MTH101: Tutorial 4

Dr. Tai-Jun Chen, Dr. Xinyao Yang

Xi'an Jiaotong-Liverpool University, Suzhou

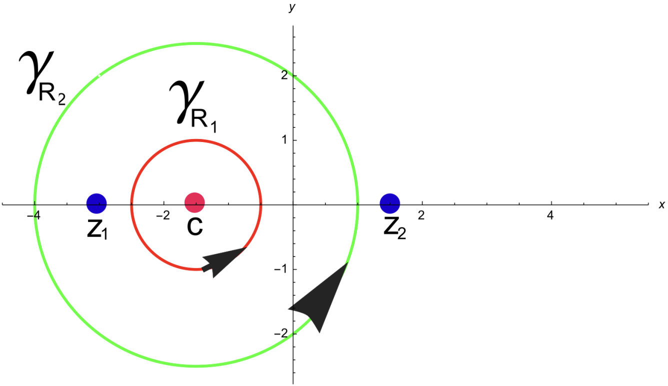
October 11, 2017

Exercise 1.1

Let γ_R be the Circle with radius R, center $c = -\frac{3}{2}$ with counterclockwise orientation.
Compute the integral

$$I = \oint_{\gamma_R} \frac{2z^2 - iz + \sqrt{3}}{(z+3)(2z-3)} dz,$$

with radius $R_1 = 1$ and $2 < R_2 < 3$.


y
x
γ_{R_2}
γ_{R_1}
z_1
c
z_2
−4
−2
2
4
2
1
−1
−2

Solution:

We have that the function

$$g(z) = \frac{2z^2 - iz + \sqrt{3}}{(z+3)(2z-3)}$$

is **not Analytic** at the points $z_1 = -3$ and $z_2 = \frac{3}{2}$.

When $R_1 = 1$ we observe that z_1 and z_2 are **outside** γ_{R_1} , then the function $g(z)$ is **Analytic** in a **Simply Connected Domain** containing γ_{R_1}.

Then by **Cauchys Integral Theorem** we have:

$$\oint_{\gamma_{R_1}} g(z)\ dz = 0.$$

When $2 < R_2 < 3$ we have that the point $z_1 = -3$ is in the **interior** of γ_{R_2} while $z_2 = \frac{3}{2}$ is **outside** of γ_{R_2}.
Then the function

$$f(z) = \frac{2z^2 - iz + \sqrt{3}}{2z - 3},$$

is **Analytic** in a **Simply Connected Domain** containing γ_{R_2}.
Then by **Cauchys Integral Formula** we have

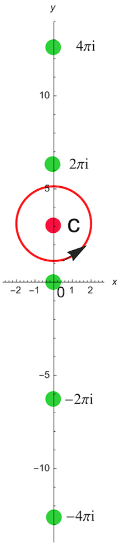
$$I = \oint_{\gamma_{R_2}} \frac{f(z)}{z - z_1} dz = 2\pi i f(z_1) = -2\pi i \frac{(18 + 3i + \sqrt{3})}{9}$$
$$= \frac{2}{3}\pi - i\left(4 + \frac{2\sqrt{3}}{9}\right)\pi.$$

Exercise 1.2

Compute the Integral

$$\oint_\gamma \frac{\cos^2 z}{(e^z - 1)(z - \pi i)} dz,$$

where γ *is counterclockwise* $|z - \pi i| = 2$.


y
4πi
10
2πi
5
C
x
−2
−1
0
1
2
−5
−2πi
−10
−4πi

Solution:
The function

$$f(z) = \frac{\cos^2 z}{e^z - 1},$$

is **Analytic** in the set

$$A = \{z \in \mathbb{C} : e^z - 1 \neq 0\} = \mathbb{C} \setminus \{2n\pi i, \quad n = 0, \pm 1, \pm 2, ...\}.$$

The point $z_0 = \pi i$ is in the **Interior** of γ, while all the points $\{2n\pi i, n = 0, \pm 1, \pm 2, ...\}$ are **outside** γ,
The function $f(z)$ is **Analytic** in a **Simply Connectes Domain** containing γ.
Then we can use **Cauchys Integral Formula**:

$$I = \oint_\gamma \frac{f(z)}{z - z_0} dz = 2\pi i f(z_0) = 2\pi i f(\pi i)$$

$$= 2\pi i \frac{\cos^2(\pi i)}{e^{\pi i} - 1} = -\pi i \cosh^2(\pi).$$

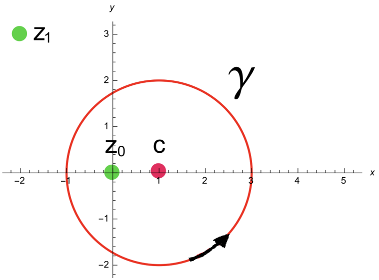
(We used the formulas $\cos iz = \cosh z$ and the fact that $e^{\pi i} = -1$.)

Exercise 1.3

Compute the Integral

$$I = \oint_{\gamma} \frac{\sinh z}{z^2(z+2-3i)} dz,$$

where γ *is* $(x-1)^2 + y^2 = 4$ *clockwise(!).*


y
x
3
2
1
-1
-2
-2
-1
1
2
3
4
5
z_1
z_0
c
γ

Solution:

The function

$$f(z) = \frac{\sinh z}{z + 2 - 3i},$$

is **Analytic** in $\mathbb{C} \setminus \{z_1\}$, with $z_1 = -2 + 3i$,
The point $z_0 = 0$ is in the **Interior** of γ while $z_1 = -2 + 3i$ is outside γ,
the function $f(z)$ is **Analytic** in a **Simply Connected Domain** containing γ,
then we can use both the **sense reversal property** and the **Cauchys Integral Formula for Derivatives**:

$$I = -\oint_{-\gamma} \frac{f(z)}{(z - z_0)^{n+1}} dz = -\frac{2\pi i}{n!} f^{(n)}(z_0),$$

with $z_0 = 0$ and $n = 1$.

Then

$$I = -2\pi i \cdot f'(0) = \frac{-2\pi i}{2-3i} = \frac{6}{13}\pi - i\frac{4}{13}\pi,$$

where we have used that

$$f'(z) = \frac{(z+2-3i)\cosh z - \sinh z}{(z+2-3i)^2},$$

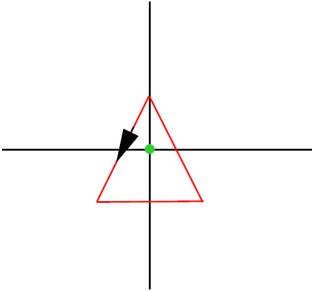
and

$$f'(0) = \frac{1}{2-3i}$$

Exercise 1.4

Integrate the given function around the triangle with vertices i, $\pm 1 - i$ counterclockwise.

$$(\cos 3z)/(6z).$$



Solution: We have the integrand $\cos 3z/(6z)$ is not analytic at $z = 0$, which is enclosed by the triangle.

We use Cauchy's integral formula and obtain that

$$\int_\gamma \frac{\cos 3z}{6z}\, dz = \int_\gamma \frac{\frac{\cos 3z}{6}}{z}\, dz = 2\pi i \left[\frac{\cos 3z}{6}\right]_{z=0} = \frac{\pi i}{3}.$$

Remark: Watch out for the coefficients.

Exercise 2.1

Determine whether the following series is convergent or divergent, choose appropriate test and justify your answer.

(1)

$$\sum_{n=0}^{\infty} \frac{(20+30i)^n}{n!}$$

(2)

$$\sum_{n=0}^{\infty} \frac{n+i}{3n^2+2i}$$

(3)

$$\sum_{n=0}^{\infty} \frac{n-i}{3n+2i}$$

Solution:

We use the Ratio test for (1):

$$\left|\frac{z_{n+1}}{z_n}\right| = \left|\frac{(20+30i)^{n+1}}{(n+1)!} \cdot \frac{n!}{(20+30i)^n}\right| = \left|\frac{20+30i}{n+1}\right| = \frac{10\sqrt{13}}{n+1}$$

from which we will see that

$$\lim_{n\to\infty}\left|\frac{z_{n+1}}{z_n}\right| = 0 < 1,$$

thus series (1) is absolutely convergent.

For series (2) we use the Comparision Test, note that

$$|z_n| = \left| \frac{n+i}{3n^2+2i} \right| > \frac{1}{3n}$$

then

$$\sum_{n=0}^{\infty} \frac{n+i}{3n^2+2i} > \frac{1}{3} \sum_{n=0}^{\infty} \frac{1}{n}$$

because the harmonic series $\sum\limits_{n=0}^{\infty} \frac{1}{n}$ is divergent, we conclude that $\sum\limits_{n=0}^{\infty} \frac{n+i}{3n^2+2i}$ is also divergent.

For series (3) we observe that

$$\lim_{n\to\infty} z_n = \frac{1}{3} \neq 0,$$

and by the Test for Divergence, we conclude that $\sum_{n=0}^{\infty} \frac{n-i}{3n+2i}$ is divergent.

Exercise 2.2

Find the center and the radius of convergence.

$$\sum_{n=0}^{\infty} \frac{(z-2i)^n}{n^n}$$

Solution:
We have $a_n = \frac{1}{n^n}$ and $z_0 = 2i$. We compute:

$$\sqrt[n]{|a_n|} = \sqrt[n]{\left|\frac{1}{n^n}\right|} = \frac{1}{n},$$

from which

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = 0$$

and the **Radius of Convergence** is $R = \infty$, the series converges for all z and the **Disk of Convergence** is the whole complex plane.