Physics 303/573

Linear Algebra 2

September 7, 2016

1 More on vectors

In the last set of notes, I tried to be abstract and general. Here are a few more concrete ways of understanding some aspects of vectors.

1.1 The interpretation of the scalar product

The scalar product of two vectors $\vec{v}, \vec{w}$ is easy to understand: it is the product of the lengths times the *cosine* of the angle between the vectors:

$$\vec{v} \cdot \vec{w} = |v||w| \cos\theta \tag{1.1}$$

Equivalently, it is the product of the length of $\vec{v}$ times the length of the projection of $\vec{w}$ along $\vec{v}$ as shown in Fig. 1a):

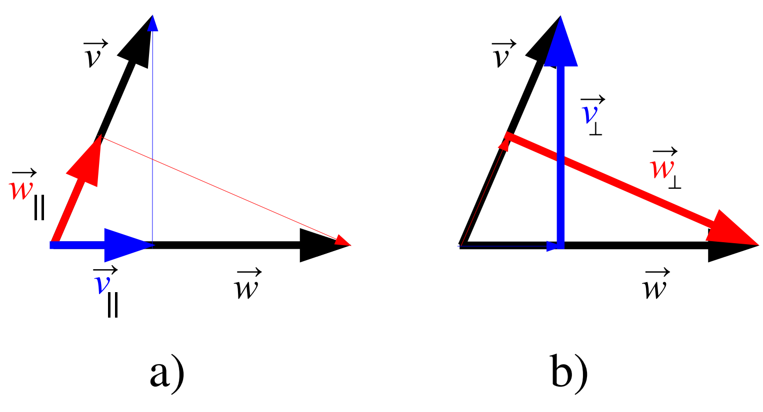

Figure 1: **a)** $\vec{v} \cdot \vec{w} = |v||w_{||}| = |w||v_{||}|$. **b)** $|\vec{v} \times \vec{w}| = |v||w_{\perp}| = |v_{\perp}||w|$. Here $\vec{v}_{\perp}$ and $\vec{v}_{||}$ are the components of $\vec{v}$ perpendicular to $\vec{w}$ and parallel to $\vec{w}$, respectively, and similarly $\vec{w}_{\perp}$ and $\vec{w}_{||}$ are the components of $\vec{w}$ perpendicular to $\vec{v}$ and parallel to $\vec{v}$, respectively.

1.2 The interpretation of the vector (or cross) product

Whereas the scalar product is a number, the vector product of two vectors is itself a vector. Its magnitude is the product of the lengths times the *sine* of the angle between the vectors:

$$|\vec{v} \times \vec{w}| = |v||w| \sin\theta \tag{1.2}$$

Equivalently, it is the product of the length of $\vec{v}$ times the length of the projection of $\vec{w}$ perpendicular to $\vec{v}$; see Figure 1b). This has a simple geometric interpretation–*it is the area of the parallelogram defined by the two vectors.*

The direction of the cross-product is perpendicular to both vectors, and by convention, is given by the right hand rule–this is equivalent to $\hat{x} \times \hat{y} = \hat{z}$. For $\vec{v} \times \vec{w}$, point your right hand in the direction of $\vec{v}$, curl your fingers in the direction of $\vec{w}$, and your thumb will point along the cross product.

1.3 Polar coordinates

I have taken considerable care not to write things in a way that depends on the choice of basis. Often we use the Cartesian basis $\hat{x}, \hat{y}, \hat{z}$; I will now discuss some other choices of basis. The simplest example of a non-Cartesian basis is in two dimensions: polar coordinates. If we have a vector $\vec{v} = v_x\hat{x} + v_y\hat{y}$ (I use $\hat{x} = \hat{x}$ as a shorthand for $\hat{e}_x$, etc.), then we can obviously write it as $\vec{v} = |v|\hat{v}$, where recall that $\hat{v}$ is the unit vector along $\vec{v}$. If we do this for the position vector that gives the position of a point in the plane as a displacement from the origin, we have:

$$\vec{r} = |r|\hat{r} \tag{1.3}$$

where $\hat{r}$ the unit vector along $\vec{r}$ depends on the direction that $\vec{r}$ points:

$$\hat{r}(\theta) = \hat{x}\cos\theta + \hat{y}\sin\theta \tag{1.4}$$

If $\vec{r}$ lies on the x-axis, then $\theta = 0$ and $\hat{r}(0) = \hat{x}$. To have an orthonormal basis, we need another perpendicular unit vector–we choose it to agree with $\hat{y}$ for $\theta = 0$:

$$\hat{\theta}(\theta) = \hat{y}\cos\theta - \hat{x}\sin\theta \tag{1.5}$$

You can easily check that $\hat{r}, \hat{\theta}$ are orthonormal. Notice that we can write

$$\begin{pmatrix} \hat{r} \\ \hat{\theta} \end{pmatrix} = \mathbf{R}(\theta) \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} \equiv \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} \tag{1.6}$$

If we have a vector $\vec{v} = v_x\hat{x} + v_y\hat{y} \equiv v_r\hat{r} + v_\theta\hat{\theta}$, then we can easily relate the components

$$\begin{pmatrix} v_r \\ v_\theta \end{pmatrix} = ([\mathbf{R}(\theta)]^{-1})^T \begin{pmatrix} v_x \\ v_y \end{pmatrix} \equiv \mathbf{R}(\theta) \begin{pmatrix} v_x \\ v_y \end{pmatrix} \equiv \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} v_x \\ v_y \end{pmatrix} \tag{1.7}$$

where we have used that $\mathbf{R}(\theta)$ is an orthogonal matrix: $\mathbf{R}^T = \mathbf{R}^{-1}$. This is a general argument: suppose $\vec{v} = v^i\hat{e}_i = v'^i\hat{e}'_i$ with $\hat{e}'_i = \mathbf{R}_i{}^j\hat{e}_j$. Then

$$v^i\hat{e}_i = v'^i\mathbf{R}_i{}^j\hat{e}_j \quad \Rightarrow \quad v^j = v'^i\mathbf{R}_i{}^j \quad \Rightarrow \quad v'^i = [\mathbf{R}^{-1}]_j{}^i v^j \equiv [\mathbf{R}^T]_j{}^i v^j = \mathbf{R}^i{}_j v^j \tag{1.8}$$

(Recall that for an orthonormal basis, we can raise and lower indices at will).

1.4 Cylindrical coordinates

The simplest generalization of polar coordinates to three dimensions is cylindrical coordinates– we simply use polar coordinates in the x, y plane and include the z direction. The position vector of a point is thus:

$$\vec{r} = R\hat{R} + z\hat{z} \tag{1.9}$$

where $R = \sqrt{x^2 + y^2}$ is the distance to the z-axis and z is of course the distance along the

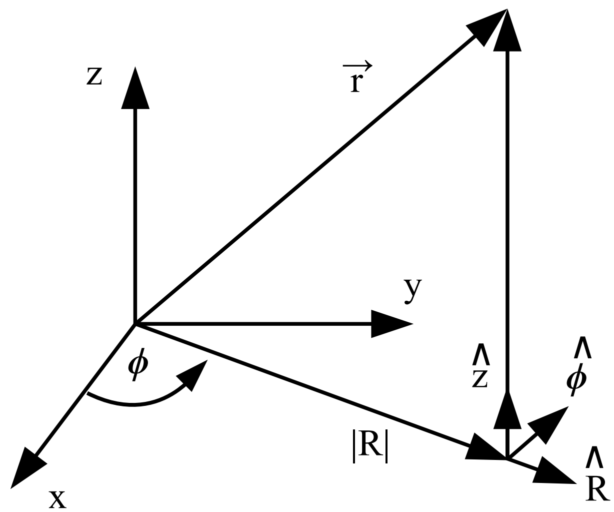

Figure 2: Cylindrical coordinates

z-axis. We can summarize cylindrical coordinates by

$$\begin{pmatrix} \hat{R} \\ \hat{\phi} \\ \hat{z} \end{pmatrix} = \mathbf{R}(\phi) \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix} \equiv \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix} \tag{1.10}$$

1.5 Spherical polar coordinates

Spherical polar coordinates describe the position vector $\vec{r} = |r|\hat{r}(\phi, \theta)$ in terms of its length r and two angles: the azimuthal angle ϕ of rotation around the z-axis (the longitude), and the polar angle (or inclination) θ down from the z-axis (the complementary angle to the latitude).

The unit vectors are:

$$\begin{aligned} \hat{r} &= (\hat{x}\cos\phi + \hat{y}\sin\phi)\sin\theta + \hat{z}\cos\theta \\ \hat{\theta} &= (\hat{x}\cos\phi + \hat{y}\sin\phi)\cos\theta - \hat{z}\sin\theta \\ \hat{\phi} &= -\hat{x}\sin\phi + \hat{y}\cos\phi \end{aligned} \tag{1.11}$$

which we can rewrite in terms of two rotation matrices $\mathbf{R}_y(\theta)$, a rotation by an angle θ around the y-axis, and $\mathbf{R}_z(\phi)$, a rotation by an angle ϕ around the z-axis:

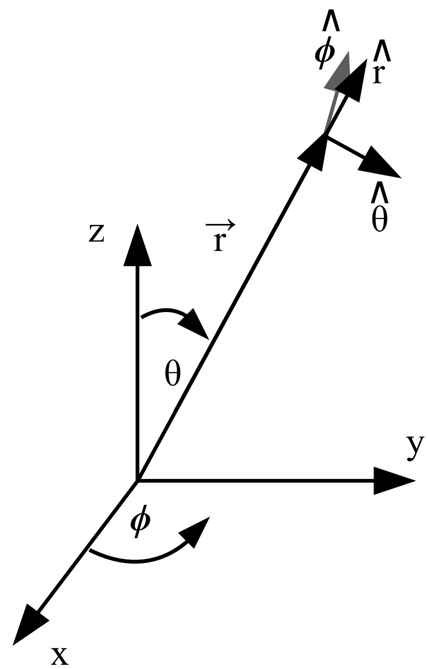

Figure 3: Spherical polar coordinates

$$\begin{pmatrix} \hat{\theta} \\ \hat{\phi} \\ \hat{r} \end{pmatrix} = \mathbf{R}_y(\theta)\mathbf{R}_z(\phi) \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix} \equiv \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix}$$
$$= \begin{pmatrix} \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \\ \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix} \tag{1.12}$$

2 Differentiation and vectors

We now consider differentiation of vectors in different coordinate systems. In general, if we have a vector $\vec{v} = v^i \hat{e}_i$, then both the components and the basis vectors can vary:

$$\frac{d\vec{v}}{dt} = \left(\frac{dv^i}{dt}\right)\hat{e}_i + v^i\left(\frac{d\hat{e}_i}{dt}\right) \tag{2.13}$$

Note that the derivative of a unit vector is always perpendicular to the unit vector.

$$\hat{e}\cdot\hat{e} = 1 \quad \Rightarrow \quad \frac{d\hat{e}}{dt}\cdot\hat{e} = 0 \tag{2.14}$$

Both the scalar and the cross product obey the Leibniz rule; this follows immediately from the index notation that I introduced last time.

We now consider vectors in different bases:

2.1 Cartesian coordinates

One advantage of Cartesian coordinates is that the basis vectors are fixed. If we have a position vector

$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z} \tag{2.15}$$

of some object that is moving, then its velocity vector is simply

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{dx}{dt}\hat{x} + \frac{dy}{dt}\hat{y} + \frac{dz}{dt}\hat{z} = v_x\hat{x} + v_y\hat{y} + v_z\hat{z} \tag{2.16}$$

Similarly, the acceleration is

$$\vec{a} = \frac{d^2\vec{r}}{dt^2} = \frac{d^2x}{dt^2}\hat{x} + \frac{d^2y}{dt^2}\hat{y} + \frac{d^2z}{dt^2}\hat{z} = a_x\hat{x} + a_y\hat{y} + a_z\hat{z} \tag{2.17}$$

In other coordinate systems, the basis vectors themselves change as a function of time.

2.2 Polar coordinates

We can find how the basis vectors change either geometrically or using the expressions (1.4) and (1.5); we find

$$\frac{d\hat{r}}{dt} = \frac{d\theta}{dt}\hat{\theta} \quad , \quad \frac{d\hat{\theta}}{dt} = -\frac{d\theta}{dt}\hat{r} \tag{2.18}$$

Thus the velocity in polar coordinates is

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}(r\hat{r}) = \frac{dr}{dt}\hat{r} + r\frac{d\theta}{dt}\hat{\theta} \tag{2.19}$$

and thus the radial component of the velocity is $\frac{dr}{dt}$ and the angular component is $r\frac{d\theta}{dt}$. The acceleration is more complicated (work this out for yourself):

$$\vec{a} = \frac{d^2\vec{r}}{dt^2} = \frac{d^2}{dt^2}(r\hat{r}) = \left(\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right)\hat{r} + \left(2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}\right)\hat{\theta} \tag{2.20}$$

Cylindrical coordinates are just the same, with a component along the fixed z-axis.

2.3 Spherical coordinates

For compactness, from now on I will use the notation of Newton, and indicate time derivatives with dots:

$$\frac{d}{dt}f(t) \equiv \dot{f}(t) \tag{2.21}$$

Spherical coordinates are sufficiently complicated that it pays off to (1.12) and the properties of rotation matrices:

$$\frac{d}{dt}\begin{pmatrix} \hat{\theta} \\ \hat{\phi} \\ \hat{r} \end{pmatrix} = (\dot{\mathbf{R}}_y(\theta)\mathbf{R}_z(\phi) + \mathbf{R}_y(\theta)\dot{\mathbf{R}}_z(\phi))\begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix}$$

$$
\begin{aligned}
&= (\dot{\mathbf{R}}_y(\theta)\mathbf{R}_z(\phi) + \mathbf{R}_y(\theta)\dot{\mathbf{R}}_z(\phi))[\mathbf{R}_y(\theta)\mathbf{R}_z(\phi)]^{-1} \begin{pmatrix} \hat{\theta} \\ \hat{\phi} \\ \hat{r} \end{pmatrix} \\
&= (\dot{\mathbf{R}}_y(\theta)\mathbf{R}_z(\phi) + \mathbf{R}_y(\theta)\dot{\mathbf{R}}_z(\phi))[\mathbf{R}_z(\phi)]^{-1}[\mathbf{R}_y(\theta)]^{-1} \begin{pmatrix} \hat{\theta} \\ \hat{\phi} \\ \hat{r} \end{pmatrix} \\
&= \left(\dot{\mathbf{R}}_y(\theta)[\mathbf{R}_y(\theta)]^{-1} + \mathbf{R}_y(\theta)\dot{\mathbf{R}}_z(\phi)[\mathbf{R}_z(\phi)]^{-1}[\mathbf{R}_y(\theta)]^{-1}\right) \begin{pmatrix} \hat{\theta} \\ \hat{\phi} \\ \hat{r} \end{pmatrix} \qquad (2.22)
\end{aligned}
$$

This may look complicated, but is actually pretty simple.

$$
\begin{aligned}
\dot{\mathbf{R}}_y(\theta)[\mathbf{R}_y(\theta)]^{-1} &= \dot{\theta} \begin{pmatrix} -\sin\theta & 0 & -\cos\theta \\ 0 & 0 & 0 \\ \cos\theta & 0 & -\sin\theta \end{pmatrix} \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \\
&= \dot{\theta} \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad (2.23)
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbf{R}_y(\theta)\dot{\mathbf{R}}_z(\phi)[\mathbf{R}_z(\phi)]^{-1}[\mathbf{R}_y(\theta)]^{-1} &= \mathbf{R}_y(\theta)\dot{\phi} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} [\mathbf{R}_y(\theta)]^{-1} \\
&= \dot{\phi} \begin{pmatrix} 0 & \cos\theta & 0 \\ -\cos\theta & 0 & -\sin\theta \\ 0 & \sin\theta & 0 \end{pmatrix} \qquad (2.24)
\end{aligned}
$$

and so

$$
\begin{aligned}
\frac{d}{dt} \begin{pmatrix} \hat{\theta} \\ \hat{\phi} \\ \hat{r} \end{pmatrix} &= \left[\dot{\theta} \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} + \dot{\phi} \begin{pmatrix} 0 & \cos\theta & 0 \\ -\cos\theta & 0 & -\sin\theta \\ 0 & \sin\theta & 0 \end{pmatrix} \right] \begin{pmatrix} \hat{\theta} \\ \hat{\phi} \\ \hat{r} \end{pmatrix} \\
&= \begin{pmatrix} -\dot{\theta}\hat{r} + \dot{\phi}(\cos\theta)\hat{\phi} \\ -\dot{\phi}(\cos\theta)\hat{\theta} - \dot{\phi}(\sin\theta)\hat{r} \\ \dot{\theta}\hat{\theta} + \dot{\phi}(\sin\theta)\hat{\phi} \end{pmatrix} \qquad (2.25)
\end{aligned}
$$

We can now apply these. For example, the velocity in spherical coordinates is

$$
\vec{v} = \dot{\vec{r}} = \frac{d}{dt}(r\hat{r}) = \dot{r}\hat{r} + r(\dot{\theta}\hat{\theta} + \dot{\phi}(\sin\theta)\hat{\phi}) \qquad (2.26)
$$

Similarly, the acceleration is

$$
\begin{aligned}
\vec{a} = \dot{\vec{v}} &= \ddot{r}\hat{r} + 2\dot{r}(\dot{\theta}\hat{\theta} + \dot{\phi}(\sin\theta)\hat{\phi}) + r[\ddot{\theta}\hat{\theta} + (\ddot{\phi}\sin\theta + \dot{\phi}\dot{\theta}\cos\theta)\hat{\phi}] \\
&\quad + r\left(\dot{\theta}(-\dot{\theta}\hat{r} + \dot{\phi}(\cos\theta)\hat{\phi}) - \dot{\phi}(\sin\theta)\dot{\phi}[(\cos\theta)\hat{\theta} + (\sin\theta)\hat{r}]\right) \\
&= (\ddot{r} - r\dot{\theta}^2 - r\dot{\phi}^2\sin^2\theta)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2\sin\theta\cos\theta)\hat{\theta} \\
&\quad + (2\dot{r}\dot{\phi}\sin\theta + 2r\dot{\theta}\dot{\phi}\cos\theta + r\ddot{\phi}\sin\theta)\hat{\phi} \qquad (2.27)
\end{aligned}
$$