Final review questions

Part I Vector fields, grad, div and curl

1. If $\boldsymbol{A}(x,y,z) = < x^2y,\ xy^2z,\ xyz >$, find div$\boldsymbol{A}$ at the point (1, -1, 2).

2. If $\boldsymbol{V}(x,y,z) = < 2xy,\ 3x^2y, -3pyz >$ and div$\boldsymbol{V}$=0 at (1,1,1), find p.

3. If $f(x,y,z) = 2xz^3 - 3x^2yz$, find grad$f$ and ||grad$f$|| at point (2,2,-1).

4. If $f(x,y,z) = 2xyz$ and $g(x,y,z) = x^2y + z$, find grad(f+g) and grad(fg) at the point (1,-1,0).

5. If $\boldsymbol{F}(x, y, z) =< x + y + 1, 1, \ -x - y >$, prove that $\boldsymbol{F}\cdot$curl$\boldsymbol{F}$=0.

6. Find the constants a, b, c so that the curl of the vector $\boldsymbol{F}(x, y, z) =< x + 2y + az, \ bx - 3y - z, 4x + cy + 2z >$ is identically equal to zero.

Part II Line integrals and surface integrals

7. Let $\boldsymbol{F}(x, y) =< x + y, -x >$ and the curve C is $C = \{(x, y) \in R^2 : y^2 + 4x^4 - 4x^2 = 0, \ x \geq 0\}$.
   (a) Show that $\boldsymbol{r}(t) =< \sin t, \ \sin 2t >, t \in [0, \pi]$ is a parametrization of C.
   (b) Calculate $\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$.

8. Calculate the work done by $\boldsymbol{F}(x,y,z) =< 1, -y,\ xyz >$ in moving a particle from (0,0,0) to (1,-1,1) along the curve $x = t, y = -t^2, z = t$ for $0 \le t \le 1$.

9. Determine whether $\boldsymbol{F} \cdot d\boldsymbol{r}$ is exact or not for the given $\boldsymbol{F}$.
   (a) $\boldsymbol{F}(x,y) =< 2x\cos 2y, -2x^2 \sin 2y >$;
   (b) $\boldsymbol{F}(x,y) =< \mathrm{e}^x \cos y + yz, xz - \mathrm{e}^x \sin y,\ xz >$.

10. Evaluate
   (a) $\int_{(0,0,0)}^{(-1,0,\pi)} (y+z)dx + (x+z)dy + (x+y)dz$,

(b) $\int_{(-1,1)}^{(4,2)} \left(y - \frac{1}{x^2}\right) dx + (x - \frac{1}{y^2}) dy.$

11. Let $S = \{(x, y) \in R^2,\ x^2 + y^2 < 1\}$ and $\boldsymbol{F}(x, y) =< y^2, x >$, verify Green's theorem.

12. Verify Green's theorem for $S = \{(x, y) \in R^2,\ x^2 + (y-1)^2 < 1\}$ and $\boldsymbol{F}(x, y) =< -x^2y, xy^2 >$.

13. For $f(x, y, z) = x^2 + y^2 + 2z^2$ and a surface $S = \{(x, y, z) \in R^3, x^2 + y^2 = 1,\ 0 \leq z \leq 1\}$, calculate $\iint_S f(x, y, z) dA$.

14. For the given $\boldsymbol{F}(x, y, z) =< 0, z, z >$ and the surface $S = \{(x, y, z) \in R^3 : z = 2 - x - y,$ $x, y, z \geq 0\}$, compute the flux across S going out from the origin.

15. Let $T = \{(x, y, z) \in R^3 : x^2 + y^2 \leq z \leq 1\}$, find the surface area of T.

16. For $\boldsymbol{F}(x, y, z) =< xy, yz, xz >$ and $T = \{(x, y, z) \in R^3 : 0 < z < 1 - x - y, 0 < y < 1 - x, 0 < x < 1\}$, calculate the surface integral on the surface S of T: $\iint_S \boldsymbol{F} \cdot \boldsymbol{n} dA$.

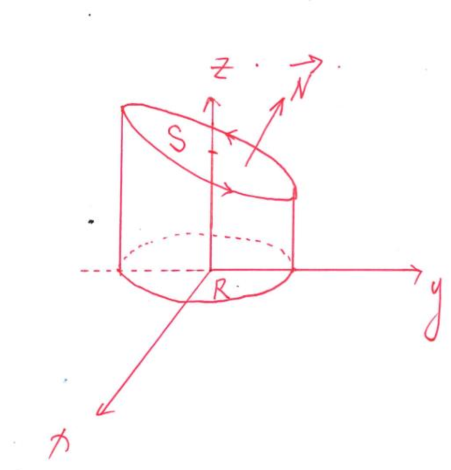
17. For $\boldsymbol{F}(x, y, z) =< x^2, -y^2, z^2 >$ and $T = \{(x, y, z) \in R^3 : x^2 + y^2 < 4, 0 < z < 2\}$, calculate the surface integral on the surface S of T: $\iint_S \boldsymbol{F} \cdot \boldsymbol{n} dA$.

18. Verify Stoke's theorem for $\boldsymbol{F}(x, y, z) =< z, x, y >$ and $S = \{(x, y, z) \in R^3 : z^2 = x^2 + y^2, 0 < z < 1\}$.

19. Use Stoke's theorem to evaluate

$$I = \oint_C -y^2 dx + x dy + z^2 dz,$$

where C is the intersection of plane $y + z = 2$ and cylinder $x^2 + y^2 = 1$. The direction of C is as denoted on the graph.



Part III Fourier series

20. A periodic function of period 2L is defined by

$$f(x) = \begin{cases} L^2, & -L \le x < 0, \\ Lx, & 0 \le x < L\,. \end{cases}$$

(a) Sketch the graph of $f(x)$ in the range $-3L \le x \le 3L$.

(b) State the values the Fourier series will converge to at $x = 0, \frac{L}{2}, L, \frac{3L}{2}$.

(c) Fin the Fourier series of $f(x)$ in $-L \le x \le L$ and give the first three non-zero terms.

21. For the 2π-periodic function

$$f(x) = \begin{cases} 1 + \dfrac{2x}{\pi}, -\pi \le x < 0, \\ 1 - \dfrac{2x}{\pi}, \quad 0 \le x < \pi. \end{cases}$$

(a) Sketch the graph for $f(x)$ in $-2\pi \le x \le 2\pi$.

(b) State the values for the Fourier series will converge to at $x = 0, \frac{\pi}{2}, \pi$.

(c) Find the Fourier series of $f(x)$ in $-\pi \le x \le \pi$.

Part IV PDEs

22. Determine the type of the following PDEs.
(1) $u_{xx} + u_{yy} = 0$.

(2) $u_{xx} + 10u_{xy} + u_{yy} = 0$.

(3) $u_{xx} + 2u_{xy} + u_{yy} = 0$.

23. Solving the following PDEs using the same method to solve ODEs.
(1) $u_{xx} + \pi^2 u = 0$.

(2) $u_{xx} - \pi^2 u = 0$.

(3) $u_x + xu = 0$.

24. Fin a function $u = u(x,t)$ satisfying

$$\begin{cases} (2+t)\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \\ u(0,t) = u(\pi,t) = 0, \quad t > 0 \\ u(x,0) = \sin x + 4\sin(2x), x \in (0,\pi) \end{cases}.$$