Physics 303/573

Notes on the Harmonic Oscillator Part II

October 12, 2016

1 The forced harmonic oscillator

The key to understanding the forced harmonic oscillator is to use the linearity of the equation. Suppose we have two solutions $x_1(t)$ and $x_2(t)$ that satisfy

$$\begin{aligned} \ddot{x}_1 + 2\beta\dot{x}_1 + \omega_0^2 x_1 &= \frac{1}{m}F_1(t) \\ \ddot{x}_2 + 2\beta\dot{x}_2 + \omega_0^2 x_2 &= \frac{1}{m}F_2(t) \end{aligned} \tag{1.1}$$

for external driving forces $F_1(t)$ and $F_2(t)$, respectively. Then if we consider a linear combination $X(t) = ax_1(t) + bx_2(t)$, we find it satisfies

$$\ddot{X}+2\beta\dot{X}+\omega_0^2 X = a\ddot{x}_1+b\ddot{x}_2+2a\beta\dot{x}_1+2b\beta\dot{x}_2+\omega_0^2[ax_1(t)+bx_2(t)] = \frac{1}{m}(aF_1(t)+bF_2(t)) \tag{1.2}$$

Also, because of the linear structure, the most general solution with a driving force $F(t)$ is a linear combination of any solution with $F(t)$ and the general *homogeneous* solution, that is the solution with no driving force.

This allows us to solve the harmonic oscillator with an arbitrary external force and initial conditions.

1.1 Harmonic driving force

Because of the linear structure, we can work with complex driving force $F(t)$ and then just take the real part at the end; the real part of the solution $x_F(t)$ with the complex driving force $F(t)$ is the solution with a driving force that is the real part of $F(t)$:

$$\ddot{x}_F + 2\beta\dot{x}_F + \omega_0^2 x_F = \frac{1}{m}F(t) \quad \Rightarrow \quad \frac{d^2}{dt^2}\mathrm{Re}(x_F) + 2\beta\frac{d}{dt}\mathrm{Re}(x_F) + \omega_0^2\mathrm{Re}(x_F) = \frac{1}{m}\mathrm{Re}(F(t)) \tag{1.3}$$

So we start with

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = \frac{1}{m} F_\omega e^{i\omega_F t} \tag{1.4}$$

where F_ω is a complex constant. Since the derivative of an exponential gives back an exponential, we know that a solution exists that is proportional to $e^{i\omega_F t}$; we put it into the equation with an arbitrary amplitude A and find:

$$x = A_{\omega_F} e^{i\omega_F t} \quad \Rightarrow \quad A_{\omega_F}(i\omega_F)^2 e^{i\omega_F t} + 2A_{\omega_F}\beta(i\omega_F)e^{i\omega_F t} + A_{\omega_F}\omega_0^2 e^{i\omega_F t} = \frac{1}{m} F_\omega e^{i\omega_F t} \tag{1.5}$$

and hence

$$A_{\omega_F}[-(\omega_F)^2 + 2i\beta\omega_F + \omega_0^2] = \frac{1}{m} F_\omega \quad \Rightarrow \quad A_{\omega_F} = \frac{1}{m} \frac{F_\omega}{\omega_0^2 - \omega_F^2 + 2i\beta\omega_F} \tag{1.6}$$

We can multiply the numerator and denominator by the complex conjugate of the denominator to find:

$$A_{\omega_F} = \frac{1}{m} \frac{F_\omega(\omega_0^2 - \omega_F^2 - 2i\beta\omega_F)}{(\omega_0^2 - \omega_F^2)^2 + 4\beta^2\omega_F^2} \tag{1.7}$$

so that we finally have the complex solution

$$x(t) = A_{\omega_F} e^{i\omega_F t} = \frac{1}{m} \frac{F_\omega(\omega_0^2 - \omega_F^2 - 2i\beta\omega_F)}{(\omega_0^2 - \omega_F^2)^2 + 4\beta^2\omega_F^2} \, e^{i\omega_F t} \tag{1.8}$$

The real solution is just the real part of (1.8).

1.1.1 Undamped limit

When there is no damping, $\beta = 0$ and we have simply

$$A_{\omega_F} = \frac{1}{m} \frac{F_\omega}{\omega_0^2 - \omega_F^2} \tag{1.9}$$

Notice that the sign of A_{ω_F} depends on whether the driving frequency ω_F is greater or less than the natural frequency ω_0:

- If $\omega_F < \omega_0$, then $A_{\omega_F} > 0$ and the motion is in phase with the driving force, that is the mass on the spring just tracks the driving force.
- If $\omega_F > \omega_0$, then $A_{\omega_F} < 0$ and the motion is out of phase with the driving force, that is the mass on the spring moves against the driving force.

Of course, in the undamped case, if $\omega_F \to \omega_0$, then $A_{\omega_F} \to \infty$ and the motion blows up.

1.1.2 The damped case

In the general damped case, we consider (1.7). The magnitude of the amplitude A (for brevity, in the rest of this section, will just write A rather than A_{ω_F}) is

$$|A| = \sqrt{AA^*} = \frac{1}{m}\frac{|F_\omega|}{\sqrt{(\omega_0^2 - \omega_F^2)^2 + 4\beta^2\omega_F^2}} \tag{1.10}$$

This reaches a maximum as a function of the driving frequency ω_F when

$$\frac{d}{d\omega_F}|A| = 0 \quad \Leftrightarrow \quad 0 = \frac{d}{d\omega_F^2}\left((\omega_0^2 - \omega_F^2)^2 + 4\beta^2\omega_F^2\right) = -2(\omega_0^2 - \omega_F^2) + 4\beta^2 \tag{1.11}$$

and hence

$$\omega_F^2 = \omega_0^2 - 2\beta^2 = \omega^2 - \beta^2 \tag{1.12}$$

This is called the resonant frequency, and is a bit lower than the frequency ω at which the spring oscillates when it is not driven. For $\beta > \frac{1}{\sqrt{2}}\omega_0$, there is no resonant frequency, and the amplitude simply decreases the higher the driving frequency.

In addition to the magnitude of the amplitude, we are interested in the phase shift between it and the driving force. Defining

$$\frac{A}{F_\omega} = \frac{|A|}{|F_\omega|}e^{-i\phi} \tag{1.13}$$

we find (using Euler's identity $e^{-i\phi} = \cos\phi - i\sin\phi$ and comparing to (1.7))

$$\tan\phi = \frac{2\beta\omega_F}{\omega_0^2 - \omega_F^2} \tag{1.14}$$

As in the undamped case, this changes sign depending on whether ω_F is less than or greater than ω_0; for $\omega_F = \omega_0$, $\tan\phi \to \infty$ and hence

$$\phi = \frac{\pi}{2} \tag{1.15}$$

that is, the motion is 90 degrees out of phase with the driving force. To find the correct quadrant of the angle, recall that

$$\sin\phi = \frac{2\beta\omega_F}{|A_{\omega_F}|} > 0 \tag{1.16}$$

1.1.3 Transients and boundary conditions

The most general motion of a driven oscillator is a linear combination of a particular solution and the general solution when the driving force is turned off; when there is any damping at all, the terms in the solution that are there without the driving force will eventually die out, and hence are called transients, but they are important for the boundary conditions. Let's consider an example of a system driven by an external force that starts at rest in its equilibrium position. The demonstrations with the rubber band and with the wok were all of this kind.

For such physical systems the driving force is real, so we need to take the real part of our solution. We can, for example, take F_ω real, so the the force is $\mathrm{Re}(F_\omega e^{i\omega_F t}) = F_\omega \cos(\omega_F t)$ Then we have a combination of the real part of (1.8) and (2.52) of the previous lecture:

$$\begin{aligned} x(t) &= \mathrm{Re}(Ae^{i\omega_F t}) + x_{max}e^{-\beta t}\cos(\omega t + \varphi) \\ &= |A|\cos(\omega_F t - \phi) + x_{max}e^{-\beta t}\cos(\omega t + \varphi) \end{aligned} \tag{1.17}$$

where recall

$$\omega = \sqrt{\omega_0^2 - \beta^2} \quad , \quad |A| = \frac{1}{m}\frac{F_\omega}{\sqrt{(\omega_0^2 - \omega_F^2)^2 + 4\beta^2\omega_F^2}} \quad , \quad \tan\phi = \frac{2\beta\omega_F}{\omega_0^2 - \omega_F^2} \tag{1.18}$$

and x_{max} and φ are the constants we want to determine. Imposing $x(0) = v(0) = 0$, we find

$$\begin{aligned} 0 &= |A|\cos\phi + x_{max}\cos\varphi \\ 0 &= -|A|\omega_f \sin\phi + x_{max}(\beta\cos\varphi + \omega\sin\varphi) \end{aligned} \tag{1.19}$$

which we can solve by

$$x_{max} = -|A|\frac{\cos\phi}{\cos\varphi} \tag{1.20}$$

with

$$\tan\varphi = -\frac{1}{\omega}(\omega_F \tan\phi + \beta) = -\frac{\beta}{\omega}\left(\frac{\omega_0^2 + \omega_F^2}{\omega_0^2 - \omega_F^2}\right) \tag{1.21}$$

An example is shown in Figure 1. When the driving frequency is not near resonance, after a few cycles the random transients die out; near resonance, the motion builds steadily to its maximum amplitude. This is just what we found in the demonstration. Notice that when the driving frequency is low $\omega_F < \omega$, the motion is (nearly) in phase with the driving force, when it is equal to the natural frequency, $\omega_F = \omega$, it is $\frac{\pi}{2}$ or 90° out of phase, and when it is high, $\omega_F > \omega$, it is (nearly) π or 180° out of phase. In the figure, the driving frequency is held constant and the natural frequency is varied, as that is easier to visualize.

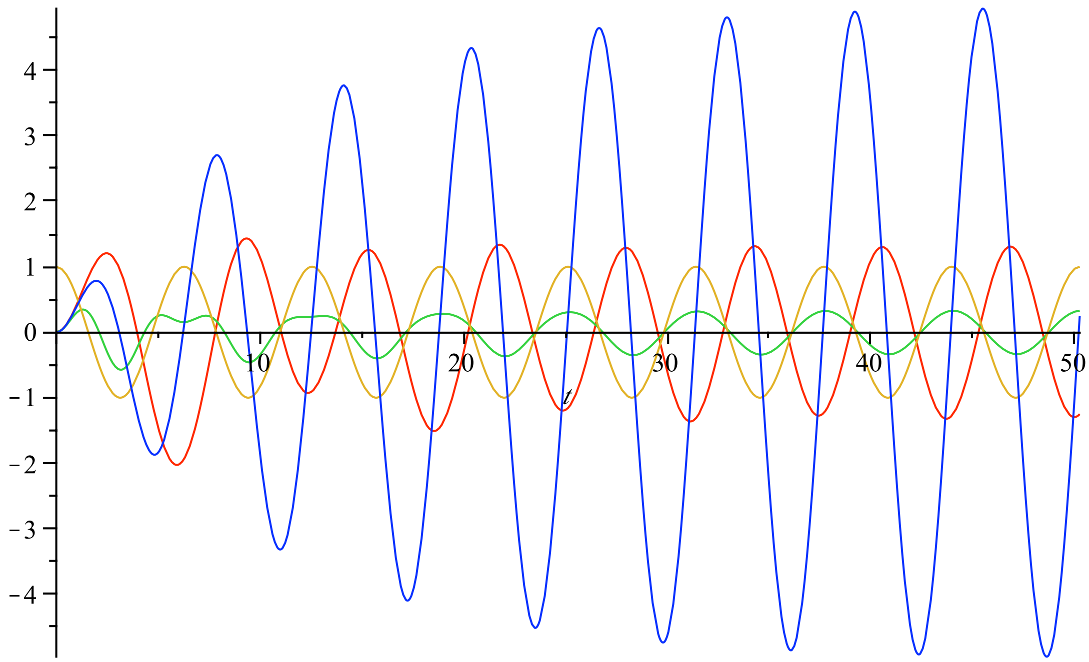

Figure 1: Transients and phase shifts in the underdamped forced harmonic oscillator. Here $\omega_F = 1$ and $\beta = \frac{1}{10}$, which is very underdamped. You can see the phase shifts of the motion as well as the transients at the beginning. Color code: The driving force is $\cos(t)$. When $\omega_F > \omega = \frac{1}{2}$, the motion is out of phase by (almost exactly) π. When $\omega_F < \omega = 2$, the motion is basically in phase. When $\omega_F = \omega = 1$, the motion is near resonance and is out of phase by $\frac{\pi}{2}$.

1.2 General periodic force: Fourier series

When the driving force is an arbitrary periodic function, we can use the linearity of the harmonic oscillator and simply take the sum of the results of the previous result. Systematically, any periodic function $F(t)$ with period T

$$F(t) = F(t+T) \quad , \quad T = \frac{2\pi}{\omega_F} \;\Leftrightarrow\; \omega_F = \frac{2\pi}{T} \tag{1.22}$$

can be expanded in a basis of simple pure frequency modes in the same way that a note on a guitar string can be described in terms of a fundamental frequency and its harmonics or overtones:

$$F(t) = \frac{1}{2}\sum_{n=-\infty}^{\infty} \tilde{F}_n e^{in\omega_F t} = \frac{1}{2}\left(\tilde{F}_0 + \sum_{n=1}^{\infty}(\tilde{F}_n + \tilde{F}_{-n})\cos(n\omega_F t) + i(\tilde{F}_n - \tilde{F}_{-n})\sin(n\omega_F t)\right) \tag{1.23}$$

This is real if the Fourier coefficients obey $\tilde{F}_{-n} = \tilde{F}_n^*$; then

$$F(t) = \frac{1}{2}\tilde{F}_0 + \sum_{n=1}^{\infty}\mathrm{Re}(\tilde{F}_n)\cos(n\omega_F t) - \mathrm{Im}(\tilde{F}_n)\sin(n\omega_F t) \tag{1.24}$$

It is easy to find the Fourier coefficients $\tilde{F}_n$ for a periodic function; note that[1]

$$\frac{1}{T}\int_0^T dt e^{in\omega t} = \begin{cases} \frac{1}{inT\omega}(e^{in\omega T}-1) = \frac{1}{in2\pi}(e^{in2\pi}-1) = 0 & n \neq 0 \\ 1 & n = 0 \end{cases} \tag{1.25}$$

Using this, it is easy to see that

$$\tilde{F}_n = \frac{2}{T}\int_0^T F(t)e^{-in\omega_F t} = \frac{\omega}{\pi}\int_0^{\frac{2\pi}{\omega_F}} F(t)e^{-in\omega_F t} \tag{1.26}$$

Notice that when the force $F(t)$ is real, we can complex conjugate and find that the Fourier coefficients automatically obey $\tilde{F}_{-n} = \tilde{F}_n^*$.

If we now go back to the harmonic oscillator, then we can use the solution (1.8) to (1.4) separately for each term in (1.23) to find

$$x_n(t) = A_{n\omega_F} e^{in\omega_F t} \tag{1.27}$$

where

$$A_{n\omega_F} = \frac{\tilde{F}_n}{2m}\frac{\omega_0^2 - (n\omega_F)^2 - 2in\beta\omega_F}{[\omega_0^2 - (n\omega_F)^2]^2 + 4n^2\beta^2\omega_F^2} \tag{1.28}$$

[1] Clearly, since $e^{in\omega t}$ is periodic, we could just as well integrate from $\frac{-T}{2}$ to $\frac{T}{2}$

Because of the linearity of the equations, as discussed at the beginning of these notes, we can find the general solution simply by adding these up; note that the $\tilde{F}_0$ term just gives a shift in the position by $x_0 = \frac{\tilde{F}_0}{2m}$:

$$x(t) = \sum_{n=-\infty}^{\infty} x_n(t) = \sum_{n=-\infty}^{\infty} A_{n\omega_F}\, e^{in\omega_F t} = \sum_{n=-\infty}^{\infty} \frac{\tilde{F}_n}{2m} \frac{\omega_0^2 - (n\omega_F)^2 - 2in\beta\omega_F}{[\omega_0^2 - (n\omega_F)^2]^2 + 4n^2\beta^2\omega_F^2}\, e^{in\omega_F t} \tag{1.29}$$

To take the real part, it is convenient to define the phases of the components of the driving force

$$\tilde{F}_n = |\tilde{F}_n| e^{-i\psi_n} \quad \Rightarrow \quad \tilde{F}_{-n} = |\tilde{F}_n| e^{i\psi_n} \quad \Rightarrow \quad F(t) = \frac{1}{2}\tilde{F}_0 + \sum_{n=1}^{\infty} |\tilde{F}_n| \cos(n\omega_F t - \psi_n) \tag{1.30}$$

as well as the phase shift of $A_{n\omega_F}$ (see 1.13):

$$\frac{A_{n\omega_F}}{\tilde{F}_n} = \frac{|A_{n\omega_F}|}{|\tilde{F}_n|} e^{-i\phi_n} \quad \Rightarrow \quad \tan\phi_n = \frac{2n\beta\omega_F}{\omega_0^2 - (n\omega_F)^2} \tag{1.31}$$

where we have used (1.14) to get the second expression. Then we can write (1.29) as

$$x(t) = \sum_{n=-\infty}^{\infty} |A_{n\omega_F}|\, e^{i(n\omega_F t - \phi_n - \psi_n)} \tag{1.32}$$

where the magnitude of $A_{n\omega_F}$ is given by (1.10):

$$|A_{n\omega_F}| = \frac{|\tilde{F}_n|}{2m} \frac{1}{\sqrt{[\omega_0^2 - (n\omega_F)^2]^2 + 4n^2\beta^2\omega_F^2}} \tag{1.33}$$

Finally, we get the real motion:

$$x(t) = \frac{\tilde{F}_0}{2m\omega_0^2} + \sum_{n=1}^{\infty} \frac{|\tilde{F}_n|}{m} \frac{\cos(\omega_F t - \phi_n - \psi_n)}{\sqrt{[\omega_0^2 - (n\omega_F)^2]^2 + 4n^2\beta^2\omega_F^2}} \tag{1.34}$$

1.3 General force: Fourier transform

To study non-periodic driving forces, we use the same basic method, but take the limit as the period $T \to \infty$. To do this, we need to make some small redefinitions; we take the limits of integration from $-\frac{T}{2}$ to $\frac{T}{2}$, we define rescaled Fourier coefficients $\tilde{f}_n = \frac{T}{2}\tilde{F}_n$, so that, plugging into (1.23,1.26), we find:

$$\tilde{f}_n = \int_{-\frac{T}{2}}^{\frac{T}{2}} dt\, F(t) e^{-i\left(\frac{2\pi n}{T}\right)t} \quad \text{and} \quad F(t) = \sum_{n=-\infty}^{\infty} \frac{\tilde{f}_n}{T} e^{i\left(\frac{2\pi n}{T}\right)t} \tag{1.35}$$

Taking the limit $T \to \infty$, and defining $s = \frac{n}{T}$, we have $\tilde{f}_n/T \to ds\tilde{f}(s)$, we find:

$$\tilde{f}(s) = \int_{-\infty}^{\infty} dt\, F(t)e^{-i2\pi st} \quad \text{and} \quad F(t) = \int_{-\infty}^{\infty} ds\, \tilde{f}(s)e^{i2\pi st} \tag{1.36}$$

or, equivalently, letting $\omega = 2\pi s$ and defining $\tilde{F}(\omega) = \tilde{f}(s)$

$$\tilde{F}(\omega) = \int_{-\infty}^{\infty} dt\, F(t)e^{-i\omega t} \quad \text{and} \quad F(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, \tilde{F}(\omega)e^{i\omega t} \tag{1.37}$$

We can now find $x(t)$ as before; using superposition and the basic solution (1.8), we have

$$x(t) = \int_{-\infty}^{\infty} d\omega\, \tilde{A}(\omega)e^{i\omega t} \tag{1.38}$$

where

$$\tilde{A}(\omega) = \frac{\tilde{F}(\omega)}{2\pi m}\frac{\omega_0^2 - \omega^2 - 2i\beta\omega}{[\omega_0^2 - \omega^2]^2 + 4\beta^2\omega^2} \tag{1.39}$$

As before, we find the real solution by taking the real part of this.

1.4 The electrical analog

We begin with the familiar laws of a circuit with a resistor, a capacitor, and an inductor in series, driven by an external time varying voltage. Ohm's law tells us that the voltage drop V across a resistance R is:

$$V = IR \tag{1.40}$$

where the current I is rate of change of the charge Q:

$$I = \dot{Q} \tag{1.41}$$

When we put a voltage across a capacitor with capacitance C, the charge stored is

$$Q = CV \tag{1.42}$$

Finally, the voltage induced by a current through an inductor with inductance L is

$$V = L\dot{I} = L\ddot{Q} \tag{1.43}$$

The voltage drop across all three, when they are in series, is simply the sum of the voltage drops, so we get:

$$L\ddot{Q} + R\dot{Q} + \frac{1}{C}Q = V(t) \tag{1.44}$$

which we recognize as a driven damped harmonic oscillator with the identifications:

Electrical		Mechanical	
Charge	Q	Position	x
Current	I	Velocity	v
Inductance	L	Mass	m
Resistance	R	Damping	c
(Capacitance)$^{-1}$	$\frac{1}{C}$	Spring constant	k
External voltage	$V(t)$	External force	$F(t)$

Table 1: The electrical–mechanical analog

Dividing through by the inductance, we get the equation

$$\ddot{Q} + 2\beta\dot{Q} + \omega_0^2 Q = \frac{1}{L}V(t) \tag{1.45}$$

where

$$\beta = \frac{R}{2L} \quad , \qquad \omega_0^2 = \frac{1}{LC} \tag{1.46}$$

We can differentiate (1.45) and get an equation for the current in terms of the derivative of the driving voltage:

$$\ddot{I} + 2\beta\dot{I} + \omega_0^2 I = \frac{1}{L}\dot{V}(t) \tag{1.47}$$

The natural frequency is $\omega = \sqrt{\omega_0^2 - \beta^2}$:

$$\omega = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} \tag{1.48}$$

The quality factor (not to be confused with the charge), is given by $\frac{\omega}{2\beta}$, which, for small damping is

$$\frac{\omega_0}{2\beta} = \frac{1}{R}\sqrt{\frac{L}{C}} \tag{1.49}$$

Everything in our analysis of the harmonic oscillator applies to the electrical analog. However, there is one thing that is slightly different. When we talk about resonance, we are interested in the frequency that maximizes the amplitude of the *current* I rather than the amplitude of the *charge* Q. Since $I = \dot{Q}$, if the amplitude of $Q(t)$ is Q_0, then the amplitude of the current is $|I_0| = \omega_F Q_0$. This means that instead of maximizing $|A|$ as in (1.10), we want to maximize $|A|\omega_F$:

$$|I_0| = \frac{V_0}{L}\frac{\omega_F}{\sqrt{(\omega_0^2 - \omega_F^2)^2 + 4\beta^2\omega_F^2}} \tag{1.50}$$

Remarkably, when we maximize this, we find that resonant frequency is the undamped frequency:

$$\omega_{res} = \omega_0 = \sqrt{\frac{1}{LC}} \tag{1.51}$$

2 The WKB approximation

The WKB approximation is very useful for certain problems in quantum mechanics, and can be introduced here. It applies to harmonic oscillators where the frequency varies slowly in time. An example might be a pendulum with a spring instead of a fixed length rope; as the length of the spring varies, the instantaneous frequency of the pendulum varies as well. If this variation is sufficiently slow, we can use the WKB approximation to describe the motion of the mass on the end of this spring pendulum.

2.1 Intuitive approach

We consider an equation of the form

$$\ddot{x} + \omega^2(t)\, x = 0 \tag{2.52}$$

where we assume that $\omega^2(t)$ varies slowly in time. We could easily include a damping term–the basic ideas would apply as well; you can try to work it out for yourself.

For slowly varying $\omega(t) \neq 0$, the solution should be approximately

$$x \sim C^{(+)} e^{i\int dt\, \omega(t)} + C^{(-)} e^{-i\int dt\, \omega(t)} \tag{2.53}$$

where $C^{(\pm)}$ are some constants; notice that for constant ω, $\int dt\, \omega(t) = \omega t$, so this reduces to the usual result for strictly constant $\omega(t) = \omega_0$. Of course, to get a real solution, we need to take $C^{(-)} = (C^{(+)})^*$.

To do better, we write

$$x = e^{\alpha(t)} \quad , \quad \alpha(t) = a(t) + i\theta(t) \tag{2.54}$$

and assume that $a(t)$ is slowly varying. Plugging this into (2.52) gives (up to an overall factor of $e^{\alpha(t)}$)

$$\ddot{\alpha} + (\dot{\alpha})^2 + \omega^2 = 0 \tag{2.55}$$

The real part of this equation is

$$\ddot{a} + (\dot{a})^2 - (\dot{\theta})^2 + \omega^2 = 0 \tag{2.56}$$

and the imaginary part is

$$\ddot{\theta} + 2\dot{a}\dot{\theta} = 0 \tag{2.57}$$

This second equation implies $\dot{a} = -\frac{1}{2}\frac{\ddot{\theta}}{\dot{\theta}}$, which we can integrate to give:

$$a(t) = c - \frac{1}{2}\ln\dot{\theta}(t) \tag{2.58}$$

where c is an integration constant; exponentiating, we have:

$$A(t) = \frac{C}{\sqrt{\dot{\theta}(t)}} \quad , \qquad C = e^c \tag{2.59}$$

Returning to the real part, we use the condition that a varies more slowly than θ to drop the $\ddot{a}$ and $(\dot{a})^2$ terms (that is we assume $\dot{\theta} >> \dot{a}$, $(\dot{\theta})^2 >> \ddot{a}$). To leading order,

$$\dot{\theta} = \pm\omega(t) \quad \Rightarrow \quad \theta_0(t) = \pm\int^t dt'\,\omega(t') \tag{2.60}$$

which combines with our expression for $A(t)$ to give an improved approximation for $x(t)$:

$$x(t) \sim \frac{C^{(+)}}{\sqrt{\omega(t)}}e^{i\int dt\,\omega(t)} + \frac{C^{(-)}}{\sqrt{\omega(t)}}e^{-i\int dt\,\omega(t)} \tag{2.61}$$

This is the standard leading term in the WKB approximation.

We can get higher corrections as follows: starting with the leading solution $\theta_0 = \pm\int dt\,\omega(t)$, we have $a_0(t) = c - \frac{1}{2}\ln\omega(t)$, which we use to get the next order corrections to $\theta(t)$ and keep iterating. Thus, the first correction gives:

$$\dot{a} = -\frac{\dot{\omega}}{2\omega} \quad , \quad \ddot{a} = -\frac{\ddot{\omega}}{2\omega} + \frac{1}{2}\left(\frac{\dot{\omega}}{\omega}\right)^2 \tag{2.62}$$

so, keeping these leading corrections to (2.56), we get

$$(\dot{\theta})^2 = \omega^2 - \frac{\ddot{\omega}}{2\omega} + \frac{3}{4}\left(\frac{\dot{\omega}}{\omega}\right)^2 \tag{2.63}$$

which we can integrate to get an improved approximation for $\theta(t)$, which in turn gives an improved approximation for $a(t)$, etc.

2.2 Systematic approach

Actually, it is conventional to organize this expansion in a more systematic way. Let us introduce a characteristic large frequency $\frac{1}{\delta}$ and define:

$$\omega = \frac{\Omega(t)}{\delta} \quad , \qquad \alpha = \frac{1}{\delta}\sum_{n=0}^{\infty} \alpha_n(t)\,\delta^n \tag{2.64}$$

Plugging this into

$$\ddot{\alpha} + (\dot{\alpha})^2 + \omega^2 = 0 \tag{2.65}$$

and multiplying through by δ^2, we get

$$\sum_{n=0}^{\infty} \ddot{\alpha}_n(t)\delta^{n+1} + \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \dot{\alpha}_n(t)\dot{\alpha}_m(t)\delta^{n+m} + \Omega^2(t) = 0 \tag{2.66}$$

We can rewrite the double sum to get:

$$\sum_{n=0}^{\infty} \ddot{\alpha}_n(t)\delta^{n+1} + \sum_{n=0}^{\infty}\sum_{m=0}^{n} \dot{\alpha}_m(t)\dot{\alpha}_{n-m}(t)\delta^{n} + \Omega^2(t) = 0 \tag{2.67}$$

Now we expand in powers of δ; the leading term is

$$(\dot{\alpha}_0(t))^2 + \Omega^2 = 0 \tag{2.68}$$

which implies that α_0 gives leading contribution to $\theta_0(t)$ that we found above:

$$\alpha_0 = \pm i \int dt\, \Omega(t) \quad \Leftrightarrow \quad \theta_0 = -\frac{i}{\delta}\alpha_0 \tag{2.69}$$

The next term is

$$\ddot{\alpha}_0(t) + 2\dot{\alpha}_0\dot{\alpha}_1 = 0 \tag{2.70}$$

which imples that α_1 just gives the leading contribution to $a(t)$ that we found above (the dependence on δ can be absorbed into the constant c:

$$\alpha_1 = c' - \frac{1}{2}\ln\Omega(t) = a_0(t) = c - \frac{1}{2}\ln\omega(t) \quad , \qquad c = c' - \frac{1}{2}\ln\delta \tag{2.71}$$

The next equation is

$$\ddot{\alpha}_1 + (\dot{\alpha}_1)^2 + 2\dot{\alpha}_0\dot{\alpha}_2 = 0 \tag{2.72}$$

which can be solved for α_2. The general equation for α_n is

$$2\dot{\alpha}_0\dot{\alpha}_n + \ddot{\alpha}_{n-1} + \sum_{m=1}^{n-1} \dot{\alpha}_m\dot{\alpha}_{n-m} = 0 \tag{2.73}$$

2.3 A related exactly soluble problem

Recall that in the previous set of notes, we considered the equation

$$(D+A)(D+B)x = \ddot{x} + (A+B)\dot{x} + (AB+\dot{B})x = 0 \quad \text{where} \quad D = \frac{d}{dt} \tag{2.74}$$

and said that the most general solution to this equation can be expressed as a solution to the system of first-order equations

$$\dot{x} + B\,x = x_A(t) \quad , \quad \dot{x}_A + A\,x_A = 0 \tag{2.75}$$

We can actually give the solution to this explicitly in terms of integrals, even when A and B are functions of time. The equation $\dot{x}_A + A\,x_A$ can be integrated to give:

$$x_A(t) = e^{-\int^t A} \tag{2.76}$$

To solve the equation for x, we substitute $x = \left(e^{-\int^t B}\right) y$; then

$$\dot{x} = \left(e^{-\int^t B}\right)(\dot{y} - B\,y) \tag{2.77}$$

and hence (2.75) reduces to

$$\dot{y} = \left(e^{\int^t B}\right)\left(e^{-\int^t A}\right) = e^{\int^t (B-A)} \tag{2.78}$$

which we can immediately integrate to

$$y = \int^t e^{\int (B-A)} \tag{2.79}$$

and hence we find the total solution is

$$x = \left(e^{-\int^t dt' B(t')}\right) \int^t dt' \left(e^{\int^{t'} dt''(B(t'')-A(t''))}\right) \tag{2.80}$$

where in the final expression (though not in the intermediate expressions (2.76)-(2.79)) I have been careful to explicitly indicate the dummy integration variables.

If we want to have no damping, then from (2.74), we need $A + B = 0$; for a mechanical system, we also require that the $\omega^2(t)$ is real, so $A^2 + \dot{A}$ must be real. However, we also want a stable system, which means that $\omega^2(t) = -(A^2 + \dot{A}) > 0.$. If we write $A = a + ib$, then

$$\omega^2 = b^2 - a^2 - \dot{a} - i(2ab + \dot{b}) \tag{2.81}$$

the condition that the frequency is real implies

$$a = -\frac{\dot{b}}{2b} \tag{2.82}$$

Plugging this back into (2.81), we find

$$\omega^2 = b^2 - \frac{3}{4}\left(\frac{\dot{b}}{b}\right)^2 + \frac{\ddot{b}}{2b} \tag{2.83}$$

Note the striking resemblance of (2.82),(2.83) to (2.62),(2.63). If we plug

$$B = -A = \frac{\dot{b}}{2b} - ib = \left(\frac{1}{2}\frac{d}{dt}\ln(b)\right) - ib \tag{2.84}$$

into our general solution (2.80), we find

$$\begin{aligned} x(t) &= \frac{1}{\sqrt{b}}\left(e^{i\int^t b}\right)\int^t b\left(e^{-2i\int^t b}\right) = \frac{1}{\sqrt{b}}\left(e^{i\int^t b}\right)\int^t \frac{i}{2}\frac{d}{dt}\left(e^{-2i\int^t b}\right) \\ &= \frac{1}{\sqrt{b}}\left(e^{i\int^t b}\right)\left[C^{(+)} + \frac{i}{2}\left(e^{-2i\int^t b}\right)\right] \\ &= \frac{C^{(+)}}{\sqrt{b}}\left(e^{i\int^t b}\right) + \frac{C^{(-)}}{\sqrt{b}}\left(e^{-i\int^t b}\right) \end{aligned} \tag{2.85}$$

where $C^{(\pm)}$ are two integration constants. Notice that except for the correction (2.83) to ω, this is precisely the form (2.61). As for the WKB approximation, to make $x(t)$ real, we need $C^{(-)} = (C^{(+)})^*$.

3 Nonlinear Harmonic motion

We began our discussion of the Harmonic oscillator by noting that it arises universally if one Taylor expands a smooth force around an equilibrium point. Let's now consider the higher terms in the expansion. Because the system is nonlinear, we can no longer simply consider the superposition of different solutions, and the initial conditions change the solution in complicated ways. Let us start by proceeding naively; for a force

$$F(x) = m(-\omega_0^2 x + \Delta f(x)) \tag{3.86}$$

where we have defined the normalized nonlinear terms by $\Delta f(x)$, we expand both the force and the solution in a small parameter λ:

$$\begin{aligned} \Delta f(x) &= \sum_{n=2}^{\infty} f_n \, \lambda^{n-1} (x(t))^n \\ x(t) &= \sum_{n=0}^{\infty} x_n(t) \lambda^n \end{aligned} \tag{3.87}$$

Plugging in the expansion for the position $x(t)$ into the expansion for force $F(x)$, we find

$$\Delta f(x) = \lambda f_2 x_0^2 + \lambda^2 (f_3 x_0^3 + 2 f_2 x_0 x_1) + \lambda^3 (f_4 x_0^4 + 3 f_3 x_0^2 x_1 + f_2 (x_1^2 + 2 x_0 x_2)) + ... \tag{3.88}$$

We can now expand the nonlinear harmonic oscillator equation:

$$\ddot{x} + \omega_0^2 x = \Delta f(x) \tag{3.89}$$

(for simplicity we consider the undamped case, but not much changes with damping), and expand it order by order in λ

$$\begin{array}{lrcl} \mathcal{O}(1): & \ddot{x}_0 + \omega_0^2 x_0 &=& 0 \\ \mathcal{O}(\lambda): & \ddot{x}_1 + \omega_0^2 x_1 &=& f_2 x_0^2 \\ \mathcal{O}(\lambda^2): & \ddot{x}_2 + \omega_0^2 x_2 &=& f_3 x_0^3 + 2 f_2 x_0 x_1 \\ \mathcal{O}(\lambda^3): & \ddot{x}_3 + \omega_0^2 x_3 &=& f_4 x_0^4 + 3 f_3 x_0^2 x_1 + f_2 (x_1^2 + 2 x_0 x_2)) \end{array} \tag{3.90}$$

etc. The $\mathcal{O}(1)$ equation is the usual linear harmonic oscillator and is easily solved, and all the subsequent equations get an effective driving force from the lower order terms.

If the boundary conditions are such that we take $x_0(t) = c_0 e^{i\omega t}$ or $x_0(t) = c_0 e^{-i\omega t}$, that is, a complex solution which is a pure exponential (which arises in quantum mechanics but is unphysical for classical oscillating masses), then (3.90) is the whole story, and each correction $x_n(t)$ has a frequency $n\omega$ induced by the forcing term.

However, if the boundary conditions are such that $x_0(t)$ has terms involving both $e^{i\omega t}$ and $e^{-i\omega t}$ simultaneously, then the driving terms which arise from powers and products of the x_n's contain admixtures of lower frequencies, and it is more useful to resum the series. The general discussion is quite complicated, so I will focus on some examples.

Consider a quadratic term λx^2; notice it breaks the symmetry $F(x) = -F(-x)$. When this symmetry is not broken, the center of the motion (the time-independent term) stays at $x_c = 0$; for $F(x) \neq -F(-x)$, which would be relevant to, *e.g.*, a rubber band, the center of the motion x_c depends on the amplitude.

$$\ddot{x} + \omega_0^2 x = \lambda x^2 \tag{3.91}$$

We write

$$x(t) = x_c + A\cos(\omega t) + ... \tag{3.92}$$

where we fix the amplitude A of the fundamental oscillation and see what we get; to leading order, we have

$$\begin{aligned}(\omega_0^2 - \omega^2)A\cos(\omega t) + \omega_0^2 x_c &= \lambda[x_c^2 + 2x_c A\cos(\omega t) + A^2\cos^2(\omega t)] \\ &= \lambda[x_c^2 + 2x_c A\cos(\omega t) + \frac{A^2}{2}(\cos(2\omega t) + 1))]\end{aligned} \tag{3.93}$$

Collecting the constant terms, we find:

$$\omega_0^2 x_c = \lambda x_c^2 + \frac{1}{2}\lambda A^2 \quad \Rightarrow \quad x_c = \frac{\omega_0^2 - \sqrt{\omega_0^4 - 2\lambda^2 A^2}}{2\lambda} = \frac{\lambda A^2}{2\omega_0^2} + \mathcal{O}(\lambda^2) \tag{3.94}$$

which shows that the center of the motion is shifted from the equilibrium point as the amplitude increases by the nonlinearity; this does not happen when $F(x) = -F(-x)$. Notice also that when the nonlinearity is too strong or the amplitude is too large, (3.94) breaks down. This makes sense: a quadratic force will dominate over the linear restoring force when it becomes strong enough, and the mass will eventually run away. This is best seen by computing the potential:

$$\frac{F(x)}{m} = -\omega_0^2 x + \lambda x^2 = -\frac{d}{dx}\frac{V(x)}{m} \quad \Rightarrow \quad V(x) = \frac{1}{2}m\omega_0^2 x^2 - \frac{1}{3}m\lambda x^3 \tag{3.95}$$

which clearly is unbounded for x large, so if the oscillations are large enough, the mass will escape the minimum at $x = 0$ and go over the maximum at $x = \frac{\omega_0^2}{\lambda}$.

Returning to (3.93) and looking at the terms linear in $\cos(\omega t)$, we find

$$(\omega_0^2 - \omega^2)A = 2\lambda x_c A \quad \Rightarrow \quad \omega^2 = \omega_0^2 - 2\lambda x_c = \omega_0^2 - \frac{\lambda^2 A^2}{\omega_0^2} + \mathcal{O}(\lambda^3) \tag{3.96}$$

so the fundamental frequency ω is shifted from ω_0 by an amount that depends on the amplitude A:

$$\omega = \omega_0[1 - \frac{\lambda^2 A^2}{2\omega_0^4} + \mathcal{O}(\lambda^3)] \tag{3.97}$$

Finally, we see that there is a term proportional to $\cos(2\omega t)$ in (3.93); this looks like a driving force and gives rise to higher harmonics which may be calculated using the methods developed above.