Xi'an Jiaotong-Liverpool University 西交利物浦大學

EEE204 Continuous and Discrete Time Signals and Systems II

2018–2019 Semester 2

Electrical and Electronic Engineering

Xi'an Jiaotong-Liverpool University

Week 10

Find the Inverse z-transform (causal)

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$$X_1(z) = \frac{z}{z^2 - 0.9z + 0.2}$$

$$\begin{aligned}
X_1(z) &= \frac{z^{-1}}{1 - 0.9z^{-1} + 0.2z^{-2}} = \frac{z^{-1}}{(1 - 0.4z^{-1})(1 - 0.5z^{-1})}, \\
&= \frac{r_1}{1 - 0.4z^{-1}} + \frac{r_2}{1 - 0.5z^{-1}}, \\
r_1 &= (1 - 0.4z^{-1}) \frac{z^{-1}}{(1 - 0.4z^{-1})(1 - 0.5z^{-1})}\bigg|_{z=0.4} = -10 \\
r_2 &= (1 - 0.5z^{-1}) \frac{z^{-1}}{(1 - 0.4z^{-1})(1 - 0.5z^{-1})}\bigg|_{z=0.5} = 10,
\end{aligned}$$

$$X_1(z) = \frac{10}{1 - 0.5z^{-1}} - \frac{10}{1 - 0.4z^{-1}}, x_1[n] = 10(0.5^n - 0.4^n)u[n].$$

Discrete-Time Fourier Transform (DTFT)

DTFT

The DTFT is a frequency-domain representation for a wide range of both finite- and infinite-length discrete-time signals $x[n]$, which is defined as follows

$$X(e^{j\omega}) \triangleq \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n},$$

or

$$X(\cdot) = \text{DTFT}\{x[\cdot]\},$$

or shorthand:

$$x[n] \overset{\text{DTFT}}{\leftrightarrow} X(e^{j\omega}).$$

$X(e^{j\omega}) \triangleq \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$

- The DTFT is a continuous function of frequency ω.
- If $h[n]$ is the impulse response of an LTI system, then the DTFT of $h[n]$ is the frequency response $X(e^{j\omega})$ of that system.
- The DTFT function $X(e^{j\omega})$ is always periodic in ω with period 2π, that is, $X(e^{j(\omega+2\pi)}) = X(e^{j\omega})$.

Example

Find the DTFT of $\delta[n]$

$$\text{DTFT}(\delta[n]) = \sum_{n=-\infty}^{\infty} \delta[n] e^{-j\omega n},$$

$$= \delta[0] e^{-j\omega 0},$$

$$= 1.$$

Find the DTFT of $\delta[n-n_0]$

$$\begin{aligned}
&\text{DTFT}(\delta[n-n_0]) \\
&= \sum_{n=-\infty}^{\infty} \delta[n-n_0] e^{-j\omega n}, \\
&= \delta[n_0 - n_0] e^{-j\omega n_0}, \\
&= e^{-j\omega n_0}.
\end{aligned}$$

Example

Find the DTFT of $x[n] = u[n] - u[n-N]$

$$
\begin{aligned}
X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} (u[n] - u[n-N])e^{-j\omega n}, \\
&= \sum_{n=0}^{N-1} e^{-j\omega n}, \\
&= \frac{1 - e^{-j\omega N}}{1 - e^{-j\omega}}, \\
&= \frac{e^{-j\omega N/2}\left(e^{j\omega N/2} - e^{-j\omega N/2}\right)}{e^{-j\omega/2}(e^{j\omega/2} - e^{-j\omega/2})}, \\
&= e^{-j\omega(N-1)/2} \cdot \frac{\sin(\omega N/2)}{\sin(\omega/2)}.
\end{aligned}
$$

Example

Find the DTFT of $x[n] = a^n u[n]$

$$
\begin{aligned}
X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} a^n u[n] e^{-j\omega n}, \\
&= \sum_{n=0}^{\infty} a^n e^{-j\omega n}, \\
&= \sum_{n=0}^{\infty} (ae^{-j\omega})^n, \\
&= \frac{1}{1 - ae^{-j\omega}}, \quad |ae^{-j\omega}| < 1 \\
&= \frac{1}{1 - ae^{-j\omega}}, \quad |a| < 1.
\end{aligned}
$$

Existence of the DTFT

- In the case of finite-length sequences, the sum defining the DTFT has a finite number of terms, thus the DTFT always exists.

- In the general case, where one or both of the limits on the sum in the definition are infinite, the DTFT sum may diverge (become infinite).

- A sufficient condition for the existence of the DTFT of a sequence $x[n]$ is

$$\left|X\left(e^{j\omega}\right)\right| \leqslant \sum_{n=-\infty}^{\infty} |x[n]| < \infty.$$

Existence of the DTFT

$$\left|X(e^{j\omega})\right| \leqslant \sum_{n=-\infty}^{\infty} |x[n]| < \infty$$

Proof:

$$\begin{aligned}
\left|X(e^{j\omega})\right| &= \left|\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}\right|, \\
&\leqslant \sum_{n=-\infty}^{\infty} \left|x[n]e^{-j\omega n}\right|, \\
&= \sum_{n=-\infty}^{\infty} |x[n]||e^{-j\omega n}|, \\
&= \sum_{n=-\infty}^{\infty} |x[n]|.
\end{aligned}$$

Existence of the DTFT

Only when a sequence $x[n]$ is absolutely summable, will the infinite sum defining the DTFT $X(e^{j\omega})$ is said to converge to a finite result for all ω.

$$\left|X(e^{j\omega})\right| \leqslant \sum_{n=-\infty}^{\infty} |x[n]| < \infty$$

Example

Find the DTFT of $x[n] = r^n e^{j\omega_0 n} u[n]$

$$
\begin{aligned}
X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} r^n e^{j\omega_0 n} u[n] e^{-j\omega n}, \\
&= \sum_{n=0}^{\infty} r^n e^{j\omega_0 n} e^{-j\omega n} = \sum_{n=0}^{\infty} \left[r e^{-j(\omega-\omega_0)} \right]^n, \\
&= \frac{1}{1 - r e^{-j(\omega-\omega_0)}}, \quad \left| r e^{-j(\omega-\omega_0)} \right| < 1 \\
&= \frac{1}{1 - r e^{-j(\omega-\omega_0)}}, \quad |r| < 1.
\end{aligned}
$$

Properties of the DTFT

Linearity

If $x_1[n] \overset{\text{DTFT}}{\longleftrightarrow} X_1(e^{j\omega})$ and $x_2[n] \overset{\text{DTFT}}{\longleftrightarrow} X_2(e^{j\omega})$ then

$$x[n] = a_1 x_1[n] + a_2 x_2[n]$$

$$\overset{\text{DTFT}}{\longleftrightarrow} a_1 X_1(e^{j\omega}) + a_2 X_2(e^{j\omega}).$$

Linearity

Proof:

$$x[n] = a_1x_1[n]+a_2x_2[n] \overset{\text{DTFT}}{\longleftrightarrow} a_1X_1(e^{j\omega})+a_2X_2(e^{j\omega}).$$

$$\begin{aligned}
X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}, \\
&= \sum_{n=-\infty}^{\infty} (a_1x_1[n] + a_2x_2[n])e^{-j\omega n}, \\
&= a_1 \sum_{n=-\infty}^{\infty} x_1[n]e^{-j\omega n} + a_2 \sum_{n=-\infty}^{\infty} x_2[n]e^{-j\omega n}, \\
&= a_1X_1(e^{j\omega}) + a_2X_2(e^{j\omega}).
\end{aligned}$$

Example

$x[n] = 0.8^n u[n] + 2(-0.5)^n u[n]$

$$
\begin{aligned}
X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}, \\
&= \sum_{n=-\infty}^{\infty} [0.8^n u[n] + 2(-0.5)^n u[n]] e^{-j\omega n}, \\
&= \sum_{n=0}^{\infty} 0.8^n e^{-j\omega n} + 2 \sum_{n=0}^{\infty} (-0.5)^n e^{-j\omega n}, \\
&= \frac{1}{1 - 0.8e^{-j\omega}} + \frac{2}{1 + 0.5e^{-j\omega}}.
\end{aligned}
$$

Time Delay

If $x[n] \overset{\text{DTFT}}{\leftrightarrow} X(e^{j\omega})$, then

$$x[n-k] \overset{\text{DTFT}}{\leftrightarrow} e^{-j\omega k}X(e^{j\omega}).$$

Time Delay

Proof: $x[n-k] \overset{\text{DTFT}}{\leftrightarrow} e^{-j\omega k} X(e^{j\omega})$.

$$
\begin{aligned}
\text{DTFT}(x[n-k]) &= \sum_{n=-\infty}^{\infty} x[n-k] e^{-j\omega n}, \\
&\overset{m=n-k}{=\!=\!=} \sum_{m=-\infty}^{\infty} x[m] e^{-j\omega(k+m)}, \\
&= e^{-j\omega k} \sum_{m=-\infty}^{\infty} x[m] e^{-j\omega m}, \\
&= e^{-j\omega k} X(e^{j\omega}).
\end{aligned}
$$

Frequency Shift

If $x[n] \overset{\text{DTFT}}{\longleftrightarrow} X(e^{j\omega})$, then

$$e^{j\omega_0 n}x[n] \overset{\text{DTFT}}{\longleftrightarrow} X(e^{j(\omega-\omega_0)}).$$

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Frequency Shift

Proof: $e^{j\omega_0 n}x[n] \overset{\text{DTFT}}{\leftrightarrow} X(e^{j(\omega-\omega_0)})$.

$$\text{DTFT}(e^{j\omega_0 n}x[n]) = \sum_{n=-\infty}^{\infty} e^{j\omega_0 n}x[n]e^{-j\omega n},$$

$$= \sum_{n=-\infty}^{\infty} x[n]e^{-j(\omega-\omega_0)n},$$

$$= X(e^{j(\omega-\omega_0)})$$

Example

$$x[n] = A\cos(\omega_0 n + \phi)(u[n] - u[n-N]) = A\cos(\omega_0 n + \phi)u_N[n]$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} = \sum_{n=-\infty}^{\infty} [A\cos(\omega_0 n + \phi)u_N[n]]e^{-j\omega n},$$

$$= \frac{A}{2}e^{j\phi} \sum_{n=-\infty}^{\infty} e^{j\omega_0 n}u_N[n]e^{-j\omega n} + \frac{A}{2}e^{-j\phi} \sum_{n=-\infty}^{\infty} e^{-j\omega_0 n}u_N[n]e^{-j\omega n},$$

$$= \frac{A}{2}e^{j\phi}e^{-j(\omega-\omega_0)(N-1)/2} \cdot \frac{\sin[(\omega-\omega_0)N/2]}{\sin[(\omega-\omega_0)/2]}$$

$$+ \frac{A}{2}e^{-j\phi}e^{-j(\omega+\omega_0)(N-1)/2} \cdot \frac{\sin[(\omega+\omega_0)N/2]}{\sin[(\omega+\omega_0)/2]}.$$

Convolution

If $x_1[n] \overset{\text{DTFT}}{\leftrightarrow} X_1(e^{j\omega})$ and $x_2[n] \overset{\text{DTFT}}{\leftrightarrow} X_2(e^{j\omega})$ then

$$x[n] = x_1[n] * x_2[n]$$

$$\overset{\text{DTFT}}{\leftrightarrow} X_1(e^{j\omega})X_2(e^{j\omega}).$$

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Convolution

Proof: $x[n] = x_1[n] * x_2[n] \overset{\text{DTFT}}{\leftrightarrow} X_1(e^{j\omega})X_2(e^{j\omega})$.

$$
\begin{aligned}
X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} &= \sum_{n=-\infty}^{\infty} \left[\sum_{k=-\infty}^{\infty} x_1[k]x_2[n-k] \right] e^{-j\omega n}, \\
&= \sum_{k=-\infty}^{\infty} x_1[k] \left[\sum_{n=-\infty}^{\infty} x_2[n-k]e^{-j\omega n} \right], \\
&= \sum_{k=-\infty}^{\infty} x_1[k]e^{-j\omega k} X_2(e^{j\omega}), \\
&= X_1(e^{j\omega})X_2(e^{j\omega}).
\end{aligned}
$$

Understand all the properties in Table 5.1 (P.391)

The DTFT v.s. the z-Transform

z-transform is defined as

$$X(z) \triangleq \sum_{n=-\infty}^{\infty} x[n]z^{-n},$$

DTFT is defined as

$$X(e^{j\omega}) \triangleq \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n},$$

That is

$$X(e^{j\omega}) = X(z)\big|_{z=e^{j\omega}},$$

if ROC of z-transform includes the unit circle.

Example

Find DTFT of $x[n] = \left(\frac{1}{2}\right)^n u[n]$

$$X(z) = \frac{1}{1 - \frac{1}{2}z^{-1}}, \text{ROC} : |z| > \frac{1}{2}$$

$$X(e^{j\omega}) = \frac{1}{1 - \frac{1}{2}e^{-j\omega}}.$$

If the ROC does not include the unit circle, then strictly speaking the DTFT does not exist!

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Uniqueness

- The DTFT is a unique relationship between $x[n]$ and $X(e^{j\omega})$.
- Two different signals cannot have the same DTFT.
- If we know a DTFT representation, we can start in either the time or frequency domain and easily write down the corresponding representation in the other domain.

Inverse DTFT

Inverse DTFT

The uniqueness property implies we can always go back and forth between the time-domain and frequency-domain representations. Considering the $X(e^{j\omega})$ is a continuous function, the inverse DTFT is defined as

$$x[n] \triangleq \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}\,\mathrm{d}\omega,$$

or

$$x(\cdot) = \mathsf{IDTFT}\{X[\cdot]\},$$

or shorthand:

$$x[n] \overset{\mathsf{IDTFT}}{\leftrightarrow} X(e^{j\omega}).$$

Inverse DTFT

$$x[n] = \frac{1}{2\pi}\int_0^{2\pi} X(e^{j\omega})e^{j\omega n}\,\mathrm{d}\omega$$

$$\begin{aligned} x[n] &= \frac{1}{2\pi}\int_0^{2\pi} X(e^{j\omega})e^{j\omega n}\,\mathrm{d}\omega, \\ &= \frac{1}{2\pi}\int_0^{2\pi} \sum_{k=-\infty}^{\infty} x[k]e^{-j\omega k}e^{j\omega n}\,\mathrm{d}\omega, \\ &= \frac{1}{2\pi}\int_0^{2\pi} \sum_{k=-\infty}^{\infty} x[k]e^{-j\omega(k-n)}\,\mathrm{d}\omega, \end{aligned}$$

Inverse DTFT

$$x[n] = \frac{1}{2\pi}\int_0^{2\pi} \sum_{k=-\infty}^{\infty} x[k] e^{-j\omega(k-n)}\, \mathrm{d}\omega$$

$$\begin{aligned}
x[n] &= \frac{1}{2\pi}\int_0^{2\pi} \sum_{k=-\infty, k\neq n}^{\infty} x[k] e^{-j\omega(k-n)}\, \mathrm{d}\omega \\
&+ \frac{1}{2\pi}\int_0^{2\pi} x[n] e^{-j\omega(n-n)}\, \mathrm{d}\omega, \\
&= \frac{1}{-j2\pi(k-n)} \sum_{k=-\infty, k\neq n}^{\infty} x[k] \left[e^{-j2\pi(k-n)} - e^{-j0(k-n)} \right] \\
&+ \frac{1}{2\pi}\int_0^{2\pi} x[n]\, \mathrm{d}\omega,
\end{aligned}$$

Inverse DTFT

$$x[n] = \frac{1}{2\pi}\int_0^{2\pi}\sum_{k=-\infty}^{\infty} x[k]e^{-j\omega(k-n)}\,\mathrm{d}\omega$$

$$\begin{aligned}
x[n] &= \frac{1}{-j2\pi(k-n)}\sum_{k=-\infty,k\neq n}^{\infty} x[k]\left[e^{-j2\pi(k-n)} - e^{-j0(k-n)}\right] \\
&\quad + \frac{1}{2\pi}\int_0^{2\pi} x[n]\,\mathrm{d}\omega, \\
&= \frac{1}{-j2\pi(k-n)}\sum_{k=-\infty,k\neq n}^{\infty} x[k](0) + x[n], \\
&= x[n].
\end{aligned}$$

Example

Find the IDTFT of $X(e^{j\omega}) = e^{-j\omega n_0}$

$$\begin{aligned} x[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}\, \mathrm{d}\omega, \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-j\omega n_0}e^{j\omega n}\, \mathrm{d}\omega, \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\omega(n-n_0)}\, \mathrm{d}\omega, \end{aligned}$$

When $n \neq n_0$,

$$x[n] = \frac{1}{j2\pi(n-n_0)}\left[e^{j\pi(n-n_0)} - e^{-j\pi(n-n_0)}\right] = 0,$$

Example

Find the IDTFT of $X(e^{j\omega}) = e^{-j\omega n_0}$

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\omega(n-n_0)}\,\mathrm{d}\omega,$$

When $n = n_0$,

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\omega(n_0-n_0)}\,\mathrm{d}\omega = 1,$$

$$x[n] = \delta[n-n_0].$$

Homework

- Page 358–400, read Chapter 5;
- Page 400, Q5.6: (a)–(c);
- Page 401, Q5.8;
- Page 401, Q5.10;
- Page 403, Q5.21: (a)–(k);
- Page 403–404, Q5.22: (a)–(h);
- Page 406–407, Q5.26: (a)–(d).

Thank you for your attention.