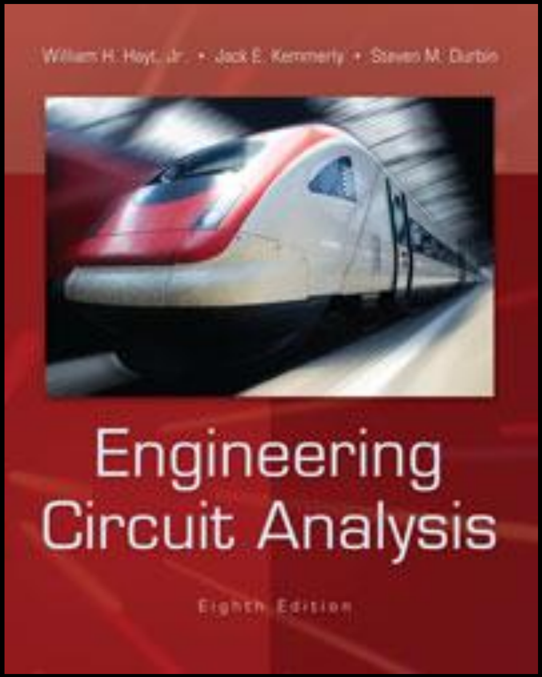
# Chapter 14
# Complex Frequency and the Laplace Transform



# Motivating Complex Frequency

An exponentially damped sinusoidal function, such as the voltage

$$v(t) = V_m e^{\sigma t} \cos(\omega t + \theta)$$

includes as "special cases"

- dc, when $\sigma = \omega = 0$: $v(t) = V_m cos(\theta) = V_0$
- sinusoidal, when $\sigma = 0$: $v(t) = V_m \cos(\omega t + \theta)$
- exponential, when $\omega = 0$: $v(t) = V_m e^{\sigma t}$

# The Complex Frequency

Any function that may be written in the form

$$f(t) = \boldsymbol{K}e^{st}$$

where **K** and **s** are complex constants (independent of time) is characterized by the complex frequency **s.**

# The DC Case

A constant voltage

$$v(t) = V_0$$

may be written in the form

$$v(t) = V_0 e^{(0)t}$$

So: the complex frequency of a dc voltage or current is zero (i.e., $\mathbf{s = 0}$).

# The Exponential Case

The exponential function

$$v(t) = V_0 e^{\sigma t}$$

is already in the required form.

The complex frequency of this voltage is therefore $\sigma$ *or*

$$\mathbf{s} = \sigma + j0$$

# The Sinusoidal Case

For a sinusoidal voltage

$$v(t) = V_m \cos(\omega t + \theta)$$

we apply Euler's identity:

$$\cos(\omega t + \theta) = \frac{1}{2}[e^{j(\omega t+\theta)} + e^{-j(\omega t+\theta)}]$$

to show that

$$v(t) = \frac{1}{2}V_m[e^{j(\omega t+\theta)} + e^{-j(\omega t+\theta)}]$$

$$= \left(\tfrac{1}{2}V_m e^{j\theta}\right)e^{j\omega t} + \left(\tfrac{1}{2}V_m e^{-j\theta}\right)e^{-j\omega t}$$

$$v(t) = \mathbf{K}_1 e^{\mathbf{s}_1 t} + \mathbf{K}_2 e^{\mathbf{s}_2 t}$$

# The Exponentially Damped Sinusoidal Case

$$v(t) = V_m e^{\sigma t} \cos(\omega t + \theta)$$
$$= \tfrac{1}{2} V_m e^{\sigma t} [e^{j(\omega t + \theta)} + e^{-j(\omega t + \theta)}]$$

$$v(t) = \tfrac{1}{2} V_m e^{j\theta} e^{j(\sigma + j\omega)t} + \tfrac{1}{2} V_m e^{-j\theta} e^{j(\sigma - j\omega)t}$$

The damped sine has two complex frequencies

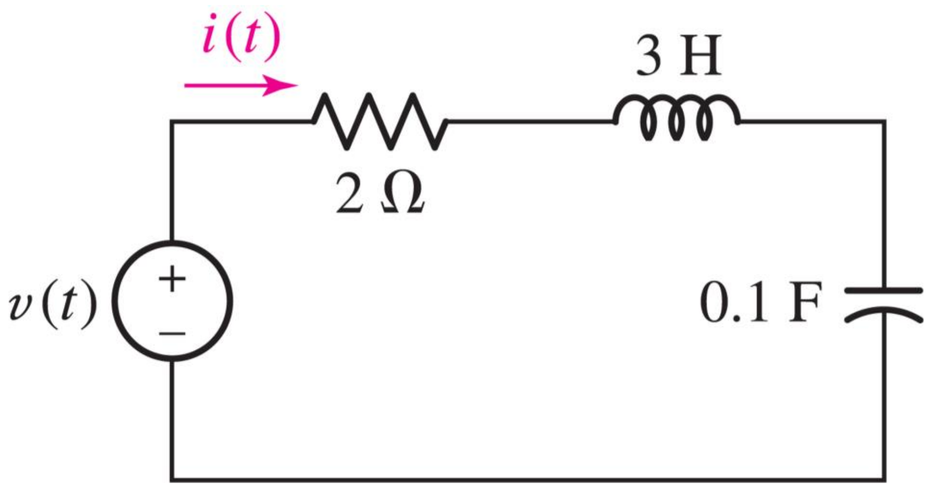
$\mathbf{s}_1 = \sigma + j\omega$ and $\mathbf{s}_2 = \sigma - j\omega$

which are complex conjugates of each other.

# Example: Forcing Function

If $v(t) = 60e^{-2t}\cos(4t + 10^\circ)$ V, solve for $i(t)$.

*Method: write* $v(t) = \mathrm{Re}\{\mathbf{V}e^{st}\}$ *with* $s = -2 + j4$




*Answer:* $5.37e^{-2t}\cos(4t - 106.6^\circ)$ A

# The Laplace Transform

The two-sided Laplace transform of a function *f(t)* is defined as

$$\mathbf{F}(\mathbf{s}) = \int_{-\infty}^{\infty} e^{-\mathbf{s}t} f(t)\, dt$$

**F(s)** is the frequency-domain representation of the time-domain waveform *f(t)*.

# The Laplace Transform

$$\mathbf{F}(\mathbf{s}) = \int_{0^-}^{\infty} e^{-\mathbf{s}t} f(t)\, dt$$

For time functions that do not exist for $t < 0$, or for those time functions whose behavior for $t < 0$ is of no interest, the time-domain description can be thought of as $v(t)u(t)$.

$$f(t) = \frac{1}{2\pi j} \int_{\sigma_0 - j\infty}^{\sigma_0 + j\infty} e^{\mathbf{s}t} \mathbf{F}(\mathbf{s})\, d\mathbf{s}$$

$$f(t) \Leftrightarrow \mathbf{F}(\mathbf{s})$$

This leads to the one-sided Laplace Transform, which from now on will be called simply the Laplace Transform.

# Example: Laplace Transform

Compute the Laplace transform of the function
*f(t) = 2u(t − 3).*

*Apply:*

$$\boxed{\mathbf{F}(\mathbf{s}) = \int_{0^-}^{\infty} e^{-\mathbf{s}t} f(t)\, dt}$$

*to show that* $F(s) = \frac{2}{s} e^{-3s}$

# Laplace Transform of the Unit Step

$$\mathcal{L}\{u(t)\} = \int_{0^-}^{\infty} e^{-st} u(t)\, dt = \int_0^{\infty} e^{-st}\, dt$$

$$= -\frac{1}{s} e^{-st} \Big|_0^{\infty} = \frac{1}{s}$$

$$u(t) \Leftrightarrow \frac{1}{s}$$

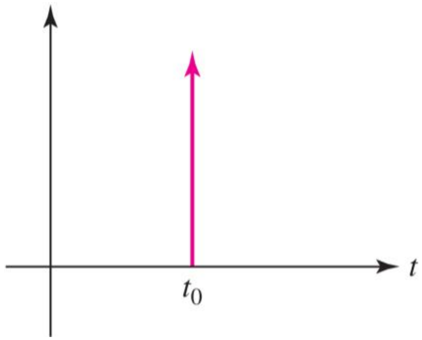
This is valid for Re(s)>0

# The Unit Impulse δ(t)

The unit impulse is defined as δ(t)=du(t)/dt

$$\delta(t - t_0) = 0 \qquad t \neq t_0$$

$$\int_{t_0-\varepsilon}^{t_0+\varepsilon} \delta(t - t_0)\, dt = 1$$




$$\mathscr{L}\{\delta(t - t_0)\} = \int_{0^-}^{\infty} e^{-\mathbf{s}t}\delta(t - t_0)\, dt = e^{-\mathbf{s}t_0}$$

$$\delta(t - t_0) \Leftrightarrow e^{-\mathbf{s}t_0}$$

# The Unit Impulse δ(t)

The value of $\int_{-\infty}^{\infty} f(t)\delta(t - t_0)\, dt$ is $f(t_0)$

This is the *sifting property* of the unit impulse.

The Laplace Transform pair is simple:

$$\delta(t) \Leftrightarrow 1$$

# Other Important Laplace Transforms

The decaying exponential: $e^{-\alpha t}u(t) \Longleftrightarrow \dfrac{1}{\mathbf{s}+\alpha}$

The ramp: $tu(t) \Leftrightarrow \dfrac{1}{\mathbf{s}^2}$

The "ramp/exponential": $te^{-\alpha t}u(t) \Leftrightarrow \dfrac{1}{(\mathbf{s}+\alpha)^2}$

# Linearity and Laplace

$$\mathcal{L}\{f_1(t) + f_2(t)\} = \int_{0^-}^{\infty} e^{-\mathbf{s}t}[f_1(t) + f_2(t)]\,dt$$
$$= \int_{0^-}^{\infty} e^{-\mathbf{s}t} f_1(t)\,dt + \int_{0^-}^{\infty} e^{-\mathbf{s}t} f_2(t)\,dt$$
$$= \mathbf{F}_1(\mathbf{s}) + \mathbf{F}_2(\mathbf{s})$$

$$\mathcal{L}\{kv(t)\} = k\mathcal{L}\{v(t)\} \qquad kv(t) \Leftrightarrow k\mathbf{V}(\mathbf{s})$$

# Example: Laplace Transform

Calculate the inverse transform of

$$F(s) = 2(s + 2)/s.$$

Method: Use linearity properties and transform pairs.

*Answer:* $f(t) = 2\delta(t) + 4u(t)$

# Laplace Transform Theorems

- Time Differentiation:

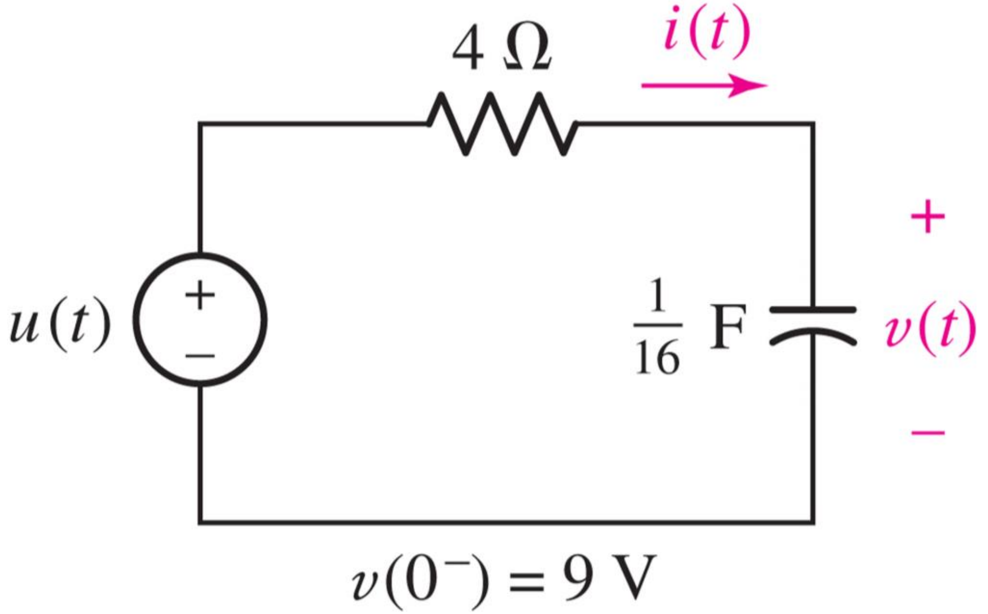
$$\frac{dv}{dt} \Leftrightarrow sV(s) - v(0^-)$$

- Time Integration:

$$\int_{0^-}^{t} v(x)\,dx \Leftrightarrow \frac{V(s)}{s}$$

# Example: Using LT Theorems

Determine $i(t)$ and $v(t)$ for $t > 0$ in the series RC circuit shown:




*Answer:* $i(t) = -2e^{-4t}u(t)$ A, $v(t) = (1 + 8e^{-4t})u(t)$ V

# Laplace Transform of Sinusoids

$$\cos(\omega t)u(t) \Leftrightarrow \frac{s}{s^2+\omega^2}$$

$$\sin(\omega t)u(t) \Leftrightarrow \frac{\omega}{s^2+\omega^2}$$

# Laplace Transform Theorems

- Initial Value Theorem:

$$\lim_{t \to 0^+} f(t) = \lim_{\mathbf{s} \to \infty} [\mathbf{sF(s)}]$$

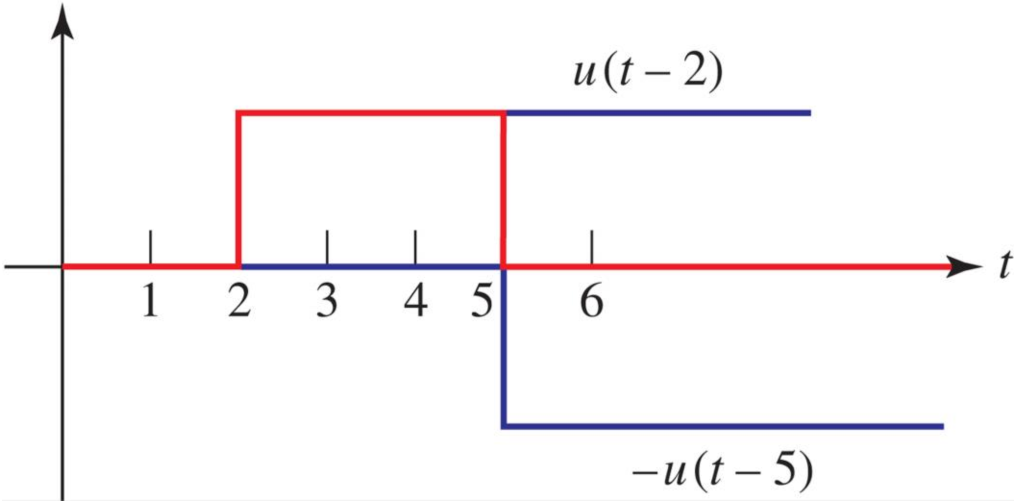
- Final Value Theorem:

$$\lim_{t \to \infty} f(t) = \lim_{\mathbf{s} \to 0} [\mathbf{sF(s)}]$$

# Example: Laplace Transform

Determine the transform of the rectangular pulse

$$v(t) = u(t - 2) - u(t - 5)$$




*Answer:*

$$V(s) = \frac{e^{-2s} - e^{-5s}}{s}$$