Chapter 3: Fourier analysis

Recommended text: Chapter 11

Fourier analysis is a way of analysing functions into a sum or integral of simpler periodic functions. There are two types of Fourier analysis:

1. Fourier series-these expand periodic functions into a sum of sine and cosine terms with the same period.
2. Fourier integrals-this is an extension of Fourier series to non-periodic functions.

Here are a few examples with possible applications of Fourier analysis:

1. Analysis of analogue electronic circuits.
2. Telephone dialling. When you press a key on a touch-tone phone the tone produced is a simple combination of two frequencies. At the telephone exchange each tone is analysed by Fourier analysis to find the two frequencies involved. This reveals the number that was pressed.
3. X-ray crystallography to reconstruct a protein's structure from the special frequencies in its diffraction pattern.
4. Predicting tides which are influenced by the rotation of the earth (1 day), the time that the moon takes to rotate about the earth (about 28 days) and the time that the earth takes to rotate about the sun (about 365 days).
5. Digitally filtering the high frequency noise out of a noisy signal.
6. Solving partial differential equations.

Fourier invented this type of analysis in 1807 when he was investigating the way that heat flows in 2D and 3D objects. He was able to solve many problems with this approach but some of his ideas took many years to be fully proven. In 1965, Turkey and Cooley invented an extremely fast numerical method of doing the calculations involved in Fourier transforms C this is called the Fast Fourier Transform (FFT) and it is available in Matlab.

3.1: Fourier series

A function $f(x)$ is said to be periodic if

$$f(x) = f(x + p)$$

for any x, and fixed period p. In this section we shall consider how such functions can be approximated using trigonometric functions. In order to simplify the analysis we shall, for the moment, restrict our attention to functions $f(\cdot)$ which are 2π periodic, such that $f(x) = f(x + 2\pi)$. With this in mind we shall attempt to approximate $f(x)$ by the 2π periodic trigonometric sum

$$f(x) \approx f_N(x) = a_0 + \sum_{n=1}^{N} a_n \cos(nx) + b_n \sin(nx). \tag{3.1}$$

Before proceeding to calculate the coefficients of this polynomial it is helpful to evaluate the following integrals:

$$I_{mn} = \int_{-\pi}^{\pi} \cos(mx)\cos(nx)dx, \quad J_{mn} = \int_{-\pi}^{\pi} \sin(mx)\sin(nx)dx$$
$$K_{mn} = \int_{-\pi}^{\pi} \sin(mx)\cos(nx)dx,$$

where m and n are positive integers. Using standard trigonometric identities we can write

$$I_{mn} = \int_{-\pi}^{\pi} \frac{1}{2}[\cos((m-n)x) + \cos((m+n)x)]dx,$$

$$J_{mn} = \int_{-\pi}^{\pi} \frac{1}{2}[\cos((m-n)x) - \cos((m+n)x)]dx,$$

$$K_{mn} = \int_{-\pi}^{\pi} \frac{1}{2}[\sin((m-n)x) + \sin((m+n)x)]dx,$$

It follows (for $m \geq 1$ and $n \geq 0$) that

$$I_{mn} = 0 \text{ for } m \neq n, \; I_{mm} = \int_{-\pi}^{\pi} \frac{1}{2}dx = \pi,$$
$$J_{mn} = 0 \text{ for } m \neq n, \; J_{mm} = \int_{-\pi}^{\pi} \frac{1}{2}dx = \pi, \quad K_{mn} = 0.$$

To summarise

$$\begin{aligned}
\int_{-\pi}^{\pi} \cos(mx)\cos(nx)dx &= \pi\delta_{mn}, \\
\int_{-\pi}^{\pi} \sin(mx)\sin(nx)dx &= \pi\delta_{mn}, \\
\int_{-\pi}^{\pi} \sin(mx)\cos(nx)dx &= 0, \qquad (3.2) \\
\int_{-\pi}^{\pi} 1\cos(mx)dx &= 2\pi\delta_{0m}, \\
\int_{-\pi}^{\pi} 1\sin(mx)dx &= 0,
\end{aligned}$$

where the Kronecker delta function δ_{mn} is defined by

$$\delta_{mn} = \begin{cases} 1, m = n, \\ 0, m \neq n. \end{cases}$$

3.2: Fourier Series for 2π-periodic functions

By taking $N \to \infty$ in (3.1), we hope to obtain an exact series representation of $f(x)$. In other words we write any 2π periodic piecewise continuous function f as an infinite series of the following form:

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx). \qquad (3.3)$$

If this is the case, coefficients a_n and b_n can be found as follows:

Multiplying the above by $\cos(mx)$, integrating between $-\pi$ and π gives

$$\begin{aligned}\int_{-\pi}^{\pi} f(x)\cos(mx)dx &= a_0 \int_{-\pi}^{\pi} \cos(mx)dx \\ &+ \sum_{n=1}^{\infty}\left[a_n\left(\int_{-\pi}^{\pi}\cos(mx)\cos(nx)dx\right)\right. \\ &+ \left. b_n\left(\int_{-\pi}^{\pi}\sin(mx)\sin(nx)dx\right)\right] \\ &= a_0 2\pi\delta_{0m} + \sum_{n=1}^{\infty} a_n \pi \delta_{mn}\end{aligned}$$

Therefore, as $m = 0$,

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx.$$

As $m \neq 0$,

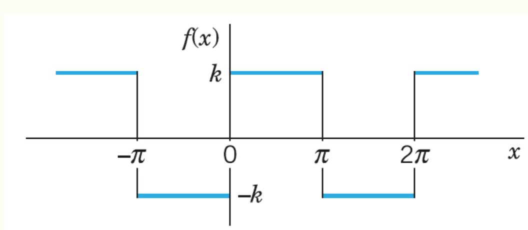
$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(mx)dx, \text{ for all } m.$$

A similar result can be derived for the coefficients b_m by multiplying (3.3) by $\sin(mx)$ and integrating between $-\pi$ and π. This gives

$$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(mx)dx, \text{ for all } m.$$

Example 1: Find the Fourier series of the period function $f(x)$ in the figure below, where $f(x)$ is

$$f(x) = \begin{cases} -k & \text{if } -\pi < x < 0 \\ k & \text{if } 0 < x < \pi \end{cases}, \quad f(x+2\pi) = f(x).$$



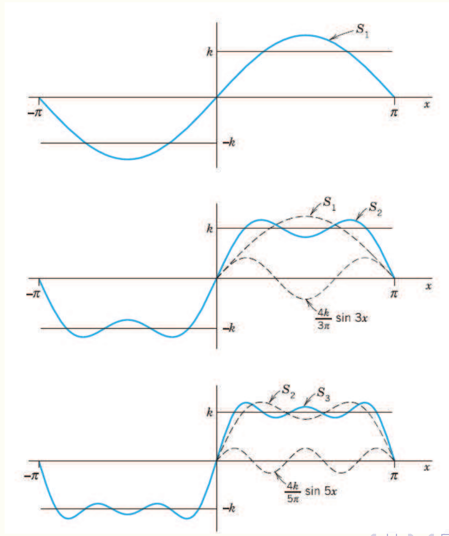
$$
\begin{aligned}
a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx \\
&= \frac{1}{2\pi}\left[{}^{0}_{-\pi} - kdx + \int_{0}^{\pi} kdx\right] \\
&= \frac{1}{2\pi}\left([-kx]_{-\pi}^{0} + [kx]_{0}^{\pi}\right) \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx \\
&= \frac{1}{\pi}\left\{\int_{-\pi}^{0} -k\cos(nx)dx + \int_{0}^{\pi} k\cos(nx)dx\right\} \\
&= \frac{1}{\pi}\left\{\left[\frac{-k\sin nx}{n}\right]_{-\pi}^{0} + \left[\frac{k\sin nx}{n}\right]_{0}^{\pi}\right\} \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx \\
&= \frac{1}{\pi}\left\{\int_{-\pi}^{0} -k\sin(nx)dx + \int_{0}^{\pi} k\sin(nx)dx\right\} \\
&= \frac{1}{\pi}\left\{\left[\frac{-k(-\cos nx)}{n}\right]_{-\pi}^{0} + \left[\frac{k(-\cos nx)}{n}\right]_{0}^{\pi}\right\} \\
&= \frac{k}{n\pi}\left\{[\cos nx]_{-\pi}^{0} + [-\cos nx]_{0}^{\pi}\right\} \\
&= \frac{k}{n\pi}(2 - 2\cos n\pi) \\
&= \frac{2k}{n\pi}(1 - \cos n\pi) \\
&= \begin{cases} 0; & \text{when } n \text{ is even} \\ \frac{4k}{n\pi}; & \text{when } n \text{ is odd} \end{cases}
\end{aligned}
$$

Hence the Fourier series of $f(x)$ is

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) = \sum_{n=1}^{\infty}\left(\frac{2k}{n\pi}(1 - \cos n\pi)\sin nx\right).$$



Remark

Note that $f(x)$ is an odd function and furthermore that its Fourier series contains no terms involving $\cos nx$. It is in general true that the Fourier series for an odd function $f(x)$ takes the form

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx), \tag{3.4}$$

while for an even function $g(x)$ takes the form

$$g(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) \tag{3.5}$$

(convince yourself that these statements are true). Remembering these facts can save time when evaluating the coefficients in a Fourier series.

3.3: Fourier Series for $2L$-periodic functions

Consider now a function $f(x)$ which has period $2L$. It is straightforward to transform the problem of finding a Fourier series for $f(x)$ on the interval $-L < x < L$ into one on the interval $-\pi < x < \pi$.

It follows that its Fourier series is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right) \right], \tag{3.6}$$

with coefficients a_0, a_n, b_n

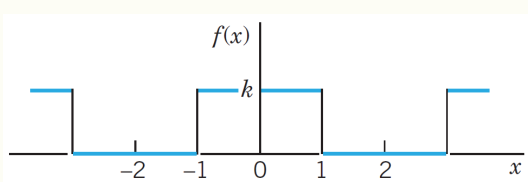
$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi}{L}x\right)dx, \quad n = 1, 2, \cdots$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi}{L}x\right)dx, \quad n = 1, 2, \cdots$$

Example 2: Find the Fourier series of the periodic function $f(x)$

$$f(x) = \begin{cases} 0 & \text{if} \quad -2 < x < -1, \\ k & \text{if} \quad -1 < x < 1, \qquad p = 2L = 4, L = 2 \\ 0 & \text{if} \quad 1 < x < 2, \end{cases}$$



Solution:

$$\begin{aligned}
a_0 &= \frac{1}{2L}\int_{-L}^{L} f(x)dx \\
&= \frac{1}{4}\int_{-2}^{2} f(x)dx \\
&= \frac{1}{4}\left[\int_{-2}^{-1} 0dx + \int_{-1}^{1} kdx + \int_{1}^{2} 0dx\right] \\
&= \frac{1}{4}\left[kx\right]_{-1}^{1} \\
&= \frac{1}{4}2k \\
&= \frac{k}{2}
\end{aligned}$$

$$\begin{aligned}
a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{1}{2}\int_{-2}^{2} f(x)\cos\left(\frac{n\pi x}{2}\right)dx \\
&= \frac{1}{2}\left[\int_{-2}^{-1} 0dx + \int_{-1}^{1} k\cos\left(\frac{n\pi x}{2}\right)dx + \int_{1}^{2} 0dx\right] \\
&= \frac{1}{2}\int_{-1}^{1} k\cos\left(\frac{n\pi}{2}x\right)dx = \frac{1}{2}\left[k\frac{\sin\left(\frac{n\pi x}{x}\right)}{\frac{n\pi}{2}}\right]_{-1}^{1} \\
&= \frac{k}{n\pi}\left[\sin\left(\frac{n\pi}{2}\right) - \sin\left(\frac{-n\pi}{2}\right)\right] \\
&= \frac{k}{n\pi}\left[\sin\left(\frac{n\pi}{2}\right) + \sin\left(\frac{n\pi}{2}\right)\right] \\
&= \frac{2k}{n\pi}\sin\left(\frac{n\pi}{2}\right)
\end{aligned}$$

$$
\begin{aligned}
b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{1}{2}\int_{-2}^{2} f(x)\sin\left(\frac{n\pi x}{2}\right)dx \\
&= \frac{1}{2}\left[\int_{-2}^{-1} 0dx + \int_{-1}^{1} k\sin\left(\frac{n\pi x}{2}\right)dx + \int_{1}^{2} 0dx\right] \\
&= \frac{1}{2}\int_{-1}^{1} k\sin\left(\frac{n\pi}{2}x\right)dx = \frac{1}{2}\left[k\frac{-\cos\left(\frac{n\pi x}{2}\right)}{\frac{n\pi}{2}}\right]_{-1}^{1} \\
&= \frac{k}{n\pi}\left[-\cos\left(\frac{n\pi}{2}\right) + \cos\left(\frac{-n\pi}{2}\right)\right] \\
&= \frac{k}{n\pi}\left[-\cos\left(\frac{n\pi}{2}\right) + \cos\left(\frac{n\pi}{2}\right)\right] \\
&= 0
\end{aligned}
$$

Hence the Fourier series of $f(x)$ is

$$
\begin{aligned}
f(x) &= a_0 + \sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi}{2}x\right) + b_n\sin\left(\frac{n\pi}{2}x\right)\right] \\
&= \frac{k}{2} + \sum_{n=1}^{\infty}\left\{\left[\frac{2k}{n\pi}\sin\left(\frac{n\pi}{2}\right)\right]\cos\left(\frac{n\pi}{2}x\right)\right\}
\end{aligned}
$$

Even and odd $2L$-periodic functions

As in (3.4) and (3.5), the Fourier series for an **odd** $2L$-periodic function $f(x)$ can be simplified as

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right), \tag{3.7}$$

where

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{2}x\right)dx.$$

The Fourier series for an **even** $2L$-periodic function $f(x)$ can be simplified as

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right), \tag{3.8}$$

where

$$a_0 = \frac{1}{L}\int_0^L f(x)dx, \quad a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi}{L}x\right)dx.$$