MTH101: Tutorial 3

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Exercise 1.1

Show that $\cosh z = \cosh x \cos y + i \sinh x \sin y$.

Solution:

We start with the definition of $\cosh z$ and writing $z = x + iy$:

$$\begin{aligned}
\cosh z &= \frac{e^{x+iy} + e^{-(x+iy)}}{2} \\
&= \frac{e^{x}(\cos y + i \sin y) + e^{-x}(\cos y - i \sin y)}{2} \\
&= \cos y \frac{e^{x} + e^{-x}}{2} + i \sin y \frac{e^{x} - e^{-x}}{2} \\
&= \cosh x \cos y + i \sinh x \sin y
\end{aligned}$$

Exercise 1.2

Find the function value in the form $u + iv$.

$$\cosh(-1+2i), \quad \cos(-2-i)$$

Solution:

By the previous exercise,

$$\cosh(-1+2i) = \cosh(-1)\cos 2 + i\sinh(-1)\sin 2,$$

For the second function value we use formula (6a) on page 634:

$$\cos z = \cos x \cosh y - i \sin x \sinh y,$$

thus,

$$\begin{aligned}\cos(-2-i) &= \cos(-2)\cosh(-1) - i\sin(-2)\sinh(-1)\\ &= \cos 2\cosh(-1) + i\sin 2\sinh(-1)\end{aligned}$$

Notice that they are equal, and this is not surprising because

$$i(-1+2i) = -i + 2i^2 = -2 - i$$

and we can employ the formula

$$\cos z = \cosh(iz).$$

Students can do similar practice checking by some specific values that

$$\sinh(iz) = i \sin z$$

Exercise 1.3

Verify that $\cos x \sinh y$ *is a harmonic function.*

Solution:

Let $u = \cos x \sinh y$, then

$$u_x = -\sin x \sinh y,\ \ u_{xx} = -\cos x \sinh y;$$

$$u_y = \cos x \cosh y,\ \ u_{yy} = \cos x \sinh y.$$

Thus

$$u_{xx} + u_{yy} = -\cos x \sinh y + \cos x \sinh y = 0$$

and u is Harmonic.

Try to find the harmonic conjugate of u on your own as a practice.

Exercise 2.1

Find the path and sketch it.

1. $z(t) = (1 + 2i)t, \quad (2 \leq t \leq 5);$
2. $z(t) = 2 + 4e^{\pi it/2}, \quad (0 \leq t \leq 2)$

Solutions:

1. Since the parametrization:

$$\begin{aligned} z(t) = x(t) + iy(t) &= (1 + 2i)t \\ &= t + i \cdot 2t \end{aligned}$$

is line segment on the straight line $y = 2x$ with initial point $(2, 4)$ and end point $(5, 10)$.

2. First, $e^{\pi it/2}$ $(0 \leq t \leq 2)$ is a unit semicircle traveled counterclockwise above the real axis.
Second, $4e^{\pi it/2}$ $(0 \leq t \leq 2)$ is a semicircle with radius 4 traveled counterclockwise above the real axis.
Finally, $2 + 4e^{\pi it/2}$ $(0 \leq t \leq 2)$ is a shift of that semicircle to the right by 2 units, that is, a upper semicircle centered at 2 with radius 4 traveled counterclockwise.

Exercise 2.2

Find a parametrization representation and sketch the path.

1. *Upper half of* $|z-2+i|=2$ *from* $(4,-1)$ *to* $(0,-1)$.
2. *Parabola* $y = 1 - \frac{1}{4}x^2, \quad (-2 \leq x \leq 2)$

Solution:

1. First, from the previous exercise we know that $z = 2 - i + 2e^{it}$ is tha parametric representation of a circle centered at $2-i$ with radius 2. Moreover, upper half is associated with $t \in [0, \pi]$.

$$z(t) = 2 - i + 2e^{it}, t \in [0, \pi]$$

2. Simply let $x = t$, and $y = 1 - \frac{1}{4}x^2 = 1 - \frac{1}{4}t^2$. Thus

$$z(t) = x(t) + iy(t) = t + i(1 - \frac{1}{4}t^2), \quad t \in [-2, 2].$$

Exercise 2.3

Find a parametrization for the **Counterclockwise** *oriented path* $\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3$ *where*

$$\gamma_1 \text{ is the segment joining } z_1 \text{ to } z_2,$$
$$\gamma_2 \text{ is the segment joining } z_2 \text{ to } z_3,$$
$$\gamma_3 \text{ is the upper semicircle with center } z_0 = 0 \text{ and radius } R = 2,$$

and

$$z_1 = -2, \quad z_2 = -3i, \quad z_3 = 2.$$

Compute the Integral

$$\oint_\gamma z dz.$$

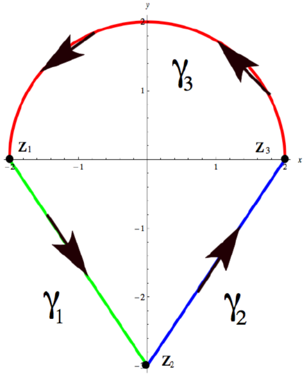

Figure: The path $\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3$

Solution We write a Parametrization $z(t) = x(t) + iy(t)$ of the 3 paths:

$$\gamma_1 : \begin{cases} x(t) = t, \\ y(t) = -(\frac{3}{2}t + 3), \end{cases} \quad \text{or} \quad z(t) = t - i(\frac{3}{2}t + 3), \quad t \in [-2, 0],$$

Along γ_1 we have $\dot{z}(t) = 1 - i\frac{3}{2}, \; t \in [-2, 0]$.

$$\gamma_2 : \begin{cases} x(t) = t, \\ y(t) = \frac{3}{2}t - 3 \end{cases} \quad \text{or} \quad z(t) = t + i(\frac{3}{2}t - 3), \; t \in [0, 2],$$

Along γ_2 we have $\dot{z}(t) = 1 + i\frac{3}{2}, t \in [0, 2]$.

$$\gamma_3 : \begin{cases} x(t) = 2\cos t, \\ y(t) = 2\sin t, \end{cases} \quad \text{or} \quad z(t) = 2\cos t + i2\sin t = 2e^{it}, \; t \in [0, \pi],$$

Along γ_3 we have $\dot{z}(t) = 2ie^{it}, t \in [0, \pi]$.

We use that

$$I = \oint_{\gamma} f(z)dz = \oint_{\gamma_1 \cup \gamma_2 \cup \gamma_3} f(z)dz = \oint_{\gamma_1} f(z)dz + \oint_{\gamma_2} f(z)dz + \oint_{\gamma_3} f(z)dz,$$

and the formula

$$\oint_{\gamma} f(z)dz = \int_a^b f(z(t))\dot{z}(t)dt,$$

where $z(t)$ is a parametrization of γ. In this case we have that $f(z(t)) = z(t)$. Then:

$$\begin{aligned}\oint_{\gamma_1} f(z)dz &= \int_{-2}^{0} \left[t - i(\frac{3}{2}t + 3)\right](1 - i\frac{3}{2})dt \\ &= (1 - i\frac{3}{2})\left[\frac{t^2}{2} - i(\frac{3}{4}t^2 + 3t)\right]_{-2}^{0} = -\frac{13}{2}.\end{aligned}$$

$$\oint_{\gamma_2} f(z)dz = \int_0^2 \left[t + i(\frac{3}{2}t - 3)\right](1 + i\frac{3}{2})dt$$
$$= (1 + i\frac{3}{2})\left[\frac{t^2}{2} + i(\frac{3}{4}t^2 - 3t)\right]_0^2 = \frac{13}{2}$$

$$\oint_{\gamma_3} f(z)dz = \int_0^\pi (2e^{it})(2ie^{it})\ dt = 4i\int_0^\pi e^{2it}dt$$
$$= 2[e^{2it}]_0^\pi = 0$$

Finally,

$$\oint_\gamma f(z)dz = -\frac{13}{2} + \frac{13}{2} + 0 = 0.$$

(The result 0 is not a coincidence, because the integrand $f(z) = z$ is analytic both in the interior and on the boundary of γ.)

Exercise 2.4

Integrate the following complex functions using appropriate method.

1. $$\int_\gamma \mathrm{Re}\ z\ dz$$

   γ *is the shortest path from* $1+i$ *to* $3+3i$.

2. $$\int_\gamma e^z\ dz$$

   γ *is the shortest path from* πi *to* $2\pi i$.

3. $$\int_\gamma \sec^2 z\ dz$$

   γ *is any path from* $\pi/4$ *to* $\pi i/4$.

Exercise 2.5

4

$$\oint_\gamma \frac{\tan \frac{1}{2}z}{z^4 - 16}\ dz$$

γ is the boundary of the square with vertices ± 1, $\pm i$ clockwise.

Solution:

1. The integrand $\mathrm{Re}\ z$ is not analytic, thus we need to using parametrization:

$$z(t) = t + it, \quad t \in [1, 3]$$

Remark: Note that parametric representations are not unique, for this exercise we can also let $z(t) = z_1 + (z_2 - z_1)t$, $t \in [0, 1]$ where z_1 is the initial point and z_2 is the end point and in this way

$$z(t) = 1 + i + (2 + 2i)t = 1 + 2t + i(1 + 2t), \quad t \in [0, 1]$$

We still continue with $z(t) = t + it,\ t \in [1, 3]$, then

$$\mathrm{Re}\ z(t) = t \quad \text{and} \quad \dot{z(t)} = 1 + i$$

and

$$\int_\gamma \mathrm{Re}\ z\ dz = \int_1^3 t(1+i)\ dt = (1+i)[\frac{t^2}{2}]_1^3 = 4 + 4i.$$

2. We know that e^z is analytic everywhere, hence we use indefinite integral and substitution of upper and lower limits, we have

$$\int_\gamma e^z\ dz = [e^z]_{\pi i}^{2\pi i} = e^{2\pi i} - e^{\pi i} = 2.$$

3. The integrand $\sec^2 z$ is not analytic at $\{z : z = \frac{\pi}{2} \pm n\pi\}$, but γ should be safe since it is from $\pi/4$ to $\pi i/4$.

$$\int_\gamma \sec^2 z \, dz = [\tan z]_{\pi/4}^{\pi i/4} = \tan(\pi i/4) - \tan(\pi/4) = i \tanh(\pi/4) - 1.$$

4. The integrand is not analytic at the points

$$z = \pi \pm 2n\pi \ , \ z = \pm 2, \pm 2i$$

which all lie outside of γ, then we can use Cauchy's integral theorem to obtain

$$\int_\gamma \frac{\tan \frac{1}{2} z}{z^4 - 16} \, dz = 0.$$