3.3: Fourier Series for $2L$-periodic functions

Consider now a function $f(x)$ which has period $2L$. It is straightforward to transform the problem of finding a Fourier series for $f(x)$ on the interval $-L < x < L$ into one on the interval $-\pi < x < \pi$.

It follows that its Fourier series is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right) \right], \tag{3.6}$$

with coefficients a_0, a_n, b_n

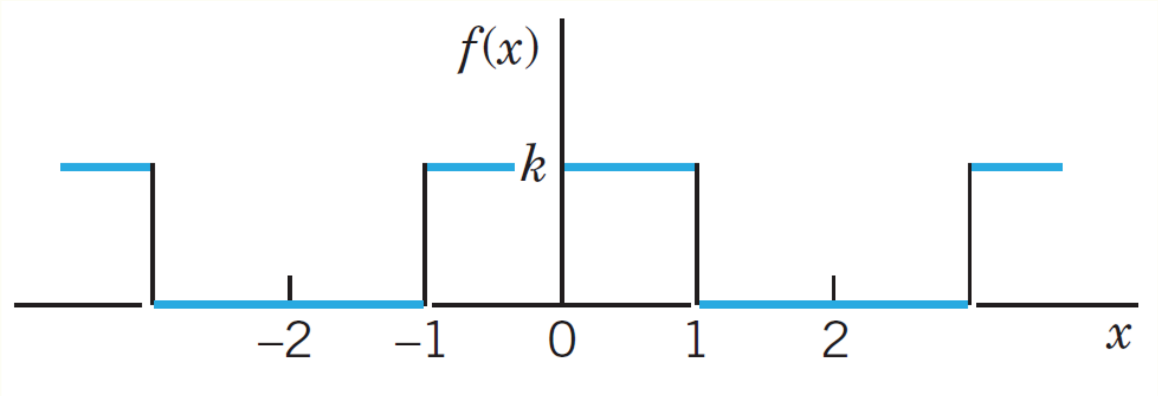
$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi}{L}x\right)dx, \ n = 1, 2, \cdots$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi}{L}x\right)dx, \ n = 1, 2, \cdots$$

Example 2: Find the Fourier series of the periodic function $f(x)$

$$f(x) = \begin{cases} 0 & \text{if} \quad -2 < x < -1, \\ k & \text{if} \quad -1 < x < 1, \\ 0 & \text{if} \quad 1 < x < 2, \end{cases} \qquad p = 2L = 4, L = 2$$



Solution:

$$
\begin{aligned}
a_0 &= \frac{1}{2L}\int_{-L}^{L} f(x)dx \\
&= \frac{1}{4}\int_{-2}^{2} f(x)dx \\
&= \frac{1}{4}\left[\int_{-2}^{-1} 0dx + \int_{-1}^{1} kdx + \int_{1}^{2} 0dx\right] \\
&= \frac{1}{4}\,[kx]_{-1}^{1} \\
&= \frac{1}{4}2k \\
&= \frac{k}{2}
\end{aligned}
$$

$$
\begin{aligned}
a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{1}{2}\int_{-2}^{2} f(x)\cos\left(\frac{n\pi x}{2}\right)dx \\
&= \frac{1}{2}\left[\int_{-2}^{-1} 0dx + \int_{-1}^{1} k\cos\left(\frac{n\pi x}{2}\right)dx + \int_{1}^{2} 0dx\right] \\
&= \frac{1}{2}\int_{-1}^{1} k\cos\left(\frac{n\pi}{2}x\right)dx = \frac{1}{2}\left[k\frac{\sin\left(\frac{n\pi x}{x}\right)}{\frac{n\pi}{2}}\right]_{-1}^{1} \\
&= \frac{k}{n\pi}\left[\sin\left(\frac{n\pi}{2}\right) - \sin\left(\frac{-n\pi}{2}\right)\right] \\
&= \frac{k}{n\pi}\left[\sin\left(\frac{n\pi}{2}\right) + \sin\left(\frac{n\pi}{2}\right)\right] \\
&= \frac{2k}{n\pi}\sin\left(\frac{n\pi}{2}\right)
\end{aligned}
$$

$$
\begin{aligned}
b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{1}{2}\int_{-2}^{2} f(x)\sin\left(\frac{n\pi x}{2}\right)dx \\
&= \frac{1}{2}\left[\int_{-2}^{-1} 0dx + \int_{-1}^{1} k\sin\left(\frac{n\pi x}{2}\right)dx + \int_{1}^{2} 0dx\right] \\
&= \frac{1}{2}\int_{-1}^{1} k\sin\left(\frac{n\pi}{2}x\right)dx = \frac{1}{2}\left[k\frac{-\cos\left(\frac{n\pi x}{2}\right)}{\frac{n\pi}{2}}\right]_{-1}^{1} \\
&= \frac{k}{n\pi}\left[-\cos\left(\frac{n\pi}{2}\right) + \cos\left(\frac{-n\pi}{2}\right)\right] \\
&= \frac{k}{n\pi}\left[-\cos\left(\frac{n\pi}{2}\right) + \cos\left(\frac{n\pi}{2}\right)\right] \\
&= 0
\end{aligned}
$$

Hence the Fourier series of $f(x)$ is

$$
\begin{aligned}
f(x) &= a_0 + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi}{2}x\right) + b_n \sin\left(\frac{n\pi}{2}x\right) \right] \\
&= \frac{k}{2} + \sum_{n=1}^{\infty} \left\{ \left[\frac{2k}{n\pi} \sin\left(\frac{n\pi}{2}\right) \right] \cos\left(\frac{n\pi}{2}x\right) \right\}
\end{aligned}
$$

Even and odd $2L$-periodic functions

As in (3.4) and (3.5), the Fourier series for an **odd** $2L$-periodic function $f(x)$ can be simplified as

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right), \tag{3.7}$$

where

$$b_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi}{2}x\right) dx.$$

The Fourier series for an **even** $2L$-periodic function $f(x)$ can be simplified as

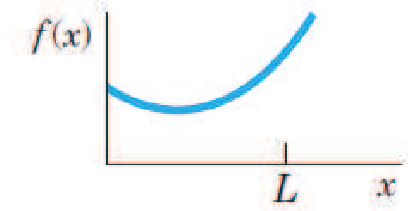
$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right), \tag{3.8}$$

where

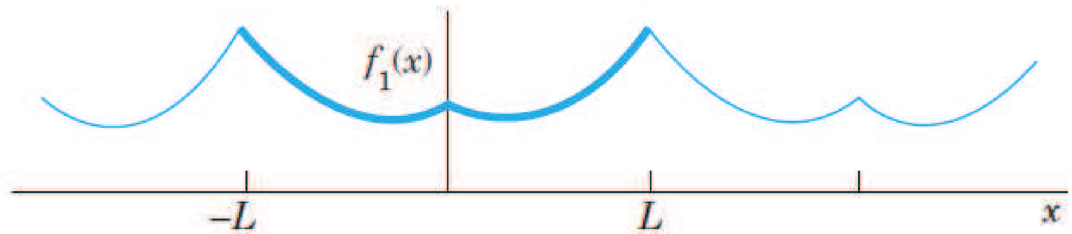
$$a_0 = \frac{1}{L}\int_0^L f(x)dx, \quad a_n = \frac{2}{L}\int_0^L f(x) \cos\left(\frac{n\pi}{L}x\right) dx.$$

3.4: Half-range expansions

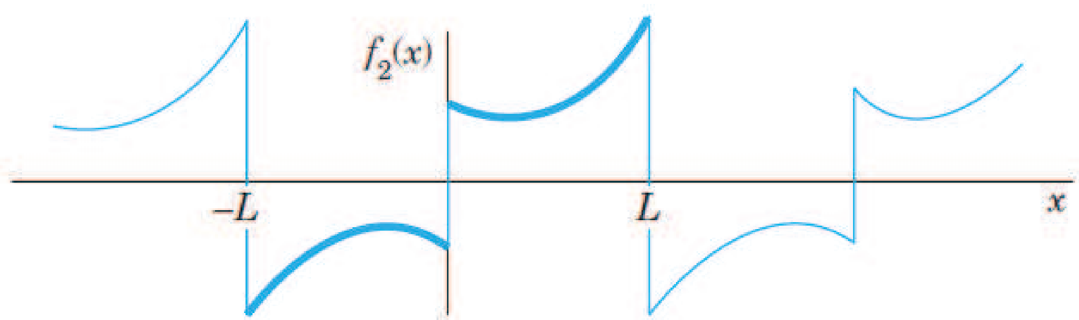
How to represent a function $f(x)$ as in the following figure (0) by a Fourier series?



(0) The given function $f(x)$



(a) $f(x)$ continued as an ***even*** periodic function of period $2L$



(b) $f(x)$ continued as an ***odd*** periodic function of period $2L$

We could extend $f(x)$ as an even function of period $2L$ as in figure (a), then its Fourier series will be

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right)$$

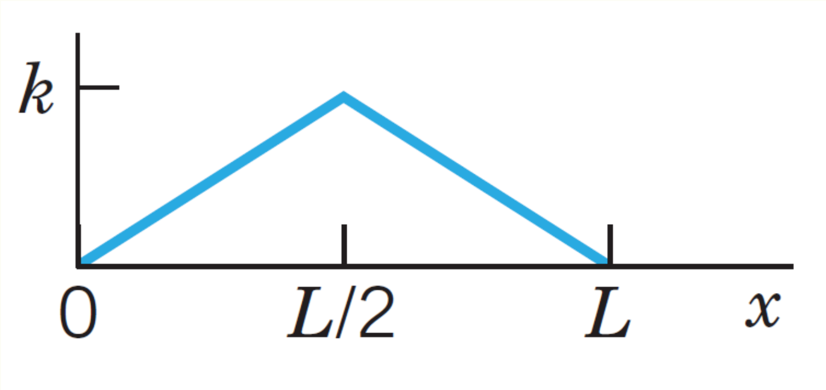
We could also extend $f(x)$ as an odd function of period $2L$ as in figure (b), then its Fourier series will be

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right)$$

Both extensions have period $2L$. This motivates the name **half-range expansions**: $f(x)$ is given only on half the range, that is, on half the interval of periodicity of length $2L$.
Let us illustrate this with an example.

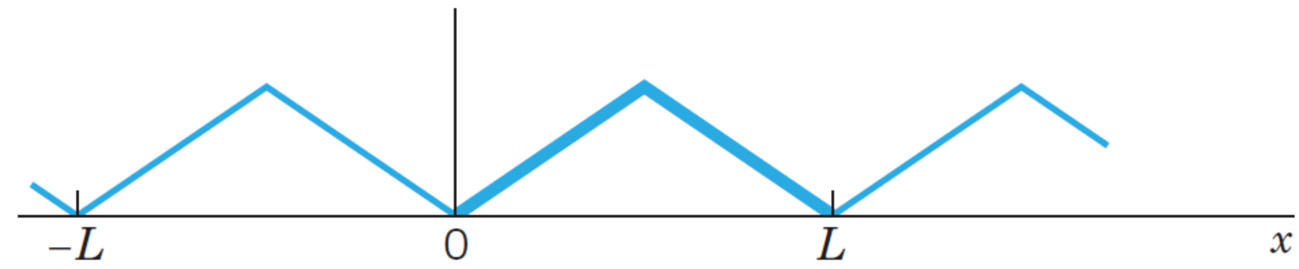
Example 3: Find the two half-expansions of the function

$$f(x) = \begin{cases} \frac{2k}{L}x & \text{if} \quad 0 < x < \frac{L}{2}, \\ \frac{2k}{L}(L-x) & \text{if} \quad \frac{L}{2} < x < L. \end{cases}$$



Solution:

(a) Even periodic extension



(a) Even extension

$$
\begin{aligned}
a_0 &= \frac{1}{L}\int_0^L f(x)dx = \frac{1}{L}\left[\int_0^{L/2}\frac{2k}{L}x dx + \int_{L/2}^{L}\frac{2k}{L}(L-x)dx\right] \\
&= \frac{2k}{L^2}\left(\int_0^{\frac{L}{2}} f(x)dx + \int_{\frac{L}{2}}^{L}(L-x)dx\right) \\
&= \frac{2k}{L^2}\left(\left[\frac{x^2}{2}\right]_0^{\frac{L}{2}} + \left[Lx - \frac{x^2}{2}\right]_{\frac{L}{2}}^{L}\right) \\
&= \frac{2k}{L^2}\left\{\left[\frac{L^2}{8}\right] + \left[\left(L^2 - \frac{L^2}{2}\right) - \left(\frac{L^2}{2} - \frac{L^2}{8}\right)\right]\right\} = \frac{k}{2}.
\end{aligned}
$$

$$
\begin{aligned}
a_n &= \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi}{L}x\right)dx \\
&= \frac{2}{L}\left[\int_0^{\frac{L}{2}} \frac{2kx}{L}\cos\left(\frac{n\pi}{L}x\right)dx + \int_{\frac{L}{2}}^{L} \frac{2k(L-x)}{L}\cos\left(\frac{n\pi}{L}x\right)dx\right] \\
&= \frac{4k}{L^2}\left[\int_0^{\frac{L}{2}} x\cos\left(\frac{n\pi}{L}x\right)dx + \int_{\frac{L}{2}}^{L}(L-x)\cos\left(\frac{n\pi}{L}x\right)dx\right] \qquad (3.9)
\end{aligned}
$$

For the first integral we obtain by integration by parts

$$
\begin{aligned}
\int_0^{\frac{L}{2}} x\cos\left(\frac{n\pi}{L}x\right)dx &= \left[\frac{Lx}{n\pi}\sin\left(\frac{n\pi}{L}x\right)\right]_0^{L/2} - \frac{L}{n\pi}\int_0^{L/2}\sin\left(\frac{n\pi}{L}x\right)dx \\
&= \frac{L^2}{2n\pi}\sin\frac{n\pi}{2} + \frac{L^2}{n^2\pi^2}\left(\cos\frac{n\pi}{2} - 1\right)
\end{aligned}
$$

Similarly, for the second integral we obtain

$$
\begin{aligned}
&\int_{\frac{L}{2}}^{L} (L-x)\cos\left(\frac{n\pi}{L}x\right)dx \\
&= \left[\frac{L}{n\pi}(L-x)\sin\left(\frac{n\pi}{L}x\right)\right]_{L/2}^{L} + \frac{L}{n\pi}\int_{L/2}^{L}\sin\left(\frac{n\pi}{L}x\right)dx \\
&= \left[0 - \frac{L}{n\pi}\left(L-\frac{L}{2}\right)\sin\frac{n\pi}{2}\right] - \frac{L^2}{n^2\pi^2}\left(\cos(n\pi) - \cos\frac{n\pi}{2}\right)
\end{aligned}
$$

We insert these two results into the formula for a_n (3.9). The sine terms cancel and so does a factor L^2. This gives

$$
a_n = \frac{4k}{n^2\pi^2}\left[2\cos\frac{n\pi}{2} - \cos(n\pi) - 1\right].
$$

Thus

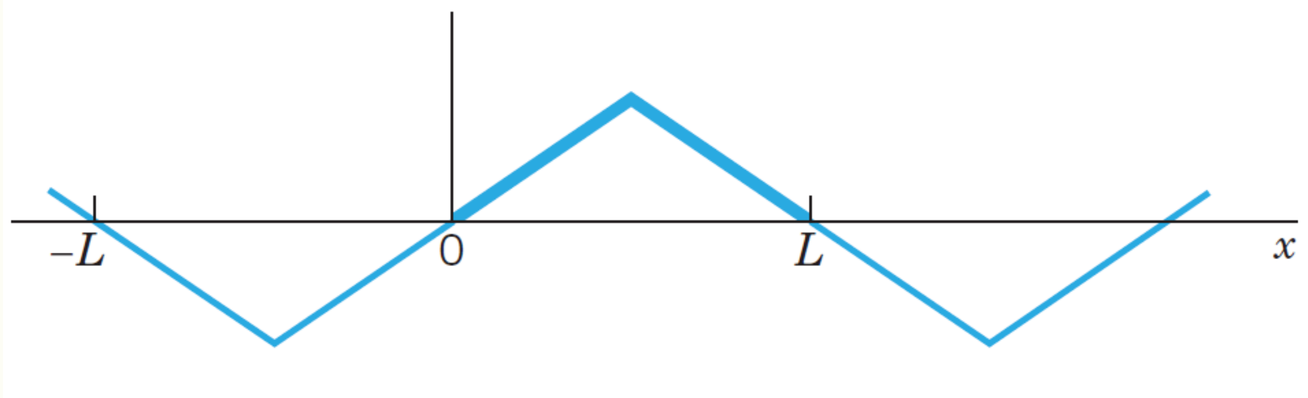
$$a_2 = -\frac{16k}{2^2\pi^2},\ a_6 = -\frac{16k}{6^2\pi^2},\ a_{10} = -\frac{16k}{10^2\pi^2}, \cdots$$

and $a_n = 0$ if $n \neq 2, 6, 10, 14, \cdots$. Hence the first half-range expansion of $f(x)$ is

$$\begin{aligned} f(x) &= \frac{k}{2} + \sum_{n=1}^{\infty} \left[\frac{4k}{n^2\pi^2} \left(2\cos\frac{n\pi}{2} - 1 - \cos(n\pi) \right) \cos\left(\frac{n\pi}{L} \right) x \right] \\ &= \frac{k}{2} - \frac{16k}{\pi^2} \left[\frac{1}{2^2} \cos\left(\frac{2\pi}{L} x \right) + \frac{1}{6^2} \cos\left(\frac{6\pi}{L} x \right) + \cdots \right] \end{aligned}$$

This Fourier cosine series represents the even periodic extension of the given function $f(x)$ of period $2L$.

(b) Odd periodic extension



(b) Odd extension

$$
\begin{aligned}
b_n &= \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{2}x\right)dx \\
&= \frac{2}{L}\left[\int_0^{\frac{L}{2}} \frac{2kx}{L}\sin\left(\frac{n\pi}{L}x\right)dx + \int_{\frac{L}{2}}^{L} \frac{2k(L-x)}{L}\sin\left(\frac{n\pi}{L}x\right)dx\right] \\
&= \frac{4k}{L^2}\left[\int_0^{\frac{L}{2}} x\sin\left(\frac{n\pi}{L}x\right)dx + \int_{\frac{L}{2}}^{L} (L-x)\sin\left(\frac{n\pi}{L}x\right)dx\right] \\
&= \frac{8k}{n^2\pi^2}\sin\frac{n\pi}{2},\ \ n = 1, 2, \cdots .
\end{aligned}
$$

Hence the other half-range periodic extension of $f(x)$ is

$$\begin{aligned} f(x) &= \sum_{n=1}^{\infty} \left[\frac{8k}{n^2\pi^2} \sin\frac{n\pi}{2} \right] \sin\left(\frac{n\pi}{L}x\right) \\ &= \frac{8k}{\pi^2} \left(\frac{1}{1^2} \sin\frac{\pi x}{L} - \frac{1}{3^2} \sin\frac{3\pi x}{L} + \frac{1}{5^2} \sin\frac{5\pi x}{L} - + \cdots \right) \end{aligned}$$

3.5: Properties of Fourier series

A Fourier series is an infinite series used to represent function values and therefore we must ask the questions:

1. Is it convergent?
2. If so, to what does it converge?
3. Can we apply differentiation to represent $f'(x)$?
4. Can we apply integration to represent $\int f(x)dx$?

Example 4:

$$f(x) = \begin{cases} 0, & -\pi < x \le 0 \\ 1, & 0 < x \le \pi \end{cases}, f(x+2\pi) = f(x)$$

The Fourier coefficients are

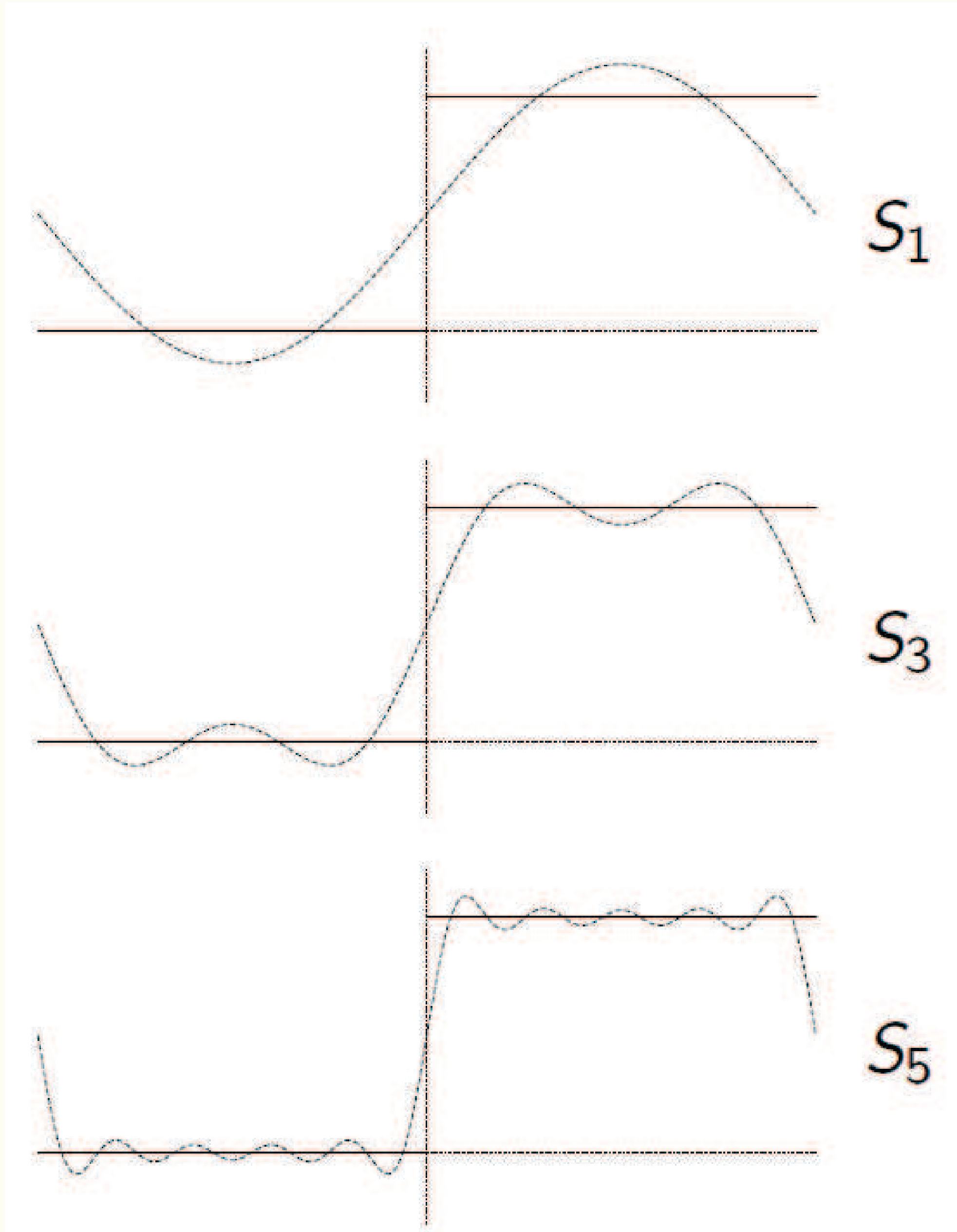
$$a_0 = 1,\ a_n = 0,\ b_n = \frac{1-(-1)^n}{n\pi},\ n = 1, 2, \cdots$$

The Fourier series is therefore

$$f(x) = \frac{1}{2} + \frac{2}{\pi}\sin x + \frac{2}{3\pi}\sin 3x + \frac{2}{5\pi}\sin 5x + \cdots .$$

Define

$$\begin{aligned} S_1 &= \frac{1}{2} + \frac{2}{\pi}\sin x \\ S_2 &= S_1 + \frac{2}{3\pi}\sin 3x \\ S_3 &= S_2 + \frac{2}{5\pi}\sin 5x \quad \text{etc} \end{aligned}$$



S_1
S_3
S_5

We see from this example that as $n \to \infty$

1. At points where $f(x)$ is continuous, say $x = a$, the sum of the Fourier series is $f(a)$.
2. At points where $f(x)$ is discontinuous, say $x = b$, the sum of the Fourier series is

$$\frac{1}{2}[f(b^+) + f(b^-)]$$

ie mid-way between the two values.

This result is Fourier's theorem.

A Fourier series can be used for integrating functions.

Theorem

If $f(x)$ is piecewise continuous in $-L < x < L$ and periodic with period $2L$ then the Fourier series can be integrated term by term.

The Fourier series for a smooth function $f(x)$ may be differentiated term by term.

Theorem

The Fourier series for $f'(x)$ can be obtained by differentiating the Fourier series for $f(x)$

ONLY

if $f(x)$ is continuous for all x.