EEE225 Advanced Electrical Circuits and Electromagnetics

Lecture 4 Steady Electric Current

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1. Currents

- Electrostatics – generated by *electric charges at rest*.
- Magnetostatics – generated by *electric charges in motion*, which constitute the ***currents***.
- There are several types of electric currents caused by the *motion of free charges*:

Governed by Ohm's law!

  - ***Conduction currents*** in conductors are caused by drift motion of conduction electrons;
  - ***Convection currents*** result from motion of electrons and/or ions in a vacuum;
  - ***Electrolytic currents*** are the result of migration of positive and negative ions.

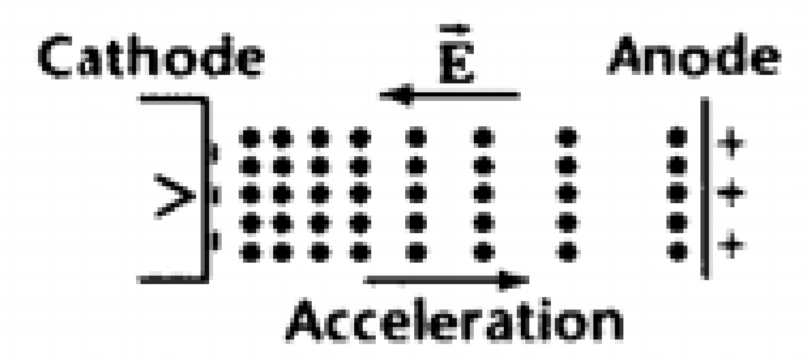
1.1 Conduction Current

- An electron which may be considered as not being attached to any particular atom is called a *free electron*.
  - A free electron has the capability of moving through a whole crystal lattice. However, the heavy, positively charged ions are relatively fixed at their regular positions in the crystal lattice and do not contribute to the current in the metal.
- Thus, the current in a metal conductor, called ***conduction current***, is simply a flow of electrons.
  - The transitory flow of charges comes to a halt in a very short time in an isolated conductor placed in an electric field.
  - To maintain a ***steady current*** within a conductor, a continuous supply of electrons at one end and removal at the other is necessary.

1.2 Convection Current

Not required

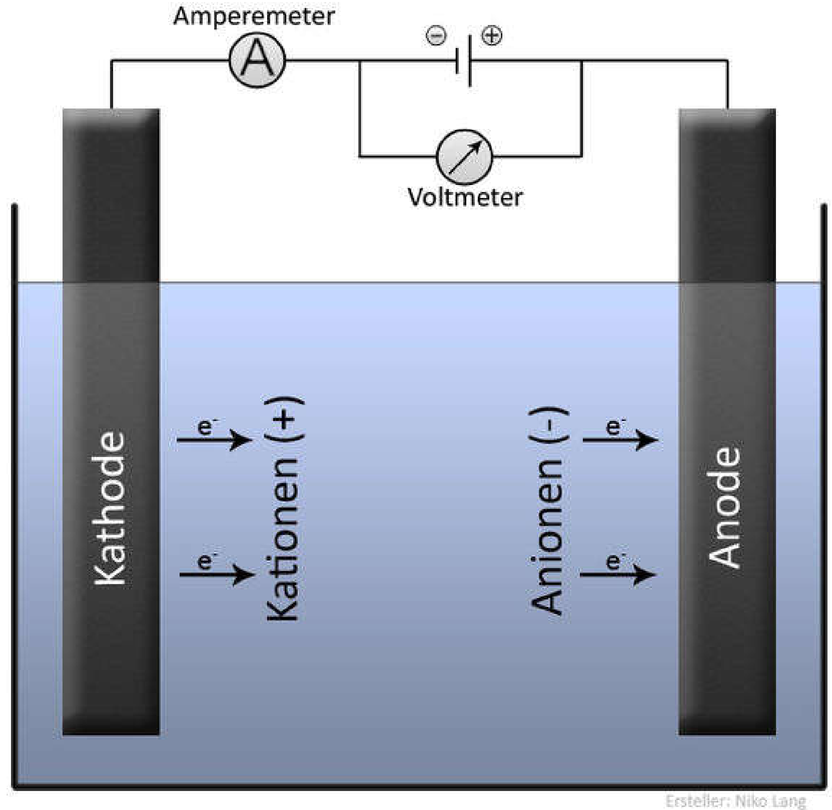
- ***Convection currents*** are the result of the motion of positively or negatively charged particles in a vacuum or rarefied gas.
- Examples:
  - Electron beams in a cathode-ray tube
  - The violent motions of charged particles in a thunderstorm
- Convection currents, the result of hydrodynamic motion involving a mass transport, are not governed by Ohm's law.



1.3 Electrolytic Current

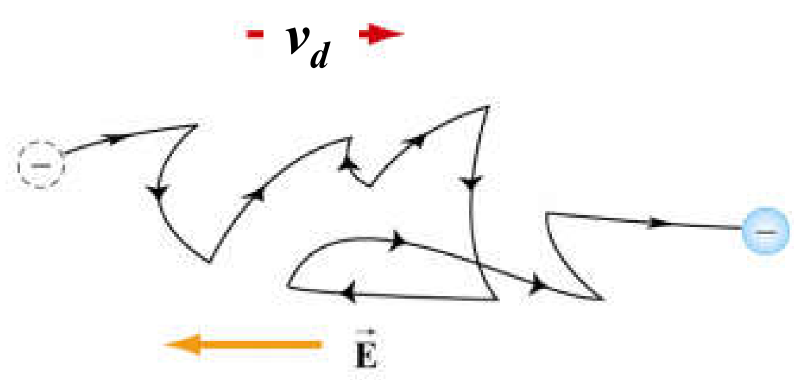
Not required

- The ***electrolyte*** in an electrolytic tank is essentially a liquid medium with a low conductivity, usually a diluted salt solution.
  - Highly conducting metallic electrodes are inserted in the solution.
  - When a voltage is applied to the electrodes, an electric field is established within the solution, and the molecules of the electrolyte are decomposed into oppositely charged ions by a chemical process called ***electrolysis***.
- Positive ions move in the direction of the electric field, and negative ions move in a direction opposite to the field, both contributing to a current flow in the direction of the field, which is the ***electrolytic current***.
- Not governed by Ohm's law either.



2.1 Conduction current

- The speed $\boldsymbol{v_d}$ at which the charge carriers are moving is known as the *drift velocity*. Physically, $\boldsymbol{v_d}$ is the average speed of the charge carriers inside a conductor when an external electric field is applied.



- Imagine: apply an electric field E to a conductor (for simplicity, care about magnitude only, i.e. ignore the direction)
  - Force applied on an electron: $\vec{F} = q\vec{E}$
  - Acceleration: $\vec{a} = \vec{F}/m_e$
  - Drift velocity: $\vec{v}_d = \vec{a}\tau = \dfrac{q\tau}{m_e}\vec{E}$

2.1 Conduction current

$$\boxed{\vec{v}_d = \frac{q\tau}{m_e}\vec{E}}$$

- For most conducting materials the drift velocity v_d is directly proportional to the electric field intensity **E**.

$$\vec{v}_d = \mu_e \vec{E} \quad (m/s)$$

  – where $\mu_e = \frac{q\tau}{m_e}$ is the electron ***mobility*** measured in ($m^2/V \cdot s$)

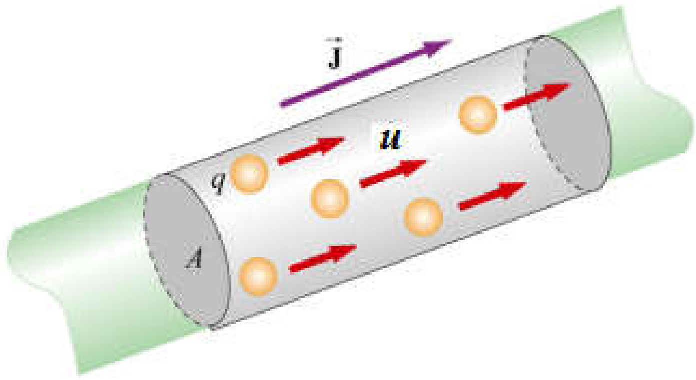
- Example: apply an electric field of 1 V/m to a copper conductor at room temperature 300 K:
    - τ (time between collisions) 3E-14 s;
    - m_e (mass of electron) 1E-30 kg;
    - q (charge of single electron) -1.6E-19 C;

  – What is the average moving speed of electrons?

Speed of electric signal is as fast as light!

2.2 Current Density

- Consider the steady motion of electrons (each of charge q, negative for electrons)



  - across an element of surface $\vec{\boldsymbol{A}} = \hat{\boldsymbol{n}}A$;
  - with a velocity $\boldsymbol{v_d}$
  - *N (concentration)* is the number of charge carriers per unit volume
  - In time Δt, the amount of charge passing through the elemental surface $\vec{\boldsymbol{A}}$ is: $\Delta Q = Nqv_dA\Delta t$
- Current is the time rate of change of charges:

$$\Delta I = \frac{\Delta Q}{\Delta t} = Nqv_dA = Nq\vec{\boldsymbol{v}}_{\boldsymbol{d}} \cdot \vec{\boldsymbol{A}}$$

- Define $\vec{\boldsymbol{J}} = Nq\vec{\boldsymbol{v}}_{\boldsymbol{d}}$ as the *volume current density*, or simple ***current density***, so $\Delta I = \vec{\boldsymbol{J}} \cdot \vec{\boldsymbol{A}}$

2.2 Current Density

$$\boxed{\Delta I = \vec{\boldsymbol{J}} \cdot \vec{\boldsymbol{A}}}$$

- The total current I flowing through an arbitrary surface S is then the flux of the **J** vector through S:

$$I = \int_S \vec{\boldsymbol{J}} \cdot \boldsymbol{d}\vec{\boldsymbol{s}} \quad (A)$$

- Noting that the product Nq is in fact free charge per unit volume, we may rewrite **J** as

$$\vec{\boldsymbol{J}} = Nq\vec{\boldsymbol{v}}_{\boldsymbol{d}}(A/m^2)$$

  – In the case of conduction currents there may be more than one kind of charge carriers (electrons, holes and ions) drifting with different velocities, the equation of **J** should be generalized to:

$$\vec{\boldsymbol{J}} = \sum_i N_i q_i \vec{\boldsymbol{v}}_{\boldsymbol{d}} \quad (A/m^2)$$

2.3 Conductivity

$$\vec{J} = Nq\vec{v}_d \quad (A/m^2)$$

$$\boxed{\vec{v}_d = \frac{q\tau}{m_e}\vec{E}}$$

- Substitute v_d in, get

$$I = \frac{q^2 N\tau}{m_e}\vec{A} \cdot \vec{E}$$

Only related to the substance's properties, so defined as:

$$\sigma = \frac{q^2 N\tau}{m_e}$$

- So $I = \sigma AE$ or $\vec{J} = \sigma\vec{E}$,
  - Where σ is a macroscopic constitutive parameter of the medium called ***conductivity***.

Example 1

- A copper wire of length l = 1 km and radius a = 3 mm carries a steady current of intensity I = 10 A. The current is uniformly distributed across the wire cross section. The time in which the electrons drift along the wire is 3.82×10^6 s.
- Find the concentration of conduction electrons in copper.

2.4 Resistivity

$$\boxed{\vec{J} = \sigma \vec{E}}$$

The current density at any point in a conducting medium is proportional to the electric field intensity. The constant of proportionality is the conductivity of the medium.

– Isotropic materials for which the linear relation holds are called ohmic (linear) media.

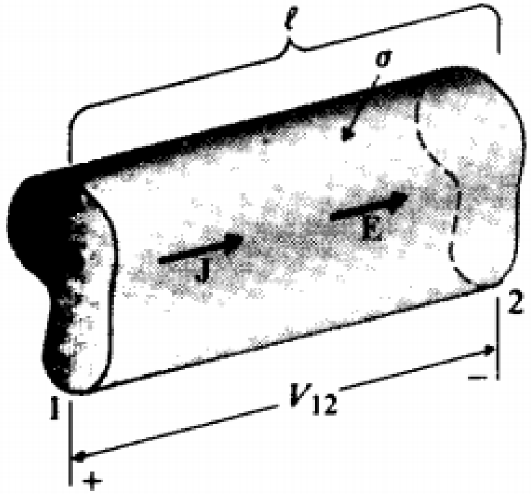
– The unit for σ is A/V·m or S/m

- The reciprocal of conductivity is called resistivity, in Ω·m.

$$\rho = \frac{1}{\sigma}$$

– Conductivity and resistivity are equivalent to each other. In this module, usually we are using conductivity.

3.1 Ohm's Law



- Within the conducting material, $\mathbf{J} = \sigma\mathbf{E}$, where both **J** and **E** are in the direction of current flow.
- The potential difference between 1 and 2 is:

$$V_{1\ 2} = El$$

- The total current is

$$I = \int_S \vec{J} \cdot d\vec{s} = JS$$

- Combine these two equations, we get

$$\frac{I}{S} = J = \sigma E = \sigma \frac{V_{1\ 2}}{l} \Rightarrow V_{1\ 2} = \left(\frac{l}{\sigma S}\right) I = RI$$

  – where $R = l / \sigma S$ is the formula for the resistance of a straight piece of homogeneous material of a uniform cross section for steady current.

Microscopic Ohm's law

$$\vec{J} = \sigma\vec{E}$$

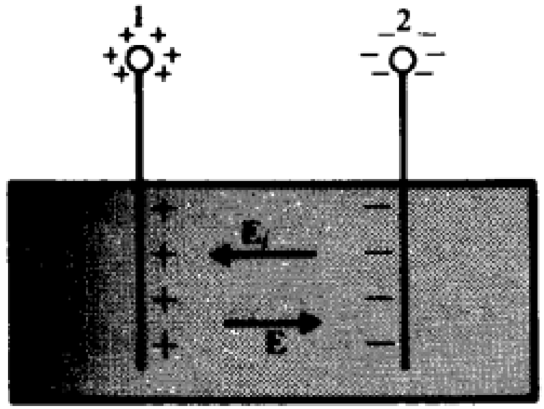
$$\Downarrow$$

Macroscopic Ohm's law

$$V = RI$$

3.2 Electromotive Force (EMF)

- In a static E-field: $\oint_C \mathbf{E} \cdot d\boldsymbol{\ell} = 0.$ ⇨ $\oint_C \frac{1}{\sigma} \mathbf{J} \cdot d\boldsymbol{\ell} = 0.$
- A steady current cannot be maintained in the same direction in a closed circuit by a *conservative* electrostatic field.
- There must exist a source of energy to maintain the steady current in a closed loop.
  - The external source may be nonelectrical (battery, generator, solar cell, thermocouple, etc.), but it has to be non-conservative.
  - The source sets up an impressed electric field $\mathbf{E}_\mathbf{i}$ inside the source (battery).
  - The line integral of $\mathbf{E}_\mathbf{i}$ from the negative to the positive electrode inside the battery is called the ***electromotive force*** (***EMF***).



Electric battery

3.2 Electromotive Force (EMF)

- EMF
  - Since $\mathbf{E_i}$ = -$\mathbf{E}$ inside the source, so

$$\mathcal{V} = \int_2^1 \mathbf{E}_i \cdot d\boldsymbol{\ell} = -\underbrace{\int_2^1 \mathbf{E} \cdot d\boldsymbol{\ell}}_{\text{Inside the source}}. \;\Longrightarrow\; \mathcal{V} = \underbrace{\int_1^2 \mathbf{E} \cdot d\boldsymbol{\ell}}_{\text{Outside the source}}$$

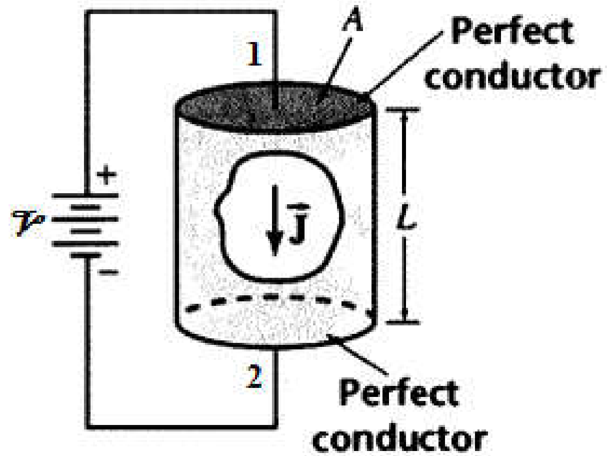
  - SI unit is volt, not a force in newtons (N)
  - Denoted by $\mathcal{V}$ or $\mathcal{E}$
  - $\mathcal{V}$ is a measure of the strength of the non-conservative source.

$$\mathcal{V} = \underbrace{\int_1^2 \mathbf{E} \cdot d\boldsymbol{\ell}}_{\text{Outside the source}} = V_{12} = V_1 - V_2.$$

  - The EMF of the source, expressed as the line integral of the conservative $\mathbf{E}$, can be interpreted as the voltage rise (potential difference) between the positive and negative terminals.

3.3 Kirchhoff's Voltage Law (KVL)

- When a resistor is connected between terminals 1 and 2 of the battery, the point form of Ohm's law must use the total electric field intensity (**E** and $\mathbf{E_i}$) like:



$$\mathbf{J} = \sigma(\mathbf{E} + \mathbf{E}_i).$$

- Therefore

$$\mathcal{V} = \oint_C (\mathbf{E} + \mathbf{E}_i) \cdot d\ell = \oint_C \frac{1}{\sigma} \mathbf{J} \cdot d\ell.$$

- If the resistor has a conductivity σ, length l, and uniform cross section S, $J = I/S$; then the right side becomes RI. So $\mathcal{V} = RI$.
- Generalized: $\boxed{\sum_j \mathcal{V}_j = \sum_k R_k I_k \qquad (\mathrm{V}).}$ → **KVL**
  - This is the Kirchhoff's voltage law, which states that, around a closed path in an electric circuit, the sum of the EMF is equal to the sum of the voltage drops across the resistances.

Continuity

- ***Principle of conservation of charge*** – in an arbitrary volume V bounded by surface S, a net charge Q exists within this region. If a net current I flows across the surface ***out*** of this region, the charge in the volume must ***decrease*** at a rate that equals the current.

$$I = \oint_S \mathbf{J} \cdot d\mathbf{s} = -\frac{dQ}{dt} = -\frac{d}{dt}\int_V \rho \, dv.$$

- Apply the Gauss's theorem, we have

$$\int_V \nabla \cdot \mathbf{J} \, dv = -\int_V \frac{\partial \rho}{\partial t} \, dv.$$

- The equation must hold regardless of the choice of V, so

$$\boxed{\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t} \qquad (\text{A/m}^3).} \longrightarrow \textbf{Equation of continuity}$$

3.4 Kirchhoff's Current Law (KCL)

- For steady current, charge density does not vary with time, so $\partial\rho/\partial t = 0$, therefore $\nabla \cdot \mathbf{J} = 0.$
- Thus, steady electric currents are *divergenceless*, or *solenoidal*.
- The integral form:

$$\nabla \cdot \mathbf{J} = 0. \Rightarrow \oint_S \mathbf{J} \cdot d\mathbf{s} = 0,$$

$$\Downarrow$$

$$\boxed{\sum_j I_j = 0 \qquad \text{(A).}} \longrightarrow \text{KCL}$$

  - This is the Kirchhoff's current law, which states that, the sum of all the currents flowing out of a junction in an electric circuit is zero.

Relaxation time

- Charges introduced in the interior of a conductor will move to the conductor surface and redistribute themselves in such a way as to make $\rho = 0$ and $\mathbf{E} = 0$ inside the conductor under equilibrium conditions. => How long does this take?

Ohm's Law $\boldsymbol{J} = \sigma \boldsymbol{E}$

Equation of continuity $\boldsymbol{\nabla} \cdot \boldsymbol{J} = -\dfrac{\partial \rho}{\partial t}$

$$\sigma \nabla \cdot \mathbf{E} = -\frac{\partial \rho}{\partial t}.$$

Gauss's law $\nabla \cdot \mathbf{E} = \rho/\epsilon,$

$$\frac{\partial \rho}{\partial t} + \frac{\sigma}{\epsilon}\rho = 0.$$

Solve!

$$\rho = \rho_0 e^{-(\sigma/\epsilon)t} \qquad (\mathrm{C/m^3}),$$

– An initial charge density ρ_0 will decay to 36.8% of its value at

$$\boxed{\tau = \frac{\epsilon}{\sigma} \qquad (\mathrm{s})} \longrightarrow \textbf{Relaxation time}$$

- Eg: for copper, a good conductor, $\tau = 1.52 \times 10^{-19}$ s, a very short time.

Example 2

- A certain amount of charge is placed within an isolated conductor. The current through a closed surface bounding the charge is observed to be $i(t) = 0.125e^{-25t}$ A.
- Determine:
  - (a) the relaxation time;
  - (b) the charge transported through the surface in time $t = 5\tau$;
  - (c) the initial charge.

3.5 Joule's Law

- The work Δw done by an electric field $\mathbf{E}$ in moving a charge q a distance Δl is $q\mathbf{E}\cdot\Delta l$, which corresponds to a power p

$$p = \lim_{\Delta t \to 0} \frac{\Delta w}{\Delta t} = q\mathbf{E}\cdot\mathbf{u},$$

  – where **u** is the drift velocity

- The total power delivered to all the charge carriers in a volume dv is

$$dP = \sum_i p_i = \mathbf{E}\cdot\left(\sum_i N_i q_i \mathbf{u}_i\right) dv = \mathbf{E}\cdot\mathbf{J}\,dv \;\Longrightarrow\; \frac{dP}{dv} = \mathbf{E}\cdot\mathbf{J} \qquad (\mathrm{W/m^3}).$$

- Thus the point function $\mathbf{E}\cdot\mathbf{J}$ is a power density under steady-current conditions.

$$\boxed{P = \int_V \mathbf{E}\cdot\mathbf{J}\,dv} \longrightarrow \text{Joule's Law}$$

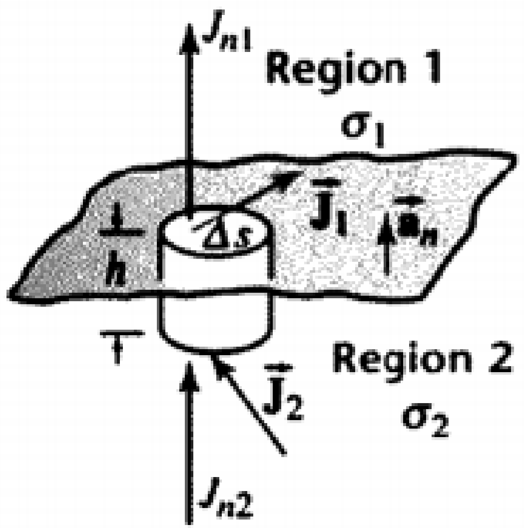
$$P = \int_L E\,d\ell \int_S J\,ds = VI = I^2R \qquad (\mathrm{W}).$$

Example 3

- A parallel-plate capacitor whose plates are 10 cm square and 0.2 cm apart contains a medium with $\varepsilon_r = 2$ and $\sigma = 4 \times 10^{-5}$ S/m. To maintain a steady current through the medium a potential difference of 120V is applied between the plates.
- Determine the electric field intensity, the volume current density, the current, and the resistance of the medium.

4. Boundary Conditions

Governing Equations for Steady Current Density	
Differential Form	Integral Form
$\nabla \cdot \mathbf{J} = 0$	$\oint_S \mathbf{J} \cdot d\mathbf{s} = 0$
$\nabla \times \left(\dfrac{\mathbf{J}}{\sigma}\right) = 0$	$\oint_C \dfrac{1}{\sigma} \mathbf{J} \cdot d\boldsymbol{\ell} = 0$



- The normal component of a divergenceless vector field is continuous, so

$$J_{1n} = J_{2n}$$

- The tangential component of a curl-free vector field is continuous across an interface, so

$$\frac{J_{1t}}{J_{2t}} = \frac{\sigma_1}{\sigma_2}.$$

Next

- Capacitors
- Inductors
- Resistors