Chapter 1.9 Distributions of Several Random Variables

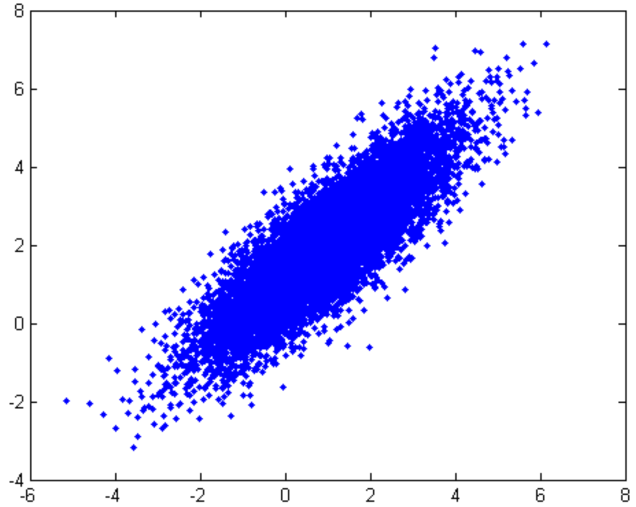
1.9.1 Discrete 2D Distributions

1.9.2 Continuous 2D Distributions

1.9.3 Covariance and correlation

1.9.4 Mean and variance of a sum of variables

1.9.5 Summary



16 March 2018

1.9.0 Joint Distributions

Often we are given a sample space S and n random variables defined on S, where n is an integer greater ≥ 2. We shall consider the discrete case here.

For example height and weight of a person: what is the probability that a person's weight is $> 80Kg$ and the height is$\leq 1.60m$, that is $P(W > 80 \cap H \leq 1.60)$. Clearly, the probability changes if the person is taller.

Another example is the number of alpha and beta radiations emitted by a particle.

We study the joint behavior of X and Y through the joint distribution (X, Y).

1.9.0 Joint Distributions

The joint distribution of (X, Y) describes the probabilities that X and Y take some values at the same time.

If the two variables are independent there is not much point in considering their joint distribution.

The important case is when the variables are dependent and the value of one affects the probability of the other.

1.9.1 Discrete 2D Distributions

A two-dimensional random variable (X, Y) has discrete distribution if it assumes finite (or countably infinite) pairs of values $(x_1, y_1), (x_2, y_2), \cdots$ with positive joint probabilities, whereas for any other pairs not given above will have zero probabilities.

1.9.1 Discrete 2D Distributions

We have the following results:

1. Probability mass function
$$f(x,y) = P(X = x, Y = y)$$
2. Cumulative distribution function
$$F(x,y) = P(X \le x, Y \le y) = \sum_{x_i \le x} \sum_{y_j \le y} f\left(x_i, y_j\right)$$
3. $\sum_i \sum_j f\left(x_i, y_j\right) = 1$

4. Marginal distribution of Y
$$f(x) = P(X = x) = \sum_{all\ y_j} f\left(x, y_j\right)$$
$$f(y) = P(Y = y) = \sum_{all\ x_i} f(x_i, y)$$

1.9.1 Discrete 2D Distributions, example (1)

Example 1

Let Y_1, Y_2 be discrete random variables with joint probability mass function $f(y_1, y_2)$ given by:

$f_{Y_1 Y_2}(0{,}0) = P(Y_1 = 0, Y_2 = 0)$

$f_{Y_2}(0) = P(Y_2 = 0)$

		$\boldsymbol{Y_1}$			
		0	1	2	Tot
$\boldsymbol{Y_2}$	0	0	0.1	0.2	0.3
	1	0.1	0.2	0	0.3
	2	0.4	0	0	0.4
	Tot	0.5	0.3	0.2	1

$f_{Y_1}(2) = P(Y_1 = 2)$

1.9.1 Discrete 2D Distributions, example (2)

Example 1

$f_{Y_1,Y_2}(y_1,y_2)$

		$\boldsymbol{Y_1}$			
		0	1	2	Tot
$\boldsymbol{Y_2}$	0	0	0.1	0.2	0.3
	1	0.1	0.2	0	0.3
	2	0.4	0	0	0.4
	Tot	0.5	0.3	0.2	1

$f_{Y_2}(y_2)$

$f_{Y_1}(y_1)$

i. Find the probability $\mathrm{F}(1,1) = P(Y_1 \leq 1, Y_2 \leq 1)$;

ii. Find the marginal probability $f_{Y_1}(y_1) = P(Y_1 = y_1)$;

iii. Find the marginal cdf of Y_1, $F_{Y_1}(y_1) = P(Y_1 \leq y_1)$.

1.9.1 Discrete 2D Distributions, example (3)

Solution

i. $F(1,1) = P(Y_1 \leq 1, Y_2 \leq 1) =$

$P(Y_1 = 0, Y_2 = 0) + P(Y_1 = 1, Y_2 = 0) +$

$P(Y_1 = 0, Y_2 = 1) + P(Y_1 = 1, Y_2 = 1) =$

$0 + 0.1 + 0.1 + 0.2 = 0.4$

	Y_1		
Y_2	0	1	2
0	0	0.1	0.2
1	0.1	0.2	0
2	0.4	0	0

1.9.1 Discrete 2D Distributions, example (4)

ii. We find the marginal distribution $f_{Y_1}(y_1)$.

$P(Y_1 = 0) = P(Y_1 = 0, Y_2 = 0) + P(Y_1 = 0, Y_2 = 1) + P(Y_1 = 0, Y_2 = 2)$
$= 0 + 0.1 + 0.4 = 0.5$

$P(Y_1 = 1) = P(Y_1 = 1, Y_2 = 0) + P(Y_1 = 1, Y_2 = 1) + P(Y_1 = 1, Y_2 = 2) =$
$0.1 + 0.2 + 0 = 0.3$

$P(Y_1 = 2) = P(Y_1 = 2, Y_2 = 0) + P(Y_1 = 2, Y_2 = 1) + P(Y_1 = 2, Y_2 = 2)$
$= 0.2 + 0 + 0 = 0.2$

		Y_1		
		0	1	2
Y_2	0	0	0.1	0.2
	1	0.1	0.2	0
	2	0.4	0	0
		$P(Y_1 = 0) = 0.5$	$P(Y_1 = 1) = 0.3$	$P(Y_1 = 2) = 0.2$

1.9.1 Discrete 2D Distributions, example (3)

	Y_1		
Y_2	0	1	2
0	0	0.1	0.2
1	0.1	0.2	0
2	0.4	0	0
	0.5	0.3	0.2

Solution

iii. Find the marginal cmf of Y_1

$$F_{Y_1}(0) = P(Y_1 \leq 0) = P(Y_1 = 0) = 0.5$$

$$F_{Y_1}(1) = P(Y_1 \leq 1) =$$

$$P(Y_1 = 0) + P(Y_1 = 1) = 0.5 + 0.3 = 0.8$$

$$F_{Y_1}(2) = P(Y_1 \leq 2) = P(Y_1 = 0) + P(Y_1 = 1) + P(Y_1 = 2)$$

$$= 0.5 + 0.3 + 0.2 = P(Y_1 \leq 1) + P(Y_1 = 2) = 0.8 + 0.2 = 1$$

1.9.1 Discrete 2D Distributions, example (5)

In the same way we can find the marginal distribution $f(y_2)$.

$$P(Y_2 = 0) = P(Y_1 = 0, Y_2 = 0) + P(Y_1 = 1, Y_2 = 0) + P(Y_1 = 2, Y_2 = 0) = 0 + \frac{1}{10} + \frac{2}{10} = \frac{3}{10}$$

$$P(Y_2 = 1) = P(Y_1 = 0, Y_2 = 1) + P(Y_1 = 1, Y_2 = 1) + P(Y_1 = 2, Y_2 = 1) = \frac{1}{10} + \frac{2}{10} + 0 = \frac{3}{10}$$

$$P(Y_2 = 2) = P(Y_1 = 0, Y_2 = 2) + P(Y_1 = 1, Y_2 = 2) + P(Y_1 = 2, Y_2 = 2) = \frac{2}{10} + 0 + 0 = \frac{1}{5}$$

		$\boldsymbol{Y_1}$			
		0	1	2	
$\boldsymbol{Y_2}$	0	0	0.1	0.2	$P(Y_2 = 0) = 0.3$
	1	0.1	0.2	0	$P(Y_2 = 1) = 0.3$
	2	0.4	0	0	$P(Y_2 = 2) = 0.4$

1.9.2 Continuous 2D Distributions

We shall consider the continuous two-dimensional (bivariate) distributions here.

For a joint probability density function $f(x, y)$:

1. Cumulative density function is $F(x, y) = \int_{Y \leq y} \int_{X \leq x} f(u, v)\, dudv$
2. $\lim\limits_{a \to \infty} \lim\limits_{b \to \infty} F(a, b) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)\, dxdy = 1$
3. Probabilities are found through the double integration of the joint pdf, i.e.

   $$P(a \leq X \leq b, c \leq Y \leq d) = \int_{c}^{d} \int_{a}^{b} f(x, y)\, dxdy.$$

1.9.2 Continuous 2D Distributions

4. Marginal densities are

$$f(x) = \int_{-\infty}^{\infty} f(x,y)\, dy$$

$$f(y) = \int_{-\infty}^{\infty} f(x,y)\, dx$$

5. The marginal cdfs

$$F_X(x) = \int_{-\infty}^{x} f_X(u)du = \int_{-\infty}^{x}\int_{-\infty}^{\infty} f(u,v)dvdu = \lim_{y\to\infty} F_{XY}(x,y)$$

$$F_Y(y) = \int_{-\infty}^{y} f_Y(v)dv = \int_{-\infty}^{y}\int_{-\infty}^{\infty} f(u,v)dudv = \lim_{x\to\infty} F_{XY}(x,y)$$

1.9.2 Continuous distributions, example 2 (1)

Consider two variables, X, Y, with joint pdf

$$f_{XY}(x,y) = \begin{cases} 4xy, if\ 0 < x \leq 1, 0 < y \leq 1 \\ 0, otherwise \end{cases}$$

Find the joint cdf, marginal pdfs and cdfs. Also $P(X \leq 0.3|Y \leq 0.5)$,

i) The joint cdf is simply

$$F_{XY}(x,y) = \int_0^x \int_0^y 4uv\, dvdu = x^2y^2, 0 < x \leq 1, 0 < y \leq 1$$

$F_{XY}(x,y) = 0, x, y \leq 0$ and $F_{XY}(x,y) = 1, x, y > 1.$

1.9.2 Continuous distributions, example 2 (2)

$$f_{XY}(x,y) = \begin{cases} 4xy, if\ 0 < x \leq 1, 0 < y \leq 1 \\ 0, otherwise \end{cases}$$

ii) The marginal pdf's are

$$f_X(x) = \int_0^1 4xv\, dv = 4x\left[\frac{y^2}{2}\right]_0^1 = 2x, 0 < x \leq 1$$

$$f_Y(y) = \int_0^1 4uy\, du = 4y\left[\frac{x^2}{2}\right]_0^1 = 2y, 0 < y \leq 1$$

Remember for discrete variables we took the sums

1.9.2 Continuous distributions, example 2 (3)

$$f_{XY}(x,y) = \begin{cases} 4xy, if\ 0 < x \leq 1, 0 < y \leq 1 \\ 0, otherwise \end{cases}$$

iii. The marginal cdf's can be simply computed from the joint cdf:

$$F_X(x) = F_{XY}(x,1) = [x^2y^2]_{y=1} = x^2, 0 < x \leq 1;$$

$$F_Y(y) = F_{XY}(1,y) = [x^2y^2]_{x=1} = y^2, 0 < y \leq 1$$

In fact, $F_X(x) = \int_0^x \int_0^1 4uv\, dv\, du = \int_0^x f_X(u)du = [u^2]_0^x = x^2;$

Naturally we can show the same for Y.

1.9.2 Continuous distributions, example 2 (4)

Recall that, $F_{XY}(x, y) = P(X \leq x, Y \leq y)$. So the marginal cdfs are given by

$$\mathrm{F_X(x)} = P(X \leq x) = \lim_{y \to \infty} P(X \leq x, Y \leq y)$$

$$= P(X \leq x, Y \leq 1) = F_{XY}(x, 1)$$

because Y must be ≤ 1. In general substitute the maximum value of the other variable or take the limit to ∞.

$$\mathrm{F}_Y(\mathrm{y}) = P(Y \leq y) = F_{XY}(1, y)$$

1.9.2 Continuous 2D Distributions, example 3

To solve this we need to use integration over a general region

Example 3

For joint probability density function

$$f(x, y) = 2; \quad 0 < x \leq y, 0 < y \leq 1$$

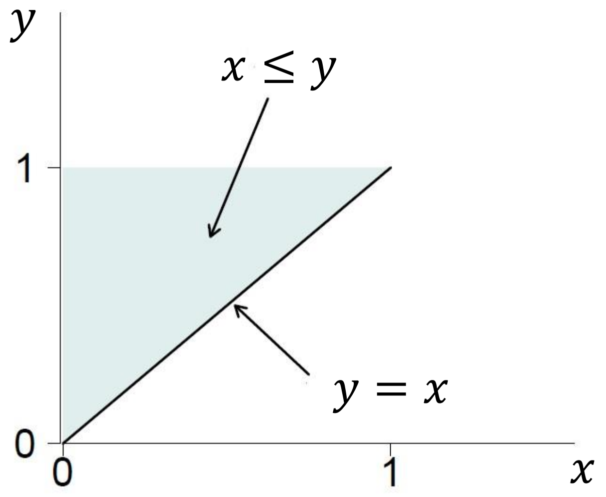
find its distribution function $F(x, y)$.

Solution

We can write the domain R as

$0 < x \leq 1, x < y \leq 1$

This is more useful because we integrate from below



1.9.2 Continuous 2D Distributions, example 3(2)

The distribution function $F(x, y)$ is

$$F(x,y) = \int_{X \le x} \int_{Y \le y} f(x,y)\, dy dx$$

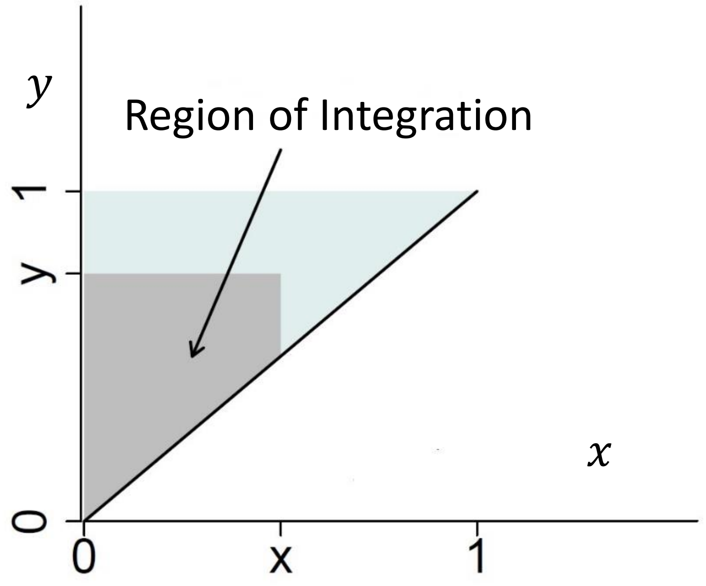
$$= \int_0^{x} \int_{x}^{y} f(x,y)\, dy dx$$

$$= 2\int_0^x \int_x^y 1\, dy dx$$

$$= 2\int_0^x [y - x]\, dx$$

$$= 2\left[yx - \tfrac{1}{2}x^2\right]$$

$$= 2yx - x^2 = x(2y - x); 0 < x < y, 0 < y < 1 \quad \blacksquare$$



1.9.2 Continuous 2D Distributions, example 3(3)

Example 3 (continued)

i. Find the marginal distribution $f_Y(y)$.

Solution

Recall that $f(x,y) = 2; \quad 0 < x \leq y, 0 < y \leq 1.$

i. Marginal distribution $f_Y(y) = \int_0^\infty f(x,y)dx =$

$\int_0^y f(x,y)dx = \int_0^y 2dx = 2y\,; 0 < y \leq 1.$

1.9.2 Continuous 2D Distributions, example 3(4)

ii) Find the marginal distribution $F_Y(y)$.

Solution 2 of (i)

i. Recall that $\mathrm{f}_{\mathrm{X,Y}}(x, y) = 2 \;\; ; \;\; 0 < x \leq y, 0 < y \leq 1$. Then

$$F_Y(y) = \lim_{x \to \infty} F_{XY}(x, y) = F_{X,Y}(x = y, y) =$$

$$[2yx - x^2]_{x=y} = y^2$$

Marginal distribution $f_Y(y) = \frac{dF_Y(y)}{dy} = 2y \,; 0 < y \leq 1.$

1.9.3 Covariance and correlation

The covariance between two variables is a measure of their association. It is defined as

$$Cov(X,Y) = E[(X-\mu_X)(Y-\mu_Y)]$$

A computationally more convenient form of the covariance is $Cov(X,Y) = E(XY) - \mu_X E(Y) - \mu_Y E(X) + \mu_X \mu_Y = E(XY) - \mu_X \mu_Y = E(XY) - E(X)E(Y)$.

Obviously, $Cov(X,Y) = Cov(Y,X)$.

1.9.3 Covariance

The covariance is thus a measure of the extent to which the values of X and Y tend to increase or decrease together:

Case 1) If X has values greater than its mean μ_X whenever Y has values greater than its mean μ_Y and X has values less than μ_X whenever Y has values less than μ_Y, then

$$(X - \mu_X)(Y - \mu_Y) > 0 \text{ and therefore } \mathrm{Cov}(X, Y) > 0.$$

Case 2) On the other hand, if values of X are above μ_X whenever values of Y are below μ_Y and vice versa, then $\mathrm{Cov}(X, Y) < 0$.

Case 3) Generally we have a mixed case comprising of (1) and (2). The sign of $\mathrm{Cov}(X, Y)$ depends on the extent of (1) and (2)

1.9.3 Covariance and correlation

When $Cov(X,Y) > 0$ the variables have positive relationship, i.e. they are small and large together.

Vice versa for $Cov(X,Y) < 0$

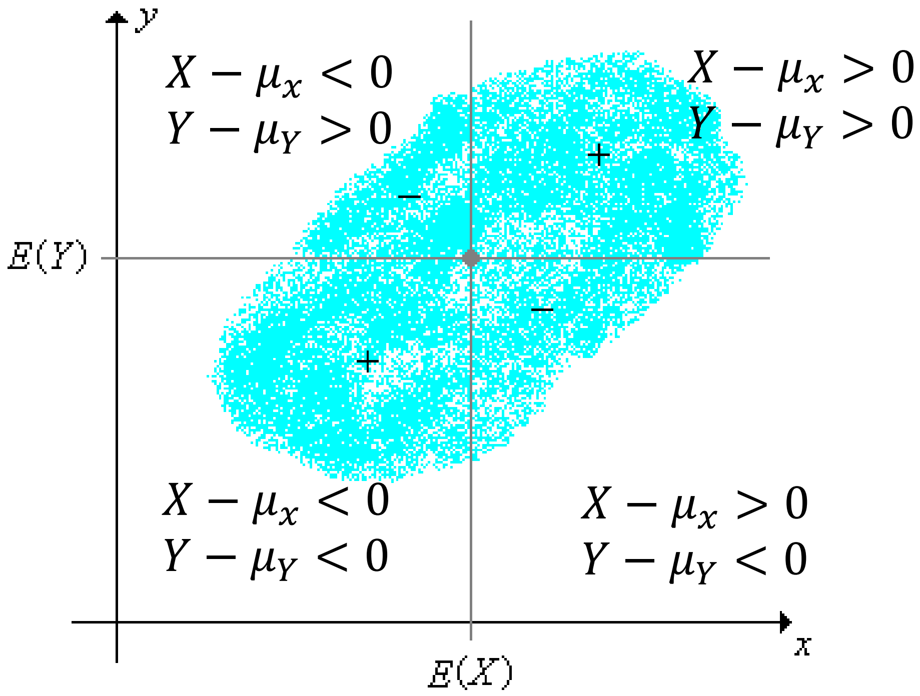


Diagram describing case (3) in which $(X - \mu_X)(Y - \mu_Y) > 0$ for most values of (X,Y) and $(X - \mu_X)(Y - \mu_Y) < 0$ for others.

1.9.3 Covariance and correlation

The covariance is measured in units (for example m*Kg) and can take any value

$$-\infty < Cov(X,Y) < \infty$$

A more useful measure of association is the correlation (coefficient)

Definition Assuming the variances $Var(X) > 0$ and $Var(Y) > 0$, the correlation of X and Y is given by

$$\rho(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sigma_X \sigma_Y}.$$

The correlation is a scaled version of covariance, $-1 \le \rho \le 1$.

It is dimensionless number, i.e. no units.

1.9.3 Covariance and correlation

1) The correlation measures the **linear association** between X and Y.

   When $\rho > 0$, the variables are positively (linearly) correlated.

   When $\rho < 0$, the variables are negatively (linearly) correlated.

   When $\rho = 0$, the variables are (linearly) uncorrelated.

2) The correlation is symmetric, i.e. $\rho(X,Y) = \rho(Y,X)$.

3) When $Cov(X,Y) = 0$:

   a) $E(XY) = E(X)E(Y)$. Derives from $Cov(X,Y) = E(XY) - E(X)E(Y) = 0$

   b) $\rho(X,Y) = \frac{Cov(X,Y)}{\sigma_X \sigma_Y} = 0$

1.9.3 Covariance and correlation

Example 6

Given the joint probability mass function table, find the correlation ρ of X and Y:

		Y	
		0	1
X	0	0.7	0.1
	1	0.1	0.1

1.9.3 Covariance and correlation: discrete

Solution The solutions are:

		Y		
		0	1	
X	0	0.7	0.1	$P(X=0) = 0.8$
	1	0.1	0.1	$P(X=1) = 0.2$
		$P(Y=0) = 0.8$	$P(Y=1) = 0.2$	

$E(X) = (0)(0.8) + (1)(0.2) = 0.2$

$E(Y) = (0)(0.8) + (1)(0.2) = 0.2$

$E(X^2) = (0^2)(0.8) + (1^2)(0.2) = 0.2$

$E(Y^2) = (0^2)(0.8) + (1^2)(0.2) = 0.2$

$E(XY) = (0)(0)(0.7) + (0)(1)(0.1) + (1)(0)(0.1) + (1)(1)(0.1) = 0.1[\neq E(X)E(Y) = 0.04]$

$$\sigma_X^2 = E(X^2) - [E(X)]^2 = 0.2 - 0.04 = 0.16;\ \sigma_Y^2 = 0.2 - 0.04 = 0.16$$

$$Cov(X,Y) = E(XY) - E(X)E(Y) = 0.1 - 0.2^2 = 0.06$$

$$\rho(X,Y) = \frac{Cov(X,Y)}{\sigma_x \sigma_y} = \frac{0.06}{0.16} = \frac{3}{8} = 0.375$$

1.9.3 Covariance and correlation: continuous

Example 7

Consider two variables with pdf

$$f_{XY}(x,y) = 4xy, 0 < x \le 1, 0 < y \le 1$$

Find the covariance and the correlation

Solution

$$f_X(X) = \int_0^1 f_{XY}(x,y)dy = 4x\left[\frac{y^2}{2}\right]_0^1 = 2x, f_Y(y) = 2y.$$

$$E(X) = \int_0^1 x f_X(x)dx = \int_0^1 2x^2 dx = \frac{2}{3} = E(Y)$$

1.9.3 Covariance and correlation: continuous

$$E(X^2) = \int_0^1 x^2 f_X(x)dx = \int_0^1 2x^3 dx = \frac{1}{2} = E(Y^2)$$

$$Var(X) = E(X^2) - [E(X)]^2 = \frac{1}{2} - \left(\frac{2}{3}\right)^2 = \frac{1}{18} = Var(Y)$$

$$E(XY) = \int_0^1 \int_0^1 xyf_{XY}(x,y)dydx = \int_0^1 \int_0^1 4x^2y^2 dydx = \frac{4}{9}$$

$$Cov(X,Y) = E(XY) - E(X)E(Y) = \frac{4}{9} - \frac{2}{3}\frac{2}{3} = 0\ [E(XY) = E(X)E(Y)]$$

$$\rho(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}} = 0$$

1.9.4 Sum of variables

Determining the distribution of a sum of variables is, in general, complicated.

For example, even the sum of two uniform variables is not uniform. In some cases it is easy, like for Poisson or Exponential variables.

However, the mean and variance are simpler to determine.

1.9.4 Mean of a sum of variables

Theorem 1

The mean of a sum of random variables equals the sum of the means, i.e.

$$E(X_1 + X_2 + \cdots + X_n) = E(X_1) + E(X_2) + \cdots + E(X_n)$$

This result is always true (for continuous/discrete, and Independent/dependent random variables).

1.9.4 Mean of a sum of variables

Proof (Continuous case)

We consider $E(X_1 + X_2)$ first.

$$E(X_1 + X_2) = \int_{\mathbb{R}} \int_{\mathbb{R}} (x_1 + x_2) f(x_1, x_2)\, dx_1 dx_2$$

$$= \int_{\mathbb{R}} x_1 \left[\int_{\mathbb{R}} f(x_1, x_2)\, dx_2 \right] dx_1 + \int_{\mathbb{R}} x_2 \left[\int_{\mathbb{R}} f(x_1, x_2)\, dx_1 \right] dx_2$$

$$= \int_{\mathbb{R}} x_1 f(x_1) dx_1 + \int_{\mathbb{R}} x_2 f(x_2) dx_2 = E(X_1) + E(X_2).$$

This can be further generalized to prove

$$E(X_1 + X_2 + \cdots + X_n) = E(X_1) + E(X_2) + \cdots + E(X_n).$$

The discrete case can also be similarly proved. ■

1.9.4 Variance of a sum of variables

Theorem 3

$$Var(aX + bY + c) \stackrel{\text{def}}{=} a^2 Var(X) + 2abCov(X, Y) + b^2 Var(Y)$$

Proof

Denoting $\mu_X = E(X)$ and $\mu_Y = E(Y)$,

$$\text{Var}(aX + bY + c) \stackrel{\text{def}}{=} E[(aX + bY) - (a\mu_X + b\mu_Y) + c\ - c]^2 =$$

$$E[a(X - \mu_X) + b(Y - \mu_Y)]^2 =$$

$$a^2 E[X - \mu_X]^2 + 2abE[(X - \mu_X)(Y - \mu_Y)]\ + b^2 E[Y - \mu_Y]^2 =$$

$$\text{a}^2\text{Var}(X) + 2\text{abCov}(X, Y) + \text{b}^2\text{Var}(Y)\ \blacksquare$$

1.9.4 Variance of a sum of variables

Corollary 1

Since the correlation $cor(X,Y) = \rho_{XY} = \frac{Cov(X,Y)}{\sigma_X \sigma_Y}$,

$$\mathrm{Var}(aX + bY + c) \stackrel{\text{def}}{=} \mathrm{a}^2\sigma_X^2 + 2ab\sigma_\mathrm{X}\sigma_Y\rho + b^2\sigma_Y^2$$

Corollary 2

When random variables X and Y are uncorrelated, then $\mathrm{Cov}(aX, bY) = ab\mathrm{Cov}(X,Y) = ab\rho = 0.$ Therefore $\mathrm{Var}(aX + bY) = \mathrm{a}^2\mathrm{Var}(X) + \mathrm{b}^2\mathrm{Var}(Y).$

1.9.4 Variance of a sum of variables

For more variables you have to expand the square. For example.

$$\mathrm{Var}(X+Y+Z) = \mathrm{Var}(X)+\mathrm{Var}(Y)+\mathrm{Var}(Z)+2\mathrm{Cov}(X,Y)+2Cov(X,Z)+2\mathrm{Cov}(Y,Z) =$$

$$E(X^2)-[E(X)]^2+E(Y^2)-[E(Y)]^2+E(Z^2)-[E(Z)]^2$$

$$+2(E(XY)-E(X)E(Y))+2(E(XZ)-E(X)E(Z))$$

$$+2(E(YZ)-E(Y)E(Z))$$

1.9.5 Summary

- Know how to obtain marginal distributions from joint pdf or joint pmf and pdf
- Know how to find covariance $\mathrm{Cov}(X, Y)$ and correlation $\rho(X, Y)$
- Understand the concept of correlation for discrete/continuous 2D distributions
- Know how to compute the mean and the variance of a sum of variables