Xi'an Jiaotong-Liverpool University
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EEE220 Instrumentation and Control System

2018-19 Semester 2

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6 May, 2019

Lecture 21

Outline

Frequency Response Methods

- ❑ **Introduction**
- ❑ **Frequency Response Plots – Polar Plot & Bode Plot**
- ❑ Frequency Response Measurements
- ❑ Performance Specifications in the Frequency Domain
- ❑ Gain Margin and Phase Margin
- ❑ Compensators
- ❑ Frequency Response Methods Using Matlab

Introduction

In this chapter, we consider the steady-state response of a system to a **sinusoidal** input signal.

$$r(t) = A\sin\omega t$$

$$R(s) = \frac{A\omega}{s^2 + \omega^2}$$

- The frequency response of a system is defined as the steady-state response of the system to a sinusoidal input signal.
- The sinusoid is a unique input signal, and the resulting output signal for a linear constant coefficient system is sinusoidal in the steady state;
- The output signal differs from the input only in **amplitude** and **phase angle**.

Frequency Response

Consider the system $Y(s) = T(s)R(s)$, with $r(t) = A \sin \omega t$. We have

$$R(s) = \frac{A\omega}{s^2 + \omega^2} \qquad T(s) = \frac{m(s)}{q(s)} = \frac{m(s)}{\prod_{i=1}^{n}(s + p_i)}$$

where $-p_i$ are assumed to be distinct poles. Then in partial forms, we have

$$Y(s) = \frac{k_1}{s + p_1} + \cdots + \frac{k_n}{s + p_n} + \frac{\alpha s + \beta}{s^2 + \omega^2}$$

Taking the inverse Laplace transform yields

$$y(t) \ = k_1 e^{-p_1 t} + \cdots + k_n e^{-p_n t} + \mathcal{L}^{-1}\{\frac{\alpha s + \beta}{s^2 + \omega^2}\}$$

where α and β are constants which are problem dependent. If the system is stable, then all p_i should have positive real parts, the steady-state output is

$$\lim_{t \to \infty} y(t) = \mathcal{L}^{-1}\{\frac{\alpha s + \beta}{s^2 + \omega^2}\}$$

So, in the limit for $y(t)$, it can be shown, for $t \to \infty$ (the steady state),

$$
\begin{aligned}
y(t) &= \frac{1}{\omega} |A\omega T(j\omega)| \sin(\omega t + \phi) \\
&= A|T(j\omega)| \sin(\omega t + \phi)
\end{aligned}
$$

where $\emptyset = \angle T(j\omega)$.

The steady-state response described above is true only for stable systems.

> Therefore, the steady-state output signal depends only on the magnitude ($|T(j\omega)\ |$) and phase ($\emptyset$) of $T(j\omega)$.

Q: Why NOT use the final value theorem to find the steady-state output?

Laplace Transform vs. Fourier Transform

Laplace transform pair

$$F(s) = \mathscr{L}\{f(t)\} = \int_0^{\infty} f(t)e^{-st}\,dt$$

$$f(t) = \mathscr{L}^{-1}\{F(s)\} = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} F(s)e^{st}\,ds,$$

$$s = j\omega \longrightarrow$$

Fourier transform pair

$$F(\omega) = \mathscr{F}\{f(t)\} = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt$$

$$f(t) = \mathscr{F}^{-1}\{F(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}\,d\omega.$$

- The Laplace transform enables us to investigate the s-plane location of the poles and zeros of a transfer function $T(s)$.

- The Fourier transform allows us to consider the frequency response including amplitude and phase characteristics of the system.

Advantages & Disadvantages of Frequency Response Method

Advantages:

- The ready availability of sinusoid test signals for various range of frequency and amplitudes. Thus, the experimental determination of the system frequency response is easily accomplished;
- The unknown transfer function of a system can often be deduced from the experimentally determined frequency response of the system;
- The design of a system in the frequency domain provides the designer with control of the bandwidth of a system, as well as some measure of the response of the system to undesired noise and disturbances.
- The transfer function describing the sinusoidal steady-state behavior of a system can be easily obtained by replacing s with $j\omega$ in the system transfer function $T(s)$.

Basic Disadvantage:

- Indirect link between the frequency and time domain.

Frequency Response Plots – Polar Plot

The transfer function of a system G(s) can be described in the frequency domain by the relation

$$G(j\omega) = G(s)|_{s=j\omega} = R(\omega) + jX(\omega),$$

where

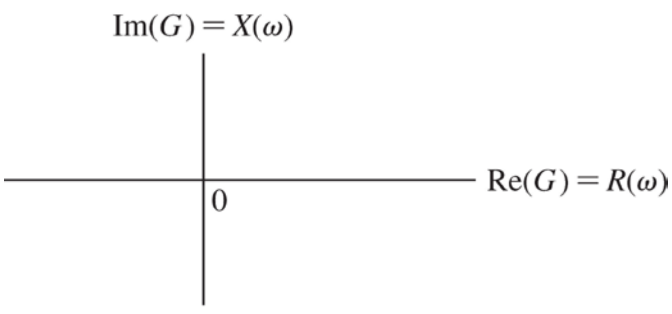
$$R(\omega) = \mathrm{Re}[G(j\omega)] \quad \text{and} \quad X(\omega) = \mathrm{Im}[G(j\omega)].$$

Alternatively, the transfer function can be represented by magnitude and phase as

$$G(j\omega) = |G(j\omega)|e^{j\phi(\omega)} = |G(j\omega)|\underline{/\phi(\omega)},$$

where

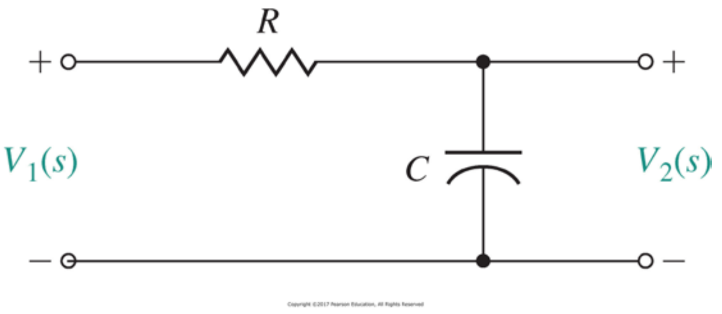
$$\phi(\omega) = \tan^{-1}\frac{X(\omega)}{R(\omega)} \quad \text{and} \quad |G(j\omega)|^2 = [R(\omega)]^2 + [X(\omega)]^2.$$



The polar plot representation of the frequency response is obtained by using the above equations.

Example 21.1: Frequency Response of an RC Filter

A simple RC filter is shown in the right figure.
The transfer function is



$$G(s) = \frac{V_2(s)}{V_1(s)} = \frac{1}{RCs+1}$$

and the sinusoidal steady-state transfer function is

$$G(j\omega) = \frac{1}{j\omega(RC)+1} = \frac{1}{j(\omega/\omega_1)+1} \qquad \text{where } \omega_1 = \frac{1}{RC}$$

Then the polar plot is obtained from the relation

$$G(j\omega) = \frac{1}{1+(\frac{\omega}{\omega_1})^2} - j\frac{\frac{\omega}{\omega_1}}{1+(\frac{\omega}{\omega_1})^2}$$

or

$$G(j\omega) = |G(j\omega)|\angle\phi(\omega),$$

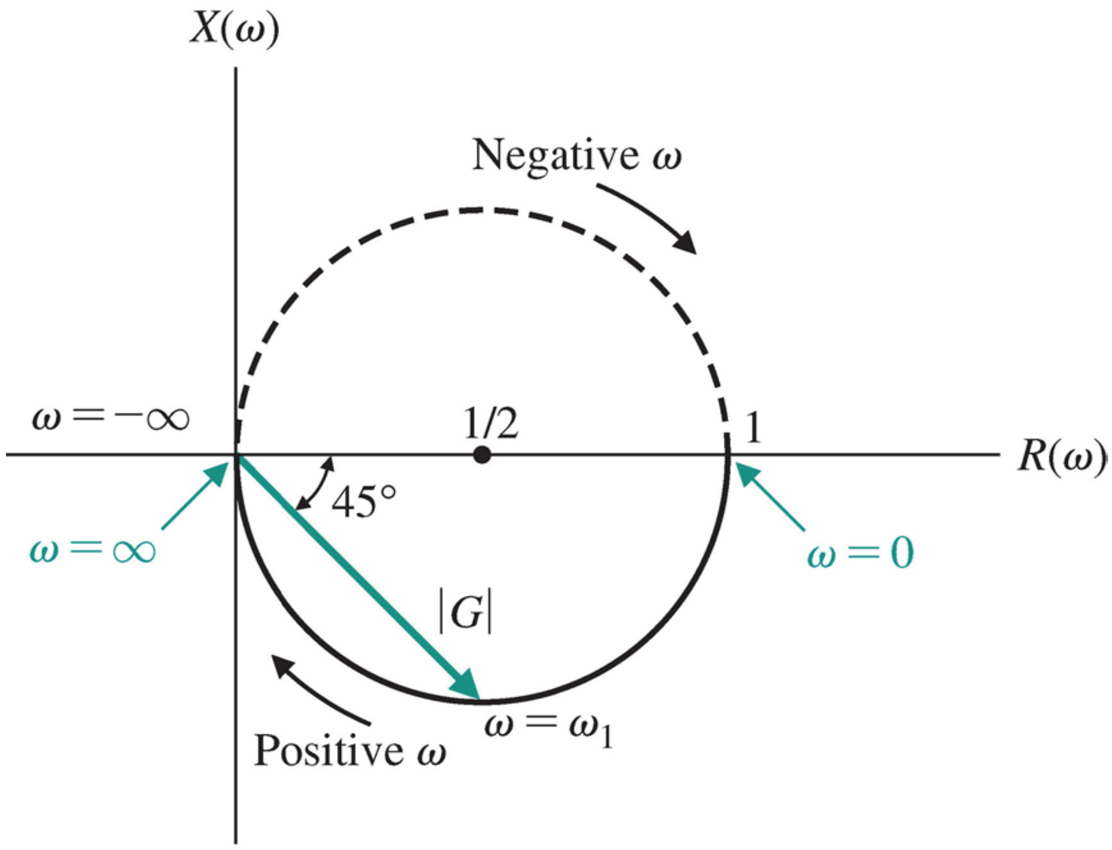
where $|G(j\omega)| = [\frac{1}{1+(\frac{\omega}{\omega_1})^2}]^{1/2}$ and $\phi(\omega) = -\tan^{-1}(\frac{\omega}{\omega_1})$

Polar Plot for the RC Filter

$$\boxed{\boldsymbol{G(j\omega) = |G(j\omega)|\angle\phi(\omega),}}$$

where $|G(j\omega)| = [\frac{1}{1+(\frac{\omega}{\omega_1})^2}]^{1/2}$

and $\phi(\omega) = -\tan^{-1}(\frac{\omega}{\omega_1})$



Example 21.2: Polar Plot of a Transfer Function

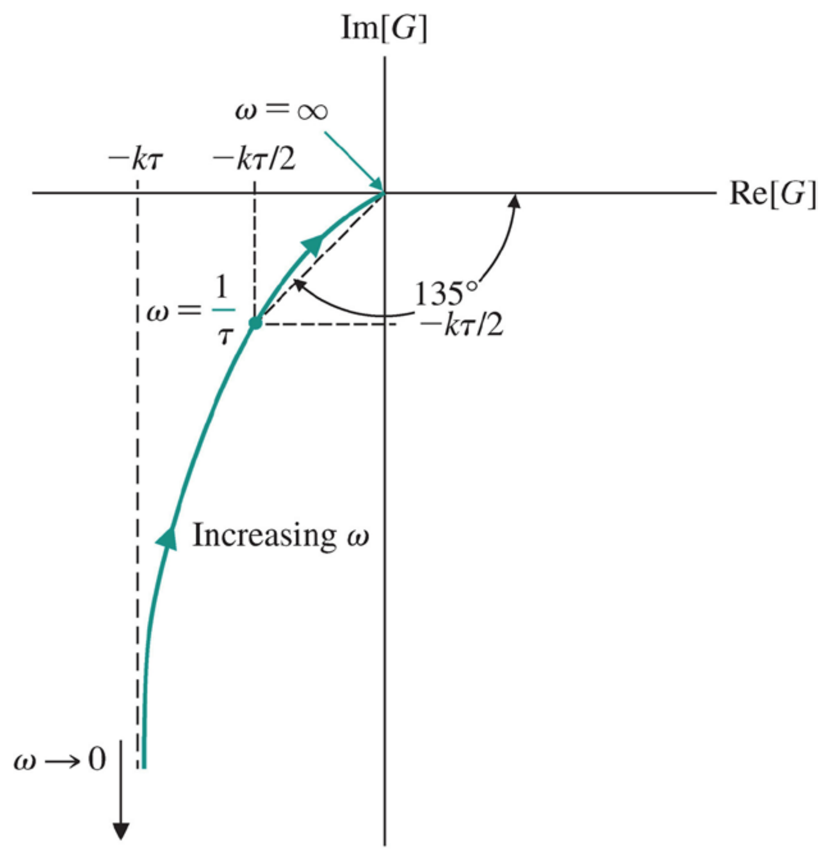
Consider a transfer function

$$G(j\omega) = \frac{K}{j\omega(j\omega\tau + 1)} = \frac{K}{j\omega - \omega^2\tau}$$

Then the magnitude and phase angle are written as

$$|G(j\omega)| = \frac{K}{(\omega^2 + \omega^4\tau^2)^{1/2}}$$

$$\phi(\omega) = -\tan^{-1}\frac{1}{-\omega\tau}.$$



Typical values:

ω	0	$1/2\tau$	$1/\tau$	∞
$\lvert G(j\omega)\rvert$	∞	$4K\tau/\sqrt{5}$	$K\tau/\sqrt{2}$	0
$\phi(\omega)$	-90°	-117°	-135°	-180°

Bode Plots

The introduction of **logarithm plots**, often called **Bode Plots** in honor of H. W. Bode, simplifies the determination of the graphical portrayal of the frequency response.

The transfer function in the frequency domain is

$$G(j\omega) = |G(j\omega)|e^{j\phi(\omega)}$$

The logarithm of the magnitude is normally expressed in terms of the logarithm to the **base 10**, so we use

$$\boxed{\textbf{Logarithm Gain} = \mathbf{20log_{10}}|\boldsymbol{G(j\omega)}|}$$

where the units are **decibels (dB)**.

For a Bode diagram, the plot of logarithmic gain in dB versus ω is normally plotted on one set of axes, and the phase $\phi(\omega)$ versus ω on another set of axes.

Bode Plot of an RC Filter

Reconsider the transfer function

$$G(j\omega) = \frac{1}{j\omega(RC) + 1} = \frac{1}{j\omega\tau + 1} \qquad \text{where } \tau = RC$$

The logarithm gain is

$$20 \log|G(j\omega)| = 20 \log\left(\frac{1}{1 + (\omega\tau)^2}\right)^{1/2} = -10 \log(1 + (\omega\tau)^2).$$

For small frequencies – that is, $\omega \ll {}^{1}/_{\tau}$, the logarithm gain is

$$20 \log|G(j\omega)| = -10 \log(1) = 0 \text{ dB}, \qquad \omega \ll 1/\tau.$$

For large frequencies – that is, $\omega \gg {}^{1}/_{\tau}$, the logarithm gain is

$$20 \log G(j\omega) = -20 \log(\omega\tau) \qquad \omega \gg 1/\tau,$$

and at $\boldsymbol{\omega = {}^{1}/_{\tau}}$ (**break frequency** or **corner frequency**), we have

$$20 \log|G(j\omega)| = -10 \log 2 = -3.01 \text{ dB}.$$

Phase angle of the transfer function is

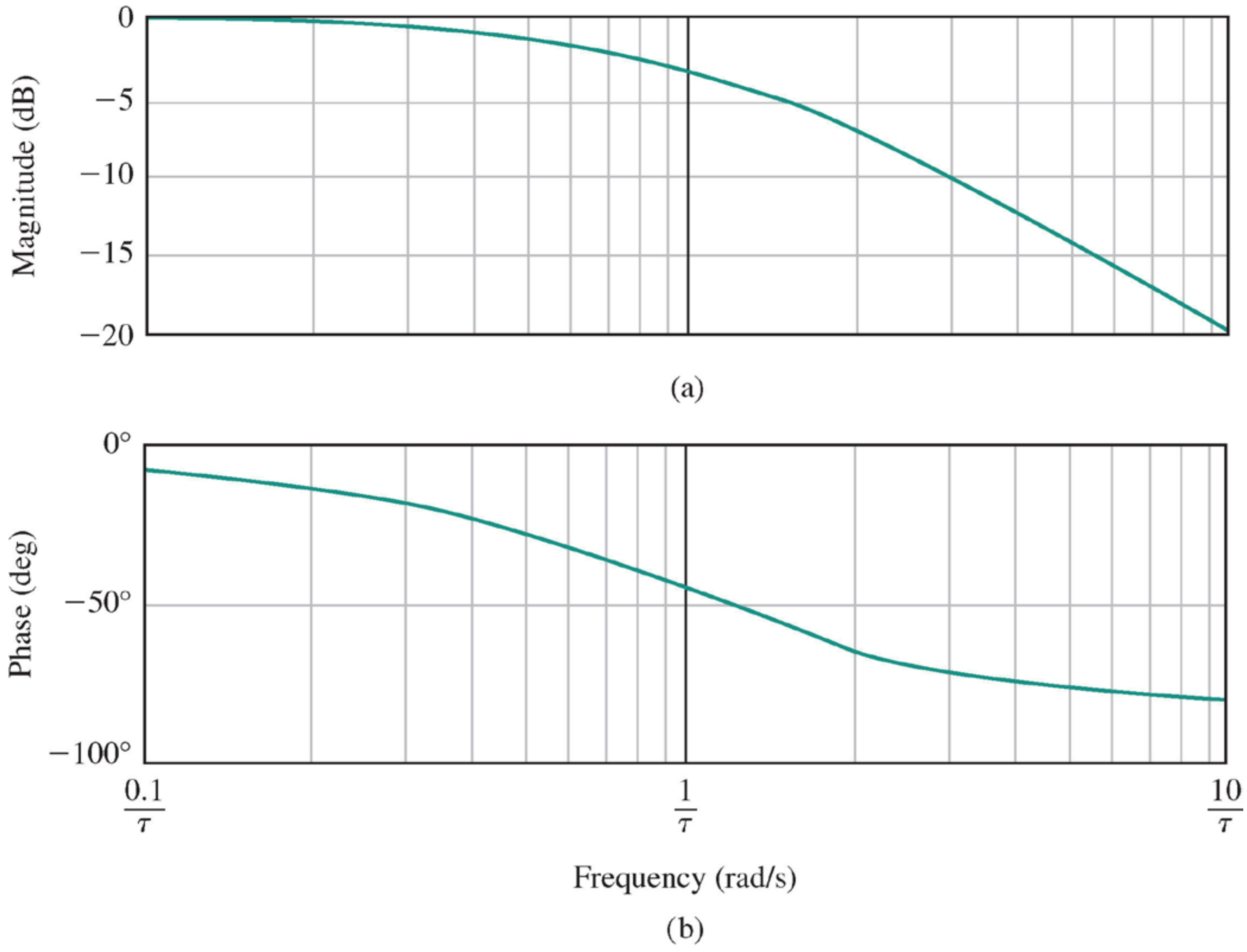
$$\phi(\omega) = -\tan^{-1}(\omega\tau).$$

- A linear scale of frequency is not the most convenient choice, we consider the use of a logarithmic scale of frequency. Then, on a set of axes where the horizontal axis is **$\log\omega$**.
- An interval of two frequencies with a ratio equal to 10 is called a **decade**, so that the range of frequencies from ω_1 to ω_2, where $\omega_2 = 10\omega_1$, is called a decade.
- The logarithmic gains, for $\omega \gg {}^{1}/_{\tau}$, over a decade of frequency is

$$\begin{aligned} 20\log|G(j\omega_1)| - 20\log|G(j\omega_2)| &= -20\log(\omega_1\tau) - (-20\log(\omega_2\tau)) \\ &= -20\log\frac{\omega_1\tau}{\omega_2\tau} \\ &= -20\log\frac{1}{10} = +20\text{ dB}; \end{aligned}$$

That is, the slope of the asymptotic line for this first-order transfer function is -20 dB/decade.

Bode plot for $G(j\omega) = 1/(j\omega\tau + 1)$; (a) magnitude plot and (b) phase plot.

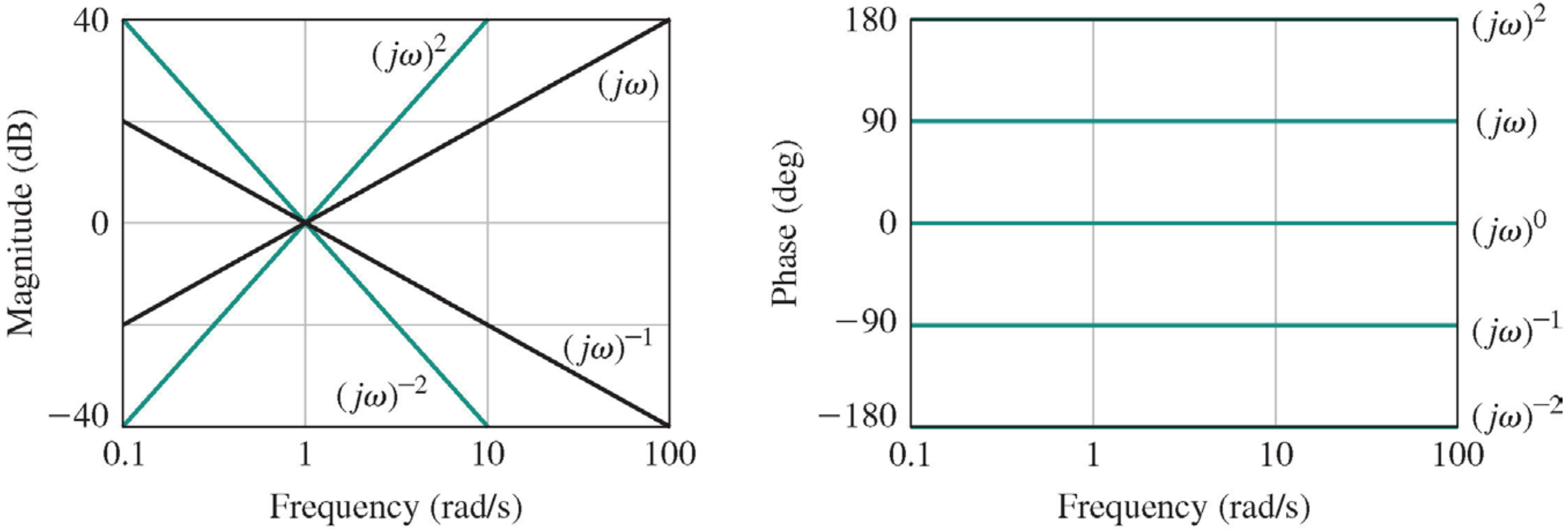


Bode Plots for Typical Transfer Functions

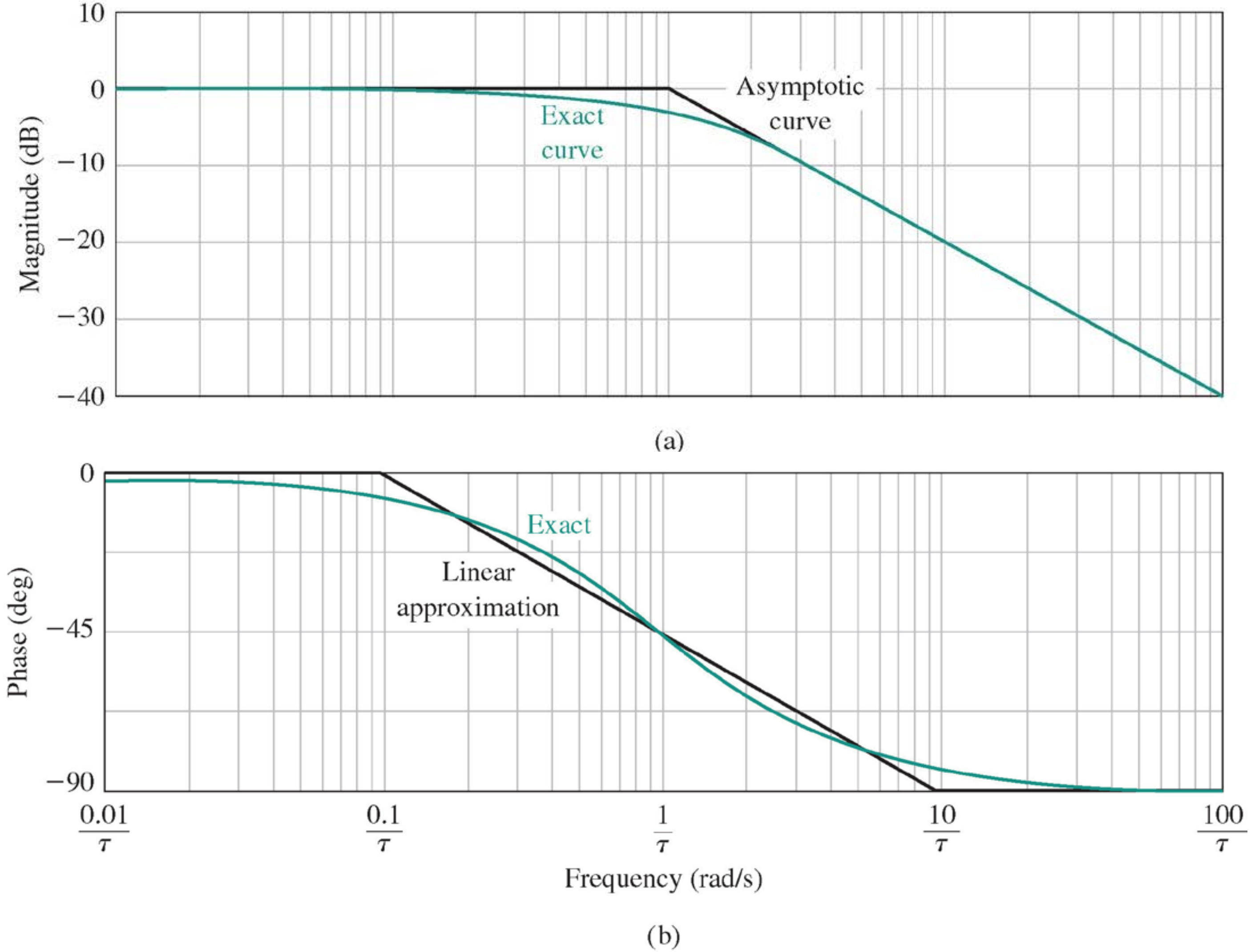
Bode plot for $(j\omega)^{\pm N}$. -- Poles (or Zeros) at the Origin.

$$20 \log \left| \frac{1}{(j\omega)^N} \right| = \pm 20N \log \omega,$$

$$\phi(\omega) = \pm 90^\circ N.$$



Bode diagram for $(1 + j\omega\tau)^{-1}$. -- Poles (or Zeros) on the Real Axis.



Bode diagram for $G(j\omega) = [1 + (2\zeta/\omega_n)\, j\omega + (j\omega/\omega_n)^2]^{-1}$.

-- Complex Conjugate Poles or Zeros.

Normalized form $[1 + j2\zeta u - u^2]^{-1}$

where $u = \omega/\omega_n$.

$$\boxed{20 \log|G(j\omega)| = -10 \log((1 - u^2)^2 + 4\zeta^2 u^2),}$$

$$\boxed{\phi(\omega) = -\tan^{-1} \frac{2\zeta u}{1 - u^2}.}$$

when $u \ll 1$ $20 \log|G(j\omega)| = -10 \log 1 = 0 \text{ dB},$

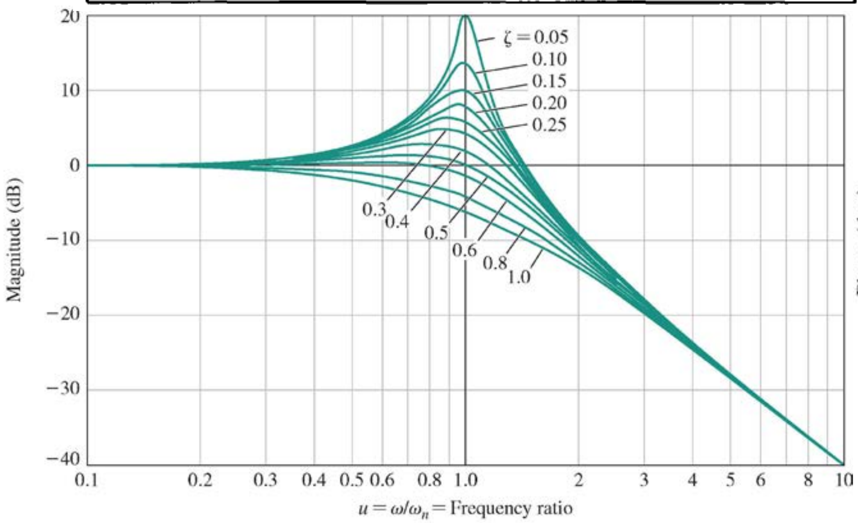
While the phase angle approaches 0°

when $u \gg 1$ $20 \log|G(j\omega)| = -10 \log u^4 = -40 \log u,$

While the phase angle approaches -180°

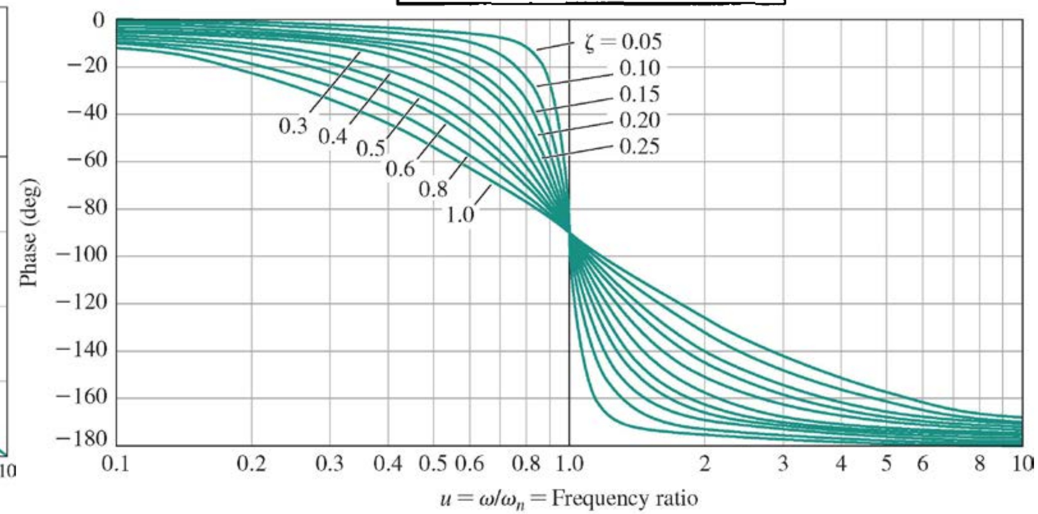
The difference between the actual magnitude curve and the asymptotic approximation is a function of the **damping ratio** and MUST be accounted for when $\zeta < 0.707$.

$$20 \log|G(j\omega)| = -10 \log((1 - u^2)^2 + 4\zeta^2 u^2),$$



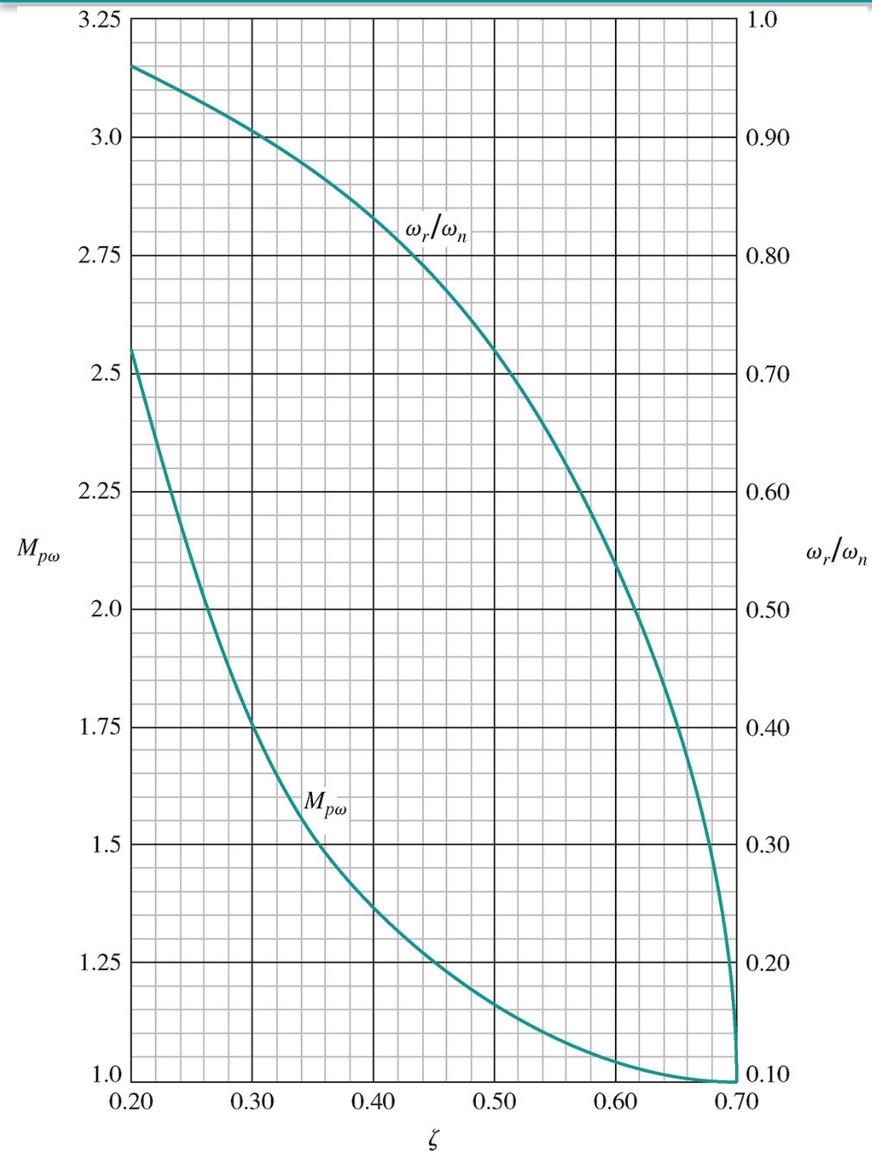
(a)

$$\phi(\omega) = -\tan^{-1}\frac{2\zeta u}{1 - u^2}.$$



(b)

The maximum value $M_{p\omega}$ of the frequency response occurs at the resonant frequency ω_r. When the damping ratio approaches zero, then ω_r approaches ω_n.

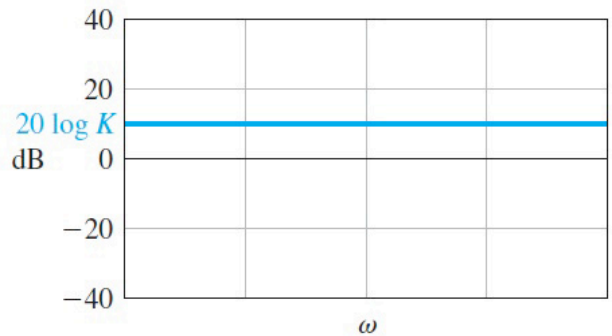
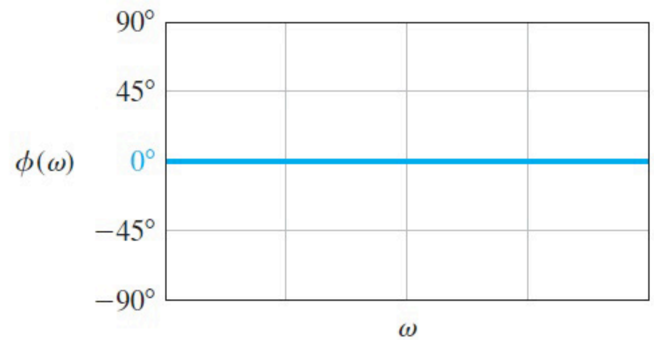
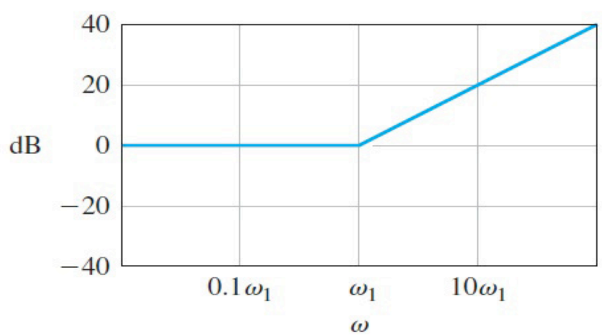
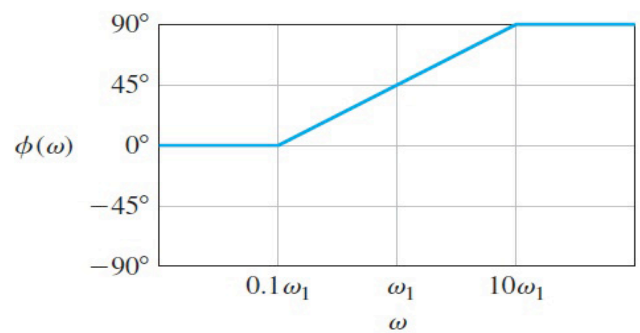
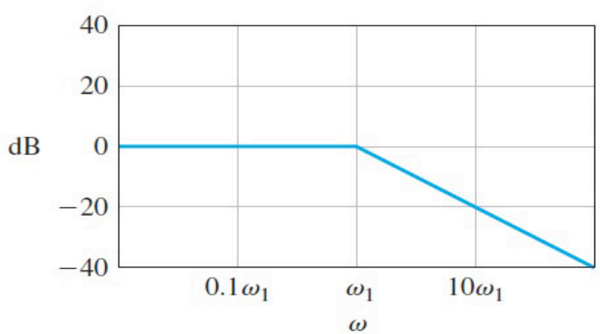
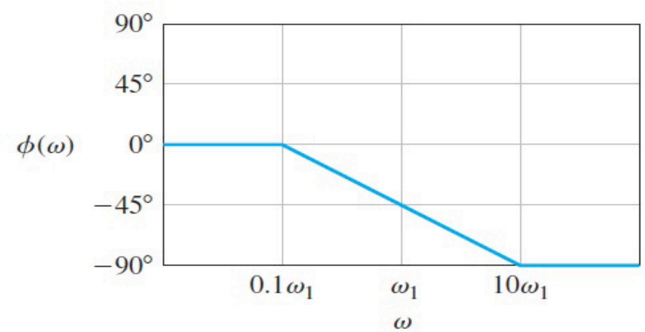


$$\boxed{\omega_r = \omega_n\sqrt{1-2\zeta^2},} \quad \zeta < 0.707,$$

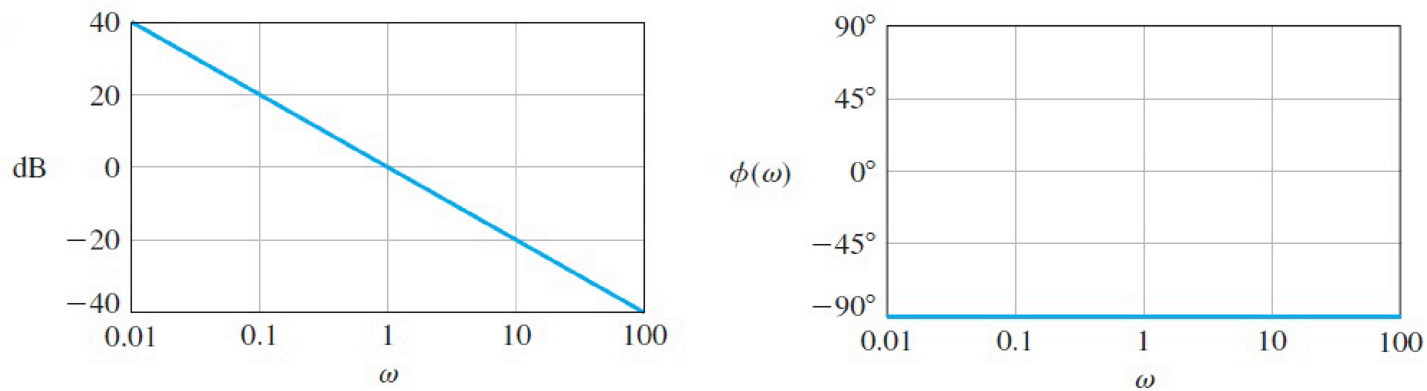
$$\boxed{M_{p\omega} = |G(j\omega_r)| = \left(2\zeta\sqrt{1-\zeta^2}\right)^{-1},} \quad \zeta < 0.707,$$

The maximum $M_{p\omega}$ of the frequency response and the resonant frequency ω_r versus ζ for a pair of conjugate poles.

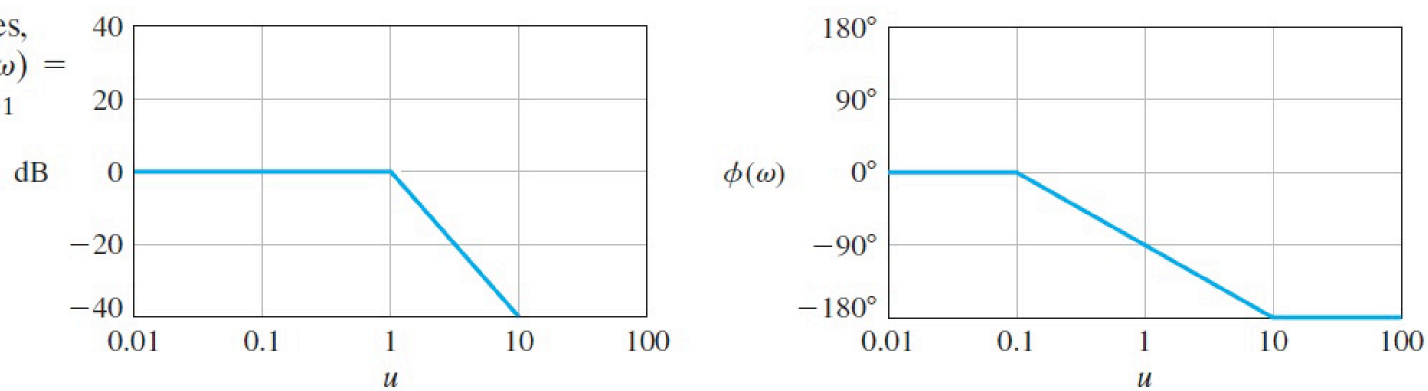
Summary: Asymptotic Curves for Basic Terms of a Transfer Function

Term	Magnitude 20 log|G|	Phase $\phi(\omega)$
1. Gain, $G(j\omega) = K$		
2. Zero, $G(j\omega) = 1 + j\omega/\omega_1$		
3. Pole, $G(j\omega) = (1 + j\omega/\omega_1)^{-1}$		

4. Pole at the origin,
$G(j\omega) = 1/j\omega$



5. Two complex poles,
$0.1 < \zeta < 1, G(j\omega) =$
$(1 + j2\zeta u - u^2)^{-1}$
$u = \omega/\omega_n$



Sketching a Bode Plot

Consider the system with the following transfer function

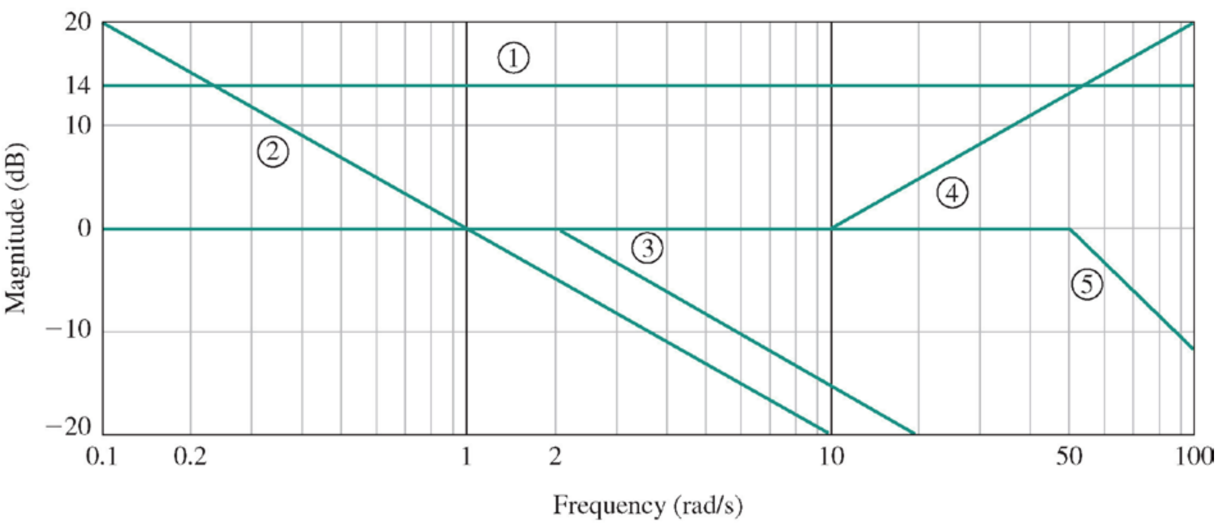
$$G(j\omega) = \frac{5(1 + j0.1\omega)}{j\omega(1 + j0.5\omega)(1 + j0.6(\omega/50) + (j\omega/50)^2)}.$$

Solutions:

There are five terms in the transfer function:

1. A constant gain $K = 5$
2. A pole at the origin
3. A pole at $\omega = 2$
4. A zero at $\omega = 10$
5. A pair of complex poles at $\omega = \omega_n = 50$

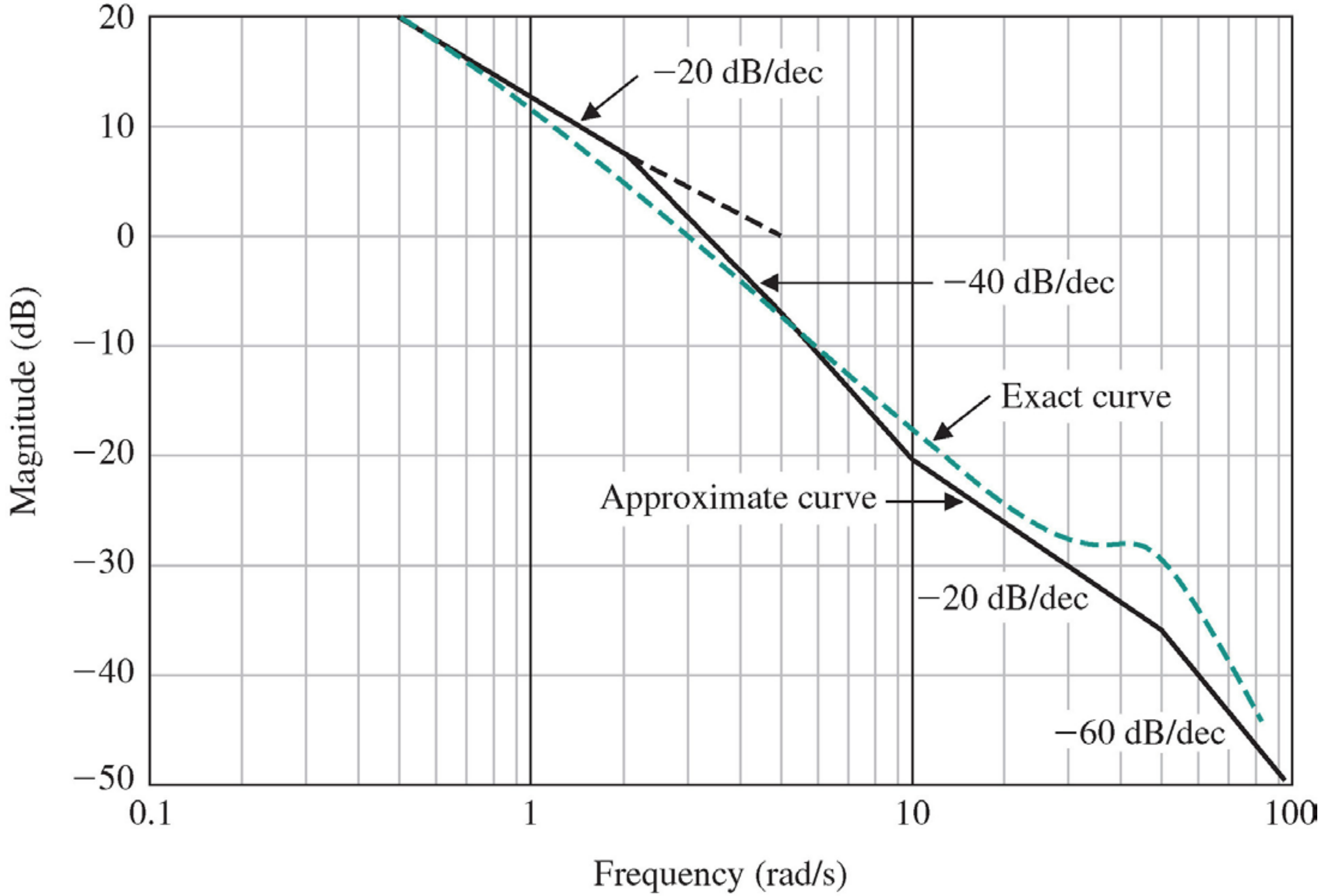
FIGURE 8.19 Magnitude asymptotes of poles and zeros used.



First, we plot the magnitude characteristic for each individual pole and zero factor and the constant gain:

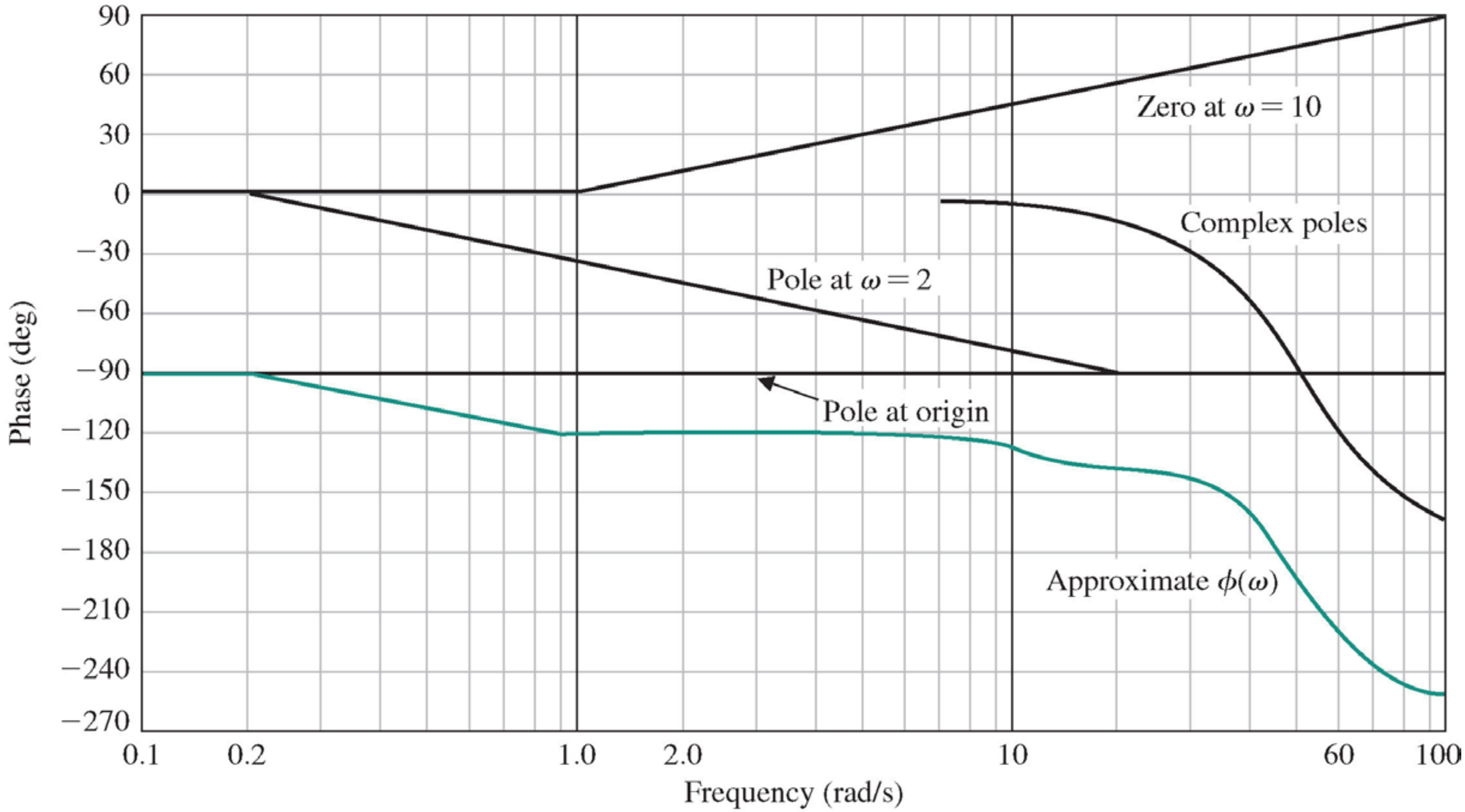
1. The constant gain is $20 \log 5 = 14$ dB, as shown in Figure 8.19.
2. The magnitude of the pole at the origin extends from zero frequency to infinite frequencies and has a slope of -20 dB/decade intersecting the 0-dB line at $\omega = 1$, as shown in Figure 8.19.
3. The asymptotic approximation of the magnitude of the pole at $\omega = 2$ has a slope of -20 dB/decade beyond the break frequency at $\omega = 2$. The asymptotic magnitude below the break frequency is 0 dB, as shown in Figure 8.19.
4. The asymptotic magnitude for the zero at $\omega = +10$ has a slope of $+20$ dB/decade beyond the break frequency at $\omega = 10$, as shown in Figure 8.19.
5. The magnitude for the complex poles is -40 dB/decade. The break frequency is $\omega = \omega_n = 50$, as shown in Figure 8.19. This approximation must be corrected to the actual magnitude because the damping ratio is $\zeta = 0.3$, and the magnitude differs appreciably from the approximation, as shown in Figure 8.20.

FIGURE 8.20 Magnitude characteristic.

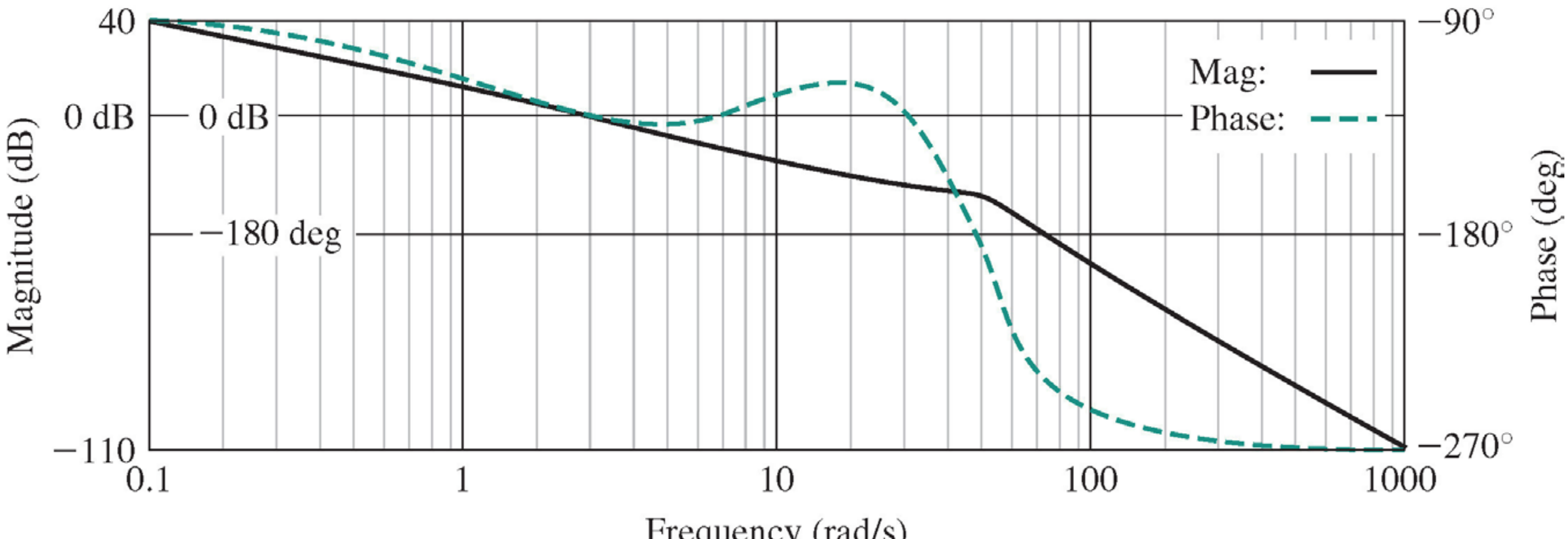


1. The phase of the constant gain is $0°$.
2. The phase of the pole at the origin is a constant $-90°$.
3. The linear approximation of the phase characteristic for the pole at $\omega = 2$ is shown in Figure 8.21, where the phase shift is $-45°$ at $\omega = 2$.
4. The linear approximation of the phase characteristic for the zero at $\omega = 10$ is also shown in Figure 8.21, where the phase shift is $+45°$ at $\omega = 10$.
5. The actual phase characteristic for the pair of complex poles is obtained from Figure 8.10 and is shown in Figure 8.21.

FIGURE 8.21 Phase characteristic.



The Bode plot of the $G(j\omega)$



Quiz 21.1

For the following transfer function

$$L(s) = G_c(s)G(s) = \frac{300(s + 100)}{s(s + 10)(s + 40)}.$$

1. determine the phase angle $\emptyset(\omega)$ when $\omega = 28.3$ rad/s;
2. Find the logarithmic gain of $L(j\omega)$ at $\omega = 28.3$ rad/s.
3. Sketch the Bode plots for this transfer function.

Thank You！