Chapter 1.8 Exponential, Normal and Rayleigh Distributions

1.8.1 Exponential Distribution

1.8.2 Normal Distribution

1.8.3 Rayleigh

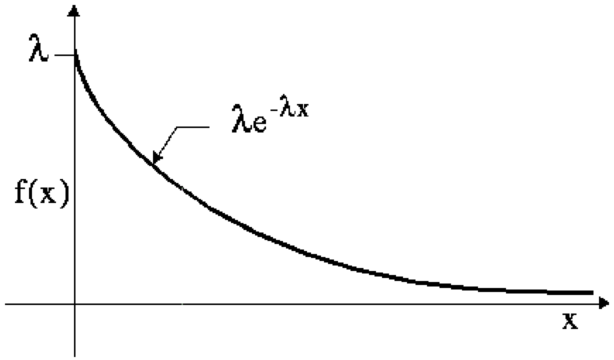
1.8.4 Summary

April 8, 2017

1.8.1 Exponential Distribution, pdf

The random variable X follows an exponential distribution if its pdf is

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & ; \quad x \geq 0 \\ 0 \quad ; & \text{otherwise} \end{cases}.$$



We write $X \sim \mathrm{Exp}(\lambda)$ for a random variable that has exponential distribution.

1.8.1 Exponential Distribution, cdf

The cumulative distribution of an exponential variable is obtained as

$$F(x) = P(X \leq x) = \int_0^x \lambda e^{-\lambda u} du = \left[-e^{-\lambda u}\right]_0^x = 1 - e^{-\lambda x}, x \geq 0$$

$$F(x) = 0 \text{ if } x < 0.$$

1.8.2 Exponential Distribution

The mean and variance of X are respectively

$E(X)=\frac{1}{\lambda}$ and $\mathrm{V(X)} = \frac{1}{\lambda^2}$.

We saw the proof for the mean in Tutorial 2 using integration by parts. For the variance one needs to apply integration by parts twice.

1.8.1 Exponential Distribution

The Exponential distribution often arises as the distribution of the length of time until some specific event occurs. For example, the length of time (starting from now) until an earthquake occurs.

We will meet again this distribution when studying the Poisson Process

1.8.1 Exponential Distribution, **Example 1**

Suppose that the length of a phone call in minutes is an exponential random variable with parameter $\lambda = \frac{1}{10}$. If someone arrives immediately ahead of you at a public telephone booth, find the probability that you will have to wait more than 10 minutes.

1.8.1 Exponential Distribution, **Example 1 solution**

Solution

Let X denote the length of the call made by the person in the booth. Then required probability is

$$P(X > 10) = 1 - \int_0^{10} 0.1e^{-0.1x}\,dx = 1 - [-e^{-0.1x}]_0^{10} = 1 - (1 - e^{-1}) = e^{-1} \approx 0.3679$$

■

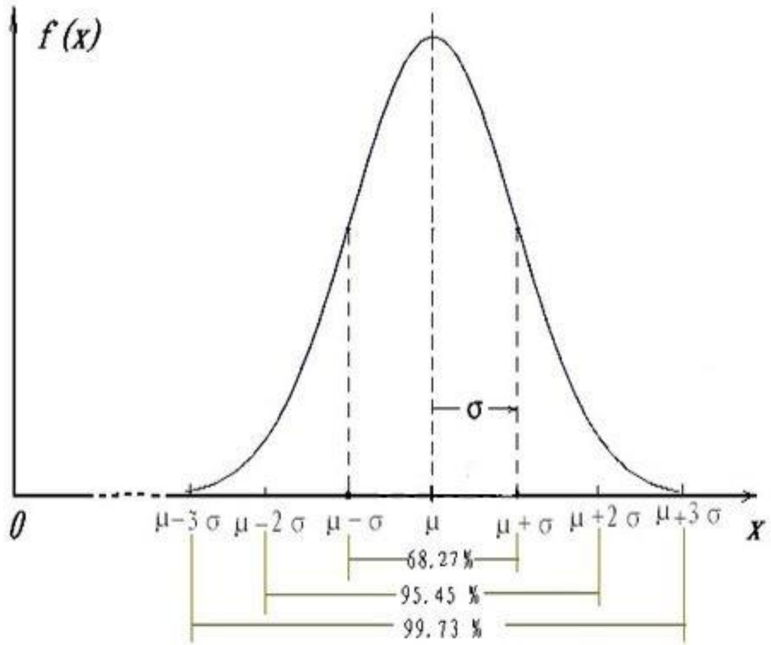
1.8.2 Normal Distribution

The random variable X follows a normal distribution if its pdf is

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2} \quad ; \quad -\infty < x < \infty.$$

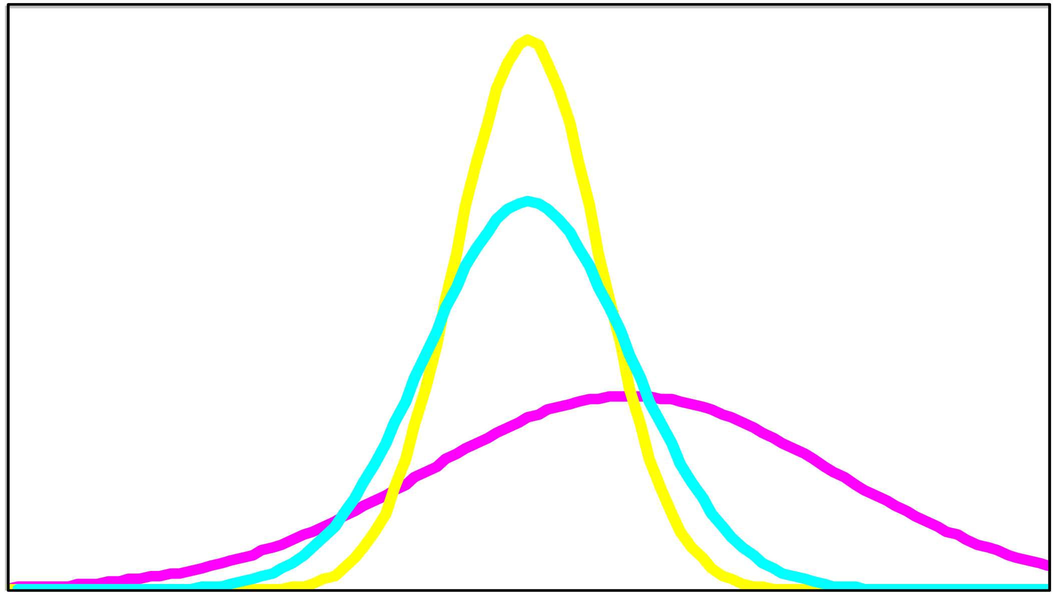
We write $X \sim \mathrm{N}(\mu, \sigma^2)$ for a random variable that has normal distribution. The mean and variance of X are respectively μ and σ^2.

The pdf is a bell-shaped curve symmetric about μ.



1.8.2 Characterisation of the Normal

- The Normal distribution is identified by the two parameters μ (the mean) and σ (the standard deviation). These change the position and spread of the distribution over the real line



By varying the parameters **μ** and **σ**, we obtain different normal distributions.

Usefulness of the Normal distribution

The Normal or Gaussian or random errors distribution is a “natural” distribution often observed in nature.
It is the limiting distribution for many other distributions.

It is assumed that “extreme values” (faraway from the mean) are rare.

1.8.2 Normal Distribution

Given pdf $f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}$, we cannot integrate f to obtain the required area. Instead we shall use statistical table.

In statistical table, we only have values corresponding to $Z \sim \mathrm{N}(0,1)$.

i.e. Z is standard normal random variable with pdf $f(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$.

Note: this was useful when computers were not available but nowadays we don't need it anymore (it is still useful in exams ☺).

1.8.2 Normal Distribution

Theorem For $X \sim \mathrm{N}(\mu, \sigma^2)$, standard normal distribution can be obtained by *standardizing* the variable as $\mathrm{z} = \frac{X-\mu}{\sigma} \sim \mathrm{N}(0,1)$.

Proof

Consider distribution function of $Z = \frac{X-\mu}{\sigma}$:

$F_Z(z) = \mathrm{P}(Z \leq z) = \mathrm{P}\left(\frac{X-\mu}{\sigma} \leq z\right) =$

$\mathrm{P}(X \leq \mu + z\sigma) = F_X(\mu + z\sigma)$

1.8.2 Normal Distribution

Consider the Normal cdf: $\mathrm{F_X(x)} = \int \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}} dx$

We change the variable to $z = \frac{(x-\mu)}{\sigma}$, with $dz = \frac{1}{\sigma} dx$. Then,

$$\mathrm{F}_X(\mathrm{x}) = \int \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}} \frac{1}{\sigma} dx = \int \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz = F_Z(z)$$

Hence $Z = \frac{X-\mu}{\sigma} \sim \mathrm{N}(0,1)$. ■

1.8.2 Normal Distribution

Practical meaning: the normal density cannot be found by integration (no closed formula solution). Then we need to do it numerically. Values have been computed for a standard normal $z \sim N(0,1)$. Therefore, if $X \sim N(\mu, \sigma^2)$ we can obtain

$P(X \leq x) = P\left(z \leq \frac{x-\mu}{\sigma}\right)$ by using the tables.

1.8.2 Normal Distribu

$$P(Z \le -2.72) = 0.0033$$

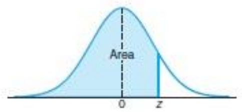


Table A.3 Areas under the Normal Curve

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
−3.4	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0002
−3.3	0.0005	0.0005	0.0005	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0003
−3.2	0.0007	0.0007	0.0006	0.0006	0.0006	0.0006	0.0006	0.0005	0.0005	0.0005
−3.1	0.0010	0.0009	0.0009	0.0009	0.0008	0.0008	0.0008	0.0008	0.0007	0.0007
−3.0	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
−2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
−2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
−2.7	0.0035	0.0034	$\boxed{0.0033}$	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
−2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
−2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
−2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
−2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
−2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
−2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
−2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
−1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
−1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
−1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
−1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
−1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
−1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
−1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
−1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
−1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
−1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
−0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
−0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
−0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
−0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
−0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
−0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
−0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
−0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
−0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
−0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641

1.8.2 Normal Distribut

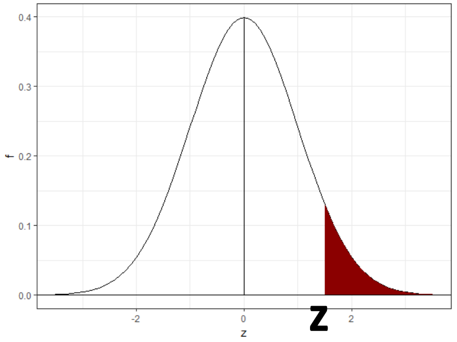
$$P(Z \leq 0.72) = 0.7642$$

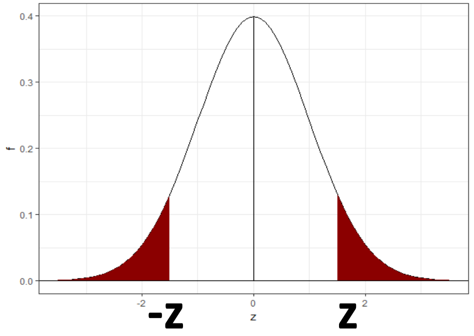
Table A.3 (continued) Areas under the Normal Curve

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	$\boxed{0.7642}$	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

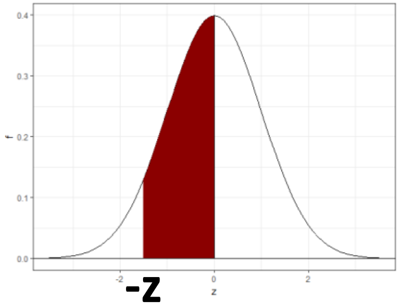
1.8.2 Normal distribution: tips & tricks

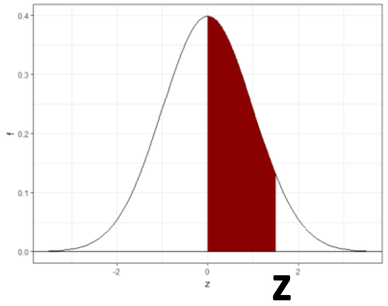
- $P(Z > z) = 1 - P(X < z) =$
- $P(Z > z) = P(Z \le -z);$





- $P(z \le 0) = P(z > 0) = 0.5;$
- $P(-z < Z \le 0) = 0.5 - P(Z \le -z);$
- $P(0 < Z \le z) = P(-z < Z \le 0) =$
  $0.5 - P(Z \le -z);$
- $P(z_1 < Z \le z_2) = P(Z \le z_2) - P(Z \le z_1);$





1.8.2 Normal Distribution

Example 3

If X is a normal random variable with parameters $\mu = 3$ and $\sigma^2 = 9$, find

i. $P(X \leq 3)$

ii. $P(X \leq 2)$

iii. $\mathrm{P}(2 < X < 5)$

iv. $\mathrm{P}(X > 0)$

v. $\mathrm{P}(|X - 3| > 6)$

1.8.2 Normal Distribution

Solution

i. $P(X < 3) = P\left(\frac{X-3}{3} \leq \frac{3-3}{3}\right) = P(Z \leq 0) = 0.5$

That is: the mean is equal to the median. This is always true for all Normal variables (N is symmetric about the mean)

Table A.3 Areas under the Normal Curve

z	.00	.01	.02	.03	.04
−3.4	0.0003	0.0003	0.0003	0.0003	0.0003
−3.3	0.0005	0.0005	0.0005	0.0004	0.0004
−3.2	0.0007	0.0007	0.0006	0.0006	0.0006
−0.4	0.3446	0.3409	0.3372	0.3336	
−0.3	0.3821	0.3783	0.3745	0.3707	
−0.2	0.4207	0.4168	0.4129	0.4090	
−0.1	0.4602	0.4562	0.4522	0.4483	
−0.0	0.5000	0.4960	0.4920	0.4880	

1.8.2 Normal Distribution

Solution

ii. $P(X < 2) = P\left(\frac{X-3}{3} \leq \frac{2-3}{3}\right) = P\left(Z \leq -\frac{1}{3}\right)$

$= P(Z \leq -0.33) = 0.3707$

Table A.3 Areas under the Normal Curve

z	.00	.01	.02	.03	.04
−3.4	0.0003	0.0003	0.0003	0.0003	0.0003
−3.3	0.0005	0.0005	0.0005	0.0004	0.0004
−3.2	0.0007	0.0007	0.0006	0.0006	0.0006
−0.4	0.3446	0.3409	0.3372	0.3336	
−0.3	0.3821	0.3783	0.3745	0.3707	
−0.2	0.4207	0.4168	0.4129	0.4090	
−0.1	0.4602	0.4562	0.4522	0.4483	
−0.0	0.5000	0.4960	0.4920	0.4880	

1.8.2 Normal Distribution

Solution

iii. $\mathrm{P}(2 < X < 5) = P\left(\frac{2-3}{3} < \frac{X-3}{3} < \frac{5-3}{3}\right)$

$= P\left(-\frac{1}{3} < Z < \frac{2}{3}\right)$

$= P\left(-\frac{1}{3} < Z < \frac{2}{3}\right)$

$= P\left(Z < \frac{2}{3}\right) - P\left(Z \leq -\frac{1}{3}\right)$

$= P(Z < 0.67) - P(Z \leq -0.33)$

$= 0.7486 - 0.3707 = 0.3779 \approx 0.378$

1.8.2 Normal Distribution

iv. $\mathrm{P}(X > 0) = P\left(\frac{X-3}{3} > \frac{0-3}{3}\right)$

$= P(Z > -1)$

$= 1 - P(Z \leq -1)$

$= 1 - 0.1587$

$= 0.8413$

≈ 0.841

1.8.2 Normal Distribution

v. $\mathrm{P}(|X-3|>6)$

$= \mathrm{P}(X-3>6 \text{ or } X-3<-6)$

$= \mathrm{P}(X-3>6)+\mathrm{P}(X-3<-6)$

$= \mathrm{P}\left(\frac{X-3}{3}>\frac{6}{3}\right)+\mathrm{P}\left(\frac{X-3}{3}<\frac{-6}{3}\right)$

$= \mathrm{P}(Z>2)+\mathrm{P}(Z<-2) \quad = 1-\mathrm{P}(Z\leq 2)+\mathrm{P}(Z<-2)$

$= 1-0.9772+0.0228 = 0.0456$

More quickly $[= 2P(Z<-2) = 2*0.0228 \; = 0.0456]$

■

1.8.2 Normal Distribution: thin tails

- Extreme values are considered "rare" for Normal variates because the "tails" of the curve are thin. That is to say:
- Values "far away" from the mean have low probability. We can measure this by considering $\mathrm{P}(|\mathrm{X}| \geq \mu \pm k\sigma)$. From the tables we find

$$\mathrm{P}(\mu - k\sigma < X \leq \mu + k\sigma) = P(Z > -k) + P(Z \leq k)$$

-

σ	2σ	3σ	4σ
61.8.27%	95.45%	99.73%	99.99%

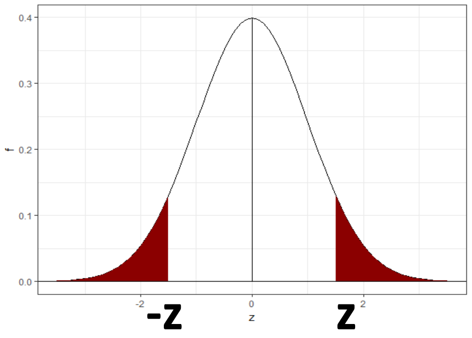
almost all values within $\pm 3\sigma$

1.8.2 Normal Distribution: problem

- How can you compute this probability quickly from the tables?
- $\mathrm{P}(\mu - k\sigma < X \leq \mu + k\sigma) = P(Z > -k) + P(Z \leq k)$

1.8.2 Normal Distribution: problem solution

- How can you compute this probability quickly from the tables?
- $\mathrm{P}(\mu - k\sigma < X \leq \mu + k\sigma) = P(Z > -k) + P(Z \leq k)$
- Since
  $P(Z > -k) + P(Z \leq k) = 1 - \big(P(Z \leq -k) + P(Z > k)\big)$
  and $P(Z \leq -k) = P(Z > k)$,
- $\mathrm{P}(\mu - k\sigma < X \leq \mu + k\sigma) = 1 - 2P(Z \leq -k)$



1.8.2 Normal Distribution

Example 4

The life, in years, of a randomly chosen battery is normally distributed with mean 2 and a standard deviation of 0.4 .

i. Explain why the life of a battery can be modeled by a normal distribution even though the life of a battery cannot possibly be negative.

ii. Find the probability that a randomly chosen battery has a life less than 1 year. Give your answer corrected to 3 significant figures.

iii. A man buys 10 randomly chosen batteries. Find the probability that at least two have lives each exceeding two years.

1.8.2 Normal Distribution

Solution

i. The probability of getting negative values is negligible since they are at least 5 standard deviations away from the mean in the normal distribution.

ii. Let $X \sim \mathrm{N}(2{,}0.4^2)$. Required probability is

$$P(X < 1) = P\left(\frac{X-2}{0.4} < \frac{1-2}{0.4}\right)$$

$$= P(Z < -2.5) = 0.0062$$

1.8.2 Normal Distribution

iii. Let $X \sim \mathrm{N}(2, 0.4^2)$. We first find

$P(X > 2) = P\left(\frac{X-2}{0.4} > \frac{2-2}{0.4}\right)$

$= P(Z > 0) = 0.5$

Let $Y \sim \mathrm{Bin}(10, 0.5)$. Required probability is

$P(Y \geq 2) = 1 - P(Y = 0) - P(Y = 1)$

$$= 1 - \binom{10}{0}(0.5)^0(0.5)^{10} - \binom{10}{1}(0.5)^1(0.5)^9$$

$= 0.98925 \approx 0.989$ ■

1.8.2 Normal Distribution: quantiles

- Since $z = \frac{x-\mu}{\sigma}$,

$$x = \mu + z\sigma$$

Example: find the third quartile of a $N(5, 2.5^2)$

Solution:

$$P(X \leq Q_3) = 0.75 \to P\left(Z \leq \frac{Q_3 - 5}{2.5}\right) = P(Z \leq z) = 0.75$$

After finding z, $Q_3 = 5 + 2.5z$

1.8.2 Normal Distribution: quantiles

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	$\boxed{0.7486}$	$\boxed{0.7517}$	0.7549

- Since we can't find exactly 0.75 we average the z's :

$$z = \frac{0.67+0.68}{2} = 0.675 \text{ and } Q_3 = 5 + 2.5 * 0.675 = 6.6875$$

1.8.3 Rayleigh distribution

- The Rayleigh distribution is the distribution of the length of a vector when the horizontal and vertical components are Normally distributed.

- I prepared a set of slides that you can use as notes for studying.

1.8.3 Rayleigh distribution: summary

- $\mathrm{R} = \sqrt{X^2 + Y^2}, X, Y \sim N(0, \sigma^2)$

$$f(r) = \frac{r}{\sigma} e^{-\frac{1}{2}\frac{r^2}{\sigma^2}}, r > 0;$$

- $F(r) = P(R \leq r) = 1 - e^{-\frac{1}{2}\frac{r^2}{\sigma^2}}, r > 0$
- $E(R) = \sigma\sqrt{\frac{\pi}{2}}, V(X) = \frac{4-\pi}{2}\sigma^2$

1.8.4 Summary

- Exponential $P(X \leq x) = 1 - e^{-\lambda x}, x > 0$
- Normal $X \sim N(\mu, \sigma^2) \rightarrow Z = \frac{x-\mu}{\sigma} \sim N(0,1)$
  - $P(X \leq x) = P\left(z \leq \frac{x-\mu}{\sigma}\right)$
  - $f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}}$
- Rayleigh $\mathrm{r} = \sqrt{X^2 + Y^2}, X, Y \sim N(0, \sigma^2)$

$$f(r) = \frac{r}{\sigma} e^{-\frac{1}{2}\frac{r^2}{\sigma^2}} \; r > 0;$$