Engineering Mathematics III
MTH201

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Recommended textbook:

Advanced Engineering Mathematics, E. Kreyszig, John Wildy and Sons, 10th edition, 2011.

Aims:

- To give students advanced mathematical tools common to engineering applications;
- To demonstrate the physical origin of applied mathematics;
- To understand the basic concepts and results in field theory;
- To familiarise with the partial differential equations of importance to engineering and science and their properties;
- To train the students' ability to think logically and independently and to acquire the skills of problem solving.

Learning outcomes:

- To have a good understanding of the use of vector fields, Fourier series, and partial differential equations;
- To solve problem by establishing appropriate mathematical models using the most relevant techniques;
- To Appreciate the importance of mathematics to engineering and sciences.

Assessment:

Assignments	2 quizzes$\times 5\% = 10\%$
Mid-term exam	20%
Final exam	70%
Total	100%

Schedule:

Week 1 to 5	Vector calculus
Week 4	Quiz 1 (during tutorial class)
Week 6	Mid-term exam revision
Week 7	Mid-term exam (details to be advised)
Week 8 to 13	Fourier analysis and Partial differential equations
Week 12	Quiz 2 (during tutorial class)
Week 14	Final exam revision

Chapter 1: Revisions on vector calculus

Section 1.1 Scalar and Vector field

In engineering, physics, mathematics, and other areas we encounter two kinds of quantities. They are scalars and vectors.

A **scalar** is a quantity that is determined by its magnitude. It takes on a numerical value, i.e., a number.
Examples: time, temperature, length, distance, speed, density, energy, and voltage.

A **vector** is a quantity that has both magnitude and direction. We can say that a vector is an arrow or a directed line segment. For a vector, it has 2 important components:
-direction;
-length.

Examples: 1. A velocity vector has length or magnitude, which is speed, and direction, which indicates the direction of motion. 2. Displacement, velocity and force (see the figure below). More formally, we have the following. We denote vectors by lowercase boldface letters $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{v}$ etc. In handwriting you may use arrows, for instance $\vec{a}, \vec{b}, \vec{v}$ etc.

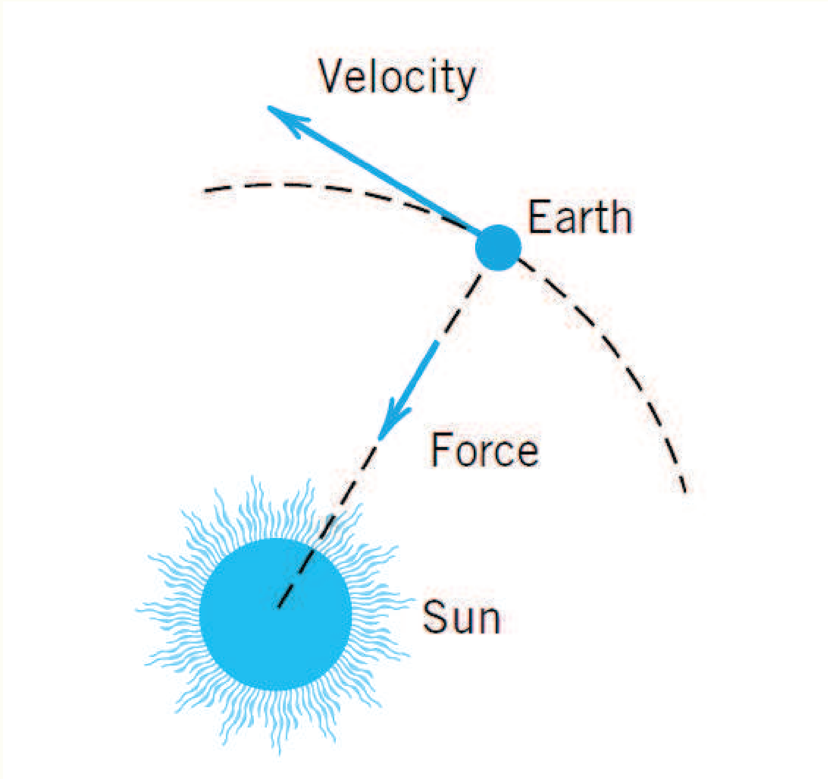


Figure: Force and velocity

A vector has a tail, called its **initial point**, and a tip, called its **terminal point**. This is motivated in the **translation** (displacement without rotation) of the triangle in the figure below, where the initial point P of the vector $\vec{a}$ is the original position of a point, and the terminal point Q is the terminal position of that point, its position after the translation. The length of the arrow equals the distance between P and Q. This is called the **length** (or magnitude) of the vector $\vec{a}$ and is denoted by $|\vec{a}|$. Another name for length is **norm** (or Euclidean norm).

A vector of length 1 is called a **unit vector**.

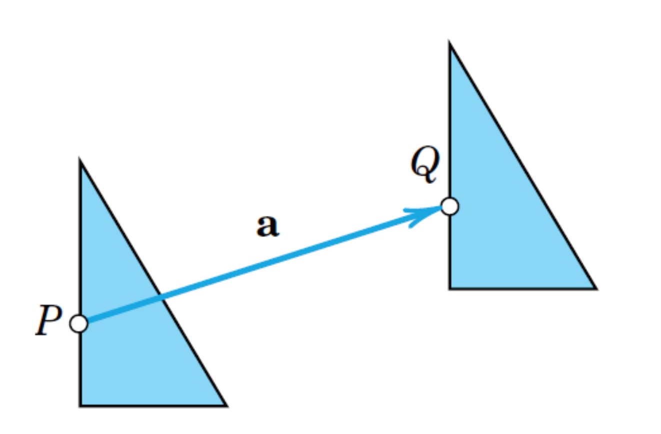


Figure: Translation

Let $\boldsymbol{a}$ be a given vector with initial point P: (x_1, y_1, z_1) and terminal point Q: (x_2, y_2, z_2). Then the three coordinate differences

$$a_1 = x_2 - x_1, \; a_2 = y_2 - y_1, \; a_3 = z_2 - z_1 \tag{1.1}$$

are called the **components** of the vector $\boldsymbol{a}$ with respect to that coordinate system, and we write simply

$$\boldsymbol{a} =< a_1, a_2, a_3 > .$$

The length $|\boldsymbol{a}|$ of $\boldsymbol{a}$ can now readily be expressed in terms of components because from (1.1) and the Pythagorean theorem we have

$$|\boldsymbol{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2} \tag{1.2}$$

A Cartesian coordinate system being given, the position vector $\boldsymbol{r}$ of a point A: (x, y, z) is the vector with the origin $(0, 0, 0)$ ad the initial point and A as the terminal point (see the figure below). Thus in components, $\boldsymbol{r} =< x, y, z >$. This can be seen directly from (1.1)with $x_1 = y_1 = z_1 = 0$.

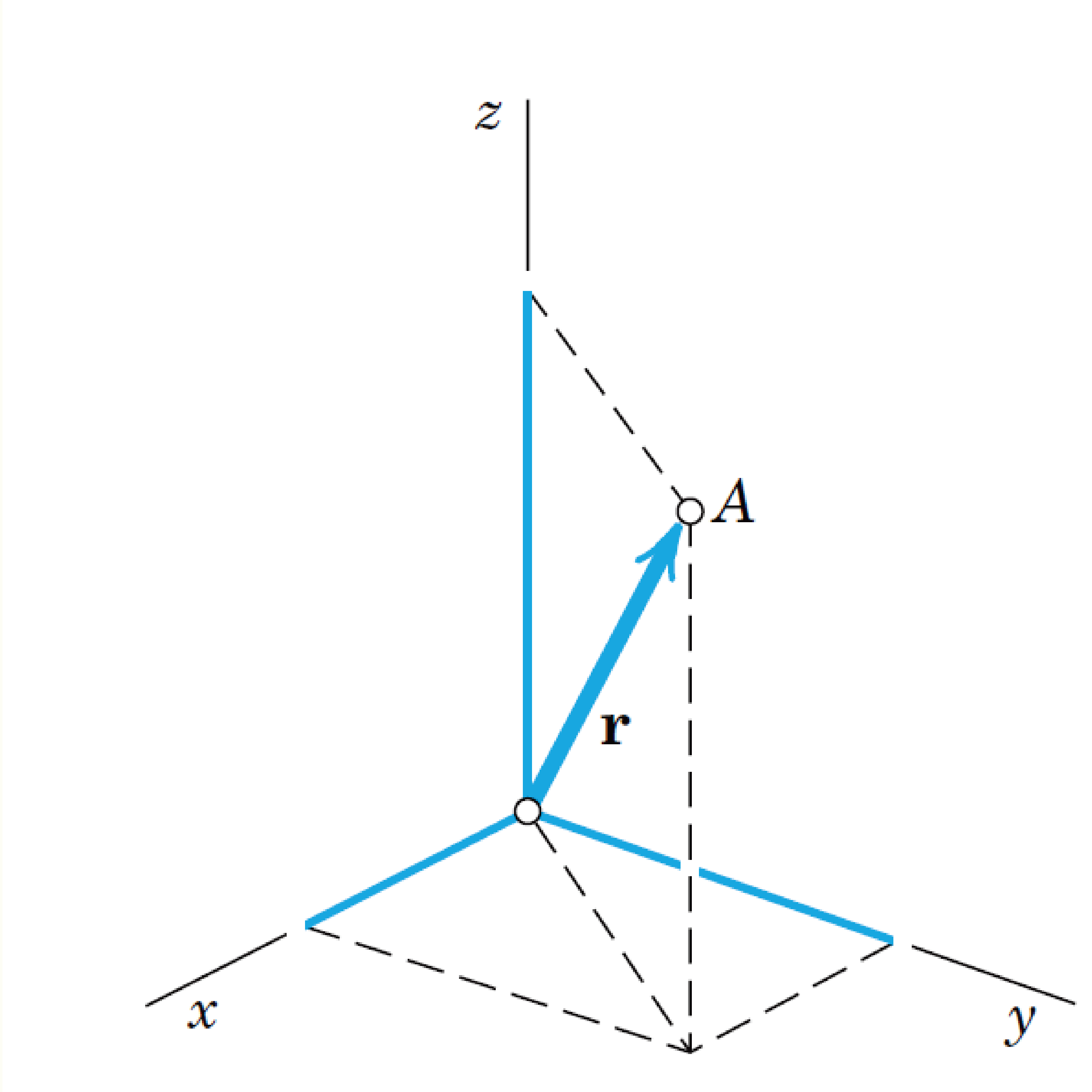
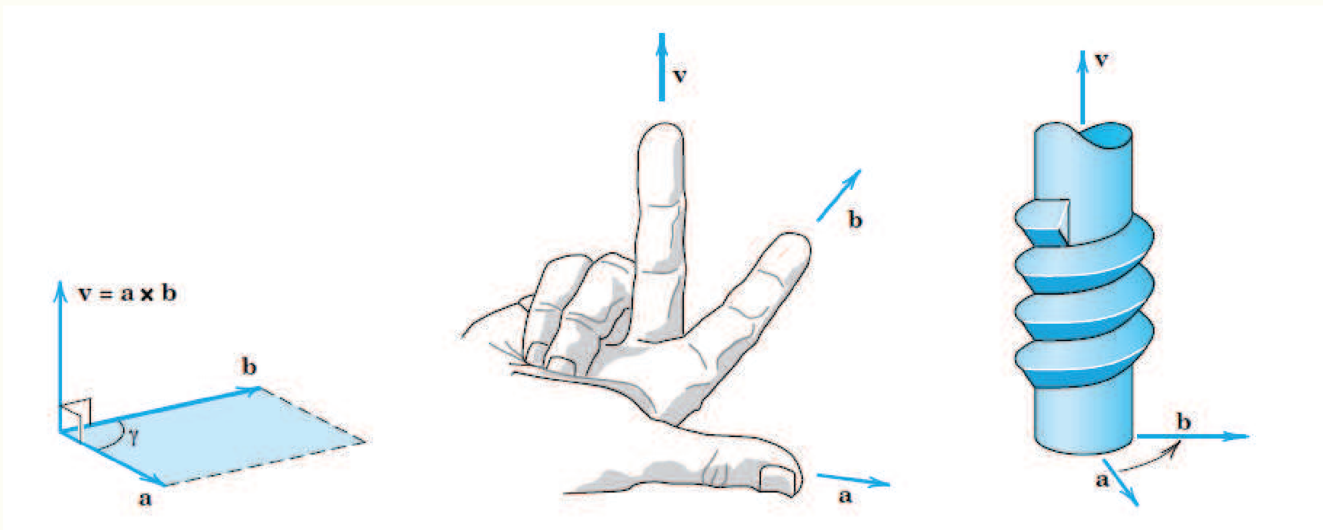


Figure: Position vector $\boldsymbol{r}$ of a point A: (x, y, z)

Section 1.2 Right-handed Triple

A triple of vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{v}$ is **right-handed** if the vectors in the given order assume the same sort of orientation as the thumb, index finger, and middle finger of the right hand when these are held as in figures below. We may also say that if $\boldsymbol{a}$ is rotated into the direction of $\boldsymbol{b}$ through the angle $\gamma (< \pi)$, then $\boldsymbol{v}$ advances in the same direction as a right-handed screw would if turned in the same way (see the three figures below).



Section 1.3 Examples of Vector Fields

Examples of vector fields are the field of tangent vectors of a curve (shown in the following figure 1.4), normal vectors of a surface (figure 1.5), and velocity field of a rotating body (fugure 1.6).Note that vector functions may also depend on time t or on some other parameters.

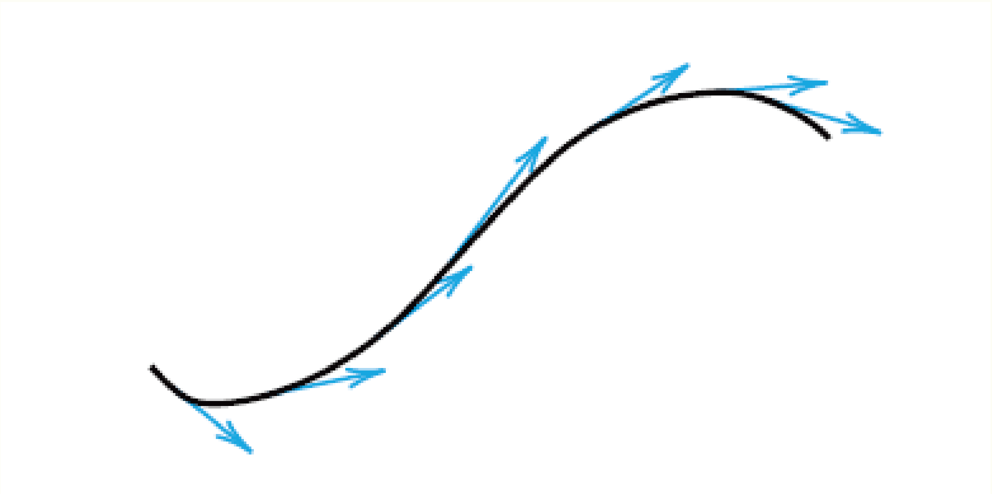


Figure: 1.4: Field of tangent vectors of a curve

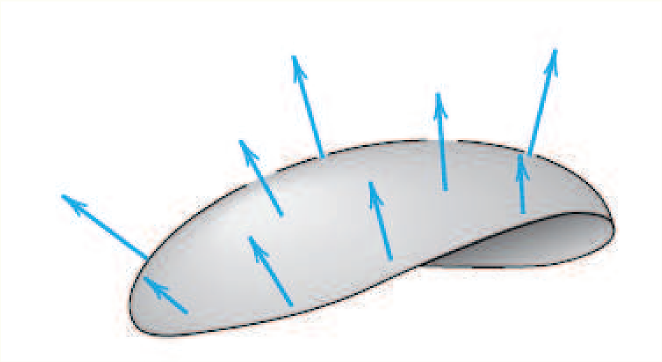


Figure: 1.5: Field of normal vectors of a surface

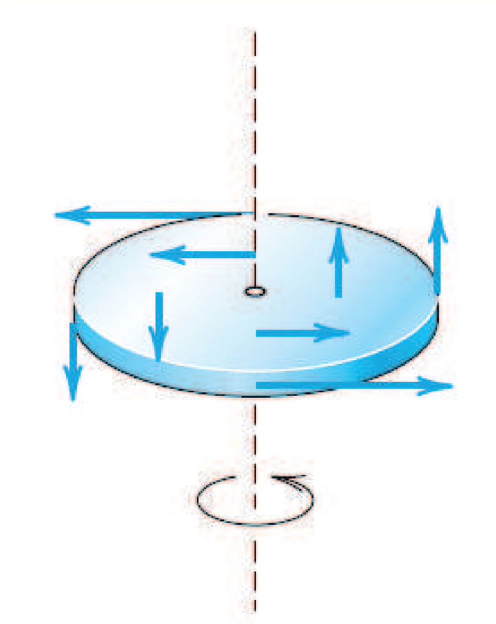


Figure: 1.6: Velocity field of a rotating body

Section 1.4 Derivative of a Vector Function

A vector function $\boldsymbol{v}(t)$ is said to be **differentiable** at a point t if the following limit exists:

$$\boldsymbol{v}'(t) = \lim_{\Delta t \to 0} \frac{\boldsymbol{v}(t + \Delta t) - \boldsymbol{v}(t)}{\Delta t}.$$

This vector $\boldsymbol{v}'(t)$ is called the **derivative** of $\boldsymbol{v}(t)$. See the following figure. Sometimes the derivative of $\boldsymbol{v}(t)$ is also denoted by $\frac{d\boldsymbol{v}(t)}{dt}$.

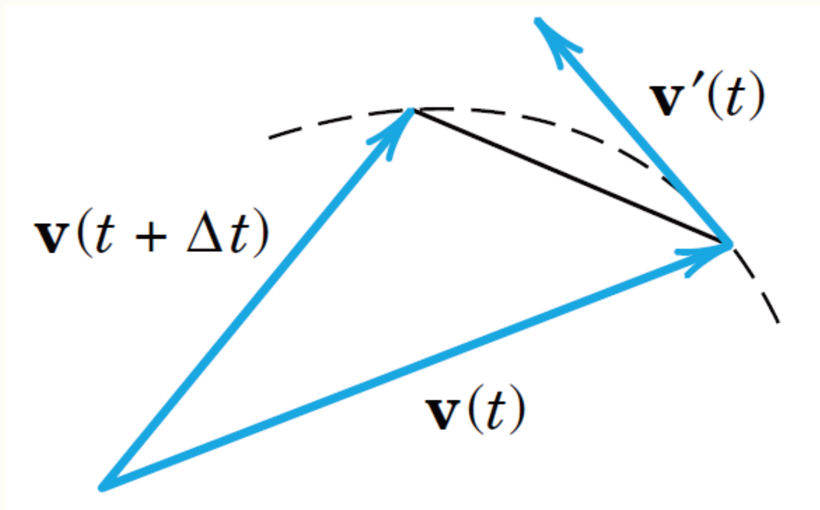


Figure: 1.7: Derivative of a vector function

Hence the derivative $\boldsymbol{v}'(t)$ is obtained by differentiating each component separately.
For instance, if $\boldsymbol{v}(t) =< t, t^2, 0 >$, then $\boldsymbol{v}'(t) =< 1, 2t, 0 >$.

Section 1.5 Concepts of Gradient

The setting is that we are given a scalar function $f(x, y, z)$ that is defined and differentiable in a domain in 3-space with Cartesian coordinates x, y, z. We denote the **gradient** of that function by grad$f$ or ∇f (nabla f). Then the gradient of $f(x, y, z)$ is defined as the vector function

$$\operatorname{grad} f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle = \frac{\partial f}{\partial x}\boldsymbol{i} + \frac{\partial f}{\partial y}\boldsymbol{j} + \frac{\partial f}{\partial z}\boldsymbol{k}.$$

Section 1.6 Concepts of Divergence

Let $\boldsymbol{F}(x, y, z)$ be a differentiable vector function, where x, y, z are Cartesian coordinates, and let F_1, F_2, F_3 be the components of $\boldsymbol{F}$. Then the function

$$\mathrm{div}\boldsymbol{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

is called the **divergence** of $\boldsymbol{F}$ or the divergence of the vector field defined by $\boldsymbol{F}$.

For example, $\boldsymbol{F} =< 3xz, 2xy, -yz^2 >= 3xz\mathbf{i} + 2xy\boldsymbol{j} - yz^2\boldsymbol{k}$, then $\mathrm{div}\boldsymbol{F} = 3z + 2x - 2yz$.

Section 1.7 Concepts of Curl

Let

$$\begin{aligned} \boldsymbol{F}(x,y,z) &= < F_1(x,y,z), F_2(x,y,z), F_3(x,y,z) > \\ &= F_1(x,y,z)\boldsymbol{i} + F_2(x,y,z)\boldsymbol{j} + F_3(x,y,z)\boldsymbol{k} \end{aligned}$$

be a differentiable vector function of the Cartesian coordinates x, y, z. Then the curl of the vector function $\boldsymbol{F}$ or of the vector field given by $\boldsymbol{F}$ is defined by the symbolic determinant

$$\begin{aligned} \mathrm{curl}\boldsymbol{F} &= \mathrm{curl} < F_1, F_2, F_3 > \\ &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} \\ &= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\boldsymbol{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\boldsymbol{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\boldsymbol{k} \end{aligned}$$

Note:
This is the formula when x, y, z are right-handed. If they are left-handed, the determinant has a minus sign in front.

Chapter 2 Vector Integral Calculus, Integral Theorems

Recommended text: Chapter 10

Section 2.1 Line Integrals

Line integral is the integration of a function $f(x)$ between two points (say, A and B) in space or in plane along a path or a curve C. If C is presented by

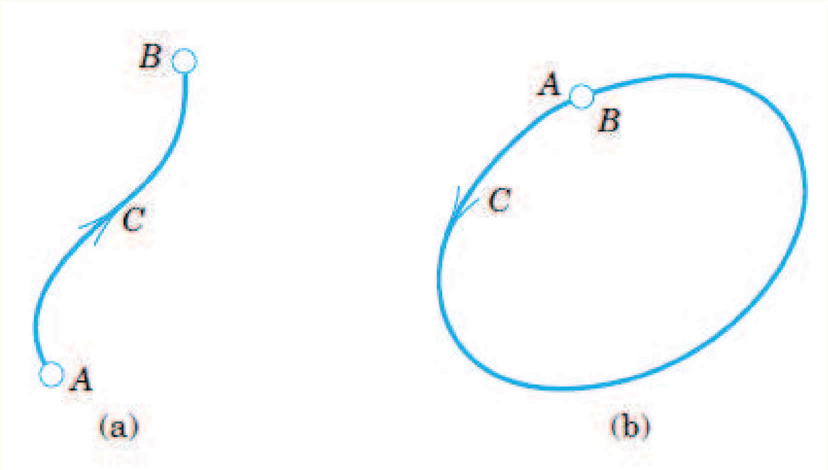


Figure: 2.1: Oriented path (curve) C: $\boldsymbol{r}(t), a \leq t \leq b$

$$\boldsymbol{r}(t) =< x(t), y(t), z(t) >= x(t)\boldsymbol{i} + y(t)\boldsymbol{j} + z(t)\boldsymbol{k},$$

where $\boldsymbol{r}(t)$ is the **parametric representation** of C and $a \leq t \leq b$, then the **line integral** of a vector function $\boldsymbol{F}(\boldsymbol{r})$ over a line C is defined by

$$\int_C \boldsymbol{F}(\boldsymbol{r}) \cdot dr = \int_a^b \boldsymbol{F}(\boldsymbol{r}(t)) \cdot \boldsymbol{r}'(t) dt, \tag{2.1}$$

where $\boldsymbol{r}'(t) = \dfrac{d\boldsymbol{r}(t)}{dt}$.

Note: From the definition, if

$\boldsymbol{F}(\boldsymbol{r}) =< F_1, F_2, F_3 >, \boldsymbol{r}(t) = [x(t), y(t), z(t)]$ then

$$\int_C \boldsymbol{F}(\boldsymbol{r}) \cdot dr = \int_a^b (F_1 x' + F_2 y' + F_3 z') dt$$

Proof of the note:

$$\boldsymbol{r}'(t) = \frac{d\boldsymbol{r}}{dt} \to \boldsymbol{r}'dt = d\boldsymbol{r}, \text{ and } d\boldsymbol{r} = d < x, y, z > = < dx, dy, dz >$$

so,

$$\int_C \boldsymbol{F}(\boldsymbol{r}) \cdot d\boldsymbol{r} = \int_C \boldsymbol{F}(\boldsymbol{r}) \cdot < dx, dy, dz > = \int_c (F_1 dx + F_2 dy + F_3 dz)$$

For $x'(t) = \frac{dx}{dt}$, $y'(t) = \frac{dy}{dt}$, $z'(t) = \frac{dz}{dt}$ we get $x'dt = dx$, $y'dt = dy$, $z'dt = dz$.

Hence,

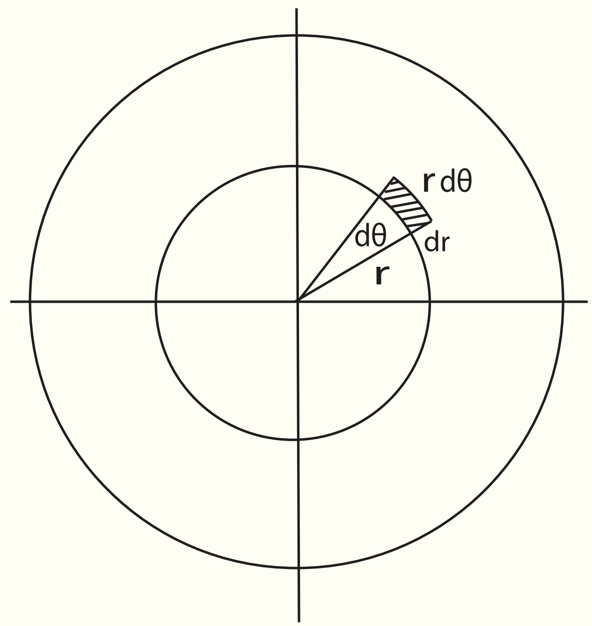
$$\begin{aligned} \int_C \boldsymbol{F}(\boldsymbol{r}) \cdot d\boldsymbol{r} &= \int_C \boldsymbol{F}(\boldsymbol{r}) \cdot < dx, dy, dz > \\ &= \int_c (F_1 dx + F_2 dy + F_3 dz) \\ &= \int_a^b (F_1 x' + F_2 y' + F_3 z') dt \end{aligned}$$

$\square$

Note: If the path of integration C is a closed path, then we write the line integral $\oint_C$.

Section 2.2 Integrals over a circular curve

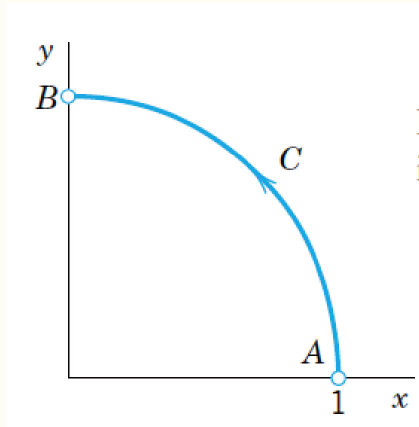
If we want to calculate an integral over a circle it is best to use polar coordinates r and θ. The diagram below shows a small shaded area where r changes by dr and θ changes by $d\theta$. This area is nearly a rectangle.



The area of this small area is nearly $rdrd\theta$, and dr measures the height of the small region.
Here θ is measured in radians and can range from 0 to 2π.
We might need to use the polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$.

Example 1

Find the value of the line integral when $\boldsymbol{F}(\boldsymbol{r}) =< -y, -xy >$ and C is the circular arc in the following figure from A to B.



Solution:

We represent C by $\boldsymbol{r}(t) =< \cos t, \sin t >= \cos t\mathbf{i} + \sin t\mathbf{j}$ where $0 \leq t \leq \frac{\pi}{2}$. Then

$x = r\cos t = \cos t$ and $y = r\sin t = \sin t$

$$
\begin{aligned}
\boldsymbol{F}(\boldsymbol{r}) &= <-y, -xy> \\
&= -y\mathbf{i} - xy\mathbf{j} \\
&= -\sin t\mathbf{i} - \cos t \sin t\mathbf{j}
\end{aligned}
$$

So

$$
\begin{aligned}
\int_C \boldsymbol{F}(\boldsymbol{r}) \cdot d\boldsymbol{r} &= \int_C \boldsymbol{F}(\boldsymbol{r}) \cdot \boldsymbol{r}' dt \\
&= \int_0^{\frac{\pi}{2}} < -\sin t, -\cos t \sin t > \cdot < -\sin t, \cos t > dt \\
&= \int_0^{\frac{\pi}{2}} (\sin^2 t - \cos^2 t \sin t) dt \\
&= \int_0^{\frac{\pi}{2}} \frac{1}{2}(1 - \cos 2t) dt - \int_1^0 u^2(-du) \\
&= \frac{\pi}{4} - 0 - \frac{1}{3} \\
&\approx 0.4521
\end{aligned}
$$

Example 2

Find the value of $\int_C \boldsymbol{F}(\boldsymbol{r}) \cdot d\boldsymbol{r}$ when $\boldsymbol{F}(\boldsymbol{r}) =< z, x, y >$ and C is the helix, $\boldsymbol{r}(t) =< \cos t, \sin t, 3t >$ where $0 \leq t \leq 2\pi$.

Solution:

$\int_C \boldsymbol{F}(\boldsymbol{r}) \cdot d\boldsymbol{r} = \int_C \boldsymbol{F}(\boldsymbol{r}) \cdot \boldsymbol{r}' dt$

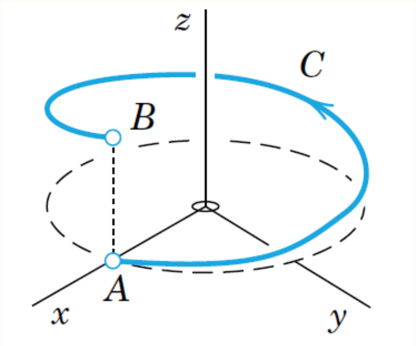
Since $\boldsymbol{r}(t) =< \cos t, \sin t, 3t >= \cos t\mathbf{i} + \sin t\mathbf{j} + 3t\mathbf{k}$, with $0 \leq t \leq 2\pi$; we have $x(t) = \cos t, y(t) = \sin t, z(t) = 3t$.

Thus,

$$\boldsymbol{F}(\boldsymbol{r}(t)) \cdot \boldsymbol{r}'(t) = (3t\mathbf{i} + \cos t\mathbf{j} + \sin t\mathbf{k}) \cdot (-\sin t\mathbf{i} + \cos t\mathbf{j} + 3\mathbf{k}).$$

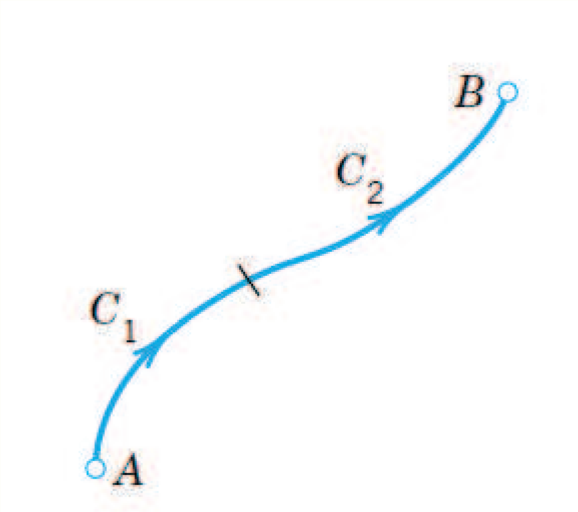
The dot product is $3t(-\sin t) + \cos^2 t + 3\sin t$. Hence (2.1) gives

$$\int_C \boldsymbol{F}(\boldsymbol{r}) \cdot d\boldsymbol{r} = \int_0^{2\pi} (-3t\sin t + \cos^2 t + 3\sin t) dt = 6\pi + \pi + 0 = 7\pi.$$



2.3 Properties of line integral

- $\int_C k\boldsymbol{F} \cdot d\boldsymbol{r} = k \int_C \boldsymbol{F} \cdot d\boldsymbol{r}$, where k is a constant
- $\int_C (\boldsymbol{F} + \boldsymbol{G}) \cdot d\boldsymbol{r} = \int_C \boldsymbol{F} \cdot d\boldsymbol{r} + \int_C \boldsymbol{G} \cdot d\boldsymbol{r}$
- $\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = \int_{C_1} \boldsymbol{F} \cdot d\boldsymbol{r} + \int_{C_2} \boldsymbol{F} \cdot d\boldsymbol{r}$



- $\int_{C^-} \boldsymbol{F} \cdot d\boldsymbol{r} = -\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$

Example 4: (page 417)

Prove the work done by the force $\boldsymbol{F}$ equals the gain in kinetic energy $\left(W = \frac{1}{2}m|v|^2\right)$.

Proof: The work done by the force $\boldsymbol{F}$ is $W = \int_C \boldsymbol{F}(\boldsymbol{r}(t)) \cdot d\boldsymbol{r}(t)$, where t is the time and $\boldsymbol{r}(t)$ is the displacement.
Therefore the velocity is

$$\boldsymbol{v}(t) = \frac{d\boldsymbol{r}(t)}{dt}, \text{ so } d\boldsymbol{r} = \boldsymbol{v}dt.$$

So

$$W = \int_C \boldsymbol{F}(\boldsymbol{r}) \cdot d\boldsymbol{r} = \int_a^b \boldsymbol{F}(\boldsymbol{r}) \cdot \boldsymbol{v}dt$$

By Newton's second law, force=mass(m)×acceleration($\boldsymbol{a}$), and the acceleration is equals to $\frac{d\boldsymbol{v}}{dt} = \boldsymbol{v}'$,
then $\boldsymbol{F} = m\boldsymbol{a} = m\boldsymbol{v}'$.

Therefore

$$\begin{aligned} W &= \int_a^b \boldsymbol{F}(\boldsymbol{r}) \cdot \boldsymbol{v} dt \\ &= \int_a^b m\boldsymbol{v}' \cdot \boldsymbol{v} dt \\ &= \int_a^b m \left(\frac{\boldsymbol{v} \cdot \boldsymbol{v}}{2} \right)' dt \\ &= \frac{m}{2} \int_a^b (\boldsymbol{v} \cdot \boldsymbol{v})' dt \\ &= \frac{m}{2} \left[|\boldsymbol{v}|^2 \right]_a^b \end{aligned}$$

Proved.

We know that $\frac{1}{2} m |\boldsymbol{v}|^2$ is in fact the kinetic energy.

Notes:

- $\int \frac{d\boldsymbol{v}}{dt} dt = \boldsymbol{v}$ or $\int \boldsymbol{v}' dt = \boldsymbol{v}$

$$(\boldsymbol{u} \cdot \boldsymbol{v})' = \boldsymbol{u}' \cdot \boldsymbol{v} + \boldsymbol{u} \cdot \boldsymbol{v}'$$

$$(\boldsymbol{v} \cdot \boldsymbol{v})' = \boldsymbol{v}' \cdot \boldsymbol{v} + \boldsymbol{v} \cdot \boldsymbol{v}'$$

- $$(\boldsymbol{v} \cdot \boldsymbol{v})' = 2(\boldsymbol{v}' \cdot \boldsymbol{v})$$

$$\boldsymbol{v}' \cdot \boldsymbol{v} = \frac{(\boldsymbol{v} \cdot \boldsymbol{v})'}{2}$$

- $\int (\boldsymbol{v} \cdot \boldsymbol{v})' dt = \int \left(|\boldsymbol{v}|^2\right)' dt = |\boldsymbol{v}|^2$

Theorem: Path dependence

The line integral $\int_c \boldsymbol{F}(\boldsymbol{r}) \cdot d\boldsymbol{r} = \int_a^b \boldsymbol{F}(\boldsymbol{r}(t)) \cdot \boldsymbol{r}'(t)dt$ generally depends not only on $\boldsymbol{F}$ and on the endpoints A and B of the path, but also the path itself along the integral is taken.

Example (page 418)

Find the values of $\int_{C_1} \boldsymbol{F}(\boldsymbol{r}) \cdot d\boldsymbol{r}$ and $\int_{C_2} \boldsymbol{F}(\boldsymbol{r}) \cdot d\boldsymbol{r}$ when $\boldsymbol{F}(\boldsymbol{r}) =< 0, xy, 0 >$, and $C_1 : \boldsymbol{r}_1(t) =< t, t, 0 >$ and $C_2 : \boldsymbol{r}_2(t) =< t, t^2, 0 >$,where $0 \leq t \leq 1$.

Solution:

$$\begin{aligned}
\int_{C_1} \boldsymbol{F}(\boldsymbol{r}) \cdot d\boldsymbol{r} &= \int_0^1 \boldsymbol{F}(\boldsymbol{r}(t)) \cdot \boldsymbol{r}'(t)dt \\
&= \int_0^1 < 0, t^2, 0 > \cdot < 1, 1, 0 > dt \\
&= \int_0^1 t^2 dt \\
&= \left[\frac{t^3}{3}\right]_0^1 \\
&= \frac{1}{3}
\end{aligned}$$

$$
\begin{aligned}
\int_{C_2} \boldsymbol{F}(\boldsymbol{r}) \cdot d\boldsymbol{r} &= \int_0^1 \boldsymbol{F}(\boldsymbol{r}(t)) \cdot \boldsymbol{r}'(t) dt \\
&= \int_0^1 [0, t^3, 0] \cdot [1, 2t, 0] dt \\
&= \int_0^1 2t^4 dt \\
&= \left[\frac{2t^5}{5} \right]_0^1 \\
&= \frac{2}{5}
\end{aligned}
$$