EE121
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Lecture #3

Boolean algebra
Combinational-circuit analysis

# Boolean algebra

- a.k.a. “switching algebra”
  - deals with boolean values -- 0, 1
- Positive-logic convention
  - analog voltages LOW, HIGH --> 0, 1
- Negative logic -- seldom used
- Signal values denoted by variables (X, Y, FRED, etc.)

## Boolean operators

- Complement: X′ (opposite of X)
- AND: X · Y
- OR: X + Y

binary operators, described functionally by truth table.

| X | Y | X AND Y |
|---|---|---|
| 0 | 0 | 0 |
| 0 | 1 | 0 |
| 1 | 0 | 0 |
| 1 | 1 | 1 |

| X | Y | X OR Y |
|---|---|---|
| 0 | 0 | 0 |
| 0 | 1 | 1 |
| 1 | 0 | 1 |
| 1 | 1 | 1 |

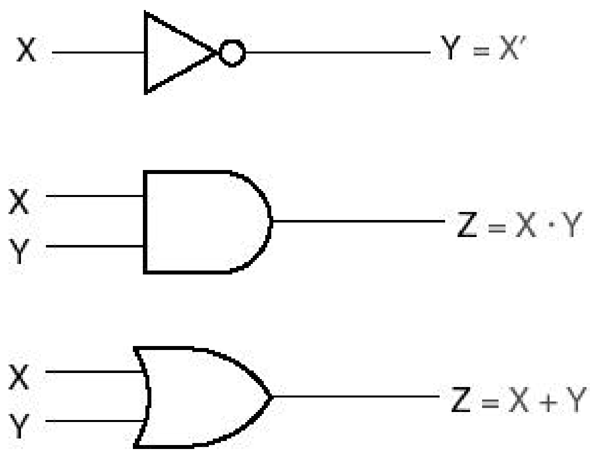
| X | NOT X |
|---|---|
| 0 | 1 |
| 1 | 0 |

- Axiomatic definition: A1-A5, A1′-A5′

## More definitions

- Literal: a variable or its complement
  - X, X′, FRED′, CS_L
- Expression: literals combined by AND, OR, parentheses, complementation
  - X+Y
  - P · Q · R
  - A + B · C
  - ((FRED · Z′) + CS_L · A · B′ · C + Q5) · RESET′
- Equation: Variable = expression
  - P = ((FRED · Z′) + CS_L · A · B′ · C + Q5) · RESET′

# Logic symbols




# Theorems

| | | | | |
|---|---|---|---|---|
| (T1) | $X + 0 = X$ | (T1′) | $X \cdot 1 = X$ | (Identities) |
| (T2) | $X + 1 = 1$ | (T2′) | $X \cdot 0 = 0$ | (Null elements) |
| (T3) | $X + X = X$ | (T3′) | $X \cdot X = X$ | (Idempotency) |
| (T4) | $(X')' = X$ | | | (Involution) |
| (T5) | $X + X' = 1$ | (T5′) | $X \cdot X' = 0$ | (Complements) |

- Proofs by perfect induction

## More Theorems

| | | | | |
|---|---|---|---|---|
| (T6) | $X + Y = Y + X$ | (T6′) | $X \cdot Y = Y \cdot X$ | (Commutativity) |
| (T7) | $(X + Y) + Z = X + (Y + Z)$ | (T7′) | $(X \cdot Y) \cdot Z = X \cdot (Y \cdot Z)$ | (Associativity) |
| (T8) | $X \cdot Y + X \cdot Z = X \cdot (Y + Z)$ | (T8′) | $(X + Y) \cdot (X + Z) = X + Y \cdot Z$ | (Distributivity) |
| (T9) | $X + X \cdot Y = X$ | (T9′) | $X \cdot (X + Y) = X$ | (Covering) |
| (T10) | $X \cdot Y + X \cdot Y' = X$ | (T10′) | $(X + Y) \cdot (X + Y') = X$ | (Combining) |
| (T11) | $X \cdot Y + X' \cdot Z + Y \cdot Z = X \cdot Y + X' \cdot Z$ | | | (Consensus) |
| (T11′) | $(X + Y) \cdot (X' + Z) \cdot (Y + Z) = (X + Y) \cdot (X' + Z)$ | | | |

- N.B. T8′, T10, T11

## Duality

- Swap 0 & 1, AND & OR
  - Result: Theorems still true
- Why?
  - Each axiom (A1-A5) has a dual (A1′-A5′)
- Counterexample:

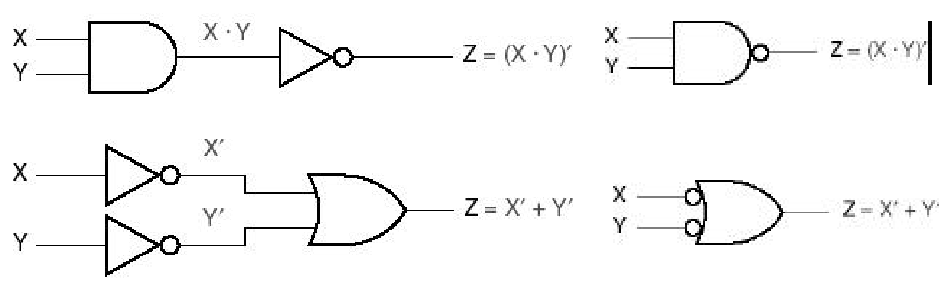
| | |
|---|---|
| $X + X \cdot Y = X$ (T9) | $X + (X \cdot Y) = X$ (T9) |
| $X \cdot X + Y = X$ (dual) | $X \cdot (X + Y) = X$ (dual) |
| $X + Y = X$ (T3′) | $(X \cdot X) + (X \cdot Y) = X$ (T8) |
| ???????????? | $X + (X \cdot Y) = X$ (T3′) |
| | parentheses, |
| | operator precedence! |

# N-variable Theorems

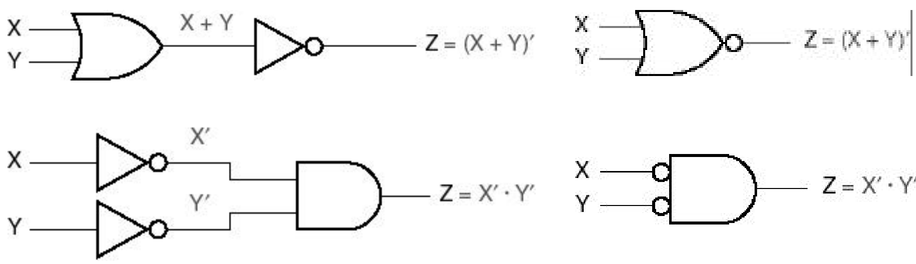
| | | |
|---|---|---|
| (T12) | $X + X + \cdots + X = X$ | (Generalized idempotency) |
| (T12′) | $X \cdot X \cdot \cdots \cdot X = X$ | |
| (T13) | $(X_1 \cdot X_2 \cdot \cdots \cdot X_n)' = X_1' + X_2' + \cdots + X_n'$ | (DeMorgan's theorems) |
| (T13′) | $(X_1 + X_2 + \cdots + X_n)' = X_1' \cdot X_2' \cdot \cdots \cdot X_n'$ | |
| (T14) | $[F(X_1, X_2, \ldots, X_n, +, \cdot)]' = F(X_1', X_2', \ldots, X_n', \cdot, +)$ | (Generalized DeMorgan's theorem) |
| (T15) | $F(X_1, X_2, \ldots, X_n) = X_1 \cdot F(1, X_2, \ldots, X_n) + X_1' \cdot F(0, X_2, \ldots, X_n)$ | (Shannon's expansion theorems) |
| (T15′) | $F(X_1, X_2, \ldots, X_n) = [X_1 + F(0, X_2, \ldots, X_n)] \cdot [X_1' + F(1, X_2, \ldots, X_n)]$ | |

- Prove using finite induction
- Most important: DeMorgan theorems

# DeMorgan Symbol Equivalence

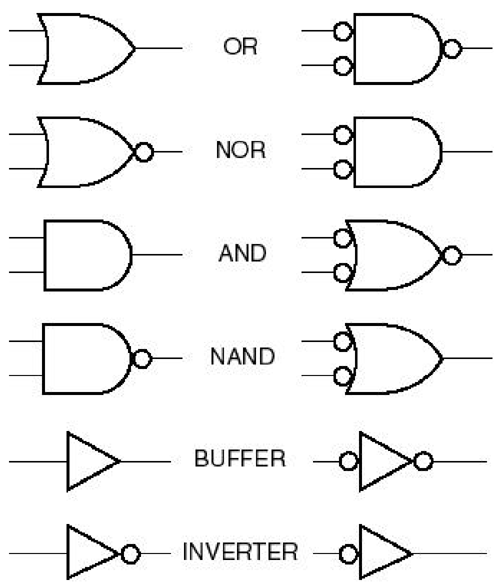


## Likewise for OR




• (be sure to check errata!)

## DeMorgan Symbols



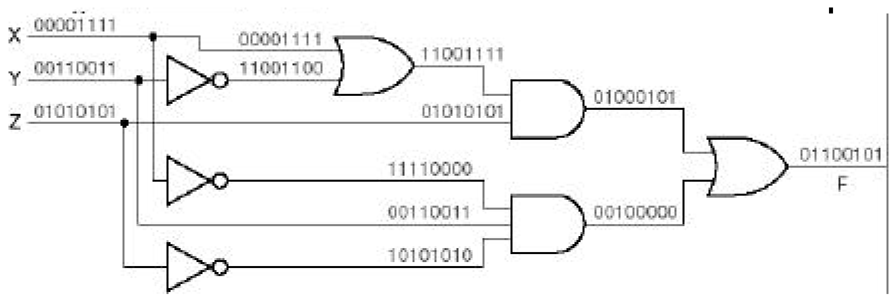
## Even more definitions (Sec. 4.1.6)

- Product term
- Sum-of-products expression
- Sum term
- Product-of-sums expression
- Normal term
- Minterm (n variables)
- Maxterm (n variables)

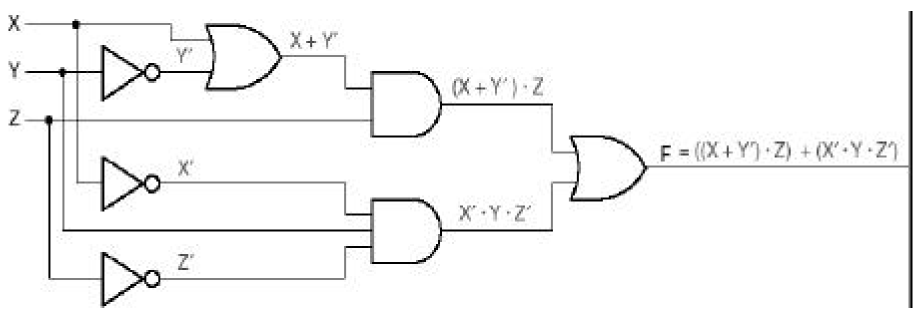
## Truth table vs. minterms & maxterms

| ***Row*** | X | Y | Z | F | ***Minterm*** | ***Maxterm*** |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | F(0,0,0) | $X' \cdot Y' \cdot Z'$ | $X + Y + Z$ |
| 1 | 0 | 0 | 1 | F(0,0,1) | $X' \cdot Y' \cdot Z$ | $X + Y + Z'$ |
| 2 | 0 | 1 | 0 | F(0,1,0) | $X' \cdot Y \cdot Z'$ | $X + Y' + Z$ |
| 3 | 0 | 1 | 1 | F(0,1,1) | $X' \cdot Y \cdot Z$ | $X + Y' + Z'$ |
| 4 | 1 | 0 | 0 | F(1,0,0) | $X \cdot Y' \cdot Z'$ | $X' + Y + Z$ |
| 5 | 1 | 0 | 1 | F(1,0,1) | $X \cdot Y' \cdot Z$ | $X' + Y + Z'$ |
| 6 | 1 | 1 | 0 | F(1,1,0) | $X \cdot Y \cdot Z'$ | $X' + Y' + Z$ |
| 7 | 1 | 1 | 1 | F(1,1,1) | $X \cdot Y \cdot Z$ | $X' + Y' + Z'$ |

## Combinational analysis

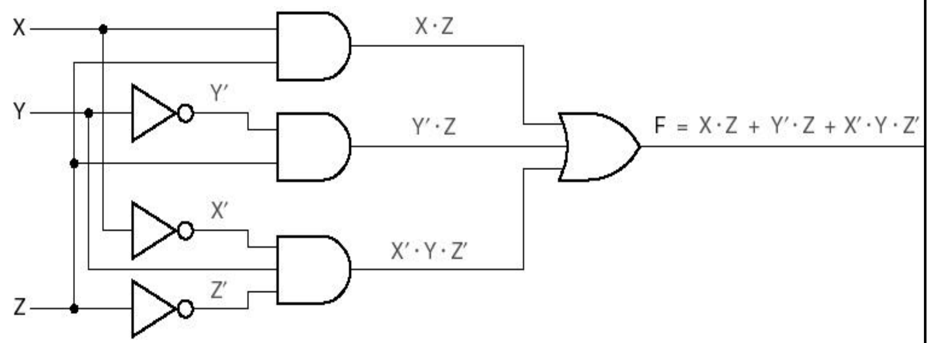



## Signal expressions




- Multiply out:
  $F = ((X + Y') \cdot Z) + (X' \cdot Y \cdot Z')$
  $\quad = (X \cdot Z) + (Y' \cdot Z) + (X' \cdot Y \cdot Z')$

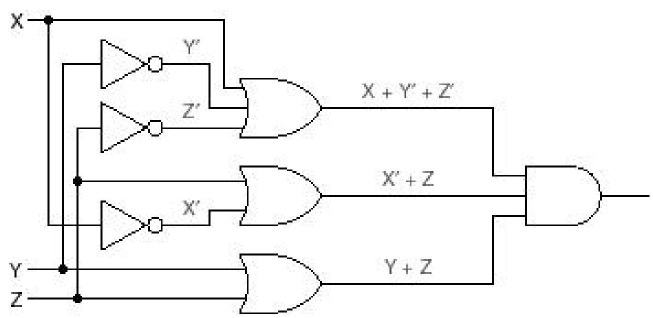
## New circuit, same function



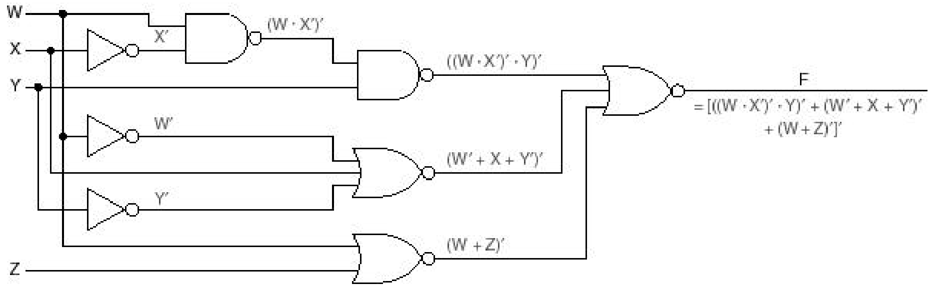

## "Add out" logic function

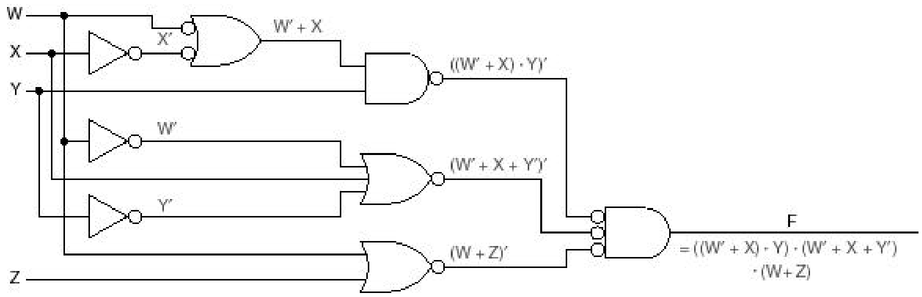
$$
\begin{aligned}
F &= ((X + Y') \cdot Z) + (X' \cdot Y \cdot Z') \\
&= (X + Y' + X') \cdot (X + Y' + Y) \cdot (X + Y' + Z') \cdot (Z + X') \cdot (Z + Y) \cdot (Z + Z') \\
&= 1 \cdot 1 \cdot (X + Y' + Z') \cdot (X' + Z) \cdot (Y + Z) \cdot 1 \\
&= (X + Y' + Z') \cdot (X' + Z) \cdot (Y + Z)
\end{aligned}
$$

- Circuit:

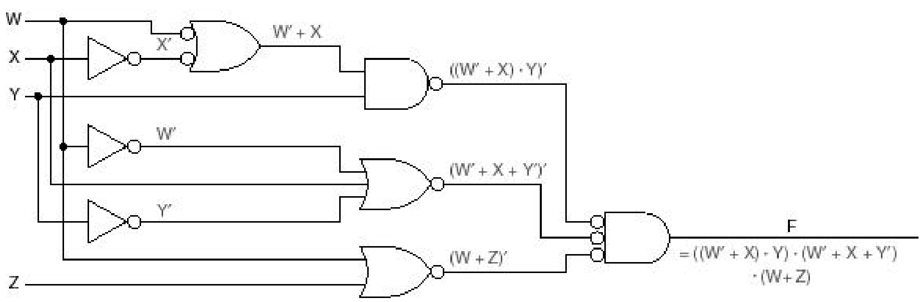


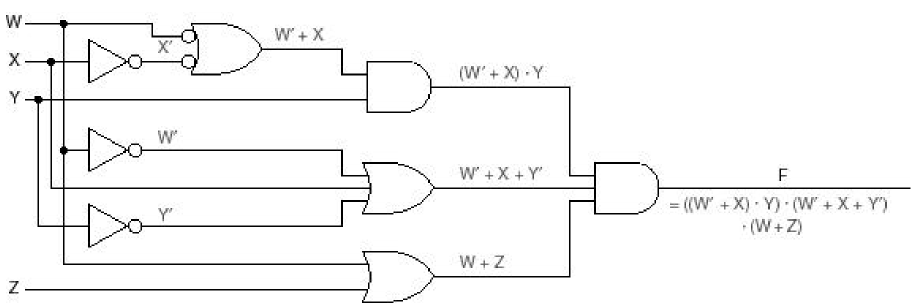
# Shortcut: Symbol substitution




# Different circuit, same function



# Another example

$G(W, X, Y, Z) = W \cdot X \cdot Y + Y \cdot Z$