MTH101: Tutorial 1

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Exercise 1.1

Verify that for any $z \in \mathbb{C}$ we have

$$|z_1 \cdot z_2| = |z_1| \cdot |z_2|.$$

Solution.

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ then we compute

$$z_1 \cdot z_2 = (x_1 + iy_1) \cdot (x_2 + iy_2) = x_1x_2 - y_1y_2 + i(x_1y_2 + x_2y_1).$$

Then

$$\begin{aligned}
|z_1 \cdot z_2| &= |x_1x_2 - y_1y_2 + i(x_1y_2 + x_2y_1)| \\
&= \sqrt{(x_1x_2 - y_1y_2)^2 + (x_1y_2 + x_2y_1)^2} \\
&= \sqrt{x_1^2x_2^2 + y_1^2y_2^2 - 2x_1x_2y_1y_2 + x_1^2y_2^2 + x_2^2y_1^2 + 2x_1x_2y_1y_2} \\
&= \sqrt{x_1^2(x_2^2 + y_2^2) + y_1^2(x_2^2 + y_2^2)} \\
&= \sqrt{(x_1^2 + y_1^2)(x_2^2 + y_2^2)} = \sqrt{(x_1^2 + y_1^2)}\sqrt{(x_2^2 + y_2^2)} \\
&= |z_1| \cdot |z_2|.
\end{aligned}$$

Exercise 1.2

Find the expression of $\arg(z)$ *and* $\operatorname{Arg}(z)$ *for the following Complex Numbers:*

$$z_1 = 1 + i, \qquad z_2 = -1 + i, \qquad z_3 = \sqrt{3} - i, \qquad z_4 = -\sqrt{3} - i.$$

Solution

1. The point $z_1 = 1 + i$ is in the first quadrant of the Complex Plane as its Real Part, $x_1 = 1$ and its Imaginary Part, $y_1 = 1$ are both positive.

We have that

$$Arg(z_1) = \arctan\left(\frac{y_1}{x_1}\right) = \arctan(1) = \frac{\pi}{4}$$

While the infinitely many values of the $arg(z_1)$ are given by

$$arg(z_1) = Arg(z_1) + 2n\pi = \frac{\pi}{4} + 2n\pi, \qquad \text{with} \qquad n = 0, \pm 1, \pm 2, \ldots$$

Solution

2. The point $z_2 = -1 + i$ is in the second quadrant of the Complex Plane as its Real Part, $x_2 = -1$ is negative and its Imaginary Part, $y_2 = 1$ is positive.

As in the previous computation we conclude:

$$Arg(z_2) = \arctan\left(\frac{y_1}{x_1}\right) + \pi = \arctan(-1) + \pi = -\frac{\pi}{4} + \pi = \frac{3\pi}{4}$$

while

$$arg(z_2) = Arg(z_2) + 2n\pi = \frac{3\pi}{4} + 2n\pi,$$

where $n = 0, \pm 1, \pm 2, \ldots$.

Solution

3. The point $z_3 = \sqrt{3} - i$, is in the fourth quadrant of the Complex Plane as its Real Part, $x_3 = \sqrt{3}$ is positive and its Imaginary Part, $y_3 = -1$ is negative.

We have that

$$Arg(z_3) = \arctan\left(\frac{y_3}{x_3}\right) = \arctan\left(-\frac{1}{\sqrt{3}}\right) = -\frac{\pi}{6}$$

While the infinitely many values of the $arg(z_3)$ are given by

$$arg(z_3) = Arg(z_3) + 2n\pi = -\frac{\pi}{6} + 2n\pi, \qquad \text{with} \qquad n = 0, \pm 1, \pm 2, \ldots$$

Solution

4. The point $z_4 = -\sqrt{3} - i$ is in the third quadrant of the Complex Plane as its Real Part, $x_4 = -\sqrt{3}$ and its Imaginary Part, $y_4 = -1$ are both negative.

As in the previous computation we conclude:

$$Arg(z_4) = \arctan\left(\frac{y_4}{x_4}\right) - \pi = \arctan\left(\frac{1}{\sqrt{3}}\right) - \pi = \frac{\pi}{6} - \pi = -\frac{5\pi}{6}$$

while

$$arg(z_4) = Arg(z_4) + 2n\pi = -\frac{5\pi}{6} + 2n\pi,$$

where $n = 0, \pm 1, \pm 2, \ldots$.

Exercise 1.3

Write in **Polar Form** *the Complex Numbers of the previous Exercise.*

Solution

1. We have

$$z_1 = 1 + i, \qquad \text{and} \qquad Arg(z_1) = \frac{\pi}{4},$$

the Polar form is given by

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1),$$

where

$$r_1 = |z_1| = \sqrt{1^2 + 1^2} = \sqrt{2}, \qquad \theta_1 = Arg(z_1) = \frac{\pi}{4},$$

then we conclude

$$z_1 = \sqrt{2}[\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)].$$

Solution

2. We have

$$z_2 = -1 + i, \qquad \text{and} \qquad Arg(z_2) = \frac{3\pi}{4},$$

the Polar form is given by

$$z_2 = r_2(\cos\theta_2 + i\sin\theta_2),$$

where

$$r_2 = |z_2| = \sqrt{(-1)^2 + 1^2} = \sqrt{2}, \qquad \theta_2 = Arg(z_2) = \frac{3\pi}{4},$$

then we conclude

$$z_2 = \sqrt{2}[\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)].$$

Solution

3. We have

$$z_3 = -\sqrt{3} + i, \qquad \text{and} \qquad Arg(z_3) = -\frac{\pi}{6},$$

the Polar form is given by

$$z_3 = r_3(\cos\theta_3 + i\sin\theta_3),$$

where

$$r_3 = |z_3| = \sqrt{(-\sqrt{3})^2 + 1^2} = 2, \qquad \theta_3 = Arg(z_3) = -\frac{\pi}{6},$$

then we conclude

$$\begin{aligned} z_3 &= 2\left[\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right] \\ \textbf{or} \quad &= 2\left[\cos\left(\frac{\pi}{6}\right) - i\sin\left(\frac{\pi}{6}\right)\right]. \end{aligned}$$

Solution

4. We have

$$z_4 = -\sqrt{3} - i, \qquad \text{and} \qquad Arg(z_4) = -\frac{5\pi}{6},$$

the Polar form is given by

$$z_4 = r_4(\cos\theta_4 + i\sin\theta_4),$$

where

$$r_4 = |z_4| = \sqrt{(-\sqrt{3})^2 + (-1)^2} = 2, \qquad \theta_4 = Arg(z_4) = -\frac{5\pi}{6},$$

then we conclude

$$\begin{aligned} z_4 &= 2\left[\cos\left(-\frac{5\pi}{6}\right) + i\sin\left(-\frac{5\pi}{6}\right)\right] \\ \textbf{or} \quad &= 2\left[\cos\left(\frac{5\pi}{6}\right) - i\sin\left(\frac{5\pi}{6}\right)\right]. \end{aligned}$$

Exercise 1.4

Write in **Exponential Form** *the Complex Numbers of the previous Exercise.*

Solution

We recall that it is easy to obtain the Exponential Form from the Polar Form:

$$z = r(\cos\theta + i\sin\theta) = re^{i\theta}.$$

Then

$$z_1 = \sqrt{2}[\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)] = \sqrt{2}e^{i\frac{\pi}{4}},$$
$$z_2 = \sqrt{2}[\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)] = \sqrt{2}e^{i\frac{3\pi}{4}},$$
$$z_3 = 2[\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)] = 2e^{-i\frac{\pi}{6}},$$
$$z_4 = 2[\cos\left(-\frac{5\pi}{6}\right) + i\sin\left(-\frac{5\pi}{6}\right)] = 2e^{-i\frac{5\pi}{6}}.$$

Exercise 1.5

Compute the following quantities:

$$z_1 \cdot z_2, \qquad \frac{z_2}{z_3}, \qquad z_3 \cdot z_4, \qquad \frac{z_1}{z_4}.$$

Solution

We have:

$$z_1 \cdot z_2 = \sqrt{2}e^{i\frac{\pi}{4}} \cdot \sqrt{2}e^{i\frac{3\pi}{4}} = \sqrt{2}\sqrt{2}e^{i\left(\frac{\pi}{4}+\frac{3\pi}{4}\right)} = 2e^{i\pi} = -2,$$

$$\frac{z_2}{z_3} = \frac{\sqrt{2}e^{i\frac{3\pi}{4}}}{2e^{-i\frac{\pi}{6}}} = \frac{\sqrt{2}}{2}e^{i\left[\frac{3\pi}{4}-\left(-\frac{\pi}{6}\right)\right]} = \frac{\sqrt{2}}{2}e^{i\left(\frac{11\pi}{12}\right)},$$

$$z_3 \cdot z_4 = 2e^{-i\frac{\pi}{6}} \cdot 2e^{-i\frac{5\pi}{6}} = 4e^{-i\pi} = 4e^{i\pi} = -4,$$

$$\frac{z_1}{z_4} = \frac{\sqrt{2}e^{i\frac{\pi}{4}}}{2e^{-i\frac{5\pi}{6}}} = \frac{\sqrt{2}}{2}e^{i\frac{13}{12}\pi} = \frac{\sqrt{2}}{2}e^{-i\frac{11}{12}\pi}.$$

Exercise 1.6

Find the solutions of the equation $z^4 = i$.

Solution

There are 4 solutions that we denote by $\omega_0, \omega_1, \omega_2, \omega_3$. We first represent i in exponential form $re^{i\theta}$, with $r = 1$ and $\theta = \frac{\pi}{2}$. Then by the formula we get

$$\omega_k = 1^{\frac{1}{4}} e^{i\frac{\frac{\pi}{2}+2k\pi}{4}} = e^{i\left(\frac{\pi}{8}+k\frac{\pi}{2}\right)}, \qquad \text{with } k = 0, 1, 2, 3.$$

In details we have

$$\begin{aligned}
k &= 0, &\qquad \omega_0 &= e^{i\frac{\pi}{8}}, \\
k &= 1, &\qquad \omega_1 &= e^{i\frac{5\pi}{8}}, \\
k &= 2, &\qquad \omega_2 &= e^{i\frac{9\pi}{8}} = e^{-i\frac{7\pi}{8}}, \\
k &= 3, &\qquad \omega_3 &= e^{i\frac{13\pi}{8}} = e^{-i\frac{3\pi}{8}}.
\end{aligned}$$

Exercise 1.7

Solve the equation: $z^4 - 6iz^2 + 16 = 0$

Solution:

We let $w = z^2$ and rewrite the given equation as $w^2 - 6iw + 16 = 0$. Applying the quadratic formula

$$\omega = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

to the equation, we first have

$$w = \frac{6i \pm \sqrt{(-6i)^2 - 4 * 16}}{2} = \frac{6i \pm \sqrt{-100}}{2} = \frac{6i \pm 10i}{2} = 8i \text{ or } -2i$$

Hence we have $z^2 = 8i$ or $z^2 = -2i$, to find the values of z we need the square roots of $8i$ and $-2i$.

$$z^2 = 8i = 8e^{\frac{\pi}{2}i} \qquad \Rightarrow z = \sqrt{8}e^{i\frac{\pi}{4}} \text{ or } \sqrt{8}e^{-i\frac{3\pi}{4}}$$

$$z^2 = -2i = 2e^{-\frac{\pi}{2}i} \qquad \Rightarrow z = \sqrt{2}e^{-i\frac{\pi}{4}} \text{ or } \ z = \sqrt{2}e^{i\frac{3\pi}{4}}$$

Exercise 1.8

Write the following Complex Functions in the form of $f = u + iv$:

1. $f(z) = |z|^2 + \bar{z} - 5z$,
2. $f(z) = \dfrac{1}{\bar{z}}$.

Solution:

(1) We let $z = x + iy$ and obtain

$$\begin{aligned} f(z) &= |z|^2 + \bar{z} - 5z \\ &= x^2 + y^2 + \overline{(x + iy)} - 5(x + iy) \\ &= x^2 + y^2 + x - iy - 5x - i5y \\ &= x^2 + y^2 - 4x - i6y, \end{aligned}$$

from which we get

$$u(x, y) = x^2 + y^2 - 4x, \qquad v(x, y) = -6y.$$

(2) For the second function we have:

$$f(z) = \frac{1}{\bar{z}} = (\bar{z})^{-1} = (x - iy)^{-1}$$
$$= \frac{x + iy}{x^2 + y^2}$$

where we have used the formula $z^{-1} = \dfrac{\bar{z}}{|z|^2}$. Then:

$$u(x, y) = \frac{x}{x^2 + y^2}, \qquad v(x, y) = \frac{y}{x^2 + y^2}$$

Remark:

Observe that in the computation of the previous exercise we have used two important rules:

$$\bar{\bar{z}} = z, \qquad |z| = |\bar{z}|$$