MTH101: Review Session II

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Complex numbers

- **Geometric form**: $z = (x, y)$;
- **Algebraic form**: $z = x + iy$;
- **Polar form**: $z = r(\cos\theta + i\sin\theta)$;
- **Exponential form**: $z = r \cdot e^{i\theta}$,

where $r = |z| = \sqrt{x^2 + y^2}$ is the distance between z and 0,

$$\theta = \text{Arg}(z) = \begin{cases} \arctan(\frac{y}{x}), & \text{if } x > 0, \\ \arctan(\frac{y}{x}) + \pi, & \text{if } x < 0 \text{ and } y \geq 0, \\ \arctan(\frac{y}{x}) - \pi, & \text{if } x < 0 \text{ and } y < 0, \\ \frac{\pi}{2}, & \text{if } x = 0 \text{ and } y > 0, \\ -\frac{\pi}{2}, & \text{if } x = 0 \text{ and } y < 0. \end{cases}$$

Note that Principal Argument $\text{Arg}(z) \in (-\pi, \pi]$, argument $\arg(z) = \text{Arg}(z) + 2n\pi,\ n = 0, \pm1, \pm2, \ldots$

Operational laws

$$z_1 \pm z_2 = (x_1 \pm x_2) + i(y_1 \pm y_2);$$
$$z_1 \cdot z_2 = (x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2);$$
$$\frac{z_1}{z_2} = z_1 \cdot \frac{1}{z_2} = (x_1 + iy_1)\left(\frac{x_2}{x_2^2 + y_2^2} + i\frac{-y_2}{x_2^2 + y_2^2}\right)$$

Conjugate of a complex number

$$\bar{z} = x - iy$$

By using the formula $|z|^2 = z \cdot \bar{z}$, we have an easier approach to calculate the multiplicative inverse:

$$\frac{1}{z_2} = \frac{\overline{z_2}}{|z_2|^2}.$$

Operations using the Exponential Form

Consider $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$, then

$$z_1 \cdot z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)},$$
$$\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)},$$
$$(z_1)^n = r_1^n e^{in\theta_1}.$$

Roots of complex number The equation $z^n = re^{i\theta}$ has exactly n roots, and they are

$$r^{\frac{1}{n}} e^{i\frac{\theta+2k\pi}{n}}, \quad \text{with } k = 0, 1, ..., n-1.$$

Complex Functions

Complex functions $f(z) : \mathbb{C} \to \mathbb{C}$ can be written as the sum of two real functions for $z = (x, y)$:

$$f(z) = u(x, y) + iv(x, y).$$

Some topology terminology

- Neighborhood (Open Disk)
- Connected
- Domain (open and connected)
- Closed curve

Analyticity

We call a function $f(z)$ is

- **Analytic at** z_0 if it is differentiable in a neighborhood of z_0;
- **Analytic in a Domain D** if it is Analytic at any points of **D**;
- **Entire** if it is Analytic on the whole complex plane $\mathbb{C}$.

Cauchy-Riemann Equations

Let $f(z) = u(x, y) + iv(x, y)$ for $z = (x, y)$, then the following statements are equivalent:

- $f(z)$ is Analytic in a domain D;
- $u_x = v_y, \quad u_y = -v_x$. at all points of D.

Moreover, if $f(z)$ is Analytic in a domain D, then both u and v satisfy **Laplace's equation**

$$\nabla^2 u = u_{xx} + u_{yy} = 0,$$
$$\nabla^2 v = v_{xx} + v_{yy} = 0.$$

Some basic complex functions

- **Polynomials**: $f(z) = c_n z^n + c_{n-1} z^{n-1} + \cdots + c_1 z + c_0$ are **entire** functions.
- **Rational functions**:$f(z) = \frac{P(z)}{Q(z)}$ are analytic whenever $Q(z) \neq 0$.
- **Exponential function**: $f(z) = e^z = e^x(\cos y + i \sin y)$ is an **entire** function (some further properties need to be memorized: periodicity, $e^{2\pi i} = 1$, $(e^z)' = e^z$ etc.)
- **Logarithm function** $f(z) = \mathrm{ln}\ z = \ln|z| + i \arg z$ is analytic except at 0 and on the negative real axis and $(\mathrm{ln} z)' = \frac{1}{z}$. In particular, the principle value of the logarithm function is $\mathrm{Ln}\ z = \ln|z| + i\mathrm{Arg}\ z$.

(Continued)

- **General Power function**: $f(z) = z^c = e^{c \ln z} = e^{c(\text{Ln } z + 2n\pi i)}$, $n = 0, \pm 1, \pm 2, \ldots$ and its principle value is $e^{c \text{ Ln}(z)}$.
- **Trigonometric Functions**: $\cos z = \frac{1}{2}(e^{iz} + e^{-iz})$, $\sin z = \frac{1}{2i}(e^{iz} - e^{-iz})$, $(\sin z)' = \cos z$, $(\cos z)' = -\sin z$, $\sin^2 z + \cos^2 z = 1$ and they are **entire** functions (Note that $\tan z$, $\cot z$, $\sec z$ and $\csc z$ are not entire!)
- **Hyperbolic Functions**: $\cosh z = \frac{1}{2}(e^{z} + e^{-z})$, $\sinh z = \frac{1}{2}(e^{z} - e^{-z})$, $(\sinh z)' = \cosh z$, $(\cosh z)' = \sinh z$, $\cosh^2 z - \sinh^2 z = 1$ and they are **entire** functions (Note that $\tanh z$, $\coth z$, $\text{sech}\, z$ and $\text{csch}\, z$ are not entire!)

Complex Integrals

$$\int_\gamma f(z)dz$$

(Note: Orientation Matters!!)

- **Integrate by parametrization** with a parametrization of $\gamma = z(t)$, $t \in [a, b]$:

$$\int_\gamma f(z)\,dz = \int_a^b f(z(t))\dot{z}(t)dt,$$

  where $\dot{z}(t)$ is the derivative of $z(\cdot)$ with respect to t.
  (If you see functions, e.g., $|z|$, $\bar{z}$, or if γ is not closed, then integrate it by parametrization.)

- **Cauchy's Integtal Theorem**
  If a function $f(z)$ is **Analytic** in a **Simply Connected Domain** D, then for every **simply closed path** γ in D we have

$$\oint_{\gamma} f(z)\, dz = 0$$

  (Check Hypotheses carefully!!)

- **Cauchy's Integral Formulas**
  If $f(x)$ is **Analytic** in a **Simply Connected Domain** D, then for any point $z_0 \in D$ and any **counterclockwise oriented simple closed path** γ that encloses the point z_0 we have

$$\oint_{\gamma} \frac{f(z)}{z - z_0}\, dz = 2\pi i f(z_0).$$

- **Cauchy's Integral Formulas for derivatives**
  If $f(x)$ is **Analytic** in a **Simply Connected Domain** D, then for any point $z_0 \in D$ and any **counterclockwise oriented simple closed path** γ that encloses the point z_0 we have

$$\oint_\gamma \frac{f(z)}{(z-z_0)^{n+1}}\,dz = \frac{2\pi i}{n!} f^{(n)}(z_0).$$

- **Residue's Theorem**
  If $f(z)$ is **analytic** in a **simply connected Domain** D except for finitely many isolated singularities z_1, z_2,..., z_n and γ is a **simple closed path** with **counterclockwise** orientation in D which encloses all the isolated singularities. Then

$$\oint_{\gamma} f(z)\, dz = 2\pi i \sum_{k=1}^{n} \mathrm{Res}_{z_k}(f).$$

  (This method will be convenient if you see an isolated singular point which is not a pole, e.g., $\oint_{\gamma} e^{1/z}\, dz$ where γ is a unit circle; or if you find more than one singularities, e.g., $\oint_{\gamma} \frac{1}{(z-2)(z-3)}\, dz$ where γ is all z such that $|z-2| = 2$.)

Classification of singularities

We need to classify the singularities, which needs the Laurent Series

Taylor's Series: In a open disk, has no negative powers:

Laurent Series: In an annulus, possibly has negative powers.

- If it has infinitly many negative power terms, the singularity is essential;
- If it has finitely many negative power terms, the singularity is a pole;
- If it has no negative power term, the singularity is removable.

1 The singularity is a pole, how to identify poles? If

$$f(z) = \frac{p(z)}{q(z)},$$

where $q(z)$ is analytic at z_0 and has a zero of order n at z_0, $p(z)$ is analytic and non-zero at z_0, then $f(z)$ has a pole of order n at z_0.

- **simple pole** $\mathrm{Res}_{z_0}(f) = \lim_{z \to z_0} (z - z_0) f(z)$, or $\mathrm{Res}_{z_0}(f) = \frac{p(z_0)}{q'(z_0)}$
- **pole of order n** $\mathrm{Res}_{z_0}(f) = \frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} [(z - z_0)^n f(z)]$.

2 The singularity z_0 is removable, e.g., $\frac{\sin z}{z}$, there is no negative power term $\quad\Rightarrow\quad b_1 = 0 \quad\Rightarrow\quad \text{Res}_{z_0}(f) = 0$.

3 The singularity is essensial, e.g., $e^{1/z}$, then you need to write the Laurent Series at z_0, and collect the coefficient of term $(z - z_0)^{-1}$.

Real Integrals

Example

$$\int_{-\infty}^{\infty} \frac{1}{x^4+1}\ dx$$

- Use Residue integration method to calculate
$$\int_{\gamma\cup[-R,R]} \frac{1}{z^4+1}\ dz$$
along the closed curve formed by a semicircle γ above the real axis and the interval $[-R, R]$ on the real axis.
- Use ML-inequality to prove that along the semicircle γ the integral is 0 as $R \to \infty$.
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$$\int_{-\infty}^{\infty} \frac{1}{x^4+1}\ dx = \lim_{R\to\infty} \int_{[-R,R]} \frac{1}{z^4+1}\ dz = 2\pi i \sum \text{Res } f(z)$$

Bibliography

1 *Kreyszig, E.* **Advanced Engineering Mathematics**. Wiley, 9th Edition.