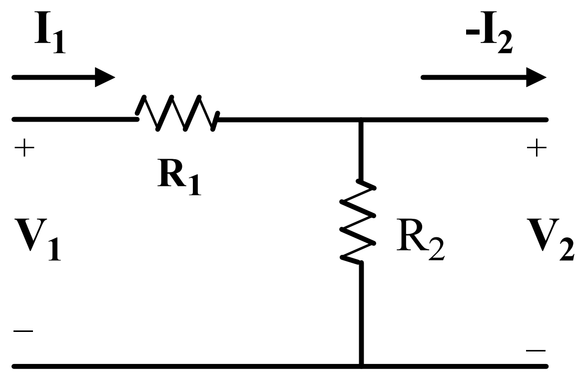
Transmission Parameters – Example 1

Given the network below with assumed voltage polarities and current directions. Find the transmission parameters.



Solution

$$V_1 = (R_1 + R_2)I_1 + R_2 I_2$$

$$V_2 = R_2 I_1 + R_2 I_2$$

From these equations we can directly evaluate the **A,B,C,D** parameters.

$$A = \frac{V_1}{V_2}\bigg|_{I_2 = 0} = \frac{R_1 + R_2}{R_2} \qquad B = \frac{V_1}{-I_2}\bigg|_{V_2 = 0} = R_1$$

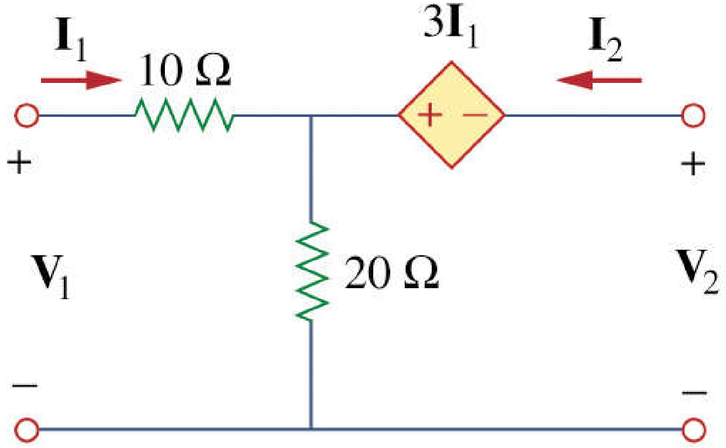
$$C = \frac{I_1}{V_2}\bigg|_{I_2 = 0} = \frac{1}{R_2} \qquad D = \frac{I_1}{-I_2}\bigg|_{V_2 = 0} = 1$$

Transmission Parameters – Example 2

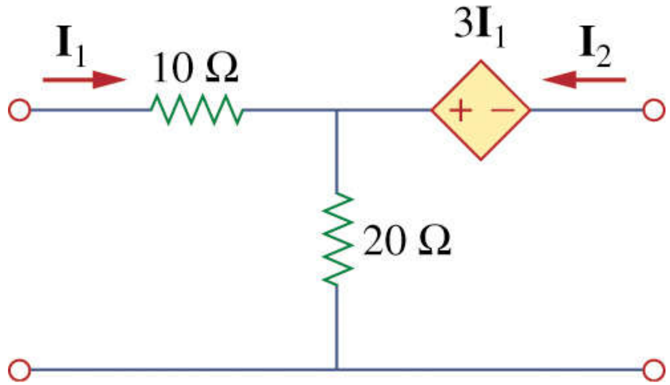
Find the transmission parameters.

Solution

$$\mathbf{A} = \left.\frac{\mathbf{V}_1}{\mathbf{V}_2}\right|_{\mathbf{I}_2 = 0} \qquad \mathbf{C} = \left.\frac{\mathbf{I}_1}{\mathbf{V}_2}\right|_{\mathbf{I}_2 = 0}$$



(a)



From Fig. (a)

$$\mathbf{V}_1 = (10+20)\mathbf{I}_1 = 30\mathbf{I}_1$$

$$\mathbf{V}_2 = 20\mathbf{I}_1 - 3\mathbf{I}_1 = 17\mathbf{I}_1$$

Thus

$$\mathbf{A} = \frac{\mathbf{V}_1}{\mathbf{V}_2} = \frac{30\mathbf{I}_1}{17\mathbf{I}_1} = 1.765, \quad \mathbf{C} = \frac{\mathbf{I}_1}{\mathbf{V}_2} = \frac{\mathbf{I}_1}{17\mathbf{I}_1} = 0.0588\ S$$

Transmission Parameters – Example 2 cont.

$$\boxed{\mathbf{B} = \frac{\mathbf{V}_1}{-\mathbf{I}_2}}\Bigg|\ \boxed{\mathbf{V}_2 = 0} \qquad \boxed{\mathbf{D} = \frac{\mathbf{I}_1}{-\mathbf{I}_2}}\Bigg|\ \boxed{\mathbf{V}_2 = 0}$$

From Fig. (b):

$$\frac{\mathbf{V}_1 - \mathbf{V}_a}{10} - \frac{\mathbf{V}_a}{20} + \mathbf{I}_2 = 0$$

$$\mathbf{V}_a = 3\mathbf{I}_1 \text{ and } \mathbf{I}_1 = (\mathbf{V}_1 - \mathbf{V}_a)/10$$

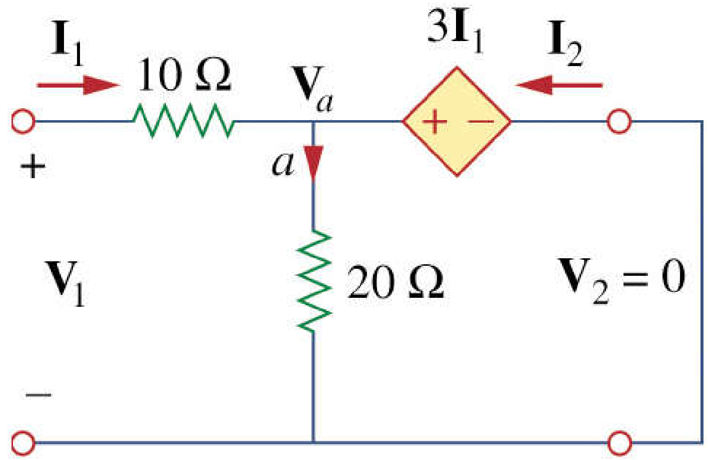
$$\Rightarrow \mathbf{V}_1 = 13\mathbf{I}_1$$

$$\Rightarrow \mathbf{I}_1 - \frac{3\mathbf{I}_1}{20} + \mathbf{I}_2 = 0 \Rightarrow \frac{17}{20}\mathbf{I}_1 = -\mathbf{I}_2$$

Therefore,

$$\mathbf{D} = -\frac{\mathbf{I}_1}{\mathbf{I}_2} = \frac{20}{17} = 1.176$$

$$\mathbf{B} = -\frac{\mathbf{V}_1}{\mathbf{V}_2} = \frac{-13\mathbf{I}_1}{(17/20)\mathbf{I}_1} = 15.29 \quad \Omega$$



(b)

$$[\mathbf{T}] = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} = \begin{bmatrix} 1.765 & 15.29\ \Omega \\ 0.0588\ \mathrm{S} & 1.176 \end{bmatrix}$$

Summary

z parameters	y parameters	T parameters (A,B,C,D)
$z_{11} = \left.\frac{V_1}{I_1}\right\vert_{I_2=0}$		
$z_{12} = \left.\frac{V_1}{I_2}\right\vert_{I_1=0}$		
$z_{21} = \left.\frac{V_2}{I_1}\right\vert_{I_2=0}$		
$z_{22} = \left.\frac{V_2}{I_2}\right\vert_{I_1=0}$		

Relationship between y and z Parameters

Given the **z** parameters, find the **y** parameters:

$$\begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{z}_{11} & \mathbf{z}_{12} \\ \mathbf{z}_{21} & \mathbf{z}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix} = [\mathbf{z}] \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix} \Rightarrow \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix} = [\mathbf{z}]^{-1} \begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{bmatrix}$$

$[\mathbf{z}]^{-1}$ is the inverse of $[\mathbf{z}]$

$$[\mathbf{y}] = [\mathbf{z}]^{-1}$$

$$\begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{y}_{11} & \mathbf{y}_{12} \\ \mathbf{y}_{21} & \mathbf{y}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{bmatrix} = [\mathbf{y}] \begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{bmatrix}$$

Relationship between y and z Parameters

$$[\mathbf{z}] = \begin{bmatrix} \mathbf{z}_{11} & \mathbf{z}_{12} \\ \mathbf{z}_{21} & \mathbf{z}_{22} \end{bmatrix} \Rightarrow \Delta_{\mathbf{z}} = \mathbf{z}_{11}\mathbf{z}_{22} - \mathbf{z}_{12}\mathbf{z}_{21}$$

$$[\mathbf{y}] = [\mathbf{z}]^{-1}$$

$$\begin{bmatrix} \mathbf{y}_{11} & \mathbf{y}_{12} \\ \mathbf{y}_{21} & \mathbf{y}_{22} \end{bmatrix} = \frac{\begin{bmatrix} \mathbf{z}_{22} & -\mathbf{z}_{12} \\ -\mathbf{z}_{21} & \mathbf{z}_{11} \end{bmatrix}}{\Delta_{\mathbf{z}}} = \frac{\begin{bmatrix} \mathbf{z}_{22} & -\mathbf{z}_{12} \\ -\mathbf{z}_{21} & \mathbf{z}_{11} \end{bmatrix}}{\mathbf{z}_{11}\mathbf{z}_{22} - \mathbf{z}_{12}\mathbf{z}_{21}}$$

$$\boxed{\begin{aligned} \mathbf{y}_{11} &= \frac{\mathbf{z}_{22}}{\mathbf{z}_{11}\mathbf{z}_{22} - \mathbf{z}_{12}\mathbf{z}_{21}} \\ \mathbf{y}_{12} &= -\frac{\mathbf{z}_{12}}{\mathbf{z}_{11}\mathbf{z}_{22} - \mathbf{z}_{12}\mathbf{z}_{21}} \\ \mathbf{y}_{21} &= -\frac{\mathbf{z}_{21}}{\mathbf{z}_{11}\mathbf{z}_{22} - \mathbf{z}_{12}\mathbf{z}_{21}} \\ \mathbf{y}_{22} &= \frac{\mathbf{z}_{11}}{\mathbf{z}_{11}\mathbf{z}_{22} - \mathbf{z}_{12}\mathbf{z}_{21}} \end{aligned}}$$

Relationship between y and z Parameters

For the **y** parameters we have

$$\mathbf{I} = \mathbf{Y}\,\mathbf{V}$$

From above

$$\boxed{\mathbf{V} = \mathbf{Y}^{-1}\mathbf{I} = \mathbf{Z}\,\mathbf{I}}$$

For the **Z** parameters we have

$$\boxed{\mathbf{V} = \mathbf{Z}\,\mathbf{I}}$$

Therefore

$$\boxed{\mathbf{Z} = \mathbf{Y}^{-1} = \begin{bmatrix} \mathbf{z}_{11} & \mathbf{z}_{12} \\ \mathbf{z}_{21} & \mathbf{z}_{22} \end{bmatrix} = \begin{bmatrix} \dfrac{\mathbf{y}_{22}}{\Delta_{\mathbf{Y}}} & \dfrac{-\mathbf{y}_{12}}{\Delta_{\mathbf{Y}}} \\ \dfrac{-\mathbf{y}_{21}}{\Delta_{\mathbf{Y}}} & \dfrac{\mathbf{y}_{11}}{\Delta_{\mathbf{Y}}} \end{bmatrix}}$$

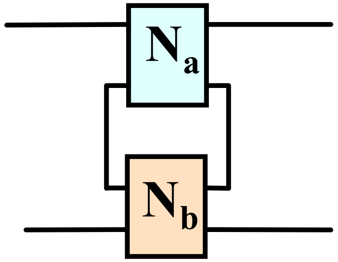
where

$$\Delta_{\mathbf{Y}} = \mathbf{y}_{11}\mathbf{y}_{22} - \mathbf{y}_{12}\mathbf{y}_{21}$$

Interconnection

Three ways that two ports are interconnected

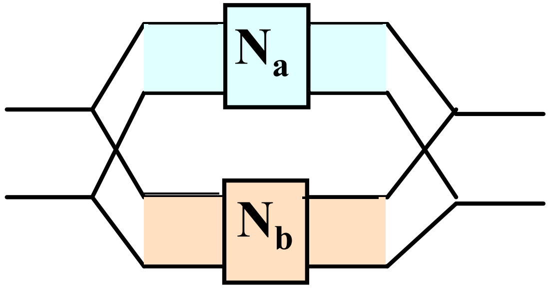
* Series



Z *parameters*

$$[\mathbf{z}] = [\mathbf{z}_a] + [\mathbf{z}_b]$$

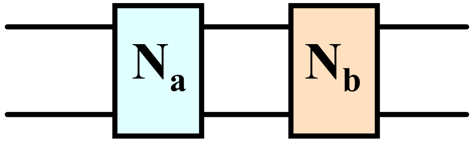
* Parallel



Y *parameters*

$$[\mathbf{y}] = [\mathbf{y}_a] + [\mathbf{y}_b]$$

* Cascade

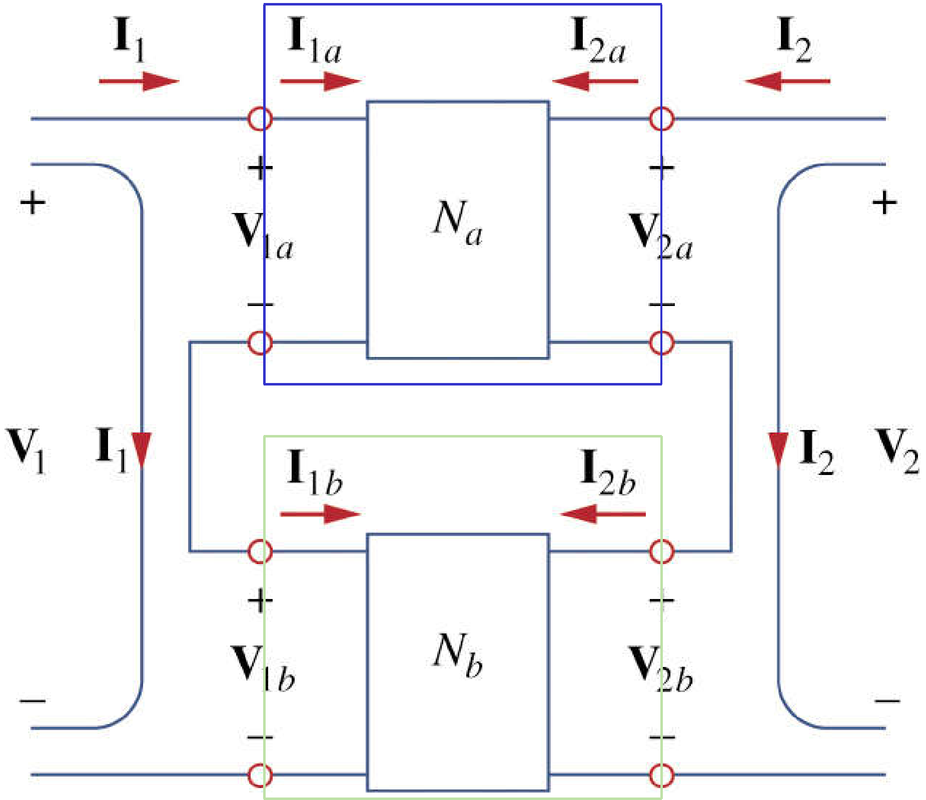


ABCD *parameters*

$$[\mathbf{T}] = [\mathbf{T}_a][\mathbf{T}_b]$$

Interconnection – Series Connection

$$\mathbf{V}_1 = \mathbf{z}_{11}\mathbf{I}_1 + \mathbf{z}_{12}\mathbf{I}_2$$
$$\mathbf{V}_2 = \mathbf{z}_{21}\mathbf{I}_1 + \mathbf{z}_{22}\mathbf{I}_2$$



For network N_a

$$\mathbf{V}_{1a} = \mathbf{z}_{11a}\mathbf{I}_{1a} + \mathbf{z}_{12a}\mathbf{I}_{2a}$$
$$\mathbf{V}_{2a} = \mathbf{z}_{21a}\mathbf{I}_{1a} + \mathbf{z}_{22a}\mathbf{I}_{2a}$$

For network N_b

$$\mathbf{V}_{1b} = \mathbf{z}_{11b}\mathbf{I}_{1b} + \mathbf{z}_{12b}\mathbf{I}_{2b}$$
$$\mathbf{V}_{2b} = \mathbf{z}_{21b}\mathbf{I}_{1b} + \mathbf{z}_{22b}\mathbf{I}_{2b}$$

From the diagram:

$$\mathbf{I}_1 = \mathbf{I}_{1a} = \mathbf{I}_{1b}, \quad \mathbf{I}_2 = \mathbf{I}_{2a} = \mathbf{I}_{2b}$$

Interconnection – Series Connection

$$\mathbf{V}_1 = \mathbf{z}_{11}\mathbf{I}_1 + \mathbf{z}_{12}\mathbf{I}_2$$
$$\mathbf{V}_2 = \mathbf{z}_{21}\mathbf{I}_1 + \mathbf{z}_{22}\mathbf{I}_2$$

From the diagram

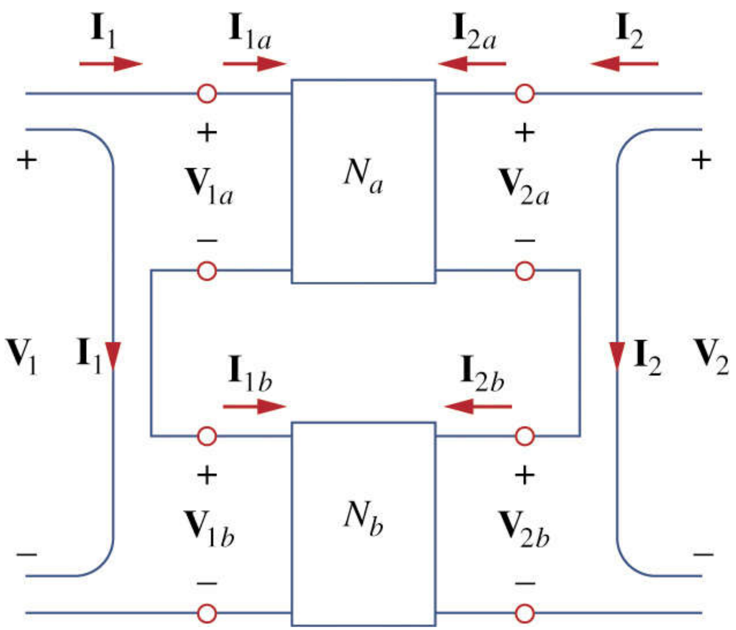
$$\mathbf{V}_1 = \mathbf{V}_{1a} + \mathbf{V}_{1b} = (\mathbf{z}_{11a} + \mathbf{z}_{11b})\mathbf{I}_1 + (\mathbf{z}_{12a} + \mathbf{z}_{12b})\mathbf{I}_2$$
$$\mathbf{V}_2 = \mathbf{V}_{2a} + \mathbf{V}_{2b} = (\mathbf{z}_{21a} + \mathbf{z}_{21b})\mathbf{I}_1 + (\mathbf{z}_{22a} + \mathbf{z}_{22b})\mathbf{I}_2$$

So

$$\begin{bmatrix} \mathbf{z}_{11} & \mathbf{z}_{12} \\ \mathbf{z}_{21} & \mathbf{z}_{22} \end{bmatrix} = \begin{bmatrix} \mathbf{z}_{11a} + \mathbf{z}_{11b} & \mathbf{z}_{12a} + \mathbf{z}_{12b} \\ \mathbf{z}_{21a} + \mathbf{z}_{21b} & \mathbf{z}_{22a} + \mathbf{z}_{22b} \end{bmatrix}$$

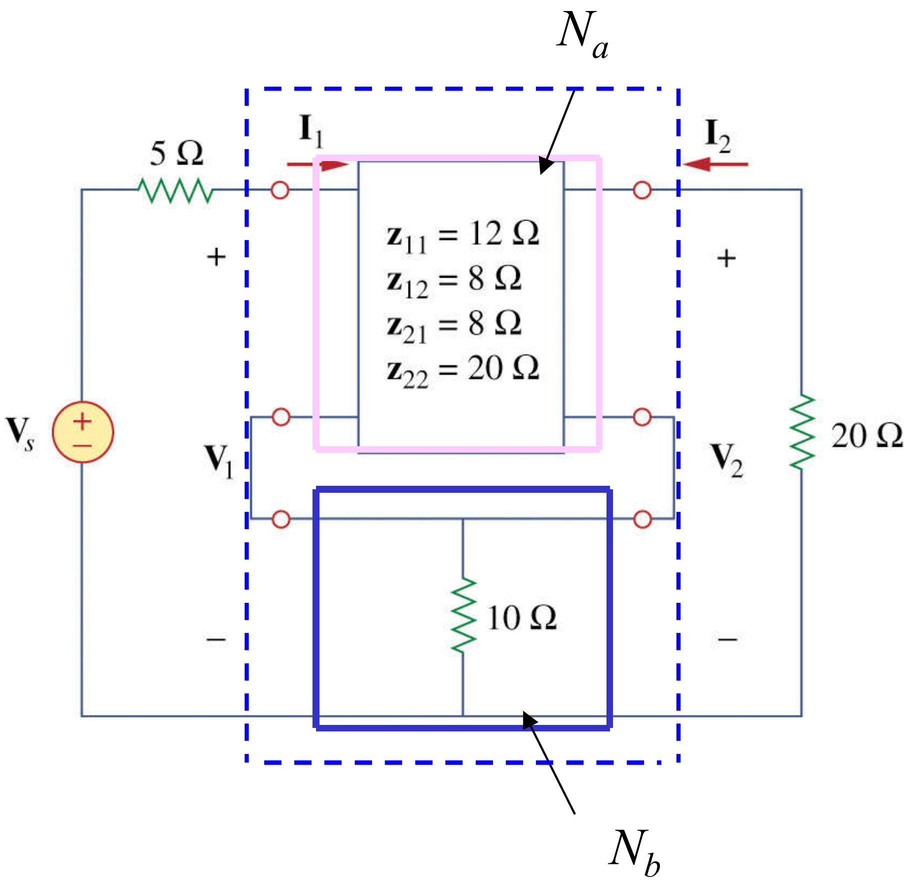
$$[\mathbf{z}] = [\mathbf{z}_a] + [\mathbf{z}_b]$$

This can be extended to *n* networks in series.



Interconnection – Series Connection Example

Find $\mathbf{V}_2/\mathbf{V}_s$ in the circuit



Solution

This can be regarded as two-ports in series.

For N_b,

$$\mathbf{z}_{12b} = \mathbf{z}_{21b} = 10 = \mathbf{z}_{11b} = \mathbf{z}_{22b}$$

Thus,

$$[\mathbf{z}] = [\mathbf{z}_a] + [\mathbf{z}_b]$$

$$= \begin{bmatrix} 12 & 8 \\ 8 & 20 \end{bmatrix} + \begin{bmatrix} 10 & 10 \\ 10 & 10 \end{bmatrix} = \begin{bmatrix} 22 & 18 \\ 18 & 30 \end{bmatrix}$$

And

$$\mathbf{V}_1 = \mathbf{z}_{11}\mathbf{I}_1 + \mathbf{z}_{12}\mathbf{I}_2 = 22\mathbf{I}_1 + 18\mathbf{I}_2$$

$$\mathbf{V}_2 = \mathbf{z}_{21}\mathbf{I}_1 + \mathbf{z}_{22}\mathbf{I}_2 = 18\mathbf{I}_1 + 30\mathbf{I}_2$$

Interconnection – Series Connection Example cont.

$$\mathbf{V}_1 = \mathbf{z}_{11}\mathbf{I}_1 + \mathbf{z}_{12}\mathbf{I}_2 = 22\mathbf{I}_1 + 18\mathbf{I}_2$$
$$\mathbf{V}_2 = \mathbf{z}_{21}\mathbf{I}_1 + \mathbf{z}_{22}\mathbf{I}_2 = 18\mathbf{I}_1 + 30\mathbf{I}_2$$

Also, at the input port $\mathbf{V}_1 = \mathbf{V}_s - 5\mathbf{I}_1$

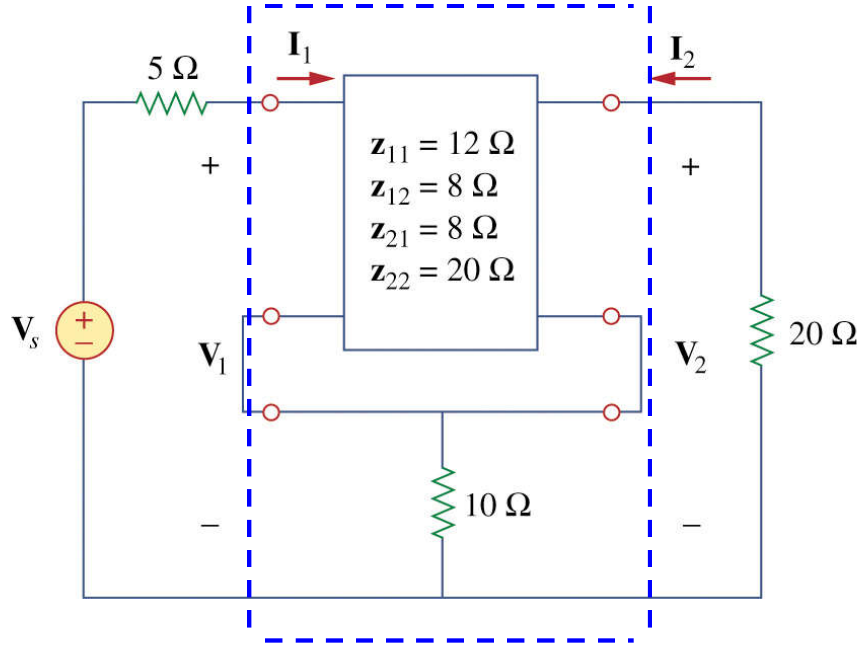
and at the output port $\mathbf{V}_2 = -20\mathbf{I}_2 \Rightarrow \mathbf{I}_2 = -\dfrac{\mathbf{V}_2}{20}$

$$\Rightarrow \mathbf{V}_s - 5\mathbf{I}_1 = 22\mathbf{I}_1 - \frac{18}{20}\mathbf{V}_2 \Rightarrow \mathbf{V}_s = 27\mathbf{I}_1 - 0.9\mathbf{V}_2$$

$$\mathbf{V}_2 = 18\mathbf{I}_1 - \frac{30}{20}\mathbf{V}_2 \Rightarrow \mathbf{I}_1 = \frac{2.5}{18}\mathbf{V}_2$$

$$\Rightarrow \mathbf{V}_s = 27 \times \frac{2.5}{18}\mathbf{V}_2 - 0.9\mathbf{V}_2 = 2.85\mathbf{V}_2$$

So: $\dfrac{\mathbf{V}_2}{\mathbf{V}_s} = \dfrac{1}{2.85} = 0.3509$



Interconnection – Parallel Connection

$$\mathbf{I}_1 = \mathbf{y}_{11}\mathbf{V}_1 + \mathbf{y}_{12}\mathbf{V}_2$$
$$\mathbf{I}_2 = \mathbf{y}_{21}\mathbf{V}_1 + \mathbf{y}_{22}\mathbf{V}_2$$

For network N_a

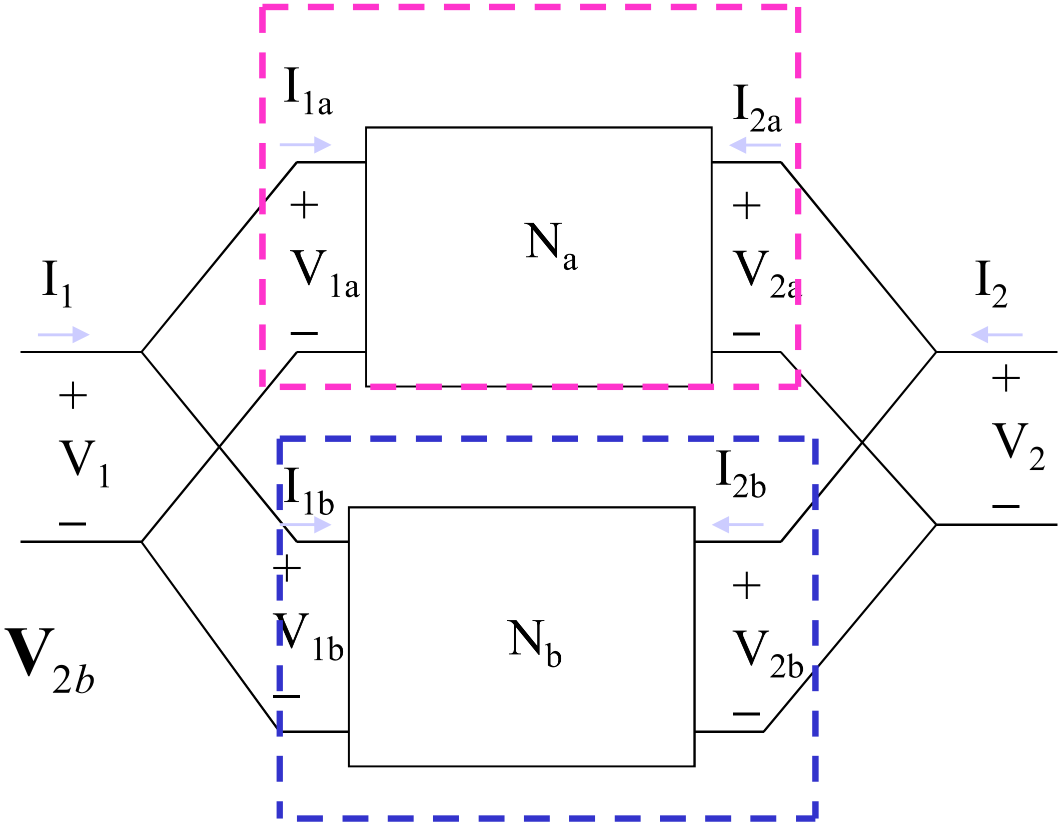
$$\mathbf{I}_{1a} = \mathbf{y}_{11a}\mathbf{V}_{1a} + \mathbf{y}_{12a}\mathbf{V}_{2a}$$
$$\mathbf{I}_{2a} = \mathbf{y}_{21a}\mathbf{V}_{1a} + \mathbf{y}_{22a}\mathbf{V}_{2a}$$

For network N_b

$$\mathbf{I}_{1b} = \mathbf{y}_{11b}\mathbf{V}_{1b} + \mathbf{y}_{12b}\mathbf{V}_{2b}$$
$$\mathbf{I}_{2b} = \mathbf{y}_{21b}\mathbf{V}_{1b} + \mathbf{y}_{22b}\mathbf{V}_{2b}$$

From the diagram :

$$\mathbf{V}_1 = \mathbf{V}_{1a} = \mathbf{V}_{1b}, \quad \mathbf{V}_2 = \mathbf{V}_{2a} = \mathbf{V}_{2b}$$
$$\mathbf{I}_1 = \mathbf{I}_{1a} + \mathbf{I}_{1b}, \quad \mathbf{I}_2 = \mathbf{I}_{2a} + \mathbf{I}_{2b}$$



Interconnection – Parallel Connection

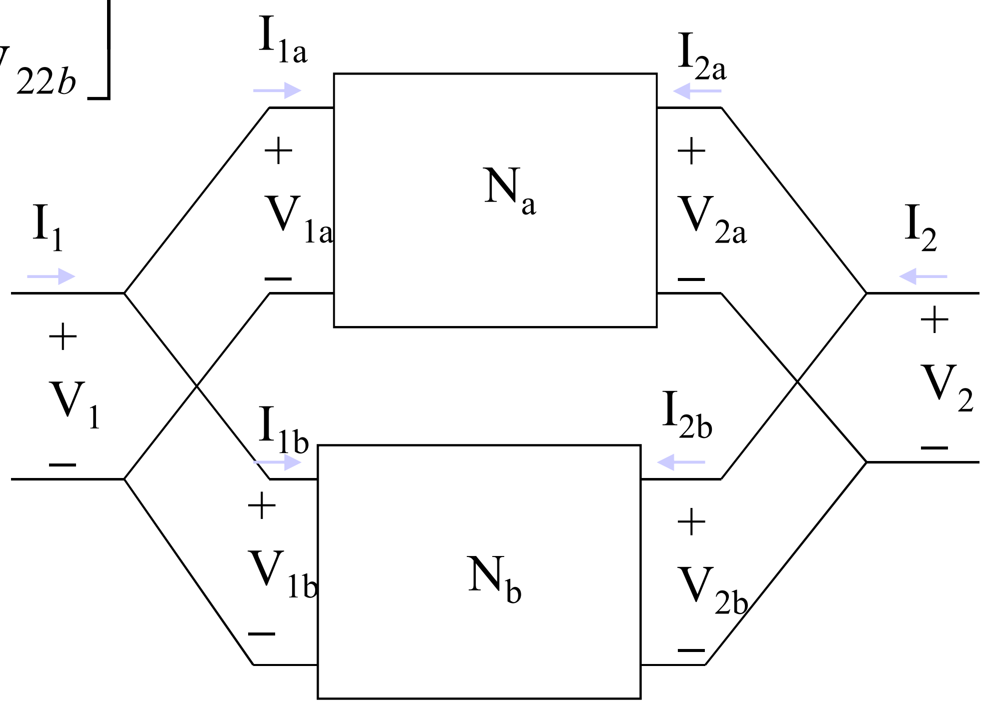
$$\mathbf{I}_1=\mathbf{I}_{1a}+\mathbf{I}_{1b}=(\mathbf{y}_{11a}+\mathbf{y}_{11b})\mathbf{V}_1+(\mathbf{y}_{12a}+\mathbf{y}_{12b})\mathbf{V}_2$$
$$\mathbf{I}_2=\mathbf{I}_{2a}+\mathbf{I}_{2b}=(\mathbf{y}_{21a}+\mathbf{y}_{21b})\mathbf{V}_1+(\mathbf{y}_{22a}+\mathbf{y}_{22b})\mathbf{V}_2$$

$$\boxed{\begin{aligned}\mathbf{I}_1&=\mathbf{y}_{11}\mathbf{V}_1+\mathbf{y}_{12}\mathbf{V}_2\\ \mathbf{I}_2&=\mathbf{y}_{21}\mathbf{V}_1+\mathbf{y}_{22}\mathbf{V}_2\end{aligned}}$$

So

$$\begin{bmatrix}\mathbf{y}_{11} & \mathbf{y}_{12}\\ \mathbf{y}_{21} & \mathbf{y}_{22}\end{bmatrix}=\begin{bmatrix}\mathbf{y}_{11a}+\mathbf{y}_{11b} & \mathbf{y}_{12a}+\mathbf{y}_{12b}\\ \mathbf{y}_{21a}+\mathbf{y}_{21b} & \mathbf{y}_{22a}+\mathbf{y}_{22b}\end{bmatrix}$$

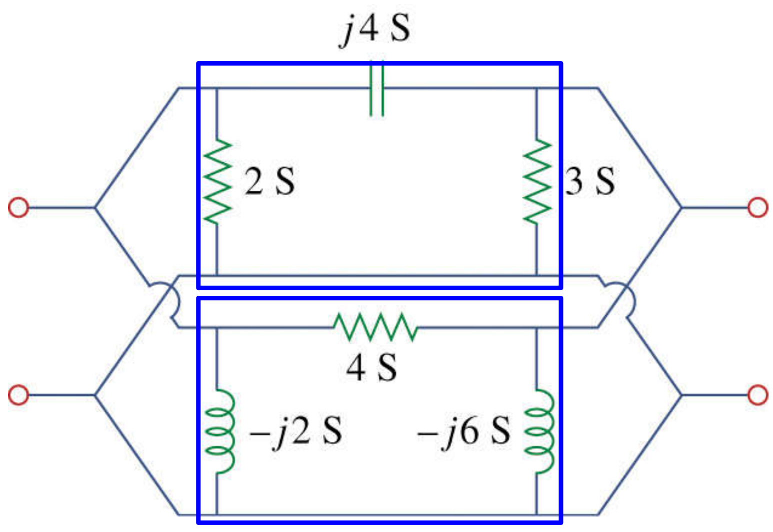
$$[\mathbf{y}]=[\mathbf{y}_a]+[\mathbf{y}_b]$$



This can be extended to n networks in parallel.

Interconnection – *Parallel Connection Example*

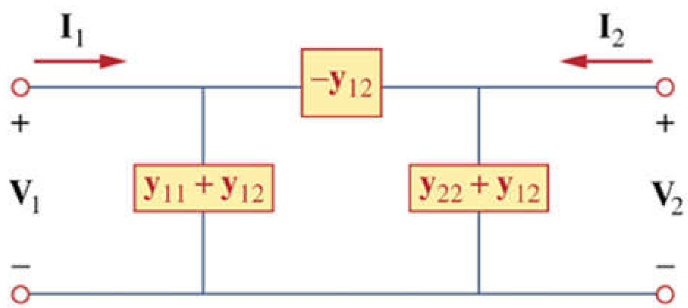
Find the y parameters of the two-port



Recall

When the two-port network is linear and has no dependent sources: $\mathbf{y}_{12} = \mathbf{y}_{21}$, Reciprocal Two-Port Network

Π-equivalent circuit for reciprocal two-port network

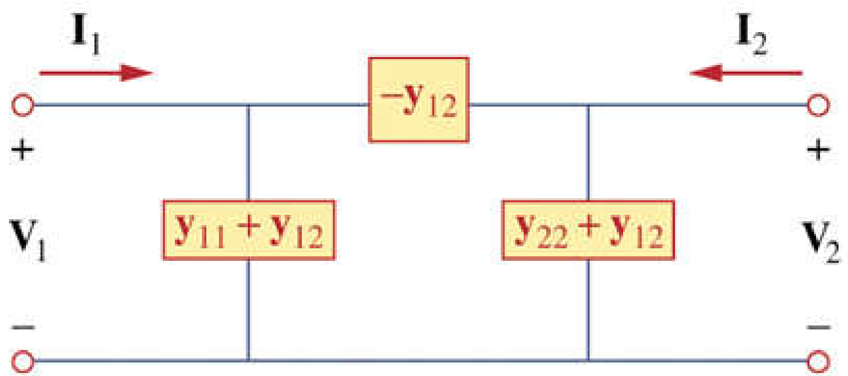


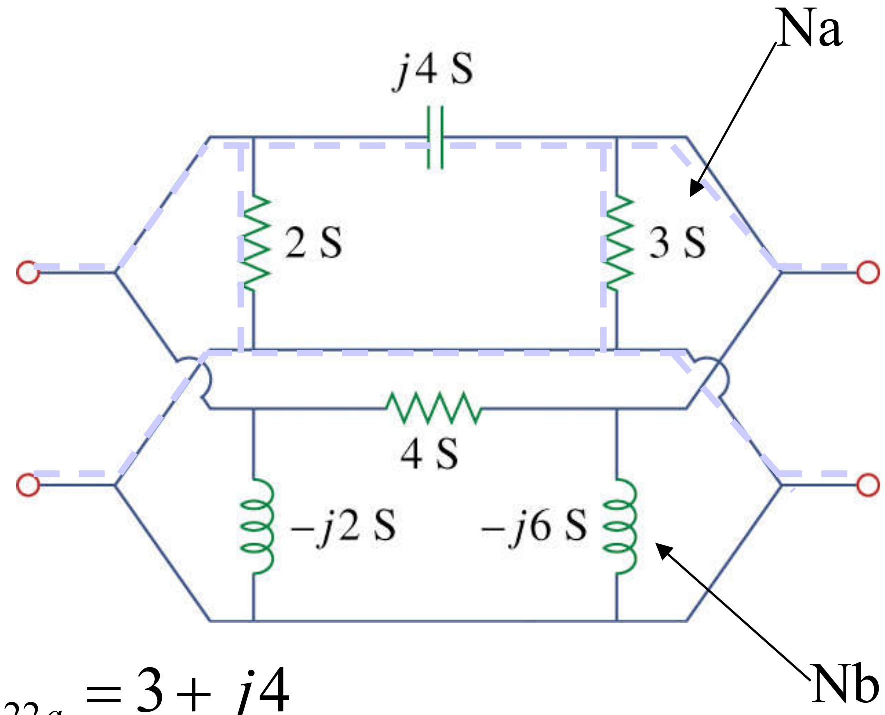
Recall

Any two-port made entirely of resistors, capacitors and inductors must be reciprocal.

Interconnection – *Parallel Connection Example cont.*

Solution





$$\mathbf{y}_{12a} = -j4 = \mathbf{y}_{21a} \qquad \mathbf{y}_{11a} = 2 + j4 \qquad \mathbf{y}_{22a} = 3 + j4$$

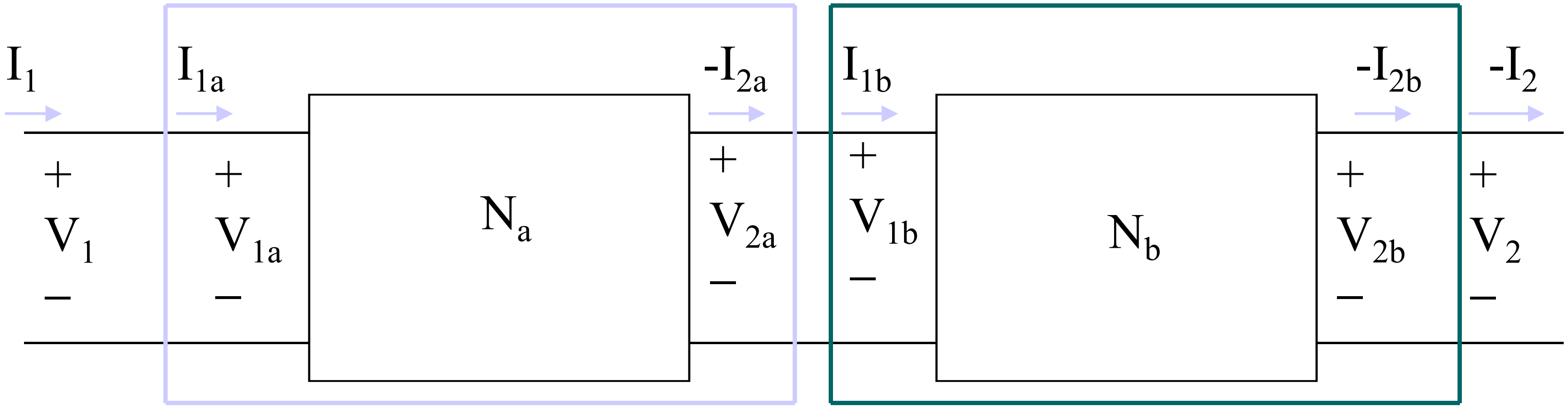
$$\mathbf{y}_{12b} = -4 = \mathbf{y}_{21b} \qquad \mathbf{y}_{11b} = 4 - j2 \qquad \mathbf{y}_{22b} = 4 - j6$$

$$[\mathbf{y}] = [\mathbf{y}_a] + [\mathbf{y}_b] = \begin{bmatrix} 6 + j2 & -4 - j4 \\ -4 - j4 & 7 - j2 \end{bmatrix} \quad \text{S}$$

Interconnection - Cascade Connection

Cascade connection of two 2-port networks: the output of one is the input of the other.

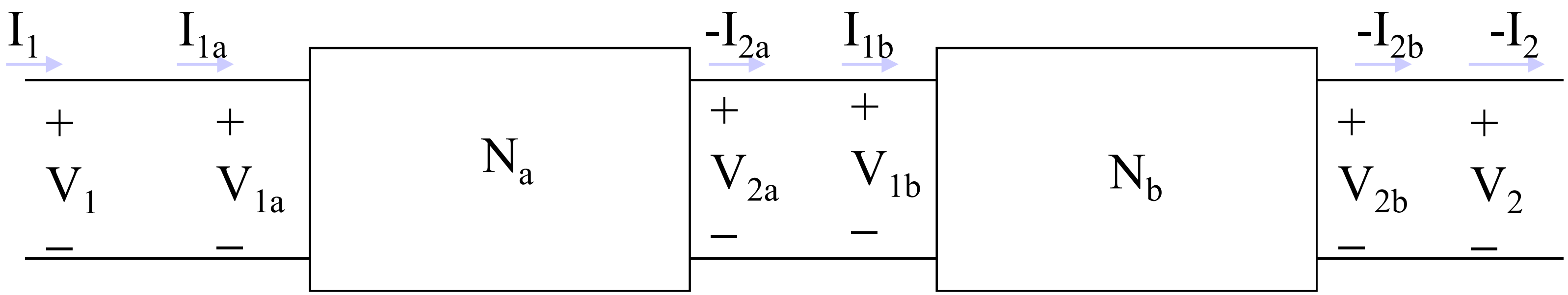
$$\mathbf{V}_1 = \mathbf{A}\mathbf{V}_2 - \mathbf{B}\mathbf{I}_2$$
$$\mathbf{I}_1 = \mathbf{C}\mathbf{V}_2 - \mathbf{D}\mathbf{I}_2$$



For the two networks

$$\begin{bmatrix} \mathbf{V}_{1a} \\ \mathbf{I}_{1a} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_a & \mathbf{B}_a \\ \mathbf{C}_a & \mathbf{D}_a \end{bmatrix} \begin{bmatrix} \mathbf{V}_{2a} \\ -\mathbf{I}_{2a} \end{bmatrix} \qquad \begin{bmatrix} \mathbf{V}_{1b} \\ \mathbf{I}_{1b} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_b & \mathbf{B}_b \\ \mathbf{C}_b & \mathbf{D}_b \end{bmatrix} \begin{bmatrix} \mathbf{V}_{2b} \\ -\mathbf{I}_{2b} \end{bmatrix}$$

Interconnection - Cascade Connection



$$\begin{bmatrix} \mathbf{V}_{1a} \\ \mathbf{I}_{1a} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_a & \mathbf{B}_a \\ \mathbf{C}_a & \mathbf{D}_a \end{bmatrix} \begin{bmatrix} \mathbf{V}_{2a} \\ -\mathbf{I}_{2a} \end{bmatrix} \qquad \begin{bmatrix} \mathbf{V}_{1b} \\ \mathbf{I}_{1b} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_b & \mathbf{B}_b \\ \mathbf{C}_b & \mathbf{D}_b \end{bmatrix} \begin{bmatrix} \mathbf{V}_{2b} \\ -\mathbf{I}_{2b} \end{bmatrix}$$

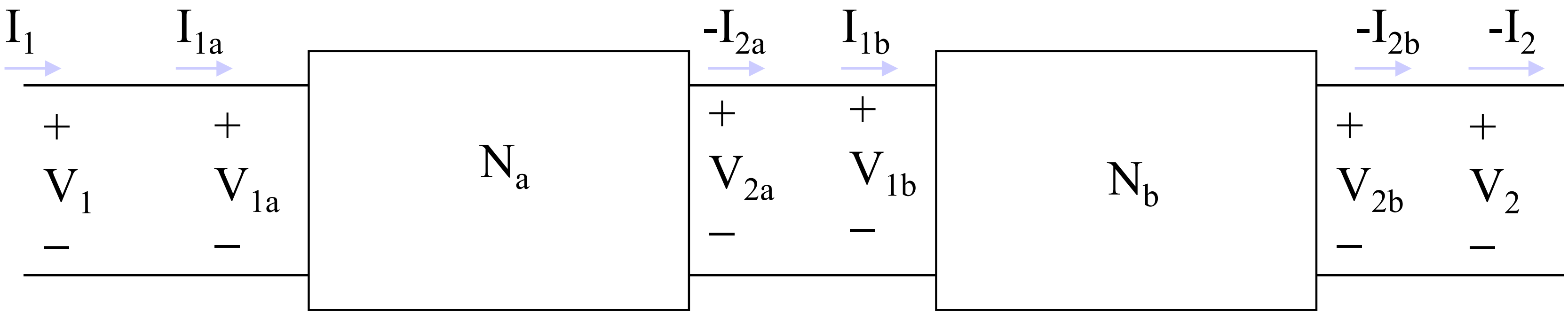
From the diagram

$$\begin{bmatrix} \mathbf{V}_1 \\ \mathbf{I}_1 \end{bmatrix} = \begin{bmatrix} \mathbf{V}_{1a} \\ \mathbf{I}_{1a} \end{bmatrix} \qquad \begin{bmatrix} \mathbf{V}_{2a} \\ -\mathbf{I}_{2a} \end{bmatrix} = \begin{bmatrix} \mathbf{V}_{1b} \\ \mathbf{I}_{1b} \end{bmatrix} \qquad \begin{bmatrix} \mathbf{V}_{2b} \\ -\mathbf{I}_{2b} \end{bmatrix} = \begin{bmatrix} \mathbf{V}_2 \\ -\mathbf{I}_2 \end{bmatrix}$$

So we have

$$\begin{bmatrix} \mathbf{V}_1 \\ \mathbf{I}_1 \end{bmatrix} = \begin{bmatrix} \mathbf{A}_a & \mathbf{B}_a \\ \mathbf{C}_a & \mathbf{D}_a \end{bmatrix} \begin{bmatrix} \mathbf{V}_{2a} \\ -\mathbf{I}_{2a} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_a & \mathbf{B}_a \\ \mathbf{C}_a & \mathbf{D}_a \end{bmatrix} \begin{bmatrix} \mathbf{A}_b & \mathbf{B}_b \\ \mathbf{C}_b & \mathbf{D}_b \end{bmatrix} \begin{bmatrix} \mathbf{V}_{2b} \\ -\mathbf{I}_{2b} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_a & \mathbf{B}_a \\ \mathbf{C}_a & \mathbf{D}_a \end{bmatrix} \begin{bmatrix} \mathbf{A}_b & \mathbf{B}_b \\ \mathbf{C}_b & \mathbf{D}_b \end{bmatrix} \begin{bmatrix} \mathbf{V}_2 \\ -\mathbf{I}_2 \end{bmatrix}$$

Interconnection - Cascade Connection



$$\begin{bmatrix} \mathbf{V}_1 \\ \mathbf{I}_1 \end{bmatrix} = \begin{bmatrix} \mathbf{A}_a & \mathbf{B}_a \\ \mathbf{C}_a & \mathbf{D}_a \end{bmatrix} \begin{bmatrix} \mathbf{A}_b & \mathbf{B}_b \\ \mathbf{C}_b & \mathbf{D}_b \end{bmatrix} \begin{bmatrix} \mathbf{V}_2 \\ -\mathbf{I}_2 \end{bmatrix}$$

$$\mathbf{V}_1 = \mathbf{A}\mathbf{V}_2 - \mathbf{B}\mathbf{I}_2$$
$$\mathbf{I}_1 = \mathbf{C}\mathbf{V}_2 - \mathbf{D}\mathbf{I}_2$$

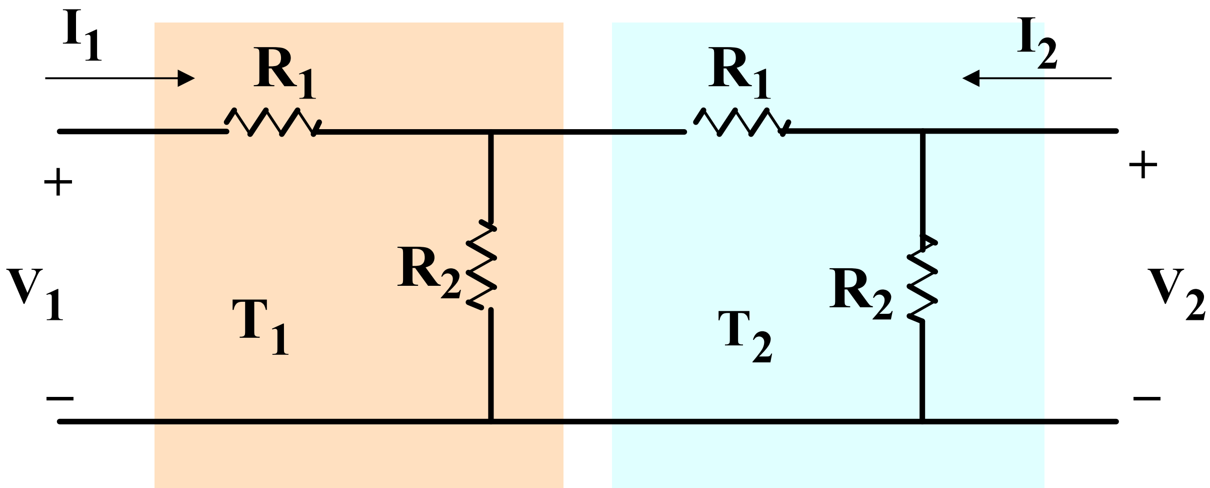
Then

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_a & \mathbf{B}_a \\ \mathbf{C}_a & \mathbf{D}_a \end{bmatrix} \begin{bmatrix} \mathbf{A}_b & \mathbf{B}_b \\ \mathbf{C}_b & \mathbf{D}_b \end{bmatrix} \quad or \quad [\mathbf{T}] = [\mathbf{T}_a][\mathbf{T}_b]$$

$$\boxed{[\mathbf{T}_a][\mathbf{T}_b] \neq [\mathbf{T}_b][\mathbf{T}_a]}$$

Interconnection - Cascade Connection (Example)

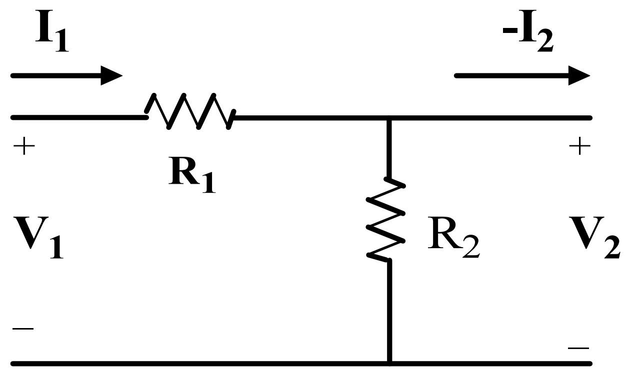
Find the A,B,C,D parameters



Transmission Parameters – Example 1 cont.

Recall

Given the network below with assumed voltage polarities and current directions. Find the transmission parameters.



Solution

$$V_1 = (R_1 + R_2)I_1 + R_2 I_2$$

$$V_2 = R_2 I_1 + R_2 I_2$$

From these equations we can directly evaluate the **A,B,C,D** parameters.

$$A = \left.\frac{V_1}{V_2}\right|_{I_2 = 0} = \frac{R_1 + R_2}{R_2} \qquad B = \left.\frac{V_1}{-I_2}\right|_{V_2 = 0} = R_1$$

$$C = \left.\frac{I_1}{V_2}\right|_{I_2 = 0} = \frac{1}{R_2} \qquad D = \left.\frac{I_1}{-I_2}\right|_{V_2 = 0} = 1$$

Transmission Parameters – Example 1 cont.

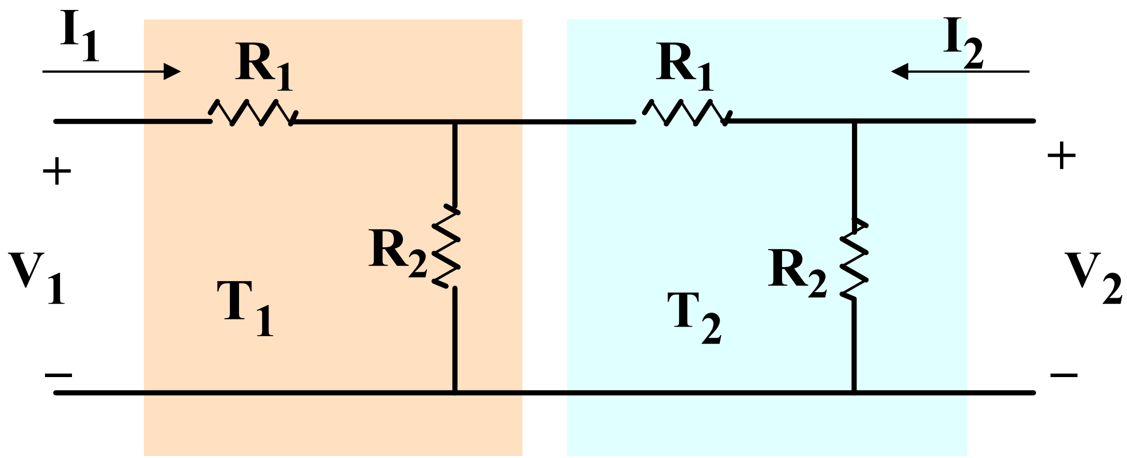
Solution

$$A = \frac{V_1}{V_2}\bigg|_{I_2 = 0} = \frac{R_1 + R_2}{R_2}$$

$$C = \frac{I_1}{V_2}\bigg|_{I_2 = 0} = \frac{1}{R_2}$$

$$B = \frac{V_1}{-I_2}\bigg|_{V_2 = 0} = R_1$$

$$D = \frac{I_1}{-I_2}\bigg|_{V_2 = 0} = 1$$



$$[T] = \begin{bmatrix} \frac{R_1 + R_2}{R_2} & R_1 \\ \frac{1}{R_2} & 1 \end{bmatrix} \begin{bmatrix} \frac{R_1 + R_2}{R_2} & R_1 \\ \frac{1}{R_2} & 1 \end{bmatrix}$$

Transmission Parameters – Example 1 cont.

$$[\mathbf{T}] = \begin{bmatrix} \dfrac{R_1 + R_2}{R_2} & R_1 \\ \\ \dfrac{1}{R_2} & 1 \end{bmatrix} \begin{bmatrix} \dfrac{R_1 + R_2}{R_2} & R_1 \\ \\ \dfrac{1}{R_2} & 1 \end{bmatrix}$$

$$[\mathbf{T}] = \begin{bmatrix} \left(\dfrac{R_1 + R_2}{R_2}\right)^2 + \dfrac{R_1}{R_2} & \dfrac{R_1 + R_2}{R_2} R_1 + R_1 \\ \\ \dfrac{R_1 + R_2}{R_2}\dfrac{1}{R_2} + \dfrac{1}{R_2} & \dfrac{R_1}{R_2} + 1 \end{bmatrix} = \begin{bmatrix} \dfrac{(R_1 + R_2)^2 + R_1^2}{R_2^2} & \dfrac{R_1^2 + 2R_1R_2}{R_2} \\ \\ \dfrac{R_1 + 2R_2}{R_2^2} & \dfrac{R_1 + R_2}{R_2} \end{bmatrix}$$

Summary

- Two-port networks consist of input port and output port.
- Four parameters were discussed to model the two-port network, they are impedance [z], admittance [y], hybrid [h] and transmission [T] parameters.
- The relationships between the four sets of parameters (especially z and y parameters).
- Two-port network can be connected in series, parallel and cascade. In series connection, z-parameters are added, in parallel, y-parameters are added, and in cascade, T-parameters are multiplied.

Quiz

- 1. When port 1 of a two-port circuit is short-circuited, $\mathbf{I_1} = 4\ \mathbf{I_2}$ and $\mathbf{V_2} = 0.25\ \mathbf{I_2}$. Which of the following is true?
  - (a) y11 = 4; (b) y12 = 16;
  - (c) y21 = 16; (d) y22 = 0.25.

- 2. A two-port is described by the following equations:

$$\mathbf{V}_1 = 50\ \mathbf{I}_1 + 10\ \mathbf{I}_2$$
$$\mathbf{V}_2 = 30\ \mathbf{I}_1 + 20\ \mathbf{I}_2$$

  Which of the following is *not* true?
  - (a) z12 = 10; (b) y12 = -0.0143
  - (c) z21 = 30; (d) A = 50.