Physics 303/573

Motion of Rigid Bodies

November 17, 2016

1 Introduction

A rigid body is a collection of particles whose separation and relative orientation is fixed. Its position can be described entirely by specifying the position of its center of mass and its orientation. Its motion is described by how its center of mass moves with respect to some reference frame, and how its orientation changes–that is, how it rotates about its center of mass.

I start by reviewing a number of concepts that we have already discussed.

2 Review of center of mass

2.1 Definition

Recall the definition of the center of mass: for a system of n-particles with masses m_i, $i = 1..n$, and positions $\vec{r}_i$, we define the weighted average of the positions:

$$\vec{r}_{cm} = \frac{\sum_{i=1}^{n} m_i \vec{r}_i}{\sum_{i=1}^{n} m_i} = \frac{1}{M}\sum_{i=1}^{n} m_i \vec{r}_i \quad , \quad M = \sum_{i=1}^{n} m_i \tag{2.1}$$

For a continuous distribution with mass density $\rho(\vec{r})$ defined as the mass per volume $V(\vec{r})$ at a point $\vec{r}$

$$\rho(\vec{r}) = \lim_{V(\vec{r})\to 0} \frac{\sum_{\vec{r}_i \in V(\vec{r})} m_i}{V(\vec{r})} \tag{2.2}$$

the center of mass is expressed in terms of the volume integral

$$\vec{r}_{cm} = \frac{\int_V \rho(\vec{r})\vec{r} d^3r}{M} \quad , \quad M = \int_V \rho(\vec{r}) d^3r \tag{2.3}$$

where V is the volume occupied by the mass. Similarly, for a continuous distribution of surface density $\sigma(\vec{r})$ defined as the mass per area $A(\vec{r})$ at a point $\vec{r}$

$$\sigma(\vec{r}) = \lim_{A(\vec{r})\to 0} \frac{\sum_{\vec{r}_i \in A(\vec{r})} m_i}{A(\vec{r})} \tag{2.4}$$

we have

$$\vec{r}_{cm} = \frac{\int_A \sigma(\vec{r})\vec{r} d^2r}{M} \quad , \quad M = \int_A \sigma(\vec{r}) d^2r \tag{2.5}$$

where A is the surface area in which the mass lies. Finally, we can consider a linear distribution with linear density $\lambda(\vec{r})$ defined as the mass per length $L(\vec{r})$ at a point $\vec{r}$

$$\lambda(\vec{r}) = \lim_{L(\vec{r})\to 0} \frac{\sum_{\vec{r}_i \in L(\vec{r})} m_i}{L(\vec{r})} \tag{2.6}$$

we have

$$\vec{r}_{cm} = \frac{\int_L \lambda(\vec{r})\vec{r}dr}{M} \quad , \quad M = \int_L \lambda(\vec{r})dr \tag{2.7}$$

where L is the interval along which the mass distribution lies.

Because integrals are just limits of sums, it is easiest to study the properties of the center of mass using the definition (2.1). A rather obvious but important property of the center of mass is that if the body has some symmetry, then the center of mass must be fixed by the symmetry; for example, if the body has spherical symmetry, so that $\rho(\vec{r}) = \rho(|r|)$, then then the center of mass is at the center of the sphere–any point away from the center has a corresponding point diametrically opposite, and hence the contribution of those two points cancels:

$$\frac{m_i\vec{r}_i + m_i(-\vec{r}_i)}{M} = 0 \tag{2.8}$$

Similarly, if the body has an axis of symmetry, so that (in, for example, cylindrical coordinates) $\rho(\vec{r}) = \rho(|R|, z)$, then the center of mass lies on the axis, and if it has a plane of symmetry then it lies in the plane.

2.2 The center of mass is independent of how it is calculated

A very important property is that **the center of mass of a collection of bodies with masses M_i is the same as the center of mass of a collection of point particles with masses M_i located at the center of masses of the bodies**. We prove this for two bodies; by induction it applies to any number of bodies. Suppose we have two bodies, M_1 and M_2, each made up of several masses m_{1i} and m_{2i}, respectively, with center of masses

$$\vec{r}_{1cm} = \frac{\sum_i m_{1i}\vec{r}_{1i}}{M_1} \quad , \quad \vec{r}_{2cm} = \frac{\sum_i m_{2i}\vec{r}_{2i}}{M_2} \tag{2.9}$$

The total center of mass is

$$\vec{r}_{cm} = \frac{\sum_i m_{1i}\vec{r}_{1i} + \sum_i m_{2i}\vec{r}_{2i}}{\sum_i m_{1i} + \sum_i m_{2i}} = \frac{M_1 \frac{\sum_i m_{1i}\vec{r}_{1i}}{M_1} + M_2 \frac{\sum_i m_{2i}\vec{r}_{2i}}{M_2}}{M_1 + M_2} = \frac{M_1\vec{r}_{1cm} + M_2\vec{r}_{2cm}}{M_1 + M_2} \tag{2.10}$$

so we see that it doesn't matter if we split the calculation up into pieces whose center of mass we find, and then find the center of mass of those.

A sample application of this property can give us the center of mass of a uniform sphere with an off-center bubble inside it; for example, suppose the bubble has radius $\frac{1}{2}R$ of the sphere and just touches the north pole, as shown in Figure 1. The center of mass of the solid sphere would be at its center; we can imagine the bubble as a sphere whose radius is $\frac{1}{2}$ the sphere, and whose density is the same in magnitude as that of the sphere, but is negative in sign, so it cancels the mass in its

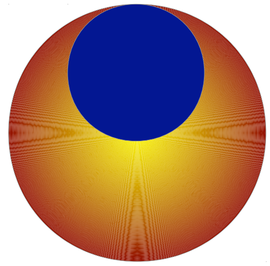

Figure 1: Sphere with a spherical bubble

volume. Let's call the mass of the solid sphere M_0; then because the volume of the bubble is $(\frac{1}{2})^3$ of the volume of the solid sphere, we treat it as sphere with mass $-\frac{1}{8}M_0$. The center of mass of the bubble is at its center as well, which is at $\frac{1}{2}R\hat{z}$

$$\vec{r}_{cm} = \frac{M_0\vec{0} - (\frac{1}{8}M_0)(\frac{1}{2}R\hat{z})}{M_0 - \frac{1}{8}M_0} = -\frac{1}{14}R\hat{z} \tag{2.11}$$

Thus in this case, the center of mass is $\frac{1}{14}$ of the radius below the center of the sphere.

2.3 C.o.m velocity and momentum (from multiparticle lecture)

The velocity of the center of mass is the same weighted average of the velocities:

$$\vec{v}_{cm} = \frac{d}{dt}\vec{r}_{cm} = \frac{1}{M}\sum_{i=1}^{n} m_i\vec{v}_i \tag{2.12}$$

The total linear momentum is by definition the sum of the linear momenta of all the particles:

$$\vec{P}_{tot} = \sum_{i=1}^{n}\vec{p}_i = \sum_{i=1}^{n} m_i\vec{v}_i \tag{2.13}$$

but this is clearly the same as the center of mass momentum:

$$\vec{P}_{tot} = \sum_{i=1}^{n} m_i\vec{v}_i = M\vec{v}_{cm} = \vec{P}_{cm} \tag{2.14}$$

Suppose that each particle feels a force; we can think of the force on the i'th particle as coming partly from an external force $\vec{F}_{(ext)i}$ and partly from its interactions with the other particles:

$$\vec{F}_i = \vec{F}_{(ext)i} + \sum_{j\neq i}^{n}\vec{F}_{ij} \tag{2.15}$$

Newton's second law tells us

$$\vec{F}_i = \dot{\vec{p}_i} \tag{2.16}$$

Using (2.14), we find

$$\dot{\vec{P}}_{cm} = \sum_{i=1}^{n} \vec{F}_i = \sum_{i=1}^{n} \left(\vec{F}_{(ext)i} + \sum_{j \neq i}^{n} \vec{F}_{ij} \right) \tag{2.17}$$

However, if the forces between the particles obey Newton's third law, $F_{ij} = -F_{ji}$; since we are summing over all unequal i, j, for each pair of particles we have $F_{ij} + F_{ji}$, and hence the double sum in (2.17) vanishes:

$$\dot{\vec{P}}_{cm} = \sum_{i=1}^{n} \vec{F}_{(ext)i} \equiv \vec{F}_{ext} \tag{2.18}$$

In particular, if there are no external forces and the particles only interact among themselves, the total momentum is conserved. This occurs in many situations of interest–for example when two particles scatter. They collide, interact in complicated ways, but nonetheless, the total momentum is conserved.

2.4 Angular momentum (from multiparticle lecture)

The instantaneous angular momentum of a particle with position $\vec{r}$ and momentum $\vec{p}$ depends on the choice of the origin of the coordinate system and is given by

$$\vec{L} = \vec{r} \times \vec{p} \tag{2.19}$$

The total angular momentum of a system is the sum of all the angular momenta:

$$\vec{L}_{tot} = \sum_{i=1}^{n} \vec{r}_i \times \vec{p}_i \tag{2.20}$$

Differentiating, we have

$$\dot{\vec{L}}_{tot} = \sum_{i=1}^{n} \dot{\vec{r}}_i \times \vec{p}_i + \sum_{i=1}^{n} \vec{r}_i \times \dot{\vec{p}}_i = \sum_{i=1}^{n} \vec{r}_i \times \vec{F}_i \tag{2.21}$$

where we have used $\dot{\vec{r}}_i \times \vec{p}_i = \vec{v}_i \times \vec{p}_i = \vec{v}_i \times m\vec{v}_i = 0$ and Newton's second law. Substituting (2.15) into (4.58), we find

$$\dot{\vec{L}}_{tot} = \sum_{i=1}^{n} \vec{r}_i \times \vec{F}_{(ext)i} + \sum_{i=1}^{n} \sum_{j \neq i}^{n} \vec{r}_i \times \vec{F}_{ij} \tag{2.22}$$

Using Newton's third law $\vec{F}_{ij} = -\vec{F}_{ji}$, we obtain

$$\dot{\vec{L}}_{tot} = \sum_{i=1}^{n} \vec{r}_i \times \vec{F}_{(ext)i} + \frac{1}{2} \sum_{i=1}^{n} \sum_{j \neq i}^{n} \vec{r}_{ij} \times \vec{F}_{ij} \tag{2.23}$$

where $\vec{r}_{ij} = \vec{r}_j - \vec{r}_i$ is the vector from the i'th particle to the j'th particle.

If the interaction between the particles is through a central force, then $\vec{F}_{ij} \propto \vec{r}_{ij}$ and the second term in (2.23) vanishes:

$$\dot{\vec{L}}_{tot} = \sum_{i=1}^{n} \vec{r}_i \times \vec{F}_{(ext)i} \equiv \vec{N} \tag{2.24}$$

where we have introduced the total torque $\vec{N}$ [1]. If there is no torque, then as long as the particles in the system interact through central forces, the total angular momentum $\vec{L}_{tot}$ is conserved.

It is useful to introduce coordinates relative to the center of mass:

$$\vec{\bar{r}}_i = \vec{r}_i - \vec{r}_{cm} \tag{2.25}$$

Note that relative coordinates are of course the same: $\vec{\bar{r}}_{ij} = \vec{r}_{ij}$. Note as well that the center of mass of these new coordinates is just the origin:

$$\sum_{i=1}^{n} m_i \vec{\bar{r}}_i = \sum_{i=1}^{n} m_i(\vec{r}_i - \vec{r}_{cm}) = M(\vec{r}_{cm} - \vec{r}_{cm}) = 0 \tag{2.26}$$

so $\vec{\bar{r}}_i$ are nothing more than the coordinates in the center of mass frame. Thus if we compute

$$\vec{\bar{p}}_i = m_i(\dot{\vec{r}}_i - \dot{\vec{r}}_{cm}) \tag{2.27}$$

we find that from the definition of the center of mass

$$\sum_{i=1}^{n} \vec{\bar{p}}_i = \vec{P}_{tot} - \vec{P}_{cm} = 0 \tag{2.28}$$

The angular momentum (2.19) can be rewritten as:

$$\vec{L}_{tot} = \sum_{i=1}^{n} (\vec{r}_{cm} + \vec{\bar{r}}_i) \times m_i(\dot{\vec{r}}_{cm} + \dot{\vec{\bar{r}}}_i) = \vec{r}_{cm} \times \vec{P}_{cm} + \sum_{i=1}^{n} \vec{\bar{r}}_i \times m_i \dot{\vec{\bar{r}}}_i \equiv \vec{L}_{orb} + \vec{L}_{spin} \tag{2.29}$$

where two terms vanish because of the definition (2.25):

$$\sum_{i=1}^{n} m_i \vec{\bar{r}}_i = \sum_{i=1}^{n} m_i \dot{\vec{\bar{r}}}_i = 0 \quad \Rightarrow \quad \sum_{i=1}^{n} \vec{\bar{r}}_i \times m_i \dot{\vec{r}}_{cm} = \sum_{i=1}^{n} \vec{r}_{cm} \times m_i \dot{\vec{\bar{r}}}_i = 0 \tag{2.30}$$

Thus the total angular momentum has an orbital component $\vec{L}_{orb}$, which is the angular momentum of a point particle with mass M (the total mass of the system) located at the center of mass, and a spin component $\vec{L}_{spin}$, which is the angular momentum of the system rotating around its center of mass.

The torque has a similar decomposition into an orbital piece $\vec{N}_{orb}$ and spin piece $\vec{N}_{spin}$, which is the torque about the center of mass:

$$\begin{aligned}
\vec{N} &= \sum_{i=1}^{n} \vec{r}_i \times \vec{F}_{(ext)i} \\
&= \sum_{i=1}^{n} (\vec{r}_{cm} + \vec{\bar{r}}_i) \times \vec{F}_{(ext)i} \\
&= \vec{r}_{cm} \times \vec{F}_{(ext)} + \sum_{i=1}^{n} \vec{\bar{r}}_i \times \vec{F}_{(ext)i} \\
&\equiv \vec{N}_{orb} + \vec{N}_{spin}
\end{aligned} \tag{2.31}$$

[1]Taylor uses $\vec{\Gamma}$ for torque and $\vec{\ell}$ for angular momentum. The notation in these notes is more common.

2.5 Kinetic energy (from multiparticle lecture)

The total kinetic energy of a system is just the sum of the kinetic energies of all the particles; if we write it in terms of the velocity of the center of mass $\vec{v}_{cm}$ and the velocities relative to the center of mass $\vec{\bar{v}}_i$, we find:

$$T_{tot} = \sum_{i=1}^{n} \frac{1}{2} m_i \vec{v}_i \cdot \vec{v}_i = \frac{1}{2} \sum_{i=1}^{n} m_i (\vec{v}_{cm} + \vec{\bar{v}}_i) \cdot (\vec{v}_{cm} + \vec{\bar{v}}_i) = \frac{1}{2} M \vec{v}_{cm} \cdot \vec{v}_{cm} + \frac{1}{2} \sum_{i=1}^{n} m_i \vec{\bar{v}}_i \cdot \vec{\bar{v}}_i \tag{2.32}$$

where the term $\sum_{i=1}^{n} m_i \vec{\bar{v}}_i \cdot \vec{v}_{cm} = 0$ for the same reason as (2.26),(2.28). Thus the total kinetic energy can be decomposed into the kinetic energy of the center of mass and the internal kinetic energy: $T_{tot} = T_{cm} + T_{int}$.

3 Moment of inertia

The kinetic energy of a body

$$T = \frac{1}{2} \sum_i m_i |\vec{v}_i|^2 \tag{3.33}$$

Suppose that the body is rigidly rotating about some axis with an angular velocity ω; then the velocity of any particle in it is given by

$$|\vec{v}_i| = \omega r_{i\perp} \tag{3.34}$$

where $r_{i\perp}$ is the perpendicular distance of the i'th particle to the axis of rotation; we can also write

$$|\vec{v}_i| = |\vec{\omega} \times \vec{r}_i| \tag{3.35}$$

where the vector $\vec{\omega}$ has magnitude the angular velocity ω and points along the axis of rotation. Then the kinetic energy (3.33) becomes

$$T = \frac{1}{2} \sum_i m_i (r_{i\perp})^2 \omega^2 \equiv \frac{1}{2} I_\omega \omega^2 \tag{3.36}$$

The expression

$$I_\omega = \sum_i m_i (r_{i\perp})^2 \tag{3.37}$$

is the **moment of inertia** of the body about the axis $\vec{\omega}$. Just as for the center of mass, we can consider continuous distributions, for example

$$I_\omega = \int_V \rho(\vec{r}) |\vec{r}_\perp|^2 d^3 r \tag{3.38}$$

where $|\vec{r}_\perp|$ is the perpendicular distance to the axis. We can write down similar formulae for surface and linear distributions.

The total angular momentum of a system is the sum of all the angular momenta:

$$\vec{L}_{tot} = \sum_i m_i \vec{r}_i \times \vec{v}_i \tag{3.39}$$

For a rigid body rotating about a fixed axis, we have:

$$\vec{L}_{tot} = \sum_i m_i \vec{r}_i \times (\vec{\omega} \times \vec{r}_i) = \sum_i m_i \left(|r_i|^2 \vec{\omega} - (\vec{\omega} \cdot \vec{r}_i) \vec{r}_i \right) \tag{3.40}$$

The component of this along the axis of rotation is

$$L_\omega = \hat{\omega} \cdot \vec{L} = \sum_i m_i \left(|r_i|^2 - (\hat{\omega} \cdot \vec{r}_i)^2 \right) \omega = I_\omega \omega \tag{3.41}$$

because the Pythagorean theorem implies

$$(r_{i\perp})^2 = |r_i|^2 - (\hat{\omega} \cdot \vec{r}_i)^2 \tag{3.42}$$

In the absence of an external torque, the angular momentum is conserved; for a body rotating about a fixed axis, we have

$$N_\omega = \dot{L}_\omega = I_\omega \dot{\omega} \tag{3.43}$$

This is an example of a correspondence between linear motion and rotational motion; with the correct identification, both can be described by equations with the same form. This means that we can understand the behavior of rotating systems from our intuition and experience with linear motion.

The correspondence is summarized in the table below:

Linear motion		**Rotational motion**	
Position	x	Angle	θ
Velocity	$v = \dot{x}$	Angular velocity	$\omega = \dot{\theta}$
Mass	m	Moment of inertia	I
Linear Momentum	$p = mv$	Angular momentum	$L = I\omega$
Force	$F = m\dot{v}$	Torque	$N = I\dot{\omega}$
Kinetic energy	$T = \frac{1}{2}mv^2$	Kinetic energy	$T = \frac{1}{2}I\omega^2$

Table 1: One-dimensional linear motion compared to rotational motion about a fixed axis

3.1 Parallel axis theorem

There are two theorems that simplify the calculation of moments of inertia.

The parallel axis theorem says that the moment of inertia about a given axis is the same as the moment of inertia of the body about a parallel axis going through its center of mass (the "spin" component of the moment of inertia) plus the moment of inertia of a point particle with the mass of the body concentrated at the center of mass (the "orbital" component of the moment of inertia):

$$I = I_{cm} + m(r_{\perp cm})^2 \tag{3.44}$$

This is most easily understood by using

$$L = I\omega = L_{spin} + L_{orbit} = I_{cm}\omega_{spin} + m(r_{\perp cm})^2 \omega_{orbit} \tag{3.45}$$

For a rigidly rotating body going around a fixed axis, ω_{spin}, the rate at which the body spins about its center of mass, is just the same as ω_{orbit}, the rate at which its center of mass is orbiting the axis, and hence (3.44) follows. One can also just check this by direct computation.

3.2 Perpendicular axis theorem

The second useful theorem applies to lamina, or bodies that lie entirely in one plane. It is called the **perpendicular axis theorem**, and states that for such a planar body, the moment of inertia along an axis perpendicular to the body is the sum of the moments of inertia along two axes lying in the plane of the body but perpendicular to each other and meeting each other at the point where the perpendicular axis pierces the plane.

The proof is just the Pythagorean theorem, as one can see in the figure below. The body lies in the $1, 2$ plane, and the axis labeled 3 is perpendicular to it. For any point in the body, the perpendicular distance to axis 3 is labeled c, whereas the perpendicular distance to axis 1 is a and the perpendicular distance to axis 2 is b. The Pythagorean theorem tells us that $a^2 + b^2 = c^2$, but that implies

$$I_3 = \sum_i m_i c_i^2 = \sum_i m_i (a_i^2 + b_i^2) = I_1 + I_2 \tag{3.46}$$

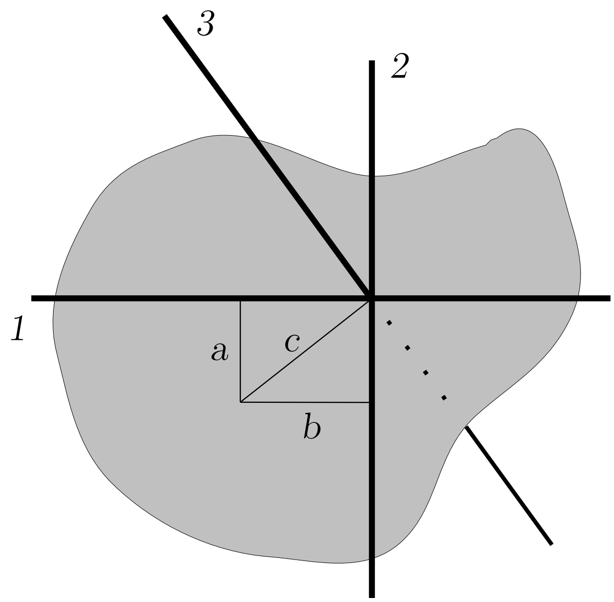

Figure 2: A planar body in the $1, 2$ plane, with the axis 3 perpendicular.

3.3 Moment of inertia tensor

If we look more carefully at (3.40), we notice that the angular momentum need *not* be parallel to the axis of rotation, and that the moment of inertia is not a number like the mass, but depends on which axis we are looking at. It is useful to rewrite (3.40) as

$$\vec{L} = \mathbb{I}\vec{\omega} \tag{3.47}$$

where $\mathbb{I}$ is a linear operator called the **moment of inertial tensor**; its components are given by

$$\mathbb{I}_{ab} = \sum_i m_i(|r_i|^2\delta_{ab} - r_{ia}r_{ib}) \tag{3.48}$$

where a, b label the different axes, and $\delta_{ab} = 1$ when $a = b$ and vanishes otherwise. In cartesian coordinates, we have

$$\mathbb{I}_{ab} = \sum_i m_i \begin{pmatrix} r_i^2 - x_i^2 & -x_iy_i & -x_iz_i \\ -y_ix_i & r_i^2 - y_i^2 & -y_iz_i \\ -z_ix_i & -z_iy_i & r_i^2 - z_i^2 \end{pmatrix} = \sum_i m_i \begin{pmatrix} y_i^2 + z_i^2 & -x_iy_i & -x_iz_i \\ -y_ix_i & x_i^2 + z_i^2 & -y_iz_i \\ -z_ix_i & -z_iy_i & x_i^2 + y_i^2 \end{pmatrix} \tag{3.49}$$

Notice that the diagonal terms are the familiar moments about the cartesian axes. One can always choose a coordinate basis to make a symmetric matrix diagonal; this means that for some choice of axes, the moment of inertia tensor just reduces to the moments about these axes, which are called the *principal axes*. Then the moment of inertia tensor has the form

$$\mathbb{I}_{ab} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \tag{3.50}$$

where the eigenvalues λ_a are called the principle moments. When the body rotates about one of the principle axes, the angular momentum lies along the axis of rotation.

The spin part of the kinetic energy of a rigid body rotating about a principle axis with angular velocity ω_a is given by (3.36): $T_a = \frac{1}{2}\lambda_a\omega_a^2$. For a general rotation, we just get the sum of the kinetic energies about all the principle axes:

$$T_{spin} = \frac{1}{2}\sum_{a=1}^{3}\lambda_a\omega_a^2 = \frac{1}{2}\sum_{a,b=1}^{3}\mathbb{I}_{ab}\omega_a\omega_b \tag{3.51}$$

where the last line follows by rotating to arbitrary axes about the center of mass.

To find the principle moments, we to diagonalize a 3×3 matrix; this is straightforward, and discussed in the book.

A sample calculation We now consider a simple but nontrivial example. Consider four identical particles with mass m positioned at

$$\vec{r}_1 = \hat{x} \quad , \quad \vec{r}_2 = \hat{y} \quad , \quad \vec{r}_3 = \hat{z} \quad , \quad \vec{r}_4 = -\hat{x} - \hat{y} - \hat{z} \tag{3.52}$$

Clearly, the center of mass is at the origin; because all the masses are equal, we have $\vec{r}_{cm} = \sum \vec{r}_i = \vec{0}$. It is easy to compute the moment of inertia tensor, as the off-diagonal terms all come from $\vec{r}_4$:

$$\mathbb{I}_{ab} = m\begin{pmatrix} 4 & -1 & -1 \\ -1 & 4 & -1 \\ -1 & -1 & 4 \end{pmatrix} \tag{3.53}$$

This may look intimidating, but by symmetry, it is easy to see that the vector $\vec{r}_4$ is an eigenvector with eigenvalue $2m$. Trying any vector perpendicular to $\vec{r}_4$ gives an eigenvector with eigenvalue $5m$–so we can choose a new orthonormal basis:

$$\hat{e}_1 = \frac{1}{\sqrt{3}}(\hat{x}+\hat{y}+\hat{z}) \quad , \quad \hat{e}_2 = \frac{1}{\sqrt{2}}(\hat{x}-\hat{y}) \quad , \quad \hat{e}_1 = \frac{1}{\sqrt{6}}(\hat{x}+\hat{y}-2\hat{z}) \tag{3.54}$$

We reach this by a rotation (using $\hat{r}_1 = \hat{x}, \hat{r}_2 = \hat{y}, \hat{r}_3 = \hat{z}$)

$$R_{ab} = \hat{e}_a \cdot \hat{r}_b \qquad \Rightarrow \qquad R_{ab} = \frac{1}{\sqrt{6}} \begin{pmatrix} \sqrt{2} & \sqrt{2} & \sqrt{2} \\ \sqrt{3} & -\sqrt{3} & 0 \\ 1 & 1 & -2 \end{pmatrix} \tag{3.55}$$

We can easily check that $\mathbf{R}^T\mathbf{R} = 1$, and we can compute the moment of inertia tensor in the new basis:

$$R_{ab}\mathbb{I}_{bc}R^T_{cd} = m \begin{pmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix} \tag{3.56}$$

4 Euler's equations

We know that the time derivative of a vector in a rotating frame is expressed in terms of the time derivative of the components and the time derivative of the basis vectors (see eq. (3.24) in the lectures on noninertial frames):

$$\frac{d\vec{v}}{dt} = \dot{\vec{v}} + \vec{\omega} \times \vec{v} \tag{4.57}$$

where $\dot{\vec{v}} = \dot{v}_a \hat{e}_a$ and $\dot{\hat{e}}_a = \vec{\omega} \times \hat{e}_a$ is the vector describing the rotation of the basis vectors.

On the other hand, we know that the time derivative of the angular momentum vector is just the torque:

$$\frac{d\vec{L}}{dt} = \vec{N} \qquad \Rightarrow \qquad \dot{\vec{L}} + \vec{\omega} \times \vec{L} = \vec{N} \tag{4.58}$$

If we assume that the basis vectors lie along the principle axes, then $\vec{L} = \sum_a \lambda_a \omega_a \hat{e}_a$, and we find Euler's equations:

$$\begin{aligned} \lambda_1\dot{\omega}_1 + (\lambda_3 - \lambda_2)\omega_2\omega_3 &= N_1 \\ \lambda_2\dot{\omega}_2 + (\lambda_1 - \lambda_3)\omega_1\omega_3 &= N_2 \\ \lambda_3\dot{\omega}_3 + (\lambda_2 - \lambda_1)\omega_1\omega_2 &= N_3 \end{aligned} \tag{4.59}$$

In particular, if there is no external torque ($\vec{N} = 0$) and $\lambda_3 > \lambda_2 > \lambda_1$, then we see that $\lambda_3 - \lambda_2$ and $\lambda_2 - \lambda_1$ are positive whereas $\lambda_1 - \lambda_3$ is negative; no matter how we choose the principle moments, two of the coefficients in (4.60) will have one sign and the other will have the opposite sign.

For vanishing torque, we can find three obvious solutions: steady rotation about the principle axes, that is, all $\dot{\omega}_a = 0$, and two out of the three $\omega_a = 0$ with the third arbitrary. Let us consider the stability of these solutions. Suppose that, for example, ω_1 is large and ω_2, ω_3 are small. Then to leading order we have:

$$\begin{aligned} \lambda_1 \dot{\omega}_1 &\approx 0 \\ \lambda_2 \dot{\omega}_2 + (\lambda_1 - \lambda_3)\omega_1\omega_3 &\approx 0 \\ \lambda_3 \dot{\omega}_3 + (\lambda_2 - \lambda_1)\omega_1\omega_2 &\approx 0 \end{aligned} \tag{4.60}$$

and hence we may treat ω_1 as an approximate constant. Differentiating the $\dot{\omega}_2$ equation and using the $\dot{\omega}_3$ equation, we find

$$\ddot{\omega}_2 + \frac{(\lambda_3 - \lambda_1)(\lambda_2 - \lambda_1)\omega_1^2}{\lambda_2\lambda_3}\omega_2 \approx 0 \tag{4.61}$$

This is an approximate harmonic oscillator equation for ω_2 if

$$\omega_{eff}^2 = \frac{(\lambda_3 - \lambda_1)(\lambda_2 - \lambda_1)\omega_1^2}{\lambda_2\lambda_3} > 0 \tag{4.62}$$

that happens if λ_1 is the largest or smallest principle moment, and hence the motion is stable for rotations about the largest or smallest principle moment, and unstable around the intermediate principle moment.

5 Euler angles

It is often convenient to introduce a set of variables that describes the orientation of a rigid body in our inertial frame. These are called Euler angles, and can be described by three successive rotations (see Lecture 2, eq. (1.12)) about the z, y, z axis:

$$\begin{aligned} \mathbf{R} &= \mathbf{R}_z(\psi)\mathbf{R}_y(\theta)\mathbf{R}_z(\phi) \\ &= \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \cos\phi\cos\theta\cos\psi - \sin\phi\sin\psi & \sin\phi\cos\theta\cos\psi + \cos\phi\sin\psi & -\sin\theta\cos\psi \\ -\cos\phi\cos\theta\sin\psi - \sin\phi\cos\psi & -\sin\phi\cos\theta\sin\psi + \cos\phi\cos\psi & \sin\theta\sin\psi \\ \cos\phi\sin\theta & \sin\phi\sin\theta & \cos\theta \end{pmatrix} \end{aligned} \tag{5.63}$$

The basis vectors $\hat{e}_a$ along the principal axes of the body are given in terms of the inertial basis vectors $\hat{e}_{a(0)}$ by

$$\hat{e}_a = R_{ab}\hat{e}_{b(0)} \tag{5.64}$$

Then

$$\dot{\hat{e}}_a = \dot{R}_{ab}\hat{e}_{b(0)} = \dot{R}_{ab}R^{-1}_{bc}\hat{e}_c = \dot{R}_{ab}R_{cb}\hat{e}_c \tag{5.65}$$

and hence the rotation of the body is then described by (see equation (3.22) in the previous set of notes) by the vector

$$\omega_a = \frac{1}{2}\epsilon_{abc}\dot{R}_{bd}R_{cd} \tag{5.66}$$

(This is equivalent to)

$$\dot{\hat{e}}_a = \vec{\omega} \times \hat{e}_a \tag{5.67}$$

Computing $\vec{\omega}$ we find:

$$\vec{\omega} = (\dot{\theta}\sin\psi - \dot{\phi}\sin\theta\cos\psi)\hat{e}_1 + (\dot{\theta}\cos\psi + \dot{\phi}\sin\theta\sin\psi)\hat{e}_2 + (\dot{\psi} + \dot{\phi}\cos\theta)\hat{e}_3 \tag{5.68}$$

From (3.51) it follows that the spin part of the kinetic energy is:

$$T = \frac{1}{2}\left(\lambda_1(\dot{\theta}\sin\psi - \dot{\phi}\sin\theta\cos\psi)^2 + \lambda_2(\dot{\theta}\cos\psi + \dot{\phi}\sin\theta\sin\psi)^2 + \lambda_3(\dot{\psi} + \dot{\phi}\cos\theta)^2\right) \tag{5.69}$$

For the special case $\lambda_1 = \lambda_2$, this simplifies to

$$T = \frac{1}{2}\left(\lambda_1(\dot{\theta}^2 + (\dot{\phi}\sin\theta)^2) + \lambda_3(\dot{\psi} + \dot{\phi}\cos\theta)^2\right) \tag{5.70}$$

These expressions allow us to write down the Lagrangian of a rigid body using Euler angles, both in the general case and for the special case when two principle moments are equal.

6 Spinning top

We consider a symmetric spinning top in a gravitational potential with mass M and two principle moments equal to each other: $\lambda_1 = \lambda_2$; let R be the distance from the pivot point to the center of mass (see figure 3). The Lagrangian is:

$$L = T - U = \frac{1}{2}\left(\lambda_1(\dot{\theta}^2 + (\dot{\phi}\sin\theta)^2) + \lambda_3(\dot{\psi} + \dot{\phi}\cos\theta)^2\right) - MgR\cos\theta \tag{6.71}$$

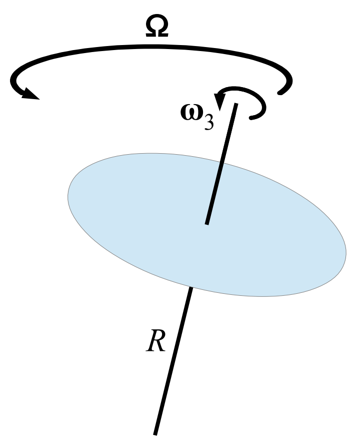

Figure 3: A spinning top. R is the distance from the pivot point to the center of mass, ω_3 is the rate at which it spins about its own axis, and Ω is the rate at which it precesses about the z-axis.

The Euler-Lagrange equations give two conservation laws for the generalized momenta conjugate to ψ, ϕ and a single dynamical equation for θ:

$$\frac{d}{dt}\left(\lambda_3(\dot{\psi} + \dot{\phi}\cos\theta)\right) = 0 \quad \Rightarrow \quad p_\psi = \lambda_3(\dot{\psi} + \dot{\phi}\cos\theta) \text{ is constant}$$

$$\frac{d}{dt}\left(\lambda_1(\dot{\phi}\sin^2\theta) + \cos\theta\, p_\psi\right) = 0 \quad \Rightarrow \quad p_\phi = \lambda_1(\dot{\phi}\sin^2\theta) + \cos\theta\, p_\psi \text{ is constant} \tag{6.72}$$

and

$$\lambda_1 \ddot{\theta} = \lambda_1 \dot{\phi}^2 \sin\theta\cos\theta - \dot{\phi}\sin\theta\, p_\psi + MgR\sin\theta \tag{6.73}$$

This is a pretty nonlinear and complicated equation for θ, particularly after we eliminate $\dot{\phi}$ in favor of p_ϕ: $\dot{\phi} = (p_\phi - \cos\theta\, p_\psi)/(\lambda_1 \sin^2\theta)$. We will first analyze a special case and then study the general behavior by finding the effective potential.

6.1 Steady precession

When the angle θ from the vertical doesn't change, equation (6.73) simplifies dramatically. Then $\Omega = \dot{\phi}$ is constant, and writing $\omega_3 = p_\psi/\lambda_3$, we find (after dividing by $\sin\theta$):

$$\lambda_1 \Omega^2 \cos\theta - \lambda_3\, \Omega\, \omega_3 + MgR = 0 \tag{6.74}$$

This is a quadratic equation for the rate of precession Ω in terms of the angle from the vertical and the rate ω_3 at which the top is spinning. Surprisingly, when the top spins fast enough so that $\omega_3^2 > (4MgR\,\lambda_1\cos\theta)/\lambda_3^2$, there are two possible rates of precession:

$$\Omega_\pm = \frac{\lambda_3\omega_3 \pm \sqrt{(\lambda_3\omega_3)^2 - 4MgR\,\lambda_1\cos\theta}}{2\lambda_1\cos\theta} \tag{6.75}$$

For large ω_3, these simplify: Ω_+ is the solution when MgR is negligible and Ω_- is the solution when Ω^2 is negligible:

$$\Omega_+ \approx \frac{\lambda_3\omega_3}{\lambda_1\cos\theta} \quad , \quad \Omega_- \approx \frac{MgR}{\lambda_3\omega_3} \tag{6.76}$$

6.2 Nutation

The Hamiltonian for the top follows from the usual prescription:

$$H = p_\theta\dot{\theta} + p_\psi\dot{\psi} + p_\phi\dot{\phi} - L = \frac{p_\theta^2}{2\lambda_1} + \frac{p_\psi^2}{2\lambda_3} + \frac{(p_\phi - p_\psi\cos\theta)^2}{2\lambda_1\sin^2\theta} + MgR\cos\theta \tag{6.77}$$

The Hamiltonian is independent of ϕ, ψ, which implies that p_ϕ, p_ψ are conserved (that is, they are constant), and this reduces the whole system to a one-dimensional system with a time dependent coordinate θ moving in an effective potential:

$$U_{eff} = \frac{(p_\phi - p_\psi\cos\theta)^2}{2\lambda_1\sin^2\theta} + MgR\cos\theta \tag{6.78}$$

where I have dropped the irrelevant constant $\frac{p_\psi^2}{2\lambda_3}$. If $p_\phi \neq p_\psi$, this potential grows and approaches asymptotes at $\theta = 0, \pi$ and is a smooth function with a single stable minimum somewhere in between. When $p_\phi = p_\psi$, at $\theta = 0$, this approaches the constant MgR and blows up at $\theta = \pi$, with a single minimum in between.

This means that whenever $p_\phi \neq p_\psi$, for given p_ϕ, p_ψ, the minimum of U_{eff} defines a unique angle θ with steady precession.[2] For any energy greater than the minimum value of U_{eff}, θ oscillates in

[2] This is not inconsistent with the result above: the quadratic equation (6.74) fixed θ and p_ψ, and found two values of p_ϕ consistent with steady precession, which is a answering a different question.

a finite range, corresponding to a wobbling motion called *nutation*. Note that the nature of the nutation depends on the ratio p_ϕ/p_ψ. When $p_\phi > p_\psi$, we have $\dot{\phi} \propto p_\phi - p_\psi \cos\theta > 0$ for all θ, and the precession is similar to the steady precession with a steady up and down wobble. However, when $p_\phi < p_\psi$, $\dot{\phi} \propto p_\phi - p_\psi \cos\theta$ changes sign depending on the value of θ, and the top motion is more complex: the direction of the precession changes for brief intervals.

This is shown nicely in Figure 10.12 in the book.