Physics 303/573

Two body system, central forces, and gravitation

October 24, 2016

1 Two body systems

Suppose we have a system with just two bodies; we may write the Lagrangian as

$$L = \frac{1}{2}[m_1(\dot{\vec{r}}_1 \cdot \dot{\vec{r}}_1) + m_2(\dot{\vec{r}}_2 \cdot \dot{\vec{r}}_2)] - U(\vec{r}_1, \vec{r}_2) \tag{1.1}$$

Suppose further that U depends only on the relative coordinate $\vec{r} = \vec{r}_1 - \vec{r}_2$ that goes from m_2 to m_1; then the system has a symmetry under simultaneous shifts of r_1 and r_2. We can see this by changing to different coordinates: the center of mass $\vec{r}_{cm}$ and the relative coordinate $\vec{r}$. Recall

$$\vec{r}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2} \equiv \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{M} \tag{1.2}$$

where we have introduced the total mass $M = m_1 + m_2$. Then

$$\vec{r}_1 = \vec{r}_{cm} + \frac{m_2\vec{r}}{M} \quad , \quad \vec{r}_2 = \vec{r}_{cm} - \frac{m_1\vec{r}}{M} \tag{1.3}$$

and

$$L = \frac{1}{2}[M(\dot{\vec{r}}_{cm} \cdot \dot{\vec{r}}_{cm}) + \frac{m_1m_2}{M}(\dot{\vec{r}} \cdot \dot{\vec{r}})] - U(\vec{r}) \equiv \frac{1}{2}[M(\dot{\vec{r}}_{cm} \cdot \dot{\vec{r}}_{cm}) + \mu(\dot{\vec{r}} \cdot \dot{\vec{r}})] - U(\vec{r}) \tag{1.4}$$

where we have introduced the reduced mass $\mu = m_1m_2/M$. Notice that the Lagrangian depends only on $\dot{\vec{r}}_{cm}$ and not on $\vec{r}_{cm}$ itself, so it is symmetric under shifts of $\vec{r}_{cm}$–these are precisely the simultaneous shifts of r_1 and r_2 mentioned above. The corresponding conserved quantity is just the total momentum $\vec{P} = M\dot{\vec{r}}_{cm}$. By going to these coordinates, we have reduced the two body problem to the problem of a free particle with mass M and a single effective particle with mass μ and potential $U(\vec{r})$ and hence a reduced Lagrangian:

$$L_{red} = \frac{1}{2}\mu(\dot{\vec{r}} \cdot \dot{\vec{r}}) - U(\vec{r}) \tag{1.5}$$

In other words, since the center of mass motion is described by a free body, we know the solution and can safely ignore it in solving for the relative motion of the two particles.

Notice that when $m_1 >> m_2$, $M \approx m_1$ and $\mu \approx m_2$. The other extreme situation occurs when $m_1 = m_2$; then $M = 2m_1$ and $\mu = \frac{1}{2}m_1$. An example is shown in Figure 1 (for concreteness, I use the gravitational force): The two bodies rotate on orbits around a common focus, always staying on opposite sides of the origin.

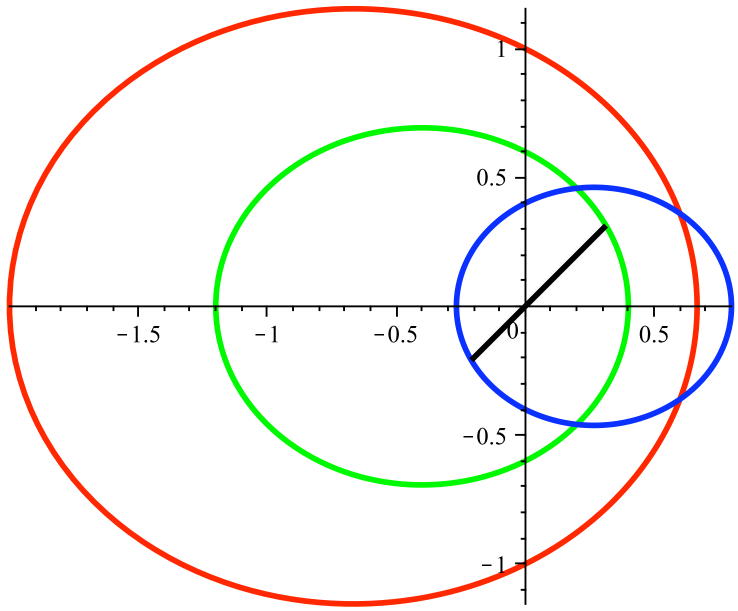


Figure 1: Orbits of two particles with m_1= .4M and m_2= .6M (which gives $\mu = .24M$), and of their relative coordinate $\vec{r}$: The **red orbit** shows $\vec{r}(\theta)$, the **green orbit** shows $\vec{r}_1(\theta)$, and the **blue orbit** shows $\vec{r}_2(\theta)$; the black cross bar shows the relative positions of m_1 and m_2 when $\theta = \frac{\pi}{4}$. As m_2 gets bigger relative to m_1, the blue orbit becomes smaller, and the green orbit approaches the red curve.

2 Central forces

We can simplify the problem further by assuming that the potential is a central potential–it depends only on the distance between the two particles (the magnitude $r = |\vec{r}|$) and not the direction of $\vec{r}$. In this case, the Lagrangian is also symmetric under rotations of $\vec{r}$, and angular momentum is conserved. Consequently, the motion lies in a plane, and we can use polar coordinates adapted to that plane.

The Lagrangian becomes:

$$L_{red} = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) - U(r) \tag{2.6}$$

A central potential gives rise to a central or radial force

$$\vec{F} = f(r)\hat{r} = -\frac{\partial U(r)}{\partial r}\hat{r} \quad , \quad r \equiv |\vec{r}| \tag{2.7}$$

The Euler-Lagrange equations that follow from (2.6) (with $U(\vec{r}) = U(r)$) are

$$\mu(\ddot{r} - r\dot{\theta}^2) = f(r) \quad , \quad \frac{d}{dt}\left(\mu r^2\dot{\theta}\right) = 0 \tag{2.8}$$

2.1 Proof that any central force is conservative

We found the force by taking the gradient of a potential, so of course it is conservative. It is however interesting to prove directly that any central force is conservative. We could prove this by explicitly calculating the curl and finding that it vanishes, but we can do this more directly by considering the work $\int \vec{F} \cdot d\vec{s}$ along an arbitrary closed curve. Because of Stokes theorem, the integral can be expressed as an integral over a surface bounded by the curve, and hence the curve can be approximated by a curve with steps as shown in Figure 2 (this would not work for computing the length of the curve): When calculate

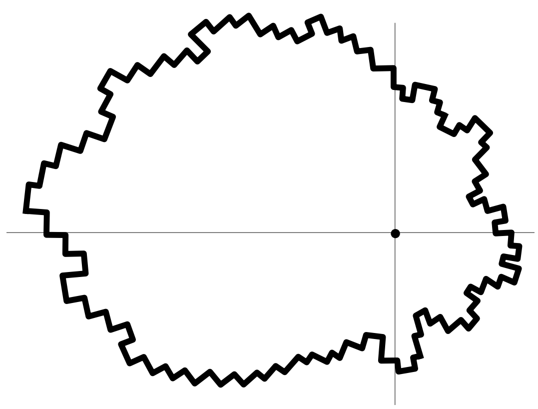


Figure 2: Approximating a curve with radial and and tangential segments

the work, since the force is radial, only the radial segments in the curve contribute; the tangential ones are perpendicular to the force and hence $\vec{F} \cdot d\vec{s} = 0$ along those segments. But because the curve is closed, and because the magnitude of the force is independent of the angle, every contribution to the line integral from a step going out is canceled by an equal and opposite contribution from a step coming back closer to the origin. Hence the total work around any closed curve must vanish. As we have assumed that the potential is time independent, the total energy (or Hamiltonian) $E = T + U$ is conserved.

2.2 Angular momentum

As mentioned above, for a central force, the angular momentum is conserved because the system is symmetric under rotations of $\vec{r}$. We have

$$\vec{L} = \vec{r} \times \vec{p} \tag{2.9}$$

where $\vec{p}$ is the linear momentum $\mu\vec{v}$. The angular momentum is clearly perpendicular to the position and velocity vectors of the particle, and vanishes if the two are parallel, that is, if the particle is moving along the position vector (radially in or out).

We can see directly that central forces conserve angular moment, because for any vector $\vec{v} \times \vec{v} = 0$, and the velocity is just $\vec{v} = \dot{\vec{r}}$:

$$\vec{L} = \vec{r} \times \mu\vec{v} \quad \Rightarrow \quad \frac{d}{dt}\vec{L} = \mu\vec{v} \times \vec{v} + \vec{r} \times \mu\vec{a} = 0 + \vec{r} \times \vec{F} \tag{2.10}$$

for a central force, $\vec{F}$ is parallel to $\vec{r}$–see (2.7) above, and hence

$$\frac{d}{dt}\vec{L} = 0 \tag{2.11}$$

As we noted above, because angular momentum is conserved, the motion is restricted to a plane perpendicular to the angular momentum, and we can always choose our coordinates to be polar coordinates in that plane. Then we write (recall equations (L2.2.18-20) in Lecture 2):

$$\vec{r} = r\hat{r} \quad , \quad \vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} \tag{2.12}$$

and

$$\vec{L} = (\mu r^2\dot{\theta})\, \hat{r} \times \hat{\theta} \tag{2.13}$$

where $\hat{r} \times \hat{\theta}$ is a unit vector perpendicular to the plane of the motion. Conservation of angular momentum means that both the direction of $\vec{L}$ as well as the magnitude L[1]

$$L = \mu r^2\dot{\theta} \tag{2.14}$$

are conserved. The equation $\dot{L} = 0$ is just the second equation in (2.8). We can now rewrite the radial part (2.8) as:

$$\mu\ddot{r} - \frac{L^2}{\mu r^3} = f(r) \tag{2.15}$$

[1] I hope that context will make it clear whether we are concerned with the magnitude of the angular momentum or the Lagrangian; in the coordinates that we use, $\vec{L} = L\hat{z}$, so we could write L_z everywhere, but it would just clutter our equations.

Note that we can do this in the Euler-Lagrange equation, but *not* in the Lagrangian–we cannot substitute the solution to some of the Euler-Lagrange equations into Lagrangian and derive correct equations if the equation involved time derivatives. We can however make this substitution in the total energy (or Hamiltonian).

2.3 Kepler's second law

Conservation of angular momentum leads to Kepler's second law, which actually holds for any central force: As a body moves under the influence of a central force, it sweeps out an equal area in equal times. Consider a particle moving for an infinitesimal time; it moves an angle $d\theta$, sweeping out a little triangular wedge with length r and width $rd\theta$; this wedge has an area $\frac{1}{2}r^2 d\theta$, and so the rate at which it is sweeping out area is:

$$\dot{A} = \frac{1}{2} r^2 \dot{\theta} = \frac{L}{2\mu} \tag{2.16}$$

Since L is conserved, this is a constant! Thus Kepler's second law is just the conservation of angular momentum in a central potential.

2.4 Effective potential

The kinetic energy of a body in polar coordinates is:

$$T = \frac{1}{2}\mu \vec{v} \cdot \vec{v} = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) \tag{2.17}$$

We can rewrite the second term in terms of the angular momentum using (2.13):

$$T = \frac{1}{2}\left(\mu\dot{r}^2 + \frac{L^2}{\mu r^2}\right) \tag{2.18}$$

Then the total energy is

$$E = T + U = \frac{1}{2}\mu\dot{r}^2 + \left(\frac{L^2}{2\mu r^2} + U(r)\right) \equiv \frac{1}{2}\mu\dot{r}^2 + U_{eff}(r) \tag{2.19}$$

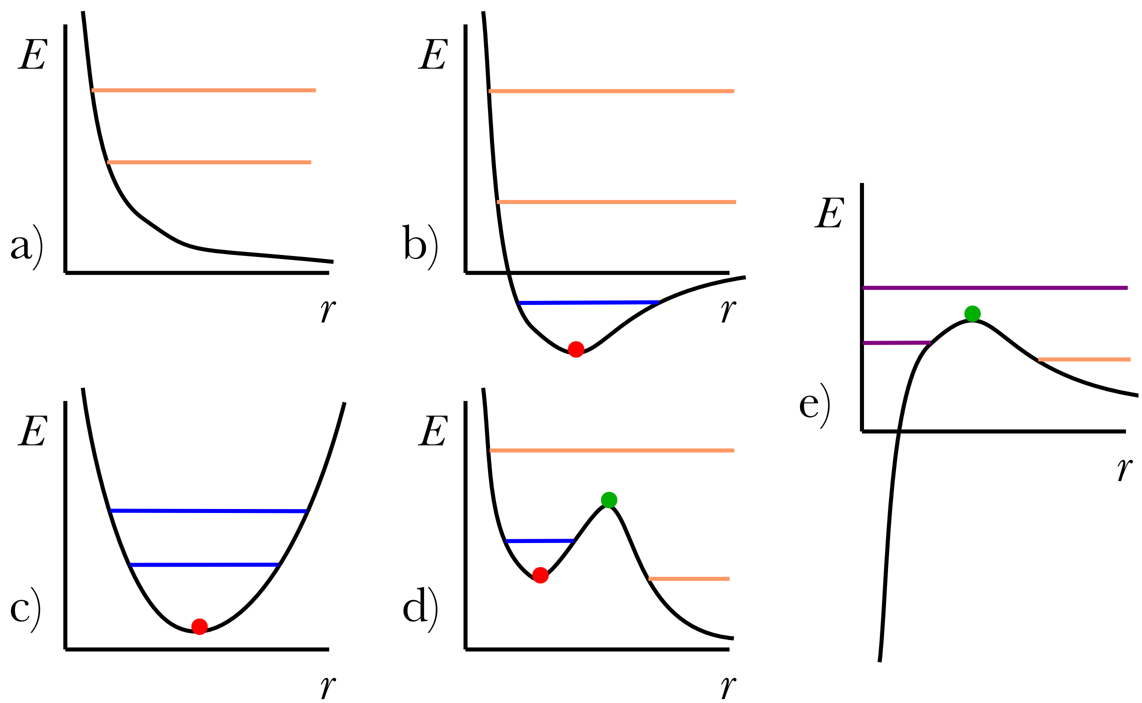
where we have defined an *effective potential*

$$U_{eff}(r) = \frac{L^2}{2\mu r^2} + U(r) \tag{2.20}$$

The angular momentum term acts like a repulsive barrier near the origin for the radial coordinate. If a particle has a given total energy and angular momentum, then we can find the points in its motion when it comes to rest and has no kinetic energy ($T = 0$) by solving the equation

$$U_{eff}(r) = \frac{L^2}{2\mu r^2} + U(r) = E \tag{2.21}$$

Depending on the shape of $U_{eff}(r)$ and the value of the total energy E, this equation may have no solutions or one or more solutions. Some examples are shown in Figure 3.



Key	
Unbounded trajectories	**orange**
Bounded trajectories (orbits)	**blue**
Stable circular orbits	**red**
Unstable circular orbits	**green**
Singular trajectories	**purple**

Figure 3: Sample effective potentials $U(r)$ and trajectories with different total energies E.

The examples are: **a)** Repulsive potential: all trajectories are unbounded. **b)** Attractive potential that goes to a constant value at infinity: there are unbounded trajectories, bounded trajectories (orbits), and stable circular orbits. **c)** Attractive potential that keeps rising at infinity: all trajectories are bounded; at the bottom of the potential, there are stable circular orbits. **d)** More complicated potential which is attractive for some range of r and repulsive for some range of r and goes to a constant value at infinity: there are unbounded trajectories, bounded trajectories (orbits), stable circular orbits and unstable circular orbits. **e)** Singular short range attractive potential: this has unstable circular orbits, unbounded trajectories and singular trajectories that fall into $r = 0$.

Note that since the effective potential depends on the angular momentum, it can change its shape as the angular momentum changes. In particular, the $L = 0$ potential generally looks very different near $r = 0$.

Just as the force is found from $-\vec{\nabla}U(r)$, we can write (2.15) as

$$\mu\ddot{r} = -\frac{d}{dr}U_{eff}(r) \tag{2.22}$$

3 Newton's Law of Gravitation

Though we are used to the idea that the same mathematical equations govern the motion of objects here on earth and the motion of the moon and the planets, it is really an astonishing and unprecedented discovery. Newton's law of universal gravitation states that the force between two bodies i, j is an attraction proportional to the product of their masses m_i, m_j directed along the vector $\vec{r}_{ij}$ connecting the two bodies, and falls off as the square of their separation:

$$\vec{F}_{ij} = -\frac{Gm_i m_j}{r_{ij}^2}\hat{r}_{ij} = -\vec{F}_{ji} \tag{3.23}$$

where F_{ij} is the force felt by i due the mass j, and the vector $\vec{r}_{ij}$ points from j to i, and $\hat{r}_{ij}$ is the unit vector along $\vec{r}_{ij}$:

$$\vec{r}_{ij} = \vec{r}_i - \vec{r}_j \quad , \quad \hat{r}_{ij} = \frac{\vec{r}_{ij}}{|r_{ij}|} \quad , \quad |r_{ij}| = \sqrt{\vec{r}_{ij}\cdot\vec{r}_{ij}} \tag{3.24}$$

The fact that the mass that enters Newton's law of gravitation (3.23) and Newton's second law is not at all obvious–it is called the principle of equivalence and has been carefully tested by what are called Eötvös experiments. The idea is that the angular frequency of a pendulum depends on the ratio of the *gravitational* mass m_g that enters Newton's law of gravitation (3.23) and the *inertial* mass m_{inert} that enters Newton's second law $\vec{F} = m_{inert}\vec{a}$:

$$\omega^2 = \frac{m_g}{m_{inert}}\frac{g}{\ell} \tag{3.25}$$

So to test the principle of equivalence, you basically measure the frequency of identical pendulums with masses made of different materials; if the gravitational mass is not identical to the inertial mass for all substances, then the frequency will depend on what material is used to make the weight. The most recent tests have established the principle of equivalence to about one part in 10^{13}.

Note that for a two particle system, we can rewrite (3.23) as

$$\vec{F} = -\frac{GM\mu}{r^2}\hat{r} \tag{3.26}$$

where $\vec{F}$ is the force on the effective particle with mass μ in the gravitational potential of a total mass M.

One can do an explicit calculation showing that the a spherically symmetric body with a total mass M exerts the same force on objects that are outside the body as a point particle of the same mass at the center of this body. We will find this result using the Gauss theorem.

3.1 The gravitational field

It is useful to define the gravitational field $\vec{g}$ of a body–it is the force that it would exert on a unit mass; for a point particle of mass M it is

$$\vec{g} = -\frac{GM}{r^2}\hat{r} \tag{3.27}$$

If we calculate the flux of the gravitational field through a sphere of radius a that is centered on the point particle, by symmetry we get the surface area times the magnitude of the field at one point:

$$\text{Flux} = \int_A \vec{g}\cdot\hat{r}dA = (4\pi a^2)\left(-\frac{GM}{a^2}\right) = -4\pi GM \tag{3.28}$$

Notice that this is independent of the the radius of the sphere and is proportional to the mass enclosed. The Gauss theorem tells us that we can rewrite this as the volume integral of the divergence of the gravitational field; what the divergence is doesn't actually matter[2], what matters is that this implies that the flux coming from a collection of particles is just the sum of the fluxes, and hence is just the $4\pi GM$ where M is the total mass enclosed. In particular, this guarantees that for a spherically symmetric distribution of mass, outside the distribution, the gravitational field is exactly the same as for a point particle of mass M located at the center of the distribution.

[2]For a point particle, it is zero except at $r = 0$.

3.2 Explicit solution and Kepler's first law

Since in a central force, the angular momentum is conserved, we are left with only the radial component of Newton's second law to solve; dividing both sides of the equation by the mass μ of the particle and using (2.8), we find:

$$\ddot{r} - r\dot{\theta}^2 = \ddot{r} - \frac{L^2}{\mu^2 r^3} = -\frac{GM}{r^2} \tag{3.29}$$

Instead of solving explicitly for $r(t)$, it is interesting to find the orbit; in polar coordinates, this is $r(\theta)$. Writing

$$\dot{r} = \frac{dr}{d\theta}\dot{\theta} = \frac{dr}{d\theta}\left(\frac{L}{\mu r^2}\right) = -\frac{L}{\mu}\frac{d}{d\theta}\left(\frac{1}{r}\right) \tag{3.30}$$

it is natural to change variables to $u(\theta) = \frac{1}{r}$; we also define $\ell = \frac{L}{\mu}$. Then we have

$$\dot{\theta} = \ell u^2 \quad , \quad \dot{r} = -\ell\frac{du}{d\theta} \quad , \quad \ddot{r} = \dot{\theta}\frac{d}{d\theta}\dot{r} = -\ell^2 u^2 \frac{d^2u}{d\theta^2} \tag{3.31}$$

Now (3.29) becomes

$$-\ell^2 u^2 \frac{d^2u}{d\theta^2} - \ell^2 u^3 = -GMu^2 \tag{3.32}$$

and hence

$$\frac{d^2u}{d\theta^2} + u = \frac{GM}{\ell^2} \tag{3.33}$$

This we recognize as a harmonic oscillator with a displaced resting point, so we can immediately write down the solution:

$$u(\theta) = A\cos\theta + \frac{GM}{\ell^2} \tag{3.34}$$

where we are free to choose the phase $\theta_0 = 0$ (that just orients the solution along the x-axis. Then going back to $r(\theta) = 1/u$, we have

$$r = \frac{1}{A\cos\theta + \frac{GM}{\ell^2}} \equiv \frac{\alpha}{1 + \epsilon\cos\theta} \tag{3.35}$$

where

$$\alpha = \frac{\ell^2}{GM} \quad , \quad \epsilon = \frac{A\ell^2}{GM} \tag{3.36}$$

We can immediately see that for $\epsilon = 0$, this a circle of radius α; ϵ is called the eccentricity, and parameterizes the shape of the orbit. For $0 < \epsilon < 1$, the orbit is closed with minimum and maximum radii

$$r_{min} = r(0) = \frac{\alpha}{1+\epsilon} \quad , \quad r_{max} = r(\pi) = \frac{\alpha}{1-\epsilon} \tag{3.37}$$

The orbit is shown in Figure 4, along with $r(\frac{\pi}{2}) = \alpha$. Here r is the distance from the origin and r' is the distance from a point which is r_{min} from the farthest end of the orbit (the origin and this point are called the foci of the ellipse) for an arbitrary point on the ellipse. We define the *semimajor axis* a as the average of r_{min} and r_{max}; it is half the longest

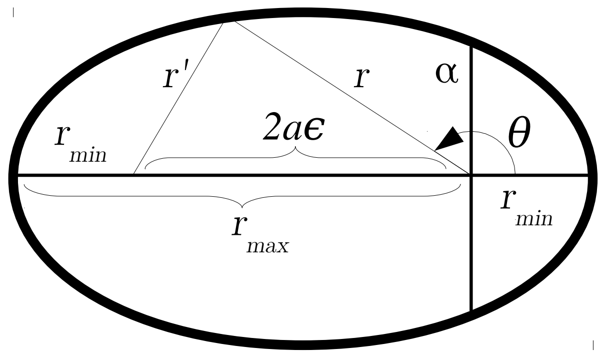


Figure 4: Elliptic orbit

distance across the orbit:

$$a = \frac{1}{2}\left(\frac{\alpha}{1+\epsilon} + \frac{\alpha}{1-\epsilon}\right) = \frac{\alpha}{1-\epsilon^2} \tag{3.38}$$

In terms of a, we have

$$r_{min} = a(1-\epsilon) \quad , \quad r_{max} = a(1+\epsilon) \tag{3.39}$$

and hence $r_{max} - r_{min} = 2a\epsilon$. Then using the pythagorean theorem, we have

$$\begin{aligned} r' &= \sqrt{(r\sin\theta)^2 + (r\cos\theta + 2\epsilon a)^2} \\ &= \sqrt{r^2 + 4r\epsilon a\cos\theta + 4\epsilon^2 a^2} \\ &= \sqrt{r^2 + 4ar(1+\epsilon\cos\theta) + 4\epsilon^2 a^2 - 4ar} \\ &= \sqrt{r^2 + 4a\alpha + 4\epsilon^2 a^2 - 4ar} \\ &= \sqrt{r^2 + 4a^2(1-\epsilon^2) + 4\epsilon^2 a^2 - 4ar} \\ &= \sqrt{r^2 + 4a^2 - 4ar} \\ &= \sqrt{(2a-r)^2} \\ &= 2a - r \end{aligned} \tag{3.40}$$

note that in the figure, $\cos\theta < 0$; we have used (3.35) to eliminate $\cos\theta < 0$ in the calculation. Hence $r + r' = 2a$; this is one definition of an ellipse (you can draw an ellipse by inserting two pins through a piece of paper into a board, tying a string to them, stretching the string with your pen, and then going around the two pins). In the last step, we have to be careful to take the positive square root; we know that $2a = r_{min} + r_{max} > r$

We have thus derived Kepler's first law: the planets move in ellipses with the sun at one of the foci.

If the eccentricity $\epsilon = 1$, then $r(\theta) \to \infty$, as $\theta \to \pm\pi$ and the orbit becomes unbounded. The physical picture is that the body, *e.g.* a comet, comes in from infinity from the left, goes around the origin, and goes back out to infinity on a parabolic orbit.

If $\epsilon > 1$, then the orbit is a hyperbola, with asymptotes at $\theta = \pm \arccos(\frac{1}{\epsilon})$. The three different situations are illustrated in Figure 5:

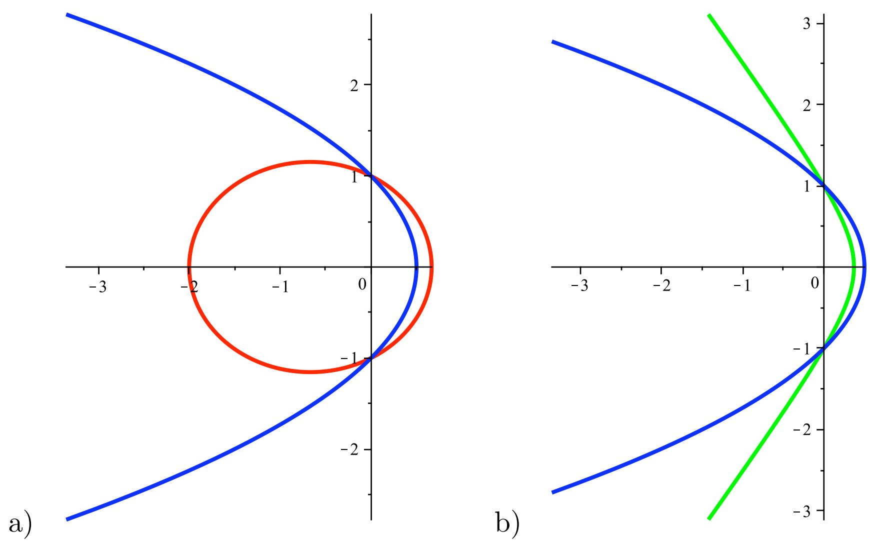


Figure 5: Sample orbits: ellipse, parabola, hyperbola: a) Ellipse $\epsilon = 0.5$, parabola $\epsilon = 1$. b) Parabola $\epsilon = 1$, hyperbola $\epsilon = 1.7$.

3.3 Kepler's third law

Kepler also found the relation between the period T of a planet's rotation and its distance from the sun. We can derive this as follows: start with (2.16) and integrate:

$$A = \int_0^T \dot{A} dt = \frac{TL}{2\mu} \tag{3.41}$$

where the A is the area πab of the ellipse[3]. The semiminor axis b can be found by the Pythagorean theorem and the expression $r_{max} - r_{min} = 2a\epsilon$ found above.

$$b^2 = a^2 - (\epsilon a)^2 \quad \Rightarrow \quad b = a\sqrt{1-\epsilon^2} = \sqrt{a\alpha} \tag{3.42}$$

and hence

$$T = \frac{2\pi\sqrt{a^3\alpha}}{\ell} \tag{3.43}$$

Using the expression (3.36) $\alpha = \frac{\ell^2}{GM}$, we find

$$T = \frac{2\pi a^{\frac{3}{2}}}{\sqrt{GM}} \tag{3.44}$$

which is Kepler's third law.

4 Velocity curves

For a circular orbit, we can rewrite Kepler's third law in terms of the the velocity of the orbiting body:

$$\frac{2\pi a}{v(a)} = \frac{2\pi a^{\frac{3}{2}}}{\sqrt{GM}} \tag{4.45}$$

which implies that the mass being orbited can be computed from the $v(a)$:

$$M = \frac{av^2(a)}{G} \tag{4.46}$$

Of course, only the mass enclosed in the orbit is felt by the body, so the *velocity curve* measures the mass distribution. Measurements of the velocities of stars in galaxies and and galaxies in galaxy clusters show that there is a large "halo" of invisible "dark" matter that has mass but doesn't interact (except perhaps very weakly) with normal matter by any force other than gravitation.

4.1 Energy

The total energy can be found from the effective potential (2.20): at the minimum (or maximum) radius, the radial velocity vanishes, and hence the energy is just the effective potential evaluated at r_{min}:

$$E = \frac{L^2}{2\mu r_{min}^2} - \frac{GM\mu}{r_{min}} = \frac{L^2(1+\epsilon)^2}{2\mu\alpha^2} - \frac{GM\mu(1+\epsilon)}{\alpha} \tag{4.47}$$

[3]In Cartesian coordinates, the equation for an ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Clearly, this is just a "squashed" unit circle, found by rescaling the coordinates x and y, and hence the area is just rescaled by a factor ab.

Using $\alpha = \frac{L^2}{GM\mu^2}$ (3.36), we find

$$E = \frac{G^2M^2\mu^3(1+\epsilon)^2}{2L^2} - \frac{G^2M^2\mu^3(1+\epsilon)}{L^2} = \frac{G^2M^2\mu^3(\epsilon^2-1)}{2L^2} \tag{4.48}$$

note that this is doesn't change if we let $\epsilon \to -\epsilon$, which shows that we would have gotten the same answer if we had used r_{max}. We can use (4.48) to express ϵ in terms of the physical parameters E, L:

$$\epsilon = \sqrt{1 + \frac{2L^2E}{G^2M^2\mu^3}} \tag{4.49}$$

We can also write

$$E = -\frac{GM\mu}{2a} \tag{4.50}$$

Thus the total energy E is the same as the total energy of a body in a circular orbit with radius $r = a$ and is half its potential energy.

5 Stability of circular orbits in general potentials

Consider a circular orbit in a general central potential; since $\dot{r} = \ddot{r} = 0$ for a circular orbit with fixed radius a Newton's second law (2.15) implies

$$-\frac{L^2}{\mu a^3} = f(a) \quad \Rightarrow \quad -\frac{\mu\ell^2}{a^3} = f(a) \tag{5.51}$$

Now consider a small deviation from a perfect circle $r = a + \delta(t)$; we could Taylor expand (2.15) directly, but it is nicer use (3.31) and rewrite (2.15) as an equation for the orbit:

$$-\mu\ell^2u^2\frac{d^2u}{d\theta^2} - \mu\ell^2u^3 = f\left(\frac{1}{u}\right) \quad \Rightarrow \quad \frac{d^2u}{d\theta^2} + u = -\frac{1}{\mu\ell^2u^2}f\left(\frac{1}{u}\right) \tag{5.52}$$

Using $u = \frac{1}{a+\delta(\theta)} = \frac{1}{a} - \frac{\delta}{a^2} + \mathcal{O}(\delta^2)$, Taylor expanding in δ and dropping all $\mathcal{O}(\delta^2)$ terms, we find

$$\left(-\frac{1}{a^2}\right)\frac{d^2\delta}{d\theta^2} + \frac{1}{a} - \frac{\delta}{a^2} = \left(-\frac{1}{\mu\ell^2}\right)\left((a+\delta)^2f(a+\delta)\right) \tag{5.53}$$

which, using (5.51) to solve for ℓ, gives

$$\frac{d^2\delta}{d\theta^2} + \delta = (-a^2)\left(\frac{1}{a^3f(a)}\right)\left((2a\delta)f(a) + a^2\frac{df(a)}{da}\delta\right) \tag{5.54}$$

and hence

$$\frac{d^2\delta}{d\theta^2} + \left(3 + \frac{a}{f(a)}\frac{df(a)}{da}\right)\delta = 0 \tag{5.55}$$

which is just a simple harmonic oscillator with

$$\omega_0^2 = 3 + \frac{a}{f(a)} \frac{df(a)}{da} \tag{5.56}$$

thus if $\left(3 + \frac{a}{f(a)} \frac{df(a)}{da}\right) > 0$, $\delta(\theta)$ will be a periodic function, and for a small perturbation, the orbit will remain nearly circular; on the other hand, if $\left(3 + \frac{a}{f(a)} \frac{df(a)}{da}\right) < 0$, ω_0 will be imaginary and $\delta(\theta)$ will grow exponentially until the approximation that δ is small breaks down. In this case, the circular orbit will be unstable.

If we consider a force of the form

$$f(r) = -cr^n \quad \Rightarrow \quad \frac{d}{dr} f(r) = -cnr^{n-1} \tag{5.57}$$

$$\omega_0^2 = 3 + n \tag{5.58}$$

and hence the orbit is stable when

$$n > -3 \tag{5.59}$$

Thus an attractive central force of the form $f(r) = -cr^n$ has stable circular orbits for all $n > -3$, which implies that an attractive potential of the form

$$U(r) = \begin{cases} \frac{c}{n+1} r^{n+1} & n \neq -1 \\ c \ln r & n = -1 \end{cases} \tag{5.60}$$

has stable circular orbits for $(n+1) > -2$. In this case, near $r = 0$, the repulsive centrifugal barrier from the L^2 term dominates in the effective potential, U_{eff}, whereas for large r, the attractive $U(r)$ dominates. For $n + 1 < 0$, this gives an effective potential of the type shown in Figure 3b); for $n + 1 \geq 0$, the effective potential will look like the one shown in Figure 3c). Notice that certain powers give nice closed orbits: $n = -2$ (the gravitational case) has $\omega_0 = 1$ (an ellipse with the origin at the focus, as expected), $n = 1$ (the harmonic oscillator) has $\omega_0 = 2$ (an ellipse centered at the origin, also expected), $n = 6$ gives $\omega = 3$ with orbits that look like a rounded triangle, etc.

The unstable case, for $n + 1 < -2$, is just the reverse: near $r = 0$, the attractive $U(r)$ dominates, whereas for large r, the repulsive centrifugal barrier from the L^2 term dominates in the effective potential; the effective potential is of the type shown in Figure 3e).

6 Perturbations of the $\frac{1}{r}$ potential and precession

In a gravitational r^{-1} potential, elliptical orbits do not precess, that is, the direction of the major axis does not change. We would like to know how small deviations from a pure r^{-1}

potential effect change nearly circular orbits; this calculation is very important, as such terms arise from the force of the various planets on each other. Indeed, the position of the planet Neptune was predicted from a careful analysis of such corrections to the orbit of Uranus. Similarly, very careful observations and calculations led to the observation of a tiny discrepancy from the prediction of Newton's laws in the orbit of Mercury. This discrepancy is explained by Einstein's theory of General Relativity, and is an important test of the theory.

We start with the result of the previous section: a slightly perturbed circular orbit with radius a has an orbit that fluctuates about the circle with an angular frequency given by (5.56)

$$\omega_0^2 = 3 + \frac{a}{f(a)}\frac{df(a)}{da} \tag{6.61}$$

We define the apsidal angle ψ as the angle between r_{min} and r_{max}; you can think of this as half of the "angular period". If $\omega_0 = 1$, the orbit doesn't precess, and $\psi = \pi$. In general, the angle will be

$$\psi = \frac{\pi}{\omega_0} \tag{6.62}$$

Thus if $\omega_0 > 1$, the orbit returns to itself sooner than expected, and $\psi < \pi$; we say the orbit regresses. If $\omega_0 < 1$, $\psi > \pi$ and the orbit advances.

For example, consider a force that deviates from r^{-2} by a small amount βr^n:

$$f(r) = -\frac{GMm}{r^2} + \beta r^n \tag{6.63}$$

Then

$$\frac{df(a)}{da} = 2\frac{GM\mu}{a^3} + \beta n a^{n-1} \tag{6.64}$$

and

$$\omega_0^2 = 3 - \frac{2\frac{GM\mu}{a^2} + n\beta a^n}{\frac{GM\mu}{a^2} - \beta a^n} = 3 - \frac{2GM\mu + \beta n a^{n+2}}{GM\mu - \beta a^{n+2}} = \frac{GM\mu - (n+3)\beta a^{n+2}}{GM\mu - \beta a^{n+2}} \tag{6.65}$$

which can be expanded in β to give

$$\omega_0^2 = 1 - \frac{(n+2)\beta a^{n+2}}{GM\mu} + \mathcal{O}(\beta^2) \tag{6.66}$$

Then (6.62) gives us

$$\psi = \pi\left(1 + \frac{(n+2)\beta a^{n+2}}{2GM\mu}\right) + \mathcal{O}(\beta^2) \tag{6.67}$$

7 Runga-Lenz vector

As noted above, in a $1/r$ potential, the axis of the orbit doesn't precess. This implies that there must be some conserved quantity and hence some associated symmetry.

The conserved quantity was discovered by J. Hermann and J. Bernoulli, and is called the Runge-Lenz vector:

$$\vec{A} = \vec{p} \times \vec{L} - GM\mu^2 \hat{r} = p^2 \vec{r} - (\vec{r} \cdot \vec{p})\vec{p} - \frac{GM\mu^2}{r}\vec{r} \tag{7.68}$$

(not to be confused with the area A) is conserved:

$$\dot{\vec{A}} = \dot{\vec{p}} \times \vec{L} - GM\mu^2 \dot{\theta}\hat{\theta} = -\frac{GM\mu}{r^2}\hat{r} \times \vec{L} - GM\mu^2\dot{\theta}\hat{\theta} = -GM\mu^2\dot{\theta}(\hat{r} \times \hat{z} + \hat{\theta}) = 0 \tag{7.69}$$

where we have used the fact that the angular momentum $\vec{L} = \mu r^2 \dot{\theta}\hat{z}$ is conserved, and the equation of motion

$$\dot{\vec{p}} = -\frac{GM\mu}{r^2}\hat{r} \tag{7.70}$$

The magnitude of the Runge-Lenz vector is not an independent quantity and can be expressed in term the total energy and the magnitude of the angular momentum:

$$\vec{A} \cdot \vec{A} = 2\mu(\vec{L} \cdot \vec{L})E + (GM\mu^2)^2 \tag{7.71}$$

However, the direction is a new conserved quantity–it points along the major axis of the ellipse that the orbit describes.

The Hamiltonian of the system is

$$H = \frac{1}{2\mu}(\vec{p} \cdot \vec{p}) - \frac{GM\mu}{r} \tag{7.72}$$

The transformations of the coordinate $\vec{r}$ is given by the Poisson bracket:

$$\delta\vec{r} = [\vec{r}, \vec{\alpha} \cdot \vec{A}] = 2(\vec{\alpha} \cdot \vec{r})\vec{p} - (\vec{r} \cdot \vec{p})\vec{\alpha} - (\vec{\alpha} \cdot \vec{p})\vec{r} \tag{7.73}$$

where $\vec{\alpha}$ is a constant vector parameter. We rewrite these transformations in Lagrangian formalism by substituting $\vec{p} = \mu\dot{\vec{r}}$:

$$\delta\vec{r} = \mu\left(2(\vec{\alpha} \cdot \vec{r})\dot{\vec{r}} - (\vec{r} \cdot \dot{\vec{r}})\vec{\alpha} - (\vec{\alpha} \cdot \dot{\vec{r}})\vec{r}\right) \tag{7.74}$$

We can check that this is a symmetry of the Lagrangian:

$$\begin{aligned}
\delta L &= \delta\left(\frac{1}{2}\mu(\dot{\vec{r}}\cdot\dot{\vec{r}}) + \frac{GM\mu}{r}\right) \\
&= \mu^2\dot{\vec{r}}\cdot\frac{d}{dt}\Big(2(\vec{\alpha}\cdot\vec{r})\dot{\vec{r}} - (\vec{r}\cdot\dot{\vec{r}})\vec{\alpha} - (\vec{\alpha}\cdot\dot{\vec{r}})\vec{r}\Big) - \frac{GM\mu^2}{r^3}\vec{r}\cdot\Big(2(\vec{\alpha}\cdot\vec{r})\dot{\vec{r}} - (\vec{r}\cdot\dot{\vec{r}})\vec{\alpha} - (\vec{\alpha}\cdot\dot{\vec{r}})\vec{r}\Big) \\
&= \mu^2\frac{d}{dt}\left((\dot{\vec{r}}\cdot\dot{\vec{r}})(\vec{\alpha}\cdot\vec{r}) - (\vec{r}\cdot\dot{\vec{r}})(\vec{\alpha}\cdot\dot{\vec{r}}) + \frac{GM}{r}\vec{\alpha}\cdot\vec{r}\right) \qquad (7.75)
\end{aligned}$$

In polar coordinates, we have:

$$\delta(r\hat{r}) = \delta r\hat{r} + r\delta\theta\hat{\theta} = \mu[2\alpha_r r(\dot{r}\hat{r} + r\dot{\theta}\hat{\theta}) - r\dot{r}(\alpha_r\hat{r} + \alpha_\theta\hat{\theta}) - (\alpha_r\dot{r} + \alpha_\theta r\dot{\theta})r\hat{r}] \qquad (7.76)$$

which implies

$$\delta r = -\mu\alpha_\theta(r^2\dot{\theta}) \quad , \quad \delta\theta = \mu(2\alpha_r r\dot{\theta} - \alpha_\theta\dot{r}) \qquad (7.77)$$

These transformations are symmetries of the Lagrangian $L = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) + GM\mu/r$ provided that we express α_r, α_θ in terms of constant parameters α_x, α_y:

$$\alpha_r = \alpha_x\cos\theta + \alpha_y\sin\theta \quad , \quad \alpha_\theta = \alpha_y\cos\theta - \alpha_x\sin\theta \qquad (7.78)$$