MTH101: Lecture 2

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Review

- Geometric Form : $z = (x, y), \qquad x, y \in \mathbb{R}$;
- Algebraic Form: $z = x + iy, \qquad x, y \in \mathbb{R}$;
- Comlex Conjugate: $\bar{z} = x - iy$;
- Modulus of a complex number: $|z| = \sqrt{x^2 + y^2}$;
- Argument of a complex number:
  $\arg(z) = \text{Arg}\,(z) + 2n\pi, \quad n = 0, \pm 1, \pm 2, ..., \ \text{Arg}\,(z) \in (-\pi, \pi]$
- Operations: Add, Substract, Multiply, Divide
- Properties of Conjugate and Modulus

Question: if $z_1 = 1 + 2i$, $z_2 = 3 + 4i$, can we say $z_2 > z_1$? or, can we say $|z_2| > |z_1|$?

Polar Form of a Complex Number

Using the Modulus and the Principal Argument of a Complex Number we introduce the Polar representation:

for any $z = x + iy \neq 0$ we write

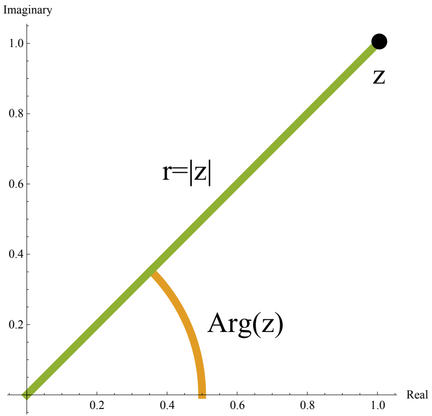
$$z = r(\cos\theta + i\sin\theta).$$

where

$$r = |z|,$$
$$\theta = Arg(z).$$

We observe that

$$r > 0, \qquad \text{and} \qquad \theta \in (-\pi, \pi].$$


Imaginary
1.0
0.8
0.6
0.4
0.2
z
r=|z|
Arg(z)
0.2
0.4
0.6
0.8
1.0
Real

Exercise

Write in Polar Form the Complex Numbers of the previous Exercise.

Example

$$-\sqrt{3} - i$$

Solution: To turn $-\sqrt{3} - i$ into exponential form we do a Cartesian to Polar conversion:

$$r = \sqrt{((-1)^2 + (-\sqrt{3})^2)} = \sqrt{(1+3)} = \sqrt{4} = 2$$

$$\theta = \tan^{-1}\left(\frac{-1}{-\sqrt{3}}\right) - \pi = -\frac{5\pi}{6}$$

So $-\sqrt{3} - i$ can also be $2(\cos\frac{-5\pi}{6} + i\sin\frac{-5\pi}{6})$

Exponential Form of a Complex Number

We make use of the **Euler's Formula**:

$$\mathbf{e}^{\mathbf{i}\theta} = \cos\theta + \mathbf{i}\sin\theta.$$

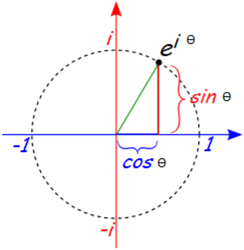
Then for any $z \neq 0$ we obtain the **Exponential Form**

$$z = r(\cos\theta + i\sin\theta) = re^{i\theta},$$

where $r = |z| > 0$ is the modulus of the complex number, and $\theta = \mathrm{Arg}(z) \in (-\pi, \pi]$ is the Principal argument of z.

Graphic representation of Euler's Formula

In fact, putting Euler's Formula on the graph produces a circle:



produces a circle of radius 1

And we can turn any complex number into $re^{i\theta}$ form (by finding the correct value of θ and the radius, r, of the circle)

Example

The number $-\sqrt{3} - i$

As in the previous example

$$r = 2$$
$$\theta = -\frac{5\pi}{6}$$

So in exponential form $-\sqrt{3} - i$ can also be $2e^{-\frac{5}{6}\pi i}$

Exercise

Write in **Exponential Form** *the Complex Numbers* z_1, z_2, z_3, z_4.

Operation using Polar Form or Exponential Form

Consider two Complex Number $z_1 = r_1 e^{i\theta_1} = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2 e^{i\theta_2} = r_2(\cos\theta_2 + i\sin\theta_2)$, then

$$z_1 \cdot z_2 = r_1 r_2 \left[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\right] = r_1 r_2 e^{i(\theta_1+\theta_2)},$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)] = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)},$$

Exercise

Compute the following quantities:

$$z_1 \cdot z_2, \qquad \frac{z_2}{z_3}, \qquad z_3 \cdot z_4, \qquad \frac{z_1}{z_4}.$$

z_1, z_2, z_3, z_4 *as in exersice from last lecture.*

n^{th} powers of a complex number

If $z = re^{i\theta} = r(\cos\theta + i\sin\theta)$, then for $n = 0, 1, 2, \ldots$

$$(z)^n = r^n[\cos(n\theta) + i\sin(n\theta)] = r^n e^{in\theta}.$$

Example

Find $i^0, i^2, i^3, i^4, \ldots$

Solution: First we write $i = (0, 1) = 1 \cdot e^{\frac{\pi}{2}i}$, then

$$\begin{aligned} n &= 0, & i^0 &= 1^0 \cdot e^{0i} = 1, \\ n &= 2, & i^2 &= 1^2 \cdot e^{\pi i} = -1, \\ n &= 3, & i^3 &= 1^3 \cdot e^{\frac{3\pi}{2}i} = -i, \\ n &= 4, & i^4 &= 1^4 \cdot e^{2\pi i} = 1. \end{aligned}$$

n^{th} Roots of a Complex Number

For a given $z \in \mathbb{C}$, the equation

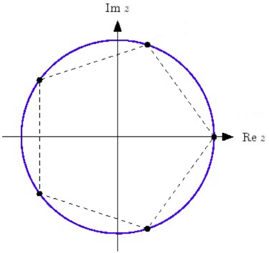
$$\omega^n = z,$$

has exactly n solutions correspond to n distinct values of ω, that is, the complex number z has n roots of order n (n^{th} roots).
If $z = re^{i\theta}$ is in **Exponential Form** then the roots are

$$r^{\frac{1}{n}} e^{i\frac{\theta+2k\pi}{n}}, \qquad \text{with } k = 0, 1, \ldots n-1.$$

Remark

All the n^{th} roots of $z = re^{i\theta}$ have the same modulus $r^{\frac{1}{n}}$ and Principal Arguments which differ by $\frac{2\pi}{n}$.
Graphically, they can be represented by the vertices of a Regular Polygon with n sides inscribed in a Circle of center O and radius $r^{\frac{1}{n}}$.
In particular, the following figure shows all 5 roots of the equation $\omega^5 = 1$



Example

Find all the solutions of the equation $\omega^6 = 1$.

Solution

There are 6 solutions that we denote by $\omega_0, \omega_1, \omega_2, \omega_3, \omega_4, \omega_5$. We first represent $z = 1$ in the form $re^{i\theta}$ with $r = 1$ and $\theta = 0$.
Then by the formula we get

$$\omega_k = 1^{\frac{1}{6}} e^{i\frac{0+2k\pi}{6}} = e^{i\frac{k\pi}{3}}, \qquad \text{with } k = 0, 1, 2, 3, 4, 5.$$

In details we have

$$\begin{aligned}
k &= 0, & \omega_0 &= 1, \\
k &= 1, & \omega_1 &= e^{i\frac{\pi}{3}}, \\
k &= 2, & \omega_2 &= e^{i\frac{2\pi}{3}}, \\
k &= 3, & \omega_3 &= e^{i\pi}, \\
k &= 4, & \omega_4 &= e^{i\frac{4\pi}{3}} = e^{-i\frac{2\pi}{3}}, \\
k &= 5, & \omega_5 &= e^{i\frac{5\pi}{3}} = e^{-i\frac{\pi}{3}}.
\end{aligned}$$

Exercise

Write the numbers $\omega_0, \ldots, \omega_5$ *using the* **Algebraic Representation**.

Bibliography

1 *Kreyszig, E.* **Advanced Engineering Mathematics**. Wiley, 10th Edition.