6. 截断误差

$$R_i^k=\frac{1}{12}\times\left[\frac{u(x_{i+1},t_k)-u(x_{i+1},t_k)}{\tau}+10\,\frac{u(x_i,t_{k+1})-u(x_i,t_k)}{\tau}+\frac{u(x_{i-1},t_{k+1})-u(x_{i-1},t_k)}{\tau}\right]$$

$$-\frac{1}{2}\times\left[\frac{u(x_{i+1},t_{k+1})-2u(x_i,t_{k+1})+u(x_{i-1},t_{k+1})}{h^2}+\frac{u(x_{i+1},t_k)-2u(x_i,t_k)+u(x_{i-1},t_k)}{h^2}\right].$$

利用微分公式

$$\frac{u(x_i,t_{k+1})-u(x_i,t_k)}{\tau}=\frac{\partial u}{\partial t}(x_i,t_{k+\frac{1}{2}})+O(\tau^2),$$

$$\frac{u(x_{i+1},t_k)-2u(x_i,t_k)+u(x_{i-1},t_k)}{h^2}=\frac{\partial^2 u}{\partial x^2}(x_i,t_k)-\frac{h^2}{12}\,\frac{\partial^4 u(x_i,t_k)}{\partial x^4}+O(h^4),$$

可得

$$R_i^k=\frac{1}{12}\times\left[\frac{\partial u}{\partial t}(x_{i-1},t_{k+\frac{1}{2}})+10\,\frac{\partial u}{\partial t}(x_i,t_{k+\frac{1}{2}})+\frac{\partial u}{\partial t}(x_{i+1},t_{k+\frac{1}{2}})\right]$$

$$-\frac{1}{2}\times\left[\frac{\partial^2 u}{\partial x^2}(x_i,t_{k+1})+\frac{\partial^2 u}{\partial x^2}(x_i,t_k)\right]+\frac{h^2}{24}\left[\frac{\partial^4 u(x_i,t_{k+1})}{\partial x^4}+\frac{\partial^4 u(x_i,t_k)}{\partial x^4}\right]$$

$$+O(\tau^2+h^4)$$

$$=\frac{1}{12}\times\left[2\,\frac{\partial u}{\partial t}(x_i,t_{k+\frac{1}{2}})+h^2\,\frac{\partial^2}{\partial x^2}\left(\frac{\partial u}{\partial t}(x_i,t_{k+\frac{1}{2}})\right)+O(h^4)+10\,\frac{\partial u}{\partial t}(x_i,t_{k+\frac{1}{2}})\right]$$

$$-\left[\frac{\partial^2 u}{\partial x^2}(x_i,t_{k+\frac{1}{2}})+O(\tau^2)\right]+\frac{h^2}{12}\left[\frac{\partial^4 u(x_i,t_{k+\frac{1}{2}})}{\partial x^4}+O(\tau^2)\right]$$

$$+O(\tau^2+h^4)$$

$$=\frac{\partial u}{\partial t}(x_i,t_{k+\frac{1}{2}})-\frac{\partial^2 u}{\partial x^2}(x_i,t_{k+\frac{1}{2}})+\frac{h^2}{12}\,\frac{\partial^2}{\partial x^2}\left[\frac{\partial u}{\partial t}-\frac{\partial^2}{\partial x^2}\right](x_i,t_{k+\frac{1}{2}})$$

$$+O(\tau^2+h^4).$$