Homework 1

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Consider a vector (given with respect to a fixed Cartesian basis). Here t means time.

$$\vec{r}(t) = \sin(\pi t)\hat{x} + \cos(\pi t)\hat{y} - \sqrt{7}\hat{z}$$

1. Problem 1

   (a) Find the length of $\vec{r}(t)$

   $$\begin{aligned}\|\vec{r}(t)\| &= \sqrt{\sin^2 \pi t + \cos^2 \pi t + (-\sqrt{7})^2} \\ &= \sqrt{1+7} \\ &= \sqrt{8}\end{aligned}$$

   (b) Find $\vec{r} \cdot \vec{w}$ where $\vec{w} = \hat{x} - 2\hat{y} + \sqrt{7}\hat{z}$

   $$\begin{aligned}\vec{r} \cdot \vec{w} &= (\sin \pi t, \cos \pi t, -\sqrt{7}) \cdot (1, -2, \sqrt{7}) \\ &= (\sin \pi t, -2\cos \pi t, -7)\end{aligned}$$

   (c) Find $\vec{r} \times \vec{w}$

   $$\begin{aligned}\vec{r} \times \vec{w} &= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \sin \pi t & \cos \pi t & -\sqrt{7} \\ 1 & 2 & \sqrt{7} \end{vmatrix} \\ &= (\sqrt{7}\cos \pi t + 2\sqrt{7}, -\sqrt{7}\sin \pi t + \sqrt{7}, \sqrt{7}\sin \pi t + \sqrt{7})\end{aligned}$$

2. Problem 2

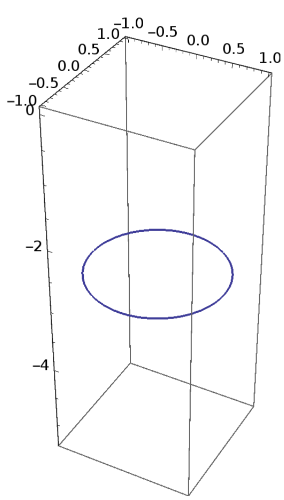
   (a) Find $\vec{v}(t) = \frac{\mathrm{d}\vec{r}}{\mathrm{d}t}$

   $$\begin{aligned}\vec{v}(t) &= \frac{\mathrm{d}\vec{r}}{\mathrm{d}t} \\ &= (\pi \cos \pi t, -\pi \sin \pi t, 0)\end{aligned}$$

   (b) What is the second derivative?

   $$\begin{aligned}\ddot{\vec{r}} &= \frac{\mathrm{d}}{\mathrm{d}t}\vec{v}(t) \\ &= (-\pi^2 \sin \pi t, -\pi^2 \cos \pi t, 0)\end{aligned}$$

3. Sketch the trajectory of the vector $\vec{r}(t)$. What kind of a shape does it sweep out?



   It is a circle.

4. Suppose that $\vec{r}(t)$ describes the motion of a body with mass m

   (a) Find the momentum.

   $$\vec{p} = m\vec{v} = (\pi m \cos \pi t, -\pi m \sin \pi t, 0)$$

   (b) Find the force that is needed to make it move along the path.

   $$\vec{F} = m\ddot{\vec{r}} = (-\pi^2 m \sin \pi t, -\pi^2 m \cos \pi t, 0)$$

5. Problem 5

   (a) Write $\vec{r}(t)$ in a sylindrical cordinate basis Clear it is a circle at a plane where $z = 0$. Thus we can write

   $$(r, \theta, z) = (1, \tan^{-1}(\frac{\cos \pi t}{\sin \pi t}), -\sqrt{7})$$

   we can also find out that it should be

   $$\begin{aligned}\begin{bmatrix} r \\ \theta \\ z \end{bmatrix} &= \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sin \pi t \\ \cos \pi t \\ -\sqrt{7} \end{bmatrix} \\ &= \begin{bmatrix} \sin \pi t \cos\theta + \cos \pi t \sin\theta \\ -\sin \pi t \sin\theta + \cos\theta \cos \pi t \\ -\sqrt{7} \end{bmatrix}\end{aligned}$$

   where now we can find that $r = 1 = \sin \pi t \cos\theta + \cos \pi t \sin\theta = \sin(\pi t\theta)$. Solve this equation we can find that $\theta = 2\pi n - \pi t + \frac{1}{2}\pi$. Take $n = 0$ we can have $\theta = -\pi t + \frac{1}{2}\pi$.

   (b) What is the time derivative $\frac{\mathrm{d}\hat{r}}{\mathrm{d}t}$ for this motion?

   $$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\hat{r} &= \cos\theta(t)\hat{x} + \sin\theta(t)\hat{y} \\ &= -\frac{\mathrm{d}\theta(t)}{\mathrm{d}t}\sin\theta(t)\hat{x} + \frac{\mathrm{d}\theta(t)}{\mathrm{d}t}\cos\theta(t)\hat{y} \\ &= \frac{\mathrm{d}\theta(t)}{\mathrm{d}t}(\cos\theta\hat{y} - \sin\theta\hat{x}) \\ &= \frac{\mathrm{d}\theta(t)}{\mathrm{d}t}\hat{\theta}\end{aligned}$$

(c) Compute the first derivative in a cylindrical coordinate basis.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\vec{r} &= \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} \\ &= -r\pi t\hat{\theta} \\ &= -\pi t\hat{\theta}\end{aligned}$$

The Problem 2 give $\dot{\vec{r}} = \pi\cos\pi t\hat{x} - \pi\sin\pi t\hat{y}$. And we can transform it into cylindrical cordinate. We can have $r = \sqrt{\pi^2\cos^2\pi t + \pi^2\sin^2\pi t} = \pi$, $\theta = \tan^{-1}(-\frac{\pi\sin\pi t}{\pi\cos\pi t}) = -\pi t$ and $z = 0$ which is agree with the answer above.

6. Problem 6

(a) Compute the length of $\vec{p}(t)$

$$\begin{aligned}\left\|\vec{p}(t)\right\| &= \sqrt{(\pi m\cos\pi t)^2 + (-\pi m\sin\pi t)^2} \\ &= \sqrt{\pi^2 m^2} = \pi m\end{aligned}$$

(b) Compute $\hat{z}\cdot(\vec{r}\times\vec{p})$

$$\begin{aligned}\hat{z}\cdot(\vec{r}\times\vec{p}) &= \hat{z}\cdot\begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k} \\ \sin\pi t & \cos\pi t & -\sqrt{7} \\ \pi m\cos\pi t & -\pi m\sin\pi t & 0\end{vmatrix} \\ &= \hat{k}\cdot(\cdots,\cdots,-\pi m\sin^2\pi t - \pi m\cos^2\pi t) \\ &= \hat{k}\cdot(\cdots,\cdots,-\pi m) \\ &= (0,0,-\pi m)\end{aligned}$$