Solutions to the resit exam of MTH201.

2017-18. S1.

1. $\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 1 \\ 0 & 2 & 3 \end{vmatrix} = \underline{\langle 1, -3, 2\rangle}.$

2. $\text{grad} f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\rangle = \underline{\langle \frac{1}{x} + e^x y,\ e^x,\ -\cos z\rangle}.$

3. Let $f(x,y,z) = 3x^2+2y-z=0$, a normal vector is $\vec{N} = \nabla f = \langle 6x, 2, -1\rangle$.

At $(1,0,3)$ $\vec{N} = \langle 6, 2, -1\rangle$.

So the upward unit normal vector is

$$\vec{n} = -\frac{\vec{N}}{\|\vec{N}\|} = -\frac{\langle 6,2,-1\rangle}{\sqrt{36+4+1}} = \underline{\frac{1}{\sqrt{41}}\langle -6, -2, 1\rangle}.$$

4. $A=3,\ B=\frac{1}{2},\ C=-1$. $\therefore AC-B^2 = -3-\frac{1}{4}<0$. So it is $\underline{\text{Hyperbolic}}$

5. $\int_C \vec{F}\cdot d\vec{r} = \int_C \vec{F}\cdot\vec{r}'(t)\,dt = \int_0^2 \langle 2t^2, 1+3\rangle\cdot\langle 2t, 0\rangle\,dt$

$= \int_0^2 4t^3\,dt = [t^4]_0^2 = \underline{16}.$

6. $\text{curl} \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^x\cos y & e^x\sin y & z \end{vmatrix} = \underline{\langle 0, 0, 2e^x\sin y\rangle}.$

7. If $\vec{F} = \langle yz+1, xz+1, xy+1\rangle = \nabla f$, then

$$\begin{cases} \frac{\partial f}{\partial x} = yz+1 \\ \frac{\partial f}{\partial y} = xz+1 \\ \frac{\partial f}{\partial z} = xy+1 \end{cases}$$

$\Rightarrow f(x,y,z) = xyz + x + g(y,z)$

$\therefore \frac{\partial f}{\partial y} = xz + \frac{\partial g}{\partial y} = xz+1$ $\therefore \frac{\partial g}{\partial y} = 1$ $\therefore g(y,z) = y + h(z)$.

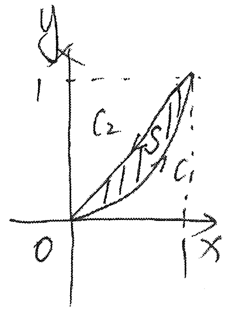
$\therefore f(x,y,z) = xyz + x + y + h(z)$.

$\therefore \frac{\partial f}{\partial z} = xy + h'(z) = xy+1 \quad \therefore h'(z) = 1 \quad \therefore h(z) = z + \text{Constant}.$

$\therefore f(x,y,z) = xyz + x+y+z + \text{Constant}.$

$\therefore \int_{(0,1,0)}^{(1,1,1)} (yz+1)\,dx + (xz+1)\,dy + (xy+1)\,dz$ is independent of path.

$= f(1,1,1) - f(0,1,0) = 1+3-1 = 3.$



8. Method 1: $A(S) = \frac{1}{2}\oint_C x\,dy - y\,dx$, where $C = C_1 + C_2$.

C_1: $y = x^2$, $\therefore dy = 2x\,dx$, $0 \le x \le 1$

C_2: $y = x$ $\therefore dy = dx$, x is from 1 to 0.

$\therefore A(S) = \frac{1}{2}\left[\int_0^1 x \cdot 2x\,dx - x^2\,dx + \int_1^0 x\,dx - x\,dx\right]$

$= \frac{1}{2}\int_0^1 x^2\,dx = \frac{1}{2}\cdot\left[\frac{x^3}{3}\right]_0^1 = \frac{1}{6}.$

Method 2: By Green's theorem:

$A(S) = \frac{1}{2}\iint_S \frac{\partial(x)}{\partial x} - \frac{\partial(-y)}{\partial y}\,dA = \iint_S dx\,dy$

$= \int_0^1 \int_{x^2}^{x} dy\,dx = \int_0^1 x - x^2\,dx = \left[\frac{1}{2}x^2 - \frac{1}{3}x^3\right]_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$ [4].

9. The characteristic equation is $\lambda^2 - 9 = 0$ $\therefore \lambda = \pm 3$.

$\therefore u = Ae^{3x} + Be^{-3x}$ is a particular solution.

Let $u(x,y) = A(y)e^{3x} + B(y)e^{-3x}$, then.

$\frac{\partial u}{\partial x} = 3A(y)e^{3x} - 3B(y)e^{-3x}.$

$\frac{\partial^2 u}{\partial x^2} = 9A(y)e^{3x} + 9B(y)e^{-3x}.$

which imply $9A(y)e^{3x}+9B(y)e^{-3x}-9(A(y)e^{3x}+B(y)e^{-3x})=0$.

Therefore, the general Solution is

$$u(x,y)=A(y)e^{3x}+B(y)e^{-3x}.$$

10. (a). $\iint_S \vec{F}\cdot\vec{n}\,dA=\iint_R \vec{F}\cdot\vec{N}\,du\,dv$, where $\vec{N}$ is the normal vector of S.

$$\vec{N}=\vec{r}_u\times\vec{r}_v=\langle 1,0,2\rangle\times\langle 0,1,-3\rangle=\begin{vmatrix}\vec{i}&\vec{j}&\vec{k}\\1&0&2\\0&1&-3\end{vmatrix}=\langle -2,3,1\rangle.$$

$$\vec{F}(\vec{r}(u,v))=\langle 0,u^2,2u-3v\rangle.$$

$$\therefore \vec{F}(\vec{r})\cdot\vec{N}=\langle 0,u^2,2u-3v\rangle\cdot\langle -2,3,1\rangle=3u^2+2u-3v.$$

$$\therefore \iint_S \vec{F}\cdot\vec{N}\,dA=\int_{-1}^{2}\int_0^1 3u^2+2u-3v\,du\,dv$$

$$=\int_{-1}^{2}\left[u^3+u^2-3uv\right]_0^1 dv=\int_{-1}^{2}1+1-3v\,dv$$

$$=\int_{-1}^{2}2-3v\,dv=\left[2v-\frac{3}{2}v^2\right]_{-1}^{2}$$

$$=4-\frac{3}{2}\times 4-\left(-2-\frac{3}{2}\right)$$

$$=\frac{3}{2}.$$

(b). By the divergence theorem:

$$\iint_S \vec{F}\cdot\vec{n}\,dA=\iiint_T \operatorname{div}\vec{F}\,dV=\iiint_T 1+0+1\,dV=2\iiint_T dV$$

$$=2\times\text{Volumn of } T=2\times\left(\frac{4}{3}\pi\cdot 4^3-\frac{4}{3}\pi\cdot 1\right)=168\pi.$$

11. By stoke's theorem:

$$\iint_S \text{curl}\vec{F}\cdot\vec{n}\,dA = \oint_C \vec{F}\cdot\vec{r}'(t)\,dt,$$

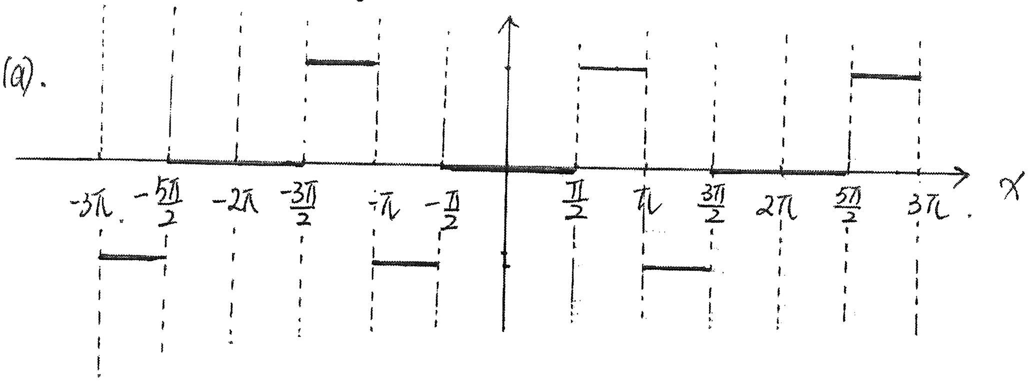
where C: $\vec{r}(t) = \langle 2\cos t, 2\sin t, 4\rangle$, $0 \le t \le 2\pi$, $\therefore \vec{r}'(t) = \langle -2\sin t, 2\cos t, 0\rangle$

$$\therefore \vec{F}(\vec{r}(t)) = \langle y, -x, yz\rangle = \langle 2\sin t, -2\cos t, 8\sin t\rangle.$$

$$\therefore \oint_C \vec{F}\cdot\vec{r}'(t)dt = \int_0^{2\pi} \langle 2\sin t, -2\cos t, 8\sin t\rangle\cdot\langle -2\sin t, 2\cos t, 0\rangle\,dt$$

$$= \int_0^{2\pi} -4\sin^2 t - 4\cos^2 t\,dt = -8\pi.$$

12. (a).



sketch of $f(x)$.

Because $f(x)$ satisfies $f(-x) = -f(x)$ and so

$f(x)$ is an odd function.

(b). The Fourier Series of $f(x)$ is

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n\cos nx + b_n\sin nx.$$

As $f(x)$ is an odd function,

$$a_0 = 0, \quad a_n = 0, \quad n \ge 1.$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx$$

Substitute for $f(x)$:

$$b_n = \frac{2}{\pi}\int_{\frac{\pi}{2}}^{\pi} \sin(nx)\,dx = \frac{2}{\pi}\left[-\frac{1}{n}\cos(nx)\right]_{\frac{\pi}{2}}^{\pi}$$

$$= \frac{-2}{n\pi}\left[\cos(n\pi) - \cos\frac{n\pi}{2}\right].$$

$$\therefore f(x) = \sum_{n=1}^{\infty} \frac{-2}{n\pi}\left[(-1)^n - \cos\frac{n\pi}{2}\right]\sin(nx).$$

Now $\cos\frac{\pi}{2} = 0$, $\cos\pi = -1$ $\cos\frac{3\pi}{2} = 0$, $\cos 2\pi = 1$, $\cos\frac{5\pi}{2} = 0$ ...

So $b_1 = -\frac{2}{\pi}(-1-0) = \frac{2}{\pi}.$

$$b_2 = \frac{-2}{2\pi}[1-(-1)] = \frac{-4}{2\pi} = -\frac{2}{\pi}.$$

$$b_3 = \frac{-2}{3\pi}(-1-0) = \frac{2}{3\pi}.$$

$$b_4 = \frac{-2}{4\pi}(1-1) = 0$$

$$b_5 = \frac{-2}{5\pi}(-1-0) = \frac{2}{5\pi}.$$

More generally, $b_n = \frac{2}{n\pi}$, if n is odd.

$b_n = -\frac{4}{n\pi}$, if $n = 2, 6, 10, \ldots$

$b_n = 0$, if $n = 4, 8, 12, \ldots$

(c). Fourier Series converges to average value at pointed discontinuity.

$f(x) = -\frac{1}{2}$ at $x = -\frac{\pi}{2}$.

0 at $x = 0$

$\frac{1}{2}$ at $x = \frac{\pi}{2}$

0 at $x = \pi$.

13. (a). Taking $\theta(x,y) = X(x)Y(y)$, then

$$\frac{\partial \theta}{\partial x} = X'Y, \quad \frac{\partial^2 \theta}{\partial x^2} = X''Y.$$

$$\frac{\partial \theta}{\partial y} = XY', \quad \frac{\partial^2 \theta}{\partial y^2} = XY''.$$

So $\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = X''Y + XY'' = 0$.

Rearranging: $\frac{X''(x)}{X(x)} = \frac{-Y''(y)}{Y(y)} = -\lambda$,

where because LHS is a function of x and RHS is a function of y.

So $\lambda =$ Constant.

$\therefore X'' + \lambda X = 0, \quad Y'' - \lambda Y = 0$.

(b). Boundary conditions given as

$\theta(x=-a, y) = 0 = X(-a)Y(y)$, So $X(-a) = 0$.

$\theta(x=a, y) = 0 = X(a)Y(y)$, So $X(a) = 0$

$\theta(x, y=0) = 0 = X(x)Y(0)$, So $Y(0) = 0$

$\theta(x, y=a) = f(x) = X(x)Y(a)$.

(c). Given $f(x) = \cos\frac{\pi x}{2a}$.

try $X(x) = \cos\frac{\pi x}{2a}$, then

$X'(x) = \frac{-\pi}{2a}\sin\frac{\pi x}{2a}$, $X''(x) = -\frac{\pi^2}{4a^2}\cos\left(\frac{\pi x}{2a}\right) = -\lambda X(x)$.

$\therefore$ Take $\lambda = \frac{\pi^2}{4a^2}$, then at $y=a$:

$\theta(x, y=a) = X(x)Y(a) = \cos\frac{\pi x}{2a}\cdot Y(a) = f(x) = \cos\frac{\pi x}{2a}$.

Therefore, $Y(a) = 1$.

(d). Equation for Y is

$$Y''(y) - \lambda Y = 0 \quad \therefore Y'' - \frac{\pi^2}{4a^2} Y = 0 .$$

Solution is $Y(y) = A e^{\frac{\pi}{2a} y} + B e^{-\frac{\pi}{2a} y}$

Boundary conditions give

$$Y(0) = 0 = A + B \qquad \therefore A = -B .$$

$$Y(a) = 1 = A e^{\frac{\pi}{2a} \cdot a} + B e^{-\frac{\pi}{2a} \cdot a} = A e^{\frac{\pi}{2}} + B e^{-\frac{\pi}{2}}$$

$$= A \left(e^{\frac{\pi}{2}} - e^{-\frac{\pi}{2}}\right)$$

Thus $A = \dfrac{1}{e^{\frac{\pi}{2}} - e^{-\frac{\pi}{2}}} = -B .$

$$\therefore Y(y) = \frac{1}{e^{\frac{\pi}{2}} - e^{-\frac{\pi}{2}}} \left(e^{\frac{\pi}{2a} y} - e^{-\frac{\pi}{2a} y}\right) .$$

Hence $\phi(x,y) = X(x) Y(y) = \cos \dfrac{\pi x}{2a} \cdot \dfrac{e^{\frac{\pi}{2a} y} - e^{-\frac{\pi}{2a} y}}{e^{\frac{\pi}{2}} - e^{-\frac{\pi}{2}}} .$