EEE108 Electromagnetism and Electromechanics

Lecture 12

Inductors and Inductance

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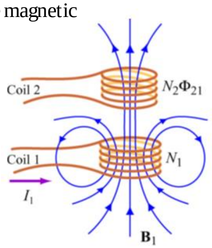
Today

- Inductors and Inductance
  - Energy Stored in Magnetic Fields
  - *RL* Circuits

Mutual Inductance

Suppose two coils are placed near each other. Some of the magnetic field lines through coil 1 will also pass through coil 2. Let Φ_{21} denote the magnetic flux through one turn of coil 2 due to I_1. By varying I_1 with time, there will be an induced emf associated with the changing magnetic flux in coil 2: $\varepsilon_{21} = -N_2 \dfrac{d\Phi_{21}}{dt}$



The rate of change of Φ_{21} in coil 2 is proportional to the time rate of change of the current in coil 1: $N_2 \dfrac{d\Phi_{21}}{dt} = M_{21}\dfrac{dI_1}{dt}$

where the proportionality constant M_{21} is called the ***mutual inductance***.

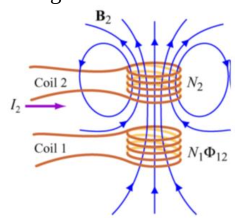
It may also be written as: $M_{21} = \dfrac{N_2 \Phi_{21}}{I_1}$

The SI unit for inductance is the henry (H): $1\,\text{henry} = 1\,\text{H} = 1\,\text{T}\cdot\text{m}^2/\text{A}$

Mutual Inductance

Similarly, the induced emf in coil 1 due to current change in coil 2:

$$\varepsilon_{12} = -N_1 \frac{d\Phi_{12}}{dt}$$



This changing flux in coil 1 is also proportional to the changing current in coil 2,

$$N_1 \frac{d\Phi_{12}}{dt} = M_{12}\frac{dI_2}{dt}$$

M_{12} also nay be written as: $M_{12} = \dfrac{N_1 \Phi_{12}}{I_2}$

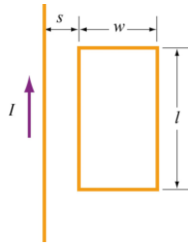
$\boxed{M_{12} = M_{21} \equiv M}$ Mutual inductance depends only on the geometrical factors of the system.

Mutual Inductance

Example

An infinite straight wire carrying current I is placed to the left of a rectangular loop of wire with width w and length l.

Determine the mutual inductance of the system.



Solution:

The total magnetic flux Φ_B through the loop:

$$\Phi_B = \int d\Phi_B = \int \mathbf{B}\cdot d\mathbf{s} = \frac{\mu_0 I l}{2\pi}\int_s^{s+w}\frac{dr}{r} = \frac{\mu_0 I l}{2\pi}\ln\left(\frac{s+w}{s}\right)$$

Thus, the mutual inductance is:

$$\boxed{M = \frac{\Phi_B}{I} = \frac{\mu_0 l}{2\pi}\ln\left(\frac{s+w}{s}\right)} \qquad (N=1)$$

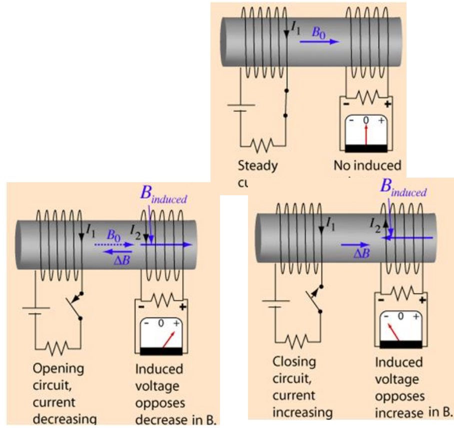
M depends only on the geometrical factors of the system (l, s, w) and is independent of the current.

Mutual Inductance

Lenz's Law

The direction of the induced current is determined by **Lenz's law**:

The induced current produces magnetic fields which tend to oppose the change in magnetic flux that induces such currents.



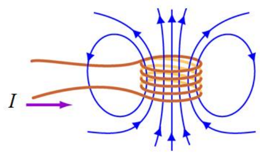
Self-Inductance

Consider a coil consisting of N turns and carrying current I.
If current is steady, magnetic flux through the loop remains constant. If I changes with time, then an induced emf arises to oppose the change.

The property of the loop in which its own magnetic field opposes any change in current is called "***self-inductance***," and the emf generated is called the ***self-induced emf*** or ***back emf***. From Faraday's law:

The self-induced emf :

$$\varepsilon_L = -N\frac{d\Phi_B}{dt}$$



The self-induced emf :

$$N\frac{d\Phi_B}{dt} = L\frac{dI}{dt}$$

where the self-inductance :

$$L = \frac{N\Phi_B}{I}$$

The SI unit is the henry (H)

Self-inductance depends only on the geometrical factors of the system.

Self-Inductance

Example -- Self-Inductance of a Solenoid

Find the self-inductance of a solenoid with N turns, length l, and radius R with a current i flowing through each turn.

Solution:

$$\boxed{L = \frac{N\Phi_B}{i}}$$

Ignoring edge effects and applying Ampere's law, the magnetic field inside a solenoid is :

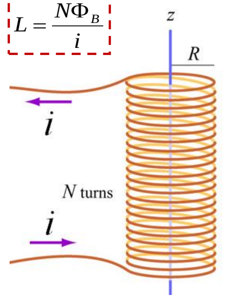
$$\mathbf{B} = \frac{\mu_0 N i}{l}\mathbf{a}_z = \mu_0 n i \mathbf{a}_z$$

The magnetic flux through each turn :

$$\Phi_B = BA = \mu_0 n i(\pi R^2)$$

Thus, the self-inductance is :

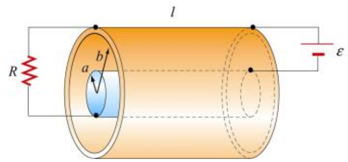
$$L = \frac{N\Phi_B}{i} = \mu_0 n^2 \pi R^2 l$$



L depends only on the geometrical factors (n, R and l) and is independent of the current i.

Self-Inductance
Example -- Self-Inductance of a Co-axial Cable

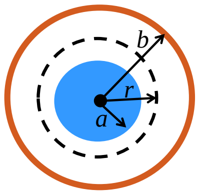
Consider a coaxial transmission line constructed of conducting cylinders of radius a and b.



Solution:

The current I flows down the inner conductor and exactly the same amount of current flows in the opposite direction in the outer conductor.

The flux density at any radius r between a and b is the same as at this radius from a long straight conductor with the same current: $B(r)=\dfrac{\mu_0 I}{2\pi r}$



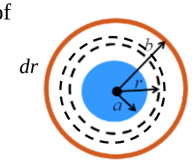
Self-Inductance
Example -- Self-Inductance of a Co-axial Cable

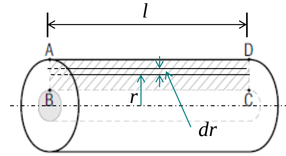
The total flux for a length of l is then integral of from the inner to outer conductor :

$$\Phi_B=\int_a^b \frac{\mu_0 I}{2\pi r}(l dr)=\frac{l\mu_0 I}{2\pi}\int_a^b \frac{dr}{r}$$
$$=\frac{l\mu_0 I}{2\pi}\ln\left(\frac{b}{a}\right)$$

The self - inductance :

$$L=\frac{\Phi_B}{I}=\frac{l\mu_0}{2\pi}\ln\left(\frac{b}{a}\right)$$





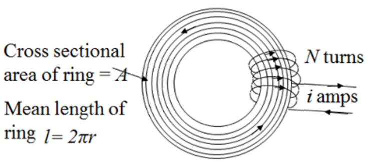
L depends only on the geometrical factors of the system.

$$B(r)=\frac{\mu_0 I}{2\pi r} \qquad d\Phi_B=\frac{\mu_0 I}{2\pi r}(l dr)$$

Self-Inductance
Example -- Self-Inductance of a Toroid

Consider a toroid with high μ material. Find the self-inductance.

Solution:



The flux created in the ring by a current i through the coil is:

$$\Phi_B=\frac{\mu N i A}{l}=\frac{\mu_0\mu_r N i A}{l}$$

If the current changes with time: $\dfrac{d\Phi_B}{dt}=\dfrac{\mu_0\mu_r NA}{l}\dfrac{di}{dt}$

Then the back emf generated : $\varepsilon_L=-N\dfrac{d\Phi_B}{dt}=-\dfrac{\mu_0\mu_r N^2 A}{l}\dfrac{di}{dt}=-L\dfrac{di}{dt}$

So the self - inductance : $L=\dfrac{\mu_0\mu_r N^2 A}{l}$

Again! L depends only on the geometrical factors of the system.

Inductors

An **inductor** is a circuit device that is designed to have a particular inductance that can store energy in a magnetic field.

An inductor's ability to store magnetic energy is measured by its **inductance**.

Typically an inductor is a conducting wire shaped as a coil, the loops helping to create a strong magnetic field inside the coil.

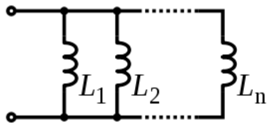
Inductor

Working principle Electromagnetic induction
First production Michael Faraday (1831)
Electronic symbol

Three basic elements:
- ❖ Resistors
- ❖ Capacitors
- ❖ Inductors

Inductors in Parallel

Inductors follow the same law as resistors: the total inductance of non-coupled inductors in parallel is equal to the reciprocal of the sum of the reciprocals of their individual inductances:



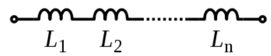
$$\frac{1}{L} = \frac{1}{L_1} + \frac{1}{L_2} + \ldots + \frac{1}{L_n}$$

This is not always true!

If the inductors are situated in each other's magnetic fields, this approach is INVALID due to mutual inductance.

Inductors in Series

The current through inductors in series stays the same, but the voltage across each inductor can be different. The sum of the potential differences (voltage) is equal to the total voltage. The total inductance:



$$L = L_1 + L_2 + \ldots + L_n$$

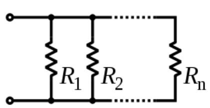
Again bear in mind: the simple relationship hold true only when there is **no mutual coupling** of magnetic fields between individual inductors.

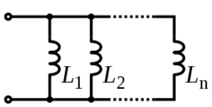
Parallel Circuits

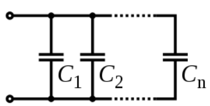
Resistors

Inductors

Capacitors







$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \ldots + \frac{1}{R_n}$$

$$\frac{1}{L} = \frac{1}{L_1} + \frac{1}{L_2} + \ldots + \frac{1}{L_n}$$

$$C = C_1 + C_2 + \ldots + C_n$$

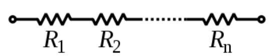
no mutual coupling

Series Circuits

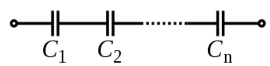
Resistors

Inductors

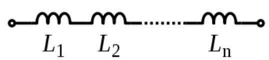
Capacitors



$$R = R_1 + R_2 + \ldots + R_n$$



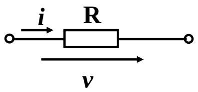
$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} + \ldots + \frac{1}{C_n}$$



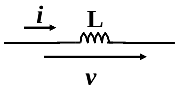
$$L = L_1 + L_2 + \ldots + L_n$$

no mutual coupling

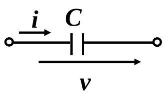
Three Basic Elements



$v = Ri$



$v = L\dfrac{di}{dt}$



$i = C\dfrac{dv}{dt}$

•Resistors are energy consumers. Inductors and capacitors are energy storage elements.

•Circuits that contain inductors and capacitors can be represented by differential equations

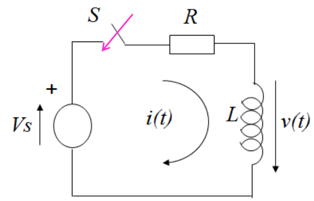
•A first order circuit has only one energy storage element, i.e. one inductor or one capacitor.

•There are two types of first order circuits:

RC circuit
RL circuit

Current Growth in an *RL* Circuit

Consider a circuit consisting of a voltage supply, switch, resistor and a coil of *N* turns, all in series.



With the switch open, no current flows in the coil and therefore there is no flux produced in the core of the coil.

At $t = 0$, the switch is closed and the current builds up with time. It is not instantaneous, because the coil produces a back emf that tries to oppose the rise of current.

Current cannot jump in an inductor.
Voltage cannot jump in a capacitor.

Current Growth in an *RL* Circuit

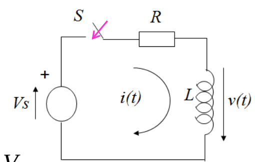
Time Constant

Kirchhoff's Circuit Rules:

$$V_s = i(t)R + v(t)$$

where $v(t) = L\dfrac{di(t)}{dt}$

$$L\frac{di(t)}{dt} + i(t)R = V_s \quad \Rightarrow \quad \frac{di(t)}{dt} + \frac{R}{L}i(t) = \frac{V_s}{L}$$



The solution of the first order differential equation is:

$$i(t) = I_0\left(1 - e^{-t/\tau}\right)$$

where $I_0 = \dfrac{V_s}{R}$ and $\tau = \dfrac{L}{R}$

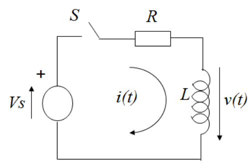
Time constant of *RL* circuit

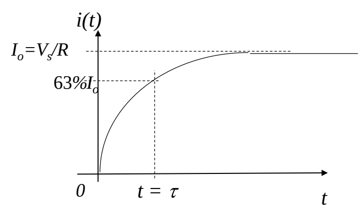
$$\tau = RC$$

Time constant of *RC* circuit

Current Growth in an RL Circuit

Current Rise in a Switched Inductor





$i(t) = I_0\left(1 - e^{-t/\tau}\right)$

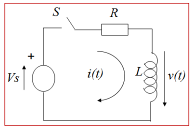
where $I_0 = \dfrac{V_s}{R}$ and $\tau = \dfrac{L}{R}$

t	0	τ	2τ	3τ	5τ
$i(t) = I_0\left(1 - e^{-t/\tau}\right)$	**0**	$\mathbf{I_0\,(1-e^{-1})}$	$\mathbf{I_0\,(1-e^{-2})}$	$\mathbf{I_0\,(1-e^{-3})}$	$\mathbf{I_0\,(1-e^{-5})}$
	0	**0.632 I_0**	**0.865 I_0**	**0.95 I_0**	**0.993 I_0**

Current Growth in an RL Circuit
Energy Stored in a Switched Inductor

The increase in the current is building up the magnetic field surrounding the coil. Energy is stored in that field.

Consider the energy supplied by the voltage source during the charge up period and where it goes…



Voltage $\quad V_s = iR + v \quad \Rightarrow \quad V_s i = i^2 R + vi$

Energy $\quad \int_0^t V_s i dt = \int_0^t i^2 R dt + \int_0^t v i dt$

Energy delivered by the source for the time period of 0 - t

Energy lost as heat in the resistor

Energy stored in the magnetic field of the inductor

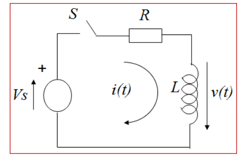
The term representing the energy stored in the coil may be written in terms of the inductance of the coil as:
$$\int_0^t v i dt = \int_0^t L\frac{di}{dt} i dt = \int_0^I L i di = \frac{1}{2}LI^2$$

The energy stored in the coil (the inductor) increases with the ***square*** of the current flowing through it and in proportion to the value of the coil inductance.

Current Growth in an RL Circuit
Energy Stored in a Switched Inductor

Energy stored in an inductor:

$$\boxed{U_B = \frac{1}{2}LI^2}$$



From the energy perspective there is an important distinction between an inductor and a resistor:

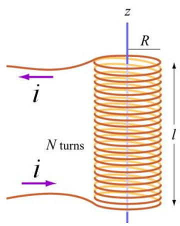
Whenever a current goes through a resistor, energy flows into the resistor and dissipates in the form of heat regardless of whether current is steady or time-dependent.

$$V_s = iR + v \quad \Rightarrow \quad V_s i = i^2 R + vi$$
$$\int_0^t V_s i dt = \int_0^t i^2 R dt + \int_0^t v i dt$$
$$v = L\frac{di}{dt}$$

Energy flows into an ideal inductor only when the current is increasing. The energy is not dissipated but stored there; it is released later when the current decreases. If the current that passes through the inductor is steady, then there is no change in energy since $di/dt = 0$.

Current Growth in an RL Circuit
Energy Density in an Inductor

In a solenoid:



$$\mathbf{B} = \frac{\mu_0 N i}{l}\mathbf{a}_z = \mu_0 n i \mathbf{a}_z \quad \Rightarrow \quad B = \mu_0 n i \quad \Rightarrow \quad i = \frac{B}{\mu_0 n}$$
$$\Phi_B = BA = \mu_0 n i (\pi R^2) \Rightarrow L = \mu_0 n^2 A l$$
$$U_B = \frac{1}{2}Li^2 = \frac{1}{2}\mu_0 n^2 A l \left(\frac{B}{\mu_0 n}\right)^2 = \frac{1}{2\mu_0} A l B^2$$

$Al = V$ volume inside the solenoid

Energy density stored in an inductor:
$$u_B = \frac{1}{2\mu_0}B^2$$

General results

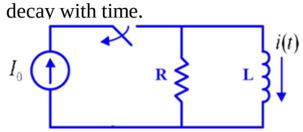
Energy density stored in a capacitor:
$$u_E = \frac{1}{2}\varepsilon E^2$$

Energy is stored in the magnetic field.

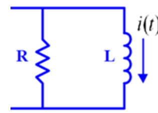
Energy is stored in the electric field.

Current Decay in an RL Circuit

The switch has been closed for long time. At t = 0, the switch is opened and the current decay with time.

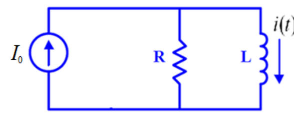


When t >= 0



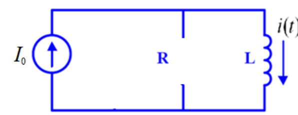
$$v_L = v_R \quad \Rightarrow \quad v_L + v_R = 0 \quad (t \geq 0)$$

so $\quad L\dfrac{di(t)}{dt} + i(t)R = 0 \quad (t \geq 0)$



Can the current source be replaced by a voltage source?

When t < 0



Current Decay in an RL Circuit

$$L\frac{di(t)}{dt}+i(t)R=0\quad(t\geq 0)$$

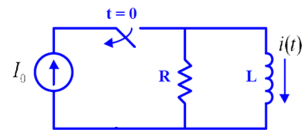
It is a first order differential equation with the initial boundary condition :

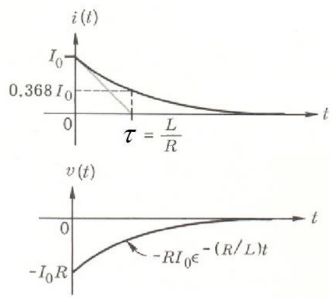
$$i=I_0\qquad(t=0)$$

The solution is :

$$i(t)=I_0e^{-t/\tau}$$

where $\tau = L/R$ called time constant.





The energy dissipated over all time by the resistor equals the initial energy stored in the inductor.

Summary

•The mutual inductance of two coils

$$M_{12}=\frac{N_{12}\Phi_{12}}{I_1}=M_{21}=\frac{N_1\Phi_{21}}{I_2}=M$$

•The induced emf in coil 2 due to the change in current in coil 1

$$\varepsilon_2=-M\frac{dI_1}{dt}$$

•The self-inductance of a coil with N turns

$$L=\frac{N\Phi_B}{I}$$

•The self-induced emf responding to a change in current inside a coil current

$$\varepsilon_L=-L\frac{dI}{dt}$$

•The current in the RL circuit

$i(t)=I_0(1-e^{-t/\tau})$ charging

$i(t)=I_0e^{-t/\tau}$ decaying

$\tau = L/R$ is the time constant.

•The magnetic energy stored in an inductor $U_B=\frac{1}{2}LI^2$

•Energy density

$$u_B=\frac{1}{2\mu_0}B^2$$

Next Week: Midterm Exam 10% of the module marks

Midterm Exam Including:

1st part of the module:

- Vector operation
- Electrostatics
- Gauss's Law
- Electrical Current
- Biot-Savart Law
- Ampere's Law
- Faraday's Law

What you have leant up to now!

Module	Date	Day	Start Time	Duration	Exam Room
EEE108	12-Apr	Thu	10:00 am	1h	Public Building-P020

Next Week: Midterm Exam

Thanks for your attendance and

Good Luck!