EEE225 Advanced Electrical Circuits and Electromagnetics

Lecture 6 Electrical Circuits - Frequency Response

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Room EE322

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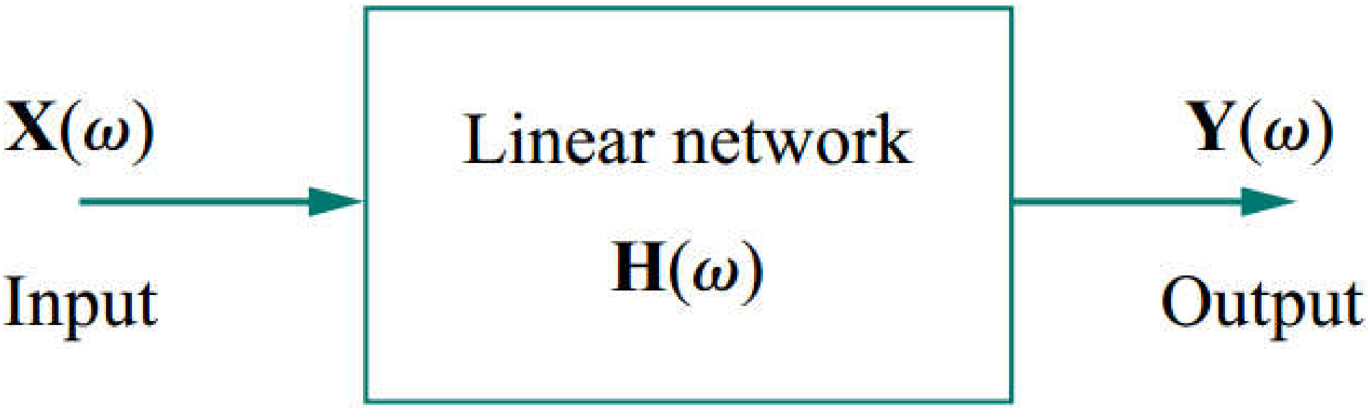
1. Frequency Response

- The ***frequency response*** of a circuit is the variation in its behavior with change in signal frequency.
  - It may also be considered as the variation of the gain and phase with frequency.
  - A circuit's response depends on the types of elements in the circuit, the way the elements are connected, ***and the impedance of the elements***.
  - The careful choice of circuit elements, their values, and their connections to other elements enables us to construct circuits that pass to the output only those input signals that reside in a desired range of frequencies – ***frequency selective circuits***.
- Frequency-selective circuits are also called ***filters***.

1.1 Transfer function

- A transfer function $H(\omega)$ is the frequency-dependent ratio of a phasor output $Y(\omega)$ (an element voltage or current) to a phasor input $X(\omega)$ (source voltage or current).



- Thus, the transfer function is:

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \begin{cases} Voltage\ gain = \dfrac{V_o(\omega)}{V_i(\omega)} \\ Current\ gain = \dfrac{I_o(\omega)}{I_i(\omega)} \end{cases}$$

1.2 The Decibel Scale

- In communication systems, *gain* is measured in *bels*. Historically, the *bel* is used to measure the ratio of two level of power or power gain G, that is,

$$G = Number\ of\ bels = \log_{10}\frac{P_2}{P_1}$$

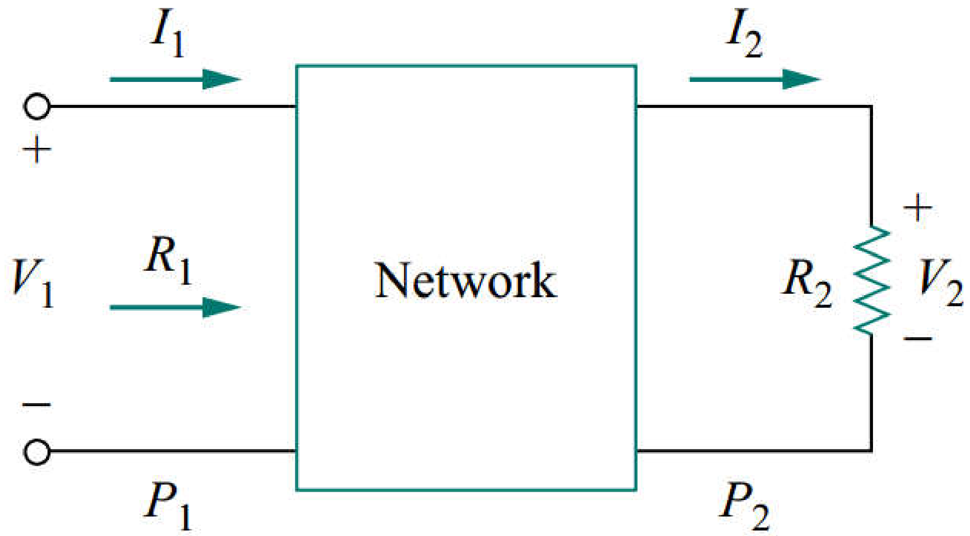
- The unit *decibel* (dB) provides us with a unit of less magnitude. It is 1/10th of a *bel* and is given by

$$G(dB) = 10\log_{10}\frac{P_2}{P_1}$$

- Properties of Logarithms:
  - 1. $\log P_1 P_2 = \log P_1 + \log P_2$
  - 2. $\log P_1/P_2 = \log P_1 - \log P_2$
  - 3. $\log P^n = n \log P$
  - 4. $\log 1 = 0$

1.2 The Decibel Scale

- The gain was originally defined based on power P; alternatively, it can be expressed in terms of voltage or current:



$$G(dB) = 10\log_{10}\frac{P_2}{P_1} = 10\log_{10}\frac{V_2^2/R_2}{V_1^2/R_1}$$

$$= 10\log_{10}\frac{V_2^2}{V_1^2} - 10\log_{10}\frac{R_2}{R_1}$$

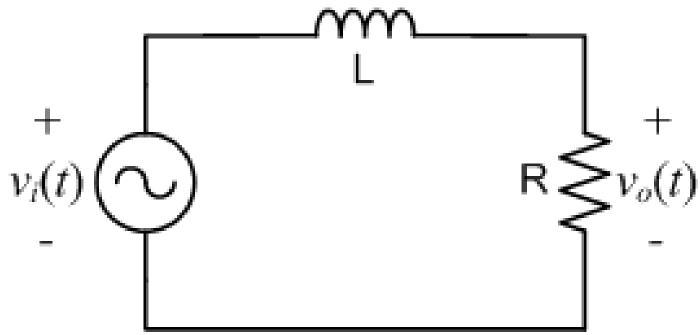
$$= 20\log_{10}\frac{V_2}{V_1} - 10\log_{10}\frac{R_2}{R_1}$$

- For the case when $R_1 = R_2$, a condition that is often assumed when comparing voltage levels, then:

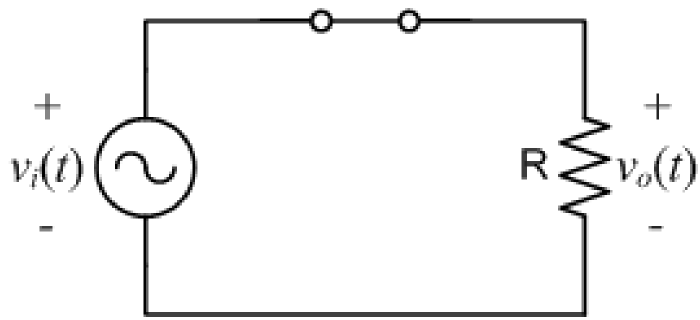
$$G(dB) = 20\log_{10}\frac{V_2}{V_1}$$

2.1 Series RL circuit - Qualitative Analysis

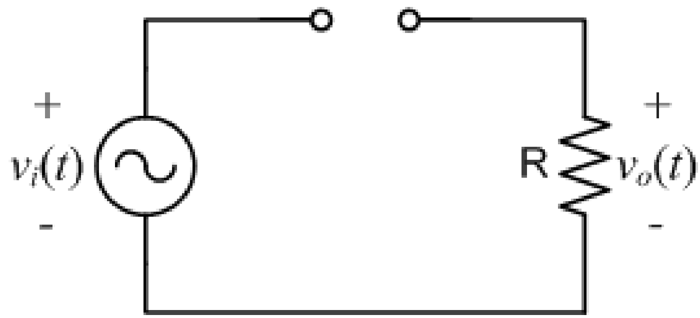
- Input: sin voltage source $v_i(t)$;
- Output: voltage on resistor $v_o(t)$;
- The frequency of the source starts very low and increases gradually.



- The impedance of the inductor is $jX_L = j\omega L$.
  - At low frequency ($X_L = \omega L << R$), the inductor's impedance is very small compared with the resistor's impedance, and the inductor effectively functions as a short circuit.

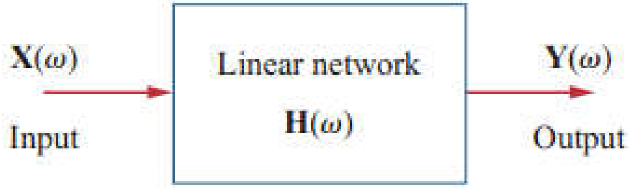


  - At high frequencies ($X_L = \omega L >> R$), the X_L is very large compared with R, and the inductor thus functions as an open circuit, effectively blocking the flow of current in the circuit.



2.1 Series RL circuit - Qualitative Analysis

- Consider the circuit:

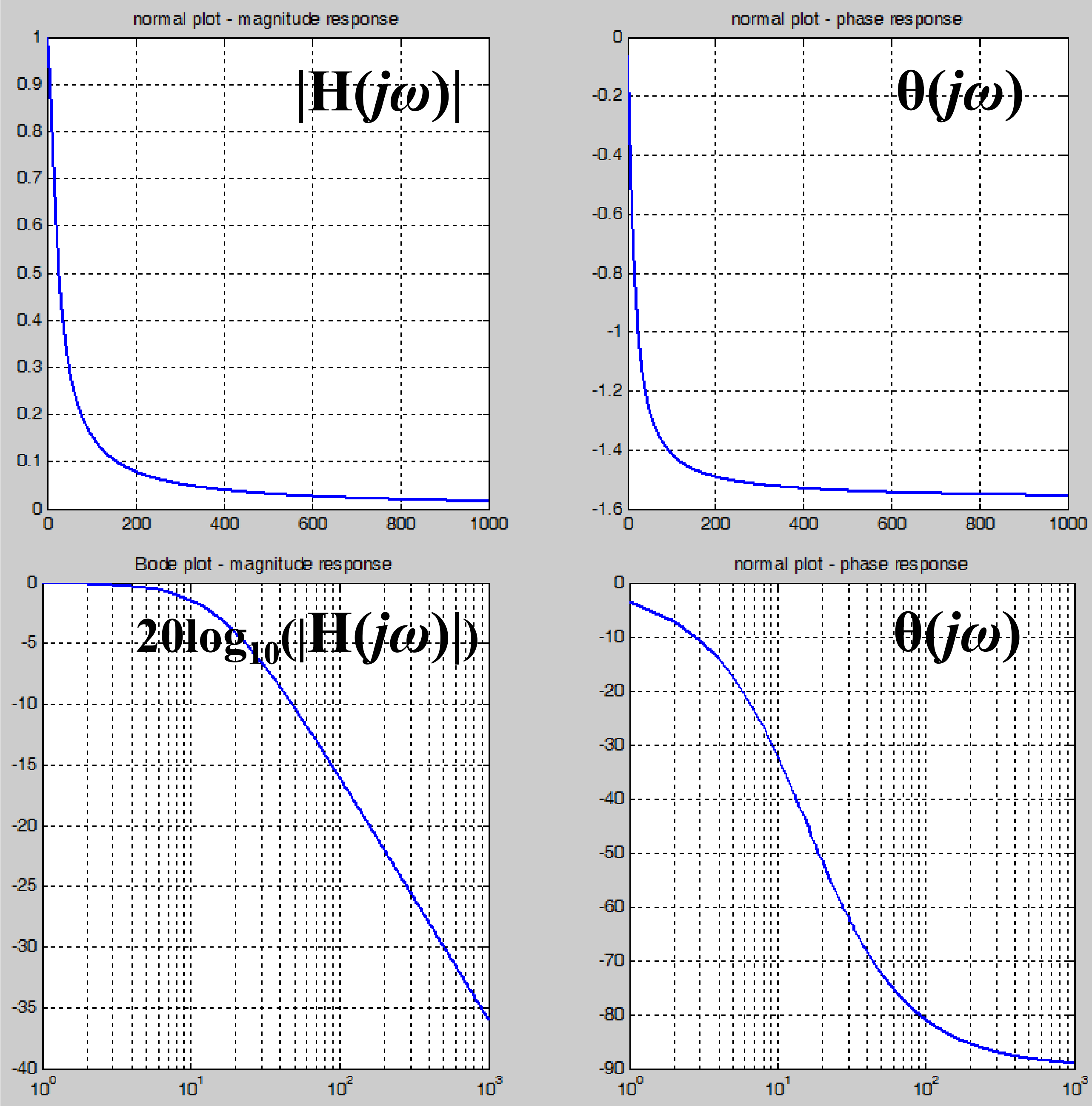


- Frequency response **H(*jω*)** is:

$$H(j\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{v_o(\omega)}{v_i(\omega)}$$

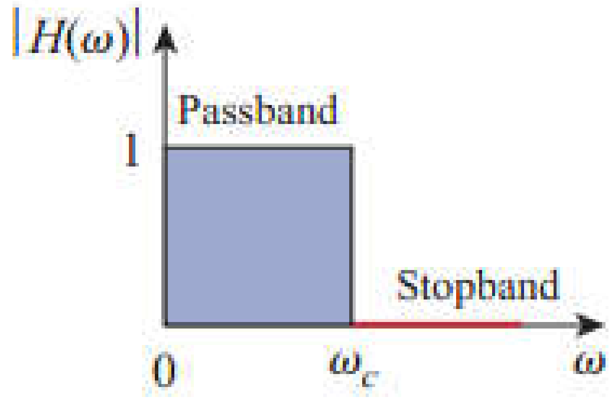
- The frequency range required in frequency response is often so wide that it is inconvenient to use a linear scale for the frequency axis. => log axis.

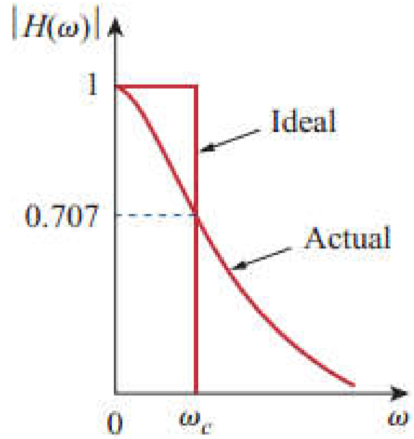
- ***Bode plot:***
  - **$20\log_{10}(|H(j\omega)|)$ VS *f* in log;**
  - **$\theta(j\omega)$ in degree VS *f* in log.**



2.1 Ideal filters & Cut-off frequency

- A lowpass filter passes low frequencies and stops high frequencies.
  - Ideally, frequency components less than ω_c can pass, and frequency components greater than ω_c cannot pass.
  - ω_c is called the cut-off frequency.
  - Actually, an RL filter (or any other practical filters) always changes gradually from pass band to the stop band.
  - ω_c is defined as the frequency for which the transfer function magnitude is decreased by the factor $1/\sqrt{2}$ from its maximum value:

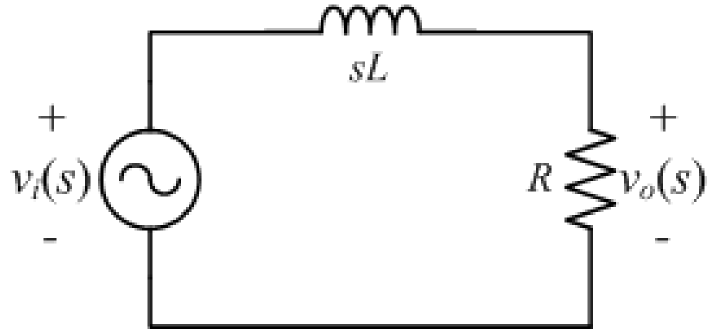




$$|H(j\omega_c)| = \frac{1}{\sqrt{2}} H_{max}$$

2.1 Series RL circuit - Quantitative Analysis

- We start to analyse the circuit in frequency domain.



$$v_i(j\omega) = v_o(j\omega) + j\omega L \frac{v_o(j\omega)}{R}$$

$$\Rightarrow \frac{v_o(\omega)}{v_i(\omega)} = \frac{1}{1 + j\omega\left(\frac{L}{R}\right)} = \frac{R/L}{j\omega + R/L}$$

- To study the frequency response, we make the substitution $s = j\omega$.

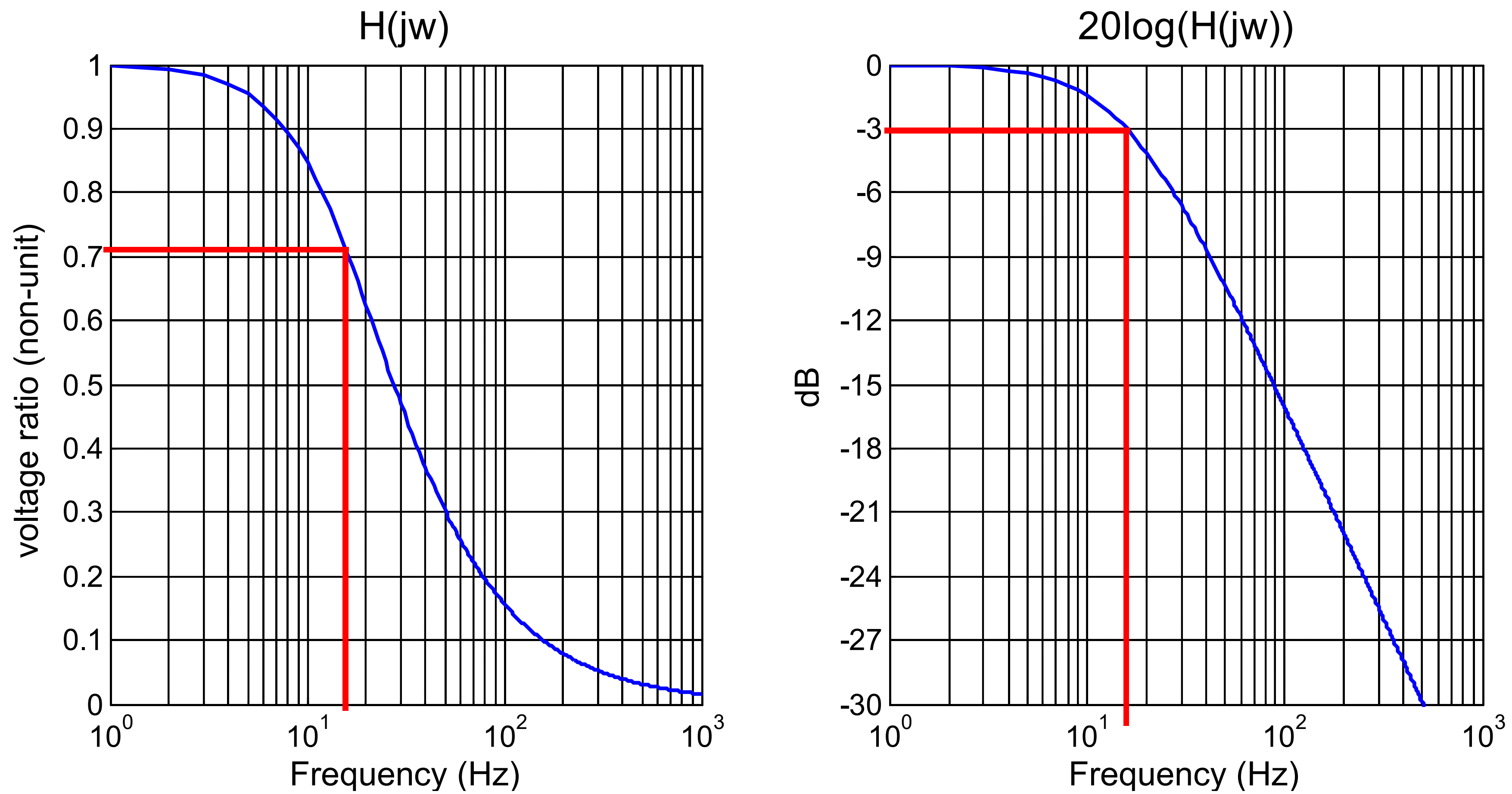
$$\Rightarrow H(j\omega) = \frac{v_o(\omega)}{v_i(\omega)} = \frac{R/L}{j\omega + R/L} = \begin{cases} |H(j\omega)| = \dfrac{R/L}{\sqrt{\omega^2 + (R/L)^2}} \\ \theta(j\omega) = -\arctan(\dfrac{\omega L}{R}) \end{cases}$$

- The max value is at $\omega = 0$, so $|H_{max}| = |H(j0)| = 1$;
- So the cut-off frequency can be solved by:

$$|H(j\omega_c)| = \frac{1}{\sqrt{2}} = \frac{R/L}{\sqrt{\omega^2 + (R/L)^2}} \Rightarrow \omega_c = \frac{R}{L}$$

2.1 Cut-off frequency – 3dB frequency

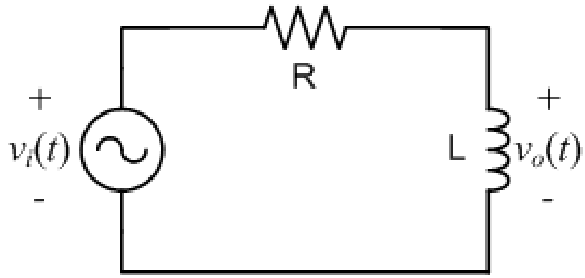
$$|H(j\omega_c)| = \frac{1}{\sqrt{2}} \Rightarrow 20log_{10}(|H(j\omega_c)|) = 20log_{10}\left(\frac{1}{\sqrt{2}}\right) = -3dB$$



$$f_c = \frac{\omega_c}{2\pi} = \frac{R}{2\pi L}$$

2.2 High pass series RL circuit

- In this case, same components, same input, but the output is changed to the voltage on inductor $v_o(t)$.



- Solve for the frequency response $H(j\omega)$:

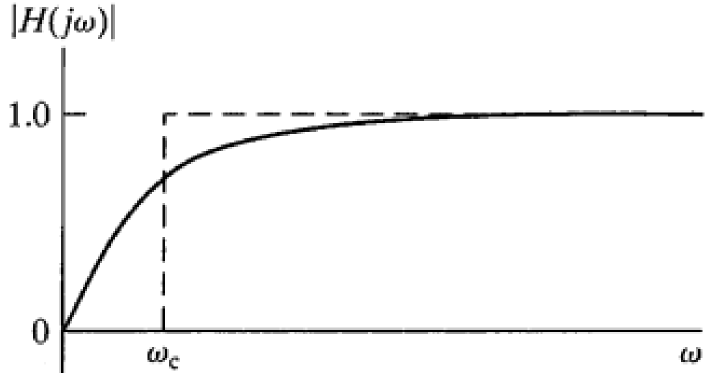
$$v_i(\omega) = v_o(\omega) + R\frac{v_o(\omega)}{\omega L} \Rightarrow \frac{v_o(\omega)}{v_i(\omega)} = \frac{1}{1 + R/j\omega L} = \frac{j\omega}{j\omega + R/L}$$

$$\Rightarrow H(j\omega) = \frac{v_o(\omega)}{v_i(\omega)} = \frac{j\omega}{j\omega + R/L} \quad \Rightarrow |H(j\omega)| = \frac{\omega}{\sqrt{\omega^2 + (R/L)^2}}$$

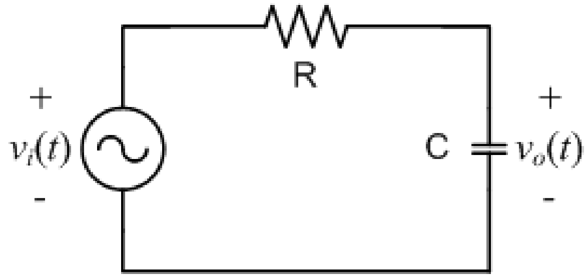
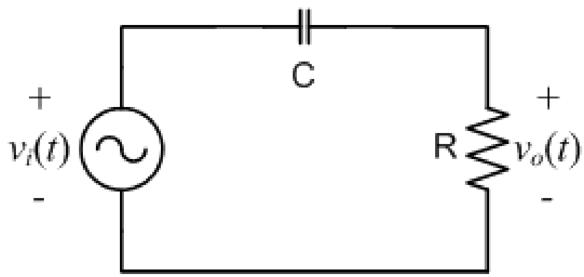
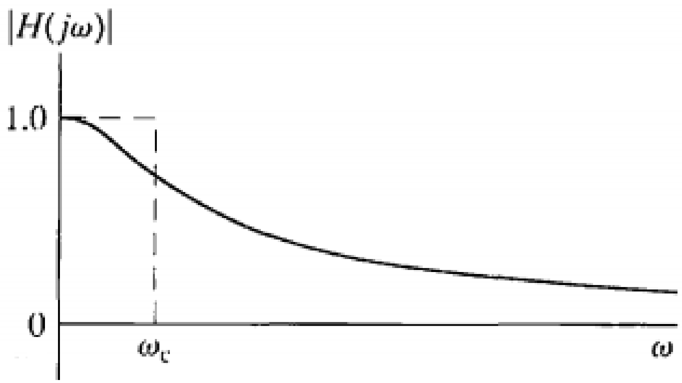
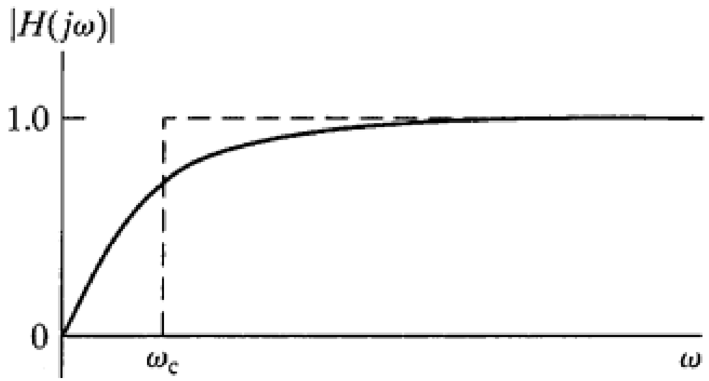
- The cut-off frequency is solved:

$$\frac{1}{\sqrt{2}} = \frac{\omega_c}{\sqrt{{\omega_c}^2 + (R/L)^2}} \quad \Rightarrow \omega_c = \frac{R}{L}$$

High pass filter !

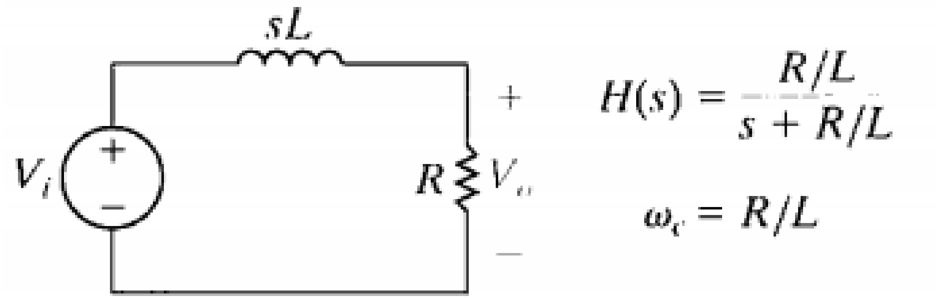


2.2 Series RC circuits

RC circuits		
Frequency response (magnitude)	$\lvert H(j\omega)\rvert = \dfrac{1/RC}{\sqrt{\omega^2 + \left(1/RC\right)^2}}$	$\lvert H(j\omega)\rvert = \dfrac{\omega}{\sqrt{\omega^2 + \left(1/RC\right)^2}}$
Cut-off frequency	$\omega_c = \dfrac{1}{RC}$	$\omega_c = \dfrac{1}{RC}$
Frequency response plot (magnitude)		

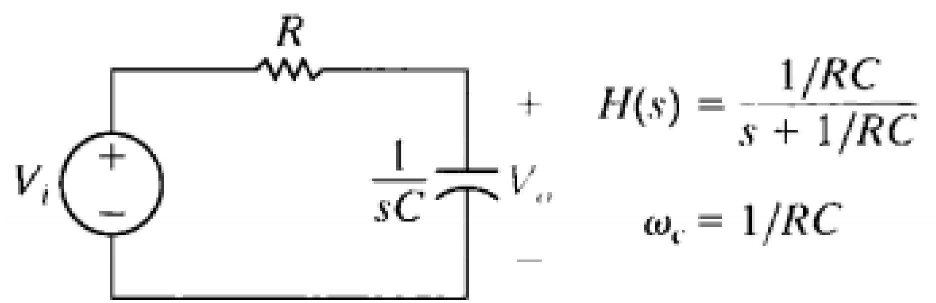
Frequency Domain and the Time Domain

- In time domain, for a first order RL or RC circuit, an important parameter is the time constant $\boldsymbol{\tau}$ that characterizes the shape of the time response.



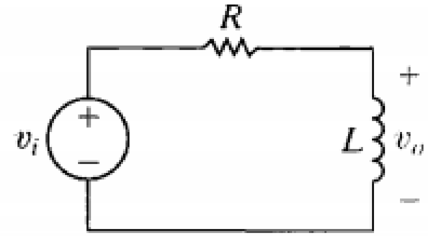
- Compare the time constants to the cut-off frequencies for these circuits and notice that:

$$\tau = \frac{1}{\omega_c}$$



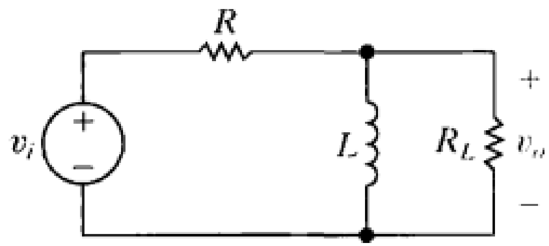
Example 1

- An RL circuit is shown in figure (A).
  - (a) Derive an expression for the circuit's transfer function;
  - (b) Use the result from (a) to determine an equation for the cut-off frequency in the series RL circuit;
  - (c) With R = 500 Ω, choose values for L that will yield a high-pass filter with a cut-off frequency of 15 kHz.



(A)

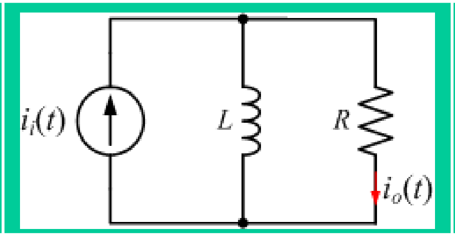
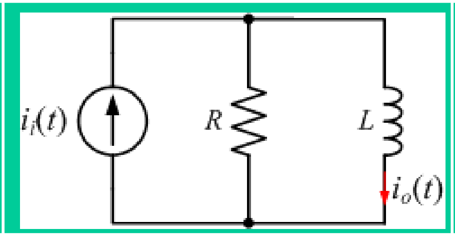
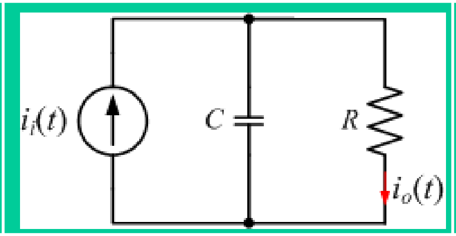
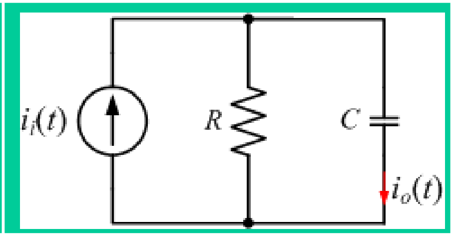
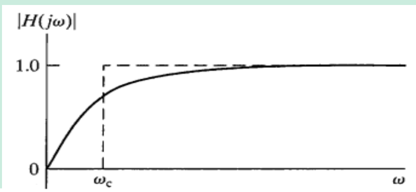
- Placing a load resistor in parallel with the inductor in the filter as shown in figure (B).
  - (d) Determine the transfer function for the new circuit;
  - (e) Sketch the magnitude plot for the loaded RL high-pass filter, using the values for R and L from (c) and letting $R_L = R$. On the same graph, sketch the magnitude plot for the unloaded RL filter of (c).



(B)

Practice 1: First order parallel circuits

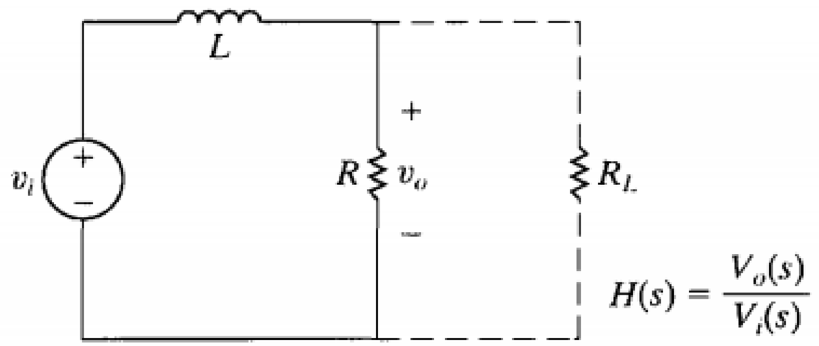
- Try to fill this table by yourselves:

Circuit				
KCL Equation in s-domain	$I_i = I_o + \frac{I_o R}{j\omega L}$			
H(s)	$H(s) = \frac{j\omega}{j\omega + R/L}$			
$\lvert H(j\omega) \rvert$	$\lvert H(j\omega)\rvert = \frac{\omega}{\sqrt{\omega^2 + (R/L)^2}}$			
ω_c	$\omega_c = \frac{R}{L}$			
Plot				

Quiz

- 1. A voltage source supplies a signal of constant amplitude, from 0 to 40 kHz, to an RC lowpass filter. A load resistor, connected in parallel across the capacitor, experiences the maximum voltage at:
  - (a) dc; (b) 10 kHz; (c) 20 kHz; (d) 40 kHz.

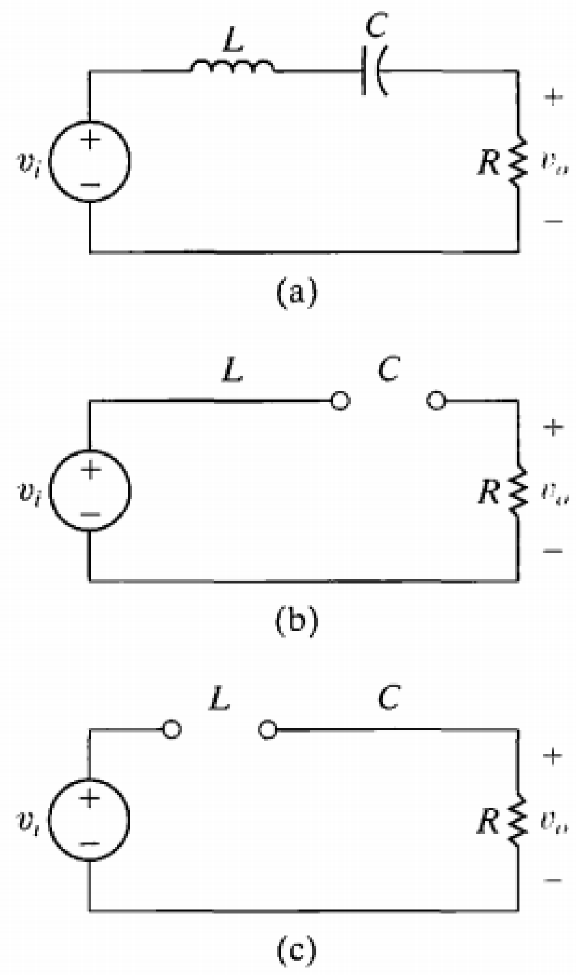
- 2. Add a load resistor R_L to the RL filter shown in the figure. Are the cut-off frequencies different? Are the peak pass-band gains different?
  - (a) Different, same; (b) Same, same;
  - (c) Different, different; (d) Same, Different.



3.1 Series RLC circuit - Qualitative Analysis

- Changes to the source frequency result in changes to the impedance of the capacitor and the inductor

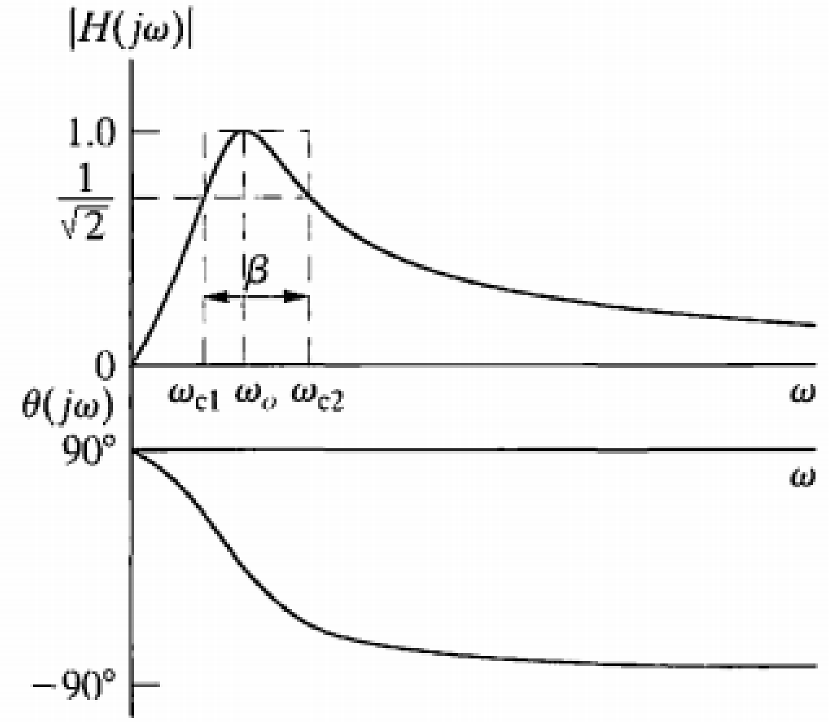
$$Z = R + j(X_L + X_C) \begin{cases} jX_L = j\omega L \\ jX_C = \dfrac{1}{j\omega C} = -j\dfrac{1}{\omega C} \end{cases}$$



  – At ω = 0, the capacitor behaves like an open circuit, and the inductor behaves like a short circuit as shown in (b).
  – At ω = ∞, the capacitor behaves like a short circuit, and the inductor behaves like an open circuit as shown in (c).
- But what happens in the frequency region between ω = 0 and ω = ∞?

3.1 Series RLC circuit - Qualitative Analysis

- Between $\omega = 0$ and $\omega = \infty$, both the capacitor and the inductor have finite impedances.
  – In this region, voltage supplied by the source will drop across both the inductor and the capacitor, but some voltage will reach the resistor.
- Since $X = X_L + X_C = \omega L - \dfrac{1}{\omega C}$
  – At some frequency, the impedance of the capacitor and the impedance of the inductor have equal magnitudes and opposite signs => they cancel out, causing the output voltage to equal the source voltage.
  – ***Resonance!***



Resonance

- Resonance occurs in any system that has a complex conjugate pair of poles; it is the cause of oscillations of stored energy from one form to another.
  - It is the phenomenon that allows frequency discrimination in communications networks.

Resonance is a condition in an RLC circuit in which the capacitive and inductive reactances are equal in magnitude, thereby resulting in a purely resistive impedance.

- Resonant circuits (series or parallel) are useful for constructing filters, as their transfer functions can be highly frequency selective.
  - They are used in many applications such as selecting the desired stations in radio and TV receivers.

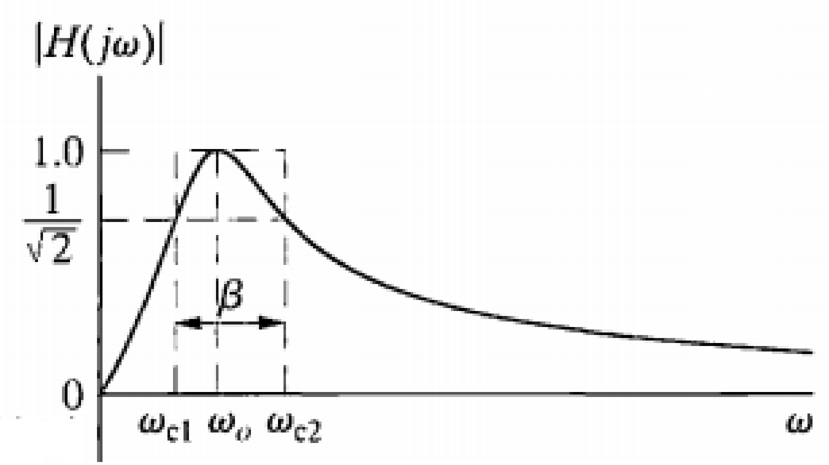
Some definitions

- Some definitions useful in the frequency response analysis:
  - Pass-band & stop-band;
  - Cut-off frequencies ω_{c1} and ω_{c2}: half-power (3dB) frequencies.
  - Center frequency ω_0, defined as the frequency for which a circuit's transfer function is purely real. It is the frequency where X_L and X_C cancelled, and the circuit is in *resonance*.

$$\omega_0 = \sqrt{\omega_{c1}\omega_{c2}}$$

  - Bandwidth β is the width of the pass-band

$$\beta = \omega_{c2} - \omega_{c1}$$



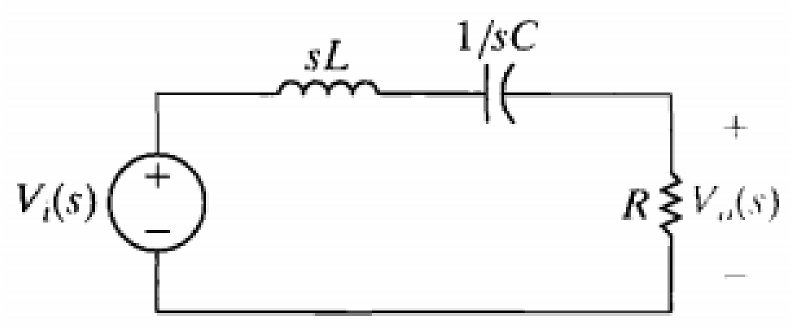
Series RLC circuit – Quantitative Analysis

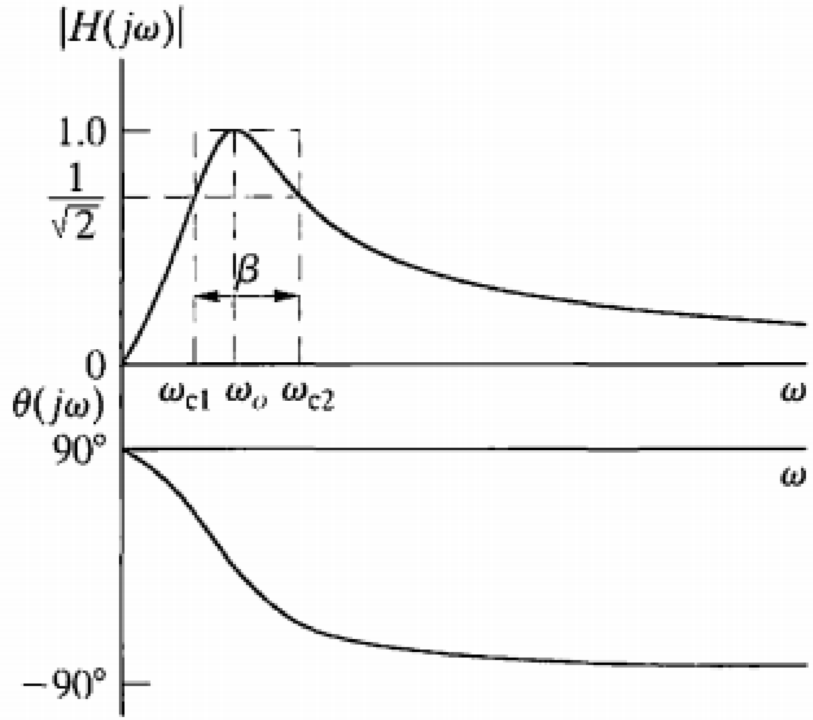
- Analyse in s-domain

$$V_i = V_o + \frac{V_o}{R}(j\omega L + \frac{1}{j\omega C})$$

$$\Rightarrow \frac{V_o}{V_i} = H(\omega) = \frac{j\omega\left(\frac{R}{L}\right)}{(j\omega)^2 + j\omega\left(\frac{R}{L}\right) + (\frac{1}{LC})}$$

$$\Rightarrow \begin{cases} |H(j\omega)| = \dfrac{\left(\frac{\omega R}{L}\right)}{\sqrt{\left[\left(\frac{1}{LC}\right) - \omega^2\right]^2 + (\frac{\omega R}{L})^2}} \\ \theta(j\omega) = 90^\circ - arctan\left[\dfrac{\frac{\omega R}{L}}{\left(\frac{1}{LC}\right) - \omega^2}\right] \end{cases}$$





Series RLC circuit – Quantitative Analysis

- The centre frequency, ω_0, is defined as the frequency for which the circuit's transfer function is purely real.

$$j\left(\omega_0 L - \frac{1}{\omega_0 C}\right) = 0 \;\Rightarrow\; \omega_0 = \frac{1}{\sqrt{LC}}$$

- The cut-off frequencies, ω_{c1} and ω_{c2}.
  - At the cut-off frequencies, the magnitude of the transfer function is $(1/\sqrt{2})H_{max}$, where $H_{max} = H(j\omega_0) = 1$.

$$\frac{1}{\sqrt{2}} = \frac{\left(\frac{\omega_c R}{L}\right)}{\sqrt{\left[\left(\frac{1}{LC}\right) - \omega_c^{\,2}\right]^2 + \left(\frac{\omega_c R}{L}\right)^2}} \;\Rightarrow\; \omega_c^{\,2} L \pm \omega_c R - \frac{1}{C} = 0$$

$$\Rightarrow\; \omega_{c1} = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \left(\frac{1}{LC}\right)},\; \omega_{c2} = \frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \left(\frac{1}{LC}\right)}$$

Series RLC circuit – Quantitative Analysis

- Bandwidth β:

$$\beta = \omega_{c2} - \omega_{c1} = \left(\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \left(\frac{1}{LC}\right)} \right) - \left(-\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + (\frac{1}{LC})} \right)$$

$$= \frac{R}{L}$$

- The quality factor Q is defined as the ratio of centre frequency to bandwidth, measuring the "sharpness" of the resonance.

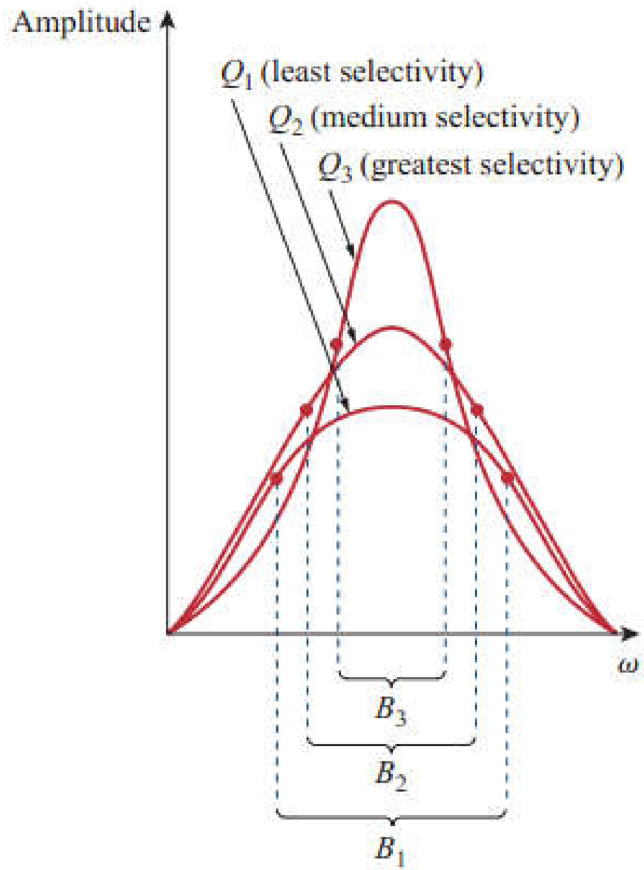
$$Q = \omega_0 / \beta$$

$$= \frac{(1/\sqrt{LC})}{(R/L)} = \sqrt{\frac{L}{CR^2}}$$

3.2 Quality factor Q

- At resonance, the reactive energy in the circuit oscillates between the inductor and the capacitor. The quality factor relates the maximum or peak energy stored to the energy dissipated in the circuit per cycle of oscillation:

$$Q = 2\pi \frac{\text{Peak energy stored in the circuit}}{\text{Energy dissipated by the circuit in one period at resonance}}$$



  - It is also regarded as a measure of the energy storage property of a circuit in relation to its energy dissipation property.

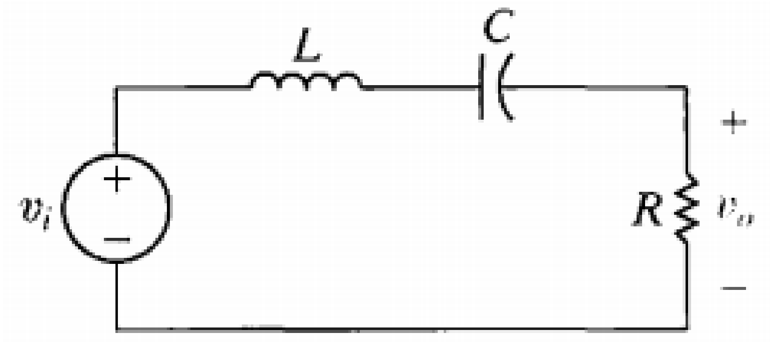
$$Q = 2\pi \frac{\frac{1}{2}LI^2}{\frac{1}{2}I^2R(1/f_0)} = \frac{2\pi f_0 L}{R} = \frac{\omega_0 L}{R} = \frac{1}{\omega_0 CR}$$

  - The relationship between β (or B) and Q is:

$$B = \frac{R}{L} = \frac{\omega_0}{Q}$$

Example 2

- Designing a Bandpass Filter
- A graphic equalizer is an audio amplifier that allows you to select different levels of amplification within different frequency regions. Using the series RLC circuit with 1μF capacitor, choose values for R and L that yield a bandpass circuit able to select inputs within the 1-10 kHz frequency band. Such a circuit might be used in a graphic equalizer to select this frequency band from the larger audio band (generally 0-20 kHz) prior to amplification.



Frequency Domain and the Time Domain

- Consider the series RLC circuit. The natural response of this circuit is characterized by the neper frequency α and the resonant frequency ω_0:

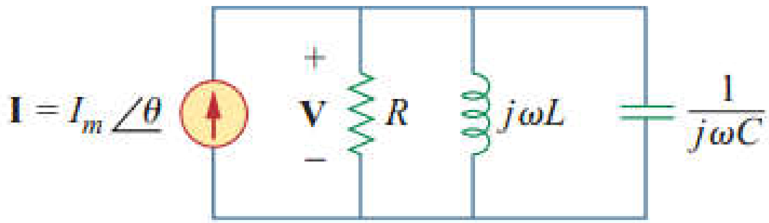
$$\alpha = \frac{R}{2L}\ rad/s \qquad\qquad \omega_0 = \sqrt{\frac{1}{LC}}\ rad/s$$

  - The same parameter ω_0 is used to characterize both the time response and the frequency response. That's why the centre frequency is also called the resonant frequency.
  - The bandwidth and the neper frequency are related by the equation: $\beta = 2\alpha$
- Recall that the natural response of a series RLC circuit may be under-damped, over-damped, or critically damped.
  - The transition from over-damped to under-damped occurs when $\omega_0^2 = \alpha^2$. At which, Q = 1/2.
  - Thus, a circuit whose frequency response contains a sharp peak at ω_0, indicating a high Q and a narrow bandwidth, will have an under-damped natural response.
  - Conversely, a circuit whose frequency response has a broad bandwidth and a low Q will have an over-damped natural response.

3.3 Parallel RLC circuit

- With current source as the input, the output is the voltage on the resistor, which equals to resistor current x resistance.
  - Start with the KCL equation:

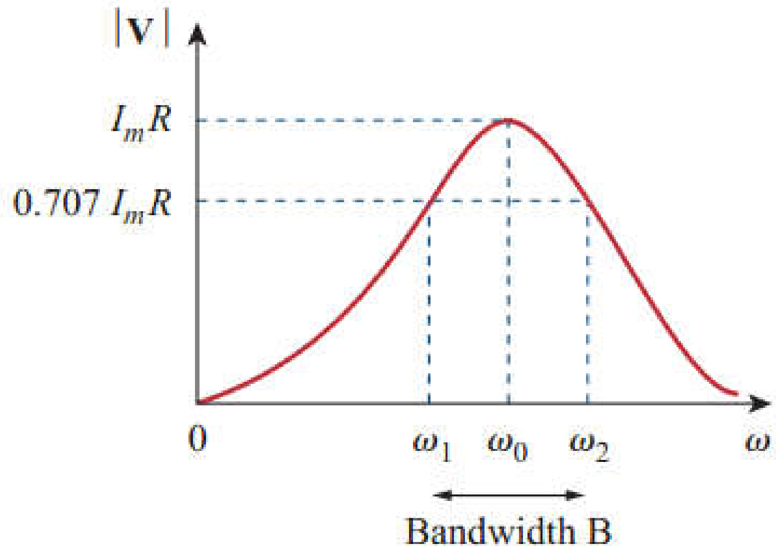
$$\mathbf{Y} = H(\omega) = \frac{\mathbf{I}}{\mathbf{V}} = \frac{1}{R} + j\omega C + \frac{1}{j\omega L}$$



  - Resonance occurs at: $\omega_0 = \frac{1}{\sqrt{LC}}$ rad/s
  - Cut-off frequencies:

$$\omega_1 = -\frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + \frac{1}{LC}}$$

$$\omega_2 = \frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + \frac{1}{LC}}$$



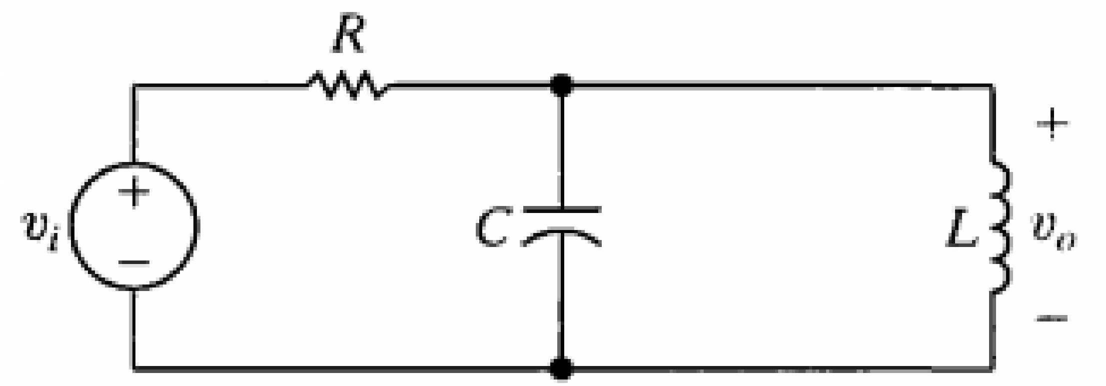
  - Bandwidth: $B = \omega_2 - \omega_1 = \frac{1}{RC}$
  - Q-factor: $Q = \frac{\omega_0}{B} = \omega_0 RC = \frac{R}{\omega_0 L}$

Summary of series and parallel RLC circuits

Characteristic	Series circuit	Parallel circuit
Resonant frequency, ω_0	$\frac{1}{\sqrt{LC}}$	$\frac{1}{\sqrt{LC}}$
Quality factor, Q	$\frac{\omega_0 L}{R}$ or $\frac{1}{\omega_0 RC}$	$\frac{R}{\omega_0 L}$ or $\omega_0 RC$
Bandwidth, B	$\frac{\omega_0}{Q}$	$\frac{\omega_0}{Q}$
Half-power frequencies, ω_1, ω_2	$\omega_0\sqrt{1+\left(\frac{1}{2Q}\right)^2} \pm \frac{\omega_0}{2Q}$	$\omega_0\sqrt{1+\left(\frac{1}{2Q}\right)^2} \pm \frac{\omega_0}{2Q}$
For $Q \geq 10$, ω_1, ω_2	$\omega_0 \pm \frac{B}{2}$	$\omega_0 \pm \frac{B}{2}$

Practice 2 - *A modified circuit with parallel L and C*

- Designing a RLC band-pass filter
  - a) Show that the RLC circuit in figure on right is also a band-pass filter by deriving an expression for the transfer function H(s).
  - b) Compute the centre frequency, ω_0.
  - c) Calculate the cut-off frequencies, ω_{c1} and ω_{0c2}, the bandwidth, β, and the quality factor, Q.
  - d) Compute values for R and L to yield a band-pass filter with a centre frequency of 5 kHz and a bandwidth of 200 Hz, using a 5 μF capacitor.



Quiz

- 3. In a series RLC circuit, which of these quality factors has the sharpest magnitude response curve near resonance?
  - (a) Q = 20; (b) Q = 12;
  - (c) Q = 8; (d) Q = 4.

- 4. How much inductance is needed to resonate at 5 kHz with a capacitance of 12 nF?
  - (a) 2,652 H; (b) 11.844 H;
  - (c) 3.333 H; (d) 84.43 mH.

Induction of Lab

- **Title: Measurements and Circuit Analysis**
  - **Time**: 11:00am – 6:00 pm (lunch break 13:00-14:00)
  - **Date**: Week 6, Thursday, Oct. 25th, 2018
  - **Room**: EE215 and EE213

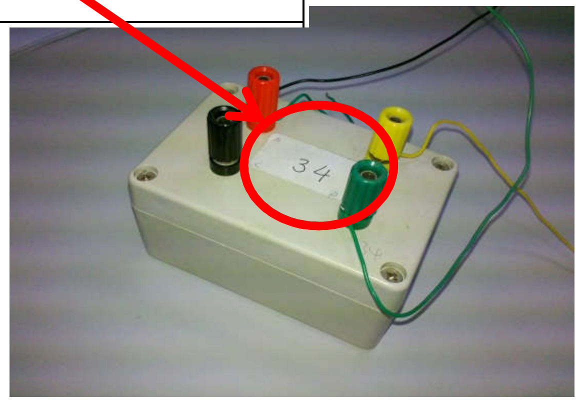
- Lab Report
  - Formal report (detailed requirement will be on ICE)
  - Soft copy (to ICE) only!
  - Individual report – one report from each student
    - Grouping information will be on ICE

Induction of Lab

- Group:
  - 2 people per group

Student Code	Surname	Forename	Chinese Name	Experiment Bench	Unknown boxes	Signature
				1	1,2,3,4,5	
				1		
				2		
				2		
				3		
				3		
				4		
				4		
				5		
				5		

Read the lab script very carefully before coming to the lab !!!



Next Lecture

- Transient circuit – DC source
  - 1^{st} order
  - 2^{nd} order