8. (1) 截断误差

$$R_i^k = \frac{1}{\tau}\left[u(x_i,t_{k+1}) - u(x_i,t_k)\right] + \frac{1}{h}\left[u(x_i,t_k) - u(x_{i-1},t_k)\right]$$

$$= \frac{\partial u}{\partial x}(x_i,t_k) + \frac{\tau}{2}\frac{\partial^2 u}{\partial x^2}(x_i,\eta_i^k) + \frac{\partial u}{\partial x}(x_i,t_k) - \frac{h}{2}\frac{\partial^2 u}{\partial x^2}(\xi_i^k,t_k)$$

$$= \frac{\tau}{2}\frac{\partial^2 u}{\partial t^2}(x_i,\eta_i^k) - \frac{h}{2}\frac{\partial^2 u}{\partial x^2}(\xi_i^k,t_k),$$

$$\eta_i^k \in (t_k, t_{k+1}),\quad \xi_i^k \in (x_{i-1}, x_i).$$

(2) 设 $\{v_i^k\}$ 为

$$\begin{cases} \frac{1}{\tau}(v_i^{k+1}-v_i^k) + \frac{1}{h}(v_i^k - v_{i-1}^k) = 0, & 1\le i\le M,\ 0\le k\le N-1 \\ v_i^0 = \varphi(x_i) + \varphi_i, & 1\le i\le M, \\ v_0^k = \psi(t_k), & 0\le k\le N \end{cases}$$

的解. 令 $\varepsilon_i^k = v_i^k - u_i^k$，则 ε_i^k 满足

$$\begin{cases} \frac{1}{\tau}(\varepsilon_i^{k+1}-\varepsilon_i^k) + \frac{1}{h}(\varepsilon_i^k - \varepsilon_{i-1}^k) = 0, & 1\le i\le M,\ 0\le k\le N-1, \\ \varepsilon_i^0 = \varphi_i, & 1\le i\le M, \\ \varepsilon_0^k = 0, & 0\le k\le N \end{cases}$$

于是有 $$\varepsilon_i^{k+1} = (1-r)\varepsilon_i^k + r\varepsilon_{i-1}^k,\quad 1\le i\le M,\ 0\le k\le N-1$$

其中 $r=\frac{\tau}{h}$. 两边取绝对值，用三角不等式.

(3) 记 $R_i^k = \frac{\tau}{2}\frac{\partial^2 u}{\partial t^2}(x_i,\eta_i^k) - \frac{h}{2}\frac{\partial^2 u}{\partial x^2}(\xi_i^k,t_k),\quad 1\le i\le M,\ 0\le k\le N-1,$

$$e_i^k = u(x_i,t_k) - u_i^k,\quad 0\le i\le M,\ 0\le k\le N-1,$$

则

$$\frac{1}{\tau}(e_i^{k+1}-e_i^k) + \frac{1}{h}(e_i^k - e_{i-1}^k) = R_i^k,\quad 1\le i\le M,\ 0\le k\le N-1,$$

$$e_i^0 = 0,\quad 1\le i\le M,$$

$$e_0^k = 0,\quad 0\le k\le N.$$