MTH101: Tutorial 9

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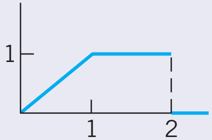
Exercise 1.1

Find the transform for the following functions.

1. $e^{-t} \sinh 4t$,

2. 

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Solution

1.

$$e^{-t}\sinh 4t = e^{-t}\left(\frac{e^{4t}-e^{-4t}}{2}\right) = \frac{e^{3t}}{2} - \frac{e^{-5t}}{2}$$

From the table 6.1 we know

$$\begin{aligned}\mathcal{L}\left[e^{-t}\sinh 4t\right] &= \frac{1}{2}\mathcal{L}\left[e^{3t}\right] - \frac{1}{2}\mathcal{L}\left[e^{-5t}\right] \\ &= \frac{1}{2}\left(\frac{1}{s-3} - \frac{1}{s+5}\right) \\ &= \frac{4}{(s+1)^2-16}.\end{aligned}$$

Or, we can use the s-Shifting theorem, $\mathcal{L}\left[e^{\alpha t}f(t)\right] = F(s-\alpha)$ for $\alpha = -1$ and $f(t) = \sinh 4t$.

Solution

2.

$$f(t) = \begin{cases} t & \text{for} \quad 0 \leq t < 1, \\ 1 & \text{for} \quad 1 \leq t < 2, \\ 0 & \text{for} \quad 2 \leq t. \end{cases}$$

Therefore,

$$\begin{aligned} F(s) &= \int_0^1 e^{-st} t dt + \int_1^2 e^{-st} \times 1 dt + \int_2^\infty e^{-st} \times 0 dt \\ &= -\frac{1}{s}\left(te^{-st}\Big|_0^1 - \int_0^1 e^{-st} dt \right) - \frac{1}{s} e^{-st}\Big|_1^2 + 0 \\ &= -\frac{1}{s}\left[e^{-s} - 0 + \frac{1}{s}\left(e^{-st}\Big|_0^1 \right) + e^{-2s} - e^{-s} \right] \\ &= -\frac{e^{-2s}}{s} + \frac{1 - e^{-s}}{s^2}. \end{aligned}$$

Exercise 1.2

Given $F(s) = \mathcal{L}[f]$, *find* $f(t)$ *for the following functions.*

$$1. \quad \frac{4s+32}{s^2-16}, \qquad\qquad 2. \quad \frac{4}{s^2-2s-3}.$$

Solution

1.

$$F(s) = \frac{4s+32}{s^2-16} = 4 \times \frac{s}{s^2-16} + 8 \times \frac{4}{s^2-16},$$

from table 6.1 we know

$$f(t) = 4\cosh 4t + 8\sinh 4t = 6e^{4t} - 2e^{-4t}.$$

Solution

2.

$$\frac{4}{s^2-2s-3} = \frac{4}{(s-1)^2-4},$$

We define $F(s) = \frac{2}{s^2-4}$, then the function we are interested can be written as $2F(s-1)$. By using the s-Shifting theorem

$$\mathcal{L}\left[e^{\alpha t}f(t)\right] = F(s-\alpha),$$

we know

$$2\mathcal{L}\left[e^{t}f(t)\right] = 2F(s-1),$$

where we can find from table 6.1 that $f(t) = \sinh 2t$. The inverse transform for the function is therefore

$$2e^{t}\sinh 2t = e^{3t} - e^{-t}.$$

Exercise 2.1

Find $f(t)$ *if* $\mathcal{L}\left[f\right]$ *equals*

$$\frac{2\left(e^{-s}-e^{-3s}\right)}{\left(s^2-4\right)}.$$

Solution.

$$\begin{aligned}\mathcal{L}^{-1}\left[\frac{2\left(e^{-s}-e^{-3s}\right)}{(s^2-4)}\right] &= \mathcal{L}^{-1}\left[\frac{2}{s^2-4}e^{-s}\right]-\mathcal{L}^{-1}\left[\frac{2}{s^2-4}e^{-3s}\right] \\ &= \sinh\left[2(t-1)\right]u(t-1)-\sinh\left[2(t-3)\right]u(t-3),\end{aligned}$$

where we use the time-shifting theorem and the fact that

$$\mathcal{L}^{-1}\left[\frac{2}{s^2-4}\right]=\sinh 2t.$$

Exercise 3.1

Find the solution to the initial value problem.

$$y'' + 4y' + 5y = \delta(t-1), \quad y(0) = 0,\ y'(0) = 3.$$

Solution

From the Laplace transformation of derivatives (Sec. 6.2) and Dirac's delta function (Sec. 6.4), we know

$$\left[s^2 Y - sy(0) - y'(0)\right] + 4\left[sY - y(0)\right] + 5Y = e^{-s}$$
$$\Rightarrow Y = \frac{3 + e^{-s}}{s^2 + 4s + 5} = 3\frac{1}{(s+2)^2 + 1} + e^{-s}\frac{1}{(s+2)^2 + 1}.$$

By using the the inverse transform of $\sin t$, s-shifting and t-shifting theorem, we know

$$y = 3e^{-2t}\sin t + u(t-1)e^{-2(t-1)}\sin(t-1).$$

Exercise 3.2

Find the solution to the initial value problem.

$$y'' + 3y' + 2y = 10\left[\sin t + \delta(t-1)\right], \quad y(0) = 1,\ y'(0) = -1.$$

Solution

From the Laplace transformation of derivatives (Sec. 6.2) and Dirac's delta function (Sec. 6.4), we know

$$\left[s^2 Y - sy(0) - y'(0)\right] + 3\left[sY - y(0)\right] + 2Y = 10\frac{1}{s^2+1} + 10e^{-s}$$

$$\Rightarrow (s^2 + 3s + 2)Y = (s+3) - 1 + 10\frac{1}{s^2+1} + 10e^{-s}$$

$$\Rightarrow Y = \frac{1}{(s+1)} + \frac{10}{(s+1)(s+2)(s^2+1)} + \frac{10e^{-s}}{(s+1)(s+2)},$$

and $y = \mathcal{L}^{-1}\left[Y\right]$. We know the first term is

$$\mathcal{L}^{-1}\left[\frac{1}{s+1}\right] = e^{-t}.$$

Solution

For the third term, since

$$\frac{10e^{-s}}{(s+1)(s+2)} = 10e^{-s}\left(\frac{1}{s+1} - \frac{1}{s+2}\right), \qquad \text{therefore,}$$

$$\Rightarrow \mathcal{L}^{-1}\left[\frac{10e^{-s}}{(s+1)(s+2)}\right] = 10\left(\mathcal{L}^{-1}\left[\frac{e^{-s}}{s+1}\right] - \mathcal{L}^{-1}\left[\frac{e^{-s}}{s+2}\right]\right)$$

$$\Rightarrow \mathcal{L}^{-1}\left[\frac{10e^{-s}}{(s+1)(s+2)}\right] = 10\left[e^{-(t-1)}u(t-1) - e^{-2(t-1)}u(t-1)\right]$$

$$\Rightarrow \mathcal{L}^{-1}\left[\frac{10e^{-s}}{(s+1)(s+2)}\right] = 10u(t-1)\left[e^{-(t-1)} - e^{-2(t-1)}\right].$$

Solution

For the second term, we can let

$$\frac{10}{(s+1)(s+2)(s^2+1)} = \left(\frac{A}{s+1} + \frac{B}{s+2} + \frac{Cs+D}{s^2+1}\right).$$

Compare the coefficients, we find that

$$\begin{cases} (A+B+C)s^3 = 0 \\ (2A+B+3C+D)s^2 = 0 \\ (A+B+2C+3D)s = 0 \\ (2A+B+2D) = 10 \end{cases} \Rightarrow \begin{cases} A = 5 \\ B = -2 \\ C = -3 \\ D = 1 \end{cases}$$

Solution

Therefore

$$\mathcal{L}^{-1}\left[\frac{10}{(s+1)(s+2)(s^2+1)}\right] = 5e^{-t} - 2e^{-2t} - 3\cos t + \sin t.$$

Adding the three terms together, we obtain

$$\begin{aligned} y(t) = \mathcal{L}^{-1}\left[Y\right] &= 6e^{-t} - 2e^{-2t} - 3\cos t + \sin t \\ &\quad + 10u(t-1)\left[e^{-(t-1)} - e^{-2(t-1)}\right]. \end{aligned}$$

Exercise 4.1

Find $\mathcal{L}\left[f\right]$ or $\mathcal{L}^{-1}\left[F(s)\right]$ for the following functions

$$1. \quad \cos^2(2t), \qquad\qquad 2. \quad \frac{3s+4}{s^4+k^2s^2}.$$

Solution

1. We can solve this problem by applying the transform of derivatives.

$$\begin{aligned} f(t) &= \cos^2(2t), \quad & f'(t) &= -4\cos(2t)\sin(2t) = -2\sin(4t) \\ f(0) &= 1, & f'(0) &= -4\cos(0)\sin(0) = 0. \end{aligned}$$

From table 6.1 and Laplace transform of derivatives we know

$$\begin{aligned} \mathcal{L}[f'] &= \frac{-8}{s^2+16} = s\mathcal{L}[f] - f(0) \\ &\Rightarrow \frac{-8}{s^2+16} = s\mathcal{L}[f] - 1 \\ &\Rightarrow \mathcal{L}[f] = \frac{1}{s}\left(\frac{-8}{s^2+16} + 1\right) = \frac{1}{s}\left(\frac{s^2+8}{s^2+16}\right). \end{aligned}$$

Solution

2. We can solve this problem by applying the transform of integrals. We know that

$$\mathcal{L}^{-1}\left[\frac{3s+4}{s^2+k^2}\right] = 3\cos(kt) + \frac{4}{k}\sin(kt)$$

Using the Laplace transform of integrals, we have

$$\begin{aligned}\mathcal{L}^{-1}\left[\frac{3s+4}{s(s^2+k^2)}\right] &= \int_0^t \left[3\cos(k\tau) + \frac{4}{k}\sin(k\tau)\right] d\tau \\ &= \frac{3}{k}\sin(kt) - \frac{4}{k^2}\left[\cos(kt) - 1\right].\end{aligned}$$

Solution

2. Similarly

$$\mathcal{L}^{-1}\left[\frac{3s+4}{s^2\left(s^2+k^2\right)}\right]=\int_0^t\left\{\frac{3}{k}\sin\left(k\tau\right)-\frac{4}{k^2}\left[\cos\left(k\tau\right)-1\right]\right\}d\tau$$
$$=-\frac{3}{k^2}\left[\cos\left(kt\right)-1\right]-\frac{4}{k^3}\sin\left(kt\right)+\frac{4t}{k^2}.$$