MTH101: Tutorial 5

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October 16, 2017

Example 1.1

Write the **Taylor Series** *with center* $z_0 = 7$ *of the Function*

$$f(z) = \frac{1}{4 - z},$$

and find its **Radius of Convergence.**

Solution: We already know that

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n, \quad \text{for } |z| < 1$$

In this case $z_0 = 7$, so we need to manipulate the function $f(z)$:

$$f(z) = \frac{1}{4-z} = \frac{1}{4-z+7-7} = \frac{1}{-3-(z-7)} = -\frac{1}{3}\left(\frac{1}{1-\left[-\frac{(z-7)}{3}\right]}\right)$$

$$= -\frac{1}{3}\sum_{n=0}^{\infty}\left[-\frac{(z-7)}{3}\right]^n = -\frac{1}{3}\sum_{n=0}^{\infty}\frac{(-1)^n}{3^n}(z-7)^n$$

which converges for $\left|-\frac{(z-7)}{3}\right| < 1$.

Then

$$f(z) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{3^{n+1}} (z-7)^n, \quad \text{for } |z-7| < 3.$$

Also we observe that the function $f(z)$ is not analytic at $z^* = 4$. Then the **Radius of Convergence** of the Taylor Series with center z_0 is given by

$$R = |z_0 - z^*| = |7 - 4| = 3,$$

and we get the same result obtained by the previous computation.

Example 1.2

Write the **Taylor Series** *with center* $z_0 = 0$ *of the function*

$$f(z) = \frac{z}{(1+z)^3},$$

and find its **Radius of Convergence.**

Solution

From the example we did in class, we know that

$$\frac{1}{(1+z)^2} = \left(\sum_{n=0}^{\infty}(-1)^{n+1}z^n\right)' = \sum_{n=1}^{\infty}(-1)^{n+1}nz^{n-1}, \quad \text{for } |z| < 1.$$

Now we observe that

$$f(z) = z \cdot \frac{1}{(1+z)^3} = -\frac{z}{2}\left(\frac{1}{(1+z)^2}\right)'$$

where

$$\left(\frac{1}{(1+z)^2}\right)' = \left(\sum_{n=1}^{\infty}(-1)^{n+1}nz^{n-1}\right)' = \sum_{n=2}^{\infty}(-1)^{n+1}n(n-1)z^{n-2}$$

for $|z| < 1$.

Then

$$f(z) = -\frac{z}{2}\left(\frac{1}{(1+z)^2}\right)' = \sum_{n=2}^{\infty} \frac{(-1)^n}{2} n(n-1) z^{n-1}, \quad \text{for } |z| < 1.$$

Example 2.1

Write the function

$$f(z) = \frac{2}{z^5} e^{\frac{3}{z}},$$

in power series with center $z_0 = 0$.

Solution

The function $f(z)$ is Analytic in the set $\mathbb{C} \setminus \{0\}$, then it is Analytic in the **Annulus**:

$$0 < |z| < \infty, \;\; (\text{or } |z| > 0, \text{or } z \neq 0).$$

Then it can be represented by a **Laurent Series** in that Annulus.
We know that

$$e^z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n, \quad \text{for all } z \in \mathbb{C},$$

then

$$e^{\frac{3}{z}} = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{3}{z}\right)^n, \quad \text{for all } z \in \mathbb{C} \setminus \{0\}.$$

Finally

$$f(z) = \frac{2}{z^5} e^{\frac{3}{z}} = \frac{2}{z^5} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{3}{z}\right)^n = \sum_{n=0}^{\infty} \frac{2(3^n)}{n!} z^{-n-5}, \quad \text{for all } z \in \mathbb{C}\setminus\{0\}.$$

Exercise 2.2

Write all the **Power Series** *with center* $z_0 = 0$ *of the function*

$$f(z) = \frac{3}{32z^4 + 2}.$$

Solution

The function $f(z)$ is similar to the sum of the **Geometric Series**

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$$

which converges for $|z| < 1$.
The Idea is to manipulate the function $f(z)$:

$$\frac{3}{32z^4+2} = \frac{3}{2}\left(\frac{1}{1+16z^4}\right) = \frac{3}{2}\left(\frac{1}{1-(-16z^4)}\right) = \frac{3}{2}\sum_{n=0}^{\infty}\left(-16z^4\right)^n$$

$$= \frac{3}{2}\sum_{n=0}^{\infty}(-1)^n 2^{4n} z^{4n}$$

which converges for $\left|-16z^4\right| < 1$.

Then

$$f(z) = \frac{3}{32z^4 + 2} = \sum_{n=0}^{\infty} 3(-1)^n 2^{4n-1} z^{4n}, \quad \text{for } |z| < \frac{1}{2}.$$

The function $f(z)$ is Analytic in the **Annulus** with center $z_0 = 0$:

$$\frac{1}{2} < |z| < +\infty,$$

then $f(z)$ can be Represented by a **Laurent Series** in that **Annulus**.
We observe that

$$|z| > \frac{1}{2} \quad \Longleftrightarrow \quad \left| \frac{1}{16z^4} \right| < 1.$$

Then we manipulate the function $f(z)$ in order to obtain a Geometric Series in powers of $\frac{1}{16z^4}$:

$$\frac{3}{32z^4+2} = \frac{3}{32z^4(1+\frac{1}{16z^4})} = \frac{3}{32z^4}\left(\frac{1}{1-(-\frac{1}{16z^4})}\right)$$
$$= \frac{3}{32z^4}\sum_{n=0}^{\infty}\left(-\frac{1}{16z^4}\right)^n$$
$$= \frac{3}{32z^4}\sum_{n=0}^{\infty}(-1)^n 2^{-4n} z^{-4n}, \quad \text{for } \left|-\frac{1}{16z^4}\right| < 1.$$

Thus,

$$f(z) = \sum_{n=0}^{\infty} 3(-1)^n 2^{-4n-5} z^{-4n-4}, \quad \text{for } |z| > \frac{1}{2}.$$

Exercise 2.3

Write all the **Power Series** *with center* $z_0 = 1$ *of the function*

$$f(z) = 1/z.$$

Solution: The geometric series is

$$\frac{1}{1-w} = \sum_{n=0}^{\infty} w^n, \quad |w| < 1$$

we need $\frac{1}{z}$ so we set $w = 1 - z$.
Then we get the Taylor series

$$\frac{1}{z} = \frac{1}{1-(1-z)} = \sum_{n=0}^{\infty} (1-z)^n$$

$$= \sum_{n=0}^{\infty} (-1)^n (z-1)^n, \quad \text{for all } |z-1| < 1.$$

Similarly, we obtain the Laurent Series converging for $|z-1|>1$ (which implies $\left|\frac{1}{z-1}\right|<1$) by the following trick, which you should remember:

$$\frac{1}{z} = \frac{1}{1-(1-z)} = \frac{1}{1-z}\cdot\frac{1}{\frac{1}{1-z}-1} = \frac{1}{z-1}\cdot\frac{1}{1-\frac{1}{1-z}}$$
$$= (z-1)^{-1}\sum_{n=0}^{\infty}\left(\frac{1}{1-z}\right)^{n}$$
$$= \sum_{n=0}^{\infty}(-1)^n(z-1)^{-n-1}, \qquad \text{for all } |z-1|>1.$$

Remark: Note that power series centered at $z_0 = 1$ must be in powers of $(z-1)$.