Physics 303/573

Projectile Motion

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1 General comments

Consider an object moving under the force of gravity and encountering air resistance. In general, Newton's law take the form

$$\vec{F} = -mg\hat{z} - f(v)\hat{v} = m\vec{a} \tag{1.1}$$

where $v = |v|$ is the magnitude of the velocity. For a smooth frictional force $f(v)$, we can Taylor expand:

$$f(v) = f(0) + f'(0)v + \frac{1}{2}f''(0)v^2 + ... \tag{1.2}$$

The book argues that $f(0) = 0$, but this need not be the case–if we consider a block sliding on a surface with friction and air resistance, then $f(0) = \mu N$ where μ is the coefficient of friction and N is the normal force (see Example 1.1 in chapter 1). However, for an object moving through a gas or liquid, it is true that $f(0) = 0$.

In general, if we can find coordinates such that Newton's laws simply become three one-dimensional systems, we say that the system is separable; for general functions $f(v)$, Newton's are not separable.

2 Linear drag

When $f(0) = 0$, the first term in the expansion is the term linear in velocity. Then Newton's laws *are* separable, and it is pretty easy to solve them:[1]

$$\vec{F} = -mg\hat{z} - \frac{m}{\tau}\vec{v} = m\vec{a} \tag{2.3}$$

and hence

$$\ddot{\vec{r}} = -\frac{1}{\tau}\dot{\vec{r}} - g\hat{z} \tag{2.4}$$

This can be integrated in two ways; integrating directly, we have[2]

$$\dot{\vec{r}} - \vec{v}_0 = -\frac{1}{\tau}(\vec{r} - \vec{r}_0) - gt\hat{z} \tag{2.5}$$

which we can solve for $\vec{r}(t)$ in terms of $\dot{\vec{r}}(t)$:

$$\vec{r}(t) = \vec{r}_0 - \tau(\dot{\vec{r}} - \vec{v}_0 + gt\hat{z}) \tag{2.6}$$

However, if we write (2.4) as

$$\dot{\vec{v}} = -\frac{1}{\tau}\vec{v} - g\hat{z} \tag{2.7}$$

we have for the x-component[3]

$$\dot{v}_x = -\frac{1}{\tau}v_x \quad \Rightarrow \quad v_x(t) = \dot{x}(t) = \dot{x}_0\, e^{-\frac{t}{\tau}} \tag{2.8}$$

and similarly

$$\dot{y}(t) = \dot{y}_0\, e^{-\frac{t}{\tau}} \tag{2.9}$$

These immediately give (using 2.6)

$$x(t) = x_0 + \tau\dot{x}_0\left(1 - e^{-\frac{t}{\tau}}\right) \quad , \quad y(t) = y_0 + \tau\dot{y}_0\left(1 - e^{-\frac{t}{\tau}}\right) \tag{2.10}$$

Note that these reach a limiting value $x \to x_0 + \tau\dot{x}_0, y \to y_0 + \tau\dot{y}_0$ as $t \to \infty$.

[1] As is discussed in the book, linear air resistance is actually a bad approximation for macroscopic objects; nonetheless, it is interesting to study.

[2] I will use $\vec{v}_0$ and $\dot{\vec{r}}_0$ interchangeably, particularly for the components $v_x(0) = v_{x0} = \dot{x}_0$, etc.

[3] I am assuming everyone knows that a first order differential equation of the form $\dot{f} = cf$ has the most general solution $f(t) = f(0)e^{ct}$.

The z component can also be integrated:

$$\dot{v}_z = -\frac{1}{\tau}v_z - g \quad \Rightarrow \quad -\frac{t}{\tau} = \int_{\dot{z}_0} \frac{dv_z}{v_z + \tau g} = \ln(v_z + \tau g) - \ln(\dot{z}_0 + \tau g) \tag{2.11}$$

which implies

$$\dot{z}(t) = (\dot{z}_0 + \tau g)e^{-\frac{t}{\tau}} - \tau g = \dot{z}_0 e^{-\frac{t}{\tau}} - g\tau(1 - e^{-\frac{t}{\tau}}) \tag{2.12}$$

We see there is a limiting velocity $v_z \to -v_{ter}$:

$$v_{ter} = g\tau \tag{2.13}$$

and hence

$$v_z(t) = -v_{ter} + (\dot{z}_0 + v_{ter})e^{-\frac{t}{\tau}} \tag{2.14}$$

Again, using (2.6) and using $v_{ter} = g\tau$, we find

$$z(t) = z_0 + \tau(\dot{z}_0 + v_{ter})(1 - e^{-\frac{t}{\tau}}) - v_{ter}t \tag{2.15}$$

Combining all these equations, we have

$$\vec{v}(t) = \vec{v}_0 - \frac{1}{\tau}(\vec{r} - \vec{r}_0) - gt\hat{z} = \vec{v}_0 e^{-\frac{t}{\tau}} - v_{ter}(1 - e^{-\frac{t}{\tau}})\hat{z} \tag{2.16}$$

and

$$\vec{r}(t) = \vec{r}_0 + \vec{v}_0\tau(1 - e^{-\frac{t}{\tau}}) - v_{ter}(t - \tau(1 - e^{-\frac{t}{\tau}}))\hat{z} \tag{2.17}$$

As we noted above, the terminal velocity $\vec{v}_{ter}$ for $t \to \infty$ lies entirely along the vertical:

$$\vec{v}_{ter} = -v_{ter}\hat{z} = -g\tau\hat{z} \tag{2.18}$$

and is independent of the initial velocity.

Suppose we choose our coordinates such that $\vec{r}_0 = 0$. The maximum height is achieved when $v_z = 0$; using (2.6) we find

$$z_{max} = h = \tau(\dot{z}_0 - gt_{max}) \tag{2.19}$$

where t_{max} is found from (2.11):

$$t_{max} = \tau \ln(1 + \frac{\dot{z}_0}{v_{ter}}) \tag{2.20}$$

and hence

$$h = \tau(\dot{z}_0 - v_{ter}\ln(1 + \frac{\dot{z}_0}{v_{ter}})) \tag{2.21}$$

Notice that at $t = 2t_{max}$, the height

$$\begin{aligned}
z(2t_{max}) &= \left(\tau \dot{z}_0 + \tau^2 g\right)\left(1 - \frac{1}{(1+\frac{\dot{z}_0}{v_{ter}})^2}\right) - \tau^2 2g\ln(1 + \frac{\dot{z}_0}{v_{ter}}) \\
&= \tau^2 g\left(1 + \frac{\dot{z}_0}{v_{ter}} - \frac{1}{1+\frac{\dot{z}_0}{v_{ter}}} - 2\ln(1 + \frac{\dot{z}_0}{v_{ter}})\right) \\
&= \tau^2 g\left(\frac{\frac{2\dot{z}_0}{v_{ter}} + \left(\frac{\dot{z}_0}{v_{ter}}\right)^2}{1+\frac{\dot{z}_0}{v_{ter}}} - 2\ln(1 + \frac{\dot{z}_0}{v_{ter}})\right) \qquad (2.22)
\end{aligned}$$

is positive for all positive values of $\frac{\dot{z}_0}{v_{ter}}$; this means that with any amount of (linear) air resistance, it takes longer for the projectile to fall from its maximum height than to reach the maximum in the first place.

The actual time t_f it takes to fall back down follows from (2.15), and is the solution to the transcendental equation

$$t_f = \left(\frac{\dot{z}_0}{g} + \tau\right)(1 - e^{-\frac{t_f}{\tau}}) \tag{2.23}$$

If we substitute this into (2.10), we find the horizontal range. Alternatively, we can solve for the time in terms of the range x_{max}; we use (2.10) to write

$$1 - e^{-\frac{t_f}{\tau}} = \frac{x_{max}}{\tau \dot{x}_0} \tag{2.24}$$

which, when substituted into (2.23), gives

$$t_f = \left(1 + \frac{\dot{z}_0}{v_{ter}}\right)\frac{x_{max}}{\dot{x}_0} \tag{2.25}$$

But (2.24) can be solved directly for t_f:

$$t_f = -\tau\ln\left(1 - \frac{x_{max}}{\tau \dot{x}_0}\right) \tag{2.26}$$

Setting (2.25) and (2.26) equal, we find a transcendental equation for x_{max}:

$$\ln\left(1 - \frac{x_{max}}{\tau \dot{x}_0}\right) + \frac{1}{\tau}\left(1 + \frac{\dot{z}_0}{v_{ter}}\right)\frac{x_{max}}{\dot{x}_0} = 0 \tag{2.27}$$

If we assume that we can expand in $\frac{1}{\tau}$, using $\ln(1-x) = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \ldots$, we find

$$-\frac{x_{max}}{\tau \dot{x}_0} - \frac{1}{2}\left(\frac{x_{max}}{\tau \dot{x}_0}\right)^2 - \frac{1}{3}\left(\frac{x_{max}}{\tau \dot{x}_0}\right)^3 - \frac{1}{4}\left(\frac{x_{max}}{\tau \dot{x}_0}\right)^4 - \ldots + \frac{1}{\tau}\left(1 + \frac{\dot{z}_0}{v_{ter}}\right)\frac{x_{max}}{\dot{x}_0} = 0 \tag{2.28}$$

which implies

$$\frac{1}{2}\left(\frac{x_{max}}{\tau \dot{x}_0}\right) + \frac{1}{3}\left(\frac{x_{max}}{\tau \dot{x}_0}\right)^2 + \frac{1}{4}\left(\frac{x_{max}}{\tau \dot{x}_0}\right)^3 + \ldots - \frac{\dot{z}_0}{v_{ter}} = 0 \tag{2.29}$$

We rewrite this as

$$x_{max} = 2\tau \dot{x}_0 \left(\frac{\dot{z}_0}{v_{ter}} - \frac{1}{3}\left(\frac{x_{max}}{\tau \dot{x}_0}\right)^2 - \frac{1}{4}\left(\frac{x_{max}}{\tau \dot{x}_0}\right)^3 + \ldots\right) \tag{2.30}$$

The leading term in $1/\tau$ gives the frictionless result (using $v_{ter} = g\tau$),

$$x_{max} = \frac{2\dot{x}_0 \dot{z}_0}{g} + ... \tag{2.31}$$

Substituting this into (2.30) and keeping the next order term we get

$$x_{max} = \frac{2\dot{x}_0 \dot{z}_0}{g}\left(1 - \frac{4}{3}\frac{\dot{z}_0}{v_{ter}} + \ldots\right) \tag{2.32}$$

Keeping the next order in (2.30), we find

$$x_{max} = \frac{2\dot{x}_0 \dot{z}_0}{g}\left(1 - \frac{4}{3}\frac{\dot{z}_0}{v_{ter}} + \frac{14}{9}\left(\frac{\dot{z}_0}{v_{ter}}\right)^2 + \ldots\right) \tag{2.33}$$

3 Quadratic drag

3.1 Dimensional analysis

Whereas linear drag is rather easy to analyze, it is harder to understand–it has to do with properties of the medium. Quadratic drag is quite natural from a purely dimensional point of view. In contrast with electromagnetic and gravitational forces, the drag force exerted on the body shouldn't depend on the mass or composition of the body, but only on its shape. However, to get units of force, we need a mass, so the drag force should be proportional to the density ρ of the medium. A force needs to have units $\propto 1/t^2$, which comes from v^2. And it needs an overall unit of length; since $\rho \propto 1/\ell^3$ and $v^2 \propto \ell^2$, we find

$$\vec{F}_{(quad\ drag)} \propto \rho A v^2 \tag{3.34}$$

here A is the cross-sectional area of the object. This is very reasonable, and can be derived from conservation of momentum (the moving object transfers momentum to the medium)–see problem 2.4.

It also happens to be the case that for macroscopic objects in air, quadratic drag is by far the most important.

3.2 Integrating Newton's laws for quadratic drag

Newton's laws now imply

$$\ddot{\vec{r}} = -g\hat{z} - \frac{c}{m} v\vec{v} \tag{3.35}$$

or, in cartesian coordinates

$$\dot{v}_x = -\frac{c}{m}\sqrt{v_x^2 + v_z^2}\ v_x \quad , \quad \dot{v}_z = -g - \frac{c}{m}\sqrt{v_x^2 + v_z^2}\ v_z \tag{3.36}$$

The book says that these can't be solved in general and considers the special cases of purely horizontal and purely vertical motion, but by a clever trick, we can get pretty far and reduce the problem to some complicated integrals.

Let's start by looking at a combination of the two equations that gets rid of the ugly square root:

$$\dot{v}_z = -g + \frac{v_z}{v_x}\dot{v}_x \tag{3.37}$$

We see that the slope of the trajectory, v_z/v_x seems to play a big role, so let's give it a new name: $w = v_z/v_x$. Then we can rewrite (3.37) as

$$\dot{w} = -\frac{g}{v_x} \tag{3.38}$$

We can also rewrite the first equation in (3.36) as

$$\dot{v}_x = -\frac{c}{m}\sqrt{1 + w^2}v_x^2 \tag{3.39}$$

Differentiating (3.38), we find

$$\ddot{w} = \frac{g\dot{v}_x}{v_x^2} \tag{3.40}$$

and hence

$$\ddot{w} = -\frac{cg}{m}\sqrt{1+w^2} \tag{3.41}$$

We integrate this equation as follows (this technique works quite often, so it is important!): multiply both sides by $\dot{w}$ to get

$$\dot{w}\ddot{w} = \frac{d}{dt}\left(\frac{1}{2}\dot{w}^2\right) = -\frac{cg}{m}\sqrt{1+w^2}\ \dot{w} \tag{3.42}$$

and hence

$$\dot{w}^2 = -\frac{2cg}{m}\int^w d\tilde{w}\,\sqrt{1+\tilde{w}^2} = -\frac{cg}{m}\left(w\sqrt{1+w^2} + \ln\left(w+\sqrt{1+w^2}\,\right) - A\right) \tag{3.43}$$

where $\tilde{w}$ is the dummy integration variable and A is an integration constant; (3.38) implies that A must be

$$A = w_0\sqrt{1+w_0^2} + \ln\left(w_0 + \sqrt{1+w_0^2}\right) + \frac{mg}{c\dot{x}_0^2} \tag{3.44}$$

Using (3.38,3.43), we have found the explicit relation between v_x and v_z.

We can integrate any equation of the form $\dot{w}^2 = f(w)$:

$$t = \pm\int_{w_0}^{w(t)} \frac{d\tilde{w}}{\sqrt{f(\tilde{w})}} \tag{3.45}$$

where the choice of the sign of the square root depends on whether we are going forward or backward in time.

Applying this to (3.43), we find

$$t = -\sqrt{\frac{m}{cg}}\int_{w_0}^{w(t)} \frac{d\tilde{w}}{\sqrt{A - \tilde{w}\sqrt{1+\tilde{w}^2} - \ln\left(\tilde{w}+\sqrt{1+\tilde{w}^2}\,\right)}} \tag{3.46}$$

Here we have negative branch of the square root, because as t increases, the body is falling, and hence $w(t) - w_0$ grows more and more negative. This integral gives us $t(w)$, and implicitly, $w(t)$. We can also find explicit integral expressions for $x(w)$ and $z(w)$ using the chain rule. We write

$$x(t) = x_0 + \int_0^t d\tilde{t}\ v_x(\tilde{t}) = x_0 + \int_{w_0}^{w(t)} \frac{d\tilde{w}}{\dot{\tilde{w}}}\ v_x(\tilde{w}) \tag{3.47}$$

and hence,

$$x(w) = x_0 + \int_{w_0}^{w} \frac{d\tilde{w}}{\dot{\tilde{w}}} \left(-\frac{g}{\dot{\tilde{w}}}\right) = x_0 - g\int_{w_0}^{w} \frac{d\tilde{w}}{\dot{\tilde{w}}^2} \tag{3.48}$$

which, using (3.43), becomes

$$x(w) = x_0 - \frac{m}{c}\int_{w_0}^{w} \frac{d\tilde{w}}{A - \tilde{w}\sqrt{1+\tilde{w}^2} - \ln\left(\tilde{w} + \sqrt{1+\tilde{w}^2}\,\right)} \tag{3.49}$$

Similarly, using $v_z = wv_x$ we find

$$z(w) = z_0 - \frac{m}{c}\int_{w_0}^{w} \frac{\tilde{w}d\tilde{w}}{A - \tilde{w}\sqrt{1+\tilde{w}^2} - \ln\left(\tilde{w} + \sqrt{1+\tilde{w}^2}\,\right)} \tag{3.50}$$

3.3 Various limits

In the book, the special cases $v_x = 0$ and $g = v_z = 0$ are solved explicitly, but we can read off a lot of precise information from our general solution. For example, we see w always decreases from w_0, eventually becoming large and negative. As this happens, from (3.43) we find

$$\frac{g^2}{v_x^2} \approx \frac{cg}{m}(w^2 + ...) = \frac{cg}{mv_x^2}(v_z^2 + ...) \tag{3.51}$$

and hence the terminal velocity is

$$v_{ter} = \sqrt{\frac{mg}{c}} \tag{3.52}$$

At the top of the motion, $w = 0$, and hence, using (3.43,3.38), we find

$$v_{top} = \sqrt{\frac{mg}{cA}} = \frac{v_{ter}}{\sqrt{A}} \tag{3.53}$$

Similarly, we can study (3.49) to find that for large (negative) w, $x(w)$ approaches an asymptote

$$x \approx x_{max} + \frac{m}{cw} \tag{3.54}$$

whereas (3.50) tells us that $z(w)$ diverges logarithmically:

$$z \approx -\frac{m}{c} \ln(-w) \tag{3.55}$$

To write this in terms of t, we can do a similar analysis of (3.46) and find

$$t \approx \sqrt{\frac{m}{cg}} \ln(-w) \tag{3.56}$$

This implies that for large t

$$x \approx x_{max} - \frac{m}{c} e^{-\sqrt{\frac{cg}{m}}t} \tag{3.57}$$

and

$$z \approx -v_{ter} t \tag{3.58}$$

(Note, however, that the factor m/c in front of the exponent in (3.57) may receive corrections due to constant terms that were neglected in (3.56); the coefficient $\sqrt{cg/m}$ *in* the exponent is correct).

4 General power-law drag

We can easily treat it in much the same way as we treated the case of quadratic drag. We now have:

$$\dot{v}_x = -\frac{c}{m}(v_x^2 + v_z^2)^\alpha v_x \quad , \quad \dot{v}_z = -g - \frac{c}{m}(v_x^2 + v_z^2)^\alpha v_z \tag{4.59}$$

Equations (3.37,3.38) are unchanged:

$$\dot{v}_z = -g + \frac{v_z}{v_x}\dot{v}_x \quad \Rightarrow \quad \dot{w} = -\frac{g}{v_x} \tag{4.60}$$

where $w = v_z/v_x$ as before. However, we now have

$$\dot{v}_x = g\frac{\ddot{w}}{\dot{w}^2} = -\frac{c}{m}(1+w^2)^\alpha (v_x)^{2\alpha+1} = -\frac{c}{m}(1+w^2)^\alpha (-\frac{g}{\dot{w}})^{2\alpha+1} \tag{4.61}$$

which implies

$$\ddot{w}(-\dot{w})^{2\alpha} = \frac{g^{2\alpha}c}{m}(1+w^2)^{\alpha}\dot{w} \tag{4.62}$$

and hence (for $\alpha \neq -1/2$)

$$\frac{(-\dot{w})^{2\alpha+1}}{2\alpha+1} = -\frac{g^{2\alpha}c}{m}\int(1+w^2)^{\alpha}dw \tag{4.63}$$

for general α, this integrates to a hypergeometric function. For integer α, which corresponds the drag proportional to an odd power of the velocity (*e.g.,* linear drag is $\alpha = 0$), the integral is trivial and can be easily evaluated. The case of quadratic drag corresponds to $\alpha = 1/2$, and was discussed above. Another special case, which we now focus on, is $\alpha = -1/2$, which corresponds to constant (speed independent) drag. Then we have

$$\ln(\dot{w}) = -\frac{c}{mg}sinh^{-1}(w) = -\frac{c}{mg}(\ln(w+\sqrt{1+w^2}) - \ln(A)) \tag{4.64}$$

and hence

$$\dot{w} = A(w+\sqrt{1+w^2})^{\beta} \tag{4.65}$$

where $\beta = -c/mg$ and the integration constant A is

$$A = -\frac{g}{\dot{x}_0(w_0+\sqrt{1+w_0^2})^{\beta}} \tag{4.66}$$

Remarkably, this is an elementary integral, and hence the system is completely solvable:

$$t = \frac{1}{A(1-\beta^2)}(w+\sqrt{1+w^2})^{-\beta}(\beta\sqrt{1+w^2}+w) \tag{4.67}$$