2.9 Stokes's theorem (page 463) Stokes's theorem allow us to transform line integrals into surface integrals and conversely. It generalizes Green's theorem in the plane.

Stokes's theorem: Let S be a piecewise smooth oriented surface in space and let the boundary of S be a piecewise smooth simple closed curve C. S is represented by $\boldsymbol{r}(u,v)$ and C by $r(s)$ with the arc lengths as the parameter. Let $\boldsymbol{F}(x,y,z)$ be a continuous vector function that has continuous first partial derivatives in a domain in space containing S. Then

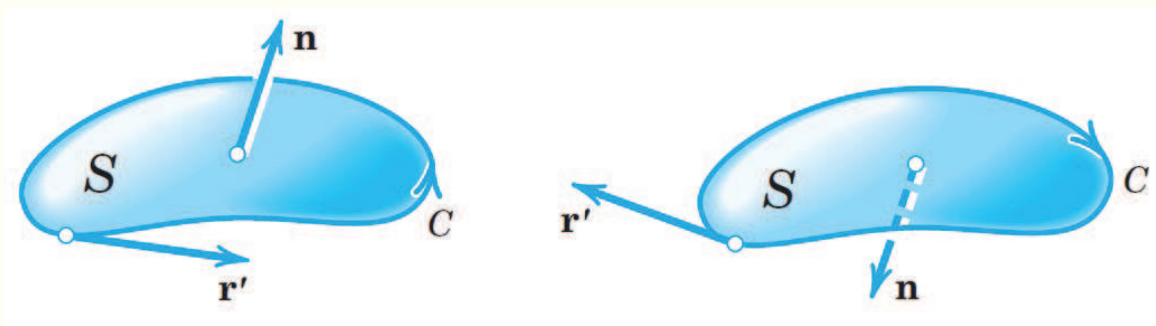
$$\iint_S (\text{curl}\boldsymbol{F}) \cdot \boldsymbol{n} dA = \oint_C \boldsymbol{F} \cdot \boldsymbol{r}'(s) ds, \tag{2.23}$$

where $\boldsymbol{n}$ is a unit normal vector of S and, depending on $\boldsymbol{n}$, the integration around C is taken in the sense shown in figure below.

Furthermore, $\boldsymbol{r}' = d\boldsymbol{r}/ds$ is the unit tangent vector and s the arc length of C.
In components, formula (2.23) becomes

$$\begin{aligned}
\iint_R \Bigg[\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) N_1 + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \right) N_2 \\
+ \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) N_3 \Bigg] du dv \\
= \oint_{\bar{C}} (F_1 dx + F_2 dy + F_3 dz). \qquad (2.24)
\end{aligned}$$

Here, $\boldsymbol{F} =< F_1, F_2, F_3 >$, $\boldsymbol{N} =< N_1, N_2, N_3 >$, $\boldsymbol{n} dA = \boldsymbol{N} du dv$, $\boldsymbol{r}' ds =< dx, dy, dz >$ and R is the region with boundary curve $\bar{C}$ in the uv-plane corresponding to S represented by $\boldsymbol{r}(u, v)$.

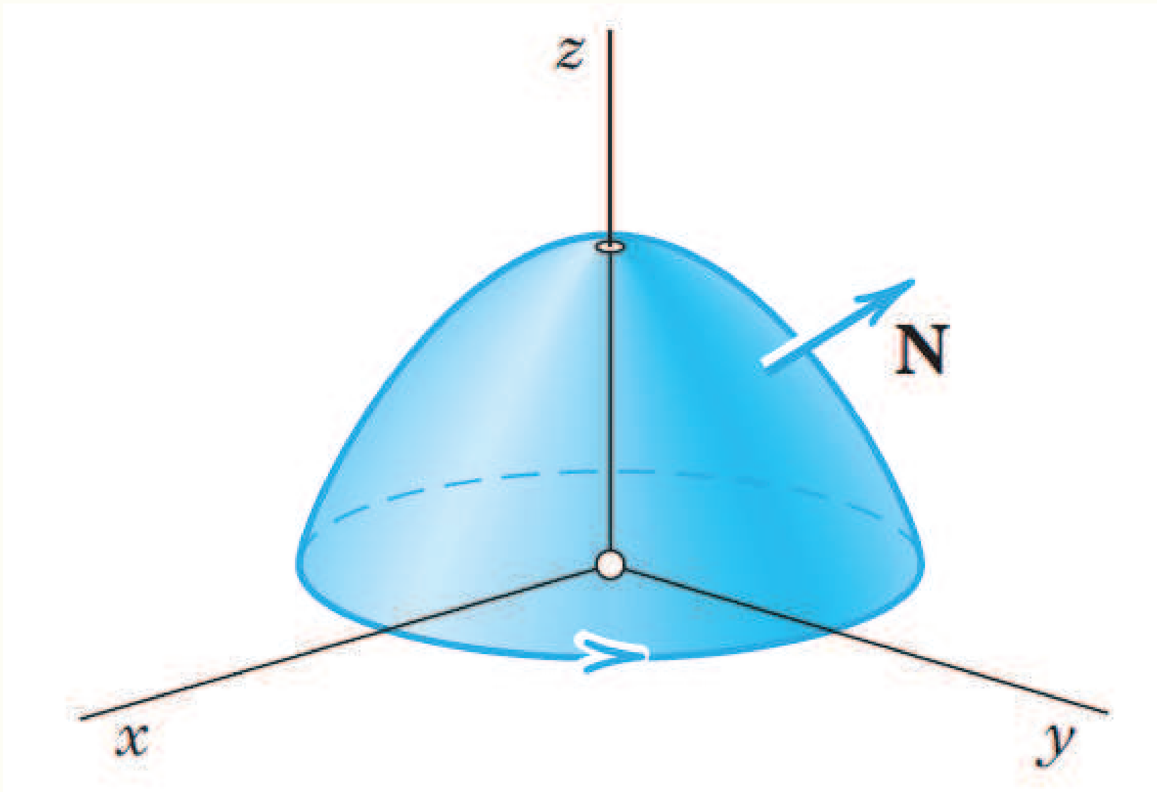


Notes: When S is a region in the xy-plane and $\boldsymbol{F} =< F_1(x,y), F_2(x,y), 0 >$, $\boldsymbol{N} =< 0, 0, N_3 >= N_3\boldsymbol{k}$, then (2.23) reduces to

$$\begin{aligned}
& \iint\limits_R \left[\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) N_3\right] dudv \\
= \; & \iint\limits_S \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA \\
= \; & \oint\limits_C (F_1 dx + F_2 dy).
\end{aligned}$$

This implies that the Stokes's theorem extends the Green's theorem in the plane, or the Green's theorem in the plane is a special case of the stokes's theorem.

Example 1: Verification of Stokes's theorem: (page 464) Let $\boldsymbol{F} =< y, z, x >$ and s be the paraboloid $z = 1 - x^2 - y^2, z \geq 0$. Verify Stokes's theorem, that is $\iint_S (\text{curl}\boldsymbol{F}) \cdot \boldsymbol{n} dA = \oint_C \boldsymbol{F} \cdot \boldsymbol{r}'(s) ds$.



Solutions: Let C be the circle $\boldsymbol{r}(s) =< \cos s, \sin s, 0 >$, oriented as in the figure. Its unit tangent vector is

$$\boldsymbol{r}'(s) =< -\sin s, \cos s, 0 > .$$

The function $F =< y, z, x >$ on C is

$$\boldsymbol{F}(\boldsymbol{r}(s)) =< \sin s, 0, \cos s > .$$

Then the right handed side is

$$\begin{aligned}
\oint_C \boldsymbol{F} \cdot \boldsymbol{r}'(s) ds &= \int_0^{2\pi} < \sin s, 0, \cos s > \cdot < -\sin s, \cos s, 0 > ds \\
&= \int_0^{2\pi} -\sin^2 s ds \\
&= -\int_0^{2\pi} \frac{1-\cos 2s}{2} ds \\
&= -\pi.
\end{aligned}$$

The left handed side is

$$\iint\limits_S (\text{curl}\boldsymbol{F}) \cdot \boldsymbol{n} dA = \iint\limits_R (\text{curl}\boldsymbol{F}) \cdot \boldsymbol{N} dxdy.$$

Since S is represented by

$$G(x, y, z) = z - f(x, y) = z - (1 - x^2 - y^2) = 0,$$

the normal vector of S is $\boldsymbol{N} = \text{grad}G =< 2x, 2y, 1 >$
For $\boldsymbol{F} =< y, z, x >$, we have

$$\text{curl}\boldsymbol{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix} =< -1, -1, -1 > .$$

Hence

$$
\begin{aligned}
\iint\limits_S (\mathrm{curl}\boldsymbol{F}) \cdot \boldsymbol{n} dA &= \iint\limits_R (\mathrm{curl}\boldsymbol{F}) \cdot \boldsymbol{N} dxdy \\
&= \iint\limits_R < -1, -1, -1 > \cdot < 2x, 2y, 1 > dxdy \\
&= \iint\limits_R (-2x - 2y - 1) dxdy, \qquad (2.25)
\end{aligned}
$$

where R is the projection of S into the xy-plane:

$$R = \{(x, y) : x^2 + y^2 = 1\}.$$

To evaluate the double integral (2.25) we use the polar coordinates $x = r\cos\theta$, $y = r\sin\theta$, so the region R can be represented by $R = \{(r, \theta), 0 \leq r \leq 1, 0 \leq \theta \leq 2\pi\}$.

Therefore (2.25) can be written into

$$\begin{aligned}
\iint_S (\mathrm{curl}\boldsymbol{F}) \cdot \boldsymbol{n} dA &= \iint_R (-2x - 2y - 1) dx dy \\
&= \int_0^{2\pi} \int_0^1 (-2r\cos\theta - 2\sin\theta - 1) r dr d\theta \\
&= \int_0^{2\pi} \left[-\frac{2}{3}(\cos\theta + \sin\theta) - \frac{1}{2} \right] d\theta \\
&= 0 + 0 - \frac{1}{2}(2\pi) \\
&= -\pi.
\end{aligned}$$

Therefore $\iint_S (\mathrm{curl}\boldsymbol{F}) \cdot \boldsymbol{n} dA = \oint_C \boldsymbol{F} \cdot \boldsymbol{r}'(s) ds$. ■

Example 2: Evaluation of a line integral by Stokes's theorem (page 467) Evaluate $\oint_C \boldsymbol{F} \cdot \boldsymbol{r}'(s)ds$, where C is the circle $x^2 + y^2 = 4, z = -3$, oriented counterclockwise as seen by a person standing at the origin, and, with respect to right-handed Cartesian coordinates,

$$\boldsymbol{F} =< y, xz^3, zy^3 >= y\mathbf{i} + xz^3\mathbf{j} - zy^3\mathbf{k}.$$

Solution:

As a surface S bounded by curve C, we can take the plane circular disk $x^2 + y^2 \leq 4$ in the plane $z = -3$. Then the unit normal vector $\boldsymbol{n}$ in Stokes's theorem points in the positive z-direction; thus $\boldsymbol{n} = \mathbf{k} =< 0, 0, 1 >$. Since $F =< y, xz^3, -zy^3 >$, then on S $\boldsymbol{F}$ has the components $F_1 = y$, $F_2 = -27x$, $F_3 = 3y^3$, that is

$$\boldsymbol{F}(\boldsymbol{r}(x, y)) =< y, -27x, 3y^3 > .$$

We thus obtain

$$
\begin{aligned}
(\text{curl}\boldsymbol{F}) \cdot \boldsymbol{n} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -27x & 3y^3 \end{vmatrix} \cdot \mathbf{k} \\
&= < 9y^2, 0, -28 > \cdot \mathbf{k} = -28
\end{aligned}
$$

Hence by the Stokes's theorem

$$
\begin{aligned}
\oint_C \boldsymbol{F} \cdot \boldsymbol{r}'(s) ds &= \iint\limits_S (\text{curl}\boldsymbol{F}) \cdot \boldsymbol{n} dA \\
&= \iint\limits_S -28 dA \\
&= -28(\text{area of } S) \\
&= -28(\pi 2^2) \\
&= -112\pi.
\end{aligned}
$$

Example 3: Work done in the displacement around a closed curve (page 467) Find the work done by the force $\boldsymbol{F} =< 2xy^3 \sin z, 3x^2y^2 \sin z, x^2y^3 \cos z >$ in the displacement around the curve of intersection of the paraboloid $z = x^2 + y^2$ and the cylinder $(x-1)^2 + y^2 = 1$.

Solution: This work is given by the line integral $\oint_C \boldsymbol{F} \cdot \boldsymbol{r}'(s)ds$, where the surface S is on the paraboloid $z = x^2 + y^2$ and its boundary is C. By Stokes's theorem

$$\oint_C \boldsymbol{F} \cdot \boldsymbol{r}'(s)ds = \iint_S (\text{curl}\boldsymbol{F}) \cdot \boldsymbol{n} dA,$$

so we now evaluate curl$\boldsymbol{F}$:

$$\begin{aligned}
\text{curl}\boldsymbol{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy^3 \sin z & 3x^2y^2 \sin z & x^2y^3 \cos z \end{vmatrix} \\
&= (3x^2y^2 \cos z - 3x^2y^2 \cos z)\mathbf{i} - (2xy^3 \cos z - 2xy^3 \cos z)\mathbf{j} \\
&\quad +(6xy^2 \sin z - 6xy^2 \sin z)\mathbf{k} = \mathbf{0}.
\end{aligned}$$

Hence $(\text{curl}\boldsymbol{F}) \cdot \boldsymbol{n} = 0$, so $\iint\limits_S (\text{curl}\boldsymbol{F}) \cdot \boldsymbol{n} dA = 0$.

In fact, we note that $\boldsymbol{F} = \text{grad} f$, where $f(x, y, z) = x^2y^3 \sin z$, so based on the theorem on page 408, we know $\text{curl}(\text{grad} f) = \mathbf{0} = \text{curl}\boldsymbol{F}$.

Therefore we know the work done by the force around the given curve is 0.

SUMMARY OF CHAPTER 2

Chapter 9 extended differential calculus to vectors, that is, to vector functions $\boldsymbol{v}(x, y, z)$ or $\boldsymbol{v}(t)$. Similarly, chapter 10 extends integral calculus to vector functions. This involves line integrals, surface integrals and triple integrals and the three 'big' theorems for transforming these integrals into one another, the theorems of Green, Gauss and Stokes.

The analog of the definite integral of calculus if the **line integral**

$$\int_C \boldsymbol{F}(\boldsymbol{r}) \cdot d\boldsymbol{r} = \int_C (F_1 dx + F_2 dy + F_3 dz) = \int_a^b \boldsymbol{F}(\boldsymbol{r}(t)) \cdot \boldsymbol{r}'(t) dt \tag{2.26}$$

where $C : \boldsymbol{r}(t) =< x(t), y(t), z(t) > \ (a \leq t \leq b)$ is a curve in space (or in the plane). Physically, (2.26) may represent the work done by a (variable) force in a displacement.

Independence of path of a line integral in a domain D means that the integral of a given function over any path C with endpoints P and Q has the same value for all paths from P to Q that lie in D; here P and Q are fixed. An integral (2.26) is independent of path in D if and only if the differential form $F_1 dx + F_2 dy + F_3 dz$ with continuous F_1, F_2, F_3 is **exact** in D. Also, if curl$\boldsymbol{F} = 0$, where $\boldsymbol{F} =< F_1, F_2, F_3 >$, has continuous first partial derivatives in a simply connected domain D, then the integral (2.26) is independent of path in D.

Integral theorems. The formula of **Green's theorem in the plane**

$$\iint\limits_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dxdy = \oint_C (F_1 dx + F_2 dy) \tag{2.27}$$

transforms **double integrals** over a region R in the $xy-$plane into line integrals over the boundary curve C of R and conversely.

Similarly, the formula of the **divergence theorem of Gauss**

$$\iiint\limits_T \mathrm{div}\boldsymbol{F} dV = \iint\limits_S \boldsymbol{F} \cdot \boldsymbol{n} dA \tag{2.28}$$

transforms **triple integrals** over a region T in space into surface integrals over the boundary surface S of T, and conversely.
Formula (2.28) implies **Green's formulas**

$$\iiint_T (f\nabla^2 g + \nabla f \cdot \nabla g) dV = \iint_S f \frac{\partial g}{\partial n} dA;$$

$$\iiint_T (f\nabla^2 g - g\nabla^2 f) dV = \iint_S \left(f \frac{\partial g}{\partial n} - g \frac{\partial f}{\partial n} \right) dA.$$

Finally, the formula of **Stokes's theorem**

$$\iint\limits_S (\mathrm{curl}\boldsymbol{F}) \cdot \boldsymbol{n} dA = \oint_C \boldsymbol{F} \cdot \boldsymbol{r}'(s) ds$$

transforms surface integrals over a surface S into line integrals over the boundary curve C of S and conversely.