Physics 303/573

Lecture 10

November 9, 2016

1 Review of the General Euler-Lagrange Equations

In the previous lecture, we found that the functional[1]

$$S[q_i] = \int_{t_1}^{t_2} dt\, L(q_i, \dot{q}_i, t) \tag{1.1}$$

(called the action functional) is extremized when the generalized coordinates q_i obey the general Euler-Lagrange Equations

$$\frac{\partial}{\partial q_i} L - \frac{d}{dt}\frac{\partial}{\partial \dot{q}_i} L = 0 \tag{1.2}$$

That is, when the coordinates $q_i(t)$ satisfy the differential equations (1.2),

$$\delta S = S[q_i + \delta q_i] - S[q_i] = O((\delta q_i)^2) \tag{1.3}$$

for an arbitrary small function $\delta q_i(t)$ that vanishes at the endpoints t_1, t_2. We saw some nice applications to finding the shortest distance between two points and to Fermat's principle, but the main application to physics that we want to consider is the application to dynamics.

2 Hamilton's principle

The key observation is that if we choose the Lagrangian to have the form

$$L(q_i, \dot{q}_i, t) = T(q_i, \dot{q}_i, t) - U(q_i, t) \tag{2.4}$$

where T is the kinetic energy of the system and U is the potential energy, then the Euler-Lagrange equations are equivalent to Newton's laws of motion. A nicer way of saying this is Hamilton's principle: The particles move along paths that extremize the action.

[1] In the previous lecture, I was careful to indicate that I was talking about the whole set of coordinates $\{q_i\}$, *etc.* I won't bother with that here.

2.1 Unconstrained particles moving in a potential

Let's consider a simple example:

$$T = \frac{1}{2}\sum_i m_i \dot{q}_i^2 \tag{2.5}$$

where q_i can represent the different coordinates of many particles and/or the multiple coordinates of particles moving in more than one dimension; the m_i for a given particle moving in different directions are of course the same.

Then the Lagrangian $L = T - U(q_i)$ gives rise to the Euler-Lagrange equations

$$-\frac{\partial U}{\partial q_i} - \frac{d}{dt} m_i \dot{q}_i = 0 \quad \Rightarrow \quad m_i \ddot{q}_i = -\frac{\partial U}{\partial q_i} \tag{2.6}$$

which is precisely Newtons law of motion with a conservative force

$$F_i = -\frac{\partial U}{\partial q_i} \tag{2.7}$$

2.2 Particle in a potential in cylindrical coordinates

In this case, we have

$$T = \frac{1}{2} m(\dot{R}^2 + R^2\dot{\phi}^2 + \dot{z}^2) \quad , \quad U = U(R, \phi, z) \tag{2.8}$$

and there are Euler-Lagrange equations–one for R:

$$\frac{d}{dt}(m\dot{R}) = -\frac{\partial U}{\partial R} + mR\dot{\phi}^2 \tag{2.9}$$

one for z:

$$\frac{d}{dt}(m\dot{z}) = -\frac{\partial U}{\partial z} \tag{2.10}$$

and one for ϕ:

$$\frac{d}{dt}(mR^2\dot{\phi}) = -\frac{\partial U}{\partial \phi} \tag{2.11}$$

If we compare to Newton's laws (using results from lecture 2 and 6)

$$\vec{F} = -\vec{\nabla} U = -\frac{\partial U}{\partial R}\hat{R} - \frac{\partial U}{\partial z}\hat{z} - \frac{1}{R}\frac{\partial U}{\partial \phi}\hat{\phi} = m\ddot{\vec{r}} = m((\ddot{R} - R\dot{\phi}^2)\hat{R} + \ddot{z}\hat{z} + (R\ddot{\phi} + 2\dot{R}\dot{\phi})\hat{\phi}) \tag{2.12}$$

we notice two things–how much simpler the Euler-Lagrange equations are, and how they conveniently group things together. The ϕ equation takes the form

$$\frac{d}{dt}(mR^2\dot{\phi}) = \dot{L}_z = -\frac{\partial U}{\partial \phi} = RF_\phi = \Gamma_z \tag{2.13}$$

where $L_z = mR^2\dot{\phi}$ is the z-component of the angular momentum and $\Gamma_z = -\frac{\partial U}{\partial \phi}$ is the z-component of the torque. This is equivalent to, but simpler, than the ϕ component of Newton's equation (2.12):

$$m(R\ddot{\phi} + 2\dot{R}\dot{\phi}) = -\frac{1}{R}\frac{\partial U}{\partial \phi} \tag{2.14}$$

We can replace $\dot{\phi}$ with L_z using:

$$\dot{\phi} = \frac{L_z}{mR^2} \tag{2.15}$$

This is particularly useful when L_z is conserved, that is, it is constant.

Then the radial equation can be rewritten in a simple form:

$$\frac{d}{dt}(m\dot{R}) = -\frac{\partial}{\partial R}(U + \frac{L_z^2}{2mR^2}) \equiv -\frac{\partial}{\partial R}U_{eff} \tag{2.16}$$

where

$$U_{eff} = U + \frac{L_z^2}{2mR^2} \tag{2.17}$$

is an "effective" potential that the radial motion sees. It takes into account the centrifugal force that a particle feels if it has a particular momentum. Notice that in this derivation, it was crucial that L_z was constant–if it had depended on R, (2.16) would be wrong.

Notice also that if U is independent of the angle, $\partial U/\partial\phi$, the z-component of the angular momentum is conserved: $\dot{L}_z = 0$. This is an example of a general phenomenon.

2.3 Lagrangians that depend on $\dot{q}$ and not q itself

Whenever the Lagrangian L depends only on $\dot{q}_i$ for some particular degree of freedom i, call it $i = 1$, and doesn't depend explicitly on q_1, the corresponding Euler-Lagrange equation is simply

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_1} = 0 \quad \Rightarrow \quad \frac{\partial L}{\partial \dot{q}_1} = p_1 = \text{a constant} \tag{2.18}$$

In other words, $\partial L/\partial\dot{q}_1$ doesn't change in time–we say it is conserved. Notice another way of stating that $L(q_i, \dot{q}_i, t)$ doesn't depend on q_1: if we shift q_1 by a constant α, since for a constant $\dot{\alpha} = 0$, $\dot{q}_1$ doesn't change, and L doesn't change. Any time that we change the variables q_i in some way and the action S stays the same, we say that the system has a symmetry. We have seen that for this simple symmetry, namely shifting q_1 by a constant, there is a corresponding conservation law–we will show later that this is a general phenomenon (the proof of this theorem is due to the great mathematician Emmy Noether and is called Noether's Theorem).

3 Lagrange multipliers and constraints

Another simplification occurs when the Lagrangian L depends only on some coordinate q_1 but not on $\dot{q}_1$. In this case the corresponding Euler-Lagrange equation imposes the *con-*

strain:

$$\frac{\partial L}{\partial q_1} = 0 \tag{3.19}$$

Indeed, we can take advantage of this observation to consider systems with constraints by simply introducing an extra coordinate that multiplies the constraint. Such an extra coordinate is called a Lagrange multiplier. We will show that that the Euler-Lagrange equations with the constraint imposed by a Lagrange multiplier are consistent with the Euler-Lagrange equations of the constrained system, where we simply solve the constraint to eliminate one of the original coordinates. We will consider only one constraint, but obviously by applying multiple constraints one after another, this is not a restriction, and our results apply to as many constraints as we wish to apply.

Consider a Lagrangian of the form

$$L_q(q_i, \dot{q}_i, t) - \lambda f(q_i) \tag{3.20}$$

Here λ is the Lagrange multiplier that imposes the constraint $f(q_i) = 0$. The Euler-Lagrange equations that follow from this are

$$\begin{aligned} \frac{\partial L_q}{\partial q_i} - \lambda \frac{\partial f}{\partial q_i} - \frac{d}{dt}\frac{\partial L_q}{\partial \dot{q}_i} &= 0 \\ f(q_i) &= 0 \end{aligned} \tag{3.21}$$

The solution to the constraint $f(q_i) = 0$ can be expressed by writing the coordinates q_i in terms of new coordinates r_a with one fewer degree of freedom, that is, if $i = 1...n$ then $a = 1...(n-1)$. So we have

$$f(q_i(r_a)) \equiv 0 \tag{3.22}$$

Differentiating, we find the relation:

$$\frac{\partial f}{\partial q_i}\frac{\partial q_i}{\partial r_a} = 0 \tag{3.23}$$

where recall that we use the summation convention–the repeated index i is summed over. We now define the constrained Lagrangian in terms of the new variables r_a as follows:

$$L_r(r_a, \dot{r}_a, t) = L_q\left(q_i(r_a), \frac{\partial q_i}{\partial r_a}\dot{r}_a, t \right) \tag{3.24}$$

Then the Euler-Lagrange equations

$$\frac{\partial L_r}{\partial r_a} - \frac{d}{dt}\frac{\partial L_r}{\partial \dot{r}_a} = 0 \tag{3.25}$$

become, using chain rule and (3.24),

$$\left(\frac{\partial L_q}{\partial q_i}\frac{\partial q_i}{\partial r_a} + \frac{\partial L_q}{\partial \dot{q}_i}\frac{\partial^2 q_i}{\partial r_a \partial r_b}\dot{r}_b \right) - \frac{d}{dt}\left(\frac{\partial L_q}{\partial \dot{q}_i}\frac{\partial q_i}{\partial r_a} \right) = 0 \tag{3.26}$$

Using the product and chain rules for differentiation, this simplifies to

$$\left(\frac{\partial L_q}{\partial q_i} - \frac{d}{dt}\frac{\partial L_q}{\partial \dot{q}_i}\right)\frac{\partial q_i}{\partial r_a} = 0 \tag{3.27}$$

which, because of (3.23) is just $\partial q_i/\partial r_a$ times the original Euler-Lagrange equation (3.21). Hence we can work with constraints imposed by Lagrange multipliers or solve the constraints and work with the reduced system.

We may worry that the original system had the constraint *and* an additional coordinate (since there is one more q_i than r_a), and hence an additional Euler-Lagrange equation. However, that equation is the one that does *not* project out the $\lambda\partial f/\partial q_i$-term, and hence simply determines λ (whose physical interpretation is a generalized constraint force).

4 Examples

We now illustrate these general results with concrete examples.

4.1 Simplified YoYo

Consider a disk with radius a, mass m, and moment of inertia I unwinding along a string as shown in figure 1. The vertical distance that the disk has dropped is y, and the angle by which it has turned is θ:

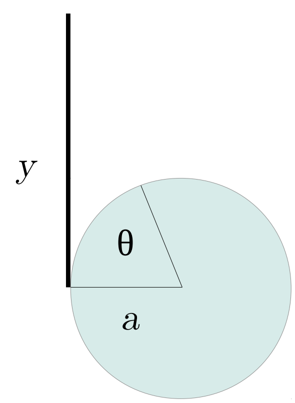

Figure 1: Simplified yoyo

Because the disk is attached to the string, there is a constraint: $y - a\theta = 0$. Because y measures the drop, the gravitational potential energy is negative: $U_g = -mgy$. Hence the Lagrangian is

$$L_{yoyo}(y, \theta, \lambda) = \frac{1}{2}m\dot{y}^2 + \frac{1}{2}I\dot{\theta}^2 + mgy - \lambda(y - a\theta) \tag{4.28}$$

The Euler-Lagrange equations (3.21) are:

$$\begin{aligned} mg - \lambda - m\ddot{y} &= 0 \\ \lambda a - I\ddot{\theta} &= 0 \\ -(y - a\theta) &= 0 \end{aligned} \tag{4.29}$$

The physical interpretation of the Lagrange multiplier is the tension in the string–the constraint force. We can solve these equations directly by adding the first two to eliminate λ and eliminating θ through the constraint to find:

$$\ddot{y} = \frac{1}{1 + I/ma^2}\, g \tag{4.30}$$

Thus the disk acts like a freely falling particle with an effective reduced gravitational acceleration

$$g_{eff} = \frac{g}{1 + I/ma^2} \tag{4.31}$$

Since physically, as the disk changes from a light hoop with a heavy axle to just a heavy hoop, $0 < I \leq ma^2$, we have $g/2 \leq g_{eff} < g$. For example, for an intermediate case such as a solid disk with $I = \frac{1}{2}ma^2$, $g_{eff} = \frac{2}{3}g$.

Alternatively, we can simply impose the constraint directly in the Lagrangain; solving for $\theta = y/a$, we find

$$L_{yoyo}(y) = \frac{1}{2}m\dot{y}^2 + \frac{1}{2}I\left(\frac{\dot{y}}{a}\right)^2 + mgy = \frac{1}{2}m\left(1 + \frac{I}{ma^2}\right)\dot{y}^2 + mgy \tag{4.32}$$

which gives (4.30) directly as Euler-Lagrange equation.

4.2 Atwood's Machine

Consider two masses m_1, m_2 connected by a string that passes over a pulley with a radius a and a moment of inertia I as shown in Figure 2:

The vertical distance from each mass to the center of the pulley is y_1, y_2, respectively, so the gravitational potential is $U_g = -m_1 g y_1 - m_2 g y_2$. The string does not stretch, so it has a fixed length $\ell = y_1 + y_2 + \pi a$. Thus the Lagrangian is

$$L_{At}(y_1, y_2, \lambda) = \frac{1}{2}m_1\dot{y}_1^2 + \frac{1}{2}m_2\dot{y}_2^2 + \frac{1}{2}I\left(\frac{\dot{y}_1}{a}\right)^2 + m_1 g y_1 + m_2 g y_2 - \lambda(y_1 + y_2 + \pi a - \ell) \tag{4.33}$$

We can write down full Euler-Lagrange equations and interpret λ as the tension pulling m_2 up, but if we are interested only in the motion, it is better to solve the constraint by writing $y_2 = -y_1 + \ell - \pi a$, which gives (dropping the subscript on y_1) the Lagrangian

$$L_{At}(y) = \frac{1}{2}(m_1 + m_2 + \frac{I}{a^2})\dot{y}^2 + (m_1 - m_2)gy + c \tag{4.34}$$

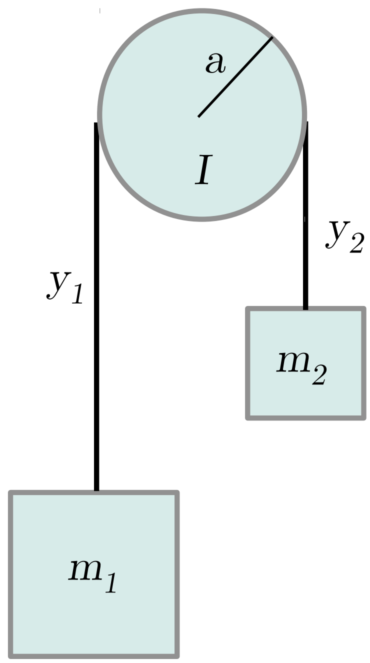

Figure 2: Atwood's Machine

where $c = m_2 g(\ell - \pi a)$ is an irrelevant constant. Then the Euler-Lagrange equation is simply

$$\ddot{y} = \left(\frac{m_1 - m_2}{m_1 + m_2 + I/a^2} \right) g \tag{4.35}$$

By choosing m_1 slightly heavier than m_2, this gives an easy way to measure g without precision timing.

4.3 Bead on a hoop

In this example, I will write down the Lagrangian after solving all the constraints, without bothering to set it up with Lagrange multipliers–you can work it out for yourself as an exercise if you like.

Consider a bead with a mass m constrained to move along a frictionless hoop with radius R that is rotating at a fixed angular velocity ω:

We can describe the motion of the bead in terms of the angle θ from the vertical. The kinetic energy has two contributions, one from the rotation of the hoop $T_\omega = \frac{1}{2}m(R\sin\theta)^2\omega^2$, and one from the motion of the bead along the hoop $T_\theta = \frac{1}{2}mR^2\dot{\theta}^2$; the potential energy is just the gravitational potential $U_g = mgR(1-\cos\theta)$, so the Lagrangian is:

$$L_{bead} = \frac{1}{2}mR^2(\dot{\theta}^2 + \omega^2\sin^2\theta) - mgR(1-\cos\theta) \tag{4.36}$$

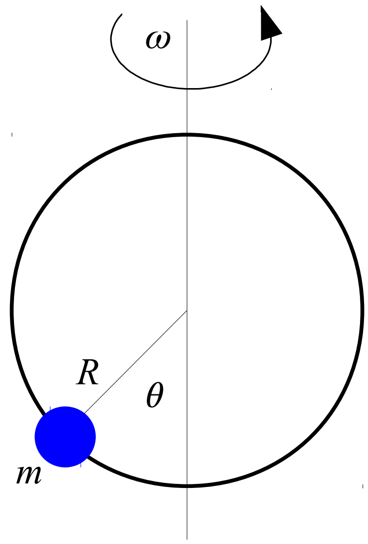

Figure 3: Bead on a hoop.

The Euler-Lagrange equation is

$$mR^2\omega^2 \sin\theta\cos\theta - mgR\sin\theta - \frac{d}{dt}(mR^2\dot{\theta}) = 0 \quad \Rightarrow \quad \ddot{\theta} = \left(\omega^2\cos\theta - \frac{g}{R}\right)\sin\theta \tag{4.37}$$

This can be solved in terms of elliptic integrals, but it is easier and gives us more insight to study the small oscillations about the equilibrium points. At equilibrium, the generalized force vanishes, so either $\sin\theta = 0$ or $\cos\theta = g/(R\omega^2)$. When $\sin\theta = 0$, we have either $\theta = 0$ or $\theta = \pi$. Near $\theta = 0$, we write $\theta = \epsilon$ and use the small angle approximation and find

$$\ddot{\epsilon} \approx -\Omega_0^2\epsilon \quad , \quad \Omega_0^2 = \frac{g}{R} - \omega^2 \quad \Rightarrow \quad \epsilon(t) = \epsilon_{max}\cos(\Omega_0 t + \phi) \tag{4.38}$$

where ϕ is an initial phase angle. This solution corresponds to small oscillations of the bead around the bottom of the hoop, and makes sense for small amplitude ϵ_{max} and if the hoop is rotating slowly enough: $\omega^2 < g/R$. Near $\theta = \pi$, we write $\theta = \pi + \epsilon$ and use $\sin(\pi+\epsilon) = -\sin\epsilon$ and $\cos(\pi+\epsilon) = -\cos\epsilon$. Then the small angle approximation gives

$$\ddot{\epsilon} \approx \Omega_\pi^2\epsilon \quad , \quad \Omega_\pi^2 = \frac{g}{R} + \omega^2 \quad \Rightarrow \tag{4.39}$$

So this is an unstable equilibrium, and there is no oscillatory behavior no matter how fast the hoop rotates.

Near $\cos\theta = g/(R\omega^2)$, we write $\theta = \Theta + \epsilon$ with $\cos\Theta = g/(R\omega^2)$; then, using trigonometric identities, we find

$$\begin{aligned}\ddot{\epsilon} &= \left(\omega^2(\cos\Theta\cos\epsilon - \sin\Theta\sin\epsilon) - \frac{g}{R}\right)(\sin\Theta\cos\epsilon + \cos\Theta\sin\epsilon) \\ &\approx -(\omega^2\sin^2\Theta)\epsilon \;=\; -(\omega^2 - \frac{g^2}{\omega^2R^2})\epsilon \;\equiv\; -\Omega_\Theta^2\epsilon \end{aligned} \tag{4.40}$$

The equilibrium is stable precisely when the equilibrium at $\theta = 0$ ceases to be stable, that is when $\omega^2 > g/R$. Then the solution looks like

$$\epsilon(t) = \epsilon_{max}\cos(\Omega_\Theta t+\phi) \quad\Rightarrow\quad \theta(t) = \Theta+\epsilon_{max}\cos(\Omega_\Theta t+\phi) = \pm\arccos\left(\frac{g}{R\omega^2}\right)+\epsilon_{max}\cos(\Omega_\Theta t+\phi) \tag{4.41}$$

for ϵ_{max} small.

5 Noether's Theorem

In section 2.3 above, we saw an example of Noether's theorem–namely, that if the Lagrangian depends only on the time derivative of a coordinate, the Euler-Lagrange equations imply that the conjugate momentum is conserved (doesn't change in time). We will now prove the general form of Noether's theorem.

5.1 Total derivatives

For simplicity, I will suppress all but one coordinate, but the arguments that I am using hold even when there are many. We first prove a simple theorem: if we add a total derivative to the Lagrangian, the Euler-Lagrange equations do not change. This is because if we add to the Lagrangian a quantity $\dot{\Delta}(q)$, which, using the chain rule is

$$\dot{\Delta} = \frac{\partial\Delta}{\partial q}\,\dot{q}$$

the fundamental theorem of calculus implies that the action changes by Δ evaluated at the endpoints:

$$L_\Delta = L_0 + \dot{\Delta}(q) \qquad\Rightarrow\qquad S_\Delta = \int_1^2 dt\, L_\Delta = S_0 + \Delta(2) - \Delta(1) \tag{5.42}$$

Since we find the Euler-Lagrange equations by extremizing with respect to $\delta q(t)$ which are fixed at the endpoints, that is $\delta q(1) = \delta q(2) = 0$, the variation of the extra term is zero. We can check directly that the Euler-Lagrange equations don't change:

$$\frac{d}{dt}\left(\frac{\partial L_\Delta}{\partial\dot{q}}\right) - \frac{\partial L_\Delta}{\partial q} = \frac{d}{dt}\left(\frac{\partial L_0}{\partial\dot{q}} + \frac{\partial\Delta}{\partial q}\right) - \left(\frac{\partial L_0}{\partial q} + \frac{\partial^2\Delta}{\partial q^2}\,\dot{q}\right) \tag{5.43}$$

which, by chain rule, is just the original Euler-Lagrange equation (the Δ-terms cancel out).

This means that if we have a Lagrangian $L_0(\dot{q})$ that depends on $\dot{q}$ but NOT on q itself as in section 2.3, even if we add a total derivative, the original conserved quantity stays conserved:

$$\frac{d}{dt}\left(\frac{\partial L_0}{\partial\dot{q}}\right) = 0 \tag{5.44}$$

In terms of the new Lagrangian L_Δ, comparing to (5.43), this becomes

$$\frac{d}{dt}\left(\frac{\partial L_\Delta}{\partial\dot{q}} - \frac{\partial\Delta}{\partial q}\right) = 0 \tag{5.45}$$

5.2 Changing coordinates

As we have seen, we can go from one set of coordinates to another to get a different description of the same dynamics. Suppose we start with $L_0(\dot{q})$, but now change coordinates to $q' = f(q)$; then we still should have a conserved quantity, but it will look different. Before, we had $L_0(\dot{q})$ didn't change when we shifted q by a constant; that is $L_0(\frac{d}{dt}(q+a)) = L_0(\dot{q})$ where a is a constant. Now $L_0(q'_a, \dot{q}'_a) = L_0(q', \dot{q}')$ for $q'_a = f(q+a)$, where we need to write $f(q+a)$ as a function of q': for example, if $q' = e^q$, we have $q'_a = e^{q+a} = e^a q'$. This leads us to the notion of a symmetry:

5.3 Symmetries

A *symmetry* is a transformation of the variables q_i that changes the Lagrangian at most by a total derivative even when we don't use the Euler-Lagrange equations.[2] A *continuous* symmetry is one that allows for transformations that change q_i by an arbitrarily small amount. We can write such *infinitesimal* transformations as:

$$\delta q_i = \alpha R_i(q_j, \dot{q}_j, t) \tag{5.46}$$

where α is a small constant parameter. In the simplest case, $L(q_i, \dot{q}_i, t)$ is invariant, but more generally, it is sufficient that there exists a quantity $K(q_i, \dot{q}_i, t)$ such that

$$\delta L = \alpha \frac{d}{dt} K \tag{5.47}$$

Now let's use chain rule to work out what this means:

$$\delta L - \alpha \dot{K} = 0 = \alpha \left(\frac{\partial L}{\partial q_i} R_i + \frac{\partial L}{\partial \dot{q}_i} \dot{R}_i - \dot{K} \right) \tag{5.48}$$

This guarantees we have a symmetry; note that we are NOT using the Euler-Lagrange equations. Now we see what this implies if we *do* use the Euler-Lagrange equations:

$$0 = \alpha \left(\left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \right) R_i + \frac{\partial L}{\partial \dot{q}_i} \dot{R}_i - \dot{K} \right) \tag{5.49}$$

and hence

$$0 = \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} R_i - K \right) \tag{5.50}$$

Thus we have shown that if the action S has a continuous symmetry, there is a quantity

$$Q = \frac{\partial L}{\partial \dot{q}_i} R_i - K \tag{5.51}$$

that is conserved, or in other words, that stays unchanged in time as the system evolves. This is Noether's theorem. Later, we will also prove the converse–if we have a conserved quantity, it generates a symmetry of the system.

Let's compute the conserved quantity Q for several examples.

[2] It is clear that symmetries form a *group*–the composition of two transformations that do not change the action clearly does not change the action, and hence defines group multiplication; the other properties are of a group are also easy to verify.

5.4 Conserved quantity for translations

We saw in section 2.3 that if the Lagrangian depends only on $\dot{q}$ but not q itself, it is invariant under the symmetry

$$\delta q = \alpha \tag{5.52}$$

then the Euler-Lagrange equations imply that $\partial L/\partial \dot{q}$ is conserved. In this case the Lagrangian itself is symmetric, so $K = 0$, and $R = 1$, so the general formula (5.51) reduces to

$$Q = \frac{\partial}{\partial \dot{q}} L \tag{5.53}$$

which is indeed the quantity we are after. For $L = \frac{1}{2} m \dot{q}^2$, this gives the linear momentum:

$$Q = m\dot{q} \tag{5.54}$$

For $L = \frac{1}{2} m(\dot{R}^2 + R^2\dot{\phi}^2 + \dot{z}^2) - U(R, z)$, which has a rotational symmetry $\delta\phi = \alpha$, this gives

$$Q = mR^2\dot{\phi} = L_z \tag{5.55}$$

5.5 Conserved quantity for rotations

For a less trivial example, consider (infinitesimal) rotations in cartesian coordinates: Here we are considering not just one transformation, but three transformations corresponding to independent infinitesimal rotations about the three different coordinate axes. Consequently, we will find three different conserved quantities: $\vec{Q}$.

$$\delta\vec{r} = \vec{\alpha} \times \vec{r} \tag{5.56}$$

Here we are considering not just one transformation, but three transformations corresponding to independent infinitesimal rotations about the three different coordinate axes. Consequently, we will find three different conserved quantities: $\vec{Q}$. These transformations preserve a Lagrangian of the form

$$L = \frac{1}{2} m\dot{\vec{r}} \cdot \dot{\vec{r}} - U(|r|) \tag{5.57}$$

Since L is invariant, $K = 0$ as before, but $\vec{R}$ is nontrivial. Using (5.51), we have:

$$\vec{Q} \equiv m\dot{\vec{r}} \times \vec{r} \tag{5.58}$$

which we recognize as $-\vec{L}$. So the conserved quantity associated to rotational symmetry is the angular momentum.

5.6 Conserved quantity with total derivatives

Let us now consider two examples with a nontrivial K. The first is a Lagrangian of a particle in two dimensions with an extra term linear in the velocity (physically, this corresponds to a particle in a constant magnetic field):

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + s(x\dot{y} - y\dot{x}) \tag{5.59}$$

Under a translation by the constants α_x, α_y

$$\delta x = \alpha_x \quad , \quad \delta y = \alpha_y \tag{5.60}$$

the Lagrangian changes by

$$\delta L = s(\alpha_x \dot{y} - \alpha_y \dot{x}) \quad \Rightarrow \quad K_x = sy \quad , \quad K_y = -sx \tag{5.61}$$

From (5.51) we find:

$$Q_x = (m\dot{x} - sy) - sy = m\dot{x} - 2sy \quad , \quad Q_y = (m\dot{y} + sx) - (-sx) = m\dot{y} + 2sx \tag{5.62}$$

We can check that the Euler-Lagrange equations imply that these are conserved:

$$\begin{aligned} s\dot{y} - \frac{d}{dt}(m\dot{x} - sy) = 0 \quad &\Rightarrow \quad m\ddot{x} - 2s\dot{y} = 0 \\ -s\dot{x} - \frac{d}{dt}(m\dot{x} + sx) = 0 \quad &\Rightarrow \quad m\ddot{y} + 2s\dot{x} = 0 \end{aligned} \tag{5.63}$$

5.7 Conserved quantity for time-translations

The final and important example is time-translation. This is a symmetry for Lagrangians with no explicit time dependence, that is, $\partial L/\partial t = 0$. Consider

$$\delta q_i = \alpha \dot{q}_i \tag{5.64}$$

It is easy to see that the chain rule implies

$$\delta L = \alpha \frac{d}{dt} L \tag{5.65}$$

Now (5.51) implies

$$Q = \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L \tag{5.66}$$

This is a conserved quantity, and is so important that it has a special name: it is called the Hamiltonian, and usually is denoted by the letter H. We will discuss it more below.

6 Generalized momenta

Wrting the Euler-Lagrange equations (1.2) as

$$\frac{d}{dt}\frac{\partial}{\partial \dot{q}_i}L = \frac{\partial}{\partial q_i}L \tag{6.67}$$

it is natural to identify $F_i \equiv \partial L/\partial q_i$ as a generalized force and $p_i \equiv \partial L/\partial \dot{q}_i$ as a generalized momentum; then (6.67) becomes the familiar

$$\dot{p}_i = F_i \tag{6.68}$$

To understand this better, we look at some examples. In section 2.1, we considered a system of unconstrained particles moving in a potential with $L = \frac{1}{2}\sum_i m_i \dot{q}_i^2 - U(q_i)$, which gives

$$p_i = m_i \dot{q}_i \quad , \qquad \not\sum_i \tag{6.69}$$

(with no summation over i)–this is just the usual momentum.

In section 2.2 we considered a particle in cylindrical coordinates with Lagrangian $\frac{1}{2}m(\dot{R}^2 + R^2\dot{\phi}^2 + \dot{z}^2) - U(R,\phi,z)$; this gives

$$p_R = m\dot{R} \quad , \qquad p_\phi = mR^2\dot{\phi} \quad , \qquad p_z = m\dot{z} \tag{6.70}$$

so p_R, p_z are components of the linear momentum, by $p_\phi = L_z$ is a component of the angular momentum. If we consider the example in section 5.3, with Lagrangian (5.59), we find

$$p_x = m\dot{x} - sy \quad . \qquad p_y = m\dot{y} + sx \tag{6.71}$$

This turns out to be the appropriate definition of linear momentum in a constant magnetic field.

So far, we have used

$$p_i = \frac{\partial L}{\partial \dot{q}} \tag{6.72}$$

to define $p_i(q_i, \dot{q}_i)$. However, we could invert this system of equations and solve for $\dot{q}_i(q_i, p_i)$. The space with coordinates $\{q_i, p_i\}$ is called *phase* space.

7 The Hamiltonian

In section 5.4, we found that (using the definition of p_i in 6.72)

$$H = p_i \dot{q}_i - L(q_i, \dot{q}_i) \tag{7.73}$$

is conserved when the Lagrangian has no explicit time dependence: $\partial L/\partial t = 0$. We want to consider $H(p_i, q_i)$, so we need to solve (6.72) for $\dot{q}_i(p_i, q_i)$ and substitute the result into (7.73). Let us see what this implies; we take p_i, q_i as our independent variables, and differentiate

$H(p,q)$ with respect to both and use the chain rule for $\dot{q}_i(p_i, q_i)$. First consider differentiating with respect to q_i:

$$\frac{\partial H}{\partial q_i} = p_j \frac{\partial \dot{q}_j}{\partial q_i} - \frac{\partial L}{\partial q_i} - \frac{\partial L}{\partial \dot{q}_j}\frac{\partial \dot{q}_j}{\partial q_i} = -\frac{\partial L}{\partial q_i} \tag{7.74}$$

where we have canceled two terms using the definition (6.72) (and relabeled the summed index $i \to j$ to avoid confusion). Using the Euler-Lagrange equations (6.67) and the definition of the generalized momentum, we find:

$$\frac{\partial H}{\partial q_i} = -\dot{p}_i \tag{7.75}$$

Next, we differentiate the Hamiltonian with respect to p_i:

$$\frac{\partial H}{\partial p_i} = \dot{q}_i + p_j \frac{\partial \dot{q}_j}{\partial p_i} - \frac{\partial L}{\partial \dot{q}_j}\frac{\partial \dot{q}_j}{\partial p_i} = \dot{q}_i \tag{7.76}$$

where again we have canceled two terms using the definition (6.72) (and relabeled the summed index $i \to j$ to avoid confusion). The two equations

$$\frac{\partial H}{\partial q_i} = -\dot{p}_i \quad , \quad \frac{\partial H}{\partial p_i} = \dot{q}_i \tag{7.77}$$

are called the Hamilton-Jacobi equations and are equivalent to relations between p_i and $\dot{q}_i$ (6.72) and the Euler-Lagrange equations (6.67). Thus the Hamiltonian is an alternative way to encode the dynamics of the system.

For an unconstrained system of particles with $L = T - U$ and

$$T(q_i, \dot{q}_i) = \frac{1}{2}\sum m_i \dot{q}_i^2 \tag{7.78}$$

we have $p_i = m_i \dot{q}_i$, which implies $\dot{q}_i = p_i/m_i$ and so we can write

$$T(q_i, p_i) = \frac{1}{2}\sum (p_i^2/m_i) \tag{7.79}$$

and hence

$$H(q_i, p_i) = \sum_i p_i \dot{q}_i - L = \sum_i p_i \dot{q}_i - T + U(q_i) = \frac{1}{2}\sum \frac{p_i^2}{m_i} + U(q_i) = T + U \tag{7.80}$$

Thus we find that the Hamiltonian is precisely the total energy.

7.1 Poisson brackets

Suppose that we want to know how some quantity $A(q_i, p_i)$ changes in time. The chain rule tells us (remember that we are using the summation convention!)

$$\dot{A} \equiv \frac{d}{dt}A(q_i, p_i) = \frac{\partial}{\partial q_i}A(q_i, p_i)\dot{q}_i + \frac{\partial}{\partial p_i}A(q_i, p_i)\dot{p}_i \tag{7.81}$$

Using the Hamilton-Jacobi equations (7.77), we have:

$$\dot{A} = \frac{\partial A}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial H}{\partial q_i} \tag{7.82}$$

We recover the Hamilton-Jacobi equations by first choosing $A = p_i$ and then $A = q_i$. This result suggests that we define the *Poisson bracket*:

$$[A, B]_{PB} \equiv \frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i} \tag{7.83}$$

Then (7.82) becomes simply:

$$\dot{A} = [A, H]_{PB} \tag{7.84}$$

The subscript gets a bit unwieldy, so I will drop it below.

The Poisson bracket is a *derivation*–that is, it obeys the Leibniz (product) rule; first consider a special case

$$\frac{d}{dt}AB = \dot{A}B + A\dot{B} \quad \Leftrightarrow \quad [H, AB] = [H, A]B + A[H, B] \tag{7.85}$$

Though we proved this using (7.84), it turns out that this property just follow from the definition of the Poisson bracket (7.83) and works for *any* three functions A, B, C:

$$[A, BC] = [A, B]C + B[A, C] \tag{7.86}$$

The Poisson bracket also obeys another version of the product rule, involving brackets of brackets. Again, we show it for H and then leave it as an exercise to prove it in general: The usual product rule for d/dt implies:

$$\frac{d}{dt}[A, B] = [\dot{A}, B] + [A, \dot{B}] \tag{7.87}$$

But using (7.84), this is equivalent to

$$[[A, B], H] = [[A, H], B] + [A, [B, H]] \tag{7.88}$$

or rearranging the terms and using the antisymmetry of the Poisson bracket:

$$[[A, B], H] + [[H, A], B] + [[B, H], A] = 0 \tag{7.89}$$

Though we proved this using (7.84), it turns out that this property just follow from the definition of the Poisson bracket (7.83) and works for *any* three functions A, B, C

$$[[A, B], C] + [[C, A], B] + [[B, C], A] = 0 \tag{7.90}$$

7.2 Noether's theorem revisited

We proved before that a continuous symmetry leads to a conserved charge Q (see section 5). Let's see how that looks in phase space. The conservation of Q becomes

$$0 = \dot{Q} = [Q, H] \tag{7.91}$$

i.e., the statement that the Poisson bracket of Q and H vanishes. Using the definition of the Poisson bracket, we thus have:

$$0 = \frac{\partial Q}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial Q}{\partial p_i}\frac{\partial H}{\partial q_i} \tag{7.92}$$

But if we have some transformations $\delta q_i, \delta p_i$, then by chain rule we have:

$$\delta H = \delta q_i \frac{\partial H}{\partial q_i} + \delta p_i \frac{\partial H}{\partial p_i} \tag{7.93}$$

Now a symmetry leaves the Hamiltonian invariant[3], so $\delta H = 0$; we can compare (7.92) with (7.93) and find

$$\delta q_i = \alpha \frac{\partial Q}{\partial p_i} \quad , \qquad \delta p_i = -\alpha \frac{\partial Q}{\partial q_i} \tag{7.94}$$

or in other words,

$$\delta q_i = \alpha[q_i, Q] \quad , \qquad \delta p_i = \alpha[p_i, Q] \tag{7.95}$$

Here α is an arbitrary constant parameter. This proves Noether's theorem in the other direction: given a conserved quantity Q, we have found a symmetry (7.95). An important consistency check is the transformation of $\dot{q}$. We start by taking the time derivative of δq_i, express it in terms of Poisson brackets, and then use the properties of Poisson brackets, in particular (7.89), to show this is equal to the variation of $\dot{q}_i$:

$$\frac{d}{dt}\delta q_i = [\delta q_i, H] = \alpha[[q_i, Q], H] = \alpha[[q_i, H], Q] = \alpha[\dot{q}_i, Q] = \delta(\dot{q}_i) \tag{7.96}$$

where we have used (7.89) and the invariance of the Hamiltonian $[Q, H] = 0$.

[3]This is in contrast to the Lagrangian, which, as explained above, can change by a total derivative.