Week 4 LTI System & Convolution

Jimin Xiao

EB Building, Room 312

Jimin.xiao@xjtlu.edu.cn

DT Convolution

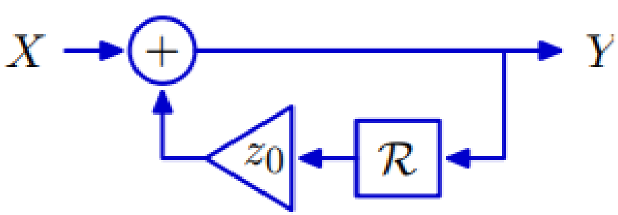
Multiple Representations of CT and DT Systems

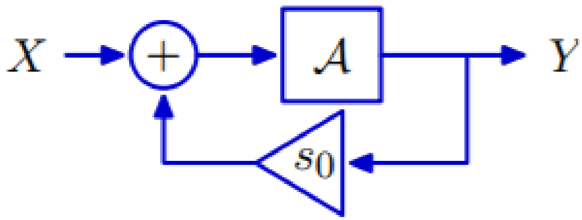
Verbal descriptions: preserve the rationale.

Difference/differential equations: mathematically compact.

$$y[n] = x[n] + z_0\, y[n-1] \qquad\qquad \dot{y}(t) = x(t) + s_0\, y(t)$$

Block diagrams: illustrate signal flow paths.





Operator representations: analyze systems as polynomials.

$$\frac{Y}{X} = \frac{1}{1 - z_0\mathcal{R}} \qquad\qquad \frac{Y}{X} = \frac{\mathcal{A}}{1 - s_0\mathcal{A}}$$

Transforms: representing diff. equations with algebraic equations.

$$H(z) = \frac{z}{z - z_0} \qquad\qquad H(s) = \frac{1}{s - s_0}$$

Convolution

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Representing a system by a single signal.

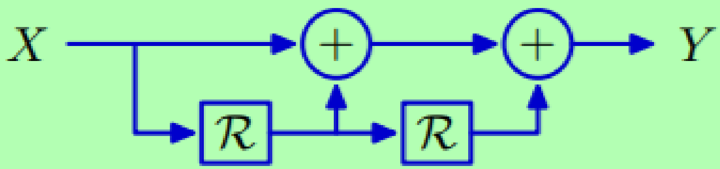
Responses to arbitrary signals

Although we have focused on responses to simple signals ($\delta[n], \delta(t)$) we are generally interested in responses to more complicated signals.

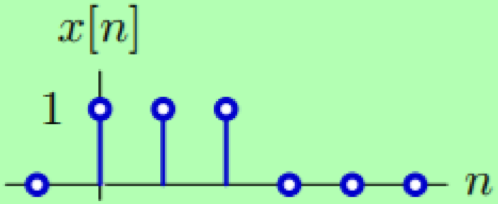
How do we compute responses to a more complicated input signals?

Check Yourself

Example: Find $y[3]$



when the input is

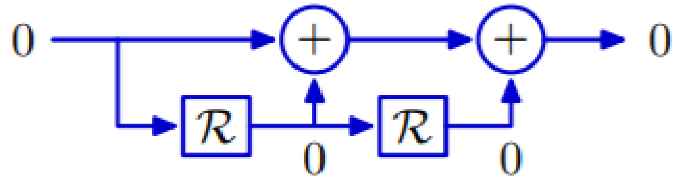


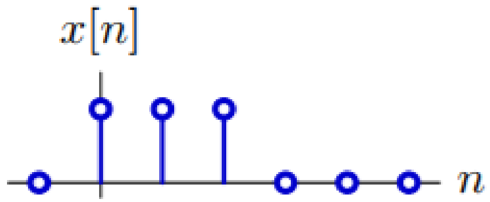
1. 1
2. 2
3. 3
4. 4
5. 5

0. none of the above

Check Yourself

Example.





Check Yourself

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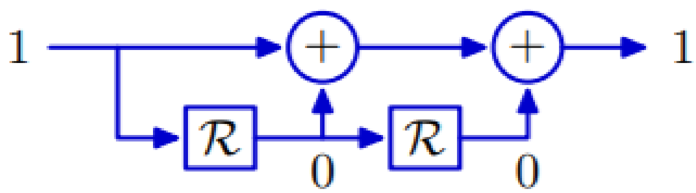
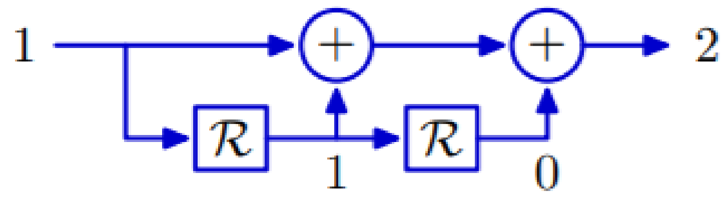
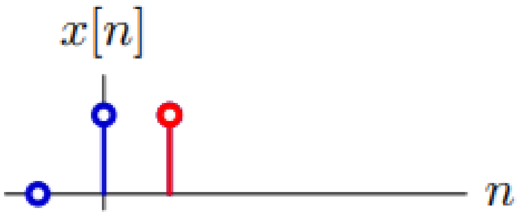
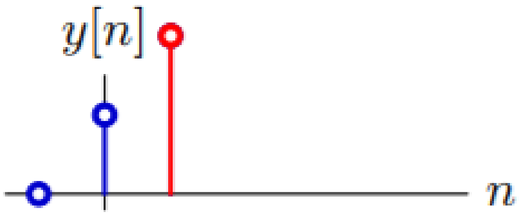


Check Yourself

Example.







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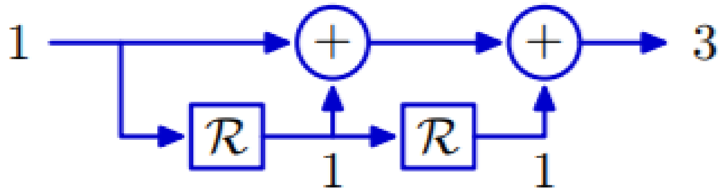

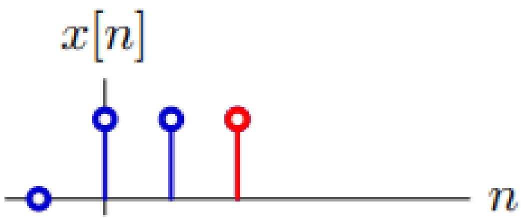

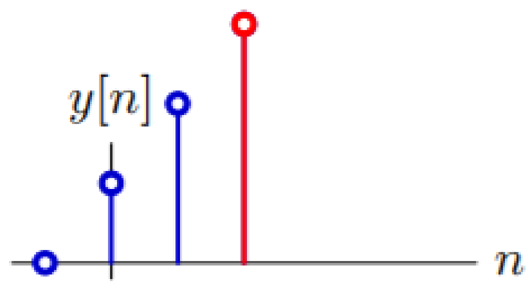

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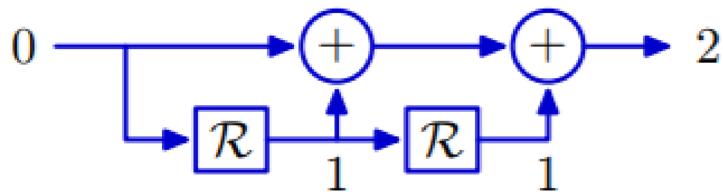
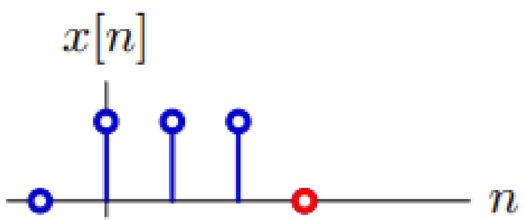
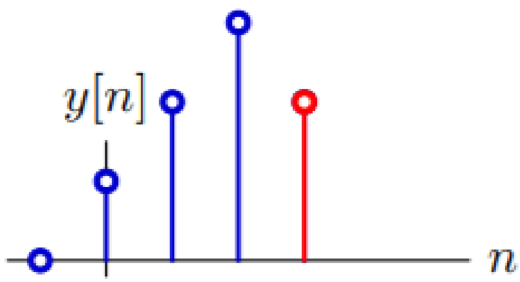
Example.







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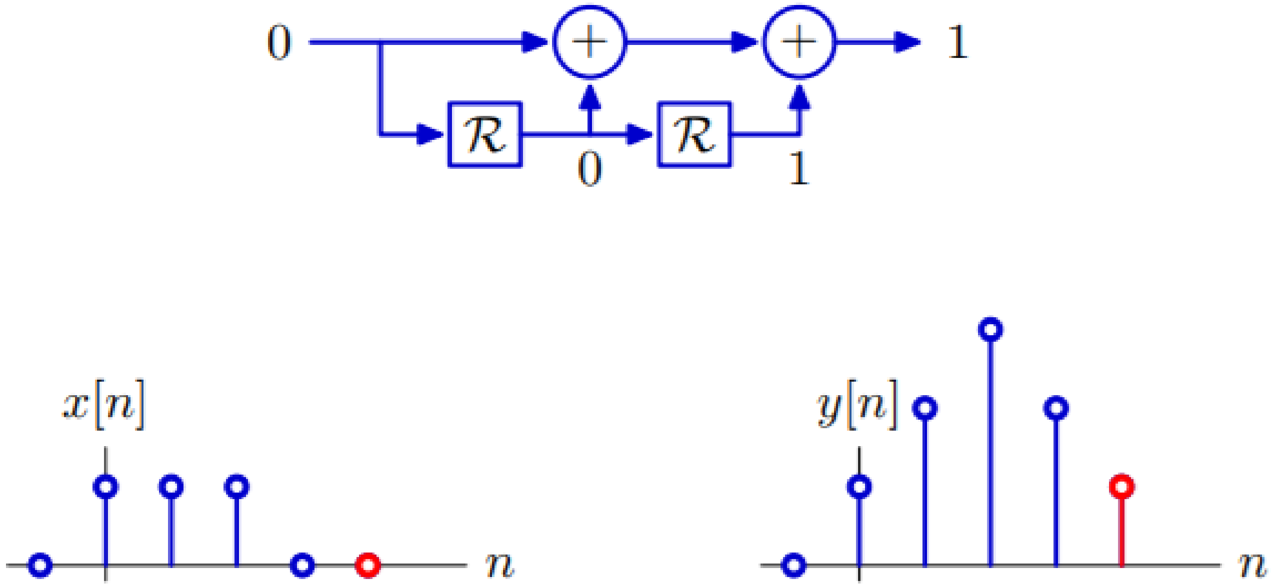

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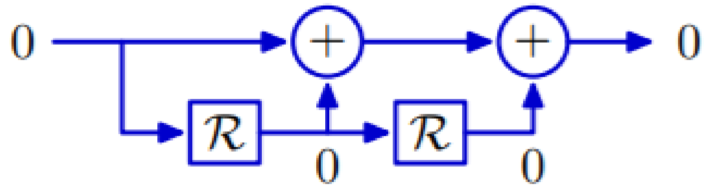
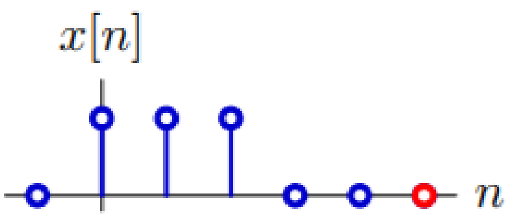
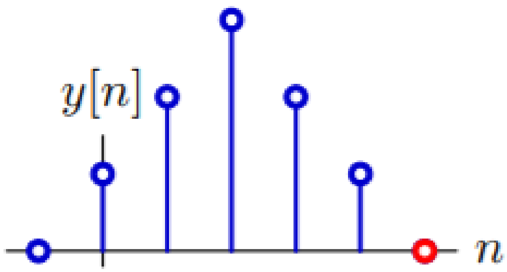
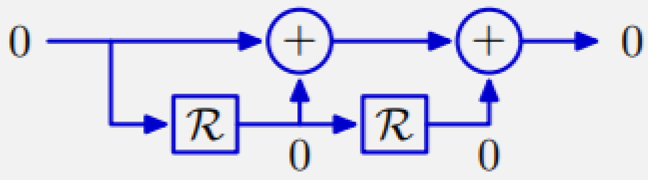


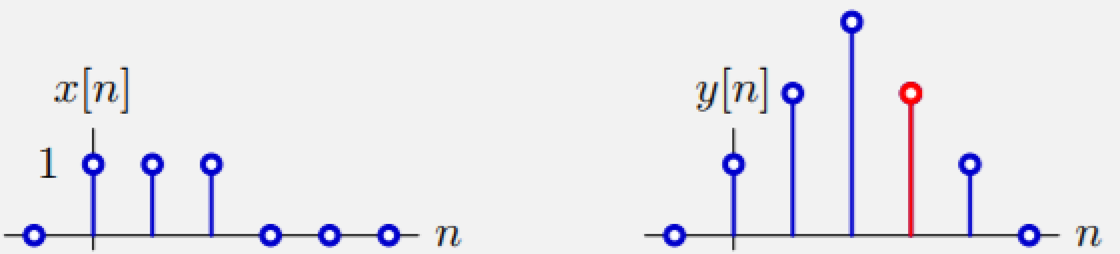




Check Yourself

What is $y[3]$? 2





1. 1
2. 2
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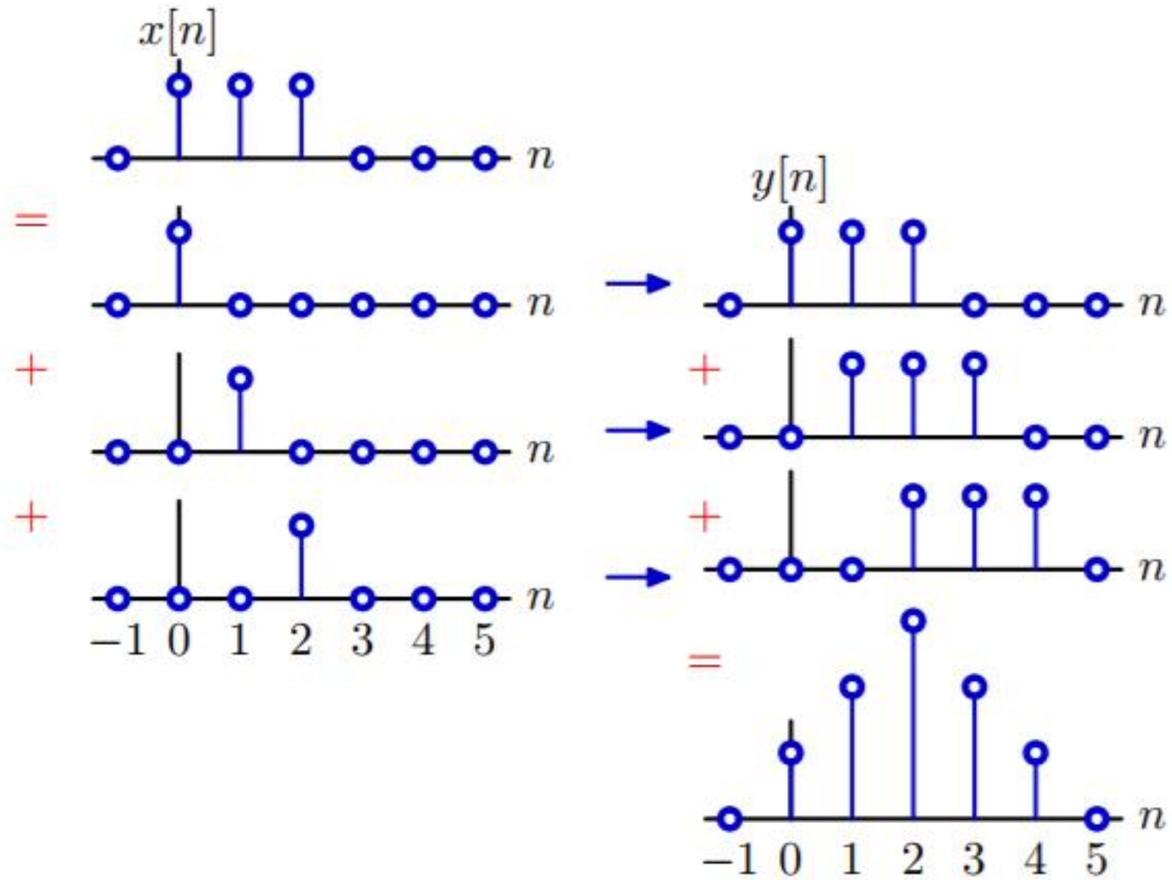
5. 5

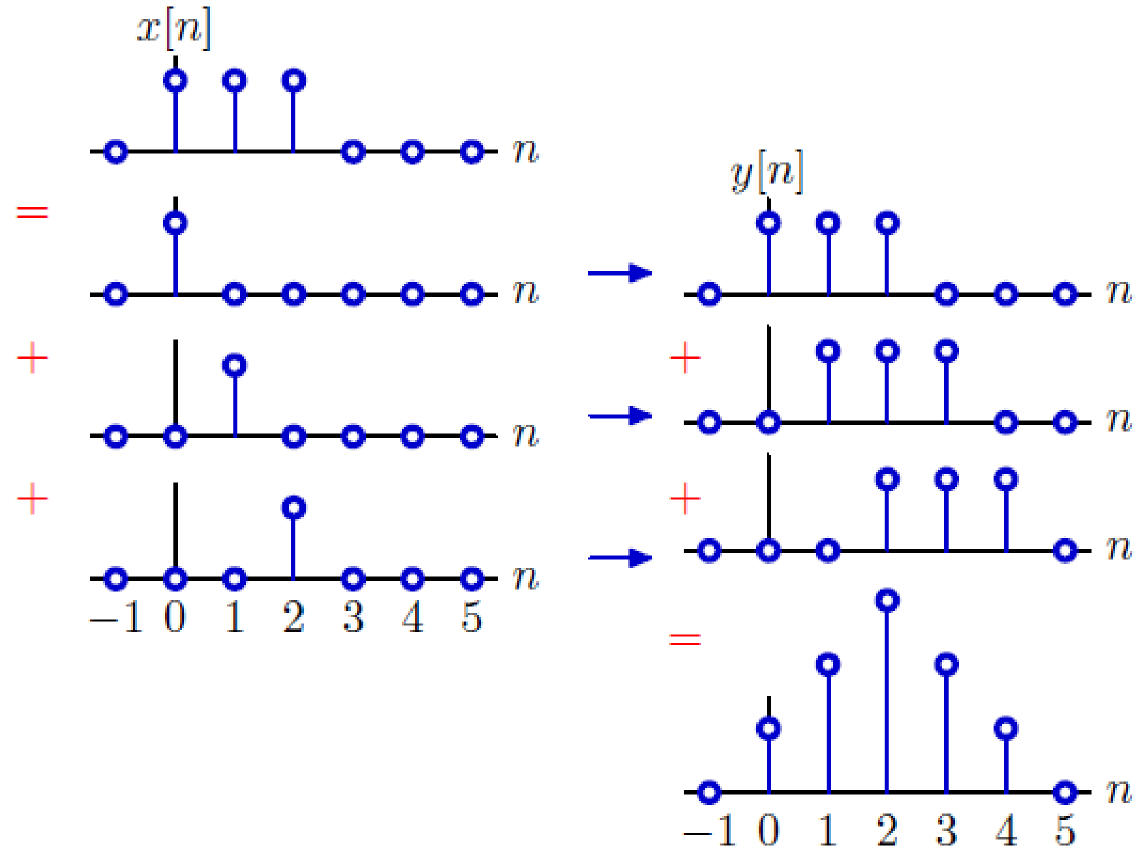
0. none of the above

Superposition

Break input into additive parts and sum the responses to the parts.



Superposition



Linear $\alpha x_1[n] + \beta x_2[n] \longrightarrow \boxed{\text{system}} \longrightarrow \alpha y_1[n] + \beta y_2[n]$

Time invariant $x[n-n_0] \longrightarrow \boxed{\text{system}} \longrightarrow y[n-n_0]$

Structure of superposition

If a system is linear and time-invariant (LTI) then its output is the sum of weighted and shifted unit-sample responses.

$$\delta[n] \longrightarrow \boxed{\text{system}} \longrightarrow h[n]$$

$$\delta[n-k] \longrightarrow \boxed{\text{system}} \longrightarrow h[n-k]$$

$$x[k]\delta[n-k] \longrightarrow \boxed{\text{system}} \longrightarrow x[k]h[n-k]$$

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \longrightarrow \boxed{\text{system}} \longrightarrow y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Convolution

Response of an LTI system to an arbitrary input.

$$x[n] \longrightarrow \boxed{\text{LTI}} \longrightarrow y[n]$$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] \equiv (x * h)[n]$$

This operation is called convolution.

Notations

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Convolution is represented with an asterisk.

$$\sum_{k=-\infty}^{\infty} x[k]h[n-k] \equiv (x * h)[n]$$

It is customary (but confusing) to abbreviate this notation:

$$(x * h)[n] = x[n] * h[n]$$

Notations

Do not be fooled by the confusing notation.

Confusing (but conventional) notation:

$$\sum_{k=-\infty}^{\infty} x[k]h[n-k] = x[n] * h[n]$$

$x[n] * h[n]$ looks like an operation of samples; but it is not!

$$x[1] * h[1] \neq (x * h)[1]$$

Convolution operates on signals not samples.

Unambiguous notation:

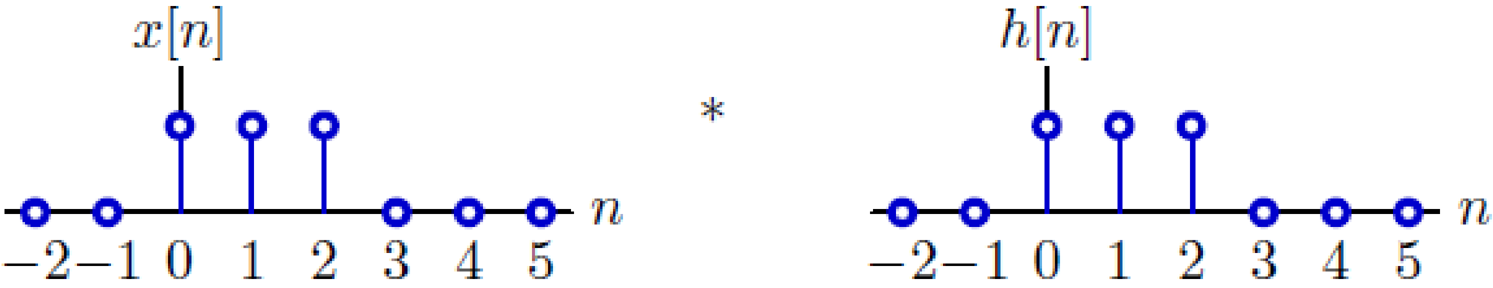
$$\sum_{k=-\infty}^{\infty} x[k]h[n-k] \equiv (x * h)[n]$$

The symbols x and h represent DT signals.

Convolving x with h generates a new DT signal $x * h$.

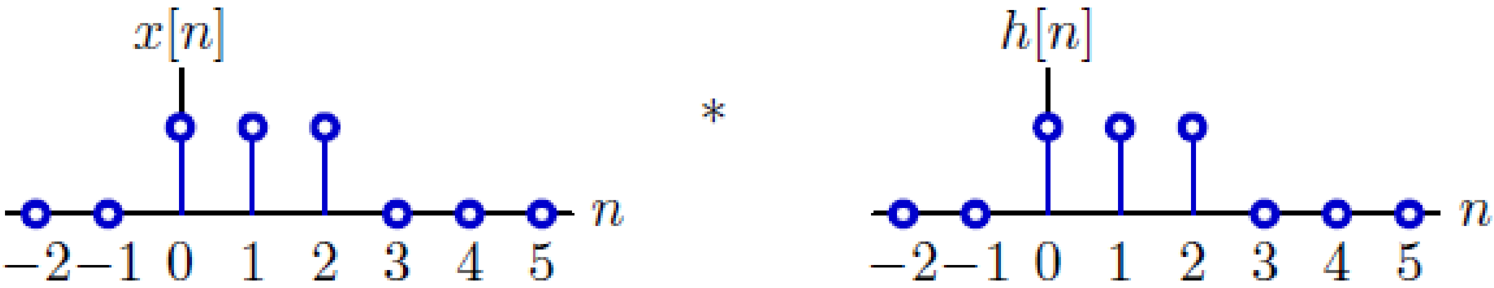
Structure of Convolution

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$



Structure of Convolution

$$y[0] = \sum_{k=-\infty}^{\infty} x[k]h[0-k]$$



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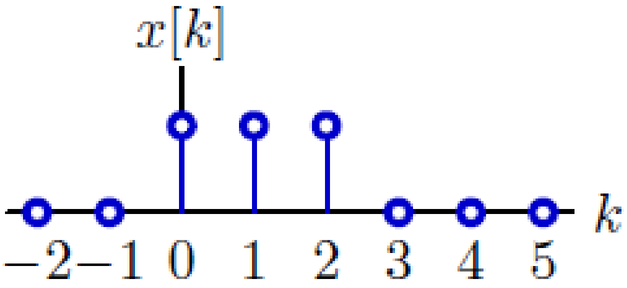

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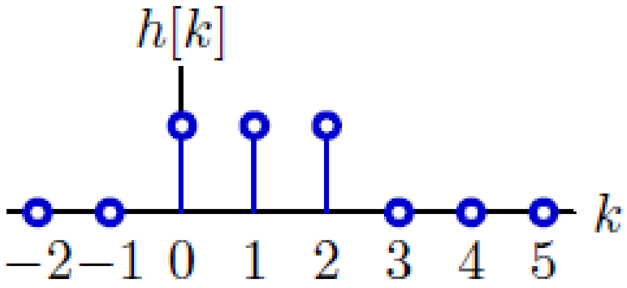

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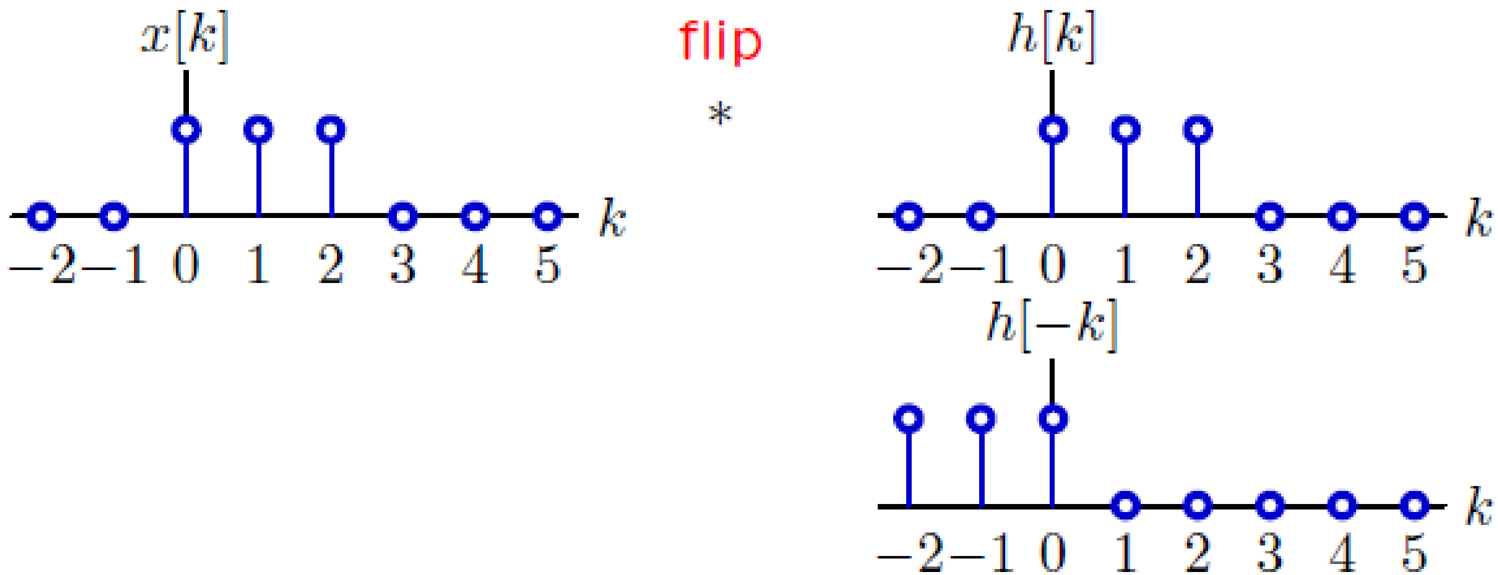

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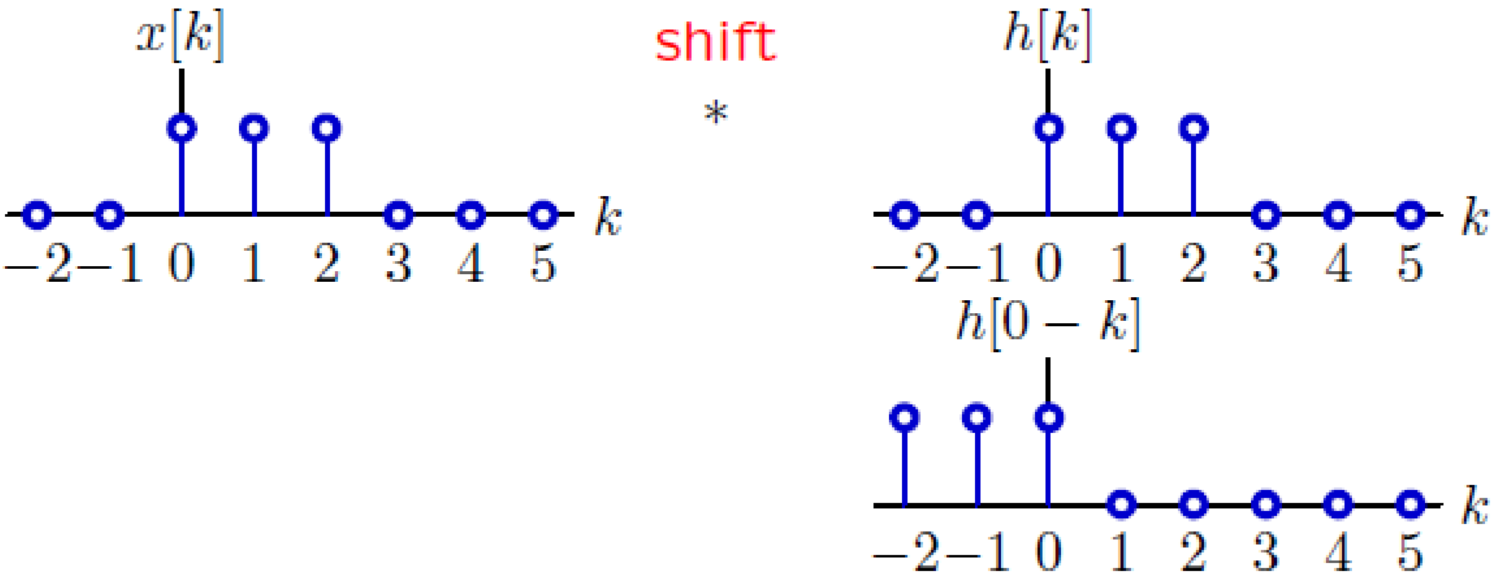

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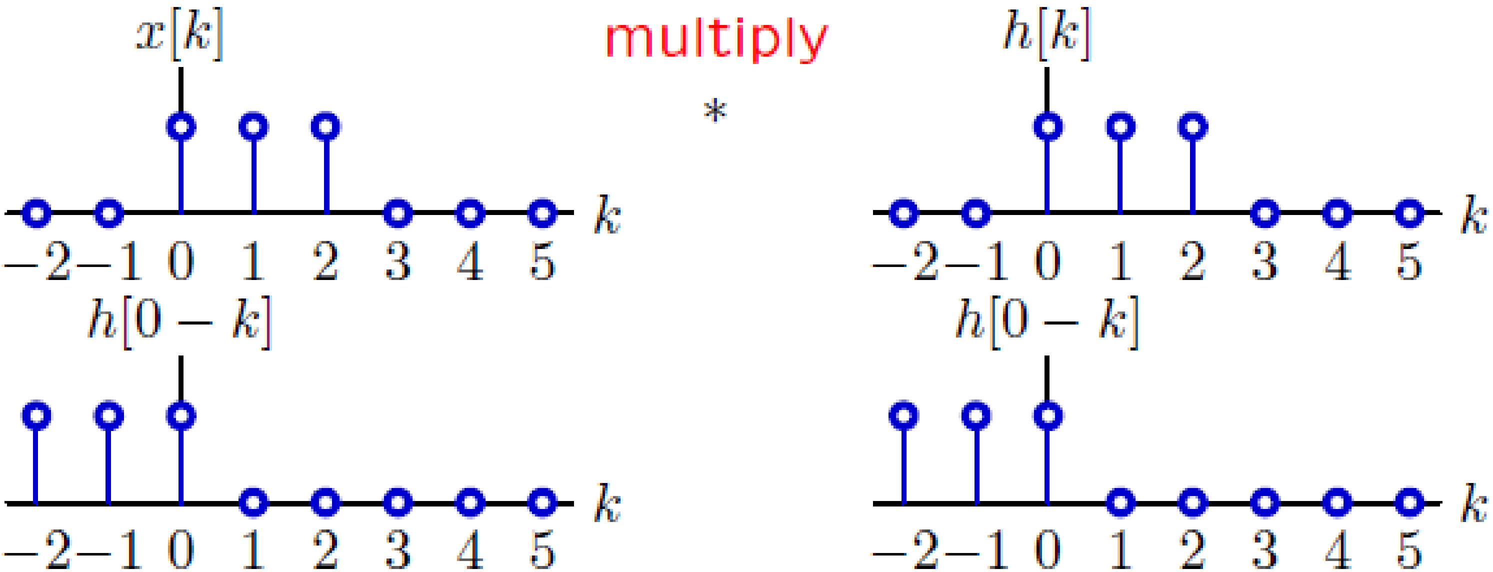

Structure of Convolution

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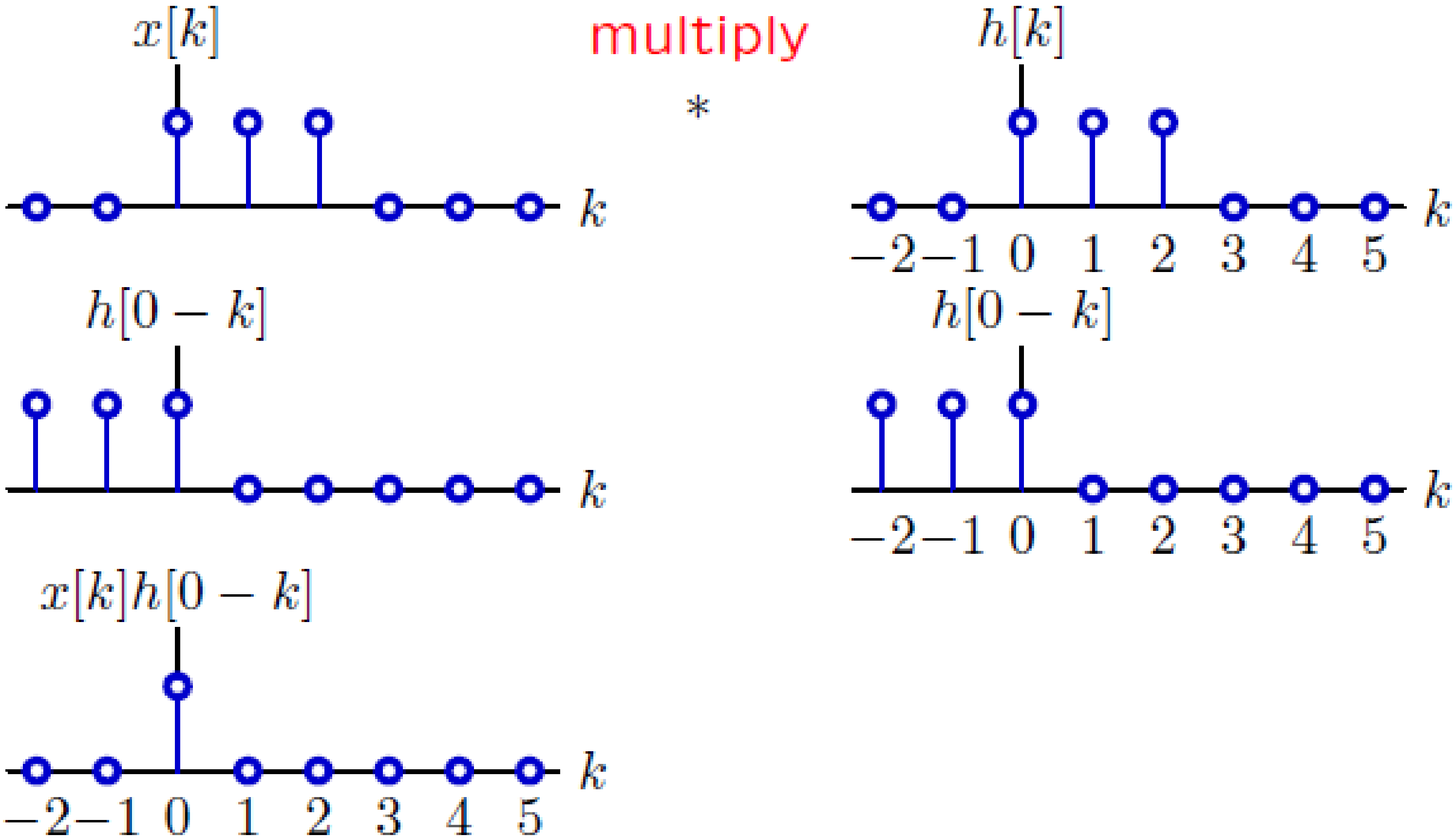

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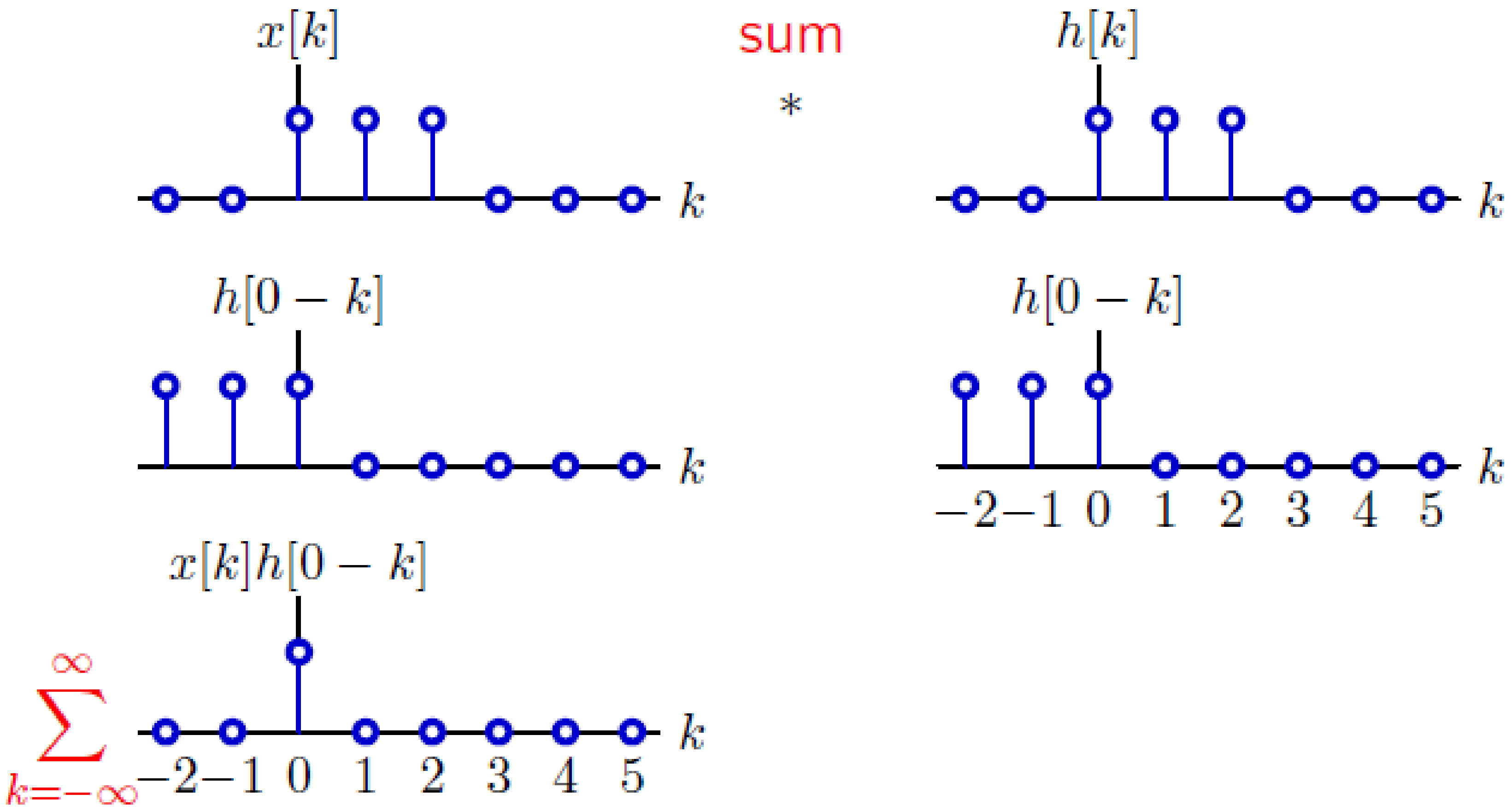

Structure of Convolution

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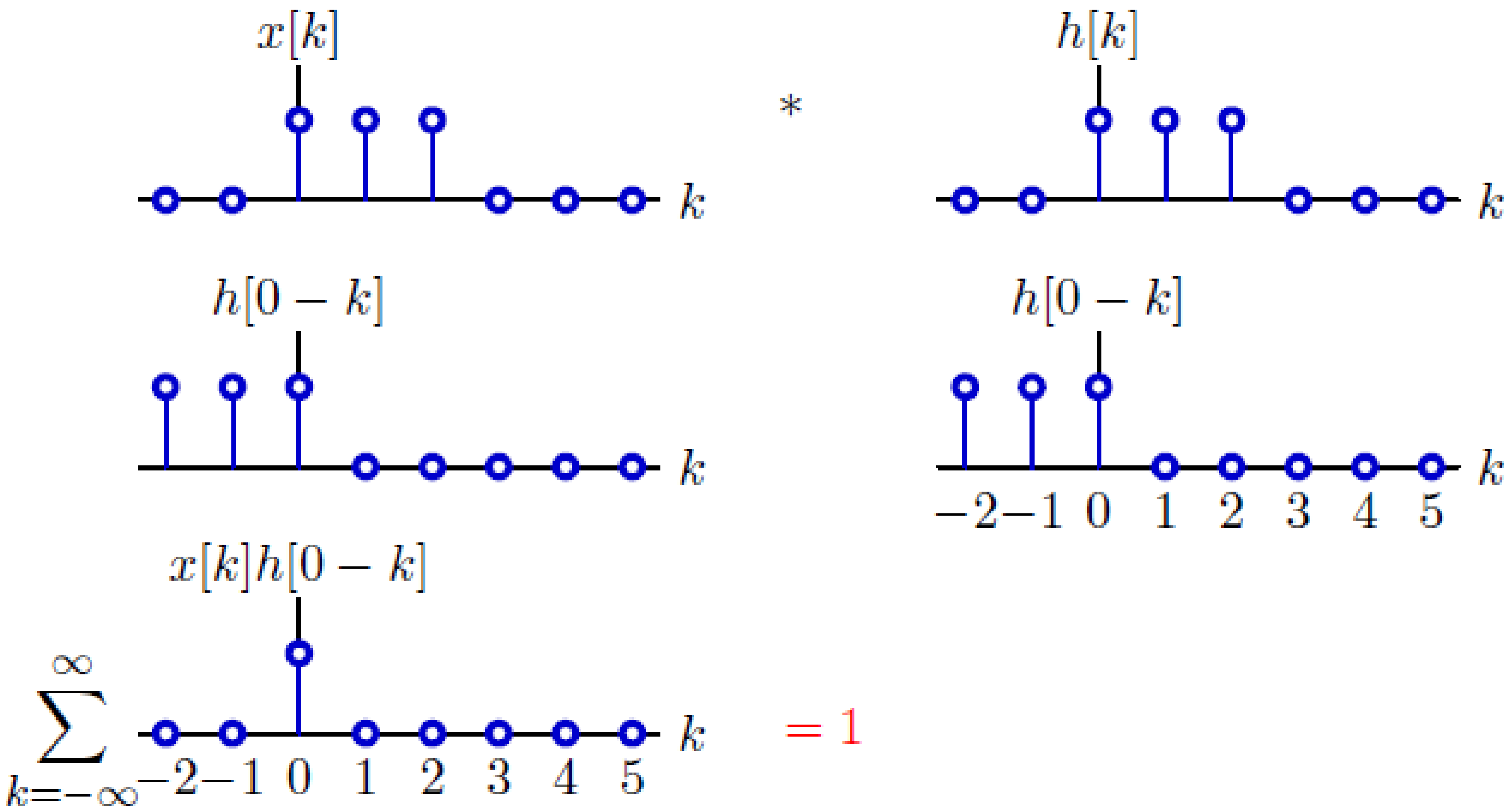
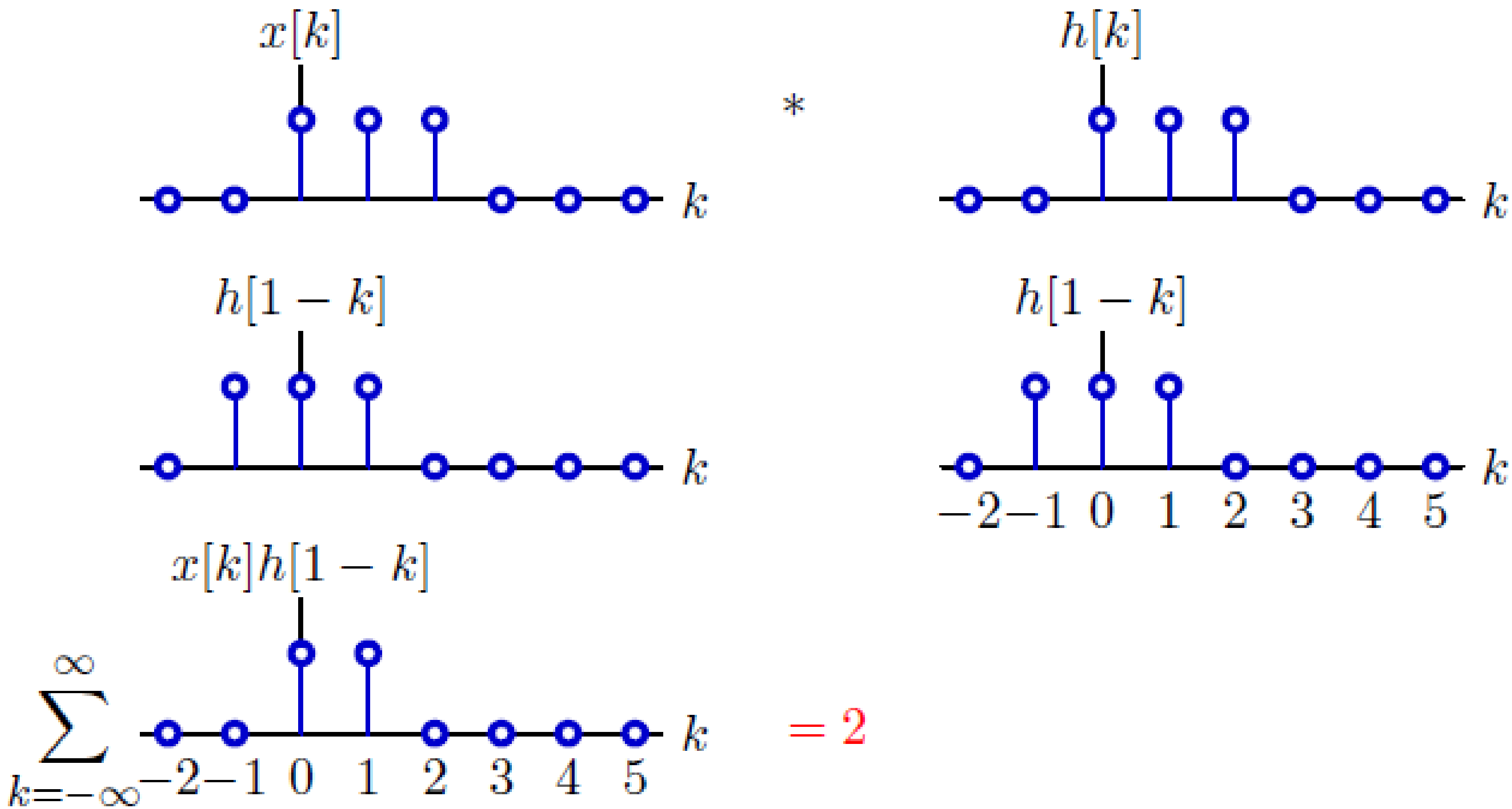


$= 1$

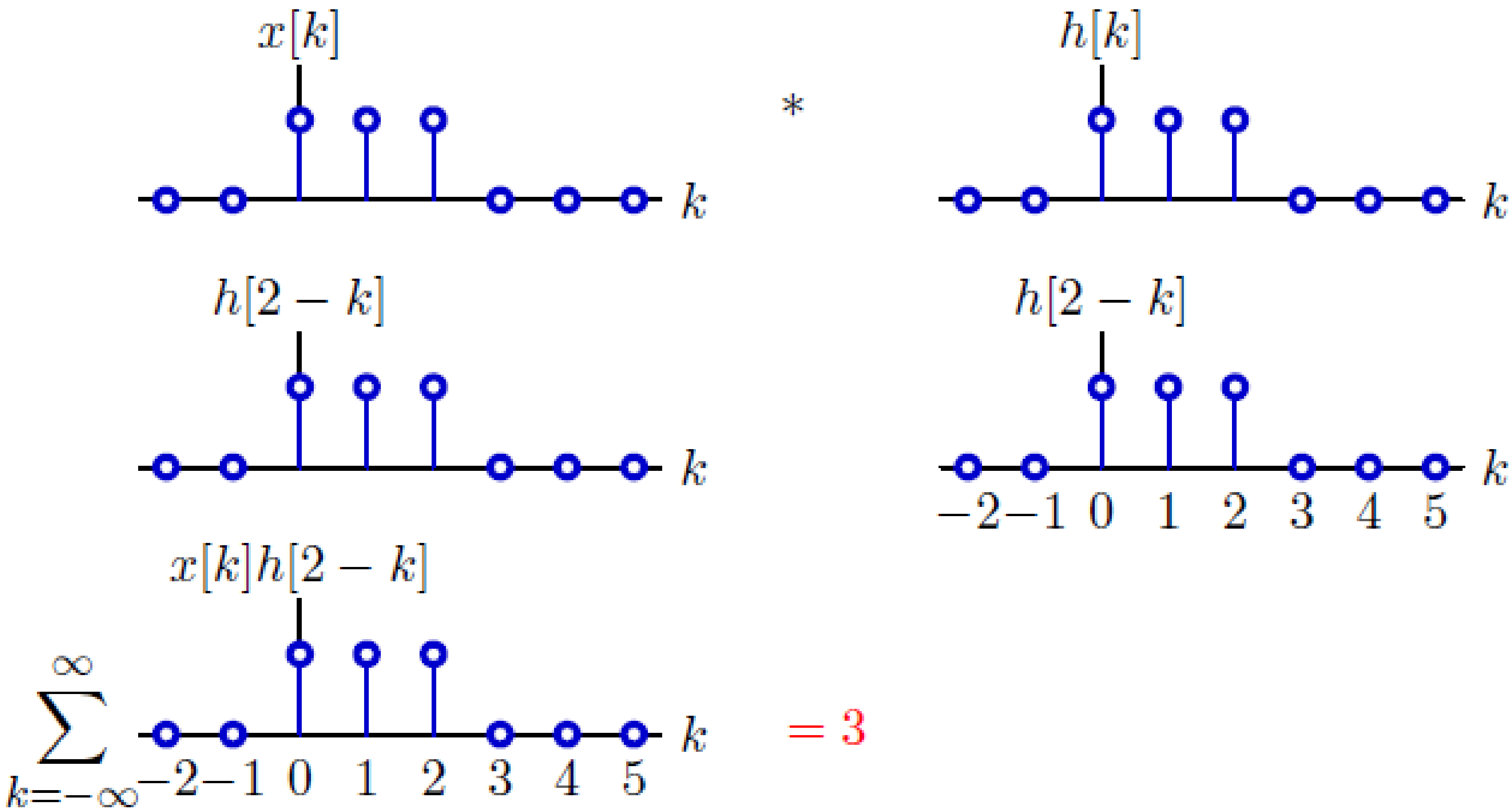
Structure of Convolution

$$y[1] = \sum_{k=-\infty}^{\infty} x[k]h[1-k]$$



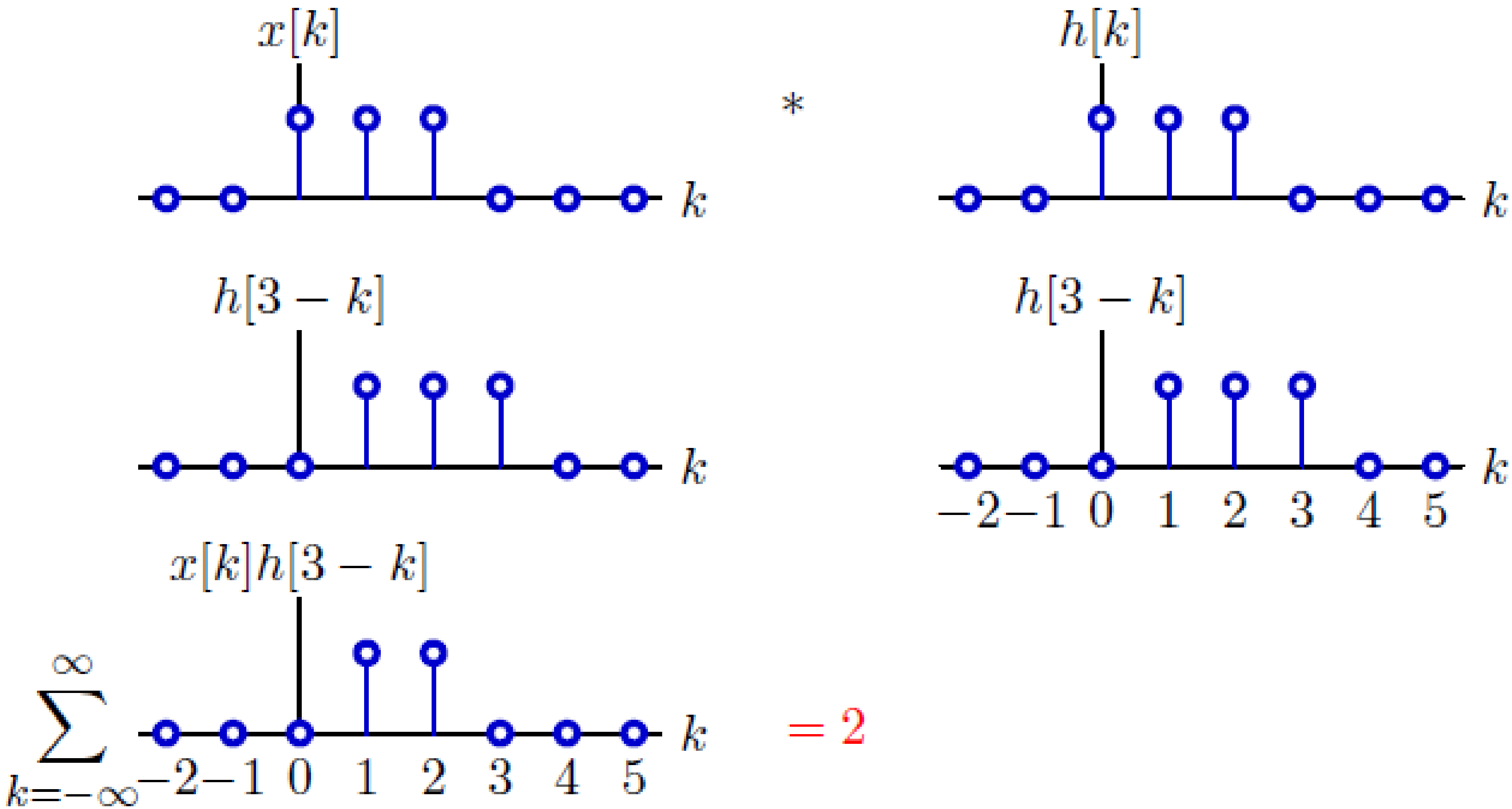
Structure of Convolution

$$y[2] = \sum_{k=-\infty}^{\infty} x[k]h[2-k]$$



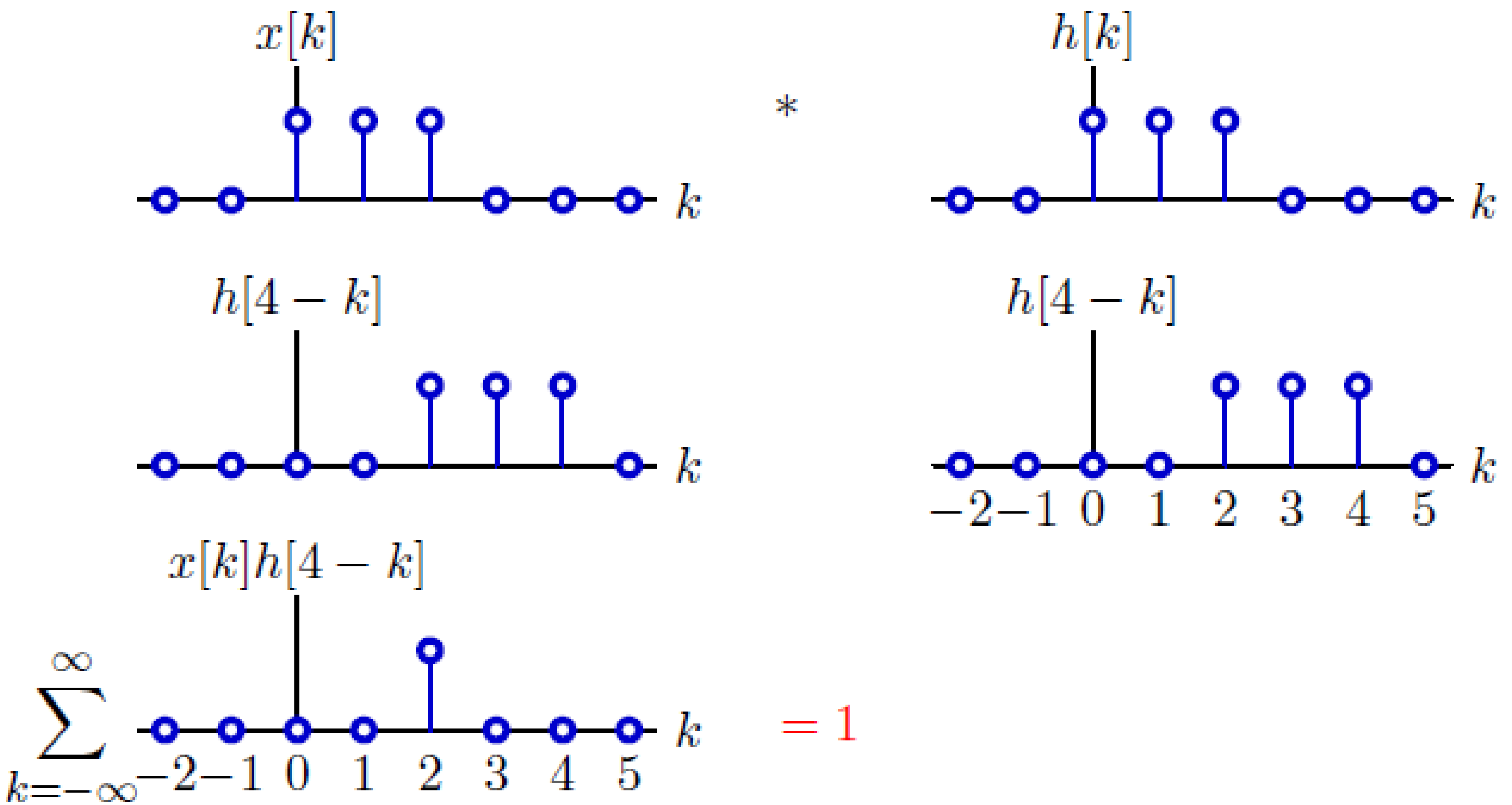
Structure of Convolution

$$y[3] = \sum_{k=-\infty}^{\infty} x[k]h[3-k]$$



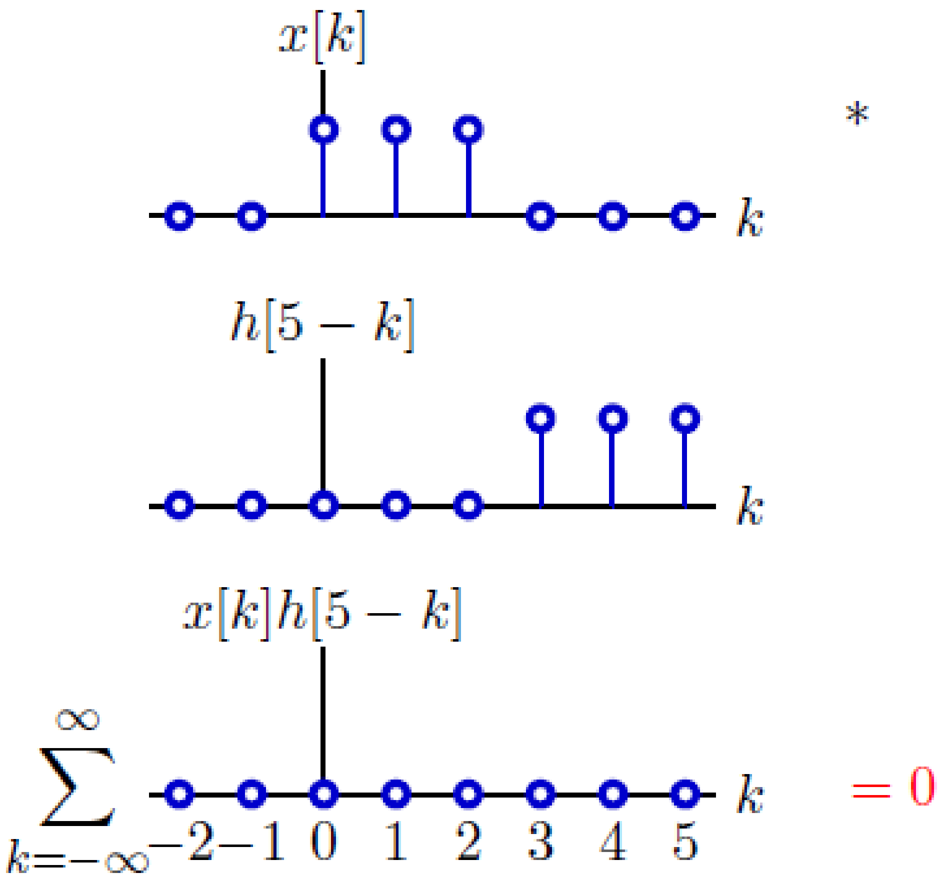
Structure of Convolution

$$y[4] = \sum_{k=-\infty}^{\infty} x[k]h[4-k]$$

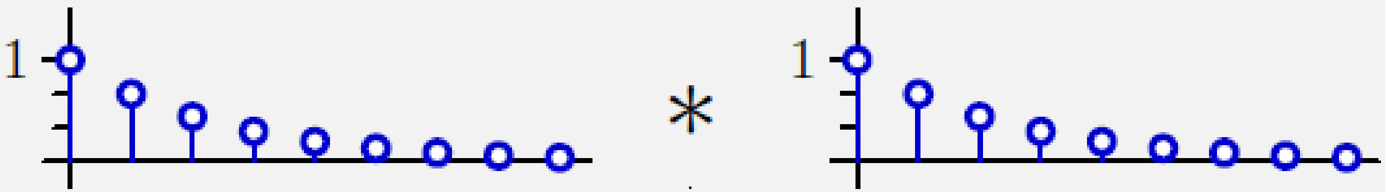


Structure of Convolution

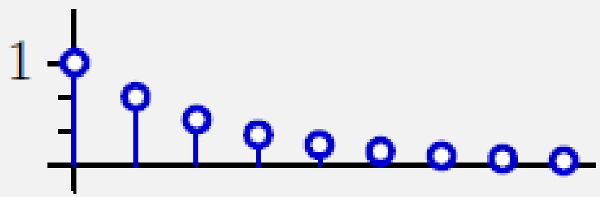
$$y[5] = \sum_{k=-\infty}^{\infty} x[k]h[5-k]$$

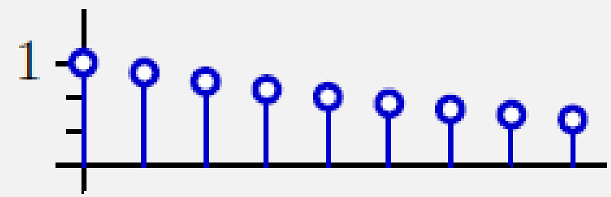


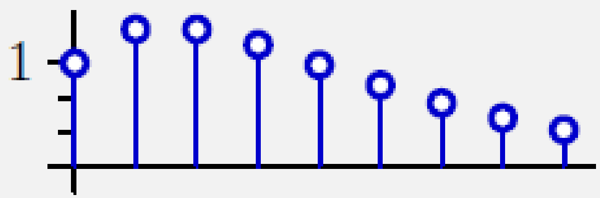
Check Yourself

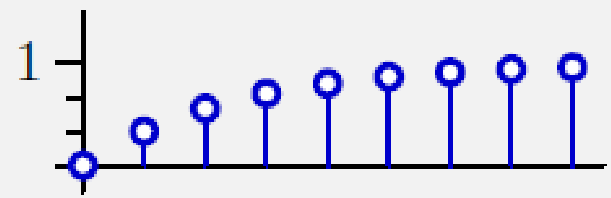


Which plot shows the result of the convolution above?

1. 

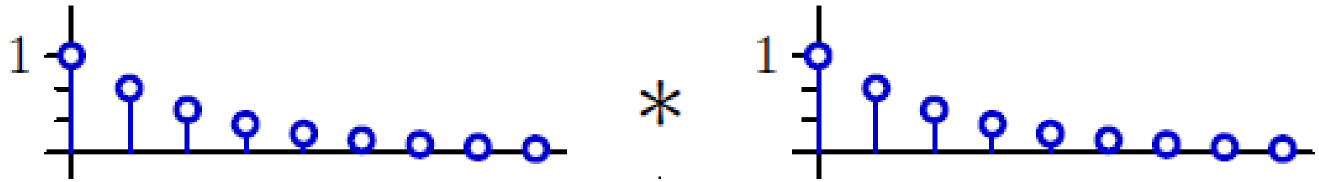
2. 

3. 

4. 

5. none of the above

Check Yourself



Express mathematically:

$$\left(\left(\frac{2}{3}\right)^n u[n]\right) * \left(\left(\frac{2}{3}\right)^n u[n]\right) = \sum_{k=-\infty}^{\infty} \left(\left(\frac{2}{3}\right)^k u[k]\right) \times \left(\left(\frac{2}{3}\right)^{n-k} u[n-k]\right)$$

$$= \sum_{k=0}^{n} \left(\frac{2}{3}\right)^k \times \left(\frac{2}{3}\right)^{n-k}$$

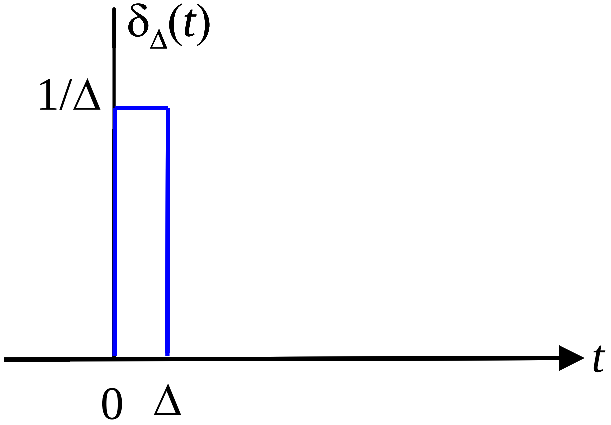
$$= \sum_{k=0}^{n} \left(\frac{2}{3}\right)^n = \left(\frac{2}{3}\right)^n \sum_{k=0}^{n} 1$$

$$= (n+1)\left(\frac{2}{3}\right)^n u[n]$$

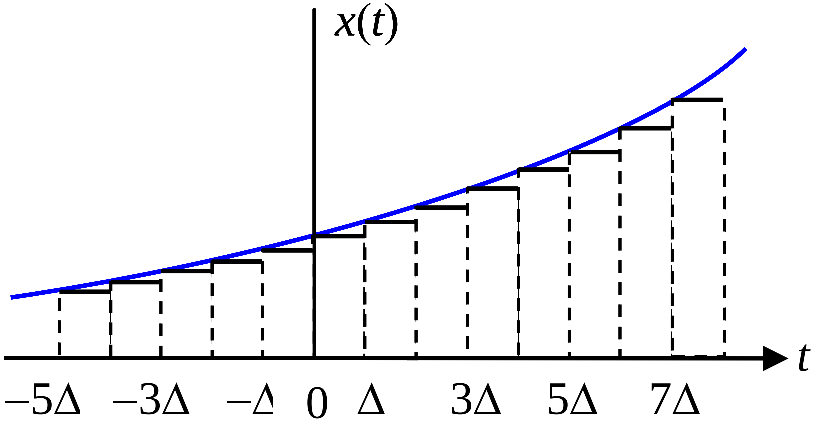
$$= 1,\ \frac{4}{3},\ \frac{4}{3},\ \frac{32}{27},\ \frac{80}{81},\ \ldots$$

CT Convolution

Representation signals using impulse signals



(a)



(b)

$$x(t) \cong x(0)\,\delta_\Delta(t)\,\Delta + x(\Delta)\,\delta_\Delta(t-\Delta)\,\Delta$$

$$+\; x(-\Delta)\,\delta_\Delta(t+\Delta)\,\Delta + \ldots$$

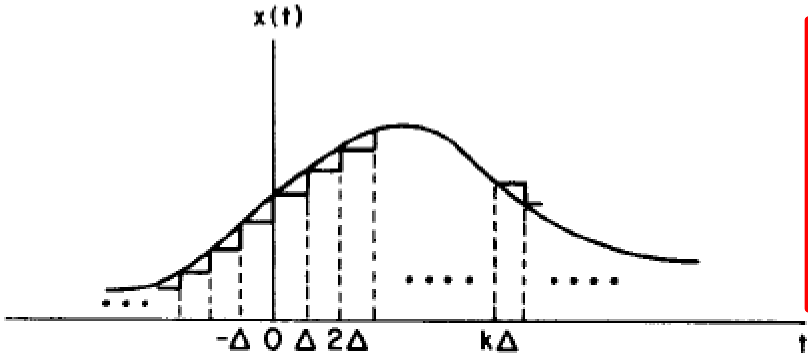
$$x(t) \cong \sum_{k=-\infty}^{+\infty} x(k\Delta)\,\delta_\Delta(t-k\Delta)\,\Delta$$

$$x(t) = \lim_{\Delta\to 0} \sum_{k=-\infty}^{+\infty} x(k\Delta)\,\delta_\Delta(t-k\Delta)\,\Delta$$

$$= \int_{-\infty}^{+\infty} x(\tau)\,\delta(t-\tau)\,d\tau$$

When $\Delta \to 0$,
$k\Delta \to \tau$,
$\delta_\Delta(t) \to \delta(t)$
$\Delta \to d\tau$

Impulse signals



$$\boxed{x(t) = \int_{-\infty}^{+\infty} x(\tau)\,\delta(t-\tau)\,d\tau}$$

Don’t confuse it with impulse signals

$x(t_1)\delta(t-t_1)$ is a delayed impulse signal

$$\int_{-\infty}^{\infty} x(t_1)\delta(t-t_1)dt =?$$

Impulse response

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The impulse response of an LTI system is the output of the system when a unit impulse is applied at the input

$$\delta(t) \quad \rightarrow \quad h(t)$$

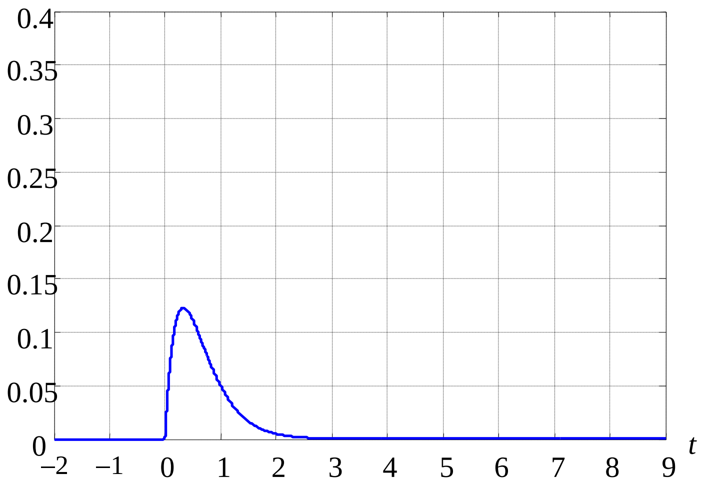
Because the system is LTI, it satisfies the linearity and the time-shifting properties.

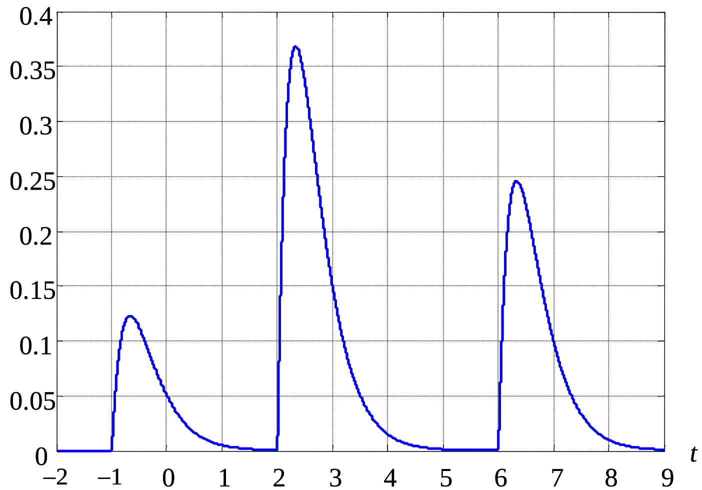
$$a\delta(t-t_0) \quad \rightarrow \quad ah(t-t_0)$$

Example

The impulse response h(t) of an LTI system is plotted. Sketch the output of the system for the input signal:

$$x(t) = \delta(t+1) + 3\delta(t-2) + 2\delta(t-6)$$





Convolution integral

$$\delta(t) \longrightarrow \boxed{\text{system}} \longrightarrow h(t)$$

$$\delta(t-\tau) \longrightarrow \boxed{\text{system}} \longrightarrow h(t-\tau)$$

$$x(\tau)\delta(t-\tau) \longrightarrow \boxed{\text{system}} \longrightarrow x(\tau)h(t-\tau)$$

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau \longrightarrow \boxed{\text{system}} \longrightarrow \boxed{y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau}$$

Convolution Integral

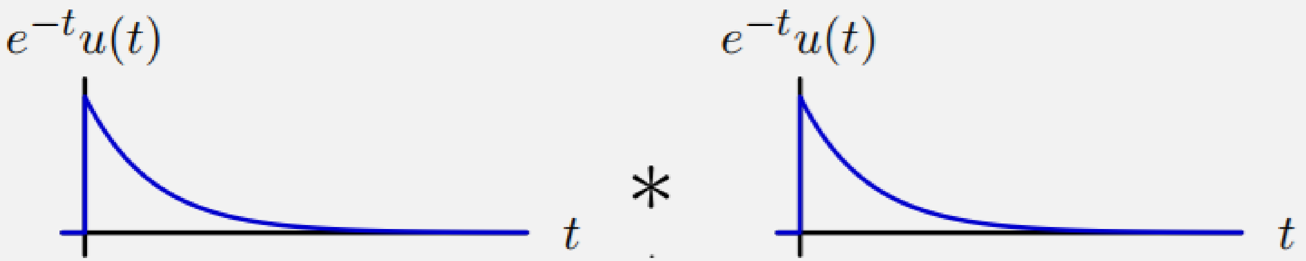
CT Convolution

Convolution of CT signals is analogous to convolution of DT signals.

DT: $$y[n] = (x * h)[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

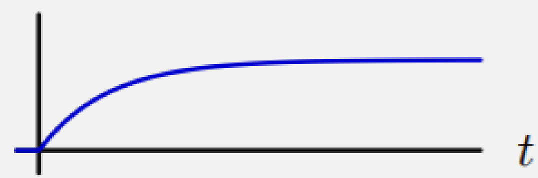
CT: $$y(t) = (x * h)(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

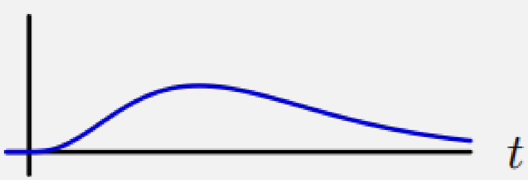
Check Yourself

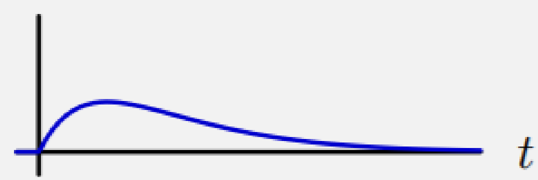


Which plot shows the result of the convolution above?

1. 

2. 

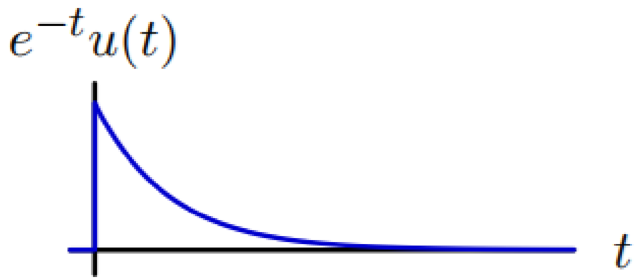
3. 

4. 

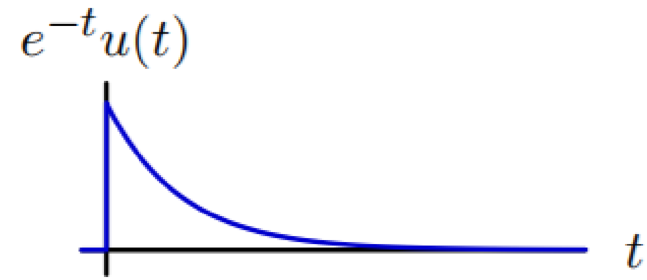
5. none of the above

Check Yourself

Which plot shows the result of the following convolution?

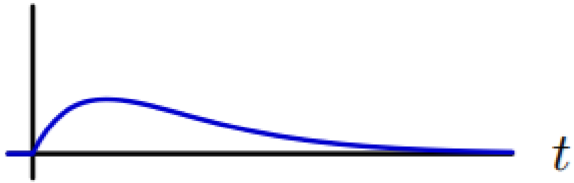


$*$



$$\left(e^{-t}u(t)\right) * \left(e^{-t}u(t)\right) = \int_{-\infty}^{\infty} e^{-\tau}u(\tau)e^{-(t-\tau)}u(t-\tau)d\tau$$

$$= \int_0^t e^{-\tau}e^{-(t-\tau)}d\tau = e^{-t}\int_0^t d\tau = te^{-t}u(t)$$



Example of convolution integral

Determine the output response of an LTI CT system when the input signal is given by $x(t) = e^{-at}u(t), a > 0$, and the impulse response is $h(t) = u(t)$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

Page: 98, Example 2.6

Graphical method

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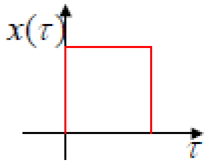
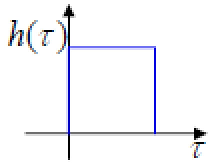
$$y(t) = x(t) * h(t) = \int_{-\infty}^{+\infty} x(\tau) h(t-\tau) d\tau$$





Graphical method

Summary of the graphical procedure

1. Fix $x(\tau)$
2. Time reversal $h(\tau) \rightarrow h(-\tau)$
3. Time shifting $h(t-\tau)$
4. Multiply $x(\tau)h(t-\tau)$
5. Integration $\int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$

Convolution properties

Commutative property

$$x(t) * h(t) = h(t) * x(t)$$

Distributive property

$$x(t) * [h_1(t) + h_2(t)] = x(t) * h_1(t) + x(t) * h_2(t)$$

Associative property

$$x(t) * [h_1(t) * h_2(t)] = [x(t) * h_2(t)] * h_1(t)$$

- The shift property

If $y(t) = x(t) * h(t)$, then $x(t-T) * h(t) = x(t) * h(t-T) = y(t-T)$,
and $x(t-T_1) * h(t-T_2) = y(t-T_1-T_2)$.

Exercises

- **3.5**
  - Determine the output y(t) for the following pairs of input signals x(t) and impulse responses h(t):

(i) $x(t) = u(t)$, $h(t) = u(t)$;

(ii) $x(t) = u(-t)$, $h(t) = u(-t)$;

(iii) $x(t) = u(t) - 2u(t-1) + u(t-2)$, $h(t) = u(t+1) - u(t-1)$;

(iv) $x(t) = \exp(2t)u(-t)$, $h(t) = \exp(-3t)u(t)$;

Exercises

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- **3.8**
  - When the unit step function, $u(t)$, is applied as the input to an LTIC system, the output produced by the system is given by $y(t) = (1 - e^{-t})u(t)$.
    - ? Determine the impulse response of the system.
- **3.10**
  - An input signal $x(t) = 1 - t;\ 0 \le t \le 1;\ x(t) = 0$ otherwise; is applied to an LTIC system whose impulse response is given by $h(t) = e^{-t}u(t)$. Calculate the output of the system

Acknowledgement

Part of the slides from MIT open courseware: Signals and system, Instructor: Prof. Dennis Freeman

http://ocw.mit.edu/courses/electrical-engineering-and-computer-science/6-003-signals-and-systems-fall-2011/lecture-videos-and-slides/MIT6_003F11_lec08.pdf