MTH101: Tutorial 10

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Exercise 1.1

Using Laplace transform (without convolution) to solve

1. $y'' + 0.04y = 0.02t^2, \quad y(0) = -25, y'(0) = 0.$
2. $y'' + 3y' + 2y = r(t), \quad y(0) = y'(0) = 0,$

$$\text{where } r(t) = \begin{cases} 1 & \text{if } 0 < t < 1, \\ 0 & \text{if } t > 1. \end{cases}$$

Solution

1. The Laplace transform for the ODE is

$$
\begin{aligned}
& s^2 Y - sy(0) - y'(0) + 0.04Y = 0.02\frac{2}{s^3} \\
\Rightarrow & \left(s^2 + 0.04\right) Y + 25s = \frac{0.04}{s^3} \\
\Rightarrow & Y = \frac{1}{s^3}\frac{0.04 - 25s^4}{(s^2 + 0.04)} = \frac{\left(1 - 25s^2\right)}{s^3} = \frac{1}{s^3} - 25 \times \frac{1}{s} \\
\Rightarrow & y(t) = \frac{t^2}{2} - 25.
\end{aligned}
$$

One can check that the initial value of this IVP is chosen in a way that the homogeneous solution of the ODE does not contribute.

Solution.

2. We first write $r(t)$ as

$$r(t) = u(t) - u(t-1).$$

The Laplace transform for the ODE becomes

$$s^2 Y - sy(0) - y'(0) + 3sY - 3y(0) + 2Y = \frac{1}{s} - \frac{e^{-s}}{s}$$
$$\Rightarrow \left(s^2 + 3s + 2\right) Y = \frac{1}{s} - \frac{e^{-s}}{s}$$
$$\Rightarrow Y = \frac{1}{s(s+1)(s+2)} - \frac{1}{s(s+1)(s+2)} e^{-s}.$$

Solution.
By using partial fraction,

$$\frac{1}{s(s+1)(s+2)} = \left[\frac{1}{2s} + \frac{1}{2(s+2)} - \frac{1}{(s+1)}\right].$$

Therefore,

$$y(t) = \left(\frac{1}{2} + \frac{1}{2}e^{-2t} - e^{-t}\right) - \left(\frac{1}{2} + \frac{1}{2}e^{-2(t-1)} - e^{-(t-1)}\right)u(t-1).$$

Exercise 2.1

Find $f(t)$ if $\mathcal{L}\left[f\right]$ equals

$$\frac{e^{-as}}{s(s-2)}.$$

Solution

We can express the function as follows

$$\frac{e^{-as}}{s(s-2)} = \frac{e^{-as}}{s}\frac{1}{(s-2)},$$

and since, $\mathcal{L}^{-1}[\frac{e^{-as}}{s}] = u(t-a), \quad \mathcal{L}^{-1}[\frac{1}{s-2}] = e^{2t},$

$$\Rightarrow f(t) = u(t-a) * e^{2t} = \int_0^t u(\tau - a)e^{2(t-\tau)}d\tau,$$

$$\Rightarrow f(t) = e^{2t}\int_a^t e^{-2\tau}d\tau = -\frac{e^{2t}}{2}e^{-2\tau}\Big|_a^t$$

$$\Rightarrow f(t) = \frac{e^{2t}}{2}\left(e^{-2a} - e^{-2t}\right) = \begin{cases} \frac{e^{2(t-a)}-1}{2} & if \;\; t > a, \\ 0 & if \;\; t < a. \end{cases}$$

Exercise 2.2

Solve the following equation by the Laplace transform.

$$y(t) - \int_0^t y(\tau)\sin 2(t-\tau)d\tau = \sin 2t.$$

Solution

The Laplace transform of the original equation is

$$
\begin{aligned}
& Y - Y \times \frac{2}{s^2+4} = \frac{2}{s^2+4} \\
\Rightarrow\ & \left(s^2+2\right) Y = 2 \\
\Rightarrow\ & Y = \frac{2}{(s^2+2)} \\
\Rightarrow\ & y = \mathcal{L}^{-1}\left[Y\right] = \sqrt{2}\sin\left(\sqrt{2}t\right).
\end{aligned}
$$

Exercise 2.3

Using Laplace transform and convolution to solve

$$y'' + 3y' + 2y = r(t), \quad y(0) = y'(0) = 0,$$

$$\text{where } r(t) = \begin{cases} 1 & \text{if } 0 < t < 1, \\ 0 & \text{if } t > 1. \end{cases}$$

Solution.
We first write $r(t)$ as $r(t) = u(t) - u(t-1)$, and to simplify the calculation, we will keep the Laplace transform of $r(t)$ as $\mathcal{L}\left[r\right]$.
The Laplace transform for the ODE becomes

$$
\begin{aligned}
& s^2 Y - sy(0) - y'(0) + 3sY - 3y(0) + 2Y = \mathcal{L}\left[r\right] \\
\Rightarrow\ & \left(s^2 + 3s + 2\right) Y = \mathcal{L}\left[r\right] \\
\Rightarrow\ & Y = \frac{\mathcal{L}\left[r\right]}{(s+1)(s+2)} = \frac{\mathcal{L}\left[r\right]}{(s+1)} - \frac{\mathcal{L}\left[r\right]}{(s+2)} \\
\Rightarrow\ & y = \mathcal{L}^{-1}\left[\mathcal{L}\left[r\right]\mathcal{L}\left[e^{-t}\right]\right] - \mathcal{L}^{-1}\left[\mathcal{L}\left[r\right]\mathcal{L}\left[e^{-2t}\right]\right] \\
\Rightarrow\ & y = r(t) * e^{-t} - r(t) * e^{-2t}.
\end{aligned}
$$

Solution.
Since

$$r(t) * e^{-at} = \int_0^t e^{-a(t-\tau)} \left[u(\tau) - u(\tau - 1)\right] d\tau$$

$$\Rightarrow r(t) * e^{-at} = \begin{cases} \int_0^t e^{-a(t-\tau)} d\tau & \text{if } t < 1 \\ \int_0^1 e^{-a(t-\tau)} d\tau & \text{if } t > 1 \end{cases}$$

$$\Rightarrow r(t) * e^{-at} = \begin{cases} \frac{e^{-at}}{a}\left(e^{at} - 1\right) & \text{if } t < 1, \\ \frac{e^{-at}}{a}\left(e^{a} - 1\right) & \text{if } t > 1. \end{cases}$$

Solution.
Therefore

$$
\begin{aligned}
y &= r(t) * e^{-t} - r(t) * e^{-2t} \\
\Rightarrow y &= \begin{cases} e^{-t}\left(e^{t}-1\right) - \frac{e^{-2t}}{2}\left(e^{2t}-1\right) & \text{if } t<1 \\ e^{-t}\left(e-1\right) - \frac{e^{-2t}}{2}\left(e^{2}-1\right) & \text{if } t>1 \end{cases} \\
\Rightarrow y &= \begin{cases} \frac{1}{2} - e^{-t} + \frac{e^{-2t}}{2} & \text{if } t<1, \\ e^{-(t-1)} - e^{-t} - \frac{e^{-2(t-1)}}{2} + \frac{e^{-2t}}{2} & \text{if } t>1, \end{cases} \\
\text{or,}\quad y &= \left(\frac{1}{2} - e^{-t} + \frac{e^{-2t}}{2}\right) - \left(\frac{1}{2} + \frac{1}{2}e^{-2(t-1)} - e^{-(t-1)}\right) u(t-1),
\end{aligned}
$$

which is exactly same as exercise 1.1.

Exercise 2.4

Solve the following initial value problem.

$$y'' + y = r(t), \quad y(0) = 0, \ y'(0) = 0,$$
$$r(t) = \begin{cases} \cos t, & \text{if } 0 \le t \le \pi, \\ 0, & \text{otherwise.} \end{cases}$$

Solution

The function $r(t)$ can be expressed as follows

$$r(t) = \cos t\,[1 - u(t-\pi)]\,,$$

but to simplify our calculation, we will keep the Laplace transform of $r(t)$ as $\mathcal{L}\left[r\right]$. Therefore, the Laplace transform of the ODE is thus

$$(s^2+1)Y = \mathcal{L}\left[r\right] \Rightarrow Y = \frac{\mathcal{L}\left[r\right]}{(s^2+1)}.$$

Solution

Since $\mathcal{L}^{-1}\left[\frac{1}{s^2+1}\right] = \sin t$,

$$y(t) = \mathcal{L}^{-1}\left[\mathcal{L}\left[r\right] \cdot \frac{1}{s^2+1}\right]$$

$$\Rightarrow y(t) = r(t) * \sin t = \int_0^t \sin\left(t-\tau\right) r(\tau) d\tau.$$

Solution

For $t < \pi$,

$$\int_0^t \sin(t-\tau) r(\tau) d\tau = \int_0^t \sin(t-\tau)\cos\tau \, d\tau$$

$$= \sin t \int_0^t \cos\tau \cos\tau \, d\tau - \cos t \int_0^t \sin\tau \cos\tau \, d\tau$$

$$= \sin t \int_0^t \left(\frac{1+\cos 2\tau}{2}\right) d\tau - \cos t \int_0^t \sin\tau \, d\sin\tau$$

$$= \sin t \left(\frac{t}{2} + \frac{\sin 2t}{4}\right) - \cos t \cdot \frac{\sin^2 t}{2}$$

$$= \frac{t\sin t}{2} + \frac{\sin^2 t \cos t}{2} - \frac{\sin^2 t \cos t}{2} = \frac{t \sin t}{2}.$$

Solution

Similarly, for $t > \pi$,

$$\begin{aligned}\int_0^\pi \sin(t-\tau) r(\tau) d\tau &= \int_0^\pi \sin(t-\tau)\cos\tau d\tau \\ &= \sin t \int_0^\pi \cos\tau \cos\tau d\tau - \cos t \int_0^\pi \sin\tau\cos\tau d\tau \\ &= \sin t \int_0^\pi \left(\frac{1+\cos 2\tau}{2}\right) d\tau - \cos t \int_0^\pi \sin\tau d\sin\tau \\ &= \sin t\left(\frac{\pi}{2}\right) = \frac{\pi}{2}\sin t.\end{aligned}$$

Therefore, we obtain

$$y(t) = \begin{cases} \frac{t}{2}\sin t & \text{if } t < \pi, \\ \frac{\pi}{2}\sin t & \text{if } t > \pi. \end{cases}$$

Exercise 3.1

Find $\mathcal{L}\left[f\right]$ for the following function.

$$f(t) = te^{-kt}\sin t.$$

Solution

We know the Laplace transform for $e^{-kt}\sin t$ is a s-shifting of Laplace transform of $\sin t$

$$\mathcal{L}\left[e^{-kt}\sin t\right] = \frac{1}{(s+k)^2+1}.$$

By definition, $\mathcal{L}\left[f(t)\right]$ is

$$\mathcal{L}\left[f(t)\right] = -\left\{\mathcal{L}\left[e^{-kt}\sin t\right]\right\}' = \frac{2(s+k)}{\left[(s+k)^2+1\right]^2}.$$

Exercise 3.2

Find $f(t)$ for the following $\mathcal{L}[f]$.

$$\mathcal{L}[f] = \frac{2s+6}{(s^2+6s+10)^2}.$$

Solution

Let $F(s) = \mathcal{L}\left[f\right]$. One can find that

$$\begin{aligned}
\int_s^{\infty} F(\tilde{s})d\tilde{s} &= \int_s^{\infty} \frac{2(\tilde{s}+3)}{\left[(\tilde{s}+3)^2+1\right]^2} d\tilde{s} \\
&= \int_s^{\infty} \frac{1}{\left[(\tilde{s}+3)^2+1\right]^2} d\left[(\tilde{s}+3)^2\right] \\
&= \left. -\frac{1}{(\tilde{s}+3)^2+1} \right|_s^{\infty} = \frac{1}{(s+3)^2+1}.
\end{aligned}$$

Since $\mathcal{L}^{-1}\left[\frac{1}{(s+3)^2+1}\right] = e^{-3t}\sin t$, and with the definition $\int_s^{\infty} F(\tilde{s})d\tilde{s} = \mathcal{L}\left[\frac{f}{t}\right]$,

$$\mathcal{L}^{-1}\left[F(s)\right] = te^{-3t}\sin t.$$