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EEE220 Instrumentation and Control System

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Lecture 18

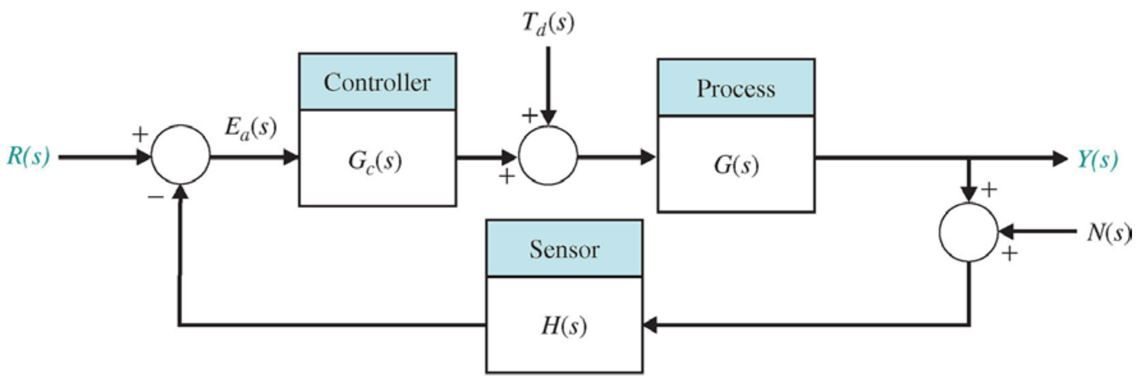
Outline

The Root Locus Method

- ❑ **The Root Locus Concept**
- ❑ **The Root Locus Procedure**
- ❑ The Root Locus Using Matlab
- ❑ Parameter Design by the Root Locus Method
- ❑ PID Controllers
  - Concept
  - PID Tuning
- ❑ Design Examples

Introduction

The relative stability and the transient response of a closed-loop control system are directly related to the location of the poles of the closed-loop transfer functions (i.e., roots of the characteristic equations) in the s-plane.



Closed-loop transfer function:

$$T(s) = \frac{G_c(s)G(s)}{1 + G_c(s)G(s)H(s)}$$

Characteristic equation:

$$\Delta(s) = 1 + G_c(s)G(s)H(s) = 0$$

Basic Concept

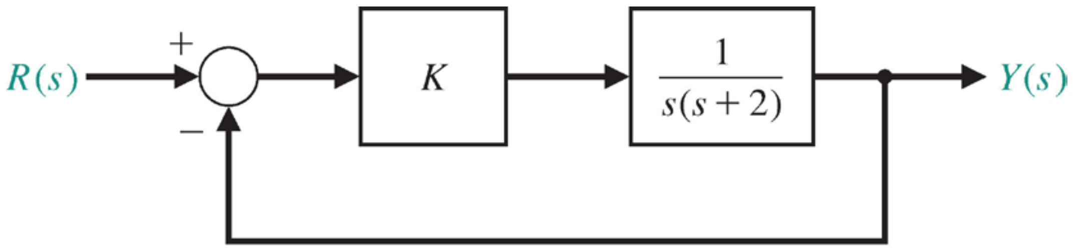
- As the system performance is closely related with the location of its poles, in practice, it is frequently necessary to adjust one or more system parameters in order to obtain suitable root locations.
- Therefore, it is worthwhile to determine how the roots of the characteristic equation of a given system migrate about the s-plane as the parameters are varied.

The root locus is the path of the roots of the characteristic equation traced out in the s-plane as a system parameter varies from zero to infinity.

- The **root locus** method was introduced by Evans in 1948 and has been developed and utilized extensively in control engineering practice. It is a graphical method for sketching the locus of roots in the s-plane as a parameter is varied.
- The root locus method provides graphical information, and therefore an approximate sketch can be used to obtain ***qualitative*** information concerning the stability and performance of the system. If the root locations are not satisfactory, the necessary parameter adjustments often can be ascertained from the root locus.

Understanding the Root Locus: Example 18.1

Consider the second-order system shown in the following figure



- The characteristic equation is

$$\Delta(s) = 1 + KG(s) = 1 + \frac{K}{s(s+2)} = 0$$

The locus of the roots are the gain K is varied is found by requiring that

$$|KG(s)| = \left|\frac{K}{s(s+2)}\right| = 1$$

and

$$\angle KG(s) = \pm 180^o, \pm 540^o, \ldots$$

- Alternatively, the characteristic equation can be written as

$$\Delta(s) = s^2 + 2s + K = 0$$

Standard form:
$$\Delta(s) = s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

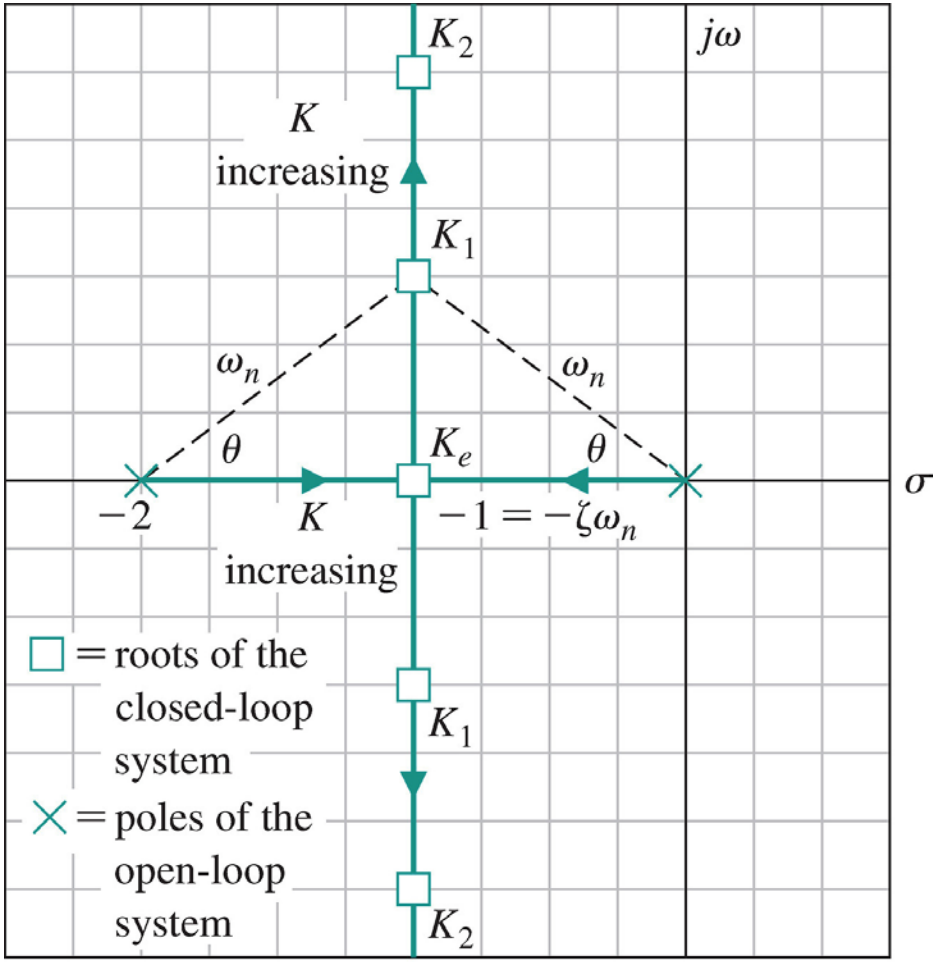
Poles of the closed-loop system are

for $K \leq 1$

$$s_1, s_2 = -1 \pm \sqrt{1-K}$$

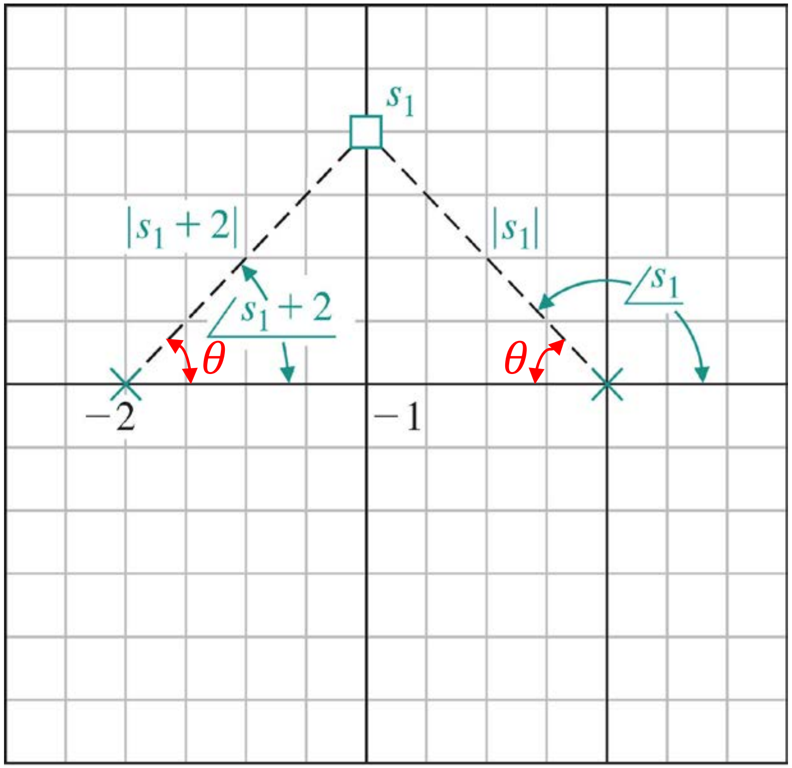
for $K > 1$

$$s_1, s_2 = -1 \pm j\sqrt{K-1}$$



The root locus is shown as green heavy lines, with arrows indicating the direction of increasing K *(*$K_e < K_1 < K_2$*).* $\theta = cos^{-1}\zeta$.

Evaluation of Angle and Gain at A Specific Root



Note: *s_1 is the vector from origin to s_1; s_1+2 is the vector from -2 to s_1. $|s_1|$ and $|s_1+2|$ are the magnitude of the vectors.*

- **Angle requirement**

  For example, at a root s_1, the angles are

$$\angle \frac{K}{s(s+2)}\Big|_{s=s_1} = -\angle s_1 - \angle(s_1+2)$$
$$= -(180^o - \theta) - \theta = -180^o$$

  Therefore, the angle requirement is satisfied at any point on the root locus.

- **Gain requirement**

  The gain K at root s_1 can be found using

$$\left|\frac{K}{s(s+2)}\right|_{s=s_1} = \frac{K}{|s_1||s_1+2|} = 1$$

Thus

$$K = |s_1||s_1+2|$$

General Form

For a closed-loop system with the following characteristic equation

$$\Delta(s) = 1 + F(s) = 0$$

The above equation may be rewritten as

$$F(s) = -1 + j0$$

In general, the function $F(s)$ may be written as

$$F(s) = \frac{K(s+z_1)(s+z_2)(s+z_3)\cdots(s+z_M)}{(s+p_1)(s+p_2)(s+p_3)\cdots(s+p_n)}$$

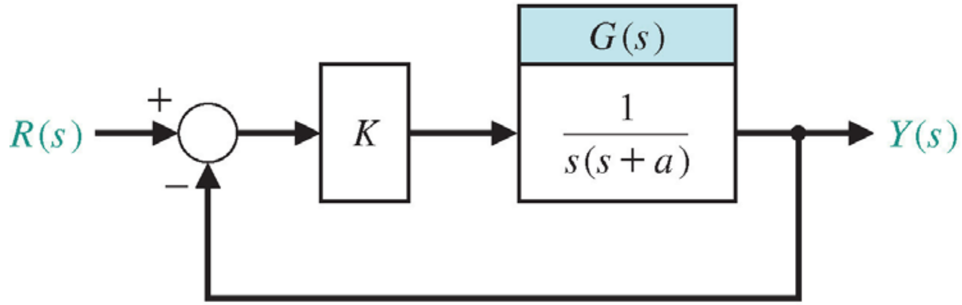
Then the magnitude and angle requirement for the root locus are

$$|F(s)| = \frac{K|s+z_1||s+z_2|\cdots}{|s+p_1||s+p_2|\cdots} = 1$$

$$\angle F(s) = \angle(s+z_1) + \angle(s+z_2) + \cdots - (\angle(s+p_1) + \angle(s+p_2) + \cdots)$$
$$= 180^o + k360^o$$

where k is an integer.

Example 18.2



Let us consider the above second-order system where $a > 0$. The effect of varying a can be effectively portrayed by rewriting the characteristic equation for the root locus form with a as the multiplying factor. **--- a is the parameter of interest!**

- The characteristic equation is

$$\Delta(s) = 1 + KG(s) = 1 + \frac{K}{s(s+a)} = 0 \quad \longrightarrow \quad s^2 + as + K = 0$$

Alternatively, it can be rewritten as

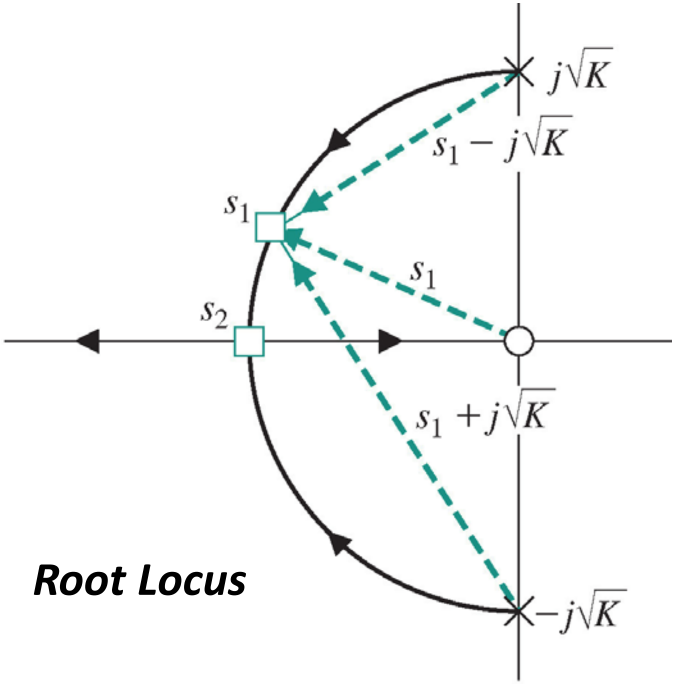
$$1 + a\frac{s}{s^2 + K} = 0$$

- Then the magnitude and angle criteria are

at root s_1
$$\frac{a|s_1|}{|s_1^2 + K|} = 1$$

$$\angle s_1 - \left(\angle s_1 + j\sqrt{K} + \angle s_1 - j\sqrt{K}\right) = \pm 180°, \pm 540°, \ldots$$

- In principle, we could construct the root locus by determining the points in the s-plane that satisfy the angle criterion.



Root Locus

- Specifically at root s_1, the angle requirement is satisfied. The magnitude of parameter a can be found by

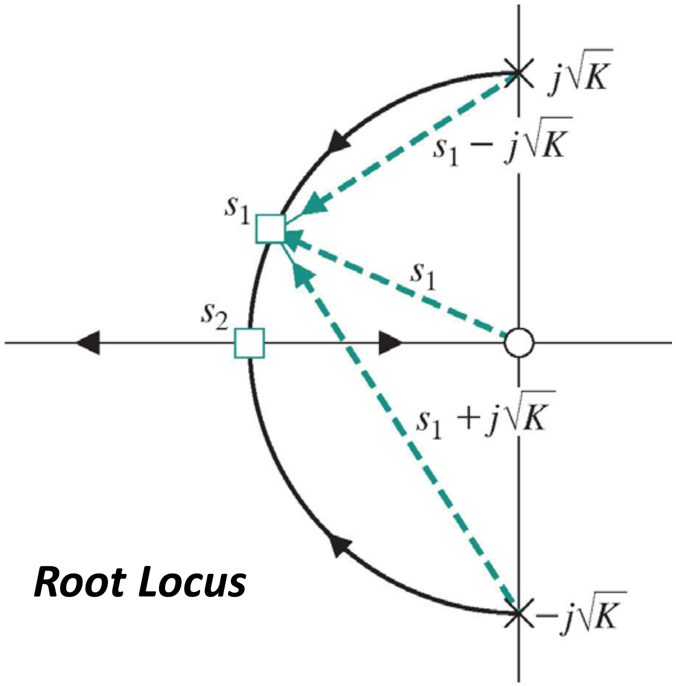
$$a = \frac{|s_1 - j\sqrt{K}||s_1 + j\sqrt{K}|}{|s_1|}$$

- The roots of the system merge on the real axis at the point $s_2 = \sigma_2$, where

$$a = \frac{|\sigma_2 - j\sqrt{K}||\sigma_2 + j\sqrt{K}|}{\sigma_2} = \frac{1}{\sigma_2}(\sigma_2^2 + K) = 2\sqrt{K}$$

therefore

$$\sigma_2 = \sqrt{K}$$



Root Locus

Any general procedure to sketch the root locus? --- YES.

The Root Locus Procedure: Step 1

To locate the roots of the characteristic equation in a graphical manner on the s-plane, we develop an orderly procedure of seven steps that facilitate the rapid sketching of the roots.

Step 1. Prepare the root locus sketch.

Begin by writing the characteristic equation as

$$\Delta(s) = 1 + F(s) = 0$$

Rearrange the equation, if necessary, so that the parameter of interest, K appears as the multiplying factor in the form,

$$1 + KP(s) = 0$$

We are often interested in determining the locus of roots when K varies as

$$0 \le K < \infty$$

Write the polynomial in the form of poles and zeros as follows:

$$1 + K\frac{\prod_{i=1}^{M}(s + z_i)}{\prod_{j=1}^{n}(s + p_j)} = 0$$

Rewrite the equation, we have

$$\prod_{j=1}^{n}(s + p_j) + K\prod_{i=1}^{M}(s + z_i) = 0$$

-> when $K = 0$, the roots of the characteristic equation are the **poles** of $P(s)$.

Conversely, as $K \to \infty$, the equation can be rewritten as

$$\frac{1}{K}\prod_{j=1}^{n}(s + p_j) + \prod_{j=1}^{M}(s + z_j) = 0$$

-> when $K \to \infty$, the roots of the characteristic equation are the **zeros** of $P(s)$.

Therefore, the locus of the roots of the characteristic equation $1 + KP(s) = 0$ begins at the poles of $P(s)$ and ends at the zeros of $P(s)$ as K increases from zero to infinity.
--- For most cases, function $P(s)$ has more poles than zeros. With n poles and M zeros and $n > M$, we have $n - M$ braches of the root locus approaching infinity.

Step 2

Step 2. Locate the segments of the real axis that are the root loci.

The Root locus on the real axis always lies in a section of the real axis to the left of an **odd** number of poles and zeros.

Example: A second-order feedback control system with characteristic equation

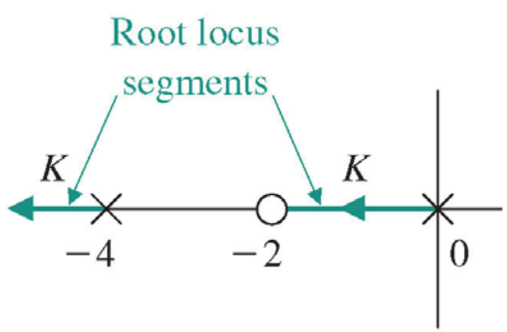
$$1 + GH(s) = 1 + \frac{K\left(\frac{1}{2}s + 1\right)}{\frac{1}{4}s^2 + s} = 0$$

Step 1. The characteristic equation can be rewritten in terms of poles and zeros

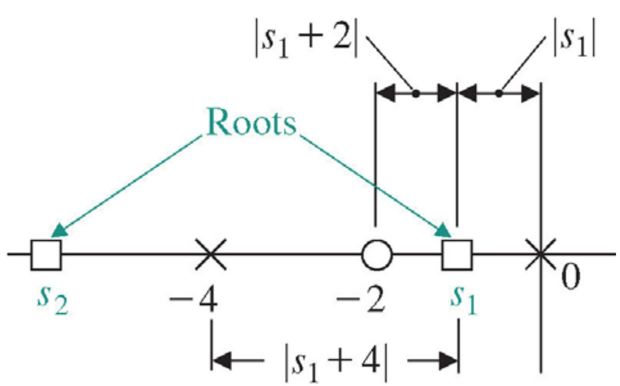
$$1 + K\frac{2(s + 2)}{s(s + 4)} = 0$$

To determine the locus of roots for the gain $0 \leq K < \infty$, we locate the poles and zeros in the s-plane.

Step 2. Locate the root loci on the real axis.



- The angle criterion is satisfied on the real axis between 0 and -2 and between -4 to $-\infty$.



- For any root s_1 on the root locus, gain K can be obtained by the magnitude criterion. For example, when $s_1 = 1$

$$\frac{(2K)|s_1 + 2|}{|s_1||s_1 + 4|} = 1 \longrightarrow K = \frac{3}{2}$$

Once K is found, the other root can be determined at $s_2 = -6$.

Step 3

Now, we determine the number of separate loci, ***SL***. Because the loci begin at poles and end at the zeros or infinity, **the number of separate loci is equal to the number of poles of $P(s)$**. --- Note that **the root loci must be symmetrical with respect to the horizontal real axis** because the complex roots must appear as pairs of complex conjugate roots.

Step 3. Determine the center σ_A and angle $\emptyset_A$ of asymptotes along which the loci proceed to the infinity.

When the number of finite zeros of $P(s)$, M is less than the number of poles n by the number $N = n - M$, then N sections of loci must end at infinity as $K \to \infty$.

The n-M linear asymptotes are centered at a point on the real axis which is often called the **asymptote centroid** given by

$$\boxed{\sigma_A = \frac{\sum poles\ of\ P(s) - \sum zeros\ of\ P(s)}{n - M} = \frac{\sum_{j=1}^{n}(-p_j) - \sum_{i=1}^{M}(-z_i)}{n - M}}$$

The **angle of the asymptotes** with respect to the real axis is

$$\boxed{\emptyset_A = \frac{2k+1}{n-M}180^o} \qquad k = 0,1,2,\ldots,(n-M-1)$$

Example of Asymptotes Calculation

Consider a unity negative feedback control system with a characteristic equation

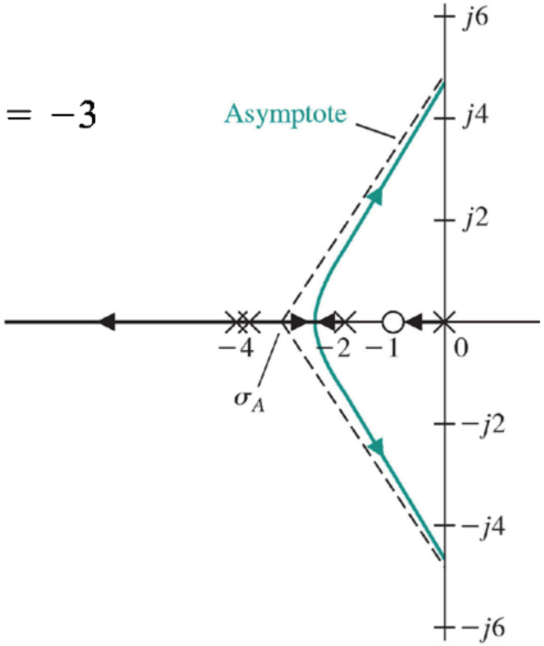
$$1 + G_c(s)G(s) = 1 + \frac{K(s+1)}{s(s+2)(s+4)^2}$$

We wish to sketch the root locus in order to determine the effect of the gain K.

- In this system, M=1, n=4, therefore, we'll have 3 asymptotes.

$$\sigma_A = \frac{(-2) + 2(-4) - (-1)}{4 - 1} = \frac{-9}{3} = -3$$

$$\phi_A = +60^\circ \quad (k = 0),$$
$$\phi_A = 180^\circ \quad (k = 1), \text{ and}$$
$$\phi_A = 300^\circ \quad (k = 2),$$



Step 4

Step 4. Determine where the root locus crosses the imaginary axis (if it does so) using Routh-Hurwitz criterion.

The actual point at which the root locus crosses the imaginary axis is readily evaluated by using the Routh- Hurwitz criterion.

Example: for a system with the following characteristic equation

$$\Delta(s) = 1 + K\frac{1}{s^3 + 2s^2 + 4s}$$

The characteristic equation can be rewritten as $\Delta(s) = s^3 + 2s^2 + 4s + K = 0$

The Routh array is

$$\begin{array}{c|cc} s^3 & 1 & 4 \\ s^2 & 2 & K \\ s^1 & \dfrac{8-K}{2} & 0 \\ s^0 & K & 0 \end{array}$$

When $K = 4$, the system will have a pair of complex poles lying on the imaginary axis. These two poles can be determined by the auxiliary polynomial

$$U(s) = 2s^2 + Ks^0 = 2s^2 + 8 = 2(s^2 + 4) = 2(s + j2)(s - j2).$$

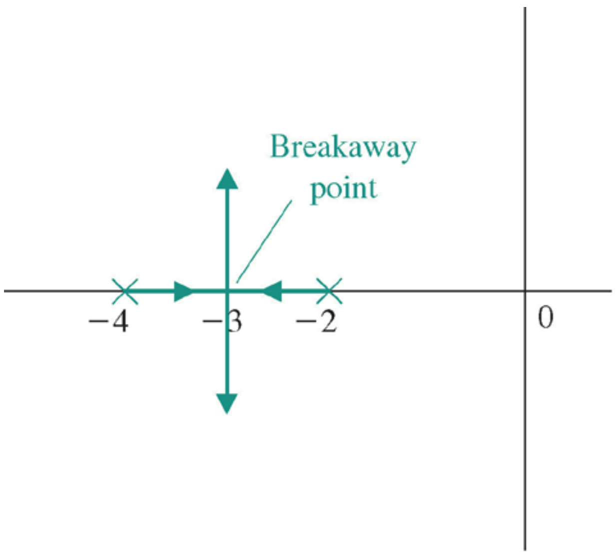
Therefore, the root locus crosses the imaginary axis at $s_{1,2} = \pm j2$.

Step 5

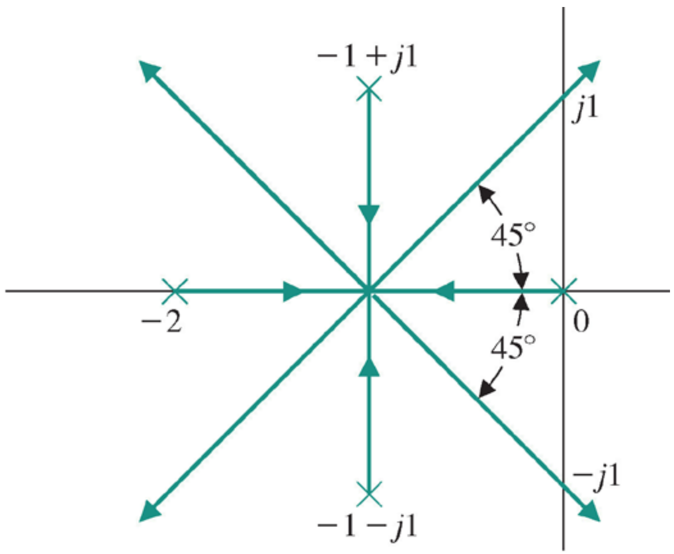
Step 5. Determine the breakaway point on the real axis (if any) at which the root locus left the real axis.

The locus leaves the real axis where there is a multiplicity of roots (typically two). In general, due to phase criterion, the tangent to the loci at the breakaway point are equally spaced over 360^o.

- For two loci at the breakaway point are spaced 180^o apart.
- For four loci at the breakaway point are spaced 90^o apart.



(a)



(b)

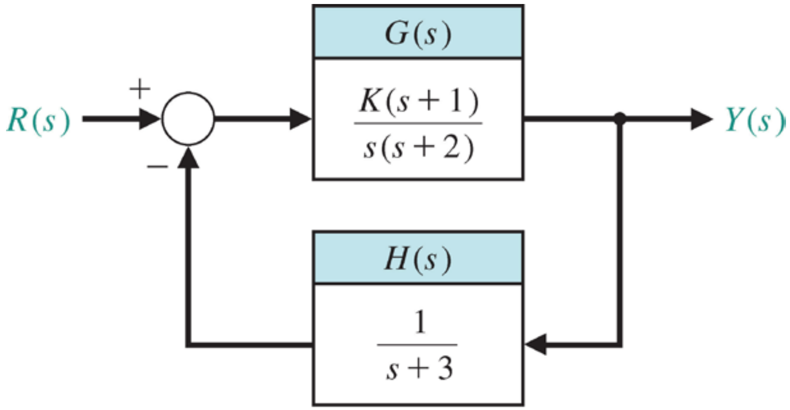
To determine the breakaway point on the real axis, we need to rearrange the characteristic equation to isolate the multiplying factor K and rewrite it as

$$p(s) = K$$

Analytically, by the breakaway point can be found by determining the maximum of $\mathrm{K} = p(s)$, i.e. we may evaluate

$$\boxed{\frac{dK}{ds} = \frac{dp(s)}{ds} = 0}$$

Example: for the following system, determine the breakaway point of root locus.



$$1 + G(s)H(s) = 1 + \frac{K(s+1)}{s(s+2)(s+3)} = 0.$$

Applying step 1-4, we will get

-2.46

To determine the breakaway point, rearrange the characteristic equation as

$$s(s+2)(s+3)+K(s+1)=0$$

$$p(s)=\frac{-s(s+2)(s+3)}{s+1}=K$$

Then we set its first derivative to be zero

$$\frac{d}{ds}\left(\frac{-s(s+2)(s+3)}{(s+1)}\right)=\frac{(s^3+5s^2+6s)-(s+1)(3s^2+10s+6)}{(s+1)^2}=0$$

$$2s^3+8s^2+10s+6=0.$$

There are three roots for the above equation, but only $s=-2.46$ is on the real axis and between -2 and -3.

The breakaway point is $s=-2.46$.

Step 6

Step 6. Determine the angle of departure of the locus from a pole and the angle of arrival of the locus at a zero, using the phase angle criterion.

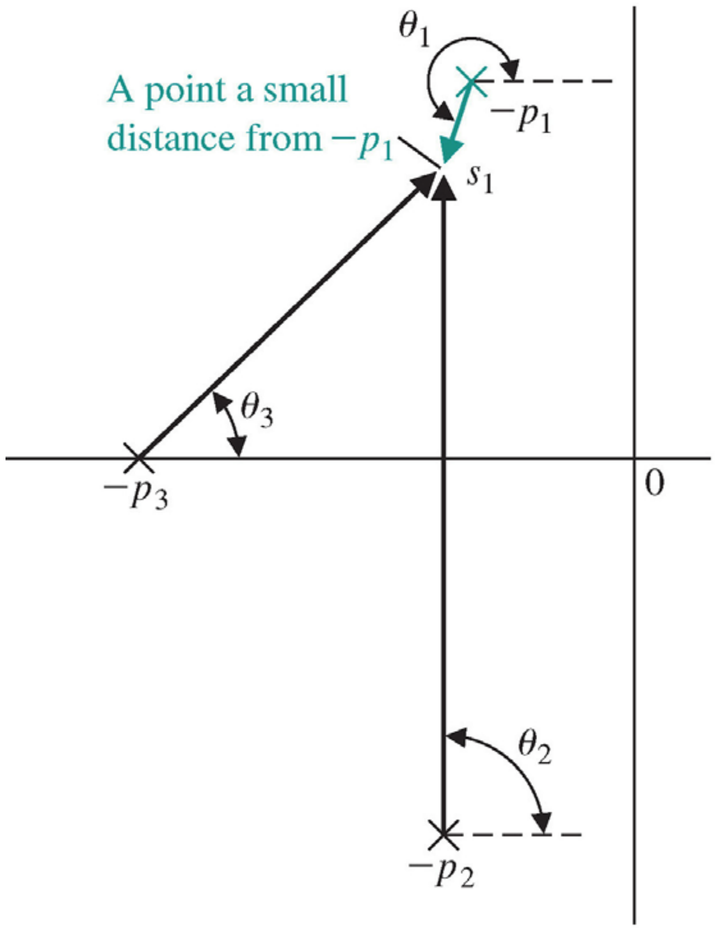
The angle of locus departure from a pole is the difference between the net angle due to all other poles and zeros and the criterion angle of $\pm 180^o(2k+1)$, and simply for the locus angle of arrival at a zero.

Example: consider the third-order loop transfer function

$$F(s) = G(s)H(s) = \frac{K}{(s + p_3)(s^2 + 2\zeta\omega_n s + \omega_n^2)}.$$

There are two complex poles and one real pole without zeros. Therefore, all the three poles will go to infinity.

What is the angle of departure from each pole?



- To determine departure angle from pole $-p_1$, choose a test point s_1 , an **infinitesimal** distance from $-p_1$. θ_1 is the angle of departure from $-p_1$.

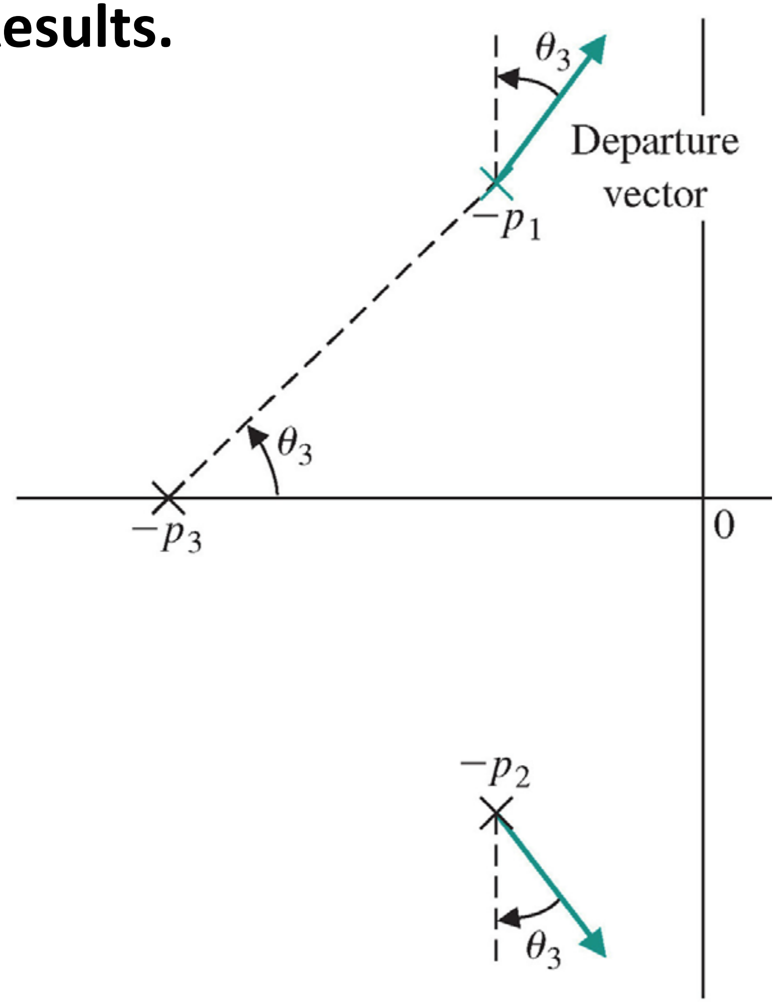
- To meet the angle requirement, we have

$$\theta_1 + \theta_2 + \theta_3 = \theta_1 + 90^\circ + \theta_3 = +180^\circ,$$

- Therefore

$$\theta_1 = 90^\circ - \theta_3;$$

Results.



Note: the departure angle at pole $-p_2$ is the negative of that at $-p_1$, because $-p_1$ and $-p_2$ are complex conjugates.

Summary

Table 7.2 Seven Steps for Sketching a Root Locus

Step	Related Equation or Rule
1. Prepare the root locus sketch.	
(a) Write the characteristic equation so that the parameter of interest, K, appears as a multiplier.	$1 + KP(s) = 0.$
(b) Factor $P(s)$ in terms of n poles and M zeros.	$1 + K\dfrac{\prod_{i=1}^{M}(s + z_i)}{\prod_{j=1}^{n}(s + p_j)} = 0.$
(c) Locate the open-loop poles and zeros of $P(s)$ in the s-plane with selected symbols.	$\times$ = poles, $\bigcirc$ = zeros Locus begins at a pole and ends at a zero.
(d) Determine the number of separate loci, SL.	$SL = n$ when $n \geq M$; n = number of finite poles, M = number of finite zeros.
(e) The root loci are symmetrical with respect to the horizontal real axis.	
2. Locate the segments of the real axis that are root loci.	Locus lies to the left of an odd number of poles and zeros.

Table 7.2 Seven Steps for Sketching a Root Locus

Step	Related Equation or Rule
3. The loci proceed to the zeros at infinity along asymptotes centered at σ_A and with angles ϕ_A.	$\sigma_A = \dfrac{\sum(-p_j) - \sum(-z_i)}{n - M}$. $\phi_A = \dfrac{2k+1}{n-M}180^\circ, k = 0, 1, 2, \ldots, (n - M - 1)$.
4. Determine the points at which the locus crosses the imaginary axis (if it does so).	Use Routh–Hurwitz criterion.
5. Determine the breakaway point on the real axis (if any).	a) Set $K = p(s)$. b) Determine roots of $dp(s)/ds = 0$ or use graphical method to find maximum of $p(s)$.
6. Determine the angle of locus departure from complex poles and the angle of locus arrival at complex zeros, using the phase criterion.	$\underline{/P(s)} = 180^\circ + k360^\circ$ at $s = -p_j$ or $-z_i$.
7. Complete the root locus sketch.	

Thank You！