MTH101: Tutorial 8

Dr. Tai-Jun Chen, Dr. Xinyao Yang

Xi'an Jiaotong-Liverpool University, Suzhou

October 30 – November 5, 2017

Exercise 1.1

Solve the initial value problem.

$$1. \quad y' + 2y = 4\cos 2x, \qquad y\left(\frac{\pi}{4}\right) = 3.$$

Solution.

1. We first find out that

$$p(x) = 2, \qquad r(x) = 4\cos 2x, \qquad h(x) = \int p dx = 2x.$$

The general solution to the ODE is thus

$$\begin{aligned} y(x) &= e^{-2x}\left(\int e^{2x} 4\cos 2x dx + c\right) \\ &= e^{-2x}\left[e^{2x}(\cos 2x + \sin 2x) + c\right] \\ &= (\cos 2x + \sin 2x) + ce^{-2x}. \end{aligned}$$

Since $y(\pi/4) = 3$, we can find that $c = 2e^{\pi/2}$.

Exercise 2.1

Use reduction of order to solve the following second order linear ODE.

$$1. \quad xy'' + 2y' + xy = 0, \quad y_1 = (\cos x)/x$$

Solution.

1. The standard form of the ODE is $y'' + \frac{2}{x}y' + y = 0$. From this form, we can find that $p(x) = 2/x$, $q(x) = 1$, and therefore if we know y_1, we can reduce the order of the ODE by letting
$y_2 = u(x)y_1$.
If we substitute this ansatz into the ODE, we can find that

$$U = u' = \frac{1}{y_1^2}e^{-\int p(x)dx} = \frac{x^2}{(\cos x)^2}e^{-2\ln x} = \frac{1}{(\cos x)^2},$$

therefore, $u = \int U(x)dx = \tan x$, and therefore $y_2 = \frac{\sin x}{x}$.
The most general to this ODE is thus $y = c_1\frac{\cos x}{x} + c_2\frac{\sin x}{x}$.

Exercise 2.2

Solve the initial value problems for the following equations.

1. $y'' + 4y' + (\pi^2 + 4)y = 0, \quad y\left(\frac{1}{2}\right) = 1, \quad y'\left(\frac{1}{2}\right) = -2,$

2. $y'' + 2k^2y' + k^4y = 0 \quad y(0) = 1, \quad y'(0) = -k^2.$

Solution.

1. The characteristic equation for it is

$$
\begin{aligned}
&\lambda^2 + 4\lambda + (\pi^2 + 4) = 0 \\
&\Rightarrow \lambda = -2 \pm i\pi \\
&\Rightarrow y = C_1 e^{(-2+i\pi)x} + C_2 e^{(-2-i\pi)x} \\
\textit{or} \quad & y = e^{-2x}\left(C_1^* \cos \pi x + C_2^* \sin \pi x\right).
\end{aligned}
$$

For this general solution, we have $y\left(\frac{1}{2}\right) = e^{-1} C_2^*$, $y'\left(\frac{1}{2}\right) = -e^{-1}\left(C_1^* \pi + 2C_2^*\right)$. With the initial values, we have $C_2^* = e$ and $C_1^* = 0$, therefore

$$y = e^{1-2x} \sin \pi x.$$

Solution.
2. The characteristic equation for it is

$$
\begin{aligned}
&\lambda^2 + 2k^2\lambda + k^4 = 0 \\
\Rightarrow &\lambda = -k^2 \\
\Rightarrow &y = (C_1 + C_2 x)\, e^{-k^2 x}.
\end{aligned}
$$

For this general solution, we have $y(0) = C_1$, $y'(0) = C_2 - k^2 C_1$. With the initial values, we have $C_2 = 0$ and $C_1 = 1$, therefore

$$y = e^{-k^2 x}.$$

Exercise 2.3

Solve the initial value problem.

1. $y'' + 6y' + 9y = e^{-x}\cos 2x, \qquad y(0) = 1, \quad y'(0) = -1.$

Solution.

1. This is a IVP with a nonhomogeneous linear ODE. We first solve the homogeneous ODE of this problem, $y'' + 6y' + 9y = 0$. Since the coefficients of the ODE are constant, we can use the ansatz $y = e^{\lambda x}$, and find the characteristic equation

$$\lambda^2 + 6\lambda + 9 = 0,$$

and there is a real double root $\lambda = -3$. Therefore, the general solution to the homogeneous ODE is thus $y_h = (c_1 + c_2 x)e^{-3x}$.

Solution.
In order to find the particular solution, one can check the table 2.1 in sec.2.7, and find that the choice of $y_p(x)$

$$y_p(x) = e^{-x}(A\cos 2x + B\sin 2x).$$

Therefore

$$\begin{aligned} y_p'(x) &= e^{-x}\left[(2B - A)\cos 2x - (B + 2A)\sin 2x\right] \\ y_p''(x) &= e^{-x}\left[-(4B + 3A)\cos 2x - (3B - 4A)\sin 2x\right]. \end{aligned}$$

Substitute them in to the ODE, one can find that $A = 0$, $B = \frac{1}{8}$, and the most general solution to the ODE is

$$y(x) = y_h(x) + y_p(x) = (c_1 + c_2 x)e^{-3x} + \frac{e^{-x}}{8}\sin 2x$$

Solution.
In order to find c_1, c_2, we need to use the initial values $y(0) = 1$, $y'(0) = -1$. We first calculate that

$$
\begin{aligned}
y(0) &= (c_1 + c_2 \cdot 0)e^0 + \frac{e^{-0}}{8} \sin 0 = c_1 \\
y'(0) &= c_2 e^0 + (-3)(c_1 + c_2 \cdot 0)e^0 - \frac{e^{-0} \sin 0}{8} + \frac{e^{-0} \cos 0}{4} \\
&= c_2 - 3c_1 + \frac{1}{4}.
\end{aligned}
$$

Therefore, $c_1 = 1$, and $c_2 = 3 - 1/4 - 1 = 7/4$, and the solution to the IVP is thus

$$y(x) = (1 + \frac{7x}{4})e^{-3x} + \frac{e^{-x}}{8} \sin 2x.$$