Physics 303/573

Some practice problems for the final exam–many more than will be in the final.

Summation convention: repeated indices are implicit summations. $q_i p_i \equiv \sum_i q_i p_i$, *etc.*

Euler-Lagrange: If $q_i(t)$ obeys the set of differential equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i} \text{ for all } i \quad \Rightarrow \quad S = \int L\,dt \text{ is extremized.}$$

Here $L(\{q_i\},\{\dot{q}_i\},t) = T - U$ is the Lagrangian and

$$\frac{\partial L}{\partial q_i} \text{ is the generalized force acting in the } i \text{ "direction".}$$

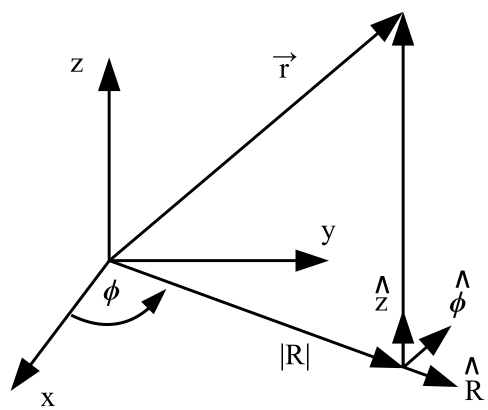
Symmetries: If there exist transformations of $q_i(t)$ with a constant parameter β,

$$\delta q_i(t) = \beta R_i(q_j(t), \dot{q}_j(t)) \text{ such that } \delta L(q_i, \dot{q}_i) = \beta \frac{d}{dt} K(q_i, \dot{q}_i) \text{ and if } \frac{\partial L}{\partial t} = 0$$

then the Noether charge $Q = \frac{\partial L}{\partial \dot{q}_i} R_i - K$ is conserved (independent of time) when the Euler-Lagrange equations are obeyed.

Conjugate momenta: $p_i = \frac{\partial L}{\partial \dot{q}_i}$ **Hamiltonian:** $H = p_i \dot{q}_i - L$

Figure 1: **Cylindrical polar coordinates:**
$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z} = R\hat{R} + z\hat{z}$.
$\hat{R} = \cos(\phi)\,\hat{x} + \sin(\phi)\,\hat{y}$, $\hat{\phi} = -\sin(\phi)\,\hat{x} + \cos(\phi)\,\hat{y}$.
Length: $r^2 = R^2 + z^2$, $R^2 = x^2 + y^2$.
Cartesian: $x = R\cos\phi$, $y = R\sin\phi$, $z = z$.
Kinetic energy in cylindrical polar coordinates: $T = \frac{1}{2}m(\dot{R}^2 + R^2\dot{\phi}^2 + \dot{z}^2)$.
Grad: $\vec{\nabla} f(R,\phi,z) = \hat{R}\frac{\partial}{\partial R} f + \hat{\phi}\frac{1}{R}\frac{\partial}{\partial \phi} f + \hat{z}\frac{\partial}{\partial z} f$
Curl: $\vec{\nabla} \times \vec{v} = \hat{R}\left(\frac{1}{R}\frac{\partial}{\partial \phi} v_z - \frac{\partial}{\partial z} v_\phi\right) + \hat{\phi}\left(\frac{\partial}{\partial z} v_R - \frac{\partial}{\partial R} v_z\right) + \hat{z}\frac{1}{R}\left(\frac{\partial}{\partial R}(R v_\phi) - \frac{\partial}{\partial \phi} v_R\right)$
Div: $\vec{\nabla} \cdot \vec{v} = \frac{1}{R}\left(\frac{\partial}{\partial R}(R v_R) + \frac{\partial}{\partial \phi} v_\phi\right) + \frac{\partial}{\partial z} v_z$



Damped harmonic oscillator: (damping proportional to the velocity):

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0 \text{ has solutions } x(t) = (x_{max})e^{-\beta t}\cos(\omega t + \varphi) \text{ where } \omega^2 = \omega_0^2 - \beta^2 .$$

The energy is $E(t) = \frac{1}{2}m(\dot{x}^2 + \omega_0^2 x^2)$.

Driven damped harmonic oscillator:

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = \frac{1}{m} F_\omega e^{i\omega_F t} \text{ has special solution } x(t) = A_{\omega_F} e^{i\omega_F t} \text{ where } A_{\omega_F} = \frac{1}{m}\frac{F_\omega}{\omega_0^2 - \omega_F^2 + 2i\beta\omega_F}$$

transient solutions are determined by boundary conditions. System is linear so solutions superpose.

Two-body system: $\vec{r}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{M}$. Total mass: $M = m_1 + m_2$.
Relative coordinate: $\vec{r} = \vec{r}_1 - \vec{r}_2$. Reduced mass: $\mu = \frac{m_1 m_2}{M}$.
Kinetic energy: $T = \frac{1}{2}(m_1|\vec{v}_1|^2 + m_2|\vec{v}_2|^2) = \frac{1}{2}(M|\vec{v}_{cm}|^2 + \mu|\vec{v}|^2)$, where $\vec{v}_1 = \frac{d\vec{r}_1}{dt}$, $\vec{v}_2 = \frac{d\vec{r}_2}{dt}$, the center of mass velocity is $\vec{v}_{cm} = \frac{d\vec{r}_{cm}}{dt}$, the relative velocity is $\vec{v} = \frac{d\vec{r}}{dt}$.

1. a) Using the definition of the curl in terms of circulation, compute the formula for the curl in **cylindrical** polar coordinates.

b) Find the circulation along a horizontal circle with radius$= a$, and center on the z-axis at a height $z = h$ for the vector field

$$\vec{v} = (R - z)\cos\phi\hat{R} + Rz\hat{\phi} + (R + z)\sin\phi\hat{z}$$

by direct computation.

c) Compute the curl of $\vec{v}$.

d) Integrate the curl of $\vec{v}$ over a flat disk bounded by the horizontal circle with radius$= a$, and center on the z-axis at a height $z = h$, and show that it agrees with the circulation that you calculated above.

2. a) Using the definition of the divergence in terms of flux, compute the formula for the divergence in **spherical** polar coordinates.

b) Find the flux through a sphere with radius$=a$ for the vector field

$$\vec{v} = r(1 + \cos\phi)\hat{r} + r\sin\phi\hat{\phi} + r\cos\theta\sin\phi\hat{\theta}$$

by direct computation. c) Compute the divergence of $\vec{v}$.

d) Integrate the divergence of $\vec{v}$ over the volume and show that it agrees with the flux that you calculated above.

3. An ant crawls with a velocity $\vec{v}$ on a turntable in the x, y plane that is turning with an angular speed Ω in the counterclockwise direction.

(a) What is the rotation vector of the turntable?

(b) What is the vector describing the coriolis force if the ant is moving in the $+\hat{R}$ direction (in cylindrical polar coordinates) from a radius R_0?

(c) What is the vector describing the coriolis force if the the ant moves tangentially to the turntable, that is, $\vec{v} = v\hat{\phi}$, from a radius R_0?

(d) What is the vector describing the coriolis force if the the ant climbs vertically along a tube on the turntable, that is, $\vec{v} = v\hat{z}$, from a radius R_0?

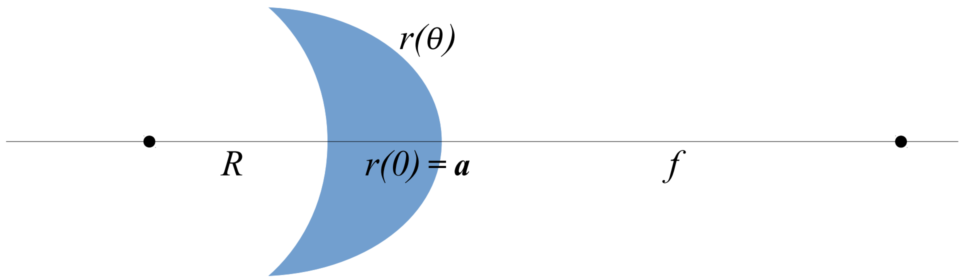
4. Consider a bead that sits on a wire whose shape, in cylindrical polar coordinates, is $z = f(R)$ for some function $f(R)$, where $R = \sqrt{x^2 + y^2}$. The wire is spinning around the z-axis with constant angular velocity ω. The force of gravity acts in the negative z direction.

(a) Write down the Lagrangian in an inertial frame in cylindrical polar coordinates, impose the constraint with a Lagrange multiplier, and find all the Euler-Lagrange equations.

(b) Find the shape $f(R)$ of the wire such that the bead will stay wherever we put it; that means the wire is an equipotential, in other words, if we move the bead along the wire, the forces do no work, in other words, the force we found in part (b) must be perpendicular to a line tangent to the wire.

(c) Suppose we had a bucket of water and were to spin it around the z-axis with constant angular velocity ω. What shape would the surface of the water settle into? Why?

5. Consider a lens with two curved surfaces as shown in the figure. One surface (on the left in the figure)



is spherical with radius R, the other surface (on the right in the figure) has a shape $r(\theta)$ that you will find. The thickness of the lens at $\theta = 0$ is given as $r(0) \equiv a$. We want to find the shape that will focus light emanating from the center of the sphere (which we choose as the origin; this is a distance R from the spherical surface), to a point a distance f from the right surface (at $\theta = 0$), that is, a distance $a + f$ from the source of the light. Assume the glass has constant index of refraction n.

a) When the light enters the glass coming from the left, what is the angle that it makes with the surface of the glass (hint–what angle does a radius make to a circle)? Since the index of refraction is constant, what path will it follow through the glass–draw a sample path from the origin to the focal point for a general angle $\theta \neq 0$.

b) Fermat's principle says that all the paths from between these two points are extremal; explain why this means that the time along all those paths is the same.

c) Now use Fermat's principle to find a quadratic equation that relates $r(\theta)$ to $\cos\theta$ and the constants n, a, R, and f. Hint: The answer actually depends only on 3 of those constants.

6. Consider a two masses m_1, m_2 connected by a spring that cannot bend but can stretch with spring constant k, and assume that the spring's unstretched length is ℓ. The masses slide without friction along the inside of a bowl whose shape is given by $z = f(x, y)$, and gravity acts on them.

a) Using a Lagrange multiplier, write down the Lagrangian (with arbitrary $f(x, y)$) for this system in cartesian coordinates.

For the rest of the problem, suppose $f = 0$.

b) Solve the constraint and write down the Lagrangian in terms x_1, x_2, y_1, y_2 and their time derivatives.

c) Rewrite the Lagrangian of part b) in terms of center of mass and relative coordinates. The Lagrangian should split into two decoupled parts, one for the center of mass coordinates, and one for the relative coordinates.

d) For the center of mass coordinate, write down the Euler-Lagrange equation and give the general solution.

e) For the part of the Lagrangian that depends only on the relative coordinates, find a symmetry and the corresponding conserved Noether charge.

f) For the relative coordinates, find the equation that determines the equilibrium point and the frequency of small oscillations around the equilibrium (in terms of the equilibrium point).

g) When $\ell = 0$, the problem simplifies greatly. In cartesian coordinates, what is the general solution?

7. a) Using the relation between an inertial frame and a frame fixed to our earth (ignoring the the orbital motion around the sun), derive the expressions for the Coriolis force and centrifugal forces as a function of the latitude.

b) In which direction will the Coriolis force act on an airplane flying due north at constant altitude at a latitude λ?

8. Consider a thin square $\ell \times \ell$ slab with a uniformly distributed mass m.

a) Thinking of it as a collection of rods stacked up on top of each other, find the moment of inertia along an axis lying on one edge.

b) Use the parallel axis and perpendicular axis theorems to find the moment of inertia along an axis perpendicular to the square passing through the center of mass.

9. Consider an **damped** driven harmonic oscillator with a mass m, a damping force with a coefficient $2m\beta$, $\beta < \omega_0$, a spring constant $m\omega_0^2$, and a driving force $F(t) = m[\sin(2\omega_0 t) + \frac{1}{2}\cos(\frac{1}{2}\omega_0 t)]$.

a) Find a particular solution.

b) Now find the transients (solutions to the equation without a driving force) that guarantee that the initial position and velocity are zero: $x(0) = v(0) = 0$.

c) Do the transients ever die out? Does the particular solution die out? If there were no damping term, would the transients die out? Would the particular solution die out?

10. Consider a planet with mass m orbiting a very heavy sun with mass $M >> m$.

a) What symmetry guarantees that the orbit is planar?

b) Write down the Lagrangian in polar coordinates in the plane of the motion.

c) Find the Hamiltonian. Given the angular momentum is L_z, what is the effective potential?

d) What is the radius of a circular orbit with angular momentum is L_z?

e) Imagine that a spherically symmetric uniform cloud of dark matter surrounds the sun out to a finite radius R_{DM} that is considerably larger than the radius you found in part d). Suppose that the mass density of this cloud is ρ; find the contribution that this cloud makes to the potential for $r < R_{DM}$ and $r > R_{DM}$.

f) What effect would this have on the perihelion precession of a slightly elliptic orbit that stays within $r < R_{DM}$?