MTH101: Tutorial 12

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December 4 – 10, 2017

Exercise 1.1

Find a general solution to the following Bessel's equation in terms of J_ν, Y_ν. Indicate whether you could use $J_{-\nu}$ instead of Y_ν. Use the indicated substitution.

1. $x^2y'' + \left(\frac{3}{16} + \frac{x}{4}\right) y = 0, \quad (y = 2u\sqrt{x},\ \sqrt{x} = z).$
2. $xy'' + 5y' + xy = 0, \quad (y = u/x^2).$
3. $y'' + xy = 0, \quad (y = u\sqrt{x}, z = \frac{2}{3}x^{\frac{3}{2}}).$

Solution

1. From the substitution, we have

$$y = 2u\sqrt{x}, \quad y' = 2u'\sqrt{x} + \frac{u}{\sqrt{x}}, \quad y'' = 2u''\sqrt{x} + \frac{2u'}{\sqrt{x}} - \frac{u}{2x^{\frac{3}{2}}},$$

and

$$z = \sqrt{x}, \quad \frac{dz}{dx} = \frac{1}{2\sqrt{x}} = \frac{1}{2z}, \quad u' = \frac{dz}{dx}\frac{du}{dz} = \frac{1}{2z} \cdot \frac{du}{dz},$$
$$u'' = \frac{1}{2z}\frac{dz}{dx}\frac{d^2u}{dz^2} - \frac{1}{2z^2}\frac{dz}{dx}\frac{du}{dz} = \frac{1}{4z^2}\frac{d^2u}{dz^2} - \frac{1}{4z^3}\frac{du}{dz}.$$

Solution

Therefore

$$x^2y'' + \left(\frac{3}{16} + \frac{x}{4}\right) y = 0$$

$$\Rightarrow z^4 \left[2z\left(\frac{1}{4z^2}\frac{d^2u}{dz^2} - \frac{1}{4z^3}\frac{du}{dz}\right) + \frac{2\cdot\frac{1}{2z}\cdot\frac{du}{dz}}{z} - \frac{1}{2}\frac{u}{z^3}\right]$$

$$+ \left(\frac{3}{16} + \frac{z^2}{4}\right) 2zu = 0$$

$$\left(\times\frac{2}{z}\right) \Rightarrow z^2\frac{d^2u}{dz^2} + z\frac{du}{dz} + \left(z^2 - \frac{1}{4}\right) u = 0, \quad (\nu = \frac{1}{2}).$$

Solution

Thus we have $u(z) = C_1 J_{1/2}(z) + C_2 Y_{1/2}(z)$, and

$$y(x) = 2\sqrt{x}\left[C_1 J_{1/2}(\sqrt{x}) + C_2 Y_{1/2}(\sqrt{x})\right],$$

since $\nu = 1/2$ is not an integer, we can use $J_{-1/2}$ instead of $Y_{1/2}$. and thus

$$y(x) = 2\sqrt{x}\left[C_1^* J_{1/2}(\sqrt{x}) + C_2^* J_{-1/2}(\sqrt{x})\right],$$

or

$$y = 2\sqrt{\frac{2}{\pi}} x^{\frac{1}{4}} [C_1^* \sin(\sqrt{x}) + C_2^* \cos(\sqrt{x})].$$

Solution

2. From the substitution, we have

$$y = \frac{u}{x^2}, \quad y' = \frac{u'}{x^2} - \frac{2u}{x^3}, \quad y'' = \frac{u''}{x^2} - \frac{4u'}{x^3} + \frac{6u}{x^4}.$$

Therefore

$$\begin{aligned}
& xy'' + 5y' + xy = 0 \\
\Rightarrow\ & \left(\frac{u''}{x} - \frac{4u'}{x^2} + \frac{6u}{x^3}\right) + 5\left(\frac{u'}{x^2} - \frac{2u}{x^3}\right) + \frac{u}{x} = 0 \\
(\times x^3) \Rightarrow\ & x^2u'' + xu' + (x^2 - 4)u = 0, \quad (\nu = 2), \\
\Rightarrow\ & u(x) = C_1 J_2(x) + C_2 Y_2(x) \\
\Rightarrow\ & y(x) = x^{-2}\left[C_1 J_2(x) + C_2 Y_2(x)\right],
\end{aligned}$$

and since $\nu = 2$ is an integer, we can not use $J_{-\nu}$ instead of Y_ν, since J_ν, $J_{-\nu}$ are not independent.

Solution

3. From the substitution, we have

$$y = u\sqrt{x}, \quad y' = u'\sqrt{x} + \frac{u}{2\sqrt{x}}, \quad y'' = u''\sqrt{x} + \frac{u'}{\sqrt{x}} - \frac{u}{4x^{\frac{3}{2}}},$$

and

$$z = \frac{2}{3}x^{\frac{3}{2}}, \quad \frac{dz}{dx} = \sqrt{x} = \left(\frac{3z}{2}\right)^{\frac{1}{3}}, \quad u' = \frac{dz}{dx}\frac{du}{dz} = \left(\frac{3z}{2}\right)^{\frac{1}{3}} \cdot \frac{du}{dz},$$

$$\begin{aligned} u'' &= \left(\frac{3z}{2}\right)^{\frac{1}{3}} \frac{dz}{dx}\frac{d^2u}{dz^2} + \frac{1}{2}\left(\frac{3z}{2}\right)^{-\frac{2}{3}} \frac{dz}{dx}\frac{du}{dz} \\ &= \left(\frac{3z}{2}\right)^{\frac{2}{3}} \frac{d^2u}{dz^2} + \frac{1}{2}\left(\frac{3z}{2}\right)^{-\frac{1}{3}} \frac{du}{dz}. \end{aligned}$$

Solution

Therefore

$$
\begin{aligned}
y'' &= u''\sqrt{x} + \frac{u'}{\sqrt{x}} - \frac{u}{4x^{\frac{3}{2}}} \\
&= \left[\left(\frac{3z}{2}\right)^{\frac{2}{3}} \frac{d^2u}{dz^2} + \frac{1}{2}\left(\frac{3z}{2}\right)^{-\frac{1}{3}} \frac{du}{dz}\right]\left(\frac{3z}{2}\right)^{\frac{1}{3}} \\
&\quad + \left(\frac{3z}{2}\right)^{\frac{1}{3}} \frac{du}{dz}\left(\frac{3z}{2}\right)^{-\frac{1}{3}} - \frac{u}{4}\left(\frac{3z}{2}\right)^{-1} \\
&= \frac{3z}{2}\frac{d^2u}{dz^2} + \frac{3}{2}\frac{du}{dz} - \frac{u}{6z},
\end{aligned}
$$

and

$$
xy = \left(\frac{3z}{2}\right)^{\frac{2}{3}} u \left(\frac{3z}{2}\right)^{\frac{1}{3}} = \frac{3z}{2}u.
$$

Solution

Therefore

$$
\begin{aligned}
y'' + xy &= 0 \\
\Rightarrow \frac{3z}{2}\frac{d^2u}{dz^2} + \frac{3}{2}\frac{du}{dz} - \frac{u}{6z} + \frac{3z}{2}u &= 0 \\
\left(\times\frac{2z}{3}\right) \Rightarrow z^2\frac{du^2}{dz^2} + z\frac{du}{dz} + \left(z^2 - \frac{1}{9}\right)u &= 0, \quad (\nu = \frac{1}{3}).
\end{aligned}
$$

Thus we have $u(z) = C_1 J_{1/3}(z) + C_2 Y_{1/3}(z)$, and

$$
y(x) = \sqrt{x}\left[C_1 J_{\frac{1}{3}}\left(\frac{2}{3}x^{\frac{3}{2}}\right) + C_2 Y_{\frac{1}{3}}\left(\frac{2}{3}x^{\frac{3}{2}}\right)\right],
$$

and since $\nu = 1/3$ is not an integer, we can use $J_{-1/3}$ instead of $Y_{1/3}$.

Exercise 1.2

Derive the Bessel's equation

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0,$$

by the following equations

$$\begin{aligned}
&(a)\ [x^{\nu}J_{\nu}(x)]' = x^{\nu}J_{\nu-1}(x),\\
&(b)\ [x^{-\nu}J_{\nu}(x)]' = -x^{-\nu}J_{\nu+1}(x),\\
&(c)\ J_{\nu-1}(x) + J_{\nu+1}(x) = \frac{2\nu}{x}J_{\nu}(x),\\
&(d)\ J_{\nu-1}(x) - J_{\nu+1}(x) = 2J_{\nu}'(x).
\end{aligned}$$

Solution

For (b), if we let $\nu = \nu - 1$, we have

$$[x^{-\nu+1}J_{\nu-1}(x)]' = -x^{-\nu+1}J_\nu(x), \tag{1.1}$$

and from (a), we know

$$\begin{aligned} J_{\nu-1}(x) &= x^{-\nu}[x^\nu J_\nu(x)]' = x^{-\nu}\left[\nu x^{\nu-1}J_\nu + x^\nu J_\nu'\right] \\ \Rightarrow J_{\nu-1} &= \nu x^{-1}J_\nu + J_\nu'. \end{aligned} \tag{1.2}$$

Solution

Substituting eq.(1.2) into eq.(1.1), we have

$$
\begin{aligned}
& \left[x^{-\nu+1}\left(\nu x^{-1} J_\nu + J_\nu'\right)\right]' = -x^{-\nu+1} J_\nu \\
\Rightarrow\ & \left[\nu x^{-\nu} J_\nu + x^{-\nu+1} J_\nu'\right]' = -x^{-\nu+1} J_\nu \\
\Rightarrow\ & -\nu^2 x^{-\nu-1} J_\nu + \nu x^{-\nu} J_\nu' + (-\nu+1) x^{-\nu} J_\nu' + x^{-\nu+1} J_\nu'' = -x^{-\nu+1} J_\nu \\
\Rightarrow\ & -\nu^2 x^{-\nu-1} J_\nu + x^{-\nu} J_\nu' + x^{-\nu+1} J_\nu'' = -x^{-\nu+1} J_\nu .
\end{aligned}
$$

If we multiply the equation by $x^{\nu+1}$, we get

$$
\begin{aligned}
& x^2 J_\nu'' + x J_\nu' - \nu^2 J_\nu = -x^2 J_\nu \\
\Rightarrow\ & x^2 J_\nu'' + x J_\nu' + \left(x^2 - \nu^2\right) J_\nu = 0.
\end{aligned}
$$