EEE108 Electromagnetism and Electromechanics

Lecture 5

Electric Potential

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Today

- Electric Potential due to Continuous Distributions
- Poisson's Equation
- Laplace Equation
- Electric Dipole

Electric Potential due to Continuous Distributions

$$\boxed{\varphi_P = \frac{1}{4\pi\varepsilon}\sum_{n=1}^{N}\frac{Q_n}{r_n}}$$

$$\varphi_L = \frac{1}{4\pi\varepsilon}\int_L \frac{\rho_L}{r}dl \quad \text{(V)} \qquad \text{Line charge}$$

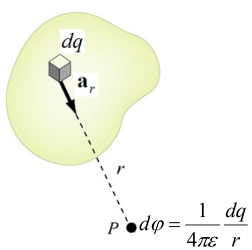
where ρ_L = linear charge density C/m

$$\varphi_S = \frac{1}{4\pi\varepsilon}\iint_S \frac{\rho_S}{r}dS \quad \text{(V)} \qquad \text{Surface charge}$$

where ρ_S = surface charge density C/m^2

$$\varphi_V = \frac{1}{4\pi\varepsilon}\iiint_V \frac{\rho_V}{r}dV \quad \text{(V)} \qquad \text{Volume charge}$$

where ρ_V = volume charge density C/m^3



Electric Potential due to Continuous Distributions

Example 1

A uniformly charged ring of radius R and charge density ρ_L.

Calculate the electric potential at a distance z from the central axis in free space.

Solution

$dl = Rd\phi \qquad dq = \rho_L dl = \rho_L Rd\phi$

The electric potential at point P:

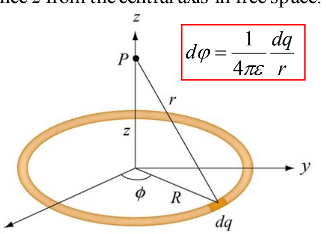
$$d\varphi = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r} = \frac{1}{4\pi\varepsilon_0}\frac{\rho_L Rd\phi}{\sqrt{R^2+z^2}}$$

$$\varphi = \int d\varphi = \frac{1}{4\pi\varepsilon_0}\frac{\rho_L R}{\sqrt{R^2+z^2}}\oint d\phi$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{Q}{\sqrt{R^2+z^2}} \qquad \boxed{\varphi = \frac{1}{4\pi\varepsilon_0}\frac{Q}{\sqrt{R^2+z^2}}}$$

where $\oint d\phi = 2\pi$ and

$Q = 2\pi R\rho_L$ (total charge)



If $z >> R$, the potential approaches its "point-charge":

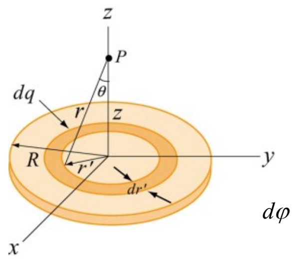
$$\varphi = \frac{1}{4\pi\varepsilon_0}\frac{Q}{z}$$

Electric Potential due to Continuous Distributions

Example 2

A uniformly charged disk of radius R and charge density ρ_S.

Calculate the electric potential at a distance z from the central axis in free space.



$$\boxed{\varphi = \frac{1}{4\pi\varepsilon_0}\frac{Q}{\sqrt{R^2+z^2}}}$$

$$d\varphi = \frac{1}{4\pi\varepsilon_0}\frac{dq'}{r} = \frac{1}{4\pi\varepsilon_0}\frac{\rho_S(2\pi r'dr')}{\sqrt{r'^2+z^2}}$$

Electric Potential due to Continuous Distributions

Example 2

A uniformly charged disk of radius R and charge density ρ_S.

Calculate the electric potential at a distance z from the central axis in free space.

Solution

Consider a circular ring of radius r' and width dr' : $dq' = \rho_S dA' = \rho_S(2\pi r'dr')$

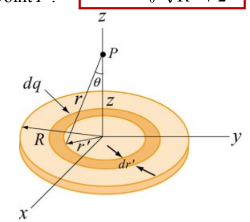
The distance from point P to the ring : $r = \sqrt{r'^2+z^2}$

The contribution of the ring to the electric potential at point P :

$$\boxed{\varphi = \frac{1}{4\pi\varepsilon_0}\frac{Q}{\sqrt{R^2+z^2}}}$$

$$d\varphi = \frac{1}{4\pi\varepsilon_0}\frac{dq'}{r} = \frac{1}{4\pi\varepsilon_0}\frac{\rho_S(2\pi r'dr')}{\sqrt{r'^2+z^2}}$$

$$\varphi = \frac{\rho_S}{4\pi\varepsilon_0}\int_0^R \frac{2\pi r'dr'}{\sqrt{r'^2+z^2}} = \frac{\rho_S}{2\varepsilon_0}\left[\sqrt{R^2+z^2}-|z|\right]$$



$$\varphi = \begin{cases} \dfrac{\rho_S}{2\varepsilon_0}\left[\sqrt{R^2+z^2}+z\right] & z \le 0 \\ \dfrac{\rho_S}{2\varepsilon_0}\left[\sqrt{R^2+z^2}-z\right] & z \ge 0 \end{cases}$$

Electric Potential due to Continuous Distributions

Example 2

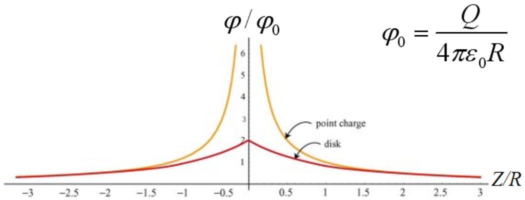
Solution Cont.

$$\varphi = \frac{\rho_S}{2\varepsilon_0}\left[\sqrt{R^2+z^2}-|z|\right]$$

If $|z| >> R$, the potential approaches its "point-charge" :

$$\varphi \approx \frac{\rho_S}{2\varepsilon_0}\frac{R^2}{2|z|} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{|z|}$$

Taylor series



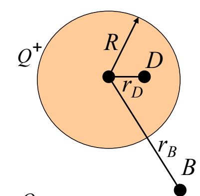
$$\varphi_0 = \frac{Q}{4\pi\varepsilon_0 R}$$

Electric Potential due to Continuous Distributions

Example 3

Potential for Uniformly Charged Non-Conducting Solid Sphere

From Gauss's Law : $\mathbf{E} = \begin{cases} \dfrac{Q}{4\pi\varepsilon_0 r^2}\mathbf{a}_r & r \ge R \\ \dfrac{Qr}{4\pi\varepsilon_0 R^3}\mathbf{a}_r & r \le R \end{cases}$



By using $\varphi_B - \varphi_A = -\int_A^B \mathbf{E} \bullet d\mathbf{l}$

Let point A is infinite, and set $\varphi_A = \varphi_\infty = 0$

Region 1 : $r > R$ $\quad \varphi_B - \varphi_A = \varphi_B = -\int_\infty^{r_B} \frac{Q}{4\pi\varepsilon_0 r^2}dr = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r_B}$

Similar in region 2 : $r < R$

$$\varphi_D - \varphi_A = \varphi_D = -\int_\infty^R \frac{Q}{4\pi\varepsilon_0 r^2}dr - \int_R^{r_D} \frac{Qr}{4\pi\varepsilon_0 R^3}dr = \frac{Q}{4\pi\varepsilon_0 R} - \frac{Q}{4\pi\varepsilon_0 R^3}\frac{1}{2}(r_D^2 - R^2)$$

$$= \frac{Q}{8\pi\varepsilon_0 R}\left(3 - \frac{r_D^2}{R^2}\right)$$

Electric Potential due to Continuous Distributions

Example 3

Potential for Uniformly Charged Non-Conducting Solid Sphere

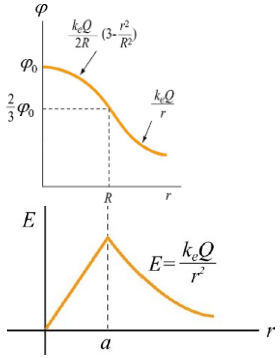
As points B and D are arbitrary, we can have:

$$\varphi = \begin{cases} \dfrac{Q}{8\pi\varepsilon_0 R}\left(3-\dfrac{r^2}{R^2}\right) = \dfrac{k_e Q}{2R}\left(3-\dfrac{r^2}{R^2}\right) & r \le R \\ \dfrac{1}{4\pi\varepsilon_0}\dfrac{Q}{r} = \dfrac{k_e Q}{r} & r \ge R \end{cases}$$

where $k_e = \dfrac{1}{4\pi\varepsilon_0}$

When $r = 0$, we have:

$$\varphi_0 = \frac{3k_e Q}{2R}$$



$a = R$

Electric Field as a Function of Electric Potential

The change of electric potential from point A to B:

$$\Delta\varphi = \varphi_B - \varphi_A = -\int_A^B \mathbf{E} \bullet d\mathbf{l}$$

Let $A = (x, y, z)$ and $B = (x + \Delta x, y, z)$

We have $\Delta\mathbf{l} = \Delta x \mathbf{a}_x$, then

$$\Delta\varphi = -\int_A^B \mathbf{E} \bullet d\mathbf{l} \cong -\mathbf{E} \bullet \Delta\mathbf{l} = -\mathbf{E} \bullet \Delta x \mathbf{a}_x = -E_x \Delta x$$

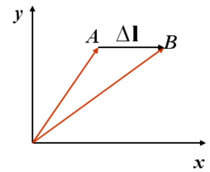
$$E_x \cong -\frac{\Delta\varphi}{\Delta x} \quad \rightarrow E_x = -\frac{\partial\varphi}{\partial x}$$

Similarly, we can get

$$E_y = -\frac{\partial\varphi}{\partial y} \text{ and } E_Z = -\frac{\partial\varphi}{\partial Z}$$

Then we have

$$\mathbf{E} = E_x\mathbf{a}_x + E_y\mathbf{a}_y + E_z\mathbf{a}_z = -\left(\frac{\partial\varphi}{\partial x}\mathbf{a}_x + \frac{\partial\varphi}{\partial y}\mathbf{a}_y + \frac{\partial\varphi}{\partial Z}\mathbf{a}_z\right)$$



With gradient operator:

$$\nabla \equiv \frac{\partial}{\partial x}\mathbf{a}_x + \frac{\partial}{\partial y}\mathbf{a}_y + \frac{\partial}{\partial Z}\mathbf{a}_z$$

$$\boxed{\mathbf{E} = -\nabla\varphi}$$

Electric Field as a Function of Electric Potential

$$\nabla = \mathbf{a}_x\frac{\partial}{\partial x} + \mathbf{a}_y\frac{\partial}{\partial y} + \mathbf{a}_z\frac{\partial}{\partial z} \quad \text{Cartesian}$$

$$\nabla = \mathbf{a}_r\frac{\partial}{\partial r} + \mathbf{a}_\phi\frac{1}{r}\frac{\partial}{\partial\phi} + \mathbf{a}_z\frac{\partial}{\partial z} \quad \text{Cylindrical}$$

$$\nabla = \mathbf{a}_R\frac{\partial}{\partial R} + \mathbf{a}_\theta\frac{1}{R}\frac{\partial}{\partial\theta} + \mathbf{a}_\phi\frac{1}{R\sin\theta}\frac{\partial}{\partial\phi} \quad \text{Spherical}$$

$$\boxed{\mathbf{E} = -\nabla\varphi}$$

$$\mathbf{E} = E_x\mathbf{a}_x + E_y\mathbf{a}_y + E_z\mathbf{a}_z = -\left(\frac{\partial\varphi}{\partial x}\mathbf{a}_x + \frac{\partial\varphi}{\partial y}\mathbf{a}_y + \frac{\partial\varphi}{\partial Z}\mathbf{a}_z\right) \quad \text{Cartesian}$$

$$\mathbf{E} = E_r\mathbf{a}_r + E_\varphi\mathbf{a}_\varphi + E_z\mathbf{a}_z = -\left(\frac{\partial\varphi}{\partial r}\mathbf{a}_r + \frac{1}{r}\frac{\partial\varphi}{\partial\phi}\mathbf{a}_\phi + \frac{\partial\varphi}{\partial Z}\mathbf{a}_z\right) \quad \text{Cylindrical}$$

$$\mathbf{E} = E_R\mathbf{a}_R + E_\theta\mathbf{a}_\theta + E_\phi\mathbf{a}_\phi = -\left(\frac{\partial\varphi}{\partial R}\mathbf{a}_R + \frac{1}{R}\frac{\partial\varphi}{\partial\theta}\mathbf{a}_\theta + \frac{1}{R\sin\theta}\frac{\partial\varphi}{\partial\phi}\mathbf{a}_\phi\right) \quad \text{Spherical}$$

Electric Field Streamlines and Equipotential Contours

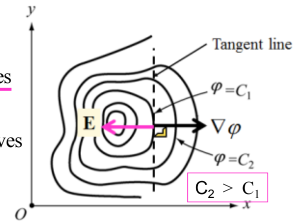
Electric potential $\varphi \rightarrow$ Scalar

Equipotential contours:

Start from two dimensions: Equipotential curves

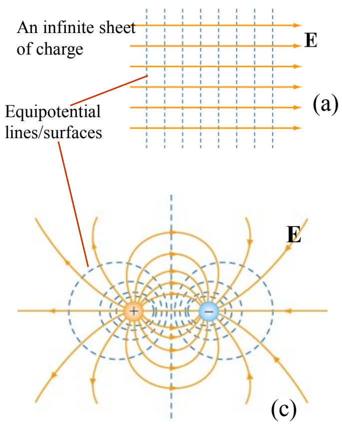
$\downarrow$

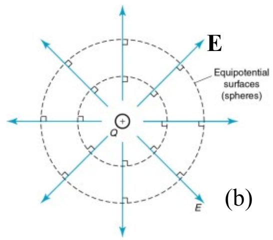
Electric potential: Constants on the curves



- Electric potential φ can change in any directions
- The maximum change rate is achieved in the perpendicular direction
- Change rate in the perpendicular direction: Gradient
- $\mathbf{E} = -\nabla\varphi \quad \rightarrow \quad \mathbf{E}$ is perpendicular to the equiotential, the direction is opposite to the direction in which the potential is increasing the most rapidly.

Electric Field Streamlines and Equipotential Contours





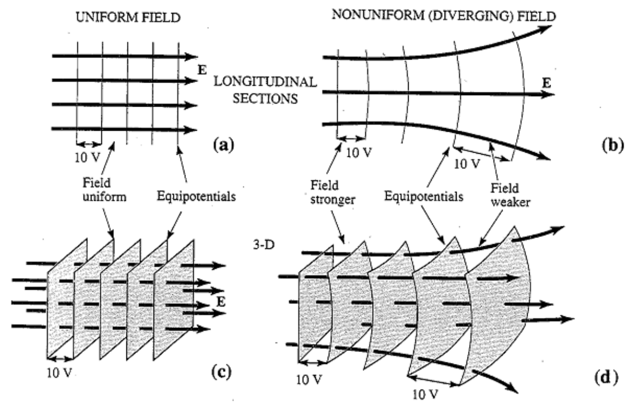
Equipotential curves and electric field streamlines for

(a) a constant electric field,

(b) a point charge, and

(c) an electric dipole.

Electric Field Streamlines and Equipotential Contours



Poisson's Equation

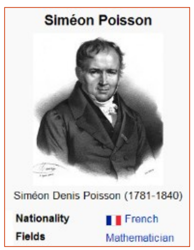
$$\begin{cases} \mathbf{E} = -\nabla\varphi \\ \nabla \bullet \mathbf{E} = \dfrac{\rho_v}{\varepsilon} \end{cases} \Rightarrow \nabla \bullet \mathbf{E} = \nabla \bullet (-\nabla\varphi) = -\nabla^2\varphi = \frac{\rho_v}{\varepsilon}$$

$\boxed{\text{Poisson's equation: } \quad \nabla^2\varphi = -\rho_v/\varepsilon}$ First published in 1813

$$\nabla^2\varphi = \nabla \bullet (\nabla\varphi) = \text{div}\,(\text{grad}\,\varphi)$$

In Cartesian coordinate system:

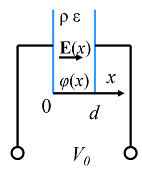
$$\nabla^2\varphi = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2}$$



Poisson's Equation

Example

A volume charge of a uniform density ρ exists in a homogeneous dielectric, of permittivity ε, between two flat metallic electrodes. The electrodes are connected to a voltage V_0, and the distance between them is d. Neglecting the fringing effects, find (a) the electric potential and (b) the electric field in the dielectric.



Solution

(a)

The potential in the dielectric varies with the distance from the electrodes only as neglecting the fringing effects (Which is equivalent to assuming that the electrodes are infinitely large.). Then Poisson's equation becomes:

$$\frac{d^2\varphi(x)}{dx^2} = -\frac{\rho}{\varepsilon} \qquad 0 < x < d$$

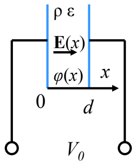
$$\text{By integrating}: \varphi(x) = -\frac{\rho x^2}{2\varepsilon} + C_1 x + C_2$$

Poisson's Equation

Example Cont.

Solution Cont.

$$\varphi(x) = -\frac{\rho x^2}{2\varepsilon} + C_1 x + C_2$$



Where C_1 and C_2 are the constants of integration.

The boundary condition: $\varphi(0) = V_0$ results in $C_2 = V_0$, and $\varphi(d) = 0$, gives

$$C_1 = \frac{\rho d}{2\varepsilon} - \frac{V_0}{d}$$

$$\text{Then}: \ \varphi(x) = \frac{\rho x(d-x)}{2\varepsilon} + V_0\left(1 - \frac{x}{d}\right)$$

(b)

$$\mathbf{E}(x) = -\nabla\varphi = -\frac{d\varphi(x)}{dx}\mathbf{a}_x = \left[\frac{\rho}{\varepsilon}\left(x - \frac{d}{2}\right) + \frac{V_0}{d}\right]\mathbf{a}_x$$

Laplace's Equation

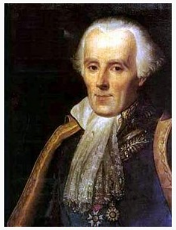
Poisson's equation $\nabla^2\varphi = -\dfrac{\rho_v}{\varepsilon}$

In charge-free space regions:

$$\nabla^2\varphi = 0$$

Laplace's equation

Pierre-Simon Laplace



1749–1827
French mathematician
and astronomer

Laplace's equation named after Pierre-Simon Laplace who first studied its properties.

Poisson's and Laplace's equations are useful for determining the electrostatic potential in regions with boundaries on which the potential is known.

Example

Given the potential field, $\varphi = 2x^2y - 5z$, and a point $P(-4, 3, 6)$, we wish to find several numerical values at point P: the potential φ, the electric field intensity $\mathbf{E}$, the direction of $\mathbf{E}$, the electric flux density $\mathbf{D}$, and the volume charge density ρ_v.

Solution

The potential at $P(-4, 3, 6)$ is

$$\varphi_P = 2(-4)^2(3) - 5(6) = 66 \text{ V}$$

electric field intensity $\mathbf{E} = -\nabla\varphi = -4xy\mathbf{a}_x - 2x^2\mathbf{a}_y + 5\mathbf{a}_z$ V/m

The value of $\mathbf{E}$ at point P $\mathbf{E}_P = 48\mathbf{a}_x - 32\mathbf{a}_y + 5\mathbf{a}_z$ V/m

The magnitude $|\mathbf{E}_P| = \sqrt{48^2 + (-32)^2 + 5^2} = 57.9$ V/m

The direction $\mathbf{a}_{E,P} = (48\mathbf{a}_x - 32\mathbf{a}_y + 5\mathbf{a}_z)/57.9$
$= 0.829\mathbf{a}_x - 0.553\mathbf{a}_y + 0.086\mathbf{a}_z$

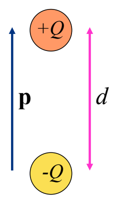
The electric flux density $\mathbf{D} = \epsilon_0\mathbf{E} = -35.4xy\,\mathbf{a}_x - 17.71x^2\,\mathbf{a}_y + 44.3\,\mathbf{a}_z$ pC/m^2

The volume cgarge density $\rho_v = \nabla\cdot\mathbf{D} = -35.4y$ pC/m^3

At P, $\rho_v = -106.2$ pC/m^3

Electric Dipole

A pair of charges of equal magnitude but opposite sign is called an ***electric dipole***.



If the charges are $+Q$ and $-Q$, separated by a distance d, the ***dipole moment vector*** $\mathbf{p}$:

$\boxed{\mathbf{p} = Qd\mathbf{a}_p}$ (charge × displacement)

The magnitude of the electric dipole is

$p = Qd \qquad Q > 0$

points from $-Q$ to $+Q$.

Electric Field Created by Electric Dipole

Determine the electric field off the ends of the dipole at $z = r$.

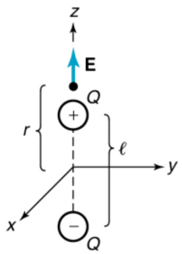
Solution

Produced by $+Q$

$$\mathbf{E} = \boxed{\frac{Q}{4\pi\varepsilon_0 (r-\frac{l}{2})^2}\mathbf{a}_z} - \boxed{\frac{Q}{4\pi\varepsilon_0 (r+\frac{l}{2})^2}\mathbf{a}_z}$$

$$= \frac{Q}{4\pi\varepsilon_0}\frac{lr}{(r^2-\frac{l^2}{4})^2}\mathbf{a}_z$$

Produced by $-Q$



When the distance is much larger than the charge separation, i.e. $r >> \frac{l}{2}$,

we have $\mathbf{E} = \frac{Ql}{4\pi\varepsilon_0 r^3}\mathbf{a}_z$ $\qquad$ $\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{R^2}\mathbf{a}_R$

Point-dipole $\qquad$ Point-charge

Electric Field Created by Electric Dipole

Determine the electric field at point P.

The horizontal (y-directed) components of the field will be canceled, only the z components (in the $-z$ direction) left.

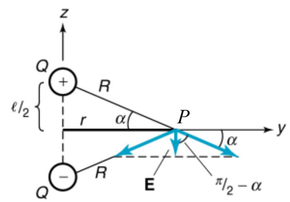
$$\mathbf{E} = -2\frac{1}{4\pi\varepsilon_0}\frac{Q}{R^2}\cos(90^0-\alpha)\mathbf{a}_z$$

where $R = \sqrt{r^2 + (\frac{l}{2})^2}$

$$\cos(90^0-\alpha) = \sin(\alpha) = \frac{l/2}{R}$$

Then

$$\mathbf{E} = -\frac{1}{4\pi\varepsilon_0}\frac{Ql}{(r^2+\frac{l^2}{4})^{3/2}}\mathbf{a}_z$$



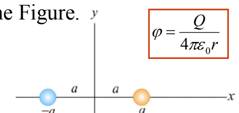
When the distance is much larger than the charge separation, i.e. $r >> \frac{l}{2}$

we have $\mathbf{E} = -\frac{Ql}{4\pi\varepsilon_0 r^3}\mathbf{a}_z$

Point-dipole

Electric Potential Created by Electric Dipole

Consider an electric dipole system shown in the Figure. Find the electric potential at an arbitrary point on the x axis in free space.



$$\boxed{\varphi = \frac{Q}{4\pi\varepsilon_0 r}}$$

Solution

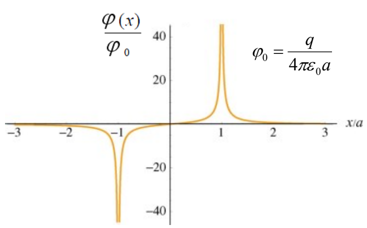
At a point on the x axis, we have

$$\varphi(x) = \frac{1}{4\pi\varepsilon_0}\frac{q}{|x-a|} + \frac{1}{4\pi\varepsilon_0}\frac{(-q)}{|x+a|} = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{|x-a|} - \frac{1}{|x+a|}\right]$$

The electric potential on y axis?

Rewrite the expression by the dimensionless electric potential as a function of x/a

$$\frac{\varphi(x)}{\varphi_0} = \frac{1}{|x/a-1|} - \frac{1}{|x/a+1|}$$



$$\varphi_0 = \frac{q}{4\pi\varepsilon_0 a}$$

$\varphi(x)$ diverges at $x/a = \pm 1$, where the charges are located.

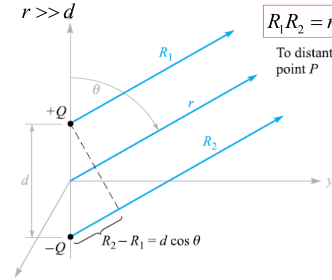
Electric Potential Created by Electric Dipole

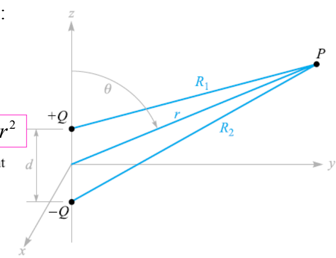
The complete expression for the potential at all points around the dipole can be shown in terms of spherical coordinates:

$$\varphi = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{R_1} - \frac{1}{R_2}\right) = \frac{Q}{4\pi\varepsilon_0}\frac{R_2 - R_1}{R_1 R_2}$$

$r >> d$

$$\boxed{R_1 R_2 = r^2}$$





$$\boxed{\varphi = \frac{Qd\cos\theta}{4\pi\varepsilon_0 r^2}}$$

Electric Field Created by Electric Dipole

$$\boxed{\varphi = \frac{Qd\cos\theta}{4\pi\varepsilon_0 r^2}}$$

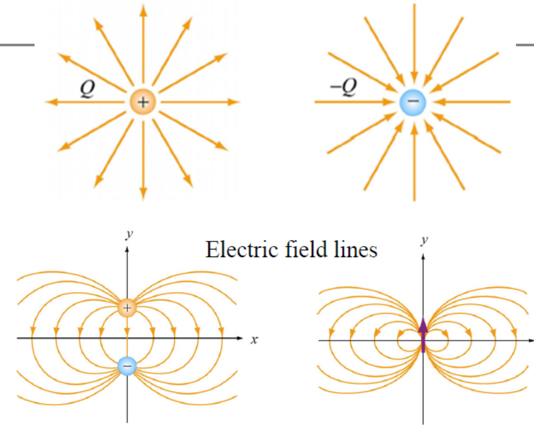
$$\mathbf{E} = -\left(\frac{\partial\varphi}{\partial R}\mathbf{a}_R + \frac{1}{R}\frac{\partial\varphi}{\partial\theta}\mathbf{a}_\theta + \frac{1}{R\sin\theta}\frac{\partial\varphi}{\partial\phi}\mathbf{a}_\phi\right)$$

$$= -\left(-\frac{Qd\cos\theta}{2\pi\varepsilon_0 R^3}\mathbf{a}_R - \frac{Qd\sin\theta}{4\pi\varepsilon_0 R^3}\mathbf{a}_\theta\right) = \frac{Qd\cos\theta}{4\pi\varepsilon_0 R^3}(2\cos\theta\mathbf{a}_R + \sin\theta\mathbf{a}_\theta)$$

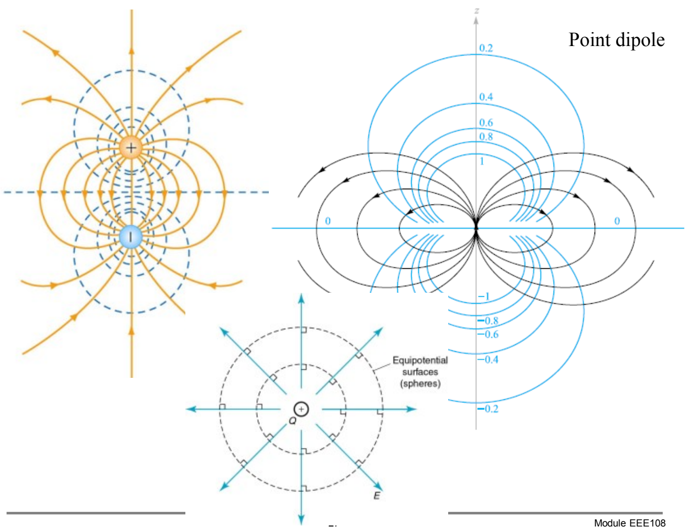
The complete expression for the field at all points around the dipole can be shown in terms of spherical coordinates:

$$\mathbf{E} = \frac{Qd}{4\pi\varepsilon_0 R^3}(2\cos(\theta)\mathbf{a}_R + \sin(\theta)\mathbf{a}_\theta)$$

The electric field depends on the product of the charge and the separation distance, which is dipole moment, p.



(a) a finite dipole (b) a point dipole.



Today Summary

- Electric Potential due to Continuous Distributions
- Poisson's Equation $\nabla^2\varphi = -\rho/\varepsilon$
- Laplace Equation $\nabla^2\varphi = 0$
- Electric Dipole $\mathbf{p} = Qd\mathbf{a}_p$

Summary of Electric Potential (1)

❑The potential at a point is the work done in bringing a unit positive charge from the zero reference (infinity) to the point.

$$\varphi = -\int_{\infty}^{P} \mathbf{E} \bullet d\mathbf{l} \quad \text{(V)}$$

❑Electrical potential difference: Voltage

$\boxed{\text{SI units: volts = J/C}}$

❑Electric Potential for a Point Charge: $\varphi = \dfrac{Q}{4\pi\varepsilon r}$

❑Superposition of Potential: $\varphi_P = \dfrac{1}{4\pi\varepsilon}\displaystyle\sum_{n=1}^{N}\dfrac{Q_n}{r_n}$

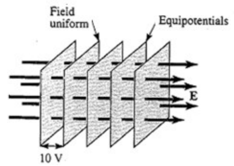
❑Electric Potential for Continuous Charge Distributions:

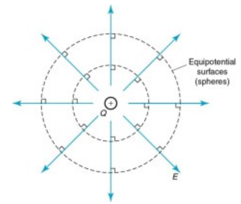
Line charge $\varphi_L = \dfrac{1}{4\pi\varepsilon}\displaystyle\int_L \dfrac{\rho_L}{r}\,dl$ Surface charge $\varphi_S = \dfrac{1}{4\pi\varepsilon}\displaystyle\iint_S \dfrac{\rho_S}{r}\,dS$

Volume charge $\varphi_V = \dfrac{1}{4\pi\varepsilon}\displaystyle\iiint_V \dfrac{\rho_V}{r}\,dV$ (V)

Summary of Electric Potential (2)

Electric field vs electric potential: $\varphi = -\int_{\infty}^{P} \mathbf{E} \bullet d\mathbf{l} \quad \Leftrightarrow \quad \mathbf{E} = -\nabla\varphi$





Potential and potential energy :

The electric energy is : $\Delta U = q_0\,\Delta\varphi$

Next

- Conductors and Dielectrics
- Capacitors and Capacitance (1)

Thanks for your attendance