MTH101: Lecture 23 – 24

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Applications

Example

Solve the ODE

$$(x^2 - x)y'' - xy' + y = 0.$$

Solution We can similarly find that

$$b(x) = \frac{x}{1-x} = x\sum_{m=0}^{\infty} x^m, \qquad b_0 = 0,$$

$$c(x) = \frac{-x}{1-x} = -x\sum_{m=0}^{\infty} x^m, \qquad c_0 = 0.$$

Applications

Solution The indicial equation is

$$r(r-1)=0 \Rightarrow r_1=1,\ r_2=0. \quad \textbf{Case 3.}$$

The first solution is $y_1(x)=\sum_{m=0}^{\infty} a_m x^{m+1}$, and we have

$$\sum_{m=0}^{\infty} m(m+1)a_m x^{m+1} - \sum_{m=0}^{\infty} m(m+1)a_m x^m$$
$$-\sum_{m=0}^{\infty}(m+1)a_m x^{m+1} + \sum_{m=0}^{\infty} a_m x^{m+1} = 0,$$
$$\Rightarrow [m(m+1)-(m+1)+1]\,a_m - (m+1)(m+2)a_{m+1} = 0.$$

Applications

Solution The recurrence relation is

$$a_{m+1} = \frac{m^2}{(m+1)(m+2)} a_m,$$

and $a_1 = a_2 = \cdots = 0$. If we choose $a_0 = 1$, $y_1(x) = x^{r_1} a_0 = x$. The second solution can be found by reduction of order. Let $y_2 = y_1 u = xu$, $y_2' = u + xu'$, $y_2'' = 2u' + xu''$. Substituting it into the ODE, we obtain

$$(x^2 - x)(xu'' + 2u') - x(xu' + u) + xu = 0,$$
$$\Rightarrow (x^2 - x)u'' + (x - 2)u' = 0.$$

Applications

Solution Therefore, by using the partial fractions and integrating, we get

$$\frac{u''}{u'} = -\frac{(x-2)}{(x^2-x)} = -\frac{2}{x} - \frac{1}{(1-x)} \quad \text{(Textbook is wrong here!)}$$

$$\Rightarrow \ln u' = \ln\left|\frac{1-x}{x^2}\right|$$

$$\Rightarrow u' = \frac{1-x}{x^2} = \frac{1}{x^2} - \frac{1}{x}, \quad u = -\frac{1}{x} - \ln x,$$

$$\Rightarrow y_2 = xu = -1 - x\ln x.$$

It is obvious that y_1 and y_2 are linearly independent, and we can use y_1 and y_2 to form a basis of the solution.

One of the most important ODEs in applied mathematics is the Bessel's equation

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0,$$

where ν is a given real number which is positive or zero. This equation often appears when the system admits cylindrical symmetry, e.g. electromagnetic waves in a cylindrical waveguide. If we divide the equation by x^2, we find that

$$y'' + \frac{1}{x}y' + \frac{(x^2 - \nu^2)}{x^2}y = 0,$$

and it has the form of Frobenius method with $b(x) = 1$ and $c(x) = x^2 - \nu^2$.

Since $b(x)$ and $c(x)$ are analyticat $x = 0$, according to Frobenius theorem, it has a solution of the form

$$y(x) = \sum_{m=0}^{\infty} a_m x^{m+r}.$$

By substituting it into the Bessel's equation, we find that

$$\sum_{m=0}^{\infty}(m+r)(m+r-1)a_m x^{m+r} + \sum_{m=0}^{\infty}(m+r)a_m x^{m+r}$$
$$+ \sum_{m=0}^{\infty} a_m x^{m+r+2} - \nu^2 \sum_{m=0}^{\infty} a_m x^{m+r} = 0.$$

The coefficient of x^r is $a_0[r(r-1)+r-\nu^2]=0$, and we find the indicial equation is

$$r^2-\nu^2=0 \quad \Rightarrow \quad (r-\nu)(r+\nu)=0.$$

and the roots are $r_1=\nu$ (≥ 0) and $r_2=-\nu$.
On the other hand, if we choose $r=\nu$, the coefficient of x^{r+1} is

$$\begin{aligned} & a_1\left[(1+r)(r)+(1+r)-\nu^2\right]=0 \\ \Rightarrow\ & a_1(2\nu+1)=0. \end{aligned}$$

Since $\nu \geq 0$, we thus have $a_1=0$.

Similarly, the coefficient of x^{r+s}, $s \geq 2$ is

$$\begin{aligned} & a_s\left[(s+r)(s+r-1)+(s+r)-\nu^2\right]+a_{s-2}=0 \\ \Rightarrow\ & s(s+2\nu)a_s+a_{s-2}=0 \\ \Rightarrow\ & a_s=-\frac{1}{s(s+2\nu)}a_{s-2}. \end{aligned}$$

Therefore,

$$a_1=a_3=a_5=\cdots=0, \quad a_{2m}=-\frac{1}{2^2m(\nu+m)}a_{2m-2},\ (m\in\mathbb{Z}^+).$$

Or,

$$a_2 = -\frac{1}{2^2(\nu+1)}a_0,$$

$$a_4 = -\frac{1}{2^2\cdot 2\cdot(\nu+2)}a_2 = \frac{1}{2^4\cdot 2\cdot(\nu+1)\cdot(\nu+2)}a_0,$$

and so on, and in general

$$a_{2m} = \frac{(-1)^m a_0}{2^{2m}m!(\nu+1)(\nu+2)\cdots(\nu+m)}, \quad m = 1, 2, \cdots.$$

Bessel Functions $J_n(x)$ for Integer $\nu = n$

If ν is integer, we denote $\nu = n$, and

$$a_{2m} = \frac{(-1)^m a_0}{2^{2m} m!(n+1)(n+2)\cdots(n+m)}, \quad m = 1, 2, \cdots .$$

Since a_0 is an arbitrary constant, in order to simplify the series, the standard choice is

$$a_0 = \frac{1}{2^n n!},$$

and

$$a_{2m} = \frac{(-1)^m}{2^{2m+n} m!(n+m)!}, \quad m = 1, 2, \cdots .$$

Bessel Functions $J_n(x)$ for Integer $\nu = n$

By inserting these coefficients and remembering $a_1 = a_3 = \cdots = 0$, we obtain a particular solution of Bessel's equation that is denoted by $J_n(x)$:

$$J_n(x) = x^n \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+n} m!(n+m)!}.$$

$J_n(x)$ is called the **Bessel function of the first kind** of order n. It converges for all x, and it converges very rapidly because of the factorials in the denominator.

Bessel Functions $J_0(x)$ and $J_1(x)$

For $n = 0$ we obtain the Bessel function of order 0:

$$J_0(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m}(m!)^2} = 1 - \frac{x^2}{2^2(1!)^2} + \frac{x^4}{2^4(2!)^2} - + \cdots ,$$

which looks similar to a cosine function.

$$\cos x = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{(2m)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - + \cdots .$$

Bessel Functions $J_0(x)$ and $J_1(x)$

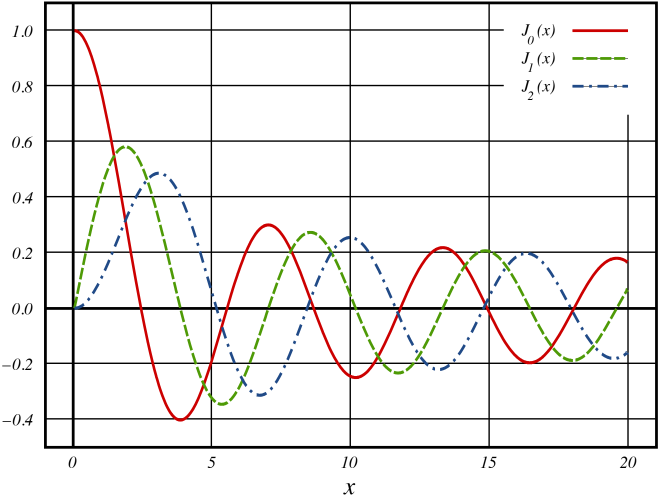
For $n = 1$ we obtain the Bessel function of order 1:

$$J_1(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{2^{2m+1} m!(m+1)!} = \frac{x}{2} - \frac{x^3}{2^3 1!2!} + \frac{x^5}{2^5 2!3!} - + \cdots ,$$

which looks similar to a sine function.

$$\sin x = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{(2m+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - + \cdots .$$

Bessel Functions $J_0(x)$ and $J_1(x)$



Remark

If we consider the standard form of the Bessel's equation

$$y'' + \frac{1}{x}y' + \frac{(x^2 - n^2)}{x^2}y = 0,$$

and consider the limit such that $|x| \gg 1$*, heuristically, we can assume the second and the last term are small and the equation come close to* $y'' + y = 0$*, and the solution is thus* $\cos x$ *and* $\sin x$ *while* y'/x *acts as a "damping term", which is responsible for the decrease in height. One can show that for large* x*,*

$$J_n(x) \approx \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right),$$

which is a surprisingly accurate approximation even for **smaller** x.

Bessel function $J_\nu(x)$ for any $\nu \geq 0$.

We can generalize the integer $\nu = n$ to any $\nu \geq 0$ by using the gamma function $\Gamma(\nu + 1)$. Remember that

$$\Gamma(\nu+1) = \int_0^\infty e^{-t} t^\nu dt, \quad \Gamma(\nu+1) = \nu\Gamma(\nu), \quad \Gamma(n+1) = n!.$$

For general ν, the appropriate choice would be

$$a_0 = \frac{1}{2^\nu \Gamma(\nu+1)}$$

$$\Rightarrow a_{2m} = \frac{(-1)^m}{2^{2m+\nu} m!(\nu+1)(\nu+2)\cdots(\nu+m)\Gamma(\nu+1)}, \quad m = 1, 2, \cdots$$

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Bessel function $J_\nu(x)$ for any $\nu \geq 0$.

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$$\Rightarrow a_{2m} = \frac{(-1)^m}{2^{2m+\nu} m!\Gamma(\nu+m+1)}, \quad m = 1, 2, \cdots .$$

The **Bessel function of the first kind of order** ν is thus

$$J_\nu(x) = x^\nu \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+\nu} m!\Gamma(\nu+m+1)},$$

which converges for all x.

General Solution. Linear Dependence

In order to find the general solution to the Bessel's equation, we will need a second solution which is independent to $J_\nu(x)$. For ν not an integer this is easy. We know there is another root $r_2 = -\nu$ which satisfies the indicial equation, and if we replace ν by $-\nu$ for $J_\nu(x)$, we have

$$J_{-\nu}(x) = x^{-\nu} \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m-\nu} m! \Gamma(-\nu + m + 1)},$$

if ν not an integer, $\Gamma(-\nu + m + 1)$ is finite and the first terms of J_ν and $J_{-\nu}$ are finite nonzero multiples of x^ν and $x^{-\nu}$, and thus are independent of each other.

General Solution. Linear Dependence

Therefore, if ν is not an integer, a general solution of Bessel's equation for all $x \neq 0$ is

$$y(x) = c_1 J_\nu(x) + c_2 J_{-\nu}(x).$$

On the other hand, it is not a general solution for an integer $\nu = n$ since in that case, $\Gamma(-\nu + m + 1) \to \infty$ for $m < \nu$ and we thus have linear dependence because

$$J_{-n}(x) = (-1)^n J_n(x), \qquad (n = 1, 2, \cdots).$$

This difficulty will be overcome in the next section, where we will introduce the second kind of Bessel functions $Y_\nu(x)$.

General Solution. Linear Dependence

(NONEXAMINABLE)

Example

Prove that for $\nu = n$,

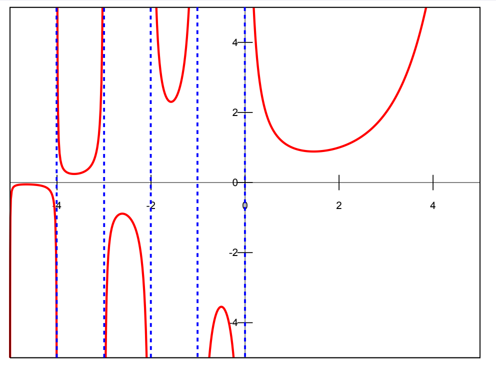
$$J_{-n}(x) = (-1)^n J_n(x), \qquad (n = 1, 2, \cdots).$$

General Solution. Linear Dependence

(NONEXAMINABLE)

Proof.

We first observe that $\Gamma(\alpha)$ becomes infinite for negative integer α:



General Solution. Linear Dependence

(NONEXAMINABLE)

Proof.

Therefore,

$$J_{-n}(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m-n}}{2^{2m-n} m! \Gamma(m-n+1)} = \sum_{m=n}^{\infty} \frac{(-1)^m x^{2m-n}}{2^{2m-n} m! \Gamma(m-n+1)},$$

since the gamma functions in the first n terms become infinite, and the coefficients become zero, and the summation starts with $m = n$. In this case, we can express $\Gamma(m-n+1) = (m-n)!$.

General Solution. Linear Dependence

(NONEXAMINABLE)

Proof.

We thus have

$$J_{-n}(x) = \sum_{m=n}^{\infty} \frac{(-1)^m x^{2m-n}}{2^{2m-n} m!(m-n)!} = \sum_{s=0}^{\infty} \frac{(-1)^{n+s} x^{2s+n}}{2^{2s+n}(n+s)!s!},$$

where we choose $s = m - n$ to rewrite the expression. Therefore,

$$J_{-n}(x) = (-1)^n x^n \sum_{s=0}^{\infty} \frac{(-1)^s x^{2s}}{2^{2s+n}(n+s)!s!} = (-1)^n J_n(x).$$

□

Properties of the Bessel Equations

Remark

There are four useful properties of the Bessel's equation

$$
\begin{aligned}
(a)\ & [x^\nu J_\nu(x)]' = x^\nu J_{\nu-1}(x), \\
(b)\ & [x^{-\nu} J_\nu(x)]' = -x^{-\nu} J_{\nu+1}(x), \\
(c)\ & J_{\nu-1}(x) + J_{\nu+1}(x) = \frac{2\nu}{x} J_\nu(x), \\
(d)\ & J_{\nu-1}(x) - J_{\nu+1}(x) = 2J_\nu'(x).
\end{aligned}
$$

Properties of the Bessel Equations

Proof.

(a)

$$x^\nu J_\nu(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+2\nu}}{2^{2m+\nu} m! \Gamma(\nu+m+1)}$$

$$\Rightarrow [x^\nu J_\nu(x)]' = \sum_{m=0}^{\infty} \frac{(-1)^m 2(m+\nu) x^{2m+2\nu-1}}{2^{2m+\nu} m! \Gamma(\nu+m+1)}$$

$$\Rightarrow [x^\nu J_\nu(x)]' = x^\nu x^{\nu-1} \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+\nu-1} m! \Gamma(\nu+m)}$$

$$\Rightarrow [x^\nu J_\nu(x)]' = x^\nu J_{\nu-1}(x).$$

$\square$

Properties of the Bessel Equations

Proof.

(b)

$$x^{-\nu}J_\nu(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+\nu} m!\Gamma(\nu+m+1)}$$

$$\Rightarrow [x^{-\nu}J_\nu(x)]' = \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m-1}}{2^{2m+\nu-1}(m-1)!\Gamma(\nu+m+1)}$$

$$\Rightarrow [x^{-\nu}J_\nu(x)]' = \sum_{s=0}^{\infty} \frac{(-1)^{s+1} x^{2s+1}}{2^{2s+\nu+1} s!\Gamma(\nu+s+2)}$$

$$\Rightarrow [x^{-\nu}J_\nu(x)]' = -x^{-\nu}\sum_{s=0}^{\infty} \frac{(-1)^{s} x^{2s+\nu+1}}{2^{2s+\nu+1} s!\Gamma(\nu+s+2)}$$

$$\Rightarrow [x^{-\nu}J_\nu(x)]' = -x^{-\nu}J_{\nu+1}(x).$$

Properties of the Bessel Equations

Proof.

(c)
Performing the differentiation of (a) and (b), we have

$$(a^*)\ \nu x^{\nu-1}J_\nu + x^\nu J_\nu' = x^\nu J_{\nu-1},$$
$$(b^*)\ -\nu x^{-\nu-1}J_\nu + x^{-\nu}J_\nu' = -x^{-\nu}J_{\nu+1}.$$

$$(a^*) - x^{2\nu}(b^*):\ 2\nu x^{\nu-1}J_\nu = x^\nu(J_{\nu-1}+J_{\nu+1})$$
$$\Rightarrow J_{\nu-1}+J_{\nu+1} = \frac{2\nu}{x}J_\nu.$$

□

Properties of the Bessel Equations

Proof.

(d)
Similarly,

$$
\begin{aligned}
(a^*) + x^{2\nu}(b^*) : 2x^\nu J'_\nu &= x^\nu (J_{\nu-1} - J_{\nu+1}) \\
&\Rightarrow J_{\nu-1} - J_{\nu+1} = 2J'_\nu.
\end{aligned}
$$

$\square$

Application

Example

We can rewrite (c) as

$$J_{\nu+1}(x) = \frac{2\nu}{x} J_\nu(x) - J_{\nu-1}(x),$$

and calculate the Bessel function of higher order from those of lower order. For example,

$$\begin{aligned} J_2(x) &= \frac{2}{x} J_1(x) - J_0(x) \\ J_3(x) &= \frac{4}{x} J_2(x) - J_1(x) = \frac{4}{x}\left[\frac{2}{x} J_1(x) - J_0(x)\right] - J_1(x) \\ &= \left(\frac{8}{x^2} - 1\right) J_1(x) - \frac{4}{x} J_0(x). \end{aligned}$$

Bibliography

1. *Kreyszig, E.* **Advanced Engineering Mathematics**. Wiley, 10th Edition.
2. *Wikipedia* **Legendre polynomials**.