EEE225 Advanced Electrical Circuits and Electromagnetics

Lecture 11Uniform Plane Waves

Dr. Zhao Wang

zhao.wang@xjtlu.edu.cn

Room EE322

Maxwell's Equations

- Finally, the Maxwell's equations are:

Law	Integral	Differential	Physical meaning
Gauss's law for **E**	$\oiint_S \boldsymbol{D} \cdot d\boldsymbol{s} = Q$	$\nabla \cdot \boldsymbol{D} = \rho$	Electric flux through a closed surface is proportional to the charged enclosed
Faraday's law	$\oint_C \boldsymbol{E} \cdot d\boldsymbol{l} = -\frac{d\Phi_B}{dt}$	$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$	Changing magnetic flux produces an electric field
Gauss's law for **B**	$\oiint_S \boldsymbol{B} \cdot d\boldsymbol{s} = 0$	$\nabla \cdot \boldsymbol{B} = 0$	The total magnetic flux through a closed surface is zero
Generalized Ampere's law	$\oint_C \boldsymbol{H} \cdot d\boldsymbol{l} = I + \varepsilon_0 \frac{d\Phi_E}{dt}$	$\nabla \times \boldsymbol{H} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t}$	Electric current and changing electric flux produces a magnetic field

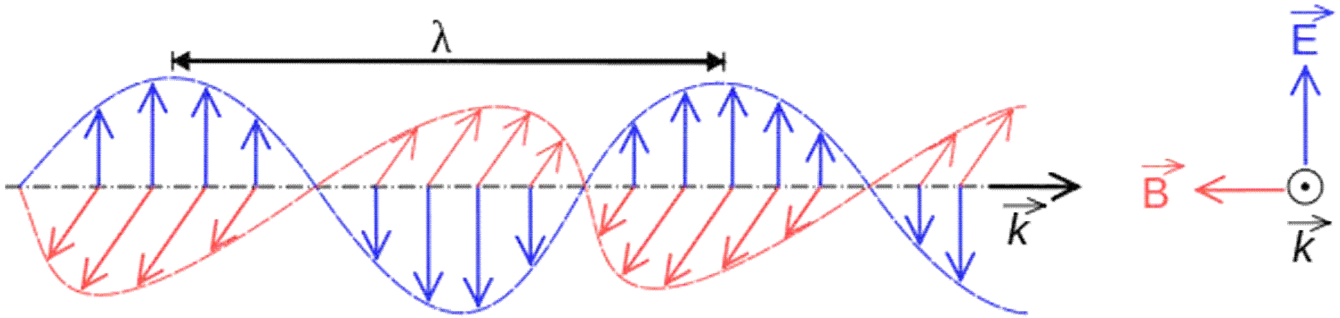
Content

Ref.4, Chap. 8

- Electromagnetic (EM) waves and spectrum
- General wave equations
  - Source free medium
  - TEM (Transverse electromagnetic) waves
  - Forward and backward travelling waves
- Plane wave in different medium
  - In a perfect dielectric medium
  - In free space
- Wave propagation
  - Bounded plane wave - Interface
  - Normal incident wave
    - Transmission and Reflection
  - Oblique incident wave

Electromagnetic Waves

- Electromagnetic wave, i.e., travelling electric and magnetic fields, is one of the most fundamental phenomena of electromagnetism, behaving as waves propagating through space. It is the consequence of general Maxwell's equations.
  - In a vacuum (free space), it propagates at a characteristic speed, the speed of light c, normally in straight lines.
  - As an electromagnetic wave, it has both ***electric*** and ***magnetic*** field components, which oscillate in a fixed relationship to one another.



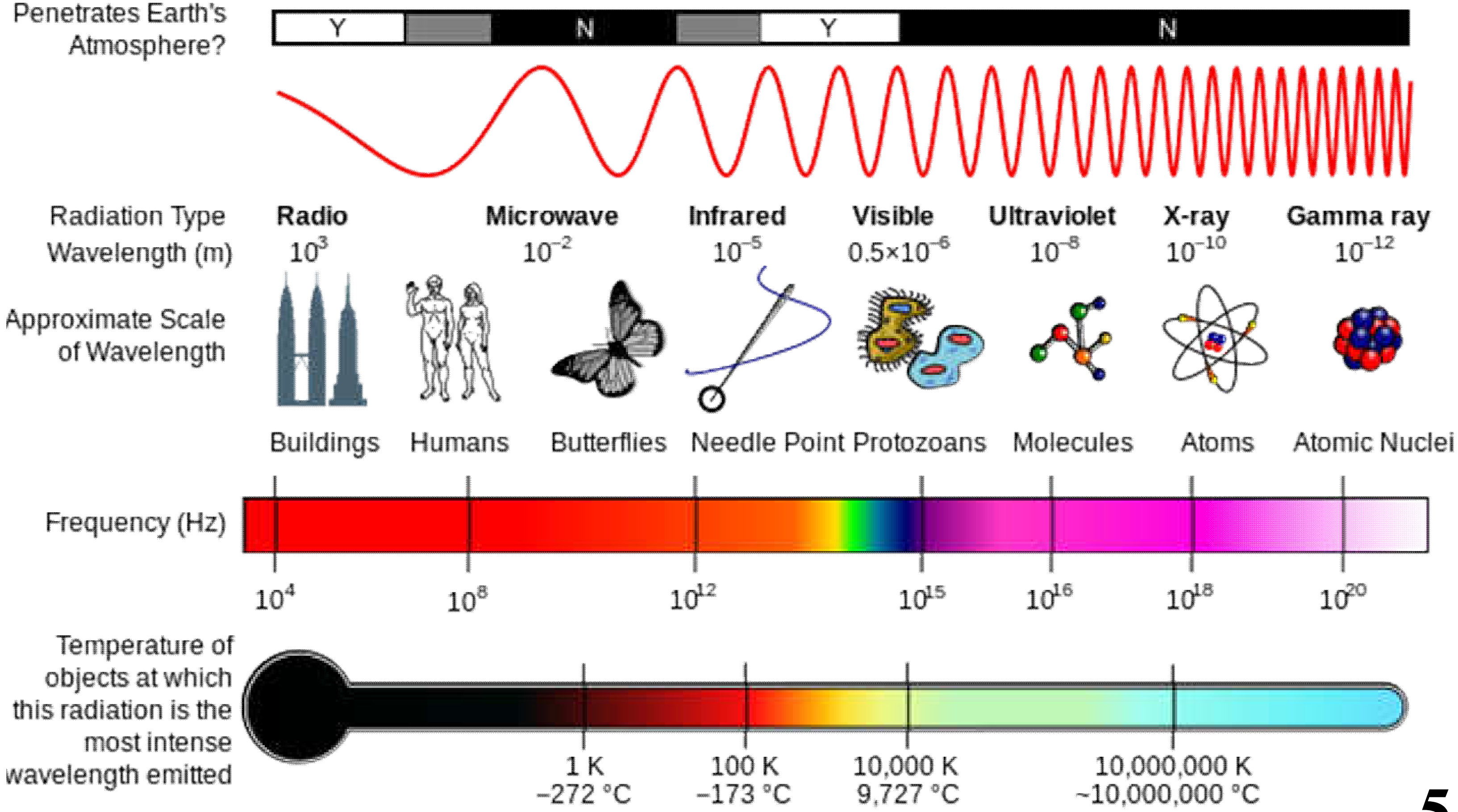
Electromagnetic Waves

- An EM wave is characterized by its ***frequency*** or ***wavelength***.
- The range of all possible frequencies of electromagnetic radiation is called the EM spectrum.

Free space

$$\lambda = \frac{c_0}{f}$$

c_0 is the speed of light, 3×10^8 m/s



Xi'an Jiaotong-Liverpool University
西交利物浦大學

General wave equation – Source free

- Consider a uniform but source-free medium having permittivity ε, permeability μ and conductivity σ. The Maxwell's equations are:

$$\nabla \times \vec{\mathbf{E}} = -\mu \frac{\partial \vec{\mathbf{H}}}{\partial t} \qquad (1)$$

$$\nabla \times \vec{\mathbf{H}} = \sigma \vec{\mathbf{E}} + \epsilon \frac{\partial \vec{\mathbf{E}}}{\partial t} \qquad (2)$$

$$\nabla \cdot \vec{\mathbf{B}} = 0 \quad \Rightarrow \quad \nabla \cdot \vec{\mathbf{H}} = 0 \qquad (3)$$

$$\nabla \cdot \vec{\mathbf{D}} = 0 \quad \Rightarrow \quad \nabla \cdot \vec{\mathbf{E}} = 0 \qquad (4)$$

  – These equations are in terms of two variables (**E** and **H**).
  – Let's try to obtain an equation in terms of one variable, say the **E** field only.

General wave equation – Derivation

- Take the curl of the equation (1), get

$$\nabla \times \nabla \times \vec{\mathbf{E}} = -\mu \nabla \times \left(\frac{\partial \vec{\mathbf{H}}}{\partial t} \right) = \nabla(\nabla \cdot \vec{\mathbf{E}}) - \nabla^2 \vec{\mathbf{E}} = -\nabla^2 \vec{\mathbf{E}}$$

- Therefore, we have

$$\nabla^2 \vec{\mathbf{E}} = \mu \frac{\partial}{\partial t} [\nabla \times \vec{\mathbf{H}}]$$

- Substitute (2) in this equation, get

$$\nabla^2 \vec{\mathbf{E}} = \mu\sigma \frac{\partial \vec{\mathbf{E}}}{\partial t} + \mu\epsilon \frac{\partial^2 \vec{\mathbf{E}}}{\partial t^2}$$

- Similarly, we can get another equation in terms of **H**:

$$\nabla^2 \vec{\mathbf{H}} = \mu\sigma \frac{\partial \vec{\mathbf{H}}}{\partial t} + \mu\epsilon \frac{\partial^2 \vec{\mathbf{H}}}{\partial t^2}$$

The presence of the first-order term in a second-order differential equation indicates that the fields decay (lose energy) as they propagate through the medium. For this reason, a conducting medium is called a ***lossy medium***.

Both are vector equations, which containing 3 components => this is a set of 6 scalar independent equations.

Wave in a perfect dielectric medium

- Perfect dielectric (or lossless medium) is the type of medium with $\sigma = 0$. Therefore, set $\sigma = 0$ in the general equation, obtain the wave equations for lossless medium as:

$$\nabla^2 \vec{\mathbf{E}} = \mu\sigma \cancel{\frac{\partial \vec{\mathbf{E}}}{\partial t}} + \mu\epsilon \frac{\partial^2 \vec{\mathbf{E}}}{\partial t^2} \qquad \nabla^2 \vec{\mathbf{E}} - \mu\epsilon \frac{\partial^2 \vec{\mathbf{E}}}{\partial t^2} = 0$$

$$\nabla^2 \vec{\mathbf{H}} = \mu\sigma \cancel{\frac{\partial \vec{\mathbf{H}}}{\partial t}} + \mu\epsilon \frac{\partial^2 \vec{\mathbf{H}}}{\partial t^2} \qquad \nabla^2 \vec{\mathbf{H}} - \mu\epsilon \frac{\partial^2 \vec{\mathbf{H}}}{\partial t^2} = 0$$

  - These equations, called the time-dependent Helmholtz equations, still represent a set of six scalar equations.
  - The absence of the first-order term signifies that the electromagnetic fields do not decay as they propagate in a lossless medium.

Six equations obtained so far

$$\frac{\partial^2 \mathrm{E}_x(x,y,z,t)}{\partial x^2} + \frac{\partial^2 \mathrm{E}_x(x,y,z,t)}{\partial y^2} + \frac{\partial^2 \mathrm{E}_x(x,y,z,t)}{\partial z^2} = \mu\varepsilon \frac{\partial^2 \mathrm{E}_x(x,y,z,t)}{\partial t^2} \quad (1)$$

$$\frac{\partial^2 \mathrm{E}_y(x,y,z,t)}{\partial x^2} + \frac{\partial^2 \mathrm{E}_y(x,y,z,t)}{\partial y^2} + \frac{\partial^2 \mathrm{E}_y(x,y,z,t)}{\partial z^2} = \mu\varepsilon \frac{\partial^2 \mathrm{E}_y(x,y,z,t)}{\partial t^2} \quad (2)$$

$$\frac{\partial^2 \mathrm{E}_z(x,y,z,t)}{\partial x^2} + \frac{\partial^2 \mathrm{E}_z(x,y,z,t)}{\partial y^2} + \frac{\partial^2 \mathrm{E}_z(x,y,z,t)}{\partial z^2} = \mu\varepsilon \frac{\partial^2 \mathrm{E}_z(x,y,z,t)}{\partial t^2} \quad (3)$$

$$\frac{\partial^2 H_x(x,y,z,t)}{\partial x^2} + \frac{\partial^2 H_x(x,y,z,t)}{\partial y^2} + \frac{\partial^2 H_x(x,y,z,t)}{\partial z^2} = \mu\varepsilon \frac{\partial^2 H_x(x,y,z,t)}{\partial t^2} \quad (4)$$

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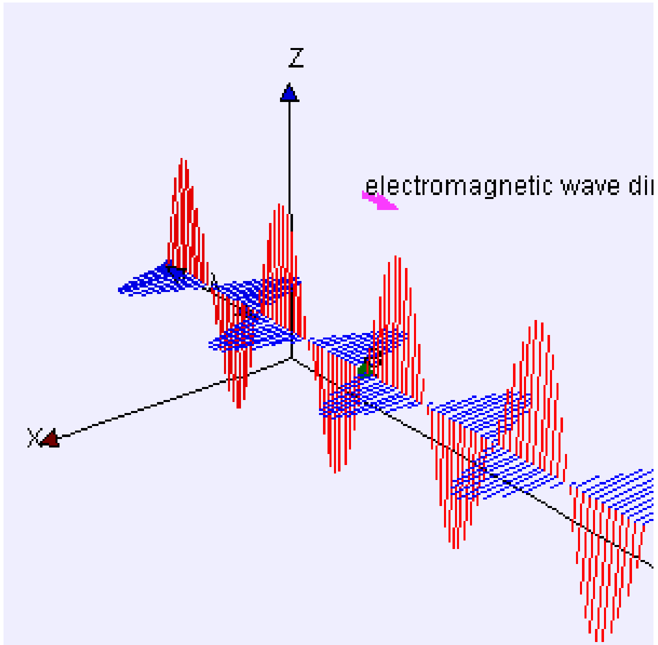
$$\frac{\partial^2 H_z(x,y,z,t)}{\partial x^2} + \frac{\partial^2 H_z(x,y,z,t)}{\partial y^2} + \frac{\partial^2 H_z(x,y,z,t)}{\partial z^2} = \mu\varepsilon \frac{\partial^2 H_z(x,y,z,t)}{\partial t^2} \quad (6)$$

TEM wave – Plane wave

- ***Plane wave***: the components of the field quantities **E** and **H** lies in a transverse plane, a plane perpendicular to the direction of propagation of the wave.
  - **E** and **H** fields have no components in the ***longitudinal direction*** (the direction of wave propagation), meaning $E_z = 0$ and $H_z = 0$. Such a wave is also called a ***TEM (transverse electromagnetic)*** wave.



Six equations obtained so far

$$\frac{\partial^2 \mathrm{E}_x(x,y,z,t)}{\partial x^2} + \frac{\partial^2 \mathrm{E}_x(x,y,z,t)}{\partial y^2} + \frac{\partial^2 \mathrm{E}_x(x,y,z,t)}{\partial z^2} = \mu\varepsilon \frac{\partial^2 \mathrm{E}_x(x,y,z,t)}{\partial t^2} \quad (1)$$

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Uniform Plane Wave Approximation

- What kind of plane wave can be considered as "uniform"?
  - Planar wavefront and uniform (constant) distributions of fields over every plane perpendicular to the direction of wave propagation.
- To a distant observer, the wavefront of a spherical wave appears to be approximately planar:
- In the family of plane waves, the ***uniform plane wave*** is one of the simplest one. A uniform plane wave propagate in the z direction, **E** and **H** are not functions of x and y, i.e.

$$\frac{\partial \vec{\mathbf{E}}}{\partial x} = 0 \quad \frac{\partial \vec{\mathbf{E}}}{\partial y} = 0$$

$$\frac{\partial \vec{\mathbf{H}}}{\partial x} = 0 \quad \frac{\partial \vec{\mathbf{H}}}{\partial y} = 0$$

Six equations obtained so far

$$\cancel{\frac{\partial^2 \mathrm{E}_x(x,y,z,t)}{\partial x^2}} + \cancel{\frac{\partial^2 \mathrm{E}_x(x,y,z,t)}{\partial y^2}} + \boxed{\frac{\partial^2 \mathrm{E}_x(x,y,z,t)}{\partial z^2} = \mu\varepsilon \frac{\partial^2 \mathrm{E}_x(x,y,z,t)}{\partial t^2} \qquad (1)}$$

$$\cancel{\frac{\partial^2 \mathrm{E}_y(x,y,z,t)}{\partial x^2}} + \cancel{\frac{\partial^2 \mathrm{E}_y(x,y,z,t)}{\partial y^2}} + \boxed{\frac{\partial^2 \mathrm{E}_y(x,y,z,t)}{\partial z^2} = \mu\varepsilon \frac{\partial^2 \mathrm{E}_y(x,y,z,t)}{\partial t^2} \qquad (2)}$$

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$$\cancel{\frac{\partial^2 H_x(x,y,z,t)}{\partial x^2}} + \cancel{\frac{\partial^2 H_x(x,y,z,t)}{\partial y^2}} + \boxed{\frac{\partial^2 H_x(x,y,z,t)}{\partial z^2} = \mu\varepsilon \frac{\partial^2 H_x(x,y,z,t)}{\partial t^2} \qquad (4)}$$

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Equations for uniform TEM waves

- For a uniform plane wave propagating in the z direction, the Helmholtz equations can be expressed in a scalar form as

$$\frac{\partial^2 E_x}{\partial z^2} - \mu\epsilon \frac{\partial^2 E_x}{\partial t^2} = 0$$

$$\frac{\partial^2 E_y}{\partial z^2} - \mu\epsilon \frac{\partial^2 E_y}{\partial t^2} = 0$$

$$\frac{\partial^2 H_x}{\partial z^2} - \mu\epsilon \frac{\partial^2 H_x}{\partial t^2} = 0$$

$$\frac{\partial^2 H_y}{\partial z^2} - \mu\epsilon \frac{\partial^2 H_y}{\partial t^2} = 0$$

- The field components are functions of *z* (the direction of propagation) and *t* (time) only.
- These equations are similar => solutions are also similar => we only need to solve one.
- Each one is a second-order differential equation with two possible solutions.

  – where E_x, E_y, H_x and H_y are the ***transverse components*** of **E** and **H**.
  – For time-harmonic fields, each wave equation can be expressed in its phasor equivalent form, such as:

$$\frac{\partial^2 E_x}{\partial z^2} - \mu\epsilon \frac{\partial^2 E_x}{\partial t^2} = 0 \quad \Rightarrow \quad \frac{d^2 \tilde{E}_x}{dz^2} + \omega^2 \mu\epsilon \tilde{E}_x = 0$$

Equations for time-harmonic uniform TEM waves

$$\frac{d^2\tilde{E}_x}{dz^2} + \omega^2\mu\epsilon\tilde{E}_x = 0$$

- For a monochromatic (single frequency) wave propagating in a uniform medium, $\omega^2\mu\varepsilon$ is a constant, define phase constant $\beta = \omega\sqrt{\mu\epsilon}$, then we can rewrite the wave equation as:

$$\frac{d^2\tilde{E}_x}{dz^2} + \beta^2\tilde{E}_x = 0$$

- There are two solutions for the x component of the E field:

$$\tilde{E}_x(z) = \hat{E}_{xf}e^{-j\beta z} \quad \text{and} \quad \tilde{E}_x(z) = \hat{E}_{xb}e^{j\beta z}$$

- So the general solution is $\tilde{E}_x(z) = \hat{E}_{xf}e^{-j\beta z} + \hat{E}_{xb}e^{j\beta z}$
  - where $\hat{E}_{xf}$ and $\hat{E}_{xb}$ are two complex constant, which can be written as

$$\hat{E}_{xf} = E_{xf}e^{j\theta_{xf}} \quad \text{and} \quad \hat{E}_{xb} = E_{xb}e^{j\theta_{xb}}$$

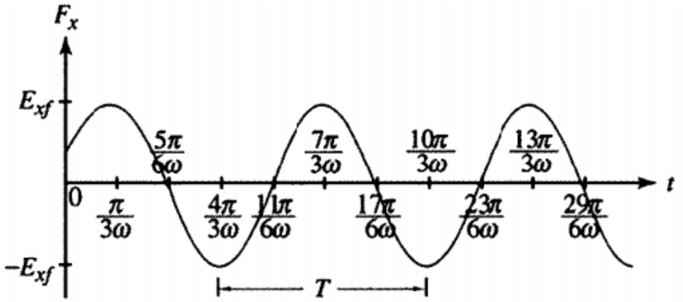
  - Then we obtain $\tilde{E}_x(z) = E_{xf}e^{-j(\beta z-\theta_{xf})} + E_{xb}e^{j(\beta z+\theta_{xb})}$

or $E_x(z,t) = E_{xf}\cos(\omega t - \beta z + \theta_{xf}) + E_{xb}\cos(\omega t + \beta z + \theta_{xb})$

The forward travelling wave

$$E_x(z,t) = E_{xf}\cos(\omega t - \beta z + \theta_{xf}) + E_{xb}\cos(\omega t + \beta z + \theta_{xb})$$

- Let's examine the first term $F_x = E_{xf}\cos(\omega t - \beta z + \theta_{xf})$ or $F_{xf}e^{-j(\beta z - \theta_{xf})}$:
  - At any given point in a transverse plane (z = constant), F_x varies sinusoidally in time.



  - The function F_x also varies with z.
    - Note that as time progresses, each point on the function moves to the right (forward direction) => this term represents a ***forward travelling wave***.

when

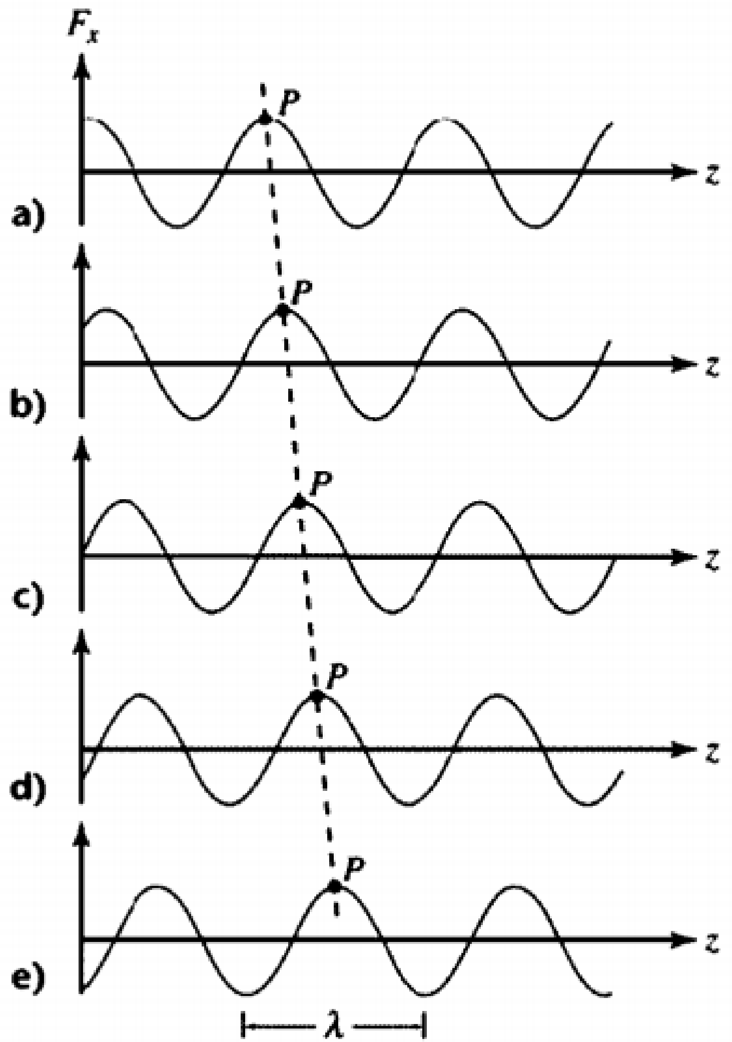
(a) $\omega t = -\theta_{xf}$

(b) $\omega t = \frac{\pi}{4} - \theta_{xf}$

(c) $\omega t = \frac{\pi}{2} - \theta_{xf}$

(d) $\omega t = \frac{3\pi}{4} - \theta_{xf}$

(e) $\omega t = \pi - \theta_{xf}$



The forward travelling wave

- At any given time (t = constant) the wave returns to its original magnitude and phase when z increases by a wavelength λ: $\beta\lambda = 2\pi$.
  - The **wavelength** is the distance between two planes when the phase difference between them at any given time is 2π radians. It is:

$$\lambda = \frac{2\pi}{\beta}$$

  - The phase velocity (phase speed) is:

$$\vec{\mathbf{u}}_p = \frac{\omega}{\beta}\vec{\mathbf{a}}_z$$

  - which can also be written as $\vec{\mathbf{u}}_p = \frac{c}{n}\vec{\mathbf{a}}_z$, where $\frac{1}{\sqrt{\mu_0\epsilon_0}} = c \approx 3 \times 10^8 \text{ m/s}$ is ***the speed of light***, and $n = \sqrt{\mu_r\epsilon_r}$ is ***the index of refraction***.
  - The phase velocity is independent of frequency!

The backward travelling wave

$$E_x(z,t) = E_{xf}\cos(\omega t - \beta z + \theta_{xf}) + E_{xb}\cos(\omega t + \beta z + \theta_{xb})$$

- The second term represents a backward travelling wave, since it moves in the negative *z* direction as time progresses.
  - Thus, the wave travels in the backward direction with a phase velocity of $-\omega/\beta\vec{\mathbf{a}}_z$.

- Similarly, we can get a solution for the y component of the **E** field, as
  - In phasor form: $\tilde{E}_y(z) = E_{yf}e^{-j(\beta z - \theta_{yf})} + E_{yb}e^{j(\beta z + \theta_{yb})}$
  - In time domain: $E_y(z,t) = E_{yf}\cos(\omega t - \beta z + \theta_{yf}) + E_{yb}\cos(\omega t + \beta z + \theta_{yb})$

- And the solutions for H_x and H_y are also similar.

Plane wave in boundless dielectric medium

- Assume that:
  - i) the dielectric medium is of infinite extend;
  - ii) there is only one wave propagating along the z direction;

  => only the forward wave is propagating.
- Then, the *x* and *y* components are:

$$\tilde{E}_x(z) = E_{xf} e^{-j(\beta z - \theta_{xf})}$$
$$\tilde{E}_y(z) = E_{yf} e^{-j(\beta z - \theta_{yf})}$$

  - Using the Maxwell's equation (1), get the x and y of **H** field as:

$$\tilde{H}_x(z) = -\sqrt{\frac{\epsilon}{\mu}}\,\tilde{E}_y(z)$$
$$\tilde{H}_y(z) = \sqrt{\frac{\epsilon}{\mu}}\,\tilde{E}_x(z)$$

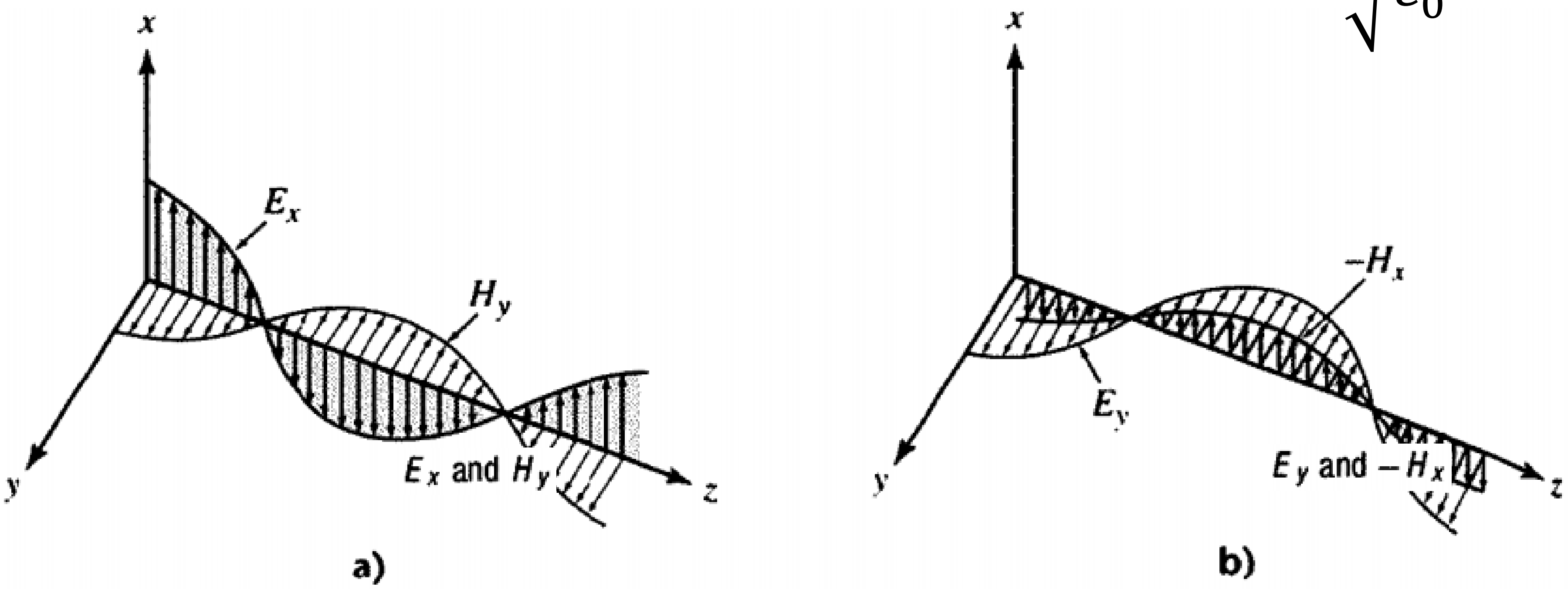
Plane wave in boundless dielectric medium

- The **E** and **H** relationship can also be written as

$$\vec{\mathbf{a}}_z \times \tilde{\mathbf{E}} = \sqrt{\frac{\mu}{\epsilon}}\, \tilde{\mathbf{H}} = \eta \tilde{\mathbf{H}}$$

  – where $\eta = \sqrt{\frac{\mu}{\epsilon}}$ has the unit of Ω. It is called the intrinsic (or wave) impedance.

  – Intrinsic impedance for the wave in free space is $\eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 377\Omega$



Example 1

- If the electric field intensity as given by $\vec{\mathbf{E}} = 377\cos(10^9 t - 5y)\vec{\mathbf{a}}_z$ V/m represents a uniform plane wave propagating in the *y* direction in a dielectric medium ($\mu = \mu_0$, $\varepsilon = \varepsilon_r\varepsilon_0$), determine:
  - a) the dielectric constant;
  - b) the velocity of propagation;
  - c) the intrinsic impedance;
  - d) the wavelength;
  - e) the magnetic field intensity.

Plane wave in free space

- Free space (or vacuum) is a special case of a dielectric medium in which $\mu = \mu_0$ and $\varepsilon = \varepsilon_0$
- We can simply replace μ with μ_0 and ε with ε_0 to get:
  - Phase constant in free space:

$$\beta_0 = \omega\sqrt{\mu_0\epsilon_0} = \frac{\omega}{c}$$

    - where $c = 1/\sqrt{\mu_0\epsilon_0} = 3 \times 10^8$ m/s is the speed of light.
  - Wave speed in free space:

$$u_p = \frac{\omega}{\beta_0} = c \longrightarrow$$

meaning that *an electromagnetic wave propagates in free space travelling with the speed of light.*

  - Wavelength in free space:

$$\lambda_0 = \frac{2\pi}{\beta_0} = \frac{c}{f}$$

  - Intrinsic impedance of free space:

$$\eta_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} = 120\pi \approx 377\ \Omega$$

Example 2

- The electric field intensity of a uniform plane wave in free space is given by $\vec{\mathbf{E}} = 94.25 \cos(\omega t + 6z)\vec{\mathbf{a}}_x$ V/m. Determine:
  - a) whether this expression satisfies the Helmholtz equation;
  - b) the velocity of propagation;
  - c) the wave frequency;
  - d) the wavelength;
  - e) the magnetic field intensity.

Quiz

- 1. The electric field of a plane wave propagating in a nonmagnetic medium is given by $\mathbf{E} = 3\sin(2\pi 10^7 t - 0.4\pi x)\mathbf{a}_y$ V/m. Determine the wavelength:
  - (a) $\lambda = 0.5$ m; (b) $\lambda = 30$ m;
  - (c) $\lambda = 1$ m; (d) $\lambda = 5$ m.

- 2. For the wave in Q1, find the relative dielectric permittivity of the medium:
  - (a) $\varepsilon_r = 36$; (b) $\varepsilon_r = 1$;
  - (c) $\varepsilon_r = 6$; (d) $\varepsilon_r = 10$.