# Week 8 Fourier Transform

Jimin Xiao

EB Building, Room 312
**jimin.xiao@xjtlu.edu.cn**
0512-81883209

## Trigonometric Fourier Series:

$$x(t) = a_0 + \sum_{n=1}^{\infty} [a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)]$$

## Exponential Fourier Series:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

# Outline

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- **From Fourier series to Fourier transform**
- **Fourier transform**
- **Fourier transform table**
- **Spectrum of a signal**
- **Properties of Fourier Transform**

# From Fourier series to Fourier transform

# Last time: Fourier Series

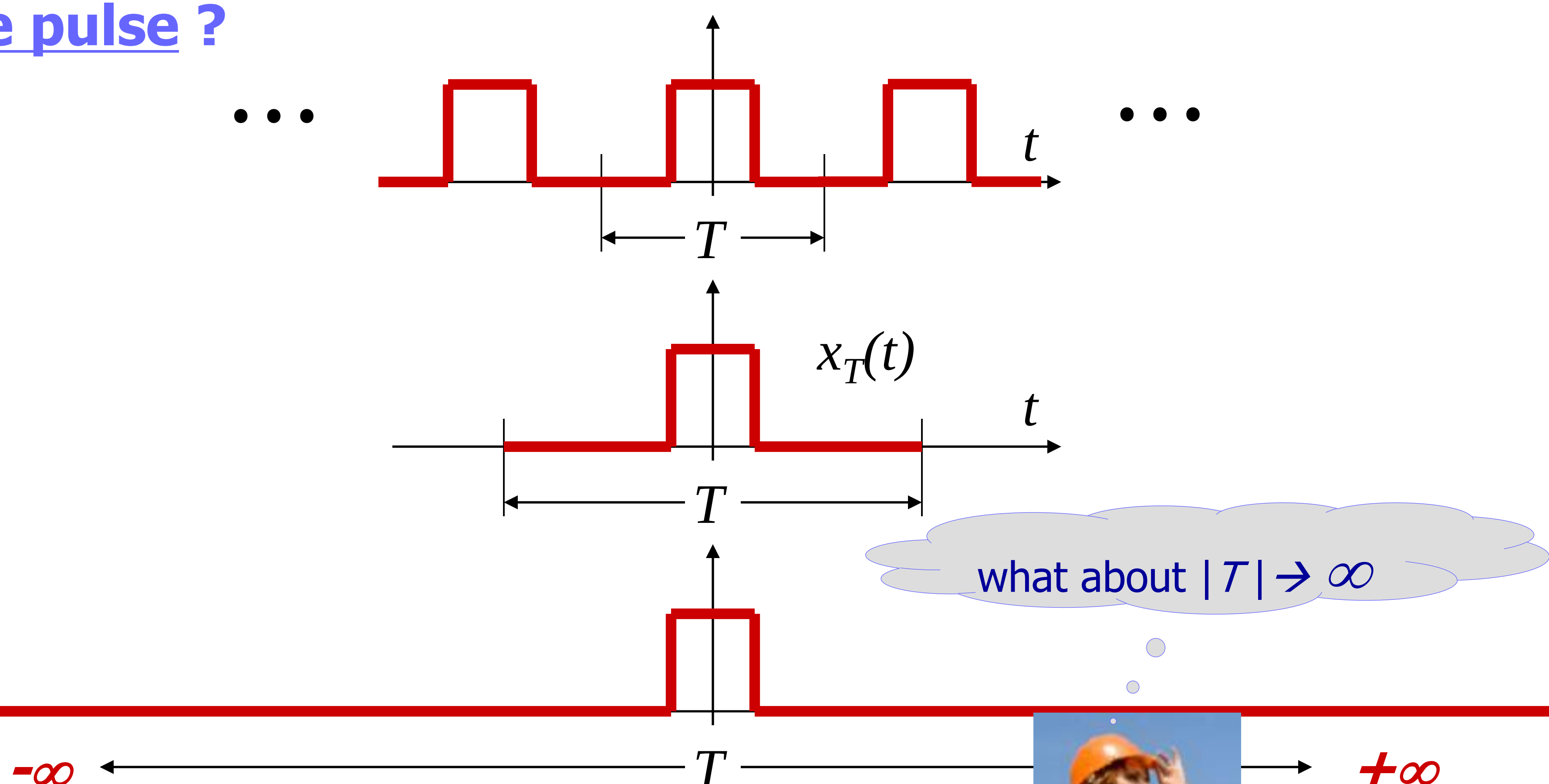
Representing periodic signals as sums of sinusoids.

→ new representations for systems as filters.

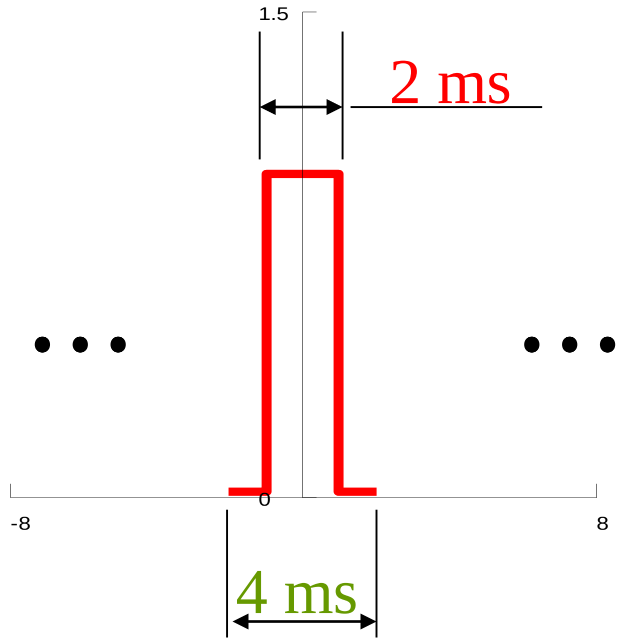
Today: generalize for aperiodic signals.

# From FS to FT

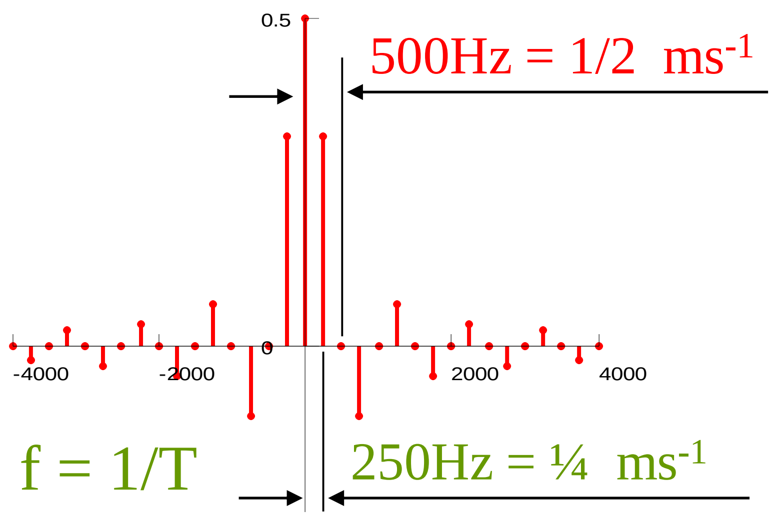
**What will happen to the spectrum of a rectangular pulse when *T* get increased, <u>while keeping the same width of the pulse</u> ?**




## F.S. of the rectangular pulse signal ($T$=4ms)

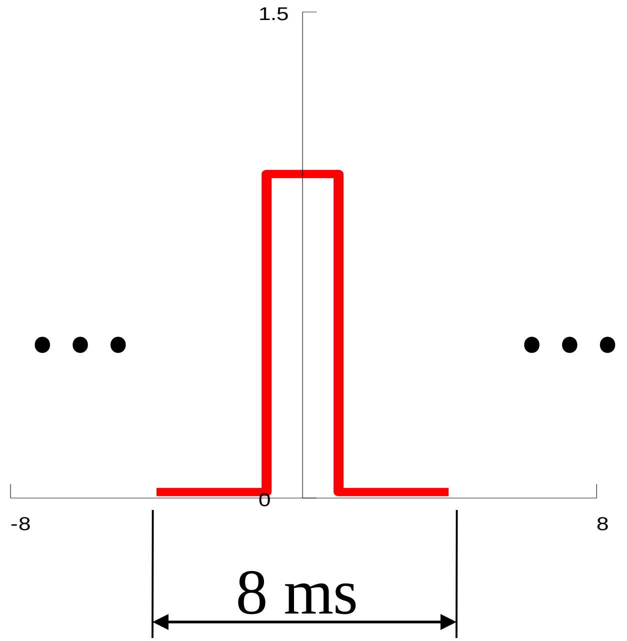


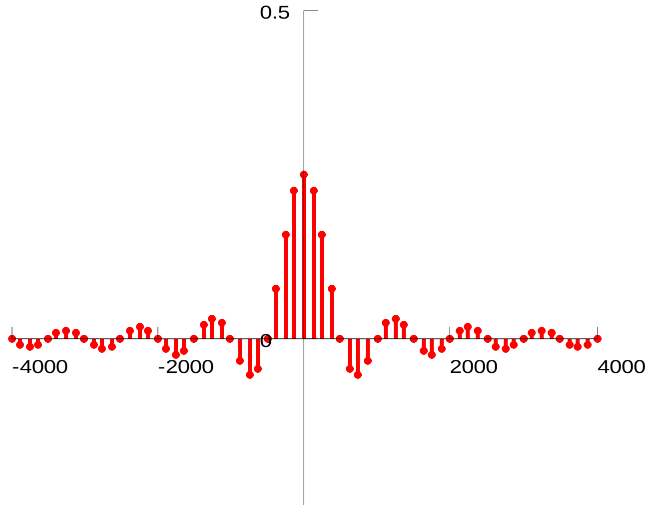

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{j2\pi n\frac{t}{T}}$$




$$C_n = \frac{2}{4} sinc(n\frac{2}{4})$$

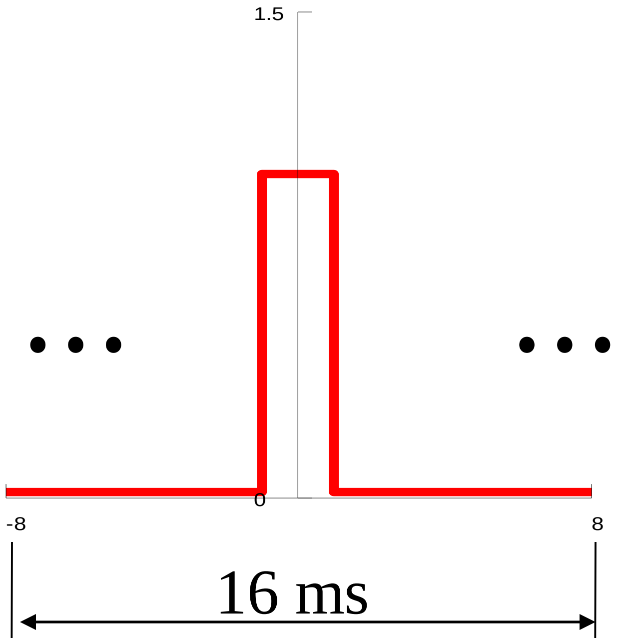
# F.S. of the rectangular pulse signal (*T*=8ms)

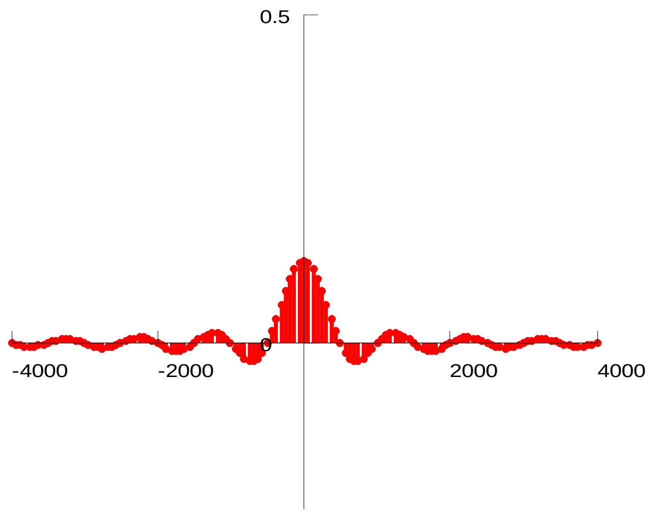



$$C_n = \frac{2}{8} sinc\left(n \frac{2}{8}\right)$$

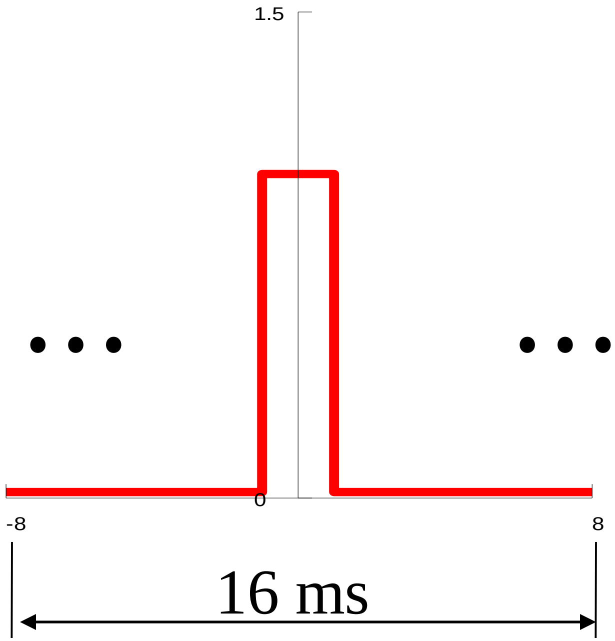
# F.S. of the rectangular pulse signal (*T*=16ms)

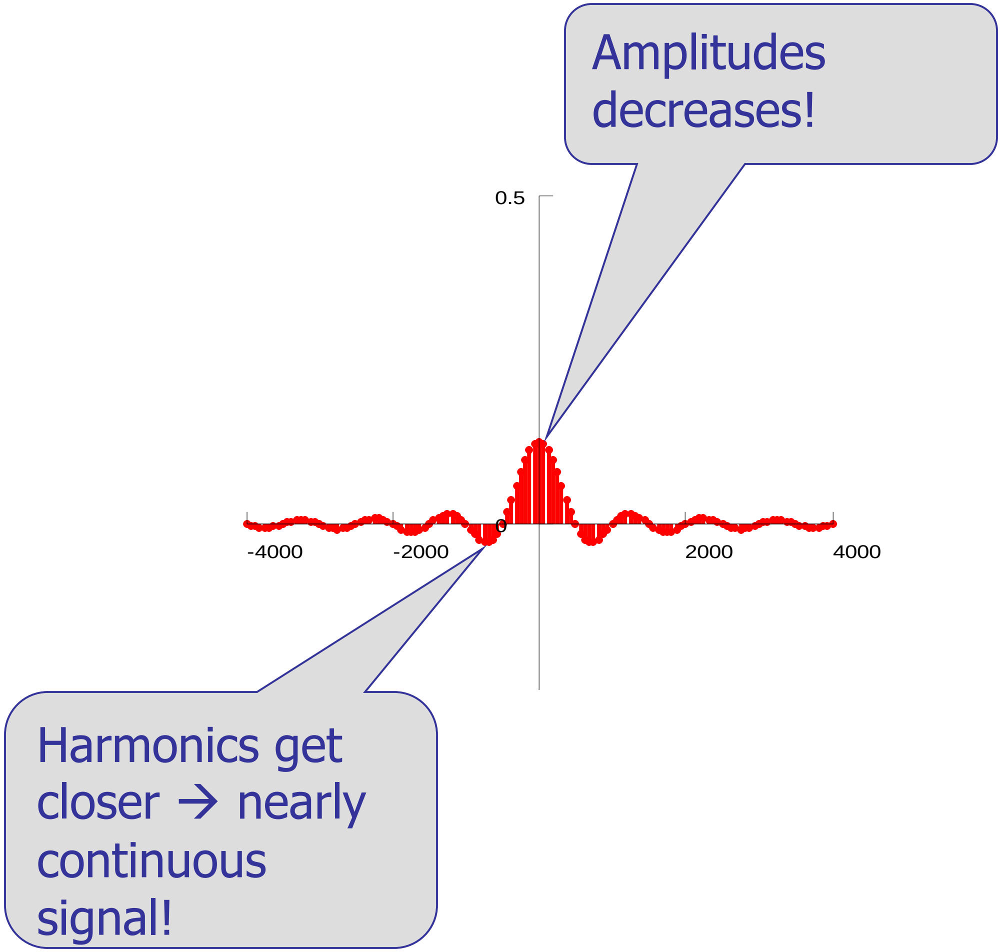



$$C_n = \frac{2}{16} sinc(n\frac{2}{16})$$

# F.S. of the rectangular pulse signal ($T$=16ms)



# From FS to FT

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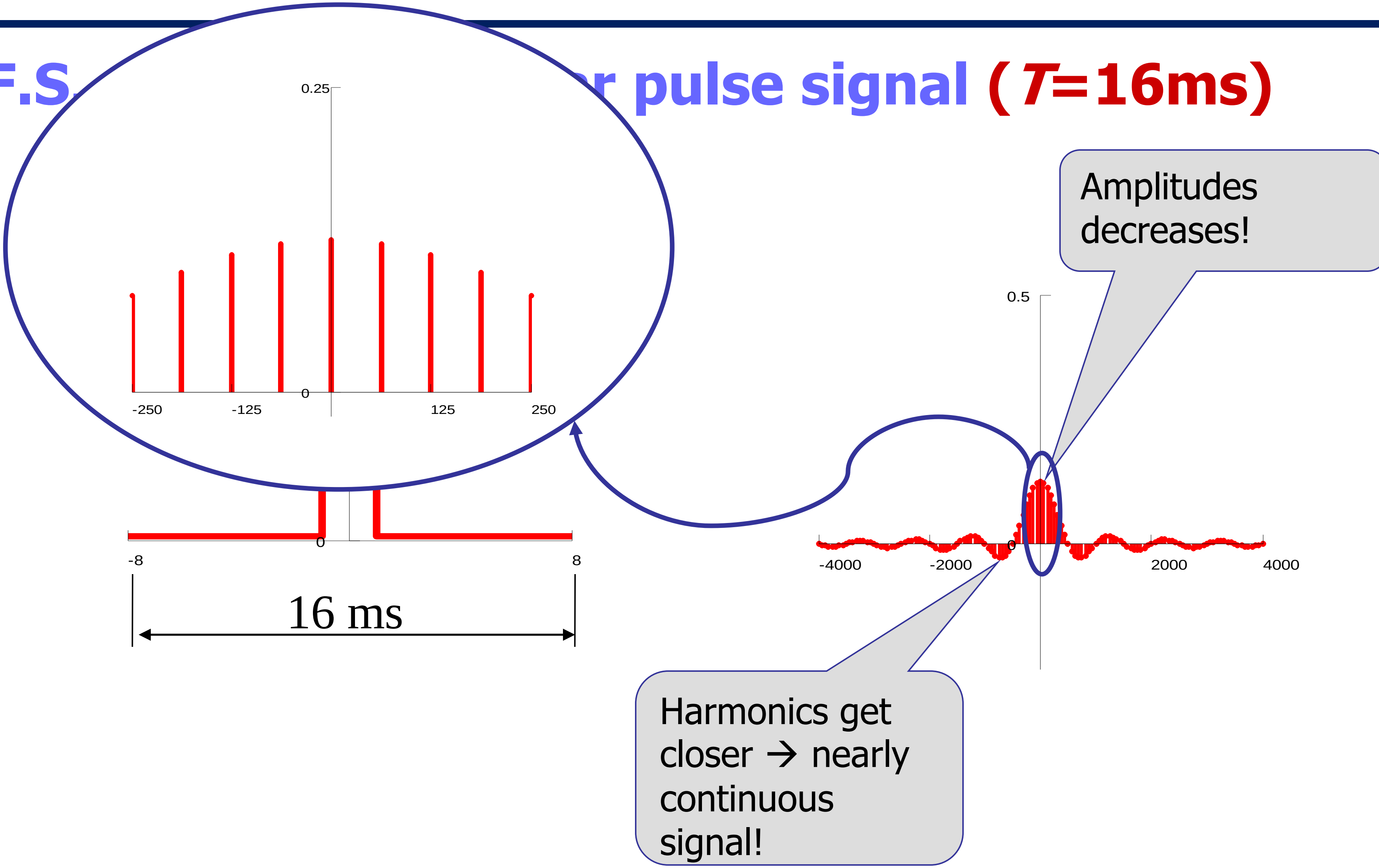
## F.S. ... r pulse signal (*T*=16ms)



# Parseval's theorem

The total average power in a periodic signal equals the sum of the average powers in all of its harmonic components.

$$P_x = \frac{1}{T}\int_T |x(t)^2|dt = \sum_{k=-\infty}^{\infty} |a_k|^2$$

# From FS to FT

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$$x_T(t) = \sum_{n=-\infty}^{\infty} c_n e^{j2\pi n \frac{t}{T}}$$

FS representation of a periodic signal

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} x_T(t) e^{-j2\pi n \frac{t}{T}} dt$$

FS coefficient

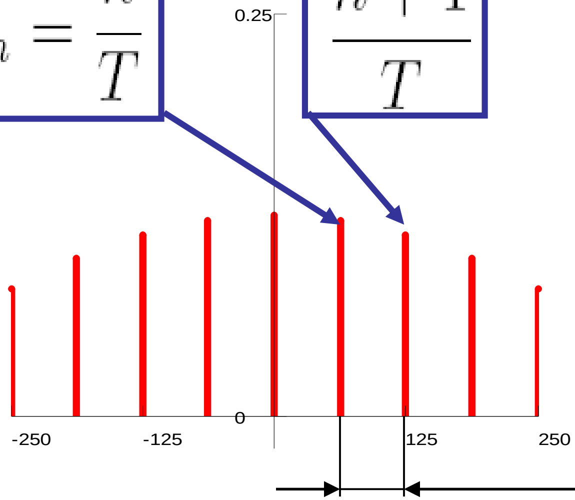
substitute into above equation

$$x_T(t) = \sum_{n=-\infty}^{\infty} \left[ \int_{-T/2}^{T/2} x_T(t) e^{-j2\pi n \frac{t}{T}} dt \right] e^{j2\pi n \frac{t}{T}} \frac{1}{T}$$

# From FS to FT

$$\boxed{f_n = \frac{n}{T}}$$

$$\boxed{\frac{n+1}{T}}$$




$$x_T(t) = \sum_{n=-\infty}^{\infty} \left[ \int_{-\frac{T}{2}}^{\frac{T}{2}} x_T(t) e^{-j2\pi t \frac{n}{T}} dt \right] e^{j2\pi t \frac{n}{T}} \frac{1}{T}$$

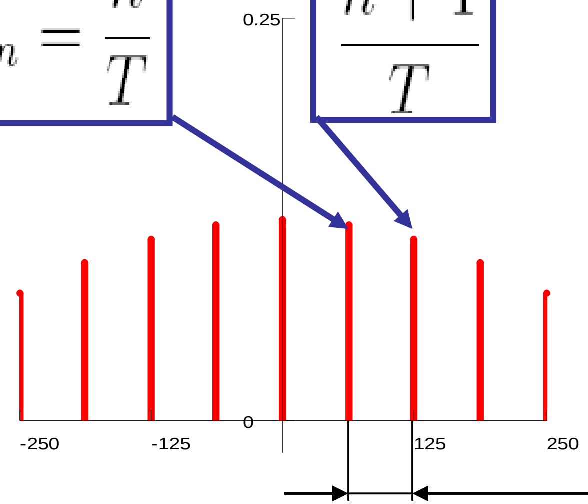
$$\Delta f = \frac{n+1}{T} - \frac{n}{T} = \frac{1}{T}$$

# From FS to FT



$$f_n = \frac{n}{T}$$

$$\frac{n+1}{T}$$




$$x_T(t) = \sum_{n=-\infty}^{\infty} \left[ \int_{-\frac{T}{2}}^{\frac{T}{2}} x_T(t) e^{-j2\pi t \frac{n}{T}} dt \right] e^{j2\pi t \frac{n}{T}} \frac{1}{T}$$

$$\Delta f = \frac{n+1}{T} - \frac{n}{T} = \frac{1}{T}$$

**What will happen when $T \rightarrow \infty$ ?**

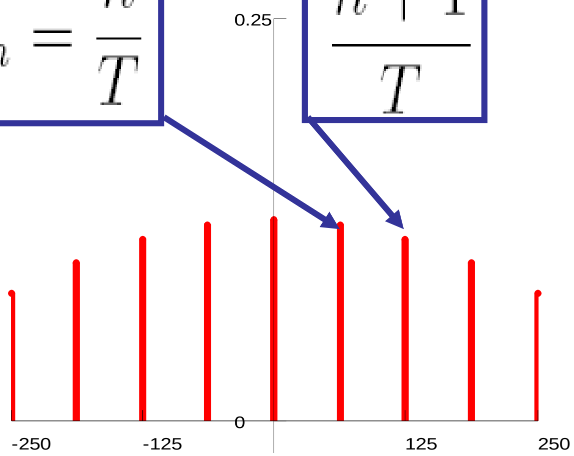
$$\left( \Delta f = \frac{1}{T} \right) \rightarrow df$$

$$\left( f_n = \frac{n}{T} \right) \rightarrow f$$

In the limit, as *T* approaches infinity, the spectrum becomes continuous.

# From FS to FT

$$\boxed{f_n = \frac{n}{T}} \qquad \boxed{\frac{n+1}{T}}$$




**What will happen when**

***T*→ ∞ ?**

$$\left(\Delta f = \frac{1}{T}\right) \rightarrow df$$

$$\left(f_n = \frac{n}{T}\right) \rightarrow f$$

$$x_T(t) = \sum_{n=-\infty}^{\infty}\left[\int_{-\frac{T}{2}}^{\frac{T}{2}} x_T(t) e^{-j2\pi t \frac{n}{T}} dt\right] e^{j2\pi t \frac{n}{T}} \frac{1}{T}$$

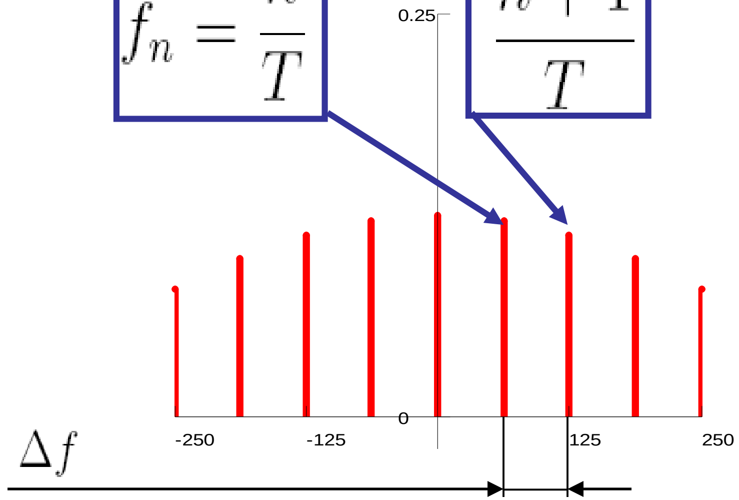
$$\int_{-\infty}^{\infty} \qquad \infty \qquad f \qquad f \qquad df$$

$$x(t) = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} x(t) e^{-j2\pi t f} dt\right] e^{j2\pi t f} df$$

# From FS to FT



$$f_n = \frac{n}{T} \qquad \frac{n+1}{T}$$




$$x_T(t) = \sum_{n=-\infty}^{\infty} \left[ \int_{-\frac{T}{2}}^{\frac{T}{2}} x_T(t) e^{-j2\pi t \frac{n}{T}} dt \right] e^{j2\pi t \frac{n}{T}} \frac{1}{T}$$

$$\int_{-\infty}^{\infty} \qquad \infty \qquad f \qquad f \qquad df$$

**What will happen when $T \rightarrow \infty$ ?**

$$\left( \Delta f = \frac{1}{T} \right) \rightarrow df$$

$$\left( f_n = \frac{n}{T} \right) \rightarrow f$$

$$\boxed{x(t) = \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} x(t) e^{-j2\pi t f} dt \right] e^{j2\pi t f} df}$$

# From FS to FT



$$x(t) = \int_{-\infty}^{\infty} \underbrace{\left[\int_{-\infty}^{\infty} x(t) e^{-j2\pi t f} dt\right]}_{} e^{j2\pi t f} df$$

It depends only on *f* → we will represent it by *X(f)*

## Forward Fourier transform :

('analysis' equation)

$$X(f) = \mathcal{F}\left\{x(t)\right\} \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} x(t) e^{-j2\pi t f} dt$$

## The Inverse Fourier transform :

('synthesis' equation)

$$x(t) = \mathcal{F}^{-1}\left\{X(f)\right\} = \int_{-\infty}^{\infty} X(f) e^{j2\pi t f} df$$

# Fourier transform pair

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| | |
|---|---|
| $G(f)=\int_{-\infty}^{\infty} g(t)e^{-j2\pi f t}\,dt$ | $X(\omega)=\int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$ |
| $g(t)=\int_{-\infty}^{\infty} G(f)e^{j2\pi f t}\,df$ | $x(t)=\boxed{\frac{1}{2\pi}}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\,d\omega$ |
| frequency (in Hz) | Angular frequency (in rad/s) |

Notation:

$$X(\omega)=F[x(t)] \quad x(t)=F^{-1}[X(\omega)] \quad x(t)\longleftrightarrow X(\omega)$$

Same

In our textbook, FT form is denoted as: $X(j\omega)$

# Fourier transform



## Forward (*analysis*) and inverse (*synthesis*) Fourier transform :

Fourier transform pair

$$x(t) \Longleftrightarrow X(f)$$

$$x(t) \xrightarrow{\mathcal{F}} X(f) \qquad X(f) \xrightarrow{\mathcal{F}^{-1}} x(t)$$


– We say that *x(t)* exists in the "time domain," and *X(f)* exists in the "frequency domain."
  - *X(f)* is just another way of looking at a function or a signal.

# Fourier transform



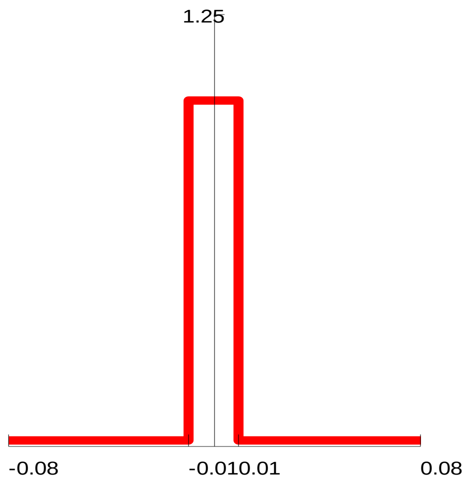
- $X(f)$ contains equivalent information to that in $x(t)$.
  - It is widely used to study linear systems.

- Allows to generalize the concept of fourier series to infinite duration and non-periodic signals.

- Introduces the concept of "continuous" frequency.

# Example

## Fourier transform of the rectangular pulse signal




$$x(t) = rect\left(\frac{t}{\tau}\right) = \begin{cases} 1, & |t| \leq -\tau/2 \\ 0, & \text{otherwise} \end{cases}$$

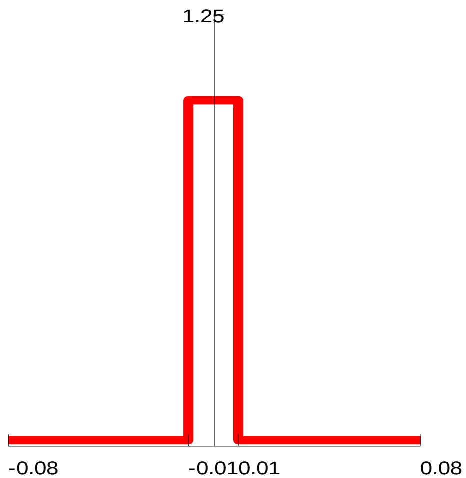
$$\begin{aligned} X(f) &= \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt \\ &= \int_{-\tau/2}^{\tau/2} e^{-j2\pi f t} dt \\ &= \left. \frac{e^{-j2\pi f t}}{-j2\pi f} \right|_{-\tau/2}^{\tau/2} \\ &= \frac{e^{-j\pi f \tau} - e^{j\pi f \tau}}{-j2\pi f} \\ &= \tau \frac{\sin(\pi f \tau)}{\pi f \tau} \\ &= \tau \, \mathrm{sinc}(f\tau) \end{aligned}$$

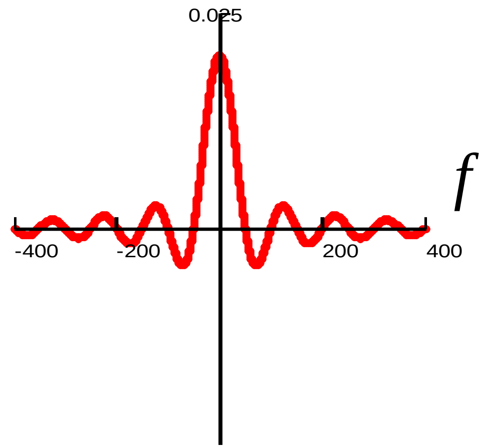
# Example

Real function

## Fourier transform of the rectangular pulse signal

$$X(f) = \tau \,\mathrm{sinc}(f\tau)$$




$$x(t) = rect\left(\frac{t}{\tau}\right) = \begin{cases} 1, & |t| \leq -\tau/2 \\ 0, & \text{otherwise} \end{cases}$$

# Fourier transform table



**Table 5.2.** CTFT pairs for elementary CT signals

| CT signals | Time domain $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\,dt$ | Frequency domain $X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$ | Comments |
|---|---|---|---|
| (1) Constant | $1$ | $2\pi\delta(\omega)$ | |
| (2) Impulse function | $\delta(t)$ | $1$ | |
| (3) Unit step function | $u(t)$ | $\pi\delta(\omega) + \frac{1}{j\omega}$ | |
| (4) Causal decaying exponential function | $e^{-at}u(t)$ | $\frac{1}{a + j\omega}$ | $a > 0$ |
| (5) Two-sided decaying exponential function | $e^{-a\lvert t\rvert}$ | $\frac{2a}{a^2 + \omega^2}$ | $a > 0$ |
| (6) First-order time-rising causal decaying exponential function | $te^{-at}u(t)$ | $\frac{1}{(a + j\omega)^2}$ | $a > 0$ |
| (7) $N$th-order time-rising causal decaying exponential function | $t^n e^{-at}u(t)$ | $\frac{n!}{(a + j\omega)^{n+1}}$ | $a > 0$ |
| (8) Sign function | $\mathrm{sgn}(t) = \begin{cases} 1 & t > 0 \\ -1 & t < 0 \end{cases}$ | $\frac{2}{j\omega}$ | |
| (9) Complex exponential | $e^{j\omega_0 t}$ | $2\pi\delta(\omega - \omega_0)$ | |
| (10) Periodic cosine function | $\cos(\omega_0 t)$ | $\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$ | |
| (11) Periodic sine function | $\sin(\omega_0 t)$ | $\frac{\pi}{j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$ | |
| (12) Causal cosine function | $\cos(\omega_0 t)u(t)$ | $\frac{\pi}{2}[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)] + \frac{j\omega}{\omega_0^2 - \omega^2}$ | |
| (13) Causal sine function | $\sin(\omega_0 t)u(t)$ | $\frac{\pi}{2j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)] + \frac{\omega_0}{\omega_0^2 - \omega^2}$ | |
| (14) Causal decaying exponential cosine function | $e^{-at}\cos(\omega_0 t)u(t)$ | $\frac{a + j\omega}{(a + j\omega)^2 + \omega_0^2}$ | $a > 0$ |
| (15) Causal decaying exponential sine function | $e^{-at}\sin(\omega_0 t)u(t)$ | $\frac{\omega_0}{(a + j\omega)^2 + \omega_0^2}$ | $a > 0$ |
| (16) Rectangular function | $\mathrm{rect}\left(\frac{t}{\tau}\right) = \begin{cases} 1 & \lvert t\rvert \le \tau/2 \\ 0 & \lvert t\rvert > \tau/2 \end{cases}$ | $\tau\,\mathrm{sinc}\left(\frac{\omega\tau}{2\pi}\right)$ | $\tau \neq 0$ |
| (17) Sinc function | $\frac{W}{\pi}\,\mathrm{sinc}\left(\frac{Wt}{\pi}\right)$ | $\mathrm{rect}\left(\frac{\omega}{2W}\right) = \begin{cases} 1 & \lvert\omega\rvert \le W \\ 0 & \lvert\omega\rvert > W \end{cases}$ | |
| (18) Triangular function | $\Delta\left(\frac{t}{\tau}\right) = \begin{cases} 1 - \frac{\lvert t\rvert}{\tau} & \lvert t\rvert \le \tau \\ 0 & \text{otherwise} \end{cases}$ | $\tau\,\mathrm{sinc}^2\left(\frac{\omega\tau}{2\pi}\right)$ | $\tau > 0$ |
| (19) Impulse train | $\sum_{k=-\infty}^{\infty}\delta(t - kT_0)$ | $\omega_0 \sum_{m=-\infty}^{\infty}\delta(\omega - m\omega_0)$ | angular frequency $\omega_0 = 2\pi/T_0$ |
| (20) Gaussian function | $e^{-t^2/2\sigma^2}$ | $\sigma\sqrt{2\pi}e^{-\sigma^2\omega^2/2}$ | |

# Fourier transform table



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| (20) Gaussian function | $e^{-t^2/2\sigma^2}$ | $\sigma\sqrt{2\pi}e^{-\sigma^2\omega^2/2}$ | |

# Example

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Determine the CTFT of the following functions and plot the corresponding magnitude and phase spectra:

4.1 $x(t) = e^{-at}u(t), a \in R^+$

4.2 $x(t) = e^{-a|t|}, a \in R^+$

4.3 $x(t) = \delta(t)$

# FT for periodic signals

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Consider a signal $x(t)$ with FT $X(\omega)$ that is a single impulse of area $2\pi$ at $\omega = \omega_0$

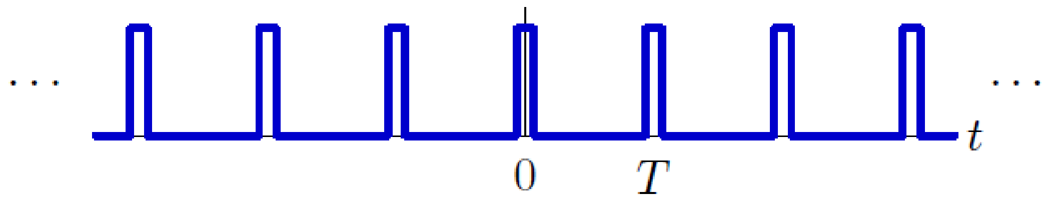
$$X(\omega) = 2\pi\delta(\omega - \omega_0)$$

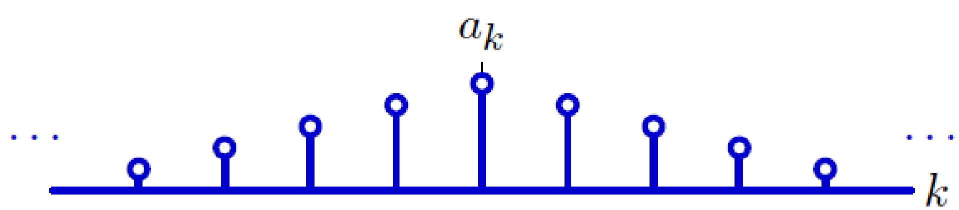
$$x(t) = \boxed{???}$$

# FT for periodic signals

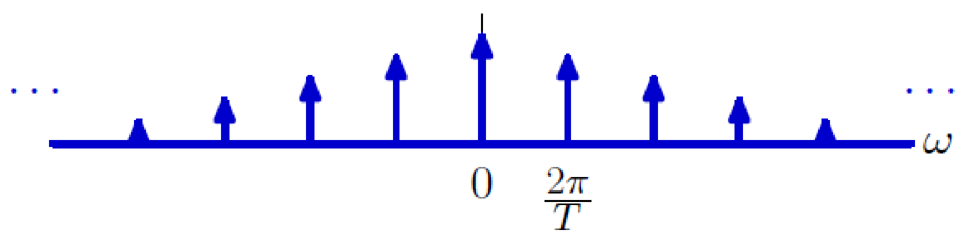
Each term in the Fourier series is replaced by an impulse.

$$x(t) = \sum_{k=-\infty}^{\infty} x_p(t - kT)$$




$$X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi \, a_k \, \delta(\omega - k\frac{2\pi}{T})$$



# Example

Find and plot FT for $x(t) = \sin(\omega_0 t)$ and $x(t) = cos(\omega_0 t)$

# Exercises

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**4.1**

**4.3**

**4.4**