Physics 303/573

Systems with more than one particle

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1 Center of mass

1.1 Definitions

Suppose that we have a system of n-particles with masses m_i, $i = 1..n$, and positions $\vec{r}_i$. It is very useful to introduce the weighted average of the positions. This is called the *center of mass* of the system, and we shall see that it is the balance point–if the system of particles is rigidly connected and hung at its center of mass, it will not tilt. The center of mass $\vec{r}_{cm}$ is defined by

$$\vec{r}_{cm} = \frac{\sum_{i=1}^{n} m_i \vec{r}_i}{\sum_{i=1}^{n} m_i} = \frac{1}{M} \sum_{i=1}^{n} m_i \vec{r}_i \tag{1.1}$$

where M is the total mass of the system:

$$M = \sum_{i=1}^{n} m_i \tag{1.2}$$

For a continuous distribution of mass described by a density $\rho(\vec{r})$, we have

$$M = \int d^3r \, \rho(\vec{r}) \tag{1.3}$$

where the integral is over all space (of course, if $\rho = 0$ outside a finite volume, then the integral reduces to an integral over that finite volume), and

$$\vec{r}_{cm} = \frac{1}{M} \int d^3r \, \rho(\vec{r}) \vec{r} \tag{1.4}$$

A useful but obvious fact is that the center of mass of two collections of masses is the just the center of mass of the two center of masses, that is, suppose

$$M_1 = \sum_{i=1}^{k} m_i \quad , \quad \vec{r}_{cm1} = \frac{1}{M_1} \sum_{i=1}^{k} m_i \vec{r}_i \quad , \quad M_2 = \sum_{i=k+1}^{n} m_i \quad , \quad \vec{r}_{cm2} = \frac{1}{M_2} \sum_{i=k+1}^{n} m_i \vec{r}_i \tag{1.5}$$

then

$$\vec{r}_{cm(tot)} = \frac{1}{M_1 + M_2}(M_1\vec{r}_{cm1} + M_2\vec{r}_{cm2}) = \frac{1}{M}\sum_{i=1}^{n} m_i\vec{r}_i \tag{1.6}$$

The velocity of the center of mass is the same weighted average of the velocities:

$$\vec{v}_{cm} = \frac{d}{dt}\vec{r}_{cm} = \frac{1}{M}\sum_{i=1}^{n} m_i\vec{v}_i \tag{1.7}$$

The total linear momentum is by definition the sum of the linear momenta of all the particles:

$$\vec{P}_{tot} = \sum_{i=1}^{n}\vec{p}_i = \sum_{i=1}^{n} m_i\vec{v}_i \tag{1.8}$$

but this is clearly the same as the center of mass momentum:

$$\vec{P}_{tot} = \sum_{i=1}^{n} m_i\vec{v}_i = M\vec{v}_{cm} = \vec{P}_{cm} \tag{1.9}$$

Suppose that each particle feels a force; we can think of the force on the i'th particle as coming partly from an external force $\vec{F}_{(ext)i}$ and partly from its interactions with the other particles:

$$\vec{F}_i = \vec{F}_{(ext)i} + \sum_{j\neq i}^{n}\vec{F}_{ij} \tag{1.10}$$

Newton's second law tells us

$$\vec{F}_i = \dot{\vec{p}}_i \tag{1.11}$$

Using (1.9), we find

$$\dot{\vec{P}}_{cm} = \sum_{i=1}^{n}\vec{F}_i = \sum_{i=1}^{n}\left(\vec{F}_{(ext)i} + \sum_{j\neq i}^{n}\vec{F}_{ij}\right) \tag{1.12}$$

However, if the forces between the particles obey Newton's third law, $F_{ij} = -F_{ji}$; since we are summing over all unequal i, j, for each pair of particles we have $F_{ij} + F_{ji}$, and hence the double sum in (1.12) vanishes:

$$\dot{\vec{P}}_{cm} = \sum_{i=1}^{n}\vec{F}_{(ext)i} \equiv \vec{F}_{ext} \tag{1.13}$$

In particular, if there are no external forces and the particles only interact among themselves, the total momentum is conserved. This occurs in many situations of interest–for example when two particles scatter. They collide, interact in complicated ways, but nonetheless, the total momentum is conserved.

1.2 Computing the center of mass

Let's consider a few examples. For two masses m_1 and m_2, we have

$$\vec{r}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2} = \vec{r}_1 + \frac{m_2}{m_1 + m_2}(\vec{r}_2 - \vec{r}_1) \tag{1.14}$$

which means that the center of mass lies a fraction $a = m_2/(m_1 + m_2)$ of the way along the vector $\vec{r}_2 - \vec{r}_1$ between the two masses:

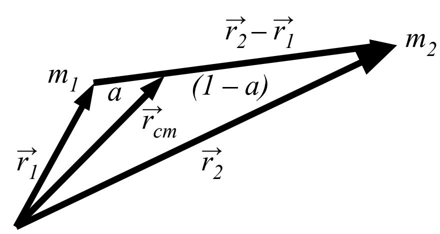

Figure 1: $\vec{r}_{cm}$ lies a fraction $a = \frac{m_2}{(m_1+m_2)}$ of the distance between m_1 and m_2.

Another easy and familiar example is the case of three equal masses. If we let one of the vertices be at the origin, and the other two be at $\vec{r}_1, \vec{r}_2$, respectively, then $\vec{r}_{cm} = \frac{1}{3}(\vec{r}_1 + \vec{r}_2)$. I leave it as an exercise for you to prove that this is the centroid of the triangle their positions define, that is, the intersection of the three lines from the vertices to the midpoints of the opposite sides.

A final example that we'll consider is a uniform solid (with density $\rho = constant$) of revolution, that is a solid whose cross-sectional radius at a height z is given by a function $R = f(z)$:

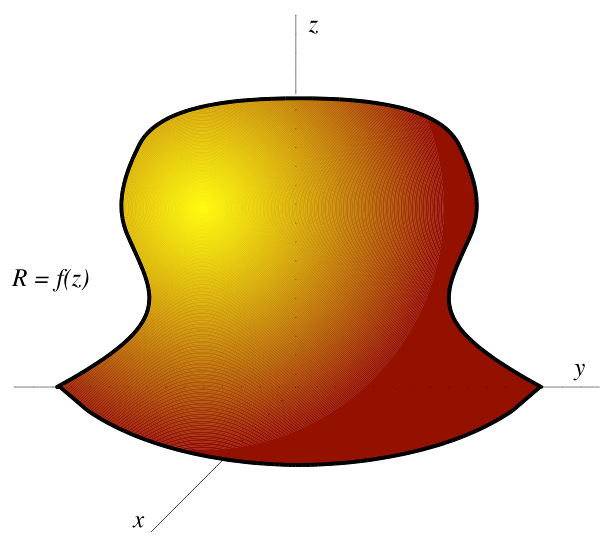

Figure 2: Solid of revolution about the z-axis; $R = \sqrt{x^2 + y^2}$.

Using symmetry, it is clear that the center of mass is on the z-axis. Furthermore, as shown in (1.6), the center of mass can be computed by subdividing the solid into clumps and taking the center of mass of their centers of mass. In this case, we slice the solid into horizontal circles of radius $R = f(z)$ and hence cross-sectional area $\pi(f(z))^2$. Then we have $x_{cm} = y_{cm} = 0$ and

$$z_{cm} = \frac{1}{M}\int_0^h dz\, z\, \rho\, \pi(f(z))^2 \quad , \quad M = \int_0^h dz\, \rho\, \pi(f(z))^2 \tag{1.15}$$

Here h is the height of the object. For the cone, we can take $f(z) = R_0(1 - z/h)$, where R_0 is the radius at the base $z = 0$, and find (canceling out obvious common factors of $\pi\rho/h^2$ in the numerator and denominator)

$$z_{cm} = \frac{\int_0^h dz\, z\, (h-z)^2}{\int_0^h dz\, (h-z)^2} = \frac{h^2(\frac{1}{2}h^2) - 2h(\frac{1}{3}h^3) + (\frac{1}{4}h^4)}{h^2(h) - 2h(\frac{1}{2}h^2) + (\frac{1}{3}h^3)} = \frac{h}{4} \tag{1.16}$$

2 Rocket Motion

Consider a rocket with instantaneous momentum $P(t) = m(t)v(t)$ in some fixed direction. It is ejecting some exhaust at a velocity v_{ex} in the reverse direction. The rate at which it is losing mass, $\frac{dm}{dt}$, is the rate at which it is ejecting the exhaust, so total momentum conservation implies

$$\frac{dP_{tot}}{dt} = \frac{dm}{dt}v_{ex} + m\frac{dv}{dt} = 0 \tag{2.17}$$

which integrates to

$$\int_{v_0}^{v(t)} dv = -v_{ex}\int_{m_0}^{m(t)} \frac{dm}{m} \quad \Rightarrow \quad v(t) = v_0 + v_{ex}\, \ln\left(\frac{m_0}{m(t)}\right) \tag{2.18}$$

We see that the final velocity depends linearly on the velocity of the exhaust, but only logarithmically on the fraction of the mass that is fuel, so having a better fuel really matters.

3 Angular momentum

The instantaneous angular momentum of a particle with position $\vec{r}$ and momentum $\vec{p}$ depends on the choice of the origin of the coordinate system and is given by

$$\vec{L} = \vec{r} \times \vec{p} \tag{3.19}$$

The total angular momentum of a system is the sum of all the angular momenta:

$$\vec{L}_{tot} = \sum_{i=1}^{n} \vec{r}_i \times \vec{p}_i \tag{3.20}$$

Differentiating, we have

$$\dot{\vec{L}}_{tot} = \sum_{i=1}^{n} \dot{\vec{r}}_i \times \vec{p}_i + \sum_{i=1}^{n} \vec{r}_i \times \dot{\vec{p}}_i = \sum_{i=1}^{n} \vec{r}_i \times \vec{F}_i \tag{3.21}$$

where we have used $\dot{\vec{r}}_i \times \vec{p}_i = \vec{v}_i \times \vec{p}_i = \vec{v}_i \times m\vec{v}_i = 0$ and Newton's second law. Substituting (1.10) into (3.21), we find

$$\dot{\vec{L}}_{tot} = \sum_{i=1}^{n} \vec{r}_i \times \vec{F}_{(ext)i} + \sum_{i=1}^{n}\sum_{j\neq i}^{n} \vec{r}_i \times \vec{F}_{ij} \tag{3.22}$$

Using Newton's third law $\vec{F}_{ij} = -\vec{F}_{ji}$, we obtain

$$\dot{\vec{L}}_{tot} = \sum_{i=1}^{n} \vec{r}_i \times \vec{F}_{(ext)i} + \frac{1}{2}\sum_{i=1}^{n}\sum_{j\neq i}^{n} \vec{r}_{ij} \times \vec{F}_{ij} \tag{3.23}$$

where $\vec{r}_{ij} = \vec{r}_j - \vec{r}_i$ is the vector from the i'th particle to the j'th particle.

If the interaction between the particles is through a central force, then $\vec{F}_{ij} \propto \vec{r}_{ij}$ and the second term in (3.23) vanishes:

$$\dot{\vec{L}}_{tot} = \sum_{i=1}^{n} \vec{r}_i \times \vec{F}_{(ext)i} \equiv \vec{N} \tag{3.24}$$

where we have introduced the total torque $\vec{N}$ [1]. If there is no torque, then as long as the particles in the system interact through central forces, the total angular momentum $\vec{L}_{tot}$ is conserved.

A particular example of this arises in planetary motion. If we put the origin of our coordinate system at the sun, the force of the sun on a planet is along the radial direction, and hence $\vec{N} = \vec{r} \times \vec{F} = 0$, and angular momentum is conserved. If we consider the area swept out by the planet as it moves for an infinitesimal time dt, we know from section 1.2 of Lecture 2 that this can be written as

$$dA = \frac{1}{2}|\vec{r} \times \vec{v}dt| = \frac{1}{2}\frac{|L|}{m}dt \tag{3.25}$$

which means that conservation of angular momentum implies Kepler's second law: Planets sweep out equal areas in equal times.

It is useful to introduce coordinates relative to the center of mass:

$$\vec{\bar{r}}_i = \vec{r}_i - \vec{r}_{cm} \tag{3.26}$$

Note that relative coordinates are of course the same: $\vec{\bar{r}}_{ij} = \vec{r}_{ij}$. Note as well that the center of mass of these new coordinates is just the origin:

$$\sum_{i=1}^{n} m_i\vec{\bar{r}}_i = \sum_{i=1}^{n} m_i(\vec{r}_i - \vec{r}_{cm}) = M(\vec{r}_{cm} - \vec{r}_{cm}) = 0 \tag{3.27}$$

so $\vec{\bar{r}}_i$ are nothing more than the coordinates in the center of mass frame. Thus if we compute

$$\vec{\bar{p}}_i = m_i(\dot{\vec{r}}_i - \dot{\vec{r}}_{cm}) \tag{3.28}$$

[1]Taylor uses $\vec{\Gamma}$ for torque and $\vec{\ell}$ for angular momentum. The notation in these notes is more common.

we find that from the definition of the center of mass

$$\sum_{i=1}^{n} \vec{\bar{p}}_i = \vec{P}_{tot} - \vec{P}_{cm} = 0 \tag{3.29}$$

The angular momentum (3.19) can be rewritten as:

$$\vec{L}_{tot} = \sum_{i=1}^{n} (\vec{r}_{cm} + \vec{\bar{r}}_i) \times m_i(\dot{\vec{r}}_{cm} + \dot{\vec{\bar{r}}}_i) = \vec{r}_{cm} \times \vec{P}_{cm} + \sum_{i=1}^{n} \vec{\bar{r}}_i \times m_i \dot{\vec{\bar{r}}}_i \equiv \vec{L}_{orb} + \vec{L}_{spin} \tag{3.30}$$

where two terms vanish because of the definition (3.26):

$$\sum_{i=1}^{n} m_i \vec{\bar{r}}_i = \sum_{i=1}^{n} m_i \dot{\vec{\bar{r}}}_i = 0 \quad \Rightarrow \quad \sum_{i=1}^{n} \vec{\bar{r}}_i \times m_i \dot{\vec{r}}_{cm} = \sum_{i=1}^{n} \vec{r}_{cm} \times m_i \dot{\vec{\bar{r}}}_i = 0 \tag{3.31}$$

Thus the total angular momentum has an orbital component $\vec{L}_{orb}$, which is the angular momentum of a point particle with mass M (the total mass of the system) located at the center of mass, and a spin component $\vec{L}_{spin}$, which is the angular momentum of the system rotating around its center of mass.

4 Kinetic energy

The total kinetic energy of a system is just the sum of the kinetic energies of all the particles; if we write it in terms of the velocity of the center of mass $\vec{v}_{cm}$ and the velocities relative to the center of mass $\vec{\bar{v}}_i$, we find:

$$T_{tot} = \sum_{i=1}^{n} \frac{1}{2} m_i \vec{v}_i \cdot \vec{v}_i = \frac{1}{2} \sum_{i=1}^{n} m_i (\vec{v}_{cm} + \vec{\bar{v}}_i) \cdot (\vec{v}_{cm} + \vec{\bar{v}}_i) = \frac{1}{2} M \vec{v}_{cm} \cdot \vec{v}_{cm} + \frac{1}{2} \sum_{i=1}^{n} m_i \vec{\bar{v}}_i \cdot \vec{\bar{v}}_i \tag{4.32}$$

where the term $\sum_{i=1}^{n} m_i \vec{\bar{v}}_i \cdot \vec{v}_{cm} = 0$ for the same reason as (3.27),(3.29). Thus the total kinetic energy can be decomposed into the kinetic energy of the center of mass and the internal kinetic energy: $T_{tot} = T_{cm} + T_{int}$.