Chapter 1.3 Probability

1 March 2018

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1.3.1 First Definition of Probability

- If the sample space S of an experiment consists of finitely many outcomes (sample points) that are equally likely, then the *probability* $P(A)$ of an event A is

$$P(A) = \frac{\text{Number of points in } A}{\text{Number of points in } S}$$

- It follows that $P(S) = 1$.

1.3.1 First Definition of Probability: example 1

Example 1

In rolling a fair die once, what is the

i. probability $P(A)$ of A of obtaining a 5 or a 6?

ii. probability $P(B)$ of B of obtaining an even number?

1.3.1 First Definition of Probability: example 1

Solution

The six outcomes are equally likely, so each has probability $\frac{1}{6}$.

i. obtaining a 5 or a 6

Event $A = \{5, 6\}$ so probability $P(A) = \frac{2}{6} = \frac{1}{3}$.

ii. obtaining an even number

Event $B = \{2, 4, 6\}$ so probability

$$P(B) = \frac{3}{6} = \frac{1}{2}. \quad \blacksquare$$

1.3.1 First Definition of Probability : example 2

Example 2

A fair die is rolled two times. What is the probability that the same number appear twice?

In each trial, the outcomes are 1, 2, 3, 4, 5, 6. For two dice, the different combinations are:

	1	2	3	4	5	6
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
5	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
6	(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

1.3.1 First Definition of Probability: example 1

Solution

In the experiment, for the same number to appear twice, there are 6 possible outcomes:

$(1,1), (2,2), (3,3), (4,4), (5,5), (6,6)$

from a total of $6 \times 6 = 36$ possible outcomes.

$\therefore$ Required probability $= \frac{6}{36} = \frac{1}{6}$.

1.3.2 Axioms of Probability

Given a sample space S, with each event A of S there is associated a number $P(S)$, called the *probability* of A, satisfying the following axioms.

1. For every A in S, $0 \leq P(A) \leq 1$.
2. For sample space S, $P(S) = 1$.
3. For any sequence of mutually exclusive events, $A_1, A_2, \cdots$ $[A_i \cap A_j = \emptyset, i \neq j]$

   $P(A_1 \cup A_2 \cup \cdots) = P(A_1) + P(A_2) + \cdots$.

For mutually exclusive events A and B,

$$P(A \cup B) = P(A) + P(B). \text{ (only if } A \cap B = \emptyset\text{)}$$

1.3.2 Axioms of Probability

Example 3

If the probability that cars to service on any workday a garage will get are $\mathrm{P}([0-9]) = 0.10, P([10-20]) = 0.20, P([21-30]) = 0.33, P([31-40]) = 0.25, P[> \ 40]) = 0.12,$

what is the probability that on a given day the garage gets at least 21 cars to service?

Solution

The events are mutually exclusive, so the probability is

$$\mathrm{P}(\mathrm{x} \geq 21) = P([21-30] \cup [31-40] \cup [> 40])$$
$$= P(x \in [21-30]) + P(x \in [31-40]) + P(x > 40) =$$

$0.33 + 0.25 + 0.12 = 0.70$ ■

What is the probability of less than 21 cars?

1.3.3 Basic Theorems of Probability

The axioms of probability enable us to build up probability theory.

1) Complementation Rule

For an event A and its complement $\bar{A}$ in a sample space S,

$$P(\bar{A}) = 1 - P(A).$$

Proof

Consider $S = A \cup \bar{A}$ where A and $\bar{A}$ are mutually exclusive.

Therefore $1 = P(S) = P(A) + P(\bar{A})$

Rearranging, $P(\bar{A}) = 1 - P(A)$. ■

1.3.3 Basic Theorems of Probability: example 4

Example 4

Five coins are tossed simultaneously. Find the probability of the event A of at least one head turns up. Assume that the coins are fair.

Solution **(at least one means = only not valid result is no heads)**

$$P(\bar{A}) = P(\text{no head}) = P(\text{all tails}) = P(TTTTT) = \left(\frac{1}{2}\right)^5 = \frac{1}{32}.$$

Therefore $P(A) = 1 - P(\bar{A}) = 1 - \frac{1}{32} = \frac{31}{32}$.

1.3.3 Basic Theorems of Probability

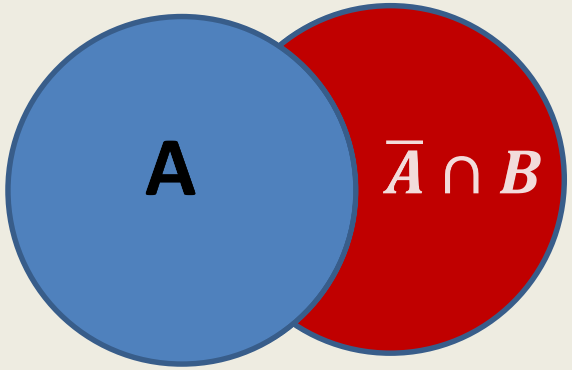
2) Addition Rule for Arbitrary Events

For events A and B in a sample space,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

1.3.3 Basic Theorems of Probability

Proof of addition rule



We partition $A \cup B$ into A and $\bar{A} \cap B$, i.e

$$A \cup B = A \cup (\bar{A} \cap B).$$

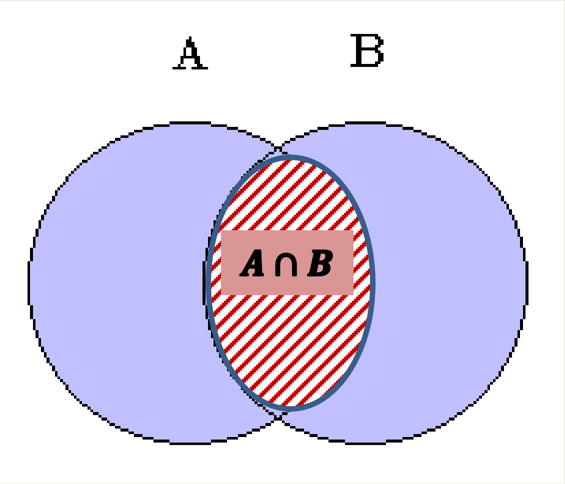
Taking probability,

$$P(A \cup B) = P(A) + P(\bar{A} \cap B). \qquad (1)$$

Similarly we partition B into $\bar{A} \cap B$ and $A \cap B$, i.e.
$B = (\bar{A} \cap B) \cup (A \cap B)$. Therefore, $\mathrm{P}(\bar{A} \cap B) = P(B) - (A \cap B)$
substituting in (1) gives $P(A \cup B) = P(A) + P(B) - (A \cap B)$ ■

1.3.3 Basic Theorems of Probability: example 5

The addition rule is telling us that if we add P(A) to P(B) we are including twice $P(A \cap B)$.



Example 5

Roll a die. Define

A = face up odd = {1, 3, 5}, B = face up prime = {2, 3, 5}

$P(A) = \frac{3}{6} = \frac{1}{2}, P(B) = \frac{3}{6} = \frac{1}{2}$ and $P(A) + P(B) = 1$

cannot be! Because $A \cup B = \{1, 2, 3, 5\}$, so $P(A \cup B) = \frac{4}{6} = \frac{2}{3}$

Now, $A \cap B = \{2, 3\}$ means that {2, 3} is both in A and B, so we have to take it away from the sum, otherwise we add it twice. In fact:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{1}{2} + \frac{1}{2} - \frac{2}{6} = \frac{2}{3}$$

1.3.3 Basic Theorems of Probability: problem

- Consider the distribution of pass/fails in a course by student gender.
- Find $P(Male \cap Pass)$
- Find $P(Male \cup Pass)$

	Pass	Fail	Tot
Male	60	30	90
Female	9	1	10
Tot	69	31	100

1.3.3 Basic Theorems of Probability: marginal probabilities

- Consider the distribution of pass/fails in a course by student program.
- We can "eliminate" one variable by summing the joint probabilities

	Pass	Fail	Tot
Math	30	20	50
Eng.	19	11	30
Science	11	9	20
Tot	60	40	100

$$P(Pass) = P(Math \cap Pass) + P(Eng \cap Pass) + P(Science \cap Pass) = \frac{30 + 19 + 11}{100} = \frac{60}{100} = 0.6$$

These are called *marginal* probabilities.

1.3.3 Marginal probabilities, example

- In Wuxi, the probabilities of rain (R = Rain, NR = No Rain) and weather (C = Cloudy, NC = Not Cloudy) $P(R \cap C) = 0.5$, $P(R \cap NC) = 0.05$, $P(NR \cap C) = 0.3$ and $P(NR \cap NC) = 0.15$. The sum is 1, because one of these must happen.
- What is the probability of Rain –regardless the weather-?

$$P(R) = P(R \cap C) + P(R \cap NC) = 0.5 + 0.05 = 0.55$$

- What is the probability of Cloudy?

$$P(C) = P(C \cap R) + P(C \cap NR) = 0.5 + 0.3 = 0.8$$

1.3.4 Conditional Probability

We often are required to find the probability of an event B under the condition that an event A occurs. This probability is called the *conditional probability* of B given A:

$$P(B|A) = \frac{P(B \cap A)}{P(A)}$$

Similarly,

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

1.3.4 Conditional Probability

Example 6

We throw two dice. Let event A be the number on die 1. Let event B be the number on die 2.

Find $P(A = 2|A + B \leq 5)$.

Solution

$$P(A = 2|A + B \leq 5) = \frac{P(A = 2 \text{ and } A + B \leq 5)}{P(A + B \leq 5)}$$

1.3.4 Conditional Probability

$$= \frac{P(A = 2 \text{ and } B \leq 3)}{P(A + B \leq 5)}$$

$$= \frac{3/_{36}}{10/_{36}} = \frac{3}{10} \quad \blacksquare$$

Another way:

Consider only events for which $(A + B \leq 5)$

$P(A = 2|A + B \leq 5)$

$= 3/10$

Table 3: sum of two dice

		Die 2 (B)					
		1	2	3	4	5	6
Die 1 (A)	1	2	3	4	5	6	7
	2	3	4	5	6	7	8
	3	4	5	6	7	8	9
	4	5	6	7	8	9	10
	5	6	7	8	9	10	11
	6	7	8	9	10	11	12

1.3.4 Conditional Probability: problem

- Consider the distribution of pass/fails in a course by student gender.
- The probability of picking a student that is male and passed is $\frac{60}{100} = 0.6$. The probability of picking a student that is female and passed is $\frac{9}{100} = 0.09$.

	Pass	Fail	Total
Male	60	30	90
Female	9	1	10
Total	69	31	100

- So boys are way better than girls?

1.3.4 Conditional Probability

1) Multiplicative Rule

If A and B are events in a sample space S and $P(A) \neq 0, P(B) \neq 0$, then

$P(A \cap B) = P(A)P(B|A) = P(B)P(A|B)$.

Proof

the result follows from the definition of conditional probability

$P(B|A) = \frac{P(A \cap B)}{P(A)}$ and similarly for $P(A|B)$.

1.3.4 Conditional Probability

Example 7

In producing screws, let A mean "screw too slim" and B mean "screw too short". Let $P(A) = 0.1$ and let the conditional probability that a slim screw is also too short be $P(B|A) = 0.2$. What is the probability that a screw that we pick randomly from the lot produced will be both too slim and too short?

Solution

$$P(A \cap B) = P(A)P(B|A) = 0.1 \times 0.2 = 0.02 \blacksquare$$

It can be shown that if $A_1, A_2, \cdots, A_n$ are events in a sample space then $P(A_1 \cap A_2 \cap \cdots \cap A_n)$

$$= P(A_1)P(A_2|A_1)P(A_3|A_1 \cap A_2) \cdots P(A_n|A_1 \cap A_2 \cap \cdots \cap A_{n-1})$$

1.3.5 Independence

1) Independence

If A and B are such that $P(A \cap B) = P(A)P(B)$

they are called <u>*independent*</u> events.

Also, $P(A|B) = P(A)$ and $P(B|A) = P(B)$ for independent events A and B. The probability of A does not depend on the occurrence or nonoccurrence of B, and conversely.

1.3.5 Independence

2) Independence of more than 2 Events

For three events, A, B, C are *independent* if

$$P(A \cap B) = P(A)P(B),$$
$$P(B \cap C) = P(B)P(C),$$
$$P(C \cap A) = P(C)P(A), \text{and}$$
$$P(A \cap B \cap C) = P(A)P(B)P(C).$$

Generalizing, we define a (finite/infinite) set of events to be independent if every finite subset of those events is independent.

1.3.5 Independence: example 8

Example 8

What is the probability of

i. Obtain two sixes rolling a die two times?

ii. Draw two Kings from a deck of 52 cards (13 cards of 4 suits).

1.3.5 Independence : example 8

Solution:

i. Let A = result first roll and B = result second roll.
The two events are independent, P(B|A) = P(B).
So $P(A = 6 \ \cap B = 6) = P(A = 6)P(B = 6|A = 6) = \frac{1}{6}\frac{1}{6} = \frac{1}{36}$

ii. Let A = first card drawn and B = second card drawn

There are 4 kings in 52 cards, so $P(A = K) = \frac{4}{52} = \frac{1}{13}$.

If A = K, there remain 3K in 51 cards, so $P(B = K|A = K) = \frac{3}{51}$.

Therefore $P(A = K \cap B = K) = \frac{1}{13}\frac{3}{51} = \frac{3}{2652} = 0.113\%$
(1.13 in 1000) the two draws are not independent

1.3.5 Independence: problem

- A company produces nails and screws. The probability of producing defective nails is 0.01 while for screws it is 0.05. They deliver a box with 20 nails and 40 screws.
  I. What is the probability that Mr Yung picks at random one piece and it is a defective screw?
  II. If Mr Yung picks a screw, what is the probability that it is defective?

1.3.6 Summary

- Axioms and Rules of Probability
  - Axioms, complementation, addition
- Conditional Probability
  - Definition, multiplication rule
- Independence
  - Definition and conditions for