Physics 303/573

Homework 8 due Monday, November 20.

1. Two particles are connected by an ideal spring with spring constant k and unstretched length $\ell = 0$. They both slide along a frictionless ramp given by the equation $z = \alpha y$.

a) Write down the Lagrangian for the system in terms of $\vec{r}_1$ and $\vec{r}_2$, imposing constraints with Lagrange multipliers.

b) Rewrite the Lagrangian in terms of $\vec{r}_{cm}$ and $\vec{r} = \vec{r}_1 - \vec{r}_2$.

c) Eliminate the Lagrange multipliers and use the constraints to eliminate $z = z_1 - z_2$ and z_{cm}.

d) Find the Euler-Lagrange equations for the resulting system.

e) Write down the most general solution for $\vec{r}_{cm}(t)$ and $\vec{r}(t)$.

2. Consider a Lagrangian

$$L = \frac{1}{2}\frac{m(\dot{x}^2 + \dot{y}^2)}{(1 + x^2 + y^2)^2}$$

a) Show that this is invariant under rotations about the $z - axis$. Find the corresponding Noether charge.

b) Rewrite L in planar polar coordinates r, θ. Show that the results of part a) are now obvious.

c) A much less obvious symmetry is the following:

$$\delta r = \alpha(1 + r^2)\cos\theta \quad , \quad \delta\theta = \alpha(r - \frac{1}{r})\sin\theta$$

Calculate the Noether charge of this symmetry assuming that $\delta L = 0$. For 20 points extra credit, prove that this is actually a symmetry.

3. A planet with angular momentum L_z and reduced mass μ is orbiting around a sun such that the total mass is M.

a) Write down the effective potential U_{eff}.

b) Sketch U_{eff}.

c) What is the radius and energy of a circular orbit?

d) What is the frequency of small oscillations around the circular orbit?

4. We found the orbit of a mass attracted by gravity to a central sun in polar coordinates:

$$r(\theta) = \frac{\alpha}{1 + \epsilon \cos\theta}$$

a) Rewrite this in cartesian coordinates x, y.

b)Show that when $0 \leq \epsilon < 1$ it can be written in the form

$$\frac{(x - x_0)^2}{a^2} + \frac{y^2}{b^2} = 1$$

c) Find x_0, a, b in terms of α, ϵ.

d) What equation do you find in the limit as $\epsilon \to 1$?

e) Can you find the correct equation for $\epsilon > 1$?

5. Problem 8.17 in the book.