Week 9 Fourier Transform (Cont.)

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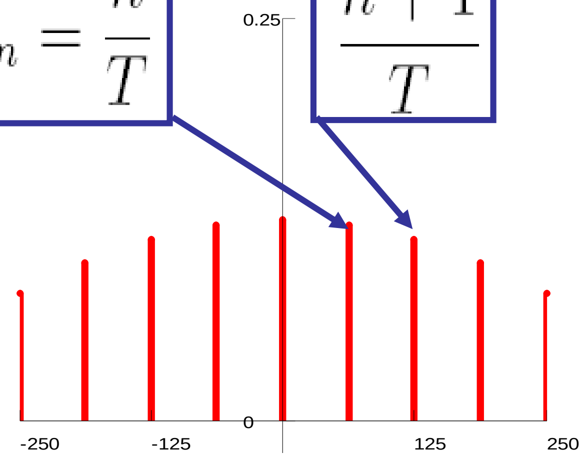
From Fourier series to Fourier transform

$$f_n = \frac{n}{T} \qquad \frac{n+1}{T}$$



$$x_T(t) = \sum_{n=-\infty}^{\infty}\left[\int_{-\frac{T}{2}}^{\frac{T}{2}} x_T(t)\, e^{-j2\pi t\frac{n}{T}}\, dt\right] e^{j2\pi t\frac{n}{T}}\, \frac{1}{T}$$

$$\int_{-\infty}^{\infty} \qquad \infty \qquad f \qquad f \qquad df$$

$$x(t) = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} x(t)\, e^{-j2\pi t f}\, dt\right] e^{j2\pi t f}\, df$$

- **What will happen when *T*→ ∞ ?**

$$\left(\Delta f = \frac{1}{T}\right) \rightarrow df$$

$$\left(f_n = \frac{n}{T}\right) \rightarrow f$$

From FS to FT

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Forward Fourier transform :

('analysis' equation)

$$X(f) = \mathcal{F}\{x(t)\} \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} x(t) e^{-j2\pi tf} dt$$

The Inverse Fourier transform :

('synthesis' equation)

$$x(t) = \mathcal{F}^{-1}\{X(f)\} = \int_{-\infty}^{\infty} X(f) e^{j2\pi tf} df$$

Properties of Fourier Transform

Properties of Fourier Transform

- Linearity
- Scaling
- Time shifting
- Frequency shifting
- Duality
- Convolution

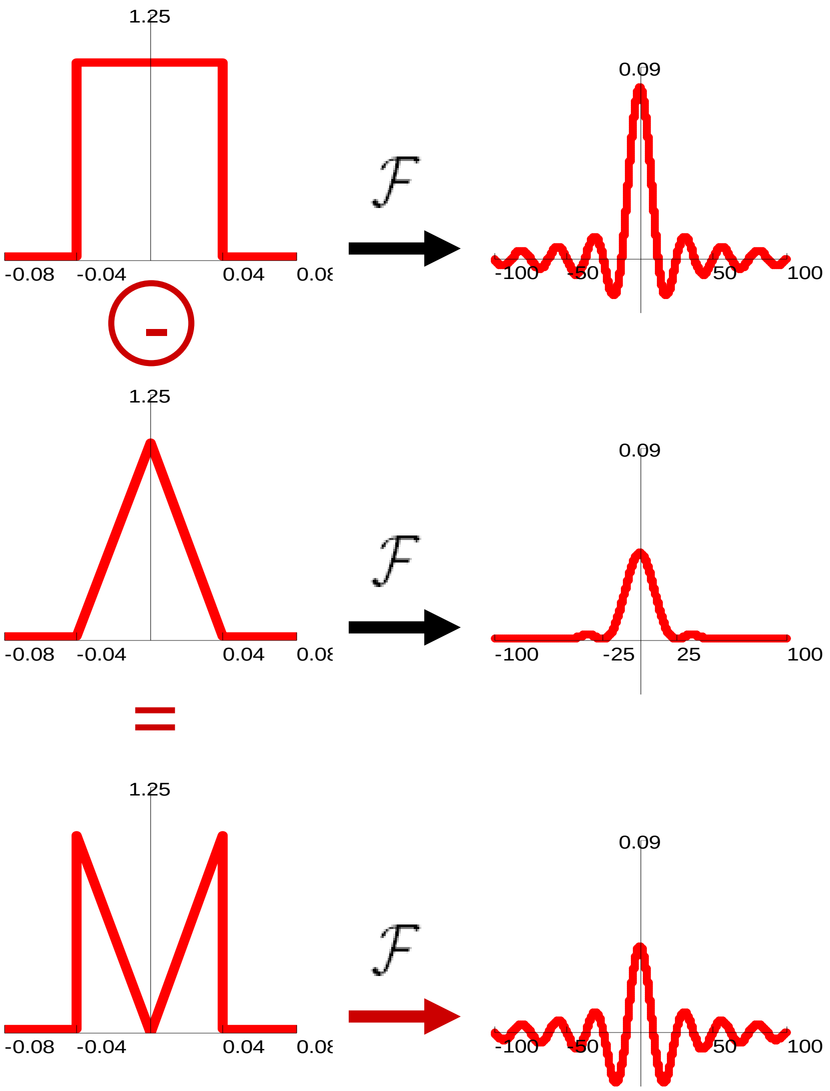
Properties of Fourier Transform

- **Linearity :**

$$\mathcal{F}\{x(t)\} = X(f)$$

$$\mathcal{F}\{y(t)\} = Y(f)$$

$$\mathcal{F}\{ax(t) + by(t)\} = aX(f) + bY(f)$$



Properties of Fourier Transform

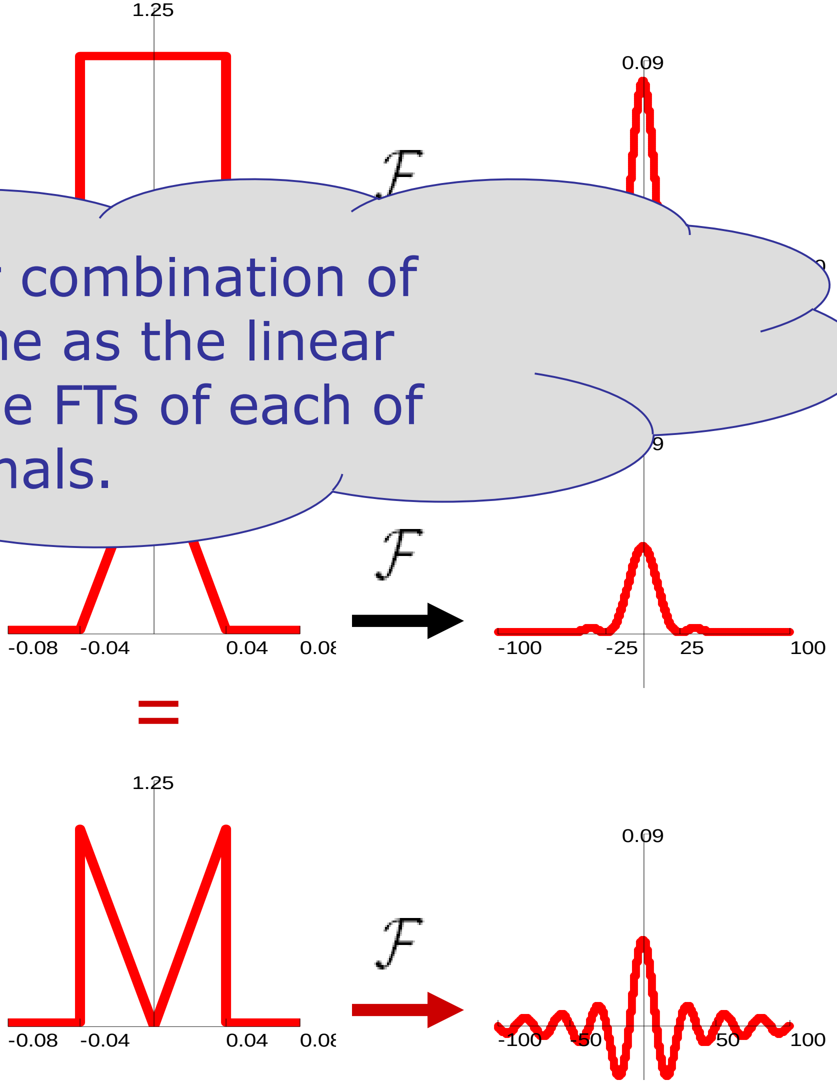
- **Linearity :**

$$\mathcal{F}\{x(t)\} = X(f)$$

The FT of a linear combination of signals is the same as the linear combination of the FTs of each of the individual signals.

$$\mathcal{F}\{y(t)\} = Y(f)$$

$$\mathcal{F}\{ax(t) + by(t)\} = aX(f) + bY(f)$$



- **Scaling Property**

$$\boxed{\mathcal{F}\{x(t)\} = X(f)}$$

What will be the FT of *x(t s)* ?

$$x(t\,s)\,, \quad s > 0$$

$$\mathcal{F}\{x(s\,t)\} = \int_{-\infty}^{\infty} x(s\,t)e^{-j2\pi f t}dt$$

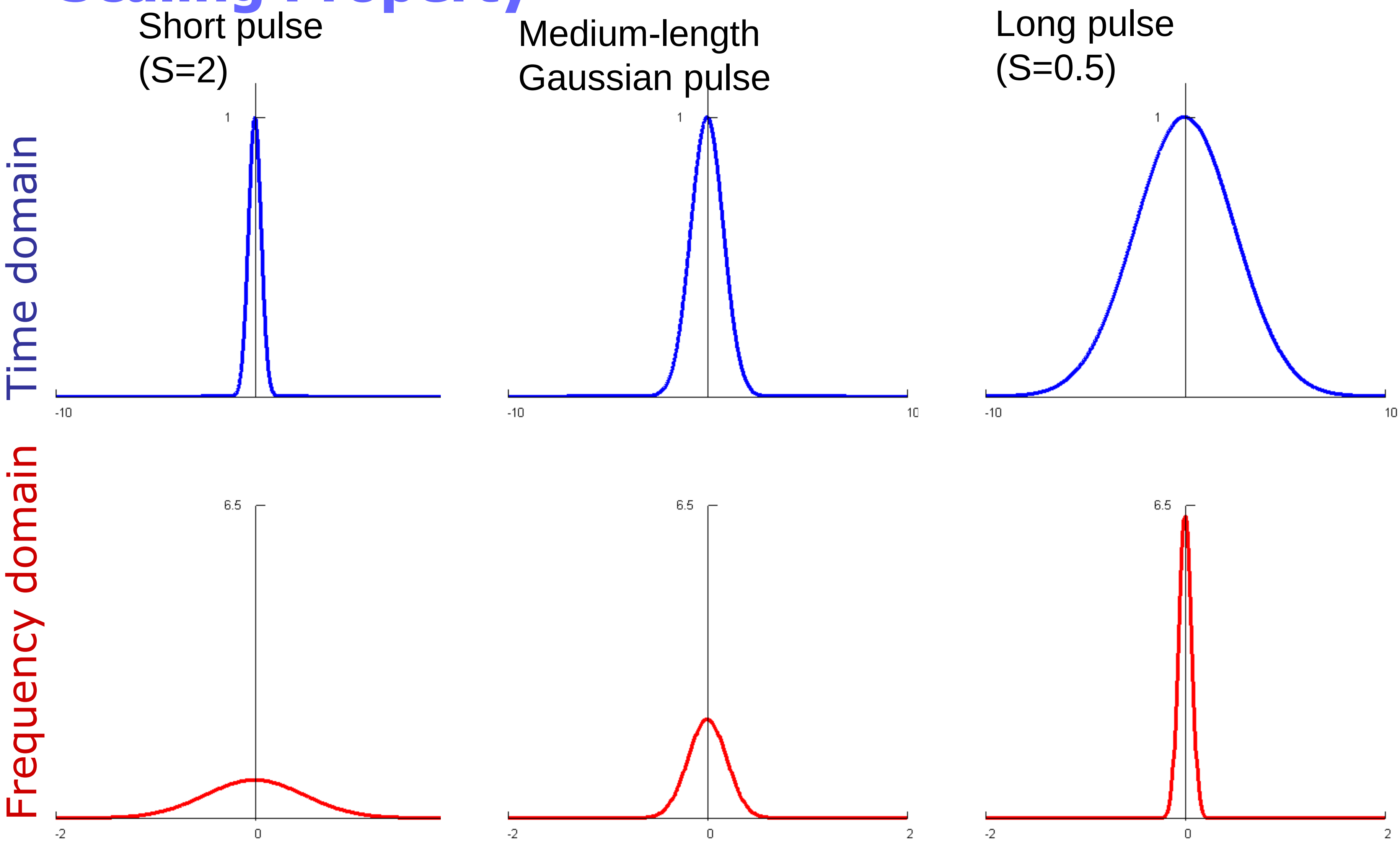
By changing variables: $u = s\,t$:

$$\mathcal{F}\{x(s\,t)\} = \frac{1}{s}\int_{-\infty}^{\infty} x(u)e^{-j2\pi f u/s}du = \frac{1}{s}X\left(\frac{f}{s}\right)$$

$$\boxed{\mathcal{F}\{x(st)\} = \frac{1}{|s|}X\left(\frac{f}{s}\right), for\ s \in R\ and\ s \neq 0}$$

Properties of Fourier Transform

- **Scaling Property**



- **Time Shifting Property**

$$\boxed{\mathcal{F}\{x(t)\} = X(f)}$$

What will be the FT of *x(t-T)* ?

$$\mathcal{F}\{x(t-T)\} = \int_{-\infty}^{\infty} x(t-T)e^{-j2\pi ft}dt$$

By changing variables: $u = t - T$:

$$\mathcal{F}\{x(t-T)\} = \int_{-\infty}^{\infty} x(u)e^{-j2\pi f(u+T)}du = X(f)e^{-j2\pi fT}$$

$$\boxed{\mathcal{F}\{x(t-T)\} = X(f)e^{-j2\pi fT}}$$

- **Time Shifting Property**

$$\boxed{\mathcal{F}\{x(t)\} = X(f)}$$

$$\mathcal{F}\{x(t-T)\} = \int_{-\infty}^{\infty} x(t-T)e^{-j2\pi ft}dt$$

By changing variables: $u = t - T$:

$$\mathcal{F}\{x(t-T)\} = \int_{-\infty}^{\infty} x(u)e^{-j2\pi f(u+T)}du = X(f)e^{-j2\pi fT}$$

$$|\mathcal{F}\{x(t-T)\}| = |X(f)|$$

$$\boxed{\mathcal{F}\{x(t-T)\} = X(f)e^{-j2\pi fT}}$$

Properties of Fourier Transform

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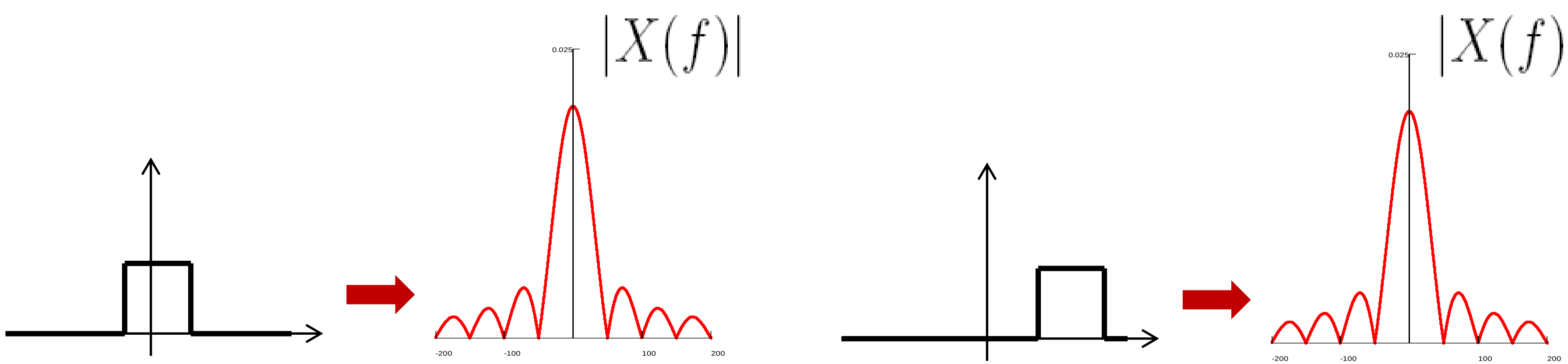
- **Time Shifting Property**
  - A shift in time domain is equivalent to a linear phase shift in frequency domain (i.e., multiplying with a complex exponential).
    - The magnitude spectrum depends only on the shape of a signal, in time domain, which is unchanged in a time shift.
    - In a time shift only the phase spectrum will be changed.

$$\text{magnitude} \quad |G(\omega)| = |\mathrm{e}^{-\mathrm{j}\omega t_0} X(\omega)| = |\mathrm{e}^{-\mathrm{j}\omega t_0}||X(\omega)| = |X(\omega)|;$$

$$\text{phase} \quad <G(\omega) = <\{\mathrm{e}^{-\mathrm{j}\omega t_0} X(\omega)\} = <\mathrm{e}^{-\mathrm{j}\omega t_0} + <X(\omega) = -\omega t_0 + <X(\omega)$$



Properties of Fourier Transform

- **Frequency shifting**

$$\mathcal{F}\{x(t)\} = X(f) \quad \Rightarrow \quad \mathcal{F}\{x(t)e^{j2\pi f_0 t}\} = \int_{-\infty}^{\infty} x(t)e^{j2\pi f_0 t}e^{-j2\pi f t}dt$$

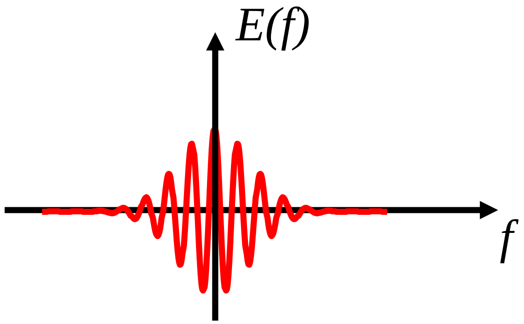
$$= \int_{-\infty}^{\infty} x(t)e^{-j2\pi(f-f_0)t}dt$$

$$= X(f-f_0)$$

$$\boxed{x(t)e^{j\omega_0 t} \xleftrightarrow{\mathcal{F}} F(\omega-\omega_0)}$$

- **Amplitude Modulation (AM):**

$$x(t) = e(t)\cos(2\pi f_0 t)$$



$X(f) = ?$

Example

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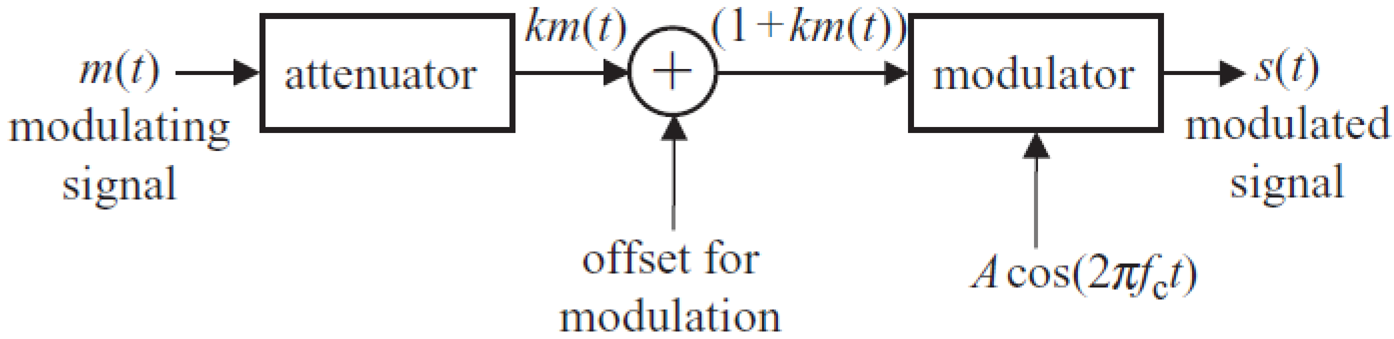
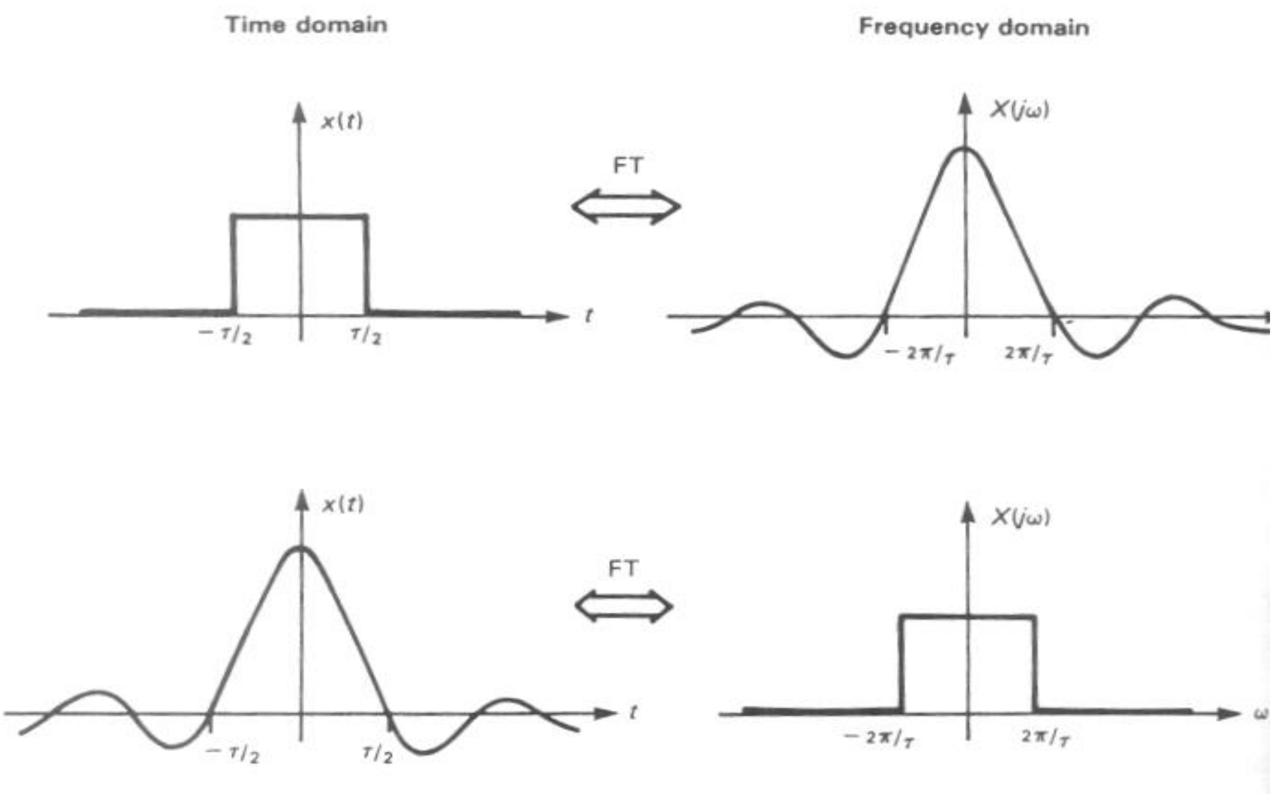
- **Evaluate the CTFT of the following signal:**



Amplitude modulation (AM) system.

Properties of Fourier Transform

- **Duality**
  - If $x(t) \Leftrightarrow X(f)$ then $y(t) = X(f)|_{f=t} \Leftrightarrow Y(f) = x(-f)$
  - Time domain and frequency domain are symmetric



$$\boxed{\text{If } x(t) \xleftrightarrow{\text{CTFT}} X(\omega), \text{ then}}$$

$$\boxed{X(t) \xleftrightarrow{\text{CTFT}} 2\pi x(-\omega)}$$

Example 4.13

Find the FT $G(\omega)$ of the signal

$$g(t) = \frac{2}{1 + t^2}$$

Hint: Recall from an example in last lecture

$$4.2\ x(t) = e^{-a|t|}, a \in R^+$$

$$x(t) = e^{-|t|} \overset{\mathcal{F}}{\leftrightarrow} X(\omega) = \frac{2}{1+\omega^2}$$

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- **Multiplication Property**

$$\boxed{\mathcal{F}\{x(t)\} = X(f)}$$

$$\boxed{\mathcal{F}\{y(t)\} = Y(f)}$$

$$\mathcal{F}\{x(t)y(t)\} = \text{?}$$

$$\mathcal{F}\{x(t)y(t)\} = \int_{-\infty}^{\infty} x(t)y(t)e^{-j2\pi ft}dt$$

By writing y(t) in terms of Y(f) ….

$$\begin{aligned}
\mathcal{F}\{x(t)y(t)\} &= \int_{t=-\infty}^{\infty} x(t)\left[\int_{\theta=-\infty}^{\infty} Y(\theta)e^{j2\pi\theta t}d\theta\right] e^{-j2\pi ft}dt \\
&= \int_{\theta=-\infty}^{\infty} Y(\theta)\left[\int_{t=-\infty}^{\infty} x(t)e^{-j2\pi t(f-\theta)}dt\right] d\theta \\
&= \int_{\theta=-\infty}^{\infty} Y(\theta)X(f-\theta)d\theta
\end{aligned}$$

- **Multiplication Property**

$$\mathcal{F}\{x(t)\} = X(f)$$

$$\mathcal{F}\{y(t)\} = Y(f)$$

Convolution

$$\int_{\theta=-\infty}^{\infty} Y(\theta)X(f-\theta)d\theta \stackrel{\text{def}}{=} X(f)*Y(f)$$

$$\mathcal{F}\{x(t)y(t)\} = X(f)*Y(f) = \int_{\theta=-\infty}^{\infty} Y(\theta)X(f-\theta)d\theta$$

Angular form: $x_1(t)x_2(t) \xleftrightarrow{\text{CTFT}} \frac{1}{2\pi}[X_1(\omega) * X_2(\omega)].$

- **The Fourier transform of the convolution of two signals in time domain ?**

$$\mathit{If}\ x_1(t) \xleftrightarrow{\text{CTFT}} X_1(\omega) \quad \mathit{and} \quad x_2(t) \xleftrightarrow{\text{CTFT}} X_2(\omega),\ \mathit{then}$$

$$x_1(t) * x_2(t) \xleftrightarrow{\text{CTFT}} X_1(\omega)X_2(\omega)$$

$$F[f_1(t) * f_2(t)] = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f_1(\tau) f_2(t-\tau)d\tau\right]e^{-j\omega t}dt$$

$$= \int_{-\infty}^{\infty} f_1(\tau)\left[\int_{-\infty}^{\infty} f_2(t-\tau)e^{-j\omega t}dt\right]d\tau$$

$$= \int_{-\infty}^{\infty} f_1(\tau)F_2(\omega)e^{-j\omega\tau}d\tau$$

$$= F_2(\omega)\int_{-\infty}^{\infty} f_1(\tau)e^{-j\omega\tau}d\tau \qquad = F_1(\omega)F_2(\omega)$$

- **FT of real-valued functions**

$$x(t) \in \mathbb{R} \qquad \mathcal{F}\{x(t)\} = X(f)$$

$$X(-f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi(-f)t} dt = \left[\int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt\right]^* = X(f)^*$$

$$X(f) = |X(f)| e^{j\angle X(f)}$$

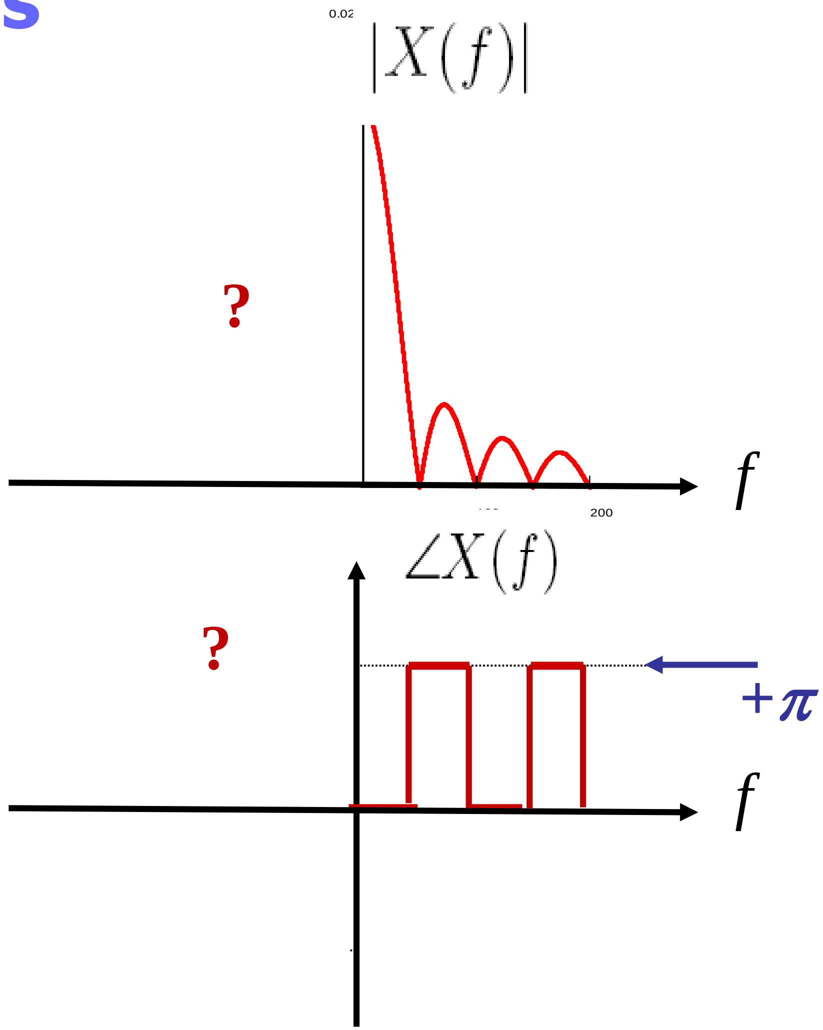
* Denotes the complex conjugate of X(f)

Hermitian symmetry

- Even symmetric magnitude spectrum
- Odd symmetric phase spectrum

Example

- **FT of real-valued functions**



- **FT of even or odd symmetry functions**

$$
\begin{aligned}
X(f) &= \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt \\
&= \int_{-\infty}^{\infty} x(t)[\cos(j2\pi f t) - j\sin(j2\pi f t)] dt \\
&= \int_{-\infty}^{\infty} x(t)\cos(j2\pi f t) dt - j\int_{-\infty}^{\infty} x(t)\sin(j2\pi f t) dt
\end{aligned}
$$

Even function → $X(f) = 2\int_{0}^{\infty} x(t)\cos(j2\pi f t) dt$ X(f) is even

Odd function → $X(f) = -j2\int_{0}^{\infty} x(t)\sin(j2\pi f t) dt$ X(f) is odd

Properties of Fourier Transform

- **Parseval energy theorem**

$$\int_{-\infty}^{\infty} | f(t) |^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} | F(\omega) |^2 d\omega$$

Parseval's theorem

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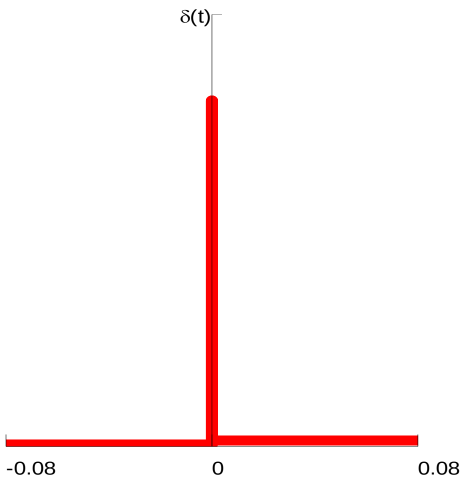
The total average power in a periodic signal equals the sum of the average powers in all of its harmonic components.

$$P_x = \frac{1}{T}\int_T |x(t)^2| dt = \sum_{k=-\infty}^{\infty} |a_k|^2$$

Example of Fourier Transform

Fourier Transform Pairs

- **FT of the Impulse Function (Delta Dirac)**



$$\mathcal{F}\{\delta(t)\} = \int_{-\infty}^{\infty} \delta(t) e^{j2\pi t f} dt$$

Revisit of impulse signal

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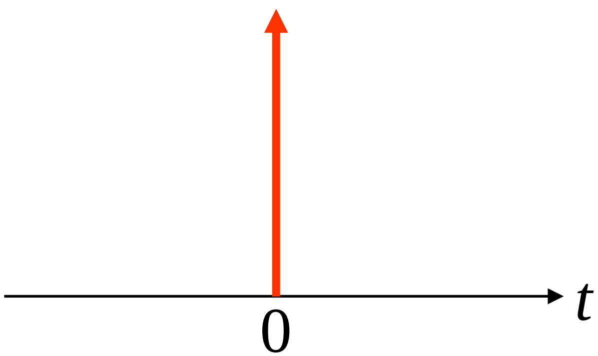
$$\delta(t)=\begin{cases}0 & t\neq 0\\ \infty & t=0\end{cases}\quad \text{and} \quad \int_{-\infty}^{\infty}\delta(t)dt=1$$



$$\boxed{\int_{-\infty}^{\infty}\delta(t-t_0)\phi(t)dt=\phi(t_0)}$$

$$\boxed{\int_{-\infty}^{\infty}\delta(t)\phi(t)dt=\phi(0)}$$

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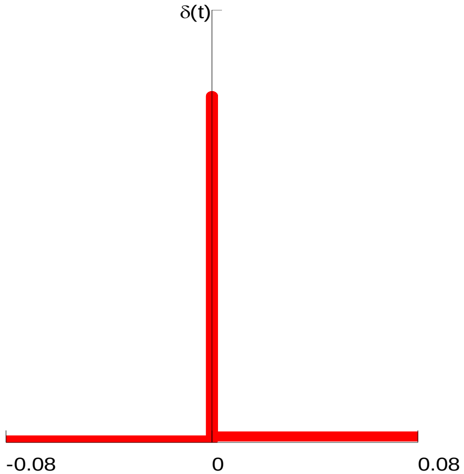
- **FT of the Impulse Function (Delta Dirac)**



$$\mathcal{F}\{\delta(t)\} = \int_{-\infty}^{\infty} \delta(t) e^{j2\pi t f} dt$$

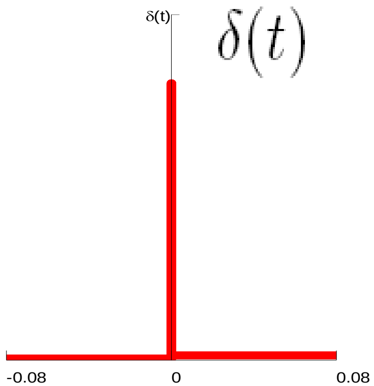
$$\mathcal{F}\{\delta(t)\} = e^{j2\pi t f}\big|_{t=0} = 1$$

$$\int_{-\infty}^{\infty} x(u)\delta(t-u)du = x(t)$$

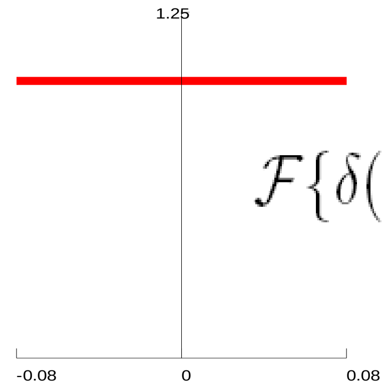
$$\boxed{\mathcal{F}\{\delta(t)\} = 1}$$

Fourier Transform Pairs (Ex. 3)

- **FT of the Impulse Function (Delta Dirac)**



$\delta(t)$



$$\mathcal{F}\{\delta(t)\} = 1$$

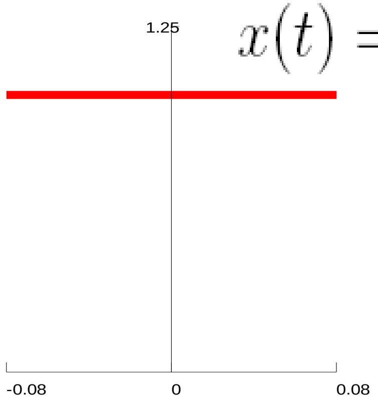
f

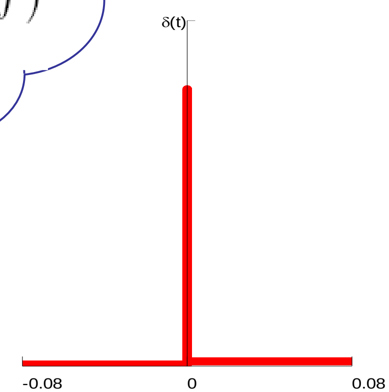
  - By using duality property :

$$\int_{-\infty}^{\infty} e^{-j2\pi f t} dt = \delta(f)$$

$$\boxed{\mathcal{F}\{1\} = \delta(f)}$$

$x(t) = 1$





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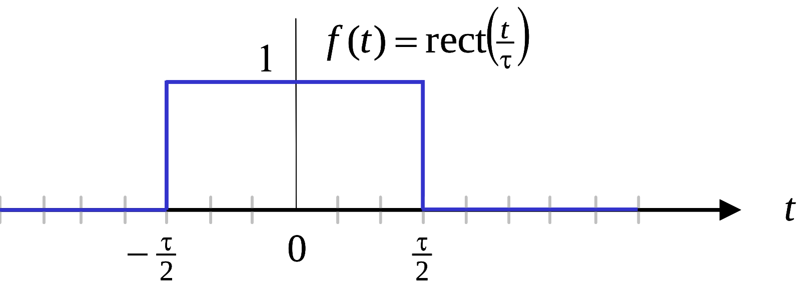
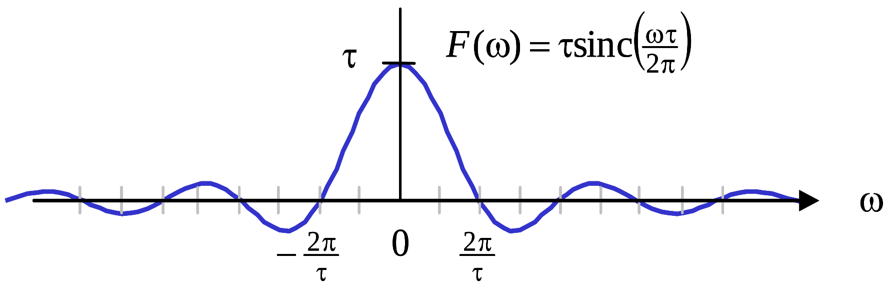
Fourier transform of 1





When $\tau \to \infty$, the rect. signal becomes a constant signal

$$F(1) = \lim_{\tau\to\infty} \tau \, sinc(\frac{\omega\tau}{2\pi})$$

$$\delta(\omega) = \lim_{\tau\to\infty} \frac{\tau}{2\pi} \, sinc(\frac{\omega\tau}{2\pi})$$

$$F(1) = \lim_{\tau\to\infty} 2\pi \frac{\tau}{2\pi} \, sinc\left(\frac{\omega\tau}{2\pi}\right) = 2\pi\delta(\omega)$$

Inverse Fourier transform

What is inverse Fourier transform of the following functions?

$$X(\omega) = \frac{1}{5 + j\omega} + \frac{8}{16 + \omega^2}$$

By using the look-up table method, pg. 329 textbook.

$$\frac{1}{5 + j\omega} \overset{CTFT}{\longleftrightarrow} e^{-5t}u(t)$$

$$\frac{8}{16 + \omega^2} \overset{CTFT}{\longleftrightarrow} e^{-4|t|}$$

Therefore, the inverse CTFT is:

$$x(t) = e^{-5t}u(t) + e^{-4|t|}$$

Inverse Fourier transform

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For more complex functions,

$$X(\omega) = \frac{1 + j\omega}{(j\omega)^2 + 5(j\omega) + 6}$$

use partial fraction expansion.

(1) Factorize

$$X(\omega) = \frac{N(\omega)}{(\mathrm{j}\omega - p_1)(\mathrm{j}\omega - p_2)\cdots(\mathrm{j}\omega - p_n)}.$$

(2) Express it in terms of n partial fractions,

$$X(\omega) = \frac{k_1}{(\mathrm{j}\omega - p_1)} + \frac{k_2}{(\mathrm{j}\omega - p_2)} + \cdots + \frac{k_n}{(\mathrm{j}\omega - p_n)},$$

$$k_r = [(\mathrm{j}\omega - p_r)X(\omega)]_{\mathrm{j}\omega = p_r},$$

(3) The inverse CTFT can be calculated as:

$$x(t) = [k_1 \mathrm{e}^{p_1 t} + k_2 \mathrm{e}^{p_2 t} + \cdots + k_n \mathrm{e}^{p_n t}]u(t).$$

Dirichlet conditions

- **Conditions for the Fourier transform of $x(t)$ to exist (Dirichlet conditions (sufficient but not necessary):**
  - $x(t)$ is single-valued with finite maxima and minima in any finite time interval
  - $x(t)$ is piecewise continuous; i.e., it has a finite number of discontinuities in any finite time interval
  - $x(t)$ is absolutely integrable

$$\int_{-\infty}^{\infty} |g(t)| dt < \infty$$

Time Domain	Frequency Domain
convolution	multiplication

$$h(t) * x(t) = H(\omega) X(\omega)$$

$$x(t) \rightarrow \boxed{\begin{array}{c}\text{Impulse Response}\\ \text{LTI System}\\ h(t)\end{array}} \rightarrow y(t)=f(t)*h(t)$$

$$X(\omega) \rightarrow \boxed{\begin{array}{c}\text{Impulse Response}\\ \text{LTI System}\\ H(\omega)\end{array}} \rightarrow Y(\omega) = F(\omega)H(\omega)$$

Transfer function of an LTI system

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$$\boxed{h(t) \xleftrightarrow{CTFT} H(\omega)}$$

Fourier transfer function:

$$\boxed{H(\omega) = \frac{Y(\omega)}{X(\omega)}}$$

Example

Suppose the CT signal

$$x(t) = \mathrm{e}^{-t}u(t)$$

is applied as input to a causal LTIC system modeled by the impulse response

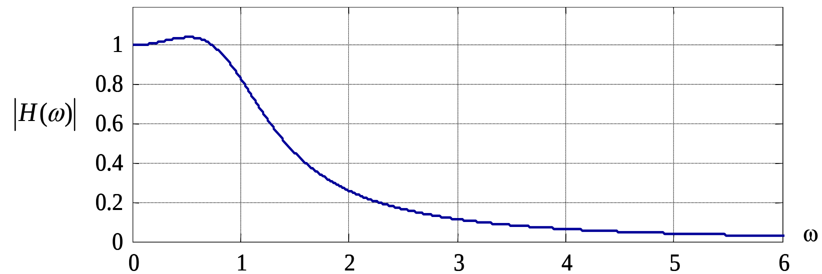
$$h(t) = \mathrm{e}^{-2t}u(t)$$

Calculate the resulting output $y(t)$

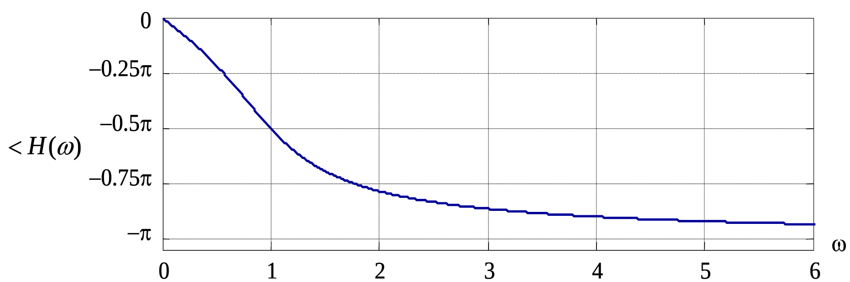
Gain and phase responses

The Fourier transfer function $H(\omega)$ provides a complete description of the LTIC system.

The magnitude spectrum $|H(\omega)|$ response function is also referred to as the *gain response* of the system



while the phase spectrum $<H(\omega)$ is referred to as the *phase response* of the system.



Exercises

Let $f(t)$ be a signal, and let $F(\omega) = \mathcal{F}\{f(t)\}$ be its Fourier transform. Find the Fourier transform of the signal $g(t) = f(2(t-3))$ as a function of the Fourier transform of the signal $f(t)$.

Exercises

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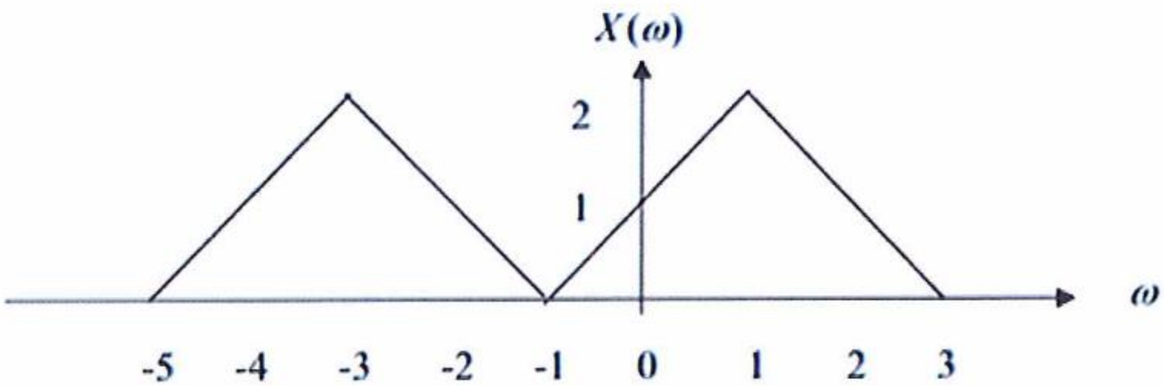
The Fourier transform of $x(t)$ is shown in the figure as $X(\omega)$. Without explicitly computing $x(t)$,



a) Compute quantities of $\int_{-\infty}^{\infty} x(t)dt$ **5**

b) Compute quantities of $\int_{-\infty}^{\infty} |x(t)|^2 dt$ **5**

c) Compute quantities of $x(0)$ **5**

Exercises

- **4.6**
- **4.19**