EEE336 Signal Processing and Digital Filtering

Lecture 8 Discrete-Time Systems in Frequency Domain

8_1 FD Analyses of Systems

(Frequency Response)

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Analyses of LTI systems in transform domain

- Frequency response of LTI discrete-time system:
  - Most discrete-time signals in practice can be represented as a linear combination of sinusoidal discrete-time signals $e^{j\omega n}$ at different angular frequencies ω;

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)e^{j\omega n}d\omega$$

  - If the response of the LTI system to $e^{j\omega n}$ is known, then the response to $x[n]$ can be determined using the superposition property;

  - Therefore, we call $e^{j\omega n}$ as ***eigen function***.

Frequency response of a Discrete-time system

- For the system as follows

$$e^{j\omega_0 n} \longrightarrow \boxed{h[n]} \longrightarrow$$

$$y[n] = \sum_{k=-\infty}^{\infty} h[k] e^{j\omega_0 (n-k)}$$

$$= \underbrace{\boxed{\left(\sum_{k=-\infty}^{\infty} h[k] e^{-j\omega_0 k} \right)}}_{\boxed{H(e^{j\omega_0})}} e^{j\omega_0 n}$$

- System input: $e^{j\omega_0 n}$, a complex exponential at a specific frequency ω_0
- System output: the same exponential, at the same frequency ω_0 but weighted by a complex amplitude that is a function of this input frequency $H(e^{j\omega_0})e^{j\omega_0 n}$;
- $H\left(e^{j\omega_0}\right) = H(\omega_0)$ is called the ***frequency response*** of the system at ω_0.

Frequency response of a Discrete-time system

$$\boxed{H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}}$$

– Called the ***frequency response*** of the LTI system, providing a frequency-domain description of the system;

– $H(e^{j\omega})$ is the Fourier transform of the impulse response $h[n]$ of the system;

- It exists if h[n] is absolutely summable;
- It is a complex function of ω with a period of 2π
- It can be expressed in real/imaginary or magnitude/phase parts

$$H(e^{j\omega}) = H_{re}(e^{j\omega}) + jH_{im}(e^{j\omega}) = \underbrace{\boxed{|H(e^{j\omega})|}}_{\text{Magnitude response}} e^{j\overbrace{\boxed{\theta(\omega)}}^{\text{Phase response}}}$$

Magnitude response

Phase response

Frequency response - Example

- Example: M-Moving-average filter is given by

$$h[n] = \begin{cases} \frac{1}{M}, 0 \le n \le M-1 \\ 0, \quad\quad otherwise \end{cases}$$

  – Its frequency response is thus given by

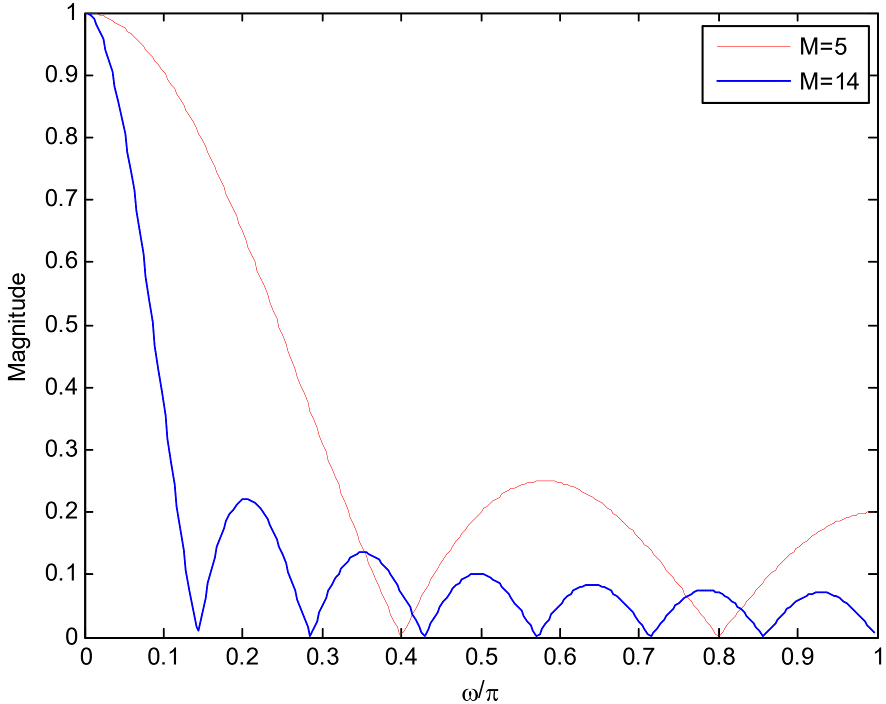
$$H(e^{j\omega}) = \frac{1}{M}\sum_{n=0}^{M-1} e^{-j\omega n} = \frac{1}{M}\left(\sum_{n=0}^{\infty} e^{-j\omega n} - \sum_{n=M}^{\infty} e^{-j\omega n}\right)$$

$$= \frac{1}{M}\left(\sum_{n=0}^{\infty} e^{-j\omega n}\right)\left(1 - e^{-jM\omega}\right) = \frac{1}{M}\frac{1 - e^{-jM\omega}}{1 - e^{-j\omega}}$$

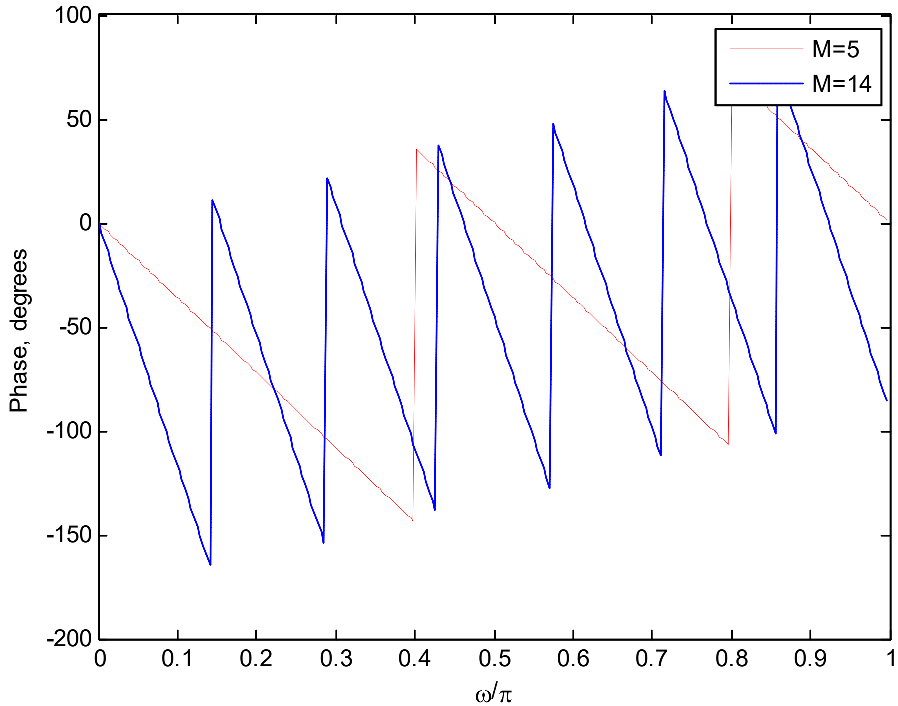
$$= \boxed{\frac{1}{M}\frac{\sin(M\omega/2)}{\sin(\omega/2)}} e^{j\boxed{\left(-\frac{(M-1)\omega}{2}\right)}}$$

Magnitude response Phase response

Frequency response - Example



Magnitude response

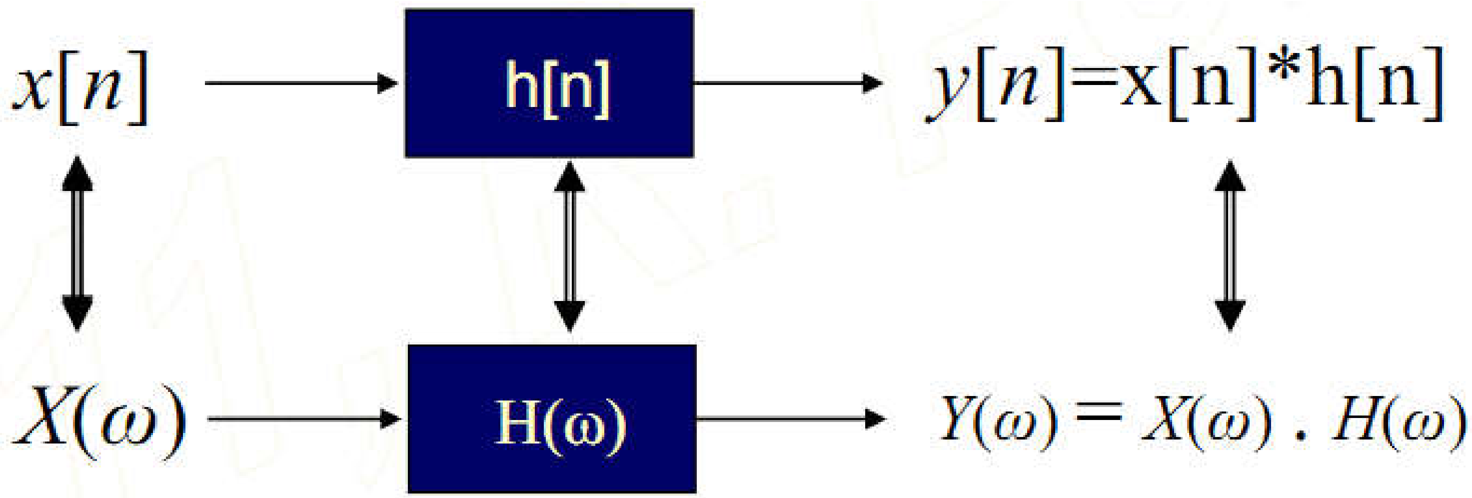


Phase response

- The magnitude has a maximum value of unity at $\omega = 0$, and has zeros at $\omega = 2\pi k/M$;
- The phase function exhibits discontinuities of π at each zero of the magnitude and is linear elsewhere with a slope of $-(M\text{-}1)/2$;
- Both magnitude and phase functions are periodic in ω with a period $2\,\pi$.

Time-Frequency Domain Relationship

- If $x[n]$ is input to an LTI system with an impulse response of $h[n]$, then the DTFT of the output is the product of $X(\omega)$ and $H(\omega)$



Transfer function of LTI

- Besides $e^{j\omega n}$, another commonly used eigen function of LTI system is

$$x[n] = z^n, -\infty < n < \infty$$

- The output of the LTI system is

$$z^n \longrightarrow \boxed{h[n]} \longrightarrow y[n] = \sum_{k=-\infty}^{\infty} h[k] z^{n-k} \quad k]$$

$$= \boxed{\left(\sum_{k=-\infty}^{\infty} h[k] z^{-k} \right)} z^n$$

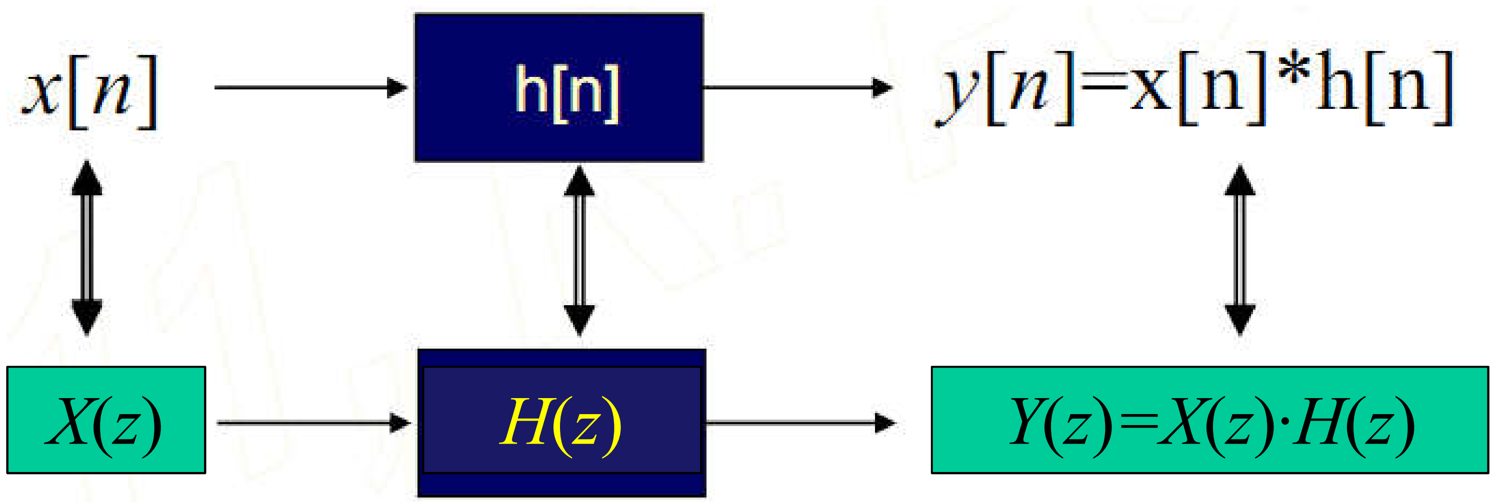
$$\downarrow$$

$$\boxed{H(z)}$$

  - Output $y[n]$ is the input z^n multiplied by $H(z)$;
  - $H(z)$ is the transfer function of LTI system.

Time-Transform Domain Relationship

- If $x[n]$ is input to an LTI system with an impulse response of $h[n]$, then the z-transform of the output is the product of $X(z)$ and $H(z)$



8_1 Wrap up

- What's the frequency response of a system?
  - The frequency response H(ω) is the DTFT of the impulse response h[n];
  - Concept of "eigen function"
- Convolution theorem of DTFT:
  - TD convolution <=> FD multiplication
- Frequency domain: DTFT
- Transform domain: z-transform

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Lecture 8 Discrete-Time Systems in Frequency Domain

8_2 FD Example - Filtering

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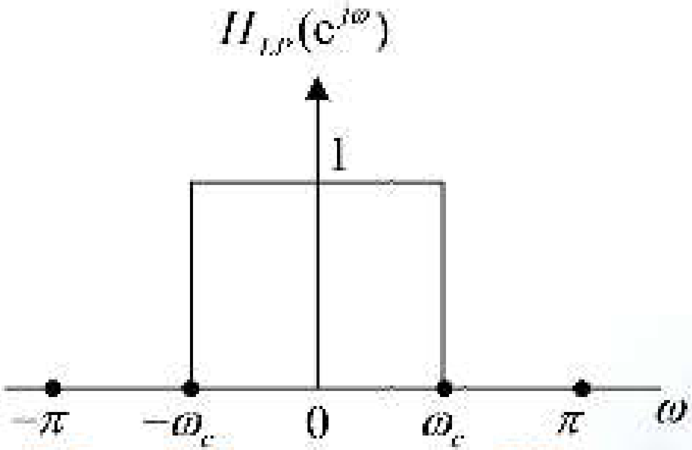
The concept of filtering

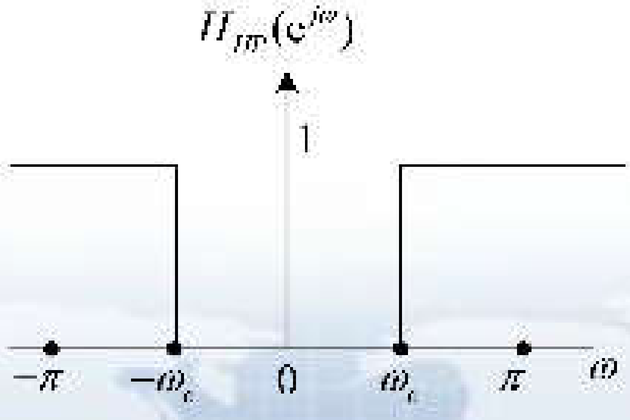
- What is filtering?
  - To pass certain frequency components in an input sequence (without any distortion, if possible);
  - To block other frequency components (without leakage, if possible).

- Why can an LTI system be used for filtering?
  - $x[n]$ is a weighted sum of the eigen function $e^{j\omega n}$ for ω in $[-\pi, \pi]$;
  - LTI output for $e^{j\omega n}$ is given by $\left|H\left(e^{j\omega}\right)\right|e^{j\omega n}$;
  - Appropriately choosing $\left|H\left(e^{j\omega}\right)\right|$ can pass or block certain frequency components in the input signal.

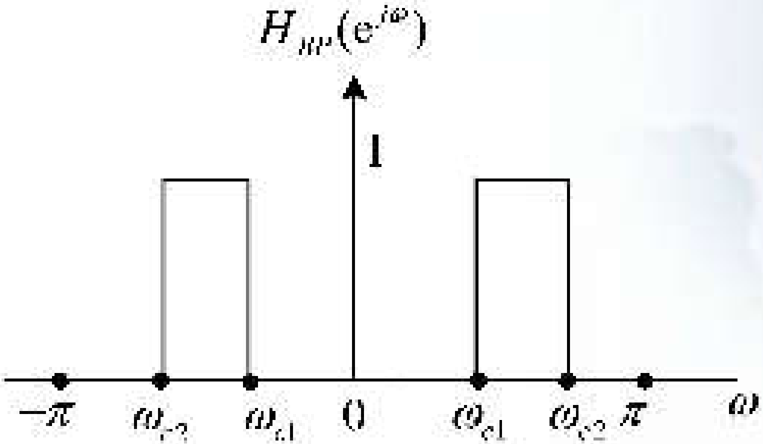
The concept of filtering



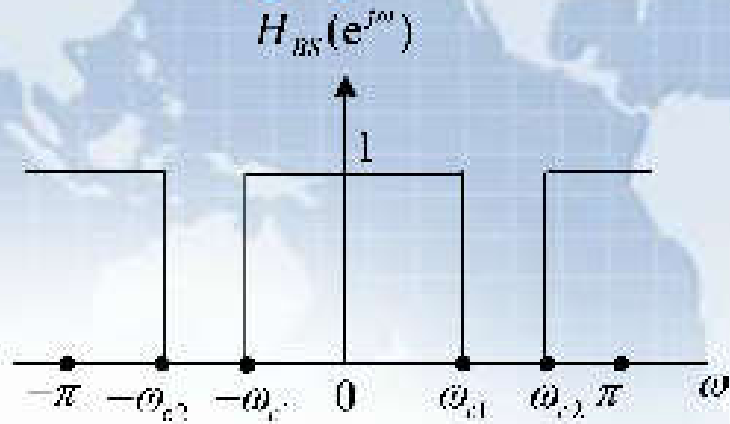
Ideal lowpass filter



Ideal highpass filter



Ideal bandpass filter



Ideal bandstop filter

Example 1 of filter – FIR filter design

- Consider an input signal $x[n] = \cos(0.1n) + \cos(0.4n)$, design an FIR filter to pass the high-frequency component and block the low-frequency component.
  - We want our filter to be as simple as possible, so let's assume that we have length 3, symmetric impulse response filter. That is, $h[0] = h[2] = a$, and $h[1] = b$.
  - Then our filter should have a frequency response of the form:

$$H(\omega) = h[0]e^{-j\omega 0} + h[1]e^{-j\omega 1} + h[2]e^{-j\omega 2}$$

$$= a + be^{-j\omega} + ae^{-j\omega 2} = a(1 + e^{-j\omega 2}) + be^{-j\omega}$$

$$= a \cdot 2e^{-j\omega}\frac{e^{j\omega} + e^{-j\omega}}{2} + be^{-j\omega}$$

$$= (2a\cos(\omega) + b)e^{-j\omega}$$

Example 1 of filter – FIR filter design (cont.)

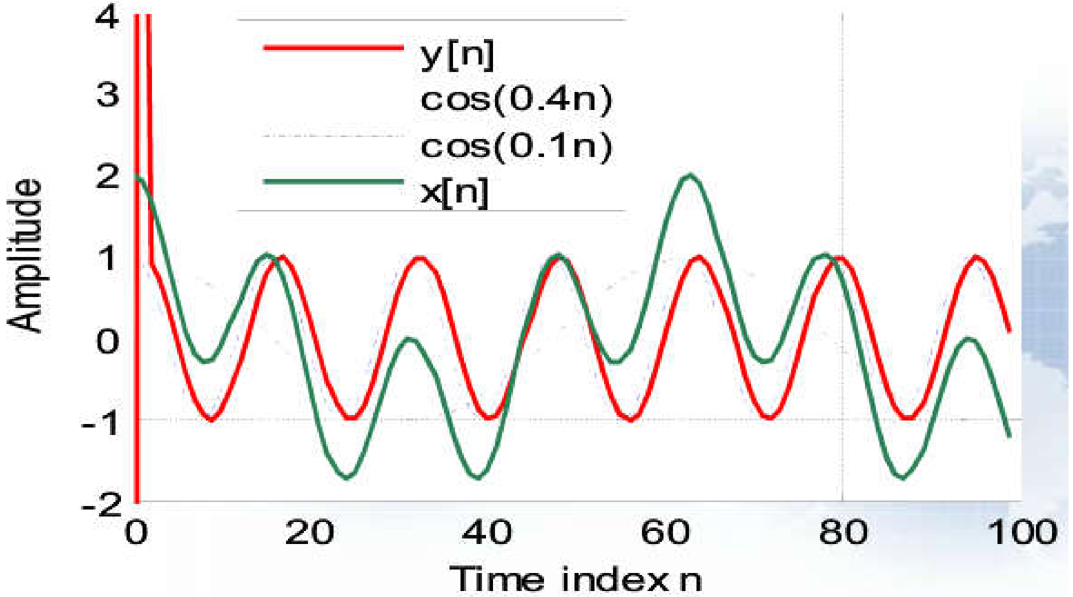
$$\boxed{H(\omega) = (2a\cos(\omega) + b)e^{-j\omega}}$$

- To block low frequency component: $2a\cos(0.1) + b = 0$
- To pass high frequency component: $2a\cos(0.4) + b = 1$

$\Rightarrow$ $a = -6.76$, $b = 13.46$

– So the system can be represented by:

$$y[n] = -6.76(x[n] + x[n-2]) + 13.46x[n-1]$$



(a) Time domain plot

n	$\cos(0.1n)$	$\cos(0.4n)$	$x[n]$	$y[n]$
0	1.0	1.0	2.0	−13.52390
1	0.9950041	0.9210609	1.9160652	13.956333
2	0.9800665	0.6967067	1.6767733	0.9210616
3	0.9553364	0.3623577	1.3176942	0.6967064
4	0.9210609	−0.0291995	0.8918614	0.3623572
5	0.8775825	−0.4161468	0.4614357	−0.0292002
6	0.8253356	−0.7373937	0.0879419	−0.4161467

(b) Time domain data

Example 2 of filter - Noise suppression

- Consider a noisy input $x[n]$ given by

$$x[n] = \boxed{s[n]} + \boxed{d[n]}$$

Signal: $s[n] = 2n * 0.9^n$

Noise: different types of noises

  - a) Apply the moving-average filter to suppress the noise $d[n]$;

$$y[n] = \frac{1}{M}\sum_{l=0}^{M-1} x[n-l]$$

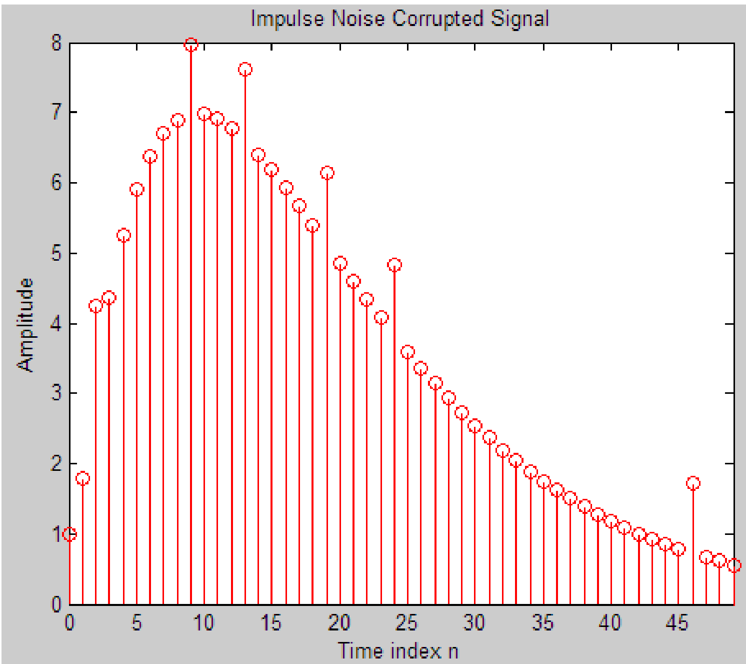
  - b) Apply the median filter

$$y[n] = med\{x[n-K], \dots, x[n+K]\}$$

which chooses the median value over the $(2K+1)$-length window $\{x[n-K], \dots, x[n+K]\}$ as the output.

Example 2 of filter - Noise suppression (cont.)

For randomly occurred noise (pepper and salt noise):

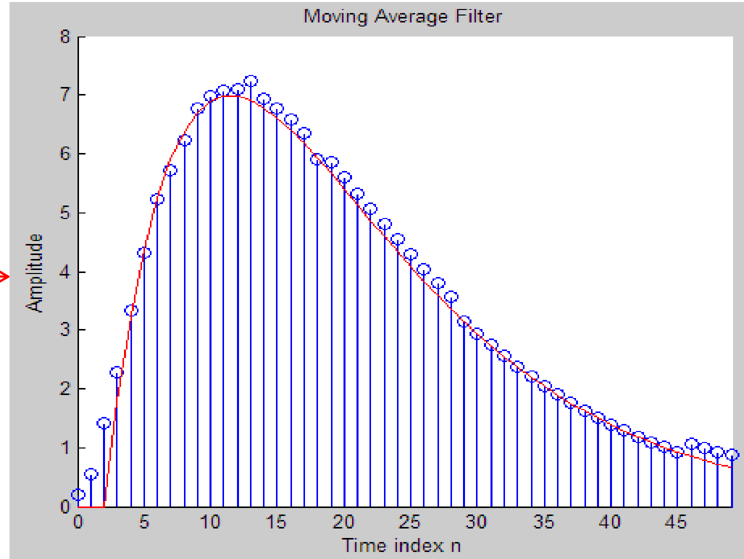


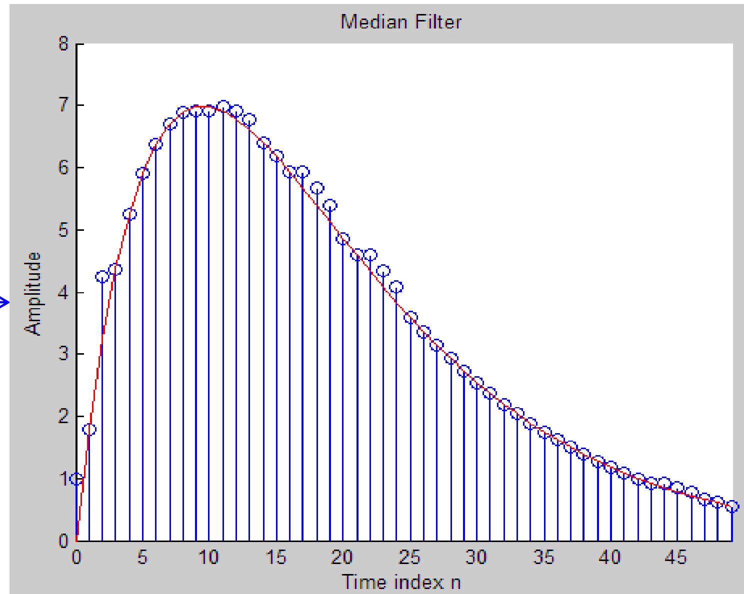
M=5

Moving average Filter

K=5

Median Filter





- Median filter is better than moving-average filter
- Median filter is NOT LTI system

Try uniformly distributed noises by yourselves!

8_2 Wrap up

- What is filtering?
- Four types of filters
  - LP, HP, BP, BS
- Examples of typical filters
  - FIR filter
  - MAF filter and Median filter

Chapter 8 Summary

- Discrete-Time System in ***frequency*** domain
  - Analyses of discrete-time systems in transform domain
    - Frequency response and DTFT
      - Example of filters
    - Time-frequency domain relationship of continuous and discrete systems
    - Time-transform domain relationship
  - Applications of LTI systems for filtering
    - Concept of filtering
    - Noise suppression