Homework 5

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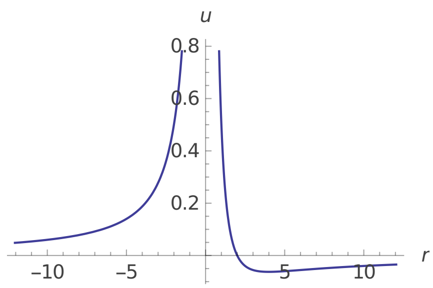
1. Consider a potential energy in one dimension of the form

$$U(r) = -\frac{A}{r} + \frac{1}{r^n}$$

   (a) Sketch this potential for the special value $A = \frac{1}{2}$, $n = 2$, i.e. $U(r) = -\frac{1}{2r} + \frac{1}{r^2}$



   (b) For $r > 0$ and arbitrary $A > 0$, and any $n > 0$, what is the extremum r_{eq} of this potential?

$$\begin{aligned}
\frac{\mathrm{d}U}{\mathrm{d}r} &= 0 \\
\frac{\mathrm{d}}{\mathrm{d}r}\left[-Ar^{-1} + r^{-n}\right] &= 0 \\
Ar^{-2} - nr^{n-1} &= 0 \\
\frac{A}{r^2} - \frac{n}{r^{n+1}} &= 0 \\
\frac{A}{r^2} &= \frac{n}{r^{n+1}} \\
r^{n-1} &= \frac{n}{A} \\
r &= \left(\frac{n}{A}\right)^{\frac{1}{n-1}}
\end{aligned}$$

   (c) For this extremum r_{eq}, write the Taylor expansion for $U(r_{\mathrm{eq}} + \delta)$ to second order in δ.

$$\begin{aligned}
U(r_{\mathrm{eq}} + \delta) &= U(r_{\mathrm{eq}}) + \frac{\mathrm{d}U}{\mathrm{d}r}(r_{\mathrm{eq}})\delta + \frac{1}{2}\frac{\mathrm{d}^2U}{\mathrm{d}r^2}(r_{\mathrm{eq}})\delta^2 + \cdots \\
&\approx U(r_{\mathrm{eq}}) + \frac{\mathrm{d}U}{\mathrm{d}r}(r_{\mathrm{eq}})\delta + \frac{1}{2}\frac{\mathrm{d}^2U}{\mathrm{d}r^2}(r_{\mathrm{eq}})\delta^2
\end{aligned}$$

and we can find

$$
\begin{aligned}
U(r_{\text{eq}}) &= -A(\frac{A}{n})^{\frac{1}{n-1}} + (\frac{A}{n})^{\frac{n}{n-1}} \\
\frac{\mathrm{d}U}{\mathrm{d}r}(r_{\text{eq}}) &= 0 \\
\frac{\mathrm{d}^2U}{\mathrm{d}r^2} &= -2Ar^{-3} + n^2r^{-n-2} + nr^{-n-2} \\
\frac{\mathrm{d}^2U}{\mathrm{d}r^2}(r_{\text{eq}}) &= -2A(\frac{A}{n})^{\frac{3}{n-1}} + n^2(\frac{A}{n})^{\frac{n+2}{n-1}} + n(\frac{A}{n})^{\frac{n+2}{n-1}}
\end{aligned}
$$

Wwhich give us:

$$
U(r_{\text{eq}} + \delta) \approx \left[-A(\frac{A}{n})^{\frac{1}{n-1}} + (\frac{A}{n})^{\frac{n}{n-1}}\right]\delta + \frac{1}{2}\left[-2A(\frac{A}{n})^{\frac{3}{n-1}} + n^2(\frac{A}{n})^{\frac{n+2}{n-1}} + n(\frac{A}{n})^{\frac{n+2}{n-1}}\right]\delta^2
$$

if we chose r_{eq} as the reference point, thus we have

$$
U = \frac{1}{2}\left[-2A(\frac{A}{n})^{\frac{3}{n-1}} + n^2(\frac{A}{n})^{\frac{n+2}{n-1}} + n(\frac{A}{n})^{\frac{n+2}{n-1}}\right]\delta^2
$$

(d) If the particle has a mass m, what is the angular frequency of small oscillations?

For U we have:

$$
\begin{aligned}
U &= \frac{1}{2}kx^2 \\
\frac{\mathrm{d}U}{\mathrm{d}x} &= kx \\
\frac{\mathrm{d}^2U}{\mathrm{d}x^2} &= k
\end{aligned}
$$

Thus we have k as:

$$
k = \frac{1}{2}[-2A(\frac{A}{n})^{\frac{3}{n-1}} + n^2(\frac{A}{n})^{\frac{n+2}{n-1}} + n(\frac{A}{n})^{\frac{n+2}{n-1}}]
$$

Thus we have

$$
\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{1}{2}\left[-2A(\frac{A}{n})^{\frac{3}{n-1}} + n^2(\frac{A}{n})^{\frac{n+2}{n-1}} + n(\frac{A}{n})^{\frac{n+2}{n-1}}\right]\frac{1}{m}}
$$

2. The mass shown from above in figure is resting on a friction less horizontal table. Each of the two identical springs has force constant k and unstretched length l_0. At equilibrium the mass rests at the origin, and the distances a are not necessarily equal to l_0. (That is, the springs may already be stretched or compressed.) Show that when the mass moves to a position (x, y), with x and y small, the potential energy has the form for an anisotripic oscillator. Show that if $a < l_0$ the equilibrium at the origin is unstable and explain why.

$$
U = \frac{1}{2}(k_x x^2 + k_y y^2)
$$

we have for r:

$$
\begin{cases}
r_1 = \sqrt{y^2 + (a+x)^2} \\
r_2 = \sqrt{y^2 + (a-x)^2}
\end{cases}
$$

For the force, we have:

$$
\begin{cases}
F_1 = -k(r_1 - l_0) \\
F_2 = -k(r_2 - l_0)
\end{cases}
$$

And we find F_x and F_y from it:

$$\begin{cases} F_x = -k(r_1 - l_0)\dfrac{x+a}{r_1} - k(r_2 - l_0)\dfrac{a-x}{r_2} \\ F_y = -k(r_1 - l_0)\dfrac{y}{r_1} - k(r_2 - l_0)\dfrac{y}{r_2} \end{cases}$$

Now we want to find the energy. If we Taylor expand the energy near the equilibrium point, which is choose as $(0,0)$ here, we have:

$$U = U(0,0) + \frac{\mathrm{d}U}{\mathrm{d}\vec{r}}(0,0)\vec{r} + \frac{1}{2}\frac{\mathrm{d}^2U}{\mathrm{d}\vec{r}}(0,0)\vec{r}^2 + \cdots$$

now notice that since we chose the equilibrium point as $(0,0)$, $U(0,0) = 0$. Moreover, since this is an equilibrium point, $\frac{\mathrm{d}U}{\mathrm{d}\vec{r}} = 0$. Thus we have:

$$U \approx \frac{1}{2}\frac{\mathrm{d}^2U}{\mathrm{d}\vec{r}}(0,0)\vec{r}^2$$

Also we can notice that $U = \int F \cdot \mathrm{d}\vec{r}$. Thus we have:

$$U \approx \frac{\mathrm{d}F}{\mathrm{d}\vec{r}}(0,0)\vec{r}^2$$

Thus we have:

$$\begin{cases} F_x = -k(\sqrt{y^2+(a+x)^2} - l_0)\dfrac{x+a}{\sqrt{y^2+(a+x)^2}} - k(\sqrt{y^2+(a-x)^2} - l_0)\dfrac{a-x}{\sqrt{y^2+(a-x)^2}} \\ F_y = -k(\sqrt{y^2+(a+x)^2} - l_0)\dfrac{y}{\sqrt{y^2+(a+x)^2}} - k(\sqrt{y^2+(a-x)^2} - l_0)\dfrac{y}{\sqrt{y^2+(a-x)^2}} \end{cases}$$

Thus we have:

$$\begin{aligned} \frac{\mathrm{d}F_x}{\mathrm{d}x} = &- \frac{k(a-x)^2\sqrt{(a-x)^2+y^2} - l}{((a-x)^2+y^2)^{\frac{3}{2}}} \\ &+ \frac{k(\sqrt{(a-x)^2+y^2} - l)}{\sqrt{(a-x)^2+y^2}} \\ &- \frac{k(\sqrt{(a+x)^2+y^2} - l)}{\sqrt{(a+x)^2+y^2}} \\ &+ \frac{k(a+x)^2(\sqrt{(a+x)^2+y^2} - l)}{((a+x)^2+y^2)^{\frac{3}{2}}} \\ &+ \frac{k(a-x)^2}{((a-x)^2+y^2)} \\ &- \frac{k(a+x)^2}{((a+x)^2+y^2)} \end{aligned}$$

and we evaluate it at $(0,0)$:

$$\begin{aligned} \frac{\mathrm{d}F_x}{\mathrm{d}x}(0,0) &= -\frac{ka^2a - l}{a^3} + \frac{k(a-l)}{a} - \frac{k(a-l)}{a} + \frac{ka^2(a-l)}{a^3} + \frac{ka^2}{a^2} - \frac{ka^2}{a^2} \\ &= -\frac{ka^2(a-l)}{a^3} + \frac{ka^2(a-l)}{a^3} = 0 \end{aligned}$$

Thus we take $k_x = 0$ as $k_x = \iint U \mathrm{d}x$ as $U = \frac{1}{2}k_x x$.

And we have

And for y:

$$\begin{aligned}\frac{\mathrm{d}F_y}{\mathrm{d}y} =& \frac{ky^2(\sqrt{(a-x)^2+y^2}-l)}{(a-x)^2+y^2)^(3/2)} \\ &+ \frac{ky^2(\sqrt{(a+x)^2+y^2}-l)}{(a+x)^2+y^2)^(3/2)} \\ &- \frac{k(\sqrt{(a-x)^2+y^2}-l)}{\sqrt{(a-x)^2+y^2}} \\ &- \frac{k(\sqrt{(a+x)^2+y^2}-l)}{\sqrt{(a+x)^2+y^2}} \\ &- \frac{ky^2}{(a-x)^2+y^2} \\ &- \frac{ky^2}{(a+x)^2+y^2)}\end{aligned}$$

And we have:

$$\frac{\mathrm{d}F_y}{\mathrm{d}y}(0,0) = -2\frac{k(a-l)}{a}$$

Thus we have:

$$U_y \approx = -\frac{k(a-l)}{a}y^2$$

Moreover, we have

$$k_y = -\frac{2k(a-l)}{a}$$

where $k_y = \iint U \mathrm{d}y$ as $U = \frac{1}{2}k_y y$.

Thus we have

$$U = \frac{1}{2}(k_x^2 + k_y^2)$$

When $a < l$, easy to see that $\frac{\mathrm{d}^2U}{\mathrm{d}^2y} > 0$ from above result, which gives a velocity to the mass, i.e., when $a < l$, U is maximum value, any tiny displacement will make move further away from the equilibrium point.

3. An undamped oscillator has period T. A bit of damping is added, and the period changes to $T\sqrt{1+q^2}$, where q is some constant.

   (a) What is the damping factor β? What is the quality factor Q?

   From the problem we have:

$$T = \frac{2\pi}{\omega_0}, T' = T\sqrt{1+q^2} = \frac{2\pi}{\omega_0}\sqrt{1+q^2} = \frac{2\pi}{\omega}$$

   Thus from T' we can find

$$\begin{aligned}\omega_0 &= \omega\sqrt{1+q^2} \\ \omega_0^2 &= \omega^2(1+q^2)\end{aligned}$$

Notice that

$$\begin{aligned}
\omega^2 &= \omega_0^2 - \beta^2 \\
\beta^2 &= \omega_0^2 - \omega^2 \\
\beta^2 &= \omega^2(1+q^2) - \omega^2 \\
\beta^2 &= \omega(1+q^2-1) \\
\beta^2 &= \omega^2 q^2 \\
\beta &= \omega q
\end{aligned}$$

and for quality factor:

$$Q = \frac{\omega}{2\beta} = \frac{\omega}{2\omega q} = \frac{1}{q}$$

(b) Suppose $q = \frac{1}{2\pi}$. What is the percentage change in the angular frequency? Approximately how many cycles are needed before the amplitude drops by a factor of e? Which effect is more noticeable?

From above, we have

$$\frac{\omega}{\omega_0} = \frac{1}{\sqrt{1+q^2}} = \frac{1}{\sqrt{1+\frac{1}{4\pi^2}}} = 0.9876$$

$$\begin{aligned}
e^{-MT'\beta} &= \frac{1}{e} \\
MT'\beta &= 1 \\
M &= \frac{1}{\beta T'} \\
M &= \frac{1}{\omega q T} \\
M &= \frac{\omega}{\omega q 2\pi} \\
M &= \frac{1}{q 2\pi} \\
M &= \frac{2\pi}{2\pi} \\
M &= 1
\end{aligned}$$

4. Consider an overdamped harmonic oscillator with a driving force $F_\omega \sin(\omega_F t)$.

(a) Find the motion without transients.

We could chose the $t = 0$ so that we have the driving force as $F\cos(\omega t)$. This just make a time shift of hour resould, which does not matter since the system is periodic. Thus we can write the force as $Fe^{i\omega t}$. Given differential equation:

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = \frac{F}{m} e^{i\omega t}$$

such a differential equation have two solution, where the final solution is a linear combination of the two solution. Now we know that the solution have a form:

$$x = Ae^{i(\omega t - \varphi)}$$

Now we just plug it into the equation:

$$
\begin{aligned}
\ddot{x} + 2\beta\dot{x} + \omega_0^2 x &= \frac{F}{m}e^{i\omega t} \\
-A\omega^2 e^{i(\omega t-\varphi)} + 2\beta A i\omega e^{i(\omega t-\varphi)} + \omega_0^2 A e^{i(\omega t-\varphi)} &= \frac{F}{m}e^{i\omega t} \\
(-\omega^2 + 2\beta\omega i + \omega_0^2)Ae^{i(\omega t-\varphi)} &= \frac{F}{m}e^{i\omega t} \\
(-\omega^2 + 2\beta\omega i + \omega_0^2)A &= \frac{F}{m}e^{i\varphi} \\
(-\omega^2 + 2\beta\omega i + \omega_0^2)A &= \frac{F}{m}(\cos\varphi + i\sin\varphi)
\end{aligned}
$$

To solve this we can first separate the real and the imaginary part:

$$
\begin{cases}
(-\omega^2 + \omega_0^2)A = \dfrac{F}{m}\cos\varphi \\
2\beta\omega A = \dfrac{F}{m}\sin\varphi
\end{cases}
$$

Thus, we could have:

$$
\begin{aligned}
\frac{\sin\varphi}{\cos\varphi} &= \frac{2\beta\omega}{-\omega^2 + \omega_0^2} \\
\tan\varphi &= \frac{2\beta\omega}{-\omega^2 + \omega_0^2}
\end{aligned}
$$

Or we could have

$$
\begin{aligned}
(-\omega^2 + 2\beta\omega i + \omega_0^2)A &= \frac{F}{m}e^{i\varphi} \\
(-\omega^2 + 2\beta\omega i + \omega_0^2)(-\omega^2 + 2\beta\omega i + \omega_0^2)^* AA^* &= \frac{FF^*}{mm^*}e^{i\varphi}e^{i\varphi}{}_* \\
[(\omega_0^2 - \omega^2) + 4\beta^2\omega^2]A^2 &= \frac{F^2}{m^2} \\
A^2 &= \frac{F^2}{m^2}\frac{1}{(\omega_0^2 - \omega^2) + 4\beta^2\omega^2} \\
A &= \frac{F}{m}\frac{1}{\sqrt{(\omega_0^2 - \omega^2) + 4\beta^2\omega^2}}
\end{aligned}
$$

(b) Find the motion if the initial position and velocity vanish.

We want to find a solution such that $x(0) = \dot{x}(0) = 0$. Thus we have $x = Ae^{-i\varphi}$. Plug it into the equation:

$$
\begin{aligned}
\ddot{x} + 2\beta\dot{x} + \omega_0^2 x &= \frac{F}{m}e^{i\omega t} \\
-A\omega^2 e^{-i\varphi} &= \frac{F}{m} \\
-A\omega^2 &= \frac{F}{m}e^{i\varphi} \\
-A\omega^2 &= \frac{F}{m}(\cos\varphi + i\sin\varphi)
\end{aligned}
$$

Now we find that the imaginary part is zero in the left side:

$$
\begin{aligned}
\sin\varphi &= 0 \\
\varphi &= 0 \text{ or } \pi
\end{aligned}
$$

Thus we have

$$-A\omega^2 = \pm\frac{F}{m}$$
$$A = \mp\frac{F}{m}\frac{1}{\omega^2}$$

Now if we want to solve it without the time shift, we can have:

$$-A\omega^2 e^{-i\varphi} = \frac{F}{m}e^{-i\frac{\pi}{2\omega}}$$
$$-A\omega^2 = \frac{F}{m}e^{i(\varphi-\frac{\pi}{2\omega})}$$
$$-A\omega^2 = \frac{F}{m}\cos(\varphi-\frac{\pi}{2\omega}) + i\cos(\varphi)$$

Thus we have $\cos(\varphi) = 0$, which give us $\varphi = \frac{\pi}{2}$ or $\frac{3\pi}{2}$. Thus we have:

$$-A\omega^2 = \frac{F}{m}\cos(\varphi-\frac{\pi}{2\omega})$$
$$-A\omega^2 = \frac{F}{m}\sin(\varphi)$$
$$-A\omega^2 = \pm\frac{F}{m}$$

Which give our same result as above. The only difference is a $\frac{T}{4}$ phase shift.

5. Consider a RLC circuit with a resistor with resistance R in series with a capacitor with capacitance C and an inductor with inductance L.

   (a) What is the natural frequency ω and what is the damping factor β?

   We have following equation for LRC:

$$V_R = IR = \dot{Q}R$$
$$V_L = L\dot{I} = \ddot{Q}L$$
$$V_C = \frac{Q}{C}$$

   Now since thsere is not input to the circuit, we have $V = 0$, which gives us:

$$\ddot{Q}L + \dot{Q}R + \frac{Q}{C} = 0$$

   where now we can change this to stander euqation of oscillations:

$$\ddot{Q} + 2\beta\dot{Q} + \omega_0^2 Q = 0 \quad \text{where } \beta = \frac{R}{2L}\ ,\ \omega_0 = \sqrt{\frac{1}{LC}}$$

   and we have

$$\omega = \sqrt{\omega_0^2 - \beta^2}$$

   (b) Suppose this is being driven by a voltage $V(t) = A\cos(\frac{\omega}{2}t) + B\sin(2\omega t)$.

   From the driving force we have:

$$\ddot{Q} + 2\beta\dot{Q} + \omega_0^2 Q = A\cos(\frac{\omega}{2}t) + B\sin(2\omega t)$$

since $(D^2 + 2\beta D + \omega_0^2 I)$ is a linear operator as differential operation is linear. Thus the solution of above equation is just a linear combination of

$$\begin{cases} \ddot{Q} + 2\beta\dot{Q} + \omega_0^2 Q = A\cos(\frac{\omega}{2}t) \\ \ddot{Q} + 2\beta\dot{Q} + \omega_0^2 Q = B\sin(2\omega t) \end{cases}$$

Thus the solution is just:

$$Q = AC_A \cos(\frac{\omega}{2}t - \varphi_A) + BC_B \sin(2\omega t - \varphi_B)$$

where

$$C_A = \frac{1}{\sqrt{(\omega_0^2 - (\frac{\omega}{2})^2)^2 + \beta^2\omega^2}}$$

and

$$C_B = \frac{1}{\sqrt{(\omega_0^2 - 4\omega^2)^2 + 16\beta^2\omega^2}}$$

To find the circuit, than we have:

$$I = \dot{Q} = -\frac{1}{2}A\omega C_A \sin(\frac{\omega}{2}t - \varphi_A) + 2B\omega C_B \cos(2\omega t - \varphi_B)$$