EEE225 Advanced Electrical Circuits and Electromagnetics

Lecture 2 Static Electric and Magnetic Fields

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- Maxwell's Equations

- 1. Gauss's law for electric field
- 2. Gauss's law for magnetic field
- 3. Electric field loop theorem
- 4. Magnetic field loop theorem -> Ampere's law

Maxwell's Equations

- Maxwell's equations are a set of four equations that describe properties of the electric and magnetic fields and relate them to their sources.

Law	Integral	Differential	Physical meaning
Gauss's law for **E**	$\oiint_S \boldsymbol{E}\cdot d\boldsymbol{s} = \frac{Q}{\varepsilon_0}$	$\nabla\cdot\boldsymbol{E} = \frac{\rho}{\varepsilon_0}$	Electric flux through a closed surface is proportional to the charges enclosed
Faraday's law	$\oint_C \boldsymbol{E}\cdot d\boldsymbol{l} = -\frac{d\Phi_B}{dt}$	$\nabla\times\boldsymbol{E} = -\frac{\partial\boldsymbol{B}}{\partial t}$	Changing magnetic flux produces an electric field
Gauss's law for **H**	$\oiint_S \boldsymbol{B}\cdot d\boldsymbol{s} = 0$	$\nabla\cdot\boldsymbol{B} = 0$	The total magnetic flux through a closed surface is zero
Generalized Ampere's law	$\oint_C \boldsymbol{H}\cdot d\boldsymbol{l} = I + \varepsilon_0\frac{d\Phi_E}{dt}$	$\nabla\times\boldsymbol{H} = \boldsymbol{J} + \frac{\partial\boldsymbol{D}}{\partial t}$	Electric current and changing electric flux produces a magnetic field

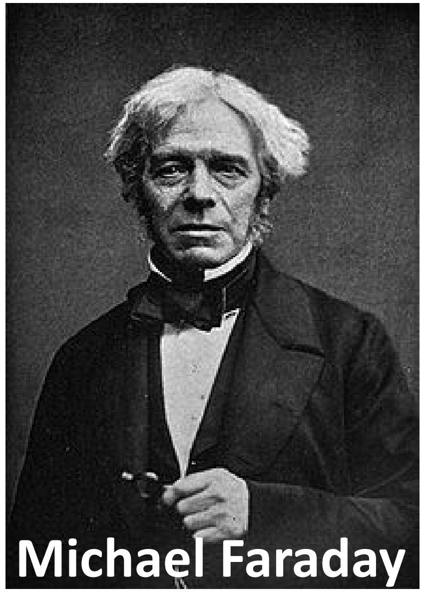
Electromagnetics – *Important people*

Carl Friedrich Gauss

André-Marie Ampère

James C. Maxwell

Hermann von Helmholtz



Michael Faraday

Maxwell's Equations for static fields

- However, the fields we learnt before were static, i.e. they are not changing according to time.

Law	Integral	Differential	Physical meaning
Gauss's law for **E**	$\oiint_S \boldsymbol{E} \cdot d\boldsymbol{s} = \frac{Q}{\varepsilon_0}$	$\boldsymbol{\nabla} \cdot \boldsymbol{E} = \frac{\rho}{\varepsilon_0}$	Electric flux through a closed surface is proportional to the charges enclosed
E-field loop theory	$\oint_C \boldsymbol{E} \cdot d\boldsymbol{l} = \mathbf{0}$	$\boldsymbol{\nabla} \times \boldsymbol{E} = \mathbf{0}$	Work done by moving a charge in the electric field along a closed loop is 0
Gauss's law for **H**	$\oiint_S \boldsymbol{B} \cdot d\boldsymbol{s} = 0$	$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$	The total magnetic flux through a closed surface is zero
Ampere's law	$\oint_C \boldsymbol{H} \cdot d\boldsymbol{l} = I$	$\boldsymbol{\nabla} \times \boldsymbol{H} = \boldsymbol{J}$	The magnetic field produced by an electric current is proportional to the current

1. Gauss's Law for Electric Field

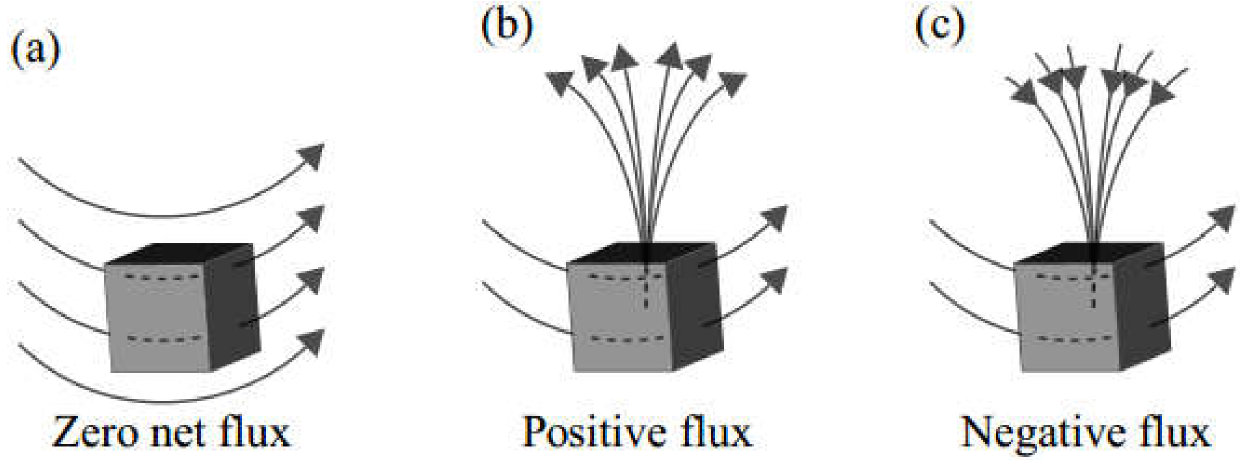
- The integral form of Gauss's Law for E-field:

$$\oiint_S \boldsymbol{E} \cdot d\boldsymbol{s} = \frac{Q}{\varepsilon_0}$$

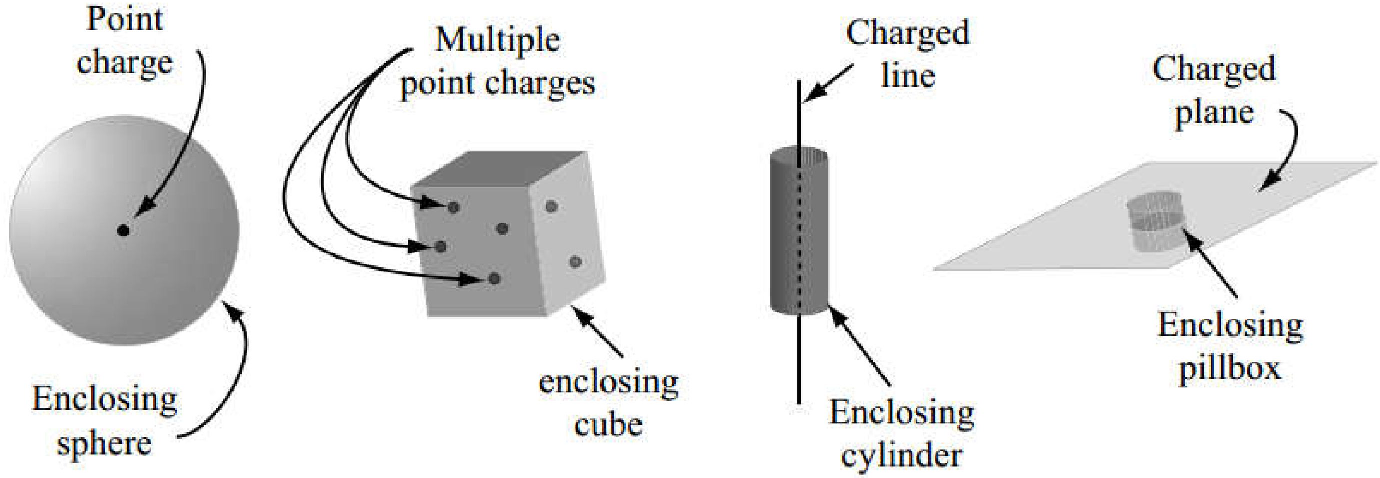
Electric charges produce an electric field, and the flux of that field passing through any closed surface is proportional to the total charge contained within that surface.

1. Gauss's Law for Electric Field

- Flux lines penetrating closed surfaces:



- Surfaces enclosing known charges:



1. Gauss's Law for Electric Field

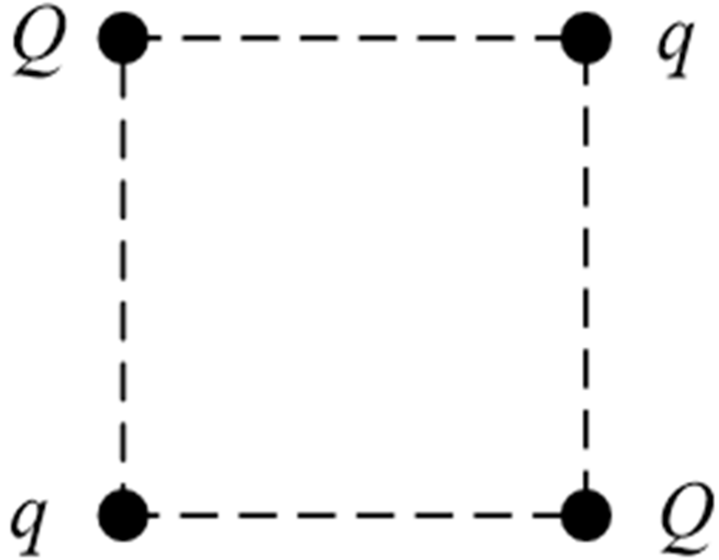
- **E** – Electric field intensity, or simply, E-field
- What is an electric field?
  - Michael Faraday: field of force;
  - James Maxwell: the space around an electrified object – a space in which electric forces act.
- A brief definition: the electric field at any location is the number of newtons of electrical force exerted on each coulomb of charge at that location.

$$\vec{E} = \lim_{q_0 \to 0} \frac{\vec{F}}{q_0} = \frac{\vec{F}}{q_0} \longrightarrow \boxed{\vec{F} = \frac{1}{4\pi\varepsilon_0} \frac{q q_0}{r^2} \hat{r}}$$

SI Unit: N/C = V/m

Quiz

- Two charges of size Q are placed on the diagonal corners of a square, and another two charges of size q are placed on the other two diagonal corners as shown below. If the total force on each charge Q is zero, find the relationship between Q and q.

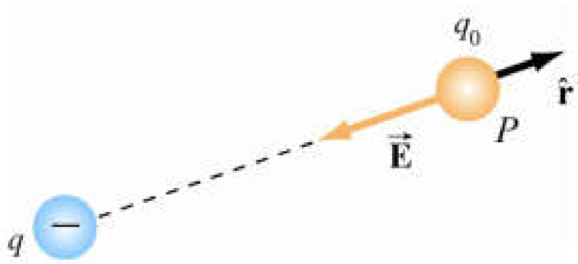


1. Gauss's Law for Electric Field

Key Concept - **E**

- For a point charge source q, the electric field intensity (short as E-field) is:

$$\vec{E} = \frac{q}{4\pi\varepsilon_0 r^2}\hat{a}_{12}$$

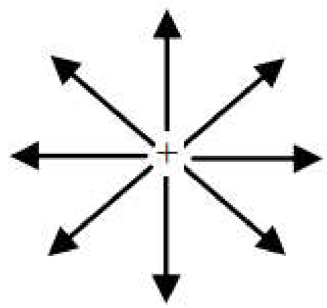


- The E-field also obeys the superposition law: the total electric field due to a group of charges is equal to the vector sum of the electric fields of individual charges.

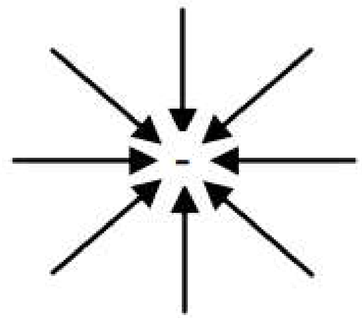
Point charge (charge $= q$)	$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}$ (at distance r from q)
Conducting sphere (charge $= Q$)	$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\hat{r}$ (outside, distance r from center) $\vec{E} = 0$ (inside)
Uniformly charged insulating sphere (charge $= Q$, radius $= r_0$)	$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\hat{r}$ (outside, distance r from center) $\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{Qr}{r_0^3}\hat{r}$ (inside, distance r from center)
Infinite line charge (linear charge density $= \lambda$)	$\vec{E} = \frac{1}{2\pi\varepsilon_0}\frac{\lambda}{r}\hat{r}$ (distance r from line)
Infinite flat plane (surface charge density $= \sigma$)	$\vec{E} = \frac{\sigma}{2\varepsilon_0}\hat{n}$

1. Gauss's Law for Electric Field

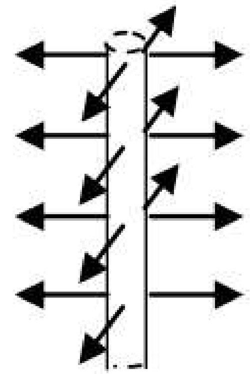
- It is often helpful to be able to visualize the electric field in the vicinity of a charged object.



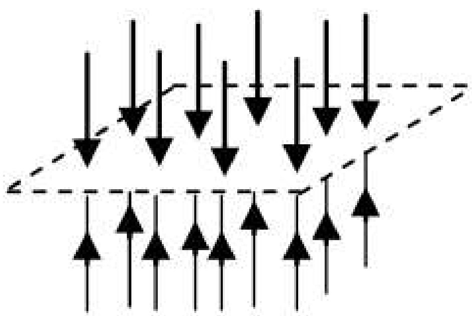
Positive point charge



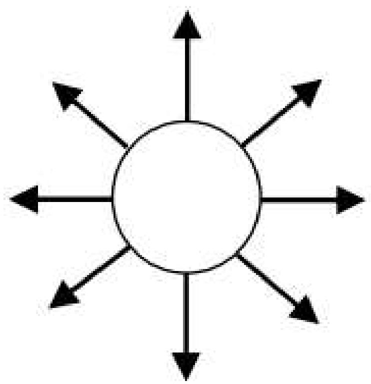
Negative point charge



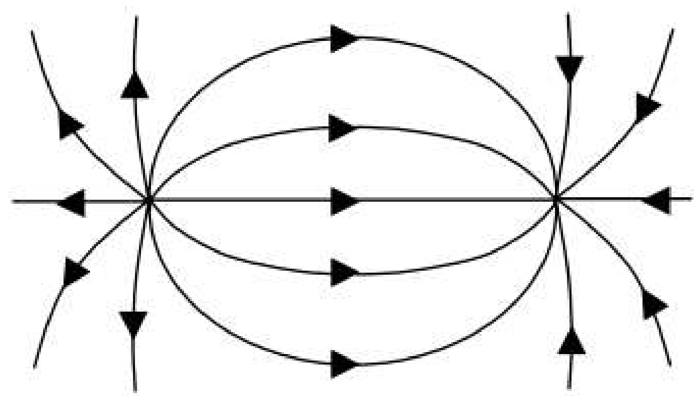
Infinite line of positive charge



Infinite plane of negative charge



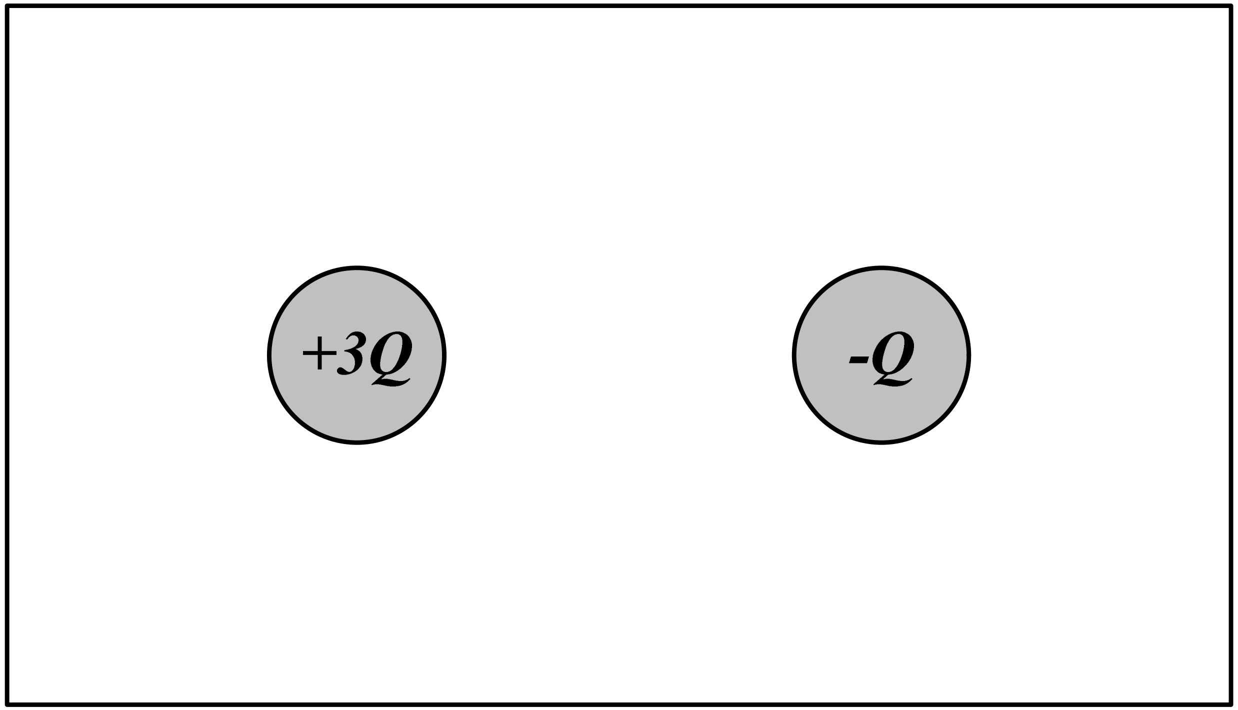
Positively charged conducting sphere



Electric dipole with positive charge on left

Quiz

- A positive point charge +3Q and a negative point charge -Q are placed in free space as shown below. Draw the electric field lines inside the boxed region.



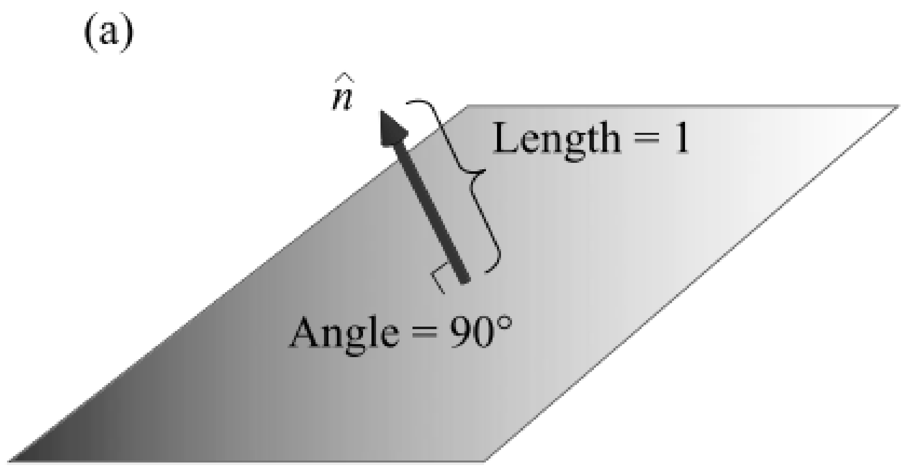
- Draw, sketch, plot, illustrate, …

1. Gauss's Law for Electric Field

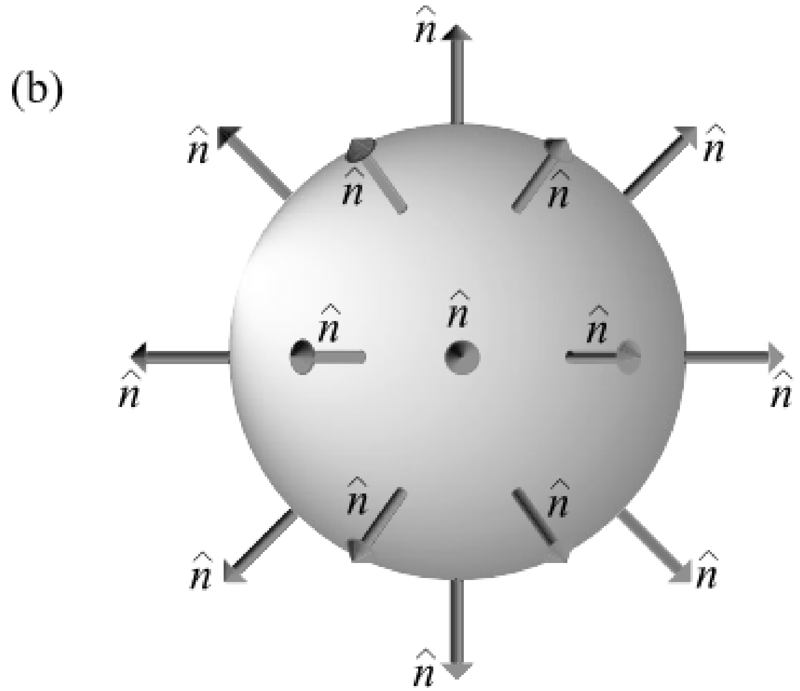
- The surface area element (segment) *ds* contains two parts:

$$d\vec{\boldsymbol{s}} = \hat{\boldsymbol{n}}ds$$

Unit normal vector   Area element



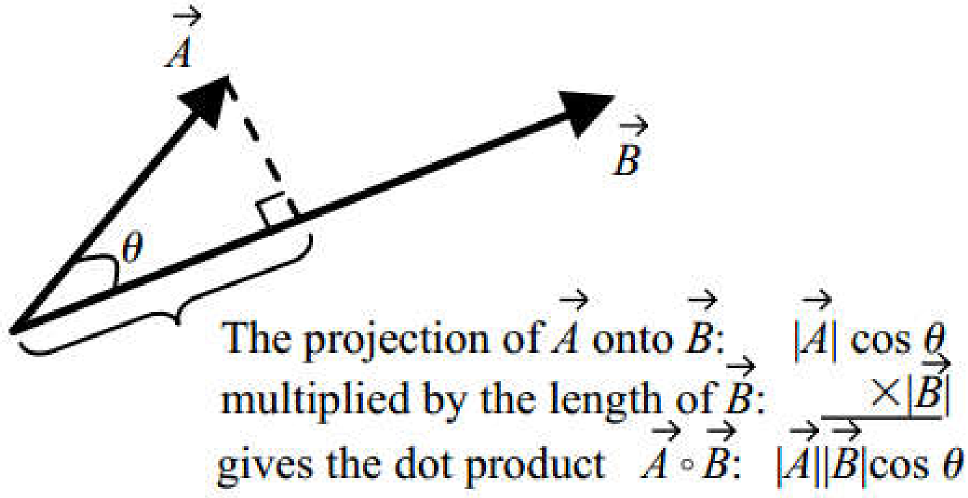
Planar surface



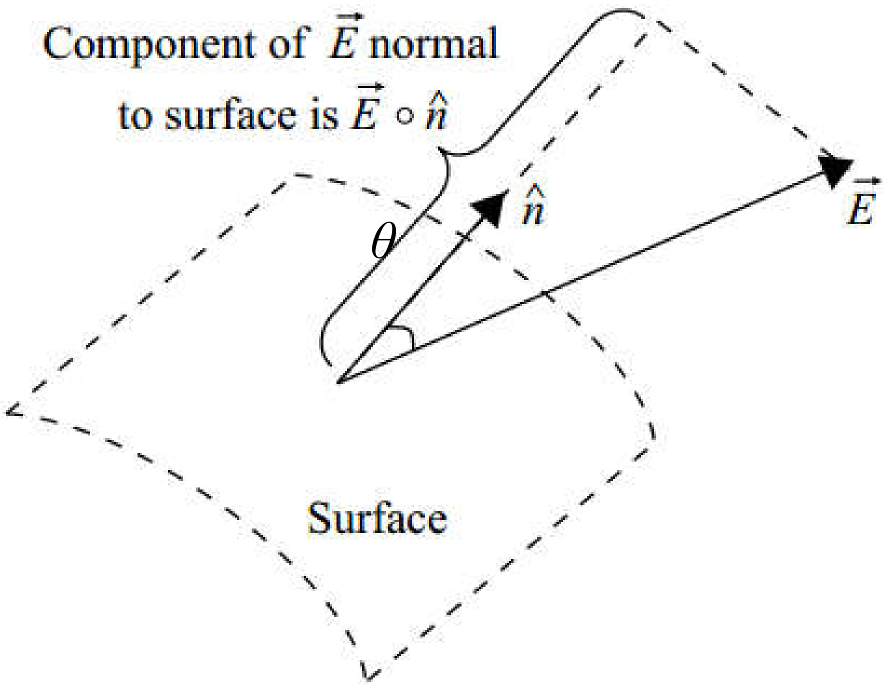
Spherical surface

1. Gauss's Law for Electric Field

- The dot product $\boldsymbol{A} \cdot \boldsymbol{B}$ represents the projection of $\boldsymbol{A}$ onto the direction of $\boldsymbol{B}$ multiplied by the length of $\boldsymbol{B}$.

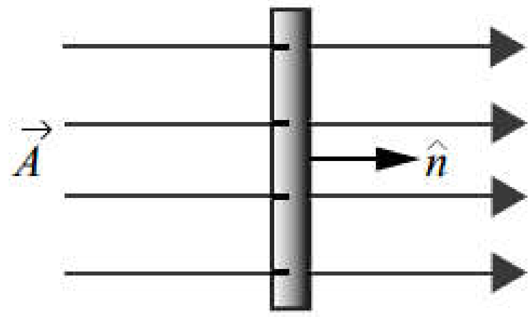


- $\boldsymbol{E} \cdot d\boldsymbol{s} = \boldsymbol{E} \cdot \hat{\boldsymbol{n}} ds$
  - $\boldsymbol{E} \cdot \hat{\boldsymbol{n}}$ represents the component of the electric field vector that is perpendicular to the surface under consideration.
  - So $\boldsymbol{E} \cdot \hat{\boldsymbol{n}} = E\cos\theta$



1. Gauss's Law for Electric Field

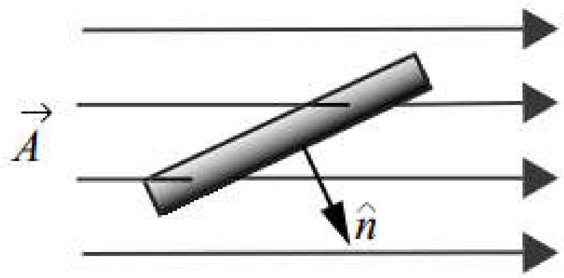
- In the simplest case of a uniform vector field **A** and a surface **S** perpendicular to the direction of the field, the flux Φ is defined as the product of the field magnitude and the area of the surface:



$$\Phi = \boldsymbol{A} \cdot \boldsymbol{S} = AS$$

  - $\boldsymbol{A}$ is perpendicular to the surface, i.e. parallel to the unit normal **n**, so $\boldsymbol{A} \cdot \hat{n} = A$.

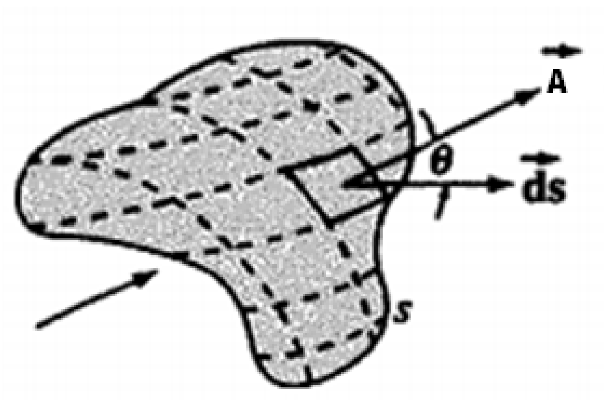
- If the vector field is uniform but is not perpendicular to the surface, the flux may be determined simply by finding the component of $\boldsymbol{A}$ perpendicular to the surface and then multiplying that value by the surface area:



$$d\Phi = \boldsymbol{A} \cdot d\hat{\mathbf{s}} = \boldsymbol{A} \cdot \hat{\mathbf{n}} ds$$

1. Gauss's Law for Electric Field

Flux through a surface



- A large, curved surface is divided into small vector areas, each one has an area of ds_i and direction $\mathbf{n_i}$;
- A vector field flows through this surface having different values and different directions at any point on the surface, illustrated as $\boldsymbol{A_i}$;
- So the flux flows through one area element is:

$$\Delta\Psi_i = \vec{\boldsymbol{A}}_i \cdot \Delta\vec{\boldsymbol{s}}_{\boldsymbol{i}}$$

- The total flux flows through the whole surface is:

$$\Psi = \sum \vec{\boldsymbol{A}}_i \cdot \Delta\vec{\boldsymbol{s}}_{\boldsymbol{i}}$$

- If the area elements are small enough, the summation becomes integration:

$$\Psi = \lim_{\Delta s_i \to 0} \sum \vec{\boldsymbol{A}}_i \cdot \Delta\vec{\boldsymbol{s}}_{\boldsymbol{i}} = \iint \vec{\boldsymbol{A}} \cdot d\vec{\boldsymbol{s}}$$

1. Gauss's Law for Electric Field

- ***Electric flux*** (Ψ), or ***displacement***, ***displacement flux***, is the number of field lines (Q) that penetrates a given surface.

- Properties of electric flux:
  - It must be independent of the medium;
  - Its magnitude solely depends on the charge from which it originates;
  - If a point charge is enclosed in an imaginary sphere of radius R, the electric flux must pass perpendicularly and uniformly through the surface of the sphere;
  - The ***electric flux density*** (**D**), the flux per unit area, is inversely proportional to R^2.
  - Therefore, for a point charge source, the electric flux density is:

$$\vec{\boldsymbol{D}} = \varepsilon_0 \vec{\boldsymbol{E}} = \frac{Q\hat{\boldsymbol{r}}}{4\pi r^2}$$

1. Gauss's Law for Electric Field

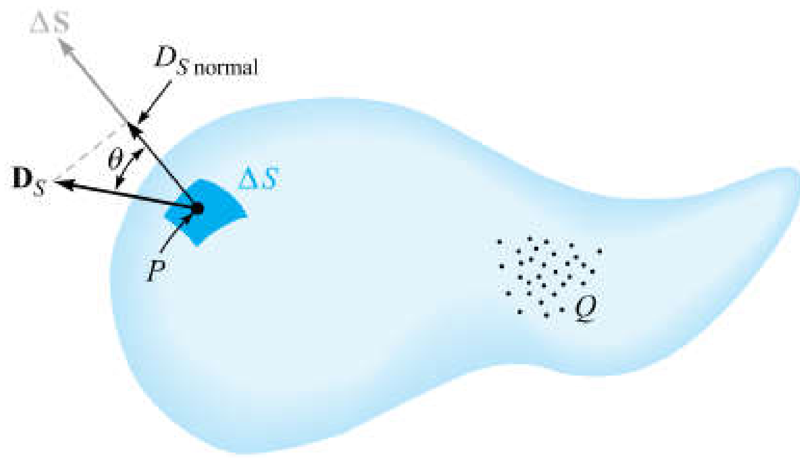
From **E** to **D**

- In free space, Gauss's law can be expressed in terms of electric field intensity:

$$\oiint \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{s}} = \frac{Q}{\varepsilon_0} = \Phi_E \neq \Psi$$

- More general Gauss's law states that the net outward flux passing through a closed surface is equal to the total charge enclosed by that surface. That is:

$$Q = \Psi = \oiint \vec{\boldsymbol{D}} \cdot d\vec{\boldsymbol{s}}$$



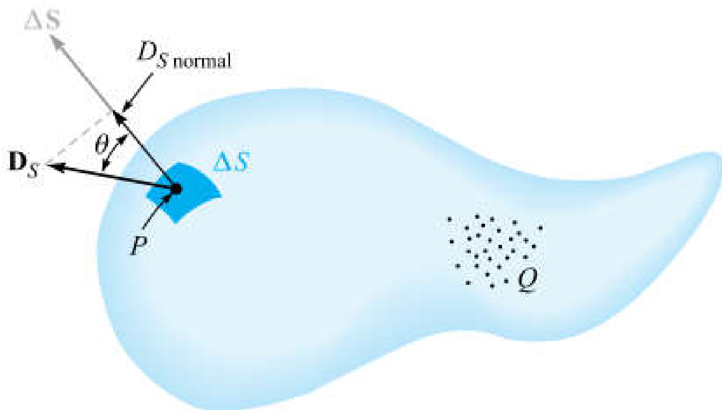
Integral form of Gauss's Law – *The total electric flux emanating from a closed surface is numerically equal to the net positive charge inside the closed surface.*

1. Gauss's Law for Electric Field

Integral equation

- Since $\boxed{\Psi = Q}$
  - where Q is the charge of a point source, which equals to the number of field lines.
- Right side of the equation can be extended to a volume of discrete or continuously distributed charges with the charge density ρ:

$$Q = \iiint_V \rho \cdot dv$$



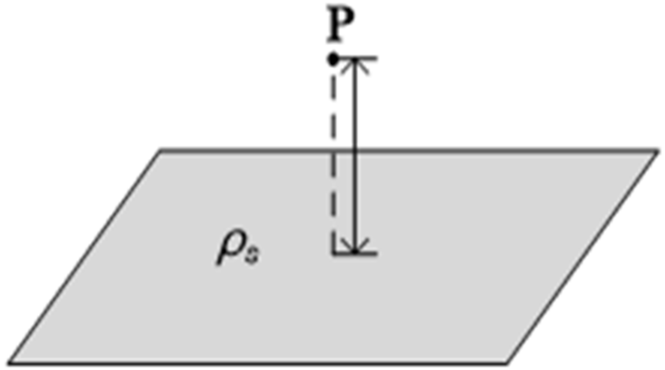
- Meanwhile, the total flux flow though a surface S which enclose this volume V is:

$$\Psi = \oint\!\!\!\oint \vec{\boldsymbol{D}} \cdot d\vec{\boldsymbol{s}}$$

$$\boxed{\oint\!\!\!\oint \vec{\boldsymbol{D}} \cdot d\vec{\boldsymbol{s}} = \iiint_V \rho \cdot dv}$$

Quiz

- 1. Find the electric field intensity E at a point P above the infinitely large conducting sheet as shown below. The surface charge density of the conducting sheet is +ρ.



- 2. Determine the **E** field both inside and outside a spherical cloud of electrons with a uniform volume charge density $\rho = -\rho_0$ (where ρ_0 is a positive quantity) for $0 \leq R \leq b$ and $\rho = 0$ for $R > b$.

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1. Gauss's Law for Electric Field

- The Differential form of Gauss's Law for E-field:

$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = \frac{\rho}{\varepsilon_0}$$

The electric field produced by electric charge diverges from positive charge and converges upon negative charge.

1. Gauss's Law for Electric Field

- "Divergence" is used to describe the mathematical operation that measures the rate at which electric field lines "flow" away from positive charge and toward points of negative electric charge.
- The mathematical definition of divergence may be understood by considering the flux through an infinitesimal surface surrounding the point of interest:

$$\operatorname{div}(\vec{A}) = \nabla \cdot \vec{A} = \lim_{\Delta V \to 0} \frac{1}{\Delta V} \oint_S \vec{A} \cdot \hat{n} ds$$

- More conveniently, it can be calculated by:
  - Cartesian: $\nabla \cdot \boldsymbol{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$
  - Cylindrical: $\nabla \cdot \boldsymbol{A} = \frac{1}{r}\frac{\partial (rA_r)}{\partial r} + \frac{1}{r}\frac{\partial A_\varphi}{\partial \varphi} + \frac{\partial A_z}{\partial z}$
  - Spherical: $\nabla \cdot \boldsymbol{A} = \frac{1}{R^2}\frac{\partial (R^2 A_R)}{\partial R} + \frac{1}{R sin\theta}\frac{\partial (A_\theta sin\theta)}{\partial \theta} + \frac{1}{R sin\theta}\frac{\partial A_\varphi}{\partial \varphi}$

1. Gauss's Law for Electric Field

- Consider the electric field of the positive point charge; the electric field lines originate on the positive charge and calculated by

$$\vec{\boldsymbol{E}} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{\boldsymbol{r}}$$

  - The divergence of the electric field is zero at all points away from the origin;
  - To evaluate the divergence at the origin, use the formal definition of divergence:

$$\nabla \cdot \vec{E} = \lim_{\Delta V \to 0}\left(\frac{1}{\Delta V}\frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\oint_S ds\right) = \lim_{\Delta V \to 0}\left(\frac{1}{\Delta V}\frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}(4\pi r^2)\right)$$

$$= \lim_{\Delta V \to 0}\left(\frac{1}{\Delta V}\frac{q}{\varepsilon_0}\right) = \frac{1}{\varepsilon_0}\lim_{\Delta V \to 0}\frac{q}{\Delta V} = \frac{\rho}{\varepsilon_0}$$

  - In the case of a point charge at the origin, the flux through an infinitesimally small surface is nonzero only if that surface contains the point charge. Everywhere else, the flux into and out of that tiny surface must be the same (since it contains no charge), and the divergence of the electric field must be zero.

1. Gauss's Law for Electric Field

- How to link the integral and differential forms together?

Recall the "Divergence Theorem", also known as "Gauss's Theorem": $\iiint_V \nabla \cdot \boldsymbol{A} dv = \oiint_S \boldsymbol{A} \cdot d\boldsymbol{s}$

- Therefore, we have

$$\oiint_S \vec{\boldsymbol{D}} \cdot d\vec{\boldsymbol{s}} = \boxed{\iiint_V \nabla \cdot \vec{\boldsymbol{D}} dv = \iiint_V \rho \cdot dv}$$

  – This must be true for any volume *v* bounded by a surface *s*, so the two integrands must be equal.

- Thus at any point in space, we have $\boxed{\nabla \cdot \vec{\boldsymbol{D}} = \rho}$
  – In free space, it can also be written as $\nabla \cdot \vec{\boldsymbol{E}} = \rho/\varepsilon_0$

Quiz

- The electric flux density in the region $r \leq 0.08$ m is

$$\boldsymbol{D} = 5r^2\hat{\boldsymbol{r}}\ \mathrm{mC/m^2}.$$

  – (a) Find the volume charge density ρ_v for $r = 0.06$ m;

  – (b) To make $\boldsymbol{D} = 0$ for $r > 0.08$ m, what surface charge density could be located at $r = 0.08$ m?

2. Gauss's Law for Magnetic Field

- The integral form of Gauss's Law for magnetic field:

$$\oiint_S \boldsymbol{B} \cdot d\boldsymbol{s} = 0$$

The total magnetic flux passing through any closed surface is zero.

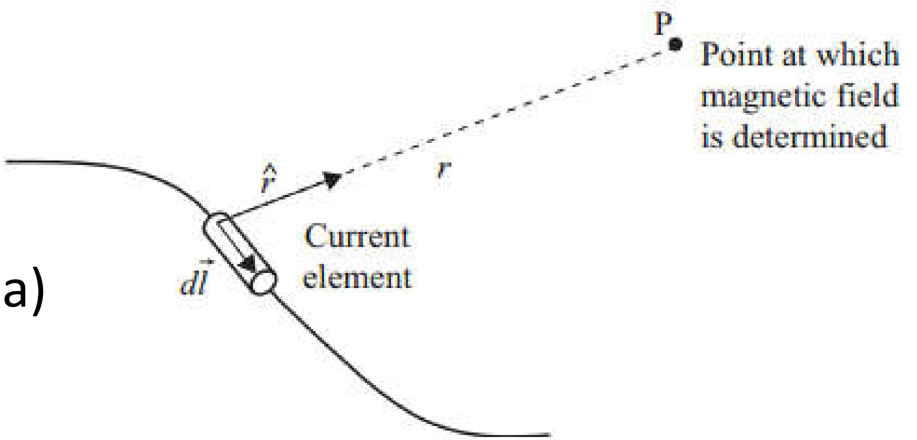
- Gauss's law for magnetic fields arises directly from the lack of isolated magnetic poles ("magnetic monopoles") in nature.
  - To date, all efforts to detect magnetic monopoles have failed, and every magnetic north pole is accompanied by a magnetic south pole, no matter how small they are;
  - In other words, if you have a real or imaginary closed surface of any size or shape, the total magnetic flux through that surface must be 0.

2. Gauss's Law for Magnetic Field

Key Concept - **B**

- **B** – "Magnetic flux density" or "Magnetic induction"
- What is a magnetic field?
  - Also a field of force;
  - More precisely – a space in which the magnetic force experienced by a moving charged particle.
- Source of a magnetic field: electric current element $Id\vec{\boldsymbol{l}}$
  - The relationship is given in Biot-Savart Law:

$$\vec{B} = \int_L d\vec{B} \qquad \boxed{d\vec{B} = \frac{\mu_0}{4\pi}\frac{Id\vec{l} \times \hat{r}}{r^2}}$$



SI Unit: N/(Cm/s) = N/(Am) = Vs/m^2 = T (Tesla)

2. Gauss's Law for Magnetic Field

Key Concept - **B**

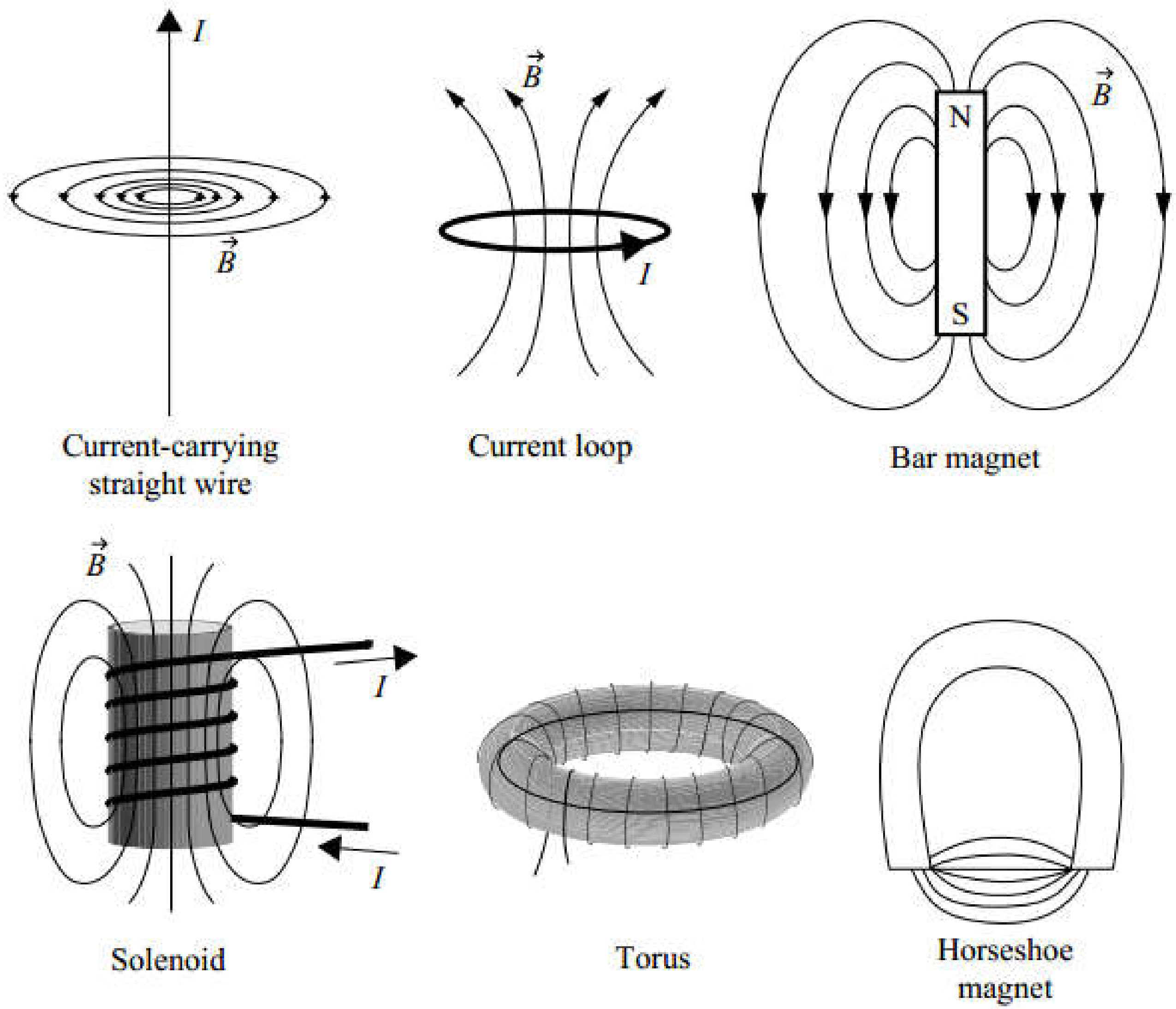
- Magnetic field equations for some typical

Infinite straight wire carrying current I (at distance r)	$\vec{B} = \dfrac{\mu_0 I}{2\pi r}\hat{\varphi}$
Segment of straight wire carrying current I (at distance r)	$d\vec{B} = \dfrac{\mu_0}{4\pi}\dfrac{I d\vec{l} \times \hat{r}}{r^2}$
Circular loop of radius R carrying current I (loop in yz plane, at distance x along x-axis)	$\vec{B} = \dfrac{\mu_0 I R^2}{2(x^2 + R^2)^{3/2}}\hat{x}$
Solenoid with N turns and length l carrying current I	$\vec{B} = \dfrac{\mu_0 N I}{l}\hat{x}$ (inside)
Torus with N turns and radius r carrying current I	$\vec{B} = \dfrac{\mu_0 N I}{2\pi r}\hat{\varphi}$ (inside)

2. *Gauss's Law for Magnetic Field*

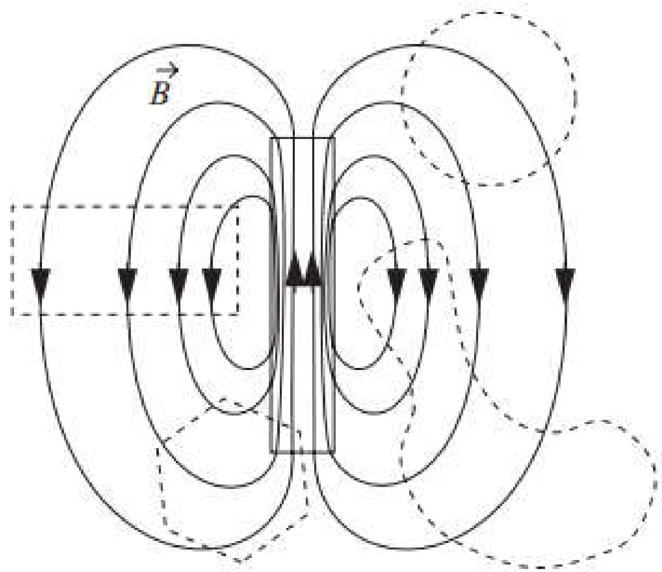
Visualization of **B** field

- It is often helpful to be able to visualize the magnetic field in the vicinity of a current source:



2. Gauss's Law for Magnetic Field

Flux through a closed surface



- Like the electric flux Φ_E, the magnetic flux Φ_B through a surface may be thought of as the "amount" of magnetic field "flowing" through the surface.
  - When you think about the number of magnetic field lines through a surface, don't forget that magnetic fields are actually continuous in space, and that "number of field lines" only has meaning once you've established a relationship between the number of lines you draw and the strength of the field.
  - No matter what shape of surface you choose, and no matter where in the magnetic field you place that surface, you'll find that the number of field lines entering the volume enclosed by the surface is exactly equal to the number of field lines leaving that volume.

    The physical reasoning behind Gauss's law should now be clear: the net magnetic flux passing through any closed surface must be zero because magnetic field lines always form complete loops.

2. Gauss's Law for Magnetic Field

Recall the "Divergence Theorem", also known as "Gauss's Theorem" again: $\iiint_V \nabla \cdot \boldsymbol{A} dv = \oiint_S \boldsymbol{A} \cdot d\boldsymbol{s}$

- Therefore, we can link the integral from to differential form:

$$\iiint_V \nabla \cdot \vec{\boldsymbol{B}} dv = \oiint_S \vec{\boldsymbol{B}} \cdot d\vec{\boldsymbol{s}} = 0$$

  - This must be true for any volume *v* bounded by a surface *s*, so the two integrands must be equal.

- Thus at any point in space, we have $\boxed{\nabla \cdot \vec{\boldsymbol{B}} = 0}$

***Differential form of Gauss's Law for magnetism –** the divergence of the magnetic field – the tendency of the magnetic field to either "flow" away or towards a point, is zero.*

2. Gauss's Law for Magnetic Field

- Take the magnetic field around a long, current-carrying wire as example:
  - The divergence of B is $\boldsymbol{\nabla} \cdot \boldsymbol{B} = \boldsymbol{\nabla} \cdot \left(\frac{\mu_0 I}{2\pi r}\hat{\varphi}\right)$
  - In cylindrical coordinates

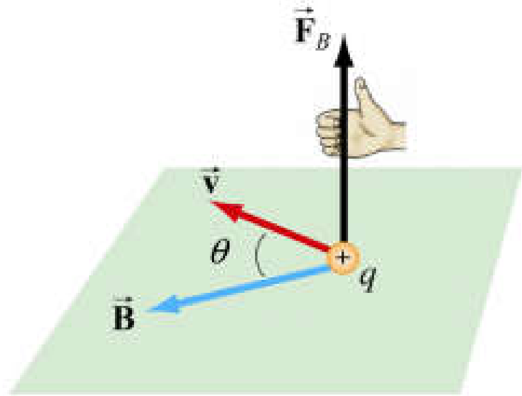
$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = \frac{1}{r}\frac{\partial}{\partial r}(rB_r) + \frac{1}{r}\frac{\partial B_\varphi}{\partial \varphi} + \frac{\partial B_z}{\partial z} = \frac{1}{r}\frac{\partial\left(\frac{\mu_0 I}{2\pi r}\right)}{\partial \varphi} = 0$$

- Vector fields with zero divergence are called "solenoidal" fields, and <u>all static magnetic fields are solenoidal</u>.

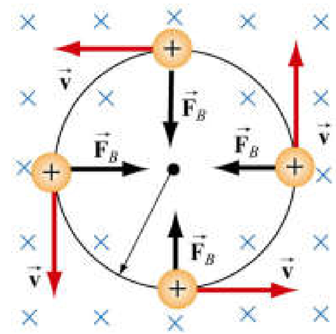
2. Gauss's Law for Magnetic Field

- 1. Magnetic forces on a moving charge:
  - Experiment shows:
    - $\mathbf{F_B}$ is perpendicular to $\boldsymbol{v}$ and $\mathbf{B}$;
    - F_B is proportional to v, q and B;

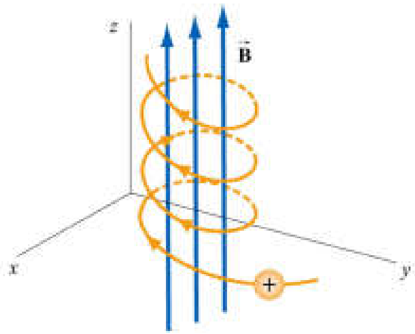
$$\Rightarrow \overrightarrow{F_B} = q(\vec{v} \times \vec{B})$$



  - This is the Lorentz force



  - What would happen if a charged particle moves through a uniform magnetic field with its initial velocity $\boldsymbol{v}$ at a right angle to $\mathbf{B}$?



  - What if the initial velocity of the charged particle has a component parallel to the magnetic field $\mathbf{B}$?

2. Gauss's Law for Magnetic Field

Lorentz force

- When a charged particle moves in electric (**E**) and magnetic (**B**) fields, it experiences a total force $\mathbf{F_L}$ exerted by both fields: $\boldsymbol{F_L} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})$

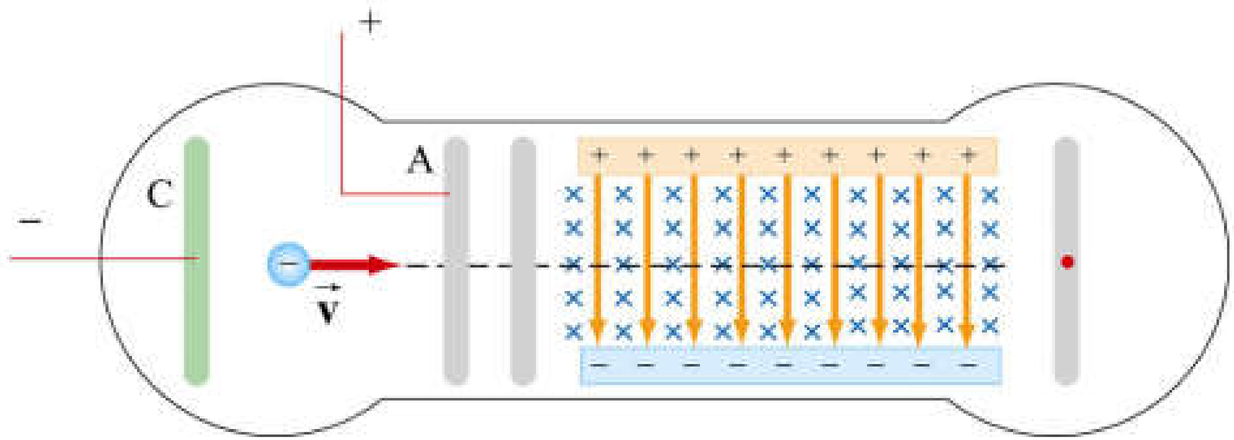
$$\vec{\mathbf{F}}_{\text{Lorentz}} = \vec{\mathbf{F}}_E + \vec{\mathbf{F}}_B = q\vec{\mathbf{E}} + q\left(\vec{\mathbf{v}} \times \vec{\mathbf{B}}\right)$$

Electric Force | Magnetic Force

Electric force:	Magnetic force:
• always in the direction of the electric field • acts on a charged particle whether or not it is moving • expends energy in displacing a charged particle	• always perpendicular to the magnetic field. • acts on it only when it is in motion. • does no work when a particle is displaced

Quiz

- In a velocity selector, by combining the two fields, particles which move with a certain velocity can be selected. This was the principle used by J. J. Thomson to measure the charge-to-mass ratio of the electrons.
  - The electrons with charge $q = -e$ and mass m are emitted from the cathode C and then accelerated toward slit A. Let the potential difference between A and C be ΔV, and the electric and magnetic field in the apparatus are set as E and B.
  - To make the electron moving in straight line, what the accelerating voltage ΔV should be?



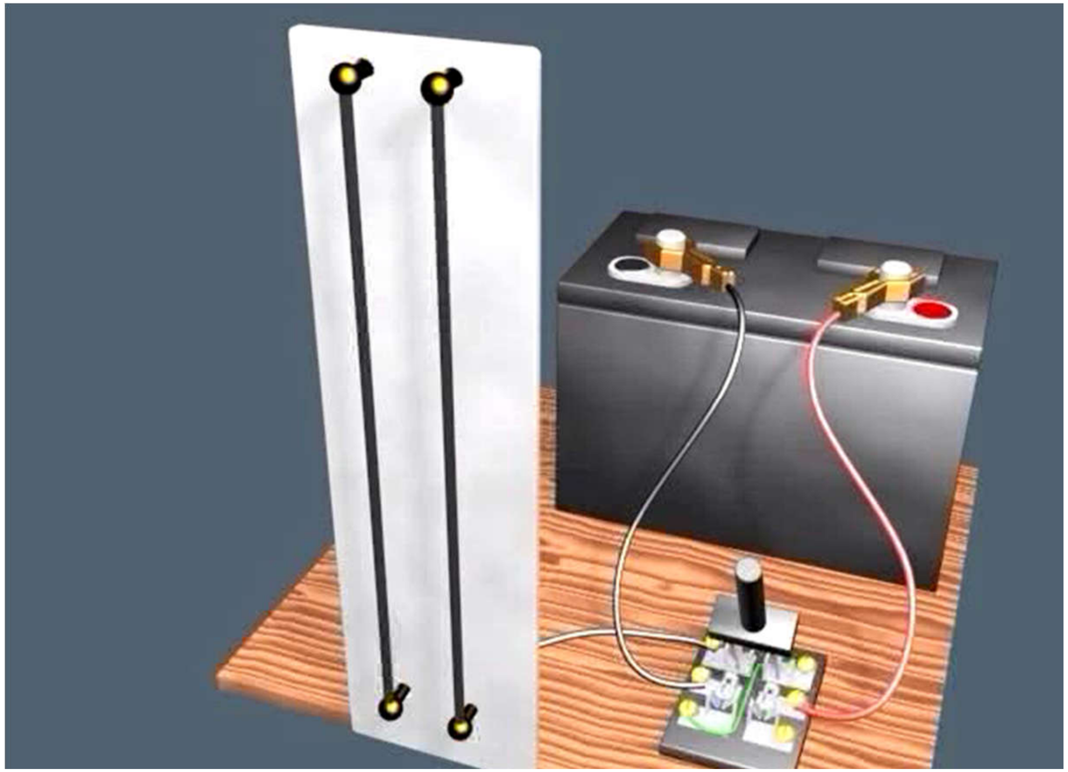
2. Gauss's Law for Magnetic Field

- 2. Magnetic forces on current carrying wires:

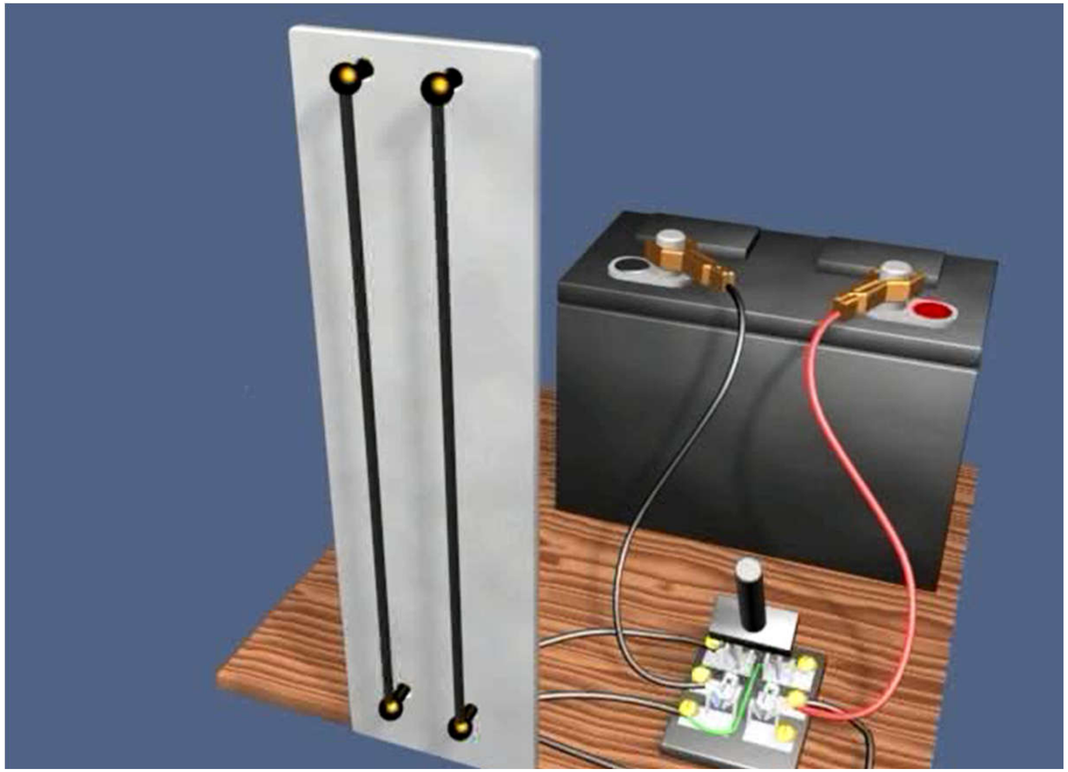
A current-carrying wire produces a magnetic field.

In a magnetic field, a wire carrying a current experiences a net force.

Two current-carrying wires to exert force on each other.



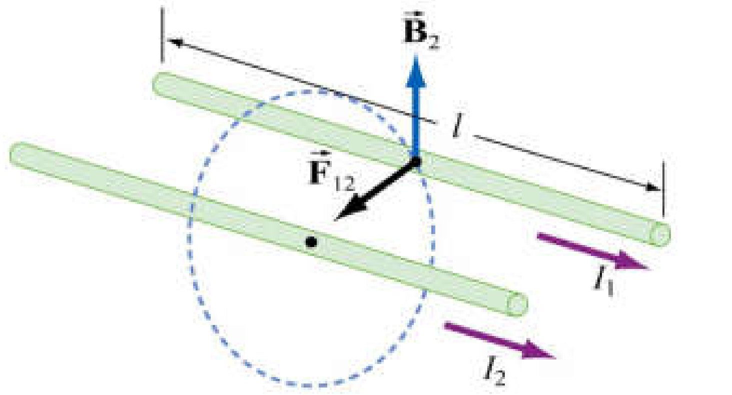
Parallel – current in same direction



Series – current in different direction

2. Gauss's Law for Magnetic Field

Two parallel wires



- The magnetic force, $\mathbf{F}_{12}$, exerted on wire 1 by wire 2 is generated by the magnetic field lines due to I_2

- At an arbitrary point *P* on wire 1, we have $\vec{\boldsymbol{B}}_2 = -(\frac{\mu_0 I_2}{2\pi a})\hat{\boldsymbol{j}}$, which points in the direction ⊥ to wire 1.

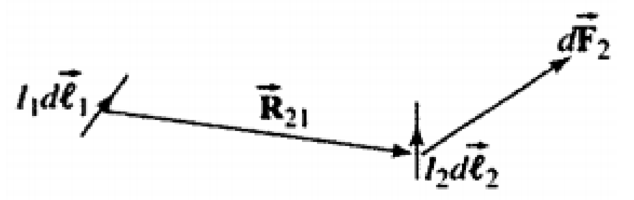
- Therefore

$$\vec{\boldsymbol{F}}_{12} = I_1 \vec{\boldsymbol{l}} \times \vec{\boldsymbol{B}}_2 = I_1 l \hat{\boldsymbol{\imath}} \times \left(-\frac{\mu_0 I_2}{2\pi a}\hat{\boldsymbol{j}}\right) = -\frac{\mu_0 I_1 I_2}{2\pi a}\widehat{\boldsymbol{k}}$$

  - Clearly $\mathbf{F}_{12}$ points toward wire 2.
  - Two parallel wires carrying currents in the same direction will attract each other.
  - If the currents flow in opposite directions, the resultant force will be repulsive.

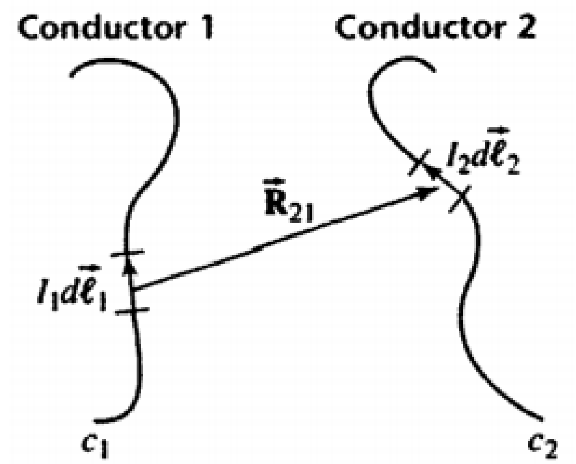
2. Gauss's Law for Magnetic Field

Ampere's force law



- Consider two current carrying elements $I_1 d\boldsymbol{l_1}$ and $I_2 d\boldsymbol{l_2}$ interact, the elemental force exerted by element 1 upon element 2 is:

$$\boldsymbol{d\vec{F}_2} = \frac{\mu_0 I_2 d\overrightarrow{\boldsymbol{l_2}}}{4\pi} \times \left[\frac{I_1 d\overrightarrow{\boldsymbol{l_1}} \times \vec{\boldsymbol{R}}_{21}}{R_{21}^3}\right]$$

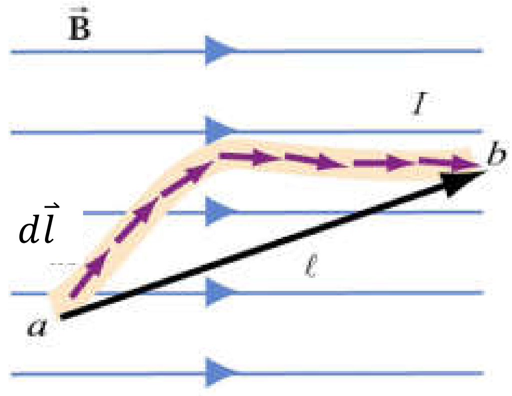


- If each current-carrying element is a part of a current-carrying conductor, the magnetic force exerted by conductor 1 upon conductor 2 is

$$\vec{\boldsymbol{F}}_2 = \frac{\mu_0}{4\pi} \int_{c_2} I_2 d\overrightarrow{\boldsymbol{l_2}} \times \int_{c_1} \frac{I_1 d\overrightarrow{\boldsymbol{l_1}} \times \vec{\boldsymbol{R}}_{21}}{R_{21}^3} = \int_{c_2} I_2 d\overrightarrow{\boldsymbol{l_2}} \times \vec{\boldsymbol{B}}_1$$

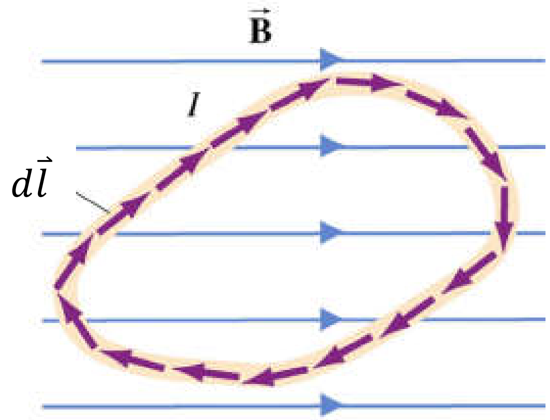
2. Gauss's Law for Magnetic Field

Ampere's force law



- In general, when a current-carrying conductor is placed in an external magnetic field **B**, the magnetic force exerted on it is:

$$\vec{F}_B = I\left(\int_a^b d\vec{l}\right) \times \vec{B} = I\vec{l} \times \vec{B}$$



- If the wire forms a closed loop of arbitrary shape, then the force on the loop becomes

$$\vec{F}_B = I\left(\oint d\vec{l}\right) \times \vec{B} = 0$$

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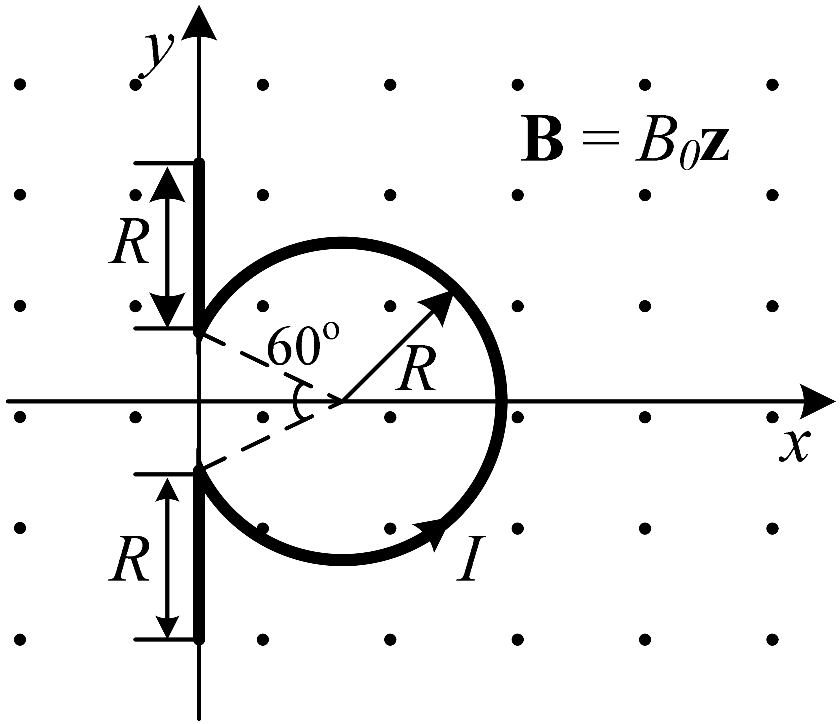
Quiz

- A wire, bent as shown below, lies in the xy plane and carries a current I. If the magnetic flux density in the region is $\mathbf{B} = B_0\mathbf{z}$, determine an expression for the magnetic force acting on the wire, explain your answer.



3. Electric Field Loop Theorem

- The integral form of the E-field loop theorem:

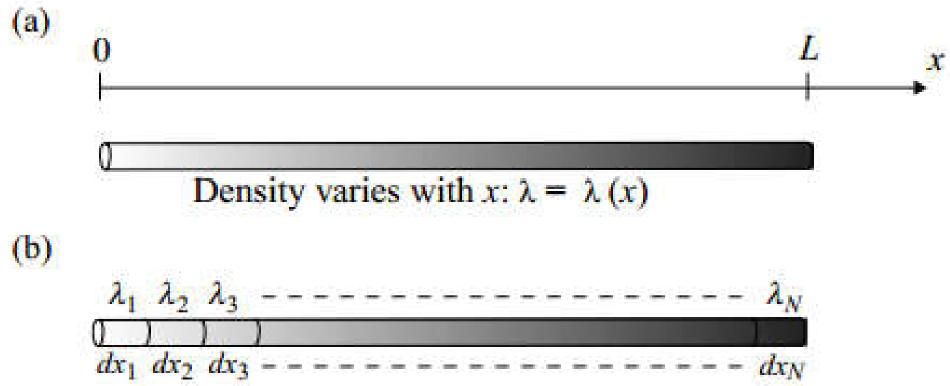
$$\oint_C \boldsymbol{E} \cdot d\boldsymbol{l} = 0$$

Work done by moving a charge around a closed loop must be zero, i.e., the static electric field is conservative.

3. Electric Field Loop Theorem

- An example of line integral:

Find the total mass of a wire for which the density varies along its length.



- 1. dividing the wire into a series of short segments over each of which the linear density λ (mass per unit length) is approximately constant;
- 2. the mass of each segment is the product of the linear density of that segment λ_i times the segment length dx_i:
- 3. the total mass of the entire wire is the sum of the segment masses:

$$\text{Mass} = \sum_{i=1}^{N} \lambda_i dx_i$$

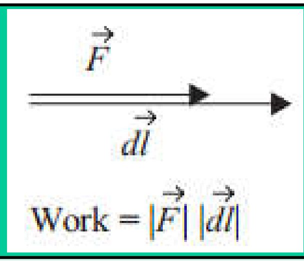
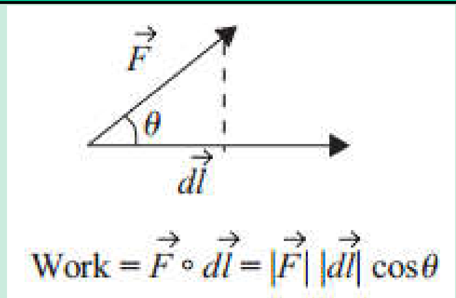
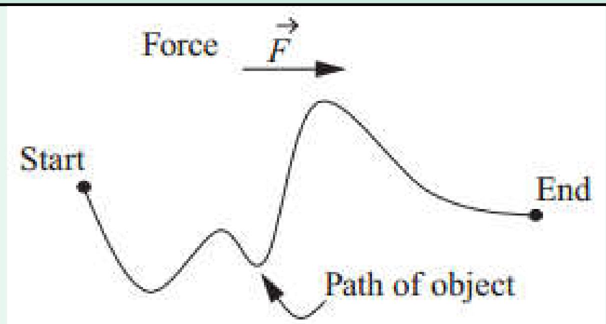
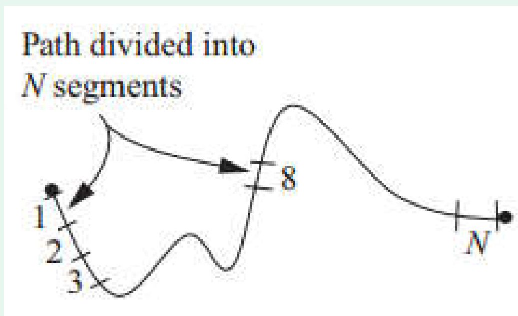
- 4. allowing the segment length to approach zero turns the summation of the segment masses into a line integral:

$$\text{Mass} = \int_0^L \lambda(x) dx$$

3. Electric Field Loop Theorem

- An example of path integral:

Consider the work done by a force as it moves an object along a path.

Straight path Parallel **F** and **L**		
Straight path **F** and **L** have an angle θ		
General case: Path is a curve, force is not constant		

1. $dW_i = \vec{F}_i \cdot d\vec{l}_i$
2. $W = \sum dW_i \Rightarrow$
   $W = \sum \vec{F}_i \cdot d\vec{l}_i$
3. $W = \int \vec{F} \cdot d\vec{l}$

3. Electric Field Loop Theorem

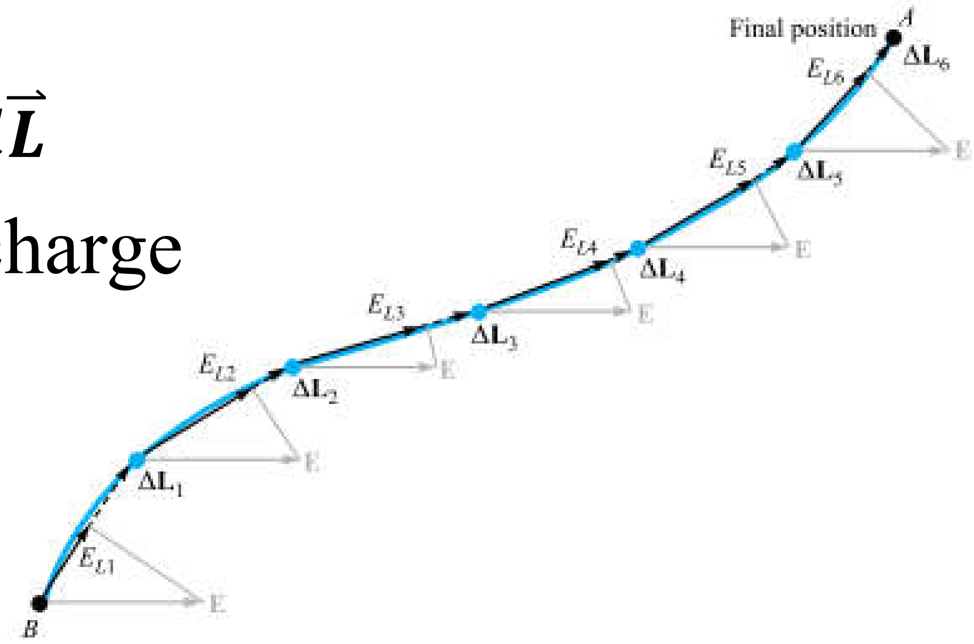
Work of moving a charge

- To move a charge q a distance $d\boldsymbol{L}$ in an electric field **E**, we must overcome the force arising from the electric field on q, which is $\vec{\boldsymbol{F}}_{\boldsymbol{E}} = q\vec{\boldsymbol{E}}$
- Therefore, the force we used is

$$\vec{\boldsymbol{F}}_{apply} = -q\vec{\boldsymbol{E}} \cdot \widehat{\boldsymbol{a}}_{\boldsymbol{L}}$$

  - where $\widehat{\boldsymbol{a}}_{\boldsymbol{L}}$ is the unit vector in the direction of $d\boldsymbol{L}$.
- So the differential work done by moving charge q is $dW = -q\vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{L}}$
- The work required to move the charge from B to A is determined by:

$$W_{AB} = -q\int_{B}^{A} \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{L}}$$



3. Electric Field Loop Theorem

Key concept - V

- A negative sign is required since we are asking for the work required by us to move the charge against the field.
- Define the ***voltage*** or ***potential difference*** V between points B and A as the work per unit of charge required to move the charge from B to A:

$$V_{AB} = \frac{W_{AB}}{q} = -\int_B^A \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{L}} = V_A - V_B$$

  – Notice V_{AB} is the potential difference between two points.

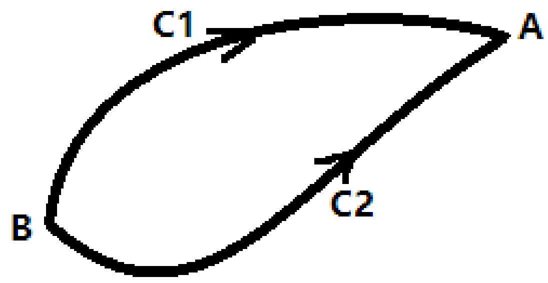
- The potential of a point, such as point A, should be based on the potential of a reference point, usually the potential at infinite distance to the source, set as $V_{infinite} = 0$. So

$$V_A = V_A - V_{infinite} = -\int_{infinite}^A \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{L}}$$

3. Electric Field Loop Theorem

Electric field circulation

- If we move the charge from B to A, then return to B, moving along the path below clockwise, the net work done is zero, so



$$\int_B^B \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{L}} = \oint_C \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{L}} = 0$$

  - The close loop integral is called the circulation of the electric field.
  - The circulation equals to zero means the static electric field is ***conservative***.

- Break the close loop C into two parts, c_1 and c_2:

$$\oint_C \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{L}} = \int_{c_1} \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{L}} - \int_{c_2} \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{L}} = 0$$

  - Meaning the voltage between two points is only determined by the relative position of the two points regardless of the path taken.

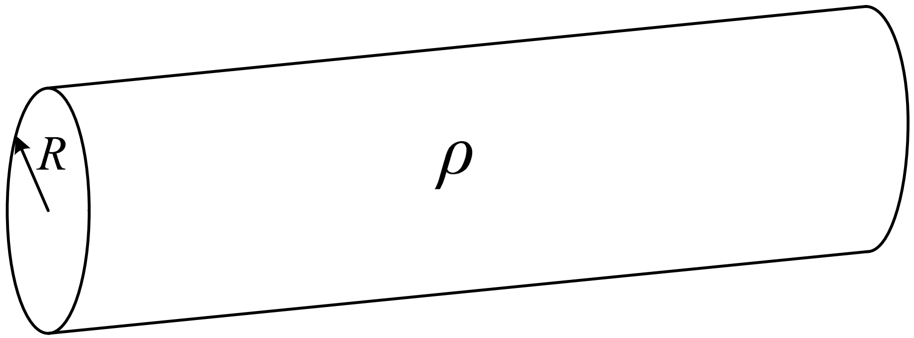
Quiz

- A cylindrical tube is filled with negatively charged chocolate crumbs with a density of

$$\rho = -1.1 \times 10^{-3}\ C/m^3$$

as shown below.

  - (a) Assume the cylindrical tube is infinitely long, find the electric field intensity **E** everywhere;
  - (b) An explosion will occur wherever the magnitude of the electric field intensity is larger than3.1×10^6 N/C. Determine the limitation of the radius of the cylindrical tube.



3. Electric Field Loop Theorem

- The differential form of the E-field loop theorem:

$$\boldsymbol{\nabla} \times \boldsymbol{E} = 0$$

The electric fields diverge away from points of positive charge and converge toward points of negative charge, such fields cannot circulate back on themselves.

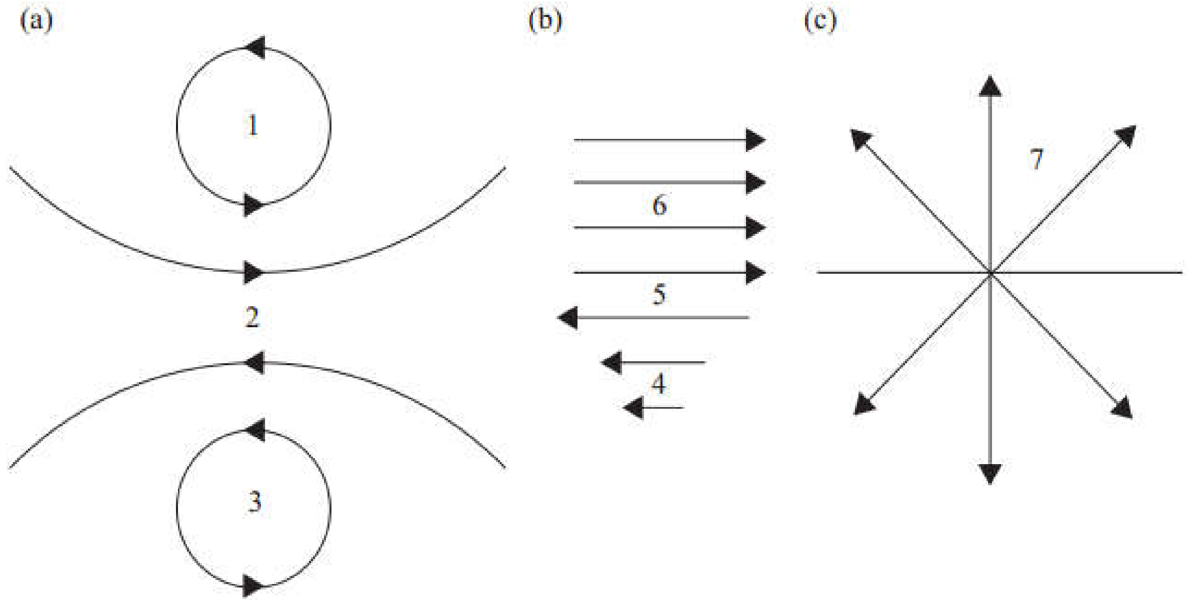
3. Electric Field Loop Theorem

Curl $\nabla \times$

- The curl of a vector field is a measure of the field's tendency to circulate about (around) a point.
  - The curl at a specified point may be found by considering the circulation per unit area over an infinitesimal path around that point.

$$\mathrm{curl}(\vec{A}) = \nabla \times \vec{A} = \lim_{\Delta S \to 0} \left(\frac{1}{\Delta S} \oint_C \boldsymbol{A} \cdot d\boldsymbol{l} \right)$$

  - The direction of the curl is the normal direction of the surface for which the circulation is a maximum.
  - Example: find the locations of large curl in each of these field:



3. Electric Field Loop Theorem

Stokes's Theorem

- How to link the integral and differential forms together:

Recall the "Curl Theorem", also known as "Stokes's Theorem": $\iint_S (\nabla \times \boldsymbol{A}) \cdot d\boldsymbol{s} = \oint_C \boldsymbol{A} \cdot d\boldsymbol{l}$

- Therefore, we have

$$\oint_C \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{L}} = \iint_S \nabla \times \vec{\boldsymbol{E}} \cdot d\boldsymbol{s} = 0 \Longrightarrow \boxed{\nabla \times \vec{\boldsymbol{E}} = 0}$$

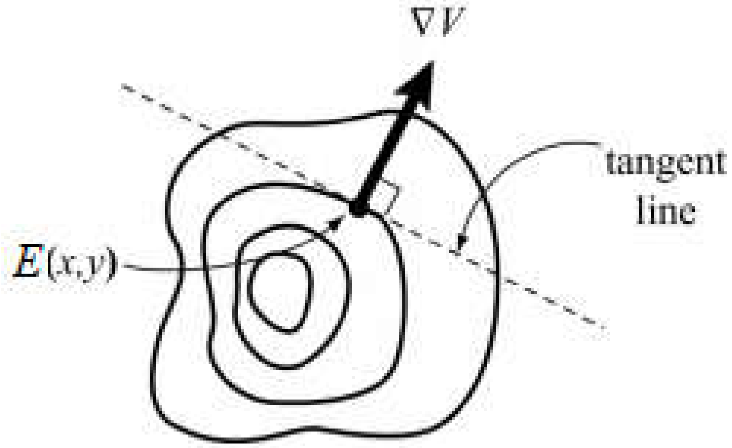
  – This must be true for any surface S bounded by a close loop C, so the two integrands must be equal.
  – Therefore, the static electric field is called ***irrotational*** field.

- If a vector field is irrotational, this vector field can be represented in terms of the gradient of a scalar field:

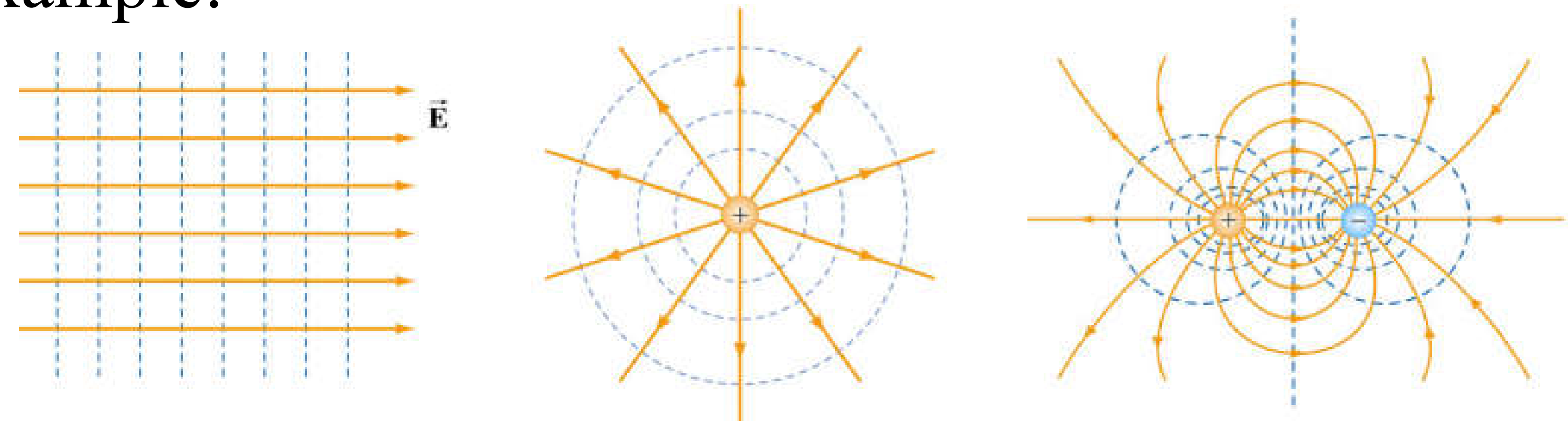
$$\vec{\boldsymbol{E}} = -\nabla V$$

3. Electric Field Loop Theorem

- The curves characterized by constant $V(x, y, z)$ are called ***equipotential lines***.
  - Since $\vec{\boldsymbol{E}} = -\nabla V$, we can show that the direction of **E** is always perpendicular to the equipotential lines through the point.



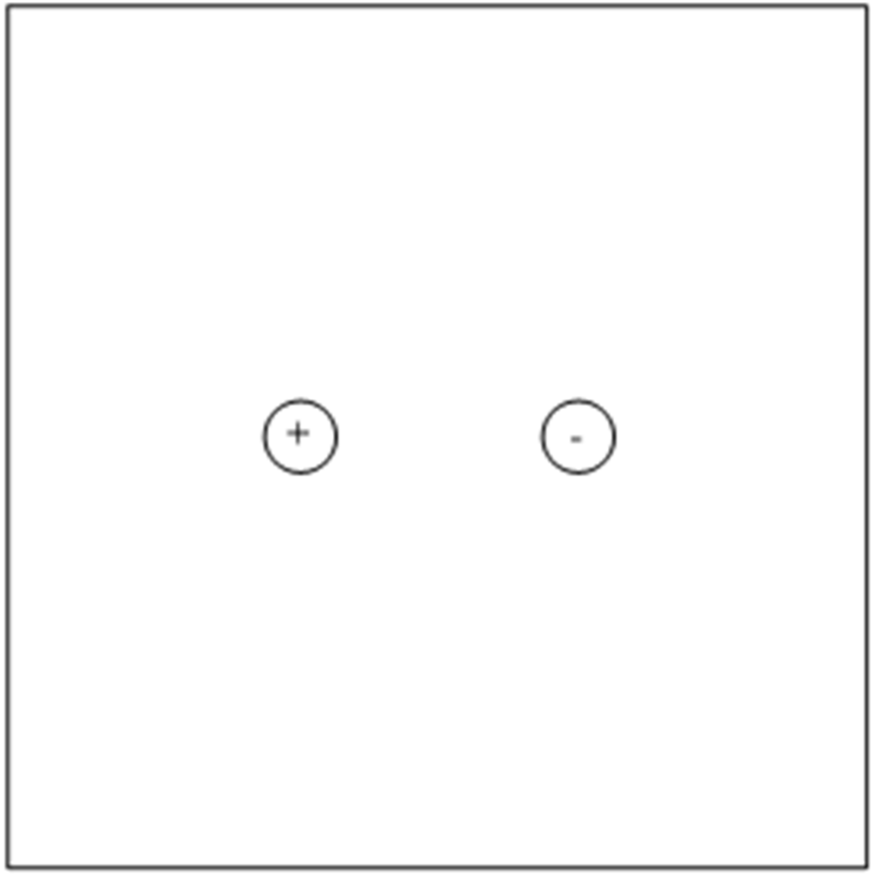
- Example:



(a) A constant **E** field (b) A point charge (b) An electric dipole

Quiz

- A positive point charge +Q and a negative point charge -Q are placed in free space as shown in Figure Q1(a). Draw the electric field lines and equipotential lines inside the boxed region.



4. Ampere's Theorem

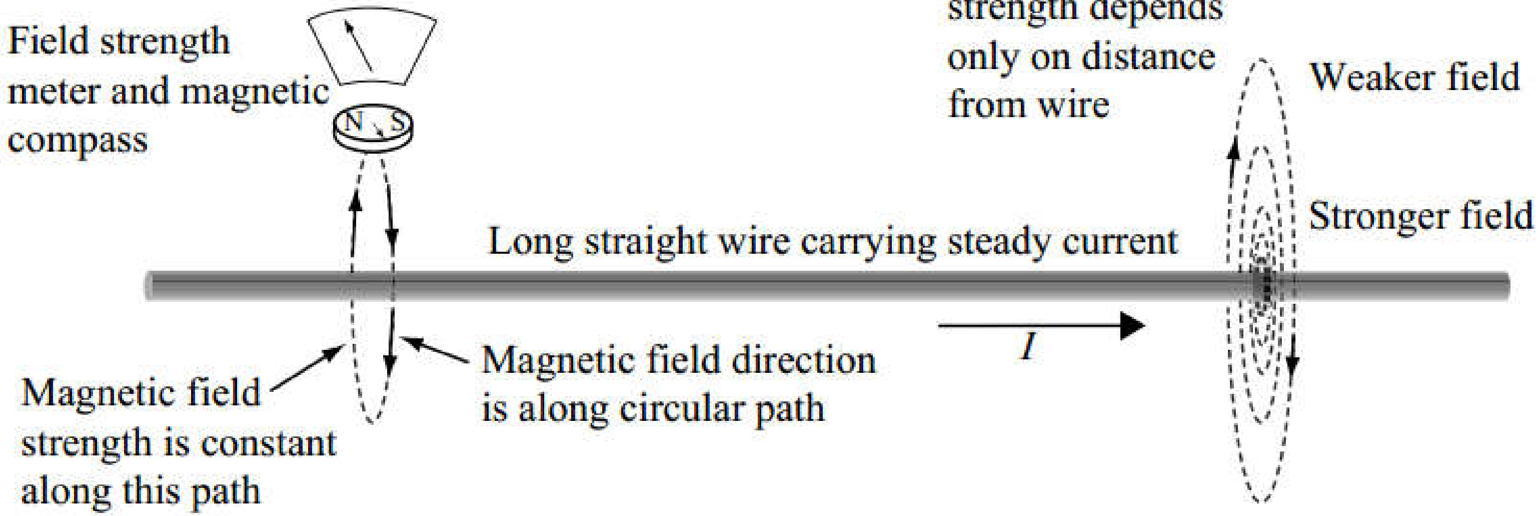
- The integral form of the magnetic-field loop theorem (more famously known as Ampere's Law):

$$\oint_C \boldsymbol{H} \cdot d\boldsymbol{l} = I_{enc}$$

An electric current through a surface produces a circulating magnetic field around any path that bounds that surface.

4. Ampere's Theorem

- The integral form of the magnetic-field loop theorem (more famously known as Ampere's Law):



- Keep track of the magnitude and direction of the magnetic field as you move around the wire in tiny increments.
  - at each incremental step, you found the component of the magnetic field ***B*** along that portion of your path *dl*, you'd be able to find ***Bdl***;
  - keeping track of each value of ***Bdl*** and then summing the results over your entire path, then making this process continuous by letting the path increment shrink toward zero, get:

$$\oint_C \boldsymbol{B} \cdot d\boldsymbol{l}$$

4. Ampere's Theorem

Field intensity $\vec{H}$

In free space:

$$\mathbf{E} \leftrightarrow \mathbf{H}$$
$$\mathbf{D} \leftrightarrow \mathbf{B}$$
$$\mathbf{D} = \varepsilon_0 \mathbf{E} \leftrightarrow \mathbf{B} = \mu_0 \mathbf{H}$$

- **E**
  - Electric field intensity
  - Unit: V/m
- **D**
  - Electric flux density
  - Unit: C/m^2

- **H**
  - Magnetic Field intensity
  - Unit: A/m
- **B**
  - Magnetic flux density
  - Unit: Tesla = Wb/m^2

$$\oint_C \boldsymbol{B} \cdot d\boldsymbol{l} = \oint_C \mu_0 \boldsymbol{H} \cdot d\boldsymbol{l} = \mu_0 I_{enc}$$

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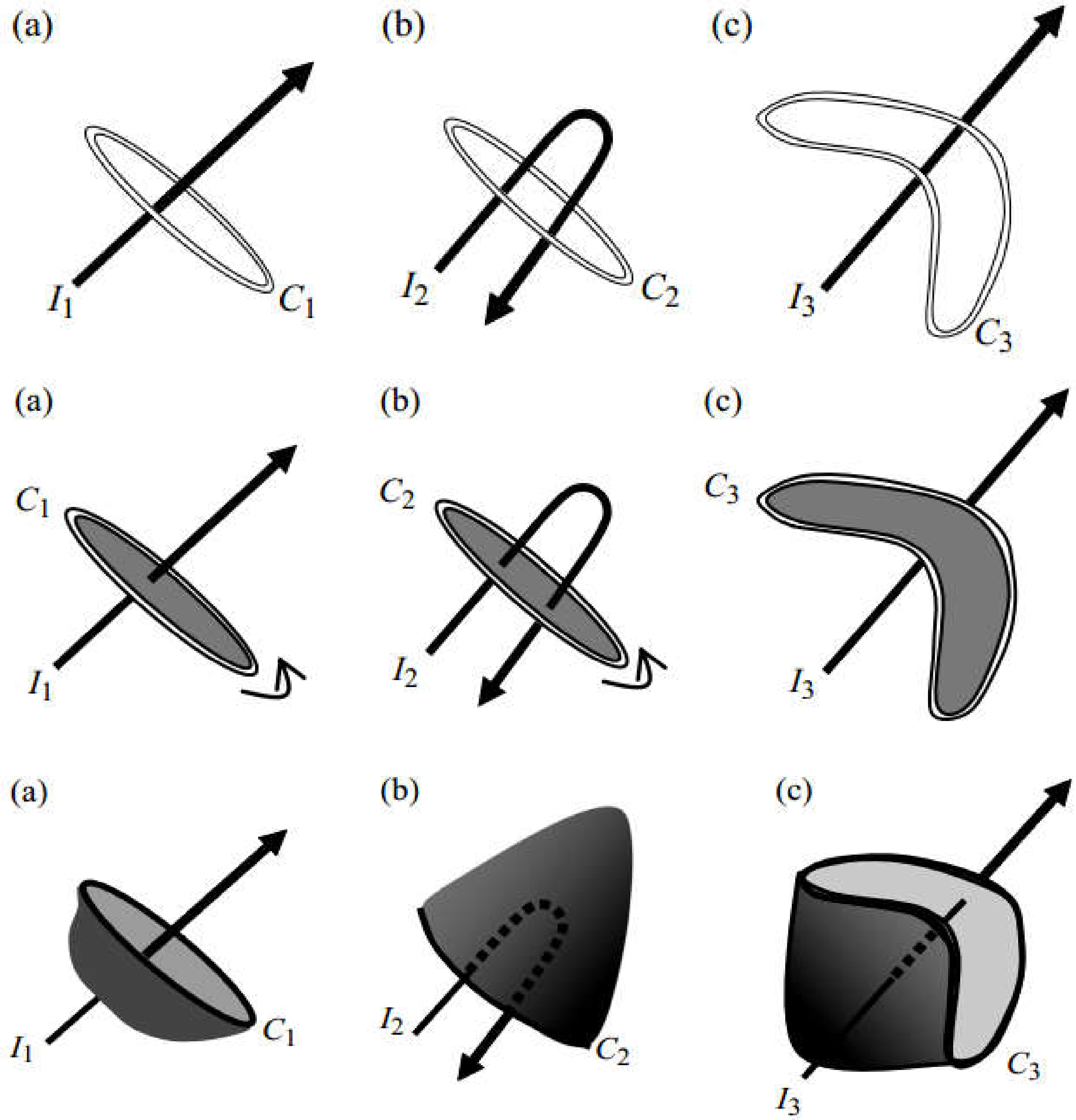
4. Ampere's Theorem

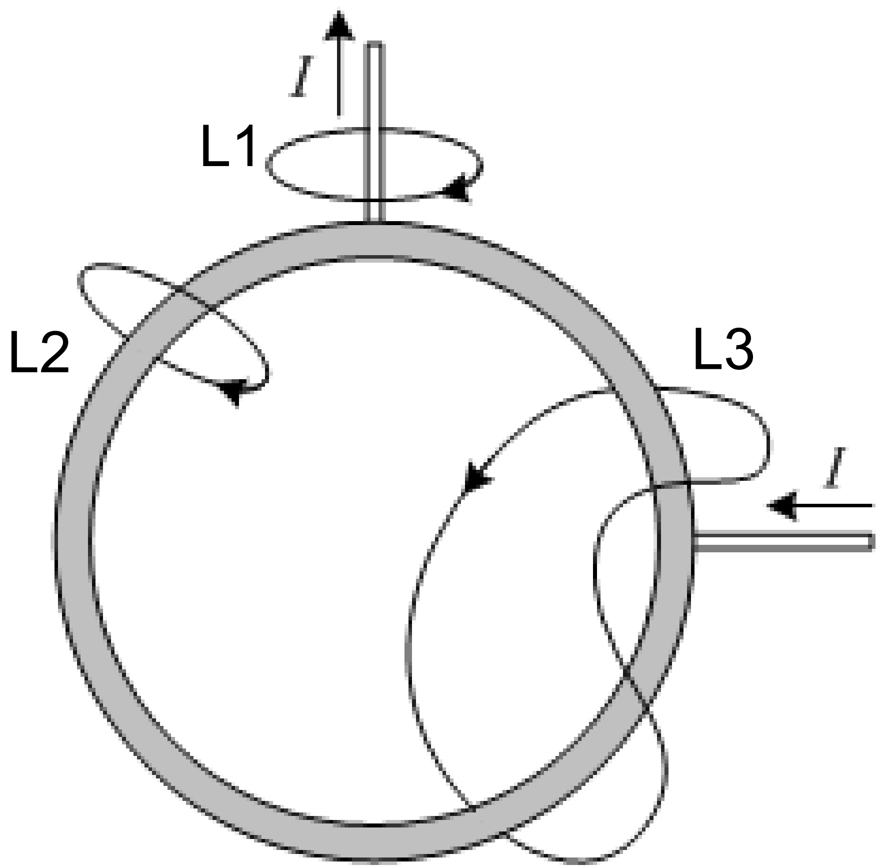
Enclosed current I_{enc}

- "Enclosing" is done by the path C around which the magnetic field is integrated.
- Eg. which of the currents are enclosed by the path?
  - The easiest way to answer that question is to imagine a membrane stretched across the path.
  - The enclosed current is the net current that penetrates the membrane.
    - if you wrap the fingers of your right hand around the path in the direction of integration, your thumb points in the direction of positive current.
    - the enclosed current is exactly the same irrespective of the shape of the surface you choose.



Quiz

- A uniform resistance wire (the grey part) has two lead wires connecting to a DC current source supplying I constantly, as shown below. It can be known that $\oint_{L_1} \boldsymbol{B} \cdot d\boldsymbol{l} = -\mu_0 I$.
- Find $\oint_{L_2} \boldsymbol{B} \cdot d\boldsymbol{l}$ and $\oint_{L_3} \boldsymbol{B} \cdot d\boldsymbol{l}$ also represented by μ_0 and I.



4. Ampere's Theorem

- The differential form of the magnetic-field loop theorem (more famously known as Ampere's Law):

$$\nabla \times \overrightarrow{\boldsymbol{H}} = \vec{\boldsymbol{J}}$$

$$\nabla \times \overrightarrow{\boldsymbol{B}} = \mu_0 \vec{\boldsymbol{J}}$$

A circulating magnetic field is produced by an electric current.

4. Ampere's Theorem

Curl $\nabla \times$

- All magnetic fields circulate back upon themselves and form continuous loops.
  - All fields that circulate back on themselves must include at least one location about which the path integral of the field is nonzero.
  - For the magnetic field, locations of nonzero curl are locations at which current is flowing.

- Considering a special Amperian loop surrounding the current

$$\nabla \times \vec{B} = \lim_{\Delta S \to 0} \left(\frac{1}{\Delta S} \oint_C \vec{\boldsymbol{B}} \cdot d\vec{\boldsymbol{l}} \right) = \lim_{\Delta S \to 0} \left(\frac{1}{\Delta S} \frac{\mu_0 I}{2\pi r} (2\pi r) \right) = \lim_{\Delta S \to 0} \left(\frac{1}{\Delta S} \mu_0 I \right)$$

  - However, $I/\Delta S$ is just the average current density over the surface ΔS, and as ΔS shrinks to zero, this becomes equal to $\boldsymbol{J}$, the current density at the origin, so

$$\nabla \times \vec{B} = \mu_0 \vec{\boldsymbol{J}}$$

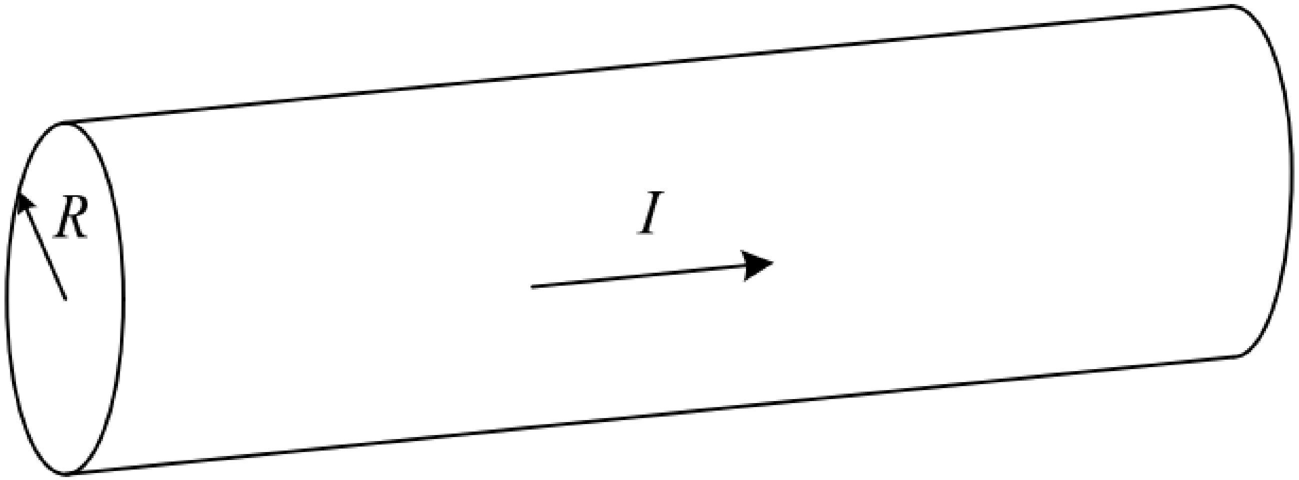
4. Ampere's Theorem

- Current density is defined as the vector current flowing through a unit cross-sectional area perpendicular to the direction of the current.
  - Unit: A/m^2
- The complexity of the relationship between the total current I through a surface and the current density ~J depends on the geometry of the situation.
  - If the current density **J** is uniform over a surface S and is everywhere perpendicular to the surface, the relationship is $I = |\boldsymbol{J}|S$
  - If **J** is uniform over a surface S but is not perpendicular to the surface, to find the total current I through S you'll have to determine the component of the current density perpendicular to the surface: $I = \boldsymbol{J} \cdot \boldsymbol{S} = \boldsymbol{J} \cdot \hat{\boldsymbol{n}}S$
  - And, if **J** is nonuniform and not perpendicular to the surface, then

$$I = \iint_S \boldsymbol{J} \cdot d\vec{\boldsymbol{s}} = \iint_S \boldsymbol{J} \cdot \hat{\boldsymbol{n}} ds$$

Quiz

- An infinitely long cylinder with the radius of R carries a steady current along the axial direction as shown below. Determine the magnetic field intensity $\boldsymbol{H}(r)$ everywhere.



Next

- Materials in electric and magnetic fields
- Boundary conditions