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EEE220 Instrumentation and Control System

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Lecture 23

Outline

Stability in the Frequency Domain

- Introduction
- Mapping Contours in the s-Plane
- The Nyquist Criterion
- Relative Stability and the Nyquist Criterion
- Stability in the Frequency Domain Using Matlab

Introduction

- **Stability** is key characteristic of a feedback control system.

- Approaches for determining stability of a system:
  - Routh-Hurwitz Criterion
  - Root Locus Method
  - Nyquist Stability Criterion

- **Nyquist Stability Criterion** is a method for investigating the stability of a system **in frequency domain**, that is, in terms of the frequency response. It was developed by H. Nyquist in 1932 and remains a fundamental approach to the investigation of the stability of linear control system.

- The Nyquist Stability Criterion is based on the Cauchy's theorem which is concerned with **mapping contours** in the complex s-plane.

Mapping Contours in the s-Plane

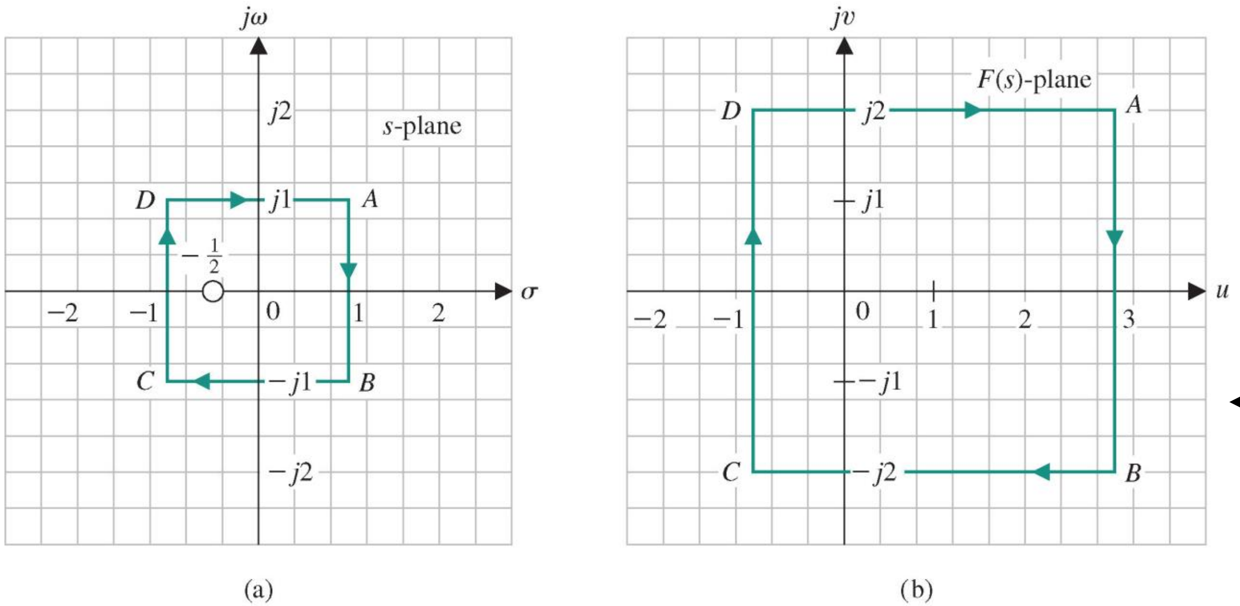
A **contour map** is a contour or trajectory in one plane mapped or translated into another plane by a relation $F(s)$ where s is a complex variable. So $F(s)$ is also complex and can be defined as $F(s) = u + jv$.

- Example: Mapping a square contour by $F(s) = 2s + 1$.

$$u + jv = F(s) = 2s + 1 = 2(\sigma + j\omega) + 1$$

therefore

$$u = 2\sigma + 1 \text{ and } v = 2\omega$$

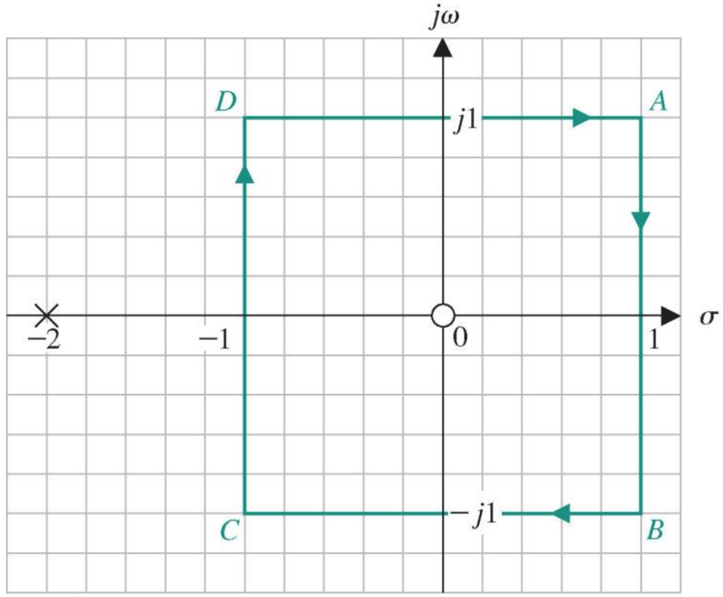


Conformal Mapping
(same shape after mapping)

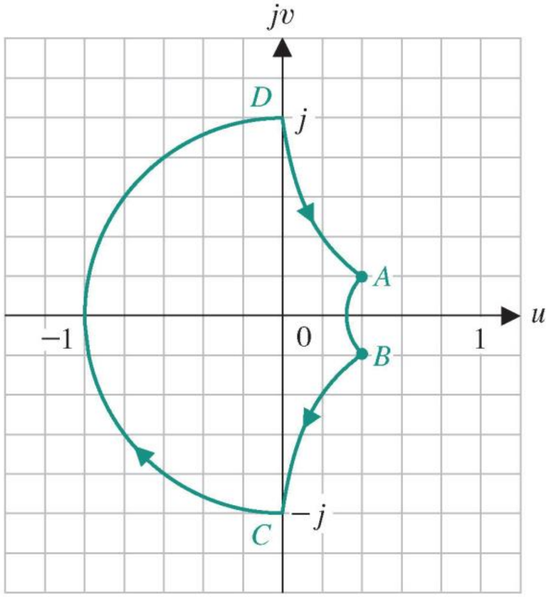
- Consider another rational function of s: $F(s) = \frac{s}{s+2}$

Table 9.1 Values of $F(s)$

	Point A		Point B		Point C		Point D	
$s = \sigma + j\omega$	$1 + j1$	1	$1 - j1$	$-j1$	$-1 - j1$	-1	$-1 + j1$	$j1$
$F(s) = u + jv$	$\frac{4 + 2j}{10}$	$\frac{1}{3}$	$\frac{4 - 2j}{10}$	$\frac{1 - 2j}{5}$	$-j$	-1	$+j$	$\frac{1 + 2j}{5}$



(a)



(b)

Cauchy's theorem

Assume that $F(s)$ has a finite number of poles and zeros and can be expressed as

$$F(s) = \frac{K\prod_{i=1}^{n}(s+z_i)}{\prod_{k=1}^{M}(s+p_k)}$$

❖ **Cauchy's Theorem (Principle of the Argument):**

If a contour Γ_s in the s-plane encircles Z zeros and P poles of $F(s)$ and does not pass through any poles or zeros of $F(s)$ and the traversal is in the clockwise direction along the contour, the corresponding contour Γ_F in the $F(s)$-plane encircles the origin of the $F(s)$ plane $N = Z - P$ times in the clockwise direction.

- In the previous example $F(s) = \frac{s}{s+2}$, the contour in the $F(s)$ plane encircles the origin once, since

$$N = Z - P = 1 - 0 = 1$$

How to Understand the Cauchy's Theorem?

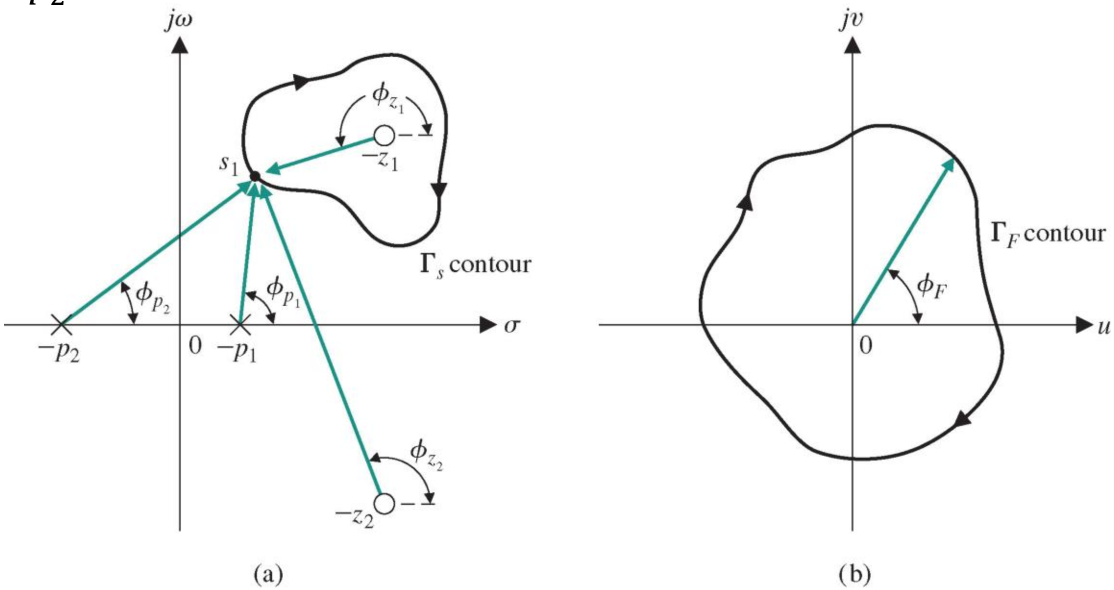
Consider the following $F(s)$. We choose a contour in s-plane which only encircles $-z_1$.

$$F(s) = \frac{(s+z_1)(s+z_2)}{(s+p_1)(s+p_2)}$$

$F(s)$ can be expressed as

$$F(s) = |F(s)|\angle F(s) = \frac{|s+z_1||s+z_2|}{|s+p_1||s+p_2|}(\angle(s+z_1) + \angle(s+z_2) - \angle(s+p_1) - \angle(s+p_2))$$
$$= |F(s)|(\phi_{z_1} + \phi_{z_2} - \phi_{p_1} - \phi_{p_2})$$

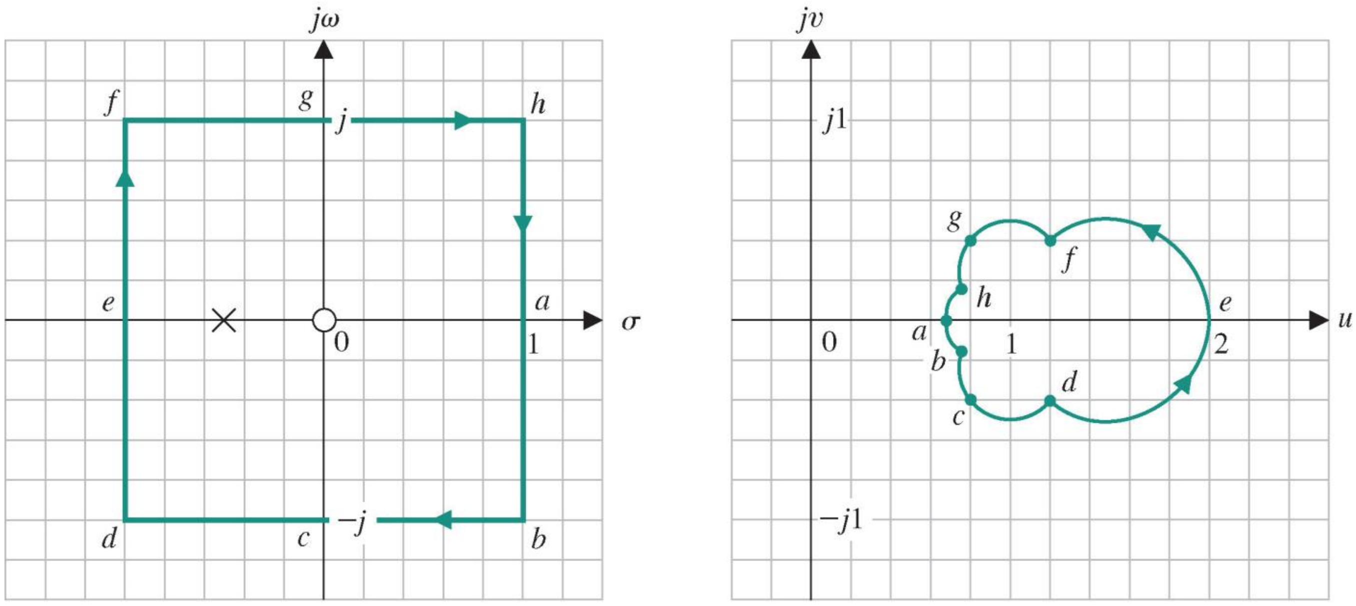
As s traverses 360^o along Γ_s, the angle $\emptyset_{z_1}$ traverses a full 360^o; while $\emptyset_{z_2}, \emptyset_{p_1}$ and $\emptyset_{p_2}$ traverse 0^o. Thus, the net angle of $F(s)$ will increase 360^o.



More generally, if there are Z zeros and P poles enclosed within Γ_s, then the net angle increase in F(s) would be equal to

$$\emptyset = 2\pi Z - 2\pi P = 2\pi N$$

Therefore, the contour Γ_F will encircle the origin N times.

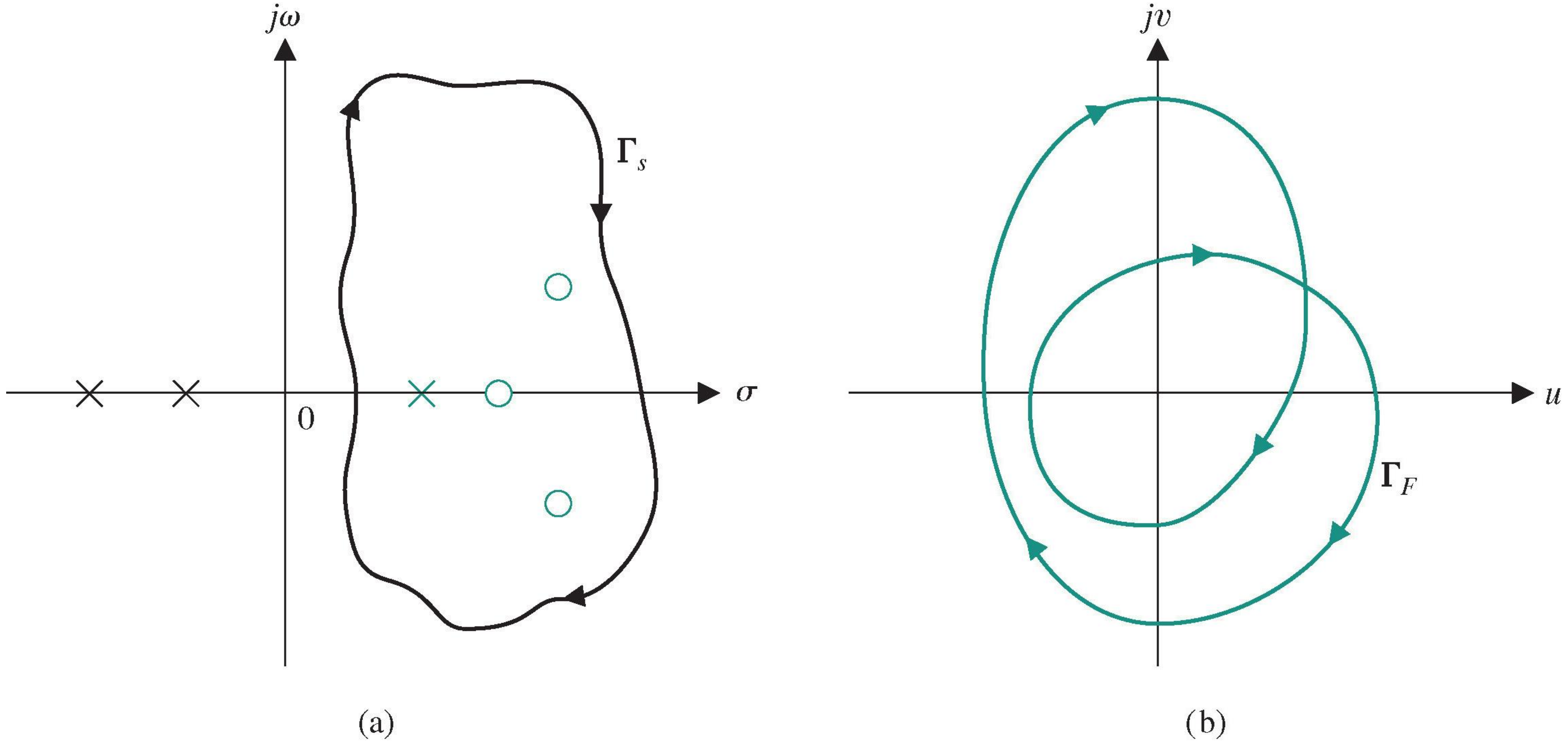


(a) (b)

For example: $F(s) = \frac{s}{s+1/2}$, $N = Z - P = 0$.

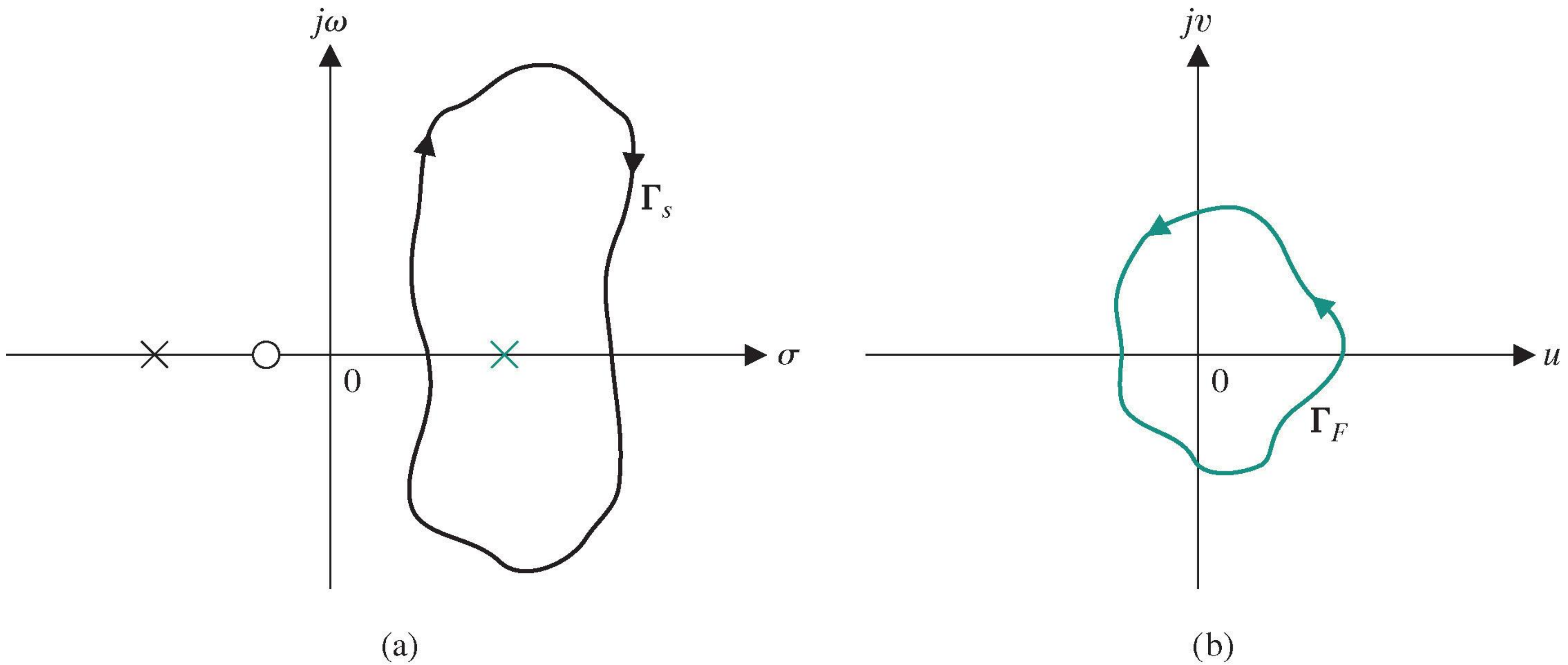
Examples

$$N = Z - P = 3 - 1 = 2$$



$$N = Z - P = 0 - 1 = -1$$

(Note the counterclockwise direction)



Introducing the Nyquist Criterion

To investigate the stability of a control system with loop transfer function

$$L(s) = \frac{N(s)}{D(s)}$$

we consider the characteristic equation

$$F(s) = 1 + L(s) = \frac{N(s) + D(s)}{D(s)} = \frac{K\prod_{i=1}^{n}(s + z_i)}{\prod_{k=1}^{M}(s + p_k)} = 0$$

The loop transfer function $L(s)$ is typically available in factored form, while $F(s)$ (which is $1 + L(s)$) is NOT – Need to Investigate the zeros of $F(s)$.

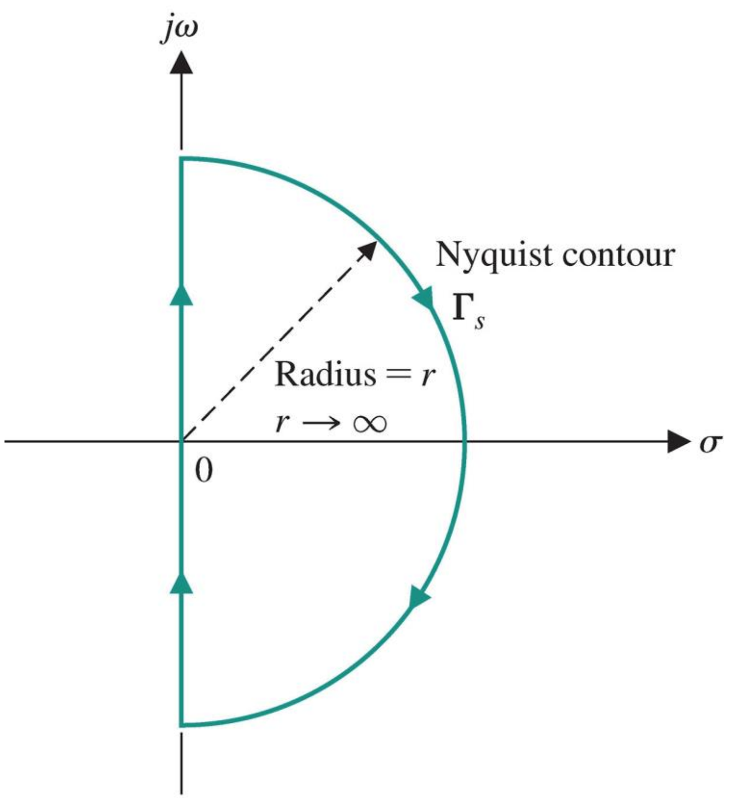
- For a system to be stable, all poles of the system (which are the zeros of $F(s)$)must lie in the left-hand s-plane.
- Choose a contour Γ_s in the s-plane that encloses the entire right-hand s-plane,
- Determine whether any zeros of F(s) lie within Γ_F by utilizing Cauchy's theorem. That is, we plot Γ_F in the F(s)-plane and determine the number of encirclements of the origin N.
- Then the number of zeros of $F(s)$ within the contour Γ_s (and therefore, the unstable zeros of $F(s)$) is

$$\boxed{Z = N + P}$$

Nyquist Contour

Nyquist contour Γ_s encloses the entire right half s-plane.

- The contour Γ_s passes along $j\omega$-axis from $-j\infty$ to $j\infty$, and this part of the contour provides the familiar $F(j\omega)$.
- The counter is completed by a semicircle path of radius r, where r approaches infinity – so this part of the contour typically maps to a point.
- The contour Γ_F is known as the **Nyquist Plot**.



Nyquist Stability Criterion

It is very convenient to change the function and represent it as

$$F'(s) = F(s) - 1 = L(s)$$

Based on this rearrangement, the **Nyquist Stability Criterion** can be stated as follows:

❖ Nyquist Stability Criterion

- A feedback system is stable if and only if the contour Γ_L in the $L(s)$-plane does not encircle the $\mathbf{(-1, 0)}$ point when the number of poles of $L(s)$ in the right-hand s-plane is zero (when $P = 0$);

- A feedback system is stable if and only if, for the contour Γ_L, the number of counterclockwise encirclements of the $\mathbf{(-1, 0)}$ point is equal to the number of poles of $L(s)$ with positive real parts (when $P \neq 0$).

Clearly,

1) If the number of poles of $L(s)$ in the right hand s-plane is zero ($P = 0$), We require for a stable system that

$$N = 0$$

So the contour Γ_L must NOT encircle the (-1,0) point;

2) If P is not zero ($P \neq 0$), then we require for a stable system that $Z = 0$, then we must have

$$N = -P$$

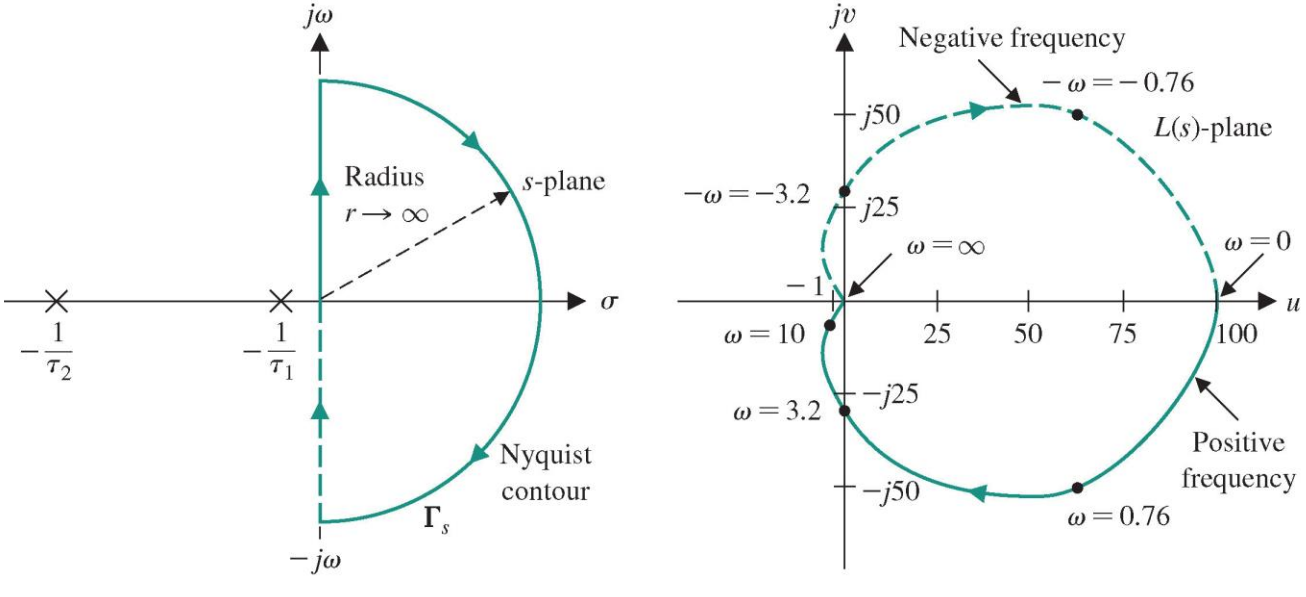
So we require P counterclockwise encirclements of the (-1, 0) point.

Examples

- **System with two real poles**

Consider a closed-loop system with the following loop transfer function

$$L(s) = \frac{K}{(\tau_1 s + 1)(\tau_2 s + 1)} \qquad \text{where } \tau_1 = 1, \tau_2 = \frac{1}{10}, K = 100$$



(a) (b)

$P = 0, N = 0$

$\downarrow$

$Z = 0$

$\downarrow$

The system is stable.

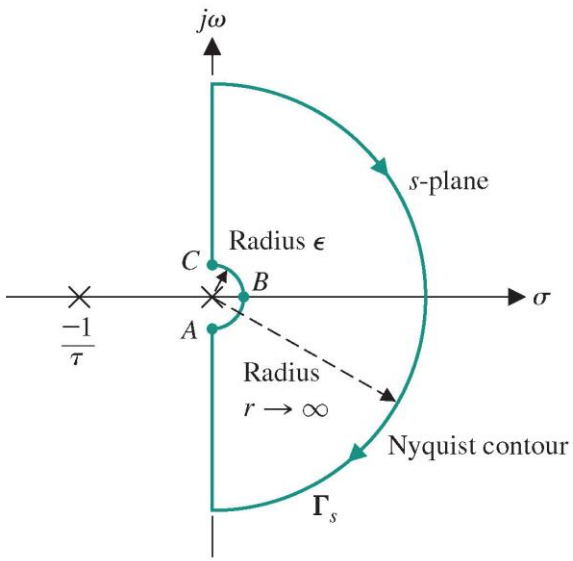
ω	0	0.1	0.76	1	2	10	20	100	∞
$\lvert L(j\omega)\rvert$	100	96	79.6	70.7	50.2	6.8	2.24	0.10	0
$\angle L(j\omega)$ (degrees)	0	−5.7	−41.5	−50.7	−74.7	−129.3	−150.5	−173.7	−180

- **System with a pole at the origin**

Consider a closed-loop system with the following loop transfer function

$$L(s) = \frac{K}{s(\tau s + 1)}$$

There is a pole at the origin. Cauchy's theorem requires that the contour Γ_s can not pass through the pole at the origin. Therefore we choose an infinitesimal detour around the pole at the origin which is a small semicircle of radius ε where $\varepsilon \to 0$.



(a)

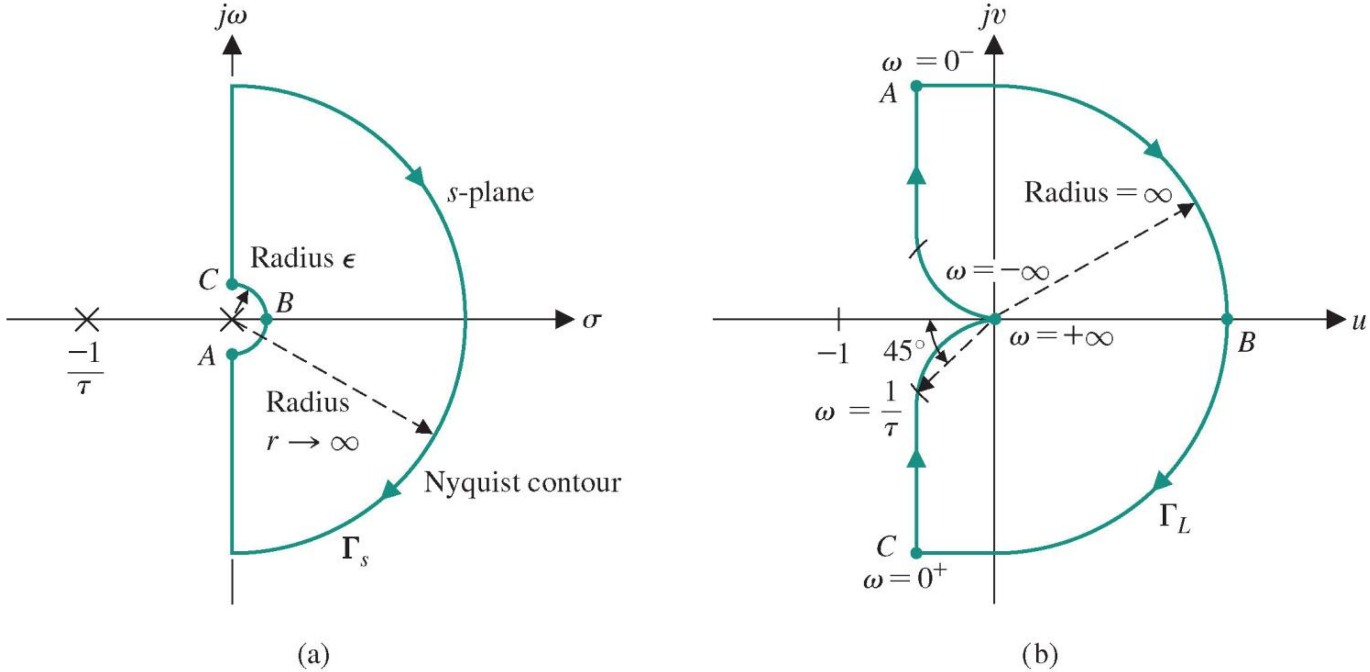
(a) The origin of the s-plane (the portion from $\omega = 0^-$ to $\omega = 0^+$)

The semicircular detour can be represented by

$$s = \varepsilon e^{j\emptyset} \qquad \text{where } \varepsilon \to 0,\ \emptyset \text{ varies from } -90^o \text{ to } 90^o$$

So the mapping for $L(s)$ from $\omega = 0^-$ to $\omega = 0^+$ along the detour is

$$\lim_{\epsilon \to 0} L(s) = \lim_{\epsilon \to 0} \frac{K}{\epsilon e^{j\phi}} = \lim_{\epsilon \to 0} \frac{K}{\epsilon} e^{-j\phi}$$



(b) The portion from $\omega = 0^+$ to $\omega = +\infty$

Since $s = j\omega$:

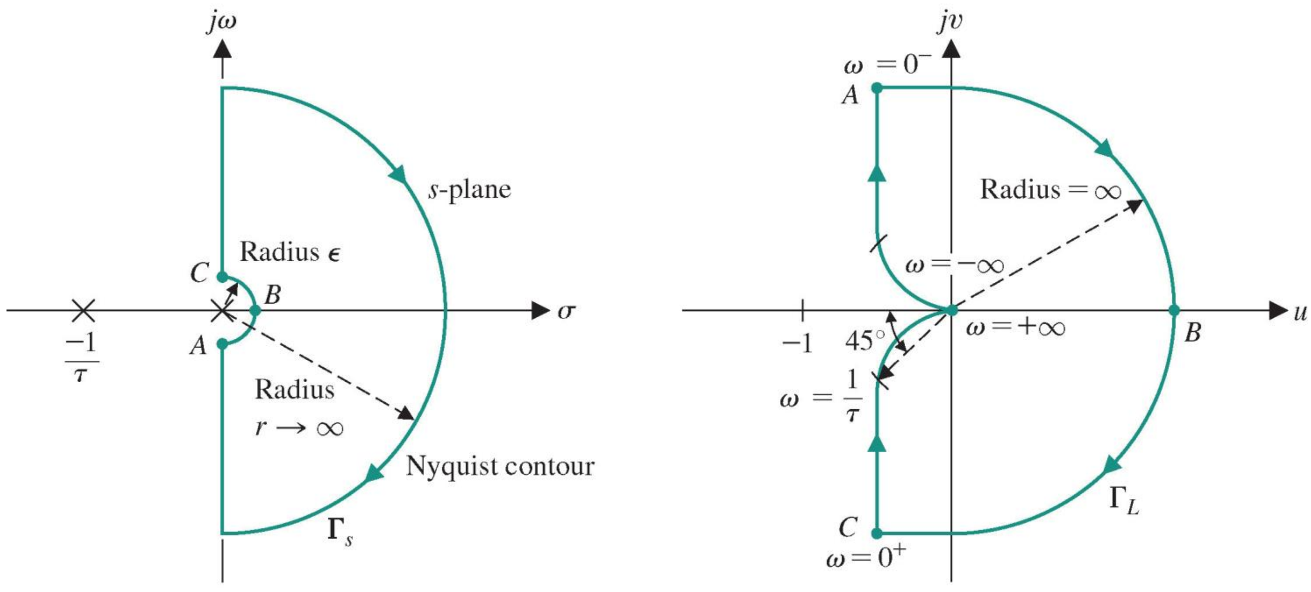
$$L(s)\big|_{s=j\omega} = L(j\omega)$$

$L(j\omega)$ can be obtained for any ω. When $\omega \to +\infty$

$$\lim_{\omega \to +\infty} L(j\omega) = \lim_{\omega \to +\infty} \frac{K}{+j\omega(j\omega\tau + 1)}$$

$$= \lim_{\omega \to \infty} \left|\frac{K}{\tau\omega^2}\right| \underline{\angle -(\pi/2) - \tan^{-1}(\omega\tau)}.$$

Therefore, the magnitude approaches zero at an angle of -180^o.



(a) (b)

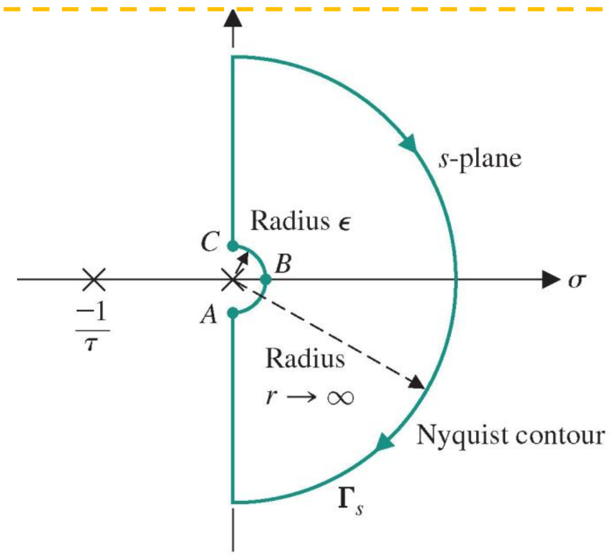
(c) The portion from $\boldsymbol{\omega} = +\infty$ to $\boldsymbol{\omega} = -\infty$

The semicircle with infinite radius can be expressed as

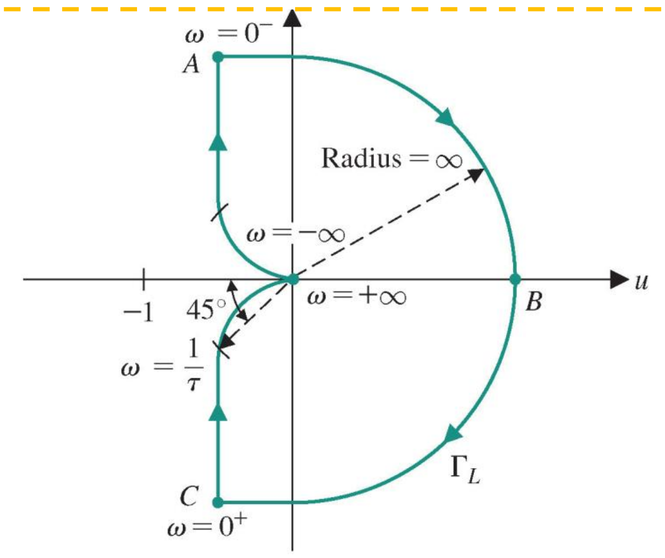
$$s = re^{j\emptyset} \qquad \text{where } r \to \infty,\ \emptyset \text{ varies from } 90^o \text{ to } -90^o$$

The mapping is

- Normally, the magnitude of $L(s)$ as $s = re^{j\emptyset}$ and $r \to \infty$ will approach zero or a constant.



(a)

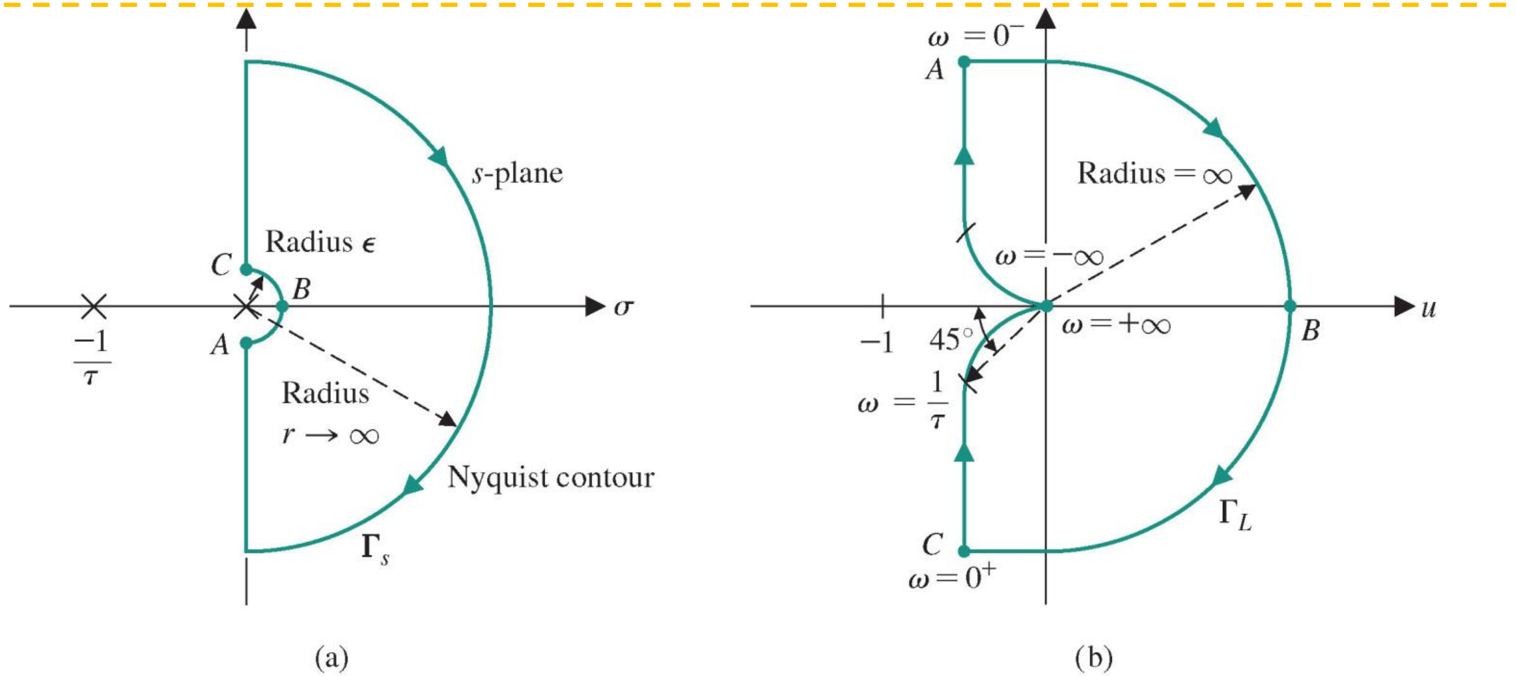


(b)

(d) The portion from $\boldsymbol{\omega = -\infty}$ to $\boldsymbol{\omega = 0^-}$

The plot of this portion is symmetrical to the portion from $\omega = +\infty$ to $\omega = 0^+$.

- Normally, the Nyquist Plot is symmetrical, therefore, it is sufficient to construct Γ_L for only positive ω.



$P = 0,\ N = 0 \rightarrow\ Z = 0$

The system is stable.

- **System with three poles**

Consider a closed-loop system with the following loop transfer function

$$L(s) = \frac{K}{s(\tau_1 s + 1)(\tau_2 s + 1)}$$

(a) The small semicircular detour around the origin (from $\boldsymbol{\omega = 0^-}$ to $\boldsymbol{\omega = 0^+}$)

This portion maps into a semicircle of infinite radius.

(b) The semicircle with infinite radius (from $\boldsymbol{\omega = +\infty}$ to $\boldsymbol{\omega = -\infty}$)

This portion maps into the origin point.

(c) The positive $j\omega$-axis (from $\omega = 0^+$ to $\omega = +\infty$)

$$s = j\omega$$

$$L(j\omega) = \frac{K}{j\omega(j\omega\tau_1 + 1)(j\omega\tau_2 + 1)}$$

$$= \frac{-K(\tau_1 + \tau_2) - jK(1/\omega)(1 - \omega^2\tau_1\tau_2)}{1 + \omega^2(\tau_1^2 + \tau_2^2) + \omega^4\tau_1^2\tau_2^2}$$

$$= \frac{K}{[\omega^4(\tau_1 + \tau_2)^2 + \omega^2(1 - \omega^2\tau_1\tau_2)^2]^{1/2}}$$

$$\times \underline{/-\tan^{-1}(\omega\tau_1) - \tan^{-1}(\omega\tau_2) - (\pi/2)}.$$

when $\omega \to +\infty$:

$$\lim_{\omega\to\infty} L(j\omega) = \lim_{\omega\to\infty} \left|\frac{1}{\omega^3\tau_1\tau_2}\right| \underline{/-(\pi/2) - \tan^{-1}(\omega\tau_1) - \tan^{-1}(\omega\tau_2)}$$

$$= \lim_{\omega\to\infty} \left|\frac{1}{\omega^3\tau_1\tau_2}\right| \underline{/-3\pi/2}.$$

Therefore Γ_L approaches zero at -270^o → It is possible to encircle the (-1, 0) point.

Actually, the intersection point of the contour with the real axis can be derived. $L(j\omega)$ can be written as real and imaginary part:

$$L(j\omega) = u + jv, \text{ where}$$

$$u = \left.\frac{-K(\tau_1 + \tau_2)}{1 + \omega^2(\tau_1^2 + \tau_2^2) + \omega^4\tau_1^2\tau_2^2}\right|_{\omega^2 = 1/\tau_1\tau_2}$$

$$= \frac{-K(\tau_1 + \tau_2)\tau_1\tau_2}{\tau_1\tau_2 + (\tau_1^2 + \tau_2^2) + \tau_1\tau_2}$$

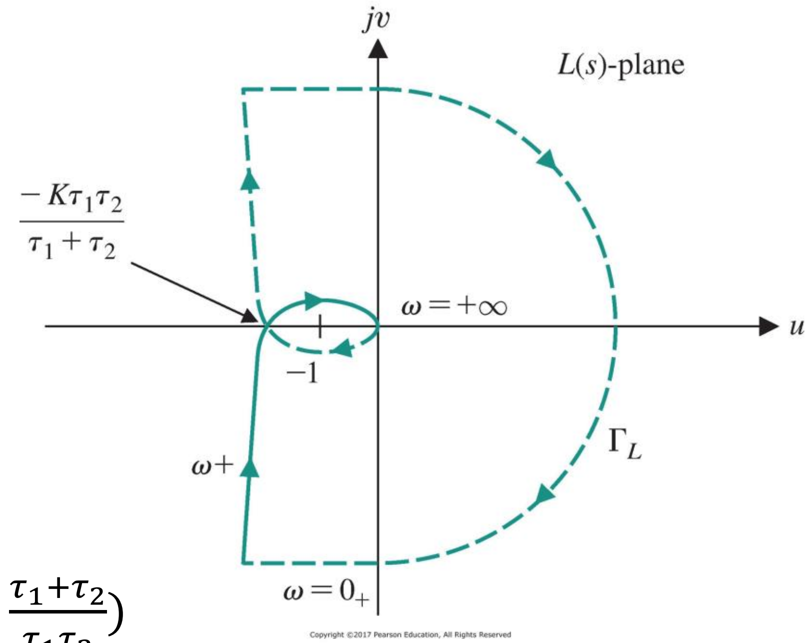
$$v = \frac{-K(1/\omega)(1 - \omega^2\tau_1\tau_2)}{1 + \omega^2(\tau_1^2 + \tau_2^2) + \omega^4\tau_1^2\tau_2^2} = 0.$$

Let $v = 0$, we have

$$\omega = 1/\sqrt{\tau_1\tau_2}$$

So, the contour crosses the real axis at

$$u = \frac{-K\tau_1\tau_2}{\tau_1 + \tau_2}$$

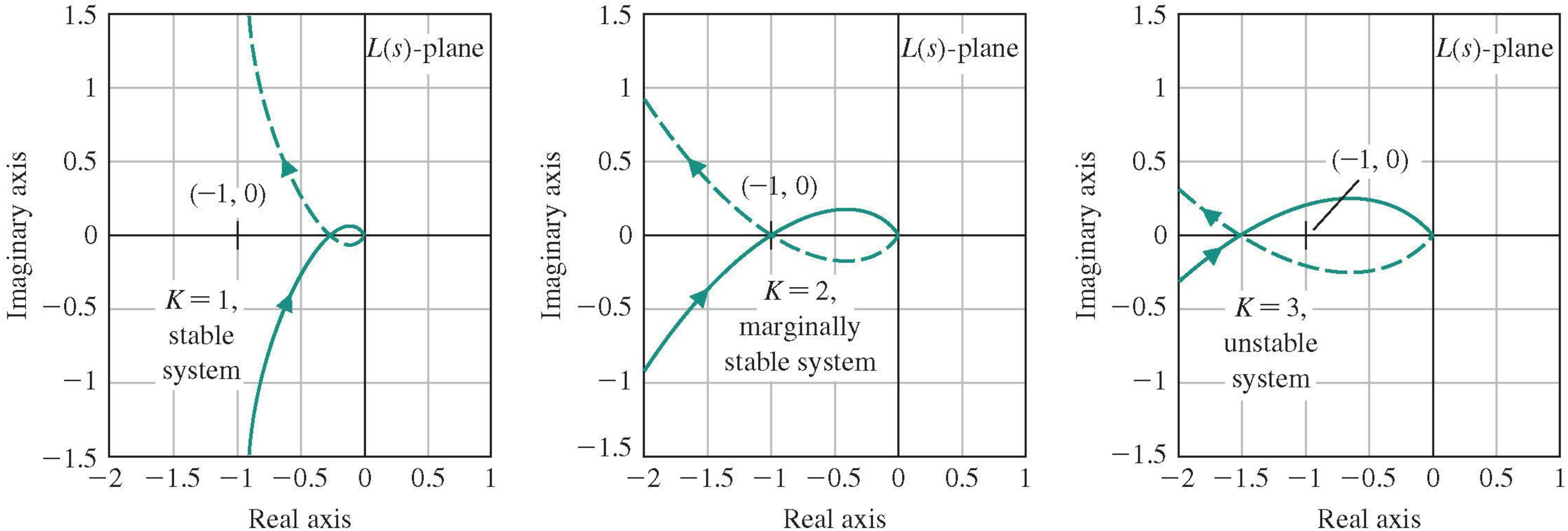


To ensure stability, it requires $\frac{-K\tau_1\tau_2}{\tau_1+\tau_2} \geq -1$. ($K \leq \frac{\tau_1+\tau_2}{\tau_1\tau_2}$)

Assume $\tau_1 = \tau_2 = 1$, the condition for the system to be stable is

$$K \leq 2$$

(a) K = 1, (b) K = 2, and (c) K = 3.



(a) (b) (c)

- **System with two poles at the origin**

Consider a closed-loop system with the following loop transfer function

$$L(s) = \frac{K}{s^2(\tau s + 1)}$$

(a) From $\boldsymbol{\omega = 0^+}$ to $\boldsymbol{\omega = +\infty}$

$s = \mathrm{j}\omega$, we have

$$L(j\omega) = \frac{K}{-\omega^2(j\omega\tau + 1)} = \frac{K}{[\omega^4 + \tau^2\omega^6]^{1/2}} \underline{/-\pi - \tan^{-1}(\omega\tau)}$$

So the angle of $L(j\omega)$ is -180^o or less, so the contour for this portion must be above the u axis.

when $\omega \to 0^+$: $$\lim_{\omega\to 0+} L(j\omega) = \lim_{\omega\to 0+} \left|\frac{K}{\omega^2}\right| \underline{/-\pi}.$$

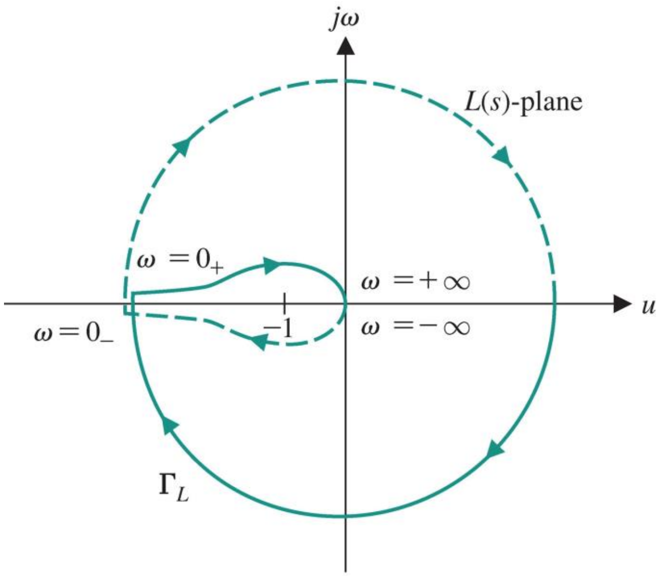
when $\omega \to +\infty$: $$\lim_{\omega\to +\infty} L(j\omega) = \lim_{\omega\to +\infty} \frac{K}{\omega^3} \underline{/-3\pi/2}.$$

(b) From $\omega = 0^-$ to $\omega = 0^+$

$s = \varepsilon e^{j\emptyset}$ where $\varepsilon \to 0$, $\emptyset$ varies from -90^o to 90^o

$$\lim_{\epsilon \to 0} L(s) = \lim_{\epsilon \to 0} \frac{K}{\epsilon^2} e^{-2j\phi}$$

The contour ranges from an angle of 180^o to an angle of -180^o and passes through a full circle.



(c) From $\omega = +\infty$ to $\omega = -\infty$

This portion maps into the origin point.

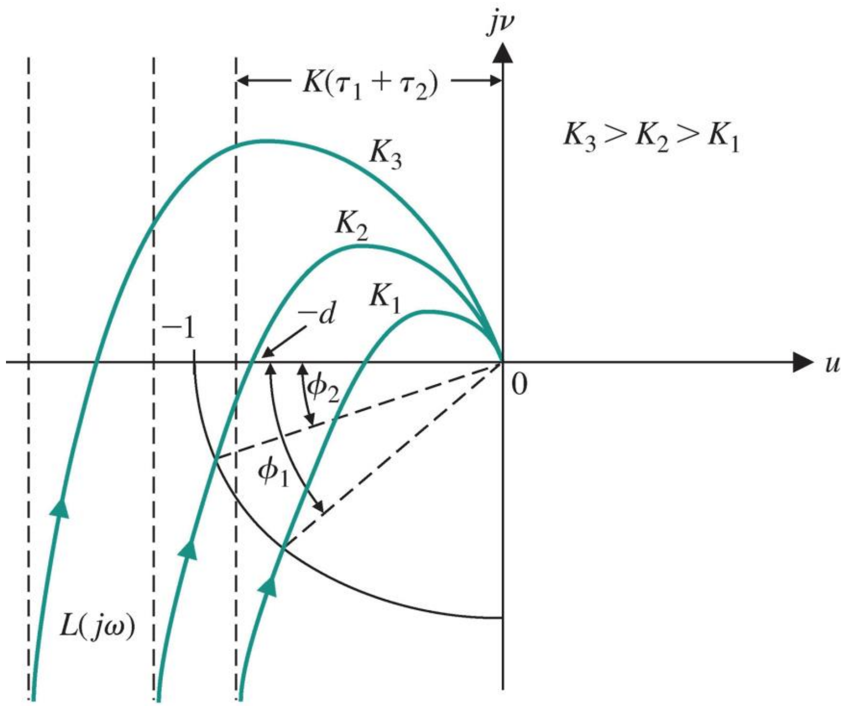
The intersection point of the contour with the u-axis can be also derived, which is less than -1, therefore, the contour encircles the (-1, 0) point **twice**, there must be **two** poles of the closed-loop system in the right-hand plane → the system is unstable.

Relative Stability and Nyquist Criterion

- Nyquist Criterion provides sufficient information concerning absolute stability, as well as **relative stability**.
- Relative stability related with the real part of the dominant complex poles; therefore, settling time is a measure of relative stability. A system with shorter settling time is considered as relatively more stable.
- Relative stability can be expressed on **Bode Plot** by using Gain Margin and Phase Margin.
- In **Nyquist Plot**, the contour $L(j\omega)$ to the critical point (-1, 0) is also a measure of relative stability. It can be also expressed as Gain Margin and Phase Margin.

Re-consider the closed-loop system with the following loop transfer function

$$L(s) = \frac{K}{s(\tau_1 s + 1)(\tau_2 s + 1)}$$



- The contour around the (-1, 0) point is mapped from $s = j\omega$, so $L(j\omega) = \frac{K}{j\omega(j\omega\tau_1+1)(j\omega\tau_2+1)}$
- The contour intersects the u-axis at $u = \frac{-K\tau_1\tau_2}{\tau_1+\tau_2}$; therefore, K_0 for which the contour passing through (-1, 0) is $K_0 = \frac{\tau_1+\tau_2}{\tau_1\tau_2}$ (marginally stable).
- The difference between K and K_0 is the gain margin.

Gain Margin and Phase Margin on Nyquist Plot

❖ **Gain Margin** is the increase in the system gain when phase = -180^o that will result in a marginally stable system with intersection of the $-1 + j0$ point on the Nyquist plot.

For example, for the system $L(s) = \frac{K_2}{s(\tau_1 s+1)(\tau_2 s+1)}$

$$G.M. = \frac{1}{|L(j\omega)|_{(when\ phase=-180^o)}} = [\frac{K_2\tau_1\tau_2}{\tau_1+\tau_2}]^{-1} = \frac{1}{d}$$

In logarithmic (decibel) form:

$$G.M. = 20\log\frac{1}{d} = -20\log d\ \text{ dB}$$

when $\tau_1 = \tau_2 = 1$ and $K_2 = 0.5$, $G.M. = 20\log 4 = 12$ dB.

❖ **Phase Margin** is the amount of phase shift of the $L(j\omega)$ at unity magnitude that will result in a marginally stable system with intersection of the $-1 + j0$ point on the Nyquist plot.

For the above system (when $K = K_2$), let magnitude = 1, find the corresponding phase, the difference between this phase and -180^o is the phase margin.

$$\emptyset_{PM} = \emptyset_2$$

Introducing Nichols Chart

Once the Nyquist Plot is obtained for the loop transfer function $L(j\omega)$, the magnitude and phase for the closed-loop transfer function $T(j\omega)$ can be derived for any ω.

$$T(j\omega) = \frac{L(j\omega)}{1 + L(j\omega)} \qquad \text{where } L(j\omega) = u + jv$$

The magnitude of the closed-loop transfer function is

$$M(\omega) = \left|\frac{u + jv}{1 + u + jv}\right| = \frac{(u^2 + v^2)^{1/2}}{[(1 + u)^2 + v^2]^{1/2}}$$

Squaring and rearranging, we obtain

$$(1 - M^2)u^2 + (1 - M^2)v^2 - 2M^2u = M^2$$

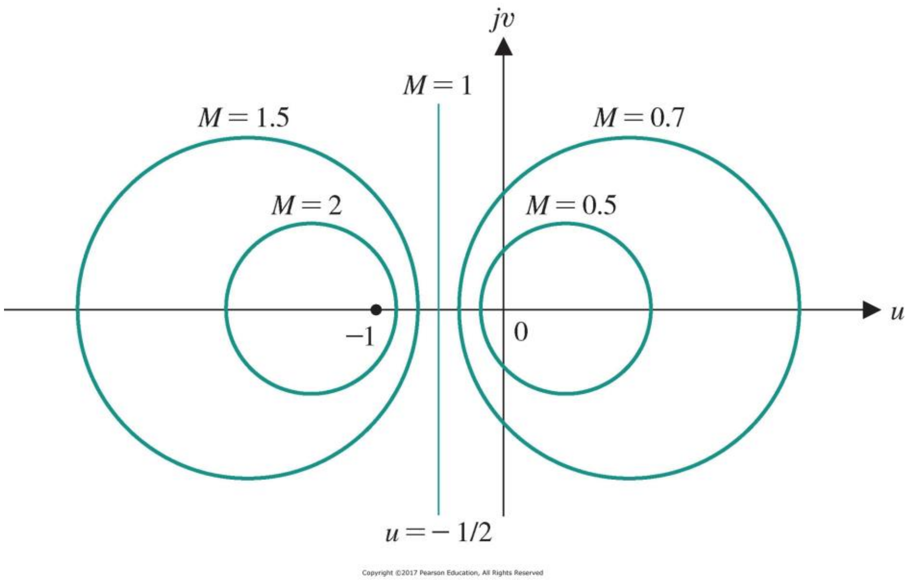
Dividing the equation by $1 - M^2$ and adding the term $[M^2/(1 - M^2)]^2$ to both sides, and rearranging, we have

$$\left(u - \frac{M^2}{1 - M^2}\right)^2 + v^2 = \left(\frac{M}{1 - M^2}\right)^2$$

It is the equation of a circle on the (u, v)-plane with the center at

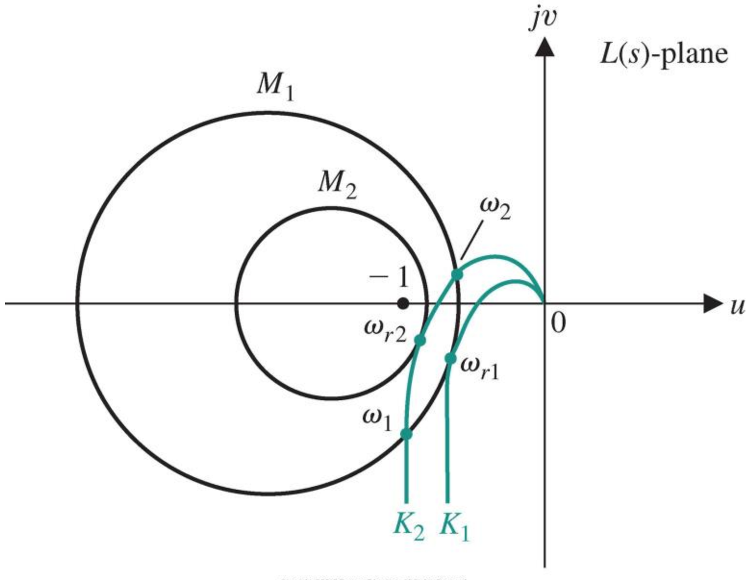
$$\left(u = \frac{M^2}{1 - M^2}, v = 0\right)$$

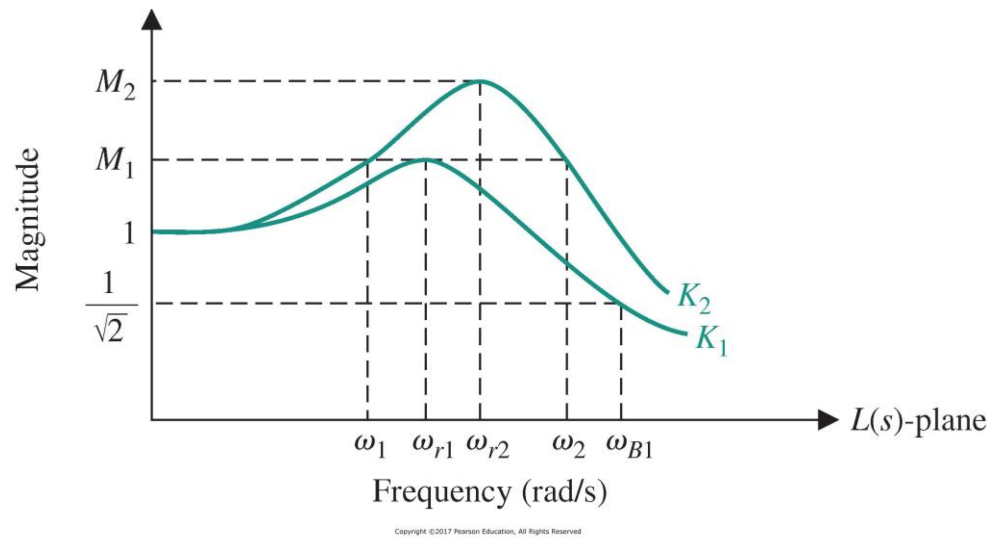
$$\left(u - \frac{M^2}{1 - M^2}\right)^2 + v^2 = \left(\frac{M}{1 - M^2}\right)^2$$



- Several constant M circles. Left of $u = -1/2$ are for $M > 1$, and the circle to the right of $u = -1/2$ are for $M < 1$.

- When $M = 1$, the circle becomes the straight line $u = -1/2$.

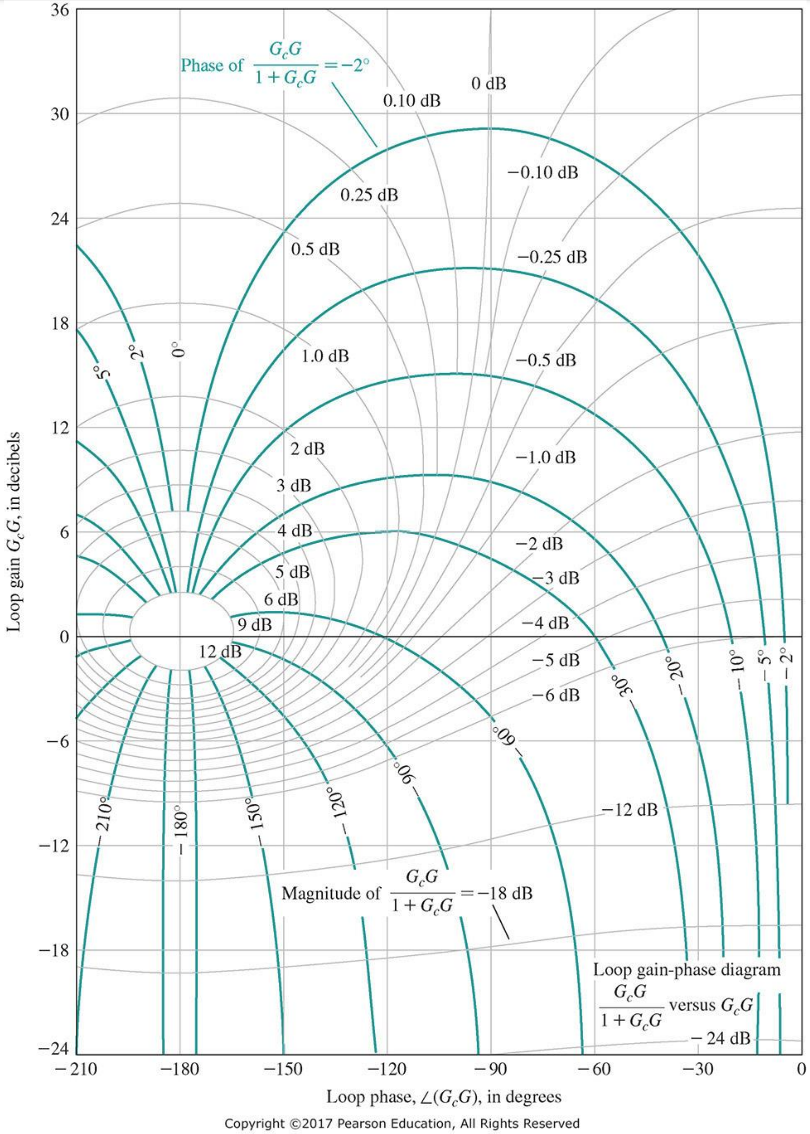




- The magnitude of $T(j\omega)$ can be read from the M circles and the Nyquist plot of $L(j\omega)$.
- The maximum magnitude, $M_{p\omega}$, is the value of the M circle that is tangent of the $L(j\omega)$-locus; the point of tangency occurs at the frequency ω_r, which is the resonant frequency.
- For the magnitude of other ω, it is the value of the M circle that intersects with the $L(j\omega)$-locus at ω.

Nichols Chart

- The phase of $T(j\omega)$ can be also expressed as N constant circles as magnitude.
- N. B. Nichols transformed the M and N circles to the log-magnitude-phase diagram, the resulting chart is called the **Nichols chart**.



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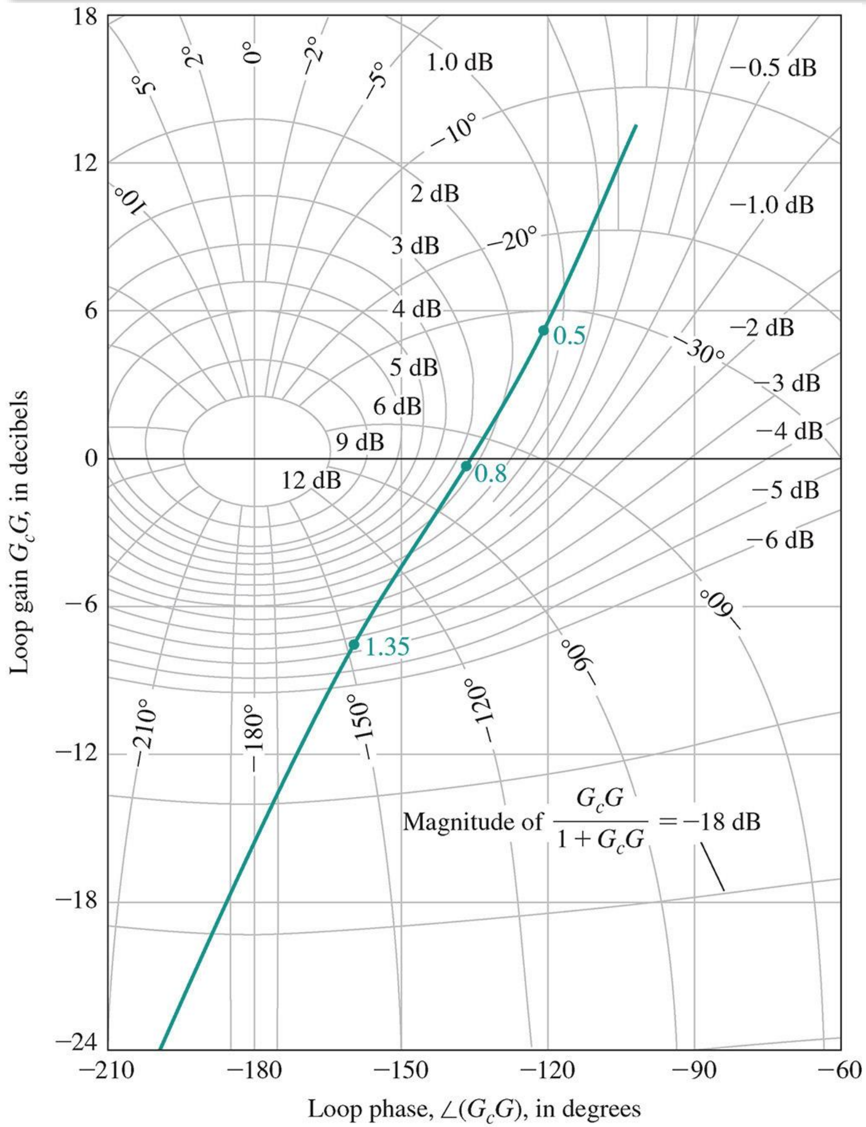
Using Nichols Chart



Green heavy line is the $L(j\omega)$-locus for system

$$L(j\omega) = \frac{1}{j\omega(j\omega+1)(0.2j\omega+1)}$$

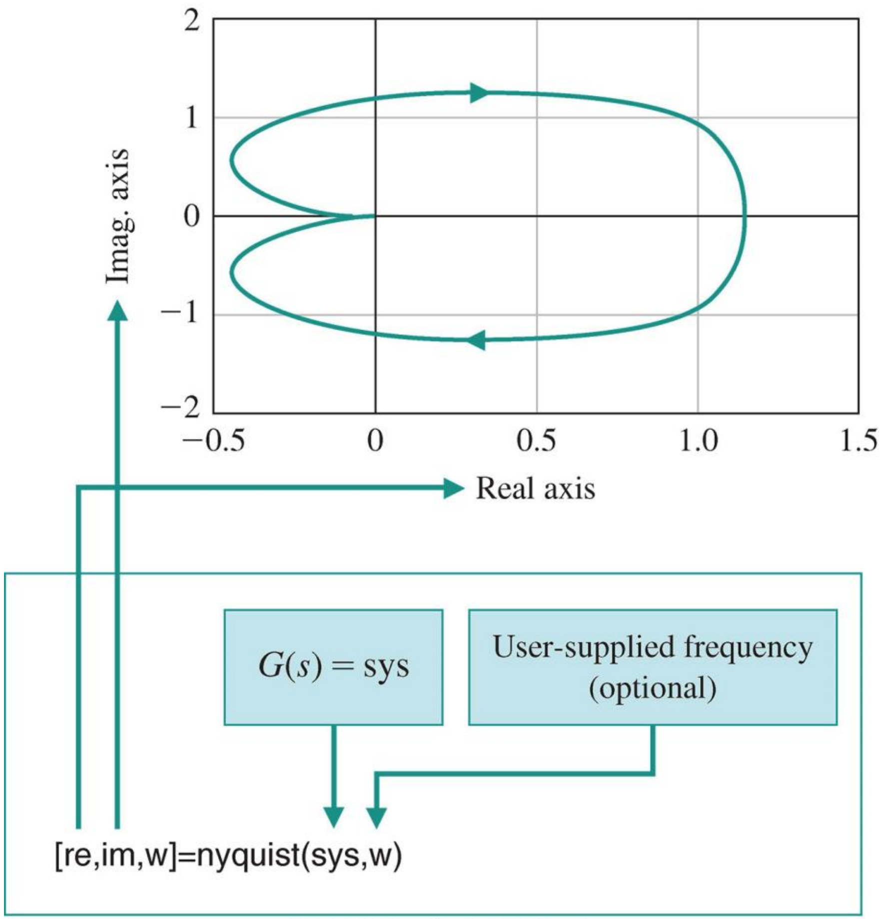
- for $\omega = 0.5$:
  $|T(j\omega)| \cong 1.4\text{ dB}, \angle T(j\omega) \cong -35^o$

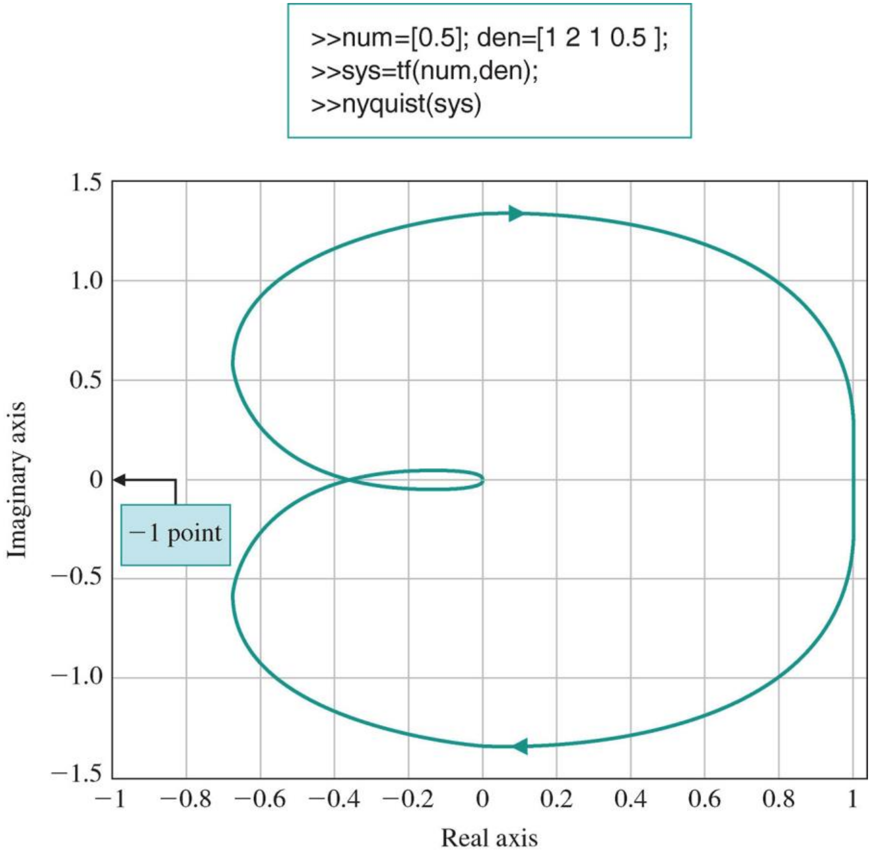
- for $\omega = 0.8$:
  (note it is also the resonant frequency)
  $M_{p\omega} = |T(j\omega)| \cong 2.5\text{ dB}, \angle T(j\omega) \cong -72^o$

- for $\omega = 1.35$:
  $|T(j\omega)| \cong \ ? \ , \angle T(j\omega) \cong \ ?$

Using Matlab for Frequency Response Analysis

The **nyquist** function.



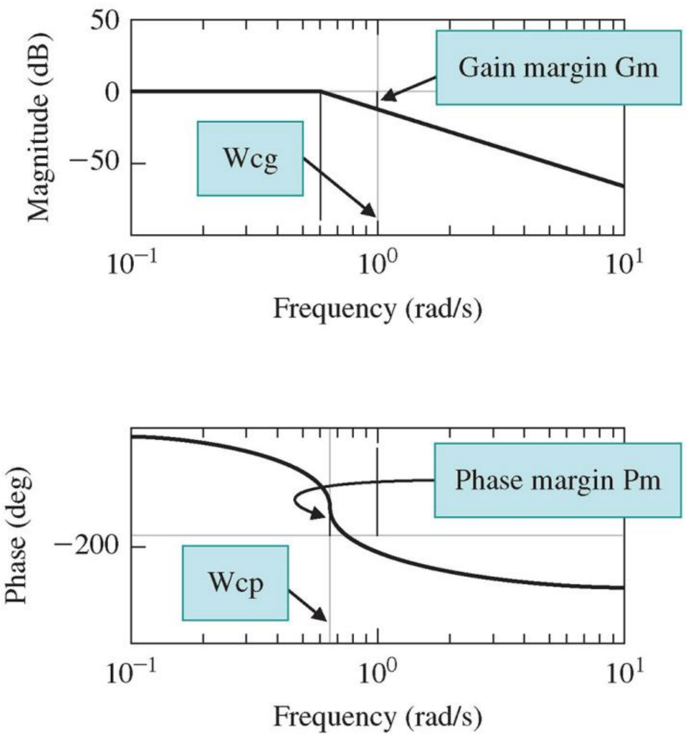


The **margin** function.

```
[mag,phase,w]=bode(sys);
[Gm,Pm,Wcg,Wcp]=margin(mag,phase,w);
```

or

```
[Gm,Pm,Wcg,Wcp]=margin(sys);
```

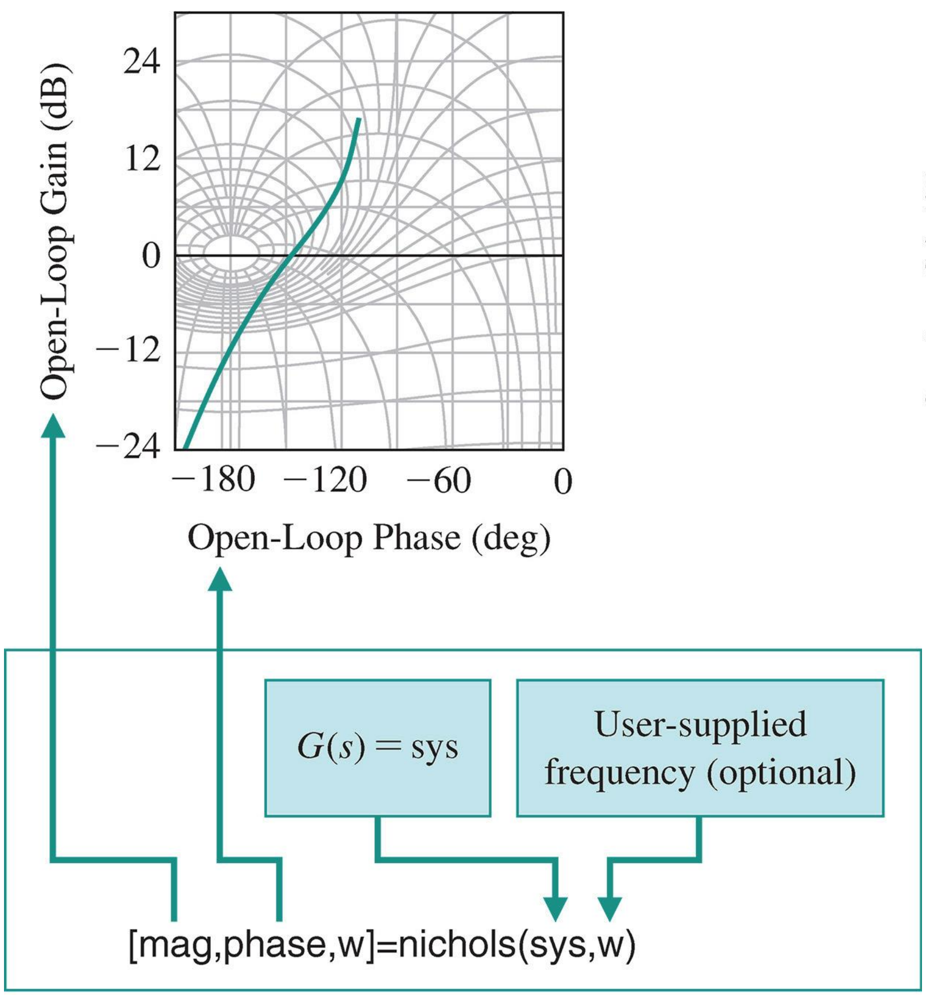


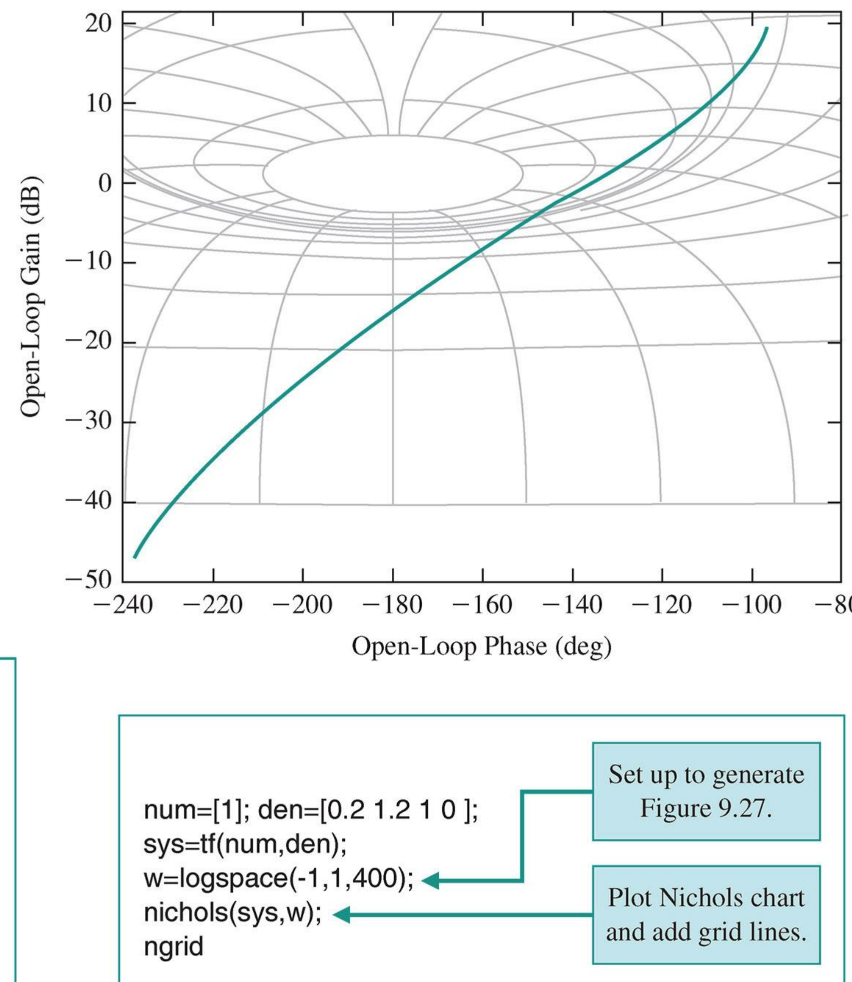
Example

```
num=[0.5]; den=[1 2 1 0.5];
sys=tf(num,den);
margin(sys);
```

Gm = gain margin (dB)
Pm = phase margin (deg)
Wcg = freq. for phase = −180
Wcp = freq. for gain = 0 dB

The **nichols** function.





Thank You！

-----------THE END-----------