Physics 303/573

Notes on the Harmonic Oscillator Part I

October 4, 2016

1 Undamped, unforced harmonic oscillator

The harmonic oscillator is fundamental in all areas of physics because it characterizes the response of a generic system in equilibrium when it has been disturbed slightly. This is because, by definition, in equilibrium, the force vanishes, and for generic systems, it is a smooth function $F(x)$ of the displacement from equilibrium. Then we can Taylor expand the force, and for sufficiently small displacements, the linear term will dominate; if the equilibrium is (locally) stable, then force will oppose the displacement, and we have

$$F(x) = F(0) + \frac{dF}{dx}x + ... := -kx + ... \tag{1.1}$$

where F is the force, k is the first coefficient in the Taylor expansion, x is some measure of the displacement, and ... indicates the higher order terms that we are ignoring for now. The force obeys Newton's laws of motion, and we call the characteristic response to the force $m\ddot{x}$, where m is the mass if the displacement x is literally the position along a line, but may have another interpretation if x is another measure of the displacement. So to linear order we have:

$$m\ddot{x} = -kx \tag{1.2}$$

We will usually write this as:

$$\ddot{x} + \omega^2 x = 0 \tag{1.3}$$

where

$$\omega = \sqrt{\frac{k}{m}} \tag{1.4}$$

is the characteristic frequency–notice that it has units of inverse time.

1.1 Energy

When $F = F(x)$, we can find the potential energy $U(x)$ such that $F(x) = -\frac{dU}{dx}$; then the total energy $E = T + U$ is conserved:

$$E = T + U(x) = \frac{1}{2}m(\dot{x})^2 + U(x)$$

is conserved because of Newton's laws:

$$\frac{dE}{dt} = \frac{d}{dt}\left(\frac{1}{2}m(\dot{x})^2 + U(x)\right) = m\dot{x}\ddot{x} + \frac{dU}{dx}\dot{x} = \dot{x}(m\ddot{x} - F(x)) = 0 \tag{1.5}$$

In our case

$$U = \frac{1}{2}kx^2 \ . \tag{1.6}$$

1.2 Examples

For a mass on spring, x is really the position along a line, and m is really the mass, and k is called the spring constant.

For a pendulum, we start with the potential energy. We consider a mass μ on a string of length ℓ; we use the angle θ from the vertical as the displacement; then the potential energy is simply the height, which is

$$U(\theta) = \mu g\ell(1 - \cos\theta) = \frac{1}{2}\mu g\ell\theta^2 + ... \tag{1.7}$$

where we have Taylor expanded the potential to be able to compare to (1.6). The kinetic energy is $\frac{1}{2}$ times the mass on the string μ times the linear velocity squared:

$$T = \frac{1}{2}\mu(\ell\dot{\theta})^2 \tag{1.8}$$

Comparing (1.7) to (1.6), we see that for the pendulum, when we use θ to measure the displacement, the spring constant is

$$k = \mu g\ell \tag{1.9}$$

Comparing (1.8) to $T = \frac{1}{2}m(\dot{x})^2$, we find that the "mass" for the pendulum is

$$m = \mu\ell^2 \tag{1.10}$$

The final equation we get is

$$\mu\ell^2\ddot{\theta} = -\mu g\ell\theta \quad \Rightarrow \quad \ddot{\theta} = -\frac{g}{\ell}\theta \tag{1.11}$$

If we hadn't Taylor expanded the potential, we would have found:

$$\ddot{\theta} = -\frac{g}{\ell} \sin\theta \tag{1.12}$$

This illustrates both how we can Taylor expand the potential (or equivalently the force), and how we can use any variable to measure the deviation from equilibrium.

1.3 Momentum

We can rewrite the equation (1.2) if we introduce the momentum $p = m\dot{x}$ as a system of first order differential equations:

$$p = m\dot{x} \quad , \quad \dot{p} = -kx \quad \Rightarrow \quad \begin{pmatrix} \dot{x} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} \frac{1}{m}p \\ -kx \end{pmatrix} \tag{1.13}$$

Recall that it is convenient to rewrite this in terms of a single complex variable. I'll do it in a slightly different way than before; let's write

$$a = x + Aip \tag{1.14}$$

where A is a constant that we need to find. Then

$$\dot{a} = \dot{x} + Ai\dot{p} = \frac{1}{m}p - Aikx = \frac{1}{m}p - Aik(a - Aip) = -Aika + \left(\frac{1}{m} - A^2k\right)p \tag{1.15}$$

which gives us an equation for a if we choose $A^2k = \frac{1}{m}$ or:

$$A = \sqrt{\frac{1}{km}} = \frac{1}{m\omega} \tag{1.16}$$

and hence we have

$$a = x + \frac{i}{m\omega}p = x + \frac{i}{\omega}\dot{x} \tag{1.17}$$

and, since $k(\frac{1}{m\omega}) = \omega$

$$\dot{a} = -i\omega a \tag{1.18}$$

Notice that (1.17) makes sense dimensionally: p has dimensions $\frac{[M][L]}{[T]}$, and ω has dimensions $\frac{1}{[T]}$, so the ratio $\frac{p}{m\omega}$ has dimensions of length. The equation (1.18) is the easiest differential equation there is; its solution is

$$a = a_0\, e^{-i\omega(t-t_0)} = x_{max}\, e^{-i(\omega t+\phi)} \tag{1.19}$$

where x_{max} is the maximum displacement and ϕ is an arbitrary phase angle. Using one of the many amazing formulae that Euler discovered, (following, in this case, the work of de Moivre), namely $e^{i\alpha} = \cos\alpha + i\sin\alpha$, we can find the motion and the momentum simply by taking the real and imaginary parts of (1.19):

$$x(t) = x_{max}\, \cos(\omega t + \phi) \tag{1.20}$$

and

$$\frac{p}{m\omega} = -x_{max}\, \sin(\omega t + \phi) \quad \Rightarrow \quad p(t) = -m\omega x_{max}\, \sin(\omega t + \phi) \tag{1.21}$$

1.4 Angular frequency, frequency, period

The quantity ω is called the angular frequency. It measure the number of radians per unit time that the motion describes–see (1.20). The period is the time it takes to execute one cycle–it is clearly

$$T = \frac{2\pi}{\omega} \tag{1.22}$$

as that is the time it takes to return the cosine and sine to its original value. We also can discuss the frequency f–the number of cycles per unit time. This is just

$$f = \frac{1}{T} = \frac{\omega}{2\pi} \tag{1.23}$$

.

1.5 Phase space

In the complex plane, if we plot $a(t)$, we get a circle of radius x_{max}; as time progresses, the angle decreases by $-\omega t$, and hence $a(t)$ moves clockwise with angular frequency ω. Since the real part of a is the position, and the imaginary part is the momentum (up to a rescaling by $\frac{1}{m\omega}$), we can think of this as a plot of the motion in x, p space–this is called phase space.

1.6 Energy in terms of a

We know that the motion preserves the radius. It is a simple calculation to show that

$$E = \frac{1}{2}k(aa^*) = \frac{1}{2}k(x^2 + \frac{1}{\omega^2}(\dot{x})^2) = \frac{1}{2}kx^2 + \frac{1}{2}m(\dot{x})^2 \tag{1.24}$$

where a^* is the complex conjugate of a and we have used (1.17). Thus the fact that the radius of the circle in the complex plane is constant is just the conservation of energy. When we have friction, we expect the mass to spiral in to the center as it loses energy.

1.7 Three dimensions

Recall that if we can find coordinates such that Newton's laws simply become three one-dimensional systems, we say that the system is separable.

A nontrivial example of a separable system is the harmonic oscillator. Consider a potential

$$U = \frac{1}{2}\sum_i k_i x^i x^i \equiv \frac{m}{2}\sum_i \omega_i^2 x^i x^i \tag{1.25}$$

where k_i are different spring constants for the different directions, and ω_i are the corresponding frequencies. Then

$$\vec{F} = -\hat{x}_i k_i x^i = m\hat{x}_i \ddot{x}^i \quad \Rightarrow \quad \ddot{x}^i + \omega_i^2 x^i = 0 \tag{1.26}$$

that is,

$$\ddot{x} + \omega_x^2 x = 0 \quad , \quad \ddot{y} + \omega_y^2 y = 0 \quad , \quad \ddot{z} + \omega_z^2 z = 0 \tag{1.27}$$

We simply solve each component separately and choose boundary conditions for each. Even though this is very simple, the resulting motion can be quite complex. In Figure 1, the motion traced out by $x = cos(\omega_x t + \phi_x)$ and $y = cos(\omega_y t + \phi_y)$ is shown for several different choices of ω_x, ω_y and ϕ_x, ϕ_y.

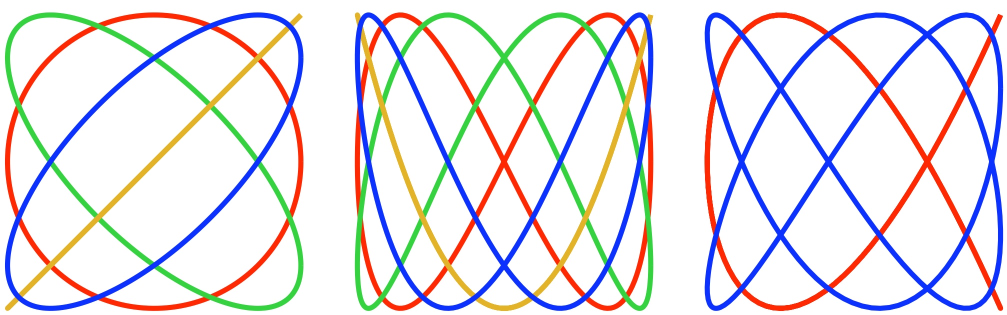

Figure 1: Lissajous figures arising from the motion of a particle in a two-dimensional harmonic oscillator with different frequencies and phases:
a) $\omega_y = \omega_x$ and $\phi_y = \phi_x$, $\phi_y = \phi_x + \frac{\pi}{4}$, $\phi_y = \phi_x + \frac{\pi}{2}$, $\phi_y = \phi_x + \frac{3\pi}{4}$
b) $\omega_y = 2\omega_x$ and $\phi_y = \phi_x$, $\phi_y = \phi_x + \frac{\pi}{4}$, $\phi_y = \phi_x + \frac{\pi}{2}$, $\phi_y = \phi_x + \frac{3\pi}{4}$
c) $\omega_y = \frac{3}{2}\omega_x$ and $\phi_y = \phi_x$, $\phi_y = \phi_x + \frac{\pi}{3}$

1.8 Constant External Force

Though this section is about unforced harmonic motion, adding a constant force is so trivial that we can look at it. For example, in one dimension, if we have

$$F = -kx + F_0 \tag{1.28}$$

it is clear that the effect is simply to shift the equilibrium point by $x_0 = \frac{F_0}{k}$, and hence our whole previous discussion goes through. This means that we can consider weights that move up and down while hanging on springs in the presence of gravity.

2 Damped Harmonic Oscillator

We now consider a system with a frictional term proportional to the velocity; for low enough velocities, this is a good approximation. Again, if the force is a smooth function of the velocity, this is the leading approximation in a Taylor expansion. Then our equation of motion becomes

$$m\ddot{x} + b\dot{x} + kx = 0 \tag{2.29}$$

where b is the frictional constant with units of $\frac{[M]}{[T]}$. We can rewrite this as

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0 \tag{2.30}$$

where $b = 2m\beta$. We will solve this in two ways–first using the same method as in (1.13) and (1.14), and then using an operator method.

2.1 Solution using a single complex variable

First, we rewrite the equation (2.30) in terms $v = \frac{p}{m} = \dot{x}$ (we used p above, but since we have scaled m out in (2.30), it is easier to use v as a system of first order differential equations:

$$\dot{x} = v \quad , \quad \dot{v} = -\omega_0^2 x - 2\beta v \quad \Rightarrow \quad \begin{pmatrix} \dot{x} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} v \\ -2\beta v - \omega_0^2 x \end{pmatrix} \tag{2.31}$$

It is convenient to rewrite this in terms of a single complex variable a; let's write

$$a = x + Aiv \tag{2.32}$$

where A is a constant that we need to find (it has a factor of $\frac{1}{m}$ scaled out compared to the A we had in (1.14)). Then:

$$\begin{aligned}\dot{a} = \dot{x} + Ai\dot{v} = v - Ai(2\beta v + \omega_0^2 x) &= v - Ai(2\beta v + \omega_0^2(a - Aiv)) \\ &= v(1 - 2i\beta A - \omega_0^2 A^2) - Ai\omega_0^2 a \end{aligned} \tag{2.33}$$

so this becomes a first order differential equation for a if we choose A to solve the quadratic equation

$$\omega_0^2 A^2 + 2i\beta A - 1 = 0 \quad \Rightarrow \quad A_\pm = \frac{-i\beta \pm \omega}{\omega_0^2} \tag{2.34}$$

where the frequency ω is shifted from the undamped frequency ω_0 by:

$$\omega = \sqrt{\omega_0^2 - \beta^2} \tag{2.35}$$

Notice that A is in general complex–this is very different from the undamped case. We now get two possible equations:

$$\dot{a}_\pm = -iA_\pm \omega_0^2 a_\pm \quad \text{where} \quad a_\pm = x + iA_\pm v = x + \frac{\beta \pm i\omega}{\omega_0^2} v \tag{2.36}$$

There are two solutions $a_\pm(t)$

$$a_\pm(t) = |a_{max}| e^{-\beta t} e^{\mp i(\omega t + \phi)} \tag{2.37}$$

where $|a|, \phi$ are integration constants, and the true frequency ω is less than the frequency ω_0 that the undamped system would have. Observe that if the damping is too strong, $\beta \geq \omega_0$, then $A_\pm$ are both pure imaginary.

• When the damping isn't too strong, $\beta < \omega_0$, the solution oscillates with angular frequency $\omega = \sqrt{\omega_0^2 - \beta^2}$ and decays exponentially with a decay constant β. This case is called *underdamped*. In this case, we have chose the integration constants so that the solutions $a_\pm(t)$ are complex conjugates of each other.

• When the damping is stronger, $\beta > \omega_0$, then we get decay with two different exponentials with time constants $\beta \pm q$: $q = i\omega = \sqrt{\beta^2 - \omega_0^2}$. This case is called *overdamped*.

• Finally, $\beta = \omega_0$ is special. This case is called *critically damped*.

We now analyze the solutions for the different cases.

2.1.1 Underdamping: $\beta < \omega_0$

In this case, $a_- = (a_+)^*$, but writing things in terms of $a_\pm$ gives us more general results. We take linear combinations of a_+ and a_-; the velocity is simply:

$$v(t) = \frac{\omega_0^2}{\omega}\left(\frac{a_+ - a_-}{2i}\right) = -\frac{\omega_0^2}{\omega}|a_{max}| e^{-\beta t} \sin(\omega t + \phi) \tag{2.38}$$

Then the position is

$$x(t) = \frac{1}{2}(a_+ + a_-) - \frac{\beta}{\omega_0^2}v(t) = |a_{max}|e^{-\beta t}(\cos(\omega t + \phi) + \frac{\beta}{\omega}\sin(\omega t + \phi)) \tag{2.39}$$

If we write $\frac{\beta}{\omega} = \tan\psi$ for some angle $0 \leq \psi < \frac{\pi}{2}$ with $\psi = 0$ in the undamped case and $\psi \to \frac{\pi}{2}$ as we approach critical damping, we find the standard form

$$x(t) = x_{max}e^{-\beta t}\cos(\omega t + \varphi) \quad \text{where} \quad x_{max} = \frac{|a_{max}|}{\cos\psi} \text{ and } \varphi = \phi - \psi \tag{2.40}$$

and

$$v(t) = -\omega x_{max}e^{-\beta t}(\sin(\omega t + \varphi) + \tan\psi\cos(\omega t + \varphi)) \tag{2.41}$$

The phase space variable that best describes the motion is the undamped variable

$$a_0(t) = x(t) + \frac{i}{\omega_0}v(t) = |a_{max}|e^{-\beta t}(\cos(\omega t + \phi) + \frac{\beta - i\omega_0}{\omega}\sin(\omega t + \phi)) \tag{2.42}$$

because then the energy is simply $\frac{1}{2}(mv^2 + kx^2) = \frac{1}{2}m(v^2 + \omega_0^2x^2) = \frac{1}{2}m\omega_0^2a_0a_0^*$.

In Figure 1, you can see the trajectories in phase space for different amounts of damping.

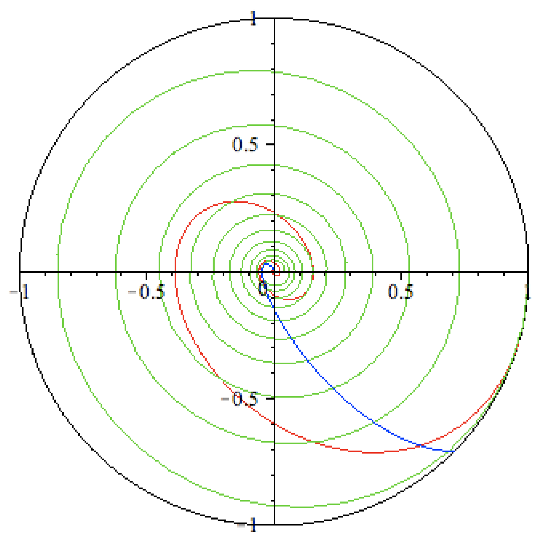

Figure 2: Trajectories in phase space with position x and velocity v on the x,y-axes, respectively. All trajectories have $\omega = 1$, so $\omega_0 = \sqrt{1+\beta^2} > \beta$, that is, the system is underdamped. Color code: $\beta = 0$, $\beta = .05$, $\beta = .3$, $\beta = 1$.

2.1.2 Overdamping: $\beta > \omega_0$

In this case, $A_\pm$ are pure imaginary and $a_\pm$ are real, and it is useful to rewrite (2.36) and (2.37) as:

$$a_\pm = x + \frac{\beta \pm q}{\omega_0^2} v = a_\pm(0) e^{-(\beta \pm q)t} \tag{2.43}$$

Now we can extract $v(t)$ as before:

$$v(t) = \frac{\omega_0^2}{q} \left(\frac{a_+ - a_-}{2} \right) = \frac{\omega_0^2}{2q} \left(a_+(0) e^{-(\beta+q)t} - a_-(0) e^{-(\beta-q)t} \right) \tag{2.44}$$

and

$$x(t) = \left(\frac{a_+ + a_-}{2} \right) - \frac{\beta}{\omega_0^2} v(t) = \frac{a_+(0)}{2q} e^{-(\beta+q)t}(q - \beta) + \frac{a_-(0)}{2q} e^{-(\beta-q)t}(q + \beta) \tag{2.45}$$

These solutions decay with the sum of two exponentials, one faster with decay constant $\beta + q$, and on slower with decay constant $\beta - q$ unless the initial conditions are tuned to give a pure exponential decay with one or the other decay.

The critically damped case is found by carefully taking the limit $q \to 0$. However, it is simpler to find it using another method.

2.2 Solution using operator factorization

We will now solve the damped harmonic oscillator another way; this method is a bit more sophisticated but it can be used for more general systems, so it is worth learning.

We start by rewriting the differential equation (2.30) in an operator notation; we introduce the operator $D = \frac{d}{dt}$ and write

$$(D^2 + 2\beta D + \omega_0^2)x = 0 \tag{2.46}$$

We can rewrite this in factored form; depending on whether we are in the underdamped, critically damped, or overdamped situation, we write

$$\begin{array}{lll} \text{Underdamped:} & \beta < \omega_0 & (D + \beta + i\omega)(D + \beta - i\omega)x = 0 \\ \text{Critically damped} & \beta = \omega_0 & (D + \beta)^2 x = 0 \\ \text{Overdamped} & \beta > \omega_0 & (D + \beta + q)(D + \beta - q)x = 0 \end{array} \tag{2.47}$$

where recall (2.35) and the definitions below imply: $\beta^2 + \omega^2 = \beta^2 - q^2 = \omega_0^2$. Notice that this method can be used for higher derivative equations and in special cases for equations that do not have constant coefficients.

2.2.1 A general method

For a general equation of the form

$$(D+A)(D+B)x = 0 \tag{2.48}$$

the solution can be found as follows; suppose $x_A(t)$ is any solution to $(D+A)x_A = 0$; then the most general solution has the form $x(t)$ where $x(t)$ is a solution to

$$(D+B)x = x_A \tag{2.49}$$

Applying this to the case when A and B are constants, if $A \neq B$ then using $(D+A)x_A = 0 \;\Rightarrow\; Dx_A = -Ax_A$, we have

$$(D+B)x_A = (B-A)x_A \tag{2.50}$$

and hence, if $(D+B)x_B = 0$,

$$(D+A)((D+B)(x_A+x_B)) = (D+A)((B-A)x_A) = (B-A)(D+A)x_A = 0 \tag{2.51}$$

So the most general solution has the form $x(t) = x_A + x_B$ where $x_{A,B}$ solve $(D+A)x_A = 0,\ (D+B)x_B = 0$ (in other words, $\dot{x}_A + Ax_A = \dot{x}_B + Bx_B = 0$). Explicitly, we have:

$$x(t) = x_A(0)e^{-At} + x_B(0)e^{-Bt} \quad , \quad v(t) = \dot{x}(t) = -(Ax_A(0)e^{-At} + Bx_B(0)e^{-Bt}) \tag{2.52}$$

where $x_A(0)$ and $x_B(0)$ are arbitrary integration constants which are related to the initial position $x(0)$ and the initial velocity $v(0)$ by

$$x(0) = x_A(0) + x_B(0) \quad , \quad v(0) = -Ax_A(0) - Bx_B(0) \tag{2.53}$$

and hence

$$x_A(0) = \frac{Bx(0)+v(0)}{B-A} \quad , \quad x_B(0) = \frac{Ax(0)+v(0)}{A-B} \tag{2.54}$$

When $A = B$, then if x_A is a solution of $(D+A)x_A = 0$,

$$(D+A)(tx_A) = x_A + t(D+A)x_A = x_A \tag{2.55}$$

and hence, satisfies (2.49) (with $B = A$); thus in this case the most general solution is

$$x(t) = x(0)e^{-At} + t(v(0) + Ax(0))e^{-At} \tag{2.56}$$

where we have expressed the integration constants in terms of the initial position $x(0)$ and the initial velocity $v(0)$.

2.3 Application to the damped harmonic oscillator

It is straightforward to apply the formalism that we have developed to the damped harmonic oscillator.

For the underdamped case, $A, B = \beta \pm i\omega$, so we when we apply (2.52) we choose the integration constants to make $x(t)$ and $v(t)$ real:

$$x(t) = x_{max} e^{-\beta t} \cos(\omega t + \varphi) \quad , \quad v(t) = -x_{max} e^{-\beta t} (\beta \cos(\omega t + \varphi) + \omega \sin(\omega t + \varphi)) \tag{2.57}$$

where we can identify

$$x(0) = x_{max} \cos\varphi \quad , \quad v(0) = -x_{max}(\beta \cos\varphi + \omega \sin\varphi) \tag{2.58}$$

This agrees with (2.40), (2.41) above. For the overdamped case, $A, B = \beta \pm q$, so (2.52) gives:

$$\begin{aligned} x(t) &= x_+(0) e^{-(\beta+q)t} + x_-(0) e^{-(\beta-q)t} \\ v(t) &= -\left[x_+(0)(\beta + q) e^{-(\beta+q)t} + x_-(0)(\beta - q) e^{-(\beta-q)t} \right] \end{aligned} \tag{2.59}$$

You should check that this agrees with (2.45),(2.44) (you need to use the definition $q^2 = \beta^2 - \omega_0^2$ and find the relation between the integration constants).

Finally, for the critically damped case, $A = \beta = \omega_0$, and (2.56) implies

$$x(t) = x(0) e^{-\beta t} + t[v(0) + \beta x(0)] e^{-\beta t} \quad , \quad v(t) = v(0) e^{-\beta t} - \beta t [v(0) + \beta x(0)] e^{-\beta t} \tag{2.60}$$

2.4 Quality factor Q

There exist two definitions of Q in the literature; these agree for small damping, but differ when the damping is comparable to the frequency.

The first definition is simply:

$$Q = \frac{\omega}{2\beta} \tag{2.61}$$

which clearly goes to infinity as the damping $\beta \to 0$.

The second definition is 2π times the ratio of the total stored energy to the energy lost in one cycle:

$$Q' = \frac{2\pi E_{max}}{\Delta E} \tag{2.62}$$

One can evaluate this using the integral of $\dot{E} = -c\dot{x}^2$. However, there is a much easier way to do this: Both the velocity and the position have the form $e^{-\beta t} f(t)$ for some periodic function $f(t) = f(t + \frac{2\pi}{\omega})$, and hence the energy has the form

$$E(t) = \frac{1}{2} m v(t)^2 + \frac{1}{2} k x(t)^2 \ \equiv \ e^{-2\beta t} g(t) \quad , \quad g(t) = g(t + \frac{2\pi}{\omega}) \tag{2.63}$$

for some periodic function $g(t)$. Hence the second definition gives

$$Q' = \frac{2\pi E(0)}{E(0) - E(\frac{2\pi}{\omega})} = \frac{2\pi g(0)}{g(0) - e^{-\frac{4\pi\beta}{\omega}} g(\frac{2\pi}{\omega})} = \frac{2\pi}{1 - e^{-\frac{4\pi\beta}{\omega}}} \tag{2.64}$$

where g drops out because it is periodic. For sufficiently small β, we can Taylor expand Q' and find it agrees with Q:

$$\frac{2\pi}{1 - e^{-\frac{4\pi\beta}{\omega}}} \approx \frac{2\pi}{1 - (1 - \frac{4\pi\beta}{\omega})} = \frac{\omega}{2\beta} \tag{2.65}$$

Notice that because $E \propto e^{-2\beta t}$ and because $g(t)$ is periodic, we can interpret Q' as

$$Q' = \frac{2\pi E(t)}{E(t) - E(t + \frac{2\pi}{\omega})} = \frac{2\pi}{1 - e^{-\frac{4\pi\beta}{\omega}}} \tag{2.66}$$

for any time t–that is, it is 2π divided by the fraction of the energy lost during a cycle, and this is independent of the time when it is measured.

2.5 An Example

In lecture, I will consider a couple of real life examples: a weight on a rubber band and a ball rolling in a wok. I will briefly describe the measurements and how we can compute ω, β, ω_0 and Q from them. The easiest measurement we can make without fancy equipment is to time a certain number of cycles; in practice, the number of cycles we choose will depend on how short the period is and how much damping there is. Suppose that N cycles take a time T_N; then the period is

$$T = \frac{T_N}{N} \tag{2.67}$$

and the angular frequency is

$$\omega = \frac{2\pi}{T} = \frac{2\pi N}{T_N} \tag{2.68}$$

We will measure the damping by counting the number of cycles M that it takes for the amplitude to drop to $\frac{1}{e}$ of the original amplitude. This corresponds corresponds to a time that is M times the period T, and hence we have:

$$e^{-MT\beta} = \frac{1}{e} \quad \Rightarrow \quad MT\beta = 1 \quad \Rightarrow \quad \beta = \frac{1}{MT} = \frac{N}{MT_N} \tag{2.69}$$

The undamped frequency ω_0 can now be calculated:

$$\omega_0 = \sqrt{\omega^2 + \beta^2} = \frac{N}{T_N}\sqrt{4\pi^2 + \frac{1}{M^2}} \tag{2.70}$$

Finally, the quality factor Q is

$$Q = (\omega)\left(\frac{1}{2\beta}\right) = \left(\frac{2\pi}{T}\right)\left(\frac{MT}{2}\right) = \pi M \tag{2.71}$$

that is, just π times the number of cycles for the amplitude to drop to $\frac{1}{e}$.