EEE336 Signal Processing and Digital Filtering

Lecture 7 Discrete-Time Signals in Frequency Domain

7_1 The Importance of FD Analyses

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Why introduce the FD analyses?

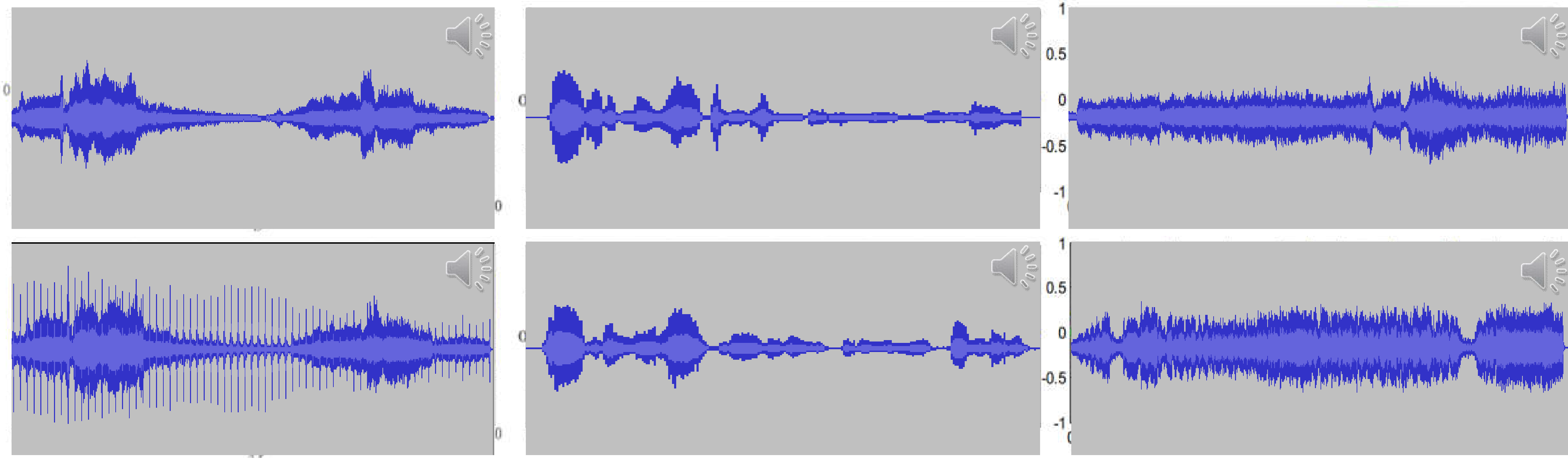
- Time domain operation are often not very informative and/or efficient in signal processing.
  - An alternative representation and characterization of signals and systems can be made in transform domain
  - Much more can be said, much more information can be extracted from a signal in the transform / frequency domain.
  - Many operations that are complicated in time domain become rather simple algebraic expressions in transform domain
  - Most signal processing algorithms and operations become more intuitive in frequency domain, once the basic concepts of the frequency domain are understood.

Signals in Time Domain

Filter out noises in the signal

Analyze female and male voices

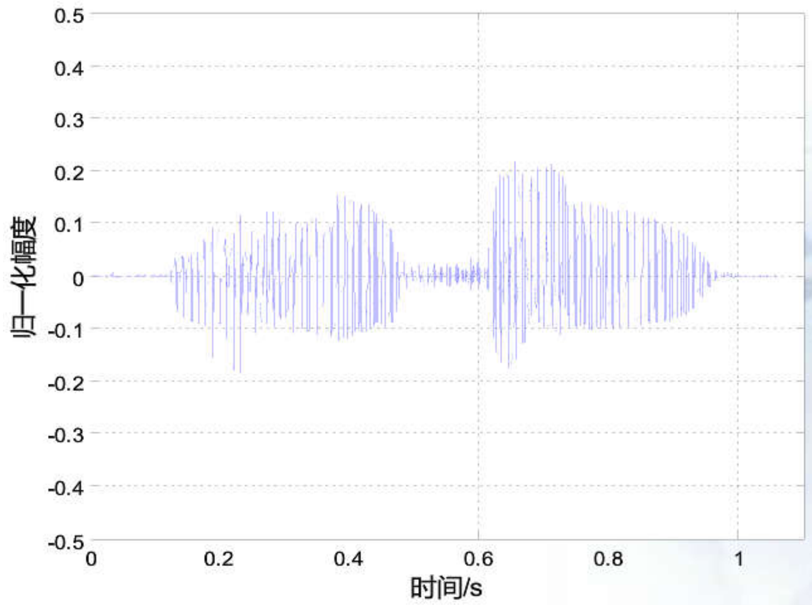
Analyze the vocal from different singer



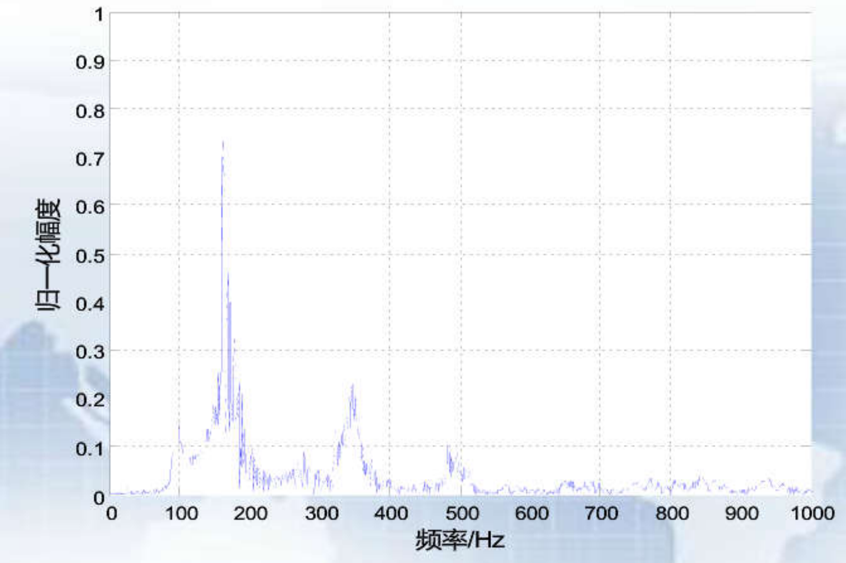
- Time domain analyses are not informative enough in these examples.

Importance in Physics

- In physics, a signal in transform domain will exhibit more properties, which facilitates us to implement signal processing
  - Eg: speech signals, image signals, etc.



Speech signal in time domain



Speech signal in frequency domain

Importance in Mathematics

- In mathematics, signal processing can be simplified in transform domain
  - Eg: s-transform, z-transform, Fourier transform
  - S-transform can make differential equation (of continuous signals) become to algebraic equation.

$$\because \frac{d^k}{dt^k}x(t) \Leftrightarrow s^k X(s), \frac{d^k}{dt^k}y(t) \Leftrightarrow s^k Y(s)$$

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k} \Leftrightarrow \left(\sum_{k=0}^{N} a_k s^k\right) Y(s) = \left(\sum_{k=0}^{M} b_k s^k\right) X(z)$$

  - Z-transform can make difference equation (of discrete signals) become to algebraic equation.

$$\because x[n-k] \Leftrightarrow z^{-k} X[z], \;\; y[n-k] \Leftrightarrow z^{-k} Y[z]$$

$$\sum_{k=0}^{N} d_k y[n-k] = \sum_{k=0}^{M} p_k x[n-k] \Leftrightarrow \left(\sum_{k=0}^{N} d_k z^{-k}\right) Y(z) = \left(\sum_{k=0}^{M} p_k z^{-k}\right) X(z)$$

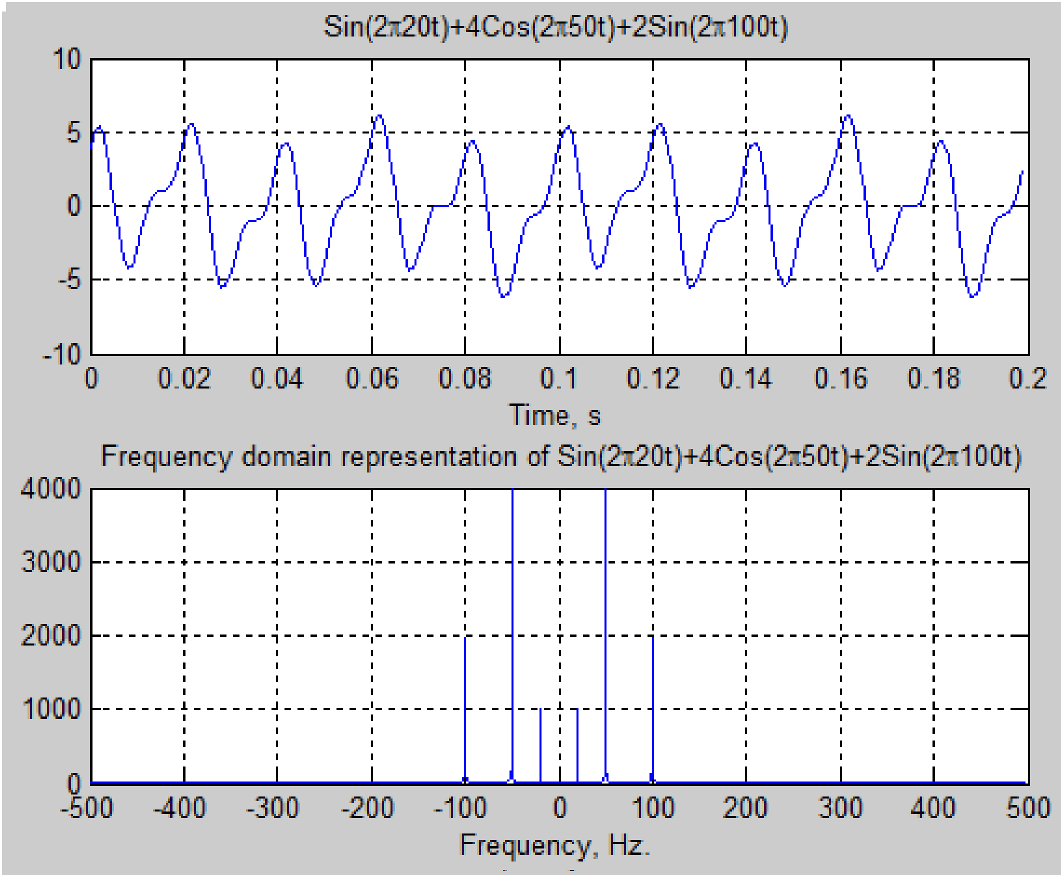
Frequency domain representation

- The frequency domain representation of a signal can be obtained through ***Fourier transforms***.

$$\sin(2\pi 50t)$$

$$\sin(2\pi 50t)+\sin(2\pi 75t)$$

$$\sin(2\pi 20t)+4\cos(2\pi 50t)+2\sin(2\pi 100t)$$



Spectrum: A compact representation of the frequency content of a signal that is composed of sinusoids

7_1 Wrap up

- Why should we introduce the frequency domain analyses?
  - In physics
  - In mathematics
  - Examples

- Next: Fourier transform families

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7_2 Fourier Analyses (transforms)

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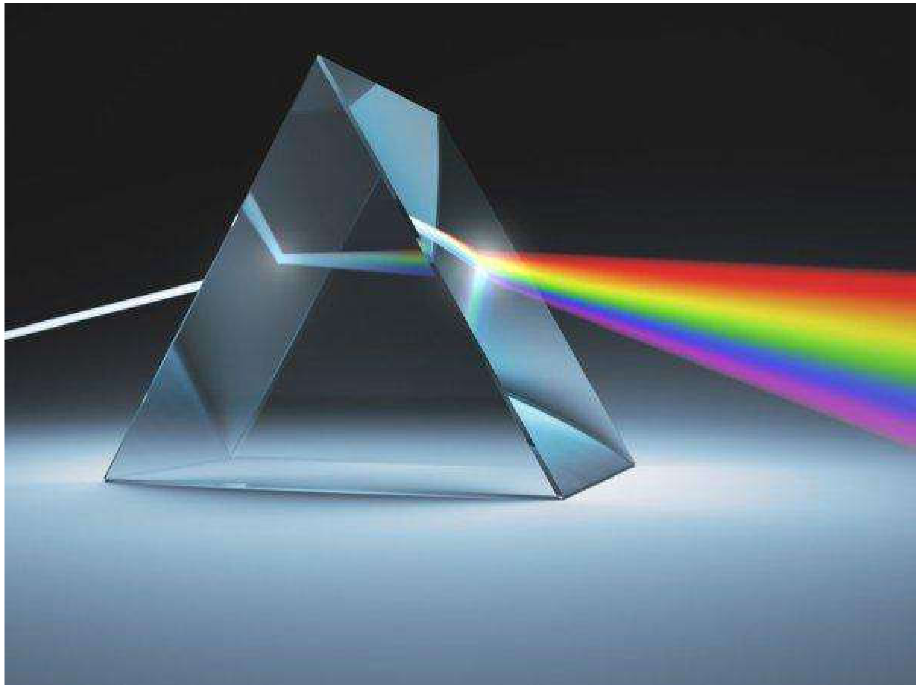
Fourier transforms



- A prism can be used to break up white light (sunlight) into the colors of the rainbow.

- Fourier transform is used to break up signals into the frequency components.
  - Continuous-Time VS Discrete-Time
  - Periodic VS Non-periodic

	CT Signals	DT Signals
Periodic	Fourier Series	Discrete FS
Non-periodic	CTFT	DTFT

Fourier WHO?

Jean B. Joseph Fourier

(1768-1830)

"An arbitrary function, continuous or with discontinuities, defined in a finite interval by an arbitrarily capricious graph can always be expressed as a sum of sinusoids"

J.B.J. Fourier

December 21, 1807

$$F[k] = \int f(t) e^{-j2\pi kt/N} dt \qquad f(t) = \frac{1}{2\pi} \sum_{i=0}^{N-1} F[k] e^{j2\pi kt/N}$$

- Fourier Series (FS)
  - Fourier's original work: A periodic function can be represented as a finite, weighted sum of sinusoids that are integer multiples of the fundamental frequency Ω_0 of the signal. These frequencies are said to be harmonically related, or simply harmonics.
- Continuous Time Fourier Transform (CTFT)
  - Extension of Fourier series to non-periodic functions: Any continuous aperiodic function can be represented as an infinite sum (integral) of sinusoids. The sinusoids are no longer integer multiples of a specific frequency.
- Discrete Time Fourier Transform (DTFT)
  - Extension of FT to discrete sequences. Any discrete function can also be represented as an infinite sum (integral) of sinusoids. While time domain is discretized, frequency domain is still continuous.
- Discrete Fourier Transform (DFT)
  - Because DTFT is defined as an infinite sum, the frequency representation is not discrete. An extension to DTFT is DFT, where the frequency variable is also discretized.
- Fast Fourier Transform (FFT)
  - Mathematically identical to DFT, however a significantly more efficient implementation. FFT is what signal processing made possible today!

Dirichlet conditions

- Dirichlet conditions: the sufficient conditions for the existence of Fourier representations of signals
  - The signal must have finite number of discontinuities
  - The signal must have finite number of extremum points within its period
  - The signal must be absolutely integrable within its period

$$\int_{t_0}^{t_0+T} |x(t)| dt < \infty$$

All periodic signals of practical interest satisfy these conditions

Fourier Series (FS)

Synthesis equation

- Any **periodic** signal x(t) whose fundamental period is T_0 (hence, fundamental frequency $f_0=1/T_0$, $\Omega_0=2\pi f_0$), can be represented as a sum of complex exponentials (sines and cosines)

k**th **harmonic

$$x(t) = C_0 + \sum_{k=1}^{\infty} C_k \cos\left(k\Omega_0 t - \theta_k\right)$$

$$C_0 = \frac{a_0}{2}, \quad C_k = \sqrt{a_k^2 + b_k^2}, \quad \theta_k = \tan^{-1}\frac{b_k}{a_k}$$

DC component

Harmonic amplitudes

Phase angles

Fourier Series

Analysis equation

- The coefficients c_k can be obtained by the ***analysis equation***

$$c_k = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} x(t)e^{-jk\Omega_0 t}\,dt$$

The limits of the integral can be chosen to cover any interval of T_0, for example, $[-T_0/2, T_0/2]$ or $[0, T_0]$.

- Represent the complex Fourier series in trigonometric forms:

$$x(t) = a_0 + \sum_{k=1}^{\infty}\left(a_k\cos(k\Omega_0 t) + b_k\sin(k\Omega_0 t)\right)$$

$$a_k = \frac{2}{T_0}\int_{T_0} x(t)\cos(k\Omega_0 t)\,dt$$

$$b_k = \frac{2}{T_0}\int_{T_0} x(t)\sin(k\Omega_0 t)\,dt$$

$$a_0 = 2c_0 \qquad a_k = c_k + c_{-k} \qquad b_k = j(c_k - c_{-k})$$

$$c_k = \frac{a_k - jb_k}{2}, \qquad c_{-k} = \frac{a_k + jb_k}{2}$$

If a signal is even, then all b_k=0;
If a signal is odd, then all a_k=0.

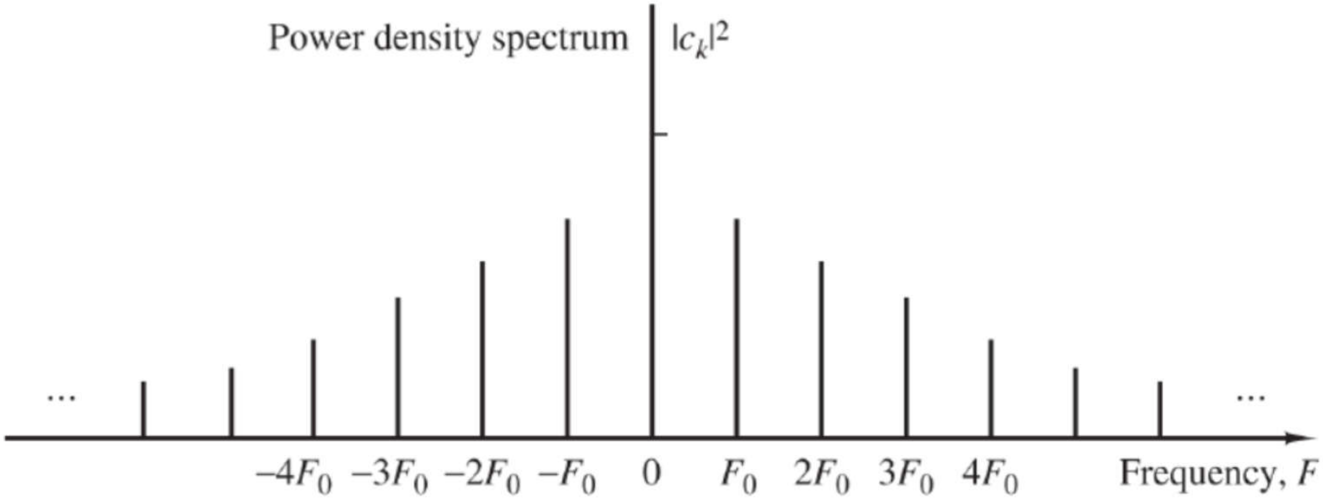
Power Density Spectrum

- A periodic signal has infinite energy and a finite average power, which is given as

$$P_x = \frac{1}{T_p}\int_{T_p} |x(t)|^2\, dt \Longrightarrow P_x = \frac{1}{T_p}\int_{T_p} |x(t)|^2\, dt = \sum_{k=-\infty}^{\infty} |c_k|^2$$

Parseval's relation for power signals

- ***Power density spectrum*** of a continuous-time periodic signal



Fourier Series

- Fourier series gives us the spectrum of the continuous time signals that are periodic with a fundamental frequency of Ω_0;
- The Fourier series of such a signal is a series of impulses at integer multiples of Ω_0;
  - These impulses in the frequency domain represent the harmonics of the signal;
  - Remember: the term $e^{j\Omega o}$ represents one spectral component at frequency Ω_0;
  - Cos(Ω_0t) has two such complex exponentials in it, at $\pm\Omega_0$. Therefore, each cosine at a particular frequency Ω_0 consists of two spectral components, one at each of $\pm\Omega_0$.
- **Fourier series are defined *only* on periodic signals.**

Continuous Time Fourier Transform (CTFT)

- Non-periodic continuous time signals can also be represented as a sum of weighted complex exponentials
  - Definition:

Analysis equation:

$$X_a(j\Omega) = \int_{-\infty}^{+\infty} x_a(t) e^{-j\Omega t} dt$$

Synthesis equation:

$$x_a(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X_a(j\Omega) e^{j\Omega t} d\Omega$$

$$x_a(t) \overset{\text{CTFT}}{\longleftrightarrow} X_a(j\Omega)$$

  - The complex exponentials are no longer discrete and integer multiples of a fundamental frequency;
  - Unlike the FS, where we represented the signal with a finite sum of harmonics, for non-periodic signals, we need a sum (integration) of continuum of frequencies.

Continuous Time Fourier Transform (CTFT)

- Variable Ω is real and denotes the continuous-time angular frequency in radians
- In general, the CTFT is a complex function of Ω in the range $-\infty < \Omega < +\infty$

$$X_a(j\Omega) = |X_a(j\Omega)| e^{j\theta_a(\Omega)}$$

Fourier spectrum *Magnitude spectrum* *Phase spectrum*

- If x(t) is real -> FT is conjugate symmetric

$$X(-\Omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} x(t) e^{j\Omega t} dt = X^*(\Omega)$$

$$|X(-\Omega)| = |X(\Omega)|, \quad \phi(-\Omega) = -\phi(\Omega)$$

- If x(t) is even (that is, symmetric) -> FT is real

Continuous Time Fourier Transform (CTFT)

Property	Signal	Fourier Transform
Linearity	$a_1x_1(t)+a_2x_2(t)$	$a_1X_1(\omega)+a_2X_2(\omega)$
Time shifting	$x(t-t_0)$	$e^{-j\omega t_0}X(\omega)$
Frequency shifting	$e^{j\omega_0 t}x(t)$	$X(\omega-\omega_0)$
Time scaling	$x(at)$	$\frac{1}{\|a\|}X\left(\frac{\omega}{a}\right)$
Time reversal	$x(-t)$	$X(-\omega)$
Duality	$X(t)$	$2\pi x(-\omega)$
Time differentiation	$\frac{dx(t)}{dt}$	$j\omega X(\omega)$
Frequency differentiation	$(-jt)x(t)$	$\frac{dX(\omega)}{d\omega}$
Integration	$\int_{-\infty}^{t}x(\tau)\,d\tau$	$\pi X(0)\delta(\omega)+\frac{1}{j\omega}X(\omega)$
Convolution	$x_1(t)*x_2(t)$	$X_1(\omega)X_2(\omega)$
Multiplication	$x_1(t)x_2(t)$	$\frac{1}{2\pi}X_1(\omega)*X_2(\omega)$
Real signal	$x(t)=x_e(t)+x_o(t)$	$X(\omega)=A(\omega)+jB(\omega)$ $X(-\omega)=X^*(\omega)$
Even component	$x_e(t)$	$\mathrm{Re}\{X(\omega)\}=A(\omega)$
Odd component	$x_o(t)$	$j\,\mathrm{Im}\{X(\omega)\}=jB(\omega)$

Examples

- 1. The Dirac delta (unit impulse) function

$$x(t)=\delta(t) \overset{\mathcal{F}}{\Leftrightarrow} X(j\Omega)=\int_{t=-\infty}^{\infty} \delta(t)e^{-j\Omega t}dt=e^{-j\Omega 0}=1$$

- 2. A shifted Dirac delta (unit impulse) function

$$x(t)=\delta(t-t_0) \overset{\mathcal{F}}{\Leftrightarrow} X(j\Omega)=\int_{t=-\infty}^{\infty} \delta(t-t_0)e^{-j\Omega t}dt=e^{-j\Omega t_0}$$

Is this expression a function of time or frequency?

- 3. A cosine function

$$x(t)=\cos(\Omega_0 t)=\frac{\left(e^{j\Omega_0 t}+e^{-j\Omega_0 t}\right)}{2} \overset{\mathcal{F}}{\Leftrightarrow} X(\Omega)=\int_{t=-\infty}^{\infty} \frac{\left(e^{j\Omega_0 t}+e^{-j\Omega_0 t}\right)}{2} e^{-j\Omega t}dt=\frac{1}{2}\left[\delta(\Omega-\Omega_0)+\delta(\Omega+\Omega_0)\right]$$

Energy density spectrum of aperiodic signals

- The total energy Ex of a finite energy CT signal $x_a(t)$ is:

$$\mathcal{E}_x = \int_{-\infty}^{\infty} |x_a(t)|^2 dt = \int_{-\infty}^{\infty} x_a(t) x_a^*(t) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X_a(j\Omega)|^2 d\Omega$$

Paseval's relation

  – The quantity $|X_a(j\Omega)|^2$ is called the energy density spectrum of $x_a(t)$ and denoted as $S_{xx}(\Omega)$
  – Energy over a specific range $[\Omega_a, \Omega_b]$ can be computed using:

$$\mathcal{E}_{x,r} = \frac{1}{2\pi} \int_{\Omega_a}^{\Omega_b} S_{xx}(\Omega) d\Omega$$

7_2 Wrap up

- Any ***periodic <u>continuous</u>*** signal x(t), with fundamental period of T_0, can be represented as a **finite and discrete** sum of complex exponentials that are integer multiples of the fundamental frequency Ω_0: FOURIER SERIES.
  - The FS is discrete in frequency domain – only a finite number of frequencies are required to construct a periodic signal.
- A ***non-periodic <u>continuous</u>*** time signal can also be represented as an **infinite and continuous** sum of complex exponentials: FOURIER TRANSFORM.
  - The CTFT is continuous in frequency domain – exponentials of a continuum of frequencies are required to reconstruct a non-periodic signal.
- Both transforms are non-periodic in frequency domain.

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7_3 Discrete-Time Fourier Transform

(Definition and Calculation)

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From CTFT to DTFT

- From CT Signals to DT Signals
  - a DT signal can be obtained from a CT signal through the process of sampling:

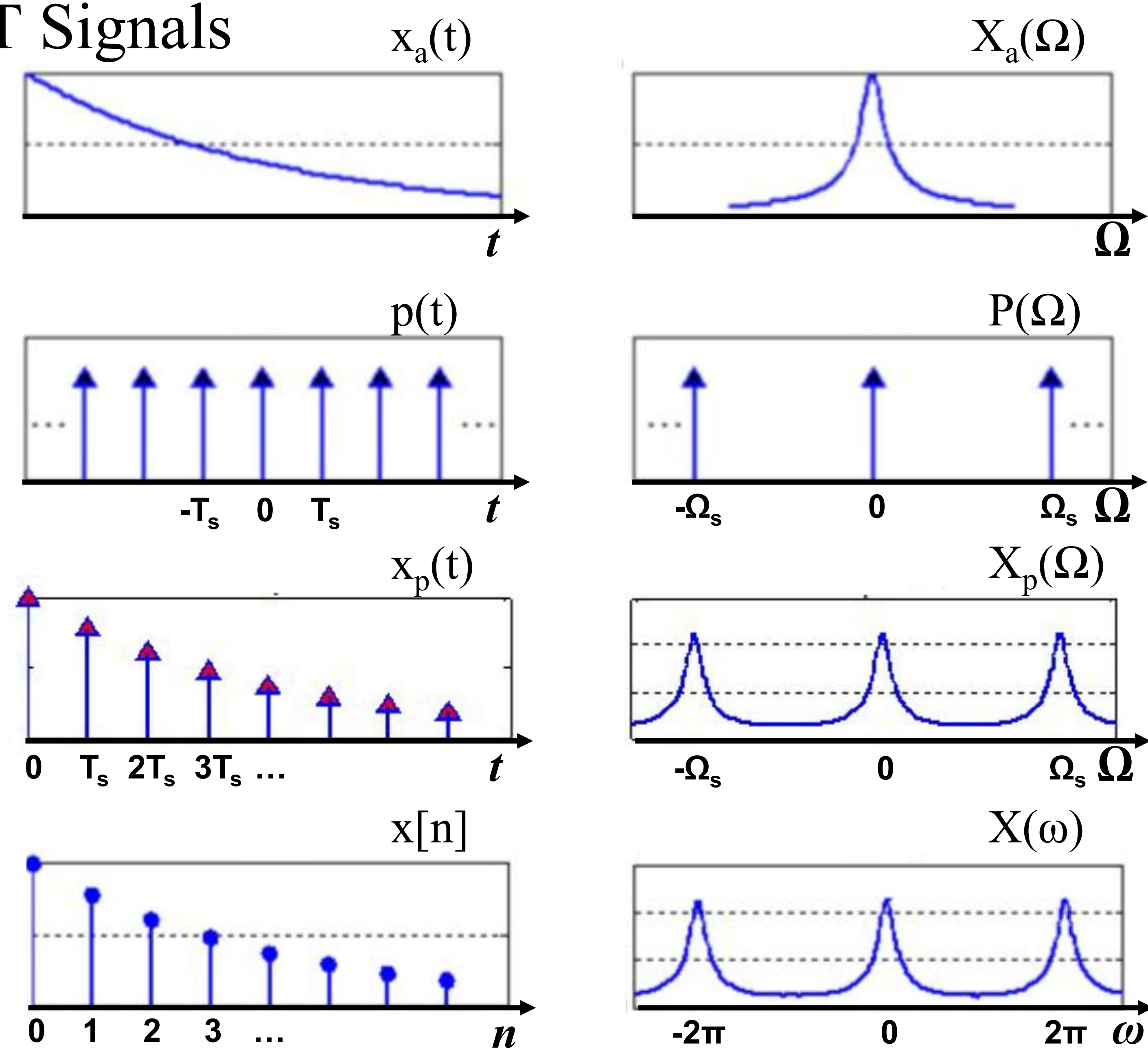
$$x_p(t) = \sum_{n=-\infty}^{\infty} x_a(nT_s)\delta(t - nT_s)$$

$$X_p(j\Omega) = \sum_{n=-\infty}^{\infty} x_a(nT_s)e^{-j\Omega T_s n}$$

$$x[n] = x_a(t)\Big|_{t=nT_s} = x_a(nT_s)$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

$$\boxed{\omega = \Omega T_s}$$



DTFT Definition

- The discrete-time Fourier transform (DTFT) $X(e^{j\omega})$ of a sequence x[n] is defined by:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

  - DTFT $X(e^{j\omega})$ of a sequence x[n] is a continuous function of ω

- Inverse Discrete-Time Fourier Transform - the Fourier coefficients {x[n]} can be computed from $X(e^{j\omega})$ using

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

DTFT Definition

- $X(e^{j\omega})$ is a complex function with the real variable ω

$$X(e^{j\omega}) = X_{re}(e^{j\omega}) + jX_{im}(e^{j\omega})$$

or

$$X(e^{j\omega}) = \left|X(e^{j\omega})\right| e^{j\theta(\omega)}, \text{ where } \quad \theta(\omega) = \arg\{X(e^{j\omega})\}$$

Fourier spectrum *Magnitude spectrum* *Phase spectrum*

- It is usually assumed that the phase function $\theta(\omega)$ is restricted to $(-\pi, \pi)$, but since it's periodic, it can be extended to $(-\infty, \infty)$

Important theorems of DTFT

- Theorem 1: DTFT is periodic with 2π.
- Theorem 2: The digital frequency 2π corresponds to the linear sampling frequency of the signal.
- Theorem 3: DTFT only exists for sequences that are absolutely summable.

Theorem 1 - Periodicity

- The DTFT of a discrete sequence is periodic with the period 2π, that is

$$\boxed{X(\omega) = X(\omega + 2\pi k) \quad \text{for any integer } k}$$

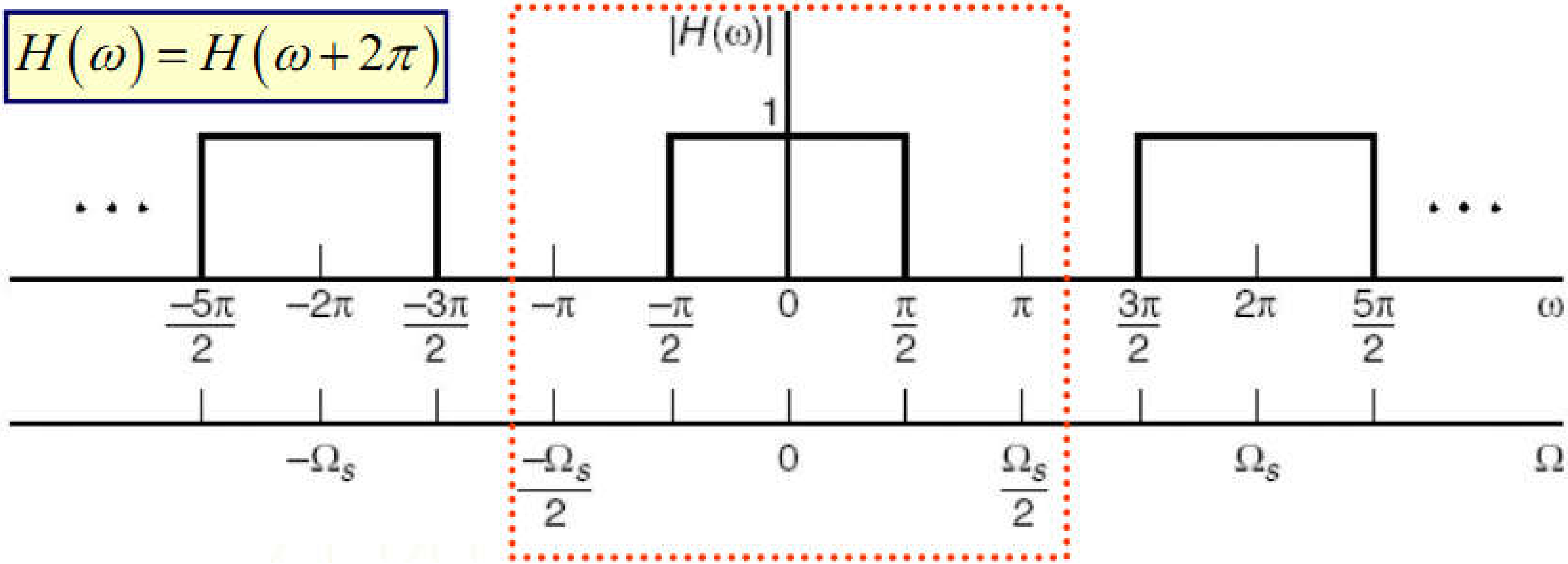
- The periodicity of DTFT can be easily verified from the definition:

$$X(\omega + 2\pi k) = \sum_{n=-\infty}^{\infty} x[n] e^{-j(\omega + 2\pi k)n}$$

$$= \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} e^{-j(2\pi k)n} = \sum_{n=-\infty}^{\infty} x[n] e^{-j(\omega)n} = X(\omega)$$

Theorem 2 - Implications of the periodicity

$$\boxed{H(\omega) = H(\omega + 2\pi)}$$

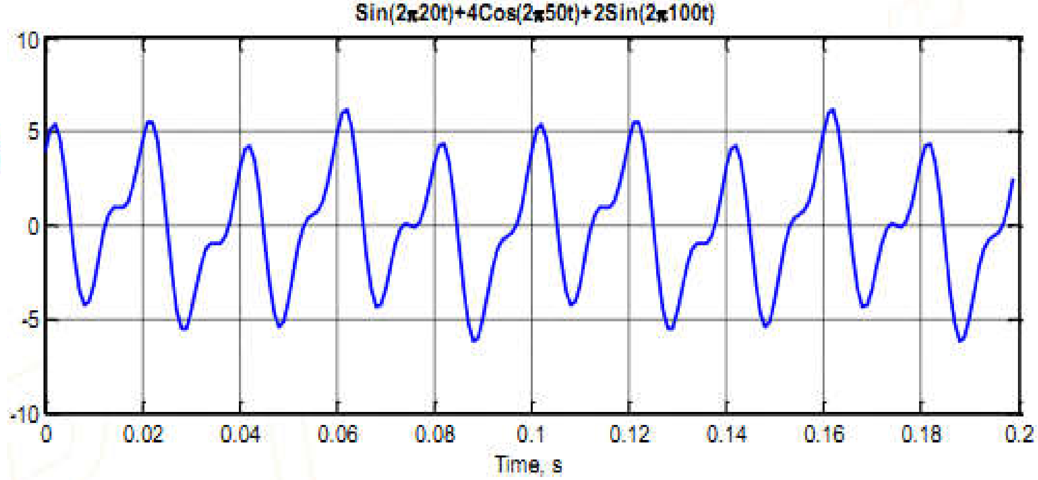


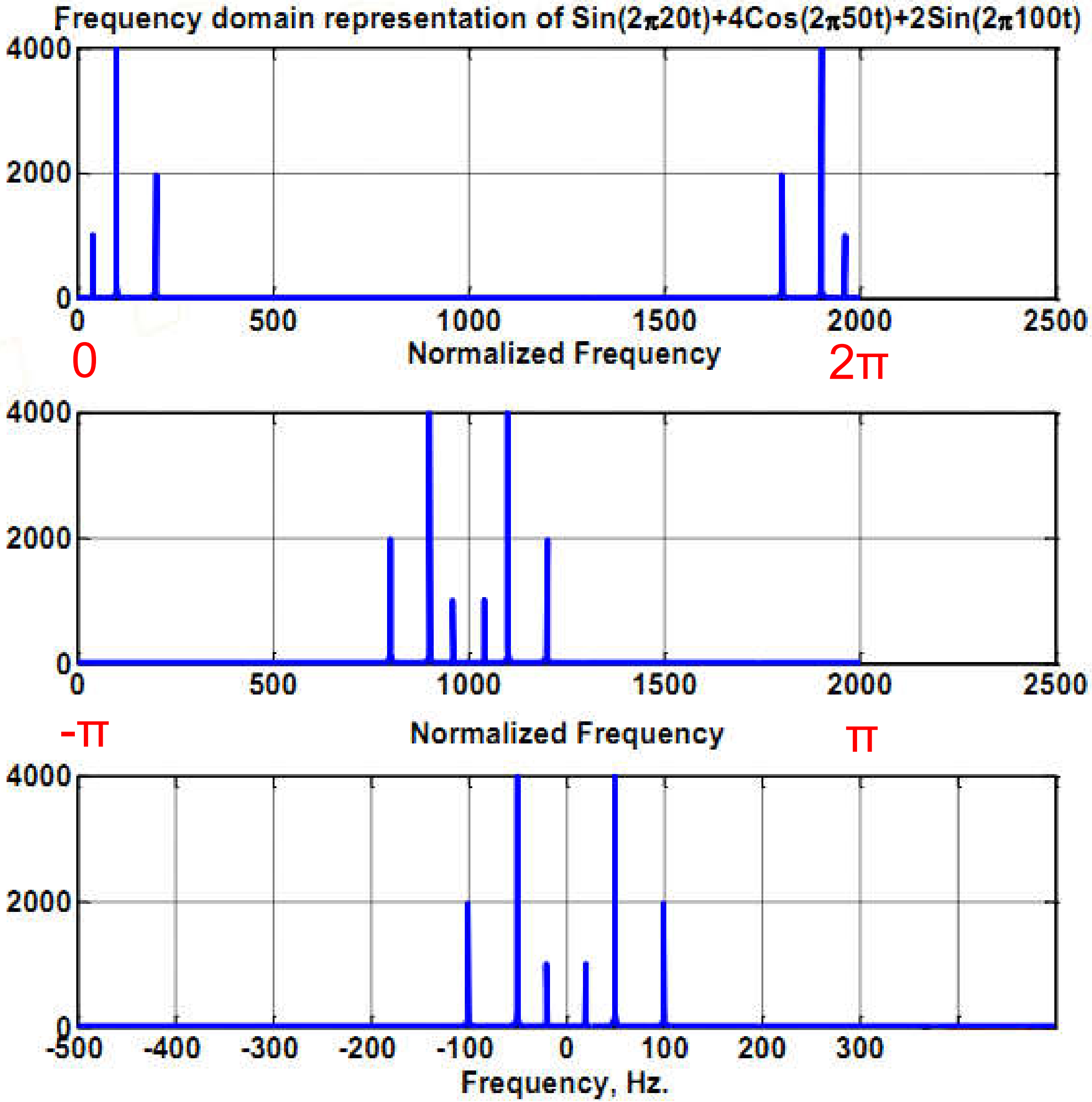
- The discrete frequency 2π corresponds to the sampling frequency Ω_s used to sample the original continuous signal x(t) to obtain x[n].

$$x(t) = A\sin(\Omega t - \theta) \quad \Rightarrow x(nT_s) = A\sin(\Omega T_s n - \theta)$$

↳ $\omega = \Omega T_s$ ➔ For $\Omega = \Omega_s$, we have $\omega = \Omega_s T_s = 2\pi f_s T_s = 2\pi$

```
t=0:0.001:2; % sampling frequency = 1000Hz
x=sin(2*pi*20*t)+4*cos(2*pi*50*t)+2*sin(2*pi*100*t);
```






```
X=abs(fft(x));
plot(X)
```

```
X2=fftshift(X);
plot(X)
```

```
f=-499.9:1000/2001:500;
plot(f,X2);
```

Theorem 3 - Existence of DTFT

- The DTFT of a sequence exists if and only if, the sequence x[n] is absolutely summable, that is

$$\sum_{n=-\infty}^{\infty} |x(n)| < \infty$$

because

$$|X(\omega)| = \left| \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \right| \le \sum_{n=-\infty}^{\infty} |x[n]| \cdot \left| e^{-j\omega n} \right| \le \sum_{n=-\infty}^{\infty} |x[n]| < \infty$$

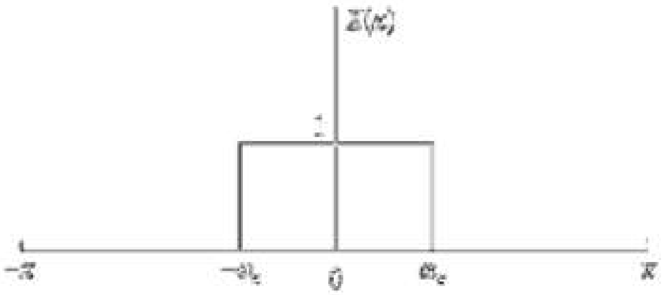
  - This is the sufficient condition for the existence of DTFT;
  - Certain sequences that do not satisfy this requirement also have DTFTs, if they satisfy "mean square convergence" :

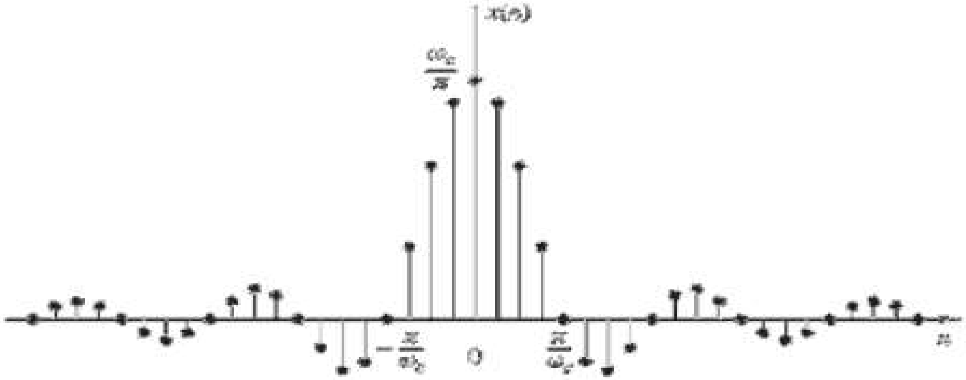
$$E_x = \sum_{n=-\infty}^{\infty} |x(n)|^2 < \infty$$

Example

$$X(\omega)=\begin{cases}1, & |\omega|\le\omega_c\\ 0, & \omega_c\le|\omega|\le\pi\end{cases} \quad \overset{\mathcal{F}^{-1}}{\Longleftrightarrow} \quad x(n)=\frac{1}{2\pi}\int_{-\pi}^{\pi}X(\omega)e^{j\omega n}d\omega=\frac{1}{2\pi}\int_{-\omega_c}^{\omega_c}e^{j\omega n}d\omega=\frac{\sin\omega_c n}{\pi n}$$

(with $n\neq 0$)





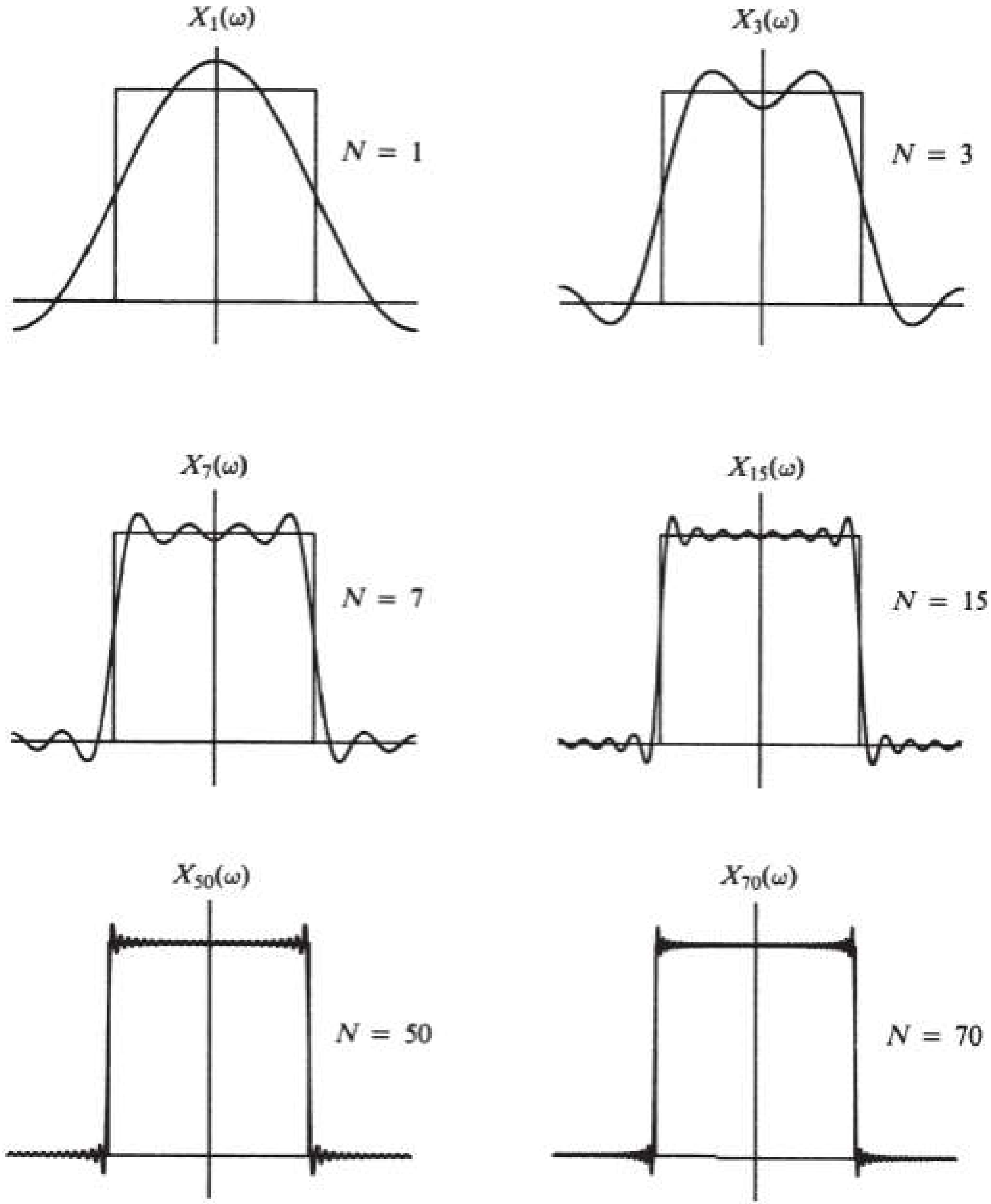
- The sequence {x(n)} is not absolutely summable $\sum_{n=-\infty}^{\infty}|x(n)| \not< \infty$
- But it is mean-square convergent $\sum_{n=-\infty}^{\infty}|x(n)|^2 < \infty$

- Let us consider the finite sum

$$X_N(\omega) = \sum_{n=-N}^{N} \frac{\sin \omega_c n}{\pi n} e^{-j\omega n}$$

  - Independent of the value of N, there are always ripples at the point of discontinuity;
  - The number of ripples increases as N increases;
  - The largest ripple remains the same for all values of N;

Gibbs Phenomenon !



7_3 Wrap up

- From CTFT to DTFT
  - Sampling in TD => Repeating in FD
  - $\omega = \Omega T_s$
- DTFT and IDTFT
  - Analyses and Syntheses equations
- Theorems
  - 1. Periodicity
  - 2. Digital frequency 2π <==> Sampling frequency Ω_s
  - 3. Existence of DTFT: Dirichlet condition

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7_4 Discrete-Time Fourier Transform

(DTFT Pairs and Properties)

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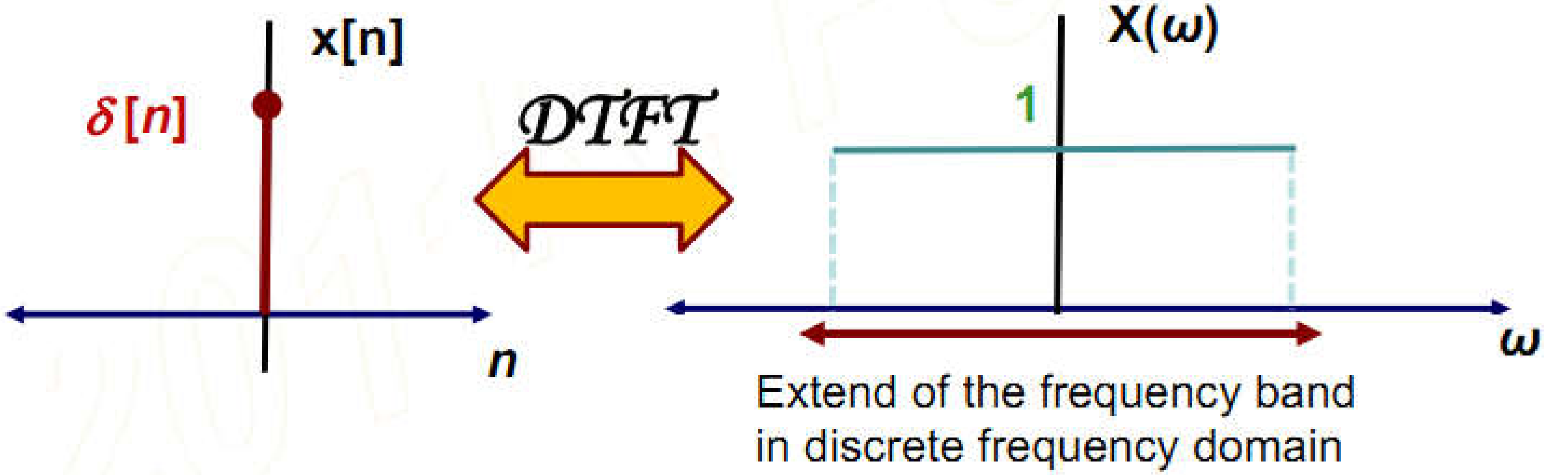
Important DTFT Pairs (1)

- 1. Impulse Function

$$\Delta(\omega) = \mathcal{F}\{\delta[n]\} = \sum_{n=-\infty}^{\infty} \delta[n] e^{-j\omega n} = 1 \cdot e^{-j\omega 0} = 1$$

- The DTFT of the impulse function is "1" over the entire frequency band.

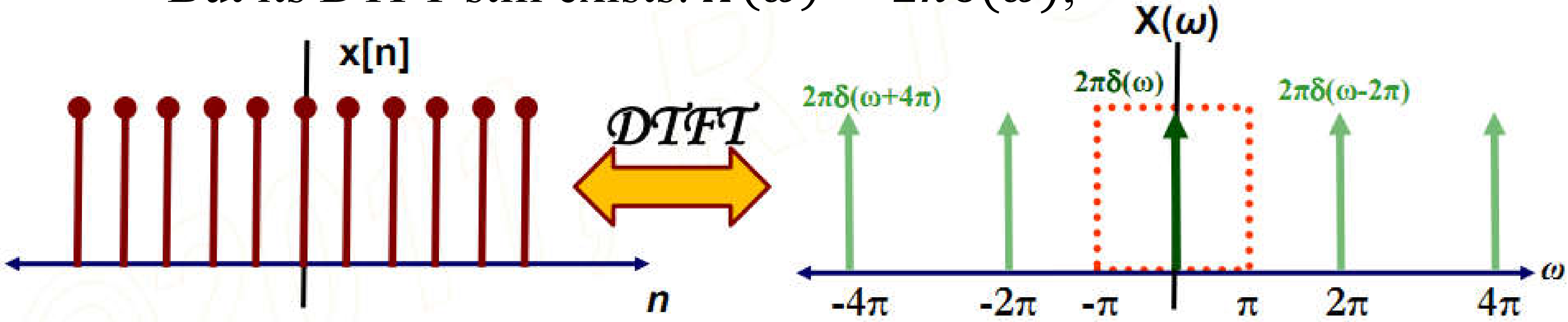


Important DTFT Pairs (2)

- 2. Constant Function

$$X(\omega) = \mathcal{F}\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

  - Note that x[n]=1 is not absolute summable;
  - But its DTFT still exists: $X(\omega) = 2\pi\delta(\omega)$;



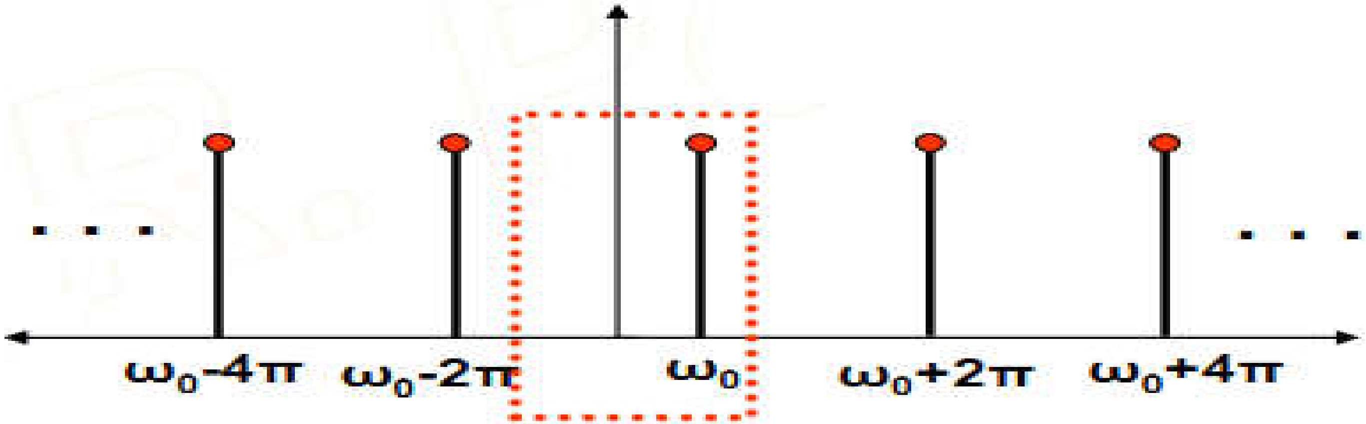
  - This transformation is correct, proved by the inverse DTFT:

$$\mathcal{F}^{-1}\left\{2\pi \sum_{m=-\infty}^{\infty} \delta(\omega - 2\pi m)\right\} = 2\pi \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[\sum_{m=-\infty}^{\infty} \delta(\omega - 2\pi m)\right] e^{j\omega n} d\omega = e^{j0n} = 1$$

Important DTFT Pairs (3)

- 3. The complex exponential

$$x[n] = e^{j\omega_0 n} \Leftrightarrow X(\omega) = 2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - \omega_0 + 2\pi k)$$

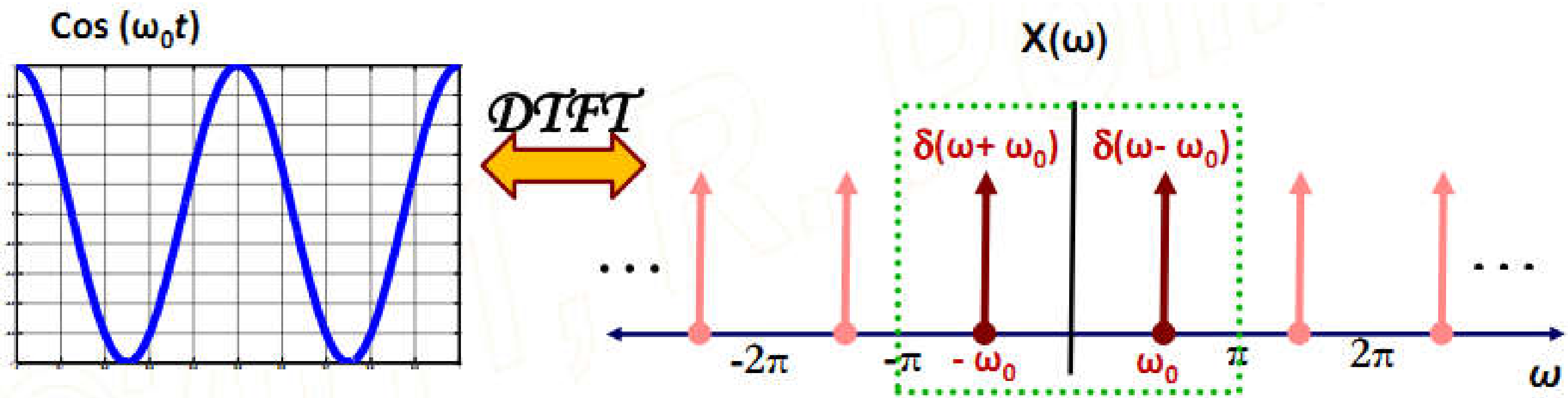


  - We are only interested in [-π, π] range, where there is only one spectral component.
  - Hence, the spectrum of a single complex exponential at a specific frequency is an impulse at that frequency.

Important DTFT Pairs (4)

- 4. The sinusoid

$$x[n] = \cos(\omega_0 n) \overset{\mathfrak{I}}{\Leftrightarrow} \pi \sum_{m=-\infty}^{\infty} \delta(\omega - 2\pi m - \omega_0) + \pi \sum_{m=-\infty}^{\infty} \delta(\omega - 2\pi m + \omega_0)$$



  - The above expression can also be obtained from the DTFT of the complex exponential through the Euler's formula.

$$e^{j\omega_0 n} \overset{\mathfrak{I}}{\Leftrightarrow} 2\pi \sum_{m=-\infty}^{\infty} \delta(\omega - \omega_0 \pm 2\pi m)$$

Important DTFT Pairs (5)

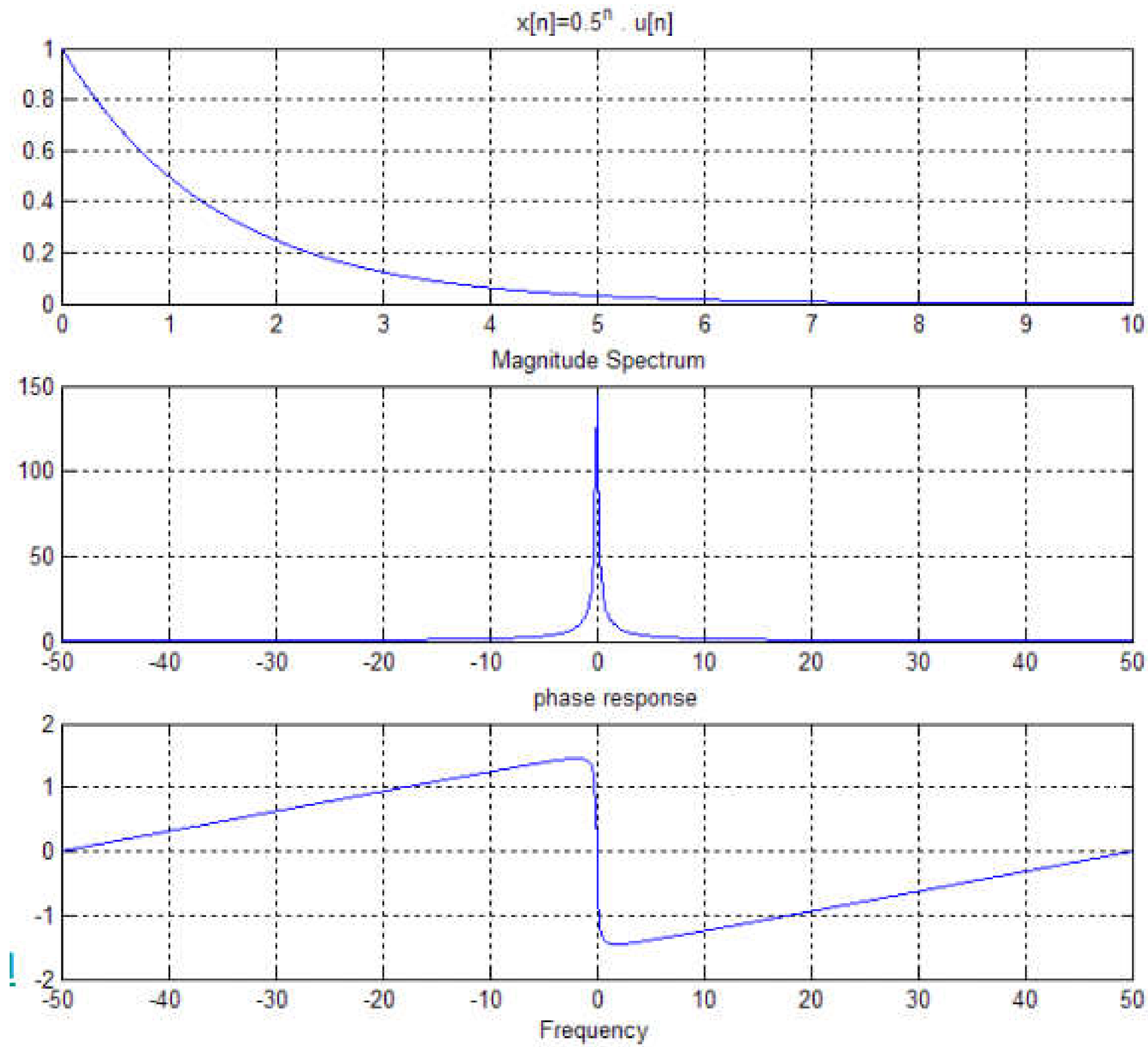
- 5. The real exponential

$$x[n] = \alpha^n \mu[n], \quad |\alpha| < 1$$

$$\overset{\Im}{\Leftrightarrow}$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \alpha^n \mu[n]\, e^{-j\omega n} = \sum_{n=0}^{\infty} \alpha^n e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} \left(\alpha e^{-j\omega}\right)^n = \frac{1}{1-\alpha e^{-j\omega}}$$



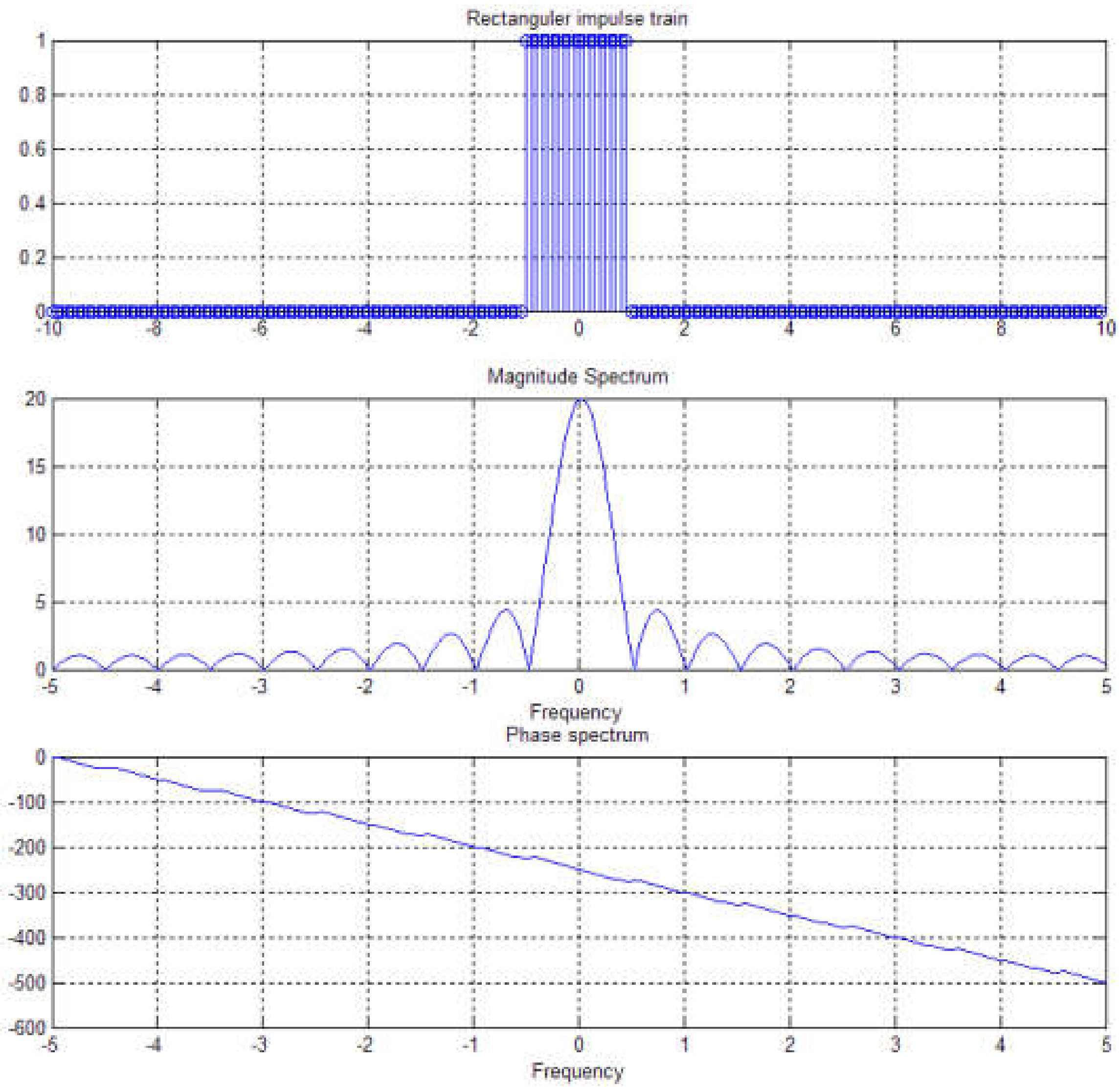
Important DTFT Pairs (6)

- 6. Rectangular pulse train

$$x[n] = \text{rect}_M[n] = \begin{cases} 1, & -M \le n \le M \\ 0, & \text{otherwise} \end{cases}$$

$$\overset{\Im}{\Leftrightarrow}$$

$$\sum_{n=-M}^{M} e^{-j\omega n} = \frac{\sin(M+1/2)\omega}{\sin(\omega/2)}, \quad \omega \neq 0$$



DTFT Properties (1)

- Linearity
  - Given $x_1[n]$ and $X_1(\omega)$ form a DTFT pair, and $x_2[n]$ and $X_2(\omega)$ form another DTFT pair i.e.

$$x_1[n] \xleftrightarrow{\text{DTFT}} X_1(\omega)$$

$$x_2[n] \xleftrightarrow{\text{DTFT}} X_2(\omega)$$

  - We can show that

$$ax_1[n] + bx_2[n] \xleftrightarrow{\text{DTFT}} aX_1(\omega) + bX_2(\omega)$$

DTFT Properties (2)

- Time-reversal: A reversal of the time domain variable causes a reversal of the frequency variable

$$x[-n] \xleftrightarrow{\text{DTFT}} X(-\omega)$$

- Proof:

DTFT Properties (3)

- Symmetric

$$x^*[n] \xleftrightarrow{\text{DTFT}} X^*(-\omega) \qquad x^*[-n] \xleftrightarrow{\text{DTFT}} X^*(\omega)$$

  - 1. If x[n] is real: $X(\omega) = X^*(-\omega)$

$$|X(\omega)| = |X(-\omega)| \qquad X_R(\omega) = X_R(-\omega)$$

$$\varphi(\omega) = -\varphi(-\omega) \qquad X_I(\omega) = -X_I(-\omega)$$

  - 2. If $x[n] = x_{even}[n] + x_{odd}[n]$

$$x_{even}[n] \xleftrightarrow{\text{DTFT}} X_{real}(\omega)$$

$$x_{odd}[n] \xleftrightarrow{\text{DTFT}} X_{imag}(\omega)$$

DTFT Properties (4)

- Time Domain Shifting (TD Delay) => FD Phase Shift

$$x[n-M] \xleftrightarrow{\text{DTFT}} X(\omega)e^{-j\omega M}$$

  – Note that the magnitude spectrum is unchanged by time shift.

- Frequency Domain Shifting => TD Phase Shift

$$e^{j\omega_0 n}x[n] \xleftrightarrow{\text{DTFT}} X(\omega-\omega_0)$$

DTFT Properties (5)

- Convolution in TD = multiplication in FD

$$x[n] * h[n] \xleftrightarrow{\text{DTFT}} X(\omega) \cdot H(\omega)$$

- Proof:

DTFT Properties (5)

- Multiplication in TD = convolution integral in FD

$$x[n] \cdot h[n] \xleftrightarrow{\text{DTFT}} \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\gamma) \cdot H(\omega - \gamma) d\gamma$$

  – h[n] can be considered as either system impulse response or another signal;
  – This property is also called the modulation theorem, since it involves the modulation of one signal x[n] with the other h[n];

DTFT Properties (6)

- Parseval Theorem: The energy of the signal , whether computed in TD or FD, is the same!

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(\omega)|^2 d\omega$$

Energy density spectrum of the signal

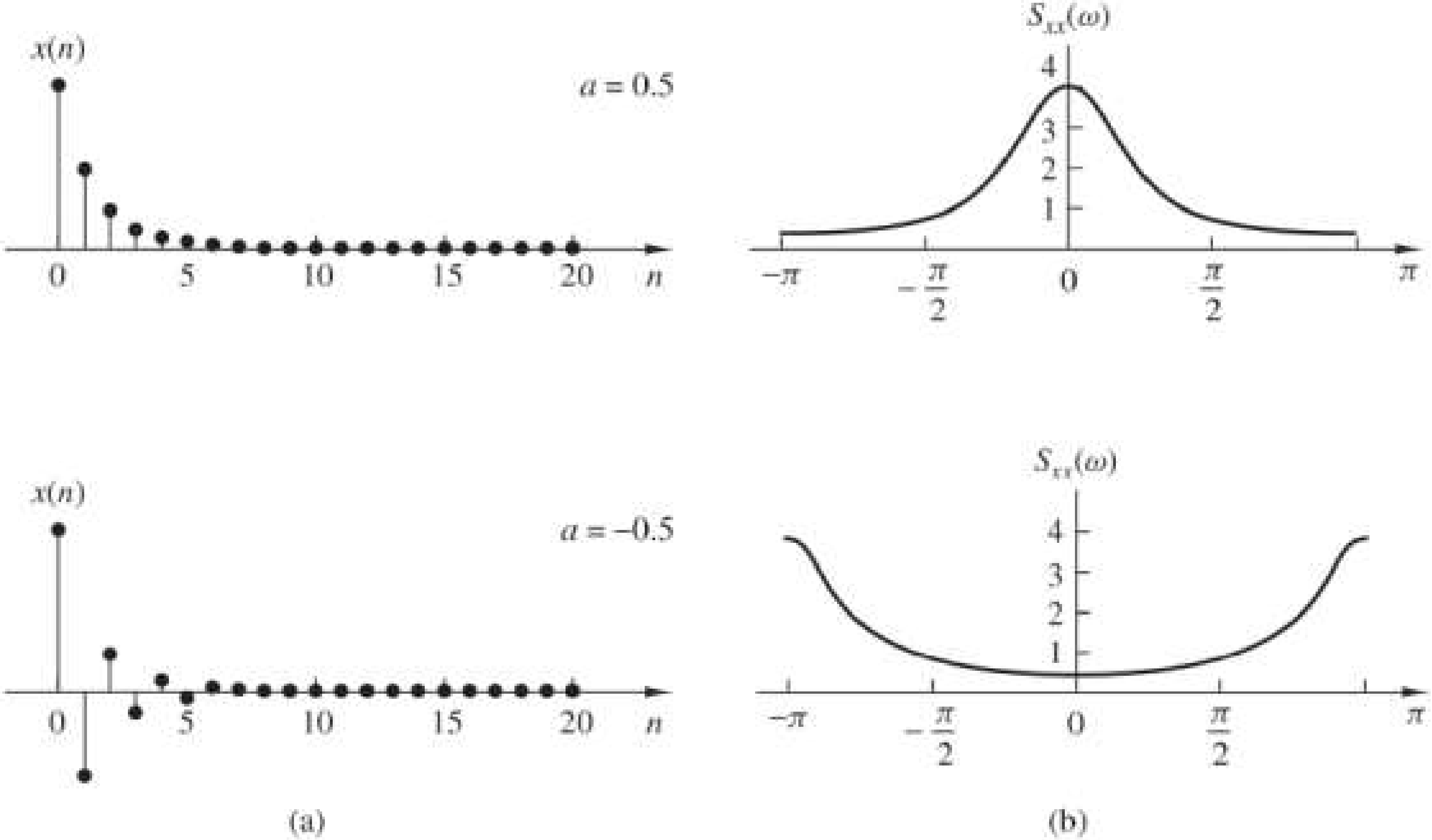
$$\sum_{n=-\infty}^{\infty} x[n]\cdot x^*[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)\cdot X^*(\omega) d\omega$$

Energy density spectrum

- Example - Determine and sketch the energy density spectrum of the signal

$$x(n) = a^n u(n), \qquad -1 < a < 1$$

- Result:



7_4 Wrap up

- DTFT Pairs:
  - Impulse, constant, exponential, sinusoids, real exponential, rectangular
- DTFT Properties:
  - Linearity, time reversal, symmetry, shifting, convolution, Parseval
- They are all the building blocks of DTFT calculation.
  - Be familiar, be able to use them freely for calculation

Chapter 7 Summary

- Discrete-Time Fourier Transform (DTFT)
  - Why do we need frequency domain analyses?
  - Fourier transforms: FS, CTFT, DFS, DTFT, DFT, FFT
  - From CTFT to DTFT
  - DTFT and IDTFT definition and calculation
  - Theorems:
    - periodicity, analog and digital frequency mapping, condition
  - Pairs:
    - $\delta[n]$, constant, exponential, sinusoids, real exponential, rectangular
  - DTFT Properties:
    - Linearity, time reversal, symmetry, shifting, convolution, Parseval