EEE336 Signal Processing and Digital Filtering

Lecture 12 Digital Filters Classification

Lect_12_1 FIR VS IIR

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FIR systems

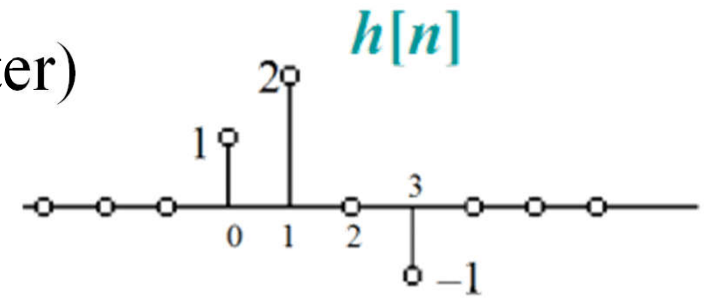
- If the impulse response h[n] of a system is of finite length, that system is referred to as a finite impulse response (FIR) system

$$h[n]=0 \text{ for } n<N_1 \text{ and } n>N_2,\quad N_1<N_2$$

- The output of such a system can then be computed as a finite convolution sum

$$y[n]=\sum_{k=-\infty}^{\infty} h[k]x[n-k]=\sum_{k=N_1}^{N_2} h[k]x[n-k]$$

  – E.g., h[n]=[**1** 2 0 -1] is a FIR system (filter)



FIR systems

- For a causal LTI system h[n]:

$$y[n] = \sum_{k=N_1}^{N_2} h[k]x[n-k] = \sum_{k=0}^{M} h[k]x[n-k] = \sum_{j=0}^{M} b_j x[n-j]$$

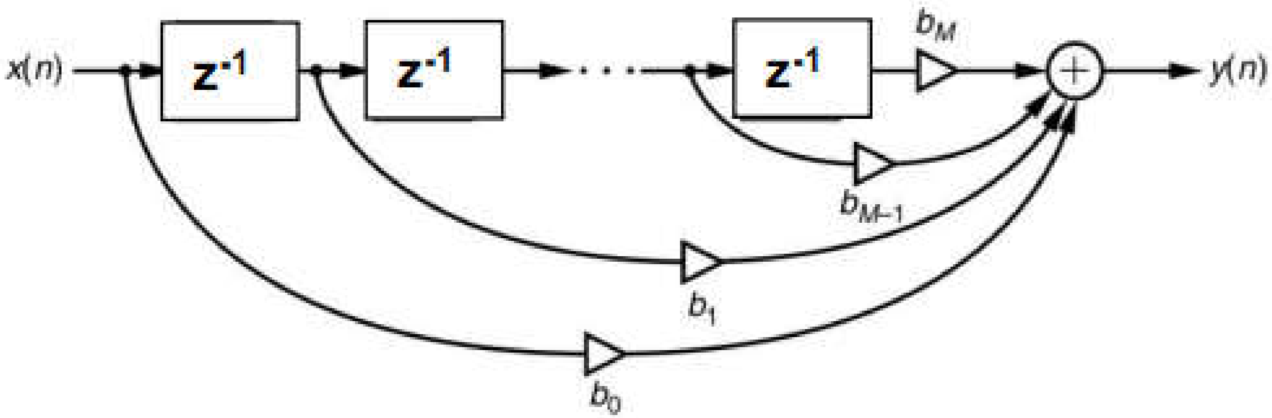
- Compare with the CCLDE:

$$y[n] + \cancel{\sum_{i=0}^{N-1} a_i y[n-i]} = \sum_{j=0}^{M} b_j x[n-j]$$

- FIR systems are also called non-recursive systems, where the output can be computed from the current and past input values only – without requiring the values of previous outputs

FIR systems

- The CCLDE representation of an FIR system can schematically be represented using the following diagram, known as the "filter structure"



- Example: Consider the system given by

$$y[n] = -0.1462x[n] + 0.2925x[n-1] + 0.7074x[n-2] - 0.1462x[n-3]$$

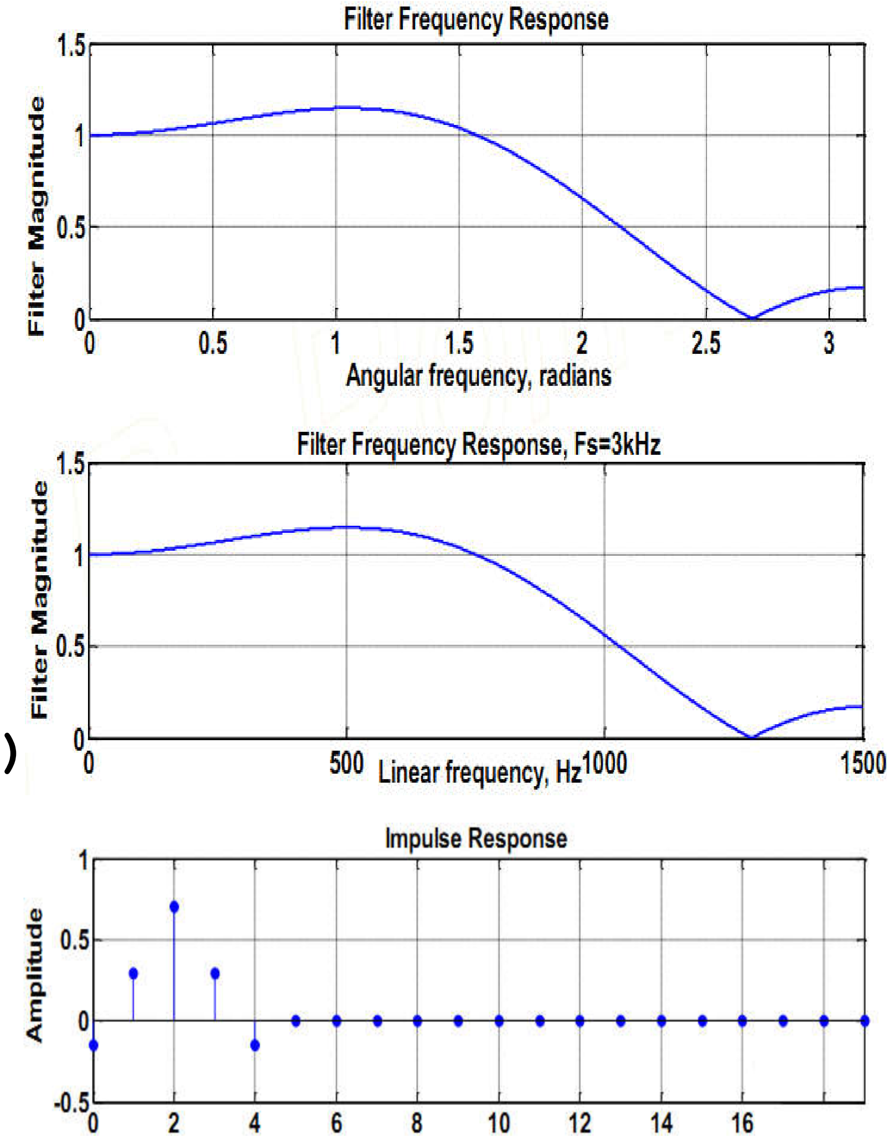
- In Matlab
  - **freqz()** – Frequency response of digital filter
  - **Impz()** – Impulse response of digital filter

```
clear;  close all
b=[-0.1462 0.2925 0.7074 0.2925 -0.1462];
a=1;

[H w]=freqz(b, a, 1024);
subplot(311);plot(w, abs(H));
title('Filter Frequency Response');

[H1 f]=freqz(b, a, 1024, 3000);
subplot(312);plot(f, abs(H1))
title('Filter Frequency Response, Fs=3kHz')

subplot(313)
impz(b,a,20); grid;
xlabel('Normalized Time')
title('Impulse Response‘)
```



IIR systems

- If the impulse response is of infinite length, then the system is referred to as an infinite impulse response (IIR) system.
  - These systems cannot be characterized by the convolution sum due to infinite sum.
  - Instead, they are typically characterized by constant coefficient linear difference equations (CCLDEs)

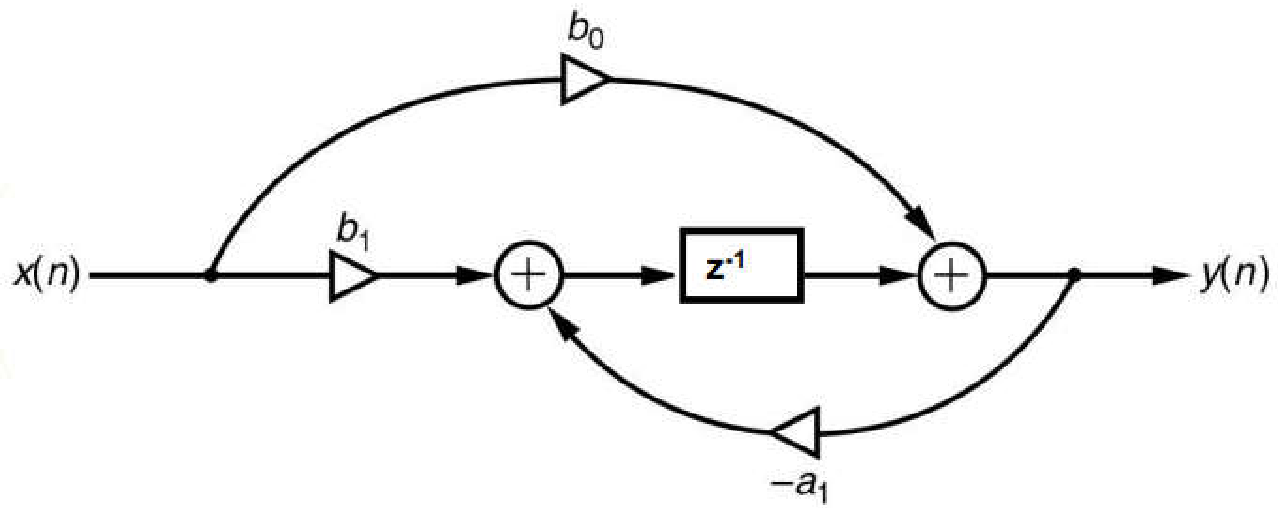
$$y[n] + \sum_{i=1}^{N-1} a_i y[n-i] = \sum_{j=0}^{M} b_j x[n-j]$$

  - There output y[n] relies on both current and previous input x[n-j], and also previous output y[n-i]. It is called a ***recursive system***
  - But not only IIR systems are recursive, some FIR systems can also be expressed in a recursive structure, such as the accumulator:

$$y[n] = \sum_{\ell=-\infty}^{n} x[\ell] \quad \Rightarrow \quad y[n] = y[n-1] + x[n]$$

IIR systems

- The filter structure of IIR systems – which has a distinct feedback (recursion) loop, has the following form:



- Stability
  - FIR system consists of finite terms => it is always stable
  - IIR systems are not guaranteed to be stable, since their h[n] consists of infinite number of terms => Their design requires stability checks (absolutely summable or z-transform ROC includes the unit circle).

12_1 Wrap up

	FIR	IIR
h[n]	Finite	Infinite
CCLDE	y[n]=sum(x[n-j])	y[n]=-sum(y[n-i])+sum(x[n-j])
recursive	Both	Recursive
stable	Yes	Depends

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Lecture 12 Digital Filters Classification

Lect_12_2 Ideal VS Practical Filters

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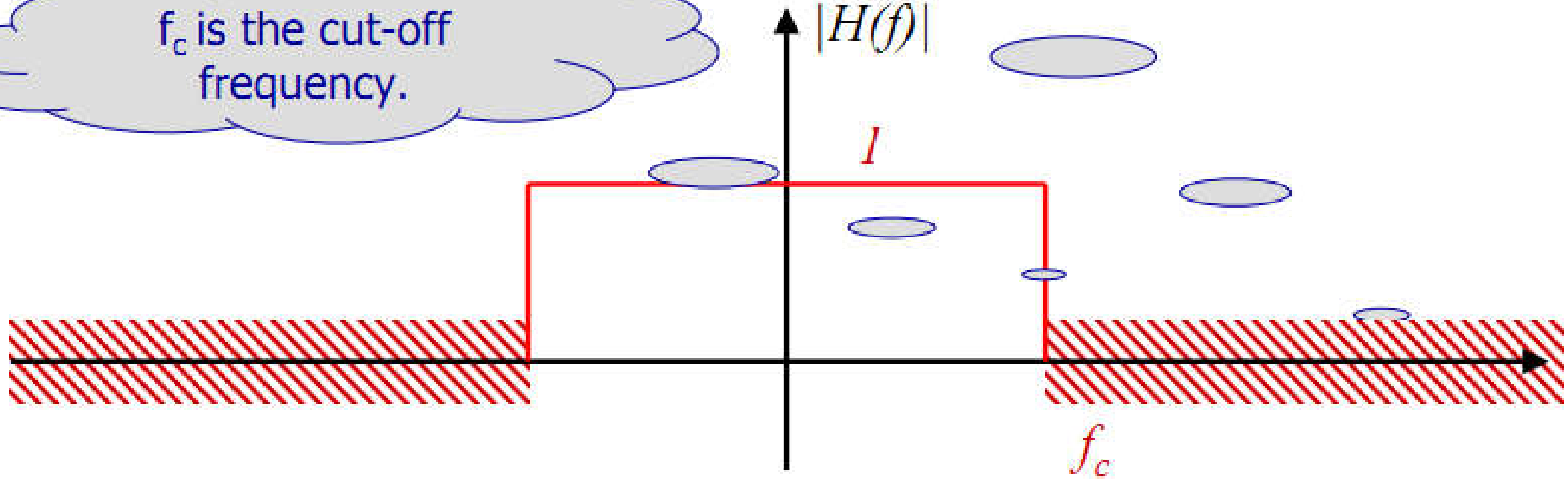
Ideal Filters

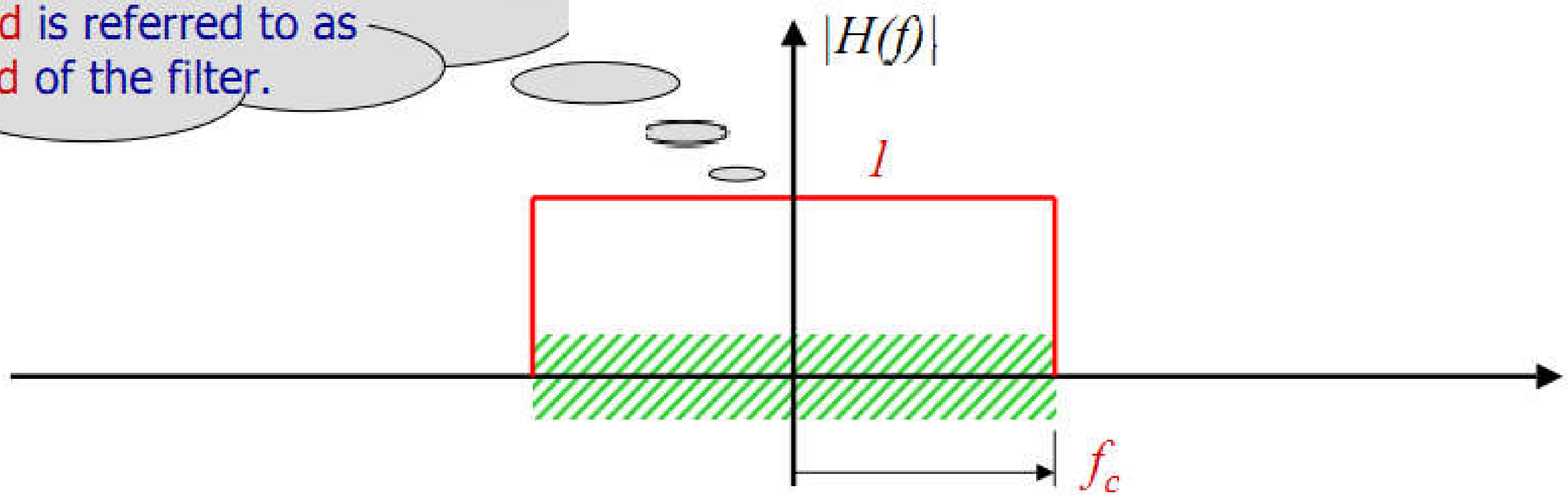
- An ideal filter is a digital filter designed to pass signal components of certain frequencies without distortion, which therefore has a frequency response equal to 1 at these frequencies, and has a frequency response equal to 0 at all other frequencies
- The range of frequencies where the frequency response takes the value of one is called the ***passband***
- The range of frequencies where the frequency response takes the value of zero is called the ***stopband***
- The transition frequency from a passband to stopband region is called the ***cut-off frequency***

The range of frequency that are blocked is referred to as the stop-band of the filter.

f_c is the cut-off frequency.

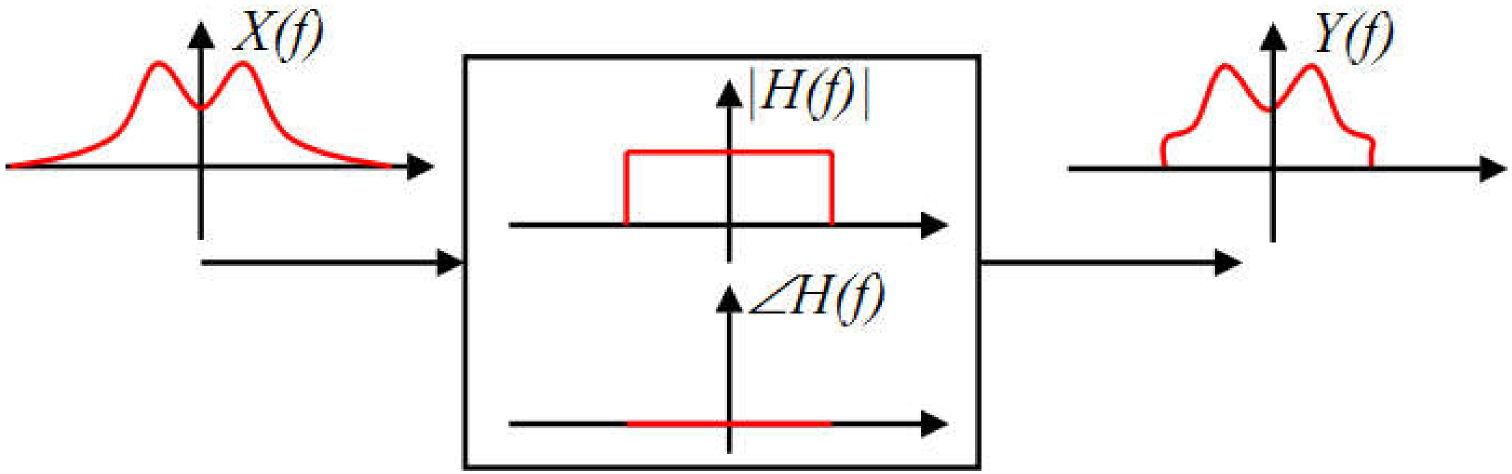


The range of frequency that is left unaffected is referred to as pass-band of the filter.



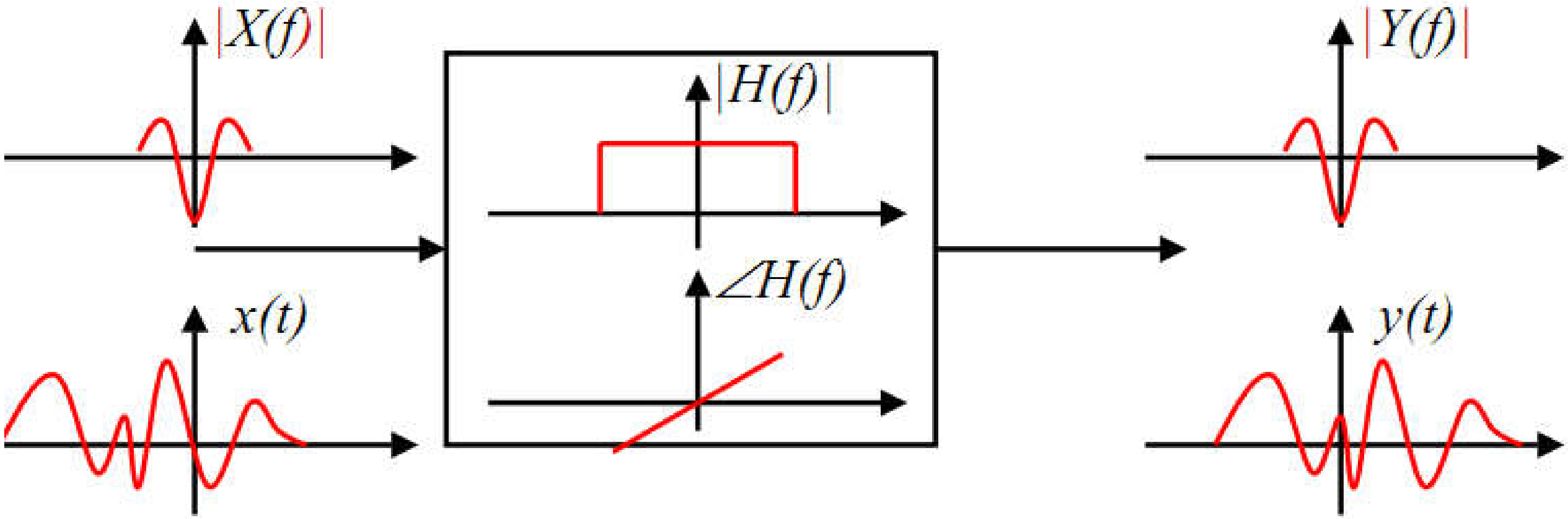
A filter example

- This filter with uniform magnitude and zero phase
  - Passes a pre-specified range of frequencies without any alteration, but completely reject the remaining frequency components.
  - The phase of it is zero for all frequencies

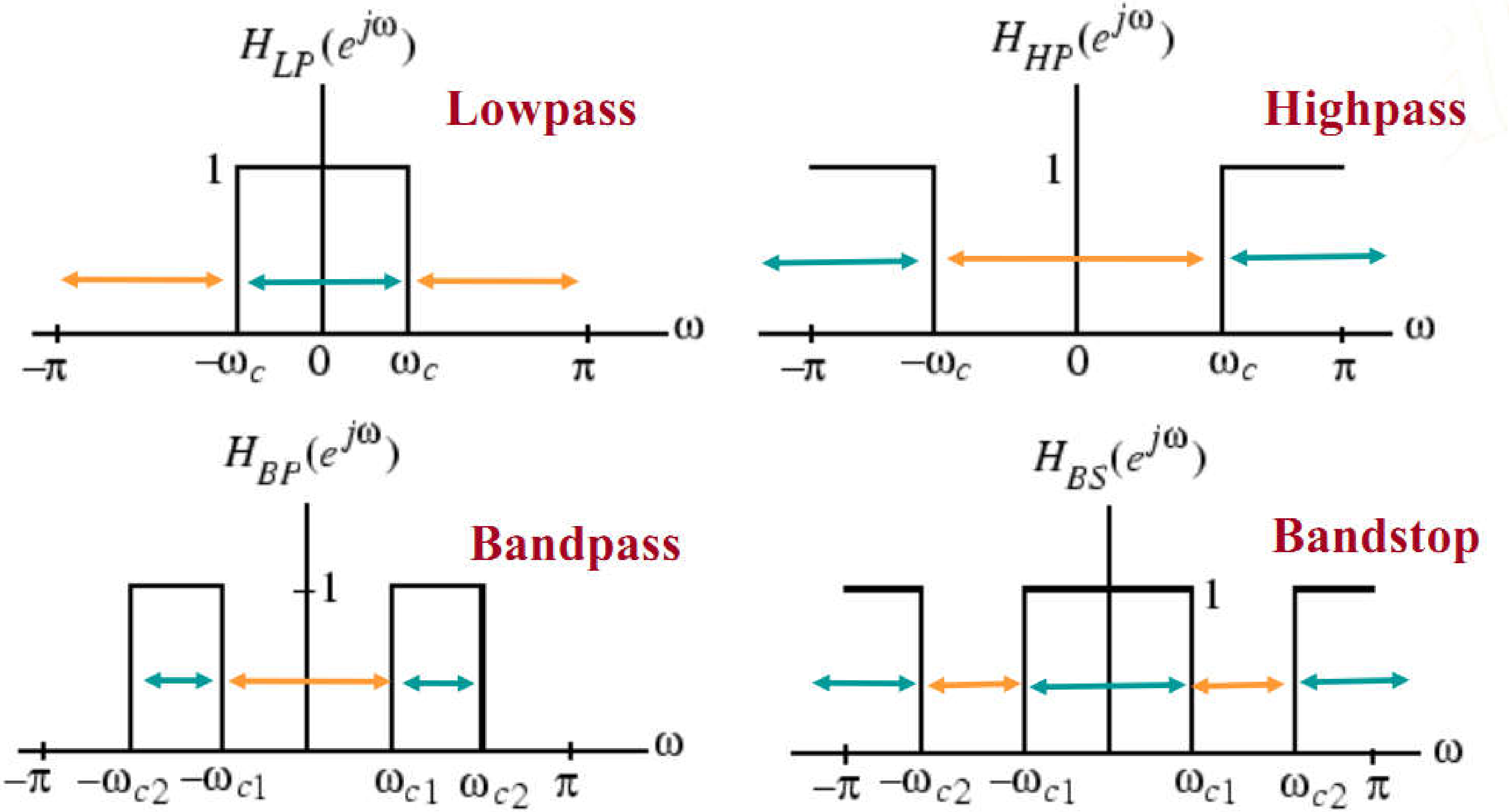


A filter example

- A linear phase response will result in a delayed output.



- The frequency responses of four common ideal filters in the [-π π] range are

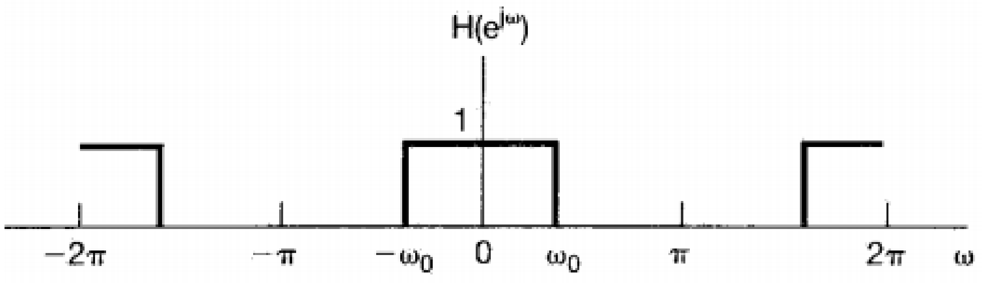


Ideal Filters

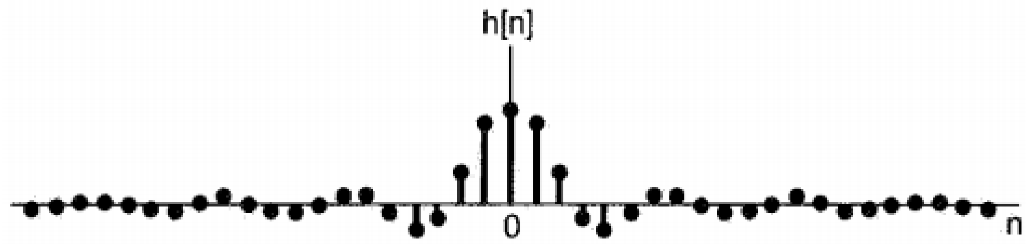
- The DTFT of a rectangular pulse is a sinc function;
- From the duality theorem, the inverse DTFT of a rectangular pulse is also a sinc function.

- Since the ideal (lowpass) filter is of rectangular shape, its impulse response must be of sinc.

$$H_{LP}(\omega)=\begin{cases}1, & 0\le|\omega|\le\omega_c\\ 0, & \omega_c\le|\omega|\le\pi\end{cases}\qquad h_{LP[n]}=\frac{\sin(\omega_c n)}{\pi n},\ -\infty<n<\infty$$



(a)



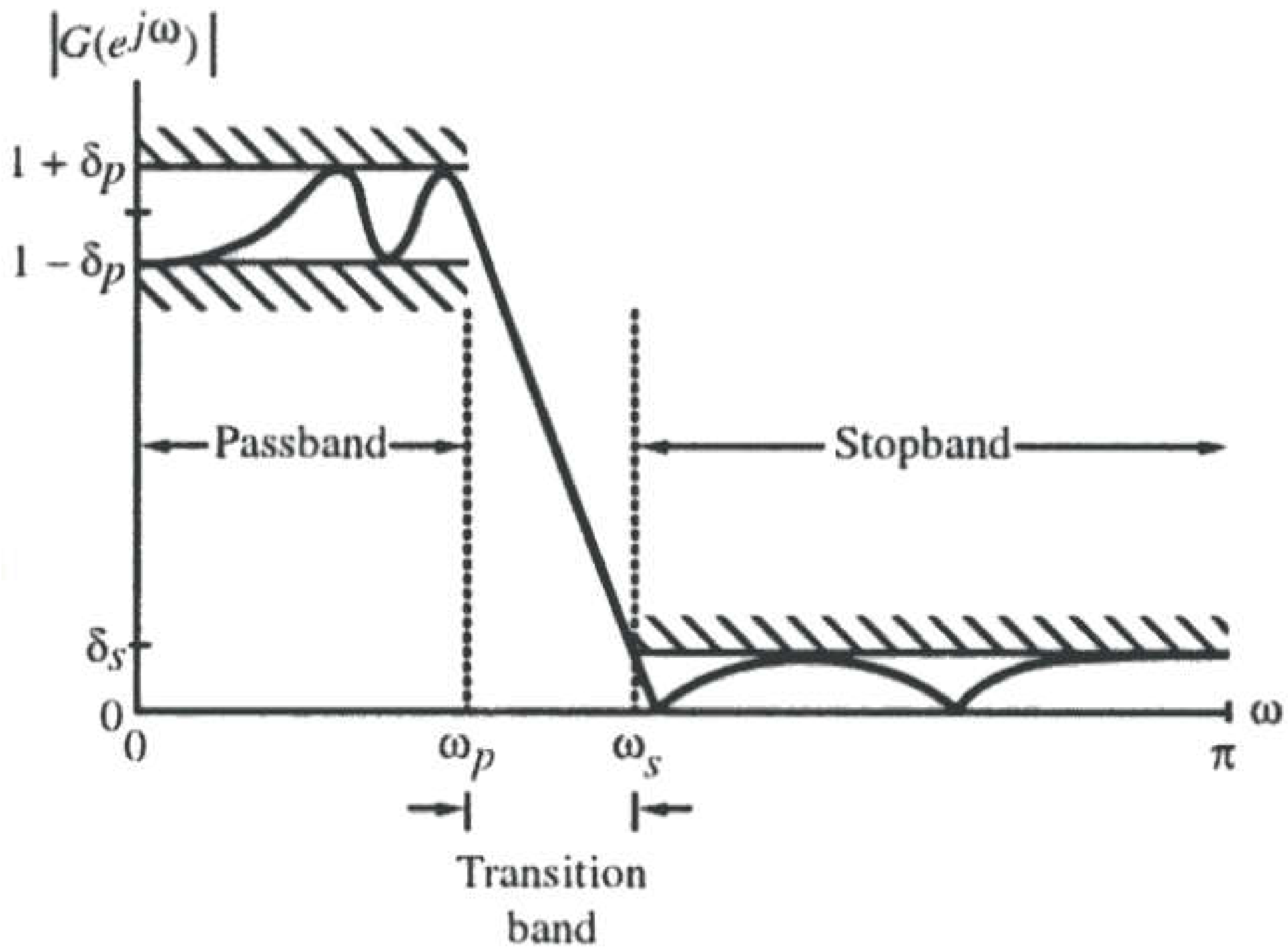
Ideal Filters

- We note the following about the impulse response of an ideal filter
  - $h_{LP}[n]$ is not absolutely summable
    - The corresponding transfer function is therefore not BIBO stable
  - $h_{LP}[n]$ is not causal, and is of doubly infinite length
    - Not realizable in time domain
  - The remaining three ideal filters are also characterized by doubly infinite, noncausal impulse responses and also are not absolutely summable

- Thus, the ideal filters with the *ideal brick wall* frequency responses cannot be realized with finite dimensional LTI filter

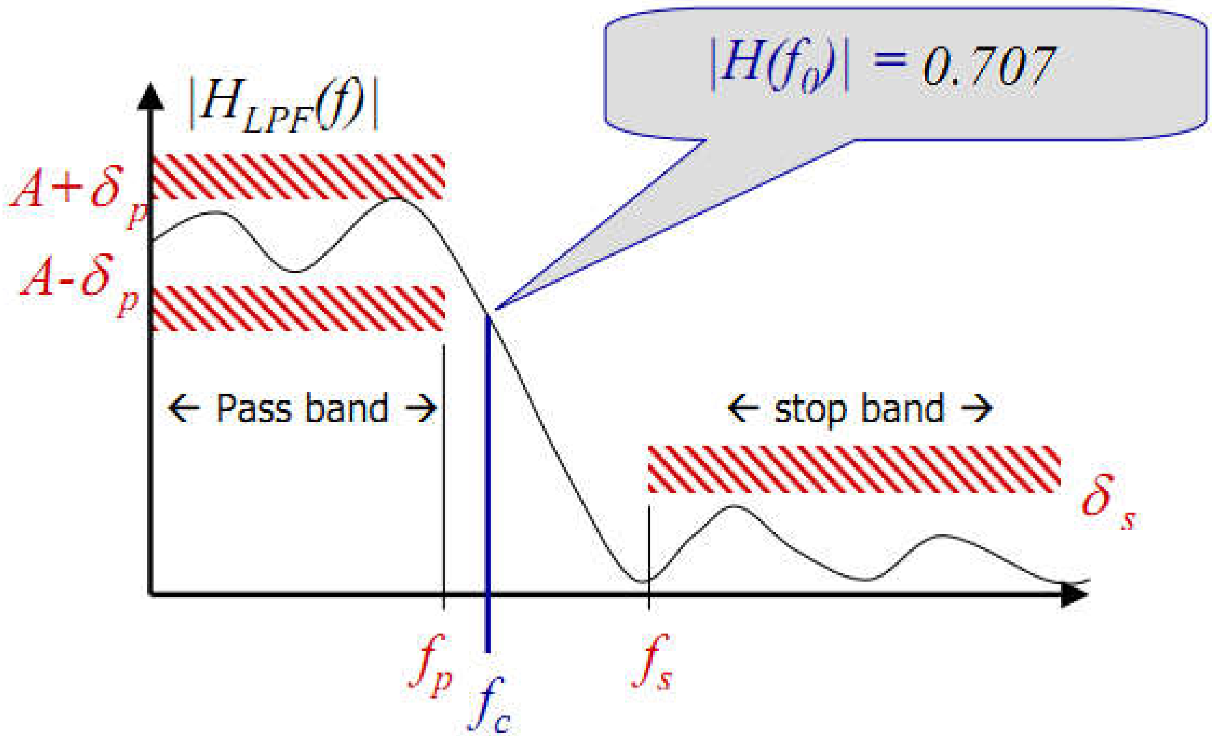
Realizable filters

- TO develop stable and realizable transfer functions
  - A finite transition band is introduced between the passband and stopband
  - This permits the magnitude response to decay slowly from its maximum value in the passband to the zero value in the stopband
  - The magnitude response is allowed to vary by a specified amount both in the passband and in the stopband



Realizable filters

- The cut-off frequency
  - The frequency at which the |H(f)| drops to 0.717A or equivalently -3dB



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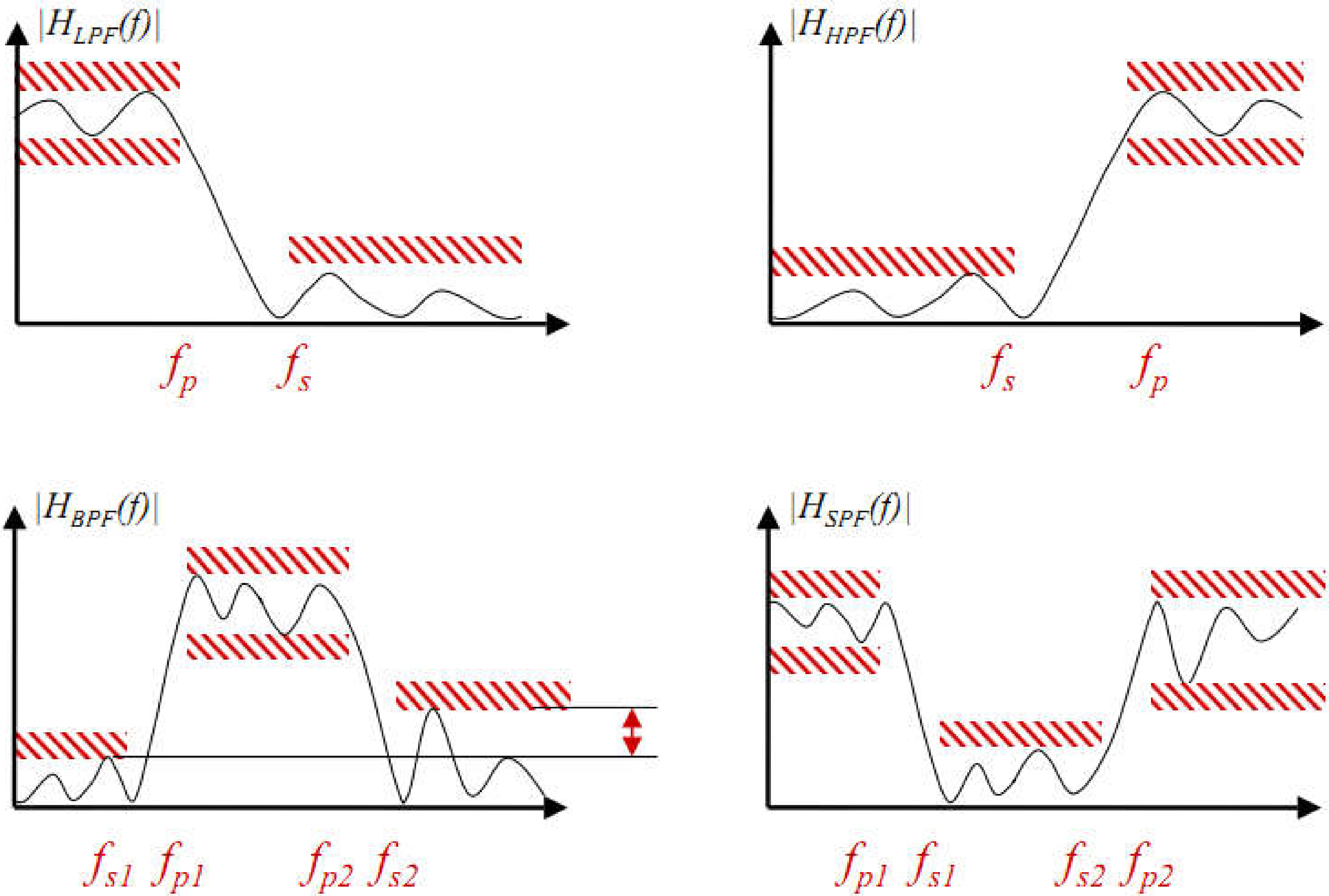
Practical filters



12_2 Wrap up

		Ideal	Realizable
$H(\omega)$	Pass band	1	$1 \pm \delta_p$
	Stop band	0	δ_s
	Transition band	ω_c	(ω_p, ω_s)
$h[n]$	Stable	No	Yes
	Causal	No	Yes

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Lecture 12 Digital Filters Classification

Lect_12_3 Classification of Phase

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Phase delay and group delay

- A frequency selective system (filter) with frequency response

$$H(\omega) = |H(\omega)|e^{j\theta(\omega)}$$

  – changes the amplitude of all frequencies by a factor of $|H(\omega)|$;
  – And adds a phase of $\theta(\omega)$ to all frequencies.

- For an input at certain frequency ω_0
  – Frequency domain: $Y(\omega_0) = X(\omega_0)|H(\omega_0)|e^{j\theta(\omega_0)}$
  – Time domain: $x[n] = Acos(\omega_0 n + \varphi)$

$$\Downarrow$$

$$y[n] = A|H(\omega_0)|cos(\omega_0 n + \theta(\omega_0) + \varphi)$$

Phase delay $\tau_p(\omega_0)$

- If the input is a sinusoidal signal of frequency ω_0:

$$x[n] = A cos(\omega_0 n + \varphi)$$

- The output is also a sinusoidal signal of the same frequency ω_0 but lagging in phase by $\theta(\omega_0)$ radians:

$$\begin{aligned} y[n] &= A|H(\omega_0)|cos(\omega_0 n + \theta(\omega_0) + \varphi) \\ &= A|H(\omega_0)|cos\left(\omega_0\left(n + \frac{\theta(\omega_0)}{\omega_0}\right) + \varphi\right) \\ &= A|H(\omega_0)|cos\left(\omega_0\left(n - \tau_p(\omega_0)\right) + \varphi\right) \end{aligned}$$

  – Where $\tau_p(\omega_0) = -\frac{\theta(\omega_0)}{\omega_0}$ is called the ***phase delay***.
    - The minus sign indicates phase lag, and τ_p is in terms of time.

Group delay $\tau_g(\omega)$

- If an input system consists of many frequency components (which most practical signals do), each component goes through different phase delays when processed by a system;

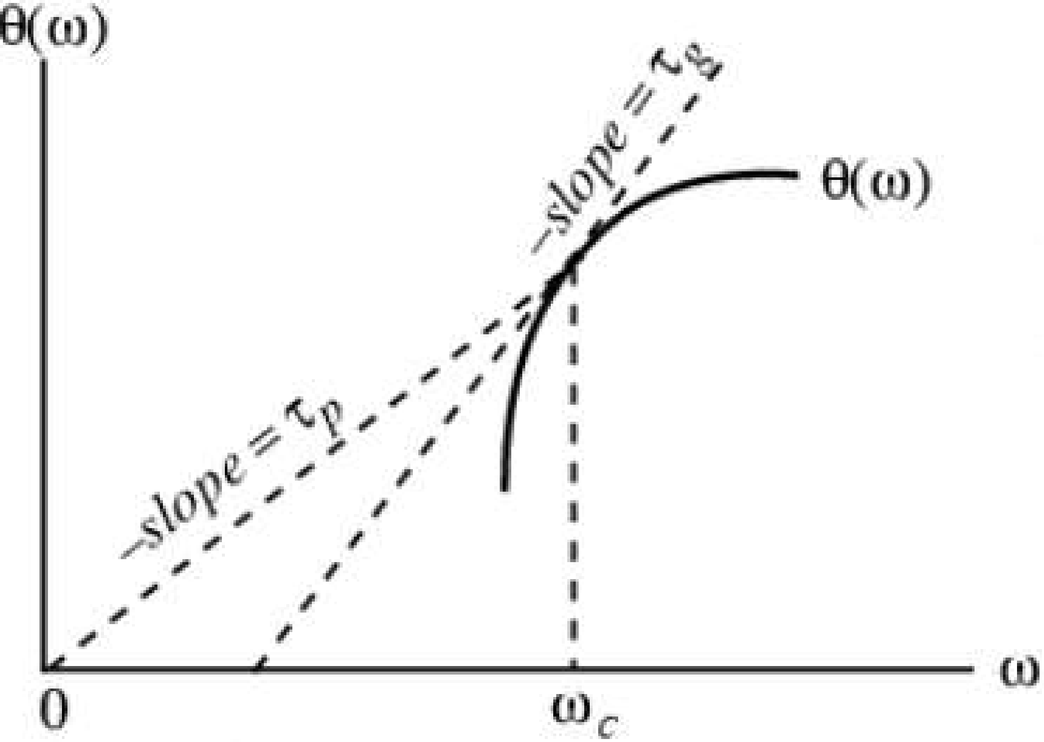
- then we can also define ***group delay***, the phase shift by which the envelope of the signal shifts:

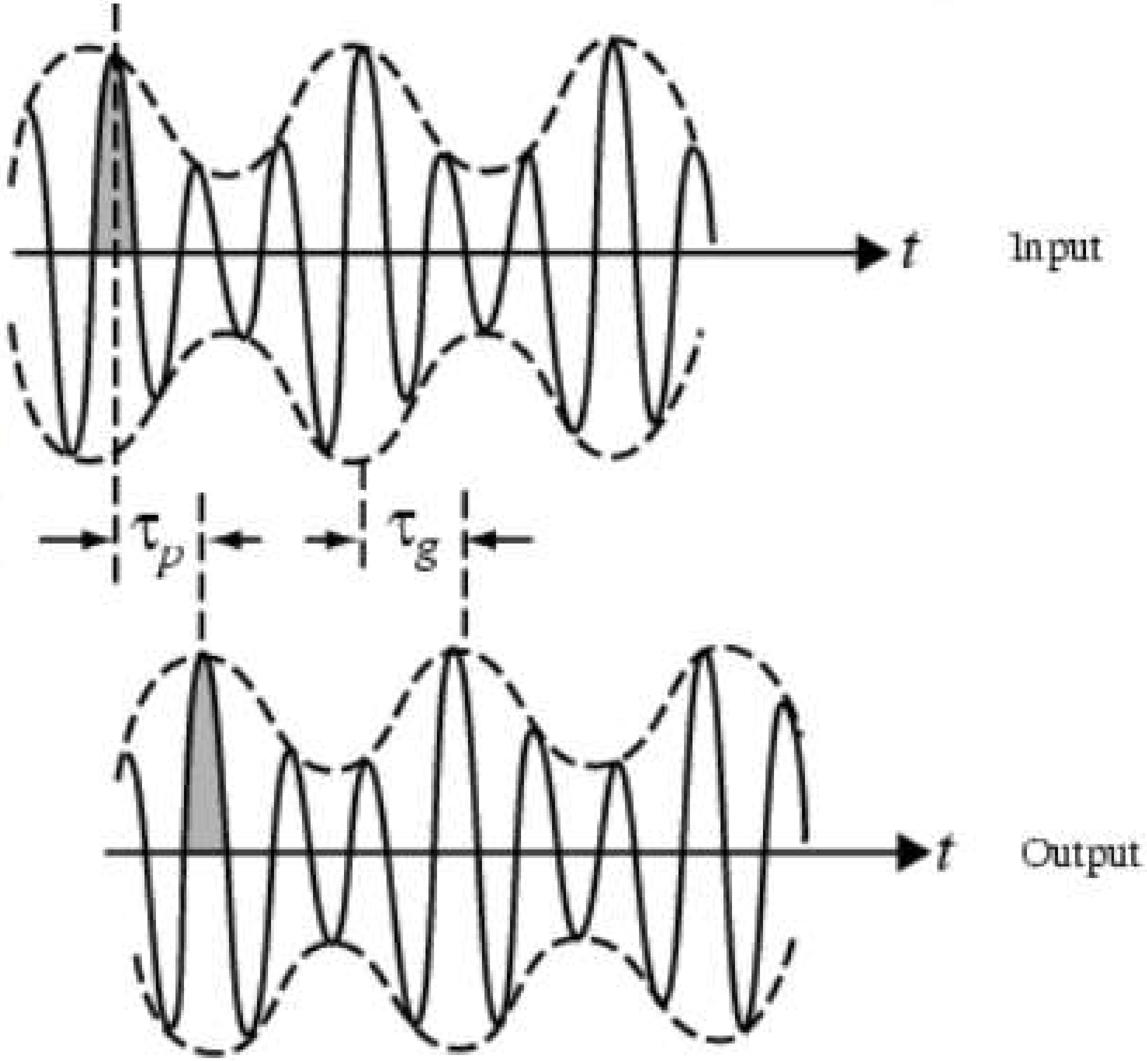
$$\tau_g(\omega) = -\frac{d\theta(\omega)}{d\omega}$$

  – Necessary assumption: the phase function is unwrapped so that its derivative exists.

Phase delay and group delay

- Note that both phase delay and group delay are slopes of the phase function, just defined slightly differently





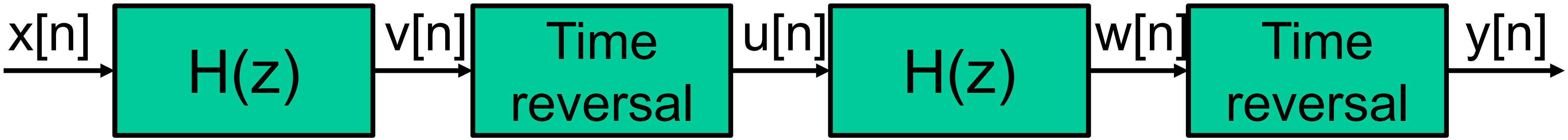
Zero-phase filters

- In many applications, it is necessary that the digital filter designed does not distort the phase of the input signal components => to make sure the frequency response of the filter does not delay any of the spectral components.

- A zero – phase transfer function has no phase component
  - the spectrum is purely real (no imaginary component) and non-negative
  - it is NOT possible to design a causal digital filter with a zero phase.

Zero-phase filters

- For non-real time processing of finite length, zero-phase filtering can be implemented by relaxing the causality requirement
- A zero-phase filtering scheme can be obtained by the following procedure:
  - Process the input data (finite length) with a causal real-coefficient filter H(z).
  - Time reverse the output of this filter and process by the same filter.
  - Time reverse once again the output of the second filter



$$Y(\omega) = H^*(\omega)V(\omega) = H^*(\omega)H(\omega)X(\omega) = \left|H(\omega)\right|^2 X(\omega)$$

Zero-phase filters

- For non-real time processing of finite length, zero-phase filtering can be implemented by relaxing the causality requirement
- A zero-phase filtering scheme can be obtained by the following procedure:
  - Process the input data (finite length) with a causal real-coefficient filter H(z).
  - Time reverse the output of this filter and process by the same filter.
  - Time reverse once again the output of the second filter

$$x[n] \longrightarrow \boxed{H(z)} \longrightarrow v[n] \qquad u[n] \longrightarrow \boxed{H(z)} \longrightarrow w[n]$$

$$u[n] = v[-n], \qquad y[n] = w[-n]$$

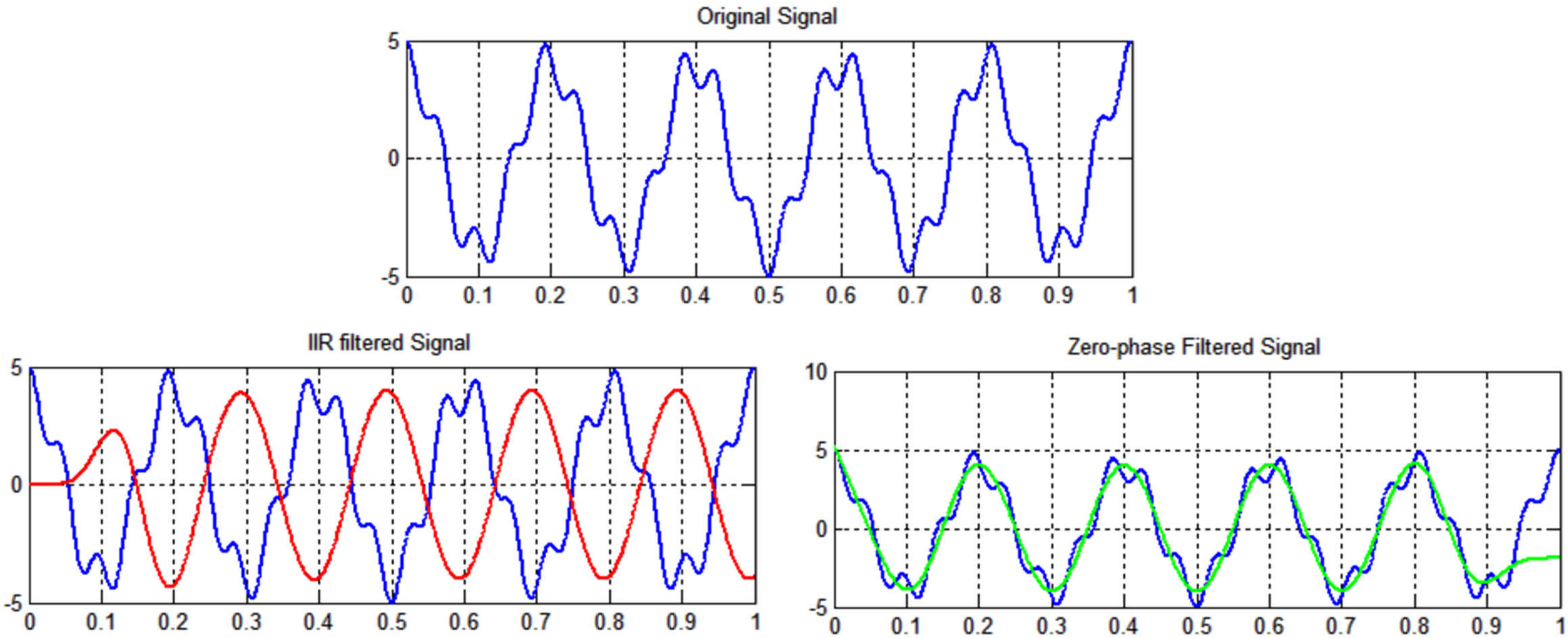
$$V(\omega) = H(\omega)X(\omega), \qquad W(\omega) = H(\omega)U(\omega)$$

$$U(\omega) = V^*(\omega), \qquad Y(\omega) = W^*(\omega) = H^*(\omega)U^*(\omega)$$

$$Y(\omega) = H^*(\omega)V^*(\omega) = H^*(\omega)H(\omega)X(\omega) = \left|H(\omega)\right|^2 X(\omega)$$

In Matlab

- The function **`filtfilt()`** implements the zero-phase filtering scheme
  - **`y=filtfilt(b,a,x)`** performs zero-phase digital filtering by processing the input data in both the forward and reverse directions.



Linear-phase filters

- For a causal transfer function with a nonzero phase response, the phase distortion can be avoided by ensuring that the transfer function has a unity magnitude and a linear-phase characteristic in the frequency band of interest

- The Fourier transform gives $Y(e^{j\omega}) = e^{-j\omega\alpha} X(e^{j\omega})$

- The frequency response is $H(e^{j\omega}) = \dfrac{Y(e^{j\omega})}{X(e^{j\omega})} = e^{-j\omega\alpha}$

  - a unity magnitude $\left|H(e^{j\omega})\right| = \left|e^{-j\omega\alpha}\right| = 1$

  - a linear phase with a group delay of α: $\theta(\omega) = \angle H(e^{j\omega}) = -\alpha\omega$

Linear-phase filters

- Note that this phase characteristic is linear for all ω in [0, 2π].
- The total delay at any frequency ω_0 is $\tau_p(\omega_0) = -\frac{\theta(\omega_0)}{\omega_0} = -\frac{-\alpha\omega_0}{\omega_0} = \alpha$
- This is identical to the group delay $d\theta(\omega)/d\omega$ evaluated at ω_0

$$\tau_g(\omega_0) = -\left.\frac{d\theta(\omega)}{d\omega}\right|_{\omega=\omega_0} = \alpha$$

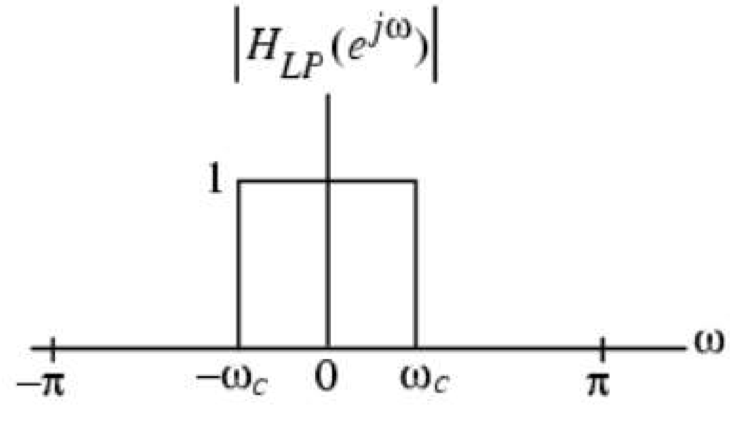
- All frequency components are delayed by α, or equivalently, the entire signal is delayed by α

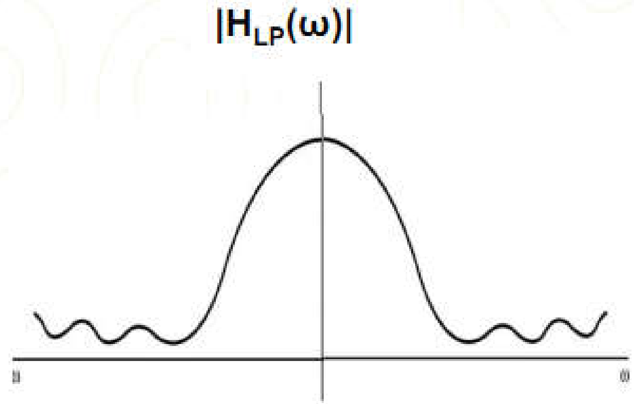
$$y[n] = x[n-\alpha]$$

  - Since the entire signal is delayed by a constant amount, there is no distortion!
  - The output of the filter simply delays the signal by a fixed amount.

Linear-phase filters

- If it is desired to pass input signal components in a certain frequency range undistorted in both magnitude and phase, then the transfer function should exhibit <u>a unity magnitude</u> response and <u>a linear-phase</u> response in the band of interest





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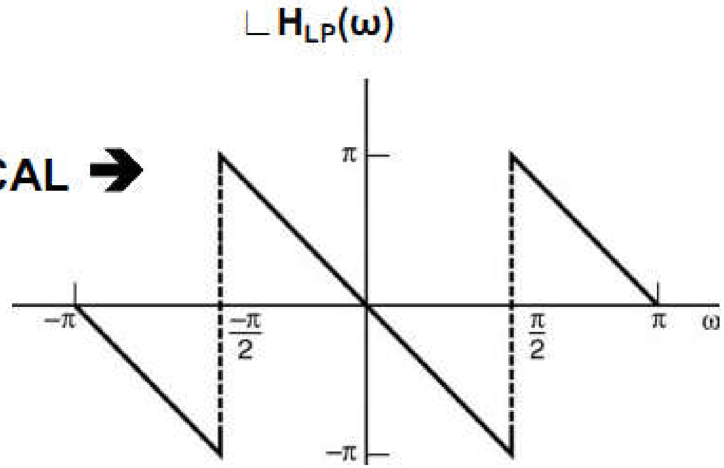
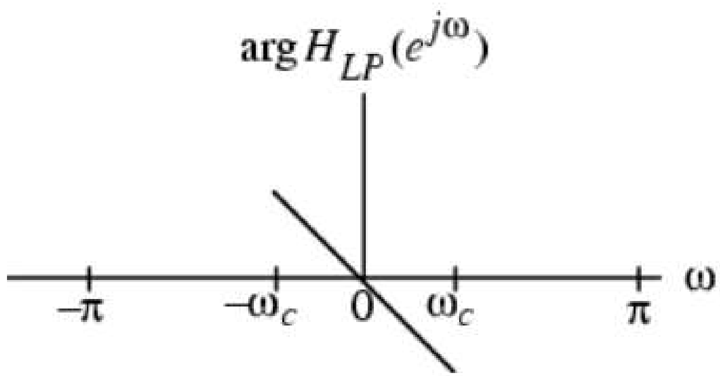


PRACTICAL →



12_3 Wrap up

- Calculating phase properties
  - Phase $\theta(\omega)$
  - Phase delay $\tau_p(\omega) = -\frac{\theta(\omega)}{\omega}$
  - Group delay $\tau_g(\omega) = -\frac{d\theta(\omega)}{d\omega}$

- Classification:
  - Zero phase: $\theta(\omega) = 0$; $\mathrm{H}(\omega)$ is real and non-negative;
    - In TD: no delay
  - Linear phase: $\theta(\omega) = -\alpha\omega$, so $\tau_p(\omega) = \tau_g(\omega) = \alpha$;
    - In TD: delay by α, y[n] = x[n-α]

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Lecture 12 Digital Filters Classification

Lect_12_4_Linear Phase FIR Filters

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Linear-phase FIR filters

- Consider a causal FIR filter of length N+1 (order N)

$$H(z) = \sum_{n=0}^{N} h[n] z^{-n} = h[0] + h[1]z^{-1} + h[2]z^{-2} + \cdots + h[M]z^{-M}$$

  - This transfer function has linear phase, if its impulse response h[n] is either

Symmetric $\boxed{h[n] = h[M-n], \quad 0 \le n \le M}$

or

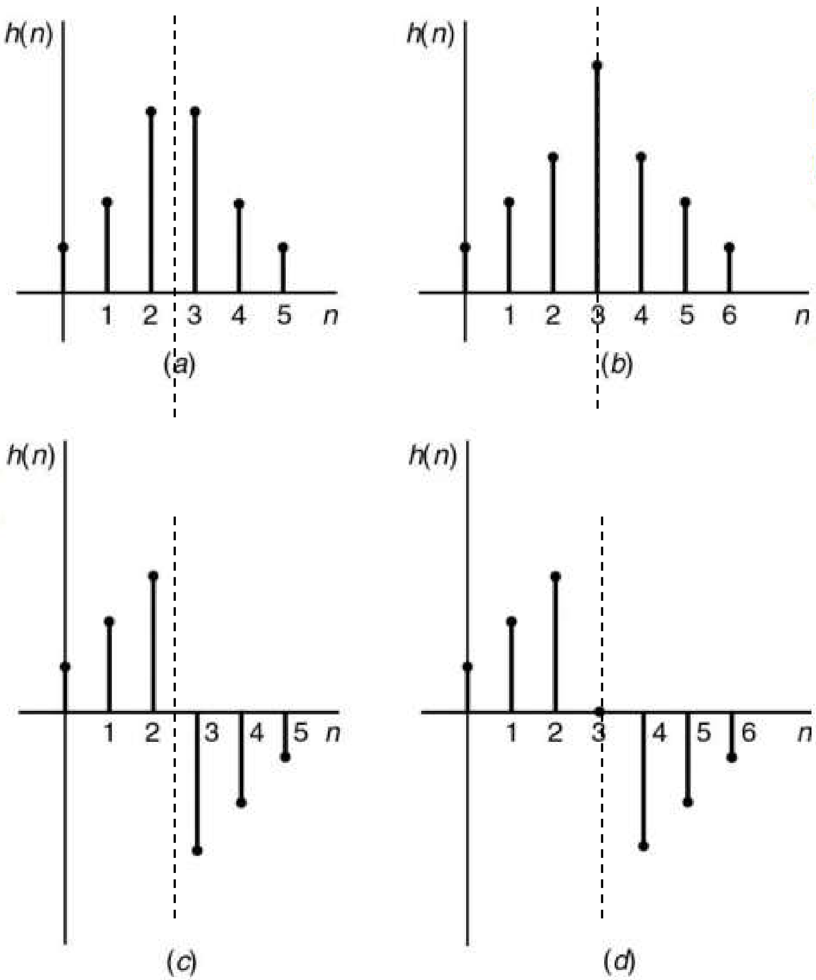
Anti-symmetric $\boxed{h[n] = -h[M-n], \quad 0 \le n \le M}$

Linear-phase FIR filters

- There are four possible scenarios: filter length even or odd, and impulse response is either symmetric or antisymmetric

FIR II: even length, symmetric (degree of filter odd)

FIR I: odd length, symmetric (degree of filter even)



FIR IV: even length, antisymmetric

FIR III: odd length, antisymmetric

Note for this case that h[M/2]=0

Linear-phase FIR filters

- Type 1: <u>**Symmetric**</u> impulse response with <u>**odd**</u> length

$h[n] = h[N-n], \quad 0 \le n \le N$ when degree N is even

Assume that N=6

$$H(z) = h[0] + h[1]z^{-1} + h[2]z^{-2} + h[3]z^{-3} + h[4]z^{-4} + h[5]z^{-5} + h[6]z^{-6}$$

$$= h[0](1+z^{-6}) + h[1](z^{-1}+z^{-5}) + h[2](z^{-2}+z^{-4}) + h[3]z^{-3}$$

$$= z^{-3}\{h[0](z^{3}+z^{-3}) + h[1](z^{2}+z^{-2}) + h[2](z+z^{-1}) + h[3]\}$$

$$H(e^{j\omega}) = e^{-j3\omega}\{2h[0]\cos(3\omega) + 2h[1]\cos(2\omega) + 2h[2]\cos(\omega) + h[3]\}$$

$$\theta(\omega) = -3\omega$$

$$H(e^{j\omega}) = e^{-jN\omega/2}\boxed{\tilde{H}(\omega)}$$

Amplitude response

Zero-phase response

The group delay is 3, indicating a constant delay of 3 samples

$$\boxed{\tilde{H}(\omega) = h[\tfrac{N}{2}] + 2\sum_{n=1}^{N/2} h[\tfrac{N}{2}-n]\cos(\omega n)}$$

Linear-phase FIR filters

- Type 2: **Symmetric** impulse response with **even** length

$$h[n]=h[N-n],\quad 0\le n\le N$$

In this case, degree N is odd.

$$H(e^{j\omega})=e^{-jN\omega/2}\boxed{\tilde{H}(\omega)}$$

$$\boxed{2\sum_{n=1}^{(N+1)/2} h[\tfrac{N+1}{2}-n]\cos(\omega(n-\tfrac{1}{2}))}$$

  - $\tilde{H}(\omega)$ is the amplitude response (also called zero-phase response), which is purely real.
  - The output is delayed by N/2 samples

Linear-phase FIR filters

- Type 3: **Anti-symmetric** impulse response with **odd** length

$$h[n] = -h[N-n], \qquad 0 \le n \le N$$

In this case, degree N is even.

$$H(e^{j\omega}) = je^{-jN\omega/2}\boxed{\tilde{H}(\omega)}$$

$$\boxed{\tilde{H}(\omega) = 2\sum_{n=1}^{N/2} h[\tfrac{N}{2}-n]\sin(\omega n)}$$

  - The phase response is of the form $\theta(\omega) = -\frac{N}{2}\omega + \frac{\pi}{2}$
  - The output is delayed by N/2 samples

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Linear-phase FIR filters

- Type 4: **Anti-symmetric** impulse response with **even** length

$$h[n] = -h[N-n], \qquad 0 \le n \le N$$

In this case, degree N is odd.

$$H(e^{j\omega}) = je^{-jN\omega/2}\boxed{\tilde{H}(\omega)}$$

$$\boxed{2\sum_{n=1}^{(N+1)/2} h[\tfrac{N+1}{2}-n]\sin(\omega(n-\tfrac{1}{2}))}$$

  - The phase response is of the form θ(ω)=-(N/2)ω + π/2
  - The output is delayed by N/2 samples
  - Note that for all cases, if $\tilde{H}(\omega)$ <0, an additional π term is added to the phase, which causes the samples to be flipped.

Zero locations of the linear-phase FIR filters

- Consider an FIR filter with symmetric impulse response

$$H(z) = \sum_{n=0}^{N} h[n]z^{-n} = \sum_{n=0}^{N} h[N-n]z^{-n}$$

$$= \sum_{m=0}^{N} h[m]z^{-N+m} = z^{-N} \sum_{m=0}^{N} h[m]z^{m} = z^{-N} H(z^{-1})$$

- The similar relation holds for anti-symmetric impulse response: $H(z) = -z^{-N} H(z^{-1})$

- Thus, if $z = \xi_0$ is a zero of $H(z)$ then so is its reciprocal $z = \xi_0^{-1} = 1/\xi_0$

Zero locations of the linear-phase FIR filters

- For an FIR filter with real impulse response:

$$H(z^*) = \sum_{n=0}^{N} h[n](z^*)^{-n} = \sum_{n=0}^{N} h[n](z^{-n})^*$$

$$= \left(\sum_{n=0}^{N} h[n]z^{-n} \right)^* = [H(z)]^*$$

- The zeroes occur in complex conjugate pairs
  - Thus, if $z = \xi_0$ is a zero of $H(z)$ then so is its conjugate $z = \xi_0^*$

Zero locations of the linear-phase FIR filters

- 1. A real zero at $z = r$
  - Must be another real zero at $z = 1/r$ (reciprocal)
  - Always appear in pair;
  - Special case: zeros at $z = 1$ and $z = -1$ can appear only singly, since such a zero is its own reciprocal.
- 2. A zero on the unit circle appear as a pair $z = e^{\pm j\varphi}$ as its reciprocal is also its complex conjugate.
- 3. A complex zero that is not on the unit circle is associated with a set of 4 zeroes:

$$z = re^{\pm j\varphi} \; and \; z = \frac{1}{r} e^{\pm j\varphi}$$

Type I Zero locations

- FIR Type I: symmetric, odd length = even order N

$$H(1) = (1)^{-N} H(1) = H(1)$$

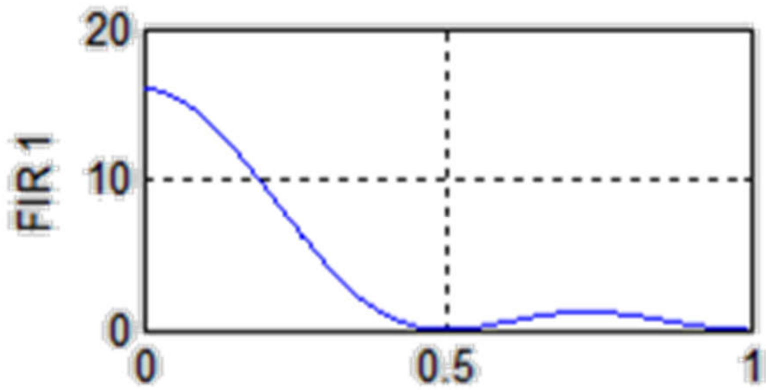
$$H(-1) = (-1)^{-N} H(-1) = H(-1)$$

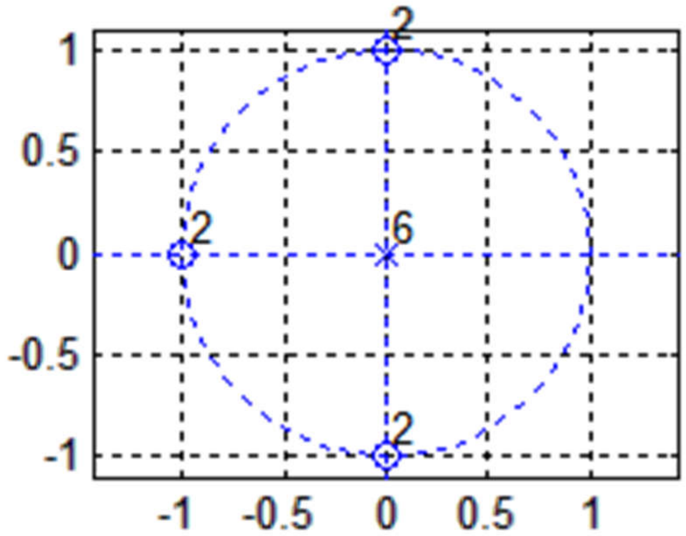
Zeroes at ± 1 are not guaranteed

- For example: N=6

```
% FIR 1
h1=[1 2 3 4 3 2 1];
```

Type 1 FIR filter: Either an even number or no zeroes at z = 1 and z=-1





Type II Zero locations

- FIR Type II: symmetric, even length = odd order N

$$H(1) = (1)^{-N} H(1) = H(1)$$

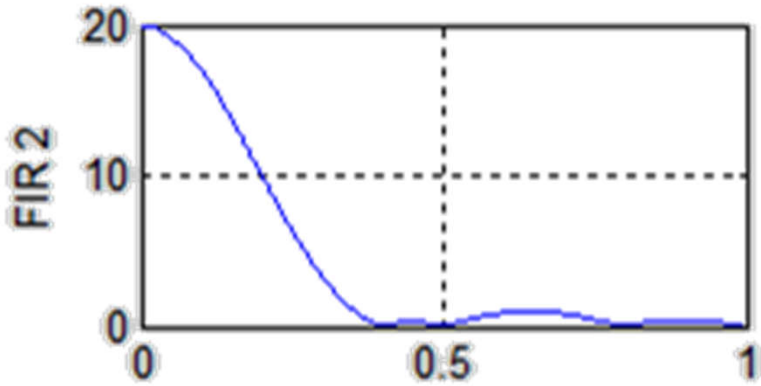
$$H(-1) = (-1)^{-N} H(-1) = -H(-1)$$

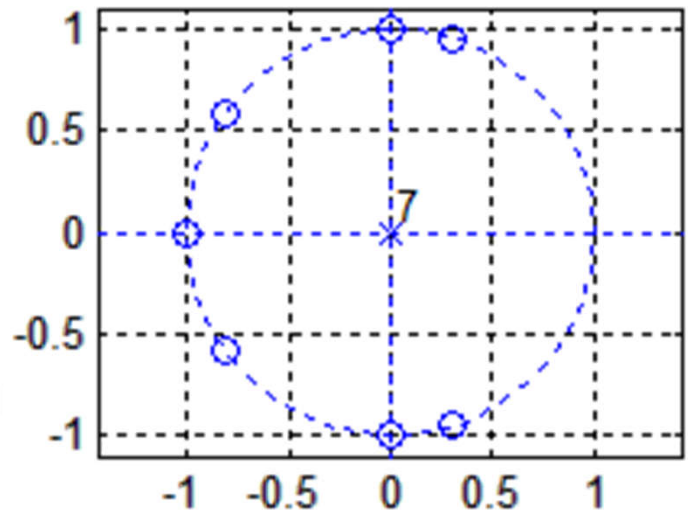
Zeroes at 1 are not guaranteed
Must have zero at z = -1

- For example: N=7

```
% FIR 2
h1=[1 2 3 4 4 3 2 1];
```

Type 2 FIR filter: Either an even number or no zeros at z = 1, and an odd number of zeros at z=-1





Type III Zero locations

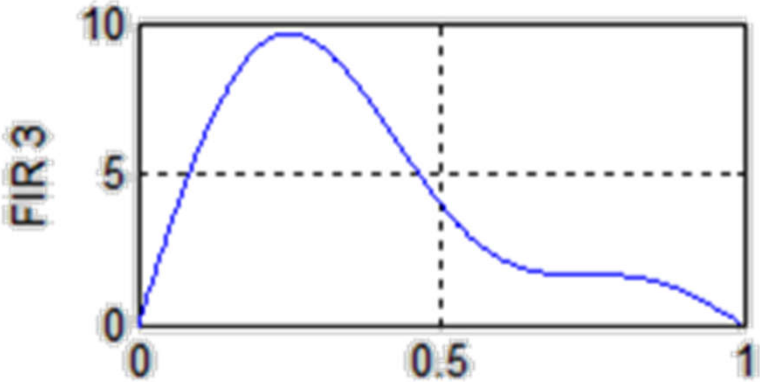
- FIR Type III: antisymmetric, odd length = even order

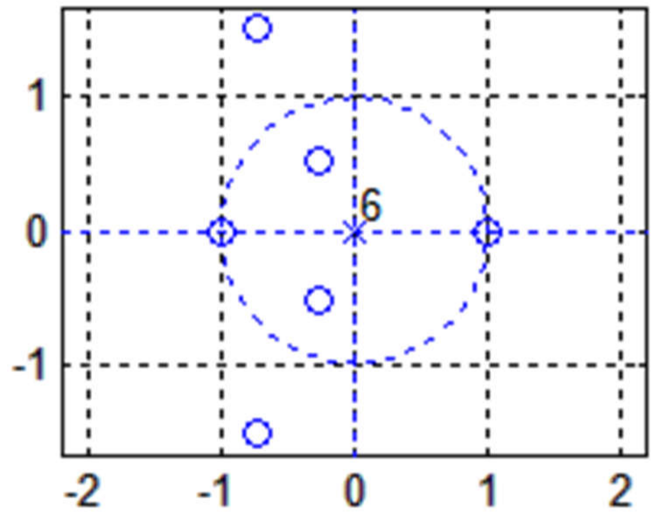
$$\left.\begin{aligned} H(1) &= -(1)^{-N}H(1) = -H(1) \\ H(-1) &= -(-1)^{-N}H(-1) = -H(-1) \end{aligned}\right\} \text{Must have zero at } z = \pm 1$$

- For example: N=7

```
% FIR 3
h1=[-1 -2 -3 0 3 2 1];
```

Type 3 FIR filter: An odd number of zeros at z = 1 and and z=-1





Type IV Zero locations

- FIR Type IV: antisymmetric, even length = odd order

$$H(1) = -(1)^{-N}H(1) = -H(1)$$

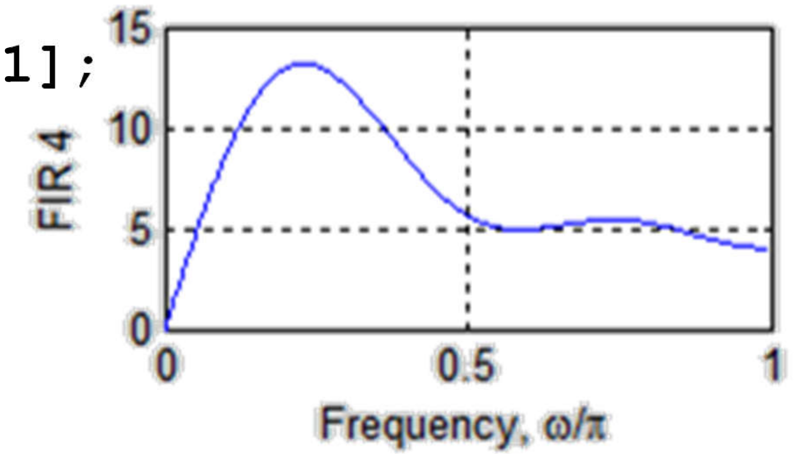
$$H(-1) = -(-1)^{-N}H(-1) = H(-1)$$

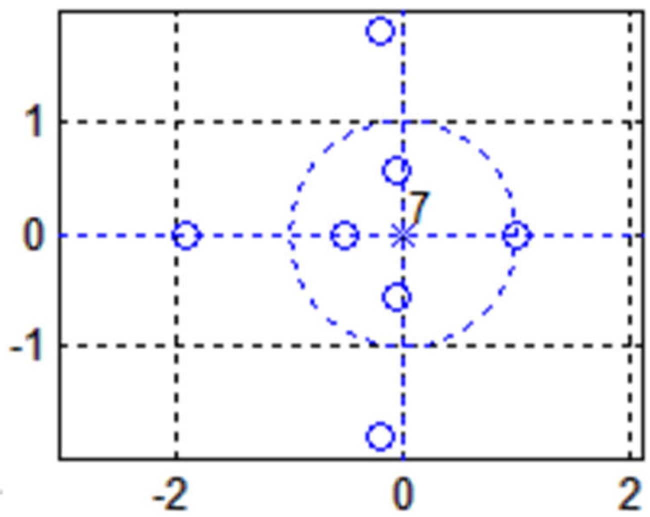
Zeroes at -1 are not guaranteed
Must have zero at z = 1

- For example: N=7

```
% FIR 4
h1=[-1 -2 -3 -4 4 3 2 1];
```

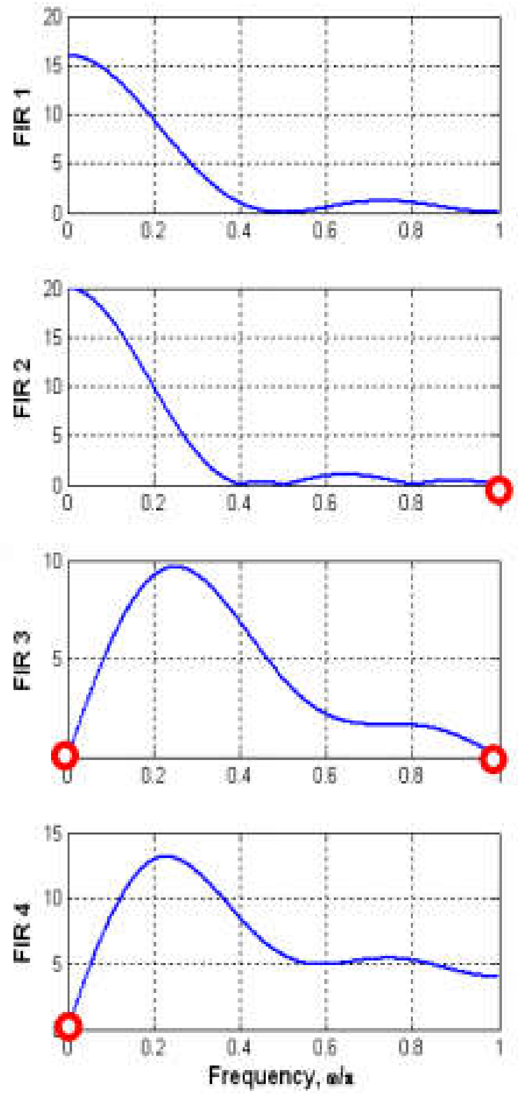
Type 4 FIR filter: An odd number of zeros at z = 1, and either an even number or no zeros at z=-1





Zero locations of the linear-phase FIR filters

- The presence of zeroes at $z=\pm 1$ leads to some limitations on the use of these linear-phase transfer functions for designing frequency-selective filters
  - A Type 2 FIR filter cannot be used to design a highpass filter since it always has a zero at $z=-1$
  - A Type 3 FIR filter has zeroes at both $z = 1$ and $z=-1$, and hence cannot be used to design either a lowpass or a highpass or a bandstop filter
  - A Type 4 FIR filter is not appropriate to design a lowpass filter due to the presence of a zero at $z = 1$
  - Type 1 FIR filter has no such restrictions and can be used to design almost any type of filter



12_4 Wrap up

- Linear-phase FIR filters
  - Why linear-phase?
  - Type 1-4
    - Phase response, group delay, and magnitude response
- Zero locations
  - Symmetry ➔ $H(z) = \pm z^{-N} H(z^{-1})$ ➔ Reciprocal is zero
  - Real coefficient ➔ $H(z^*) = [H(z)]^*$ ➔ Conjugate is zero
  - Zeroes on z-plane: singly, pair, group of 4
  - Zeroes location of FIR Type 1-4
    - Zeroes at ± 1
    - Types of filters: LP, HP, BP or BS

EEE336 Signal Processing and Digital Filtering

Lecture 12 Digital Filters Classification

Lect_12_5 Allpass, Min-phase and Max-phase Filters

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Allpass transfer functions

- An IIR transfer function A(z) with unity magnitude response for all frequencies is called an allpass transfer function

$$|A(e^{j\omega})|^2 = 1, \quad \text{for all } \omega$$

- An Mth order causal real-coefficient allpass transfer function is of the form

$$A_M(z) = \pm \frac{d_M + d_{M-1}z^{-1} + \cdots + d_1 z^{-M+1} + z^{-M}}{1 + d_1 z^{-1} + \cdots + d_{M-1} z^{-M+1} + d_M z^{-M}}$$

$$\boxed{D_M(z) = 1 + d_1 z^{-1} + \cdots + d_{M-1} z^{-M+1} + d_M z^{-M}}$$

$$A_M(z) = \pm \frac{z^{-M} D_M(z^{-1})}{D_M(z)}$$

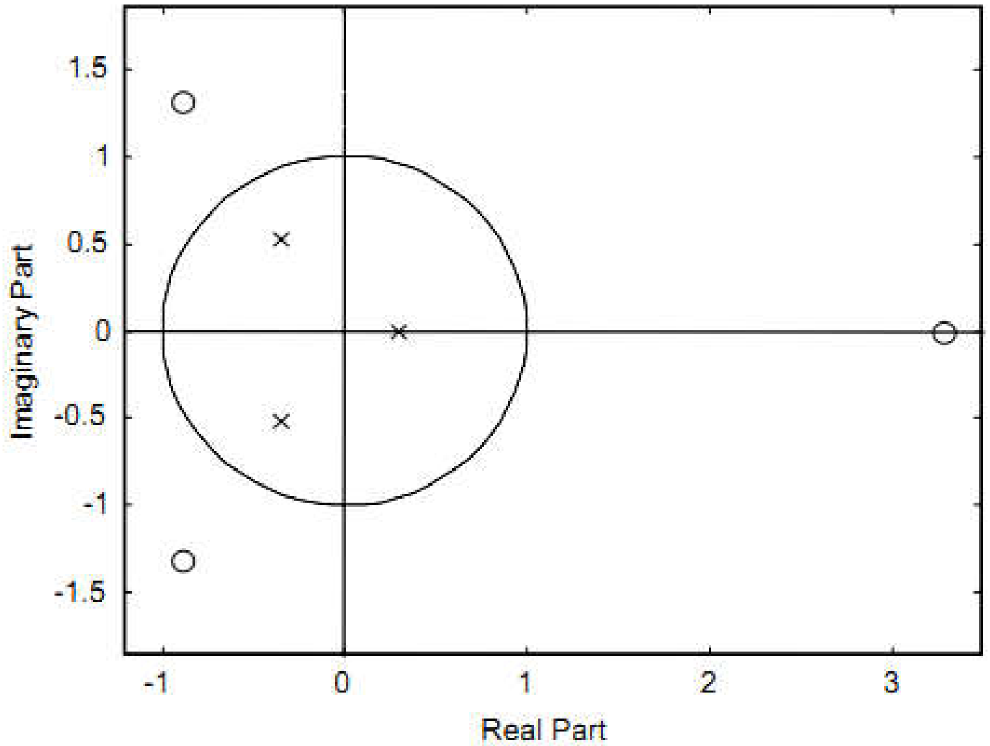
Note that if $z=re^{j\phi}$ is a pole of $A_M(z)$ then it has a zero at $z=(1/r)e^{-j\phi}$

Allpass transfer functions

- The numerator of a real-coefficient allpass transfer function $D_M(z^{-1})$ is said to be the mirror-image polynomial of the denominator $D_M(z)$, and vice versa.
- Poles and zeros exhibit mirror-image symmetry in the z-plane
  – Example: The pole-zero diagram of a third order allpass function

$$A_3(z) = \frac{-0.2 + 0.18z^{-1} + 0.4z^{-2} + z^{-3}}{1 + 0.4z^{-1} + 0.18z^{-2} - 0.2z^{-3}}$$

$$|A(z)| \begin{cases} < 1, & \text{for } |z| > 1 \\ = 1, & \text{for } |z| = 1 \\ > 1, & \text{for } |z| < 1 \end{cases}$$



Allpass transfer functions

- Phase of the allpass filter

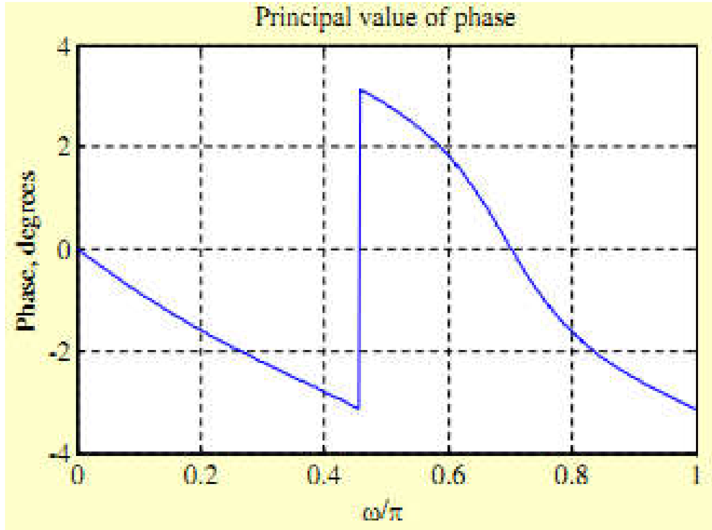
- T(ω) denotes the group delay function of A(z)

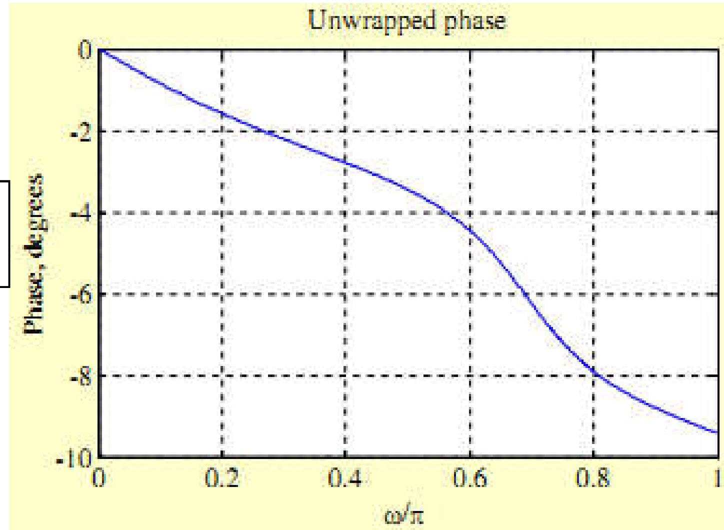
$$\tau(\omega) = -\frac{d}{d\omega}[\theta_c(\omega)]$$

Discontinuou

Unwrap

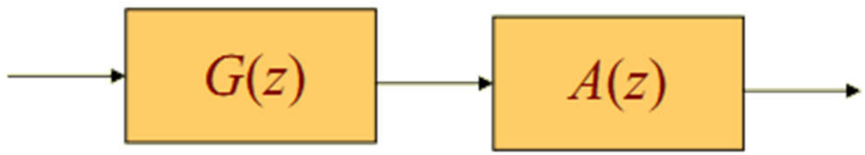
Continuous





Allpass transfer functions

- A simple application is a delay equalizer
  - G(z) : the transfer function of a digital filter
  - The nonlinear phase response of G(z) can be corrected by cascading it with an allpass filter A(z) so that the overall cascade has a constant group delay in the band of interest



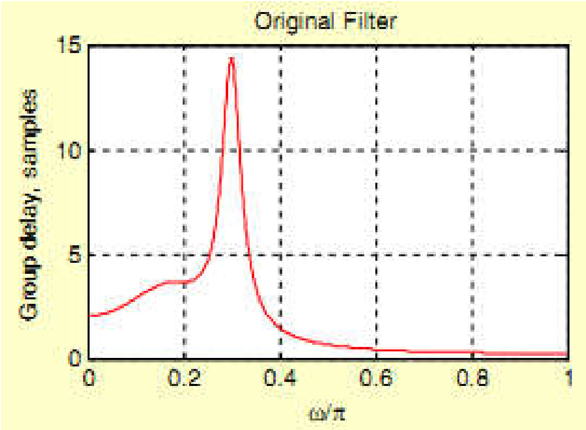
Since $|A(e^{j\omega})| = 1$, we have

$$|G(e^{j\omega})A(e^{j\omega})| = |G(e^{j\omega})|$$

  - Overall group delay is then given by the sum of the group delays of G(z) and A(z)
  - The allpass section is designed so that the overall group delay is approximately constant in the frequency range of interest

An example

- Figure below shows the group delay of a 4th order elliptic filter with satisfactory magnitude response

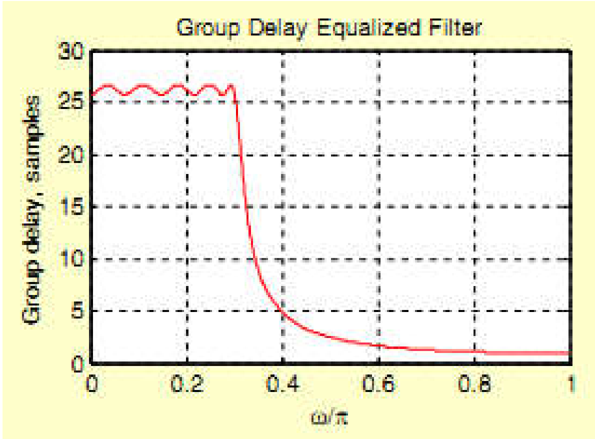


$\omega_p = 0.3\pi$,

$\delta_p = 1\text{dB}$,

$\delta_s = 35\text{dB}$

- Cascading it with an 8th order allpass section designed to equalize the group delay, get

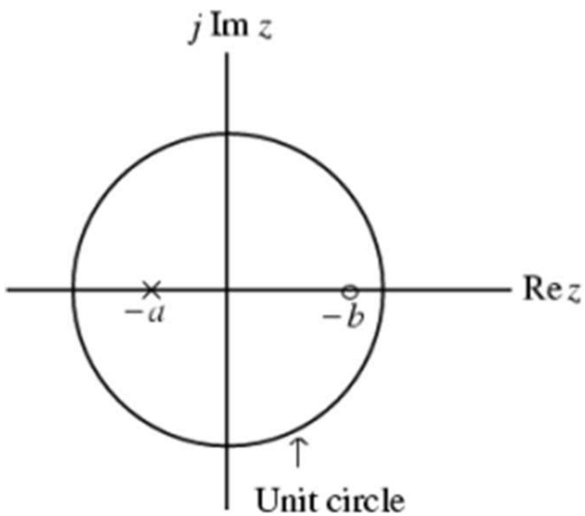


Minimum-Phase and Maximum-Phase Filters

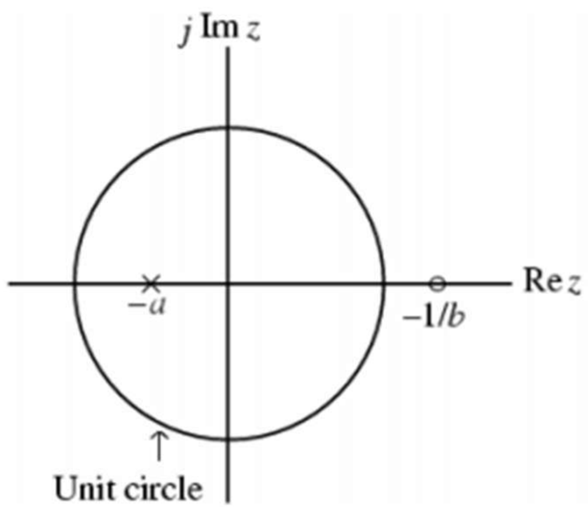
- Consider the two 1st order transfer functions:

$$H_1(z) = \frac{z+b}{z+a}, \quad H_2(z) = \frac{bz+1}{z+a}, \quad |a|<1, \quad |b|<1$$

  - Both transfer functions have a pole inside the unit circle at the same location z = -a and are stable
  - But the zero of H1(z) is inside the unit circle at z = -b, whereas, the zero of H2(z) is at z = -1/b situated in a mirror-image symmetry



$H_1(z)$



$H_2(z)$

Minimum-Phase and Maximum-Phase Filters

magnitude function

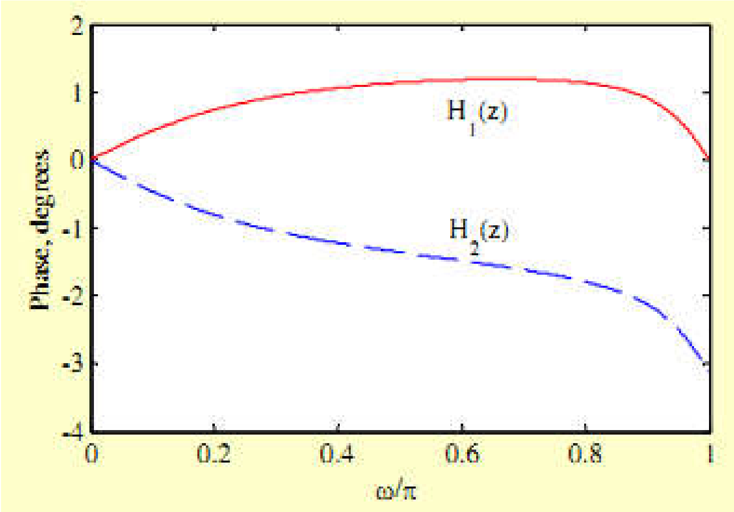
$$H_1(z)H_1(z^{-1}) = H_2(z)H_2(z^{-1})$$

phase functions

$$\arg[H_1(e^{j\omega})] = \tan^{-1}\frac{\sin\omega}{b+\cos\omega} - \tan^{-1}\frac{\sin\omega}{a+\cos\omega}$$

$$\arg[H_2(e^{j\omega})] = \tan^{-1}\frac{b\sin\omega}{1+b\cos\omega} - \tan^{-1}\frac{\sin\omega}{a+\cos\omega}$$

Unwrapped phase responses of the two transfer functions for a=0.8 and b=-0.5



A causal stable transfer function with all zeros outside the unit circle has an excess phase compared to a causal transfer function with identical magnitude but having all zeros inside the unit circle

Minimum-Phase and Maximum-Phase Filters

- A causal stable transfer function with all zeros inside the unit circle is called a minimum-phase transfer function
- A causal stable transfer function with all zeros outside the unit circle is called a maximum-phase transfer function
- A causal stable transfer function with zeros inside and outside the unit circle is called a mixed-phase transfer function
- Any mixed-phase transfer function can be expressed as the product of a minimum-phase transfer function and a stable allpass transfer function

$$H(z) = H_m(z)A(z)$$

12_5 Wrap up

- Allpass filters
  - Magnitude and phase
  - Zeroes and poles
- Minimum-phase, maximum-phase and mix-phase filters
  - Zeroes locations
  - Conversion among each other using allpass filter

Chapter 12 Summary

- Types of transfer functions
  - Time domain: Impulse response -> FIR / IIR
  - Frequency domain:
    - Magnitude response -> LPF, HPF, BPF, BSF
    - Phase characteristics -> Zero phase, linear phase
- Classification based on magnitude characteristics
  - Ideal filter VS Realizable filters
  - Allpass transfer functions
- Classification based on phase characteristics
  - Phase delay and group delay
  - Zero-phase and linear-phase filters
  - Linear phase FIR filters (4 types)
  - Minimum and maximum phase filters