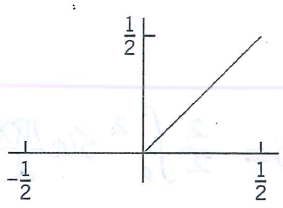
MTH201 Name________________ ID__________ Nov. 2017

Tutorial 8 Fourier series of 2L-periodic functions

Is the following function even or odd or neither? Find the expression of *f(x)* at first and then find their Fourier series. Show the details of your work.

1. 

The function is neither even nor odd.

$$f(x) = \begin{cases} 0, & -\frac{1}{2} \le x < 0 \\ x, & 0 \le x < \frac{1}{2}. \end{cases} \quad f(x+1) = f(x), \; L = \frac{1}{2}.$$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = \frac{1}{2L}\int_0^L x\,dx = \int_0^{\frac{1}{2}} x\,dx = \left[\frac{1}{2}x^2\right]_0^{\frac{1}{2}} = \frac{1}{8}.$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi}{L}x\,dx = 2\int_{-\frac{1}{2}}^{\frac{1}{2}} f(x)\cos 2n\pi x\,dx = 2\left[\int_0^{\frac{1}{2}} x\cos 2n\pi x\,dx + \int_{-\frac{1}{2}}^{0} 0\cdot\cos 2n\pi x\,dx\right]$$

Let $u = x$, $v' = \cos 2n\pi x$, $\therefore u' = 1$, $v = \frac{1}{2n\pi}\sin 2n\pi x$.

$$= 2\cdot\left[x\cdot\frac{1}{2n\pi}\sin 2n\pi x\right]_0^{\frac{1}{2}} - 2\int_0^{\frac{1}{2}}\frac{1}{2n\pi}\sin 2n\pi x\,dx = 0 - \frac{1}{n\pi}\int_0^{\frac{1}{2}}\sin 2n\pi x\,dx$$

$$= -\frac{1}{n\pi}\cdot\frac{-1}{2n\pi}\left[\cos 2n\pi x\right]_0^{\frac{1}{2}} = \frac{1}{2n^2\pi^2}\cdot(\cos n\pi - 1).$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx = 2\int_{-\frac{1}{2}}^{\frac{1}{2}} f(x)\sin 2n\pi x\,dx = 2\left\{\int_{-\frac{1}{2}}^{0} 0\sin 2n\pi x\,dx + \int_0^{\frac{1}{2}} x\sin 2n\pi x\,dx\right]$$

$= 2\int_0^{\frac{1}{2}} x\sin 2n\pi x\,dx$. Let $u = x$, $v' = \sin 2n\pi x$, $\therefore u' = 1$, $V = \frac{-1}{2n\pi}\cdot\cos 2n\pi x$

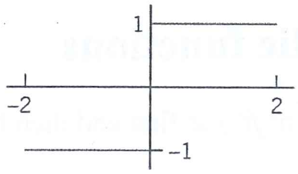
$$= 2\cdot\left[x\cdot\frac{-1}{2n\pi}\cos 2n\pi x\right]_0^{\frac{1}{2}} - 2\int_0^{\frac{1}{2}}\frac{-1}{2n\pi}\cos 2n\pi x\,dx$$

$$= 2\cdot\left(\frac{-1}{4n\pi}\cos n\pi - 0\right) + \frac{1}{n\pi}\cdot\frac{1}{2n\pi}\left[\sin 2n\pi x\right]_0^{\frac{1}{2}} = -\frac{1}{2n\pi}\cos n\pi.$$

Therefore, the Fourier series of $f(x)$ is

$$f(x) = \frac{1}{8} + \sum_{n=1}^{\infty}\left[\frac{1}{2n^2\pi^2}(\cos n\pi - 1)\cos 2n\pi x - \frac{1}{2n\pi}\cos n\pi\,\sin 2n\pi x\right]$$

2.



Because the graph of the function is symmetric about the origin, this is an odd function.

$f(x)=\begin{cases}-1, & -2\le x<0\\ 1, & 0\le x<2.\end{cases}$ $f(x+4)=f(x)$, $2L=4 \therefore L=2$.

Because f(x) is odd. So $a_n=0$, $n=0,1,2,\cdots$.

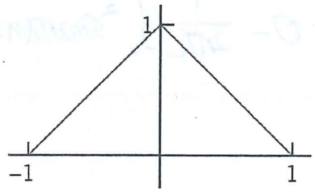
$$b_n=\frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi}{L}x\,dx=\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx=\frac{2}{2}\int_0^2 \sin\frac{n\pi x}{2}dx$$

$$=-\frac{2}{n\pi}\left[\cos\frac{n\pi x}{2}\right]_0^2=\frac{2}{n\pi}(1-\cos n\pi),\quad n=1,2,\cdots.$$

Therefore, the Fourier series of f(x) is

$$f(x)=\sum_{n=1}^{\infty} b_n\sin\frac{n\pi x}{L}=\sum_{n=1}^{\infty}\frac{2}{n\pi}(1-\cos n\pi)\sin\frac{n\pi x}{2}.$$

3.



Because the graph of the function is symmetric about the y-axis, this is an even function.

$f(x)=\begin{cases}x+1, & -1\le x<0\\ 1-x, & 0\le x<1.\end{cases}$ $2L=2$, $\therefore L=1$.

Since f(x) is even, $b_n=0$, $n=1,2,\cdots$.

$$a_0=\frac{1}{2L}\int_{-L}^{L} f(x)dx=\frac{1}{L}\int_0^L f(x)dx=\int_0^1 1-x\,dx=\left[x-\frac{1}{2}x^2\right]_0^1=\frac{1}{2}.$$

$$a_n=\frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi}{L}x\,dx=\frac{2}{L}\int_0^L f(x)\cos\frac{n\pi}{L}x\,dx=\int_0^1(1-x)\cos\frac{n\pi x}{1}dx.$$ Let $u=1-x$ $v'=\cos n\pi x$ $\therefore u'=-1$, $v=\frac{1}{n\pi}\sin n\pi x$.

$$=\left[x\cdot\frac{1}{n\pi}\sin n\pi\right]_0^1-\int_0^1\frac{1}{n\pi}\sin n\pi x\,(-1)dx=0+\int_0^1\frac{1}{n\pi}\sin n\pi x\,dx$$

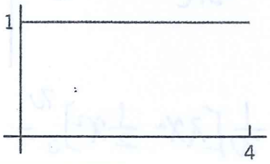
$$=\frac{1}{n\pi}\cdot\frac{-1}{n\pi}\cdot\left[\cos n\pi x\right]_0^1=\frac{1}{n^2\pi^2}(1-\cos n\pi)$$

Therefore, the Fourier series of f(x) is

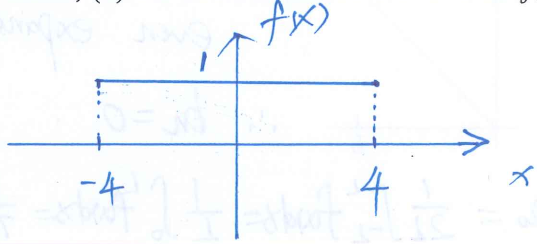
$$\frac{1}{2}+\sum_{n=1}^{\infty}\frac{1}{n^2\pi^2}(1-\cos n\pi)\cos n\pi x.$$

Half-range expansions: find (a) the Fourier cosine series, (b) the Fourier sine series. Sketch $f(x)$ and its two periodic extensions. Show the details.

4.



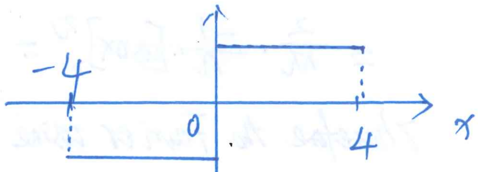
(a).



The even periodic extension of $f(x)$ is $f(x)=1$, $x\in\mathbb{R}$.

So the Fourier cosine series is $f(x)=1$.

(b). The Fourier sine series is when we expand the given function to an odd function, so



$$f(x)=\begin{cases}-1, & -4\le x<0\\ 1, & 0\le x<4.\end{cases} \quad f(x+8)=f(x),\ 2L=8 \quad \therefore L=4.$$

Because $f(x)$ is odd, so $a_0=0$, $a_n=0$, $n=1,2,\cdots$.

$$b_n=\frac{1}{L}\int_{-L}^{L}f(x)\sin\frac{n\pi x}{L}dx=\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx=\frac{2}{4}\int_0^4\sin\frac{n\pi x}{4}dx.$$

$$=\frac{1}{2}\cdot\frac{4}{n\pi}\left[-\cos\frac{n\pi x}{4}\right]_0^4=\frac{2}{n\pi}(1-\cos n\pi).\quad n=1,2,3,\cdots.$$

Therefore the Fourier sine series of $f(x)$ is

$$f(x)=\sum_{n=1}^{\infty}b_n\sin\frac{n\pi}{4}x=\sum_{n=1}^{\infty}\frac{2}{n\pi}(1-\cos n\pi)\sin\frac{n\pi x}{4}.$$

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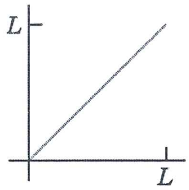
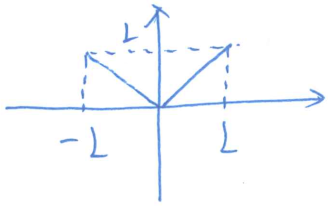
5.



(a): Even periodic expansion.



$$f(x)=\begin{cases} -x, & -L\le x<0 \\ x, & 0\le x<L \end{cases}, \quad f(x+2L)=f(x).$$

As $f(x)$ is even, $b_n=0$, $n=1,2,\cdots$.

$$a_0=\frac{1}{2L}\int_{-L}^{L} f(x)\,dx=\frac{1}{L}\int_0^L f(x)\,dx=\frac{1}{L}\int_0^L x\,dx=\frac{1}{L}\left[\frac{x^2}{2}\right]_0^L=\frac{L}{2}.$$

$$a_n=\frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx=\frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx=\frac{2}{L}\int_0^L x\cos\frac{n\pi x}{L}\,dx. \quad \text{Let } u=x,\ v'=\cos\frac{n\pi x}{L}.$$

$$\therefore u'=1,\ v=\frac{L}{n\pi}\sin\frac{n\pi x}{L}.$$

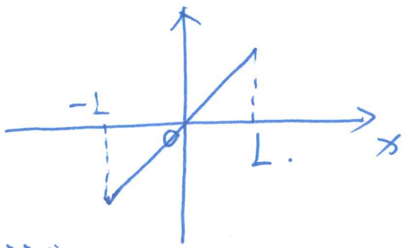
$$=\frac{2}{L}\cdot\left[x\cdot\frac{L}{n\pi}\sin\frac{n\pi x}{L}\right]_0^L-\frac{2}{L}\int_0^L\frac{L}{n\pi}\sin\frac{n\pi x}{L}\,dx=0-\frac{2}{n\pi}\int_0^L\sin\frac{n\pi x}{L}\,dx$$

$$=\frac{-2}{n\pi}\cdot\frac{-L}{n\pi}\cdot\left[\cos\frac{n\pi x}{L}\right]_0^L=\frac{2L}{n^2\pi^2}\cdot\left[\cos(n\pi)-1\right].$$

$\therefore$ The cosine Fourier series is $f(x)=\frac{L}{2}+\sum_{n=1}^{\infty}\frac{2L}{n^2\pi^2}(\cos(n\pi)-1)\cos\frac{n\pi x}{L}$.

(b). Odd periodic expansion:

$f(x)=x$, $-L<x<L$, $f(x+2L)=f(x)$.



As $f(x)$ is odd, $a_0=0$, $a_n=0$, $n=1,2,\cdots$.

$$b_n=\frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx=\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx=\frac{2}{L}\int_0^L x\sin\frac{n\pi x}{L}\,dx \quad \text{let } u=x,\ v'=\sin\frac{n\pi x}{L}.$$

$$\therefore u'=1,\ v=\frac{-L}{n\pi}\cos\frac{n\pi x}{L}.$$

$$=\frac{2}{L}\cdot\left[x\cdot\frac{-L}{n\pi}\cos\frac{n\pi x}{L}\right]_0^L-\frac{2}{L}\cdot\int_0^L\frac{-L}{n\pi}\cos\frac{n\pi x}{L}\,dx$$

$$=\frac{-2}{n\pi}\cdot L\cos n\pi+\frac{2}{n\pi}\int_0^L\cos\frac{n\pi x}{L}\,dx$$

$$=\frac{-2L}{n\pi}\cos n\pi+\frac{2}{n\pi}\cdot\frac{L}{n\pi}\cdot\left[\sin\frac{n\pi x}{L}\right]_0^L=\frac{-2L}{n\pi}\cos n\pi$$

$\therefore$ The sine Fourier series is $f(x)=\sum_{n=1}^{\infty}\frac{-2L}{n\pi}\cos n\pi\cdot\sin\frac{n\pi x}{L}$.