EEE319 Optimisation
Lecture 4 Linear Programming (2)

Prof. Xinheng Wang

xinheng.wang@xjtlu.edu.cn

Office: EE512

Outline

- Last week
  - Solved linear maximization problems
  - Slack variables
  - Transformation from minimization to maximization

- This week
  - Tableau

Simplex Method

- Last week, linear optimisation problem was solved by Simplex Method, by using a step by step inserting variable, and exchanging variable approach.
- Tableau is another approach that we are going to look at today.
- Before that, let us have a look how Simplex Method deals with $\geq$ inequality constraints and equality (=) constraints.

Simplex Method

- Minimization from an example

Min $c = 2x + 3y + 4z$

s.t. $4x + 2y + z \geq 10$

$$x + y - z \geq 5$$

$$x, y, z \geq 0$$

For $\geq$ constraints, a non-negative surplus variable is going to be introduced. In addition, instead of "+", the surplus variable is subtracted with "-".

Simplex Method

- The constraints can be written as following, with the conversion of inequality into equality

$$4x + 2y + z - s_1 = 10$$

$$x + y - z - s_2 = 5$$

- The reason to call this variable *surplus* is because it is the amount by which left-hand side exceeds right hand-side.

Simplex Method

- If we set the value of variables *x*, *y*, and *z* to zero to have a basic feasible solution

$$c = 2x + 3y + 4z = 0$$

- The value of surplus variables has to be -10 and -5, respectively, which conflicts the definition of surplus variable, which is positive.

$$4x + 2y + z - s_1 = 10$$

$$x + y - z - s_2 = 5$$

- How to solve this problem?

Simplex Method

- An artificial variable is introduced

$$c = 2x + 3y + 4z = 0$$

- This artificial variable doesn't have any physical meaning. The only purpose is to obtain a basic feasible solution. Therefore, two constraints will become:

$$4x + 2y + z - s_1 + a_1 = 10$$

$$x + y - z - s_2 + a_2 = 5$$

$$x, y, z, s_1, s_2, a_1, a_2 \geq 0$$

Simplex Method

- This artificial variable is needed to be reflected in the objective function as well, by using a very large positive coefficient number *M*. For a minimization problem, a term Ma_i is added for each artificial variable a_i; for a maximization problem, a term $-Ma_i$ is added for each artificial variable a_i. Then the objective function will become:

$$c = 2x + 3y + 4z + Ma_1 + Ma_2$$

- Here are the equations with surplus variables and artificial variables.

$$c = 2x + 3y + 4z + Ma_1 + Ma_2$$
$$4x + 2y + z - s_1 + a_1 = 10$$
$$x + y - z - s_2 + a_2 = 5$$
$$x, y, z, s_1, s_2, a_1, a_2 \geq 0$$

- Artificial variable is also used for "=" constraints.

Simplex Method

- An artificial variable is also introduced for equality constraint
- Example

  Max $z = 3x_1 + 5x_2$

  s.t. $x_1 + x_2 \geq 2$

  $x_2 \leq 6$

  $3x_1 + 2x_2 = 18$

  $x_1, x_2 \geq 0$
- This will become as follows, after introducing the variables

  Max $z = 3x_1 + 5x_2 - Ma_1 - Ma_2$

  s.t. $x_1 + x_2 - s_1 + a_1 = 2$

  $x_2 + s_2 = 6$

  $3x_1 + 2x_2 + a_2 = 18$

  $x_1, x_2 \geq 0$

  where s_1 is surplus variable, s_2 is slack variable, a_1 and a_2 are artificial variables.

Simplex Method - Tableau

- Tableau approach by an example

Max $P = 70x_1 + 50x_2$

s.t. $4x_1 + 3x_2 \leq 240$

$2x_1 + x_2 \leq 100$

- A few key steps

1) Inequality to equality with all variables;
2) Select the column with the “least negative” value;
3) Select the rows with the “least positive” value;
4) Push the intersected variable to basic by setting the coefficient to 1;
5) Push the coefficient of other variables to zero;
6) Repeat

Simplex Method - Tableau

- Tableau approach by an example

Max $P = 70x_1 + 50x_2$

s.t. $4x_1 + 3x_2 \leq 240$

$2x_1 + x_2 \leq 100$

- Step 1 inequality to equality with all variables, starting from constraints

$$
\begin{aligned}
4x_1 + 3x_2 + s_1 \qquad\qquad &= 240 \\
2x_1 + x_2 + \qquad +s_2 \quad &= 100 \\
-70x_1 - 50x_2 + \qquad +P \quad &= 0
\end{aligned}
$$

Simplex Method - Tableau

- Step 2 establish an initial tableau by making a matrix with all of the coefficients into the matrix.

$$4x_1 + 3x_2 + s_1 \qquad\qquad\qquad = 240$$
$$2x_1 + x_2 + \qquad\quad +s_2 \qquad = 100$$
$$-70x_1 - x_2 + \qquad\quad +P \qquad = 0$$

Basic variables	x_1	x_2	s_1	s_2	P	Right hand side
s_1	4	3	1	0	0	240
s_2	2	1	0	1	0	100
P	-70	-50	0	0	1	0

This tableau represents the initial solution;

$x_1 = 0; x_2 = 0; s_1 = 240; s_2 = 100; P = 0$

Basic variable: only one coefficient is 1 and all others are zero in one column.

Simplex Method - Tableau

Essential idea of Simplex Method: Replace the basic variable with non-basic variable.

Simplex Method - Tableau

- Step 3 Select the pivot column – by selecting the column with the least negative value in the objective function.

$$4x_1 + 3x_2 + s_1 \qquad = 240$$
$$2x_1 + x_2 + \qquad +s_2 \qquad = 100$$
$$-70x_1 - x_2 + \qquad +P \qquad = 0$$

Basic variables	x_1	x_2	s_1	s_2	P	Right hand side
s_1	4	3	1	0	0	240
s_2	2	1	0	1	0	100
P	-70	-50	0	0	1	0

- Pivot: a fixed point supporting something that turns or balances; The central or most important person or thing in a situation

Simplex Method - Tableau

- Step 4 Select the pivot row – by selecting the row with the least non-negative of the ratio, which is value of the right-hand side over value in pivot column

$$4x_1 + 3x_2 + s_1 \qquad\qquad = 240$$
$$2x_1 + x_2 + \qquad +s_2 \qquad = 100$$
$$-70x_1 - x_2 + \qquad +P \qquad = 0$$

Basic variables	x_1	x_2	s_1	s_2	P	Right hand side
s_1	4	3	1	0	0	240
s_2	2	1	0	1	0	100
P	-70	-50	0	0	1	0

240/4=60

100/2=50

Simplex Method - Tableau

- Step 5 calculate the new values by dividing every number in the pivot row by the pivot number (to make the coefficient to 1)

Basic variables	x_1	x_2	s_1	s_2	P	Right hand side
s_1	4	3	1	0	0	240
s_2	1	1/2	0	1/2	0	50
P	-70	-1	0	0	1	0

Simplex Method - Tableau

- Step 6 make the other values in pivot column zero, by row operations.

Basic variables	x_1	x_2	s_1	s_2	P	Right hand side
s_1	0	1	1	-2	0	40
s_2	1	1/2	0	1/2	0	50
P	0	-15	0	35	1	3500

- $\boldsymbol{R}_1$-4$\boldsymbol{R}_2$
- $\boldsymbol{R}_3$ +70$\boldsymbol{R}_2$

Simplex Method - Tableau

- Step 6 x_1 becomes the basic variable

Basic variables	x_1	x_2	s_1	s_2	P	Right hand side
s_1	0	1	1	-2	0	40
x_1	1	1/2	0	1/2	0	50
P	0	-15	0	35	1	3500

- In this case, x_1=50,$x_2 = 0, s_1 = 40, s_2 = 0, P = 3500$
- This is the end of first process. Next is to repeat these steps

Simplex Method - Tableau

- Repeat previous steps

Basic variables	x_1	x_2	s_1	s_2	P	Right hand side
s_1	0	1	1	-2	0	40
x_1	1	1/2	0	1/2	0	50
P	0	-15	0	35	1	3500

Simplex Method - Tableau

- Repeat previous steps

Basic variables	x_1	x_2	s_1	s_2	P	Right hand side
s_1	0	1	1	-2	0	40
x_1	1	1/2	0	1/2	0	50
P	0	-15	0	35	1	3500

Simplex Method - Tableau

- Repeat previous steps

Basic variables	x_1	x_2	s_1	s_2	P	Right hand side
s_1	0	1	1	-2	0	40
x_1	1	0	-1/2	3/2	0	30
P	0	0	15	5	1	4100

- -1/2 R_1+ $\boldsymbol{R_2}$
- $15R_1$+ $\boldsymbol{R_3}$

Simplex Method - Tableau

- Repeat previous steps

Basic variables	x_1	x_2	s_1	s_2	P	Right hand side
x_2	0	1	1	-2	0	40
x_1	1	0	-1/2	3/2	0	30
P	0	0	15	5	1	4100

- $x_1 = \mathbf{30}, x_2 = \mathbf{40}, s_1 = s_2 = \mathbf{0}$, P=4100
- On objective row, no other values are negative. The iteration is terminated.

Summary

- Tableau to solve linear programming

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