Xi'an Jiaotong-Liverpool University
西交利物浦大学

EEE220 Instrumentation and Control System

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Lecture 12

Outline

State Variable Models: part 2/2

- ❑ Introduction
- ❑ State Variables
- ❑ State-space Modeling
- ❑ State Space Representation in Matrix Form
- ❑ Time-domain response (Solution of State-space Models)
- ❑ **Conversion between State-space Model and Transfer Function**
- ❑ **Analysis of the State-space Models using Matlab**

Covert State-space Model to Transfer Function

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$$

since $$\mathbf{X}(s) = [s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{x}(0) + [s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B}\mathbf{U}(s)$$

$$\mathbf{Y}(s) = \mathbf{C}[s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{x}(0) + \mathbf{C}[s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B}\mathbf{U}(s) + \mathbf{D}\mathbf{U}(\mathrm{s})$$

Remember definition of transfer function requires that the initial conditions be set to zero, $\mathbf{x}(0) = 0$, thus:

$$\mathbf{Y}(s) = (\mathbf{C}[s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B} + \mathbf{D})\mathbf{U}(\mathrm{s})$$

Then transfer function between y(t) and u(t) is:

$$\mathbf{G}(s) = \mathbf{C}[s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B} + \mathbf{D}$$

$$\mathbf{G}(s) = \mathbf{C}[s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B} + \mathbf{D}$$

In general, if a linear system has q inputs and p outputs, then:

$$\mathbf{Y}(s) = \mathbf{G}(s)\mathbf{U}(s)$$

$$\mathbf{G}(s) = \begin{bmatrix} G_{11} & \dots & G_{1q} \\ G_{12} & \dots & G_{2q} \\ \vdots & & \vdots \\ G_{p1} & \dots & G_{pq} \end{bmatrix}$$

The transfer function between jth input and ith output is:

$$G_{ij}(s) = \frac{Y_i(s)}{U_j(s)}$$

Characteristic Equation from State Equations

$$\mathbf{G}(s) = \mathbf{C}[s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B} + \mathbf{D}$$

Note: for a 2 × 2 matrix M=$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, assume its inverse matrix is M^{-1}, (i.e., $\mathrm{M}^{-1}\mathrm{M} = \mathrm{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$)

Its adjugate is adj(M)=$\begin{bmatrix} \mathrm{d} & -b \\ -c & \mathrm{a} \end{bmatrix}$, its determinant is det(M)=$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \mathrm{ad} - \mathrm{bc}$.

Then $\mathrm{M}^{-1} = \frac{adj(\mathrm{M})}{\det(\mathrm{M})}$.

$$\mathbf{G}(s) = \mathbf{C}\frac{adj(s\mathbf{I} - \mathbf{A})}{det(s\mathbf{I} - \mathbf{A})}\mathbf{B} + \mathbf{D} = \frac{\mathbf{C}[adj(s\mathbf{I} - \mathbf{A})]\mathbf{B} + |s\mathbf{I} - \mathbf{A}|\mathbf{D}}{|s\mathbf{I} - \mathbf{A}|}$$

Setting the denominator of the transfer function matrix **G**(s) to be zero, we get the **characteristic equation:**

$$|s\mathbf{I} - \mathbf{A}| = \mathbf{0}$$

- Hence, it is clear that stability is decided by the pole location in the complex plane.
- Performance is also decided by the pole location.
- Specifically, we call $\boldsymbol{A}$ system matrix, $\mathbf{B}$ input matrix, $\mathbf{C}$ output matrix, $\mathbf{D}$ feedthrough matrix

Example 12.1

Obtain Transfer function for the system:

$$\frac{dx_1}{dt} = -2x_2(t) + 3u(t) \Rightarrow \dot{x_1} = -2x_2 + 3u$$
$$\frac{dx_2}{dt} = 3x_1(t) - 5x_2(t) \Rightarrow \dot{x_2} = 3x_1 - 5x_2$$
$$y(t) = v_o(t) = 2x_2(t) \Rightarrow y = 2x_2$$

$$A = \begin{bmatrix} 0 & -2 \\ 3 & -5 \end{bmatrix}, \qquad B = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$
$$C = \begin{bmatrix} 0 & 2 \end{bmatrix}, \qquad D = [0]$$

$$\mathbf{G}(s) = \frac{\mathbf{C}[adj(s\mathbf{I} - \mathbf{A})]\mathbf{B} + |s\mathbf{I} - \mathbf{A}|\mathbf{D}}{|s\mathbf{I} - \mathbf{A}|}$$

$$\frac{\mathbf{Y}(s)}{\mathbf{U}(s)} = \frac{\begin{bmatrix} 0 & 2 \end{bmatrix}\left(\mathrm{adj}\begin{bmatrix} s & 2 \\ -3 & s+5 \end{bmatrix}\right)\begin{bmatrix} 3 \\ 0 \end{bmatrix}}{\det\begin{vmatrix} s & 2 \\ -3 & s+5 \end{vmatrix}} = \frac{\begin{bmatrix} 0 & 2 \end{bmatrix}\begin{bmatrix} s+5 & -2 \\ 3 & s \end{bmatrix}\begin{bmatrix} 3 \\ 0 \end{bmatrix}}{s^2 + 5s + 6} = \frac{18}{s^2 + 5s + 6}$$

Covert Transfer Function to State-space Model

How to obtain the state space model from the transfer function without a clear knowledge of the physical system?

Method 1: to develop graphic model of the system and use this model to determine state variables.

$$\boxed{G(s) = \frac{Y(s)}{U(s)} = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}} \qquad n \geq m$$

$$G(s) = \frac{b_m s^{-(n-m)} + b_{m-1}s^{-(n-m+1)} + \cdots + b_1 s^{-(n-1)} + b_0 s^{-n}}{1 + a_{n-1}s^{-1} + \cdots + a_1 s^{-(n-1)} + a_0 s^{-n}}.$$

Recall **Mason's Signal-flow Gain Formula**:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{\sum_k P_k \Delta_k}{\Delta}.$$

When all the feedback loops are touching and all the forward paths touch the feedback loops, then:

$$G(s) = \frac{\sum_k P_k}{1 - \sum_{q=1}^{N} L_q} = \frac{\text{Sum of the forward-path factors}}{1 - \text{sum of the feedback loop factors}}.$$

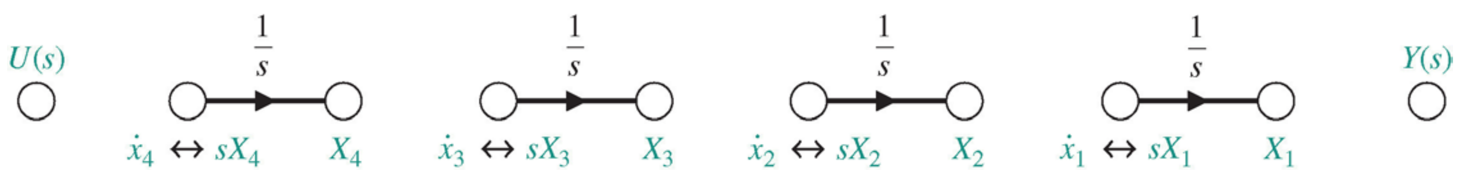
Simple Case

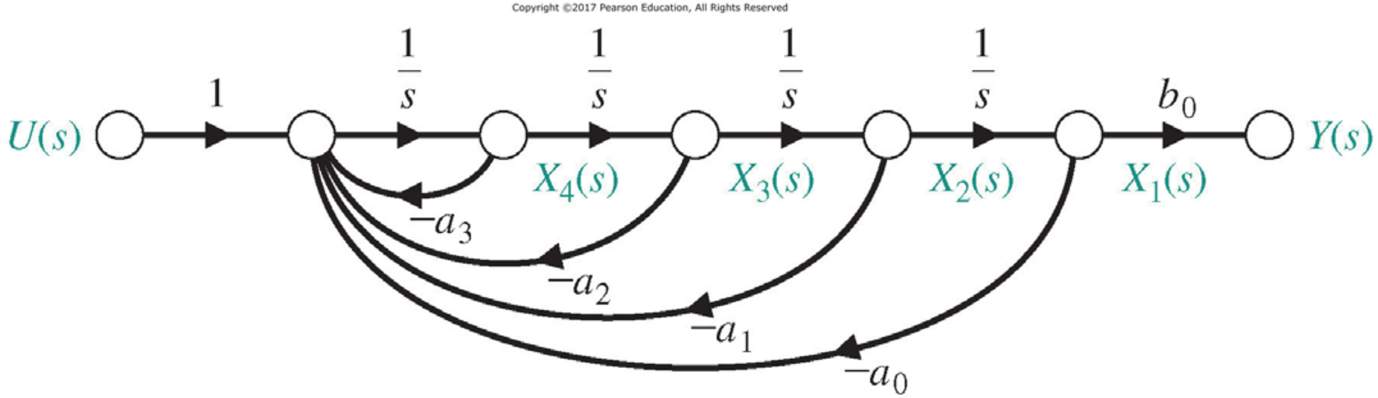
To illustrate the derivation of signal-flow graph from transfer function, let's consider a simple case, when $n = 4$, and $b_m \ldots b_2, b_1 =, 0$:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_0}{s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0}$$

$$= \frac{b_0 s^{-4}}{1 + a_3 s^{-1} + a_2 s^{-2} + a_1 s^{-3} + a_0 s^{-4}}.$$

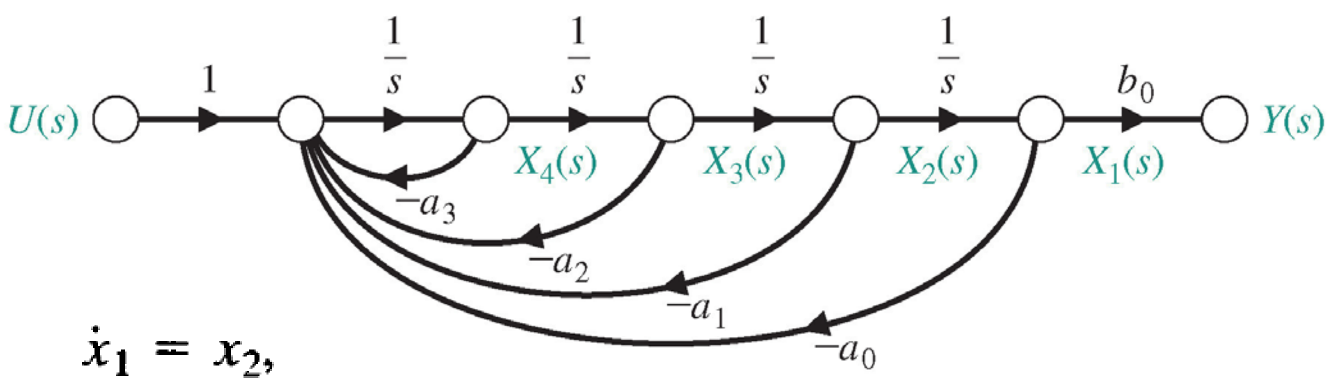
The system is fourth order, hence we need to identify four state variables: $x_1(t), x_2(t), x_3(t), x_4(t)$





$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_0}{s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0}$$

$$= \frac{b_0 s^{-4}}{1 + a_3 s^{-1} + a_2 s^{-2} + a_1 s^{-3} + a_0 s^{-4}}.$$



$$\dot{x}_1 = x_2,$$

$$\dot{x}_2 = x_3,$$

$$\dot{x}_3 = x_4,$$

$$\dot{x}_4 = -a_0 x_1 - a_1 x_2 - a_2 x_3 - a_3 x_4 + u;$$

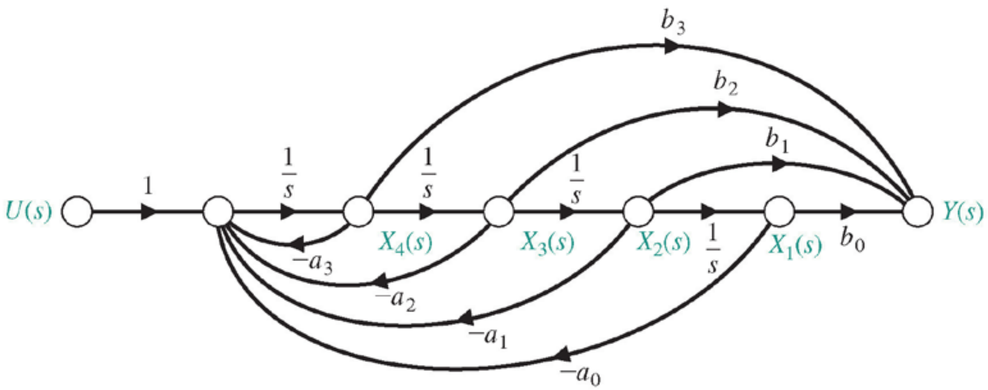
$$y = b_0 x_1.$$

Now consider the numerator is a polynomial in s:

$$G(s) = \frac{\sum_k P_k}{1 - \sum_{q=1}^{N} L_q}$$

$$G(s) = \frac{b_3 s^3 + b_2 s^2 + b_1 s + b_0}{s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0}$$

$$= \frac{b_3 s^{-1} + b_2 s^{-2} + b_1 s^{-3} + b_0 s^{-4}}{1 + a_3 s^{-1} + a_2 s^{-2} + a_1 s^{-3} + a_0 s^{-4}}.$$



$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = x_3, \qquad \dot{x}_3 = x_4,$$

$$\dot{x}_4 = -a_0 x_1 - a_1 x_2 - a_2 x_3 - a_3 x_4 + u.$$

In this equation, $x_1, x_2, \ldots x_n$ are the n **phase variables**.

y(t)?

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = x_3, \qquad \dot{x}_3 = x_4,$$

$$\dot{x}_4 = -a_0 x_1 - a_1 x_2 - a_2 x_3 - a_3 x_4 + u.$$

$$y = b_0 x_1 + b_1 x_2 + b_2 x_3 + b_3 x_4$$

A, B, C, D?

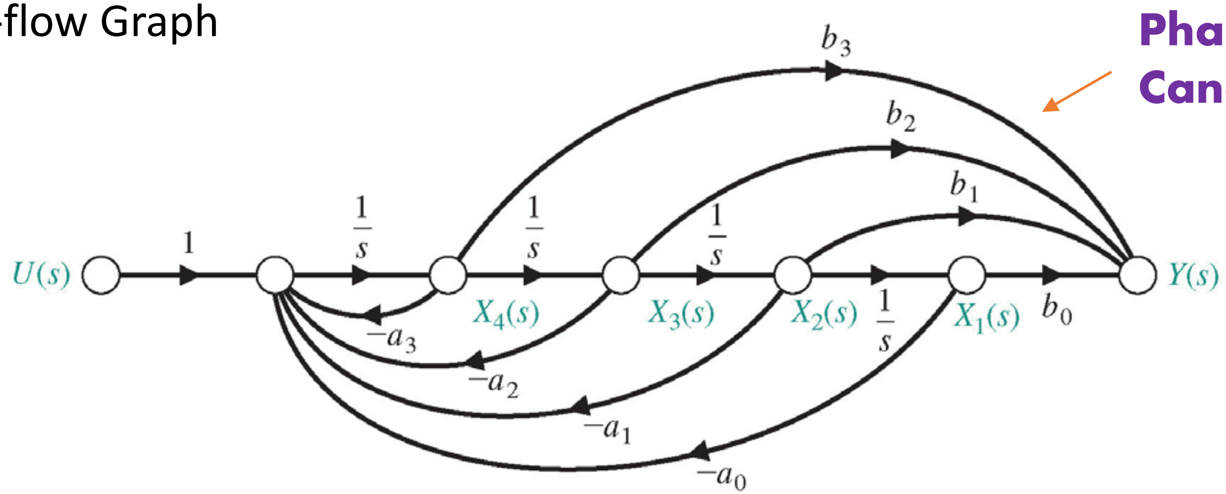
$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u,$$

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u(t). \qquad y(t) = \mathbf{C}\mathbf{x} = \begin{bmatrix} b_0 & b_1 & b_2 & b_3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

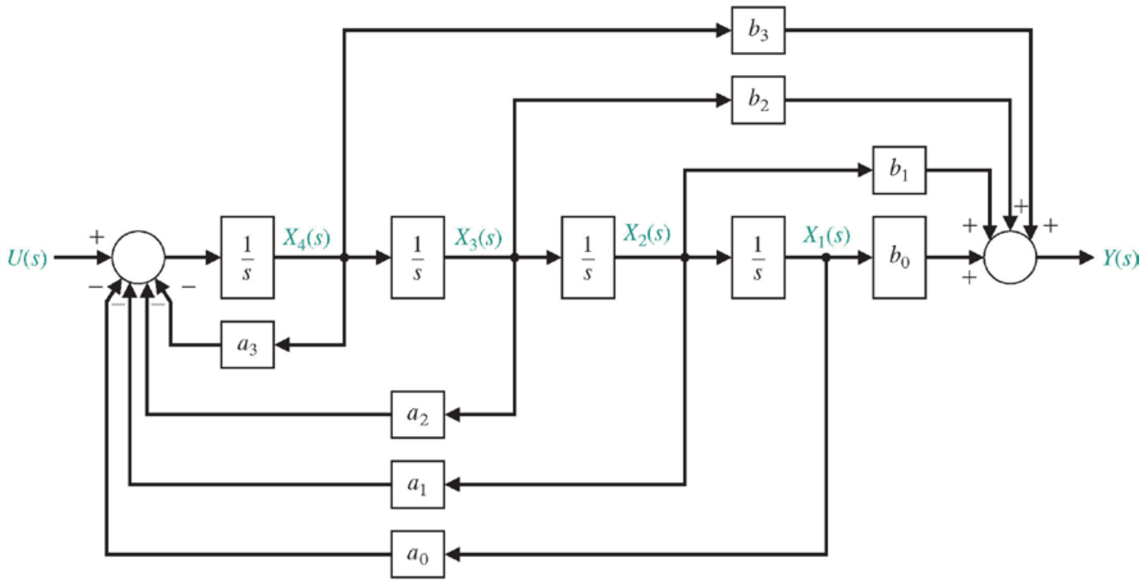
General form of State-space Model

- Signal-flow Graph

Phase Variable Canonical Form

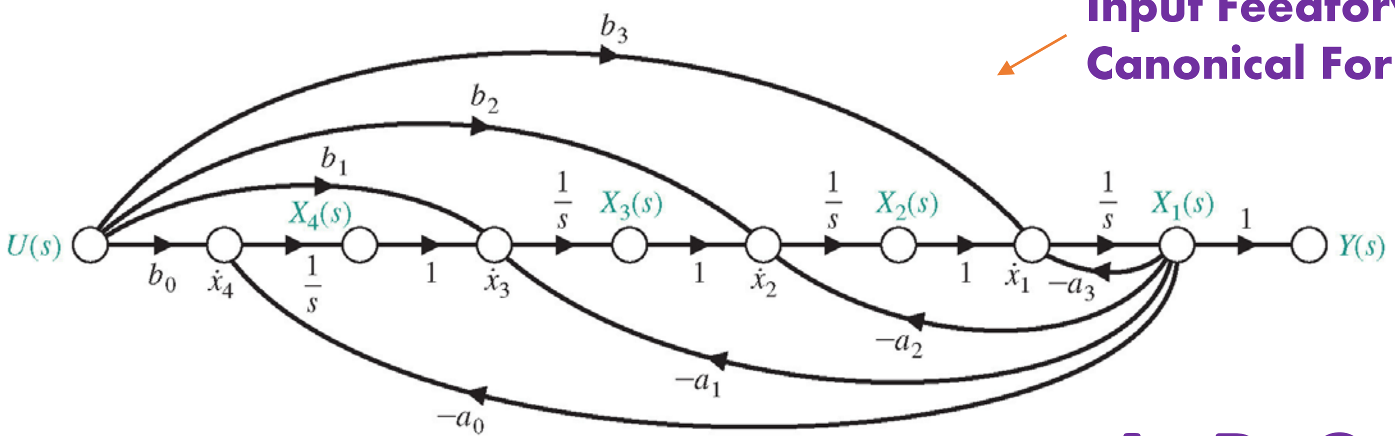


- Block Diagram



Other Forms

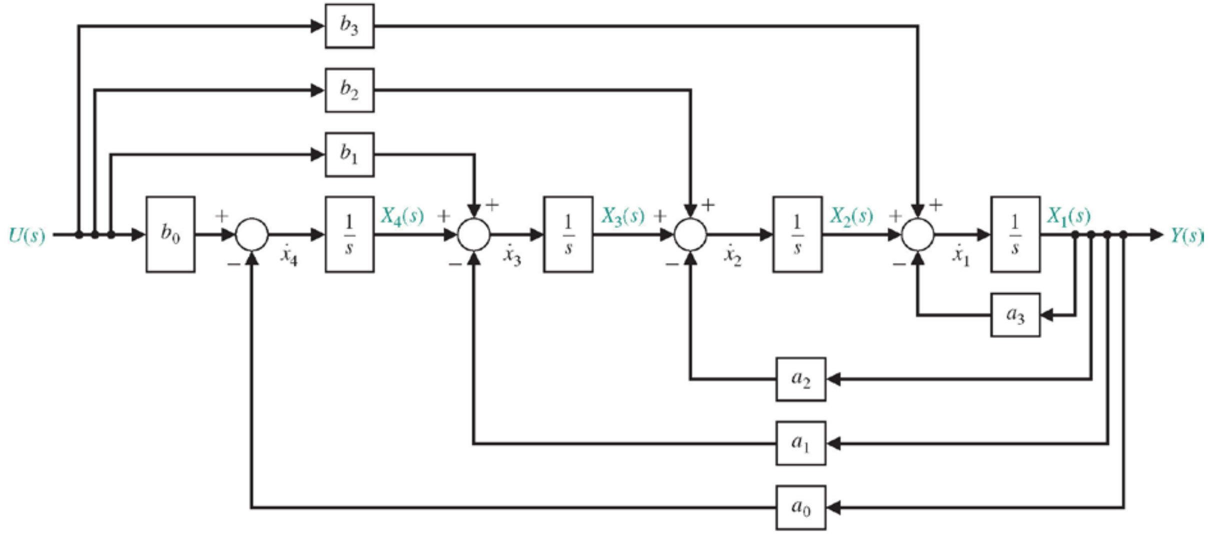
- Signal-flow Graph



Input Feedforward Canonical Form

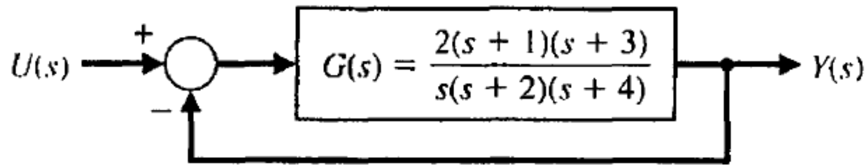
- Block Diagram

A, B, C, D?



Example 12.2

$$T(s) = \frac{Y(s)}{U(s)} = \frac{2s^2 + 8s + 6}{s^3 + 8s^2 + 16s + 6}.$$



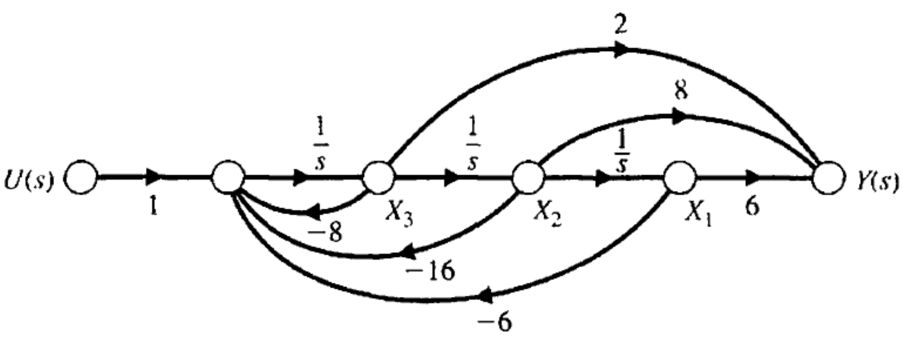
Applying the Phase variable state model:

Multiplying the numerator and denominator by s^{-3}, we have

$$T(s) = \frac{Y(s)}{U(s)} = \frac{2s^{-1} + 8s^{-2} + 6s^{-3}}{1 + 8s^{-1} + 16s^{-2} + 6s^{-3}}.$$

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -16 & -8 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = [6 \quad 8 \quad 2] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$



Method 2: State-space Model can be also obtained by introducing an intermediate variable $Z(s)$.

For simplicity, assume $n = 4$:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_3 s^3 + b_2 s^2 + b_1 s + b_0}{s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0} \frac{Z(s)}{Z(s)}.$$

$$U(s) = [s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0] Z(s).$$

Then taking inverse Laplace transform of both equations:

$$y = b_3 \frac{d^3 z}{dt^3} + b_2 \frac{d^2 z}{dt^2} + b_1 \frac{dz}{dt} + b_0 z$$

$$u = \frac{d^4 z}{dt^4} + a_3 \frac{d^3 z}{dt^3} + a_2 \frac{d^2 z}{dt^2} + a_1 \frac{dz}{dt} + a_0 z.$$

$$y = b_3\frac{d^3z}{dt^3} + b_2\frac{d^2z}{dt^2} + b_1\frac{dz}{dt} + b_0 z \qquad u = \frac{d^4z}{dt^4} + a_3\frac{d^3z}{dt^3} + a_2\frac{d^2z}{dt^2} + a_1\frac{dz}{dt} + a_0 z.$$

Define the four state variables as follows:

$$\begin{aligned} x_1 &= z \\ x_2 &= \dot{x}_1 = \dot{z} \\ x_3 &= \dot{x}_2 = \ddot{z} \\ x_4 &= \dot{x}_3 = \dddot{z}. \end{aligned}$$

Then the differential equation can be written equivalently as

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= x_3, \\ \dot{x}_3 &= x_4, \end{aligned}$$

and

$$\dot{x}_4 = -a_0x_1 - a_1x_2 - a_2x_3 - a_3x_4 + u,$$

and the corresponding output equation is

$$y = b_0x_1 + b_1x_2 + b_2x_3 + b_3x_4.$$

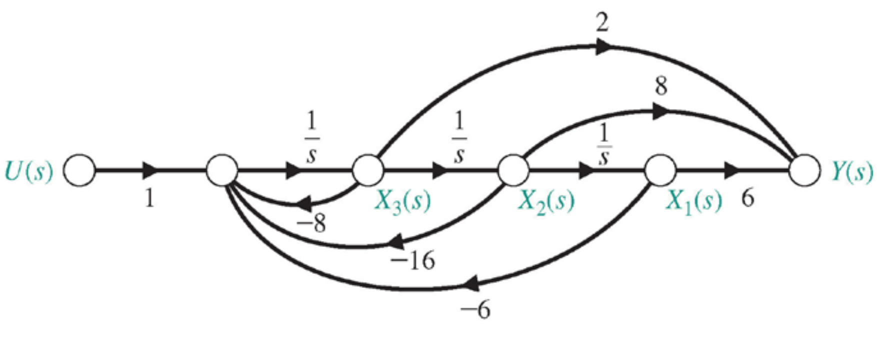
Example 12.3

Consider a closed-loop transfer function

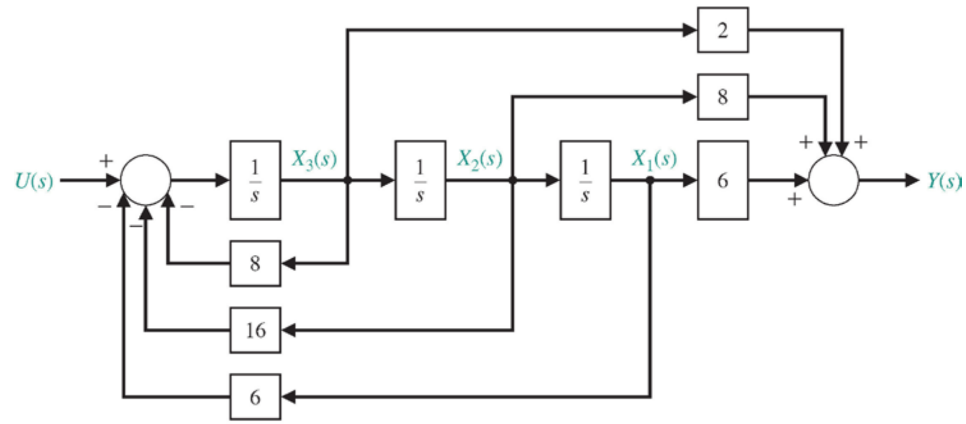
$$T(s) = \frac{Y(s)}{U(s)} = \frac{2s^2 + 8s + 6}{s^3 + 8s^2 + 16s + 6}$$

Multiplying the numerator and denominator by s^{-3}, we have

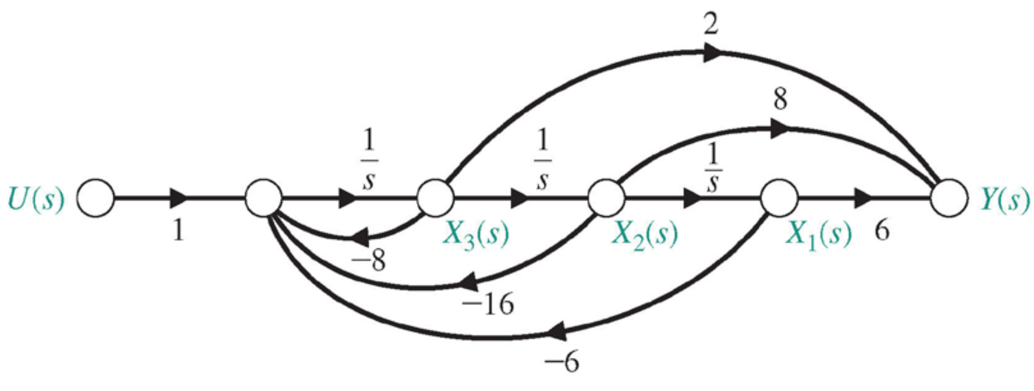
$$T(s) = \frac{Y(s)}{U(s)} = \frac{2s^{-1} + 8s^{-2} + 6s^{-3}}{1 + 8s^{-1} + 16s^{-2} + 6s^{-3}}$$



(a)



(b)



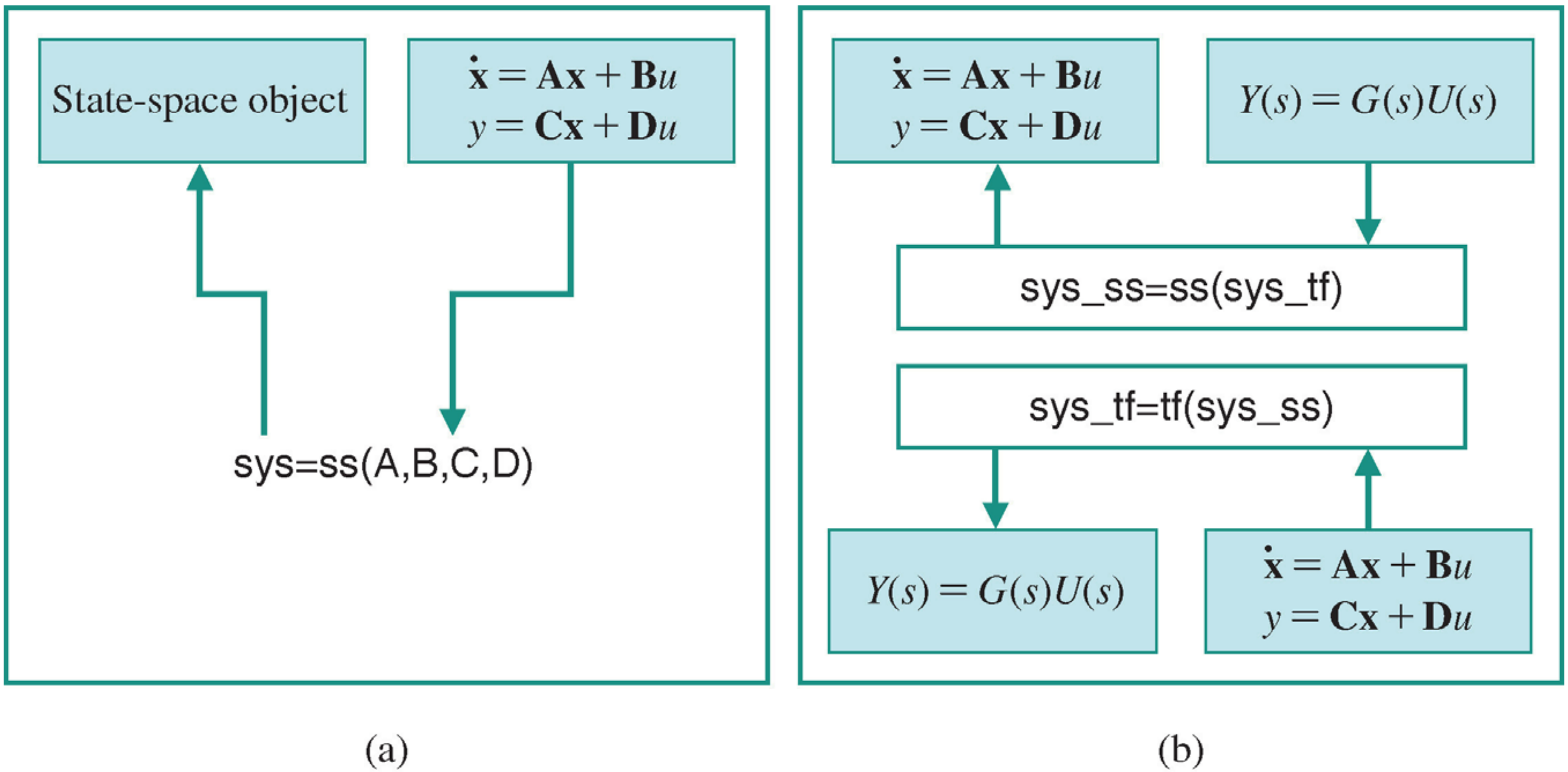
(a)

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -16 & -8 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 6 & 8 & 2 \end{bmatrix} \mathbf{x}(t) + [0]u(t)$$

Simulation by Matlab

- Covert between state space model and transfer function (ss, tf)



```
>> num = [2 8 6]; den = [1 8 16 6]; sys_tf = tf(num, den)

sys_tf =

     2 s^2 + 8 s + 6
  ----------------------
  s^3 + 8 s^2 + 16 s + 6

Continuous-time transfer function.

>>
```

Please note: a transfer function can be converted to various state space models by choosing different sets of state variables; therefore, it is possible that when using the ss function, the state space model generated will be different, depending on the specific software and version.

```
>> sys_ss = ss(sys_tf)

sys_ss =

  A =
         x1     x2     x3
   x1    -8     -4   -1.5
   x2     4      0      0
   x3     0      1      0

  B =
        u1
   x1    2
   x2    0
   x3    0

  C =
         x1     x2     x3
   y1     1      1   0.75

  D =
        u1
   y1    0

Continuous-time state-space model.
```

- Compute the state transition matrix by given A and t (expm)

$$\mathbf{\Phi}(t)=\exp(\mathbf{A}t)$$

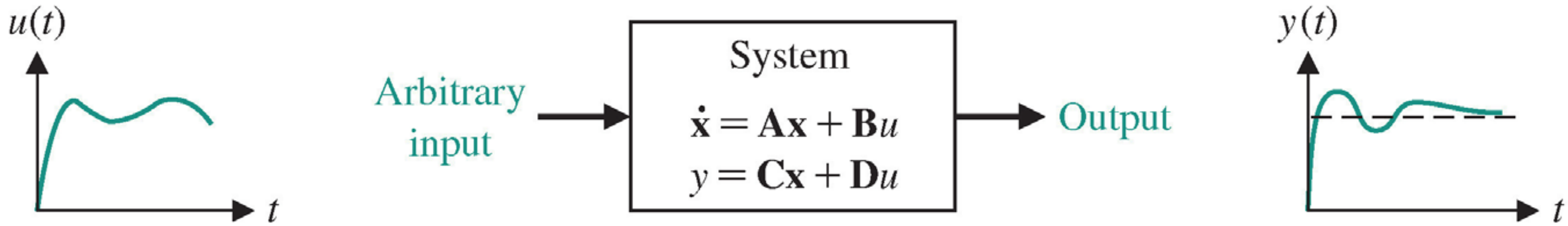
```
>>A=[0 -2; 1 -3]; dt=0.2; Phi=expm(A*dt)

Phi =

   0.9671   -0.2968
   0.1484    0.5219
```

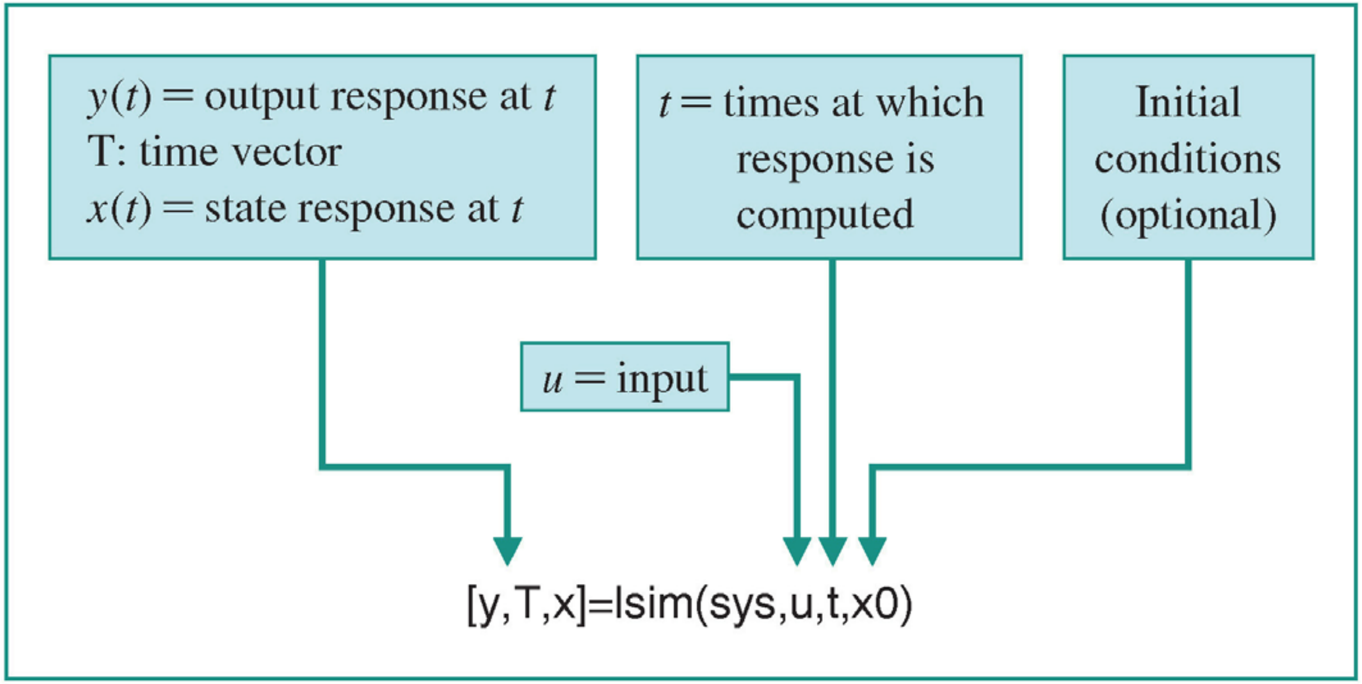
State transition matrix for a Δt of 0.2 second

If the initial conditions are $x_1(0) = x_2(0) = 1$ and the input $u(0) = 0$, the system state at $t = 0.2$ is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_{t=0.2} = \begin{bmatrix} 0.9671 & -0.2968 \\ 0.1484 & 0.5219 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_{t=0} = \begin{pmatrix} 0.6703 \\ 0.6703 \end{pmatrix}$$

- Calculate the output and state response (lsim)



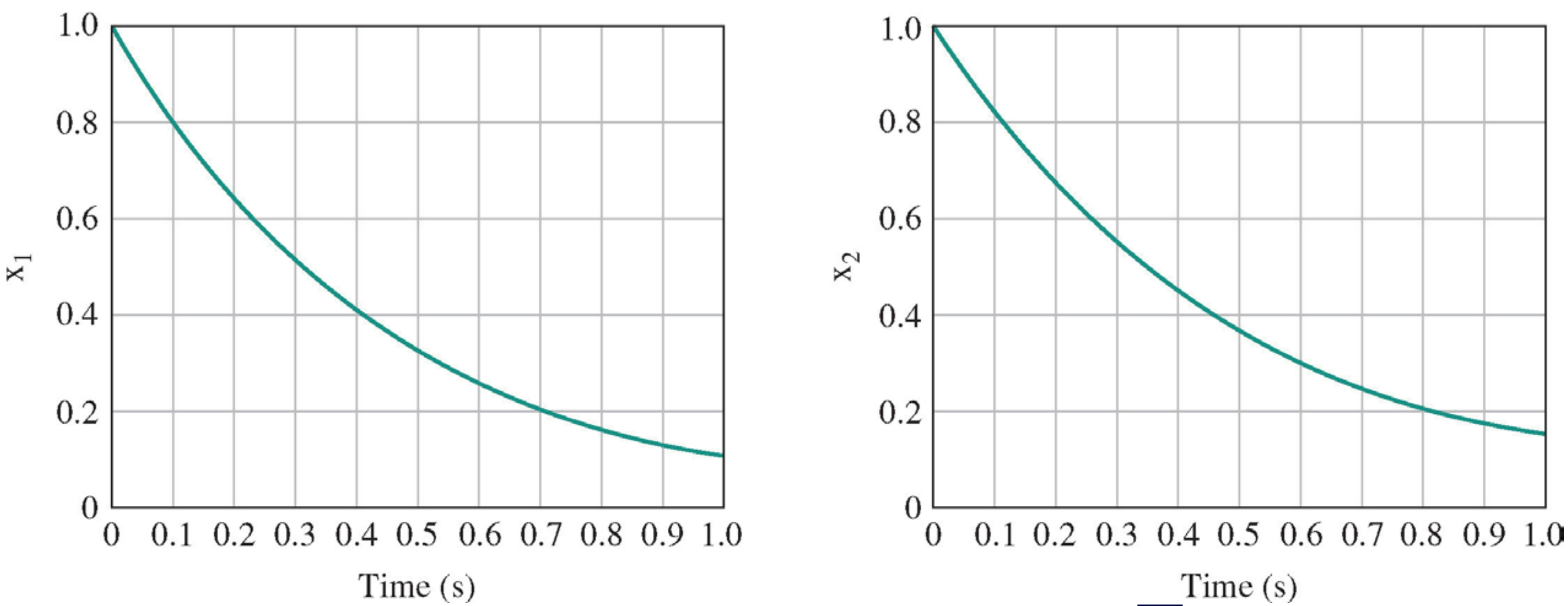


```
A=[0 -2;1 -3]; B=[2;0]; C=[1 0]; D=[0];
sys=ss(A,B,C,D);
x0=[1 1];
t=[0:0.01:1];
u=0*t;
[y,T,x]=lsim(sys,u,t,x0);
subplot(121), plot(T,x(:,1))
xlabel('Time (s)'), ylabel('x_1')
subplot(122), plot(T,x(:,2))
xlabel('Time (s)'), ylabel('x_2')
```

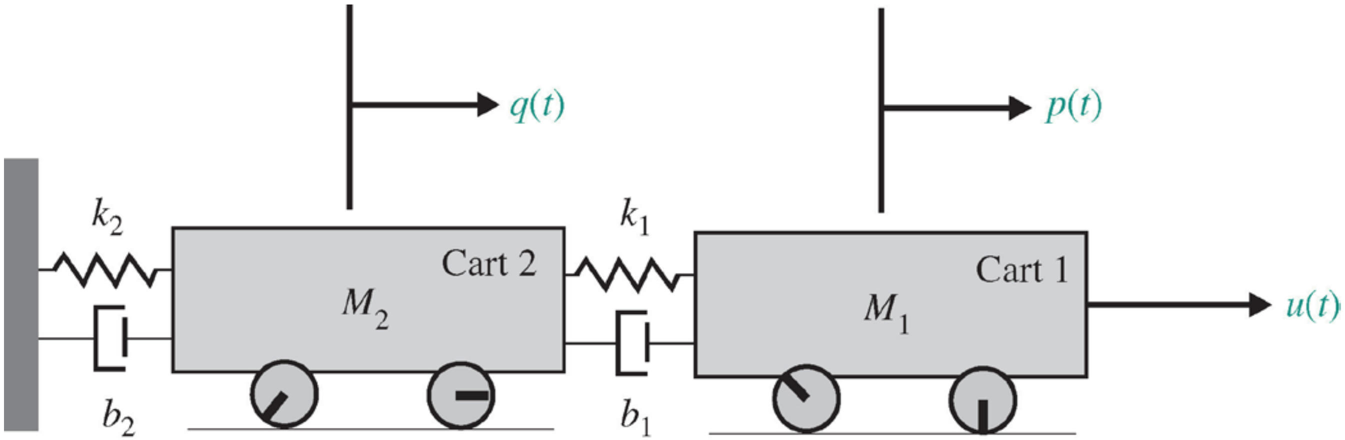
State-space model

Initial conditions

Zero input



Practical Example: Simulate the Two Rolling Carts System



$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k_1}{M_1} & \frac{k_1}{M_1} & -\frac{b_1}{M_1} & \frac{b_1}{M_1} \\ \frac{k_1}{M_2} & -\frac{k_1 + k_2}{M_2} & \frac{b_1}{M_2} & -\frac{b_1 + b_2}{M_2} \end{bmatrix}, \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{M_1} \\ 0 \end{bmatrix},$$

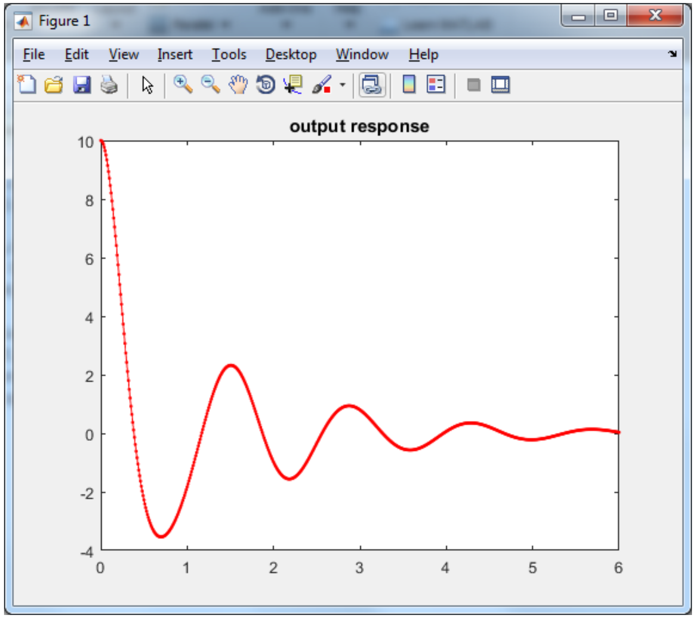
$$y = [1 \quad 0 \quad 0 \quad 0]\mathbf{x} = \mathbf{C}\mathbf{x}. \qquad \mathbf{D} = [0]$$

Suppose that the two rolling carts have the following parameter values: $k_1 = 150$ N/m; $k_2 = 700$ N/m; $b_1 = 15$ N s/m; $b_2 = 30$ N s/m; $M_1 = 5$ kg; and $M_2 = 20$ kg. The

Initial conditions: p(0)=10, q(0)=0, $\dot{p}(0) = 0, \dot{q}(0) = 0.$

- Obtain unforced response of system.

```
>> A = [0 0 1 0; 0 0 0 1; -30 30 -3 3; 7.5 -42.5 0.75 -2.25];
>> B = [0; 0; 0.2; 0];
>> C = [1 0 0 0];
>> D = [0];
>> sys = ss(A,B,C,D);
>> x0 = [10 0 0 0];
>> t = [0:0.01:6];
>> u = 0*t;
>> [y, T, x] = lsim(sys, u, t, x0);
>> figure,plot(T, y, 'r.-'); title('output response');
>> figure,subplot(211);plot(T, x(:,1), 'b.-'); title('x1 response');
>> subplot(212);plot(T, x(:,2), 'b.-'); title('x2 response');
>> figure,subplot(211);plot(T, x(:,3), 'b.-'); title('x3 response');
subplot(212);plot(T, x(:,4), 'b.-'); title('x4 response');
>>
```



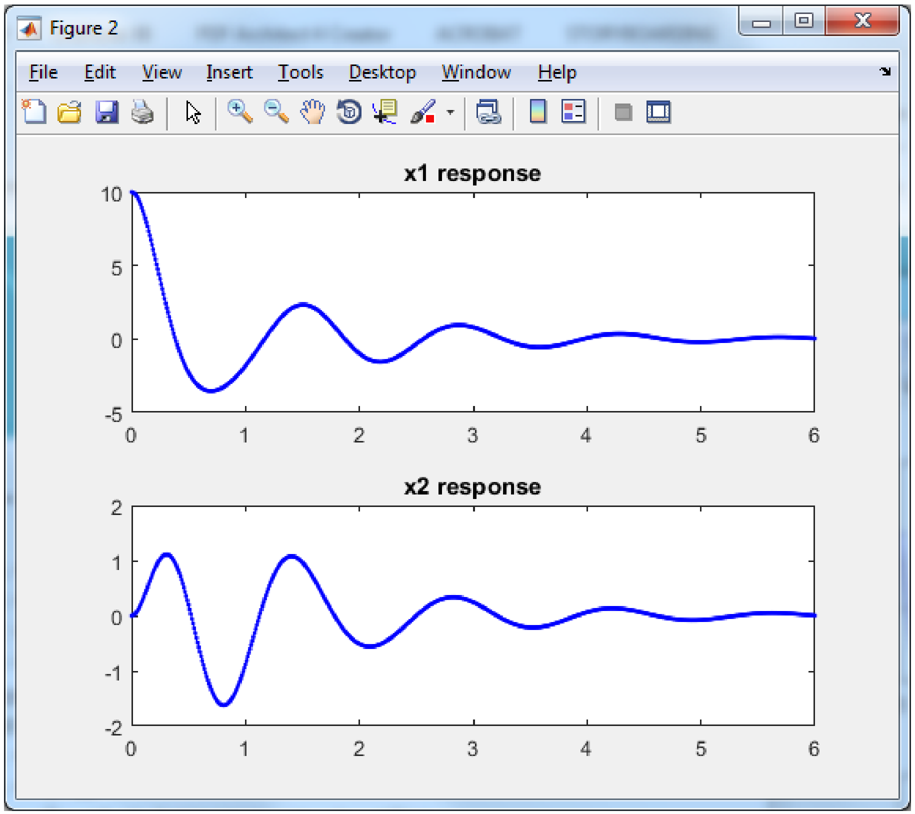


Figure 2
File Edit View Insert Tools Desktop Window Help
x1 response
10
5
0
-5
0 1 2 3 4 5 6
x2 response
2
1
0
-1
-2
0 1 2 3 4 5 6

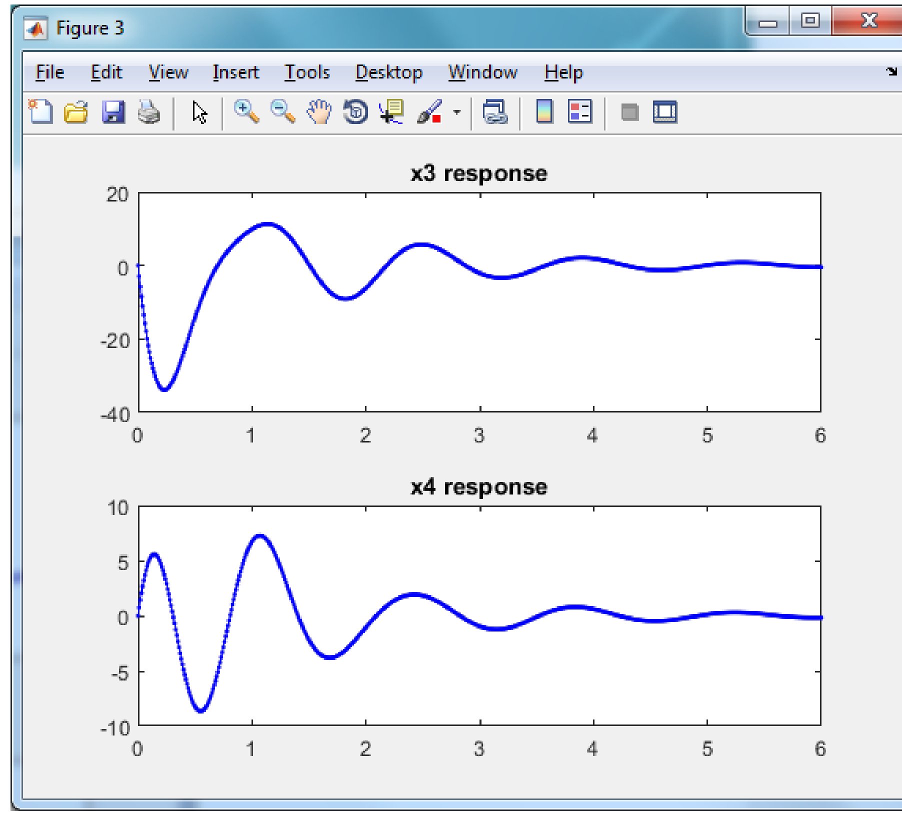


Figure 3
File Edit View Insert Tools Desktop Window Help
x3 response
20
0
-20
-40
0 1 2 3 4 5 6
x4 response
10
5
0
-5
-10
0 1 2 3 4 5 6

Quiz 12.1

Obtain a state-space model & block diagram for the system with the following transfer equation:

$$\mathbf{G}(s) = \frac{s + 2}{s^2 + 7s + 12}$$

Thank You！