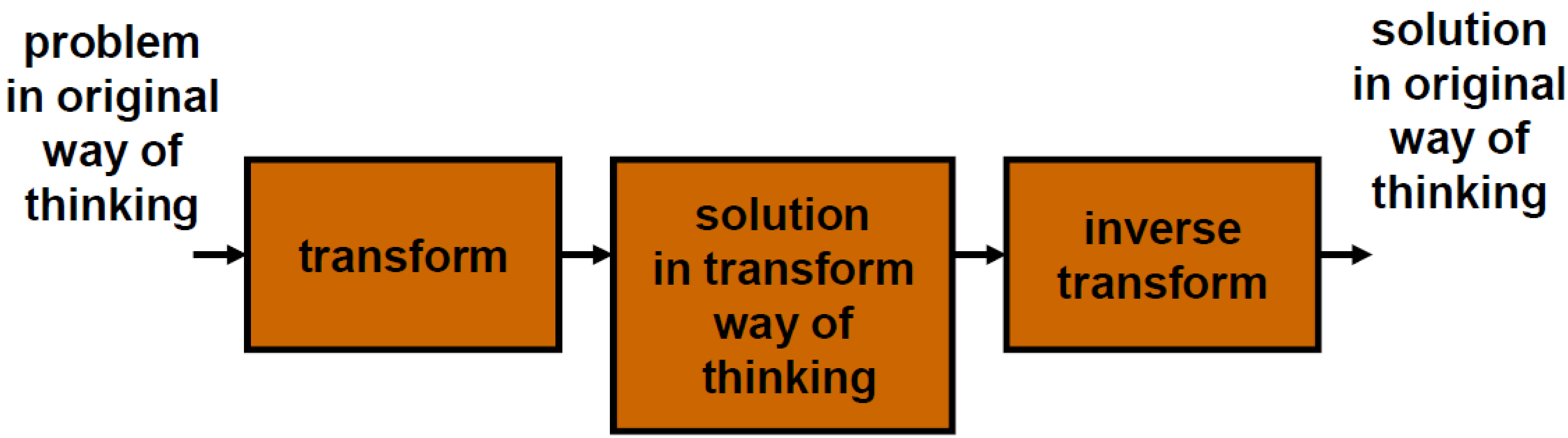
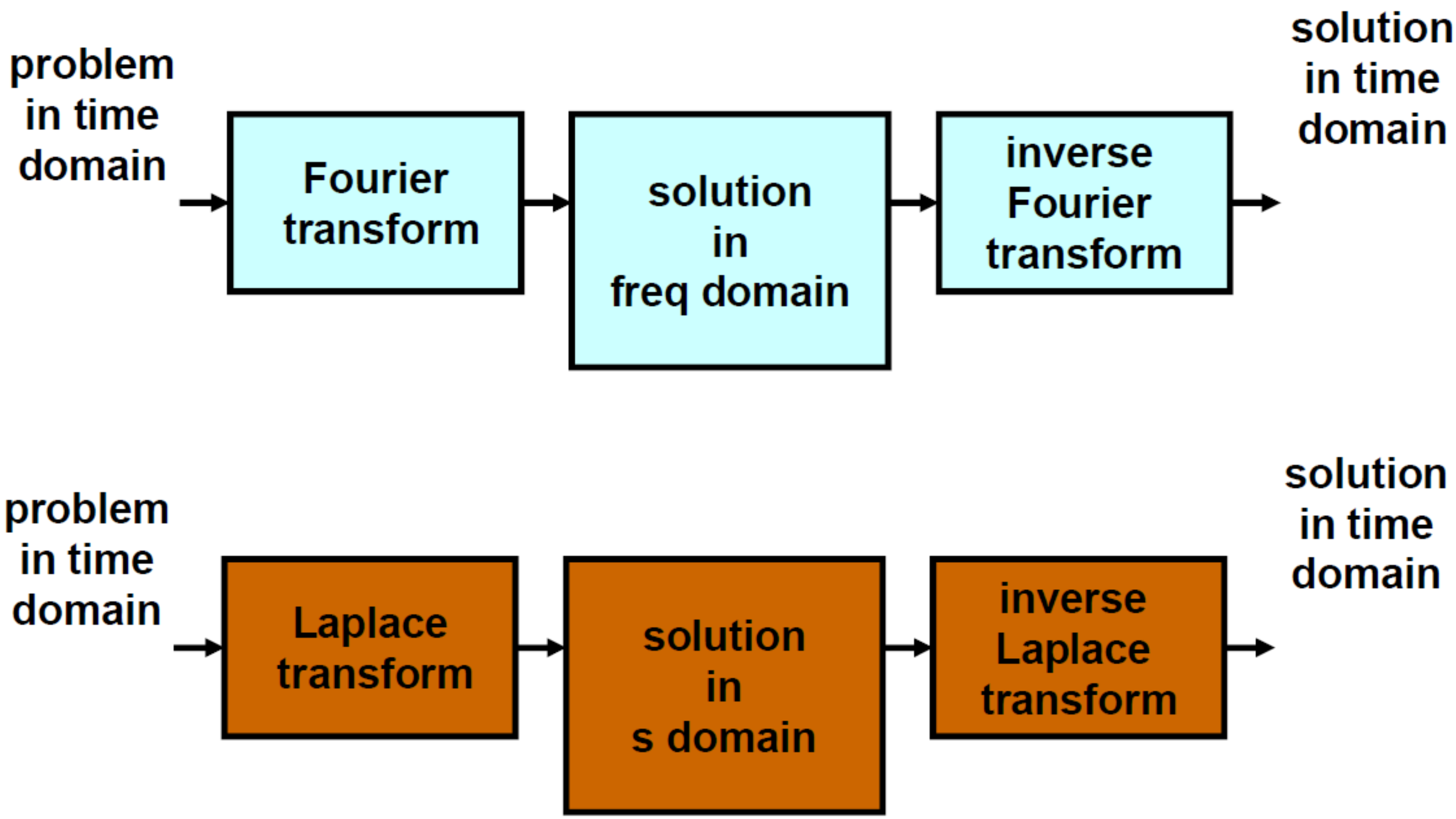
Applications of Laplace Transform

Transforms

- Transform -- a mathematical conversion from one way of thinking to another to make a problem easier to solve

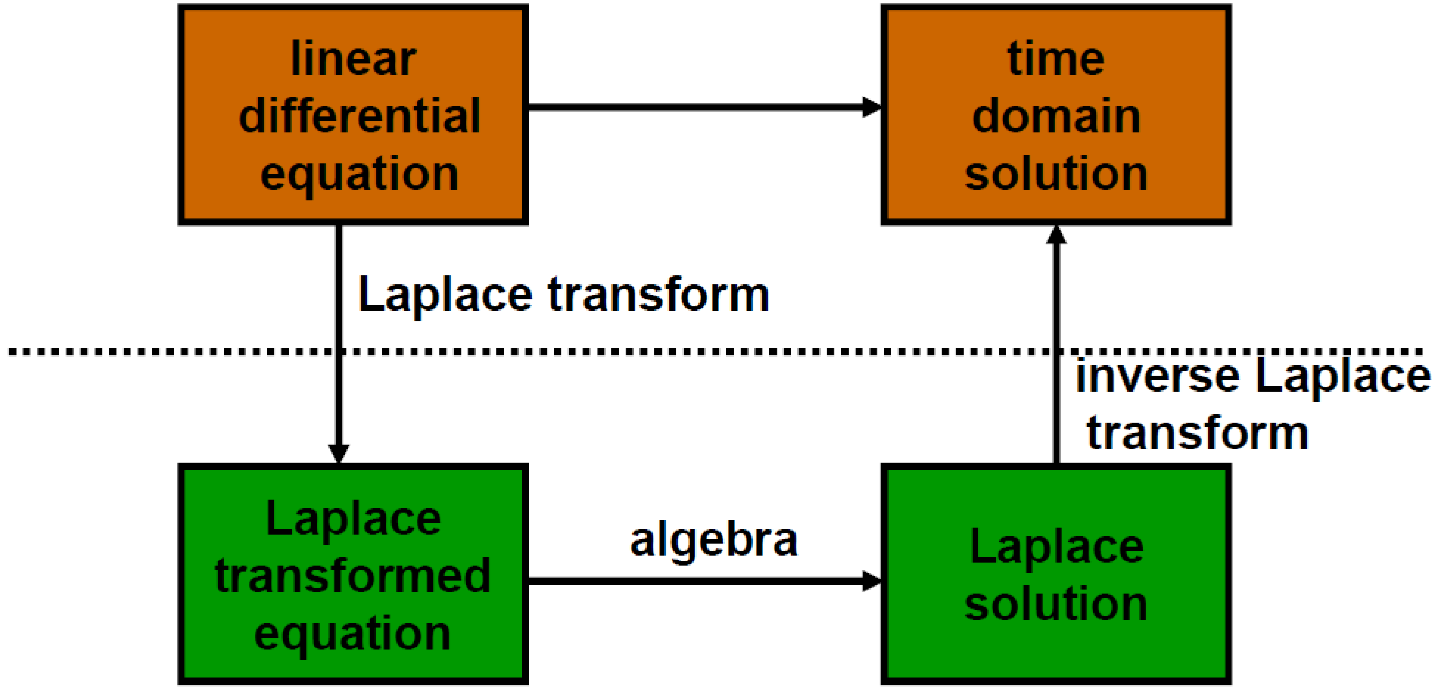


Transforms


problem in time domain
Fourier transform
solution in freq domain
inverse Fourier transform
solution in time domain
problem in time domain
Laplace transform
solution in s domain
inverse Laplace transform
solution in time domain

Laplace transformation

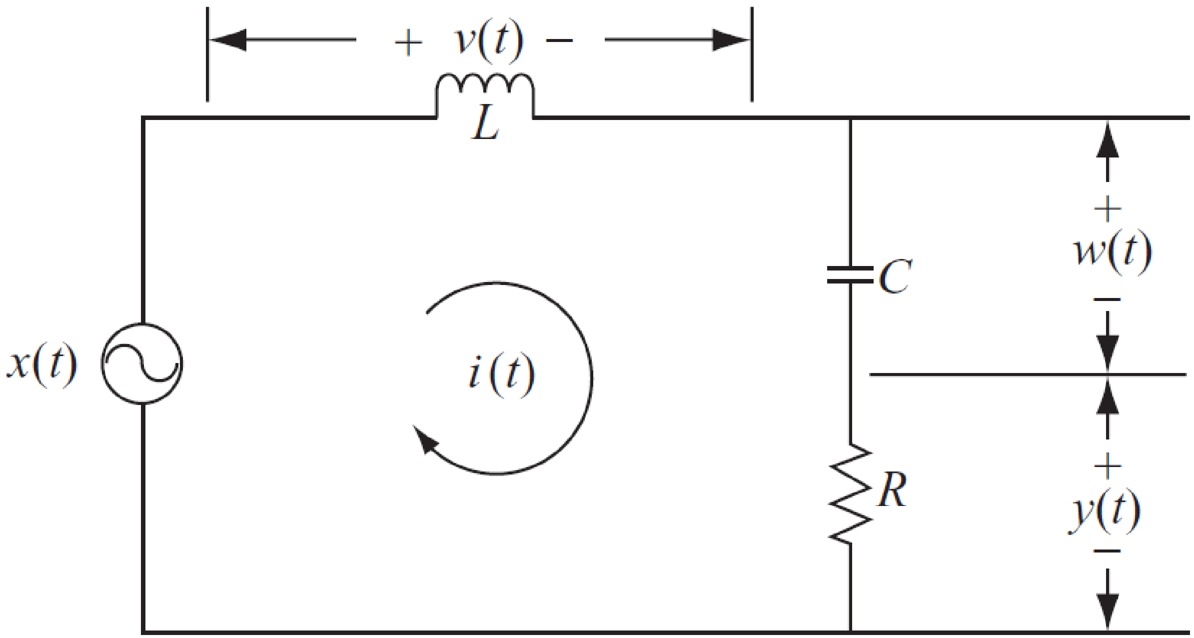
time domain



Laplace domain or
complex frequency domain

Solve differential equations

Assume *L=0 H, C=1/20 F, R=5 Ω*,
for an initial condition $y(0_{-}) = 2$V and a sinusoidal voltage $x(t) = \sin(2t)u(t)$ applied at the input of the RC circuit, calculate the output voltage $y(t)$.



Solve differential equations

It can be modeled by a constant-coefficient differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} + 4y(t) = \frac{\mathrm{d}x}{\mathrm{d}t}$$

Take Laplace transform of each term on both sides

$$X(s) = L\{x(t)\} = L\{\sin(2t)u(t)\} = \frac{2}{s^2+4}$$

$$\mathrm{L}\left\{\frac{\mathrm{d}x}{\mathrm{d}t}\right\} = sX(s) - x(0^-) = \frac{2s}{s^2+4}$$

$$\mathrm{L}\left\{\frac{\mathrm{d}y}{\mathrm{d}t}\right\} = sY(s) - y(0^-) = sY(s) - 2$$

Solve differential equations

$$[sY(s) - 2] + 4Y(s) = \frac{2s}{s^2+4}$$

$$Y(s) = \frac{2s^2+2s+8}{(s+4)(s^2+4)} \equiv \frac{A}{(s+4)} + \frac{Bs+C}{(s^2+4)}$$

$$A = \left[(s+4)\frac{2s^2+2s+8}{(s+4)(s^2+4)}\right]_{s=-4} = \frac{32}{20} = 1.6$$

Similar way to obtain *B* and *C*

$$Y(s) = \frac{1.6}{(s+4)} + \frac{0.4s+0.4}{(s^2+4)} = \frac{1.6}{(s+4)} + 0.4\frac{s}{(s^2+4)} + 0.2\frac{2}{(s^2+4)}$$

$$y(t) = [1.6e^{-4t} + 0.4\cos(2t) + 0.2\sin(2t)]u(t)$$

Solve differential equations

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$$\frac{d^2 y}{dt^2} + 3\frac{dy}{dt} + 2y(t) = \delta(t); \quad y(0^-) = \dot{y}(0^-) = 0;$$

$$\left[s^2 Y(s) - s\,\underbrace{y(0^-)}_{=0} - \underbrace{\dot{y}(0^-)}_{=0} \right] + 3\left[sY(s) - \underbrace{y(0^-)}_{=0} \right] + 2Y(s) = 1$$

or, $(s^2 + 3s + 2)Y(s) = 1$ or $Y(s) = \frac{1}{(s^2+3s+2)} = \frac{1}{(s+1)(s+2)} = \frac{1}{s+1} - \frac{1}{s+2}$.

Calculating the inverse Laplace transform, we obtain

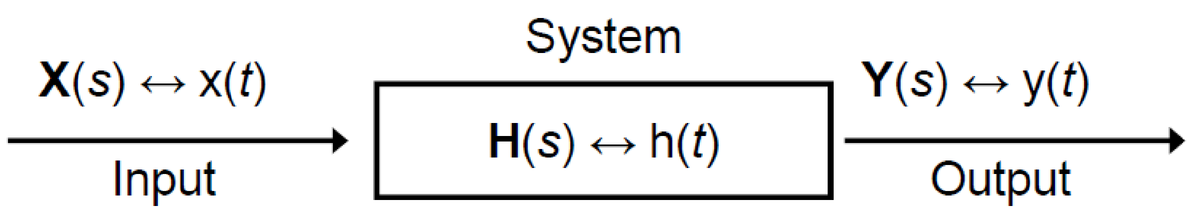
$$y(t) = e^{-t}u(t) - e^{-2t}u(t) = \left(e^{-t} - e^{-2t}\right)u(t).$$

Transfer Function

- The *transfer function*, **H**(*s*), is the ratio of the output variable of a system to its input variable

$$\mathbf{H}(s) = \frac{\mathbf{Y}(s)}{\mathbf{X}(s)} = \frac{\text{Output}}{\text{Input}}$$

- The transfer function is portrayed in block diagram form as



- **H**(*s*) is a complex quantity and is a function of frequency, $s = j\omega$

Impulse Respose

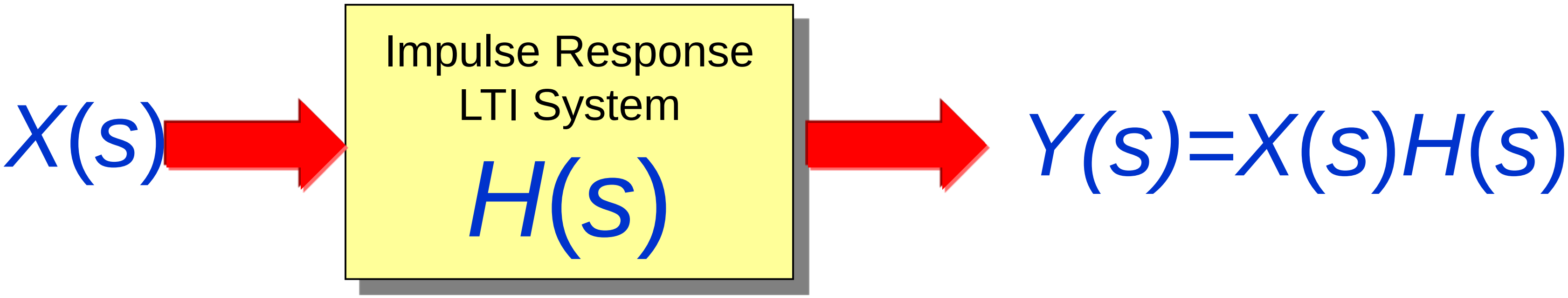
- The transfer function to find the system output to an arbitrary input using simple multiplication in the ***s*** domain

$$\mathbf{Y}(s) = \mathbf{H}(s)\ \mathbf{X}(s)$$

- In the time domain, such an operation would require use of the *convolution integral* with the **impulse response**

$$y(t) = \int_{-\infty}^{+\infty} h(\tau)\, x(t-\tau)\, d\tau = \int_{-\infty}^{+\infty} h(t-\tau)\, x(\tau)\, d\tau$$

System analysis by using LT



The causality and stability of a LTI system can be determined from the Laplace transfer function *H(s)*

Zeros and poles

If a rational transfer function H(s) can be written as the following form:

$$X(s) = \frac{N(s)}{D(s)} = \frac{b_m s^m + b_{m-1}s^{m-1} + b_{m-2}s^{m-2} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \cdots + a_1 s + a_0}.$$

Zeros The zeros of the transfer function $H(s)$ of an LTIC system are the *finite* locations in the complex s-plane where $|H(s)| = 0$.

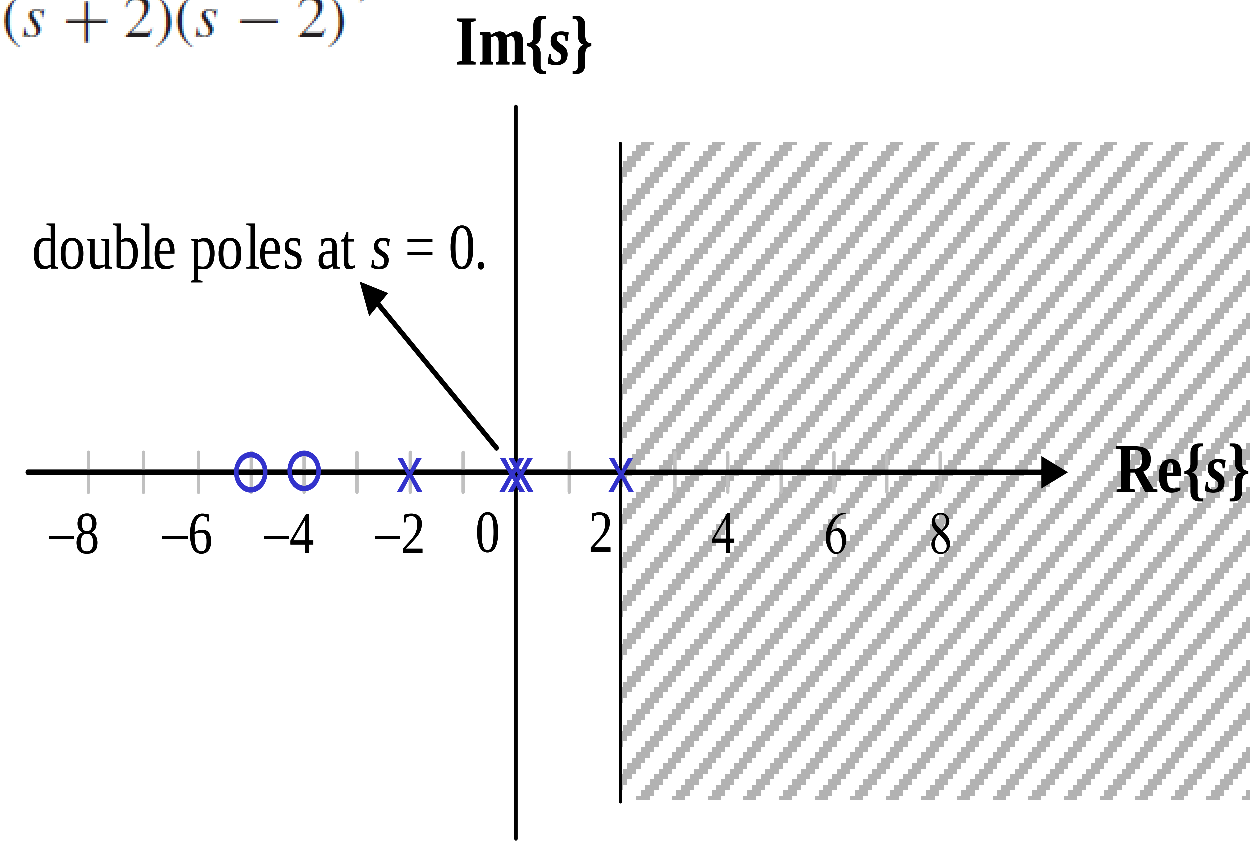
Poles The poles of the transfer function $H(s)$ of an LTIC system are the locations in the complex s-plane where $|H(s)|$ has an infinite value.

$$H(s) = \frac{N(s)}{D(s)} = \frac{b_m(s - z_1)(s - z_2)\cdots(s - z_m)}{(s - p_1)(s - p_2)\cdots(s - p_n)}.$$

Zeros and poles

Determine the poles and zeros of the following LTIC systems:

(i) $H_1(s) = \dfrac{(s+4)(s+5)}{s^2(s+2)(s-2)}$;



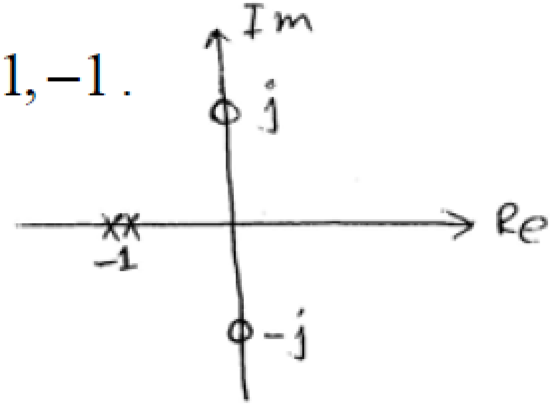
Zeros and poles

Determine the poles and zeros of the following LTIC systems:

$$H(s) = \frac{s^2+1}{s^2+2s+1};$$

Two zeros: at $s = j, -j$

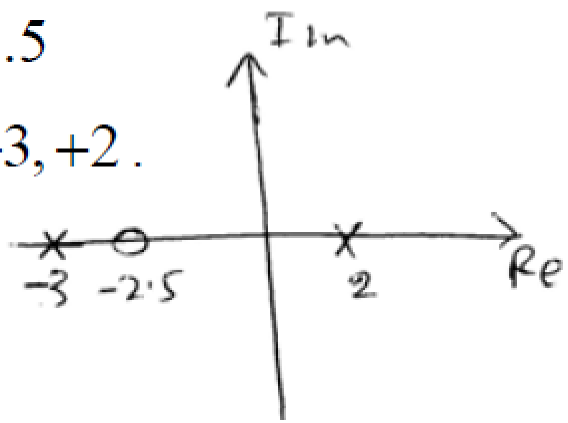
Two poles: at $s = -1, -1$.



$$H(s) = \frac{2s+5}{s^2+s-6};$$

One zero: at $s = -2.5$

Two poles: at $s = -3, +2$.

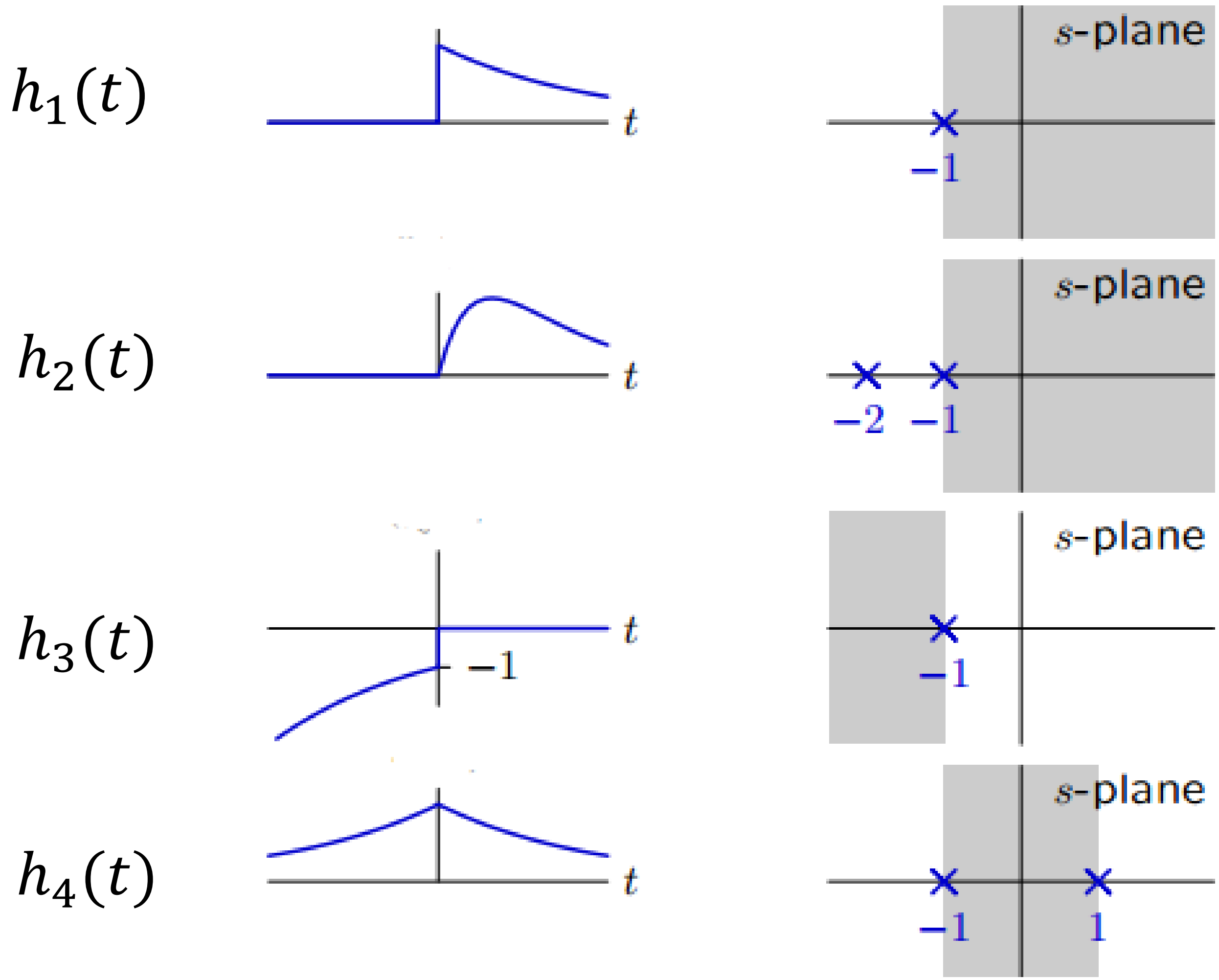


Causality

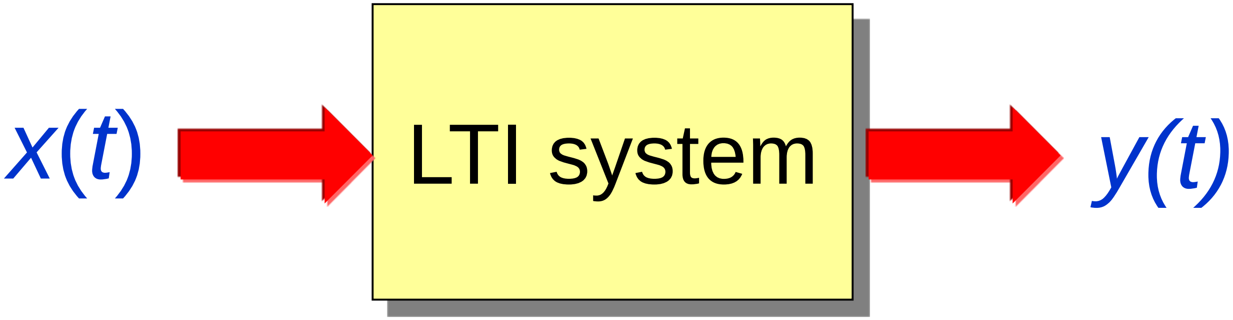
The ROC associated with the system function for a causal system is a right-half plane.

For a system with a rational system function, causality of the system is equivalent to the ROC being the right-half plane to the right of the rightmost pole.

Causality



Stable systems



$$|x(t)| \le B_x < \infty \quad \text{for } t \in (-\infty, \infty);$$

$$|y(t)| \le B_y < \infty \quad \text{for } t \in (-\infty, \infty);$$

A system is referred to as bounded-input, bounded-output (BIBO) stable if an arbitrary bounded-input signal always produces a bounded-output signal.

Stability

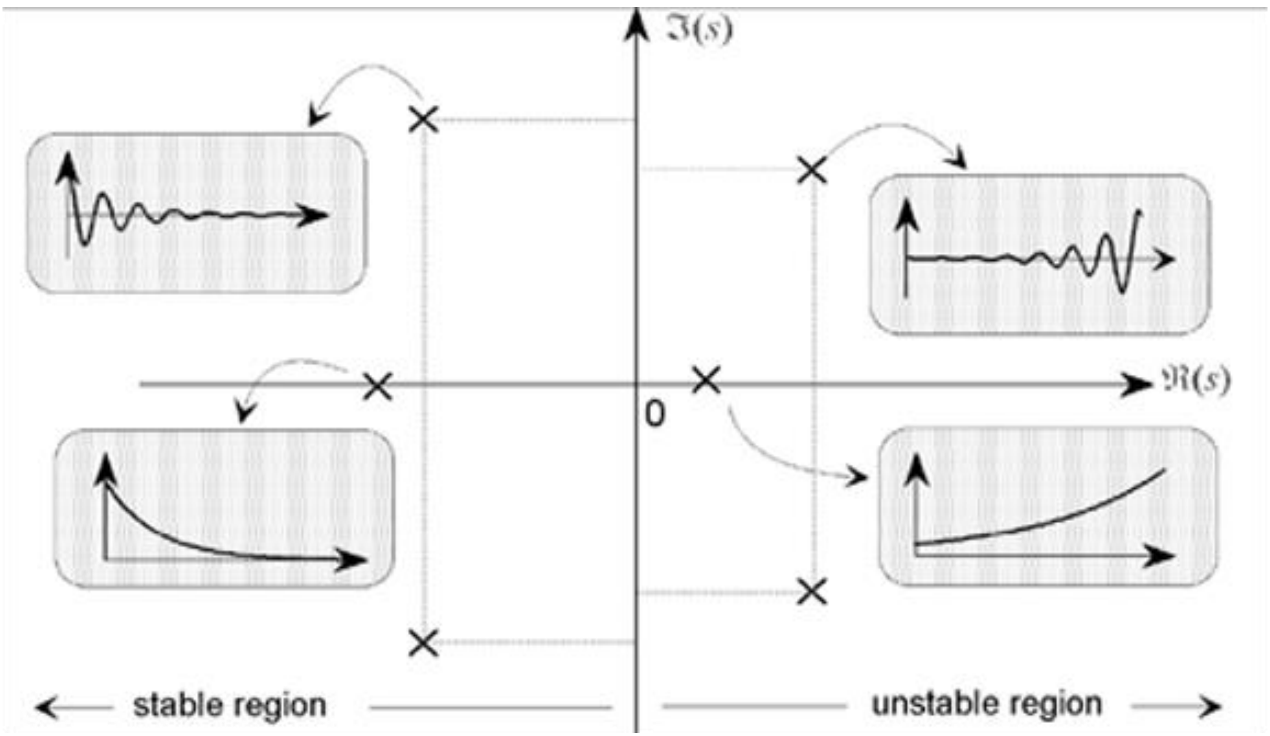
A LTI system is stable if and only if the ROC of its system function $H(s)$ includes the entire $j\omega$-axis [i.e., $\mathcal{R}e\{s\} = 0$]

$$stability = h(t)\ integrable\ \rightarrow H(j\omega)\ converge$$

$$\rightarrow ROC\ of\ H(s)\ includes\ j\omega - axis$$

Stability

A causal system with rational system function H(s) is stable if and only if all the poles of H(s) lie in the left-half of the s-plan –i.e., all of the poles have negative real parts.



https://sites.google.com/site/wainert3hmainer/school/root-locus

Zeros and poles

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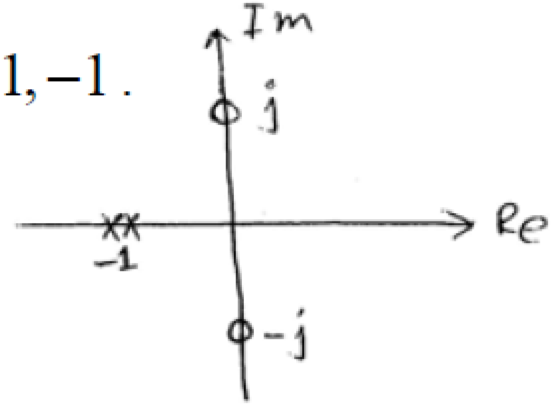
Assuming that the systems are causal, determine if the systems are BIBO stable:

$$H(s) = \frac{s^2+1}{s^2+2s+1};$$

Two zeros: at $s = j, -j$

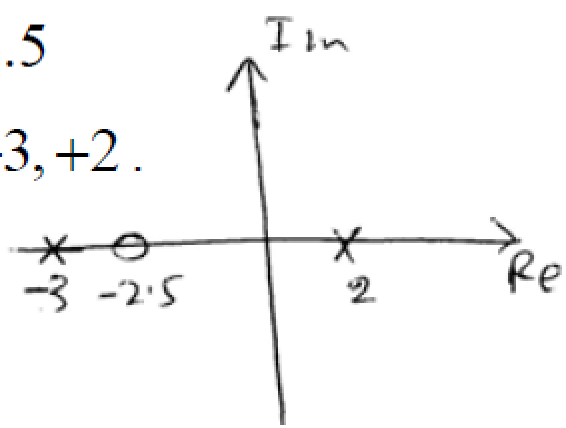
Two poles: at $s = -1, -1$.



$$H(s) = \frac{2s+5}{s^2+s-6};$$

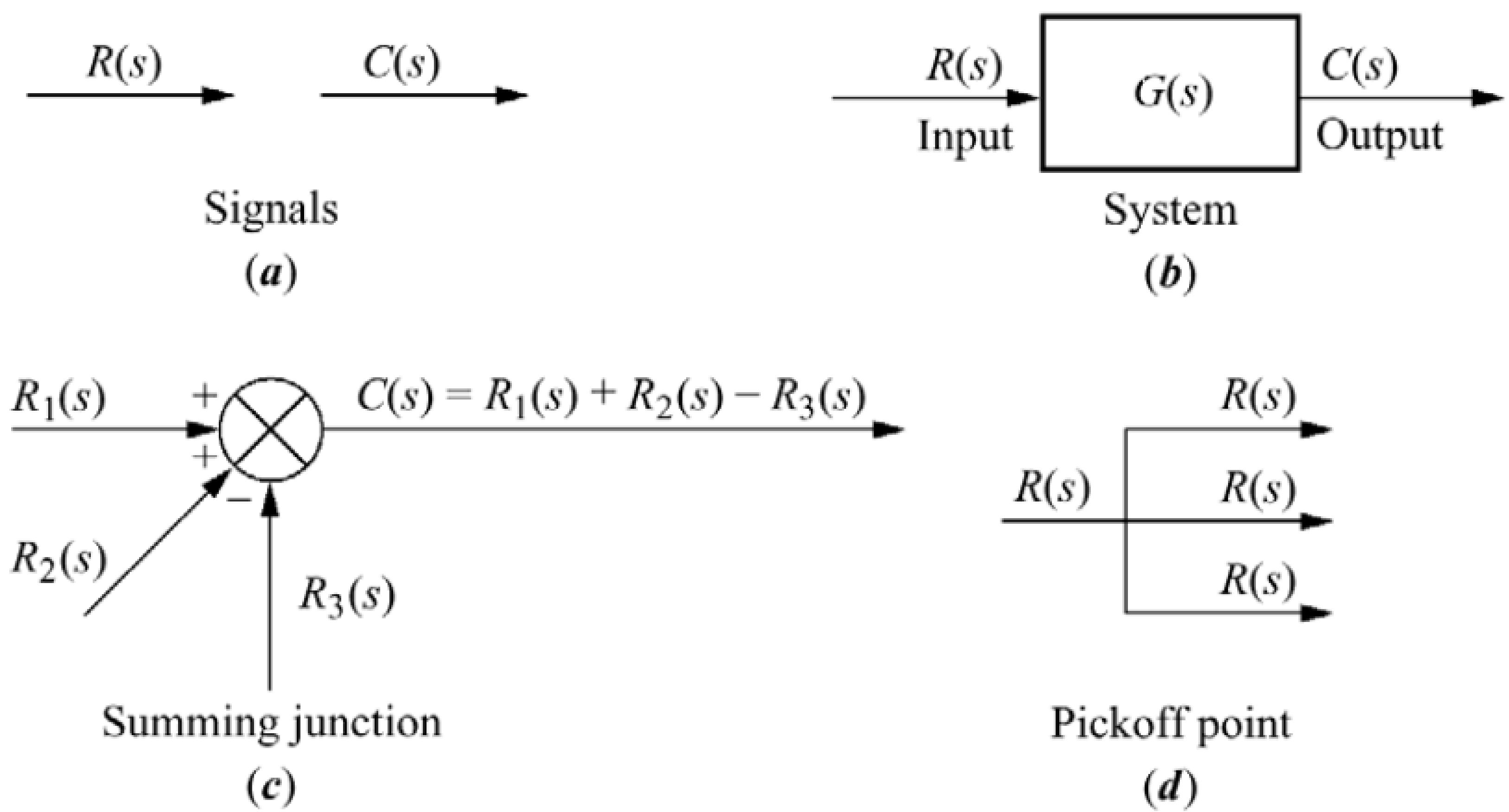
One zero: at $s = -2.5$

Two poles: at $s = -3, +2$.



Block diagram representations

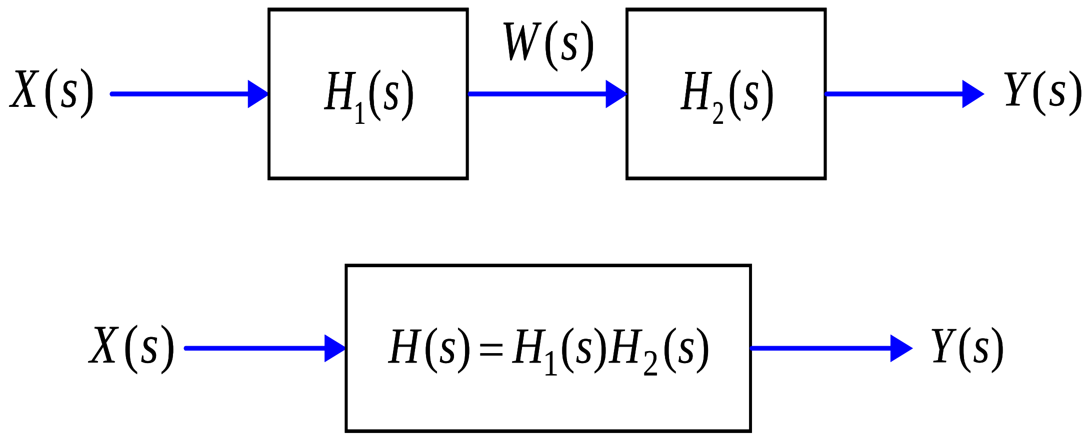
Components for Linear Time Invariant System(LTIS):



http://opencourseware.kfupm.edu.sa/colleges/ces/me/me413/files%5C2-Chapters_Chapter_10__10_2_Block_Diagram_Reduction_V2.pdf

Block diagram representations

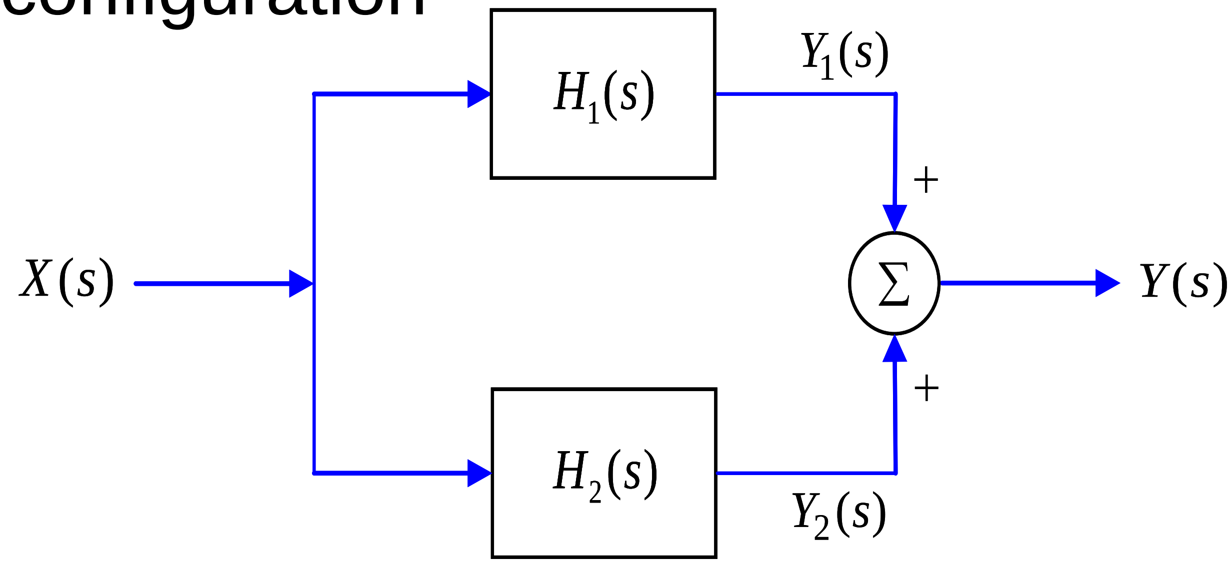
Cascaded configuration

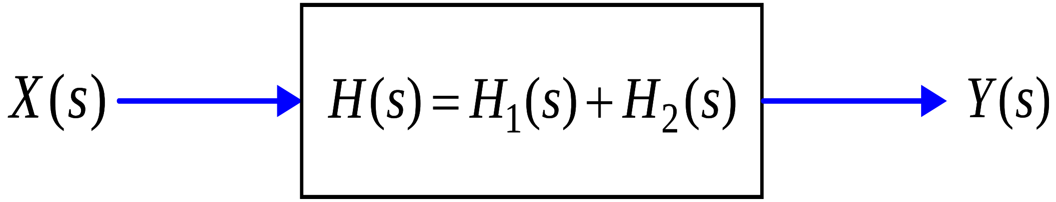


$$h(t) = h_1(t) * h_2(t) \xleftrightarrow{L} H(s) = H_1(s)H_2(s)$$

Block diagram representations

Parallel configuration

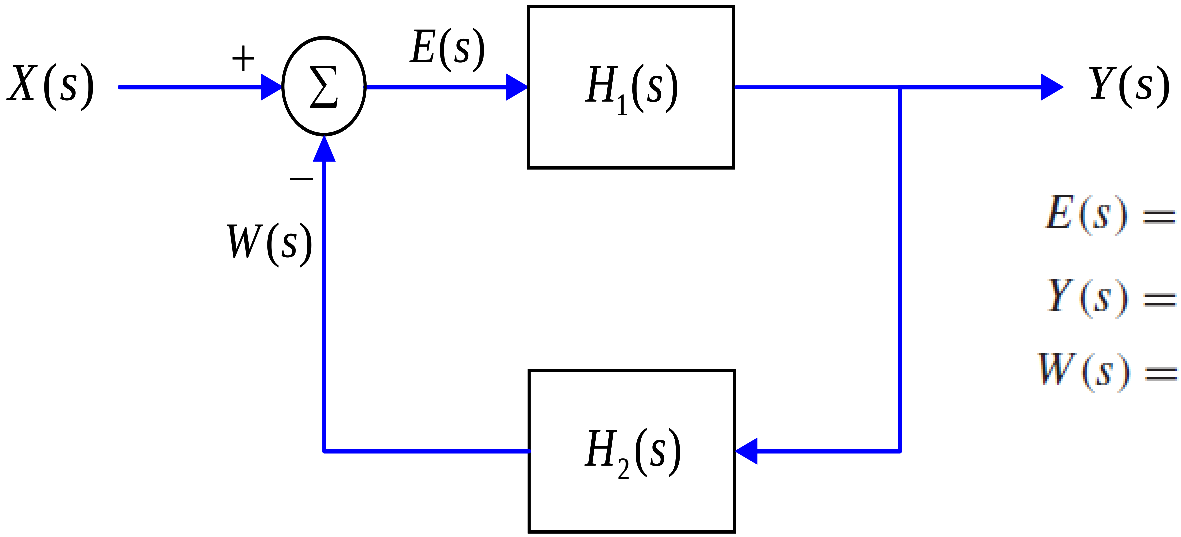




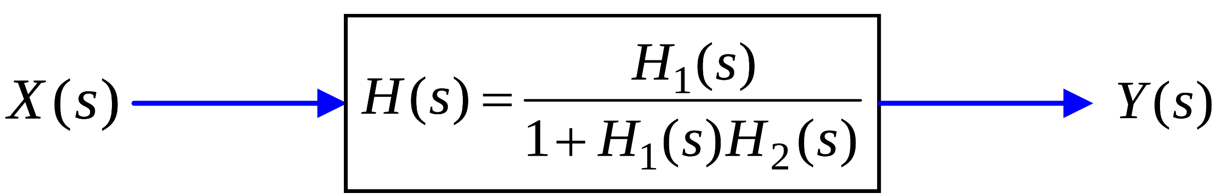
$$h(t)=h_1(t)+h_2(t)\xleftrightarrow{L}H(s)=H_1(s)+H_2(s)$$

Block diagram representations

Feedback configuration



$$E(s) = X(s) - W(s).$$
$$Y(s) = E(s)H_1(s);$$
$$W(s) = Y(s)H_2(s).$$



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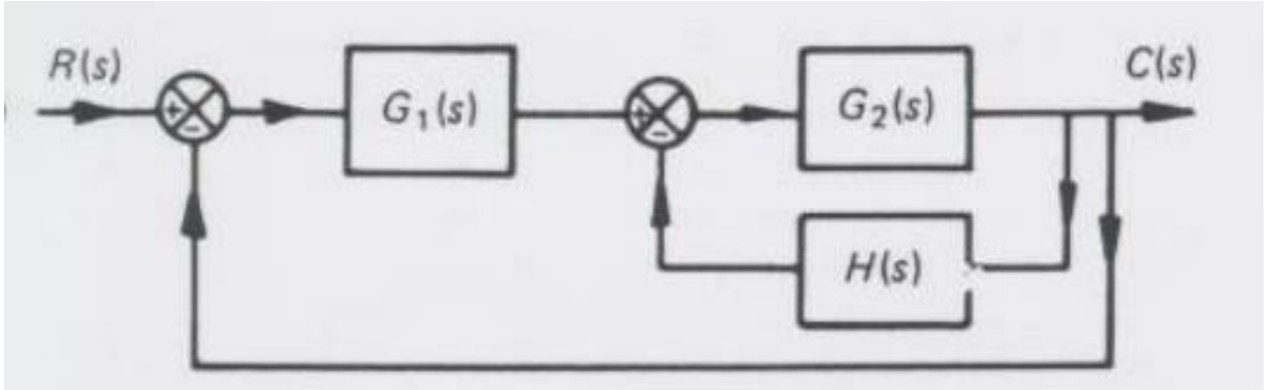
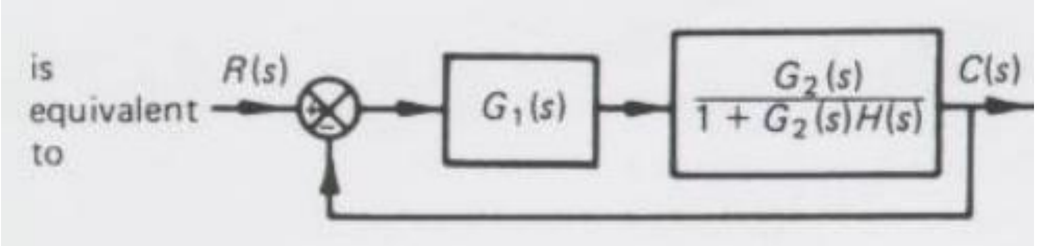
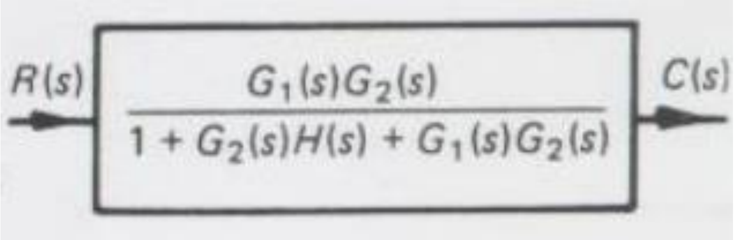
Reduce this diagram







Q6 The transfer function of the system as shown in Fig. Q6 is $H(s) = Y(s)/X(s)$ for

$$H_1(s) = 2, H_2(s) = \frac{10}{s}, H_3(s) = \frac{0.1}{s+20} \quad \text{and} \quad H_4(s) = \frac{2}{s+4}$$

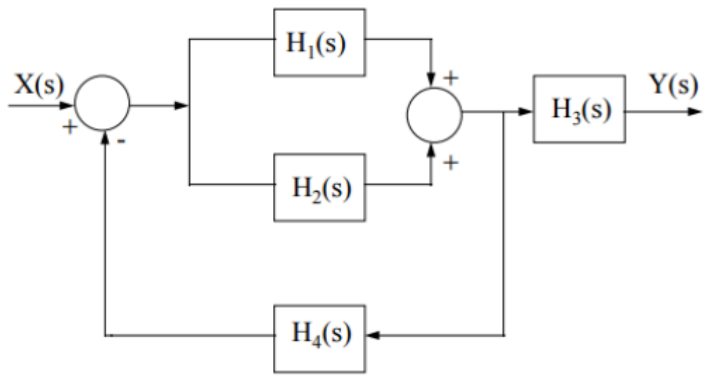


Fig. Q6

a) Simplify the block diagram to find the transfer function $H(s)$. **15**

b) Plot the location of the poles, and determine if the system is stable or not. **5**