Tutorial 10 D'Alembert's solution of wave equation and types of PDEs

1. Using the d'Alembert's solution, solve

$$\frac{\partial^2 y}{\partial t^2} = 9\frac{\partial^2 y}{\partial x^2}, \quad \text{for } 0 < x < 1, t > 0;$$

Where $y(0,t) = y(1,t) = 0$ for $t \geq 0$, and $y(x,0) = \sin(2\pi x)$, $y_t(x,0) = \sin(3\pi x)$.

The d'Alembert's solution of wave equation is

$$y(x,t) = \frac{1}{2}[F(x+ct) + F(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds,$$

where $y(x,0) = f(x)$, $y_t(x,t) = g(x)$ and F is the odd periodic extension of f.

Here $c = 3$, and the period of F $2L = 2$ $\therefore L = 1$.

Since $\sin(2\pi x)$ is already odd and periodic, so $F(x) = f(x) = \sin(2\pi x)$.

$$\therefore y(x,t) = \frac{1}{2}[\sin 2\pi(x+3t) + \sin 2\pi(x-3t)] + \frac{1}{2\times 3}\int_{x-3t}^{x+3t} \sin 3\pi s\,ds.$$

$$= \frac{1}{2}[\sin(2\pi x + 6\pi t) + \sin(2\pi x - 6\pi t)] + \frac{1}{6}\cdot\frac{-1}{3\pi}\cdot[\cos 3\pi s]_{x-3t}^{x+3t}$$

$$= \frac{1}{2}\times 2\cdot\sin\left(\frac{2\pi x + 6\pi t + 2\pi x - 6\pi t}{2}\right)\cos\left(\frac{2\pi x + 6\pi t - 2\pi x + 6\pi t}{2}\right) - \frac{1}{18\pi}[\cos 3\pi(x+3t) - \cos 3\pi(x-3t)]$$

$$= \sin 2\pi x \cos 6\pi t - \frac{1}{18\pi}\cdot(-2)\sin\frac{3\pi x + 9\pi t + 3\pi x - 9\pi t}{2}\sin\frac{3\pi x + 9\pi t - 3\pi x + 9\pi t}{2}$$

$$= \sin 2\pi x \cos 6\pi t + \frac{1}{9\pi}\cdot\sin 3\pi x \sin 9\pi t.$$

Therefore, the solution of the wave equation is

$$y(x,t) = \sin(2\pi x)\cos(6\pi t) + \frac{1}{9\pi}\sin(3\pi x)\sin(9\pi t)$$

2. Find the type of the following PDEs.
   (1) $u_{xx} + 4u_{yy} = 0$
   (2) $u_{xx} + 2u_{xy} + u_{yy} = 0$
   (3) $u_{xx} + 5u_{xy} + 4u_{yy} = 0$
   (4) $xu_{xx} - yu_{xy} = 0$
   (5) $u_{xx} - 4u_{xy} + 5u_{yy} = 0$

(1). $A=1,\ C=4,\ B=0$. $\therefore AC-B^2 = 4>0$.

So the PDE is elliptic.

(2). $A=1,\ B=1,\ C=1,\ AC-B^2=0$.

So the PDE is parabolic.

(3). $A=1,\ B=\frac{5}{2},\ C=4,\quad AC-B^2 = 4-\frac{25}{4} = -\frac{9}{4}<0$.

So the PDE is hyperbolic.

(4). $A=x,\ B=-\frac{y}{2},\ C=0,\quad AC-B^2 = 0-\frac{y^2}{4}$

$\therefore$ if $y=0$, then $AC-B^2=0$, the PDE is parabolic.

if $y\neq 0$, then $AC-B^2=-\frac{y^2}{4}<0$, the PDE is hyperbolic.

(5). $A=1,\ B=-2,\ C=5,\quad AC-B^2=5-4=1>0$.

So the PDE is elliptic.