Tutorial 1 Revision on vector calculus and line integral

Differentiation

1. Find the first and second derivatives of $\boldsymbol{r} =< 3cos2t, 3sin2t, 4t >$.

$\vec{r}'(t) = <-6\sin 2t,\ 6\cos 2t,\ 4>.$

$\vec{r}''(t) = <-12\cos 2t,\ -12\sin 2t,\ 0>.$

2. Find the first partial derivatives of $\boldsymbol{v}_1 =< e^x \text{cos}y, e^x \text{sin}y >$ and $\boldsymbol{v}_2 =< \text{cos}x\ \text{cosh}y, -\text{sin}x\ \text{sinh}y >$.

$\dfrac{\partial \vec{v}_1}{\partial x} = <e^x\cos y,\ e^x\sin y).$ $\dfrac{\partial \vec{v}_1}{\partial y} = <-e^x\sin y,\ e^x\cos y>.$

$\dfrac{\partial \vec{v}_2}{\partial x} = <-\sin x\cosh y,\ -\cos x\sinh y>.$ $\dfrac{\partial \vec{v}_2}{\partial y} = <\cos x\sinh y,\ -\sin x\cosh y>.$

Gradient

Find the gradient of the following functions *f*:

1. $f = (x-1)(4y-2)$

$\nabla f = <4y-2,\ 4(x-1)>.$

2. $f = 2x^2 + 5y^2$

$\nabla f = <4x,\ 10y>.$

3. $f = \frac{x}{y}$

$\nabla f = <\frac{1}{y},\ -\frac{x}{y^2}>.$

4. $f = (x-2)^2 + (2y+4)^2$

$\nabla f = <2(x-2),\ 4(2y+4)>.$

5. $f = x^5 + y^5$

$\nabla f = <5x^4,\ 5y^4>.$

6. $f = \frac{x^2+y^2}{x^2-y^2}$

$\nabla f = \langle -\frac{4xy^2}{(x^2-y^2)^2}, \frac{4x^2y}{(x^2-y^2)^2} \rangle.$

Velocity fields

Given the velocity potential f of a flow, find the velocity $\boldsymbol{v} = \nabla f$ of the field and its value $\boldsymbol{v}(P)$ at P.

1. $f = x^2 - 6x - y^2, P: (-1,5)$

$\vec{v} = \nabla f = \langle 2x-6, -2y \rangle, \quad \vec{v}(p) = \langle -8, -10 \rangle.$

2. $f = \cos x \cosh y, P: (\pi/2, \ln 2)$

$\vec{v} = \nabla f = \langle -\sin x \cosh y, \cos x \sinh y \rangle, \quad \vec{v}(p) = \langle -\frac{5}{4}, 0 \rangle.$

3. $f = x\left(1 + \frac{1}{x^2+y^2}\right), P: (1,1)$

$\vec{v} = \nabla f = \langle 1 - \frac{x^2-y^2}{(x^2+y^2)^2}, -\frac{2xy}{(x^2+y^2)^2} \rangle, \quad \vec{v}(p) = \langle 1, -\frac{1}{2} \rangle.$

4. $f = e^x \cos y, P: (1, \pi/2)$

$\nabla f = \langle e^x \cos y, -e^x \sin y \rangle, \quad \vec{v}(p) = \langle 0, -e \rangle.$

Divergence

Find div$\boldsymbol{v}$ and its value at P.

1. $\boldsymbol{v} = \langle 2x^2, -3y^2, 8z^2 \rangle, P: \left(3, \frac{1}{2}, 0\right)$

$\operatorname{div} \vec{v} = 4x - 6y + 16z, \quad \operatorname{div} \vec{v}(p) = 9.$

2. $\boldsymbol{v} = \langle 0, \ \sin(x^2yz), \ \cos(xy^2z) \rangle, P: (1, \frac{1}{2}, -\pi)$

$\operatorname{div} \vec{v} = x^2y \cos(x^2yz) - xy^3\sin(xy^2z), \quad \operatorname{div} \vec{v}(p) = \frac{\sqrt{2}}{8}.$

3. $\boldsymbol{v} = \frac{\langle x,y \rangle}{x^2-y^2}, x \neq y$

$\operatorname{div} \vec{v} = 0.$

4. $\boldsymbol{v} = \langle v_1(y,z), \ v_2(z,x), \ v_3(x,y) \rangle, P: (3,1,-1)$

$\operatorname{div} \vec{v} = 0.$

5. $v = < x^2yz,\ xy^2z,\ xyz^2 >, P: (-1,3,-2)$

$\operatorname{div} \vec{v} = 6xy$. $\operatorname{div} \vec{v}(P) = 36$.

6. $v = \frac{<-x,-y,-z>}{(x^2+y^2+z^2)^{\frac{3}{2}}}$

$\operatorname{div} \vec{v} = 0$.

Curl

Find curl *v* for *v* given with respect to right-handed Cartesian coordinates. Show the details of your work.

1. $v = < 4y^2,\ 3x^2, 0 >$ 1. $\operatorname{curl} \vec{v} = \nabla \times \vec{v} = <0, 0, 6x-8y>$.
2. $v = xyz < x^2,\ y^2,\ z^2 >$ 2. $\operatorname{curl} \vec{v} = < x(z^3-y^3),\ y(x^3-z^3),\ z(y^3-x^3) >$.
3. $v = \frac{<x,y,z>}{(x^2+y^2+z^2)^{\frac{3}{2}}}$ 3. $\operatorname{curl} \vec{v} = 0$
4. $v = < 0,0, e^{-x}\text{siny} >$ 4. $\operatorname{curl} \vec{v} = < e^{-x}\cos y,\ e^{-x}\sin y,\ 0 >$.
5. $v = < e^{-z^2},\ e^{-x^2},\ e^{-y^2} >$
   5. $\operatorname{curl} \vec{v} = < -2ye^{-y^2},\ -2ze^{-z^2},\ -2xe^{-x^2} >$.

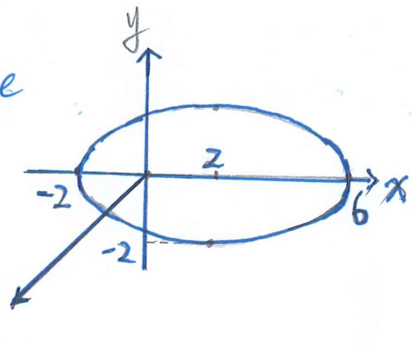
Parametric representations

What curves are represented by the following? Sketch them. (page 390, Q1-4, 8)

1. $< 2 + 4\text{cost},\ 2\text{sint},\ 0 >$

We know $x = 2+4\cos t$ $y = 2\sin t$, $z = 0$, so we have

$$\begin{cases} \left(\frac{x-2}{4}\right)^2 + \left(\frac{y}{2}\right)^2 = 1 \\ z = 0 \end{cases} \quad \text{i.e.} \quad \begin{cases} \frac{(x-2)^2}{16} + \frac{y^2}{4} = 1 \\ z = 0 \end{cases}$$



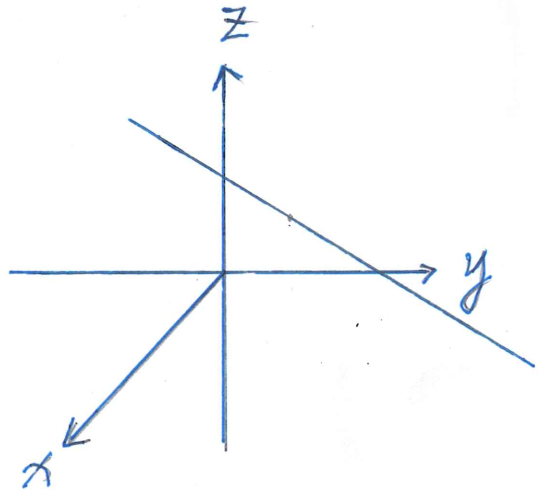
Since $z \equiv 0$, this is an ellipse in the xy-plane

With center $(2,0,0)$, the semi-major axis $a=4$ and semi-minor axis is $b=2$.

2. $< a + t,\ b + 3t,\ c - 5t >$

$x = a+t$, $y = b+3t$, $z = c-5t$, so we have

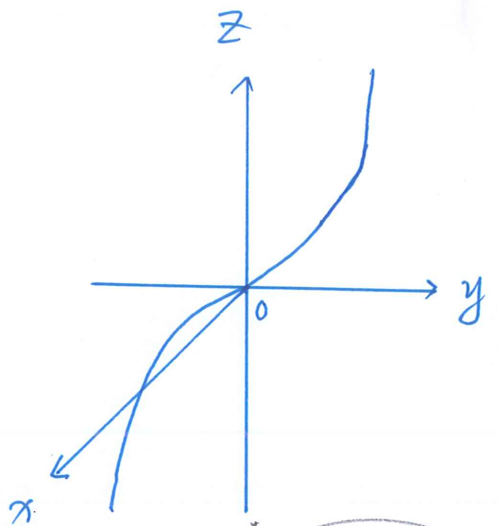
$$x-a = \frac{y-b}{3} = \frac{z-c}{-5}.$$



This is a straight line in space passing through the point (a, b, c) and whose direction vector is $\vec{n} = <1, 3, -5>$.

3. $< 0,\ t,\ 2t^3 >$

Since $x \equiv 0$ and $z = 2y^3$, this is a cubic parabola on the yz-plane.

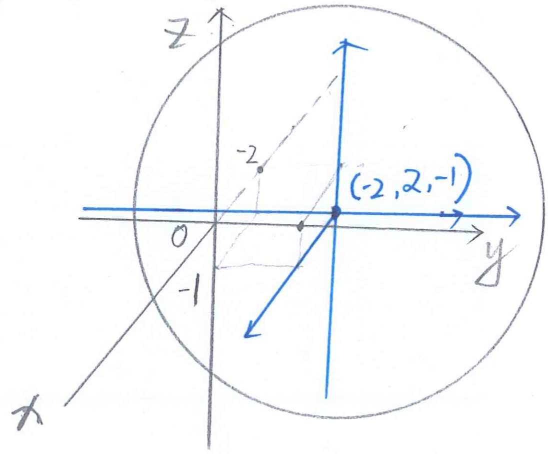


4. $< -2,\ 2 + 5\cos t, -1 + 5\sin t >$

$x = -2$. $y = 2 + 5\cos t$, $z = -1 + 5\sin t$.

So we have $\begin{cases} x = 2. \\ (y-2)^2 + (z+1)^2 = 25. \end{cases}$

This is a circle in the plane $x = -2$ with center $(-2, 2, -1)$ and radius 5.

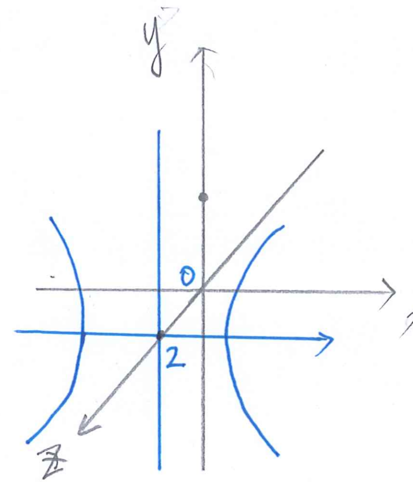


5. $< \cosh t,\ \sinh t,\ 2 >$

$\begin{cases} x = \cosh t = \dfrac{e^t + e^{-t}}{2} \\ y = \sinh t = \dfrac{e^t - e^{-t}}{2} \\ z = 2 \end{cases} \Rightarrow \begin{cases} x + y = e^t \\ x - y = e^{-t} \\ z \equiv 2. \end{cases}$

$\therefore \begin{cases} x + y = \frac{1}{x-y} \\ z = 2 \end{cases} \quad \therefore \begin{cases} x^2 - y^2 = 1 \\ z = 2. \end{cases}$

This is a hyperbola in the plane $z = 2$ with major and minor semi-axises $a = b = 1$.



Find a parametric representation (page 390, Q11, 12, 15, 17-19)

1. Circle in the plane $z = 2$ with center (1,-1) and passing through the origin.

The circle passes through the origin and has center $(1, -1, 2)$ in space, So the radius of the circle equals the distance between the center and the origin $(0, 0, 2)$ in the plane $z = 2$, i.e.

$$\sqrt{(1-0)^2 + (-1-0)^2 + (2-2)^2} = \sqrt{1+1} = \sqrt{2}.$$

Therefore $(x-1)^2 + (y+1)^2 = 2$. Using the ~~pto~~ polar coordinate system, the circle can be represented as:

$x - 1 = \sqrt{2}\cos t$, $y + 1 = \sqrt{2}\sin t$, so the parametric representation of the circle is $< 1 + \sqrt{2}\cos t, \sqrt{2}\sin t - 1,\ 2 >$.

2. Circle in the yz-plane with center (4,0) and passing through (0,3). $(y-4)^2+z^2=25$.

Radius of the circle is $\sqrt{(0-4)^2+(3-0)^2+(0-0)^2}=\sqrt{16+9}=5$.

In polar coordinate system: $\begin{cases} x=0 \\ y-4=5\cos t \\ z-0=5\sin t \end{cases}$ i.e. $\begin{cases} x=0 \\ y=4+5\cos t \\ z=5\sin t \end{cases}$

So the parametric representation of the circle is $\langle 0, 4+5\cos t, 5\sin t \rangle$.

3. Straight line $y = 2x - 1, z = 3x$.

Let $x=t$, then the straight line can be expressed by $\langle t, 2t-1, 3t \rangle$.

4. Ellipse $\frac{1}{3}x^2 + y^2 = 1,\ z = y$.

Let $\begin{cases} \frac{x}{\sqrt{3}} = \cos t \\ y=\sin t \\ z=y=\sin t \end{cases}$. then the parametric representation of the ellipse is $\langle \sqrt{3}\cos t, \sin t, \sin t \rangle$.

5. Helix $x^2 + y^2 = 25, z = 2\arctan\frac{y}{x}$.

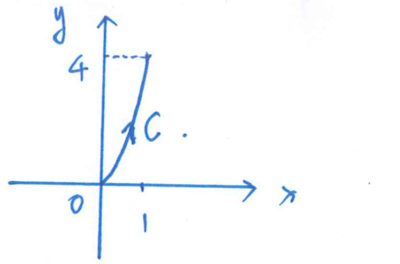
Let $\begin{cases} x=5\cos t \\ y=5\sin t \\ z=2\arctan(\tan t)=2t \end{cases}$. $\therefore \langle 5\cos t, 5\sin t, 2t \rangle$

6. Hyperbola $x^2 - y^2 = 1, z = -2$.

From question 5 in last part, we know the parametric representation for that hyperbola is $\langle \cosh t, \sinh t, -2 \rangle$.

Line integral – work

The line integral $\int_C \mathbf{F}(\mathbf{r})\cdot d\mathbf{r}$ of the vector field $\boldsymbol{F}$ along the curve C gives the work done by the field on an object moving along the curve through the field. Calculate $\int_C \mathbf{F}(\mathbf{r})\cdot d\mathbf{r}$ for the given data. If $\boldsymbol{F}$ is a force, this gives the work done by the force in the displacement along C. Show the details.

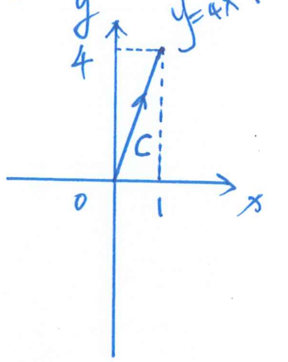


1. $F =< y^2, -x^2 >, C: y = 4x^2$ from (0,0) to (1,4).

$C: \vec{r}(t) = \langle t, 4t^2 \rangle,\ 0 \le t \le 1 \quad \therefore \vec{r}'(t) = \langle 1, 8t \rangle.$

$\vec{F}(\vec{r}(t)) = \langle (4t^2)^2, -t^2 \rangle \quad \therefore \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) = \langle 16t^4, -t^2 \rangle \cdot \langle 1, 8t \rangle = 16t^4 - 8t^3.$

$\therefore \int_C \vec{F} \cdot d\vec{r} = \int_0^1 16t^4 - 8t^3 dt = \left[\frac{16}{5}t^5 - 2t^4 \right]_0^1 = \frac{16}{5} - 2 = \frac{6}{5}.$



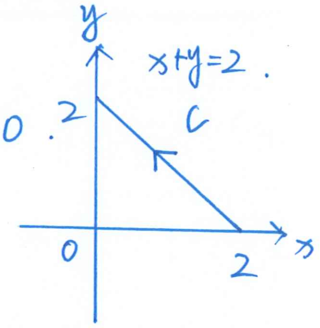
2. F as in question 1, C from (0,0) straight to (1,4). Compare the results.

$C: \vec{r}(t) = \langle t, 4t \rangle,\ \vec{r}'(t) = \langle 1, 4 \rangle.\ 0 \le t \le 1.$

$\therefore \vec{F}(\vec{r}(t)) = \langle 16t^2, -t^2 \rangle, \quad \therefore \vec{F}(\vec{r}) \cdot \vec{r}' = 16t^2 - 4t^2 = 12t^2.$

$\therefore \int_C \vec{F} \cdot d\vec{r} = \int_0^1 12t^2 dt = [4t^3]_0^1 = 4.$

We can see that the line integral $\int_C \vec{F} \cdot d\vec{r}$ is path dependent in the plane.



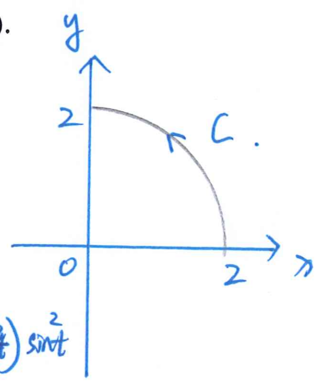
3. $F =< xy,\ x^2y^2 >$, C from (2,0) straight to (0,2).

$C: \vec{r}(t) = \langle t, 2-t \rangle,\ \vec{r}'(t) = \langle 1, -1 \rangle,$ t is from 2 to 0.

$\vec{F}(\vec{r}(t)) = \langle t(2-t),\ t^2(2-t)^2 \rangle.$

$\therefore \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) = t(2-t) - t^2(2-t)^2 = -t^4 + 4t^3 - 5t^2 + 2t.$

$\therefore \int_C \vec{F} \cdot d\vec{r} = \int_2^0 -t^4 + 4t^3 - 5t^2 + 2t\, dt = \left[-\frac{1}{5}t^5 + t^4 - \frac{5}{3}t^3 + t^2 \right]_2^0 = -\frac{4}{15}.$



4. F as in question 3, C is the quarter-circle from (2,0) to (0,2) with center (0,0).

$C: \vec{r}(t) = \langle 2\cos t, 2\sin t \rangle.\quad 0 \le t \le \frac{\pi}{2}.$

$\vec{r}'(t) = \langle -2\sin t, 2\cos t \rangle.$

$\therefore \vec{F}(\vec{r}(t)) = \langle 4\sin t\cos t,\ 16\sin^2 t\cos^2 t \rangle.$

$\therefore \vec{F}(\vec{r}) \cdot \vec{r}'(t) = -8\sin^2 t\cos t + 32\cos^3 t\sin^2 t = -8\sin^2 t\cos t + 32\cos t(\cancel{1-\cos^2 t})\sin^2 t$

$= 24\sin^2 t\cos t - 32\cos t\sin^4 t.$

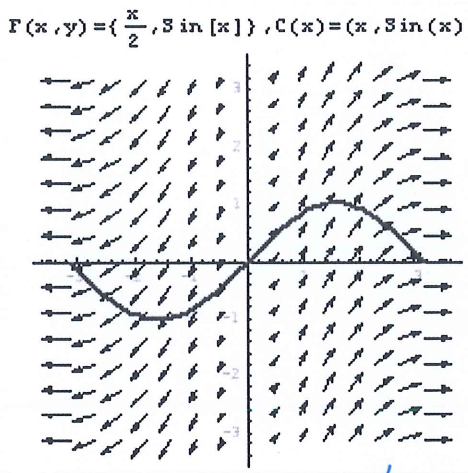
$\therefore \int_C \vec{F} \cdot d\vec{r} = \int_0^{\frac{\pi}{2}} 24\sin^2 t\cos t\,dt - 32\int_0^{\frac{\pi}{2}} \cos t\sin^4 t\, dt$

$= 24\int_0^{\frac{\pi}{2}} \sin^2 t\, d(\sin t) - 32\int_0^{\frac{\pi}{2}} \sin^4 t\, d(\sin t)$

$= 24 \cdot \left[\frac{1}{3}\sin^3 t\right]_0^{\frac{\pi}{2}} - 32\left[\frac{1}{5}\sin^5 t\right]_0^{\frac{\pi}{2}} = 8 - \frac{32}{5} = \frac{8}{5}.$

Line integral – work done by an airplane

Consider a vector field $\boldsymbol{F} =< \frac{x}{2},\ \sin x >$ which is defined on the plane. Suppose that t is the time, $\boldsymbol{F}$ is a force field, say the wind, and an airplane is moving over the curve $C: \boldsymbol{r}(t) =< t, \sin t >$ from the initial point (0,0) to the terminal point (2, sin2). See the figure blow. Calculate the work done by the wind on this airplane along the path C.



$C:\ \vec{r}(t)=\langle t, \sin t\rangle,\ 0\le t\le 2.\ \therefore\ \vec{r}'(t)=\langle 1, \cos t\rangle.$

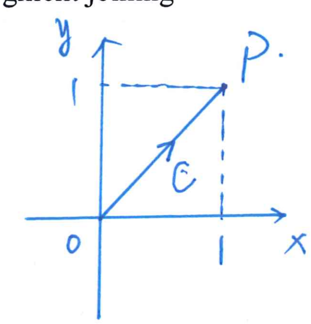
$\therefore\ \vec{F}(\vec{r}(t))=\langle \frac{t}{2}, \sin t\rangle.\ \therefore\ \vec{F}(\vec{r}(t))\cdot\vec{r}'(t)=\frac{t}{2}+\sin t\cos t.$

$\therefore\ \int_C \vec{F}(\vec{r})\cdot d\vec{r}=\int_0^2 \frac{t}{2}+\sin t\cos t\, dt$

$=\left[\frac{1}{4}t^2+\frac{1}{2}\sin^2 t\right]_0^2=1+\frac{1}{2}\sin^2 2.$

Line integral

1. Calculate the line integral for the vector field $\boldsymbol{F} =< xy, y^2 >$ over the segment joining the points from O: (0,0) to P: (1,1).



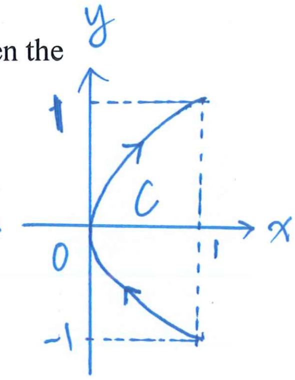
We know the path C can be represented by

$\vec{r}(t)=\langle t, t\rangle,\ 0\le t\le 1.\ \therefore\ \vec{r}'(t)=\langle 1, 1\rangle.$

$\vec{F}(\vec{r}(t))=\langle t^2, t^2\rangle.\ \therefore\ \vec{F}(\vec{r}(t))\cdot\vec{r}'(t)=2t^2.$

$\therefore\ \int_C \vec{F}(\vec{r})\cdot d\vec{r}=\int_0^1 2t^2\,dt=\frac{2}{3}.$

2. Determine the line integral of $F =< x^2, xy >$ along the parabola $x = y^2$ between the points (1,-1) and (1,1).

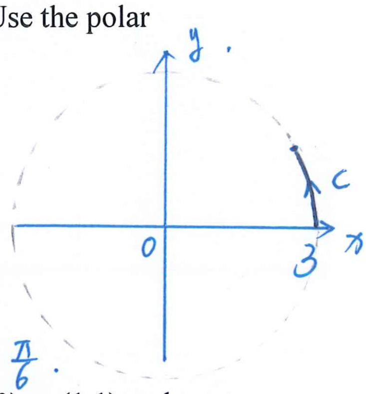


The parametric representation of the parabola is: $\vec{r}(t) = \langle t^2, t \rangle$.
$-1 \le t \le 1$.

$\therefore \vec{F}(\vec{r}(t)) = \langle t^4, t^3 \rangle$. $\therefore \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) = \langle t^4, t^3 \rangle \cdot \langle 2t, 1 \rangle = 2t^5 + t^3$

$\therefore \int_C \vec{F}(\vec{r}) \cdot d\vec{r} = \int_{-1}^{1} 2t^5 + t^3 dt = \left[\frac{1}{3}t^6 + \frac{1}{4}t^4\right]_{-1}^{1} = 0$.

3. Determine the line integral for $G =< -\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2} >$ over the circle in the plane with center (0,0) and radius 3 from the point (3,0) to the point $(\frac{3\sqrt{3}}{2}, \frac{3}{2})$. Hint: Use the polar coordinate system.

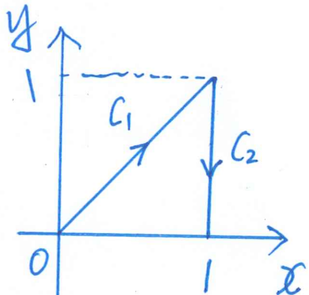


The path C: $\vec{r}(t) = \langle 3\cos t, 3\sin t \rangle$, $0 \le t \le \frac{\pi}{6}$, which implies $\vec{r}(\frac{\pi}{6}) = 3(\frac{\sqrt{3}}{2}, \frac{1}{2})$ and $\vec{r}(0) = (3, 0)$.

$\therefore \vec{G}(\vec{r}(t)) = \langle -\frac{3\sin t}{9(\cos^2 t + \sin^2 t)}, \frac{3\cos t}{9(\cos^2 t + \sin^2 t)} \rangle = \frac{1}{3}\langle -\sin t, \cos t \rangle$.

$\therefore \int_C \vec{F}(\vec{r}) \cdot d\vec{r} = \int_0^{\frac{\pi}{6}} \frac{1}{3}\langle -\sin t, \cos t \rangle \cdot \langle -3\sin t, 3\cos t \rangle dt = \int_0^{\frac{\pi}{6}} 1\, dt = \frac{\pi}{6}$.

4. Calculate the work done by $F =< x, y^2 >$ on a particle moving from (0,0) to (1,1) and then to (1,0) along the straight line segments joining the points.



Suppose that the path we use is denoted by C, and C can be decomposed by 2 paths. One is the straight line starting at (0,0) and ending at (1,1); the other one is the straight line segment starting at (1,1) and ending at (1,0).

That is $C = C_1 + C_2$, where $C_1: \vec{r}_1(t) = \langle t, t \rangle$, $0 \le t \le 1$

$C_2: \vec{r}_2(t) = \langle 1, 1-t \rangle$, $0 \le t \le 1$.

By the property of line integrals, we get

$\int_C \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r}$

$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_0^1 \vec{F}(\vec{r}_1(t)) \cdot \vec{r}_1\,'(t)\, dt = \int_0^1 \langle t, t^2 \rangle \cdot \langle 1, 1 \rangle dt = \int_0^1 t + t^2 dt = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$.

$\int_{C_2} \vec{F} \cdot d\vec{r} = \int_0^1 \vec{F}(\vec{r}_2(t)) \cdot \vec{r}_2\,'(t)\, dt = \int_0^1 \langle 1, (1-t)^2 \rangle \cdot \langle 0, -1 \rangle dt = \int_0^1 -(1-t)^2 dt = -\frac{1}{3}$.

$\therefore \int_C \vec{F} \cdot d\vec{r} = \frac{5}{6} - \frac{1}{3} = \frac{1}{2}$.