EEE336 Signal Processing and Digital Filtering

Lecture 15 IIR Filters Design

Lect_15_1 Why IIR and Bilinear Transformation

Zhao Wang

Zhao.wang@xjtlu.edu.cn

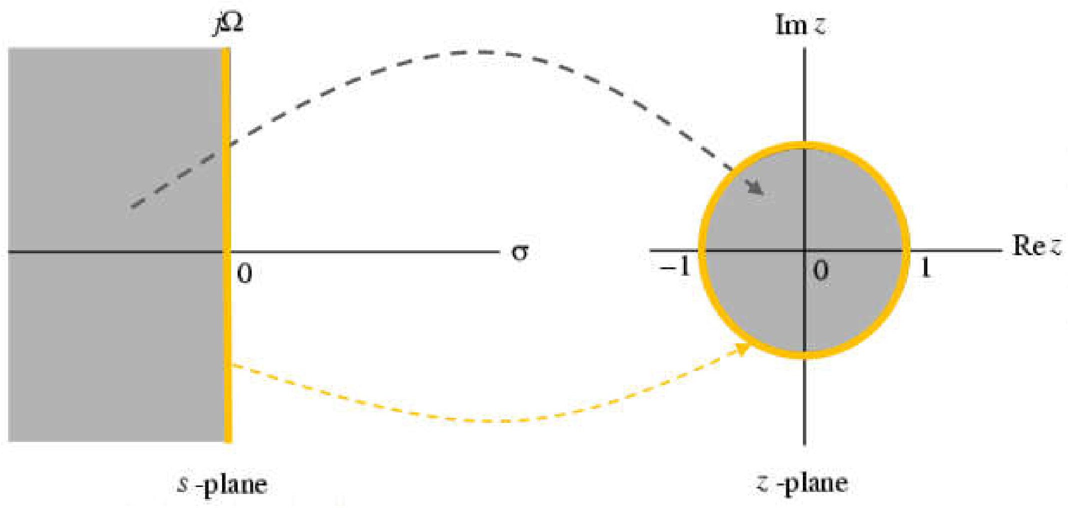
Room EE322

FIR vs IIR

- Major disadvantage of the FIR filter: Long filter lengths
  - IIR filters yield much shorter filters for the same specs: computationally efficient.
  - However, two potential concerns of IIR filters must be addressed:
- Classic IIR filter design:
  - 1. Convert the digital filter specifications into an analogue prototype lowpass filter specifications
  - 2. Determine the analogue lowpass filter transfer function $H_a(\Omega)$ and corresponding $H_a(s)$
    - Butterworth / Chybshev / Elliptic
  - 3. Transform $H_a(s)$ into the desired digital transfer function $H(z)$
    - Bilinear and inverse bilinear transformations for mapping s-plane to z-planes

Bilinear transformation

- From s-plane to z-plane



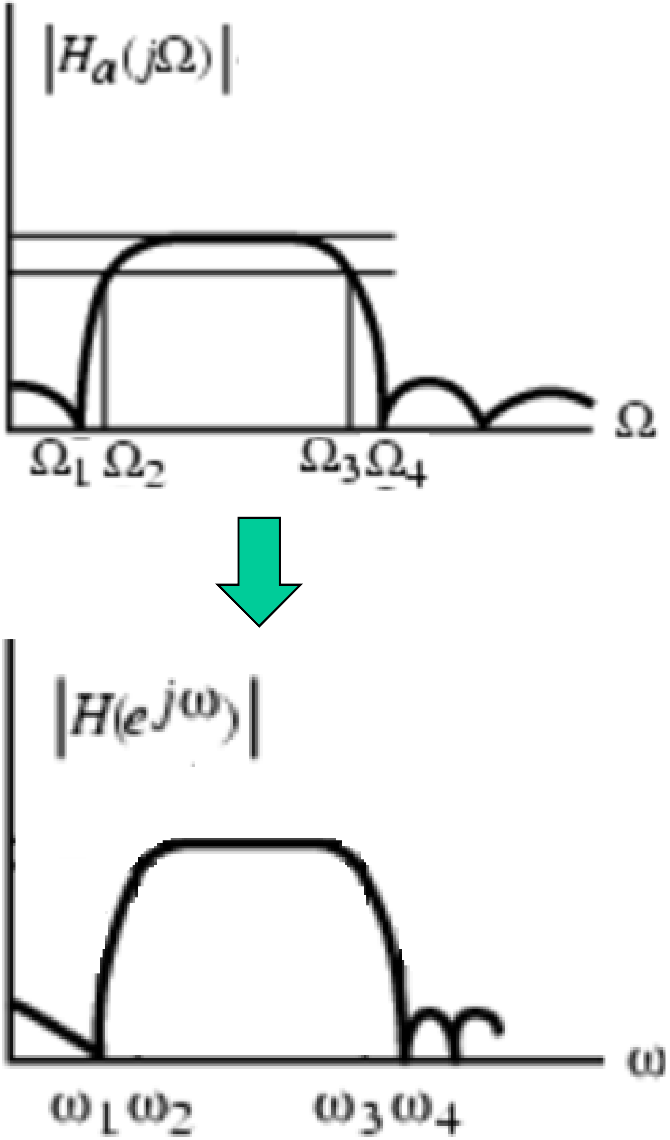
$$z = \frac{1 + \frac{T_s}{2}s}{1 - \frac{T_s}{2}s} = \frac{1+s}{1-s}$$

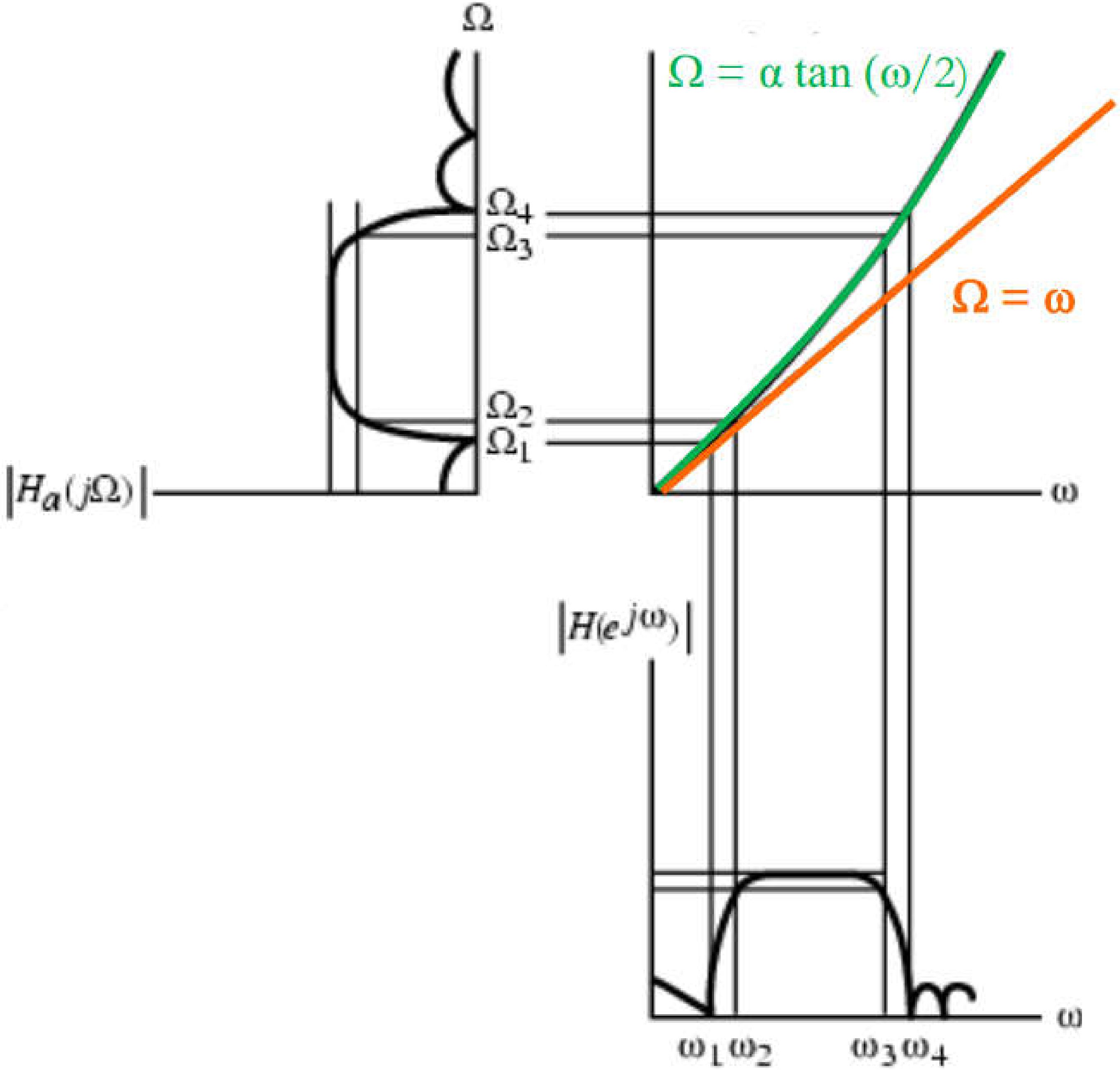
$$s = \frac{2}{T_s}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right) = \frac{1 - z^{-1}}{1 + z^{-1}}$$

The parameter Ts often does not play a role in the design, and therefore Ts=2 is chosen for convenience.

- The jΩ axis corresponds to the unit circle
- The left half of the s-plane corresponds to inside the unit circle in the z-plane
- The stability requirement of the analog filters carry to digital filters:
  - Analog: The poles of the filter frequency response must be on the left half plane
  - Digital: The poles of the filter frequency response must be inside the unit circle, i.e., the ROC must include the unit circle.

Bilinear transformation





Bilinear transformation

- Since, the frequency response is defined on the unit circle,

$$z = e^{j\omega} \Leftrightarrow s = j\Omega$$

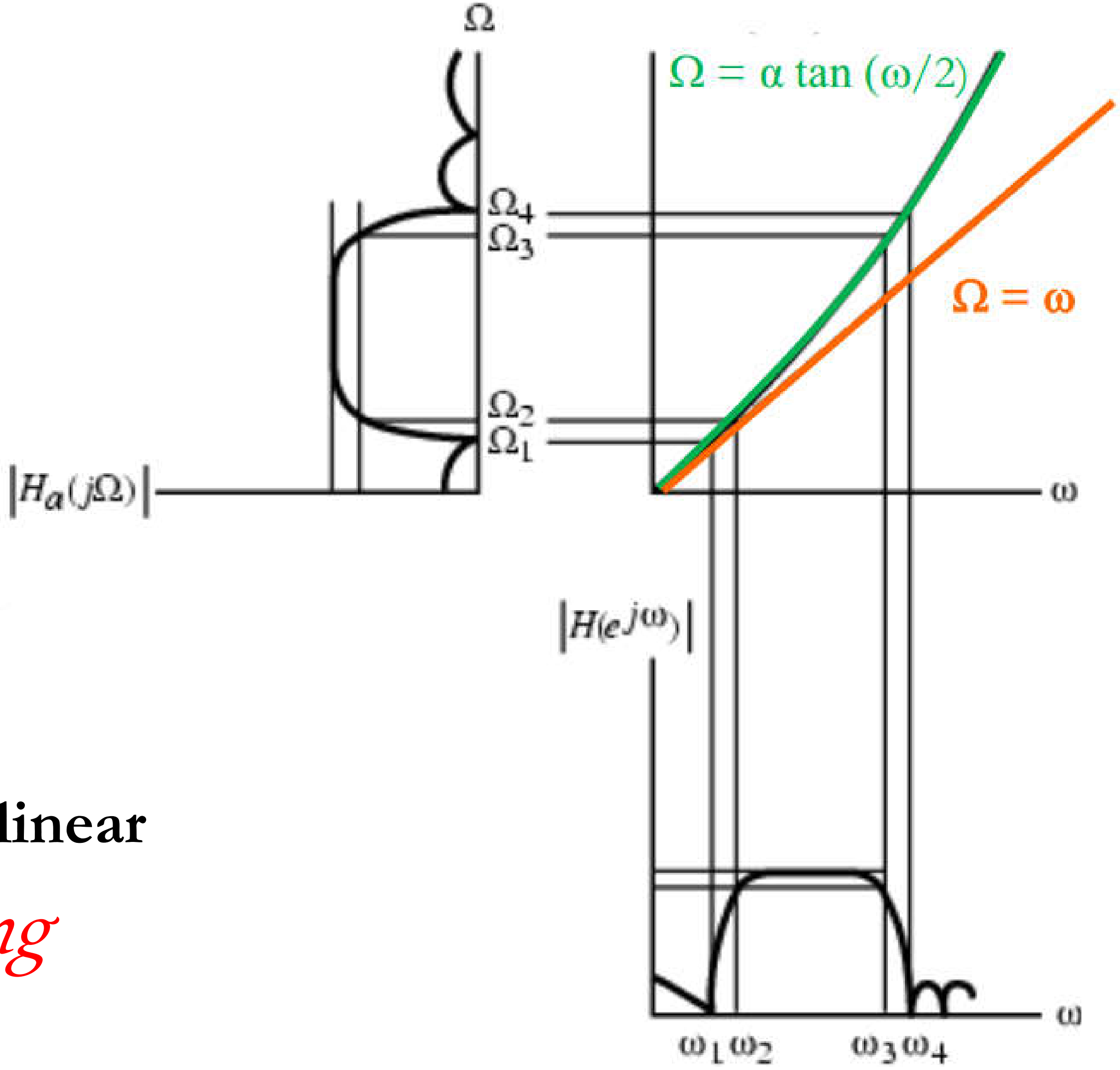
$$\boxed{s = \frac{2}{T_s}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

$$\Downarrow$$

$$\boxed{j\Omega = \frac{2}{T_s}\left(\frac{1-e^{-j\omega}}{1+e^{-j\omega}}\right)}$$

$$\Downarrow$$

$$\boxed{\Omega = \frac{2}{T_s}\tan\frac{\omega}{2} \Leftrightarrow \omega = 2\tan^{-1}\frac{\Omega T_s}{2}}$$



This mapping is (highly) nonlinear

=> Frequency warping

Bilinear transformation

- Steps in the design of a digital filter –
  - Prewarp ω_p, ω_s to find their analog equivalents Ω_p, Ω_s $\Omega = \frac{2}{T_s} tan \frac{\omega}{2}$
  - Design the analog filter $H_a(s)$
  - Design the digital filter H(z) by applying bilinear transformation to $H_a(s)$

$$s = \frac{2}{T_s}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right)$$

- How to design the analogue filter?
  - Butterworth filter – maximally flat in passband
  - Chebychev (type I and type II) filters – Equiripple in passband or stopband
  - Elliptic filter – Sharper transition band but very nonlinear phase and nonequiripple

- All of these filters are defined for lowpass characteristic
  - **Spectral transformations** are then used to convert lowpass to any one of highpass, bandpass or bandstop

15_1 Wrap up

- Compare FIR and IIR
  - Advantages and disadvantages
- Bilinear transformation
  - S-plane ←→ Z-plane: bilinear transformation
  - Relationship between s/z → relationship between ω/Ω
  - Design procedure (steps)
    - Linear frequency spec (fp and fs) → digital frequency spec (ωp and ωs) → analogue frequency spec (Ωp and Ωs)
    - Design analogue filter Ha(s)
    - Get H(z) based on Ha(s) using bilinear transformation

EEE336 Signal Processing and Digital Filtering

Lecture 15 IIR Filters Design

Lect_15_2_Analogue Filters_1

Butterworth Filters

Zhao Wang

Zhao.wang@xjtlu.edu.cn

Room EE322

The Butterworth Approximation

- The magnitude-square response of an Nth order analogue lowpass Butterworth filter:

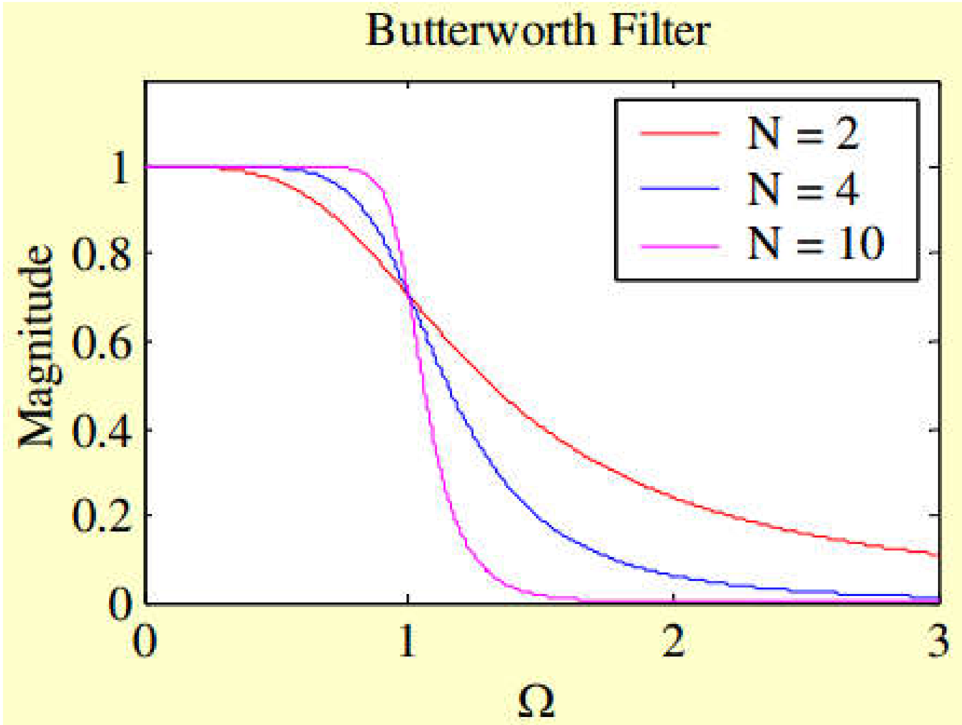
$$|H(\Omega)|^2 = \frac{1}{1 + (\Omega/\Omega_c)^{2N}}$$

  - Ω_c is the 3-dB cutoff frequency ($20log|H(\Omega_c)| = -3\ dB$), N is the filter order;
  - The most interesting property of this function is that the first 2N-1 derivatives of this function are zero at **Ω**=0.

$$\frac{d^k}{d\Omega^k}\left(\left|H_a(j\Omega)\right|^2\right)\Big|_{\Omega=0} = 0, \quad 1 \le k \le 2N-1$$

  - Order N increasing:
    - Reducing transition band;
    - Increasing smoothness near Ω=0.

The Butterworth approximation is also called a *maximally flat* approximation.



The Butterworth Approximation

- If we substitute for s using $s = j\Omega$ (remember this is for the square of the magnitude)

It is an all-pole function. Where are the poles?

$$|H(s)|^2 = |H(\Omega)|^2 = \frac{1}{1+(\Omega/\Omega_c)^{2N}} = \left|\frac{1}{1+(s/j\Omega_c)^{2N}}\right|$$

- Solving $\left(\frac{s}{j\Omega_c}\right)^{2N} = -1$ gets $s = j\Omega_c(-1)^{1/2N}$
  - Considering: $j = e^{j\frac{\pi}{2}}$ and $-1 = e^{j\pi}$, the poles are $p_l = \Omega_c e^{j\frac{\pi(2l+1+N)}{2N}}$

Where $0 \le l \le 2N-1$

- Example: for N = 4 the poles lie at

$$\Omega_c e^{j\pi/8},\quad \Omega_c e^{j3\pi/8},\quad \Omega_c e^{j5\pi/8},\quad \Omega_c e^{j7\pi/8}$$

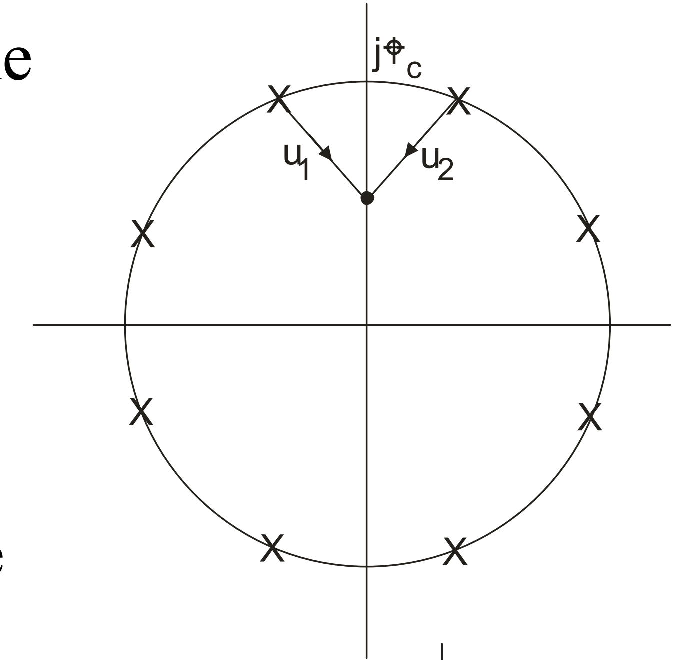
$$\Omega_c e^{j9\pi/8},\quad \Omega_c e^{j11\pi/8},\quad \Omega_c e^{j13\pi/8},\quad \Omega_c e^{j15\pi/8}$$

The Butterworth Approximation

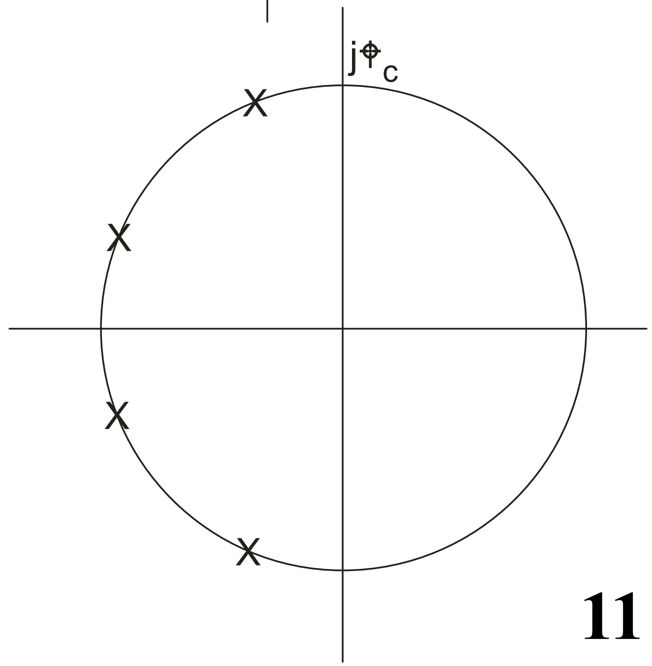
- The magnitude of the function at a point on the $j\Omega$ axis is given the product of distances:

$$\frac{1}{|\vec{u}_1||\vec{u}_2| \;\; ...\text{etc.}}$$

  - but these distances are equal in pairs.



- Thus, if we discard half of them we obtain the square root of the magnitude evaluated on the $j\Omega$ axis.
  - These pole are all in the left plane. So this filter is stable.
  - It is a 4th order Butterworth filter.
  - The poles lie on a circle of radius Ω_c.



Butterworth Filter

- In general, transfer function of an analogue Butterworth lowpass filter can be obtained using

$$H_a(s) = \frac{\Omega_c^N}{\prod_{l=1}^{N}(s - p_l)}$$

  – where $p_l = \Omega_c e^{j\frac{\pi(2l+1+N)}{2N}}, 0 \leq l \leq N - 1.$

N=1: $$H_a(s) = \frac{\Omega_c}{s + \Omega_c}$$

N=2: $$H_a(s) = \frac{\Omega_c^2}{s^2 + \sqrt{2}\Omega_c s + \Omega_c^2}$$

N=3 $$H_a(s) = \frac{\Omega_c^3}{(s + \Omega_c)(s^2 + \Omega_c s + \Omega_c^2)}$$

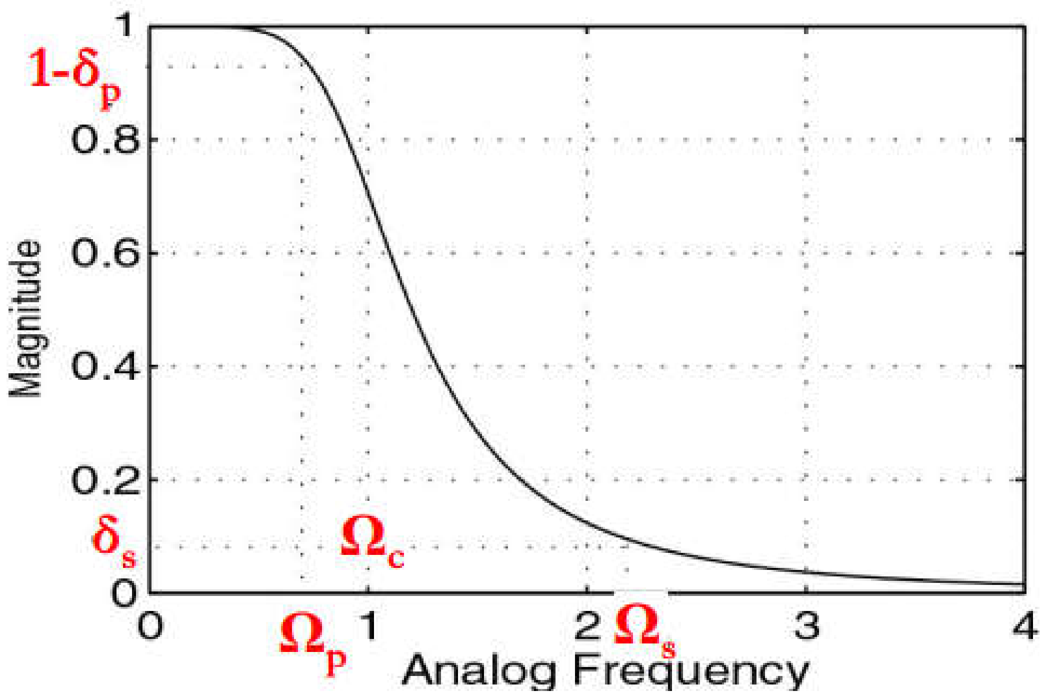
Butterworth Filter Design

- Two parameters completely characterizing a Butterworth lowpass filter are Ω_c and N

$$H_a(s) = \frac{\Omega_c^N}{\prod_{l=1}^{N}(s - p_l)}$$

where $p_l = \Omega_c e^{j\frac{\pi(2l+1+N)}{2N}}$

- To design a Butterworth filter, we thus need to find out Ω_c and N. They are determined from the specified band edges Ω_p and Ω_s, and minimum passband magnitude $1-\delta_p$, and maximum stopband ripple δ_s.



Analogue Filter Specifications

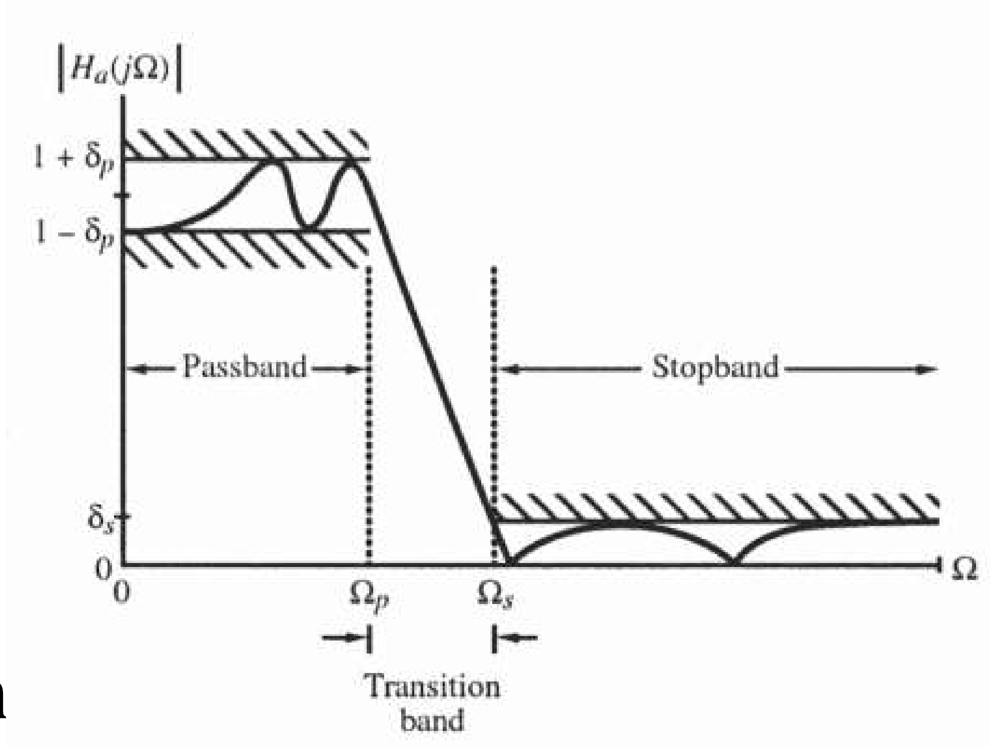
- $|H(e^{j\Omega})|\approx 1$, with an error $\pm\delta_p$ in the passband:

$$1-\delta_p \le |H_a(j\Omega)| \le 1+\delta_p, \quad |\Omega| \le \Omega_p$$

- $|H(e^{j\Omega})|\approx 0$, with an error δ_s in the stopband:

$$|H_a(j\Omega)| \le \delta_s, \quad \Omega_s \le |\Omega| < \infty$$

Ω_p – passband edge frequency
Ω_s – stopband edge frequency
δ_p - peak ripple value in the passband
δ_s - peak ripple value in the stopband



- Peak passband ripple:

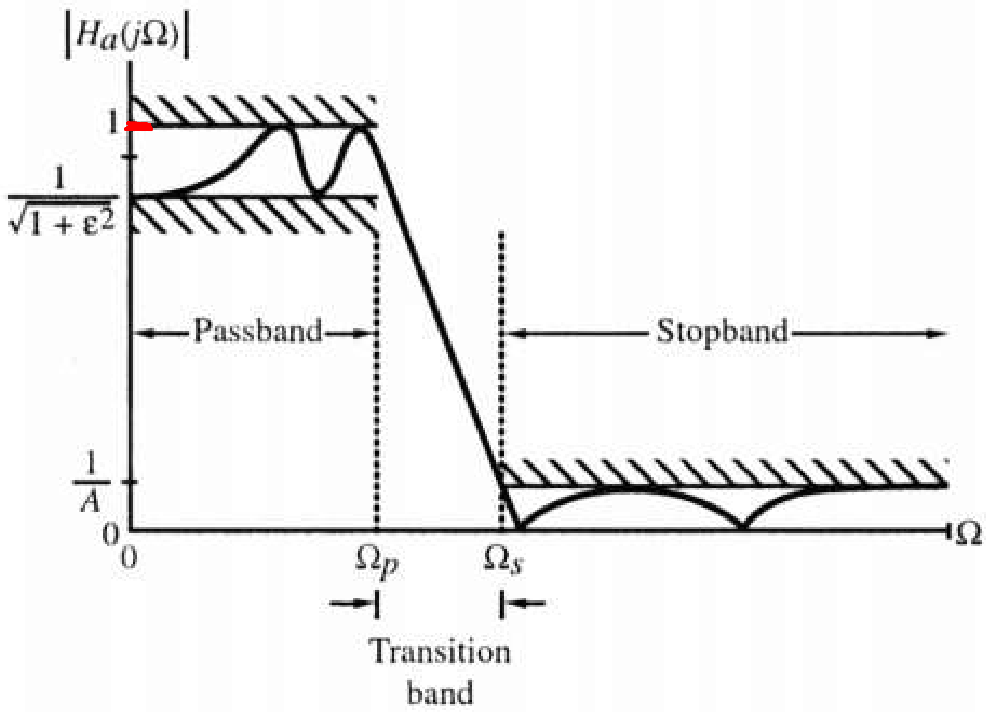
$$\alpha_p = -20\log_{10}(1-\delta_p) \text{ dB}$$

- Minimum stopband attenuation

$$\alpha_s = -20\log_{10}(\delta_s) \text{ dB}$$

Analogue Filter Specifications

- Magnitude specification may alternatively be given in a normalized form:
  - The maximum value of the magnitude in the passband is assumed to be unity



- Maximum passband deviation, given by the minimum value of the magnitude in the passband:

$$\frac{1}{\sqrt{1+\varepsilon^2}} = 1 - \delta_p$$

- Maximum stopband magnitude:

$$\frac{1}{A} = \delta_s$$

Ω_p – passband edge frequency
Ω_s – stopband edge frequency

Analogue Filter Specifications

- Many design formulas for analog filters can be streamlined through two new parameters:

LPF

  - Selectivity Factor (Transition ratio): $r = \frac{\Omega_p}{\Omega_s}, 0 < r \leq 1$
    - Note that for an ideal filter, there is no transition band => $\Omega_p = \Omega_s$ => $\boldsymbol{r}=1$
    - Selectivity is then a measure of how far the edge frequencies are from each other: the closer they are (the smaller the transition band) the higher the selectivity
  - Discrimination Factor: $d = \boxed{\left[\frac{\frac{1}{(1-\delta_p)^2}-1}{\left(\frac{1}{\delta_s}\right)^2-1}\right]^{1/2}} = \frac{\varepsilon}{\sqrt{A^2-1}}, 0 \leq d \ll 1$
    - For $\delta_p=0$ or $\delta_s=0$ (no pass or stop band ripple) => d=0
      Hence, for an ideal filter, the discrimination factor is 0.
    - Discrimination factor is then a measure of ripple in the filter characteristic. Less ripple => small discrimination

Butterworth Filter Design

- Butterworth filter design: filter order N and filter cutoff frequency Ω_c:
  - To determine the filter order N:

$$\left|H(\Omega)^2\right|_{\Omega=\Omega_p} = \frac{1}{1+(\Omega_p/\Omega_c)^{2N}} = (1-\delta_p)^2$$

$$\left|H(\Omega)^2\right|_{\Omega=\Omega_s} = \frac{1}{1+(\Omega_s/\Omega_c)^{2N}} = (\delta_s)^2$$

$$\Rightarrow \quad \boxed{N = \frac{1}{2}\,\frac{\log\left(\frac{1}{(1-\delta_p)^2}-1\right)-\log\left(\frac{1}{\delta_s^2}-1\right)}{\log\left(\frac{\Omega_p}{\Omega_s}\right)} = \frac{\log(d)}{\log(r)}}$$

  - Once N is determined, we find the value of $\Omega=\Omega_c$ for which $H(\Omega)$ drops 3 dB

$$\boxed{\Omega_c = \frac{\Omega_p}{\left(\frac{1}{(1-\delta_p)^2}-1\right)^{1/2N}}}$$

$\Omega_p \le \Omega_c \le \Omega_s$

  - Once $H(\Omega)$ is obtained, it is usually written in partial fraction form to obtain zeros and poles.

$$\boxed{H(s) = \frac{K}{(s-s_1)(s-s_2)\cdots(s-s_{2N-1})}}$$

$$\left.\begin{matrix} N \\ \Omega_s \end{matrix}\right\} \rightarrow \underline{H_a(s)}$$

Butterworth Filter Design

N

- Example: Determine the lowest order of a Butterworth lowpass filter with a 1-dB cutoff frequency at 1 kHz and a minimum attenuation of 40 dB at 5 kHz.

Ω_p Ω_s

$$\alpha_p = 1\,dB = -20\log_{10}(1-\delta_p) \longrightarrow \underline{1-\delta_p = 0.89}$$

$$\alpha_s = 40\,dB = -20\log_{10}\delta_s \longrightarrow \underline{\delta_s = 0.01}$$

1. $r = \dfrac{\Omega_p}{\Omega_s} = 0.2$

2. $d = \sqrt{\dfrac{\left(\frac{1}{1-\delta_p}\right)^2 - 1}{\left(\frac{1}{\delta_s}\right)^2 - 1}} = 0.005$

$$\Rightarrow N = \frac{\log d}{\log r} = 3.3 \Rightarrow N = 4$$

Designing a Digital LPF using Butterworth Approximation

- 1. Prewarp ω_p, ω_s to find their analog equivalents Ω_p, Ω_s;
- 2. Design the analog filter: Determine N and Ω_c:
  - a) From δ_p, δ_s, Ω_p and Ω_s obtain the order of the filter N
    - Note that the order N must be integer, so the value obtained from this expression must be rounded up to exceed the specifications
  - Use N, δ_p, and Ω_p to calculate the 3dB cutoff frequency Ωc
  - Determine the corresponding H(s) and its poles
- 3. Apply bilinear transformation to obtain H(z)

$\frac{\Omega_p}{\Omega_s} \to r$

$\delta_p, \delta_s \to d$

$\to N$

Ω_c

$H_a(s)$

$\xleftarrow{BT}$

Example

3. $H_a(s) \longrightarrow H(z)$

- Design an analogue IIR lowpass filter using the Butterworth approximation that meets the following specs:

$$f_p = 2\text{kHz},\ f_s = 3\text{kHz},\ \alpha_p < 2\text{dB}\ \alpha_s > 50\text{dB},\ f_T = 10\text{kHz}.$$

1. $\Omega_p \leftarrow \omega_p \leftarrow f_p$ $\quad \frac{f_p}{f_T} = \frac{\omega_p}{2\pi} \Rightarrow \omega_p = 0.4\pi$ (rad)

$\Omega = \tan\frac{\omega}{2}$ $(T_s = 2) \longrightarrow \Omega_p = 0.7265$ (rad/s).

$\Omega_s = 1.3764$ ( )

2. Design Butterworth $\alpha_p \longrightarrow 1 - \delta_p = 0.794$ $\quad \alpha_s \rightarrow \delta_s = 0.0032$

$r = \frac{\Omega_p}{\Omega_s} \longrightarrow N = 16$

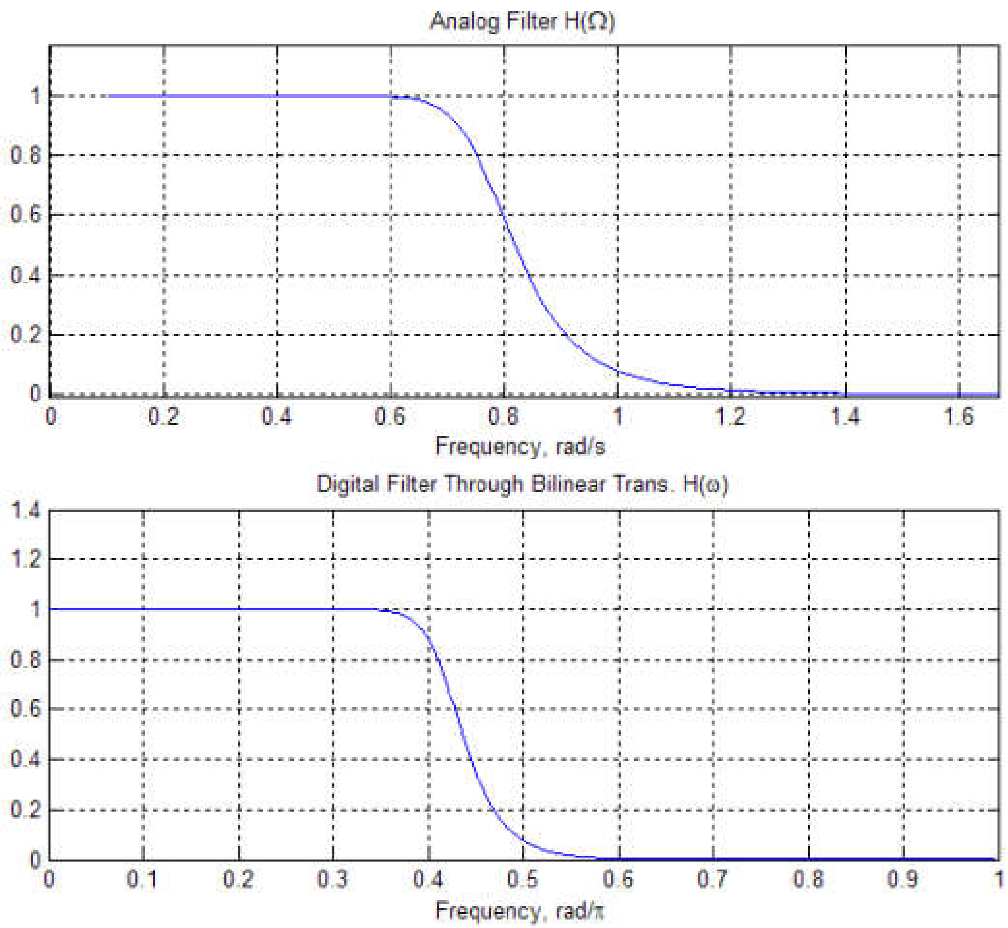
$d = \left(\frac{\delta_p}{\delta_s}\right) \nearrow$

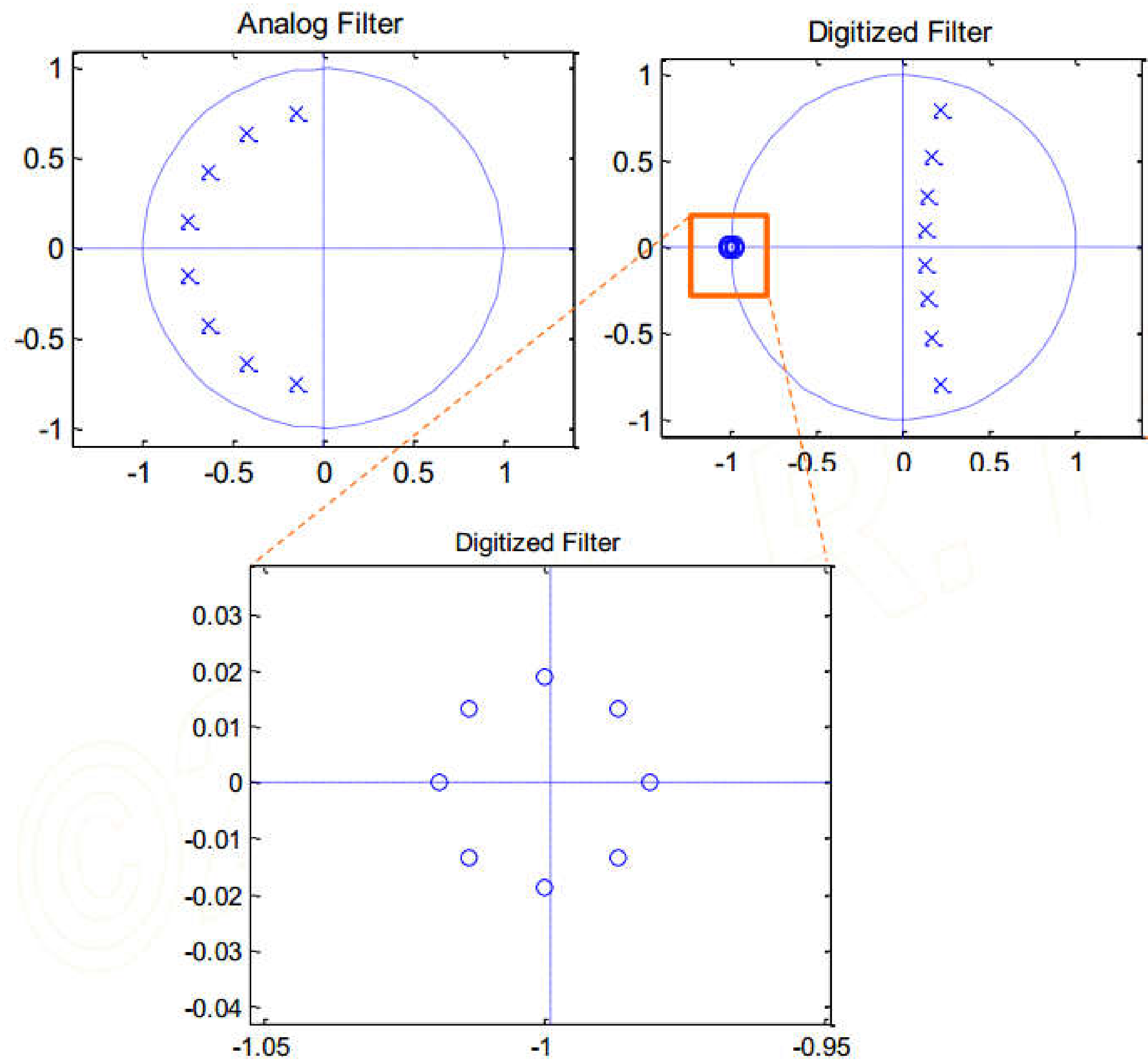
$\downarrow$

$\Omega_c =$ ___ $\Big\} \Rightarrow H_a(s)$

Example

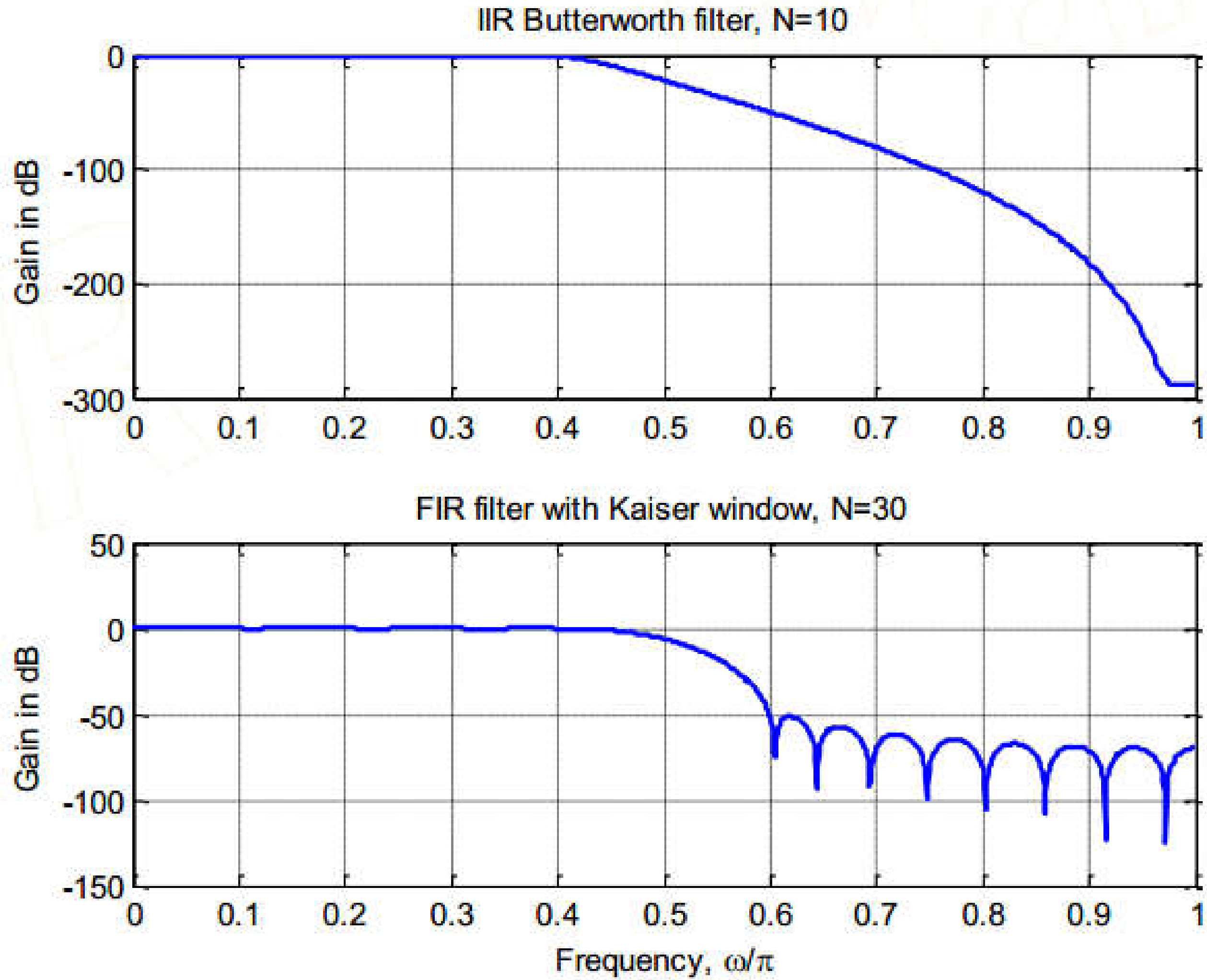
- Results are plotted here:





Compare to FIR Filter

- Let's design an FIR filter with the exact same specs, using a Kaiser window



15_2 Wrap up

- Design Procedure:
  - 1. Prewarp ω_p, ω_s to find their analog equivalents Ω_p, Ω_s;
  - 2. Design the analog filter: Determine N and Ω_c:
    - a) From δ_p, δ_s, Ω_p and Ω_s obtain the order of the filter N
      - Note that the order N must be integer, so the value obtained from this expression must be rounded up to exceed the specifications
    - b) Use N, δ_p, and Ω_p to calculate the 3dB cutoff frequency Ωc
    - c) Determine the corresponding H(s) and its poles
  - 3. Apply bilinear transformation to obtain H(z)

EEE336 Signal Processing and Digital Filtering

Lecture 15 IIR Filters Design

Lect_15_3 Other Analogue Filters

Zhao Wang

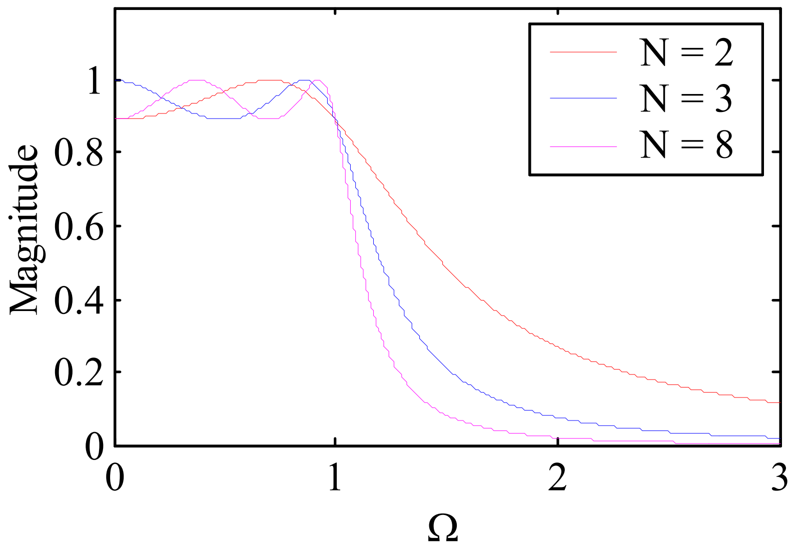
Zhao.wang@xjtlu.edu.cn

Room EE322

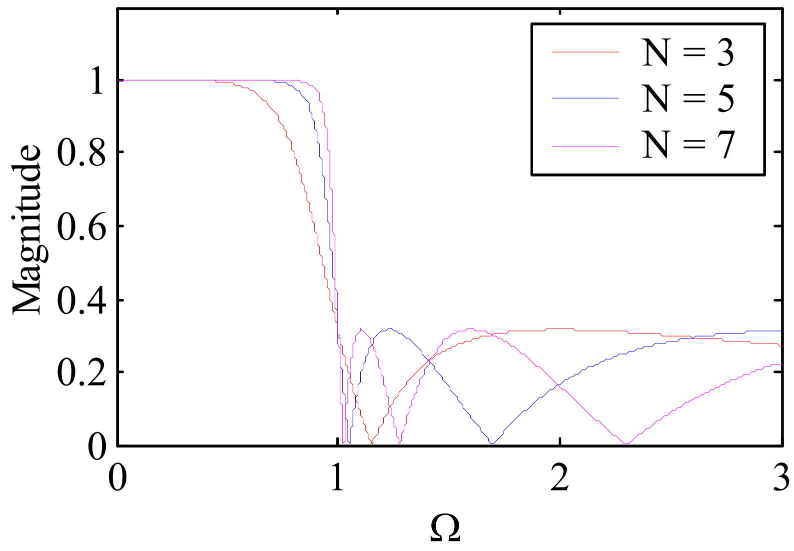
Chebyshev Approximation

- The (almost) flat passband and stopband characteristics of Butterworth filter come at the cost of *wide transition band*.
- Chebyshev filters offer a sharper cut-off than Butterworth filters of the same order, at the expense of ripple in pass or stop band. Two types of Chebyshev filters:
  - Type I has equiripple in the passband, and monotonic behavior in the stopband
  - Type II has equiripple in the stopband, and monotonic behavior in the passband

Type 1 Chebyshev Filter



Type 2 Chebyshev Filter



Chebyshev Approximation

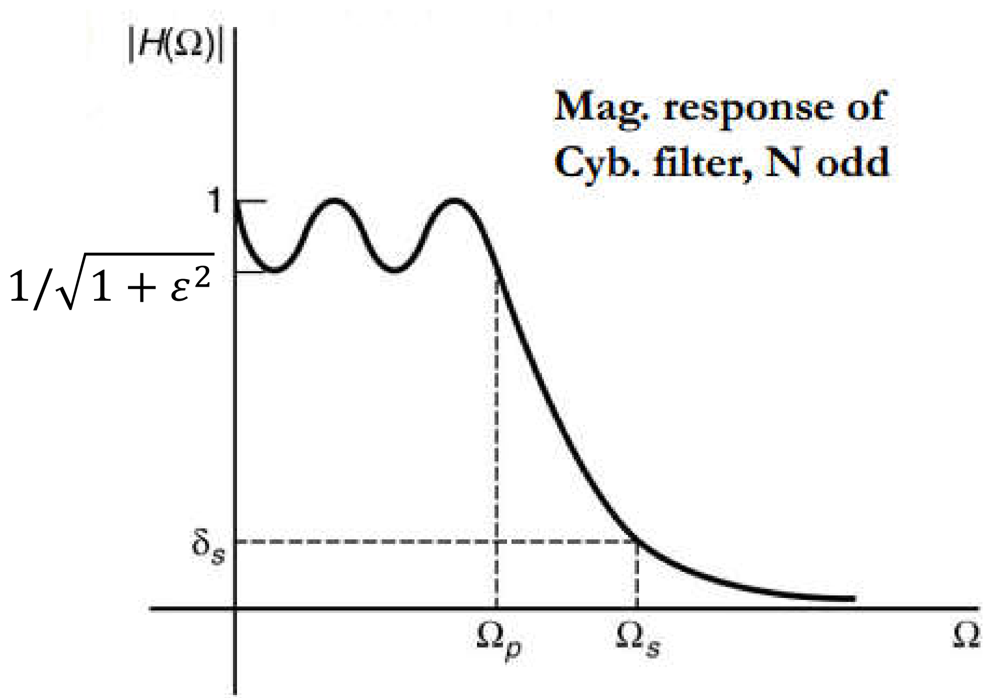
- The magnitude-square response of an N-th order ***analog lowpass Type 1 Chebyshev filter*** is given by

$$\left|H_a(\Omega)\right|^2 = \frac{1}{1+\varepsilon^2 T_N^2(\Omega/\Omega_p)}$$

where $T_N(\Omega)$ is the ***Chebyshev polynomial*** of order N:

$$T_N(\Omega) = \begin{cases} \cos(N\cos^{-1}\Omega), & |\Omega| \le 1 \\ \cosh(N\cosh^{-1}\Omega), & |\Omega| > 1 \end{cases}$$

ε is a user defined parameter that controls ripple amount.



$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$

$$\cosh^{-1}(x) = \ln(x + \sqrt{x^2+1})$$

Xi'an Jiaotong-Liverpool University
西交利物浦大學

Chebyshev Approximation

- When $\Omega = \Omega_s$, at the edge of stopband, the magnitude equals to $\delta_s = 1/A$, then

$$|H_a(j\Omega_s)|^2 = \frac{1}{1+\varepsilon^2 T_N^2(\Omega_s/\Omega_p)} = \frac{1}{A^2} = \delta_s^2$$

- Solving the above we get

$$N = \frac{\cosh^{-1}\left(\sqrt{A^2-1}/\varepsilon\right)}{\cosh^{-1}(\Omega_s/\Omega_p)} = \frac{\cosh^{-1}(1/d)}{\cosh^{-1}(1/r)}$$

  – Order N is chosen as the nearest integer greater than or equal to the above value

$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$

$$\cosh^{-1}(x) = \ln(x + \sqrt{x^2+1})$$

Chebyshev Filter Design

- Designing a Chebyshev filter requires that the appropriate filter order and cutoff frequency be determined so that the filter will satisfy the specs.
  - Given the four parameters: passband and stopband edge frequencies (Ω_p, Ω_s) , and passband and stopband ripples (δ_p , δ_s) , determine the filter order:

$$\boxed{\varepsilon = \sqrt{\frac{1}{(1-\delta_p)^2} - 1}}$$

$$\boxed{N = \frac{\cosh^{-1}\left(\sqrt{(1/\delta_s)^2 - 1}/\varepsilon\right)}{\cosh^{-1}\left(\Omega_s/\Omega_p\right)}}$$

  - No close form formula exists for computing Ωc.
    - Therefore, for Type I, take $\Omega_c = \Omega_p$.
  - Compute the Chebyshev polynomial T_N
  - Determine the poles of the filter => Obtain the analogue filter
  - Apply bilinear transformation to obtain the digital filter

Example

- Determine the lowest order of a Chebyshev LPF with a 1-dB cutoff frequency at 1 kHz and a minimum attenuation of 40 dB at 5 kHz.

Elliptical Approximation

- Elliptical approximation provides much sharper transition band at the expense of equiripple in both bands, and nonlinear behavior in the passband.
  - The square-magnitude response is given by:

$$\left|H_a(j\Omega)\right|^2 = \frac{1}{1+\varepsilon^2 R_N^2(\Omega/\Omega_p)}$$

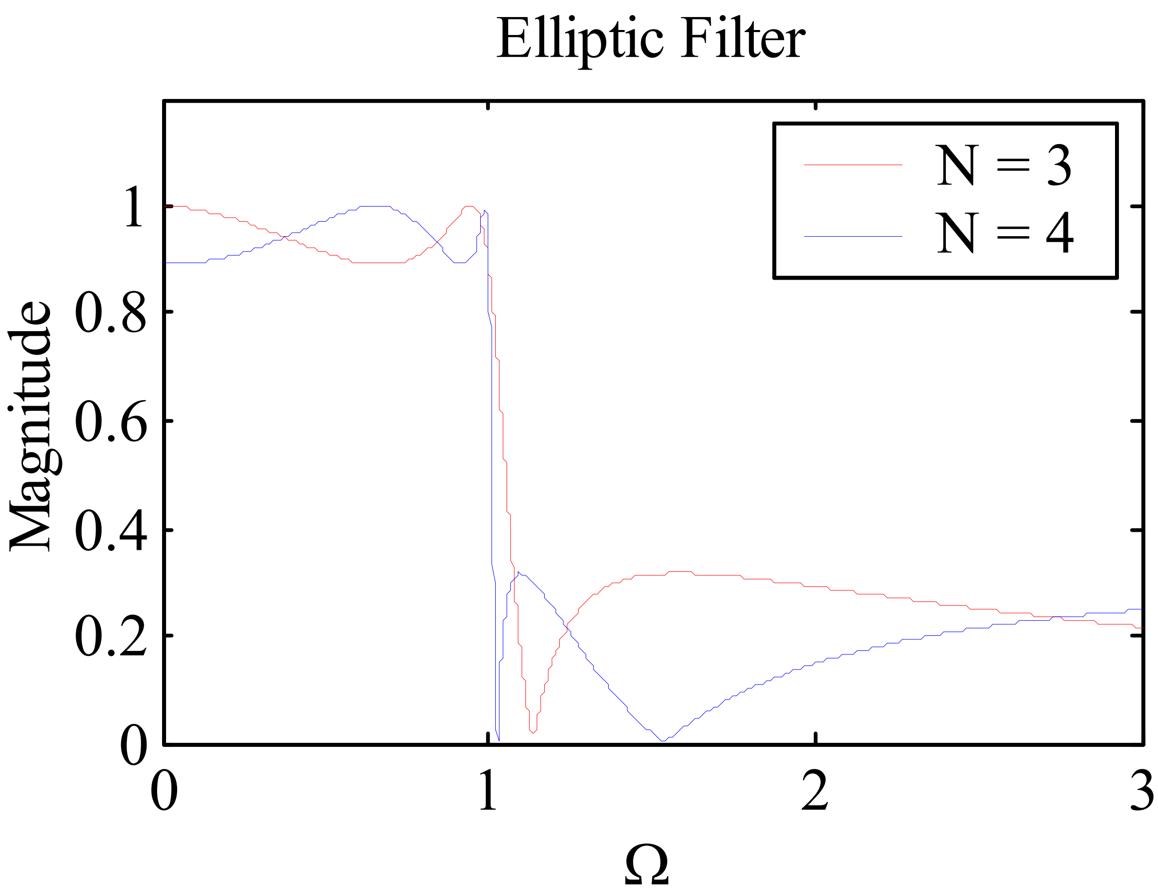
  - where $R_N(\Omega)$ is a rational function of order N satisfying:

$$R_N\left(\frac{1}{\Omega}\right) = \frac{1}{R_N(\Omega)}$$

  - with the roots of its numerator lying in the interval $0 < \Omega < 1$ and the roots of its denominator lying in the interval $1 < \Omega < \infty$.

Elliptical Approximation

- Typical magnitude response plots with $\Omega_p = 1$ are shown below

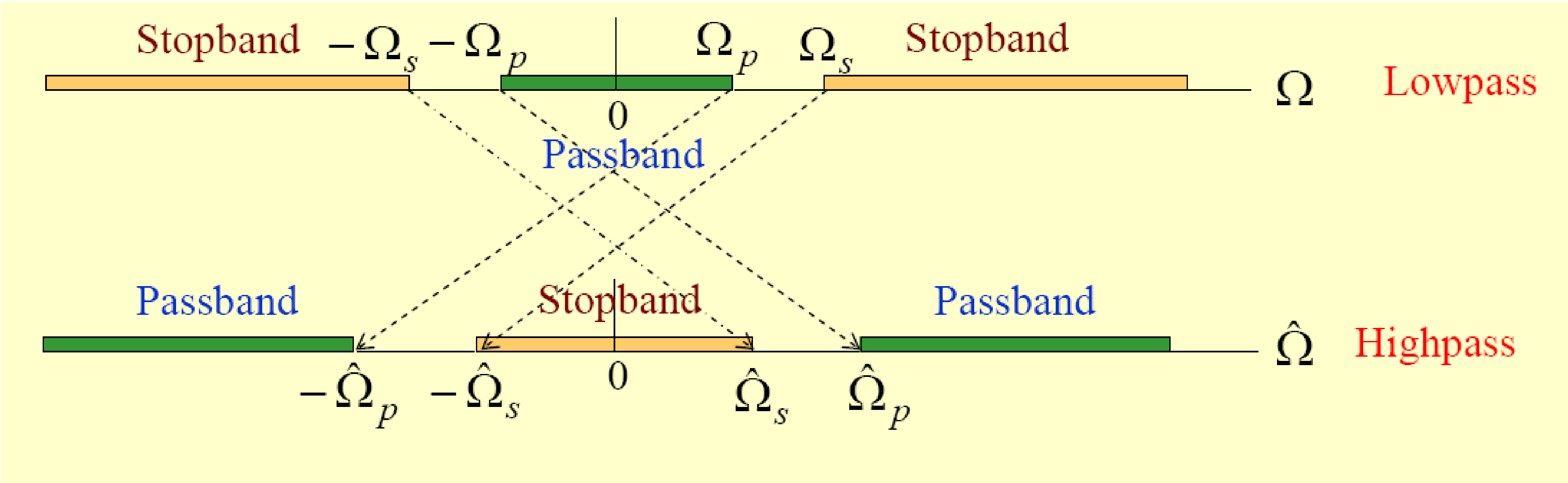


Spectral transformation

- To design an LPF:
  - 1. Prewarp ω_p, ω_s to find their analog equivalents Ω_p, Ω_s;
  - 2. Design the analog filter: Determine N and Ω_c:
  - 3. Apply bilinear transformation to obtain H(z)
- But what about other types of filters?
- => Spectral transformations: Process for converting lowpass filters into highpass, bandpass or bandstop filters.
  - In fact, spectral transformations can be used to convert a LPF into another LPF with a different cutoff frequency.
  - The transformation can be done in either analog or discrete domain.

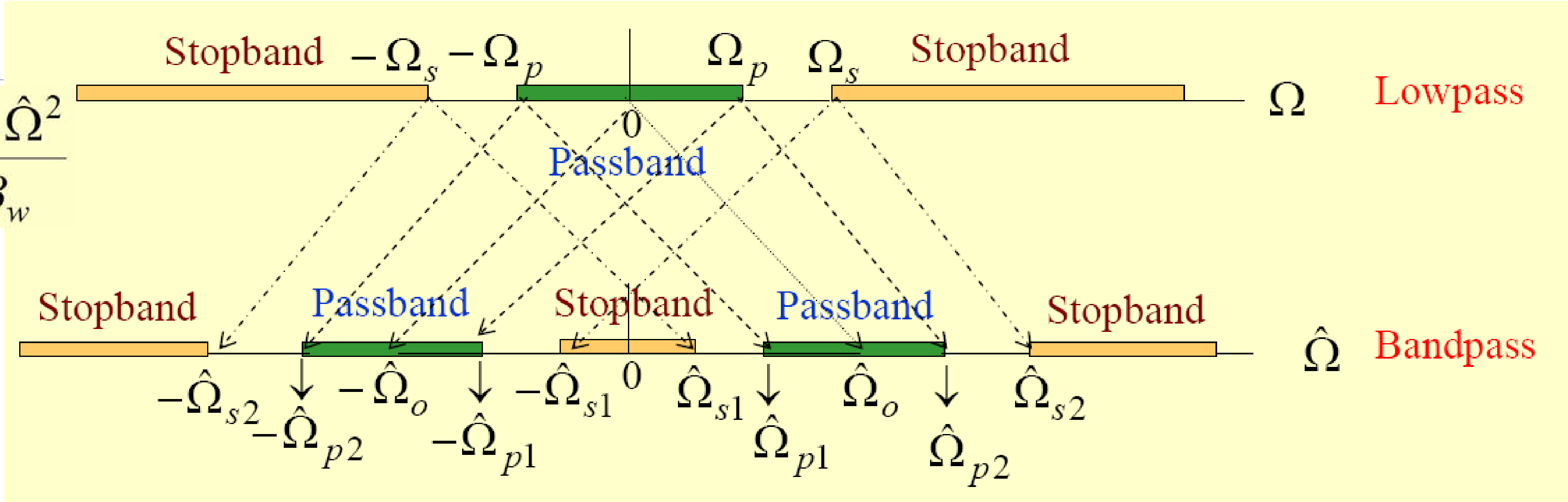
Analogue HPF Design

- Spectral transformation: $s = \dfrac{\Omega_p \widehat{\Omega}_p}{\hat{s}}$
  - where Ω_p is the passband edge frequency of $H_{LP}(s)$ and $\widehat{\Omega}_p$ is the passband edge frequency of $H_{HP}(\hat{s})$
- On the imaginary axis, the transformation is $\Omega = -\dfrac{\Omega_p \widehat{\Omega}_p}{\widehat{\Omega}}$



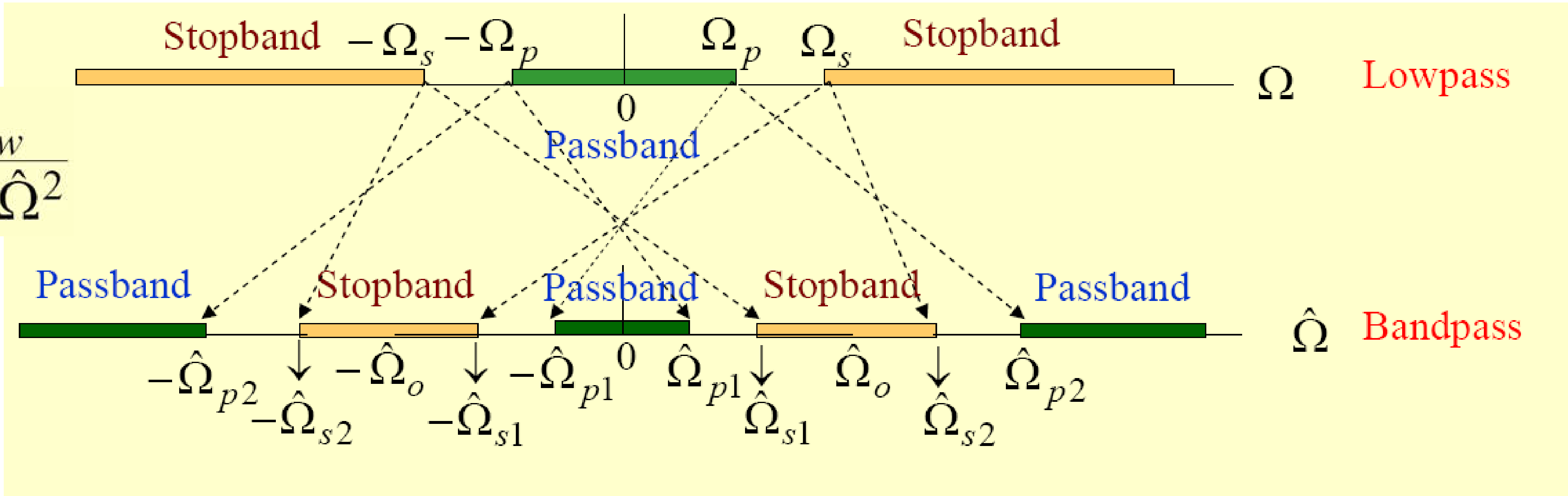
BPF:

$$\Omega = -\Omega_p \frac{\hat{\Omega}_o^2 - \hat{\Omega}^2}{\hat{\Omega} B_w}$$



BSF:

$$\Omega = \Omega_s \frac{\hat{\Omega} B_w}{\hat{\Omega}_o^2 - \hat{\Omega}^2}$$



Xi'an Jiaotong-Liverpool University
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Chapter 15 Summary

- Design IIR Filters:
  - 1. Specification:
    - Frequency: f_p, f_s → ω_p, ω_s → Ω_p, Ω_s
    - Ripples: α_p, α_s → δ_p, δ_s
    - Select the analogue filter type
  - 2. Design the analog filter:
    - a) δ_p, δ_s → discrimination factor d

      Ω_p, Ω_s → selectivity factor r
    - b) d, r → order N

      Use N, δ_p, and Ω_p to calculate the 3dB cutoff frequency Ω_c
    - c) Determine the poles and corresponding H(s)
  - 3. Apply bilinear transformation to obtain H(z)