Xi'an Jiaotong-Liverpool University
西交利物浦大學

EEE204 Continuous and Discrete Time Signals and Systems II

2018–2019 Semester 2

Electrical and Electronic Engineering

Xi'an Jiaotong-Liverpool University

Week 4

Find the Maximum Sampling Period

$x_1(t) = 5\,\mathrm{sinc}(200t)$

$$x_1(t) = 5\,\mathrm{sinc}(200t) = \frac{1}{40}\frac{200\pi}{\pi}\,\mathrm{sinc}\left(\frac{200\pi t}{\pi}\right),$$

$$= \frac{1}{40}\frac{\sin(200\pi t)}{\pi t},$$ refer to Page 295.

$$X_1(\omega) = \frac{1}{40}\,\mathrm{rect}\left(\frac{\omega}{400\pi}\right) = \begin{cases} \frac{1}{40}, & |\omega| < 200\pi, \\ 0, & |\omega| > 200\pi. \end{cases}$$

$$\mathcal{F}\left[\frac{W}{\pi}\,\mathrm{sinc}\left(\frac{Wt}{\pi}\right)\right] = \mathrm{rect}\left(\frac{\omega}{2W}\right) = \begin{cases} 1, & |\omega| < W, \\ 0, & |\omega| > W. \end{cases}$$

Find the Maximum Sampling Period

Xi'an Jiaotong-Liverpool University 西交利物浦大学

$x_1(t) = 5\ \mathrm{sinc}(200t)$

The transformation pair refers to Page 329 row 9 in Table 4.2.

$$X_1(\omega) = \frac{1}{40}\,\mathrm{rect}\left(\frac{\omega}{400\pi}\right) = \begin{cases} \frac{1}{40}, & |\omega| \leqslant 200\pi, \\ 0, & |\omega| > 200\pi. \end{cases}$$

The maximum frequency is given by $\frac{200\pi}{2\pi} = 100$ Hz. Based on Nyquist theorem, the maximum sampling period is $T_s = \frac{1}{f_s} = \frac{1}{2 \times 100} = 5$ ms.

Find the Maximum Sampling Period

$x_2(t) = 5\,\mathrm{sinc}(200t) + 8\sin(100\pi t)$

$$x_2(t) = \underbrace{5\underbrace{\mathrm{sinc}(200t)}_{\text{max frequency is 100 Hz}} + 8\underbrace{\sin(100\pi t)}_{\text{max frequency is 50 Hz}}}_{\text{max frequency is 100 Hz}},$$

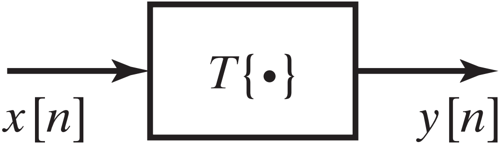
Therefore, maximum sampling period is given by

$$T_s = \frac{1}{f_s} = \frac{1}{2 \times 100} = 5 \text{ ms.}$$

$$\mathcal{F}[\sin(\omega_0 t)] = \frac{\pi}{j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$$

Discrete-time(DT) Systems

A discrete-time system is defined mathematically as a transformation or operator that maps an input sequence with values $x[n]$ into an output sequence with values $y[n]$.



This can be denoted as

$$y[n] = T\{x[n]\}.$$

Memoryless Systems

Memoryless Systems

A system is referred to as memoryless if the output $y[n]$ at every value of n depends only on the input $x[n]$ at the same value of n.

An example of a memoryless system is a system for which $x[n]$ and $y[n]$ are related by

$$y[n] = (x[n])^2, \quad \text{for each value of } n.$$

The following example is memoryless if $n_d = 0$

$$y[n] = x[n - n_d].$$

If n_d is positive, it is called a time-delay system while if n_d is negative, it is called a time advance system.

Linear Systems

The class of linear systems is defined by the principle of superposition. If $y_1[n]$ and $y_2[n]$ are the responses of a system when $x_1[n]$ and $x_2[n]$ are the respective inputs, then the system is linear if and only if (iff)

$$T\{ax_1[n] + bx_2[n]\} = aT\{x_1[n]\} + bT\{x_2[n]\},$$

for arbitrary constants a and b. This equation can be generalised to the superposition of many inputs. Specifically, if

$$x[n] = \sum_k a_k x_k[n],$$

then the output of a linear system will be

$$y[n] = \sum_k a_k y_k[n], \quad y_k[n] = T\{x_k[n]\}.$$

The Accumulator System

The system defined by the input-output equation

$$y[n] = \sum_{k=-\infty}^{n} x[k],$$

is called the accumulator system, since the output at time n is the accumulation or sum of the present and all previous input samples.

Is the accumulator system a linear system or non-linear system?

The accumulator system is a linear system and we can prove it as follows.

The Accumulator System

We begin by defining two arbitrary inputs $x_1[n]$ and $x_2[n]$ and their corresponding outputs

$$y_1[n] = \sum_{k=-\infty}^{n} x_1[k],$$

$$y_2[n] = \sum_{k=-\infty}^{n} x_2[k],$$

When the input is $x_3[n] = ax_1[n] + bx_2[n]$, the output can be shown as:

$$y_3[n] = \sum_{k=-\infty}^{n} x_3[k].$$

Xi'an Jiaotong-Liverpool University 西交利物浦大学

The Accumulator System

$$
\begin{aligned}
y_3[n] &= \sum_{k=-\infty}^{n} x_3[k], \\
&= \sum_{k=-\infty}^{n} (ax_1[k] + bx_2[k]), \\
&= a \sum_{k=-\infty}^{n} x_1[k] + b \sum_{k=-\infty}^{n} x_2[k], \\
&= ay_1[n] + by_2[n].
\end{aligned}
$$

Thus, the accumulator system satisfies the superposition principle for all inputs and is therefore linear.

Consider the system defined by

$$w[n] = \log_{10}(|x[n]|).$$

This system is not linear. To prove this, we only need to find one counterexample which violates the superposition principle.

Let $x_1[n] = 1$ and $x_2[n] = 10$, the output for $x_1[n] + x_2[n] = 1 + 10 = 11$ is

$$\log_{10}(1+10) = \log_{10}(11) \neq \log_{10}(1) + \log_{10}(10) = 1.$$

On the other hand, $w_2[n] = \log_{10}(10) = 1 \neq 10w_1[n] = 10\log_{10}(1) = 0$.

Time Invariance

Suppose we have

$$y[n] = T\{x[n]\}.$$

Then the system is said to be time invariant if, for all n_0, the input sequence with values $x_1[n] = x[n - n_0]$ produces the output sequence with values $y_1[n] = y[n - n_0]$,

$$y[n - n_0] = T\{x[n - n_0]\}.$$

A linear system which is also time-invariant is called linear time-invariant (LTI) system.

The Accumulator System

Consider the same accumulator system

$$y[n]=\sum_{k=-\infty}^{n} x[k] \rightarrow y[n-n_0]=\sum_{k=-\infty}^{n-n_0} x[k].$$

We define $x_1[n]=x[n-n_0]$ and we find

$$y_1[n]=\sum_{k=-\infty}^{n} x_1[k]=\sum_{k=-\infty}^{n} x[k-n_0],$$

$$\xlongequal{k_1=k-n_0} \sum_{k_1=-\infty}^{n-n_0} x[k_1] \xlongequal{k=k_1} y[n-n_0].$$

Therefore, the accumulator is a time-invariant system.

The Compressor System

Consider the compressor system

$$y[n] = x[Mn], \text{with } M > 1 \text{ a positive integer}$$

Consider the response $y_1[n]$ to the input $x_1[n] = x[n - n_0]$.

$$y_1[n] = x_1[Mn] = x[Mn - n_0].$$

While delaying the output $y[n]$ by n_0 samples yields

$$y[n - n_0] = x[M(n - n_0)] \neq y_1[n].$$

Thus, the compressor is a time-variant system.

The Compressor System

We now use a counterexample that violates the time-invariant property.

Consider $x[n] = \delta[n], M = 2, n_0 = 1, x_1[n] = x[n - n_0] = x[n - 1] = \delta[n - 1]$,

$$y_1[n] = x_1[Mn] = x_1[2n] = \delta[2n - 1] = 0,$$

$$\begin{aligned} y[n - n_0] &= x[M(n - n_0)] = x[2(n - 1)], \\ &= \delta[2n - 2] \neq y_1[n]. \end{aligned}$$

Thus, the compressor is a time-variant system.

Causality

Causality

A system is causal if, for every choice of n_0, the output sequence value at the index $n = n_0$ depends only on the input sequence values for $n \leqslant n_0$.

$y[n] = (x[n])^2$ is a causal system.

$y[n] = x[n - n_d]$ is a causal system if $n_d \geqslant 0$.

$y[n] = \sum_{k=-\infty}^{n} x[k]$ is a causal system.

$w[n] = \log_{10}(|x[n]|)$ is a causal system.

$y[n] = x[Mn]$ is a non-causal system.

Difference Systems

The system defined by the relationship

$$y[n] = x[n+1] - x[n]$$

is referred to as the forward difference system.

This system is non-causal since the current value of the output depends on a future value of the input.

The backward difference system defined as

$$y[n] = x[n] - x[n-1]$$

has an output that depends only on the present and past values of the input, the system is causal.

Stability

A system is stable in the bounded-input, bounded-output (BIBO) sense if and only if every bounded input sequence produces a bounded output sequence. The input $x[n]$ is bounded if there exists a fixed positive finite value B_x such that

$$|x[n]| \leqslant B_x < \infty, \text{ for all } n.$$

Stability requires that, for every bounded input, there exists a fixed positive finite value B_y such that

$$|y[n]| \leqslant B_y < \infty, \text{ for all } n.$$

Stability

Example

$y[n] = (x[n])^2$ is stable.

$y[n] = x[n - n_d]$ is stable.

$y[n] = \sum_{k=-\infty}^{n} x[k]$ is unstable.

$w[n] = \log_{10}(|x[n]|)$ is unstable.
$\log_{10}(|x[n]|) = -\infty$ when $x[n] = 0$

$y[n] = x[Mn]$ is stable.

The Accumulator System

We will make a counterexample to show the accumulator system is unstable. Consider $x[n] = u[n]$, which is clearly bounded by $B_x = 1$. For this input, the output of the accumulator is

$$y[n] = \sum_{k=-\infty}^{n} u[k] = \begin{cases} 0, & n < 0, \\ n+1, & n \geqslant 0. \end{cases}$$

There is no finite choice for B_y such that $(n+1) \leqslant B_y < \infty$ for all n; thus the system is unstable.

A DT system is described using the following input-output relationship:

$$y[n] = x[n] - 0.9x[n-3].$$

Describe this system in terms of causality, memory, stability and LTI.

Example

Linearity: For $x_1[n]$ applied as the input, the output $y_1[n]$ is given by

$$y_1[n] = x_1[n] - 0.9x_1[n-3].$$

For $x_2[n]$ applied as the input, the output $y_2[n]$ is given by

$$y_2[n] = x_2[n] - 0.9x_2[n-3].$$

For $x_3[n] = ax_1[n] + bx_2[n]$ applied as the input, the output $y_3[n]$ is given by

$$y_3[n] = (ax_1[n]+bx_2[n])-0.9(ax_1[n-3]+bx_2[n-3]).$$

Example

$$
\begin{aligned}
y_3[n] &= (ax_1[n] + bx_2[n]) - 0.9(ax_1[n-3] + bx_2[n-3]), \\
&= a(x_1[n] - 0.9x_1[n-3]) + b(x_2[n] - 0.9x_2[n-3]), \\
&= ay_1[n] + by_2[n].
\end{aligned}
$$

Therefore, the system is linear.

The invariance: For inputs $x_1[n]$ and $x_2[n] = x_1[n-n_0]$, the outputs are given by

$$
\begin{aligned}
x_1[n] &\to y_1[n] = x_1[n] - 0.9x_1[n-3], \\
x_2[n] &\to y_2[n] = x_2[n] - 0.9x_2[n-3].
\end{aligned}
$$

The second equation implies that

$$
y_2[n] = x_1[n-n_0] - 0.9x_1[n-3-n_0].
$$

We also notice that
$y_1[n-n_0] = x_1[n-n_0] - 0.9x_1[n-n_0-3] = y_2[n]$.
The system is time-invariant.

Stability: Assuming that the input is bounded $|x[n]| \leqslant B_x$, the output

$$
\begin{aligned}
|y[n]| &= |x[n] - 0.9x[n-3]|, \\
&\leqslant |x[n]| + |-0.9x[n-3]|, \\
&\leqslant 1.9B_x,
\end{aligned}
$$

is also bounded.

Therefore, the system is BIBO stable.

Causality: Since the output does not require future values of the input, the system is causal.

Memory: Since the output requires past values of the input, the system is not memoryless.

- Page 38–56, read section 1.5 regarding DT signals;
- Page 58, Q1.15: (a)–(b);
- Page 58–59, Q1.16: (a)–(b);
- Page 59, Q1.18: (a)–(c);
- Page 59, Q1.19: (b)–(c);
- Page 62, Q1.28: Except (d), (f).

Thank you for your attention.