4.4 D'Alembert's solution of the wave equation. Characteristics

In the last section, we have solved the wave equation by the method of separating variables and obtained

$$
\begin{aligned}
u(x,t) &= \sum_{n=1}^{\infty} B_n \left[\sin \frac{n\pi x}{L} \cos \frac{cn\pi t}{L}\right] \\
&= \frac{1}{2}\left[\sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{L}(x+ct) + \sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{L}(x-ct)\right] \\
&= \frac{1}{2}\left[f^*(x+ct) + f^*(x-ct)\right],
\end{aligned}
$$

when the initial velocity is zero i.e. $u_t(x,0) = g(x) = 0$.

In this section we will show that this solution can be immediately obtained by transforming the wave equation in a suitable way.

We introduce two new independent variables

$$v = x + ct, \quad w = x - ct.$$

Then u becomes a function of v and w: $u = u(v, w)$ and

$$\frac{\partial v}{\partial x} = 1,\ \frac{\partial v}{\partial t} = c,\ \frac{\partial w}{\partial x} = 1,\ \frac{\partial w}{\partial t} = -c.$$

Therefore

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial w}\frac{\partial w}{\partial x}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial v} + \frac{\partial u}{\partial w}\right) = \frac{\partial}{\partial x}(u_v + u_w) \\
&= \frac{\partial(u_v + u_w)}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial(u_v + u_w)}{\partial w}\frac{\partial w}{\partial x} \\
&= \frac{\partial(u_v + u_w)}{\partial v} + \frac{\partial(u_v + u_w)}{\partial w} \\
&= u_{vv} + u_{wv} + u_{vw} + u_{ww} = u_{vv} + 2u_{vw} + u_{ww}.
\end{aligned}$$

Similarly,

$$
\begin{aligned}
\frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right) = \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial v}\frac{\partial v}{\partial t} + \frac{\partial u}{\partial w}\frac{\partial w}{\partial t}\right) \\
&= \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial v}c + \frac{\partial u}{\partial w}(-c)\right) = \frac{\partial}{\partial t}(cu_v - cu_w) \\
&= \frac{\partial(cu_v - cu_w)}{\partial v}\frac{\partial v}{\partial t} + \frac{\partial(cu_v - cu_w)}{\partial w}\frac{\partial w}{\partial t} \\
&= \frac{\partial(cu_v - cu_w)}{\partial v}c + \frac{\partial(cu_v - cu_w)}{\partial w}(-c) \\
&= c^2 u_{vv} - c^2 u_{wv} - c^2 u_{vw} + c^2 u_{ww} \\
&= c^2 u_{vv} - 2c^2 u_{vw} + c^2 u_{ww}.
\end{aligned}
$$

Hence

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

becomes

$$c^2 u_{vv} - 2c^2 u_{vw} + c^2 u_{ww} = c^2 (u_{vv} + 2u_{vw} + u_{ww})$$

Thus we obtain

$$u_{vw} = 0$$

i.e.

$$\frac{\partial^2 u}{\partial w \partial v} = 0.$$

By two successive integrations, first with respect to w and then with respect to v, we have

$$\frac{\partial u}{\partial v} = h(v) \text{ and } u(v, w) = \int h(v)dv + \psi(w) = \phi(v) + \psi(w),$$

where $h(v)$ and $\psi(w)$ are arbitrary functions and $\phi(v) = \int h(v)dv$. By $v = x + ct$, $w = x - ct$, we thus have

$$u(x, t) = \phi(v) + \psi(w) = \phi(x + ct) + \psi(x - ct).$$

This is known as **d'Alembert's solution** of the wave equation.

D'Alembert's Solutions satisfying the initial conditions

We assume the initial conditions of the wave equation is

$$u(x,0) = f(x), \quad u_t(x,0) = g(x).$$

By differentiating $u(x,t) = \phi(x+ct) + \psi(x-ct)$ with respect to t we have

$$u_t(x,t) = c\phi'(x+ct) - c\psi'(x-ct)$$

where primes denote derivatives with respect to the entire arguments $x+ct$ and $x-ct$, respectively. Therefore we have

$$u(x,0) = \phi(x) + \psi(x) = f(x),$$

$$u_t(x,0) = c\phi'(x) - c\psi'(x) = g(x). \tag{4.11}$$

Dividing (4.11) by c and integrating with respect to x from x_0 to x, we obtain

$$\phi'(x) - \psi'(x) = \frac{1}{c} g(x)$$

$$\phi(x) - \psi(x) = \phi(x_0) - \psi(x_0) + \frac{1}{c} \int_{x_0}^{x} g(s) ds \quad (4.12)$$

Add this to

$$\phi(x) + \psi(x) = f(x)$$

we have

$$2\phi(x) = f(x) + \phi(x_0) - \psi(x_0) + \frac{1}{c} \int_{x_0}^{x} g(s) ds$$

$$\phi(x) = \frac{1}{2} \left[f(x) + \phi(x_0) - \psi(x_0) + \frac{1}{c} \int_{x_0}^{x} g(s) ds \right]$$

$$\phi(x) = \frac{1}{2} f(x) + \frac{1}{2} [\phi(x_0) - \psi(x_0)] + \frac{1}{2c} \int_{x_0}^{x} g(s) ds$$

Similarly, subtraction of (4.12) from

$$\phi(x) + \psi(x) = f(x)$$

we obtain

$$2\psi(x) = f(x) - \phi(x_0) + \psi(x_0) - \frac{1}{c}\int_{x_0}^{x} g(s)ds$$

$$\psi(x) = \frac{1}{2}\left[f(x) - \phi(x_0) + \psi(x_0) - \frac{1}{c}\int_{x_0}^{x} g(s)ds\right]$$

$$\psi(x) = \frac{1}{2}f(x) - \frac{1}{2}[\phi(x_0) - \psi(x_0)] - \frac{1}{2c}\int_{x_0}^{x} g(s)ds$$

Now we have obtained

$$\phi(x) = \frac{1}{2}f(x) + \frac{1}{2}[\phi(x_0) - \psi(x_0)] + \frac{1}{2c}\int_{x_0}^{x} g(s)ds$$
$$\psi(x) = \frac{1}{2}f(x) - \frac{1}{2}[\phi(x_0) - \psi(x_0)] - \frac{1}{2c}\int_{x_0}^{x} g(s)ds$$

Now replace x by $x + ct$ for $\phi(x)$ and x by $x - ct$ for $\psi(x)$, then

$$\phi(x+ct) = \frac{1}{2}f(x+ct) + \frac{1}{2}[\phi(x_0) - \psi(x_0)] + \frac{1}{2c}\int_{x_0}^{x+ct} g(s)ds$$
$$\psi(x-ct) = \frac{1}{2}f(x-ct) - \frac{1}{2}[\phi(x_0) - \psi(x_0)] - \frac{1}{2c}\int_{x_0}^{x-ct} g(s)ds$$

Therefore the solution $u(x,t)$ is

$$u(x,t) = \phi(x+ct)+\psi(x-ct) = \frac{1}{2}[f(x+ct)+f(x-ct)]+\frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds.$$

If the initial velocity is zero, that is, $u_t(x, 0) = g(x) = 0$, then

$$u(x, t) = \frac{1}{2}[f(x + ct) + f(x - ct)],$$

which agrees with that obtained in previous section.

Characteristics: types and normal forms of PDEs

The idea of d'Alembert's solution is just a special instance of the method of characteristics. This concerns PDEs of the form

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} = F(x, y, u, u_x, u_y).$$

There are three types of PDEs, depending on the discriminant $AC - B^2$, as follows.

Type	Defining condition	Example
Hyperbolic	$AC - B^2 < 0$	Wave equation
Parabolic	$AC - B^2 = 0$	Heat equation
Elliptic	$AC - B^2 > 0$	Laplace equation