EEE225 Advanced Electrical Circuits and Electromagnetics

Lecture 1 Introduction

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Room EE322

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Module Code, Title and Guider

Module Code: EEE225

Module Title: Advanced Electrical Circuits and Electromagnetics

Module Leader: Zhao WANG

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Office Hour: Tuesday 13:30-15:30

Wednesday 13:30-15:30

Timetable, Test and Exam

- Timetable
  - 09:00 – 11:00, Monday
  - 11:00 – 13:00, Tuesday

- Assessment (5 Credits)
  - Laboratory 1 20% (week 6)
  - Laboratory 2 20% (week 11)
  - Final Examination 60%

No resit opportunity

Attendance

- Your attendance of every lecture will be recorded on ICE;

- If a student's attendance record falls below the threshold, he/she will not be able to attend the **resit** exam of this module;
  - The standard is 4 lectures of a 2.5 credit module, and 8 lectures of a 5 credit module.
  - EEE225 is a 5 credit module, so 8 lectures, at least.

About this module

- What have you learnt in this area?
  - EEE103: Electrical Circuits (part 1)
  - EEE108: Electromagnetics (part 1)
- What will be learnt in this module?
  - Circuits part
  - EM part
- How to learn this module?
  - Lectures
  - Problem sheets and tutorials
  - External resources

Electrical Circuits

- Ch.1 Introduction
- Ch.2 Basic Components and Ohm's Law
- Ch.3 Voltage and Current Laws: KCL, KVL, loop and node, series and parallel
- Ch.4 Basic Nodal and Mesh analysis
- Ch.5 Source transformation: Thevenin and Norton equivalents
- Ch.6 Operational Amplifier
- Ch.7 Capacitor and Inductors
- Ch.8 Basic RL and RC circuits
- Ch.9 The RLC circuits
- Ch.10 Sinusoidal steady-state analysis
- Ch.11 AC Circuit Power analysis
- Ch.12 Polyphase circuits (three-phase)
- Ch.13 Magnetically coupled circuits
- Ch.14 Complex Frequency and the Laplace Transform
- Ch.15 Circuit analysis in s-domain
- Ch.16 Frequency response
- Ch.17 Two-port networks

EEE207

Electromagnetics

- Ch.1 Vector analysis
- Ch.2 Coulomb's Law and E-field Intensity
- Ch.3 Electric Flux Density, Gauss's Law and Divergence
- Ch.4 Energy and Potential
- Ch.5 Current and Conductors
- Ch.6 Dielectrics and Capacitance
- Ch.7 Poisson's and Laplace's equations
- Ch.8 The steady Magnetic Field
- Ch.9 Magnetic Forces, materials and inductance
- Ch.10 Time-varying fields and Maxwell's equations
- Ch.12 The Uniform Plane Wave
- Ch.13 Plane wave reflection and dispersion
- Ch.14 Guided waves and radiation

EEE209

Module Syllabus

Week	Monday		Tuesday
1	Introduction	Math review 1	Review of EEE103 and EEE108
2	Electrostatics		Magnetostatics
3	Steady currents		Capacitor, inductor, resistor
4	Circuits - review		Transient - 1st
5	Transient - 2nd		Transient - driven
6	Frequency response		Circuit review + Lab 1
8	Two-port networks		Maxwell's Equations
9	Electromagnetics		Plane waves
10	Transmission Lines		More about waves
11	EM review + Lab 2		Waveguides
12	Magnetically Coupled Circuits		Magnetically Coupled Circuits
13	Three phase system		Three phase system
14	Final Review		Final Review

Resources:

- Reference Books:

Title	Author	ISBN/Publisher
Electric Circuits, 9th Ed.	James W. Nilsson, Susan A. Riedel	9787121157349
Engineering Circuit Analysis, 8th Ed.	William H. Hayt, Jr. Jack E. Kemmerly, Steven M. Durbin	9787121171376
Fundamentals of Electric Circuits	Charles K. Alexander, Matthew N. O. Sadiku	9787302159841
Engineering Electromagnetics	W.Hayt, J.Buck	9787302204077
Electromagnetic Filed Theory Fundamentals	B.S.Guru, H.R.Hiziroglu	9787111158318
Field and Wave Electromagnetics	D.K.Cheng	9787302152125

- Lecture, Tutorial Notes, Lab/Practices : ICE
- Past exam papers: Library
- Online resources:
  - MIT open courses: Circuits and Electronics by Anant Agarwal
    Electricity and Magnetism by Walter Lewin

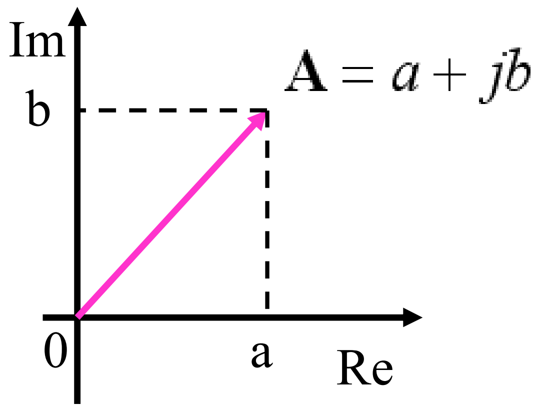
Today's Lecture – Math review

- Complex numbers
- Scalar and Vector
- Coordinate Systems
- Integral and Differential

Review – 1. Complex Numbers

Three forms

Rectangular/Cartesian Form

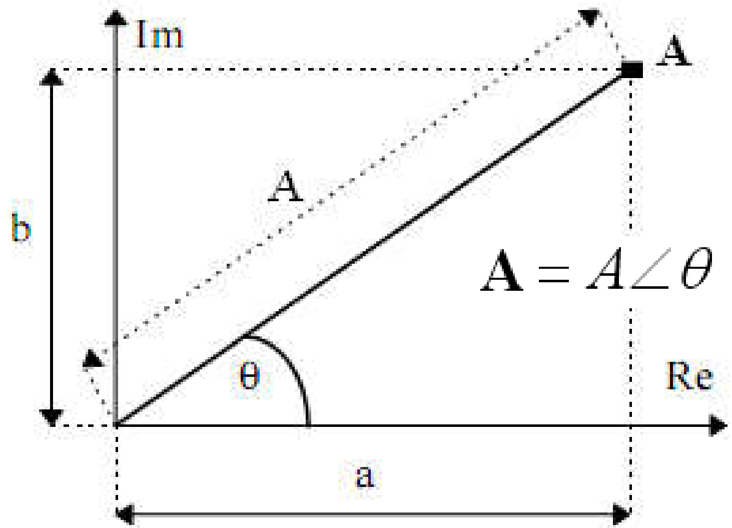


Exponential Form

$$\mathbf{A} = Ae^{j\theta}$$

magnitude phase

Polar Form



Relationships

$$\mathbf{A} = a + jb$$

$$\mathbf{A} = A\cos\theta + jA\sin\theta = A\angle\theta$$

$$\mathbf{A} = Ae^{j\theta}$$

$$A = \sqrt{\left(a^2 + b^2\right)} \quad \text{and} \quad \theta = \tan^{-1}\frac{b}{a}$$

Euler's Identity:

$$e^{j\theta} = \cos\theta + j\sin\theta$$

Replacing θ by $-\theta$:

$$e^{-j\theta} = \cos\theta - j\sin\theta$$

Complex Number – *in four quadrants*

$$\tilde{A} = a + jb, \qquad \theta = \tan^{-1}\frac{b}{a} \qquad \text{(1st Quadrant)}$$

$$\tilde{B} = -a + jb, \qquad \theta = 180^0 - \tan^{-1}\frac{b}{a} \qquad \text{(2nd Quadrant)}$$

$$\tilde{C} = -a - jb, \qquad \theta = 180^0 + \tan^{-1}\frac{b}{a} \qquad \text{(3rd Quadrant)}$$

$$\tilde{D} = a - jb, \qquad \theta = 360^0 - \tan^{-1}\frac{b}{a} \qquad \text{(4th Quadrant)}$$

a and b are positive.

Complex Number – Operations

Rectangular/Cartesian Form

- Being equal: $\mathbf{A} = a + jb$ and $\mathbf{B} = c + jd$ If **A**=**B**: $a = c$ and $b = d$
- Conjugate: $\mathbf{A} = a + jb$ and $\mathbf{A}^* = a - jb$
- Addition/Subtraction: $\mathbf{A} = a + jb$ and $\mathbf{B} = c + jd \Rightarrow \mathbf{A} \pm \mathbf{B} = (a \pm c) + j(b \pm d)$
- Multiplication: $\mathbf{A} = a + jb$ and $\mathbf{B} = c + jd$
- $\mathbf{A}^*\mathbf{B} = (ac + jjbd) + j(bc + ad) = (ac - bd) + j(bc + ad)$
- Division: $\mathbf{A} = a + jb$ and $\mathbf{B} = c + jd$

$$\frac{\mathbf{A}}{\mathbf{B}} = \frac{a + jb}{c + jd} = \frac{(a + jb)(c - jd)}{(c + jd)(c - jd)} = \frac{(a + jb)(c\text{-}jd)}{c^2 + d^2}$$

$$= \frac{(ac + bd) + j(bc - ad)}{B^2}$$

Complex Number – *Operations*

Exponential Form

- Being equal: $\mathbf{A} = Ae^{j\theta}$ and $\mathbf{B} = Be^{j\beta}$ If $\mathbf{A} = \mathbf{B}$: $A = B$ and $\theta = \beta$

- Conjugate: $\mathbf{A} = Ae^{j\theta}$ and $\mathbf{A}^* = Ae^{-j\theta}$

- Multiplication: $\mathbf{A} = Ae^{j\theta}$ and $\mathbf{B} = Be^{j\beta}$

$$\mathbf{A} * \mathbf{B} = Ae^{j\theta} * Be^{j\beta} = ABe^{j(\theta+\beta)}$$

- Division: $\mathbf{A} = Ae^{j\theta}$ and $\mathbf{B} = Be^{j\beta}$

$$\mathbf{A} / \mathbf{B} = \frac{Ae^{j\theta}}{Be^{j\beta}} = \frac{A}{B} e^{j(\theta-\beta)}$$

Complex Number – Operations

- Being equal: $\mathbf{A} = A\angle\theta$ and $\mathbf{B} = B\angle\beta$ If $\mathbf{A} = \mathbf{B}$: $A = B$ and $\theta = \beta$

- Conjugate: $\mathbf{A} = A\angle\theta$ and $\mathbf{A}^* = A\angle -\theta$

- Multiplication: $\mathbf{A} = A\angle\theta$ and $\mathbf{B} = B\angle\beta$

$$\mathbf{A} * \mathbf{B} = A\angle\theta * B\angle\beta = AB\angle(\theta+\beta)$$

- Division: $\mathbf{A} = A\angle\theta$ and $\mathbf{B} = B\angle\beta$

$$\mathbf{A}/\mathbf{B} = \frac{A\angle\theta}{B\angle\beta} = \frac{A}{B}\angle(\theta-\beta)$$

Complex Number – *Operations*

Example

If $\mathbf{A} = 2 + j5$, and $\mathbf{B} = 4 - j6$, find $(\mathbf{A}+\mathbf{B})/(\mathbf{A}-\mathbf{B})$

Solution :

$$\mathbf{A}+\mathbf{B} = (2+4) + j(5-6) = 6 - j$$

$$\mathbf{A}-\mathbf{B} = (2-4) + j(5-(-6)) = -2 + j11$$

$$\frac{\mathbf{A}+\mathbf{B}}{\mathbf{A}-\mathbf{B}} = \frac{6-j}{-2+j11} = \frac{(6-j)(-2-j11)}{(-2+j11)(-2-j11)}$$

$$= \frac{-12-j66+j2-11}{(-2)^2+11^2} = \frac{-23-j64}{125}$$

$$= -0.184 - j0.512$$

Complex Number – Summary

- ***Three forms*:** Rectangular, Polar, and Exponential forms.
- ***Addition and subtraction*:** have to be done using Cartesian form.
- ***Multiplication and division*:** can be done using Cartesian, polar and exponential form; it is significantly easier to use polar and exponential form.
- You should become familiar with the three different forms and rapid conversion from one form to another.

Review – 2. Scalar and vector

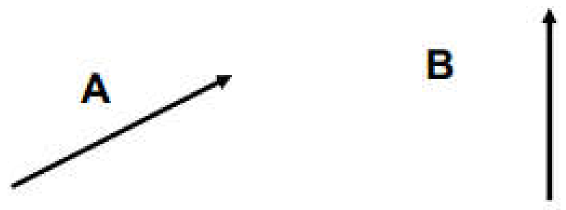
- A scalar is completely specified by its magnitude (positive or negative, together with its unit)
  - Require two things:
    - A value (positive or negative)
    - Appropriate unit
  - Example:
    - Mass: 5 kg, 1.53 ton.
    - Distance: 65 km, 0.05 mm.
    - Temperature: 20 ^{o}C, -32 ^{o}F
    - Voltage: 10 kV, -3 mV.

Review – 2. Scalar and vector

- A vector is specified by both its magnitude (with unit) and a direction.
  - Require three things:
    - A value
    - Appropriate unit
    - **A direction!**
  - Example:
    - Velocity: 25 mph West
    - Force: 10 N up
  - Notations
    - Most widely used: bold
      - Eg: **F**, **A**
    - Also written as:
      - $\vec{A}$, $\hat{A}$, $\bar{A}$ or even <u>A</u>
    - The magnitude of a vector **A** is written as
      - |**A**| or A

Review - Vector Representation

- Vectors are represented graphically or quantitatively:
  - Graphically: through arrows with the orientation representing the direction and length representing the magnitude



  - Quantitatively: a right-hand coordinate system having orthogonal axes is usually chosen:
    - Cartesian (Rectangular)
    - Circular cylindrical
    - Spherical

Review - Vector Representation (cont.)

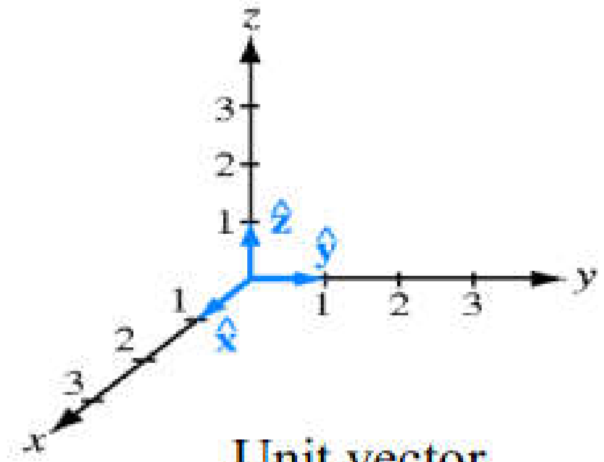
- Represent a vector $\vec{A}$ quantitatively by the magnitude and the direction:

$$\vec{A} = \hat{a}|\vec{A}| = \hat{a}A, \text{where } \hat{a} = \frac{\vec{A}}{|\vec{A}|} = \frac{\vec{A}}{A}$$

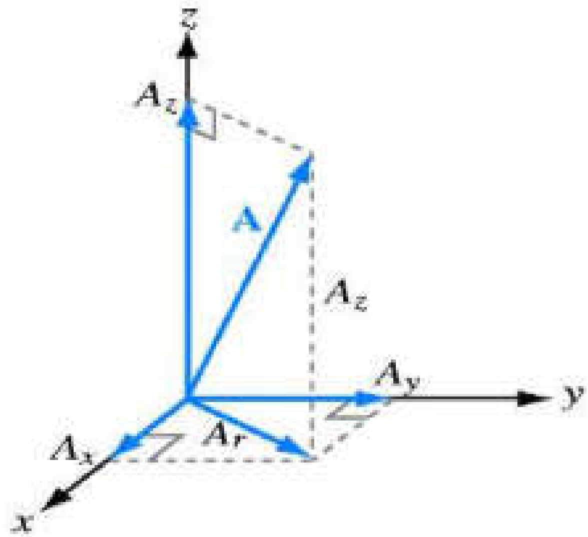
  - $\hat{a}$ is called unit vector, with the same direction as $\vec{A}$, but magnitude is 1.
  - In Cartesian coordinates, $\vec{A} = \hat{x}A_x + \hat{y}A_y + \hat{z}A_z$

$$A = \sqrt{{A_x}^2 + {A_y}^2 + {A_z}^2}$$

  - Unit vectors are: $\hat{x}, \hat{y}$ and $\hat{z}$



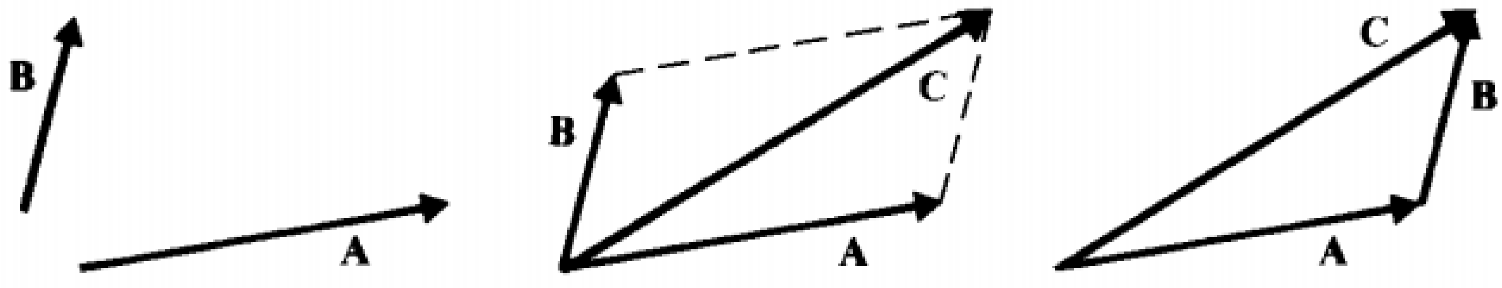
Unit vector



Components of **A**

Review – Vector algebra

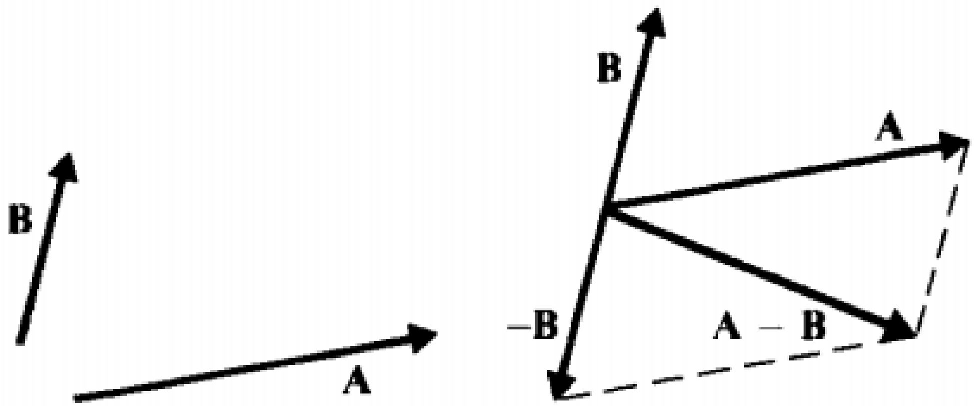
- Addition
  - Two vectors **A** and **B**, which are not in the same direction nor in opposite directions, determine a plane.
  - Their sum is another vector **C** (**C** = **A** + **B**) in the same plane.
    - 1. Parallelogram rule: The resultant **C** is the diagonal vector of the parallelogram formed by **A** and **B** drawn from the same point.
    - 2. Head-to-tail rule: The head of **A** connects to the tail of **B**. Their sum **C** is the vector drawn from the tail of **A** to the head of **B**, and vectors **A**, **B** and **C** form a triangle.



(a) Two vectors, **A** and **B**. (b) Parallelogram rule. (c) Head-to-tail rule.

Review – Vector algebra

- Subtraction
  - The difference **D**, as the result of vector subtraction **D** = **A** - **B**, is another vector also in the plane determined by **A** and **B**.
  - The vector subtraction is simply defined based on the vector addition: **D** = **A** – **B** = **A** + (-**B**)

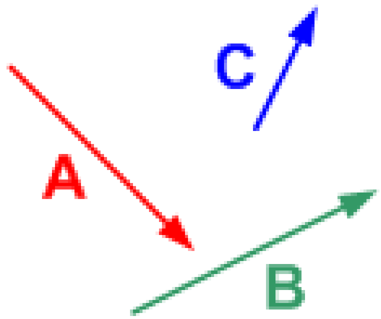


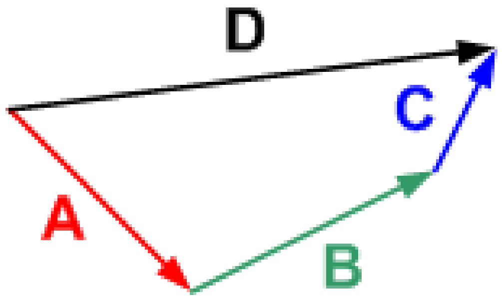
(a) Two vectors, **A** and **B**.

(b) Subtraction of vectors, **A** – **B**.

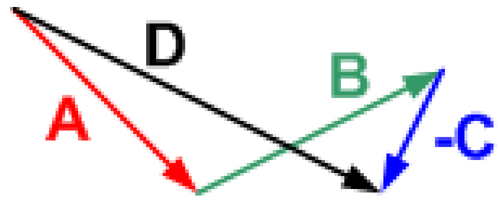
Review – Vector algebra

- Note:
  - Commutative law: **A** + **B** = **B** + **A**
  - Associative law: **A** + (**B** + **C**) = (**A** + **B**) + **C**
- Graphical representation of multiple vectors addition and subtraction:





D = **A** + **B** + **C**



D = **A** + **B** - **C**

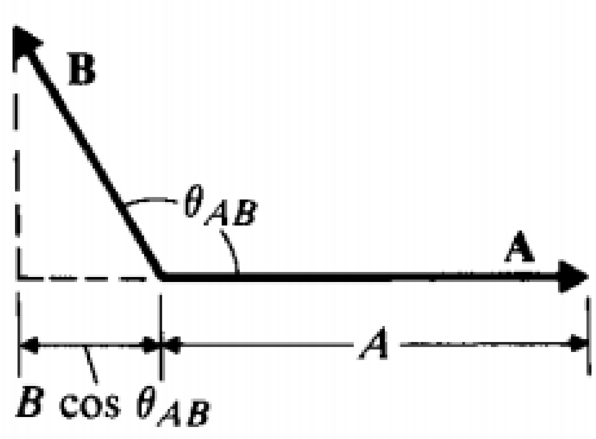
- In Cartesian coordinates:
  - $\vec{A} = \hat{x}A_x + \hat{y}A_y + \hat{z}A_z$ and $\vec{B} = \hat{x}B_x + \hat{y}B_y + \hat{z}B_z$
  - Then: $\vec{A} \pm \vec{B} = \hat{x}(A_x \pm B_x) + \hat{y}\left(A_y \pm B_y\right) + \hat{z}(A_z \pm B_z)$

Review – Vector algebra

- Scalar or Dot product
  - The result is a "scalar", the operator is a "dot".

$$\vec{A} \cdot \vec{B} \triangleq AB\cos\theta_{AB}$$

    - θ_{AB} is the smaller angle between **A** and **B**, which is always less then 180°.



- Note:
  - Commutative: **A** · **B** = **B** · **A**
  - Distributive: **A** · (**B** + **C**) = (**A** · **B**) + (**A** · **C**)

Special examples:
A·A = A^2
A ⊥ **B** => **A·B** = 0

- In Cartesian coordinates:
  - $\vec{A} = \hat{x}A_x + \hat{y}A_y + \hat{z}A_z$ and $\vec{B} = \hat{x}B_x + \hat{y}B_y + \hat{z}B_z$
  - Then: $\vec{A} \cdot \vec{B} = A_xB_x + A_yB_y + A_zB_z$

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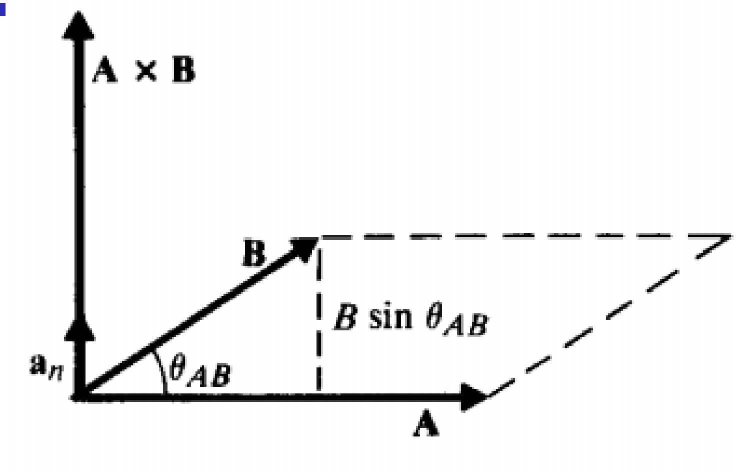
Review – Vector algebra

- Vector or Cross Product
  - The result is a "vector", the operator is a "cross".

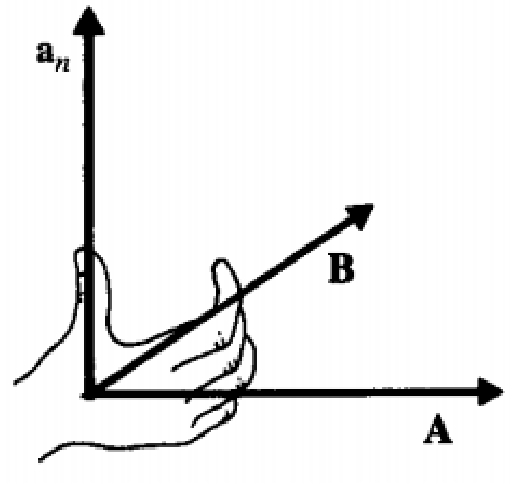
$$\vec{A} \times \vec{B} \triangleq \hat{n} AB sin\theta_{AB}$$

    - $\hat{\boldsymbol{n}}$ is the unit vector normal to the plane containing **A** and **B**.

- Note:
  - Not commutative: **A** X **B** ≠ **B** X **A**
  - Distributive: **A** X (**B** + **C**) = (**A** X **B**) + (**A** X **C**)
  - Not associative: **A** X (**B** X **C**) ≠ (**A** X **B**) X **C**



(a) $\mathbf{A} \times \mathbf{B} = \mathbf{a}_n |AB \sin \theta_{AB}|$.



(b) The right-hand rule.

Special examples:
A X A = 0
A // **B** => **A** X **B** = 0

Review – Vector algebra

- In Cartesian coordinates:
  - $\vec{A} = \hat{x}A_x + \hat{y}A_y + \hat{z}A_z$ and $\vec{B} = \hat{x}B_x + \hat{y}B_y + \hat{z}B_z$
  - Then:

$$\begin{aligned}\vec{A}\times\vec{B} &= \left(\hat{x}\times\hat{x}A_xB_x\right)+\left(\hat{x}\times\hat{y}A_xB_y\right)+\left(\hat{x}\times\hat{z}A_xB_z\right)\\ &\quad+\left(\hat{y}\times\hat{x}A_yB_x\right)+\left(\hat{y}\times\hat{y}A_yB_y\right)+\left(\hat{y}\times\hat{z}A_yB_z\right)\\ &\quad+\left(\hat{z}\times\hat{x}A_zB_x\right)+\left(\hat{z}\times\hat{y}A_zB_y\right)+\left(\hat{z}\times\hat{z}A_zB_z\right)\end{aligned}$$

  - This reduce to:

$$\vec{A}\times\vec{B} = \hat{x}\left(A_yB_z - A_zB_y\right)+\hat{y}\left(A_zB_x - A_xB_z\right)+\hat{z}\left(A_xB_y - A_yB_x\right)$$

  - This may be expressed concisely in determinant form as:

$$\vec{A}\times\vec{B} = \begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ A_x & A_y & A_z\\ B_x & B_y & B_z\end{vmatrix}$$

Review – Vector algebra

- Product of three vectors
  - Scalar triple product

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})=\mathbf{B}\cdot(\mathbf{C}\times\mathbf{A})=\mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$$

    - The result is a scalar which can be calculated by:

$$\vec{A}\cdot(\vec{B}\times\vec{C})=\begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

  - Vector triple product

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C})=\mathbf{B}(\mathbf{A}\cdot\mathbf{C})+\mathbf{C}(\mathbf{A}\cdot\mathbf{B})$$

    - This is known as the “back-cab” rule.
    - Notice the “BAC - CAB” on the right side of the equation.

Scalar and Vector – Summary

- Difference between scalar and vector:
  - A scalar is defined by 2 properties: value and unit
  - A vector is defined by 3 properties: direction, value, and unit.

- Vector representation
  - Graphically or quantitatively

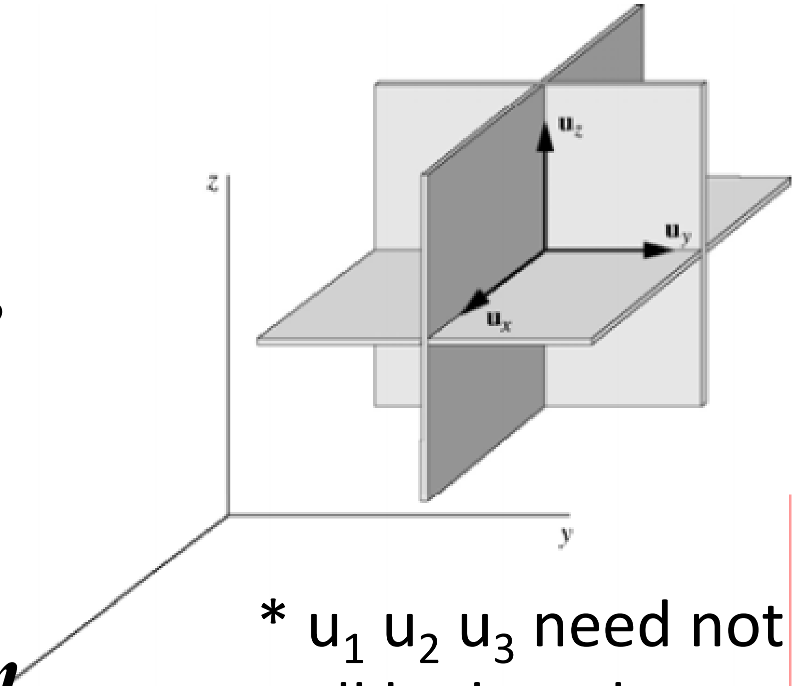
in different coordinate systems

- Vector algebra
  - Addition and subtraction
  - Production: dot (scalar) product and cross (vector) product

Review – 3. Coordinate Systems (CS)

- In a 3D space, a point can be located as the intersection of three surfaces.

- Assume that the three families of surfaces are described by u_1=constant, u_2=constant, and u_3=constant. When these three surfaces are mutually **perpendicular** to one another, we have an ***orthogonal coordinate system***.



* u_1 u_2 u_3 need not all be lengths

- Nonorthogonal CSs are not used because they actually complicate problems.

Review – Properties of Orthogonal CSs

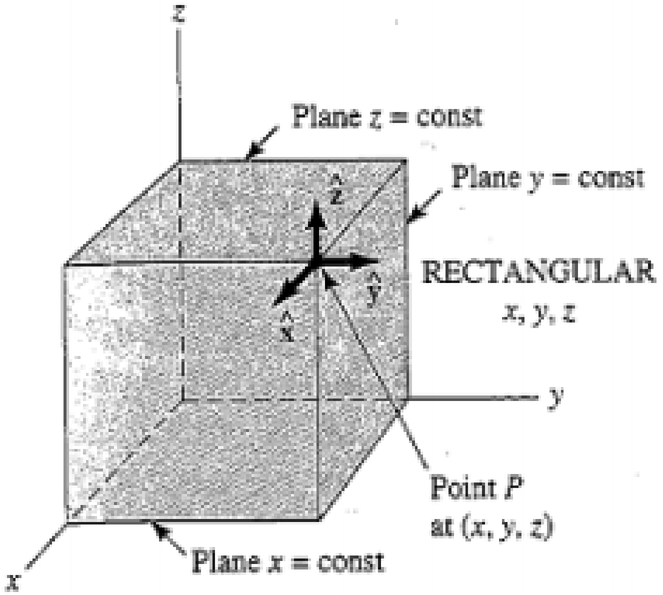
- Let $\boldsymbol{a_{u_1}}$, $\boldsymbol{a_{u_2}}$ and $\boldsymbol{a_{u_3}}$ be the unit vectors in the three coordinate directions, they are called base vectors.

- Properties:
  - $\boldsymbol{a_{u_1}} \cdot \boldsymbol{a_{u_1}} = \boldsymbol{a_{u_2}} \cdot \boldsymbol{a_{u_2}} = \boldsymbol{a_{u_3}} \cdot \boldsymbol{a_{u_3}} = 1$
  - $\boldsymbol{a_{u_1}} \cdot \boldsymbol{a_{u_2}} = \boldsymbol{a_{u_2}} \cdot \boldsymbol{a_{u_3}} = \boldsymbol{a_{u_3}} \cdot \boldsymbol{a_{u_1}} = 0$
  - $\boldsymbol{a_{u_1}} \times \boldsymbol{a_{u_2}} = \boldsymbol{a_{u_3}}$; $\boldsymbol{a_{u_2}} \times \boldsymbol{a_{u_3}} = \boldsymbol{a_{u_1}}$; $\boldsymbol{a_{u_3}} \times \boldsymbol{a_{u_1}} = \boldsymbol{a_{u_2}}$

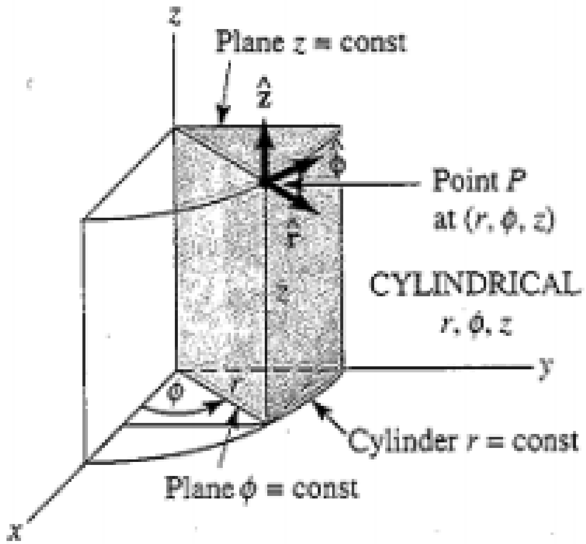
- Any vector **B** can be written as the sum of its components in the three orthogonal directions, as follows:

$$\boldsymbol{B} = \boldsymbol{a_{u_1}} B_{u_1} + \boldsymbol{a_{u_2}} B_{u_2} + \boldsymbol{a_{u_3}} B_{u_3}$$

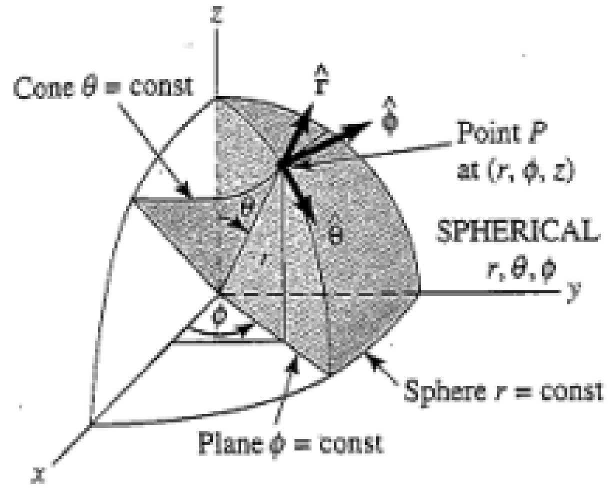
Review – Commonly used CS



(a) – Cartesian



(b) – Cylindrical



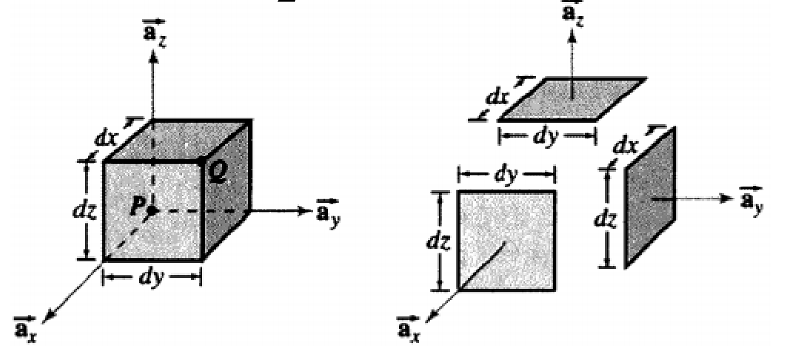
(c) – Spherical

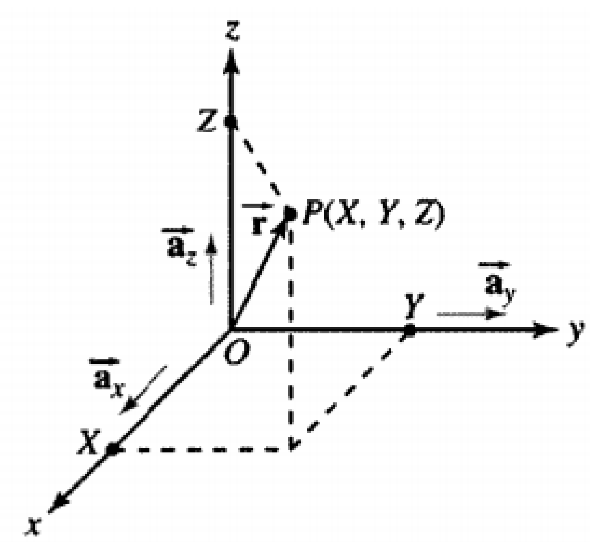
- In Cartesian CS, directions of unit vectors are independent of their positions;
- In Cylindrical and Spherical systems, directions of unit vectors depend on positions.

Review – Cartesian coordinate

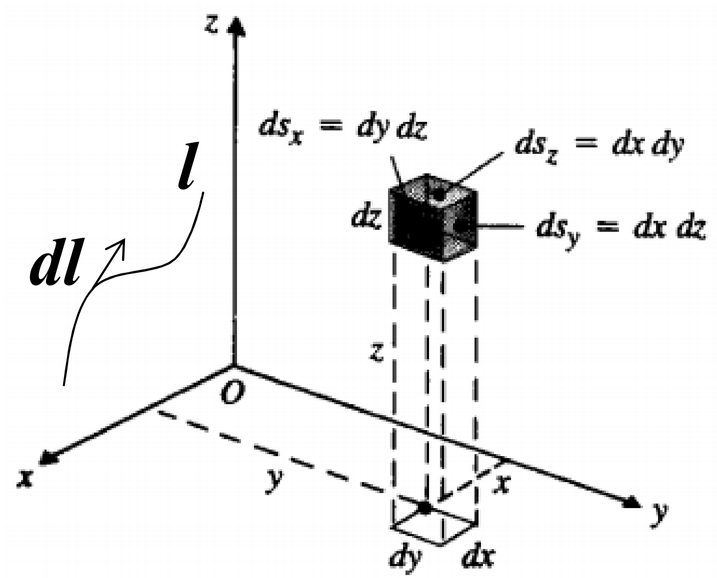
- Cartesian (rectangular) CS is generated by the intersection of 3 planes: A_x = constant, A_y = constant and A_z = constant.

$$\vec{\boldsymbol{A}} = \hat{\boldsymbol{x}}A_x + \hat{\boldsymbol{y}}A_y + \hat{\boldsymbol{z}}A_z$$





Projection of a point



A differential line segment and a differential volume

A differential line element:

$$\boldsymbol{dl} = \hat{\boldsymbol{x}}dx + \hat{\boldsymbol{y}}dy + \hat{\boldsymbol{z}}dz$$

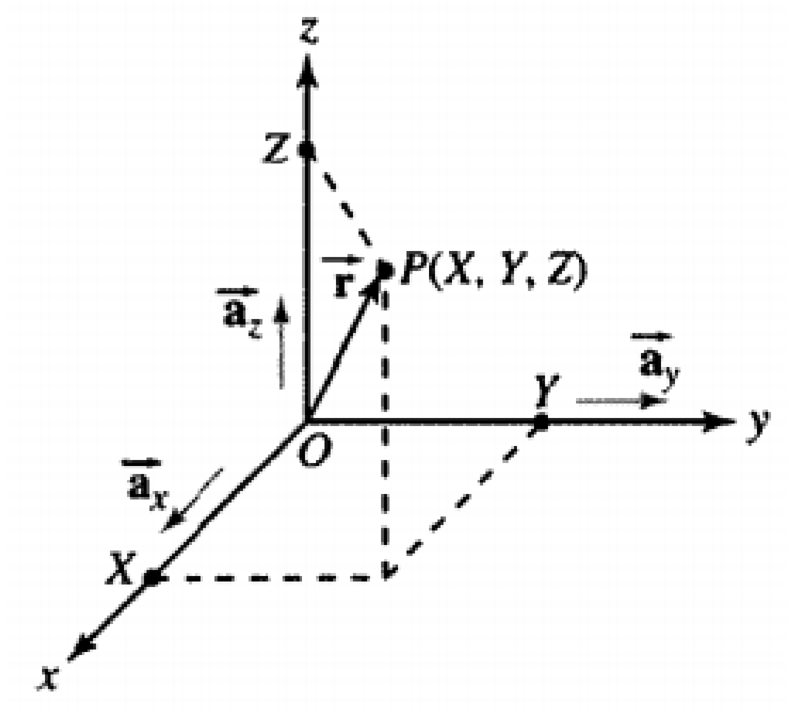
3 differential surface element:

$$\boldsymbol{ds_x} = \hat{\boldsymbol{x}}dydz$$
$$\boldsymbol{ds_y} = \hat{\boldsymbol{y}}dzdx$$
$$\boldsymbol{ds_z} = \hat{\boldsymbol{z}}dxdy$$

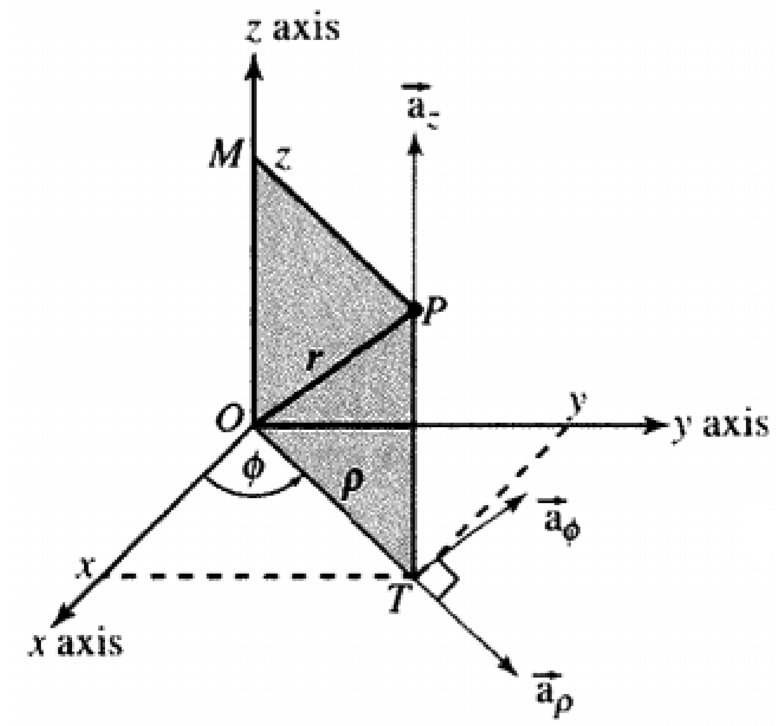
The differential volume element:

$$dv = dx\,dy\,dz$$

Cartesian C.S.
(Coordinate System)



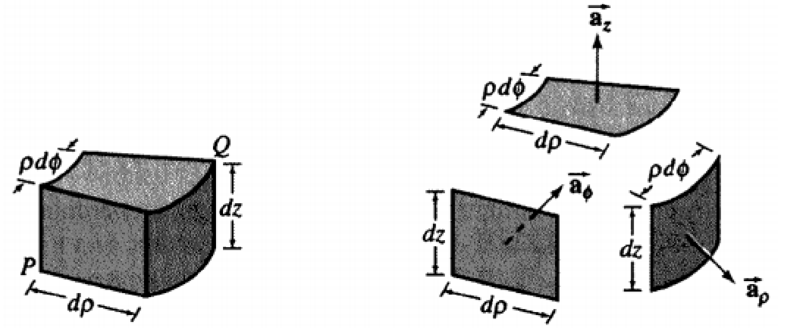
Cylindrical C.S.
(Coordinate System)



Review – Cylindrical coordinate

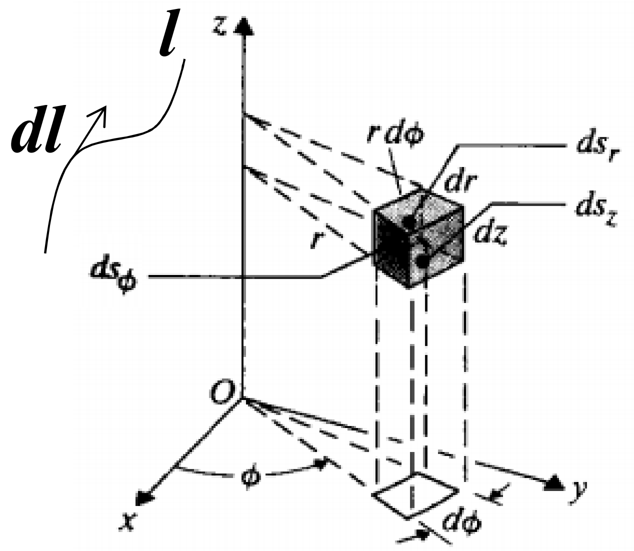
- Cylindrical CS is generated by the intersection of 2 planes and 1 cylinder: A_r=constant; A_φ=constant; A_z=constant.

$$\vec{\boldsymbol{A}} = \hat{\boldsymbol{r}}A_r + \hat{\boldsymbol{\varphi}}A_\varphi + \hat{\boldsymbol{z}}A_z$$



$$\vec{A} = \hat{r}A_r + \hat{\varphi}A_\varphi + \hat{z}A_z$$

Projection of a point



A differential line segment and a differential volume

A differential line element:

$$\boldsymbol{dl} = \hat{\boldsymbol{r}}dr + \hat{\boldsymbol{\varphi}}rd\varphi + \hat{\boldsymbol{z}}dz$$

(sometimes, use $\boldsymbol{\rho}$ to replace $\boldsymbol{r}$)

3 differential surface element:

$$\boldsymbol{ds_r} = \hat{\boldsymbol{r}}\ rd\varphi dz$$

$$\boldsymbol{ds_\varphi} = \hat{\boldsymbol{\varphi}}\ drdz$$

$$\boldsymbol{ds_z} = \hat{\boldsymbol{z}}\ rdrd\varphi$$

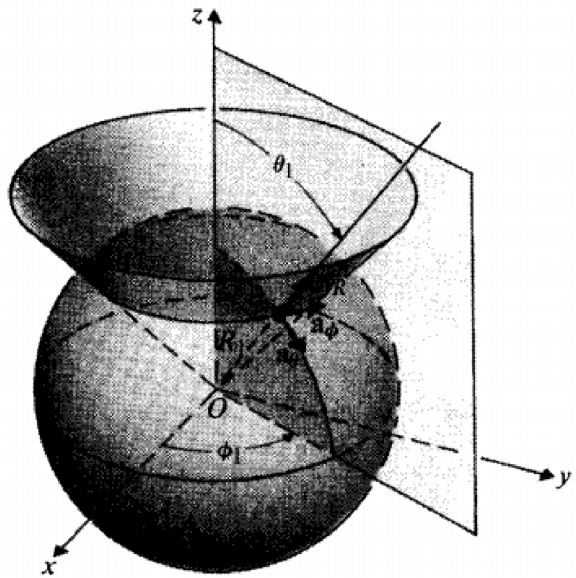
The differential volume element:

$$dv = r\,dr\,d\varphi\,dz$$

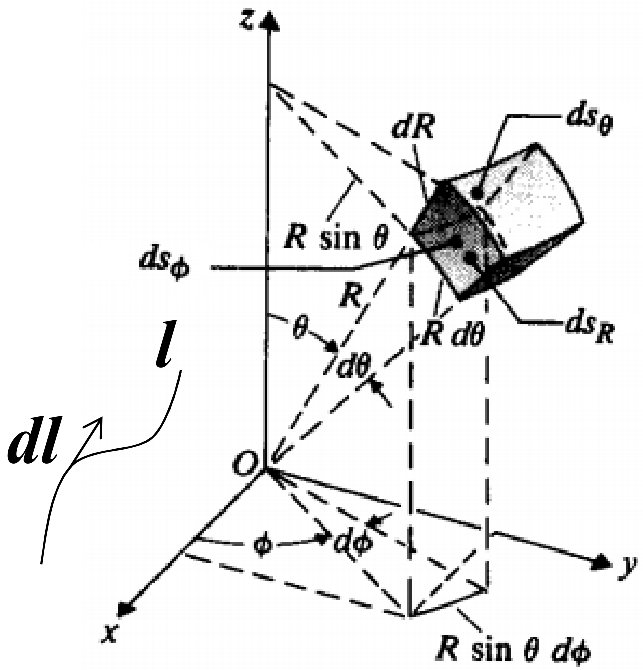
Review – Spherical coordinate

- Spherical CS is generated by the intersection of 1 plane, 1 cone and 1 sphere: A_R=constant; A_θ=constant; A_φ=constant.

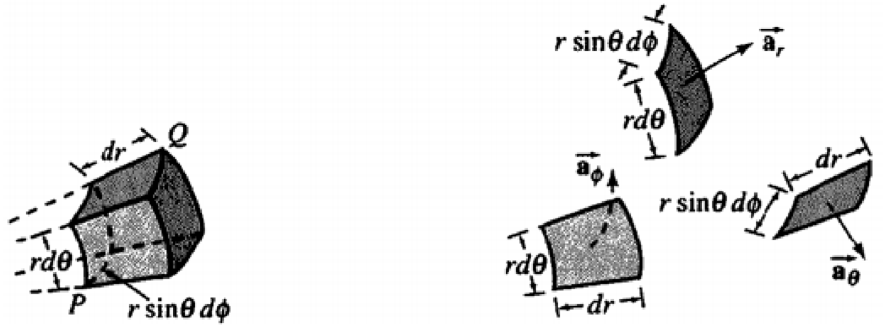
$$\vec{A} = \widehat{\boldsymbol{R}}A_R + \widehat{\boldsymbol{\theta}}A_\theta + \widehat{\boldsymbol{\varphi}}A_\varphi$$



Projection of a point



A differential line segment and a differential volume



A differential line element:

$$d\boldsymbol{l} = \widehat{\boldsymbol{R}}d\boldsymbol{R} + \widehat{\boldsymbol{\theta}}Rd\theta + \widehat{\boldsymbol{\varphi}}Rsin\theta d\varphi$$

(sometimes, use ***r*** to replace ***R***)

3 differential surface element:

$$d\boldsymbol{s_R} = \widehat{\boldsymbol{R}}\ R^2 sin\theta d\theta d\varphi$$

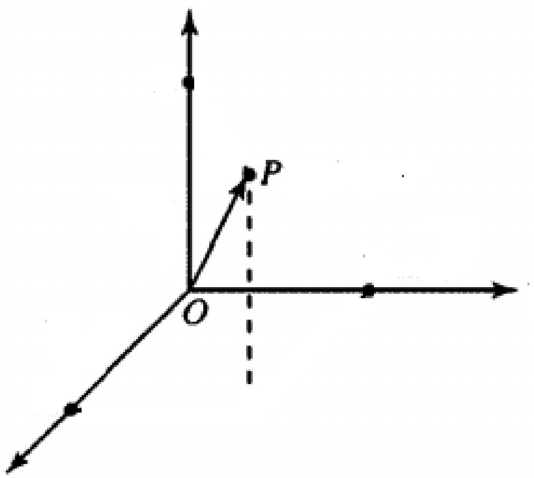
$$d\boldsymbol{s_\theta} = \widehat{\boldsymbol{\theta}}\ Rsin\theta dR d\varphi$$

$$d\boldsymbol{s_z} = \widehat{\boldsymbol{\varphi}}\ RdRd\theta$$

The differential volume element:

$$dv = R^2\ sin\theta\ dR\ d\theta\ d\varphi$$

Review – *Coordinate Variables and the Transformations*



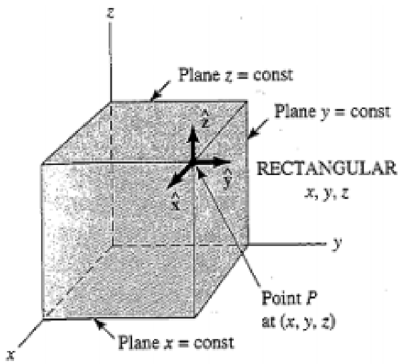
- Cartesian: x, y, z
- Cylindrical: r, φ, z
- Spherical: R, θ, φ

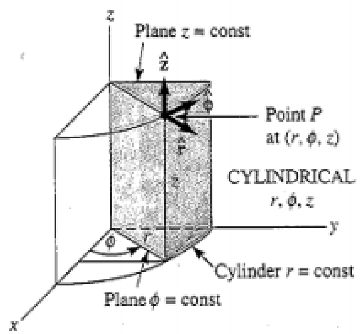
$$
\begin{aligned}
x &= r\cos\phi = R\sin\theta\cos\phi \\
y &= r\sin\phi = R\sin\theta\sin\phi \\
z &= z = R\cos\theta
\end{aligned}
$$

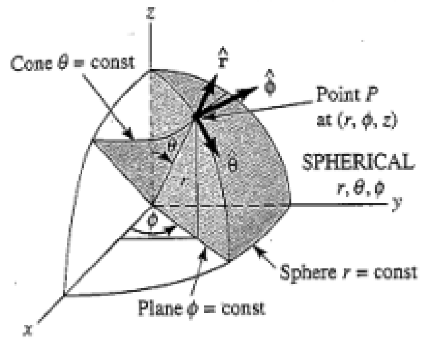
Cartesian　Cylindrical　Spherical

Review – Unit vectors and the Transformations

- Cartesian: $\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}}$
- Cylindrical: $\hat{\boldsymbol{r}}, \hat{\boldsymbol{\varphi}}, \hat{\boldsymbol{z}}$
- Spherical: $\hat{\boldsymbol{R}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\varphi}}$







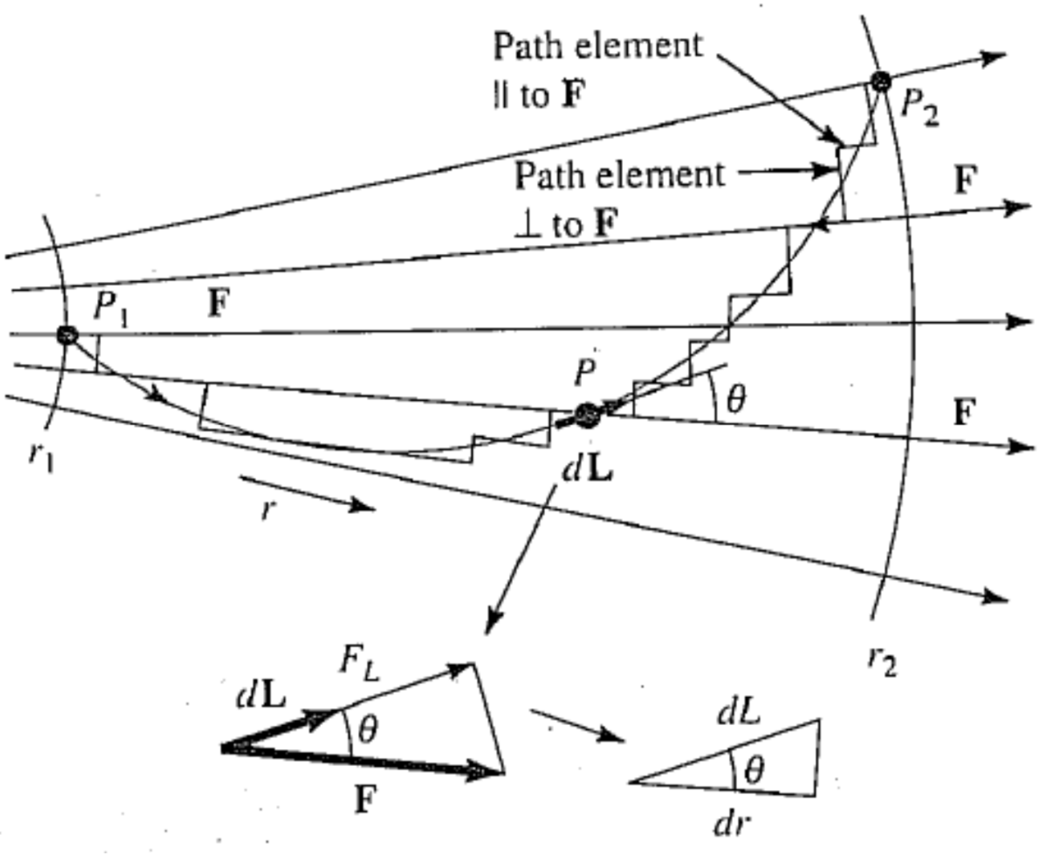
$$
\begin{array}{ccccc}
\boxed{\begin{array}{c}\hat{\mathbf{x}} \\ \hat{\mathbf{y}} \\ \hat{\mathbf{z}}\end{array}} & = & \boxed{\begin{array}{l}\hat{\mathbf{r}}\cos\phi - \hat{\boldsymbol{\varphi}}\sin\phi \\ \hat{\mathbf{r}}\sin\phi + \hat{\boldsymbol{\varphi}}\cos\phi \\ \hat{\mathbf{z}}\end{array}} & = & \boxed{\begin{array}{l}\hat{\mathbf{R}}\sin\theta\cos\phi + \hat{\boldsymbol{\theta}}\cos\theta\cos\phi - \hat{\boldsymbol{\varphi}}\sin\phi \\ \hat{\mathbf{R}}\sin\theta\sin\phi + \hat{\boldsymbol{\theta}}\cos\theta\sin\phi + \hat{\boldsymbol{\varphi}}\cos\phi \\ \hat{\mathbf{R}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta\end{array}} \\
\uparrow & & \uparrow & & \uparrow \\
\text{Cartesian} & & \text{Cylindrical} & & \text{Spherical}
\end{array}
$$

Coordinate Systems – Summary

- The orthogonal coordinate systems
  - Cartesian (rectangular) CS
  - Cylindrical CS
  - Spherical CS
- The differential elements in the three CSs
  - Line elements
  - Surface elements
  - Volume elements
- The transformations among the CSs
  - Transform in 2D and 3D

Review – Integral relations for vectors

- ***Line Integral***: integrate along a curve in 3D space
  - In a vector field **F** (eg. force field);
  - Work done by moving from point P_1 to P_2 along a curved path ***l***.



At any point *P* the product of path length ***dL*** and component of **F** parallel to it is given by:

$$\boldsymbol{F} \cdot d\boldsymbol{L} = F cos\theta dL = F_L dL$$

So, the work done by the force in moving an object a distance ***dL*** is:

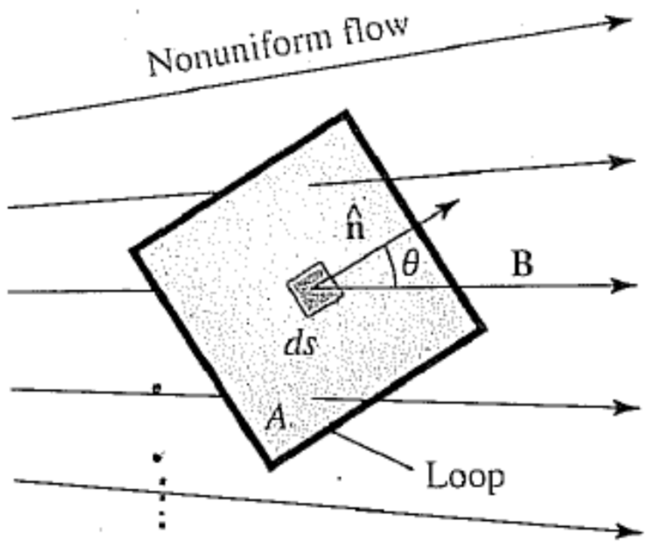
$$dW = \boldsymbol{F} \cdot d\boldsymbol{L} = F cos\theta dL$$

Summing the contributions of work done *dW* as *P* moves from P_1 to P_2 gives:

$$W = \int_{P_1}^{P_2} dW = \int_{P_1}^{P_2} \boldsymbol{F} \cdot d\boldsymbol{L}$$

Review – Integral relations for vectors

- ***Surface Integral***: integrate across a surface in 3D space
  - In a vector field **B** (eg. rate of the water flow);
  - Flux flowing through a surface **A.**



$$d\psi = \mathbf{B} \cdot \hat{\mathbf{n}}\, ds$$
$$\psi = \iint_A \mathbf{B} \cdot d\mathbf{s}$$

A is defined as a vector of magnitude A (area of the surface) and direction perpendicular to its surface;
B is a vector of magnitude B and direction of the flow;
θ is the angle between **B** and the perpendicular to **A** (defined as **n**);
So the incremental flux $d\Psi$ through a surface area element ds (with normal **n**) is given by:

$$d\psi = \boldsymbol{B} \cdot d\boldsymbol{A} = \boldsymbol{B} \cdot \hat{\boldsymbol{n}} ds = \boldsymbol{B} \cdot d\boldsymbol{s}$$

Integrate the increment flux $d\Psi$ over the whole area **A** gives:

$$\psi = \int_{areaA} \boldsymbol{B} \cdot \hat{\boldsymbol{n}} ds = \iint_A \boldsymbol{B} \cdot \hat{\boldsymbol{n}} ds$$

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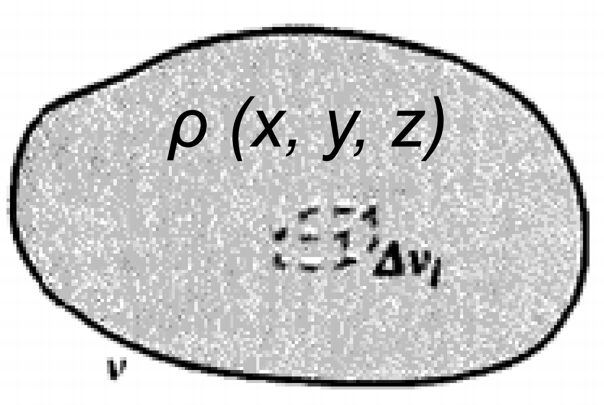
Review – Integral relations for vectors

- ***Volume Integral***: integrate on a volume in 3D space
  - In a scalar field ρ (eg. mass density);
  - Total mass with a volume V.



In Cartesian coordinate:
ρ is defined as a scalar function of position (x, y, z);
dv is the differential element of the volume V;

So the differential mass *dm* on a volume element dv is given by:

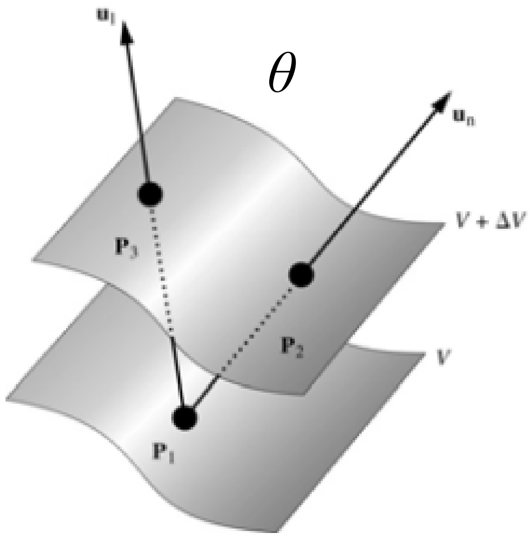
$$dm = \rho dv = \rho dxdydz$$

Integrate the increment flux *dΨ* over the whole area **A** gives:

$$m = \int_{volumeV} \rho dv = \iiint_V \rho dv$$

Review – Differential relations for vectors

- ***Gradient*** of a scalar function
  - Gradient of a scalar field is a vector field which points in the direction of the greatest rate of increase of the scalar field, and whose magnitude is the greatest rate of change.
  - It is denoted by ∇f or grad(*f*), where ∇ is a vector differential operator called "del."



Two equipotential surfaces with potentials V and V+ΔV. Select 3 points such that distances between them have $P_1P_2 \leq P_1P_3$, i.e. $\Delta n \leq \Delta l$.

$$\Rightarrow \frac{\Delta V}{\Delta n} \geq \frac{\Delta V}{\Delta l}$$

Assume that separation between surfaces is small:

$$\Rightarrow \nabla V = \frac{\Delta V}{\Delta n}\boldsymbol{u_n} \rightarrow \frac{\Delta V}{\Delta n}\boldsymbol{u_n}$$

Projection of the gradient in the $\mathbf{u}_l$ direction:

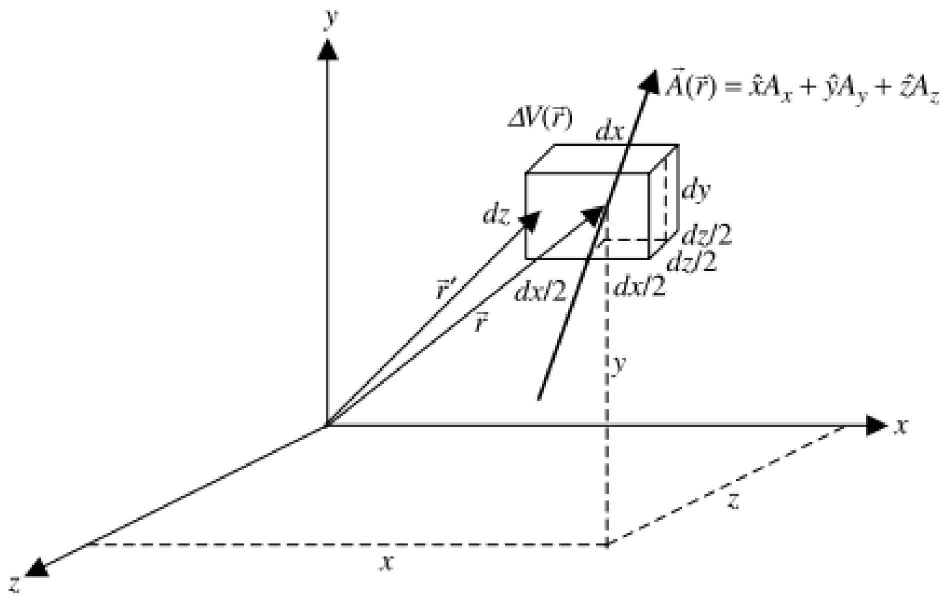
$$\frac{\Delta V}{\Delta l}\boldsymbol{u_l} \rightarrow \frac{dV}{dl}\boldsymbol{u_l} = \frac{dV}{dn}\frac{dn}{dl} = \frac{dV}{dn}cos\theta = \frac{dV}{dl}\boldsymbol{u_n}\cdot\boldsymbol{u_l} = \nabla V\cdot\boldsymbol{u_l}$$

Review – Differential relations for vectors

- Gradient in different coordinate systems:
  - Cartesian: $\nabla V = \left(\frac{\partial V}{\partial x}\hat{x},\ \frac{\partial V}{\partial y}\hat{y},\ \frac{\partial V}{\partial z}\hat{z}\right)$
  - Cylindrical: $\nabla V = \left(\frac{\partial V}{\partial r}\hat{r},\ \frac{1}{r}\frac{\partial V}{\partial \varphi}\hat{\varphi},\ \frac{\partial V}{\partial z}\hat{z}\right)$
  - Spherical: $\nabla V = \left(\frac{\partial V}{\partial R}\hat{R},\ \frac{1}{R}\frac{\partial V}{\partial \theta}\hat{\theta},\ \frac{1}{R\sin\theta}\frac{\partial V}{\partial \varphi}\hat{\varphi}\right)$
- Gradient operator:
  - Cartesian: $\nabla = \frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}$
  - Cylindrical: $\nabla = \frac{\partial}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial}{\partial \varphi}\hat{\varphi} + \frac{\partial}{\partial z}\hat{z}$
  - Spherical: $\nabla = \frac{\partial}{\partial R}\hat{R} + \frac{1}{R}\frac{\partial}{\partial \theta}\hat{\theta} + \frac{1}{R\sin\theta}\frac{\partial}{\partial \varphi}\hat{\varphi}$

Review – Differential relations for vectors

- **Divergence** of a vector field
  - The divergence of a vector at a point is the net outflux of that vector per unit volume. Thus it gives a measure of the strength of the sources that produce the vector field.
  - It is denoted by $\nabla \cdot \boldsymbol{A}$ or div(**A**), where ∇ is the same vector differential operator "del."



In Cartesian coordinates:
A(**r**) is the spatial distributed vector field;

The divergence is defined as:

$$\nabla \cdot \boldsymbol{A}(\boldsymbol{r}) = \lim_{\Delta V(\boldsymbol{r}) \to 0} \frac{\oiint_S \hat{\boldsymbol{n}} \cdot \boldsymbol{A}(\boldsymbol{r}) ds}{\Delta V(\boldsymbol{r})}$$

which can be calculated by:

$$\nabla \cdot \boldsymbol{A}(\boldsymbol{r}) = \left(\hat{\boldsymbol{x}}\frac{\partial}{\partial x} + \hat{\boldsymbol{y}}\frac{\partial}{\partial y} + \hat{\boldsymbol{z}}\frac{\partial}{\partial z}\right) \cdot \left(\hat{\boldsymbol{x}}A_x + \hat{\boldsymbol{y}}A_y + \hat{\boldsymbol{z}}A_z\right)$$

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Review – Differential relations for vectors

- Derivation in Cartesian coordinates:
  - A(r) is the spatial distributed vector field;

$$\boldsymbol{\nabla} \cdot \boldsymbol{A}(\boldsymbol{r}) = \lim_{\Delta V(\boldsymbol{r}) \to 0} \frac{\oiint_S \hat{\boldsymbol{n}} \cdot \boldsymbol{A}(\boldsymbol{r}) ds}{\Delta V(\boldsymbol{r})}$$

  - Integrate the closed-surface integral over each of six faces, and add them up get:

$$\oiint_S \hat{\boldsymbol{n}} \cdot \boldsymbol{A}(\boldsymbol{r}) ds = \iint_{S_{x1}+S_{x2}+S_{y1}+S_{y2}+S_{z1}+S_{z2}} \hat{\boldsymbol{n}} \cdot \boldsymbol{A}(\boldsymbol{r}) ds$$

$$= \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \right) dxdydz = \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \right) dV$$

  - Therefore:

$$\boldsymbol{\nabla} \cdot \boldsymbol{A}(\boldsymbol{r}) = \lim_{\Delta V(\boldsymbol{r}) \to 0} \frac{\oiint_S \hat{\boldsymbol{n}} \cdot \boldsymbol{A}(\boldsymbol{r}) ds}{\Delta V(\boldsymbol{r})} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$= \left(\hat{\boldsymbol{x}} \frac{\partial}{\partial x} + \hat{\boldsymbol{y}} \frac{\partial}{\partial y} + \hat{\boldsymbol{z}} \frac{\partial}{\partial z} \right) \cdot \left(\hat{\boldsymbol{x}} A_x + \hat{\boldsymbol{y}} A_y + \hat{\boldsymbol{z}} A_z \right)$$

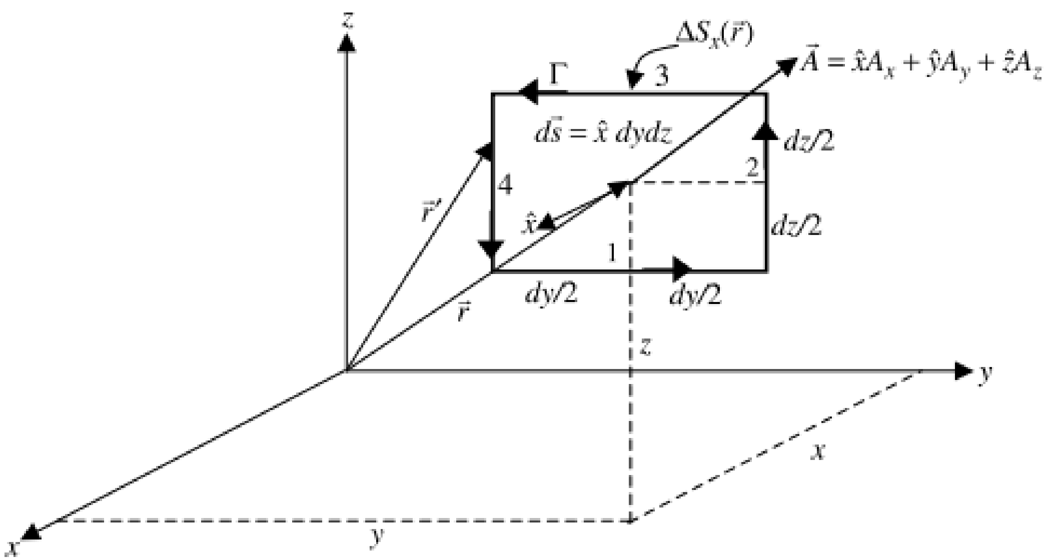
Review – Differential relations for vectors

- Divergence in different coordinate systems:
  - Cartesian: $\nabla \cdot \boldsymbol{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$
  - Cylindrical: $\nabla \cdot \boldsymbol{A} = \frac{1}{r}\frac{\partial (rA_r)}{\partial r} + \frac{1}{r}\frac{\partial A_\varphi}{\partial \varphi} + \frac{\partial A_z}{\partial z}$
  - Spherical: $\nabla \cdot \boldsymbol{A} = \frac{1}{R^2}\frac{\partial (R^2 A_R)}{\partial R} + \frac{1}{Rsin\theta}\frac{\partial (A_\theta sin\theta)}{\partial \theta} + \frac{1}{Rsin\theta}\frac{\partial A_\varphi}{\partial \varphi}$

- Some “divergence rules”:
  - $\nabla \cdot a = 0$
  - $\nabla \cdot (\boldsymbol{A_1} + \boldsymbol{A_2}) = \nabla \cdot \boldsymbol{A_1} + \nabla \cdot \boldsymbol{A_2}$
  - $\nabla \cdot c\boldsymbol{A} = c\nabla \cdot \boldsymbol{A}$

Review – *Differential relations for vectors*

- ***Curl*** of a vector field
  - Curl is a vector field with magnitude equal to the maximum "circulation" at each point and oriented perpendicularly to this plane of circulation for each point. More precisely, the magnitude of curl is the limiting value of circulation per unit area.
  - It is denoted by $\nabla \times \boldsymbol{A}$ or curl(**A**) or rot(**A**).



In Cartesian coordinates:
A(**r**) is the spatial distributed vector field;
The curlis defined as:

$$\nabla \times \boldsymbol{A}(\boldsymbol{r}) = \lim_{\Delta S(\boldsymbol{r}) \to 0} \frac{\hat{n} \oint \boldsymbol{A}(\boldsymbol{r}) ds}{\Delta S(\boldsymbol{r})}$$

which can be calculated by:

$$\nabla \times \boldsymbol{A}(\boldsymbol{r}) = \begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

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Review – Differential relations for vectors

- Curl in different coordinate systems:
  - Cartesian: $\nabla \times \boldsymbol{A}(\boldsymbol{r}) = \begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$
  - Cylindrical: $\nabla \times \boldsymbol{A}(\boldsymbol{r}) = \frac{1}{r} \begin{vmatrix} \hat{\boldsymbol{r}} & r\hat{\boldsymbol{\varphi}} & \hat{\boldsymbol{z}} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \varphi} & \frac{\partial}{\partial z} \\ A_r & rA_\varphi & A_z \end{vmatrix}$
  - Spherical: $\nabla \times \boldsymbol{A}(\boldsymbol{r}) = \frac{1}{R^2 sin\theta} \begin{vmatrix} \hat{\boldsymbol{R}} & R\hat{\boldsymbol{\theta}} & Rsin\theta\hat{\boldsymbol{\varphi}} \\ \frac{\partial}{\partial R} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \varphi} \\ A_R & RA_\theta & Rsin\theta A_\varphi \end{vmatrix}$

Review – Laplacian Operator

- Laplacian operator – a second-order differential operator that occurs frequently in the study of field theory.
  - Symbolically written as Δ or ∇^2
  - Defined as the divergence of a gradient of a scalar function.

$$\Delta f \triangleq \nabla^2 f = \nabla \cdot (\nabla f)$$

- In different coordinates:
  - Cartesian: $\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$
  - Cylindrical: $\Delta f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \varphi^2} + \frac{\partial^2 f}{\partial z^2}$
  - Spherical: $\Delta f = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial f}{\partial R}\right) + \frac{1}{R^2 sin\theta}\frac{\partial}{\partial \theta}\left(sin\theta\frac{\partial f}{\partial \theta}\right) + \frac{1}{R^2 sin^2\theta}\frac{\partial^2 f}{\partial \varphi^2}$

Integral and Differential

- Integrals in different CSs
  - Line integral
  - Surface integral
  - Volume integral

- Differentials
  - Gradient
  - Divergence
  - Curl
  - Laplacian

Next

- Review the knowledge of EEE103 and EEE108
  - Especially the parts will be used in this module
- Complete the pre-module test independently
  - It's not an exam, and won't take any part in your final mark;
  - The purpose of this test is just to provide me some information;
  - So, please do it independently!