2.5 Green's theorem in the plane (page 433)

Double integrals over a plane region may be transformed into line integrals over the boundary of the region and conversely. This is of practical interest because if may simplify the evaluation of an integral.

Green's theorem: Suppose the functions $F_1(x, y)$, $F_2(x, y)$ and their partial derivatives are single-valued, finite and continuous inside and on the boundary C of some simply connected region R in the xy-plane. Green's theorem in a plane (sometimes called the divergence theorem in two dimensions) then states

$$\oint (F_1 dx + F_2 dy) = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx dy \qquad (2.6)$$

and so relates the line integral around C to a double integral over the enclosed region R.

Proof: Without loosing generalization, we can prove Green's theorem first for a special region R that can be represented in both figures

$$a \le x \le b, \quad u(x) \le y \le v(x), \qquad \text{(The left region in figure 2.3)}$$

$$c \le y \le d, \quad p(x) \le x \le q(y), \qquad \text{(The right region in figure 2.3)}$$

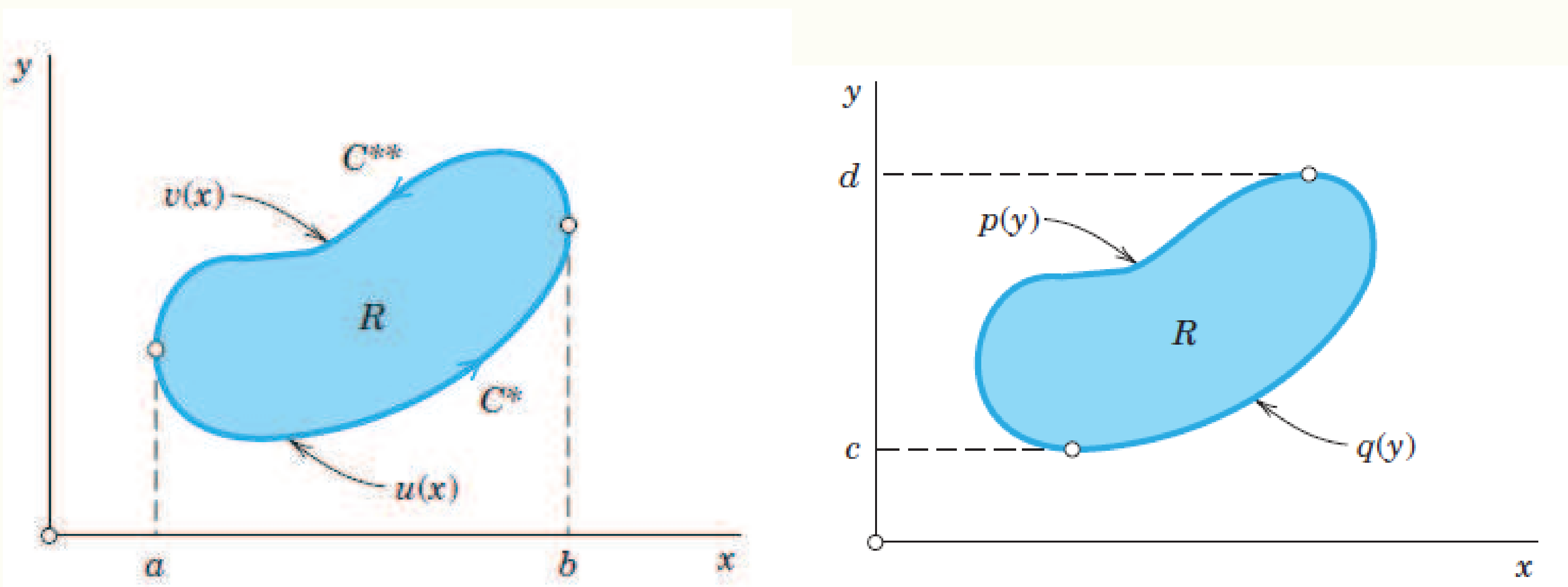

Figure: 2.3: Example of a special region

Step 1: let's start with the second term on the right hand side of (2.6). From the left region of figure (2.3) we know

$$\iint_R \frac{\partial F_1}{\partial y} dxdy = \int_a^b \left[\int_{u(x)}^{v(x)} \frac{\partial F_1}{\partial y} dy \right] dx, \tag{2.7}$$

We integrate the inner integral:

$$\int_{u(x)}^{v(x)} \frac{\partial F_1}{\partial y} dy = [F_1(x,y)]_{y=u(x)}^{y=v(x)} = F_1(x, v(x)) - F_1(x, u(x)).$$

By inserting this into (2.7) we find

$$\begin{aligned} \iint_R \frac{\partial F_1}{\partial y} dxdy &= \int_a^b F_1(x, v(x)) dx - \int_a^b F_1(x, u(x)) dx \\ &= -\int_b^a F_1(x, v(x)) dx - \int_a^b F_1(x, u(x)) dx. \end{aligned}$$

Since $y = v(x)$ represents the curve C^{**} (left region in figure 2.3) and $y = u(x)$ represents C^*, the last two integrals may be written as line integrals over C^{**} and C^*, therefore

$$\iint_R \frac{\partial F_1}{\partial y} dx dy = -\int_{C^{**}} F_1(x, y) dx - \int_{C^*} F_1(x, y) dx = -\oint_C F_1(x, y) dx. \tag{2.8}$$

This proves (2.6) if $F_2 = 0$.

This result remains valid if C has portions parallel to the y-axis (such as $\tilde{C}$ and $\tilde{\tilde{C}}$ in figure 2.4).

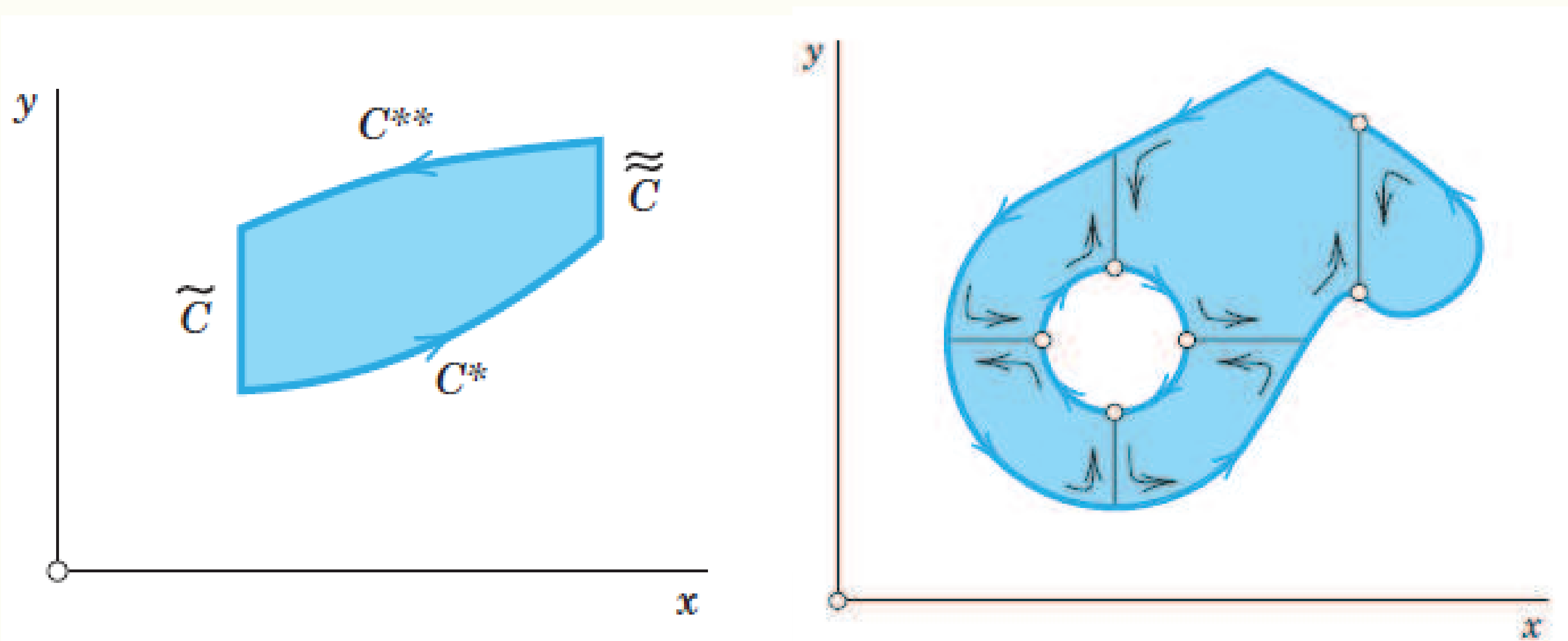

Figure: 2.4: Proof of Green's theorem

Indeed, the integrals over these portions are zero because in (2.8) on the right we integrate with respect to x (where $x = constant$). Hence we may add these integrals to the integrals over C^* and C^{**} to obtain the integral over the whole boundary C in (2.8).

Step 2: We now treat the first term on the right hand-side of (2.6). We will use the second representation of the special region (the right region in figure 2.3). Then by changing a direction of integration,

$$\begin{aligned}\iint_R \frac{\partial F_2}{\partial x} dx dy &= \int_c^d \int_{p(y)}^{q(y)} \left[\frac{\partial F_2}{\partial x} dx\right] dy \\ &= \int_c^d F_2(q(y), y) dy - \int_c^d F_2(p(y), y) dy \\ &= \int_c^d F_2(q(y), y) dy + \int_d^c F_2(p(y), y) dy \\ &= \oint F_2(x, y) dy.\end{aligned}$$

Together with (2.7), this gives (2.6) and proves Green's theorem for special regions.

Step 3: We now prove the theorem for a region R that itself is not a special region but can be subdivided into finitely many special regions (the right region in figure 2.4). In this case we apply the theorem to each subregion and then add the results; the left-hand members add up to the integral over R while the right-hand members add up to the line integral over C plus integrals over the curves introduced for subdividing R. The simple **key observation** now is that each of the latter integrals occurs twice, taken once in each direction. Hence they cancel each other, leaving us with the line integral over C.

Example 1 (page 436)

Let $F_1 = -y$ and $F_2 = x$, evaluate $\oint_C (F_1 dx + F_2 dy)$.

Solution:

$$
\begin{aligned}
\oint_C (F_1 dx + F_2 dy) &= \oint_C (-y dx + x dy) \\
&= \iint_R \left[\frac{\partial x}{\partial x} - \frac{\partial(-y)}{\partial y} \right] dx dy \\
&= \iint_R (1+1) dx dy \\
&= 2 \iint_R dx dy,
\end{aligned}
$$

where $\iint\limits_R dxdy$ is the area of the region R. Let $\iint\limits_R dxdy$ be equal to A, then

$$
\begin{aligned}
\oint_C (-ydx + xdy) &= 2A \\
A &= \frac{1}{2}\oint_C (-ydx + xdy) \\
A &= \frac{1}{2}\oint_C (xdy - ydx). \qquad (2.9)
\end{aligned}
$$

Example 2(page 436)

Find the area of plane bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Solution: By the polar coordinates for the ellipse $x = a\cos\theta$ and $y = b\sin\theta$, then we have $\frac{dx}{d\theta} = -a\sin\theta$, $\frac{dy}{d\theta} = b\cos\theta$. By (2.9) we know the area of the ellipse is

$$\begin{aligned} A &= \frac{1}{2}\oint_C (xdy - ydx) = \frac{1}{2}\int_0^{2\pi}(xdy - ydx) \\ &= \frac{1}{2}\int_0^{2\pi}[(a\cos\theta)(b\cos\theta d\theta) - (b\sin\theta)(-a\sin\theta d\theta)] \\ &= \frac{1}{2}\int_0^{2\pi}(ab\cos^2\theta + ab\sin^2\theta)d\theta \\ &= \frac{1}{2}\int_0^{2\pi} abd\theta = \frac{1}{2}ab\int_0^{2\pi} d\theta = ab\pi. \end{aligned}$$

Example 3: (page 437)

Find the area of plane bounded by the cardioid (see the figure blow), $r = a(1 - \cos\theta)$, where $0 \leq \theta \leq 2\pi$.

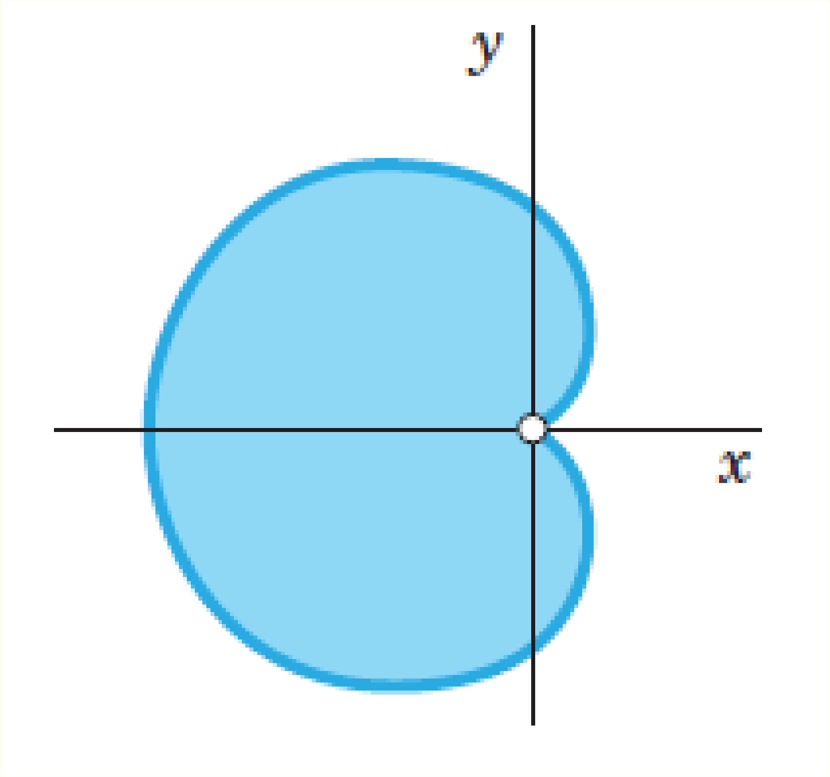

Figure: 2.6: Cardioid

Solution: Let r and θ be polar coordinates defined by $x = r\cos\theta,\ y = r\sin\theta$. Then

$$dx = \cos\theta dr - r\sin\theta d\theta, \quad dy = \sin\theta dr + r\cos\theta d\theta,$$

and (2.9) becomes

$$\begin{aligned}
A &= \frac{1}{2}\oint_C [(r\cos\theta)(\sin\theta dr + r\cos\theta d\theta) - \\
&\qquad\qquad (r\sin\theta)(\cos\theta dr - r\sin\theta d\theta)] \\
&= \frac{1}{2}\oint_C r^2 d\theta \\
&= \frac{1}{2}\oint_C [a(1-\cos\theta)]^2 d\theta \\
&= \frac{a^2}{2}\int_0^{2\pi} (1 - 2\cos\theta + \cos^2\theta) d\theta \\
&= \frac{a^2}{2}\left[2\pi + 0 + \int_0^{2\pi} \cos^2\theta d\theta\right] = \frac{3\pi a^2}{2}.
\end{aligned}$$

2.6 Parametric representation of a surface (page 433)

Representation of surfaces

Representation of a surface S in $xyz-$space are $z = f(x, y)$ or $g(x, y, z) = 0$. For example,

$$z = \sqrt{a^2 - x^2 - y^2} \text{ or } x^2 + y^2 + z^2 - a^2 = 0, (z \geq 0)$$

represents a hemisphere of radius a and center 0.

Similar to a curve in space, the surface S can also be represented by $[x, y, f(x, y)]$, where x and y are taken as parameters for S.

It will be more practical to use a parametric representation. Since surfaces are two-dimensional, we need two parameters, which we call u and v. Thus a **parametric representation** of a surface S in space is of the form

$$\boldsymbol{r}(u,v) =< x(u,v), y(u,v), z(u,v) >= x(u,v)\mathbf{i}+y(u,v)\mathbf{j}+z(u,v)\mathbf{k},$$

where (u,v) varies in some region R of $uv-$plane. This mapping maps every point (u,v) in R onto the point of S with position vector $\boldsymbol{r}(u,v)$. See the figure on next page for illustration.

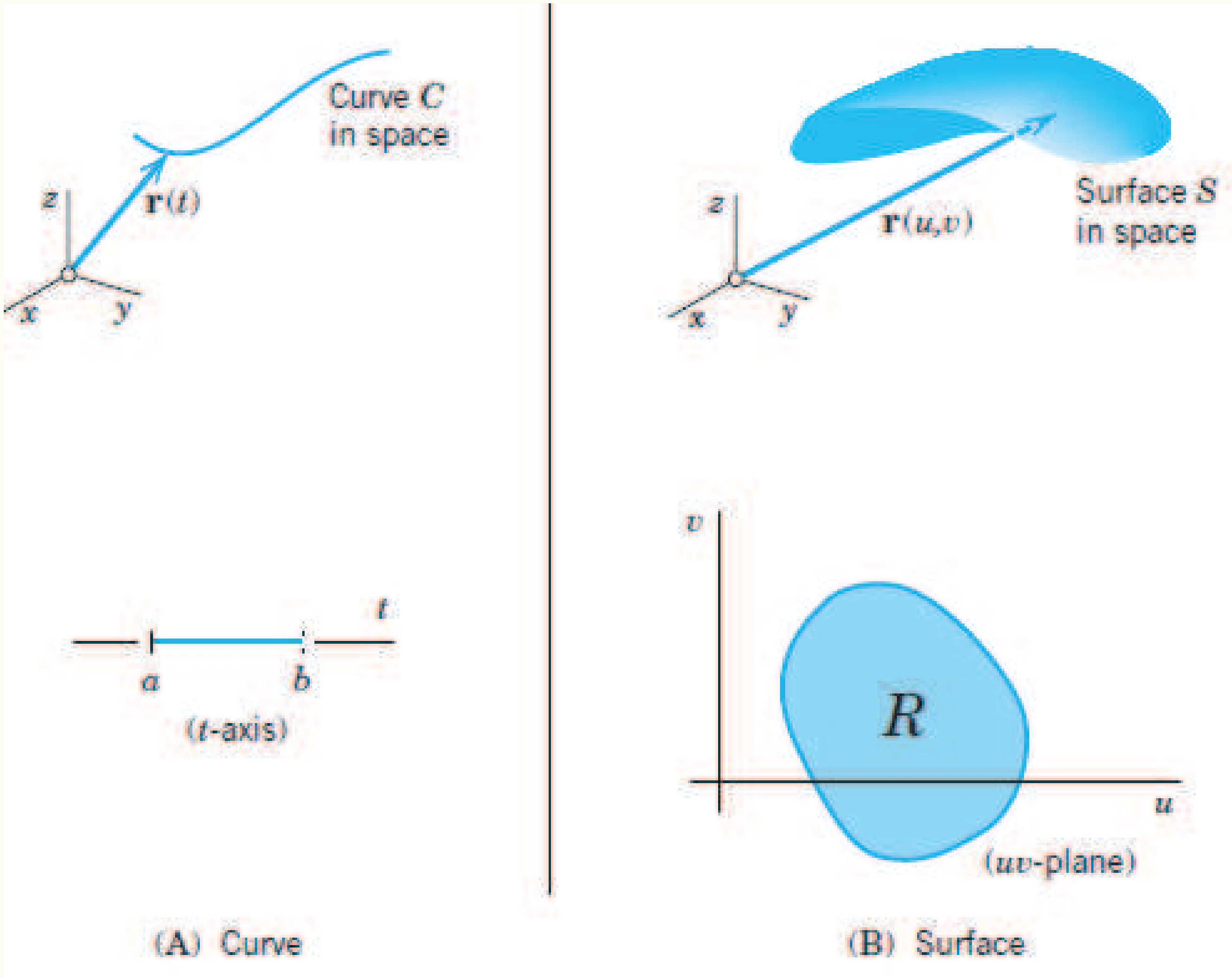

Figure: 2.7: Parametric representations of a curve and a surface

Example 1: (page 440) Find the parametric representation of a circular cylinder $x^2 + y^2 = a^2, -1 \leq z \leq 1$, whose radius is a, height is 2.

Solution:
A parametric representation is

$$\boldsymbol{r}(u, v) =< a\cos u, a\sin u, v >$$

The components of $\boldsymbol{r} = (u, v)$ are

$$x = a\cos u, y = a\sin u, z = v$$

The parameters u, v vary in the rectangle R:
$0 \leq u \leq 2\pi, -1 \leq v \leq 1$ in the $uv-$plane.
The curves $u =$constants are vertical straight lines.
The curves $v =$constants are parallel circles.

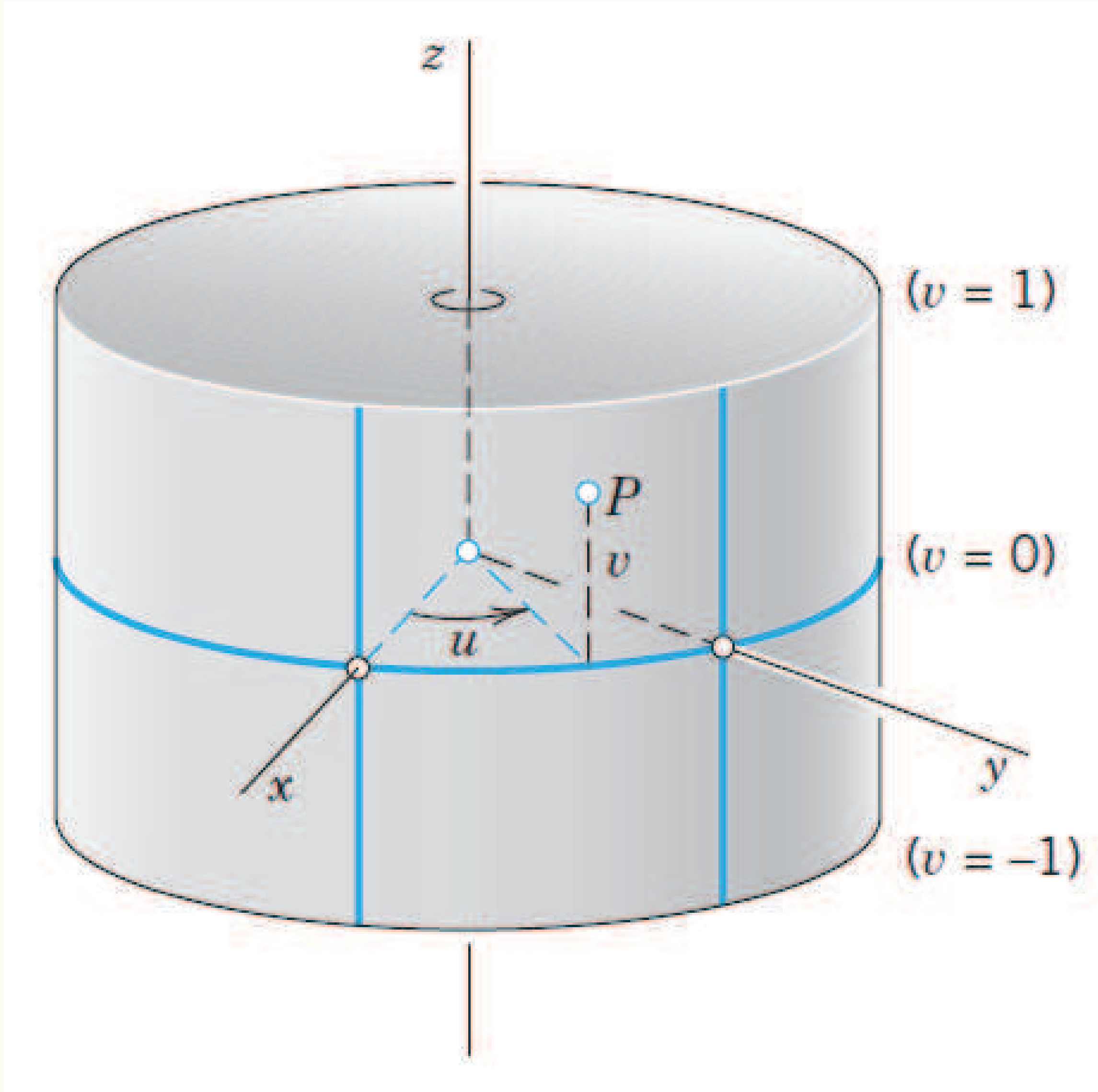

Figure: 2.8: Parametric representations of a cylinder

Example 2: (page 440) Parametric representation of a sphere.

A sphere $x^2 + y^2 + z^2 = a^2$ can be represented in the form

$$\boldsymbol{r}(u, v) = a \cos v \cos u\mathbf{i} + a \cos v \sin u\mathbf{j} + a \sin v\mathbf{k},$$

where the parameters u, v vary in the rectangle R in the $uv-$plane given by the inequalities $0 \leq u \leq 2\pi,\ -\frac{\pi}{2} \leq v \leq \frac{\pi}{2}$.

The components of $\boldsymbol{r} = \boldsymbol{r}(u, v)$ are

$$x = a \cos v \cos u,\ \ y = a \cos v \sin u,\ \ z = a \sin v.$$

Another parametric representation of the sphere used in mathematics is

$$\boldsymbol{r}(u, v) = a \cos u \sin v\mathbf{i} + a \sin u \sin v\mathbf{j} + a \cos v\mathbf{k},$$

where $0 \leq u \leq 2\pi, 0 \leq v \leq \pi$.

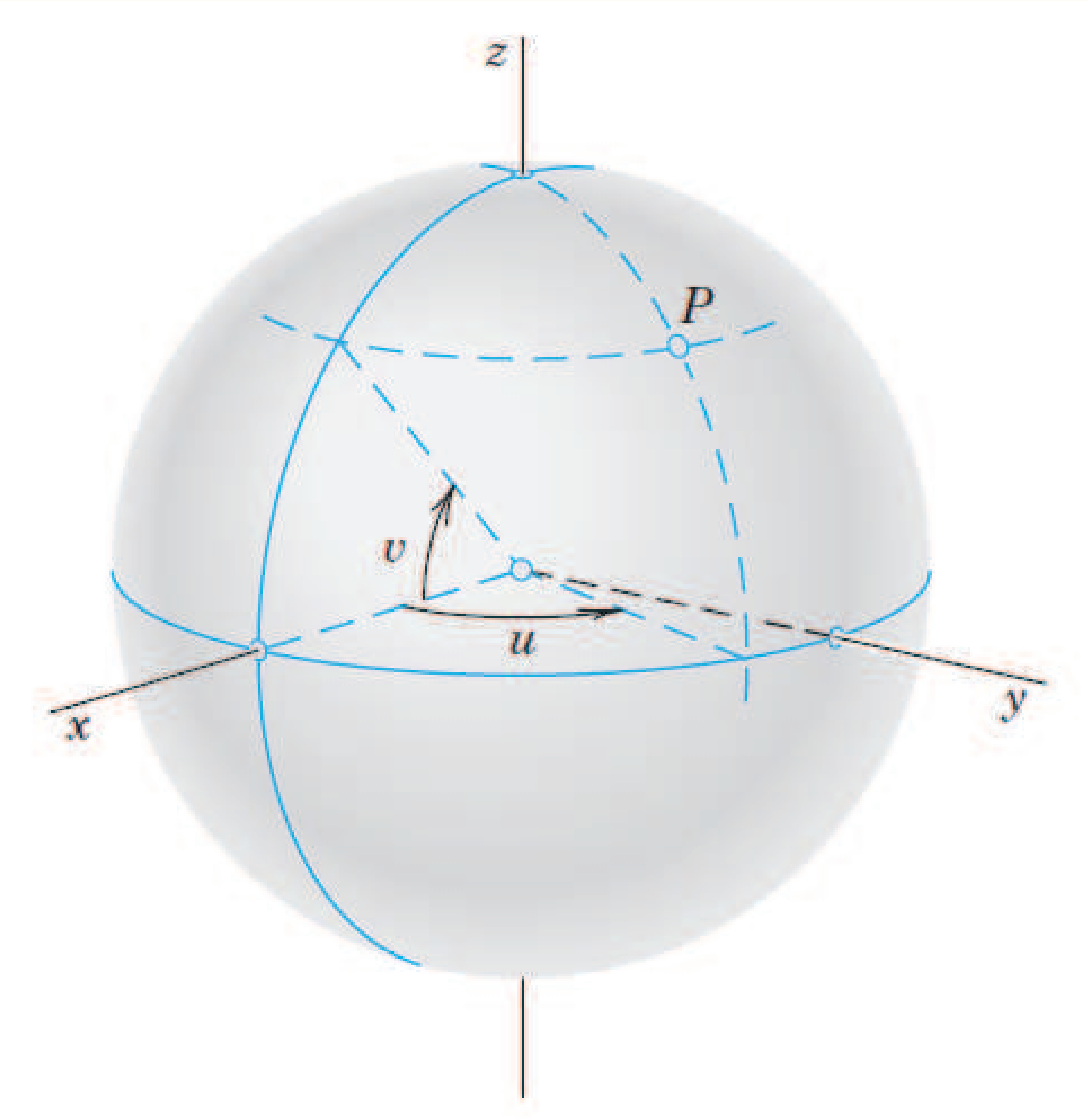

Figure: 2.9: Parametric representations of a sphere

Tangent plane and surface normal

The tangent vectors of all the curves on a surface S through a point P of S form a plane, called the **tangent plane**. Exceptions are points where S has an edge or a cusp (like a cone), so that S cannot have a tangent plane at such a point.

A vector perpendicular to the tangent plane is called **normal vector** of S at P.
Let a surface S is given by the parametric representation

$$\boldsymbol{r}(u,v) =< x(u,v), y(u,v), z(u,v) >= x(u,v)\mathbf{i}+y(u,v)\mathbf{j}+z(u,v)\mathbf{k}.$$

We can get a curve C on S by taking a pair of differentiable functions $u = u(t)$, $v = v(t)$ whose derivatives $u' = du/dt$ and $v' = dv/dt$ are continuous. Then C has the position vector $\boldsymbol{r}(t) = \boldsymbol{r}(u(t), v(t))$. By differentiation and the use of the chain rule we obtain a tangent vector of C on S

$$\boldsymbol{r}'(t) = \frac{d\boldsymbol{r}}{dt} = \frac{\partial \boldsymbol{r}}{\partial u}u' + \frac{\partial \boldsymbol{r}}{\partial v}v'.$$

Hence the partial derivatives $\boldsymbol{r}_u$ and $\boldsymbol{r}_v$ at P are tangential to S at P. Then $\boldsymbol{r}_u$ and $\boldsymbol{r}_v$ span the tangent plane of S at P. Hence their cross product gives a normal vector $\boldsymbol{N}$ of S at P

$$\boldsymbol{N} = \boldsymbol{r}_u \times \boldsymbol{r}_v \neq 0.$$

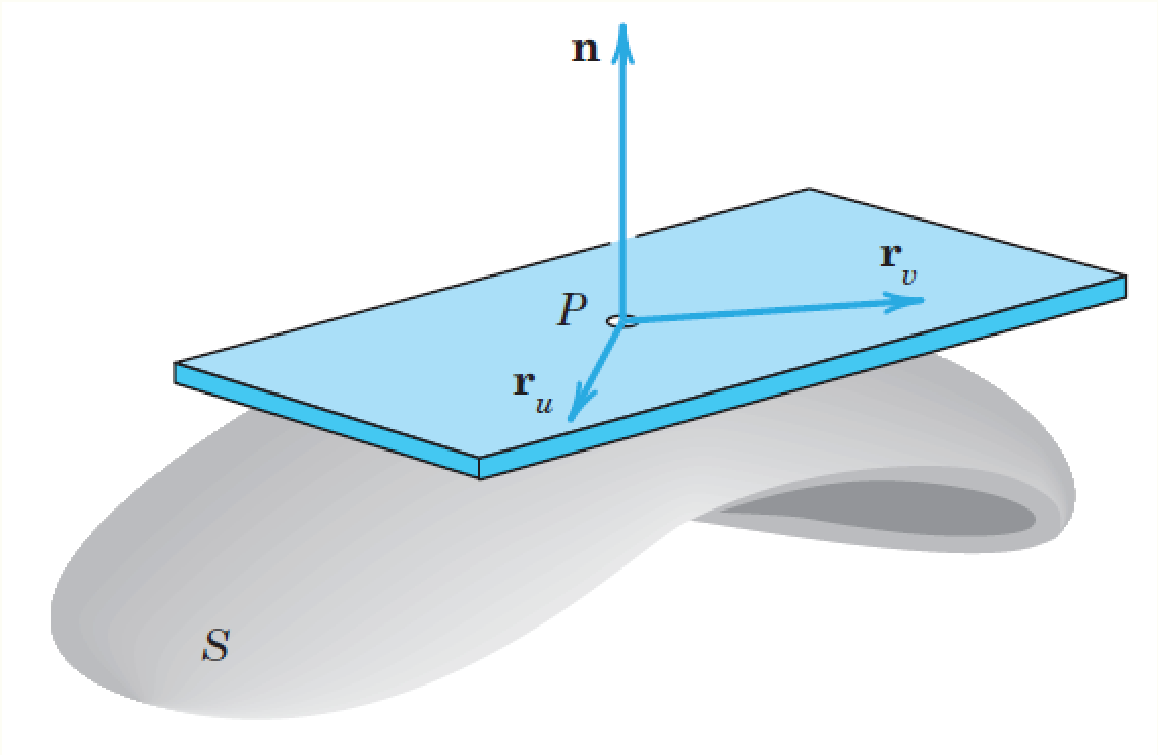

Figure: 2.10: Tangent plane and normal vector

$\boldsymbol{N} = \boldsymbol{r}_u \times \boldsymbol{r}_v \neq 0$. means that $\boldsymbol{r}_u$ and $\boldsymbol{r}_v$ are linearly independent at each point of S, then S has, at every point P, a unique tangent plane passing through P and spanned by $\boldsymbol{r}_u$ and $\boldsymbol{r}_v$, and a unique normal whose direction depends continuously on the points of S. A normal vector is given by $\boldsymbol{N} = \boldsymbol{r}_u \times \boldsymbol{r}_v \neq 0$ and the corresponding **unit normal vector** $\boldsymbol{n}$ by

$$\boldsymbol{n} = \frac{1}{\boldsymbol{N}} = \frac{1}{|\boldsymbol{r}_u \times \boldsymbol{r}_v|}\boldsymbol{r}_u \times \boldsymbol{r}_v. \tag{2.10}$$

If S is represented by $g(x, y, z) = 0$, then for a point P on S with grad$g =< \frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z} >\neq 0$, the corresponding unit normal vector $\boldsymbol{n}$ is given by

$$\boldsymbol{n} = \frac{1}{|\text{grad} g|}\text{grad} g \tag{2.11}$$

Note:

- A surface S is called a **smooth surface** if its surface normal depends continuously on the points of S.
- A surface S is called a **piecewise smooth surface** if it consists of finitely many smooth positions. For example, a sphere is smooth, and the surface of a cube is piecewise smooth.

Example 3: (page 442) Unit normal vector of a sphere.
Find the unit normal vector of the sphere

$$g(x, y, z) = x^2 + y^2 + z^2 - a^2 = 0.$$

Solution: By (2.11), the unit normal vector of the sphere is

$$\begin{aligned}
\boldsymbol{n} &= \frac{1}{|\text{grad} g|}\text{grad} g \\
&= \frac{1}{\sqrt{(2x)^2 + (2y)^2 + (2z)^2}} < 2x, 2y, 2z > \\
&= \frac{1}{\sqrt{x^2 + y^2 + z^2}} < x, y, z > \\
&= \frac{1}{a} < x, y, z > \\
&= \frac{x}{a}\mathbf{i} + \frac{y}{a}\mathbf{j} + \frac{z}{a}\mathbf{k}.
\end{aligned}$$

We see that $\boldsymbol{n}$ has the direction of the position vector $< x, y, z >$ of the corresponding point.

Example 4: (page 442) Unit normal vector of a helicoid.
Find the unit normal vector of the helicord

$$\boldsymbol{r}(u,v) =< u\cos v, u\sin v, v >$$

Solution: The normal vector of a surface is given by $\boldsymbol{N} = \boldsymbol{r}_u \times \boldsymbol{r}_v$, where

$$\boldsymbol{r}_u =< \cos v, \sin v, 0 >, \quad \boldsymbol{r}_v =< -u\sin v, u\cos v, 1 >,$$

So the normal vector is

$$\begin{aligned} \boldsymbol{N} &= \boldsymbol{r}_u \times \boldsymbol{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos v & \sin v & 0 \\ -u\sin v & u\cos v & 1 \end{vmatrix} \\ &= < \sin v, -\cos v, u > \end{aligned}$$

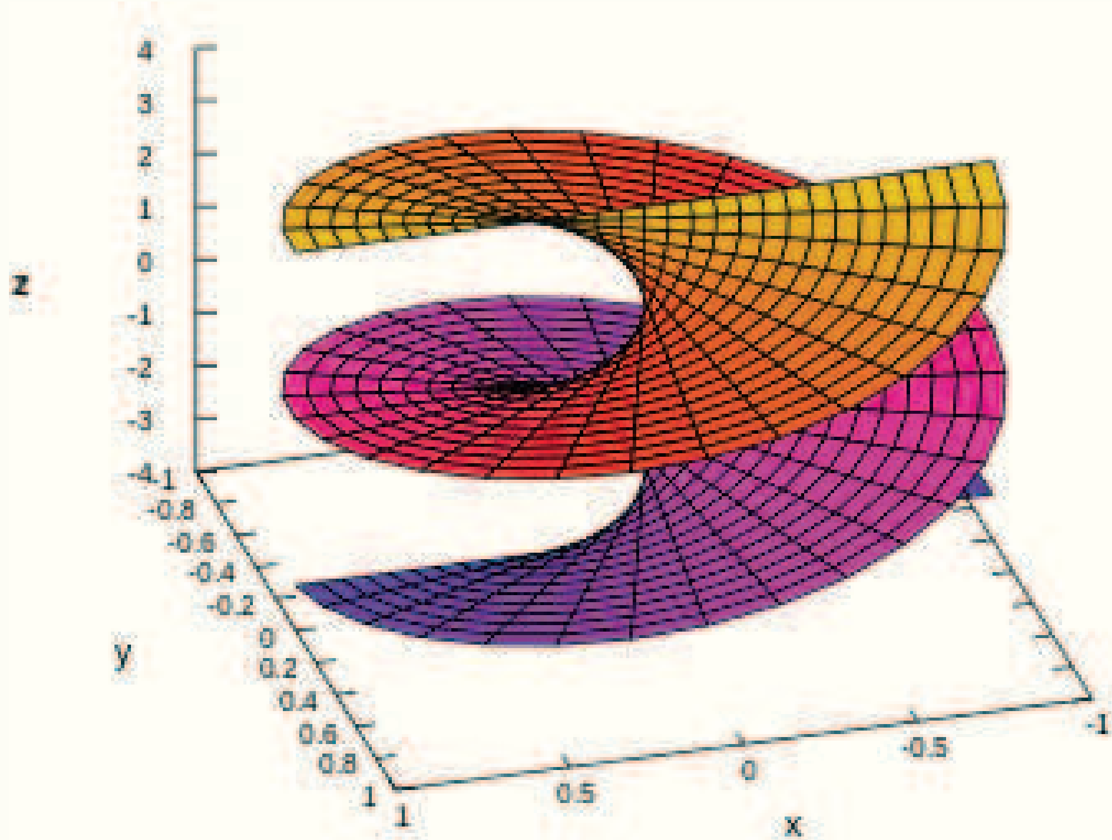

Figure: Helicoid