Xi'an Jiaotong-Liverpool University
西交利物浦大学

EEE220 Instrumentation and Control System

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Lecture 9

Outline

Control Systems:

Mathematical Models of Systems: part 1/2

- ❑ Differential Equations of Physical Systems
- ❑ The Transfer Function of Linear Systems
- ❑ Block Diagram Models
- ❑ Signal-Flow Graph Models
- ❑ Simulation Tool

Overview

To understand and control complex systems, one must obtain quantitative **mathematical models** of these systems.

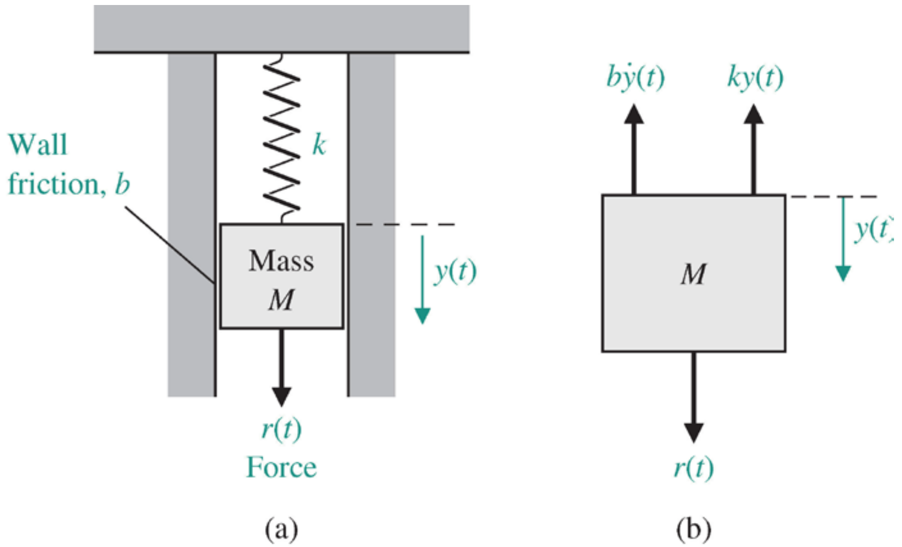
Mathematical models of physical systems are key elements in the design and analysis of control systems.

- The dynamic behavior is generally described by **ordinary differential equations (ODEs)**;
- Linearization approximations allow the use of **Laplace transform**;
- **Transfer functions**, which can be organized into **block diagrams** and **signal-flow graphs**, are very convenient and natural tools for designing and analyzing complicated control systems.

Six Step Approach to Dynamic System Modeling

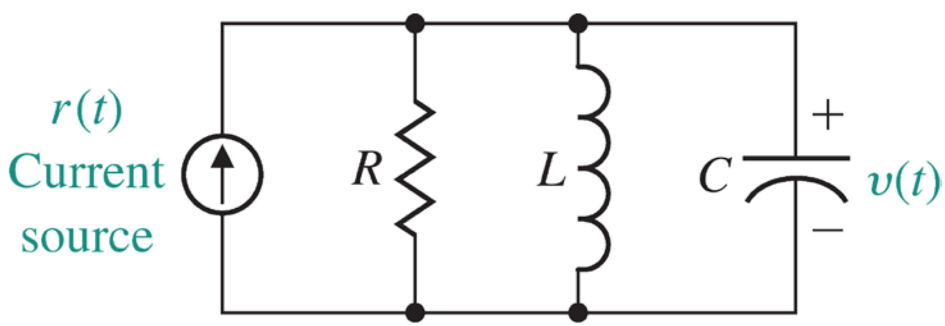
1. Define the system and its components
2. Formulate the mathematical model and list the necessary assumptions
3. Write the differential equations describing the model
4. Solve the equations for the desired output variables
5. Examine the solutions and the assumptions
6. If necessary, reanalyze or redesign the system

Differential Equation of Physical Systems



- Mechanical System

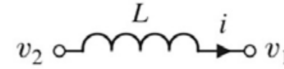
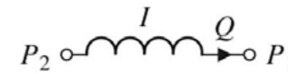
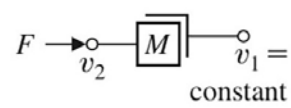
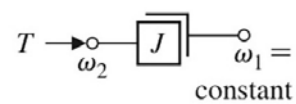
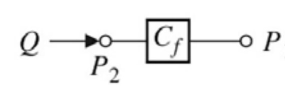
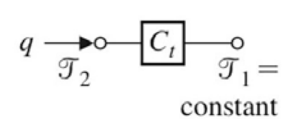
(Spring-mass-damper system)



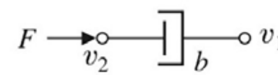
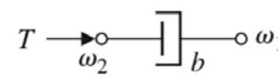
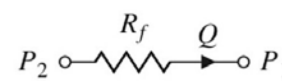
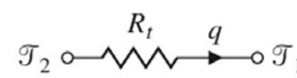
- Electrical System

(RLC circuit)

Governing Differential Equations for Ideal Elements

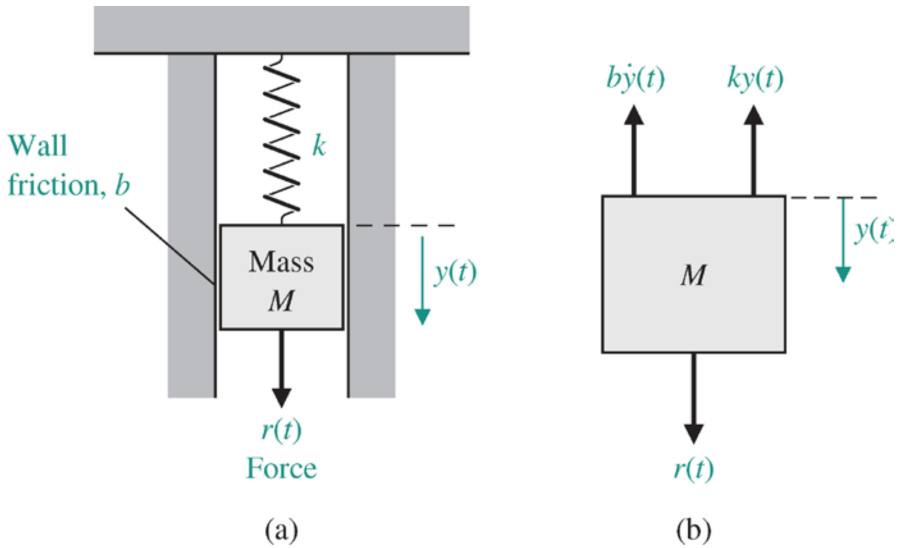
Type of Element	Physical Element	Governing Equation	Energy E or Power $\mathscr{P}$	Symbol
Inductive storage	Electrical inductance	$v_{21} = L\frac{di}{dt}$	$E = \frac{1}{2}Li^2$	
	Translational spring	$v_{21} = \frac{1}{k}\frac{dF}{dt}$	$E = \frac{1}{2}\frac{F^2}{k}$	
	Rotational spring	$\omega_{21} = \frac{1}{k}\frac{dT}{dt}$	$E = \frac{1}{2}\frac{T^2}{k}$	
	Fluid inertia	$P_{21} = I\frac{dQ}{dt}$	$E = \frac{1}{2}IQ^2$	
Capacitive storage	Electrical capacitance	$i = C\frac{dv_{21}}{dt}$	$E = \frac{1}{2}C{v_{21}}^2$	
	Translational mass	$F = M\frac{dv_2}{dt}$	$E = \frac{1}{2}M{v_2}^2$	
	Rotational mass	$T = J\frac{d\omega_2}{dt}$	$E = \frac{1}{2}J{\omega_2}^2$	
	Fluid capacitance	$Q = C_f\frac{dP_{21}}{dt}$	$E = \frac{1}{2}C_f{P_{21}}^2$	
	Thermal capacitance	$q = C_t\frac{d\mathscr{T}_2}{dt}$	$E = C_t\mathscr{T}_2$	

Governing Differential Equations for Ideal Elements (cont'd)

Type of Element	Physical Element	Governing Equation	Energy E or Power $\mathcal{P}$	Symbol
Energy dissipators	Electrical resistance	$i = \frac{1}{R} v_{21}$	$\mathcal{P} = \frac{1}{R} {v_{21}}^2$	
	Translational damper	$F = b v_{21}$	$\mathcal{P} = b {v_{21}}^2$	
	Rotational damper	$T = b \omega_{21}$	$\mathcal{P} = b {\omega_{21}}^2$	
	Fluid resistance	$Q = \frac{1}{R_f} P_{21}$	$\mathcal{P} = \frac{1}{R_f} {P_{21}}^2$	
	Thermal resistance	$q = \frac{1}{R_t} \mathcal{T}_{21}$	$\mathcal{P} = \frac{1}{R_t} \mathcal{T}_{21}$	

- *Through-variable:* F = force, T = torque, i = current, Q = fluid volumetric flow rate, q = heat flow rate.
- *Across-variable:* v = translational velocity, ω = angular velocity, v = voltage, P = pressure, $\mathcal{T}$ = temperature.
- *Inductive storage:* L = inductance, $1/k$ = reciprocal translational or rotational stiffness, I = fluid inertance.
- *Capacitive storage:* C = capacitance, M = mass, J = moment of inertia, C_f = fluid capacitance, C_t = thermal capacitance.
- *Energy dissipators:* R = resistance, b = viscous friction, R_f = fluid resistance, R_t = thermal resistance.

ODE for Mechanical System



(a) Spring-mass-damper system.
(b) Free-body diagram.

Second-order linear constant-coefficient (time-invariant) system

$$\boxed{M\cdot\frac{d^2}{dt^2}y(t) + b\cdot\frac{d}{dt}y(t) + k\cdot y(t) = r(t)}$$

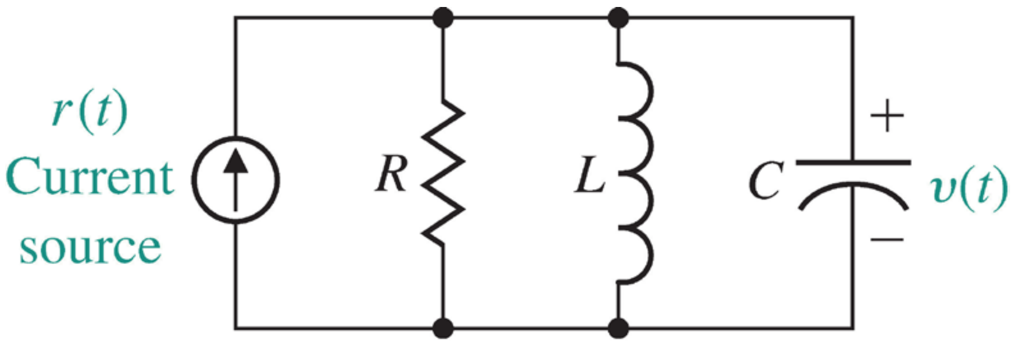
$$y(0) = y_0$$

$$y(t) = K_1 e^{-\alpha_1 t}\sin(\beta_1 t + \theta_1)$$

Using Newton's laws:

- Model wall friction as a viscous damper, that is, the friction force is linearly proportional to the velocity of the mass;
- M is the mass; b is the friction constant; k is the spring constant of ideal spring;

ODE for Electrical System



RLC circuit.

Second-order linear constant-coefficient (time-invariant) system

Using Kirchhoff's laws.

$$\frac{v(t)}{R} + C \cdot \frac{d}{dt} v(t) + \frac{1}{L} \cdot \int_0^t v(t)\, dt = r(t)$$

$$r(0) = r_0$$

$$v(t) = K_2 e^{-\alpha_2 t} \cos(\beta_2 t + \theta_2)$$

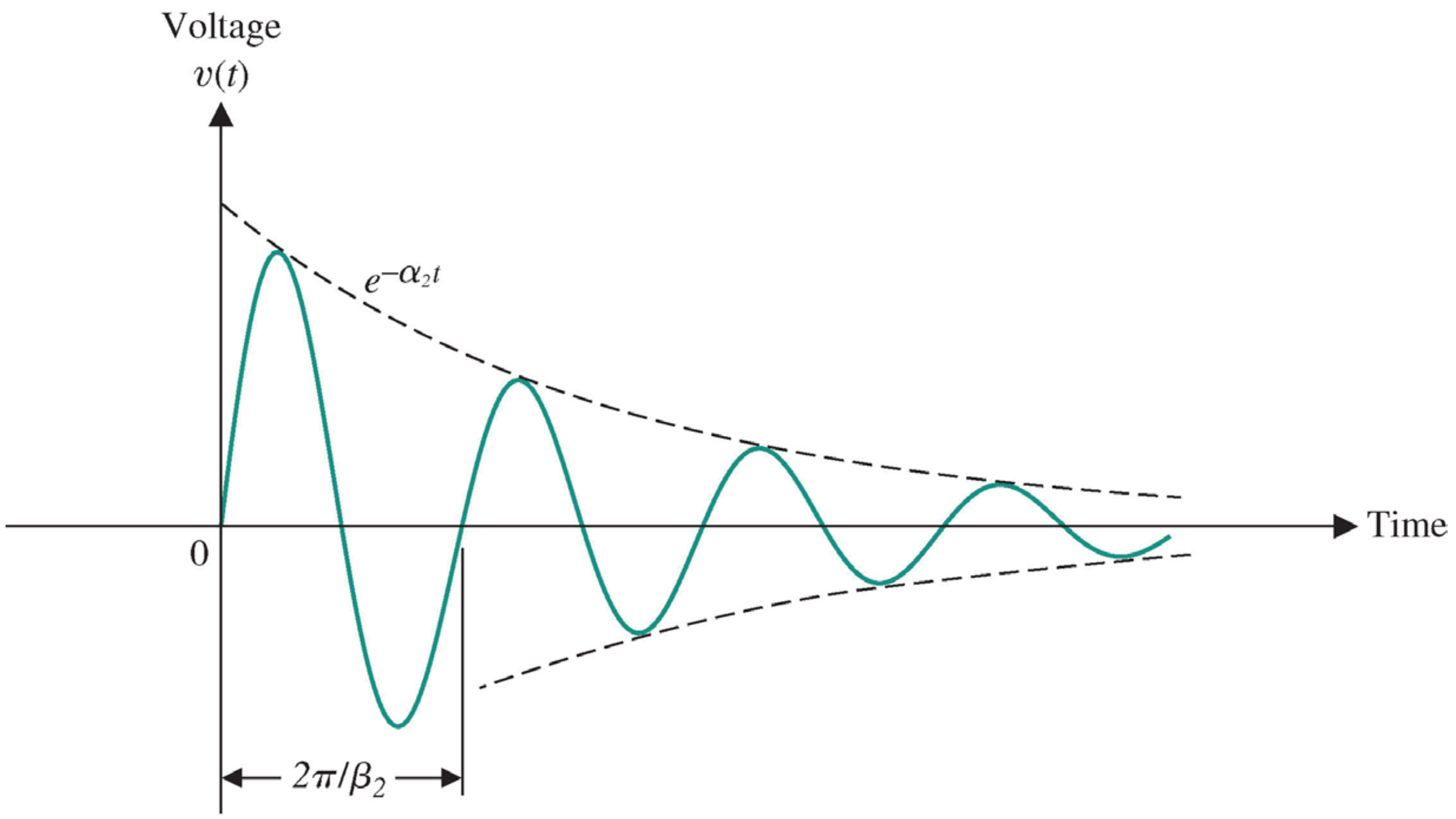
If assume:

$$\int_0^t v(t) dt = y(t)$$

Then we have:

$$\boxed{C \cdot \frac{d^2}{dt^2} y(t) + \frac{1}{R}\frac{d}{dt} y(t) + \frac{1}{L} y(t) = r(t)}$$

Second-order System Response



Linear Approximations of Physical Systems

Linear system- Necessary Conditions:

Principle of Superposition: when a system at rest is subject to an excitation $x_1(t)$, it provides a response $y_1(t)$; when a system at rest is subject to an excitation $x_2(t)$, it provides a response $y_2(t)$; For a linear system, it's necessary that the excitation $x_1(t)+x_2(t)$ results in a response $y_1(t)+y_2(t)$;

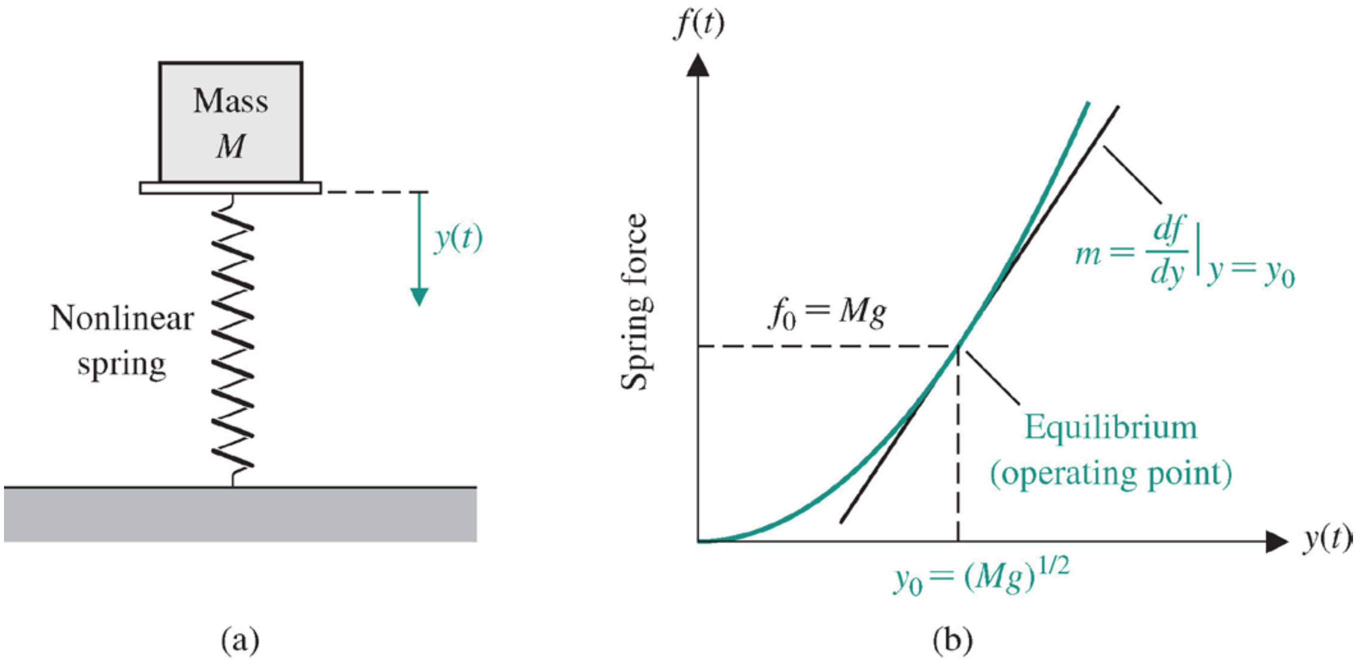
Homogeneity: consider a system with an input $x(t)$ that results in $y(t)$; For a linear system, it's necessary that the response to input $\beta x(t)$ must be equal to $\beta y(t)$.

The linearity of many mechanical and electrical elements can be assumed over a reasonably large range of the variables; this is not usually the case for thermal and fluid elements, which are more frequently nonlinear in character.

Nonlinear System

A great majority of physical system are linear within some range of the variables. In general, systems ultimately become nonlinear as the variables are increased without limit.

For example, the spring in spring-mass-damper system would behave nonlinearly, eventually overextended and break if the force continually increased.



The Laplace Transform

The Laplace transform can be used for
linear time-invariant (LTI) systems.

***time-invariant**: coefficients of system don't change with time (they are constants).

Definition:

$$L\{f(t)\} = F(s) = \int_{0^-}^{\infty} f(t)e^{-st}dt$$

Inverse Laplace transform:

$$f(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} F(s)e^{+st}ds$$

For $f(t)$ to be transformable, it is sufficient that $\int_{0^-}^{\infty}|f(t)|e^{-\sigma_1 t}dt < \infty$.

The Laplace transform method substitute relatively easily solved algebraic equations for the more difficult differential equations. The time-response of a system can be obtained by solving the algebraic equations of variable of interest.

Some Important Theorems

Property	Time Domain	Frequency Domain
1. Time delay	$f(t-T)\cdot u(t-T)$	$e^{-(s\cdot T)}\cdot F(s)$
2. Time scaling	$f(at)$	$\frac{1}{a}\cdot F\left(\frac{s}{a}\right)$
3. Frequency differentiation	$t\cdot f(t)$	$-\frac{d}{ds}F(s)$
4. Frequency shifting	$f(t)\cdot e^{-(a\cdot t)}$	$F(s+a)$
5. Frequency Integration	$\frac{f(t)}{t}$	$\int_0^{\infty} F(s)\,ds$

Initial value theorem

$$f(0)=\lim_{s\to\infty} s\,f(s)$$

Final value

$$f(\infty)=\lim_{s\to 0} s\,f(s)$$

First order differential equations

$$\mathcal{L}\left\{\frac{df(t)}{dt}\right\} = \mathcal{L}\{f'(t)\} = sF(s) - f(0)$$

Second order differential equations

$$\mathcal{L}\left\{\frac{d^2 f(t)}{dt^2}\right\} = \mathcal{L}\{f''(t)\} = s^2 F(s) - s\,f(0) - f'(0)$$

nth order differential equations

$$\mathcal{L}\left\{\frac{d^n f(t)}{dt^n}\right\} = \mathcal{L}\{f^{(n)}(t)\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - sf^{n-2}(0) - sf^{n-1}(0)$$

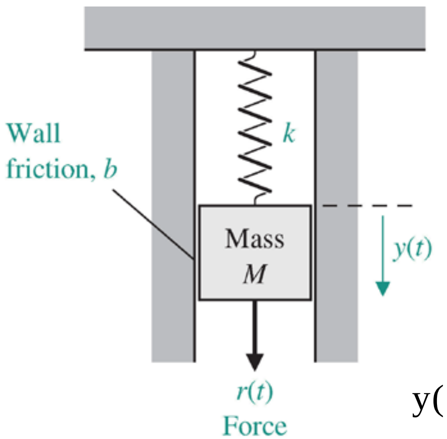
Important Laplace Pairs

$f(t)$	$F(s)$
Step function, $u(t)$	$\frac{1}{s}$
e^{-at}	$\frac{1}{s+a}$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
t^n	$\frac{n!}{s^{n+1}}$
$f^{(k)}(t) = \frac{d^k f(t)}{dt^k}$	$s^k F(s) - s^{k-1} f(0^-) - s^{k-2} f'(0^-) - \ldots - f^{(k-1)}(0^-)$
$\frac{1}{a^2+\omega^2} + \frac{1}{\omega\sqrt{a^2+\omega^2}} e^{-at} \sin(\omega t - \phi)$, $\phi = \tan^{-1}\frac{\omega}{-a}$	$\frac{1}{s[(s+a)^2+\omega^2]}$
$1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin\left(\omega_n\sqrt{1-\zeta^2}\,t + \phi\right)$, $\phi = \cos^{-1}\zeta, \zeta < 1$	$\frac{\omega_n^2}{s(s^2+2\zeta\omega_n s+\omega_n^2)}$
$\frac{\alpha}{a^2+\omega^2} + \frac{1}{\omega}\left[\frac{(\alpha-a)^2+\omega^2}{a^2+\omega^2}\right]^{1/2} e^{-at}\sin(\omega t + \phi)$. $\phi = \tan^{-1}\frac{\omega}{\alpha - a} - \tan^{-1}\frac{\omega}{-a}$	$\frac{s+\alpha}{s[(s+a)^2+\omega^2]}$

$f(t)$	$F(s)$
$\int_{-\infty}^{t} f(t)\, dt$	$\frac{F(s)}{s} + \frac{1}{s}\int_{-\infty}^{0} f(t)\, dt$
Impulse function $\delta(t)$	1
$e^{-at}\sin \omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$
$e^{-at}\cos \omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$
$\frac{1}{\omega}[(\alpha-a)^2+\omega^2]^{1/2}e^{-at}\sin(\omega t+\phi)$, $\phi = \tan^{-1}\frac{\omega}{\alpha-a}$	$\frac{s+\alpha}{(s+a)^2+\omega^2}$
$\frac{\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin \omega_n\sqrt{1-\zeta^2}t, \zeta < 1$	$\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$

Laplace Transform Application

Reconsider the mass-spring-damper system:



$$Y(s) = \frac{(M s + b)\cdot y_o}{M s^2 + b s + k} = \frac{p(s)}{q(s)}$$

$\boldsymbol{q(s)}$: characteristic equation, its roots determine the character of time response.

$$y(s) = \frac{\left(s + \frac{b}{M}\right)\cdot(y_o)}{\left[s^2 + \left(\frac{b}{M}\right)\cdot s + \frac{k}{M}\right]} = \frac{\left(s + 2\cdot\zeta\cdot\omega_n\right)}{s^2 + 2\cdot\zeta\cdot\omega_n s + \omega_n^2}$$

$$s_1 = -\left(\zeta\cdot\omega_n\right) + \omega_n\cdot\sqrt{\zeta^2 - 1}$$

$$\omega_n = \sqrt{\frac{k}{M}} \qquad \zeta = \frac{b}{\left(2\cdot\sqrt{k\cdot M}\right)}$$

$$s_2 = -\left(\zeta\cdot\omega_n\right) - \omega_n\cdot\sqrt{\zeta^2 - 1}$$

ζ: the damping ratio;
ω_n is the natural frequency.

- $\zeta > 1$: the roots are real and the system is **overdamped**;
- $\zeta < 1$: the roots are complex and the system is **underdamped**;
- $\zeta = 1$: the roots are repeated and real, and condition is called **critical damping**.

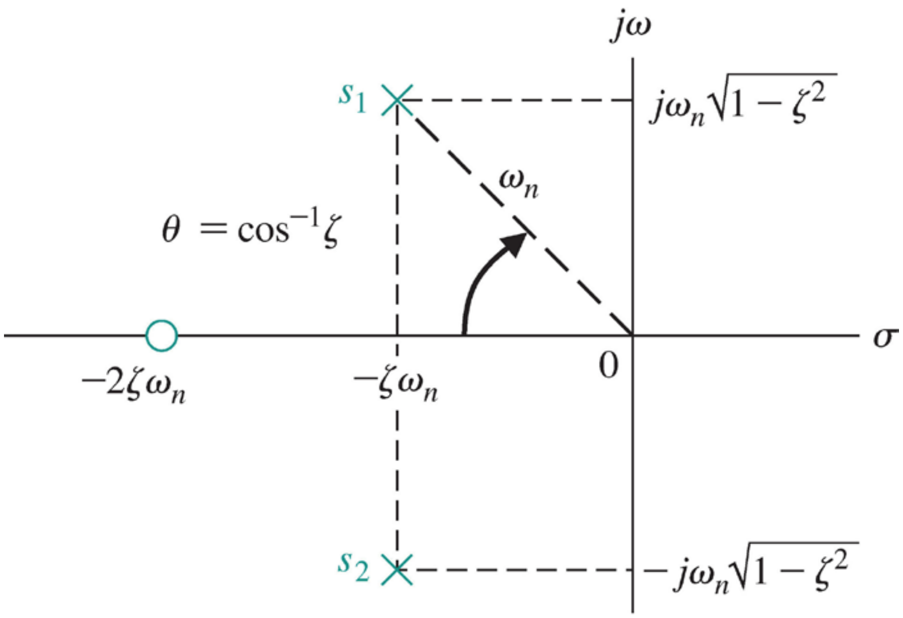
Representation in s-Plane

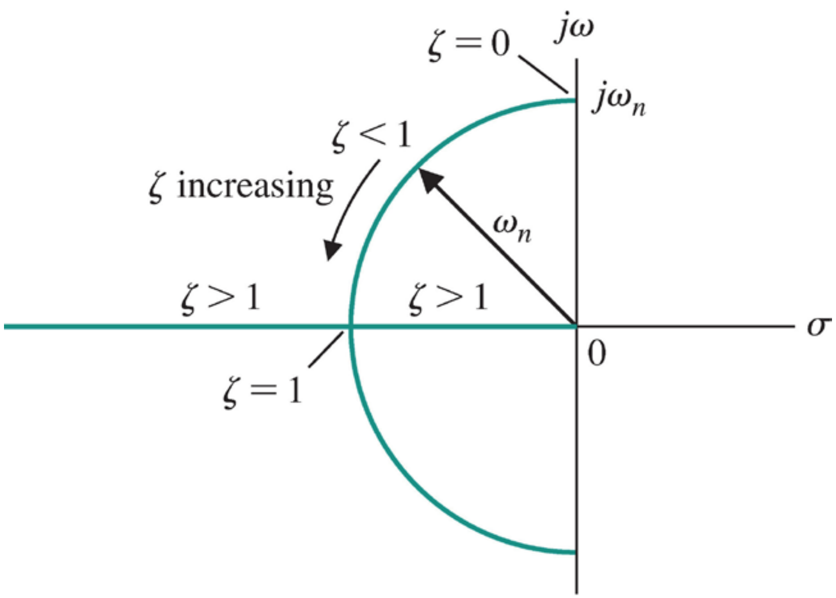
when ζ < 1, s1 and s2 are complex conjugates:

$$s1 = -(\zeta \cdot \omega_n) + j \cdot \omega_n \cdot \sqrt{1 - \zeta^2}$$

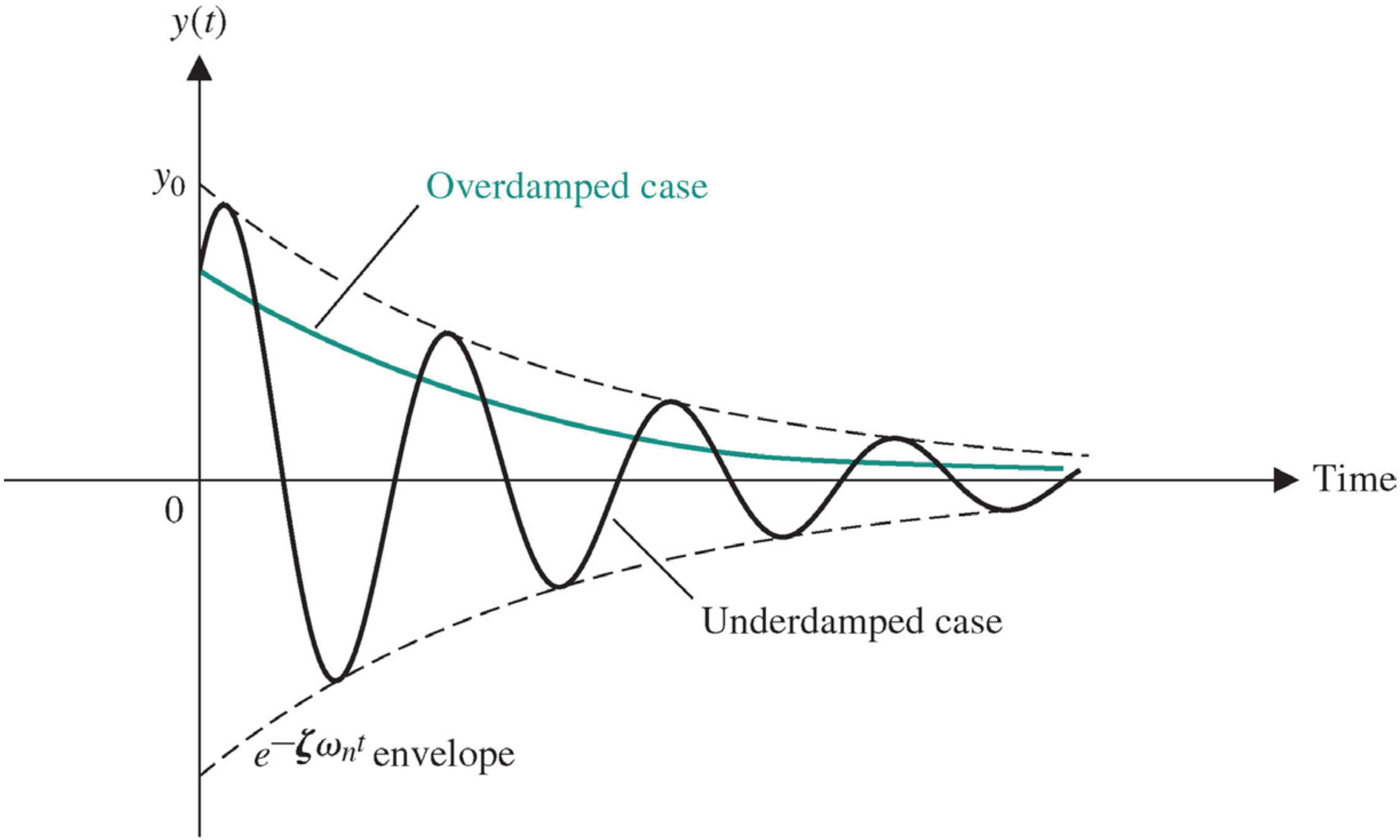
$$s2 = -(\zeta \cdot \omega_n) - j \cdot \omega_n \cdot \sqrt{1 - \zeta^2}$$

The transient response is increasing oscillatory as the roots approach the imaginary axis when ζ approaches zeros.





System Response



Obtaining Time Response from Inverse Laplace Transform

For a specific case, $k/M = 2$ and $b/M = 3$, expending $Y(s)$ in a **partial fraction expansion**:

$$Y(s) = \frac{(s+3)y_0}{(s+1)(s+2)}. \longrightarrow Y(s) = \frac{k_1}{s+1} + \frac{k_2}{s+2},$$

When $y_0 = 1$:

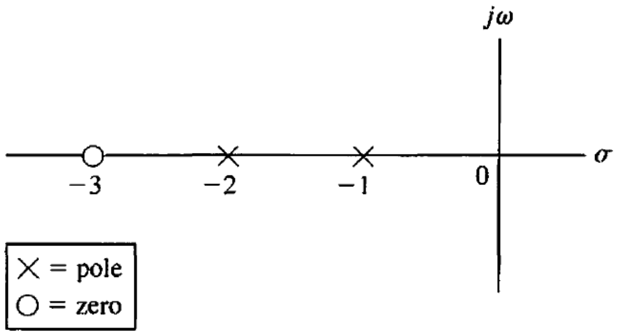
$$k_1 = \left.\frac{(s-s_1)p(s)}{q(s)}\right|_{s=s_1}$$

$$= \left.\frac{(s+1)(s+3)}{(s+1)(s+2)}\right|_{s_1=-1} = 2$$

$$k_2 = \left.\frac{(s-s_2)p(s)}{q(s)}\right|_{s=s_2}$$

$$= \left.\frac{(s+2)(s+3)}{(s+1)(s+2)}\right|_{s_2=-2} = -1$$

An s-plane pole and zero plot.



ζ ? Overdamped or underdamped?

The inverse Laplace transform is:

$$y(t) = \mathscr{L}^{-1}\left\{\frac{2}{s+1}\right\} + \mathscr{L}^{-1}\left\{\frac{-1}{s+2}\right\}.$$

The we find:

$$y(t) = 2e^{-t} - 1e^{-2t}.$$

Steady-state (final) value is:

Final Value Theorem: $$\boxed{\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s),}$$

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = 0.$$

Example 9.1: Solution of An ODE

If ODE of a system is: $$\frac{d^2}{dt^2}y(t) + 4\frac{d}{dt}y(t) + 3y(t) = 2r(t)$$

with the initial conditions and input: $y(0) = 1, \frac{d}{dt}y(0) = 0,\ r(t) = 1$ (step function)

How to find the time-domain response of the system?

1. Perform Laplace transform on the ODE:

$$(s^2Y(s) - sy(0) - \frac{d}{dt}y(0)) + 4(sY(s) - y(0)) + 3Y(s) = 2R(s)$$

Since $y(0) = 1, \frac{d}{dt}y(0) = 0$, R(s) = $\frac{1}{s}$:

$$Y(s) = \frac{s+4}{s^2+4s+3} + \frac{2}{s(s^2+4s+3)}$$

2. Expand the equation into partial fractions, which will yield:

$$Y(s) = \left(\frac{\frac{3}{2}}{s+1} + \frac{-\frac{1}{2}}{s+3}\right) + \left(\frac{-1}{s+1} + \frac{\frac{1}{3}}{s+3} + \frac{\frac{2}{3}}{s}\right)$$

3. Derive time-domain equation by using inverse Laplace transform:

$$y(t) = \left(\frac{3}{2}e^{-t} - \frac{1}{2}e^{-3t}\right) + \left(-e^{-t} + \frac{1}{3}e^{-3t} + \frac{2}{3}\right) = \frac{1}{2}e^{-t} - \frac{1}{6}e^{-3t} + \frac{2}{3}$$

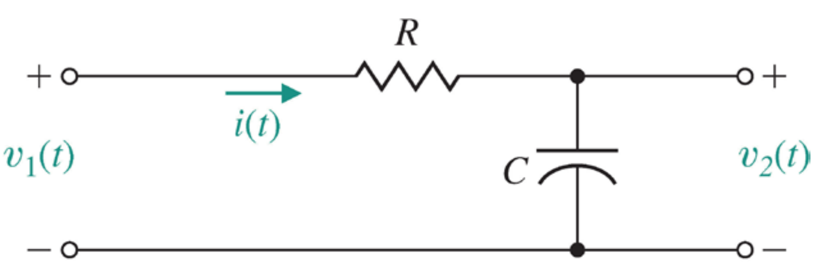
<u>steady-state response:</u>

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} Y(s) = \frac{2}{3}$$

Transfer Function

The **Transfer Function (TF)** of a linear system is defined as the ratio of the Laplace transform of the output to the Laplace transform of the input variable, with all initial conditions assumed to be zero.

- TF represents the relationship describing the dynamics of the system under consideration;
- TF may be defined only for **LTI** systems.



$$V_1(s) = \left(R + \frac{1}{Cs}\right) \cdot I(s) \qquad Z_1(s) = R$$

$$V_2(s) = \left(\frac{1}{Cs}\right) \cdot I(s) \qquad Z_2(s) = \frac{1}{Cs}$$

$$\frac{V_2(s)}{V_1(s)} = \frac{\frac{1}{Cs}}{R + \frac{1}{Cs}} = \frac{Z_2(s)}{Z_1(s) + Z_2(s)}$$

$$G(s) = \frac{V_2(s)}{V_1(s)} = \frac{1/Cs}{R + 1/Cs} = \frac{1}{RCs+1} = \frac{1}{\tau s+1} = \frac{1/\tau}{s + 1/\tau}$$

$\tau = RC$: time constant of the network.

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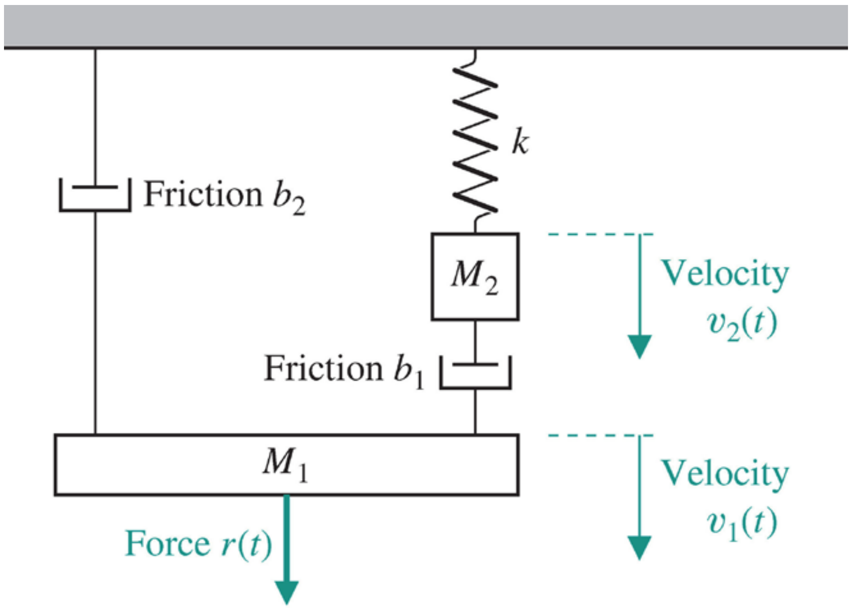
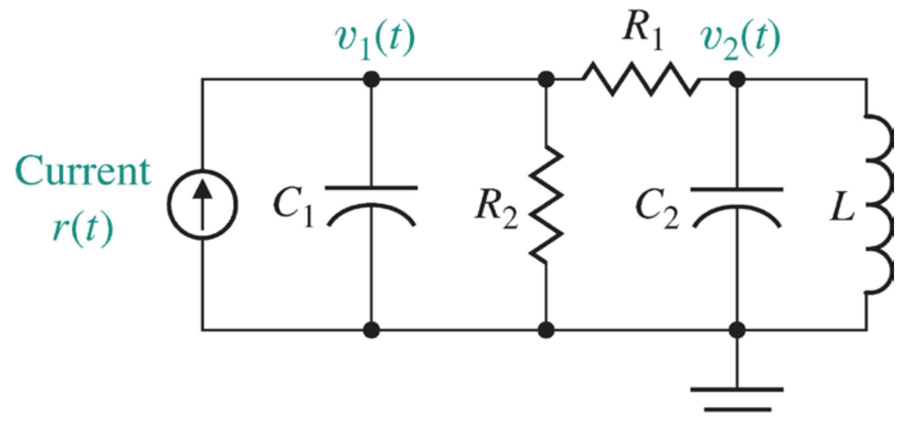
Transfer Function of A System

Mechanical vs. Electrical System: Analogue



(a) Two-mass mechanical system.



(b) Two-node electric circuit

$C_1 = M_1, C_2 = M_2,$
$L = 1/k, R_1 = 1/b_1,$
$R_2 = 1/b_2$

→ **TF is the same!**

Assuming that the initial conditions are zero:

$$M_1 s V_1(s) + (b_1 + b_2)V_1(s) - b_1 V_2(s) = R(s),$$

and

$$M_2 s V_2(s) + b_1(V_2(s) - V_1(s)) + k\frac{V_2(s)}{s} = 0.$$

Rearranging the equations, we obtain:

$$(M_1 s + (b_1 + b_2))V_1(s) + (-b_1)V_2(s) = R(s),$$

$$(-b_1)V_1(s) + \left(M_2 s + b_1 + \frac{k}{s}\right)V_2(s) = 0,$$

Matrix form:

$$\begin{bmatrix} M_1 s + b_1 + b_2 & -b_1 \\ -b_1 & M_2 s + b_1 + \frac{k}{s} \end{bmatrix}\begin{bmatrix} V_1(s) \\ V_2(s) \end{bmatrix} = \begin{bmatrix} R(s) \\ 0 \end{bmatrix}.$$

Assuming the velocity of M1 (v_1) is the output variable:

$$V_1(s) = \frac{(M_2 s + b_1 + k/s)R(s)}{(M_1 s + b_1 + b_2)(M_2 s + b_1 + k/s) - {b_1}^2}.$$

Then the Transfer function of the system is:

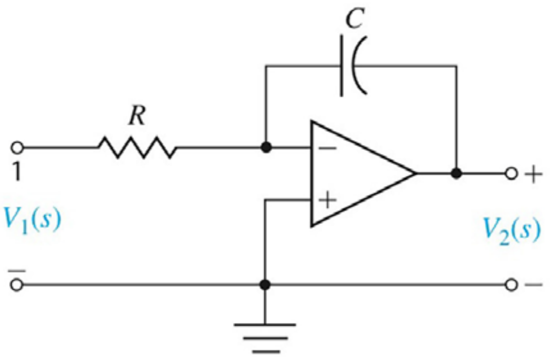
$$G(s) = \frac{V_1(s)}{R(s)} = \frac{(M_2 s + b_1 + k/s)}{(M_1 s + b_1 + b_2)(M_2 s + b_1 + k/s) - {b_1}^2}$$

$$= \frac{(M_2 s^2 + b_1 s + k)}{(M_1 s + b_1 + b_2)(M_2 s^2 + b_1 s + k) - {b_1}^2 s}.$$

Transfer Function of Op-Amp Circuits

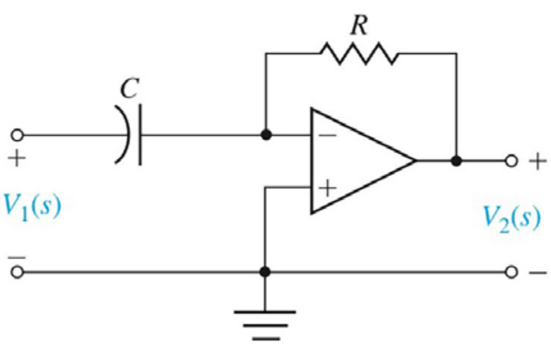
Element or System | **$G(s)$**

1. Integrating circuit, filter

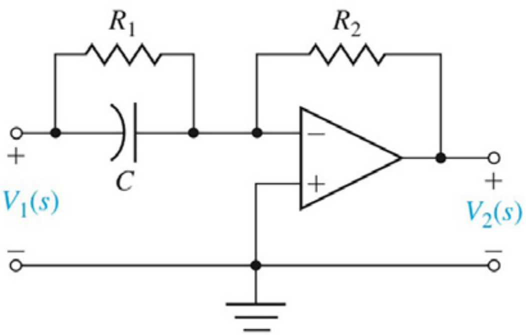
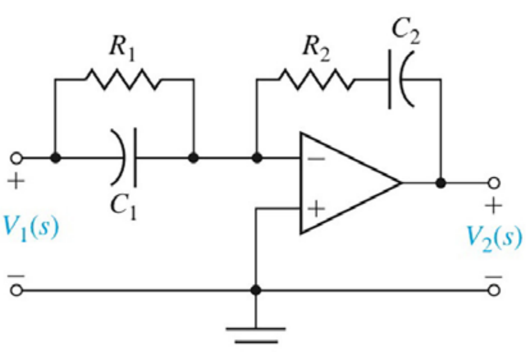


$$\frac{V_2(s)}{V_1(s)} = -\frac{1}{RCs}$$

2. Differentiating circuit



$$\frac{V_2(s)}{V_1(s)} = -RCs$$

Element or System	$G(s)$
3. Differentiating circuit 	$\frac{V_2(s)}{V_1(s)} = -\frac{R_2(R_1Cs + 1)}{R_1}$
4. Integrating filter 	$\frac{V_2(s)}{V_1(s)} = -\frac{(R_1C_1s + 1)(R_2C_2s + 1)}{R_1C_2s}$

Thank You !