Xi'an Jiaotong-Liverpool University
西交利物浦大学

EEE204 Continuous and Discrete Time Signals and Systems II

2018–2019 Semester 2

Electrical and Electronic Engineering

Xi'an Jiaotong-Liverpool University

Week 2

Example

Consider the following DT sequence

$$x[n] = \begin{cases} e^{-n}, & n \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

Determine if the signal is a power or an energy signal.

The total energy of the DT sequence is calculated as follows:

$$\sum_{n=-\infty}^{+\infty} |x[n]|^2 = \sum_{n=0}^{+\infty} |e^{-n}|^2,$$

$$= \sum_{n=0}^{+\infty} e^{-2n} = \frac{1}{1 - e^{-2}},$$

$$\approx 1.157 < \infty,$$

$x[n]$ is an energy signal.

Periodic Sequence

Periodic Sequence

- A DT sequence is periodic if

$$x[n] = x[n+N];\ \forall n,\ N \in Z^{+},\ N \geqslant 1$$

- The smallest integer N_0 is called the fundamental period
- The reciprocal $f = \dfrac{1}{N}$ is called the digital frequency
    - What is the range for f?

Since:

$$N \in Z^{+};\ N \geqslant 1,\ 0 < \frac{1}{N} \leqslant 1 \rightarrow 0 < f \leqslant 1.$$

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- Sinusoidal sequence

  $x[n] = \cos(2\pi f n)$; $\forall n \in Z$, f is digital frequency

- For sinusoidal sequence to be period:

$$\begin{aligned} \cos[2\pi f(n+N)] &= \cos(2\pi f n + 2\pi f N), \\ ? &= \cos(2\pi f n) \end{aligned}$$

  If f is a rational number, for some N, fN can be integer,

  Therefore, $\cos(2\pi f n + 2\pi f N) = \cos(2\pi f n)$.

- A discrete-time sinusoidal is periodic only if its digital frequency f is a rational number.

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Period

Discrete-time signals are periodic with respect to the digital frequency as well, the fundamental period is one.

Proof:

$$\begin{aligned}\cos[2\pi(f+1)n] &= \cos(2\pi fn + 2\pi n),\\ &= \cos(2\pi fn).\end{aligned}$$

All periodic signals are power signals.

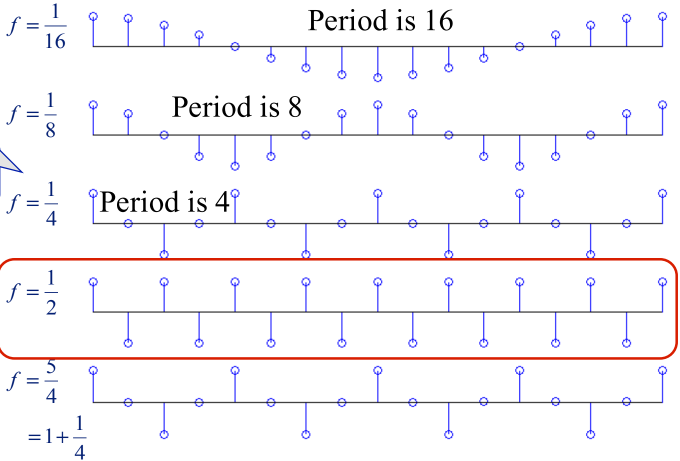
Sinusoidal Sequence

Example

$$x[n] = \cos(2\pi f n), \forall n \in Z$$

digital frequency

The digital frequency is periodic with period 1



$$x[n] = \cos\left(2\pi \frac{1}{2} n\right) = \cos(\pi n) = (-1)^n$$

The highest rate of oscillation in a DT sinusoidal is attained when $f = 1/2$.

Example

Consider the following DT sequence

$$x[n] = 5\cos(\pi n/2).$$

Determine the fundamental period of the signal and if the signal is a power or an energy signal.

$$x[n] = 5\cos(\pi n/2) = 5\cos\left(2\pi \cdot \frac{1}{4}n\right).$$

The digital frequency $f = \dfrac{1}{4}$, the smallest integer to make fN an integer is therefore $N = 4$, which is the fundamental period.

Example

$x[n] = 5\cos(\pi n/2)$

We know that all periodic signals are power signals, so $x[n]$ is a power signal.

The average power of $x[n]$ is given by

$$
\begin{aligned}
P &= \frac{1}{4}\sum_{n=0}^{3} 25\cos^2(\pi n/2), \\
&= \frac{25}{4}\sum_{n=0}^{3} \frac{1}{2}[1+\cos(\pi n)], \\
&= \frac{25}{8}\sum_{n=0}^{3} [1+(-1)^n],
\end{aligned}
$$

Example

$x[n] = 5\cos(\pi n/2)$

$$
\begin{aligned}
P &= \frac{25}{8}\sum_{n=0}^{3}[1+(-1)^n], \\
&= \frac{25}{8}\left[\sum_{n=0}^{3}1+\sum_{n=0}^{3}(-1)^n\right], \\
&= \frac{25}{8}(4+0) = \frac{25}{2}.
\end{aligned}
$$

In general, a periodic DT sinusoidal signal has an average power $\frac{A^2}{2}$, where A is the amplitude of the signal.

Example

$x[n] = 5\cos(\pi n/2)$

The second approach which applies the Euler's formula.

The average power of $x[n]$ is given by

$$
\begin{aligned}
P &= \frac{1}{4}\sum_{n=0}^{3} 25\cos^2(\pi n/2), \\
&= \frac{25}{4}\sum_{n=0}^{3}\frac{1}{2}[1+\cos(\pi n)], \\
&= \frac{25}{8}\sum_{n=0}^{3}\left[1+\frac{e^{i\pi n}+e^{-i\pi n}}{2}\right],
\end{aligned}
$$

Example

$x[n] = 5\cos(\pi n/2)$

$$
\begin{aligned}
P &= \frac{25}{8}\sum_{n=0}^{3}\left[1 + \frac{e^{i\pi n} + e^{-i\pi n}}{2}\right], \\
&= \frac{25}{8}\left[\sum_{n=0}^{3} 1 + \frac{1}{2}\sum_{n=0}^{3} e^{i\pi n} + \frac{1}{2}\sum_{n=0}^{3} e^{-i\pi n}\right], \\
&= \frac{25}{8}\left(4 + \frac{1}{2}\cdot\frac{1-e^{i\pi 4}}{1-e^{i\pi}} + \frac{1}{2}\cdot\frac{1-e^{-i\pi 4}}{1-e^{-i\pi}}\right),
\end{aligned}
$$

Example

$x[n] = 5\cos(\pi n/2)$

$$P = \frac{25}{8}\left(4 + \frac{1}{2}\cdot\frac{1-e^{i\pi 4}}{1-e^{i\pi}} + \frac{1}{2}\cdot\frac{1-e^{-i\pi 4}}{1-e^{-i\pi}}\right),$$

$$= \frac{25}{8}\left[4 + \frac{1}{2}\cdot\frac{1-\cos(4\pi)-i\sin(4\pi)}{1-e^{i\pi}}\right.$$

$$\left. + \frac{1}{2}\cdot\frac{1-\cos(4\pi)+i\sin(4\pi)}{1-e^{-i\pi}}\right],$$

$$= \frac{25}{8}\left(4 + \frac{1}{2}\cdot\frac{1-1}{1-e^{i\pi}} + \frac{1}{2}\cdot\frac{1-1}{1-e^{-i\pi}}\right) = \frac{25}{2}.$$

Power of DT Sinusoid

Consider the following DT sequence

$$x[n] = A_1 \sin(\omega_1 n + \phi_1).$$

Assuming $\omega_1 = \dfrac{m_1}{N_1} \cdot 2\pi$ and determine the power of the signal.

$$(0 \leqslant m_1 < N_1, m_1 \in Z^+ \cup \{0\}, N_1 \in Z^+).$$

General Case

Power of DT Sinusoid $x[n] = A_1 \sin(\frac{m_1}{N_1} \cdot 2\pi n + \phi_1)$

1. $m_1 = 0$, then $\omega_1 = 0, x[n] = A_1 \sin\phi_1$, for $\forall n$

$$P_1 = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2,$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} A_1^2 \sin^2\phi_1,$$

$$= \lim_{N\to\infty} \frac{2N+1}{2N+1} A_1^2 \sin^2\phi_1,$$

$$= A_1^2 \sin^2\phi_1.$$

General Case

Power of DT Sinusoid $x[n] = A_1 \sin(\frac{m_1}{N_1} \cdot 2\pi n + \phi_1)$

2. $m_1 = 1, N_1 = 1$, then
   $x[n] = A_1 \sin(2\pi n + \phi_1) = A_1 \sin \phi_1$, for $\forall n$

$$P_1 = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2,$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} A_1^2 \sin^2 \phi_1,$$

$$= \lim_{N\to\infty} \frac{2N+1}{2N+1} A_1^2 \sin^2 \phi_1,$$

$$= A_1^2 \sin^2 \phi_1.$$

Power of DT Sinusoid $x[n] = A_1 \sin(\frac{m_1}{N_1} \cdot 2\pi n + \phi_1)$

3. $m_1 = 1, N_1 = 2$, then
   $x[n] = A_1 \sin(\pi n + \phi_1) = (-1)^n A_1 \sin\phi_1$, for $\forall n$

$$
\begin{aligned}
P_1 &= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2, \\
&= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} A_1^2 \sin^2\phi_1, \\
&= \lim_{N\to\infty} \frac{2N+1}{2N+1} A_1^2 \sin^2\phi_1, \\
&= A_1^2 \sin^2\phi_1.
\end{aligned}
$$

Power of DT Sinusoid $x[n] = A_1 \sin(\frac{m_1}{N_1} \cdot 2\pi n + \phi_1)$

4. For $1 \leqslant m_1 < N_1, N_1 > 2$, then N_1 is the fundamental period.

$$
\begin{aligned}
P_1 &= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2 = \frac{1}{N_1} \sum_{n=0}^{N_1-1} |x[n]|^2, \\
&= \frac{1}{N_1} \sum_{n=0}^{N_1-1} A_1^2 \sin^2(\omega_1 n + \phi_1), \\
&= \frac{A_1^2}{N_1} \sum_{n=0}^{N_1-1} \frac{1 - \cos(2\omega_1 n + 2\phi_1)}{2}, \\
&= \frac{A_1^2}{N_1} \sum_{n=0}^{N_1-1} \left[\frac{1}{2} - \frac{1}{4} e^{j(2\omega_1 n + 2\phi_1)} - \frac{1}{4} e^{-j(2\omega_1 n + 2\phi_1)} \right],
\end{aligned}
$$

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Power of DT Sinusoid $x[n] = A_1 \sin(\frac{m_1}{N_1} \cdot 2\pi n + \phi_1)$

$$
\begin{aligned}
P_1 &= \frac{A_1^2}{N_1} \sum_{n=0}^{N_1-1} \left[\frac{1}{2} - \frac{1}{4} e^{j(2\omega_1 n + 2\phi_1)} - \frac{1}{4} e^{-j(2\omega_1 n + 2\phi_1)} \right], \\
&= \frac{A_1^2}{2N_1} \sum_{n=0}^{N_1-1} 1 - \frac{A_1^2 e^{j2\phi_1}}{4N_1} \sum_{n=0}^{N_1-1} e^{j2\omega_1 n} - \frac{A_1^2 e^{-j2\phi_1}}{4N_1} \sum_{n=0}^{N_1-1} e^{-j2\omega_1 n}, \\
&= \frac{A_1^2}{2} - \frac{A_1^2 e^{j2\phi_1}}{4N_1} \cdot \frac{1 - e^{j2\omega_1 N_1}}{1 - e^{j2\omega_1}} - \frac{A_1^2 e^{-j2\phi_1}}{4N_1} \cdot \frac{1 - e^{-j2\omega_1 N_1}}{1 - e^{-j2\omega_1}}, \\
&= \frac{A_1^2}{2} - \frac{A_1^2 e^{j2\phi_1}}{4N_1} \cdot \frac{1 - e^{j2m_1 \cdot 2\pi}}{1 - e^{j2\omega_1}} - \frac{A_1^2 e^{-j2\phi_1}}{4N_1} \cdot \frac{1 - e^{-j2m_1 \cdot 2\pi}}{1 - e^{-j2\omega_1}}, \\
&= \frac{A_1^2}{2} - 0 - 0 = \frac{A_1^2}{2}. \quad (\text{since } e^{j2m_1 \cdot 2\pi} = e^{-j2m_1 \cdot 2\pi} = 1)
\end{aligned}
$$

$x[n] = A_1 \sin(\omega_1 n + \phi_1)$, $\omega_1 = \dfrac{m_1}{N_1} \cdot 2\pi$

Thus, the power of $x[n]$ is as follows:

$$P_1 = \begin{cases} A_1^2 \sin^2 \phi_1, & m_1 = 0, \\ A_1^2 \sin^2 \phi_1, & m_1 = 1, N_1 = 1, \\ A_1^2 \sin^2 \phi_1, & m_1 = 1, N_1 = 2, \\ \dfrac{A_1^2}{2}, & 1 \leqslant m_1 < N_1, N_1 > 2. \end{cases}$$

Time Domain Transformation

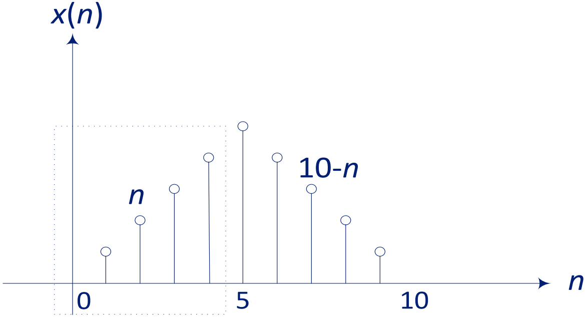
Plot $x[\alpha n + \beta]$ from $x[n]$

- Express $x[\alpha n + \beta]$ as $x\left[\alpha\left(n + \frac{\beta}{\alpha}\right)\right]$.
- Scale the signal $x[n]$ by $|\alpha|$. The resulting waveform represents $x[|\alpha|n]$.
- If α is negative, invert the scaled signal $x[|\alpha|n]$ with respect to the $n = 0$ axis, which produces the waveform for $x[\alpha n]$.
- Shift the waveform for $x[\alpha n]$ by $|\frac{\beta}{\alpha}|$ time units (left-hand side if positive, right-hand side otherwise), which will result in the required representation.

Example

Plot $x[-2n-2]$ for $x[n]$ as

$$x[n] = \begin{cases} n, & 0 \leqslant n < 5 \\ 10-n, & 5 \leqslant n < 10 \\ 0, & \text{otherwise} \end{cases}$$



Example

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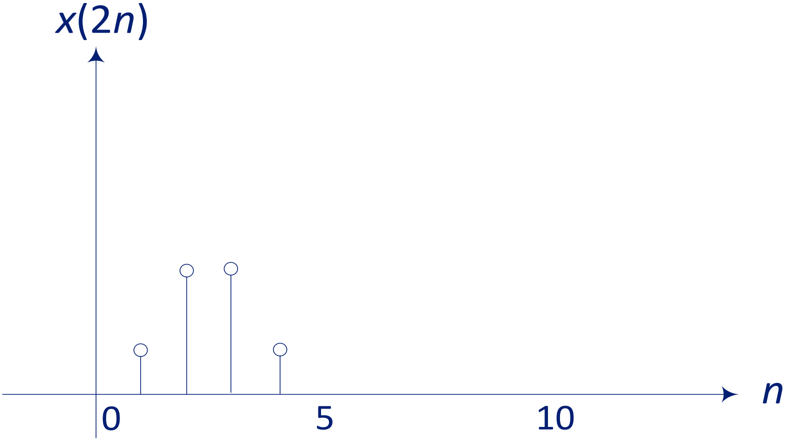
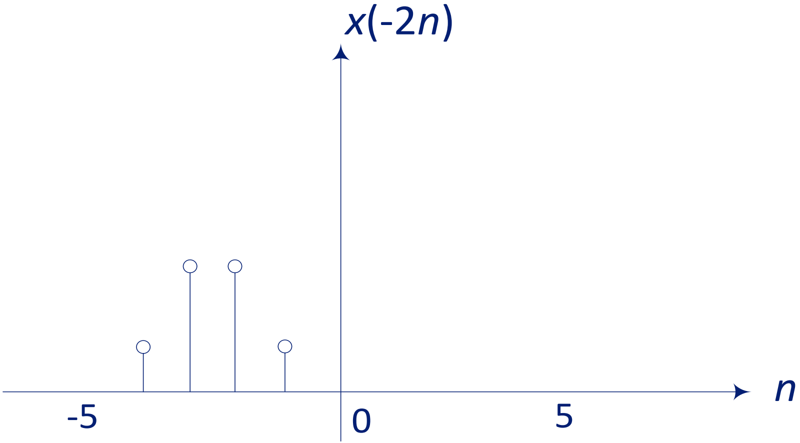
Plot $x[-2n-2] = x[-2(n+1)]$

1. Compress $x[n]$ by a factor of 2 to obtain $x[2n]$.



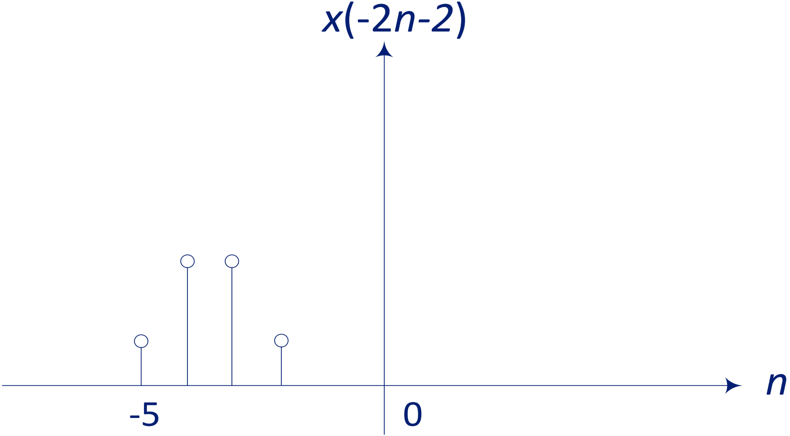
Plot $x[-2(n+1)]$

2. Time-reverse $x[2n]$ to obtain $x[-2n]$.



Plot $x[-2(n+1)]$

3. Shift $x[-2n]$ towards the left-hand side by one time unit to obtain $x[-2n-2]$.



Example

Analytical Solution

Plot $x[-2n-2]$

$$x[n] = \begin{cases} n, & 0 \leqslant n < 5 \\ 10 - n, & 5 \leqslant n < 10 \\ 0, & \text{otherwise} \end{cases}$$

$$x[-2n-2] = \begin{cases} -2n - 2, & 0 \leqslant -2n - 2 < 5 \\ 10 - (-2n - 2), & 5 \leqslant -2n - 2 < 10 \\ 0, & \text{otherwise} \end{cases}$$

Analytical Solution

Plot $x[-2n-2]$

$$x[-2n-2] = \begin{cases} -2n-2, & 0 \leqslant -2n-2 < 5 \\ 10-(-2n-2), & 5 \leqslant -2n-2 < 10 \\ 0, & \text{otherwise} \end{cases}$$

$$x[-2n-2] = \begin{cases} -2n-2, & -3.5 < n \leqslant -1 \\ 2n+12, & -6 < n \leqslant -3.5 \\ 0, & \text{otherwise} \end{cases}$$

Example

Analytical Solution

Plot $x[-2n-2]$

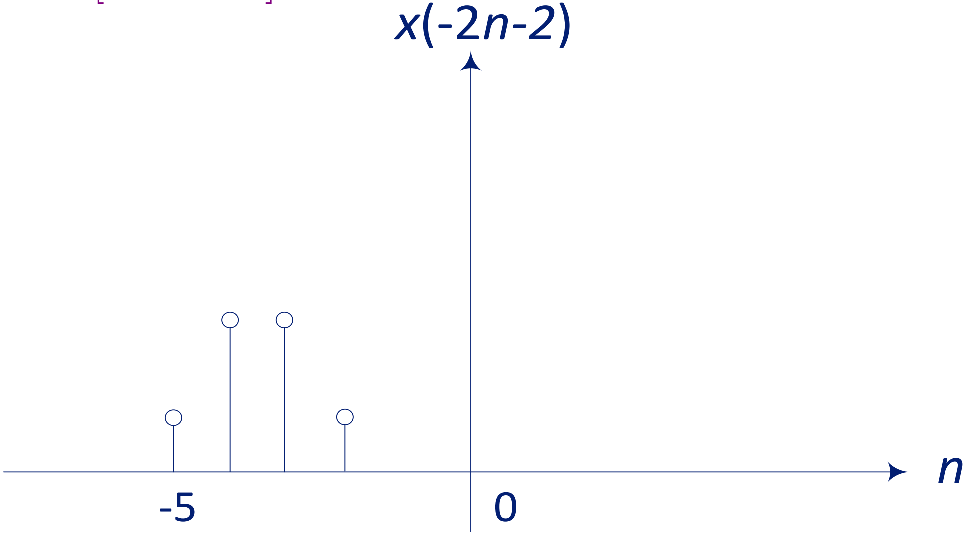
$$x[-2n-2] = \begin{cases} -2n-2, & -4 < n \leqslant -1 \\ 2n+12, & -6 < n \leqslant -4 \\ 0, & \text{otherwise} \end{cases}$$

- When $n = -5$, $x = 2 \times (-5) + 12 = 2$.
- When $n = -4$, $x = 2 \times (-4) + 12 = 4$.
- When $n = -3$, $x = -2 \times (-3) - 2 = 4$.
- When $n = -2$, $x = -2 \times (-2) - 2 = 2$.
- When $n = -1$, $x = -2 \times (-1) - 2 = 0$.

Example

Analytical Solution

Plot $x[-2n-2]$



- Page 7–9, read content about transformation of DT signals
- Page 11–14 read content about periodicity of DT signals
- Page 21–30, read section 1.3.2–1.3.3
- Page 57, Q1.3: (d)–(f);
- Page 57, Q1.6:(b)–(c);
- Page 57, Q1.9: (c)–(e).
- Page 58, Q1.11–Q1.12.

Thank you for your attention.