Solutions to the final exam of MTH201.

S1 2017-18.

1. $\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 0 & 1 \\ 1 & 1 & 2 \end{vmatrix} = \langle -1, -5, 3\rangle.$

$\therefore \|\vec{a}\times\vec{b}\| = \sqrt{1+25+9} = \underline{\sqrt{35}}.$

2. $\operatorname{div}\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = \underline{3e^{y} - \sin y + 2z}.$

3. $\operatorname{curl}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2z + axz^2 & 2xyz & xy^2 + 2x^2z \end{vmatrix} = \langle 2xy - 2xy,\ y^2 + 2axz - y^2 - 4xz,\ 2yz - 2yz\rangle$

$= \langle 0, 0, 0\rangle$

$\therefore 2axz - 4xz = 0 \quad \therefore 2a = 4 \quad \therefore \underline{a = 2}.$

4. $\vec{r}_u = \langle -2\sin u, 2\cos u, 0\rangle, \quad \vec{r}_v = \langle 0, 0, 1\rangle$

$\therefore$ A normal vector $\vec{N} = \vec{r}_u\times\vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2\sin u & 2\cos u & 0 \\ 0 & 0 & 1 \end{vmatrix} = \underline{\langle 2\cos u, 2\sin u, 0\rangle}.$

5. $A = 1,\ B = -1,\ C = 1, \quad \therefore AC - B^2 = 1 - 1 = 0.$

So the type of the PDE is <u>parabolic</u>.

6. $\vec{F} = \langle \sin y, x\cos y\rangle = \nabla f$, so $\begin{cases} \frac{\partial f}{\partial x} = \sin y & ① \\ \frac{\partial f}{\partial y} = x\cos y. & ② \end{cases}$

From ①, we have $f(x,y) = x\sin y + h(y)$ $\therefore \frac{\partial f}{\partial y} = x\cos y + h'(y) = x\cos y$.

$\therefore h(y) = $ Constant. $\therefore f(x,y) = x\sin y + $ Constant.

$\therefore \int_{(0,0)}^{(1,\frac{\pi}{4})} \sin y\,dx + x\cos y\,dy = f(1,\frac{\pi}{4}) - f(0,0) = \frac{\sqrt{2}}{2} - 0 = \underline{\frac{\sqrt{2}}{2}}.$

7. $\int_C \vec{F}\cdot d\vec{r} = \int_C \vec{F}(\vec{r}(t))\cdot \vec{r}\,'(t)\,dt = \int_0^1 \langle 4t^2, 2t(t-1)\rangle \cdot \langle 2, 1\rangle\, dt$

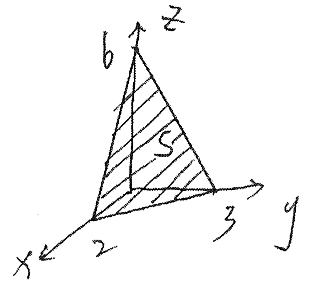
$= \int_0^1 8t^2 + 2t^2 - 2t\,dt = \int_0^1 10t^2 - 2t\,dt = \left[\frac{10}{3}t^3 - t^2\right]_0^1 = \frac{10}{3} - 1 = \frac{7}{3}.$

8. The region R bounded by C is the disk $x^2+y^2 \le 9$, so let's change to polar coordinates after applying Green's theorem:

$$\oint_C (3y - e^{\sin x})\,dx + (7x + \sqrt{y^4+1})\,dy = \iint_R \frac{\partial}{\partial x}(7x+\sqrt{y^4+1}) - \frac{\partial}{\partial y}(3y - e^{\sin x})\,dA$$

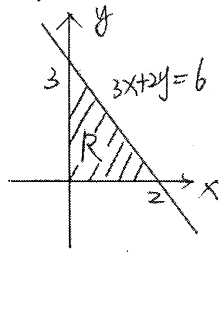
$$= \int_0^{2\pi}\int_0^3 7-3\; r\,dr\,d\theta = 4\int_0^{2\pi}\int_0^3 r\,dr\,d\theta = 36\pi.$$

9. (a)



(b). $z = 6-3x-2y$, ∴ $z_x = -3$, $z_y = -2$.

∴ Area of $S = \iint_R \sqrt{z_x^2+z_y^2+1}\,dx\,dy = \iint_R \sqrt{9+4+1}\,dx\,dy = \sqrt{14}\iint_R dx\,dy$. [2].



$= \sqrt{14}\int_0^2\int_0^{3-\frac{3}{2}x} dy\,dx = \sqrt{14}\int_0^2 3 - \frac{3}{2}x\,dx$

$= \sqrt{14}\left[3x - \frac{3}{4}x^2\right]_0^2 = \sqrt{14}(6-3) = 3\sqrt{14}.$

(c). $\iint_S \vec{F}\cdot\vec{n}\,dA = \iint_R \vec{F}\cdot\vec{N}\,dx\,dy = \iint_R \langle z, y, x\rangle\cdot\langle 3, 2, 1\rangle\,dx\,dy$

$= \iint_R 3z + 2y + x\,dx\,dy$, $z = 6-3x-2y$

$= \iint_R x+2y+3(6-3x-2y)\,dx\,dy = \int_0^2\int_0^{3-\frac{3}{2}x} 18 - 8x - 4y\,dy\,dx.$

$$= \int_0^2 \left[(18-8x)y - 2y^2\right]_0^{3-\frac{3}{2}x} dx$$

$$= \int_0^2 (18-8x)(3-\tfrac{3}{2}x) - 2(3-\tfrac{3}{2}x)^2\, dx$$

$$= \int_0^2 36 - 33x + \tfrac{15}{2}x^2\, dx$$

$$= \left[\tfrac{5}{2}x^3 - \tfrac{33}{2}x^2 + 36x\right]_0^2 = 26.$$

10. Apply the divergence theorem, we have

$$\iint_S \langle y(x-z),\ x^2,\ y^2+xz\rangle \cdot \vec{n}\, dA = \iiint_T \operatorname{div} \vec{F}\, dV$$

$$= \iiint_T y + 0 + x\, dV = \int_0^a \int_0^a \int_0^a x+y\, dz\, dy\, dx$$

$$= \int_0^a \int_0^a a(x+y)\, dy\, dx = \int_0^a \left[axy + \tfrac{1}{2}ay^2\right]_0^a dx$$

$$= \int_0^a a^2x + \tfrac{1}{2}a^3\, dx = \left[\tfrac{1}{2}a^2x^2 + \tfrac{1}{2}a^3x\right]_0^a$$

$$= \tfrac{1}{2}a^4 + \tfrac{1}{2}a^4 = a^4.$$

11. To find the boundary curve C, we solve the equations

$$\begin{cases} x^2+y^2+z^2 = 4 \\ x^2+y^2 = 1 \end{cases},\quad \text{So we get } z^2 = 3 \text{ and so } z = \sqrt{3}.$$

Thus, curve C is the circle given by the equations: $x^2+y^2=1$, $z=\sqrt{3}$.

A vector equation of C is

$$\vec{r}(t) = \langle \cos t,\ \sin t,\ \sqrt{3}\rangle,\ \therefore \vec{r}'(t) = \langle -\sin t,\ \cos t,\ 0\rangle.$$

Also, we have $\vec{F}(\vec{r}(t)) = \langle xz, yz, xy \rangle$

$$= \langle \sqrt{3}\cos t, \sqrt{3}\sin t, \cos t \sin t \rangle.$$

Therefore, by stokes's theorem,

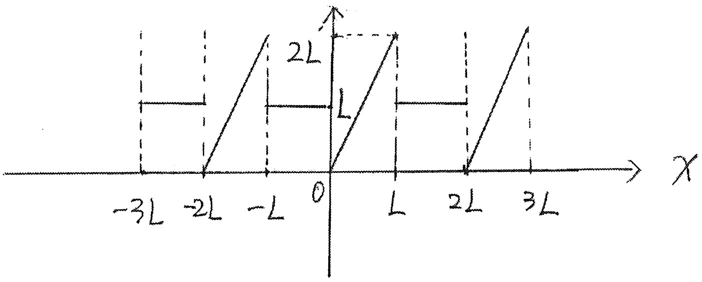
$$\iint_S \operatorname{curl} \vec{F} \cdot \vec{n}\, dA = \oint_C \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\, dt$$

$$= \int_0^{2\pi} \langle \sqrt{3}\cos t, \sqrt{3}\sin t, \cos t \sin t \rangle \cdot \langle -\sin t, \cos t, 0 \rangle\, dt$$

$$= \int_0^{2\pi} -\sqrt{3}\cos t \sin t + \sqrt{3}\sin t \cos t + 0\, dt$$

$$= 0.$$

12. (a).



(b). At $x=0$, $\frac{1}{2}[f(0^-)+f(0^+)] = \frac{1}{2}(L+0) = \frac{L}{2}$.

At $x=\frac{L}{2}$, $f(\frac{L}{2}) = L$.

At $x=L$, $\frac{1}{2}[f(L^-)+f(L^+)] = \frac{1}{2}(2L+L) = \frac{3L}{2}$

At $x=\frac{3L}{2}$, $f(\frac{3L}{2}) = L$.

(c). $a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = \frac{1}{2L}\int_{-L}^{0} L\,dx + \frac{1}{2L}\int_0^L 2x\,dx$

$$= \frac{1}{2L}[Lx]_{-L}^{0} + \frac{1}{2L}[x^2]_0^L$$

$$= \frac{1}{2L}(0+L^2) + \frac{1}{2L}(L^2-0) = L.$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi}{L}x\,dx$$

$$= \frac{1}{L}\left[\int_{-L}^{0} L\cos\frac{n\pi}{L}x\,dx + \int_{0}^{L} 2x\cos\frac{n\pi}{L}x\,dx\right]$$

$$= \frac{L}{n\pi}\left[\sin\frac{n\pi x}{L}\right]_{-L}^{0} + \frac{1}{L}\int_0^L 2x\,\frac{L}{n\pi}\,d\left(\sin\frac{n\pi x}{L}\right)$$

$$= 0 + \frac{1}{L}\left[\frac{2xL}{n\pi}\sin\frac{n\pi x}{L}\right]_0^L - \frac{1}{L}\int_0^L \frac{2L}{n\pi}\sin\frac{n\pi x}{L}\,dx$$

$$= 0 - \frac{2}{n\pi}\cdot\frac{L}{n\pi}\left[-\cos\frac{n\pi x}{L}\right]_0^L$$

$$= -\frac{2L}{n^2\pi^2}(1-\cos n\pi) = \frac{2L}{n^2\pi^2}\left[(-1)^n - 1\right].$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi}{L}x\,dx$$

$$= \frac{1}{L}\int_{-L}^{0} L\sin\frac{n\pi x}{L} + \frac{1}{L}\int_0^L 2x\sin\frac{n\pi x}{L}\,dx$$

$$= \frac{L}{n\pi}\left[-\cos\frac{n\pi}{L}x\right]_{-L}^{0} + \frac{1}{L}\int_0^L 2x\,\frac{-L}{n\pi}\,d\left(\cos\frac{n\pi}{L}x\right)$$

$$= \frac{L}{n\pi}\left[\cos n\pi - 1\right] - \frac{2}{n\pi}\left\{\left[x\cos\frac{n\pi}{L}x\right]_0^L - \int_0^L \cos\frac{n\pi x}{L}\,dx\right\}$$

$$= \frac{L}{n\pi}\left[(-1)^n - 1\right] - \frac{2}{n\pi}\left\{L(-1)^n - \frac{L}{n\pi}\left[\sin\frac{n\pi x}{L}\right]_0^L\right\}$$

$$= \frac{L}{n\pi}\left[(-1)^n - 1\right] - \frac{2L}{n\pi}(-1)^n = \frac{L}{n\pi}\left[-(-1)^n - 1\right] = -\frac{L}{n\pi}\left[(-1)^n + 1\right].$$

$$\therefore f(x) = L + \sum_{n=1}^{\infty}\left\{\frac{2L}{n^2\pi^2}\left[(-1)^n - 1\right]\cos\frac{n\pi}{L}x + \frac{-L}{n\pi}\left[(-1)^n + 1\right]\sin\frac{n\pi}{L}x\right\}.$$

When $n=1$ $\quad a_1 = \frac{2L}{\pi^2}\cdot(-2) = \frac{-4L}{\pi^2},\; b_1 = \frac{-L}{\pi}\cdot 0 = 0$

when $n=2$. $\quad a_2 = \frac{2L}{4\pi^2}\cdot 0 = 0. \quad b_2 = \frac{-L}{2\pi}\cdot 2 = \frac{-L}{\pi}.$

So the first three non-zero terms are:

$$L + \frac{-4L}{\pi^2}\cos\frac{\pi x}{L} + \frac{-L}{\pi}\cdot\sin\frac{2\pi}{L}x.$$

13. (a). $u(x,t) = X(x)T(t) \rightarrow \frac{\partial^2 u}{\partial t^2} = X(x)T''(t)$, $\frac{\partial^2 u}{\partial x^2} = X''(x)T(t)$.

$\therefore X(x)T''(t) - 4X''(x)T(t) + \frac{X(x)T(t)}{l^2} = 0$.

Divide $X(x)T(t)$ on both sides:

$$\frac{T''(t)}{T(t)} - \frac{4X''(x)}{X(x)} = -\frac{1}{l^2}.$$

$\therefore$ Let $\frac{T''(t)}{T(t)} + \frac{1}{l^2} = \frac{4X''(x)}{X(x)} = -\alpha_n^2$, then

$$X''(x) + \frac{\alpha_n^2}{4}X(x) = 0$$

$$T''(t) + \left(\frac{1}{l^2} + \alpha_n^2\right)T(t) = 0$$

(b). $X''(x) + \frac{\alpha_n^2}{4}X(x) = 0$ $\therefore \lambda^2 + \frac{\alpha_n^2}{4} = 0$ $\therefore \lambda = \pm\frac{\alpha_n}{2}i$

$\therefore X(x) = A_n \cos\frac{\alpha_n}{2}x + B_n \sin\frac{\alpha_n}{2}x$.

By the boundary conditions

$u(0,t) = u(l,t) = 0 \rightarrow X(0)T(t) = X(l)T(t) = 0$ for any t,

$\therefore X(0) = X(l) = 0$

$$\therefore \begin{cases} A_n + B_n \cdot 0 = 0 & ① \Rightarrow A_n = 0. \\ A_n \cos\frac{\alpha_n l}{2} + B_n \sin\frac{\alpha_n l}{2} = 0 & ② \end{cases}$$

From ②, $B_n \sin\frac{\alpha_n l}{2} = 0$ $\therefore B_n = 0$ is not acceptable.

$\therefore \sin\frac{\alpha_n l}{2} = 0$ $\therefore \frac{\alpha_n l}{2} = n\pi$, $\therefore \alpha_n = \frac{2n\pi}{l}$.

$\therefore X(x) = B_n \sin\frac{2n\pi}{2l}x = B_n \sin\frac{n\pi x}{l}$.

(C). $T''(t) + \left(\frac{1}{L^2} + \alpha_n^2\right) T(t) = 0$

$\therefore T''(t) + \left(\frac{1}{L^2} + \frac{4n^2\pi^2}{L^2}\right) T(t) = 0$

$\therefore T(t) = C_n \cos \frac{\sqrt{4n^2\pi^2+1}}{t} t + D_n \sin \frac{\sqrt{4n^2\pi^2+1}}{L} t$.

$\therefore$ The general solution is

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} \left[C_n \cos \frac{\sqrt{4n^2\pi^2+1}}{L} t + D_n \sin \frac{\sqrt{4n^2\pi^2+1}}{L} t\right] [2]$$

$$\therefore \frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} \cdot \left[- C_n \frac{\sqrt{4n^2\pi^2+1}}{L} \sin \frac{\sqrt{4n^2\pi^2+1}}{L} t + D_n \frac{\sqrt{4n^2\pi^2+1}}{L} \cos \frac{\sqrt{4n^2\pi^2+1}}{L} t\right]$$

By the initial conditions:

$$u(x,0) = \sum_{n=1}^{\infty} \frac{1}{n^3} \sin \frac{n\pi x}{L}.$$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = \sum_{n=1}^{\infty} \frac{\sqrt{4n^2\pi^2+1}}{n^3 L} \sin \frac{n\pi x}{L}.$$

We have $\sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} C_n = \sum_{n=1}^{\infty} \frac{1}{n^3} \sin \frac{n\pi x}{L}$

$\sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} D_n \frac{\sqrt{4n^2\pi^2+1}}{L} = \sum_{n=1}^{\infty} \frac{\sqrt{4n^2\pi^2+1}}{n^3 L} \sin \frac{n\pi x}{L}$.

$\therefore B_n C_n = \frac{1}{n^3}$, $\quad B_n D_n = \frac{1}{n^3}$.

$$\therefore u(x,t) = \sum_{n=1}^{\infty} \left[B_n C_n \sin \frac{n\pi x}{L} \cos\left(\frac{\sqrt{4n^2\pi^2+1}}{L} t\right) + B_n D_n \sin \frac{n\pi x}{L} \sin\left(\frac{\sqrt{4n^2\pi^2+1}}{L} t\right)\right]$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^3} \sin \frac{n\pi x}{L} \left[\cos\left(\frac{\sqrt{4n^2\pi^2+1}}{L} t\right) + \sin\left(\frac{\sqrt{4n^2\pi^2+1}}{L} t\right)\right]$$

$$= \sum_{n=1}^{\infty} \frac{\sqrt{2}}{n^3} \sin \frac{n\pi x}{L} \sin\left(\frac{\sqrt{4n^2\pi^2+1}}{L} t + \frac{\pi}{4}\right)$$