Tutorial 1 Revision on vector calculus and line integral

Differentiation

1. Find the first and second derivatives of $\boldsymbol{r} =< 3cos2t, 3sin2t, 4t >$.
2. Find the first partial derivatives of $\boldsymbol{v}_1 =< e^x \cos y, e^x \sin y >$ and $\boldsymbol{v}_2 =< \cos x \cosh y, -\sin x \sinh y >$.

Gradient

Find the gradient of the following functions *f:*

1. $f = (x-1)(4y-2)$
2. $f = 2x^2 + 5y^2$
3. $f = \frac{\mathrm{x}}{\mathrm{y}}$
4. $f = (x-2)^2 + (2y+4)^2$
5. $f = x^5 + y^5$
6. $f = \frac{\mathrm{x}^2+\mathrm{y}^2}{\mathrm{x}^2-\mathrm{y}^2}$

Velocity fields

Given the velocity potential *f* of a flow, find the velocity $\boldsymbol{v} = \nabla f$ of the field and its value $\boldsymbol{v}(P)$ at P.

1. $f = x^2 - 6x - y^2, P: (-1,5)$
2. $f = \cos x \cosh y, P: (\pi/2, \ln 2)$
3. $f = x\left(1 + \frac{1}{x^2+y^2}\right), P: (1,1)$
4. $f = e^x \cos y, P: (1, \pi/2)$

Divergence

Find div$\boldsymbol{v}$ and its value at P.

1. $\boldsymbol{v} =< 2x^2, -3y^2, 8z^2 >, P: \left(3, \frac{1}{2}, 0\right)$
2. $\boldsymbol{v} =< 0,\ \sin(x^2yz),\ \cos(xy^2z) >, P: (1, \frac{1}{2}, -\pi)$
3. $\boldsymbol{v} = \frac{<x,y>}{x^2-y^2}, x \neq y$
4. $\boldsymbol{v} =< v_1(y,z),\ v_2(z,x),\ v_3(x,y) >, P: (3,1,-1)$
5. $\boldsymbol{v} =< x^2yz,\ xy^2z,\ xyz^2 >, P: (-1,3,-2)$
6. $\boldsymbol{v} = \frac{<-x,-y,-z>}{(x^2+y^2+z^2)^{\frac{3}{2}}}$

Curl

Find curl $\boldsymbol{v}$ for $\boldsymbol{v}$ given with respect to right-handed Cartesian coordinates. Show the details of your work.

1. $\boldsymbol{v} = < 4y^2,\ 3x^2, 0 >$
2. $\boldsymbol{v} = xyz < x^2,\ y^2,\ z^2 >$
3. $\boldsymbol{v} = \frac{<x,y,z>}{(x^2+y^2+z^2)^{\frac{3}{2}}}$
4. $\boldsymbol{v} = < 0,0, e^{-x}\sin y >$
5. $\boldsymbol{v} = < e^{-z^2},\ e^{-x^2},\ e^{-y^2} >$

Parametric representations

What curves are represented by the following? Sketch them. (page 390, Q1-4, 8)

1. $< 2 + 4\cos t,\ 2\sin t,\ 0 >$
2. $< a + t,\ b + 3t,\ c - 5t >$
3. $< 0,\ t,\ 2t^3 >$
4. $< -2,\ 2 + 5\cos t, -1 + 5\sin t >$
5. $< \cosh t,\ \sinh t,\ 2 >$

Find a parametric representation (page 390, Q11, 12, 15, 17-19)

1. Circle in the plane $z = 2$ with center (1,-1) and passing through the origin.
2. Circle in the yz-plane with center (4,0) and passing through (0,3).
3. Straight line $y = 2x - 1, z = 3x$.
4. Ellipse $\frac{1}{3}x^2 + y^2 = 1,\ z = y$.
5. Helix $x^2 + y^2 = 25, z = 2\arctan\frac{y}{x}$.
6. Hyperbola $x^2 - y^2 = 1, z = -2$.

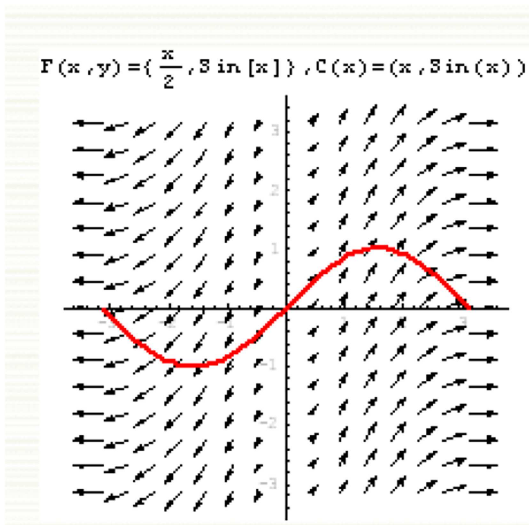
Line integral – work

The line integral $\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}$ of the vector field $\boldsymbol{F}$ along the curve C gives the work done by the field on an object moving along the curve through the field. Calculate $\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}$ for the given data. If $\boldsymbol{F}$ is a force, this gives the work done by the force in the displacement along C. Show the details.

1. $\boldsymbol{F} = < y^2, -x^2 >, C: y = 4x^2$ from (0,0) to (1,4).
2. $\boldsymbol{F}$ as in question 1, C from (0,0) straight to (1,4). Compare the results.
3. $\boldsymbol{F} = < xy,\ x^2y^2 >$, C from (2,0) straight to (0,2).
4. $\boldsymbol{F}$ as in question 3, C is the quarter-circle from (2,0) to (0,2) with center (0,0).

Line integral – work done by an airplane

Consider a vector field $\boldsymbol{F} =< \frac{x}{2},\ \sin x >$ which is defined on the plane. Suppose that *t* is the time, $\boldsymbol{F}$ is a force field, say the wind, and an airplane is moving over the curve $C: \boldsymbol{r}(t) =< t, \sin t >$ from the initial point (0,0) to the terminal point (2, sin2). See the figure blow. Calculate the work done by the wind on this airplane along the path *C*.



Line integral

1. Calculate the line integral for the vector field $\boldsymbol{F} =< xy, y^2 >$ over the segment joining the points from O: (0,0) to P: (1,1).
2. Determine the line integral of $\boldsymbol{F} =< x^2, xy >$ along the parabola $x = y^2$ between the points (1,-1) and (1,1).
3. Determine the line integral for $\boldsymbol{G} =< -\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2} >$ over the circle in the plane with center (0,0) and radius 3 from the point (3,0) to the point $(\frac{3\sqrt{3}}{2}, \frac{3}{2})$. Hint: Use the polar coordinate system.
4. Calculate the work done by $\boldsymbol{F} =< x, y^2 >$ on a particle moving from (0,0) to (1,1) and then to (1,0) along the straight line segments joining the points.