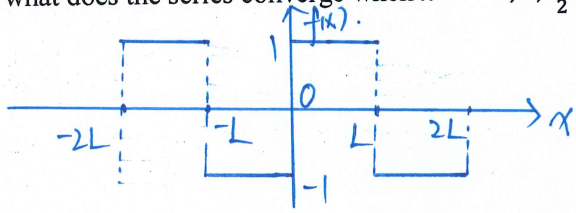
MTH201 Name________________ID____________ Nov. 2017

Tutorial 8 Fourier's theorem

1. $f(x+2L)=f(x)$ for all x and $f(x)=\begin{cases}-1, & -L<x<0,\\ 1, & 0\le x\le L,\end{cases}$,
   (a) Sketch the function $f(x)$ in the range $-2L<x<2L$.
   (b) To what does the series converge when $x=-L, 0, \frac{L}{2}, \frac{3L}{2}$.

(a).



(b) At $x=-L$, $f(x)$ is discontinuous, So the Fourier Series converges to $\frac{1}{2}[f(-L^-)+f(-L^+)]$
$=\frac{1}{2}(1-1)=0$

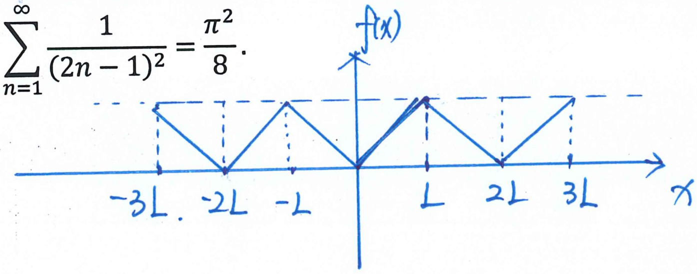
At $x=0$, - - - - - - - - - - - - - - $\frac{1}{2}[f(0^-)+f(0^+)]=\frac{1}{2}(-1+1)=0$.

At $x=\frac{L}{2}$, $f(x)$ is continuous, So the Fourier Series converges to $f(\frac{L}{2})=1$.

At $x=\frac{3}{2}L$, - - - - - - - - - - - - - - $f(\frac{3L}{2})=-1$.

2. The function $f(x)$ is defined $f(x)=\begin{cases}-x, & -L\le x<0\\ x, & 0\le x<L\end{cases}$ and $f(x+2L)=f(x)$.
   (a) Sketch $f(x)$ in $-3L<x<3L$.
   (b) State the values the Fourier series will converge to at $x=-\frac{L}{2}, 0, \frac{L}{3}, L$.
   (c) Find the Fourier series of $f(x)$ and give the first three non-zero terms.
   (d) By choosing an appropriate value for x in the Fourier series for $f(x)$, show that
   $$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}=\frac{\pi^2}{8}.$$

(a).



(b). Because $f(x)$ is continuous everywhere, so the Fourier Series converges to the value of $f(x)$ everywhere.

At $x=-\frac{L}{2}$. $f(x)=\frac{L}{2}$.

At $x=0$, $f(0)=0$

At $x=\frac{L}{3}$ $f(\frac{L}{3})=\frac{L}{3}$.

At $x=L$, $f(L)=L$.

(c). Because $f(-x)=f(x)$, $f(x)$ is an even function, so $b_n=0$, $n=1,2,3,\cdots$.

$$a_0=\frac{1}{2L}\int_{-L}^{L}f(x)dx=\frac{1}{L}\int_0^L f(x)\,dx=\frac{1}{L}\int_0^L x\,dx=\frac{1}{L}\left[\frac{1}{2}x^2\right]_0^L=\frac{L}{2}$$

$$a_n=\frac{1}{L}\int_{-L}^{L}f(x)\cos\frac{n\pi}{L}x\,dx=\frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx=\frac{2}{L}\int_0^L x\cos\frac{n\pi x}{L}dx.$$ Let $u=x$, $v'=\cos\frac{n\pi x}{L}$. $u'=1$, $v=\frac{L}{n\pi}\cdot\sin\frac{n\pi}{L}x$.

$$=\frac{2}{L}\cdot\left[x\cdot\frac{L}{n\pi}\sin\frac{n\pi}{L}x\right]_0^L-\frac{2}{L}\int_0^L\frac{L}{n\pi}\sin\frac{n\pi}{L}x\,dx$$

$$=\frac{2}{L}(0-0)-\frac{2}{n\pi}\int_0^L\sin\frac{n\pi}{L}x\,dx$$

$$=\frac{-2}{n\pi}\cdot\frac{L}{n\pi}\cdot(-1)\cdot\left[\cos\frac{n\pi}{L}x\right]_0^L=\frac{2L}{n^2\pi^2}(\cos n\pi-1).$$

Therefore, the Fourier series of $f(x)$ is

$$f(x)=\frac{L}{2}+\sum_{n=1}^{\infty}\frac{2L}{n^2\pi^2}(\cos n\pi-1)\cos\frac{n\pi}{L}x.$$

$n=0$, $a_0=\frac{L}{2}$.

$n=1$, $a_1\cos\frac{\pi}{L}x=\frac{-2L}{\pi^2}\cos\frac{\pi}{L}x$

$n=2$, $a_2\cos\frac{2\pi}{L}x=0$

$n=3$, $a_3\cos\frac{3\pi}{L}x=\frac{-2L}{9\pi^2}\cos\frac{3\pi}{L}x$.

So the first three non-zero terms are, $\frac{L}{2}$, $\frac{-2L}{\pi^2}\cos\frac{\pi x}{L}$, $\frac{-2L}{9\pi^2}\cos\frac{3\pi}{L}x$.

(d). Take $x=0$, then the Fourier series becomes

$$f(0)=\frac{L}{2}+\sum_{n=1}^{\infty}\frac{2L}{n^2\pi^2}(\cos n\pi-1),\quad \cos n\pi=\begin{cases}1, & n=2m \text{ even.}\\ -1, & n=2m-1, \text{ odd.}\end{cases}$$

$$\therefore f(0)=0=\frac{L}{2}+\sum_{m=1}^{\infty}\frac{2L}{(2m-1)^2\pi^2}(-1-1)$$

$$\therefore \sum_{m=1}^{\infty}\frac{-4L}{(2m-1)^2\pi^2}=-\frac{L}{2}\quad \therefore \sum_{m=1}^{\infty}\frac{1}{(2m-1)^2}=\frac{\pi^2}{8}.$$

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3. The function $f(x)$ is defined $f(x) = \begin{cases} x, & -L \le x < 0 \\ L, & 0 \le x < L \end{cases}$, $f(x+2L) = f(x)$.

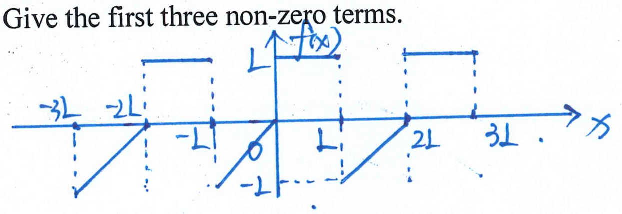
(a) Sketch $f(x)$ in $-3L<x<3L$.

(b) To what does the series converge when $x = -\frac{L}{2}, 0, \frac{L}{2}, L$.

(c) Find its Fourier series.

(d) Give the first three non-zero terms.

(a).



(b). At $x = -\frac{L}{2}$, $f(x)$ is continuous, so the Fourier series converges to $f(-\frac{L}{2}) = -\frac{L}{2}$.

At $x = 0$, $f(x)$ is discontinuous, - - - - - - - - - $\frac{1}{2}[f(0^-)+f(0^+)] = \frac{1}{2}(0+L) = \frac{L}{2}$.

At $x = \frac{L}{2}$, $f(x)$ is continuous, - - - - - - - - $f(\frac{L}{2}) = L$.

At $x = L$, $f(x)$ is discontinuous, - - - - - - - - $\frac{1}{2}[f(L^-)+f(L^+)] = \frac{1}{2}(L-L) = 0$

(c). $a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx = \frac{1}{2L}\int_{-L}^{0} x\,dx + \frac{1}{2L}\int_0^L L\,dx = \frac{1}{2L}\cdot\left[\frac{1}{2}x^2\right]_{-L}^{0} + \frac{1}{2L}\cdot[Lx]_0^L = -\frac{L}{4} + \frac{L}{2} = \frac{L}{4}$.

$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi}{L}x\,dx = \frac{1}{L}\left(\int_{-L}^{0} x\cos\frac{n\pi x}{L}dx + \int_0^L L\cos\frac{n\pi}{L}x\,dx\right) = \frac{1}{L}(I + II)$.

$I = \int_{-L}^{0} x\cos\frac{n\pi x}{L}dx$, Let $u = x$, $v' = \cos\frac{n\pi x}{L}$ $\therefore$ $u' = 1$, $v = \frac{L}{n\pi}\sin\frac{n\pi x}{L}$.

$= \left[x\cdot\frac{L}{n\pi}\sin\frac{n\pi}{L}x\right]_{-L}^{0} - \int_{-L}^{0}\frac{L}{n\pi}\sin\frac{n\pi x}{L}dx$

$= 0 - \frac{L}{n\pi}\cdot\frac{-L}{n\pi}\cdot\left[\cos\frac{n\pi}{L}x\right]_{-L}^{0} = \frac{L^2}{n^2\pi^2}(1-\cos n\pi)$.

$II = \int_0^L L\cos\frac{n\pi}{L}x\,dx = L\cdot\frac{L}{n\pi}\left[\sin\frac{n\pi}{L}x\right]_0^L = 0$.

$\therefore$ $a_n = \frac{1}{L}\cdot(I+II) = \frac{1}{L}\cdot\frac{L^2}{n^2\pi^2}(1-\cos n\pi) = \frac{L}{n^2\pi^2}\cdot(1-\cos n\pi)$.

$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi}{L}x\,dx = \frac{1}{L}\int_{-L}^{0} x\sin\frac{n\pi}{L}x\,dx + \frac{1}{L}\int_0^L L\sin\frac{n\pi}{L}x\,dx$

$= \frac{1}{L}(I + II)$.

$I = \int_{-L}^{0} x \sin\frac{n\pi}{L}x\,dx$, Let $u = x$, $v' = \sin\frac{n\pi x}{L}$.

then $u' = 1$, $v = -\frac{L}{n\pi}\cos\frac{n\pi}{L}x$.

$= \left[x \cdot \frac{-L}{n\pi}\cos\frac{n\pi}{L}x\right]_{-L}^{0} - \int_{-L}^{0} \frac{-L}{n\pi}\cos\frac{n\pi}{L}x\,dx$

$= 0 - (-L)\frac{-L}{n\pi}\cos n\pi + \int_{-L}^{0} \frac{L}{n\pi}\cos\frac{n\pi}{L}x\,dx$

$= \frac{-L^2}{n\pi}\cos n\pi + \frac{L}{n\pi}\cdot\frac{L}{n\pi}\left[\sin\frac{n\pi}{L}x\right]_{L}^{0}$

$= -\frac{L^2}{n\pi}\cos n\pi$.

$II = \int_0^L L \sin\frac{n\pi}{L}x\,dx = -L\cdot\frac{L}{n\pi}\left[\cos\frac{n\pi x}{L}\right]_0^L = -\frac{L^2}{n\pi}\cdot(\cos n\pi - 1)$.

$\therefore b_n = \frac{1}{L}(I + II) = \frac{1}{L}\cdot\frac{L^2}{n\pi}(1 - 2\cos n\pi). = \frac{L}{n\pi}(1 - 2\cos n\pi)$.

So the Fourier series of $f(x)$ is

$$f(x) = \frac{L}{4} + \sum_{n=1}^{\infty} \frac{L}{n^2\pi^2}(1 - \cos n\pi)\cos\frac{n\pi}{L}x + \frac{L}{n\pi}(1 - 2\cos n\pi)\sin\frac{n\pi}{L}x.$$

(d). $n = 0$, $a_0 = \frac{L}{4}$.

$n = 1$, $a_1\cos\frac{\pi}{L}x + b_1\sin\frac{\pi}{L}x = \frac{L}{\pi^2}2\cos\frac{\pi}{L}x + \frac{L}{\pi}\cdot 3\sin\frac{\pi}{L}x$.

$n = 2$, $a_2\cos\frac{2\pi}{L}x + b_2\sin\frac{2\pi}{L}x = \frac{L}{4\pi^2}0\cos\frac{2\pi}{L}x + \frac{L}{2\pi}(-1)\sin\frac{2\pi}{L}x = -\frac{L}{2\pi}\sin\frac{2\pi}{L}x$.

So the first three non-zero terms are

$\frac{L}{4}$, $\frac{2L}{\pi^2}\cos\frac{\pi}{L}x + \frac{3L}{\pi}\sin\frac{\pi}{L}x$, $\frac{-L}{2\pi}\sin\frac{2\pi}{L}x$.