MTH101: Lecture 12

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Singularities

Definition

We say the function $f(z)$ is **Singular** *or has a* **Singularity** *at $z = z_0$, if the function f is not analytic (or not defined) in z_0, but in any neighborhood of z_0 there exist points at which f is analytic.*

Definition

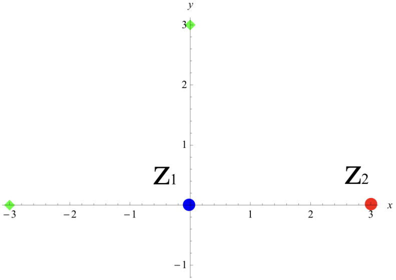
The point z_0 is called **Isolated Singularity**, *if for some $\delta > 0$ the function f is analytic in $\{z \in \mathbb{C} : 0 < |z - z_0| < \delta\}$.*

Example

The function

$$f(z) = \frac{1}{z(3-z)},$$

is Analytic in $\mathbb{C} \setminus \{z_1 = 0, z_2 = 3\}$. The points z_1 and z_2 are **Isolated Singularities.**

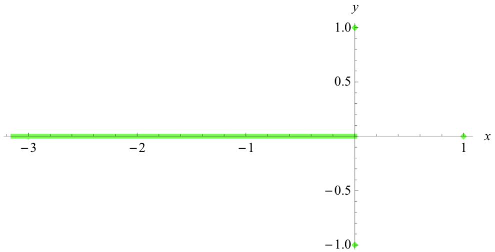


Example

The function

$$f(z) = \text{Ln } z,$$

is Analytic in $\mathbb{C} \setminus \{z \in \mathbb{C} : y = 0, x \leq 0\}$. There are no(!) **Isolated Singularities**.



Definition

If z_0 is an **Isolated Singularity** *for f, then by Laurents theorem we have that the function $f(z)$ can be represented by a* **Laurent Series**

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n},$$

in the **Annulus**

$$\{z \in \mathbb{C} : \quad 0 < |z - z_0| < \delta\}.$$

Classification of isolated singularities

Let $z_0 \in \mathbb{C}$ be an **Isolated Singularity** of $f(z)$ and consider the Laurent Series of $f(z)$:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} \text{ (Principal Part)}$$

in the **Annulus** in which converges:

$$\{z \in \mathbb{C} : 0 < |z - z_0| < R\}.$$

Csae 1

If the **Principal Part** of the Laurent Series of $f(z)$ has finitely many terms:

$$\frac{b_1}{z-z_0}+\frac{b_2}{(z-z_0)^2}+\cdots+\frac{b_m}{(z-z_0)^m}, \quad \text{with } b_m \neq 0$$

that is $b_n = 0$ for $k > m$, then the **Isolated Singularity** z_0 is called a **Pole of Order m**.

In particular, if $m = 1$, then z_0 is called a **Simple Pole**.

Case 2

If the **Principal Part** of the Laurent Series of f (z) has infinitely many terms:

$$\sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n},$$

that is, if there exist infinitely many indexes n such that the term $b_n \neq 0$, then z_0 is called **Essential Singularity**.

Case 3

If the **Principal Part** of the Laurent Series of $f(z)$ has no terms, that is if

$$b_n = 0, \quad \text{for all } n \geq 1,$$

then z_0 is called a **Removable Singularity**.
By defining

$$f(z_0) := a_0,$$

we make f to be analytic in the open Disk:

$$\{z \in \mathbb{C} : |z - z_0| < \delta\}.$$

Example

Study the singularity $z = 2$ of the function:

$$f(z) = \frac{1}{z(z-2)^2}.$$

Solution

The function $f(z)$ has two **Isolated Singularities** $z_1 = 0$ and $z_2 = 2$, then it admits a Laurent Series with center $z_0 = 2$ in the Annulus:

$$0 < |z-2| < 2.$$

We need to write the Laurent Series and study its **Principal Part**. We use the Geometric Series and write f as a function of $z-2$ for $z \neq 0, 2$:

$$\begin{aligned} f(z) &= \frac{1}{z(z-2)^2} = \frac{1}{(z-2)^2}\frac{1}{z-2+2} = \frac{1}{(z-2)^2}\frac{1}{2}\left(\frac{1}{1+\frac{z-2}{2}}\right) \\ &= \frac{1}{2}\frac{1}{(z-2)^2}\left[\frac{1}{1-(-\frac{z-2}{2})}\right] = \frac{1}{2}\frac{1}{(z-2)^2}\sum_{n=0}^{\infty}\left(-\frac{z-2}{2}\right)^n \\ &= \frac{1}{2}\frac{1}{(z-2)^2}\sum_{n=0}^{\infty}\frac{(-1)^n}{2^n}(z-2)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}(z-2)^{n-2}. \end{aligned}$$

Then

$$f(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}}(z-2)^{n-2},$$

and the series converges for $|-\frac{z-2}{2}| < 1$ and $z \neq 0$, that is in the Annulus $0 < |z-2| < 2$.

The **Principal Part** of the Laurent series is:

$$\frac{1}{2}(z-2)^{-2} - \frac{1}{4}(z-2)^{-1}.$$

The **Principal part** has finitely many terms (only two terms), in fact the coefficients satisfy

$$b_1 = -\frac{1}{4}, \quad b_2 = \frac{1}{2}, \quad \ldots, \quad b_n = 0 \quad \text{for all } n \geq 3.$$

then z_0 is a **Pole of order 2** (since the biggest n such that $b_n \neq 0$ is $n = 2$).

Example

Study the singularity of the function

$$f(z) = \sin \frac{1}{z}.$$

Solution

The function $f(z)$ has only one **Isolated Singularity** for $z_0 = 0$, then it admits a Laurent Series in the Annulus:

$$0 < |z| < +\infty.$$

We need to write the Laurent Series and study its **Principal Part**.
We use the Series of $\sin(z)$ with center $z_0 = 0$:

$$\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1}, \quad \text{for all } z \in \mathbb{C},$$

then, for $z \neq 0$, we use the previous series and replace z by $\frac{1}{z}$:

$$\sin \frac{1}{z} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \left(\frac{1}{z}\right)^{2n+1}, \quad \text{for all } z \neq 0.$$

Then

$$\sin\frac{1}{z}=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\frac{1}{z^{2n+1}},\quad \text{for all } z\neq 0.$$

We observe that all the powers of the series are negative, this means that the **Principal Part** of the Laurent Series is

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\frac{1}{z^{2n+1}}.$$

It has infinitely many terms, then $z_0=0$ is an **Essential Singularity**.

Example

Study the singularity of the function

$$f(z) = \frac{\sin z}{z}.$$

Solution

The function $f(z)$ has only one **Isolated Singularity** for $z_0 = 0$, then it admits a Laurent Series in the Annulus:

$$0 < |z| < +\infty.$$

We need to write the Laurent Series and study its **Principal Part**. We use the Series of $\sin(z)$ with center $z_0 = 0$:

$$f(z) = \frac{1}{z}\sin z = \frac{1}{z}\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n}, \quad \text{for all } z \neq 0$$

We observe that there are no negative powers in the series, then the **Principal Part** has no terms.

This means that $z_0 = 0$ is a **Removable Singularity**.

Moreover $f(z)$ can be extended to a function $g(z)$ which is analytic in the Set

$$|z| < \infty, \quad \text{that is in } \mathbb{C}.$$

In details

$$g(z) := \begin{cases} \frac{\sin z}{z}, & \text{if } z \neq 0, \\ a_0 = 1, & \text{if } z = 0. \end{cases}$$

where a_0 is the coefficient with $n = 0$ of the Power Series.
We observe that the above power Series is the Taylor Series of the function $g(z)$.

Zeros of analytic functions

Definition

Let $f(z)$ be an analytic function in the open disk $|z - z_0| < R$. We say that z_0 is a **Zero of Order n** *if*

$$f(z_0) = f'(z_0) = \dots = f^{(n-1)}(z_0) = 0, \quad \text{and } f^{(n)}(z_0) \neq 0.$$

Remark

If z_0 is a **Zero of Order n** *for $f(z)$ then its Taylor Series is given by*

$$f(z) = \sum_{k=n}^{\infty} a_k (z - z_0)^k = \sum_{k=n}^{\infty} \frac{f^{(k)}(z_0)}{k!} (z - z_0)^k,$$

where $a_n \neq 0$.

Example

The function $f(z) = (z-2)^3$ has a zero of order 3 at $z_0 = 2$. We have:

$$f'(z) = 3(z-2)^2, \quad f''(z) = 6(z-2), \quad f'''(z) = 6,$$

from which

$$f(2) = f'(2) = f''(2) = 0, \quad f'''(2) = 6 \neq 0.$$

Theorem

If $g(z)$ is Analytic at z_0 and has a **Zero of order n** *at z_0 and if the function $h(z)$ is Analytic at z_0 and $h(z_0) \neq 0$ then the function*

$$f(z) = \frac{h(z)}{g(z)},$$

has a **Pole of Order n** *at z_0.*

Example

The function $f(z) = \frac{z}{(z-1)^3}$ has an isolated singularity at $z_0 = 1$. It can be written in the following way:

$$f(z) = \frac{h(z)}{g(z)}, \quad \text{with } h(z) = z, \; g(z) = (z-1)^3.$$

The functions $h(z)$ and $g(z)$ are Analytic at z_0. Moreover, $h(z_0) = 1 \neq 0$ and $g(z)$ has a **Zero of order 3** at $z_0 = 1$ since

$$g(1) = g'(1) = g''(1) = 0, \quad g'''(1) \neq 0.$$

Then from the previous theorem $f(z)$ has a **Pole of Order 3** at $z_0 = 1$.

Bibliography

1 *Kreyszig, E.* **Advanced Engineering Mathematics**. Wiley, 10th Edition.