MTH101: Tutorial 6

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November 3, 2017

Example 1.1

Determine the location and order of the zeros.

- $(z+8i)^4$
- $(z^4-81)^3$
- $\sin^4 \frac{1}{2} z$

Solution:
(1) $z+8i$ has simple zero at $-8i$, hence $(z+8i)$ has zero of order 4 at $-8i$, one can check this by

$$f(-8i)=(-8i+8i)^4=0;\ f'(-81i)=4(-8i+8i)^3=0;$$
$$f''(-8i)=12(-8i+8i)^2=0;\ f'''(-8i)=24(-8i+8i)=0;$$
$$\text{But } f^{(4)}(-8i)=24\neq 0.$$

(2) z^4-81 has simple zeros at ± 3 and $\pm 3i$. Hence the given function has third-order zeros (zeros of order 3) at these points.
Remark: Note that we demonstrated that if g has a zero of first order (simple zero) at z_0, then g^n (n a positive integer) has a zero of order n at z_0.

(3) $\sin \frac{1}{2}z$ has simple zeros at $\pm 2n\pi$, $(n = 0, 1, 2, ...)$, hence the given function has zeros of order 4 at these points.

Example 1.2

Determine the location of the isolated singularities and also state the order for poles.

- $\dfrac{1}{(z+2i)^2} - \dfrac{z}{z-i} + \dfrac{z+1}{(z-i)^2}$
- $\tan \pi z$
- $\dfrac{\sin z}{z^4}$

Solution:
(1) The denominator of the first term has a zero of order 2 at $-2i$, and the numerator is analytic and nonzero, thus it has a pole of order 2 at $-2i$; the second term and the third one both have singularity at i, of order 1 and 2, respectively. Thus the function has pole of order 2 at i.
(2) The function can be written as

$$\frac{\sin \pi z}{\cos \pi z};$$

thus it has simple poles at $z = (2n+1)/2$, $n = 0, \pm 1, \pm 2, \ldots$ because $\cos \pi z$ has simple zeros at these z.

(3) The function

$$f(z) = \frac{\sin z}{z^4}$$

has a singularity at $z = 0$. However, since both $\sin z$ and z^4 are 0 for $z_0 = 0$, we cannot use the regular method to determine the order of that pole.

We know that

$$\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1},$$

we then multiply it by z^{-4} and get

$$\frac{\sin z}{z^4} = z^{-4} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n-3}$$

Let $2n-3<0$ we have that $n<\frac{3}{2}$, thus $n=0,1$ and the first a few terms in the Laurent series centered at 0 is

$$\frac{1}{z^3}-\frac{1}{3!z}+\frac{z}{5!}-\frac{z^3}{7!}\cdots$$

we notice that all $b_i=0$, $i>3$ and $b_3\neq 0$, therefore 0 is a pole of oder 3.

Example 2.1

Compute the integral

$$\int_\gamma f(z)\, dz$$

where γ *is the counterclockwise circle with center* 0 *and radius* 2 *and*

$$f(z) = ze^{1/z} + \frac{z}{z+1}$$

Solution:
The function $f(z)$ has isolated singularities at $z_0 = 0$ and $z_1 = -1$. Both are in side γ, then we can use the **Residue Theorem:**

$$\oint_{\gamma} f(z)\ dz = 2\pi i[\operatorname*{Res}_{z=0} f(z) + \operatorname*{Res}_{z=-1} f(z)].$$

We must compute the residues.

First Step: Residue at $z_0 = 0$.
In this case we use the definition:
$\operatorname*{Res}_{z=0} f(z)$ is the coefficient of the term of order -1 of the Laurent Series of the function $f(z)$ in the Annulus $0 < |z| < R$.
That coefficient is denoted by b_1 while R is chosen such that $f(z)$ is Analytic in $0 < |z| < R$.
The function $ze^{1/z}$ can be represented by a Laurent series

$$ze^{1/z} = z\sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{1}{z}\right)^n = \sum_{n=0}^{\infty} \frac{1}{n!} z^{-n+1}, \qquad 0 < |z| < |\infty|$$

Then the coefficient of the term of order -1, that is of the term z^{-1}, is $\frac{1}{2}$, then

$$\operatorname*{Res}_{z=0} f(z) = \frac{1}{2}$$

Second Step: Residue at $z_1 = -1$.
$z_1 = -1$ is a simple zero of the denominator of $\frac{z}{z+1}$, the numerator is analytic and nonzero at -1, thus -1 is a simple pole of the function $\frac{z}{z+1}$. Therefore,

$$\operatorname*{Res}_{z=-1} f(z) = \lim_{z\to-1}\left[(z+1)\frac{z}{z+1}\right] = -1$$

Finally,

$$\oint_\gamma f(z)\, dz = 2\pi i[\operatorname*{Res}_{z=0} f(z) + \operatorname*{Res}_{z=-1} f(z)] = 2\pi i(\frac{1}{2} - 1) = -\pi i.$$

Example 2.2

Compute the real integral

$$\int_{-\infty}^{\infty} \frac{1}{(x-1)(x^2+1)}\, dx$$

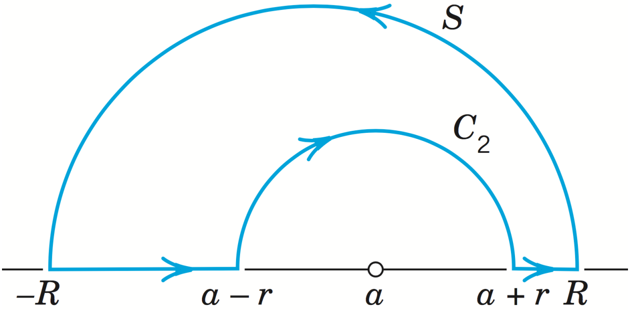
Solution:
This is an improper integral because the infinite bounds and the "bad" points at $x = 1$ on the real axis.
We first redefine the integral, denote the integrand by $f(x)$, and

$$\int_{-\infty}^{\infty} f(x)\ dx = \lim_{R\to\infty}[\lim_{r\to 0}\int_{-R}^{1-r} f(x)\ dx + \lim_{r\to 0}\int_{1+r}^{R} f(x)\ dx]$$

Consider the complex version of the integral

$$\int_{[-R,1-r]\cup[1+r,R]} \frac{1}{(z-1)(z^2+1)}\ dz$$

It would be convenient if we design a closed path.


S
C_2
$-R$
$a-r$
a
$a+r$
R

Step 1:
In the whole region bounded by $S \cup [-R, 1-r] \cup [1+r, R] \cup C_2$, there is one singular point $z = i$, which is a simple pole of the function, thus

$$\begin{aligned}
\oint_{S\cup[-R,1-r]\cup[1+r,R]\cup C_2} f(z)\ dz &= 2\pi i \operatorname*{Res}_{z=i} f(z) \\
&= 2\pi i \frac{1}{[(z-1)(z+i)]_{z=i}} \\
&= \frac{\pi}{i-1}
\end{aligned}$$

Step 2:
Use ML-inequality and triangle inequality we obtain that

$$\lim_{R\to\infty}\int_S f(z)\ dz = 0.$$

In details, we have that on the circle with radius R,

$$\left|\frac{1}{(z-1)(z^2+1)}\right| < \frac{1}{|z|^3-|z|^2-|z|-1} = \frac{1}{R^3-R^2-R-1} = M$$

and

$$L = \frac{1}{2}\cdot 2\pi R = \pi R,$$

thus,

$$0 < \lim_{R\to\infty}\left|\int_S \frac{1}{(z-1)(z^2+1)}\ dz\right| < \lim_{R\to\infty}\frac{\pi R}{R^3-R^2-R-1} = 0.$$

Step 3:
Use Theorem 1 in section 16.4 on the text, since $z = 1$ is a simple pole on the real axis

$$\begin{aligned}\lim_{r\to 0}\int_{-C_2} f(z)\ dz &= \pi i \operatorname*{Res}_{z=1} f(z)\\ &= \pi i \left[\frac{1}{z^2+1}\right]_{z=1} = \frac{\pi i}{2}.\end{aligned}$$

Finally,

$$\begin{aligned}\int_{-\infty}^{\infty} f(x)\ dx &:= \lim_{R\to\infty}\lim_{r\to 0}\int_{[-R,1-r]\cup[1+r,R]} f(z)\ dz\\ &= \oint_{S\cup[-R,1-r]\cup[1+r,R]\cup C_2} f(z)dz - \int_S f(z)dz - \int_{C_2} f(z)dz\\ &= \frac{\pi}{i-1} + \frac{\pi i}{2}\end{aligned}$$

Remark:
From this exercise and the class example we did on week 5, we can conclude that:
Assumed that the function $f(x)$ is a real rational function whose denominator is of order at least 2 degree higher than the order of the numerator, then the improper real integral can be calculated by the following formula:

$$\int_{-\infty}^{\infty} f(x)\,dx = \underbrace{2\pi i \sum \mathrm{Res}\ f(z)}_{\text{Residues in the upper half plane}} + \underbrace{\pi i \sum \mathrm{Res}\ f(z)}_{\text{Residues of simple pole on the real ax}}$$