MTH101: Lecture 17 – 18

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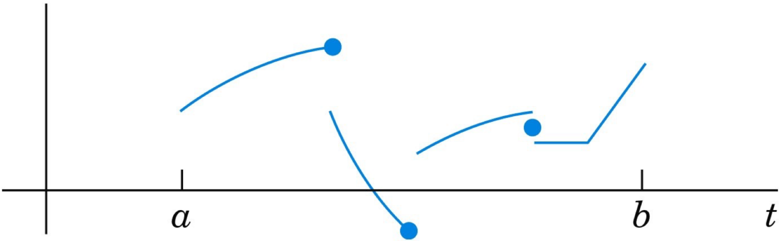
Definition

We say a function f is **piecewise continuous** *on an infinite interval, e.g. $\mathbb{R}_{\geq 0} = [0, \infty)$, if it is piecewise continuous on any finite subinterval $[a, b]$.*

It is piecewise continuous on a finite interval $[a, b]$ if:

1. *$[a, b]$ can be divided into finitely many open subintervals (a_j, a_{j+1}), $a = a_0 < a_1 < \cdots < a_n = b$, on which f is continuous.*
2. *For $j = 0, 1, \ldots, n-1$, $\lim\limits_{t \to a_j^+} f(t)$ and $\lim\limits_{t \to a_{j+1}^-} f(t)$ exist and are finite.*
3. *At the border points a_j the function f may take any (finite) values.*


a
b
t

Theorem

If $f(t)$ is defined and piecewise continuous on $\mathbb{R}_{\geq 0}$ and satisfies $|f(t)| \leq Me^{kt}$ for some $M, k \in \mathbb{R}$, then $\mathcal{L}\left[f\right]$ exists and is defined at all points $s > k$.

If $\mathcal{L}\left[f\right] = \mathcal{L}\left[g\right]$ then $f(t) = g(t)$ "almost everywhere", i.e. the set of exceptions $\{t : f(t) \neq g(t)\}$ is somehow smaller than any interval $[a, a+\varepsilon]$.

Proof.

$$\left|\mathcal{L}\left[f\right]\right| = \left|\int_0^\infty e^{-st} f(t) dt\right| \leq \int_0^\infty \left|f(t)\right| e^{-st} dt$$
$$\leq \int_0^\infty M e^{kt} e^{-st} dt = \frac{M}{s-k}.$$

□

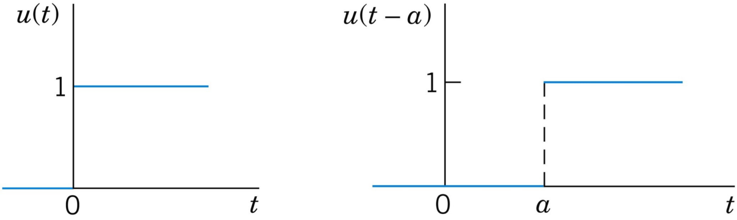
Definition

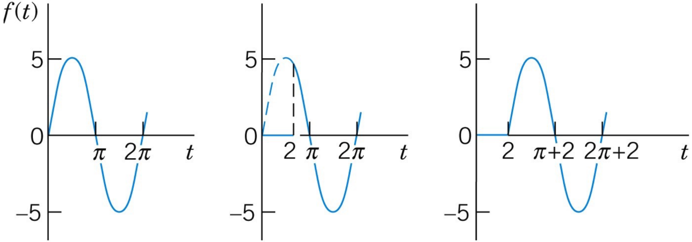
Heaviside function (or unit step function) $u(t-\alpha)$ **is defined as follows:**

$$u(t-\alpha) = \begin{cases} 0 & \textit{if} \quad t < \alpha, \\ 1 & \textit{if} \quad t > \alpha. \end{cases}$$

Remark

The Heaviside function is a typical **engineering function**, *which often involve functions that are either* **off** *or* **on***!*


u(t)
1
0
t
u(t − a)
1
0
a
t



(A) $f(t) = 5\sin t$ (B) $f(t)u(t-2)$ (C) $f(t-2)u(t-2)$

Remark

Laplace transform for the Heaviside function is as follows:

$$\mathcal{L}\left[u(t-\alpha)\right] = \int_0^\infty e^{-st}u(t-\alpha)dt = \int_\alpha^\infty e^{-st}dt$$
$$= -\frac{e^{-st}}{s}\bigg|_{t=\alpha}^{\infty} = \frac{e^{-s\alpha}}{s}.$$

Theorem

Second shifting theorem, t-Shifting

If $\mathcal{L}\left[f(t)\right] = F(s)$, *then*

$$\mathcal{L}\left[f(t-\alpha)u(t-\alpha)\right] = e^{-\alpha s}F(s),$$

$$or, \quad f(t-\alpha)u(t-\alpha) = \mathcal{L}^{-1}\left[e^{-\alpha s}F(s)\right].$$

Proof.

By definition,

$$e^{-\alpha s}F(s) = e^{-\alpha s}\int_0^\infty e^{-s\tau} f(\tau)d\tau = \int_0^\infty e^{-s(\tau+\alpha)} f(\tau)d\tau.$$

If we define $t = \tau + \alpha$, the integral becomes

$$e^{-\alpha s}F(s) = \int_\alpha^\infty e^{-st} f(t-\alpha)dt = \int_0^\infty e^{-st} f(t-\alpha)u(t-\alpha)dt.$$

$\square$

Example

Find the Laplace transform for the following function.

$$f(t) = \begin{cases} 2 & \text{if} \quad 0 < t < 1, \\ \frac{1}{2}t^2 & \text{if} \quad 1 < t < \frac{1}{2}\pi, \\ \cos t & \text{if} \quad t > \frac{1}{2}\pi. \end{cases}$$

Example

Solution

Step 1

The function can be expressed with the help of Heaviside function

$$f(t) = 2\left[1 - u(t-1)\right] + \frac{1}{2}t^2\left[u(t-1) - u(t - \frac{1}{2}\pi)\right]$$
$$+ (\cos t)\, u(t - \frac{1}{2}\pi).$$

Example

Solution

Step 2

For each term in $f(t)$, we need to write it in the form $f(t-\alpha)u(t-\alpha)$.

$$\mathcal{L}\left[2\left[1-u(t-1)\right]\right] = 2\mathcal{L}\left[1\right] - 2\mathcal{L}\left[u(t-1)\right] = \frac{2}{s} - \frac{2}{s}e^{-s},$$

$$\mathcal{L}\left[\frac{1}{2}t^2u(t-1)\right] = \mathcal{L}\left[\left[\frac{1}{2}(t-1)^2 + (t-1) + \frac{1}{2}\right]u(t-1)\right]$$

$$= \left(\frac{1}{s^3} + \frac{1}{s^2} + \frac{1}{2s}\right)e^{-s},$$

Example

Solution

Step 2

$$
\begin{aligned}
-\mathcal{L}\big[\frac{1}{2}t^2u(t-\frac{1}{2}\pi)\big] &= -\mathcal{L}\big[\left[\frac{1}{2}\left(t-\frac{1}{2}\pi\right)^2+\frac{\pi}{2}\left(t-\frac{\pi}{2}\right)+\frac{\pi^2}{8}\right] \\
&\qquad \times u\left(t-\frac{1}{2}\pi\right)\big] \\
&= -\left(\frac{1}{s^3}+\frac{\pi}{2s^2}+\frac{\pi^2}{8s}\right)e^{-\frac{\pi s}{2}},
\end{aligned}
$$

Example

Solution

Step 2

$$\begin{aligned}\mathcal{L}\left[(\cos t)u\left(t-\frac{1}{2}\pi\right)\right] &= \mathcal{L}\left[-\sin\left(t-\frac{1}{2}\pi\right)u\left(t-\frac{1}{2}\pi\right)\right]\\ &= -\frac{1}{s^2+1}e^{-\frac{\pi s}{2}}.\end{aligned}$$

The solution is the summation of all the terms.

$$\begin{aligned}\mathcal{L}[f] = \frac{2}{s} - \frac{2}{s}e^{-s} + \left(\frac{1}{s^3}+\frac{1}{s^2}+\frac{1}{2s}\right)e^{-s}\\ -\left(\frac{1}{s^3}+\frac{\pi}{2s^2}+\frac{\pi^2}{8s}\right)e^{-\frac{\pi s}{2}} - \frac{1}{s^2+1}e^{-\frac{\pi s}{2}}\end{aligned}$$

Example

Find the inverse transform for

$$F(s) = \frac{e^{-s}}{s^2 + \pi^2} + \frac{e^{-2s}}{s^2 + \pi^2} + \frac{e^{-3s}}{(s+2)^2}.$$

Example

Solution

Without the exponential term, the inverse transform for the three terms are $\sin \pi t/\pi$, $\sin \pi t/\pi$, *and* te^{-2t}, *where we use the s-shifting theorem for the last term. Hence, by the t-shifting theorem, we have*

$$f(t) = \frac{1}{\pi}\sin[\pi(t-1)]u(t-1) + \frac{1}{\pi}\sin[\pi(t-2)]u(t-2) + (t-3)e^{-2(t-3)}u(t-3).$$

Laplace Transform of Derivatives

Laplace transform is a crucial tool for solving ODE, and in order to use it, we need to consider Laplace transform for the derivatives.

Theorem

Let $f(t)$, $f'(t)$, $\cdots$, $f^{(n-1)}(t)$ are continuous for $t \geq 0$ and satisfy the growth restriction condition

$$\left| f^{(i)}(t) \right| \leq M_i e^{k_i t}, \qquad \text{for } i = 0, 1, \cdots, n-1,$$

where M_i and k_i are some constants. If $f^{(n)}(t)$ is piecewise continuous on $t \geq 0$, then the transform for $f^{(n)}(t)$ satisfies

$$\mathcal{L}\left[f^{(n)}\right](s) = s^n \mathcal{L}\left[f\right] - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0) .$$

Laplace Transform of Derivatives

Theorem

In particular, for $n = 1$ and $n = 2$,

$$\mathcal{L}\left[f'\right] = s\mathcal{L}\left[f\right] - f(0), \tag{3.1}$$

$$\mathcal{L}\left[f''\right] = s^2\mathcal{L}\left[f\right] - sf(0) - f'(0). \tag{3.2}$$

Laplace Transform of Derivatives

Proof.

We first prove eq. (3.1) by assuming $f'(t)$ is **continuous** for $t \geq 0$.

$$\begin{aligned}\mathcal{L}\left[f'\right] &= \int_0^\infty e^{-st} f'(t)dt = \left[e^{-st} f(t)\right]\Big|_0^\infty + s\int_0^\infty e^{-st} f(t)dt.\\ &= 0 - f(0) + s\mathcal{L}\left[f\right] = s\mathcal{L}\left[f\right] - f(0).\end{aligned}$$

$\square$

The proof can be easily generalized to the case that $f'(t)$ is only piecewise continuous. In this case the interval of integration need to be broken into finitely many intervals where $f'(t)$ is continuous in each of them.

Laplace Transform of Derivatives

Proof.

The proof of eq. (3.2) follows eq. (3.2).

$$\begin{aligned}\mathcal{L}\left[f''\right] &= s\mathcal{L}\left[f'\right] - f'(0) = s\left[s\mathcal{L}\left[f\right] - f(0)\right] - f'(0)\\ &= s^2\mathcal{L}\left[f\right] - sf(0) - f'(0).\end{aligned}$$

The proof can be easily generalized to the case $n = n$ by mathematical induction, and thus

$$\mathcal{L}\left[f^{(n)}\right](s) = s^n\mathcal{L}\left[f\right] - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0)\ .$$

$\square$

Laplace Transform of Integrals

The derivatives and integrals (antiderivatives) are inverse operations, and we can easily derive the Laplace transform of integral of a function.

Theorem

Let $F(s)$ denotes the Laplace transform of a function $f(t)$. By definition, $f(t)$ has to be piecewise continuous for $t \geq 0$ and satisfies growth restriction $|f(t)| \leq Me^{kt}$. Then, for $s > k > 0$ and $t > 0$,

$$\mathcal{L}\Big[\int_0^t f(\tau)d\tau\Big] = \frac{1}{s}F(s), \quad or, \quad \int_0^t f(\tau)d\tau = \mathcal{L}^{-1}\Big[\frac{1}{s}F(s)\Big].$$

Laplace Transform of Integrals

Proof.

Let $g(t) \equiv \int_0^t f(\tau)d\tau$. By the assumption that $f(t)$ is piecewise continuous, $g(t)$ is continuous. Therefore,

$$\begin{aligned} |g(t)| &= \left| \int_0^t f(\tau)d\tau \right| \leq \int_0^t |f(\tau)|\, d\tau \\ &\leq \int_0^t Me^{k\tau}\, d\tau = \frac{M}{k}\, e^{k\tau}\Big|_0^t < \frac{M}{k}e^{kt}, \end{aligned}$$

and this shows that $g(t)$ satisfies the growth restriction.

Laplace Transform of Integrals

Proof.

Now, since $g(t)$ is continuous and satisfies the growth restriction, and $g'(t) = f(t)$, which is piecewise continuous, we can use eq. (3.1) and find

$$\mathcal{L}\left[f\right] = \mathcal{L}\left[g'\right] = s\mathcal{L}\left[g\right] - g(0) = s\mathcal{L}\left[g\right].$$

For the last equality we used the fact that $g(0) = 0$. We can rewrite the equation with the definition of $F(s)$ and $g(t)$, and finally reach

$$F(s) = s\mathcal{L}\left[\int_0^t f(\tau)d\tau\right], \quad or, \quad \mathcal{L}\left[\int_0^t f(\tau)d\tau\right] = \frac{1}{s}F(s).$$

The idea of using Laplace transform to solve nonhomogeneous linear ODEs is as follows. Consider the following initial value problem

$$y'' + ay' + by = r(t) \qquad y(0) = K_0,\ y'(0) = K_1,$$

where a, b, K_0, K_1 are constants.
To solve this problem with Laplace transform, there are three steps.
Step 1: We first transform the ODE into functions of s:

$$\begin{aligned} &[s^2 Y - sy(0) - y'(0)] + a[sY - y(0)] + bY = R(s), \\ \Rightarrow\ &(s^2 + as + b)Y = (sK_0 + K_1 + aK_0) + R(s). \end{aligned}$$

Step 2: Then we solve $Y(s)$ by

$$Y(s) = \left[(s+a)K_0 + K_1\right] Q(s) + Q(s)R(s),$$

where $R(s) = \mathcal{L}\left[r(t)\right]$, and

$$Q(s) \equiv \frac{1}{s^2 + as + b} = \frac{1}{\left(s + \frac{1}{2}a\right)^2 + b - \frac{1}{4}a^2}.$$

Step 3: After finding $Y(s)$, the solution can be found by inverse transform

$$y(t) = \mathcal{L}^{-1}\left[\left[(s+a)K_0 + K_1\right] Q(s)\right] + \mathcal{L}^{-1}\left[Q(s)R(s)\right].$$

Notice that in general the last term is not easy to be solved, and we will introduce the so-called **convolution** to deal with it later.

Shifted Data Problem

Example

Using Laplace transform to solve the following ODE

$$y'' + y = 2t, \quad y\left(\frac{\pi}{4}\right) = \frac{\pi}{2}, \quad y'\left(\frac{\pi}{4}\right) = 2 - \sqrt{2}.$$

Example

Solution: This is an ODE with initial conditions given at $t = t_0 > 0$. *In order to use Laplace transform, we need to set* $t = \tilde{t} + t_0$, *and the initial conditions happen at* $\tilde{t} = 0$.

Shifted Data Problem

Example

Solution: With $t_0 = \frac{1}{4}\pi$*, the problem becomes*

$$\tilde{y}''(t) + \tilde{y} = 2\left(\tilde{t} + \frac{1}{4}\pi\right), \quad \tilde{y}(0) = \frac{1}{2}\pi, \quad \tilde{y}'(0) = 2 - \sqrt{2}.$$

The Laplace transform becomes

$$s^2\tilde{Y} - s \cdot \frac{1}{2}\pi - \left(2 - \sqrt{2}\right) + \tilde{Y} = \frac{2}{s^2} + \frac{\frac{1}{2}\pi}{s}$$

$$\Rightarrow \tilde{Y} = \frac{2}{\left(s^2+1\right)s^2} + \frac{\frac{1}{2}\pi}{\left(s^2+1\right)s} + \frac{\frac{1}{2}\pi s}{\left(s^2+1\right)} + \frac{\left(2-\sqrt{2}\right)}{\left(s^2+1\right)}$$

Shifted Data Problem

Example

Solution: By using Laplace transform of integral, we know

$$\mathcal{L}^{-1}\left[\frac{1}{s^2+1}\right] = \sin t,$$

$$\mathcal{L}^{-1}\left[\frac{1}{s\left(s^2+1\right)}\right] = \int_0^t \sin\tau d\tau = (1-\cos t)$$

$$\mathcal{L}^{-1}\left[\frac{1}{s^2\left(s^2+1\right)}\right] = \int_0^t \left(1-\cos\tau\right) d\tau = t-\sin t.$$

Shifted Data Problem

Example

Solution: Therefore

$$\begin{aligned}\tilde{y} &= 2(\tilde{t} - \sin\tilde{t}) + \frac{1}{2}\pi\,(1 - \cos\tilde{t}) + \frac{1}{2}\pi\cos\tilde{t} + (2 - \sqrt{2})\sin\tilde{t} \\ &= 2\tilde{t} + \frac{1}{2}\pi - \sqrt{2}\sin\tilde{t},\end{aligned}$$

therefore

$$y = 2t - \sqrt{2}\sin\left(t - \frac{1}{4}\pi\right) = 2t - \sin t + \cos t.$$

Dirac's Delta Function

In engineering, we often faced with problem of describing a quantity that is zero everywhere except at a single point (e.g. short impulses), while at that point it is infinite, the integral over any interval containing that point has a finite value. For this purpose, we need to introduce Dirac's delta function.

Definition

Dirac's delta function $\delta(x-c)$ is defined as follows

$$\delta(x-c) = \begin{cases} \infty, & x = c, \\ 0, & otherwise \end{cases}$$

$$f(c) = \int_a^b f(x)\delta(x-c)dx, \quad for\ c \in (a, b).$$

Dirac's Delta Function

Definition

In particular, if we let $f(x) = 1$, we will have

$$\int_a^b \delta(x-c)dx = 1, \quad \textit{for } c \in (a,b),$$

$$\textit{or} \int_{-\infty}^{\infty} \delta(x-c)dx = 1.$$

The only problem is that **no such function exists**!

Dirac's Delta Function

In practical, we can approximate Dirac's delta function as the limit of a sequence of functions, a distribution.

Remark

$$f_k(t-a) = \begin{cases} \frac{1}{k} & \text{if } a \le t \le a+k \\ 0 & \text{otherwise}, \end{cases}$$

one can check for any k, the integral of f_k equals to one if the region $(a, a+k)$ is in the integral interval. We thus have

$$\delta(t-a) = \lim_{k \to 0} f_k(t-a),$$

which satisfies all the property of the delta function.

Dirac's Delta Function

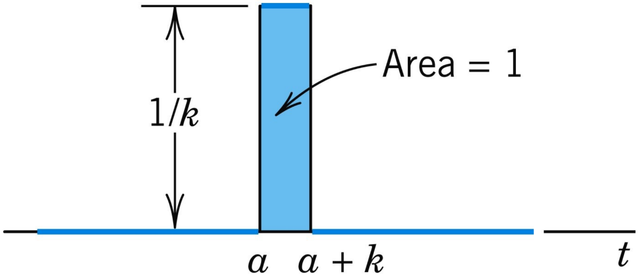


Figure: $f_k(t - a)$

Dirac's Delta Function

The Laplace transform of $\delta(t-a)$ can be obtained with the help of Heaviside function.

Claim

$$\mathcal{L}\left[\delta(t-a)\right] = e^{-as}$$

Proof.

We first express the function $f_k(t-a)$ as follows:

$$f_k(t-a) = \frac{1}{k}\left[u(t-a) - u(t-(a+k))\right].$$

Dirac's Delta Function

Proof.

By definition

$$\begin{aligned}\mathcal{L}\left[\delta(t-a)\right] &= \mathcal{L}\left[\lim_{k\to 0} f_k(t-a)\right] = \lim_{k\to 0} \mathcal{L}\left[f_k(t-a)\right] \\ &= \lim_{k\to 0} \frac{1}{k}\left[\frac{e^{-as}}{s} - \frac{e^{-(a+k)s}}{s}\right] = \lim_{k\to 0} \frac{e^{-as} - e^{-(a+k)s}}{ks} \\ &= e^{-as}.\end{aligned}$$

$\square$

Example

Solve the following ODE

$$y'' + 3y' + 2y = \delta(t-1), \quad y(0) = 0, \quad y'(0) = 0.$$

Example

Solution:
The Laplace transform of the equation is

$$s^2 Y + 3sY + 2Y = e^{-s}$$
$$\Rightarrow Y = \frac{e^{-s}}{(s+1)(s+2)} = \left(\frac{1}{s+1} - \frac{1}{s+2}\right) e^{-s}$$

Example

Solution:

Therefore

$$y = \mathcal{L}^{-1}\left[\frac{1}{(s+1)}e^{-s}\right] - \mathcal{L}^{-1}\left[\frac{1}{(s+2)}e^{-s}\right]$$

$$\Rightarrow y = e^{-(t-1)}u(t-1) - e^{-2(t-1)}u(t-1).$$

More on Partial Functions

Remark

If $g(x)$, $h(x)$ are polynomials, with $g(x)$ of lower degree than $h(x)$, by fundamental theorem of algebra, we can write $h(x)$ as

$$h(x) = c(x-\alpha_1)^{a_1}\cdots(x-\alpha_k)^{a_k}\times \left[(x-\beta_1)^2+\gamma_1^2\right]^{b_1}\cdots\left[(x-\beta_l)^2+\gamma_l^2\right]^{b_l},$$

with real constants α, β, γ, and integers constants a, b.

More on Partial Functions

Remark

The function $g(x)/h(x)$ can be written in the form

$$\frac{g(x)}{h(x)} = \sum_{i=1}^{k} \sum_{j=1}^{a_i} \frac{A_{ij}}{(x-\alpha_i)^j} + \sum_{i=1}^{l} \sum_{j=1}^{b_i} \frac{B_{ij}x + C_{ij}}{\left[(x-\beta_i)^2 + \gamma_i^2\right]^j},$$

with unique set of real constants A_{ij}, B_{ij}, C_{ij}.

Bibliography

1 *Kreyszig, E.* **Advanced Engineering Mathematics**. Wiley, 10th Edition.