Xi'an Jiaotong-Liverpool University
西交利物浦大学

EEE220 Instrumentation and Control System

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Lecture 17

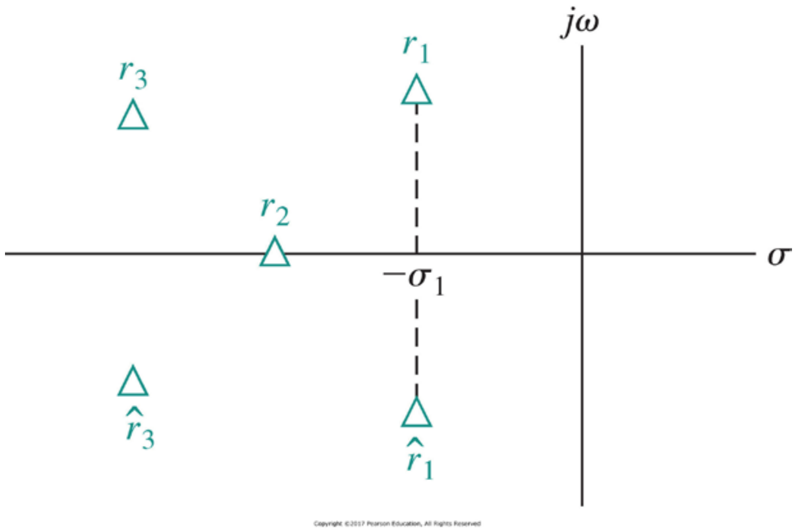
Outline

The Stability of Linear Feedback Systems

- The Concept of Stability
- The Routh-Hurwitz Stability Criterion
- **The Relative Stability of Feedback Control Systems**
- **The Stability of State Variable Systems**
- **System Stability Using Matlab**

Relative Stability

- The Routh-Hurwitz criterion ascertain the **absolute stability** of a system by determining whether any of the roots of the characteristic equation lie in the right half of the s-plane;
- However, if a system satisfies the Routh-Hurwitz criterion and is stable, it is desirable to determine the **relative stability** (the degree of stability, or how close the system is to instability);
- The relative stability can be determined by as the property that is measured by the relative real part of each root or pair of roots.

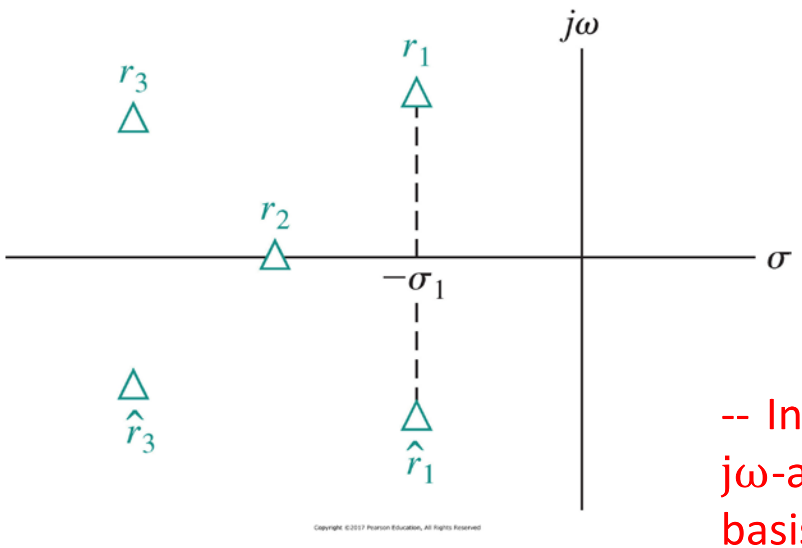


In this figure, root r_2 is relatively more stable than the roots $r_1, \hat{r}_1$.

The investigation of the relative stability is important because the location of the closed-loop poles in the s-plane determines the performance of the system.

For Examining Relative Stability: Axis Shift

- This approach is the extension of Routh-Hurwitz criterion to ascertain relative stability;
- The approach can be accomplished by utilizing a change of variable, which shifts the $j\omega$-axis in the s-plane in order to utilize the Routh-Hurwitz criterion.



In this figure, it can be noticed that a shift of the $j\omega$-axis in the s-plane to $-\sigma_1$ *will result in the roots appearing on the shifted axis (-marginally stability).*

-- In practice, the correct magnitude to shift the jω-axis must be obtained on a trial-and-error basis. Then, without solving q(s) (5th order in this case), we may determine the real-part of the dominant roots.

Example 17.1

Consider the third-order system with the following characteristic equation

$$q(s) = s^3 + 4s^2 + 6s + 4$$

To determine relative stability:

1. Apply Routh-Hurwitz criterion on this characteristic equation, the system is stable (absolute stability);
2. As a first try, we can shift the $j\omega$-axis by $\frac{1}{2}$, in other words, let us assume $s_n = s + \frac{1}{2}$, then the new characteristic equation can be obtained. Applying Routh-Hurwitz criterion, we'll find that the system is still stable after shifting the $j\omega$-axis by $\frac{1}{2}$;
3. Then we try shifting the $j\omega$-axis by 1, i.e., we assume $s_n = s + 1$, new characteristic equation now is

$$(s_n - 1)^3 + 4(s_n - 1)^2 + 6(s_n - 1) + 4 = s_n{}^3 + s_n{}^2 + s_n + 1.$$

$$(s_n - 1)^3 + 4(s_n - 1)^2 + 6(s_n - 1) + 4 = s_n{}^3 + s_n{}^2 + s_n + 1.$$

Then the Routh array is established as

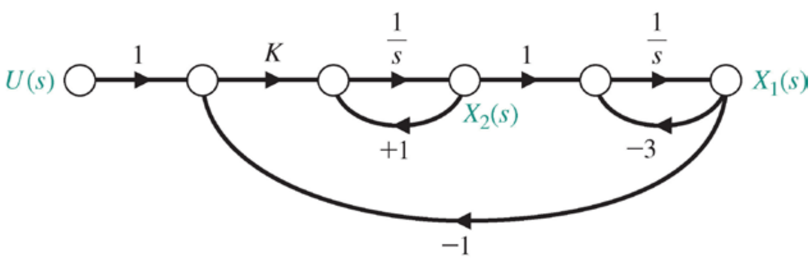
$$\begin{array}{c|cc} s_n^3 & 1 & 1 \\ s_n^2 & 1 & 1 \\ s_n^1 & 0 & 0 \\ s_n^0 & 1 & 0 \end{array}$$

There is a row with all zeros in the Routh array, indicating that there are a pair of roots on the shifted imaginary axis. These two roots can be obtained from the auxiliary polynomial

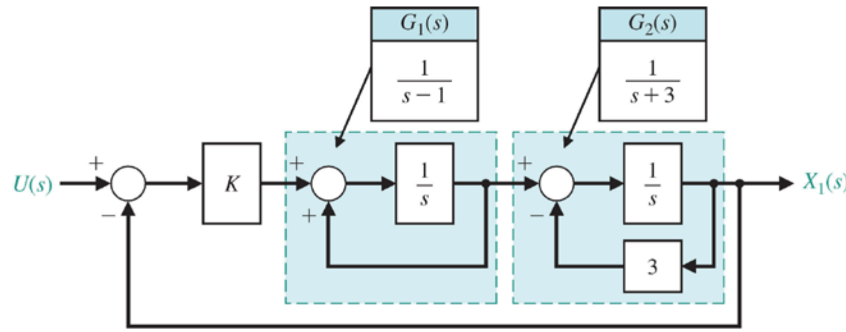
$$U(s_n) = s_n{}^2 + 1 = (s_n + j)(s_n - j) = (s + 1 + j)(s + 1 - j).$$

Stability of State Variable System

-- If the system is represented by signal-flow graph (a) or block diagram (b), stability can be assessed by firstly obtaining the transfer function of the system, then applying Routh-Hurwitz criterion to the characteristic equation.



(a)

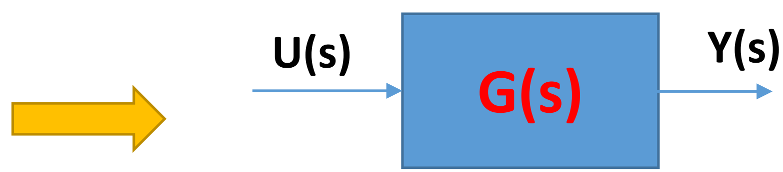


(b)

How About A System Represented by State-space Model?

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(\mathrm{t}) + \mathbf{B}\mathbf{u}(\mathrm{t})$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(\mathrm{t}) + \mathbf{D}\mathbf{u}(\mathrm{t})$$



$$\mathbf{G}(s) = \mathbf{C}[s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B} + \mathbf{D}$$

Characteristic Equation from State-space Model

Transfer function $\mathbf{G}(s) = \mathbf{C}[s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B} + \mathbf{D}$

$$\mathbf{G}(s) = \mathbf{C}\frac{adj(s\mathbf{I} - \mathbf{A})}{det(s\mathbf{I} - \mathbf{A})}\mathbf{B} + \mathbf{D} = \frac{\mathbf{C}[adj(s\mathbf{I} - \mathbf{A})]\mathbf{B} + |s\mathbf{I} - \mathbf{A}|\mathbf{D}}{|s\mathbf{I} - \mathbf{A}|}$$

Note: for a 2×2 matrix M=$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, assume its inverse matrix is M^{-1}, (i.e., $\mathrm{M}^{-1}\mathrm{M} = \mathrm{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$)

Its adjugate is adj(M)=$\begin{bmatrix} \mathrm{d} & -b \\ -c & \mathrm{a} \end{bmatrix}$, its determinant is det(M)=$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \mathrm{ad} - \mathrm{bc}$.

Then $\mathrm{M}^{-1} = \frac{adj(\mathrm{M})}{\det(\mathrm{M})}$.

Setting the denominator of the transfer function matrix **G**(s) to be zero, we get the **characteristic equation:**

$$\Delta(\boldsymbol{s}) = |s\mathbf{I} - \mathbf{A}| = \mathbf{0}$$

$\boldsymbol{n}$: **order of the system**
$\boldsymbol{A}$: $\boldsymbol{n} \times \boldsymbol{n}$ **matrix**
$\boldsymbol{s}\mathbf{I} - \mathbf{A}$: $\boldsymbol{n} \times \boldsymbol{n}$ **matrix**
$|\boldsymbol{s}\mathbf{I} - \mathbf{A}|$: $\boldsymbol{n}$**-th order polynomial**

-----No need to obtain the transfer function which requires determination of an inverse matrix. --- we only need to look into the denominator, which is the determinant of an matrix ($|s\boldsymbol{I} - \boldsymbol{A}|$).

Supplements: Determinant of Matrix

For a 2×2 matrix (2 rows and 2 columns):

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

The determinant is:

$$|A| = ad - bc$$

"The determinant of A equals a times d minus b times c"

It is easy to remember when you think of a cross:

- Blue is positive (+ad),
- Red is negative (−bc)

For a 3×3 Matrix

For a 3×3 matrix (3 rows and 3 columns):

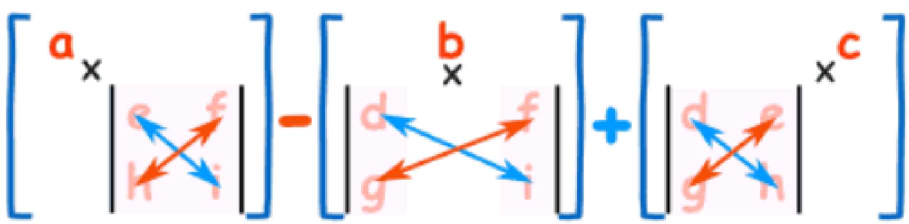
$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

The determinant is:

$$|A| = a(ei - fh) - b(di - fg) + c(dh - eg)$$

"The determinant of A equals ... etc"

It may look complicated, but **there is a pattern**:



Example:

$$B = \begin{bmatrix} 4 & 6 \\ 3 & 8 \end{bmatrix}$$

$$\begin{aligned} |B| &= 4\times8 - 6\times3 \\ &= 32-18 \\ &= 14 \end{aligned}$$

Example:

$$C = \begin{bmatrix} 6 & 1 & 1 \\ 4 & -2 & 5 \\ 2 & 8 & 7 \end{bmatrix}$$

$$\begin{aligned} |C| &= 6\times(-2\times7 - 5\times8) - 1\times(4\times7 - 5\times2) + 1\times(4\times8 - (-2\times2)) \\ &= 6\times(-54) - 1\times(18) + 1\times(36) \\ &= \mathbf{-306} \end{aligned}$$

Example 17.2

A system is described by the following model, determine its stability.

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ -3 & -1 & 0 \\ -2 & -1 & -2 \end{bmatrix} \mathbf{x}$$

Solutions:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ -3 & -1 & 0 \\ -2 & -1 & -2 \end{bmatrix},$$

$$\Delta(s) = |s\mathbf{I} - \mathbf{A}| = \left| \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ -3 & -1 & 0 \\ -2 & -1 & -2 \end{bmatrix} \right|$$

$$= \begin{vmatrix} s & -1 & 0 \\ 3 & s+1 & 0 \\ 2 & 1 & s+2 \end{vmatrix} = s^3 + 3s^2 + 5s + 6$$

$$\Delta(s) = s^3 + 3s^2 + 5s + 6$$

The Routh array is

$$\begin{array}{c|cc} s^3 & 1 & 5 \\ s^2 & 3 & 6 \\ s^1 & 3 & 0 \\ s^0 & 6 & \end{array}$$

No sign change in the first column, the system is stable.

Example 17.3

For the following system, choose values of k to make it stable.

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 0 & -1 \\ k & 0 & 2 \\ -k & -2 & -k \end{bmatrix} \mathbf{x},$$

Solutions:

$$[s\boldsymbol{I} - \boldsymbol{A}] = \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 0 & -1 \\ k & 0 & 2 \\ -k & -2 & -k \end{bmatrix} = \begin{bmatrix} s & 0 & 1 \\ -k & s & -2 \\ k & 2 & s+k \end{bmatrix}$$

$$\Delta(s) = \det\left(\begin{bmatrix} s & 0 & 1 \\ -k & s & -2 \\ k & 2 & s+k \end{bmatrix}\right) = \mathrm{s}\begin{vmatrix} s & -2 \\ 2 & s+k \end{vmatrix} - 0\begin{vmatrix} -k & -2 \\ k & s+k \end{vmatrix} + 1\begin{vmatrix} -k & s \\ k & 2 \end{vmatrix}$$

$$= s(s^2 + ks + 4) - 2k - ks = s^3 + ks^2 + (4-k)s - 2k$$

$$\Delta(s) = s^3 + ks^2 + (4-k)s + 2k$$

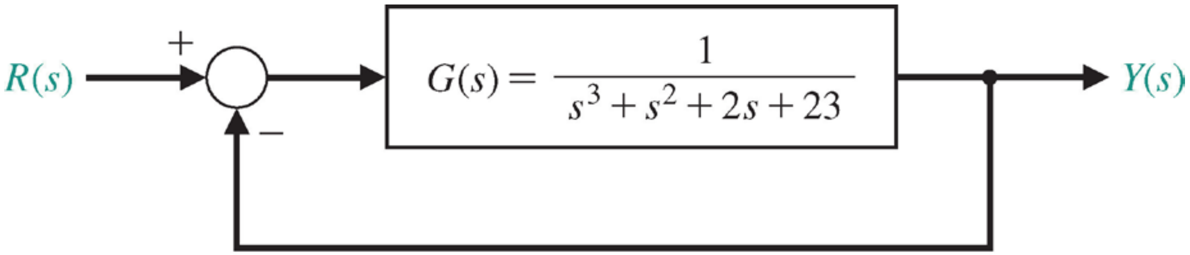
The Routh array is

$$\begin{array}{c|cc}
s^3 & 1 & 4-k \\
s^2 & k & -2k \\
s^1 & \dfrac{\begin{vmatrix} 1 & 4-k \\ k & -2k \end{vmatrix}}{-k} = 6-k & 0 \\
s^0 & -2k &
\end{array}$$

Therefore, any value of k will lead to instability.

Stability Analysis Using Matlab

pole function: compute poles of the closed-loop control system.



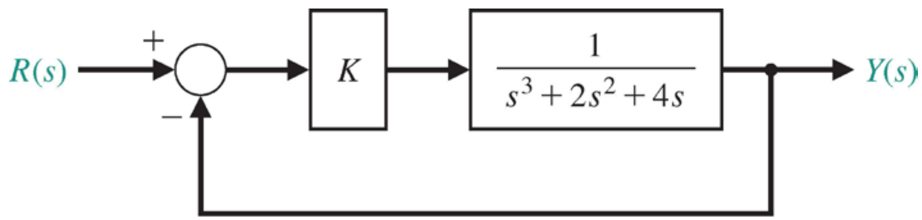

```
>>numg=[1]; deng=[1 1 2 23]; sysg=tf(numg,deng);
>>sys=feedback(sysg,[1]);
>>pole(sys)


ans =


     -3.0000
      1.0000 + 2.6458i
      1.0000 - 2.6458i
```

Unstable poles

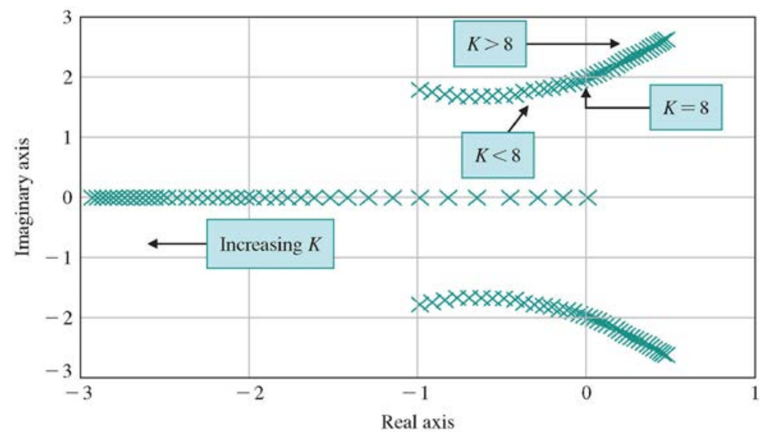
roots function: calculate root locations of $q(s) = s^3 + 2s^2 + 4s + K$ for $0 \le K \le 20$




```
% This script computes the roots of the characteristic
% equation q(s) = s^3 + 2 s^2 + 4 s + K for 0<K<20
%
K=[0:0.5:20];
for i=1:length(K)
   q=[1 2 4 K(i)];
   p(:,i)=roots(q);
end
plot(real(p),imag(p),'x'), grid
xlabel('Real axis'), ylabel('Imaginary axis')
```

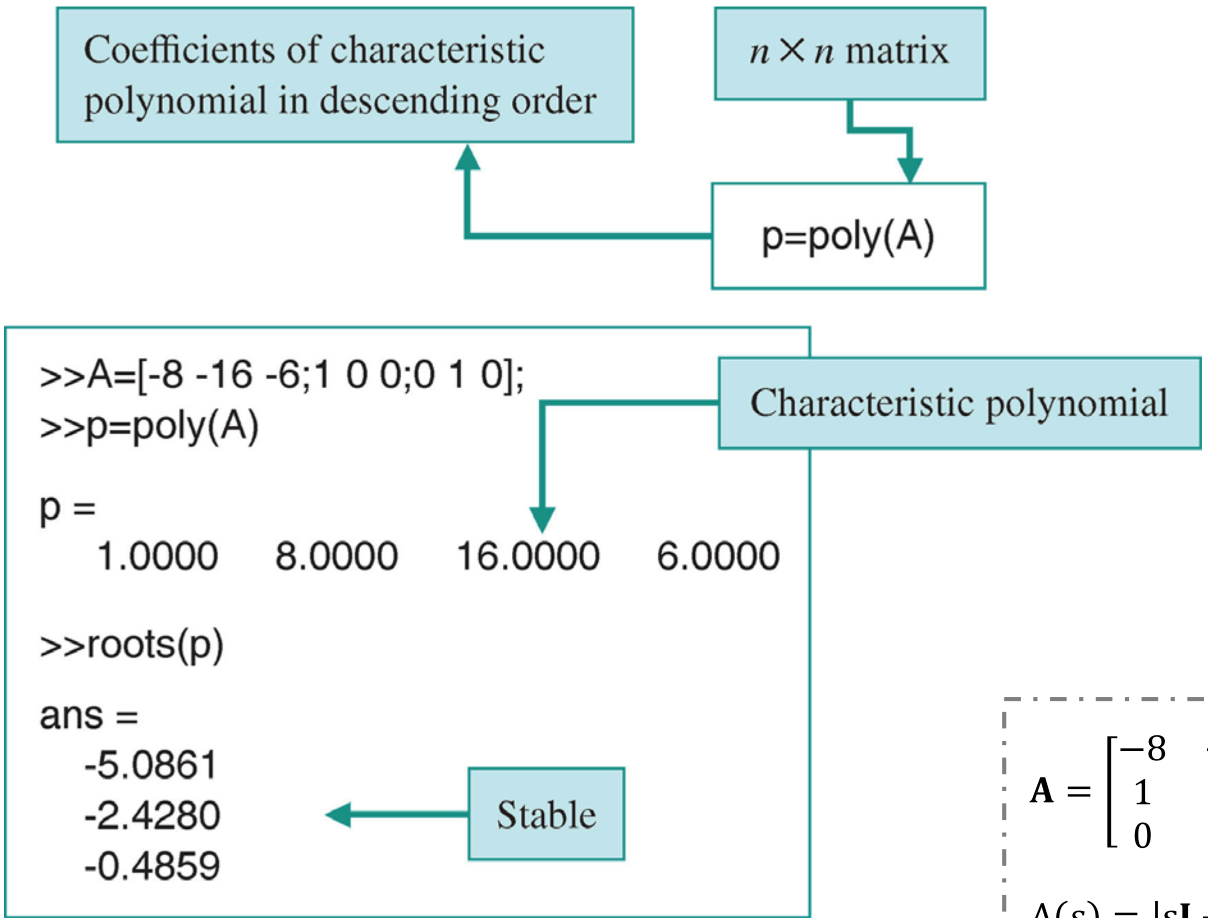
Loop for roots as a function of K

(b)



(a)

Computing the characteristic polynomial of **A** with the **poly** function.



$$\mathbf{A} = \begin{bmatrix} -8 & -16 & -6 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\Delta(s) = |s\mathbf{I} - \mathbf{A}| = s^3 + 8s^2 + 16s + 6$$

Quiz 17.1

For the following system, determine its stability.

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 3 \\ -3 & -4 & -5 \end{bmatrix} \mathbf{x},$$

Quiz 17.2

For the following system, find the value of k for which the system is stable.

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 3 \\ -9 & -k & -3 \end{bmatrix} \mathbf{x},$$

Thank You !