Xi'an Jiaotong-Liverpool University
西交利物浦大学

EEE204 Continuous and Discrete Time Signals and Systems II

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Electrical and Electronic Engineering

Xi'an Jiaotong-Liverpool University

Week 9

Inverse z-transform Example

Find the impulse response $h[n]$ of the causal system having system transfer function

$$H(z) = \frac{1+5z^{-1}}{1-2z^{-1}}, \text{ ROC: } |z| > 2$$

Look-up table approach,

$$H(z) = \frac{1}{1-2z^{-1}} + \frac{5z^{-1}}{1-2z^{-1}} \rightarrow h[n] = 2^n u[n] + 5 \cdot 2^{n-1} u[n-1].$$

"Long division" approach, which answer is correct for $h[n]$?

$$H(z) = -\frac{5}{2} + \frac{7}{2} \cdot \frac{1}{1-2z^{-1}} \rightarrow h[n] = -\frac{5}{2}\delta[n] + \frac{7}{2} \cdot 2^n u[n].$$

Inverse z-transform Example

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$$2^n u[n] + 5 \cdot 2^{n-1} u[n-1] = -\frac{5}{2}\delta[n] + \frac{7}{2} \cdot 2^n u[n]?$$

$$\begin{aligned}
h[n] &= 2^n u[n] + 5 \cdot 2^{n-1} u[n-1], \\
&\xlongequal{u[n-1]=u[n]-\delta[n]} 2^n u[n] + 5 \cdot 2^{n-1}(u[n] - \delta[n]), \\
&= 2^n u[n] + 5 \cdot 2^{n-1} u[n] - 5 \cdot 2^{n-1}\delta[n], \\
&= (1 + 5 \cdot 2^{-1}) \cdot 2^n u[n] - 5 \cdot 2^{0-1}\delta[n], \\
&= -\frac{5}{2}\delta[n] + \frac{7}{2} \cdot 2^n u[n].
\end{aligned}$$

The second form is preferable because this system has one pole, at $z = 2$, so it is preferable to use the form that has exactly one term for each pole.

Inverse z-transform

Methods for inverse z-transform:

- Table lookup, using properties.
- Contour integration.
- Series expansion into powers of z and z^{-1}.
- Partial-fraction expansion (PFE) and table lookup.

Practical problems requiring inverse z-transform?

- Given a system function $H(z)$, e.g., described by a pole-zero plot, find $h[n]$.
- When performing convolution via z-transforms: $Y(z) = H(z)X(z)$, leading to $y[n]$.

Partial-fraction Expansion and Table Lookup

Partial-fraction expansion

General strategy: suppose we have a "complicated" z-transform $X(z)$. If we can express $X(z)$ as a linear combination of "simple" functions $X_k(z)$ whose inverse z-transform is known, then we can use linearity to find $x[n]$.

$$X(z) = \alpha_1 X_1(z) + \cdots + \alpha_K X_K(z).$$

$$\Rightarrow x[n] = \alpha_1 x_1(n) + \cdots + \alpha_K x_K(n).$$

In principle one can apply this strategy to any $X(z)$. But whether "simple" $X_k(z)$'s inversion can be found will depend on the particular form of $X(z)$.

Fortunately, for the class of rational z-transforms, a decomposition into simple terms is always possible, using the partial-fraction expansion (PFE) method.

Step 1: Decompose $X(z)$ into proper form + polynomial

$$X(z) = \frac{N(z)}{D(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \cdots + a_N z^{-N}}$$

Such a rational function is called proper iff $a_N \neq 0$ and $M < N$.

We can always rewrite an improper rational function ($M \geqslant N$) as the sum of a polynomial and a proper rational function. If $M \geqslant N$, then

$$\boxed{\frac{P_M(z^{-1})}{P_N(z^{-1})} = P_{M-N}(z^{-1}) + \frac{P_{N-1}(z^{-1})}{P_N(z^{-1})}}.$$

Decompose $X(z)$ Example

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$X(z) = 1 + z^{-2}/(1+2z^{-1})$

$$
\begin{aligned}
X(z) = \frac{1+z^{-2}}{1+2z^{-1}} &= \frac{1}{2}z^{-1} + \left[\frac{1+z^{-2}}{1+2z^{-1}} - \frac{1}{2}z^{-1}\right], \\
&= \frac{1}{2}z^{-1} + \left[\frac{1+z^{-2} - \frac{1}{2}z^{-1}(1+2z^{-1})}{1+2z^{-1}}\right], \\
&= \frac{1}{2}z^{-1} + \frac{1-\frac{1}{2}z^{-1}}{1+2z^{-1}} = \frac{1}{2}z^{-1} - \frac{1}{4} + \left[\frac{1-\frac{1}{2}z^{-1}}{1+2z^{-1}} + \frac{1}{4}\right], \\
&= -\frac{1}{4} + \frac{1}{2}z^{-1} + \left[\frac{1-\frac{1}{2}z^{-1} + \frac{1}{4}(1+2z^{-1})}{1+2z^{-1}}\right], \\
&= -\frac{1}{4} + \frac{1}{2}z^{-1} + \frac{\frac{5}{4}}{1+2z^{-1}}.
\end{aligned}
$$

Decompose $X(z)$ Example

$X(z) = 1 + z^{-2}/(1 + 2z^{-1})$

$$\begin{aligned} X(z) &= \frac{1 + z^{-2}}{1 + 2z^{-1}}, \\ &= \underbrace{-\frac{1}{4} + \frac{1}{2}z^{-1}}_{\text{polynomial}} + \underbrace{\frac{\frac{5}{4}}{1 + 2z^{-1}}}_{\text{proper form}}. \end{aligned}$$

In general, this is always possible using long division.

The polynomial part is trivial to invert. Therefore, from now on we focus on proper rational functions.

Step 2: Find roots of denominator (poles)

The MATLAB `roots` command is useful here, or the quadratic formula when $N = 2$.

We call the **roots** $p_1, \cdots, p_N$, since these roots are the **poles** of $X(z)$.

Step 3: PFE for distinct roots

If the poles $p_1, \cdots, p_N$ are all different (distinct) then the expansion we seek has the form

$$X(z) = \frac{r_1}{1 - p_1 z^{-1}} + \cdots + \frac{r_N}{1 - p_N z^{-1}},$$

where the r_ks are real or complex numbers called residues. For distinct roots:

$$\boxed{r_k = (1 - p_k z^{-1}) X(z) \Big|_{z = p_k}.}$$

Proof: evaluate the LHS and RHS at $z = p_k$.

$$(1 - p_k z^{-1}) X(z) = \frac{r_1 (1 - p_k z^{-1})}{1 - p_1 z^{-1}} + \cdots + r_k + \cdots + \frac{r_N (1 - p_k z^{-1})}{1 - p_N z^{-1}}.$$

Find Residues Example

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$$X(z) = \frac{7 - 17z^{-1}}{1 - 5z^{-1} + 6z^{-2}}, r_k = (1 - p_k z^{-1})X(z)\Big|_{z=p_k}$$

$$\begin{aligned}
X(z) &= \frac{7 - 17z^{-1}}{1 - 5z^{-1} + 6z^{-2}} = \frac{7 - 17z^{-1}}{(1 - 2z^{-1})(1 - 3z^{-1})}, \\
&= \frac{r_1}{1 - 2z^{-1}} + \frac{r_2}{1 - 3z^{-1}}, \\
&= \frac{(1 - 2z^{-1})X(z)\Big|_{z=p_1}}{1 - 2z^{-1}} + \frac{(1 - 3z^{-1})X(z)\Big|_{z=p_2}}{1 - 3z^{-1}}, \\
&= \frac{\frac{7-17z^{-1}}{1-3z^{-1}}\Big|_{z=2}}{1 - 2z^{-1}} + \frac{\frac{7-17z^{-1}}{1-2z^{-1}}\Big|_{z=3}}{1 - 3z^{-1}}, \\
&= \frac{3}{1 - 2z^{-1}} + \frac{4}{1 - 3z^{-1}}.
\end{aligned}$$

Step 4: Inverse z-transform

Assuming $x[n]$ is causal (i.e., ROC $= \{|z| > \max_k |p_k|\}$), the inverse z-transform of

$$X(z) = \frac{r_1}{1 - p_1 z^{-1}} + \cdots + \frac{r_N}{1 - p_N z^{-1}},$$

is

$$\boxed{x[n] = r_1 p_1^n u[n] + \cdots + r_N p_N^n u[n].}$$

The discrete-time signal corresponding to a rational function in proper form with distinct roots is a weighted sum of geometric progression signals.

Partial-fraction Expansion

Complex conjugate pairs

In the usual case where the polynomial coefficients are real, any complex poles occur in conjugate pairs. Furthermore, the corresponding residues in the PFE also occur in complex-conjugate pairs.

Proof: Let p and p^* denote a complex-conjugate pair of roots. Suppose $X(z) = \frac{Y(z)}{(1-pz^{-1})(1-p^*z^{-1})}$ where $Y(z)$ is a ratio of polynomials in z with real coefficients. Then

$$r_1 = (1 - pz^{-1})X(z)\Big|_{z=p} = \frac{Y(z)}{1 - p^*z^{-1}}\Bigg|_{z=p} = \frac{Y(p)}{1 - p^*/p}.$$

Partial-fraction Expansion

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Complex conjugate pairs

$$X(z) = \frac{Y(z)}{(1 - pz^{-1})(1 - p^* z^{-1})}.$$

$$r_1 = (1 - pz^{-1})X(z)\Big|_{z=p} = \frac{Y(z)}{1 - p^* z^{-1}}\Bigg|_{z=p} = \frac{Y(p)}{1 - p^*/p}.$$

$$\begin{aligned}
r_2 &= (1 - p^* z^{-1})X(z)\Big|_{z=p^*} = \frac{Y(z)}{1 - pz^{-1}}\Bigg|_{z=p^*}, \\
&= \frac{Y(p^*)}{1 - p/p^*} = \left[\frac{Y^*(p^*)}{1 - p^*/p}\right]^*, \\
&= \left[\frac{Y(p)}{1 - p^*/p}\right]^* = r_1^*, \text{ since } Y^*(p^*) = Y(p).
\end{aligned}$$

Find inverse z-transform of

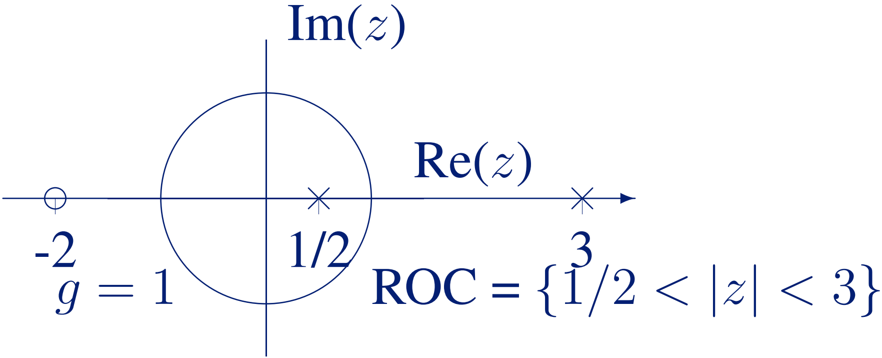
$$X(z) = \frac{r}{1 - pz^{-1}} + \frac{r^*}{1 - p^* z^{-1}},$$

$$\begin{aligned}
x[n] &= [rp^n + r^*(p^*)^n]u[n], \\
&= 2\ \text{real}(rp^n)u[n], \\
&= 2\ \text{real}(|r|e^{j\phi}|p|^n e^{j\omega_0 n})u[n], \\
&= 2\ \text{real}\{|r||p|^n[\cos(\omega_0 n + \phi) + j\sin(\omega_0 n + \phi)]\}u[n], \\
&= 2|r||p|^n \cos(\omega_0 n + \phi)u[n],
\end{aligned}$$

where $p = |p|e^{j\omega_0}$ and $r = |r|e^{j\phi}$.

PFE Example 1

Find the signal $x[n]$ whose z-transform has the following pole-zero plot



$$X(z) = \frac{z+2}{(z-\frac{1}{2})(z-3)}.$$

PFE Example 1

$$X(z) = \frac{z+2}{(z-\frac{1}{2})(z-3)} = \frac{z^{-1}+2z^{-2}}{(1-\frac{1}{2}z^{-1})(1-3z^{-1})},$$

$$= \frac{z^{-1}+2z^{-2}}{1-\frac{7}{2}z^{-1}+\frac{3}{2}z^{-2}},$$

$$= \frac{4}{3} + \left[\frac{z^{-1}+2z^{-2}}{1-\frac{7}{2}z^{-1}+\frac{3}{2}z^{-2}} - \frac{2}{3/2}\right],$$

$$= \frac{4}{3} + \frac{z^{-1}+2z^{-2}-\frac{4}{3}\left[1-\frac{7}{2}z^{-1}+\frac{3}{2}z^{-2}\right]}{1-\frac{7}{2}z^{-1}+\frac{3}{2}z^{-2}},$$

$$= \frac{4}{3} + \frac{-\frac{4}{3}+\frac{17}{3}z^{-1}}{1-\frac{7}{2}z^{-1}+\frac{3}{2}z^{-2}},$$

PFE Example 1

$$
\begin{aligned}
X(z) &= \frac{4}{3} + \frac{-\frac{4}{3} + \frac{17}{3}z^{-1}}{1 - \frac{7}{2}z^{-1} + \frac{3}{2}z^{-2}}, \\
&= \frac{4}{3} + \frac{-\frac{4}{3} + \frac{17}{3}z^{-1}}{(1 - \frac{1}{2}z^{-1})(1 - 3z^{-1})}, \\
&= \frac{4}{3} + \frac{r_1}{1 - \frac{1}{2}z^{-1}} + \frac{r_2}{1 - 3z^{-1}}, \\
r_1 &= (1 - \frac{1}{2}z^{-1}) \frac{-\frac{4}{3} + \frac{17}{3}z^{-1}}{(1 - \frac{1}{2}z^{-1})(1 - 3z^{-1})}\Bigg|_{z=\frac{1}{2}} = -2, \\
r_2 &= (1 - 3z^{-1}) \frac{-\frac{4}{3} + \frac{17}{3}z^{-1}}{(1 - \frac{1}{2}z^{-1})(1 - 3z^{-1})}\Bigg|_{z=3} = \frac{2}{3},
\end{aligned}
$$

PFE Example 1

$$X(z) = \frac{4}{3} + \frac{r_1}{1 - \frac{1}{2}z^{-1}} + \frac{r_2}{1 - 3z^{-1}},$$

$$r_1 = -2, r_2 = \frac{2}{3},$$

$$X(z) = \frac{4}{3} + \frac{-2}{1 - \frac{1}{2}z^{-1}} + \frac{\frac{2}{3}}{1 - 3z^{-1}},$$

$$x(n) = \frac{4}{3}\delta[n] - 2\left(\frac{1}{2}\right)^n u[n] \underbrace{-\frac{2}{3}3^n u[-n-1]}_{\text{anti-causal}}.$$

ROC: $\left\{\frac{1}{2} < |z| < 3\right\}$.

PFE

General PFE formula for single poles, for proper form with $M < N$:

$$X(z) = \frac{B(z)}{A(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{\prod_{k=1}^{N}(1 - p_k z^{-1})},$$
$$= \frac{r_1}{1 - p_1 z^{-1}} + \cdots + \frac{r_N}{1 - p_N z^{-1}},$$

where the residue is given by:

$$\boxed{r_k = (1 - p_k z^{-1}) X(z)\Big|_{z=p_k} = \frac{B(z)}{\prod_{l \neq k}(1 - p_l z^{-1})}\Bigg|_{z=p_k}}$$

PFE

If p_k is a repeated (2^{nd}-order) pole:

$$X(z) = \cdots + \frac{r_{k,1}}{1 - p_k z^{-1}} + \frac{r_{k,2}}{(1 - p_k z^{-1})^2} + \cdots,$$

$$r_{k,1} = -\frac{1}{p_k} \frac{\mathrm{d}}{\mathrm{d}z^{-1}} (1 - p_k z^{-1})^2 X(z) \Big|_{z=p_k},$$

$$r_{k,2} = (1 - p_k z^{-1})^2 X(z) \Big|_{z=p_k}.$$

For p_k is a repeated $L > 2$ order, $l = 1, \cdots, L$

$$r_{k,l} = \frac{1}{(L-l)!(-p_k)^{L-l}} \frac{\mathrm{d}^{L-l}(1 - p_k z^{-1})^L X(z)}{\mathrm{d}(z^{-1})^{L-l}} \Bigg|_{z=p_k}.$$

PFE Example 2

Find the impulse response of the system described by the following **diffeq**:

$$y[n] = \frac{4}{3}y[n-1] - \frac{7}{12}y[n-2] + \frac{1}{12}y[n-3] + x[n] - x[n-3].$$

Take z-transforms on both sides:

$$Y(z) = \frac{4}{3}z^{-1}Y(z) - \frac{7}{12}z^{-2}Y(z) + \frac{1}{12}z^{-3}Y(z) + X(z) - z^{-3}X(z).$$

$$\begin{aligned} H(z) = \frac{Y(z)}{X(z)} &= \frac{1 - z^{-3}}{1 - \frac{4}{3}z^{-1} + \frac{7}{12}z^{-2} - \frac{1}{12}z^{-3}}, \\ &= 12 + \left[\frac{1 - z^{-3}}{1 - \frac{4}{3}z^{-1} + \frac{7}{12}z^{-2} - \frac{1}{12}z^{-3}} - 12\right], \end{aligned}$$

PFE Example 2

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$$
\begin{aligned}
H(z) &= 12 + \left[\frac{1 - z^{-3}}{1 - \frac{4}{3}z^{-1} + \frac{7}{12}z^{-2} - \frac{1}{12}z^{-3}} - 12\right], \\
&= 12 + \frac{1 - z^{-3} - 12(1 - \frac{4}{3}z^{-1} + \frac{7}{12}z^{-2} - \frac{1}{12}z^{-3})}{1 - \frac{4}{3}z^{-1} + \frac{7}{12}z^{-2} - \frac{1}{12}z^{-3}}, \\
&= 12 + \frac{-11 + 16z^{-1} - 7z^{-2}}{1 - \frac{4}{3}z^{-1} + \frac{7}{12}z^{-2} - \frac{1}{12}z^{-3}}, \\
&= 12 + \frac{-11 + 16z^{-1} - 7z^{-2}}{\left(1 - \frac{1}{2}z^{-1}\right)^2 \left(1 - \frac{1}{3}z^{-1}\right)}, \\
&= 12 + \frac{r_{1,1}}{1 - \frac{1}{2}z^{-1}} + \frac{r_{1,2}}{\left(1 - \frac{1}{2}z^{-1}\right)^2} + \frac{r_2}{1 - \frac{1}{3}z^{-1}},
\end{aligned}
$$

PFE Example 2

$$H(z) = 12 + \frac{r_{1,1}}{1-\frac{1}{2}z^{-1}} + \frac{r_{1,2}}{\left(1-\frac{1}{2}z^{-1}\right)^2} + \frac{r_2}{1-\frac{1}{3}z^{-1}},$$

$$r_{1,1} = -\frac{1}{1/2}\frac{\mathrm{d}}{\mathrm{d}z^{-1}}\,(1-\frac{1}{2}z^{-1})^2[H(z)-12]\bigg|_{z=\frac{1}{2}},$$

$$= -2\frac{\mathrm{d}}{\mathrm{d}z^{-1}}\,\frac{-11+16z^{-1}-7z^{-2}}{\left(1-\frac{1}{3}z^{-1}\right)}\Bigg|_{z=\frac{1}{2}},$$

$$= -2\,\frac{(16-14z^{-1})(1-\frac{1}{3}z^{-1})-(-11+16z^{-1}-7z^{-2})(-\frac{1}{3})}{\left(1-\frac{1}{3}z^{-1}\right)^2}\Bigg|_{z=\frac{1}{2}},$$

$$= 114.$$

PFE Example 2

$$H(z) = 12 + \frac{r_{1,1}}{1-\frac{1}{2}z^{-1}} + \frac{r_{1,2}}{\left(1-\frac{1}{2}z^{-1}\right)^2} + \frac{r_2}{1-\frac{1}{3}z^{-1}},$$

$$r_{1,2} = (1-\frac{1}{2}z^{-1})^2[H(z)-12]\Big|_{z=\frac{1}{2}},$$

$$= \frac{-11+16z^{-1}-7z^{-2}}{(1-\frac{1}{3}z^{-1})}\Bigg|_{z=\frac{1}{2}} = -21,$$

$$r_2 = (1-\frac{1}{3}z^{-1})[H(z)-12]\Big|_{z=\frac{1}{3}},$$

$$= \frac{-11+16z^{-1}-7z^{-2}}{(1-\frac{1}{2}z^{-1})^2}\Bigg|_{z=\frac{1}{3}} = -104,$$

PFE Example 2

$$H(z) = 12 + \frac{r_{1,1}}{1-\frac{1}{2}z^{-1}} + \frac{r_{1,2}}{\left(1-\frac{1}{2}z^{-1}\right)^2} + \frac{r_2}{1-\frac{1}{3}z^{-1}},$$

$$r_{1,1} = 114, r_{1,2} = -21, r_2 = -104,$$

$$H(z) = 12 + \frac{114}{1-\frac{1}{2}z^{-1}} + \frac{-21}{\left(1-\frac{1}{2}z^{-1}\right)^2} + \frac{-104}{1-\frac{1}{3}z^{-1}},$$

$$h[n] = 12\delta[n] + 114\left(\frac{1}{2}\right)^n u[n] - 21(n+1)\left(\frac{1}{2}\right)^n u[n]$$

$$-104\left(\frac{1}{3}\right)^n u[n], \text{ from look-up table}$$

$$=12\delta[n] + \left[(93-21n)\left(\frac{1}{2}\right)^n - 104\left(\frac{1}{3}\right)^n\right]u[n].$$

Homework

- Page 798, Q10.9;
- Page 801, Q10.23;
- Page 802, Q10.24: (a)–(c);
- Page 802, Q10.25: (a)–(c);
- Page 802, Q10.26: (a)–(c);
- Page 802, Q10.27.

Thank you for your attention.