Xi'an Jiaotong-Liverpool University
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EEE220 Instrumentation and Control System

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Lecture 11

Outline

State Variable Models: part 1/2

- ❑ Introduction
- ❑ State Variables
- ❑ State-space Modeling
- ❑ State Space Representation in Matrix Form
- ❑ Time-domain response (Solution of State-space Models)
- ❑ Conversion between State-space Model and Transfer Function
- ❑ Analysis of the State-space Models using Matlab

Introduction

In the preceding classes, we studied several useful approaches to the analysis of design of control system including (suppose the system order is n):

- Ordinary differential equations (ODEs): nth-order differential equations; time-domain description of system;
- Transfer function: Laplace transform of ODEs; frequency-domain description;
- Block diagram & signal-flow graph: graphic ways derived from transfer functions.

In this chapter, we'll learn

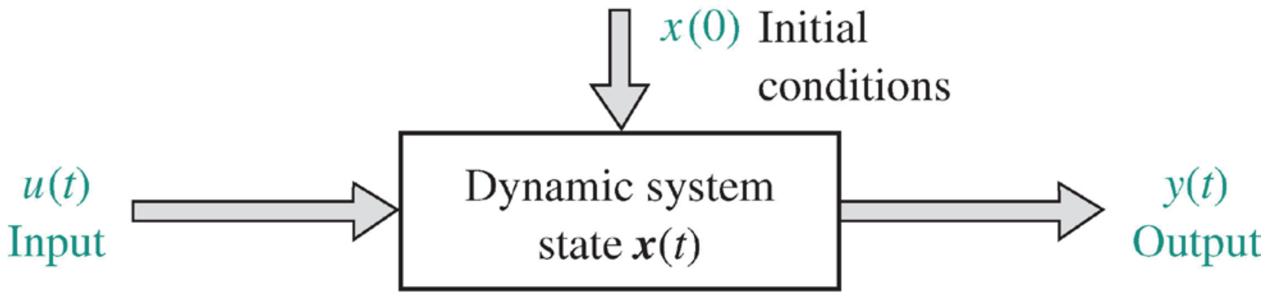
- **State-space model** (or **state variable model**) which
  - is a set of 1^{st} order differential equations therefore a time-domain description of a system and can be further represented in a convenient matrix-vector form;
  - powerful mathematical & computational tools available for analyzing and designing a system using state-space model;
  - can be extended to nonlinear, time-varying & multiple input-output system.

State-space model is an essential basis for modern control theory.

State of A System

The **State** of a system is a set of variables whose values, together with the input signals and the equations describing the dynamics, will provide the future state and output of the system;

It is the minimum information needed about the system in order to determine its future behavior.

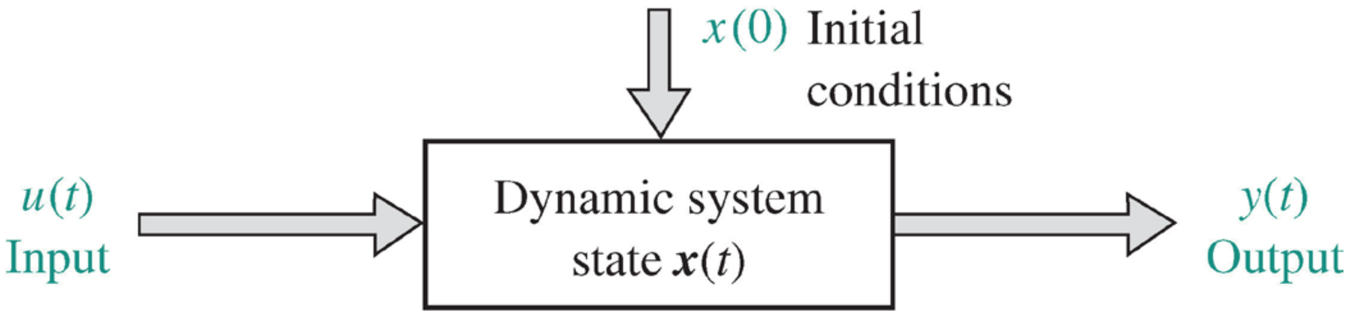


State variables: $[x_1(t), x_2(t), x_3(t), \ldots x_n(t)]$

Input signals: $[u_1(t), u_2(t), \ldots u_m(t)]$

Output signals: $[y_1(t), y_2(t), \ldots y_p(t)]$

State Variables



- The **state variables** of a system are defined as a **minimal set of variables**, which are set of variables of smallest possible size that together with any input to the system is sufficient to determine the future behavior (i.e., output) of the system.
- At any initial time $t = t_0$, the state variables $x_1(t_0), x_2(t_0)$, $x_3(t_0)$, … $x_n(t_0)$ defines the initial states of the system;
- Once the inputs of the system for $t \geq t_0$ and the initial states just defined are specified, the state variables should completely define the future behavior of the system.

State-Space Modeling

- **State differential equations**: first-order differential equations written in terms of the state variables $(x_1(t), x_2(t), \ldots x_n(t))$ and the input $(u_1(t), u_2(t), \ldots u_m(t))$;

- A **state-space model** represents a system by a series of first-order differential state equations and algebraic output equations; It simplifies analysis of complex systems with multiple inputs and outputs;

- State-space models are numerically efficient to solve, can handle complex systems, allow for a more geometric understanding of systems, and form the basis of modern control theory.

Example 11.1

Consider the following system where $u(t)$ is the input and $\dot{x}(t)$ is the output:

$$\dddot{x} + 5\ddot{x} + 3\dot{x} + 2x = u$$

How to generate state-space model?

- Changing variables, let $x_1 = x$, $x_2 = \dot{x}$, $x_3 = \ddot{x}$,

$$\dddot{x} = -5\ddot{x} - 3\dot{x} - 2x + u \longrightarrow \left\{ \begin{aligned} \dot{x}_3 &= -5x_3 - 3x_2 - 2x_1 + \mathrm{u} \\ \dot{x}_2 &= x_3 \\ \dot{x}_1 &= x_2 \end{aligned} \right.$$

State equations

- Obtaining output:

$$\left. y = \dot{x} = x_2 \right\}$$

Output equation

The system has 1 input ($u(t)$), 1 output ($y(t)$) and 3 state variables ($x_1(t), x_2(t), x_3(t)$).

State-Space Modeling: General Form

General state-space models have the following form:

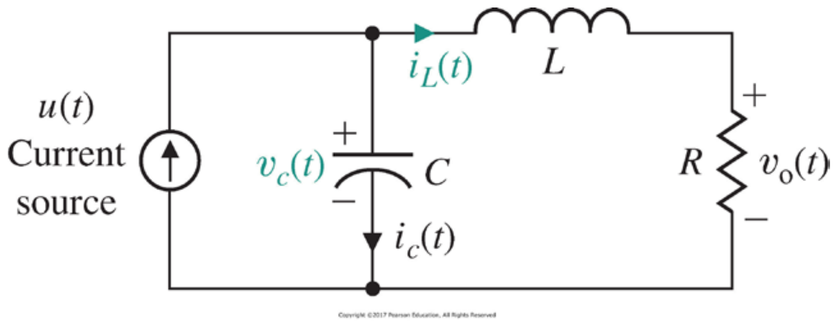
State equations

$$\left\{\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2, \dots x_n, u_1, u_2, \dots u_m,) \\ \dot{x}_2 &= f_2(x_1, x_2, \dots x_n, u_1, u_2, \dots u_m,) \\ &\vdots \\ \dot{x}_n &= f_n(x_1, x_2, \dots x_n, u_1, u_2, \dots u_m,) \end{aligned}\right.$$

Output equations

$$\left\{\begin{aligned} y_1 &= h_1(x_1, x_2, \dots x_n, u_1, u_2, \dots u_m,) \\ y_2 &= h_2(x_1, x_2, \dots x_n, u_1, u_2, \dots u_m,) \\ &\vdots \\ y_p &= h_p(x_1, x_2, \dots x_n, u_1, u_2, \dots u_m,) \end{aligned}\right.$$

Example 11.2: RLC Circuit



Ordinary Differential Equations:

$$i_c(t) = C\frac{dv_c}{dt} = u(t) - i_L(t)$$

$$L\frac{di_L}{dt} = -Ri_L(t) + v_c(t)$$

Output:

$$v_0(t) = Ri_L(t)$$

- Choose variables, let $x_1(t) = v_c(t)$, $x_2(t) = i_L(t)$

$$\frac{dx_1}{dt} = -\frac{1}{C}x_2(t) + \frac{1}{C}u(t)$$

$$\frac{dx_2}{dt} = \frac{1}{L}x_1(t) - \frac{R}{L}x_2(t)$$

$$y(t) = v_o(t) = Rx_2(t)$$

$$\longrightarrow$$

$$\dot{x}_1 = -\frac{1}{C}x_2 + \frac{1}{C}u$$

$$\dot{x}_2 = \frac{1}{L}x_1 - \frac{R}{L}x_2$$

$$y = Rx_2$$

The state variables describing a system are not a unique set, several alternative sets of state variables can be chosen.

State Variables Selection

State variables are typically has associated with energy storage:

- Each state Variable has memory;
- Each state variable has an "initial condition";

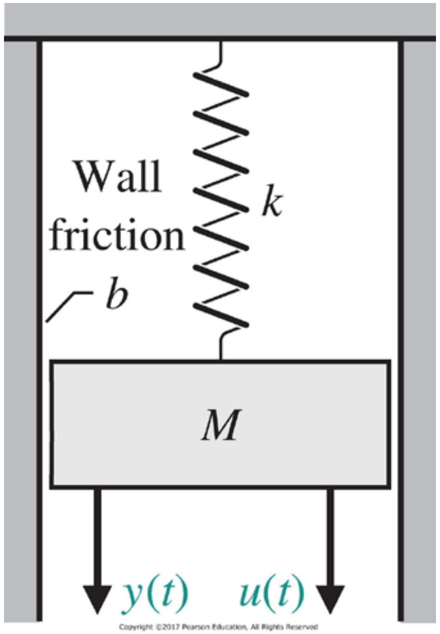
For a system, the number of state variables required is equal to the number of independent energy-storage elements.

- In electric circuits, the energy storage devices are the **capacitors** and **inductors**. They contain all of the state information or "memory" in the system. State variables:
  - Voltage across capacitors
  - Current through inductors

- In mechanical systems, energy is stored in **springs** and **masses**. State variables:
  - Spring displacement
  - Mass position and velocity

- **Resistors** (in electric circuits) and **dampers** (mechanical systems) are energy dissipaters, **they don't store energy**.

Example 11.3: Spring-Mass-Damper System



Independent energy-storage elements:

- 1 spring;
- 1 mass.

Number of state variables: 2

- let $x_1 = y(t)$, $x_2 = \dot{x}_1$, therefore, x_1 and x_2 will represent displacement and velocity respectively.

$$M\frac{d^2y(t)}{dt^2} + b\frac{dy(t)}{dt} + \mathrm{ky(t)} = \mathrm{u(t)} \longrightarrow \dot{x}_2 = \frac{-b}{M}x_2 - \frac{k}{M}x_1 + \frac{1}{M}u$$

$$\dot{x}_1 = x_2$$

Output: $y = x_1$

State-space Representation in Matrix Form

State Equations:

$$\dot{x}_1(t) = a_{11}x_1(t) + a_{12}x_2(t) + \cdots + a_{1n}x_n(t) + b_{11}u_1(t) + \cdots + b_{1m}u_m(t)$$

$$\dot{x}_2(t) = a_{21}x_1(t) + a_{22}x_2(t) + \cdots + a_{2n}x_n(t) + b_{21}u_1(t) + \cdots + b_{2m}u_m(t)$$

$$\vdots$$

$$\dot{x}_n(t) = a_{n1}x_1(t) + a_{n2}x_2(t) + \cdots + a_{nn}x_n(t) + b_{n1}u_1(t) + \cdots + b_{nm}u_m(t)$$

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix} \qquad \mathbf{u}(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_m(t) \end{pmatrix}$$

$$\boxed{\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)}$$

$$\mathbf{y}(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_p(t) \end{pmatrix}$$

Output Equations: $\boxed{\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)}$

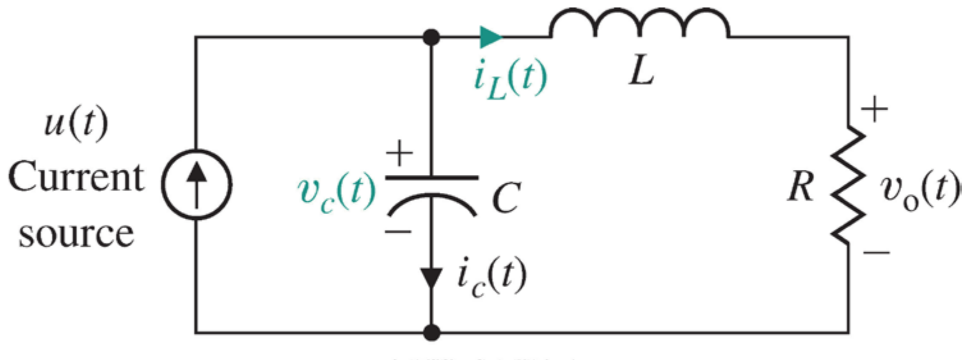
Dimensions: $\mathbf{x}(n \times 1), \mathbf{u}(m \times 1), \mathbf{y}(p \times 1);$
$\mathbf{A}(n \times n), \mathbf{B}(n \times m), \mathbf{C}(p \times n), \mathbf{D}(p \times m).$

**note: boldface small letter: vector; boldface capital letter: matrix.*

Examples

$$\boxed{\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)}$$

$$\boxed{\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)}$$



$$\dot{x}_1 = -\frac{1}{C}x_2 + \frac{1}{C}u$$

$$\dot{x}_2 = \frac{1}{L}x_1 - \frac{R}{L}x_2$$

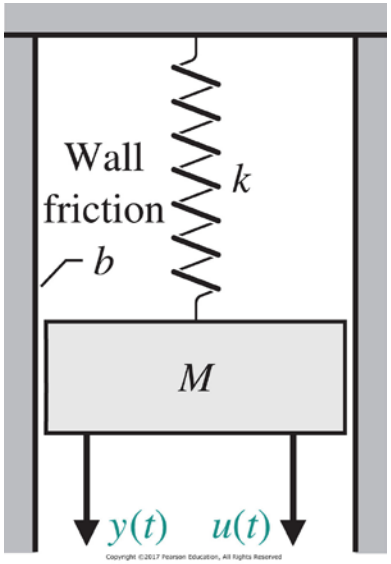
$$y = Rx_2$$

$$A = \begin{bmatrix} 0 & -\frac{1}{C} \\ \frac{1}{L} & -\frac{R}{L} \end{bmatrix}, \qquad B = \begin{bmatrix} \frac{1}{C} \\ 0 \end{bmatrix}$$

$$C = [0 \quad R], \qquad D = [0]$$

$$\boxed{\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)}$$

$$\boxed{\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)}$$



$$\dot{x}_1 = x_2$$

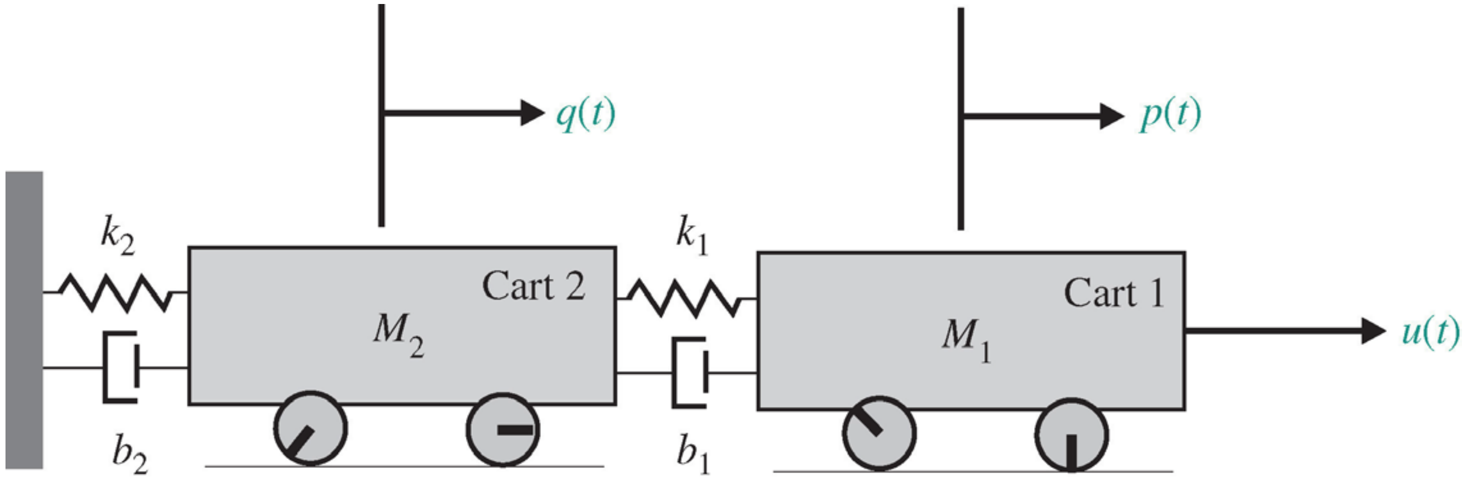
$$\dot{x}_2 = \frac{-b}{M}x_2 - \frac{k}{M}x_1 + \frac{1}{M}u$$

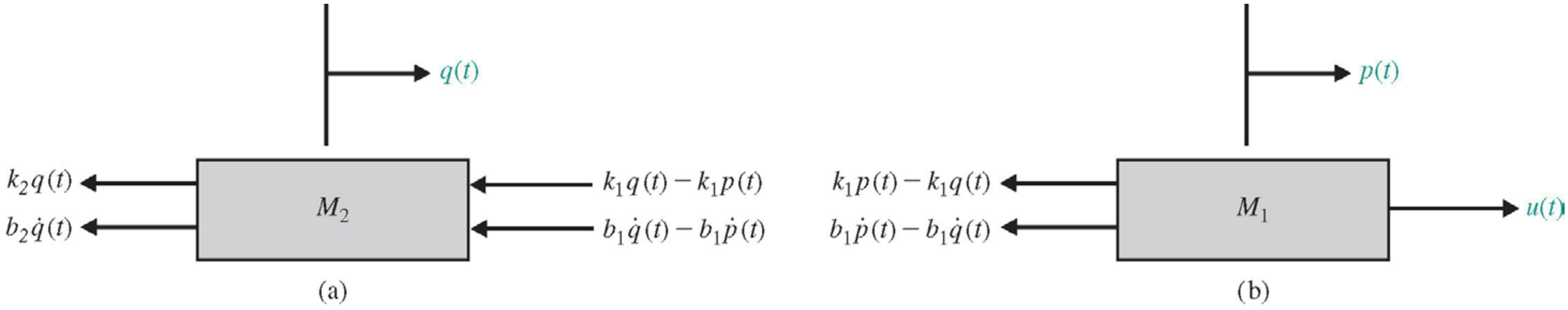
$$y = x_1$$

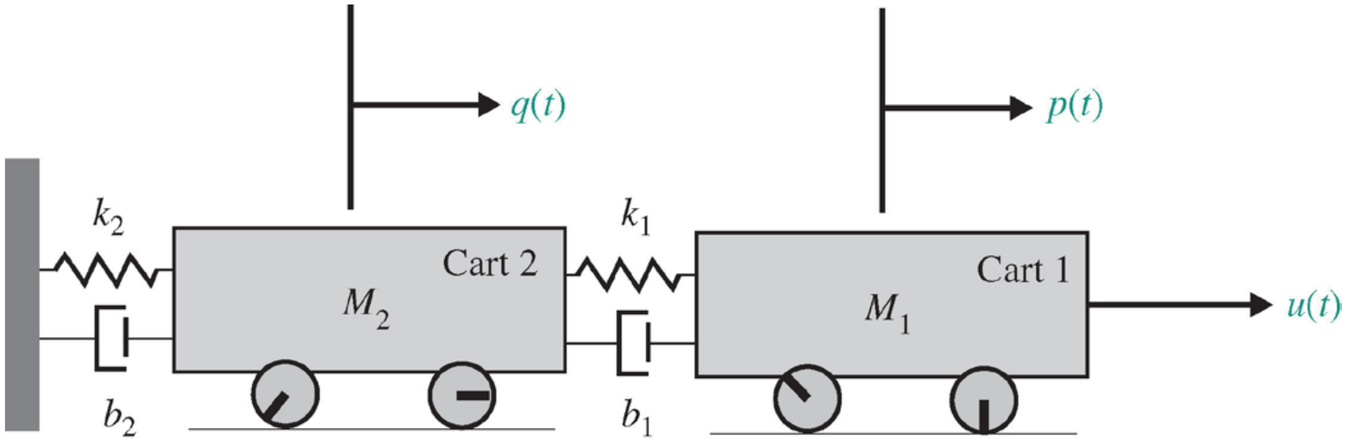
$$A = \begin{bmatrix} 0 & 1 \\ -\frac{k}{M} & -\frac{b}{M} \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \qquad D = [0]$$

Example 11.4: Two Rolling Carts







where p, q are position of carts; M1 and M2 are mass of carts;
k1, k2: spring coefficients; b1, b2: damper coefficients. u: external force.

For Mass 1: $$M_1\ddot{p} + b_1\dot{p} + k_1 p = u + k_1 q + b_1\dot{q},$$

For Mass 2: $$M_2\ddot{q} = k_1(p - q) + b_1(\dot{p} - \dot{q}) - k_2 q - b_2\dot{q}.$$

How many state variables required?

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

$$x_1 = p, \qquad x_3 = \dot{x}_1 = \dot{p},$$
$$x_2 = q. \qquad x_4 = \dot{x}_2 = \dot{q}.$$

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} p \\ q \\ \dot{p} \\ \dot{q} \end{pmatrix}$$

$$\dot{x}_3 = \ddot{p} = -\frac{b_1}{M_1}\dot{p} - \frac{k_1}{M_1}p + \frac{1}{M_1}u + \frac{k_1}{M_1}q + \frac{b_1}{M_1}\dot{q},$$

$$\dot{x}_4 = \ddot{q} = -\frac{k_1 + k_2}{M_2}q - \frac{b_1 + b_2}{M_2}\dot{q} + \frac{k_1}{M_2}p + \frac{b_1}{M_2}\dot{p},$$

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k_1}{M_1} & \frac{k_1}{M_1} & -\frac{b_1}{M_1} & \frac{b_1}{M_1} \\ \frac{k_1}{M_2} & -\frac{k_1 + k_2}{M_2} & \frac{b_1}{M_2} & -\frac{b_1 + b_2}{M_2} \end{bmatrix}, \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{M_1} \\ 0 \end{bmatrix},$$

$$y = [1 \quad 0 \quad 0 \quad 0]\mathbf{x} = \mathbf{C}\mathbf{x}. \qquad \mathbf{D} = [0]$$

Solution of State Equations

Consider the first-order differential equation (**one-dimension**):

$$\dot{x} = ax + bu$$

With Laplace transform,

$$sX(s) - x(0) = aX(s) + bU(s)$$

$$X(s) = \frac{x(0)}{s-a} + \frac{b}{s-a}U(s)$$

Inverse Laplace transform satisfies, note : $\mathcal{L}^{-1}(f \cdot g) = \mathcal{L}^{-1}(f) * \mathcal{L}^{-1}(g)$:

$$x(t) = e^{at}x(0) + be^{at} * u(t)$$

Therefore:

$$x(t) = e^{at}x(0) + \int_0^t e^{a(t-\tau)}bu(\tau)\,d\tau$$

Solutions for Matrix Form

$$\boxed{\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)}$$

$$s\mathbf{X}(s) - \mathbf{x}(0) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{U}(s),$$

$$\mathbf{X}(s) = [s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{x}(0) + [s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B}\mathbf{U}(s)$$

And we let $[s\mathbf{I} - \mathbf{A}]^{-1} = \mathbf{\Phi}(s)$, it is the Laplace form of

$\mathbf{\Phi}(t) = \exp(\mathbf{A}t)$ ---- ***fundamental/state transition matrix,***
Describes the unforced response of the system (when u(t)=0).

The matrix exponential function is defined as:

$$e^{\mathbf{A}t} = \exp(\mathbf{A}t) = \mathbf{I} + \mathbf{A}t + \frac{\mathbf{A}^2 t^2}{2!} + \cdots + \frac{\mathbf{A}^k t^k}{k!} + \ \ldots$$

Then:

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{x}(0) + \int_0^t \mathbf{\Phi}(t-\tau)\mathbf{B}\mathbf{u}(\tau)\, d\tau$$

Solution to the matrix:

$$\mathbf{x}(t) = \exp(\mathbf{A}t)\,\mathbf{x}(0) + \int_0^t \exp[\mathbf{A}(t-\tau)]\,\mathbf{B}\mathbf{u}(\tau)\, d\tau$$

Fundamental/State Transition Matrix

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{x}(0) + \int_0^t \mathbf{\Phi}(t-\tau)\mathbf{B}\mathbf{u}(\tau)\,d\tau$$

$$\mathbf{\Phi}(t) = \exp(\mathbf{A}t)$$

The solution to the unforced system (when u(t)=0):

$$\begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_3(t) \end{pmatrix} = \begin{bmatrix} \emptyset_{11} & \cdots & \emptyset_{1n} \\ \emptyset_{12} & \cdots & \emptyset_{2n} \\ \vdots & & \vdots \\ \emptyset_{n1} & \cdots & \emptyset_{nn} \end{bmatrix} \begin{pmatrix} x_1(0) \\ x_2(0) \\ \vdots \\ x_3(0) \end{pmatrix}$$

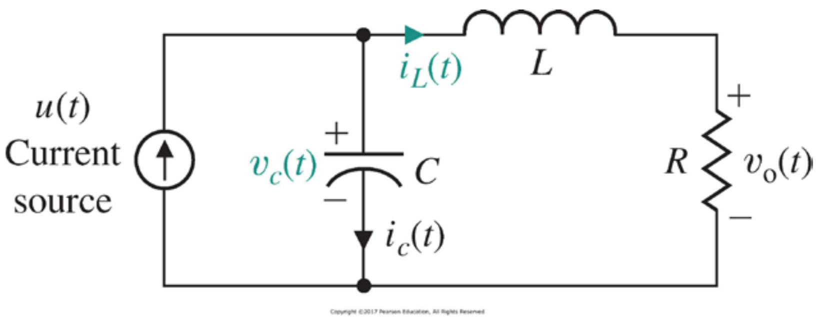
$-\ \emptyset_{ij}$ is the response of the ith state variable due to an initial condition on the jth state variable when there are zero initial conditions on all the other variables.

Example 11.5: Time Response of Unforced System

$$\underline{\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{x}(0)}$$

$$[s\mathbf{I} - \mathbf{A}]^{-1} = \mathbf{\Phi}(s)$$

$$\mathbf{\Phi}(t) = \exp(\mathbf{A}t) = \mathcal{L}^{-1}(\mathbf{\Phi}(s))$$ --- ***fundamental/state transition matrix***



$$\mathbf{A} = \begin{bmatrix} 0 & -\frac{1}{C} \\ \frac{1}{L} & -\frac{R}{L} \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} \frac{1}{C} \\ 0 \end{bmatrix}$$

$$\mathbf{C} = [0 \quad R], \qquad \mathbf{D} = [0]$$

Assume C = ½, L = 1, R = 3, then:

$$\mathbf{A} = \begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix} \longrightarrow s\mathbf{I} - \mathbf{A} = \begin{bmatrix} s & 2 \\ -1 & s+3 \end{bmatrix}$$

$$\mathbf{\Phi}(s) = [s\mathbf{I} - \mathbf{A}]^{-1} = \frac{adj\left(\begin{bmatrix} s & 2 \\ -1 & s+3 \end{bmatrix}\right)}{\det\left(\begin{bmatrix} s & 2 \\ -1 & s+3 \end{bmatrix}\right)} = \begin{bmatrix} \frac{s+3}{s^2+3s+2} & \frac{-2}{s^2+3s+2} \\ \frac{1}{s^2+3s+2} & \frac{s}{s^2+3s+2} \end{bmatrix}$$

$$\boldsymbol{\Phi}(t)=\mathcal{L}^{-1}(\boldsymbol{\Phi}(s)) = \mathcal{L}^{-1}\left(\begin{bmatrix} \frac{s+3}{s^2+3s+2} & \frac{-2}{s^2+3s+2} \\ \frac{1}{s^2+3s+2} & \frac{s}{s^2+3s+2} \end{bmatrix}\right)$$

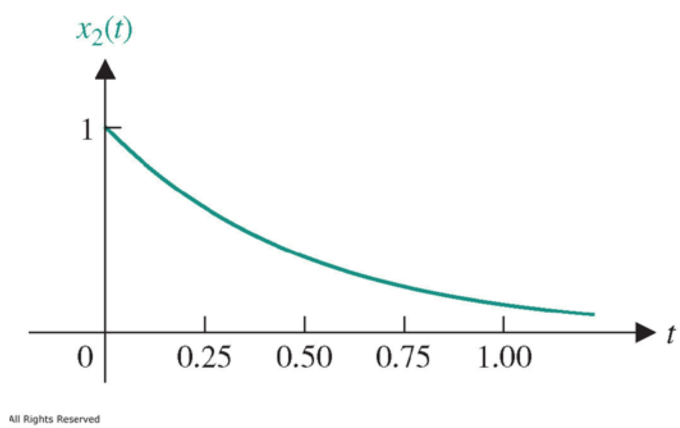
$$= \begin{bmatrix} \left(\frac{2}{s+1} + \frac{-1}{s+2}\right) & \left(\frac{-2}{s+1} + \frac{2}{s+2}\right) \\ \left(\frac{1}{s+1} + \frac{-1}{s+2}\right) & \left(\frac{-1}{s+1} + \frac{2}{s+2}\right) \end{bmatrix}$$

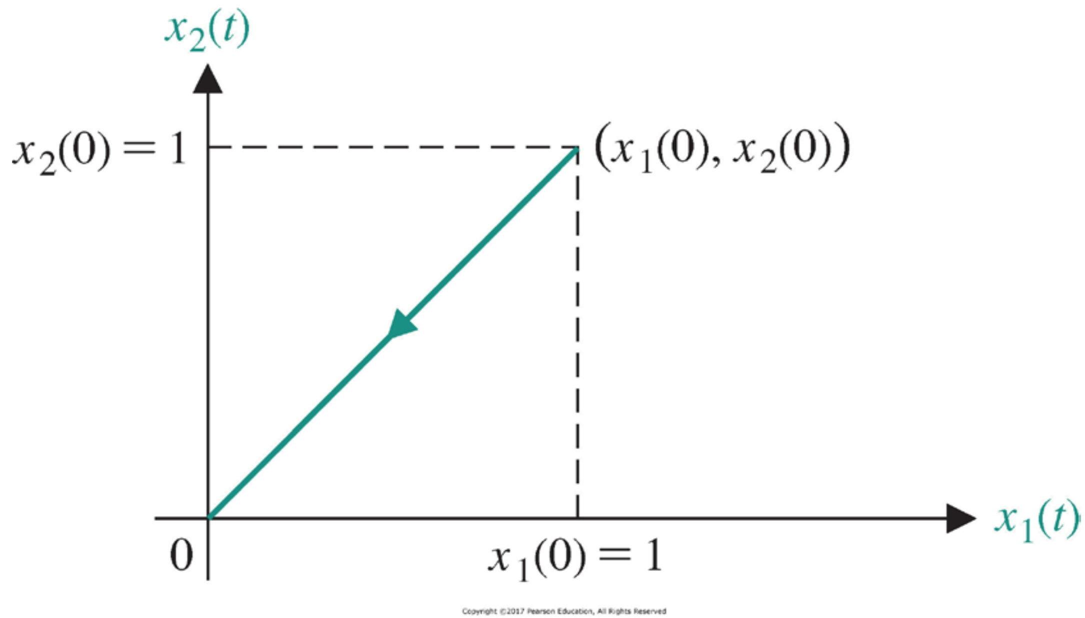
$$= \begin{bmatrix} (2e^{-t} - e^{-2t}) & (-2e^{-t} + 2e^{-2t}) \\ (e^{-t} - e^{-2t}) & (-e^{-t} + 2e^{-2t}) \end{bmatrix}$$

Assume $x_1(0) = x_2(0) = 1$:

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{bmatrix} (2e^{-t} - e^{-2t}) & (-2e^{-t} + 2e^{-2t}) \\ (e^{-t} - e^{-2t}) & (-e^{-t} + 2e^{-2t}) \end{bmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{bmatrix} e^{-2t} \\ e^{-2t} \end{bmatrix}$$

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{bmatrix} e^{-2t} \\ e^{-2t} \end{bmatrix}$$





How about $y(t)$?

Quiz 11.1

Consider the system with the mathematical model given by the differential equation:

$$5\frac{d^3y(t)}{dt^3} + 10\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 2y(t) = u(t)$$

Obtain a state variable model of this system.

Quiz 11.2

The state-space model of a system is:

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 6 \\ -1 & -5 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

(1) Find the state-transition matrix;

(2) If initial condition is $\mathbf{x}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, find the time response of the state variables.

Thank You !