EEE225 Advanced Electrical Circuits and Electromagnetics

Lecture 7 Transient Analysis – 1^{st} order circuits

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Content

- Review of capacitor and inductor
  - Capacitor
  - Inductor
- First order circuit
  - Transient and steady-state
  - Source free circuit –> natural response
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    - Solving the 1st order differential equation
    - RL, RC
    - Complete response

Review of Capacitor and Inductor

- Keywords
  - Passive elements
    - Active elements
  - Storage elements
    - Dissipative elements
  - Linear element
    - Non-linear elements

Capacitor – C

- A capacitor constructed of two parallel conducting plates of area A, separated by a distance d, has a capacitance

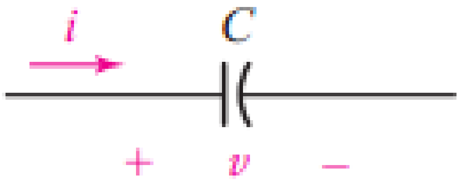
$$C = \frac{\varepsilon A}{d} = \frac{Q}{V}$$

  - where ε is the permittivity, Q is the charges stored in the capacitor, and V is the voltage between the two plates.
  - Unit: F (farad) as one coulomb per volt

- C is define by the voltage-current relationship

$$i = C\frac{dv}{dt}$$

  - where v and i are functions of time.



Capacitor – C

- The power delivered to a capacitor

$$p = vi = Cv\frac{dv}{dt}$$

- If the capacitor starts with zero-energy and zero-voltage

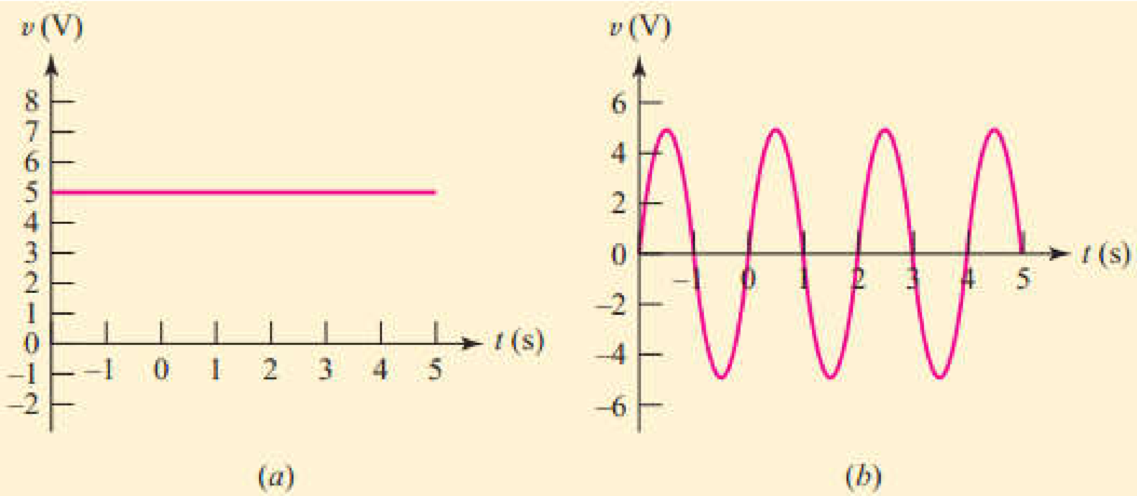
$$w_C(t) = \frac{1}{2}Cv^2$$

- Examples (Ref. 1, p219)

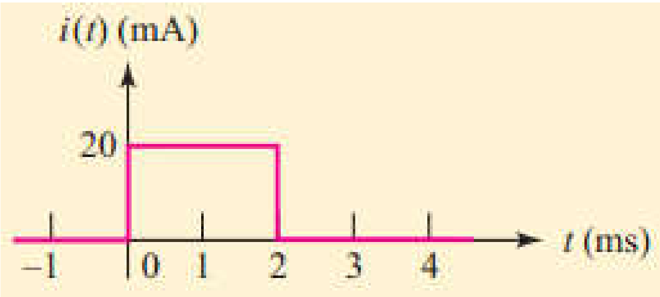
The energy is stored in the electric field between the plates.

Capacitor – Example

- Determine the current ***i*** flowing through the capacitor for the two voltage waveforms following figures if $C = 2$ F.



- Find the capacitor voltage that is associated with the current shown below. The value of the capacitance is 5 μF.



Inductor – L

- A physical inductor may be constructed by winding a length of wire into a coil. For example, an inductor that has the form of a long helix of very small pitch has an inductance of

$$L = \frac{\mu N^2 A}{l}$$

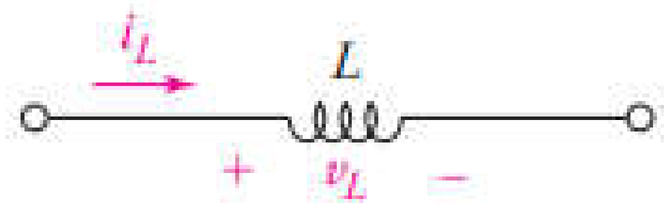
  – Where A is the cross-sectional area, l is the axial length of the helix, N is the number of complete turns of wire, and μ is the permeability of the material inside the helix,

  – Unit: H (henry) as one volt-second per ampere

- L is also define by the voltage-current relationship

$$v = L\frac{di}{dt}$$

  – where v and i are functions of time.



Inductor – L

The energy is stored in the magnetic field around the coil.

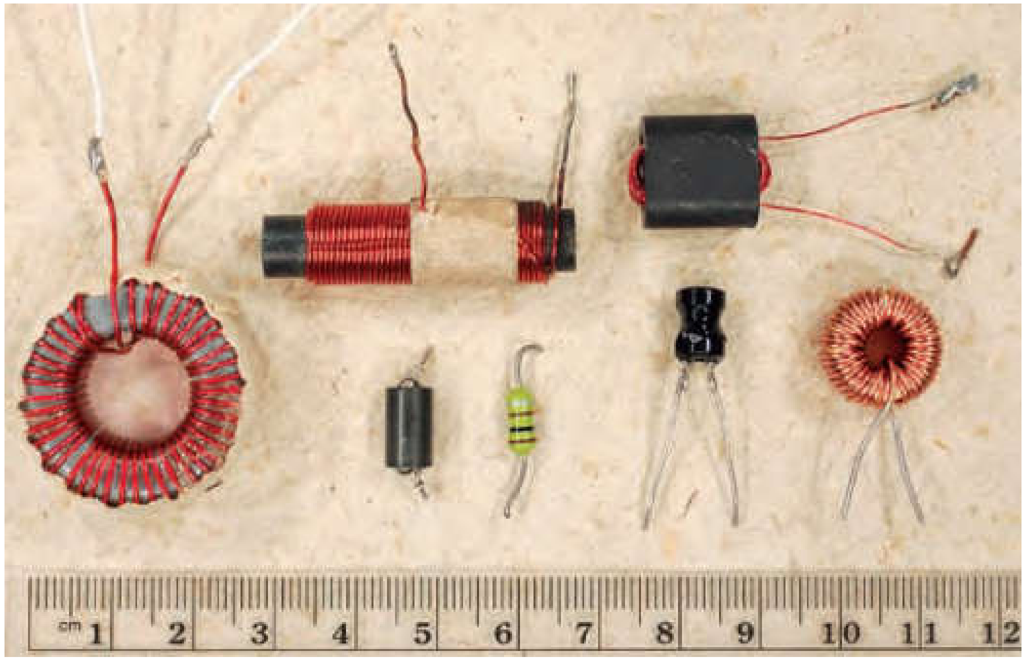
- The power delivered to an inductor

$$p = vi = Li\frac{di}{dt}$$

- If the inductor starts with zero-energy and current

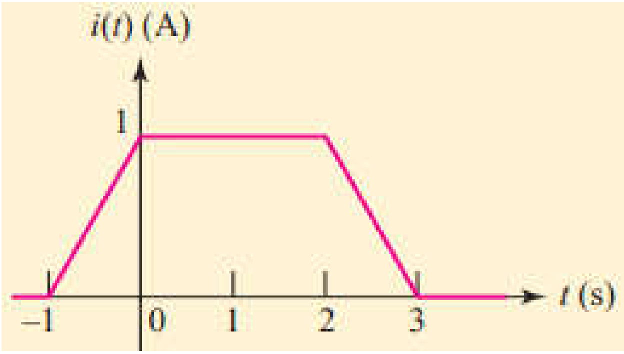
$$w_L(t) = \frac{1}{2}Li^2$$

- Examples (Ref. 1, p219)



Inductor – Example

- Given the waveform of the current in a 3 H inductor as shown below, determine the inductor voltage and sketch it.



- The current waveform of figure above has equal rise and fall times of duration. Calculate the maximum positive and negative voltages across the same inductor if the rise and fall times, respectively, are changed to
  - (a) 1 ms, 1 ms; (b) 12 μs , 64 μs ;
  - (c) 1 ns, 1 ns; (d) 0s, 0s.

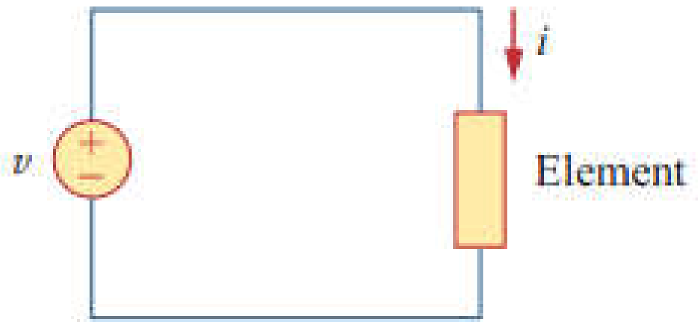
Summary

- The capacitor is the only device other than a battery that can store electrical charge.
  - A capacitor is an open circuit to dc.
  - The voltage on a capacitor cannot change abruptly.

- An inductor is an electrical component that opposes any change in electrical current.
  - An inductor acts like a short circuit to dc.
  - The current through an inductor cannot change instantaneously.

Quiz

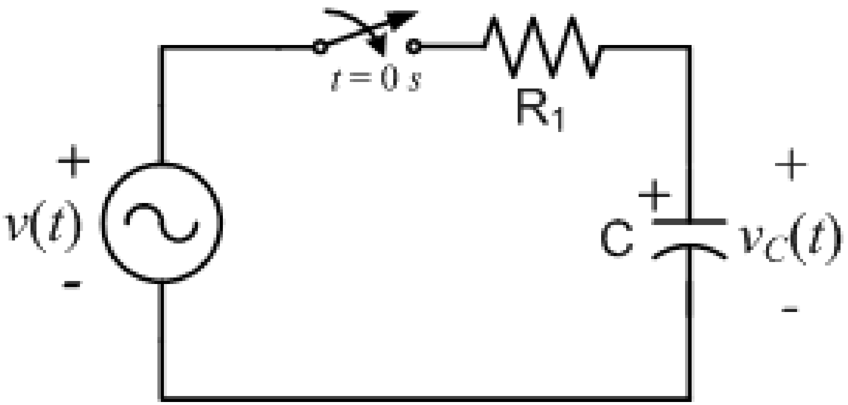
- 1. In the following figure, if $i = cos4t$ and $v = sin4t$, the element is:
  - (a) a resistor;
  - (b) a capacitor;
  - (c) an inductor.

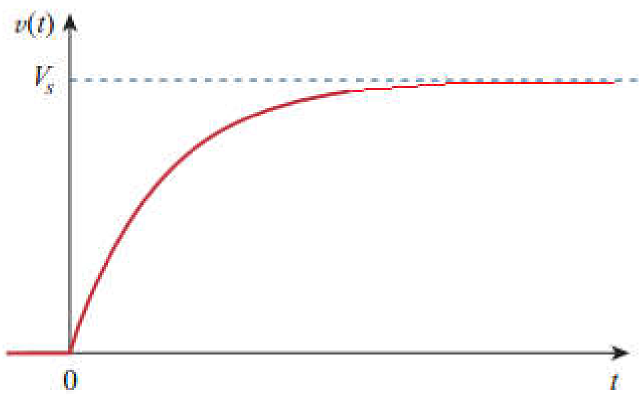


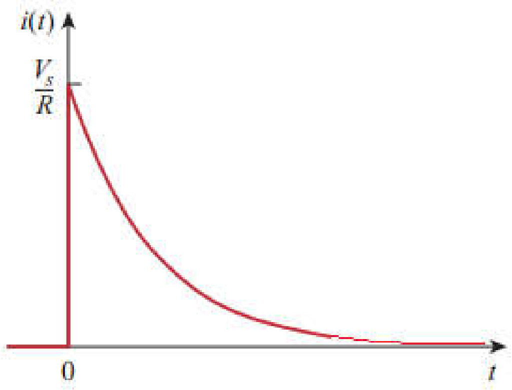
- 2. If the current through a 10 mH inductor increases from zero to 2 A, how much energy is stored in the inductor?
  - (a) 40 mJ; (b) 20 mJ;
  - (c) 10 mJ; (d) 5 mJ.

Transient

- Transient (n.)
  - When a d.c. voltage is applied to a capacitor C and resistor R connected in series, there is a short period immediately after the voltage is connected, during which the current flowing in the circuit and voltages across C and R are changing. These changing values are called ***transients***.
- Steady-state
  - After a longer while, when the capacitor C is fully charged, it acts as open-circuit, and there is no current flow in the circuit.

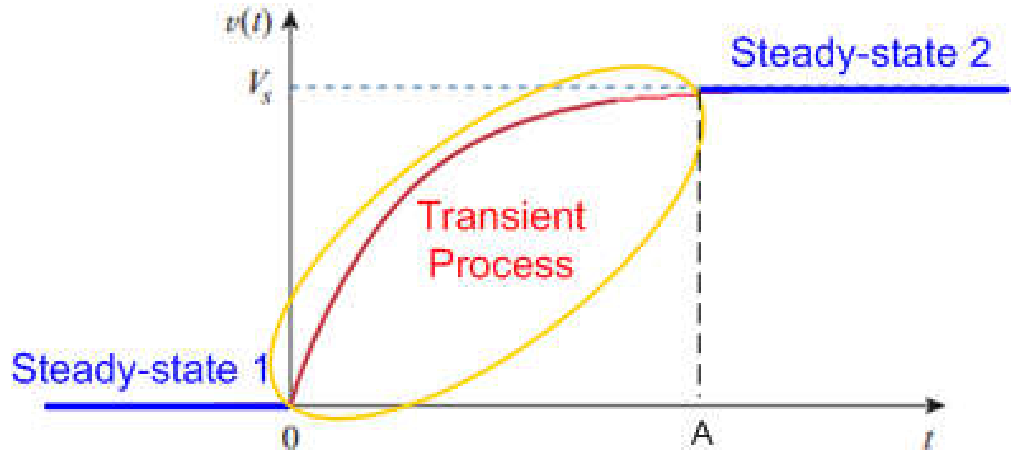






Transient

1. After the switching, there is a transient process in the circuit.
2. The transient characteristics of the circuit describes the behavior of the circuit during the transition from one steady state condition to another.



$t = 0$: the instant of switching

$\boxed{t = 0}$

$0 < t < A$: transient process

$\boxed{t = A}$

time

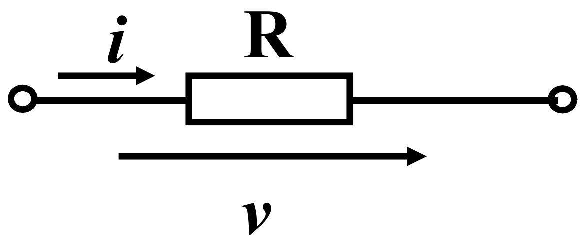
$t < 0$: original steady state

$t = 0$-: final values of the original steady state

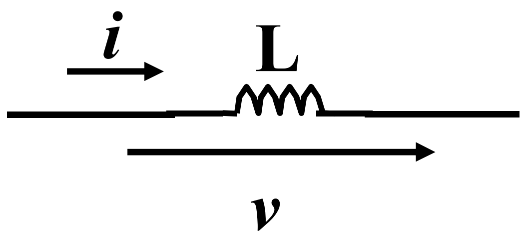
$t = 0_+$: initial values after switching, used in the differential equations as the initial values

$t > A$: new steady state

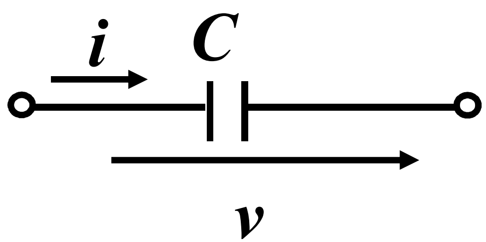
Review of R, L and C



$$v = Ri$$



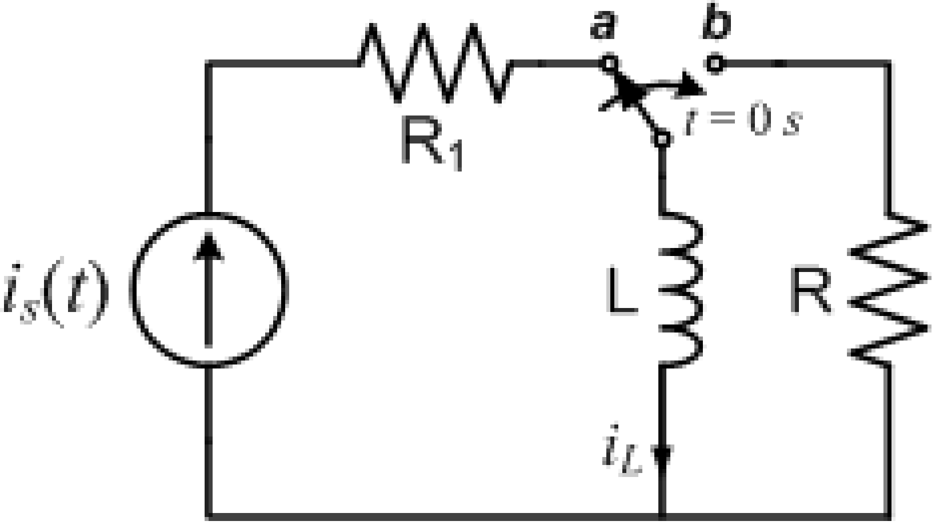
$$v = L\frac{di}{dt}$$



$$i = C\frac{dv}{dt}$$

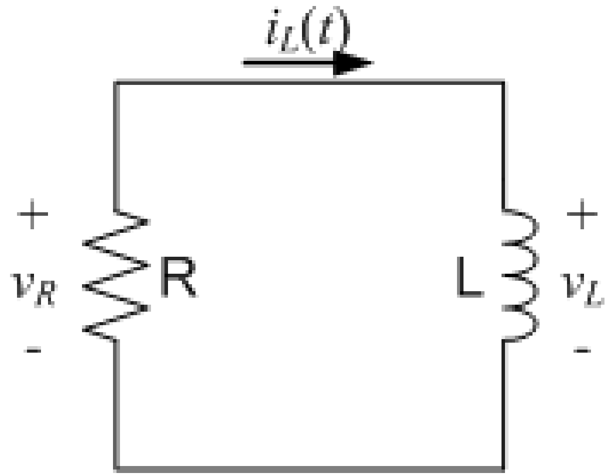
- Inductors and capacitors are energy storage elements
- Circuits that contain inductors and capacitors can be represented by differential equations.
- The order of the circuit is equal to the number of capacitors AND inductors in the circuit.
- A first order circuit has only one energy storage element, i.e. one inductor OR one capacitor.
- There are two types of first order circuits: RL circuit and RC circuit

Source free circuit - RL circuit



- $t < 0$, the circuit has a current source, a resistor R_1, a switch and an inductor;
  - Assuming the circuit is in steady-state, so the inductor is working as an ideal wire, and the current $i_L(t) = i_s(t)$, when $t < 0$.
- The switch is moved from ***a*** to ***b*** at $t = 0$;
- $t > 0$, there is no source in the circuit => source-free circuit.
  - The response is due to the initial energy stored in the inductor (or capacitor) and the physical characteristics of the circuit, not due to some external voltage or current source;
  - So called "***natural response***" - *refers to the behavior (in terms of voltages and currents) of the circuit itself, with no external sources of excitation.*

Source free circuit - RL circuit



- Considering the simple RL circuit
- Assuming the value of $i_L(t)$ at $t = 0$ as I_0;
  - In other words, $i_L(0) = I_0$.
- Applying the KVL:

$$Ri_L + v_L = Ri_L + L\frac{di_L}{dt} = 0 \Rightarrow \frac{di_L}{dt} + \frac{R}{L}i_L = 0$$

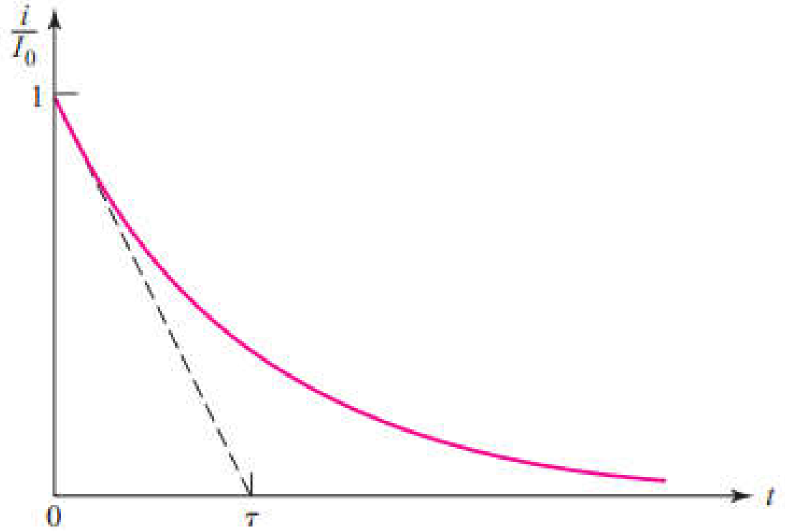
  - Find an expression for $i_L(t)$ which satisfies this equation and also has the value I_0 at $t = 0$.
- Solving the equation, get:

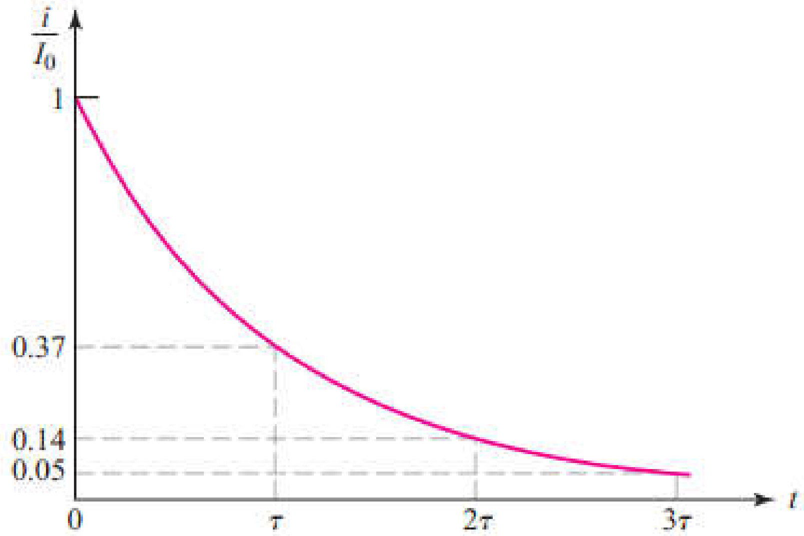
$$i_L(t) = I_0 e^{-Rt/L} = I_0 e^{-t/\tau}$$

  - where $\tau = L/R$, is the time constant for the RL circuit.

Source free circuit - RL circuit

- Time constant τ.
  - i is the current in circuit, $i(t)=i_L(t)$.





$$\frac{d}{dt}\frac{i}{I_0}\bigg|_{t=0} = -\frac{R}{L}e^{-Rt/L}\bigg|_{t=0} = -\frac{R}{L}$$

$$\left(\frac{R}{L}\right)\tau = 1$$

t	$e^{-t/\tau}$
τ	3.6788×10^{-1}
2τ	1.3534×10^{-1}
3τ	4.9787×10^{-2}
4τ	1.8316×10^{-2}
5τ	6.7379×10^{-3}

Source free circuit - RL circuit

- At $t = 0^-$, the energy stored in the inductor is

$$w_L(t) = \frac{1}{2}L{I_0}^2$$

- After switching, the power being dissipated in the resistor is

$$p_R = i^2R = {I_0}^2Re^{-2Rt/L}$$

- So the total energy turned into heat in the resistor is

$$w_R = \int_0^\infty p_R dt = {I_0}^2R\int_0^\infty e^{-2Rt/L}dt$$

$$= {I_0}^2R\left(\frac{-L}{2R}\right)e^{-2Rt/L}\Big|_0^\infty = \frac{1}{2}L{I_0}^2$$

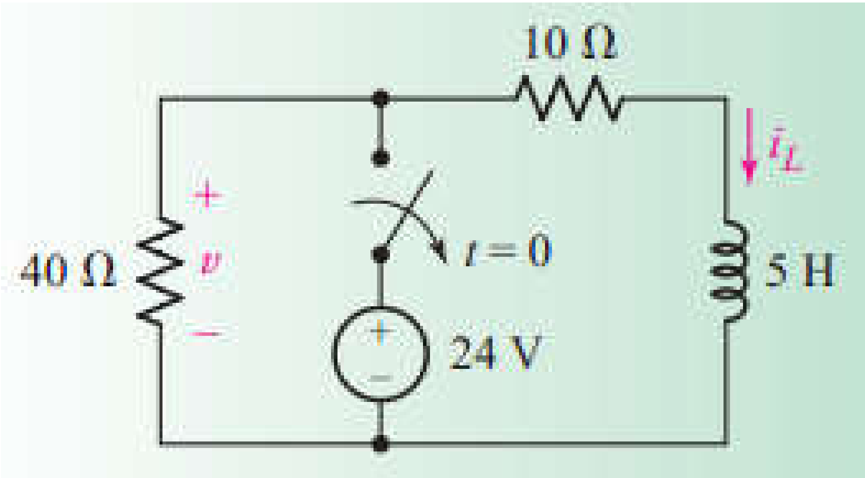
- That means all the energy initially stored in the inductor is dissipated in the resistor.

Initial condition

- First, be careful with the polarity of voltage across the capacitor and the direction of the current through the inductor.
  - Keep in mind that v and i are defined strictly according to the passive sign convention.
- Second, keep in mind that the capacitor voltage is always continuous so that $v_C(0^+)=v_C(0^-)$, and the inductor current is always continuous so that $i_L(0^+)=i_L(0^-)$.
  - where $t=0^-$ denotes the time just before a switching event and $t=0^+$ is the time just after the switching event, assuming that the switching event takes place at $t=0$.
- Thus, in finding initial conditions, we first focus on those variables that cannot change abruptly: capacitor voltage and inductor current.

Initial condition - example

- For the circuit shown in the figure, find the current labelled i_L and the voltage labelled v at $t = 200$ ms.

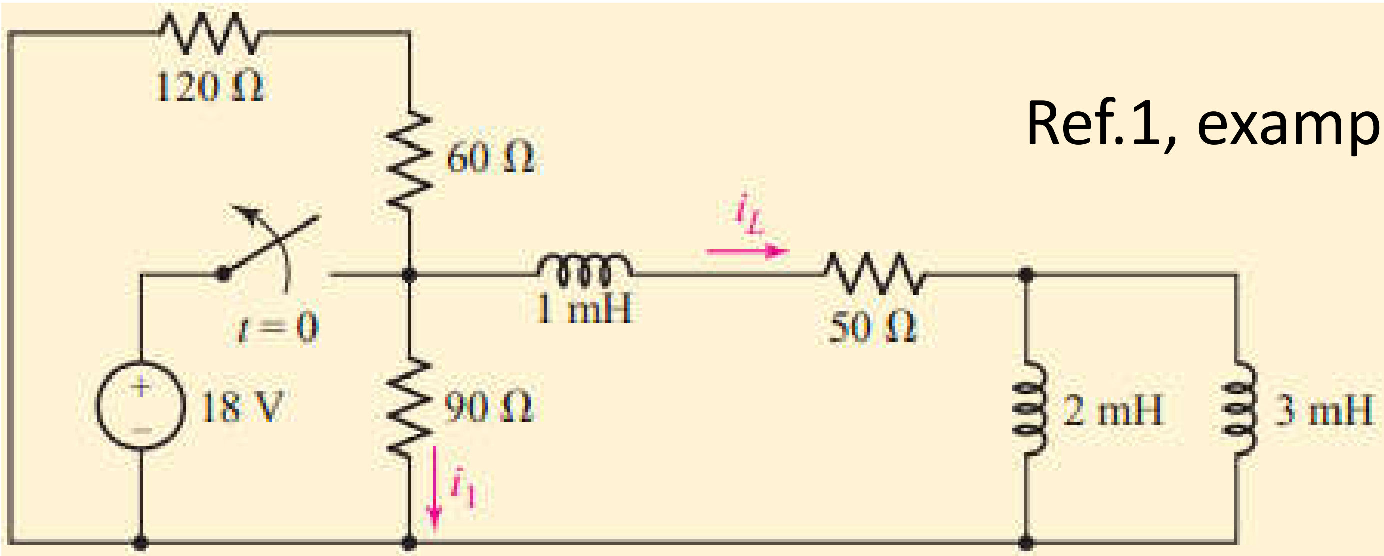


Ref.1, Example 8.2

Equivalent resistance

- Regardless of how many resistors we have in the circuit, we obtain a single time constant (either τ = L/R or τ = RC) when only one energy storage element is present.
- The value needed for R is in fact the Thévenin equivalent resistance seen by our energy storage element L or C.

- Example: Determine both i_1 and i_L in the circuit for t > 0.



Ref.1, example 8.4

Source free circuit – RC circuit

- For $t > 0$, they are source free circuits with only R and C.
- Assuming the value of $v_C(t)$ at $t = 0$ as V_0;
- Applying KCL get

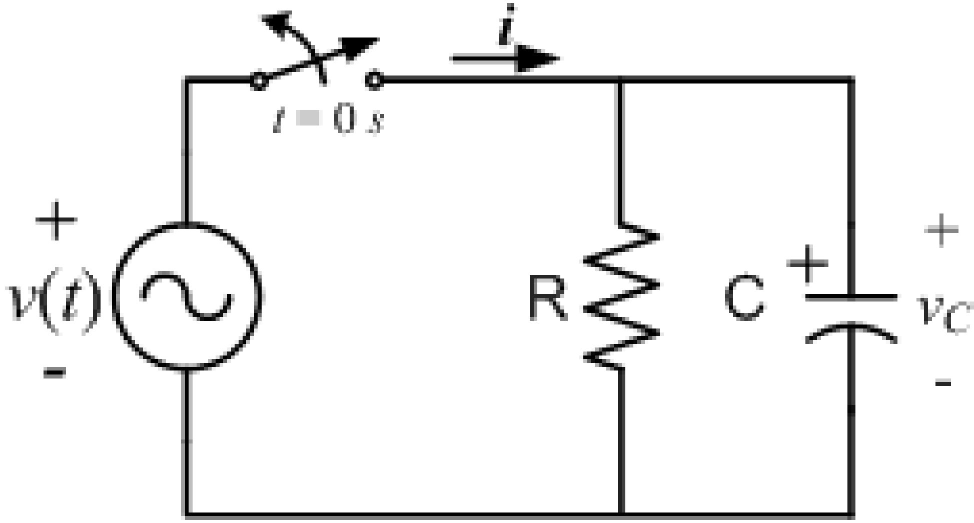
$$i_C + i_R = C\frac{dv_C}{dt} + \frac{v_C}{R} = 0$$

$$\frac{dv_C}{dt} + \frac{v_C}{RC} = 0$$

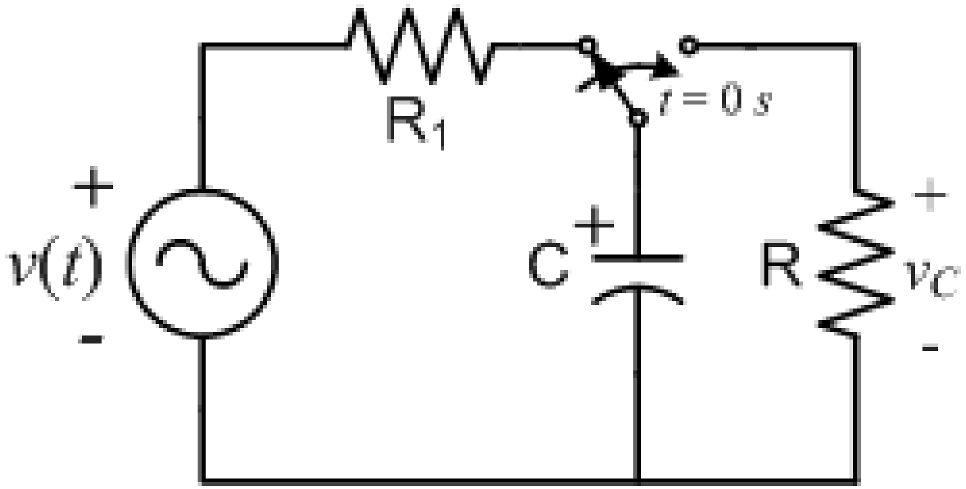
- Solving the equation get

$$v_C(t) = V_0 e^{-t/RC} = V_0 e^{-t/\tau}$$

  - where $\tau = \text{RC}$, is the time constant for the RC circuit.



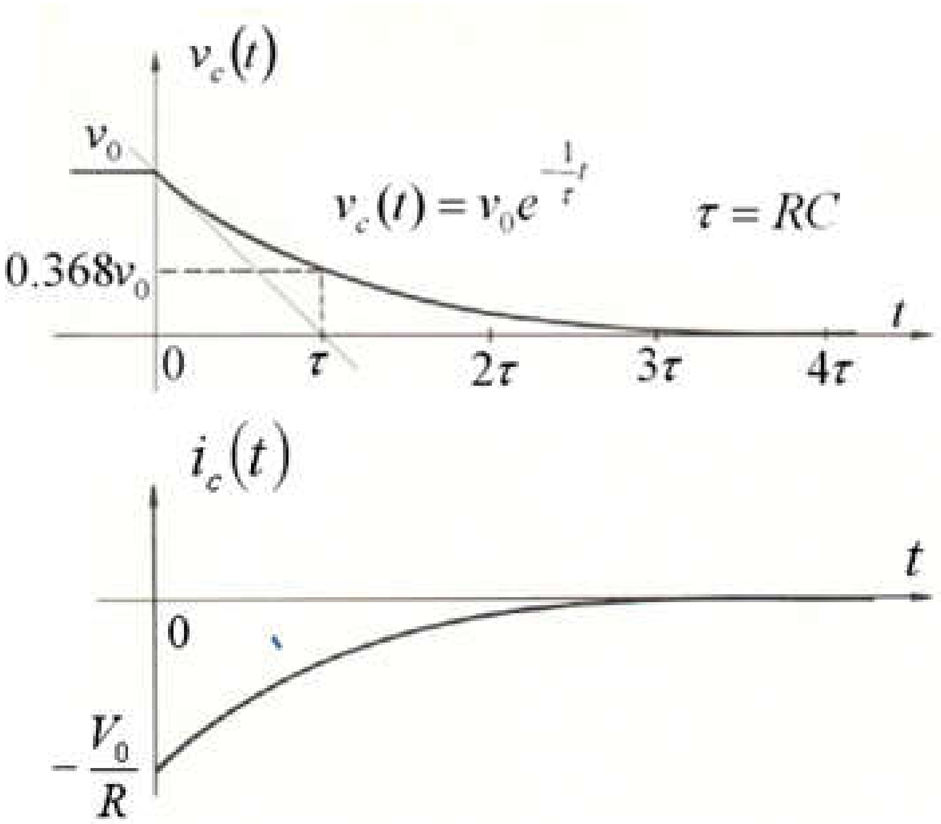
(a)

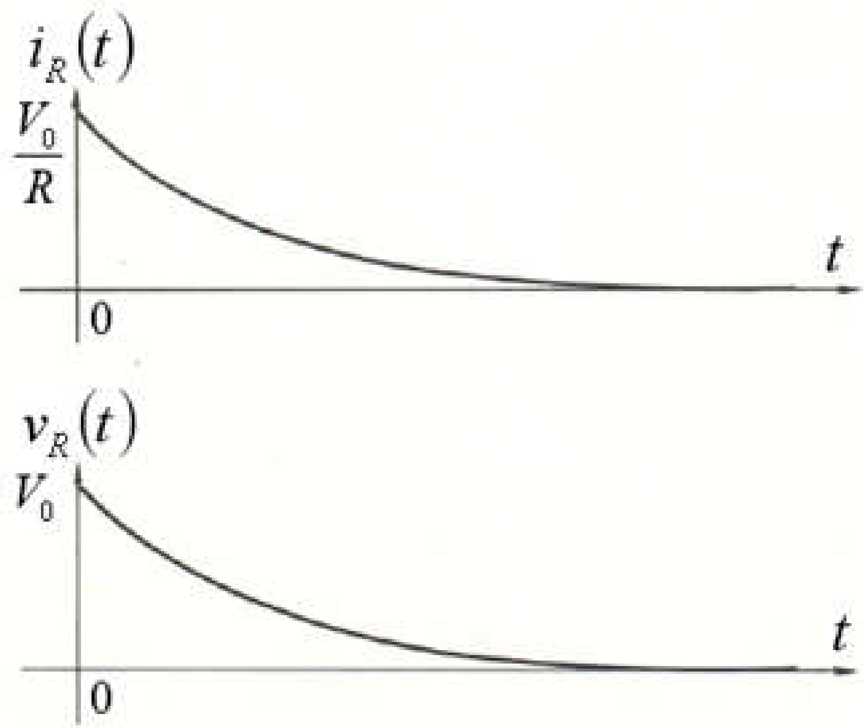


(b)

Source free circuit – RC circuit

$$v_C(t) = V_0 e^{-t/RC} = V_0 e^{-t/\tau}$$

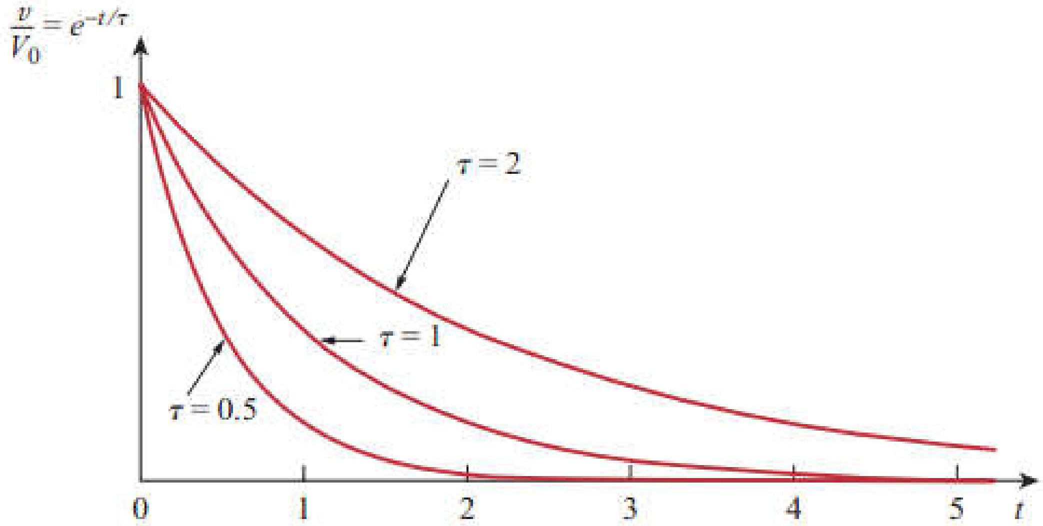




Voltage and current of the capacitor **Voltage and current of the resistor**

Source free circuit – RC circuit

- Time constant τ = RC
  - the voltage $v_C(t)$ is less than 1 percent of V_0 after 5τ.
  - Thus, it is customary to assume that the capacitor is fully discharged after five time constants. In other words, it takes 5τ for the circuit to reach its final state or steady state when no changes take place with time.
- The smaller the time constant, the more rapidly the voltage decreases, that is, the faster the response.



Source free circuit - summary

RL: $$\frac{di_L}{dt} + \frac{R}{L} i_L = 0$$

RC: $$\frac{dv_C}{dt} + \frac{1}{RC} v_C = 0$$

$$\frac{dx(t)}{dt} + \frac{x(t)}{\tau} = 0$$

Source free 1st order differential equation

- General solution for the source free first order differential equation is:

  $x(t) = x_0 e^{-t/\tau}$ ⟶ Also called the "natural response" of the circuit

  – where x_0: *Initial value*

  – τ: time constant, unit: *second*

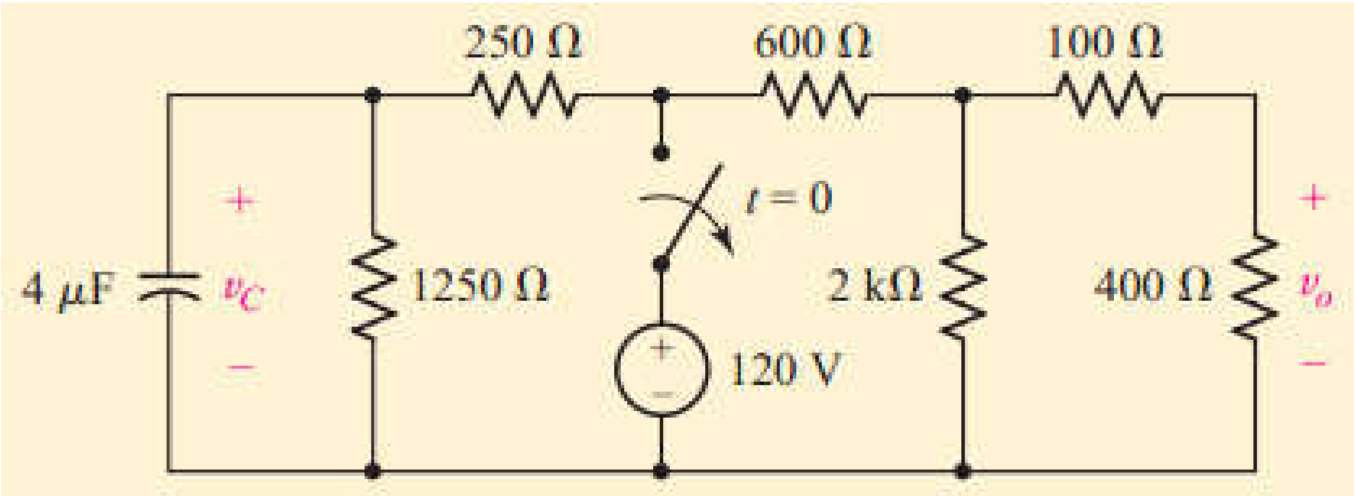
  RL: τ = L/R

  RC: τ = RC

  – Time constant represents the time it would take the voltage or current to get to it's final value if the initial rate stayed constant.

Example

- Find values of v_C and v_o in the circuit at t equal to
  - (a) 0^-; (b) 0^+; (c) 1.3 ms.

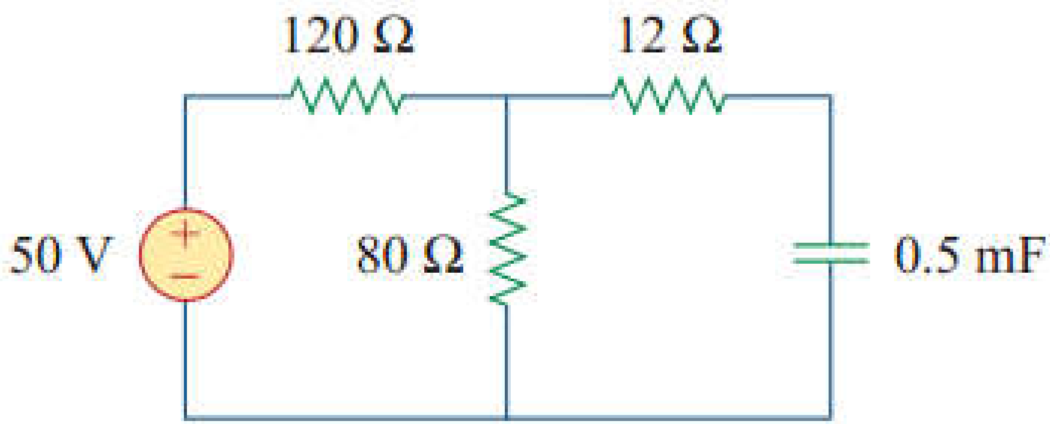


Ref.1, practice 8.6

Quiz

- A source free RL circuit has $R = 2\ \Omega$ and $L = 4$ H. The time needed for the inductor current to decrease to 36.8% of its steady-state value is:
  - (a) 0.5 s; (b) 1 s; (c) 2 s; (d) 4 s.

- The time constant for the RC circuit shown below is:
  - (a) 30 ms;
  - (b) 6 ms;
  - (c) 5.5 ms;
  - (d) 66 ms.

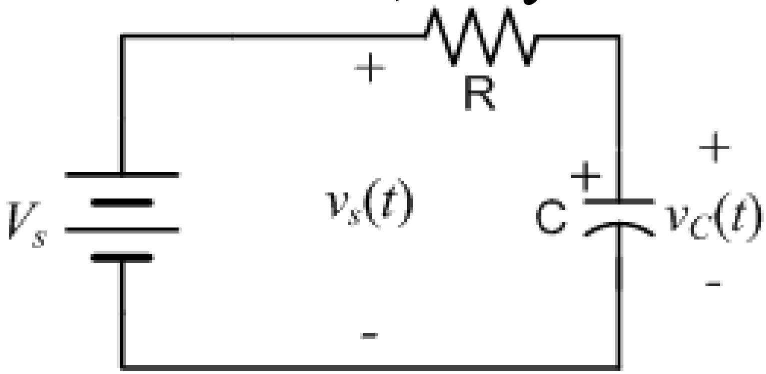


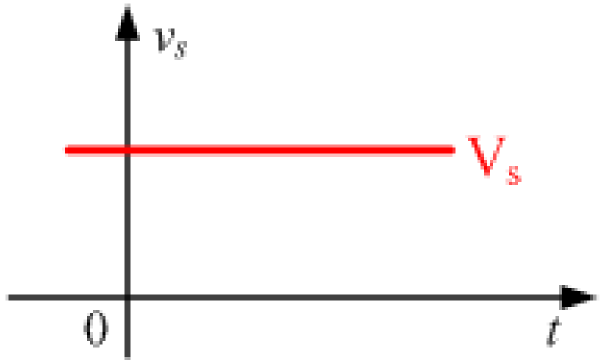
Driven circuit – circuit with source

- Source-free circuit: the circuit without independent source
  - In previous cases, we were confronted with the circuits containing sources and switches; at t = 0, switches are moved in order to remove all the sources from the circuit, while leaving known amounts of energy stored in the L or C.
  - In other words, we have been solving problems in which energy sources are suddenly ***removed*** from the circuit.
- Driven circuit: the circuit with independent source
  - Now we must consider that type of response which results when energy sources are suddenly ***applied*** to a circuit.
  - The operation of a switch in series with a battery is thus equivalent to a forcing function in the form of “step-function”.

"DC sources" VS "Step function"

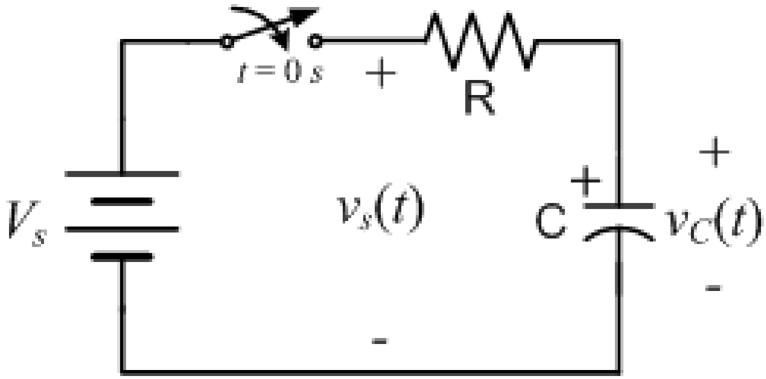
- DC sources include DC voltage source and DC current source. In this module, they are constant values all the time.

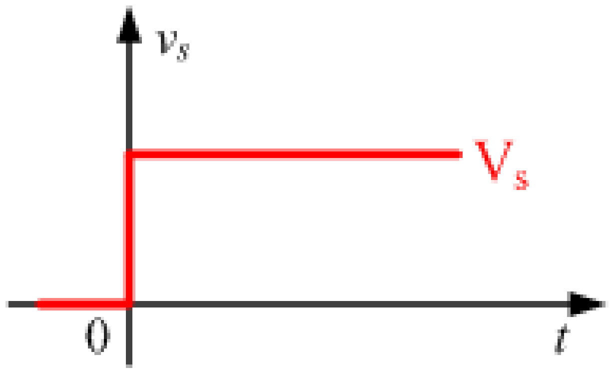




$$v_s(t) = V_s$$

- Step function: the DC sources are applied to the circuit at some time (usually at $t = 0$ s).





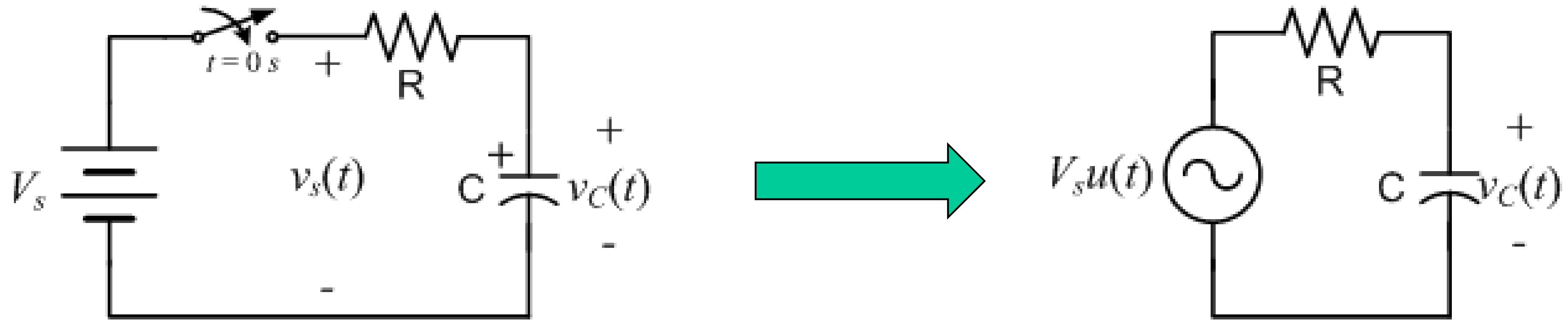
$$v_s(t) = V_s u(t)$$

  - The unit step function $u(t)$ is 0 for negative values of t and 1 for positive values of t.

Driven circuit – 1st order differential equation

Ref.3 Sec.7.5-7.6

- When the dc source of an RC circuit is suddenly applied, the voltage or current source can be modelled as a step function, and the response is known as a step response.

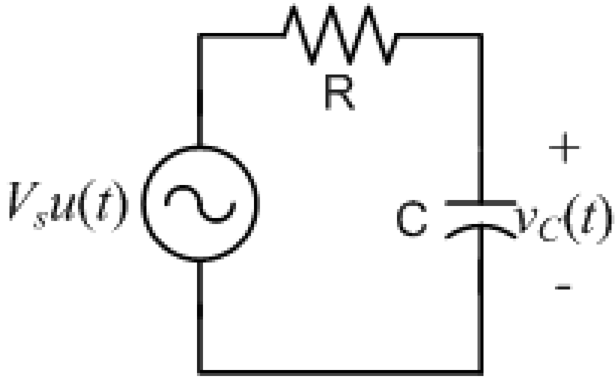


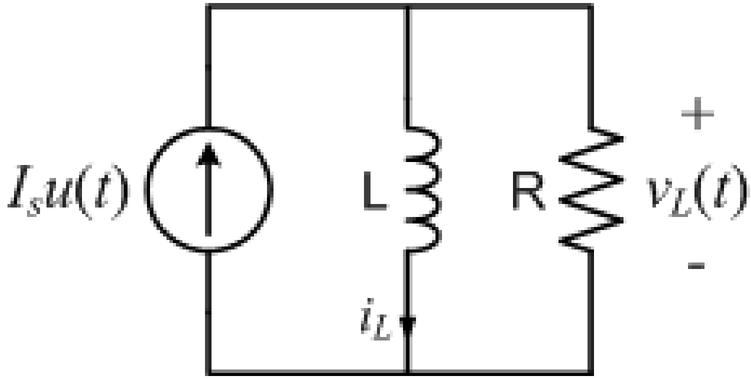
- Initial condition: $v_C(0^-) = v_C(0^+) = V_0$
- Applying KCL get $$C\frac{dv_C}{dt} + \frac{v_C - V_s u(t)}{R} = 0$$

$$\Rightarrow \frac{dv_C}{dt} + \frac{v_C}{RC} = \frac{V_s}{RC}u(t) = \frac{V_s}{RC} \quad \text{for } t > 0$$

Driven circuit – 1st order differential equation

- Compare the RC and RL circuits:





$$\frac{dv_C}{dt} + \frac{v_C}{RC} = \frac{V_s}{RC}$$

$$\frac{di_L}{dt} + \frac{i_L}{L/R} = \frac{I_s}{L/R}$$

$x(t) = v_C(t)$
$\tau = RC$
$f(t) = V_s$

$x(t) = i_L(t)$
$\tau = L/R$
$f(t) = I_s$

$$\frac{dx(t)}{dt} + \frac{x(t)}{\tau} = \frac{f(t)}{\tau}$$

x(t) is the unknown expression;
f(t) is the forcing function (source).

Solving the 1st order differential equation

$$\frac{dx(t)}{dt}+\frac{x(t)}{\tau}=\frac{f(t)}{\tau}$$

$$dx(t)+\frac{x(t)}{\tau}dt=\frac{f(t)}{\tau}dt$$

$$\boxed{dx(t)e^{t/\tau}+\frac{x(t)}{\tau}e^{t/\tau}dt}=\frac{f(t)}{\tau}e^{t/\tau}dt\boxed{=d\left[x(t)e^{t/\tau}\right]}$$

$$x(t)e^{t/\tau}=\int\frac{f(t)}{\tau}e^{t/\tau}dt\ +K$$

$$x(t)=e^{-t/\tau}\int\frac{f(t)}{\tau}e^{t/\tau}dt\ +Ke^{-t/\tau}$$

K should be determined by the initial condition of x(t) – the value of $x(0^+)$

Solving the 1st order differential equation

- The forcing function $f(t)$ can be different forms:
  - Step function with constant value X_f after t > 0, such as V_s and I_s;
  - Exponential form $f(t) = Ae^{bt}$;
  - Sinusoidal form $f(t) = Asin(\omega t)+Bcos(\omega t)$.
- Solve the differential equation with $f(t) = X_f$, a constant.

$$x(t) = e^{-t/\tau} \int \frac{X_f}{\tau} e^{t/\tau} dt \ + Ke^{-t/\tau}$$

$$= X_f e^{-t/\tau} \cdot e^{t/\tau} + Ke^{-t/\tau} = X_f + Ke^{-t/\tau}$$

- At t = 0, $X_f + Ke^{-t/\tau}\Big|_{t=0} = X_f + K = x(0)$

$$\Rightarrow K = x(0) - X_f$$

- Finally, $x(t) = X_f + \left[x(0) - X_f\right]e^{-t/\tau}$

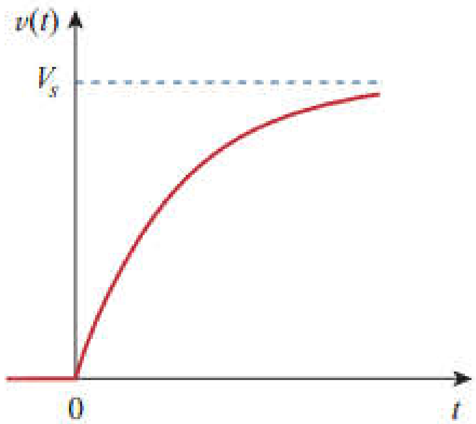
Results for RC circuit

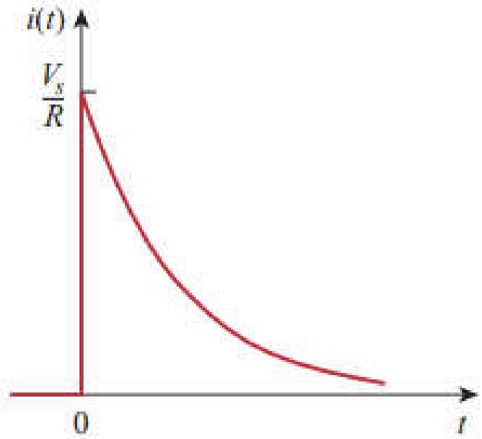
- Considering $x(t) = X_f + [x(0) - X_f]e^{-t/\tau}$ in an RC circuit:
  - $x(t)$ is the voltage on capacitor $v_C(t)$;
  - $x(0)$ is the initial voltage of the capacitor at $t = 0$;
  - X_f is the external source V_s;
  - Time constant $\tau = RC$.

$$\Rightarrow v_C(t) = V_s + [V_0 - V_s]e^{-t/RC} = \begin{cases} V_0 & t < 0 \\ V_s + [V_0 - V_s]e^{-t/RC} & t > 0 \end{cases}$$

Capacitor voltage must be continuous

- Assume the capacitor is uncharged initially: $V_0 = 0$ V.





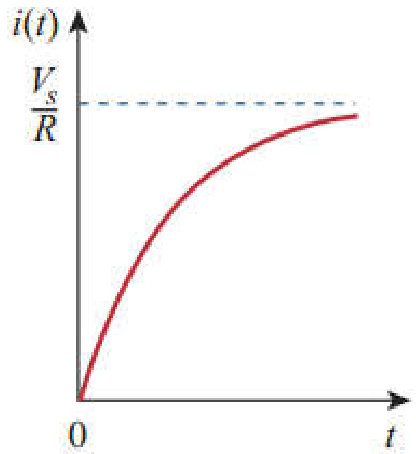
Results for RL circuit

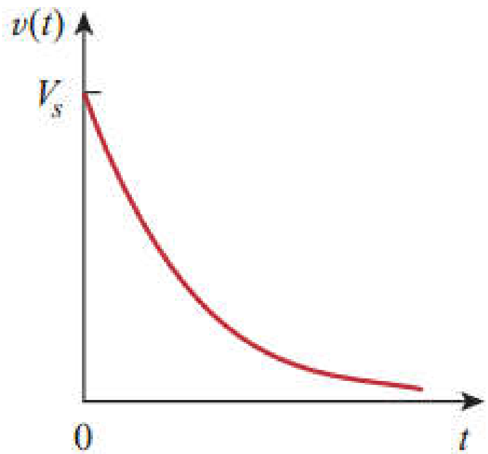
- Considering $x(t) = X_f + \left[x(0) - X_f\right]e^{-t/\tau}$ in an RL circuit:
  - $x(t)$ is the current on inductor $i_L(t)$;
  - $x(0)$ is the initial current of the inductor at $t = 0$;
  - X_f is the external source I_s;
  - Time constant $\tau = L/R$.

Inductor current must be continuous

$$\Rightarrow i_L(t) = I_s + [I_0 - I_s]e^{-Rt/L} = \begin{cases} I_0 & t < 0 \\ I_s + [I_0 - I_s]e^{-Rt/L} & t > 0 \end{cases}$$

- Assume the inductor current is zero initially: $I_0 = 0$ A.





Complete response

- Let's take the capacitor voltage as an example

$$v_C(t) = V_s + [V_0 - V_s]e^{-t/RC}$$

- Classically there are two ways of decomposing this into two components.

Method 1:

$$\boxed{\text{Complete response} = \underset{\text{stored energy}}{\text{natural response}} + \underset{\text{independent source}}{\text{forced response}}}$$

$$v = v_n + v_f$$

$$v_n = V_o e^{-t/\tau} \qquad v_f = V_s(1 - e^{-t/\tau})$$

Method 2:

$$\boxed{\text{Complete response} = \underset{\text{temporary part}}{\text{transient response}} + \underset{\text{permanent part}}{\text{steady-state response}}}$$

$$v = v_t + v_{ss}$$

$$v_t = (V_o - V_s)e^{-t/\tau} \qquad v_{ss} = V_s$$

A more convenient way of finding step response

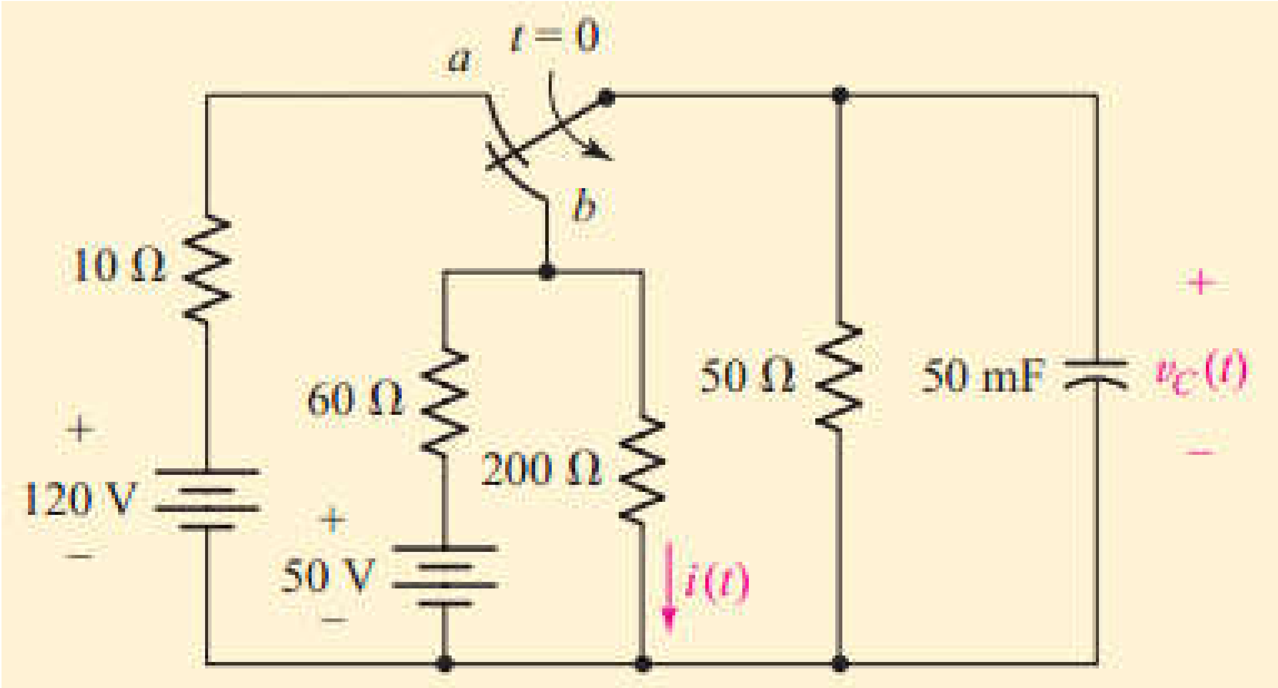
$$\boxed{x(t) = x(\infty) + [x(0) - x(\infty)]e^{-t/\tau}}$$

- $x(0)$ is the initial voltage at $t = 0^+$;
- $x(\infty)$ is the final or steady-state value.

- Thus, to find the step response of RC circuit requires three things:
  - 1. The initial capacitor voltage $v_C(0)$;
  - 2. The final capacitor voltage $v_C(\infty)$;
  - 3. The time constant τ = RC.
- Similarly, to find the step response of RL circuit requires three things:
  - 1. The initial capacitor voltage $i_L(0)$;
  - 2. The final capacitor voltage $i_L(\infty)$;
  - 3. The time constant τ = L/R.

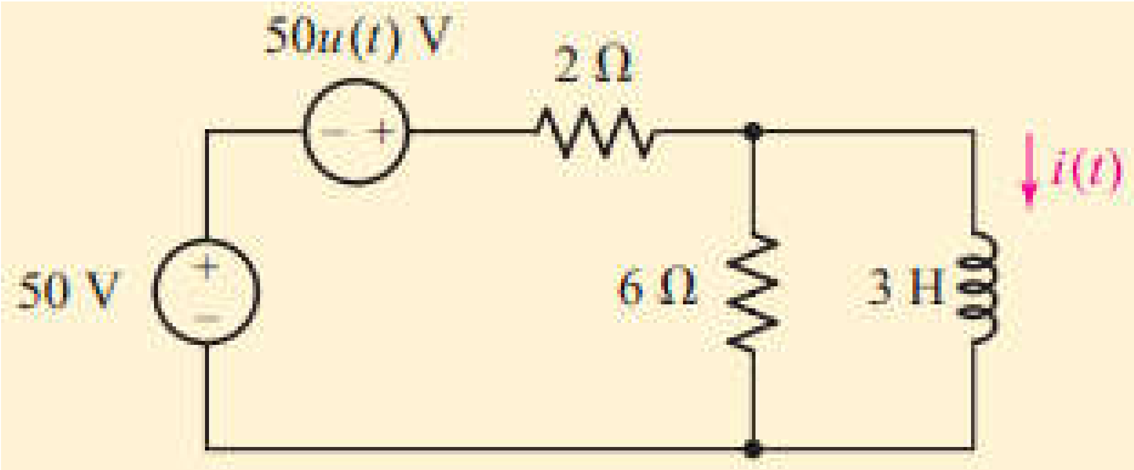
Example - RC

- Find the capacitor voltage $v_C(t)$ of the following figure.



Example - RL

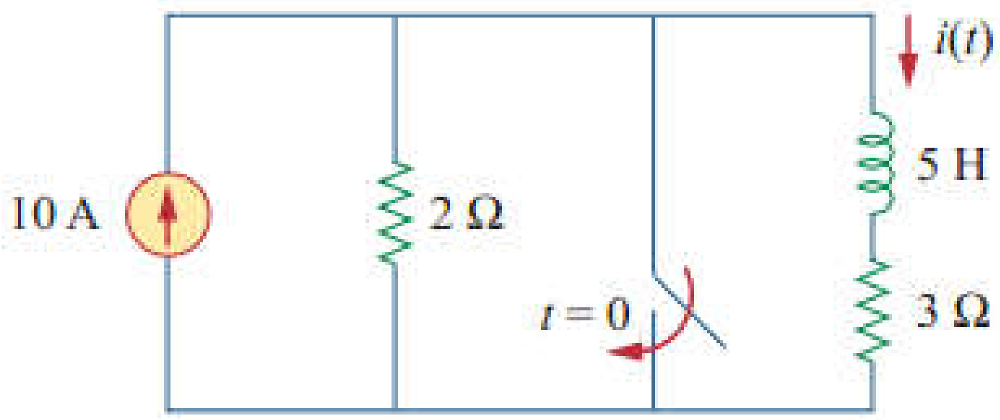
- Determine $i(t)$ for all values of time in the following circuit.



Quiz

- An RL circuit has $R = 2\ \Omega$ and $L = 4$ H. The time needed for the inductor current to charge to 36.8% of its steady-state value is:
  - (a) 0.9 s; (b) 1 s; (c) 2 s; (d) 4 s.

- For the circuit as shown, the inductor currents before $t = 0$ is:
  - (a) 10 A; (b) 6 A; (c) 4A; (d) 0A.
- For the same circuit, the inductor currents at $t = \infty$ is:
  - (a) 10 A; (b) 6 A; (c) 4A; (d) 0A.



Next Lecture

- Transient circuit – DC source
  - 2nd order