Xi'an Jiaotong-Liverpool University
西交利物浦大学

EEE204 Continuous and Discrete Time Signals and Systems II

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Electrical and Electronic Engineering

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Week 8

What is the difference between a causal signal and a causal system?

Causal Systems

A system is causal if, for every choice of n_0 , the output sequence value at the index $n = n_0$ depends only on the input sequence values for $n \leqslant n_0$.

$y[n] = (x[n])^2$ is a causal system.

$y[n] = x[n - n_d]$ is a causal system if $n_d \geqslant 0$.

$y[n] = \sum_{k=-\infty}^{n} x[k]$ is a causal system.

$w[n] = \log_{10}(|x[n]|)$ is a causal system.

$y[n] = x[Mn], M > 1$ is a non-causal system.

Causal Signals

In practical signal processing applications, DT input signals start at time $n = 0$. Signals that start at $n = 0$ are referred to as causal signals. The causal DT exponential function is given by,

$$x[n] = e^{sn}u[n] = \begin{cases} e^{sn}, & n \geqslant 0 \\ 0, & n < 0. \end{cases},$$

where we have used the unit step function to incorporate causality in the complex exponential functions.

Causal Signals

The same concept can be extended to derive causal implementations of sinusoidal and other non-causal signals.

$$x[n] = \sin(\omega n + \phi)u[n].$$

A causal DT signal is zero for $n < 0$.

A non-causal DT signal has values present for $n < 0$.

Anti-causal DT signals are zero for $n > 0$.

Rational z-transform

All of the previous examples had z-transforms that were rational functions, i.e., a ratio of two polynomials in z or z^{-1}.

$$X(z) = \frac{N(z)}{D(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \cdots + a_N z^{-N}} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}.$$

Without loss of generality, we assume $a_0 \neq 0$ and $b_0 \neq 0$, so we can rewrite

$$X(z) = \frac{b_0}{a_0} \cdot \frac{z^{-M}}{z^{-N}} \cdot \frac{z^M + \frac{b_1}{b_0} z^{M-1} + \cdots + \frac{b_M}{b_0}}{z^N + \frac{a_1}{a_0} z^{N-1} + \cdots + \frac{a_N}{a_0}} \triangleq \frac{b_0}{a_0} z^{N-M} \frac{N'(z)}{D'(z)}.$$

Rational z-transform

$$X(z) = \frac{b_0}{a_0}\frac{z^{-M}}{z^{-N}}\frac{z^M + \frac{b_1}{b_0}z^{M-1} + \cdots + \frac{b_M}{b_0}}{z^N + \frac{a_1}{a_0}z^{N-1} + \cdots + \frac{a_N}{a_0}} \triangleq \frac{b_0}{a_0}z^{N-M}\frac{N'(z)}{D'(z)}.$$

$N'(z)$ has M finite roots at $z_1, \cdots, z_M$, and $D'(z)$ has N finite roots at $p_1, \cdots, p_N$. So we can rewrite $X(z)$:

$$X(z) = \frac{b_0}{a_0}z^{N-M}\frac{(z-z_1)(z-z_2)\cdots(z-z_M)}{(z-p_1)(z-p_2)\cdots(z-p_N)}$$

or

$$\boxed{X(z) = Gz^{N-M}\frac{\prod_{k=1}^{M}(z-z_k)}{\prod_{k=1}^{N}(z-p_k)},}$$

where $G \triangleq \dfrac{b_0}{a_0}$.

Rational z-transform

$$\boxed{X(z) = Gz^{N-M}\frac{\prod_{k=1}^{M}(z - z_k)}{\prod_{k=1}^{N}(z - p_k)}}.$$

$X(z)$ has M finite zeros at $z_1, \cdots, z_M$.

$X(z)$ has N finite poles at $p_1, \cdots, p_N$.

If $N > M$, $X(z)$ has $N - M$ zeros at $z = 0$.

If $N < M$, $X(z)$ has $M - N$ poles at $z = 0$.

There can also be poles or zeros at $z = \infty$, depending if $X(\infty) = \infty$ or $X(\infty) = 0$.

Counting all of the above, there will be the same number of poles and zeros.

Pole-zero Plot

$$\boxed{X(z) = G z^{N-M} \frac{\prod_{k=1}^{M}(z - z_k)}{\prod_{k=1}^{N}(z - p_k)}.}$$

Due to the boxed form above, $X(z)$ is completely determined by its pole-zero locations up to the scale factor G.

The scale factor only affects the amplitude (or units) of the signal or system, whereas the poles and zeros affect the behaviour.

A pole-zero plot is a graphic description of rational $X(z)$, up to the scale factor. Use $\circ$ for zeros and $\times$ for poles.

Multiple poles or zeros indicated with adjacent number.

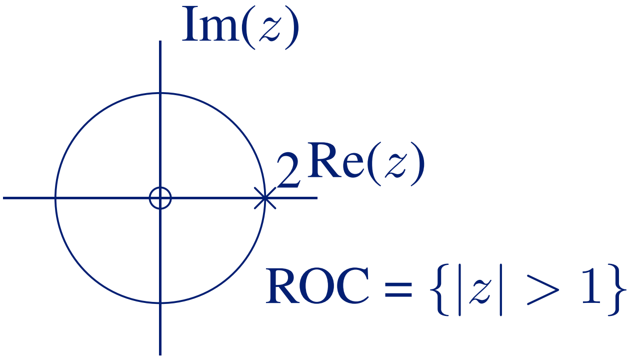
By definition, the ROC will NOT contain any poles.

Pole-zero Plot Example 1

$x[n] = nu[n]$(unit ramp signal)

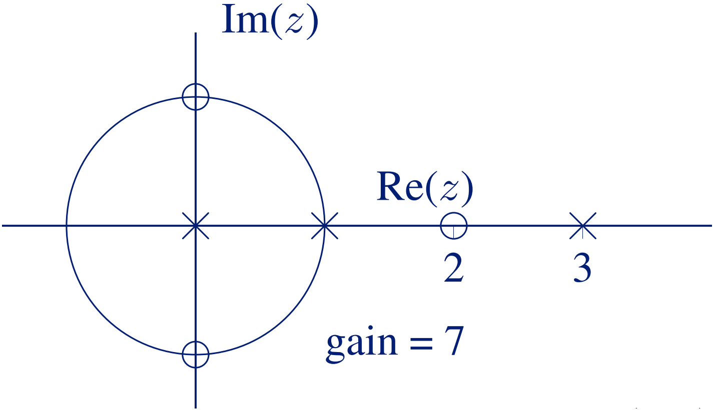
Previously showed that

$$X(z) = \frac{z}{(z-1)^2}, \{|z| > 1\}.$$



Pole-zero Plot Example 2

What are possible ROC's and $X(z)$?



$$X(z) = 7 \cdot \frac{(z-j)(z+j)(z-2)}{(z-0)(z-1)(z-3)} = 7 \cdot \frac{(1-jz^{-1})(1+jz^{-1})(1-2z^{-1})}{(1-z^{-1})(1-3z^{-1})},$$

ROC: $\{0 < |z| < 1\}, \{1 < |z| < 3\}$, or $\{|z| > 3\}$.

Single real pole:

$$x[n] = p^n u[n] \overset{z}{\leftrightarrow} X(z) = \frac{z}{z-p}.$$

Signal decays if pole is inside unit circle.

Signal blows up if pole is outside unit circle.

Signal alternates sign if pole is in left half plane, since $(-|p|)^n = (-1)^n |p|^n$.

Double real poles:

$$x[n] = np^n u[n] \overset{z}{\leftrightarrow} X(z) = -z\frac{\mathrm{d}}{\mathrm{d}z}\frac{z}{z-p} = \frac{pz}{(z-p)^2}.$$

Generalisation to multiple real poles?

You will learn this topic in more details from either EEE336 Signal Processing and Digital Filtering (A/Prof. Zhao Wang) or EEE411 Advanced Signal Processing (me).

How do we find an impulse response or general output for any input?

LTI System

As noted previously:

$$x[n] \to \boxed{\text{LTI } h[n]} \to y[n] = x[n]*h[n] \overset{z}{\leftrightarrow} \boxed{Y(z) = H(z)X(z).}$$

- Forward direction: transform $h[n]$ and $x[n]$, multiply, then inverse transform.
- Reverse engineering: put in known signal $x[n]$ with transform $X(z)$; observe output $y[n]$; compute transform $Y(z)$. Divide the two to get the system function or transfer function $\boxed{H(z) = Y(z)/X(z).}$
- The third rearrangement $\boxed{X(z) = Y(z)/H(z)}$ is also useful sometimes.

LTI System

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Impulse Response

Now apply these ideas to the analysis of LTI systems that are described by general linear constant-coefficient difference equations(LCCDE) (or just **diffeq** systems) to find impulse response $h[n]$: (Two approaches)

$$y[n] = -\sum_{k=1}^{N} a_k y[n-k] + \sum_{k=0}^{M} b_k x[n-k].$$

Applying linearity and shift properties taking z-transform of both sides of the above:

$$Y(z) = -\sum_{k=1}^{N} a_k z^{-k} Y(z) + \sum_{k=0}^{M} b_k z^{-k} X(z).$$

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Impulse Response

$$Y(z) = -\sum_{k=1}^{N} a_k z^{-k} Y(z) + \sum_{k=0}^{M} b_k z^{-k} X(z).$$

$$\left(1 + \sum_{k=1}^{N} a_k z^{-k}\right) Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z).$$

So, defining $a_0 \triangleq 1$,

$$\boxed{H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}.}$$

This is a rational system function. We can also see why studying rational z-transforms is very important.

Inversion of the z-transform

Methods for inverse z-transform:

- Table lookup, using properties.
- Contour integration.
- Series expansion into powers of z and z^{-1}.
- Partial-fraction expansion (PFE) and table lookup.

Practical problems requiring inverse z-transform?

- Given a system function $H(z)$, e.g., described by a pole-zero plot, find $h[n]$.
- When performing convolution via z-transforms: $Y(z) = H(z)X(z)$, leading to $y[n]$.

Table Lookup

Table Lookup Example

Find impulse response $h[n]$ for a system described by the following input-output relationship:

$$y[n] = x[n] - x[n-1].$$

Since the system is LTI, let $x[n] = \delta[n]$,

$$y[n] = h[n] = \delta[n] - \delta[n-1].$$

Or, write z-transforms:

$$Y(z) = X(z) - z^{-1}X(z) \rightarrow Y(z) = (1 - z^{-1})X(z).$$

$$H(z) = \frac{Y(z)}{X(z)} = 1 - z^{-1} \rightarrow h[n] = \delta[n] - \delta[n-1].$$

Table Lookup Example

Find impulse response $h[n]$ for a system described by the following input-output relationship:

$$y[n] = -y[n-2] + x[n].$$

It is not easy to use the first approach, write z-transforms:

$$Y(z) = -z^{-2}Y(z) + X(z) \to (1+z^{-2})Y(z) = X(z).$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1+z^{-2}} \to h[n] = \cos(n\pi/2)u[n].$$

Note that there is more than one choice for the inverse z-transform since ROC never discussed. Why did I choose the causal sequence? Because all LTI systems described by difference equations are causal.

Contour Integration

Contour Integration

$$\boxed{x[n] = \frac{1}{2\pi j} \oint X(z) z^{n-1} \, \mathrm{d}z.}$$

The integral is a contour integral over a closed path C that must

- enclose the origin,
- lie in the ROC of $X(z)$.

Cauchy residue theorem,

$$\frac{1}{2\pi j} \oint z^{n-1-k} \, \mathrm{d}z = \begin{cases} 1, & k = n \\ 0, & k \neq n \end{cases} = \delta[n-k]$$

Contour Integration

$$\boxed{x[n] = \frac{1}{2\pi j}\oint X(z)z^{n-1}\,\mathrm{d}z.}$$

At first sight, the above contour integration may appear to be a formidable task.

However, when the z-transform is a rational function whose contour integration can be easily evaluated by using the residue theorem.

According to the residue theorem,

$$\boxed{x[n] = \frac{1}{2\pi j}\oint X(z)z^{n-1}\,\mathrm{d}z = \sum_{k=1}^{P}\mathrm{Res}_{z\to p_k}\left[X(z)z^{n-1}\right].}$$

Contour Integration

$$\boxed{x[n] = \frac{1}{2\pi j}\oint X(z)z^{n-1}\,\mathrm{d}z = \sum_{k=1}^{P} \mathrm{Res}_{z\to p_k}[X(z)z^{n-1}],}$$

where $\mathrm{Res}_{z\to p_k}[X(z)z^{n-1}]$ and P are the residue of pole p_k and the number of poles $X(z)z^{n-1}$, respectively.

For a first-order pole

$$\mathrm{Res}_{z\to p_k}[X(z)z^{n-1}] = \lim_{z\to p_k}[(z-p_k)X(z)z^{n-1}].$$

For a pole of order m,

$$\mathrm{Res}_{z\to p_k}[X(z)z^{n-1}] = \frac{1}{(m-1)!}\lim_{z\to p_k}\frac{\mathrm{d}^{m-1}}{\mathrm{d}z^{m-1}}[(z-p_k)^m X(z)z^{n-1}].$$

Contour Integration Example

Find the inverse z-transform of

$$X(z) = \frac{1}{2(z-1)\left(z+\frac{1}{2}\right)}$$

For $n = 0$,

$$x[n] = \frac{1}{2\pi j}\oint X(z)z^{n-1}\,\mathrm{d}z = \sum_{k=1}^{P} \text{Res}_{z\to p_k}[X(z)z^{n-1}],$$

$$x[0] = \sum_{k=1}^{3} \text{Res}_{z\to p_k}[X(z)z^{-1}] = \left.\frac{1}{2(z-1)\left(z+\frac{1}{2}\right)}\right|_{z=0}$$

$$+ \left.\frac{1}{2z\left(z+\frac{1}{2}\right)}\right|_{z=1} + \left.\frac{1}{2z(z-1)}\right|_{z=-\frac{1}{2}} = -1 + \frac{1}{3} + \frac{2}{3} = 0.$$

Contour Integration Example

Find the inverse z-transform of

$$X(z) = \frac{1}{2(z-1)\left(z+\frac{1}{2}\right)}$$

For $n > 0$, which is equivalent to $n \geqslant 1$,

$$x[n] = \frac{1}{2\pi j}\oint X(z)z^{n-1}\,\mathrm{d}z = \sum_{k=1}^{P} \mathrm{Res}_{z\to p_k}[X(z)z^{n-1}],$$

$$x[n] = \sum_{k=1}^{2} \mathrm{Res}_{z\to p_k}[X(z)z^{n-1}],$$

$$= \left.\frac{z^{n-1}}{2(z+\frac{1}{2})}\right|_{z=1} + \left.\frac{z^{n-1}}{2(z-1)}\right|_{z=-\frac{1}{2}} = \frac{1}{3} - \frac{1}{3}\left(-\frac{1}{2}\right)^{n-1}.$$

Contour Integration Example

For $n < 0$, from the initial-value theorem (P.775 Table 10.1)

$$x[0] = \lim_{z\to\infty} X(z) = \lim_{z\to\infty} \frac{1}{2(z-1)\left(z+\frac{1}{2}\right)} = 0.$$

This is the same as we worked out $x[0] = 0$ before, which means

$$x[n] = 0, \text{ for } n < 0$$

.

Therefore, for any value of n, we have

$$x[n] = \left[\frac{1}{3} - \frac{1}{3}\left(-\frac{1}{2}\right)^{n-1}\right] u[n-1].$$

Series Expansion

Series Expansion

If we can expand the z-transform into a power series (considering its ROC), then "by the uniqueness of the z-transform:"

$$\boxed{\text{if } X(z) = \sum_{n=-\infty}^{\infty} c_n z^{-n} \text{ then } x[n] = c_n,}$$

i.e., the signal sample values in the time-domain are the corresponding coefficients of the power series expansion.

Series Expansion Example 1

Find impulse response $h[n]$ for the system described by:

$$y[n] = 2y[n-3] + x[n].$$

Take z-transforms on both sides:

$$Y(z) = 2z^{-3}Y(z) + X(z) \rightarrow (1 - 2z^{-3})Y(z) = X(z).$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1-2z^{-3}} = \sum_{n=0}^{\infty} \left(2z^{-3}\right)^n.$$

$$H(z) = 1 + 2z^{-3} + 2^2 z^{-6} + \cdots$$

$$h[n] = \delta[n] + 2\delta[n-3] + 2^2\delta[n-6] + \cdots = \sum_{k=0}^{\infty} 2^k \delta[n-3k].$$

Series Expansion Example 2

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The previous case was easy since the power series was just the familiar geometric series.

In general one must use tedious long division if the power series is not easy to find. For example,

$$H(z) = \frac{2}{(z-1)^2} = \frac{2}{z^2 - 2z + 1}.$$

$$H(z) = 2z^{-2} + 4z^{-3} + 6z^{-4} + \cdots .$$

$$h[n] = 2\delta[n-2] + 4\delta[n-3] + 6\delta[n-4] + \cdots$$
$$= \sum_{k=0}^{\infty} 2(k+1)\delta[n-k-2] = 2(n-1)u[n-2].$$

Series Expansion Example 3

Find impulse response $h[n]$ for the system described by:

$$y[n] = 2y[n-3] + x[n] + 5x[n-1].$$

Take z-transforms on both sides:

$$Y(z) = 2z^{-3}Y(z) + X(z) + 5z^{-1}X(z).$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1+5z^{-1}}{1-2z^{-3}} = \frac{1}{1-2z^{-3}} + \frac{5z^{-1}}{1-2z^{-3}}.$$

$$h[n] = \sum_{k=0}^{\infty} 2^k \delta[n-3k] + 5\sum_{k=0}^{\infty} 2^k \delta[n-1-3k].$$

Series Expansion Example 4

Find impulse response $h[n]$ for the system described by:

$$y[n] = 2y[n+3] + x[n+1].$$

Take z-transforms on both sides:

$$Y(z) = 2z^3 Y(z) + zX(z) \rightarrow (1-2z^3)Y(z) = zX(z).$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z}{1-2z^3} = z\sum_{n=0}^{\infty}\left(2z^3\right)^n = \sum_{n=0}^{\infty} 2^n z^{3n+1}.$$

$$h[n] = \sum_{k=0}^{\infty} 2^k \delta[n+(3k+1)].$$

But we still need a systematic method for general cases.

We will discuss PFE in more details next week.

Homework

- Page 757–763, read section 10.3;
- Page 797, Q10.4;
- Page 797, Q10.5: (a)–(c);
- Page 798, Q10.6: (a)–(d);
- Page 798, Q10.7–Q10.8;
- Page 798, Q10.10;
- Page 799, Q10.11.

Thank you for your attention.