Homework 8

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1. Two particles are connected by an ideal spring with spring constant k and unstretched length $l = 0$. They both slide along a frictionless ramp given by the equation $z = \alpha y$.

   (a) Write down the Lagrangian for the system in terms of $\vec{r}_1$ and $\vec{r}_2$, imposing constrants with Lagrange multipliers

$$L = \frac{1}{2}m_1\dot{\vec{r}}_1^2 + \frac{1}{2}m_2\dot{\vec{r}}_2^2 - mgz_1 - mgz_2 - \lambda_1(z_1 - \alpha y_2) - \lambda_2(z_2 - \alpha y_2) - \frac{1}{2}k(\vec{r}_1 - \vec{r}_2)^2$$

   (b) Rewrite the Lagrangian in terms of $\vec{r}_{\text{cm}}$ and $\vec{r} = \vec{r}_1 - \vec{r}_2$.
   First we defined:

$$\vec{r}_{\text{cm}} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}\ ,\ M = m_1 + m_2\ ,\ \mu = \frac{m_2 m_2}{m_1 + m_2}$$

   Thus, we can have

$$\vec{r}_1 = \vec{r}_{\text{cm}} + \frac{m_2\vec{r}}{M}$$
$$\vec{r}_2 = \vec{r}_{\text{cm}} - \frac{m_1\vec{r}}{M}$$

   Thus, we have:

$$\begin{aligned} L &= \frac{1}{2}m_1\dot{\vec{r}}_1^2 + \frac{1}{2}m_2\dot{\vec{r}}_2^2 - mgz_1 - mgz_2 - \lambda_1(z_1 - \alpha y_2) - \lambda_2(z_2 - \alpha y_2) - U \\ &= \frac{1}{2}m_1(\vec{r}_{\text{cm}} + \frac{m_2\vec{r}}{M})^2 + \frac{1}{2}m_2(\vec{r}_{\text{cm}} - \frac{m_1\vec{r}}{M})^2 - mgz_1 - mgz_2 - \lambda_1(z_1 - \alpha y_2) - \lambda_2(z_2 - \alpha y_2) - U \\ &= \frac{1}{2}M\dot{\vec{r}}_{\text{cm}}^2 + \frac{1}{2}\mu\dot{\vec{r}}^2 - mgz_1 - mgz_2 - \lambda_1(z_1 - \alpha y_2) - \lambda_2(z_2 - \alpha y_2) - \frac{1}{2}k\vec{r}^2 \end{aligned}$$

   (c) Eliminate the Lagrange multipliers and use the constraints to eliminate $z = z_1 - z_2$ and $z_c m$.
   From above Lagrange we have:

$$L = \frac{1}{2}M(\dot{x}_{\text{cm}}^2 + \dot{y}_{\text{cm}}^2 + \dot{z}_{\text{cm}}^2) + \frac{1}{2}\mu(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz_1 - mgz_2 - \lambda_1(z_1 - \alpha y_)) - \lambda_2(z_2 - \alpha y_2) - U$$

   Now we want to find z_{cm}:

$$z_{\text{cm}} = \frac{m_1 z_1 + m_2 z_2}{M} = \frac{m_1\alpha y_1 + m_2\alpha y_2}{M}$$
$$z = z_1 - z_2 = \alpha y_1 - \alpha y_2 = \alpha y$$

Therefore, we plug in $z = \alpha y$

$$\begin{aligned}
L &= \frac{1}{2}M(\dot{x}_{\text{cm}}^2 + \dot{y}_{\text{cm}}^2 + (\frac{m_1\alpha\dot{y}_1 + m_2\alpha\dot{y}_2}{M})^2) + \frac{1}{2}\mu(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mg\alpha y_1 - mg\alpha y_2 - U \\
&= \frac{1}{2}M(\dot{x}_{\text{cm}}^2 + \dot{y}_{\text{cm}}^2 + \alpha^2\dot{y}_{\text{cm}}^2) + \frac{1}{2}\mu(\dot{x}^2 + \dot{y}^2 + \alpha^2\dot{y}^2) - m_1 g\alpha y_1 - m_2 g\alpha y_2 - U \\
&= \frac{1}{2}M(\dot{x}_{\text{cm}}^2 + \dot{y}_{\text{cm}}^2 + \alpha^2\dot{y}_{\text{cm}}^2) + \frac{1}{2}\mu(\dot{x}^2 + \dot{y}^2 + \alpha^2\dot{y}^2) - \alpha g M y_{\text{cm}} - \frac{1}{2}k(x^2 + y^2 + \alpha^2 y^2)
\end{aligned}$$

(d) Find the Euler-Lagrange equation for the resulting system

From above we have:

$$L = \frac{1}{2}M(\dot{x}_{\text{cm}}^2 + \dot{y}_{\text{cm}}^2 + \alpha^2\dot{y}_{\text{cm}}^2) + \frac{1}{2}\mu(\dot{x}^2 + \dot{y}^2 + \alpha^2\dot{y}^2) - \alpha g M y_{\text{cm}} - \frac{1}{2}k(x^2 + y^2 + \alpha^2 y^2)$$

And for Euler-Lagrange equation we have:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q}$$

For $\vec{r}_{\text{cm}}$, we have:

$$\begin{aligned}
x_{\text{cm}}:\ \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}_{\text{cm}}} &= \frac{\partial L}{\partial x_{\text{cm}}} \\
M\ddot{x}_{\text{cm}} &= 0 \\
y_{\text{cm}}:\ \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{y}_{\text{cm}}} &= \frac{\partial L}{\partial y_{\text{cm}}} \\
-M(1+\alpha^2)\ddot{y}_{\text{cm}} &= \alpha g M
\end{aligned}$$

For $\vec{r}$, we have:

$$\begin{aligned}
x:\ \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}} &= \frac{\partial L}{\partial x} \\
\mu\ddot{x} &= -kx \\
y:\ \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{y}} &= \frac{\partial L}{\partial y} \\
\mu(1+\alpha^2)\ddot{y} &= -(\alpha^2+1)ky
\end{aligned}$$

(e) Write down the most general solution for $\vec{r}_{\text{cm}}(t)$ and $\vec{r}(t)$

For $\vec{r}_{\text{cm}}$, we have:

$$\begin{aligned}
x_{\text{cm}} &= v_x(0)x + x(0) \\
y_{\text{cm}} &= -\frac{\alpha a}{1+\alpha^2}(\tilde{v}_y(0)y + \tilde{y}(0))
\end{aligned}$$

For $\vec{r}$, we have:

$$\begin{aligned}
x &= C_{x1}e^{\sqrt{\frac{k}{\mu}}ix} + C_{x2}e^{-\sqrt{\frac{k}{\mu}}ix} \\
y &= C_{y1}e^{\sqrt{\frac{k}{\mu}}iy} + C_{y2}e^{-\sqrt{\frac{k}{\mu}}iy}
\end{aligned}$$

where

$$\begin{aligned}
z_{cm} &= \alpha y_{\text{cm}} \\
z &= \alpha y
\end{aligned}$$

2. Consider a Lagrangian

$$L = \frac{1}{2}\frac{m(\dot{x}^2+\dot{y}^2)}{(1+x^2+y^2)^2}$$

(a) Show that this is invariant under rotations about the z-axis. Find the corresponding Noether charge.

We want to find

$$Q = \frac{\partial L}{\partial \dot{q}_i} R_i - K$$

is conserved.

For rotation, we have

$$\begin{aligned}\delta x &= \alpha y \\ \delta y &= -\alpha x\end{aligned}$$

Which give us R

$$\begin{aligned}R_x &= y \\ R_y &= -x\end{aligned}$$

If we have $\delta L = 0$, we can use $K = 0$

$$\begin{aligned}Q &= \frac{\partial L}{\partial \dot{x}} R_x + \frac{\partial L}{\partial \dot{y}} R_y \\ &= \frac{m\dot{x}y}{(1+x^2+y^2)^2} - \frac{m\dot{y}x}{(1+x^2+y^2)^2} \\ &= \frac{m(\dot{x}y+\dot{y}x)}{(1+x^2+y^2)^2} \\ &= -\frac{L_z}{(1+x^2+y^2)^2}\end{aligned}$$

is conserved, which give us L_z is conserved, i.e., invariant under rotations.

(b) Rewrite L in planar polar coordinates r, θ. Show that the results of part a are now obvious.

$$L = \frac{1}{2}\frac{m(\dot{r}^2+r^2\dot{\theta}^2)}{(1+r^2)^2}$$

To check the results we just plug it into Lagrangian's equation for θ:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\frac{mr^2\dot{\theta}}{(1+r^2)^2} &= 0 \\ \frac{mr^2\dot{\theta}}{(1+r^2)^2} &= \mathrm{Const}\end{aligned}$$

where $mr^2\dot{\theta} = L_z$.

(c) A mush less obvious symmetry is the following:

$$\delta r = \alpha(1+r^2)\cos\theta\ ,\ \delta\theta = \alpha(r-\frac{1}{r})\sin\theta$$

Calculate the Noether charge of this symmetry assuming that $\delta L = 0$. For 20 points extra credit, prove taht this is actually a symmetry.

First we still want to use

$$L = \frac{1}{2}\frac{m\dot{r}^2}{(1+r^2)^2}$$

Form $\delta r = \alpha(1+r^2)\cos\theta$, $\delta\theta = \alpha(r - \frac{1}{r})\sin\theta$, we have

$$R_r = (1+r^2)\cos\theta \ , \ R_\theta = (r - \frac{1}{r})\sin\theta$$

Since $\delta L = 0$, we find $K = 0$
Now we have:

$$\begin{aligned} Q_r &= \frac{\partial L}{\partial \dot{r}}(1+r^2)\cos\theta \\ &= \frac{m\dot{r}}{(1+r^2)^2}(1+r^2)\cos\theta \\ &= \frac{m\dot{r}\cos\theta}{(1+r^2)} \\ Q_\theta &= \frac{\partial L}{\partial \dot{\theta}}(r - \frac{1}{r})\sin\theta \\ &= \frac{mr^2\dot{\theta}}{(1+r^2)^2}(r - \frac{1}{r})\sin\theta \\ &= \frac{mr\dot{\theta}(r^2-1)\sin\theta}{(1+r^2)^2} \end{aligned}$$

To show this is a symmetry, we find the change in L:

$$\delta L = \sum(\frac{\partial L}{\partial q_i}\delta q_i + \frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i)$$

For r, we have:

$$\begin{aligned} \frac{\partial L}{\partial r} &= \frac{(1+r^2)mr\dot{\theta}^2 - 2rm(\dot{r}^2 + r^2\dot{\theta}^2)(1+r^2)}{(1+r^2)^4} \\ \frac{\partial L}{\partial \dot{r}} &= \frac{m\dot{r}}{(1+r^2)^2} \\ \delta r &= \alpha(1+r^2)\cos\theta \\ \delta\dot{r} &= \frac{\mathrm{d}}{\mathrm{d}t}\alpha(1+r^2)\cos\theta = 2\alpha r\dot{r}\cos\theta - \alpha(1+r^2)\dot{\theta}\sin\theta \end{aligned}$$

For θ, we have:

$$\begin{aligned} \frac{\partial L}{\partial \theta} &= 0 \\ \frac{\partial L}{\partial \dot{\theta}} &= \frac{mr^2\dot{\theta}}{(1+r^2)^2} \\ \delta\theta &= \alpha(r - \frac{1}{r})\sin\theta \\ \delta\dot{\theta} &= \alpha(1 + \frac{1}{r^2})\dot{r}\sin\theta + \alpha(r - \frac{1}{r})\dot{\theta}\cos\theta \end{aligned}$$

Now we want to plug this in:

$$\begin{aligned}
\delta L =&\frac{\partial L}{\partial r}\delta r + \frac{\partial L}{\partial \dot{r}}\delta \dot{r} + \frac{\partial L}{\partial \theta}\delta\theta + \frac{\partial L}{\partial \dot{\theta}}\delta\dot{\theta} \\
=&(\frac{(1+r^2)mr\dot{\theta}^2 - 2rm(\dot{r}^2 + r^2\dot{\theta}^2)(1+r^2)}{(1+r^2)^4)}(\alpha(1+r^2)\cos\theta) \\
&+(\frac{m\dot{r}}{(1+r^2)^2})(2\alpha r\dot{r}\cos\theta - \alpha(1+r^2)\dot{\theta}\sin\theta) \\
&+(\frac{mr^2\dot{\theta}}{(1+r^2)^2})(\alpha(1+\frac{1}{r^2})\dot{r}\sin\theta + \alpha(r-\frac{1}{r})\dot{\theta}\cos\theta) = & 0
\end{aligned}$$

There will be lots of cancellation and end up with zero, which means there will no change for L, which means it is symmetry.

3. A planet with angular momentum L_z and reduced mass μ is orbiting around a sun such that the total mass is M.

   (a) Write down the effective potential U_{eff}.

   Let say we have a potential in form of gravity:

   $$U = -\frac{\xi}{r}$$

   where ξ is a constent. In gravity, $\xi = Gm_1m_2$.
   Now we find a effective potential for a central force:

   $$U_{\text{eff}} = \frac{L_z^2}{2\mu r^2} + U(r) = \frac{L_z^2}{2\mu r^2} - \frac{\xi}{r}$$

   (b) Sketch U_{eff}.

   (c) What is the radius and energy of a circular orbit?

For a circular, we want to have U_{eff} be a minmum:

$$\begin{aligned}\frac{\partial U_{\text{eff}}}{\partial r} &= 0\\ -\frac{L_z^2}{\mu r^3} + \frac{\xi}{r^2} &= 0\\ \frac{L_z^2}{\mu r^3} &= \frac{\xi}{r^2}\\ L_z^2 &= r\xi\mu\\ \frac{L_z^2}{\xi\mu} &= r\end{aligned}$$

To find the energy, we just plug it in:

$$\begin{aligned}E &= U + T\\ &= \frac{1}{2}\mu\dot{r}^2 + \frac{L_z^2}{2\mu r^2} - \frac{\xi}{r}\\ &= \frac{\xi^2\mu}{2L_z^2} - \frac{\xi^2\mu}{L_z^2}\\ &= -\frac{\xi^2\mu}{2L_z^2}\end{aligned}$$

(d) What is the frequenct of small oscillations around the circular orbit?

Knowing the radius, we can find the period by using the Kepler's laws:

$$\begin{aligned}2\mu A &= TL_z\\ 2\mu\pi\frac{L_z^4}{\xi^2\mu^2} &= TL_z\\ \frac{2\pi L_z^3}{\xi^2\mu} &= T\end{aligned}$$

4. We found the orbit of a mass attracted by gravity to a central sun in polar coordinates:

$$r(\theta) = \frac{\alpha}{1 + \epsilon\cos\theta}$$

(a) Rewrite the cartesian coordinates x, y.

Easy to see that

$$r = \sqrt{x^2 + y^2}\ ,\ \cos\theta = \frac{x}{\sqrt{x^2 + y^2}}$$

Now we just plug in:

$$\begin{aligned}\sqrt{x^2 + y^2} &= \frac{\alpha}{1 + \epsilon\frac{x}{\sqrt{x^2+y^2}}}\\ \sqrt{x^2 + y^2} &= \frac{\alpha\sqrt{x^2 + y^2}}{\sqrt{x^2 + y^2} + \epsilon x}\\ 1 &= \frac{\alpha}{\sqrt{x^2 + y^2} + \epsilon x}\\ \alpha &= \sqrt{x^2 + y^2} + \epsilon x\end{aligned}$$

(b) Show that when $\epsilon \in [0, 1)$ it can be written in the From

$$\frac{(x - x_0)^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\begin{aligned}
\alpha &= \sqrt{x^2 + y^2} + \epsilon x \\
\alpha - \epsilon x &= \sqrt{x^2 + y^2} \\
\alpha^2 - 2\epsilon\alpha x + \epsilon^2 x^2 &= x^2 + y^2 \\
\alpha^2 &= x^2 + y^2 + 2\epsilon\alpha x - \epsilon^2 x^2 \\
\alpha^2 &= (1 - \epsilon^2)x^2 + y^2 + 2\epsilon\alpha x \\
\frac{\alpha^2}{(1-\epsilon^2)} &= x^2 + \frac{2\epsilon\alpha x}{(1-\epsilon^2)} + \frac{y^2}{(1-\epsilon^2)} \\
\frac{\alpha^2}{(1-\epsilon^2)} + \frac{\epsilon\alpha}{(1-\epsilon^2)^2} &= x^2 + \frac{2\epsilon\alpha x}{(1-\epsilon^2)} + \frac{\epsilon\alpha}{(1-\epsilon^2)^2} + \frac{y^2}{(1-\epsilon^2)} \\
\frac{\alpha^2}{(1-\epsilon^2)} + \frac{\epsilon^2\alpha^2}{(1-\epsilon^2)^2} &= (x + \frac{\epsilon\alpha}{(1-\epsilon^2)})^2 + \frac{y^2}{(1-\epsilon^2)} \\
1 &= \frac{(x + \frac{\epsilon\alpha}{(1-\epsilon^2)})^2}{\frac{\alpha^2}{(1-\epsilon^2)} + \frac{\epsilon^2\alpha^2}{(1-\epsilon^2)^2}} + \frac{\frac{y^2}{(1-\epsilon^2)}}{\frac{\alpha^2}{(1-\epsilon^2)} + \frac{\epsilon^2\alpha^2}{(1-\epsilon^2)^2}} \\
\frac{(x + \frac{\epsilon\alpha}{(1-\epsilon^2)})^2}{\frac{\alpha^2}{(1-\epsilon^2)} + \frac{\epsilon^2\alpha^2}{(1-\epsilon^2)^2}} + \frac{y^2}{\alpha^2 + \frac{\epsilon^2\alpha^2}{(1-\epsilon^2)}} &= 1 \\
\frac{(x + \frac{\epsilon\alpha}{(1-\epsilon^2)})^2}{\frac{\alpha^2 - \alpha^2\epsilon^2 + \epsilon^2\alpha^2}{(1-\epsilon^2)^2}} + \frac{y^2}{\frac{\alpha^2 - \alpha^2\epsilon^2 + \epsilon^2\alpha^2}{(1-\epsilon^2)}} &= 1 \\
\frac{(x + \frac{\epsilon\alpha}{(1-\epsilon^2)})^2}{\frac{\alpha^2}{(1-\epsilon^2)^2}} + \frac{y^2}{\frac{\alpha^2}{(1-\epsilon^2)}} &= 1
\end{aligned}$$

(c) Find x_0, a, b in terms of α, ϵ.

From above we can find that

$$\begin{aligned}
x_0 &= -\frac{\epsilon\alpha}{(1-\epsilon^2)} \\
a^2 &= \frac{\alpha^2}{(1-\epsilon^2)^2} \\
b^2 &= \frac{\alpha^2}{(1-\epsilon^2)}
\end{aligned}$$

(d) What equation do you find in the limit as $\epsilon \to 1$?

From above we have

$$\begin{aligned}
\alpha^2 - 2\epsilon\alpha x + \epsilon^2 x^2 &= x^2 + y^2 \\
\alpha^2 - 2\alpha x + x^2 &= x^2 + y^2 \\
\alpha^2 - 2\alpha x &= y^2
\end{aligned}$$

(e) Can you find the correct equation for $\epsilon > 1$?

Same part b we can find:

$$\alpha^2 = -(\epsilon^2 - 1)x^2 + y^2 + 2\epsilon\alpha x$$

Notice the sign change, $(\epsilon^2 - 1) > 0$ now. Now follow the same step of part b we have

$$\frac{(x - \frac{\epsilon\alpha}{(\epsilon^2-1)})^2}{\frac{\alpha^2}{(1-\epsilon^2)^2}} - \frac{y^2}{\frac{\alpha^2}{(\epsilon^2-1)}} = 1$$

5. If you did Problem 4.41 you met the virial theorem for a circular orbit of a particle in a central force with $U = kr^n$. Here is a more general form of the gheorem that applies to any periodic orbit of a particle.

(a) Find the time derivative of the quantity $G = \vec{r} \cdot \vec{p}$ and, by integrating from time 0 to t, show that

$$\frac{G(t) - G(0)}{t} = 2\langle T \rangle + \langle \vec{F} \cdot \vec{r} \rangle$$

where $\vec{F}$ is the net force on the particle and $\langle f \rangle$ denotes the average over time of any quantity f

First we want to find the time derivative of G:

$$\begin{aligned}\frac{\mathrm{d}G}{\mathrm{d}t} &= \dot{\vec{r}} \cdot \vec{p} + \vec{r} \cdot \dot{\vec{p}} \\ &= \vec{v}^2 m + \vec{r} \cdot \vec{F} \\ &= 2T + \vec{r} \cdot \vec{F}\end{aligned}$$

Now we find

$$\begin{aligned}\frac{1}{t}\int_0^t \mathrm{d}G &= \frac{1}{t}\int_0^t (2T + \vec{r} \cdot \vec{F})\mathrm{d}t \\ \frac{G(t) - G(0)}{t} &= \frac{2\int_0^t T\mathrm{d}t}{t} + \frac{\int_0^t \vec{\cdot}\vec{F}}{t} \\ \frac{G(t) - G(0)}{t} &= 2\langle T \rangle + \langle \vec{F} \cdot \vec{r} \rangle\end{aligned}$$

(b) Explain why, if the particle's orbit is periodic and if we make t sufficiently large, we can make the left-hand side of this equation as small as we please. That is, the left side approaches zero as $t \to \infty$.

Since G is a bounded function over $\mathbb{R}$, we have

$$\lim_{t\to\infty} \frac{G(t) - G(0)}{t} = 0$$

(c) Use this result to prove that if $\vec{F}$ comes from the potential energy $U = kr^n$, then $\langle T \rangle = \frac{n\langle U \rangle}{2}$, if now $\langle f \rangle$ denotes the time average over a very long time.

First we find force:

$$\vec{F} = -\nabla U = -knr^{n-1}\hat{r}$$

Thus we have

$$\vec{F} \cdot \vec{r} = -knr = -nU$$

Plug it into above result:

$$
\begin{aligned}
0 = \frac{G(t) - G(0)}{t} &= 2\langle T \rangle + \langle \vec{F} \cdot \vec{r} \rangle \\
2\langle T \rangle &= -\langle \vec{F} \cdot \vec{r} \rangle \\
2\langle T \rangle &= -\langle -nU \rangle \\
\langle T \rangle &= \frac{n\langle U \rangle}{2}
\end{aligned}
$$