Physics 303/573

Motion in non-inertial frames

November 13, 2017

1 Accelerated origin of coordinates

So far, we have studied Newton's laws of motion primarily in an inertial frame. It is often useful to use a reference frame that is accelerated. If the position of a body in an inertial frame is $\vec{r}_0$, then in another frame whose origin is at $\vec{R}$, the position will be $\vec{r}$:

$$\vec{r}_0 = \vec{R} + \vec{r} \tag{1.1}$$

and hence, differentiating with respect to time, we have

$$\vec{v}_0 = \vec{V} + \vec{v} \quad , \quad \vec{a}_0 = \vec{A} + \vec{a} \tag{1.2}$$

Then Newton's second law becomes

$$\vec{F}_0 = m\vec{a}_0 = m\vec{A} + m\vec{a} \qquad \Rightarrow \qquad m\vec{a} = \vec{F}_0 - m\vec{A} \equiv \vec{F}_{\mathit{eff}} \tag{1.3}$$

where $-m\vec{A}$ is a *fictitious force* that the accelerated observer perceives just because the frame is accelerated. An example of this is provided by a plumb bob (a weight on a string) hanging in an accelerating train car; the inertial observer on the platform sees the weight accelerated by the tension in the string, and so the string hangs at an angle, whereas the accelerated observer feels a fictitious force in the direction opposite the acceleration that makes the string hang at the observed angle. A slightly less trivial example is a pendulum on an accelerating train car. Then

$$\vec{F}_{\mathit{eff}} = m\vec{g} - m\vec{A} + \vec{T} \equiv m\,\vec{g}_{\mathit{eff}} + \vec{T} \tag{1.4}$$

where $\vec{g}_{\mathit{eff}} = \vec{g} - \vec{A}$, and $\vec{T}$ is the tension that ensures that the length of the pendulum is constant. Since $\vec{g}$ points down and $\vec{A}$ is horizontal, the magnitude $g_{\mathit{eff}} = \sqrt{g^2 + A^2}$, and without any more work, we see that the angular frequency of small oscillations is

$$\Omega = \sqrt{\frac{g_{\mathit{eff}}}{\ell}} = \sqrt{\frac{\sqrt{g^2 + A^2}}{\ell}} \tag{1.5}$$

where ℓ is the length of the pendulum.

2 Tides

Another application of accelerated frames is an explanation of tides. We focus on the motion of the earth and the moon around each other; the rotation of the earth explains how the tides change, but we are interested in why the oceans bulge the way they do in the first place.

The earth's acceleration about the moon is given by Newton's law:

$$\vec{A} = -\frac{GM_☾ \vec{R}}{R^3} \tag{2.6}$$

where $M_☾$ is the mass of the moon and $\vec{R}$ is the vector from the center of the moon to the center of the earth.

Consider a drop of seawater with mass m and position $\vec{r}$ relative to the center of the earth (see Figure 1.)

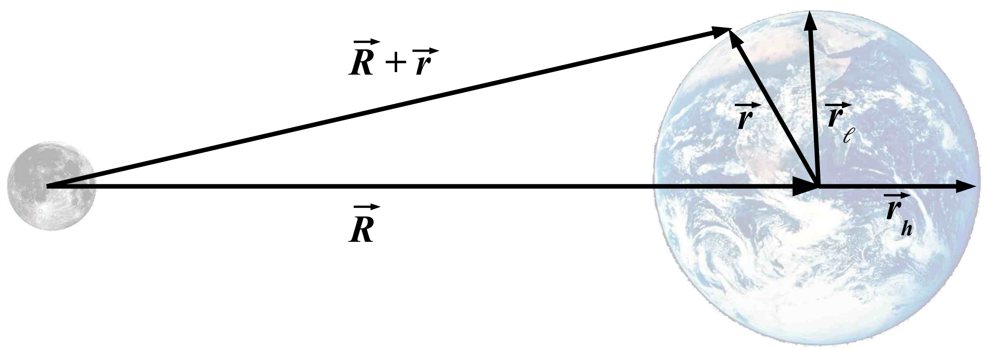

Figure 1: Earth and moon, not to scale

In the accelerated frame, Newton's law for the drop is:

$$m\vec{a} = \vec{F}_{\mathit{eff}} = -mg\hat{r} - m\vec{A} - \frac{GmM_☾(\vec{R}+\vec{r})}{|\vec{R}+\vec{r}|^3} - \vec{F}_{other} \tag{2.7}$$

where mg is the gravitational force exerted by the earth on the drop and $\vec{F}_{other}$ are all other forces that are not accounted for in the equation and which are assumed not to depend on the orientation of $\vec{r}$. The combination

$$\vec{F}_{tidal} = -m\vec{A} - \frac{GmM_☾(\vec{R}+\vec{r})}{|\vec{R}+\vec{r}|^3} = GmM_☾\left(\frac{\vec{R}}{|R|^3} - \frac{\vec{R}+\vec{r}}{|\vec{R}+\vec{r}|^3}\right) \tag{2.8}$$

is called the tidal force. Since the distance to the moon is much greater than the radius of the earth (by about a factor of 60), we have $|R| \gg |r|$, and we may Taylor expand:

$$\begin{aligned}
\vec{F}_{tidal} &= GmM_{☾}\left(\frac{\vec{R}}{|R|^3} - \frac{\vec{R}+\vec{r}}{|\vec{R}+\vec{r}|^3}\right) \\
&= \frac{GmM_{☾}}{|R|^2}\left(\hat{R} - \frac{\hat{R}+\frac{\vec{r}}{|R|}}{\left|\hat{R}+\frac{\vec{r}}{|R|}\right|^3}\right) \\
&= \frac{GmM_{☾}}{|R|^2}\left(\hat{R} - \frac{\hat{R}+\frac{\vec{r}}{|R|}}{\left(1+2\frac{\vec{r}\cdot\hat{R}}{|R|}+\frac{|r|^2}{|R|^2}\right)^{\frac{3}{2}}}\right) \\
&\approx \frac{GmM_{☾}}{|R|^2}\left[\hat{R} - \left(\hat{R}+\frac{\vec{r}}{|R|}\right)\left(1-\frac{3}{2}\left[2\frac{\vec{r}\cdot\hat{R}}{|R|}\right]\right)\right] \qquad (2.9)
\end{aligned}$$

so we find:

$$\vec{F}_{tidal} \approx GmM_{☾}\,\frac{3(\vec{r}\cdot\hat{R})\hat{R}-\vec{r}}{|R|^3} \qquad (2.10)$$

When the drop is on the side of the earth facing the moon, $\hat{R} = -\hat{r}$; when the drop is on the side of the earth away from the moon ($\vec{r}_h$ in Figure 1), $\hat{R} = \hat{r}$; in both cases, the tidal force is

$$\vec{F}_{tidal} \approx GmM_{☾}\,\frac{2\vec{r}_h}{|R|^3} \qquad (2.11)$$

which means that it points away from the center of the earth and decreases the effect of the earth's gravitational field. Thus the water level on the sides of the earth facing toward or away from the moon should be higher.

On the other hand, when the drop is halfway in between ($\vec{r}_\ell$ in Figure 1), $\vec{r}\cdot\hat{R} = 0$, and the tidal force is

$$\vec{F}_{tidal} \approx -GmM_{☾}\,\frac{\vec{r}_\ell}{|R|^3} \qquad (2.12)$$

that is, it adds to the earth's gravitational field and the water level should be lower. This leads to the correct conclusion that high tide and low tide occur twice a day.

We can make an estimate of the size of the tides fairly easily. The key observation is that if there were any forces parallel to the surface of the sea, that would cause the water to flow until they stopped acting (of course, we are ignoring smaller effects such as wind and currents which do precisely that). This means that $\vec{F}_{eff} = -mg\hat{r} + \vec{F}_{tidal}$ (the sum of the tidal and gravitational forces) is always perpendicular to the surface (this makes sense because $mg \gg |F_{tidal}|$). Consequently, if we consider the work done as we move along the surface, it must be zero. This means that the surface of the sea is an *equipotential surface*, that is, the potential energy is the same. The tidal potential is found by integrating (2.8) so that $\vec{F}_{tidal} = -\vec{\nabla} U_{tidal}$: (Since we are looking at the force on the water droplet with mass m, we take the gradient with respect to its coordinate r; we are treating the vector R from the center of the moon to the center of the earth as fixed.)

$$U_{tidal} = -GmM_{☾} \frac{3(\vec{r} \cdot \hat{R})^2 - \vec{r} \cdot \vec{r}}{2|R|^3} \tag{2.13}$$

The difference in the potential at the high tide point (when $\vec{r}_h \cdot \hat{R} = \pm r_h$) and at the low tide point (when $\vec{r}_\ell \cdot \hat{R} = 0$) is

$$\begin{aligned} \Delta U_{tidal} = U_{tidal}(r_h) - U_{tidal}(r_\ell) &= -\frac{GmM_{☾}}{2R^3} \left(3(\pm r_h)^2 - r_h^2 - (3 \cdot 0 - (r_\ell)^2)\right) \\ &= -GmM_{☾} \left(\frac{r_h^2}{R^3} + \frac{r_\ell^2}{2R^3} \right) \end{aligned} \tag{2.14}$$

To leading order $r_h \approx r_\ell \approx R_\oplus$, where $R_\oplus$ is the radius of the earth. Then the height difference between high and low tide $h := r_h - r_\ell$ can be found by setting the total difference in the potential to zero: $\Delta U_{total} = \Delta U_{tidal} + mgh = 0$; using $g = \frac{GM_\oplus}{R_\oplus^2}$ gives

$$0 = \Delta U_{total} = -GmM_{☾} \left(\frac{r_h^2}{R^3} + \frac{r_\ell^2}{2R^3} \right) + mgh \approx -3GmM_{☾} \frac{R_\oplus^2}{2R^3} + \frac{GmM_\oplus}{R_\oplus^2} h \tag{2.15}$$

and hence

$$h = \frac{3M_{☾} R_\oplus^4}{2M_\oplus R^3} \tag{2.16}$$

Plugging in the mass of the moon $M_{☾}$, the mass of the earth $M_\oplus$, the radius of the earth $R_\oplus$ and the distance from the center of the moon to the center of the earth R, we find $h \approx .5m$. Of course, the local geography also plays a big role, and the sun contributes a similar but somewhat smaller effect that we have neglected. (The sun adds to the tides when it is aligned with or against the sun, that is at new moon and full moon, and makes tides smaller when it is a half-moon).

3 Rotating frames

Next we consider motion from the perspective of a rotating observer. Then we have

$$\vec{r}_0 = r_{(0)m} \hat{e}_{(0)m} = \vec{r} = r_m \hat{e}_m \tag{3.17}$$

where the basis vectors and components in the rotating system are related to the inertial system by the rotation matrix $R_{mn} = \hat{e}_m \cdot \hat{e}_{(0)n}$ (this follows the discussion in section 2.5 in the notes for lecture 1; see also lecture 2):

$$\hat{e}_m = R_{mn}\hat{e}_{(0)n} \quad , \quad r_m = R_{mn}r_{(0)n} \tag{3.18}$$

where R_{mn} are the components of the rotation matrix $\mathbf{R}$; as rotations are orthogonal matrices, we have:

$$\mathbf{R}^T = \mathbf{R}^{-1} \quad \Leftrightarrow \quad R_{mn}R_{mp} = \delta_{np} \tag{3.19}$$

Differentiating (3.17), we find

$$\vec{v}_0 \equiv \dot{\vec{r}}_0 = \dot{\vec{r}} = \dot{r}_m\hat{e}_m + r_m\dot{\hat{e}}_m = \vec{v} + r_m\dot{R}_{mn}\hat{e}_{(0)n} = \vec{v} + r_m\dot{R}_{mn}R_{pn}\hat{e}_p \tag{3.20}$$

where $\vec{v}$ is the velocity in the rotating frame:

$$\vec{v} = \dot{r}_m\hat{e}_m \tag{3.21}$$

Because rotations are orthogonal (3.19), the product rule implies that the matrix $\dot{R}_{mn}R_{pn}$ is antisymmetric:

$$0 = \frac{d}{dt}(R_{mn}R_{mp}) = \dot{R}_{mn}R_{mp} + R_{mn}\dot{R}_{mp} \quad \Rightarrow \dot{R}_{mn}R_{mp} = -\dot{R}_{mp}R_{mn} \tag{3.22}$$

and we can write this matrix using the totally antisymmetric tensor ϵ_{mnp} defined in (L1.2.18) in Lecture 1:[1]

$$\dot{R}_{mn}R_{pn} = \epsilon_{mpq}\Omega_q \quad \Leftrightarrow \quad \Omega_q = \frac{1}{2}\epsilon_{mpq}\dot{R}_{mn}R_{pn} \tag{3.23}$$

where Ω_q are the components of a vector $\vec{\Omega}$ that points along the axis of the rotation. Using this definition in (3.20), we have

$$\vec{v}_0 = \vec{v} + r_m\hat{e}_p\Omega_q\,\epsilon_{mpq} = \vec{v} + \Omega_q r_m\hat{e}_p\,\epsilon_{qmp} \tag{3.24}$$

The expression for the cross-product given in Lecture 1 (see (L1.2.22)), gives us

$$\vec{v}_0 = \vec{v} + \vec{\Omega} \times \vec{r} \tag{3.25}$$

Differentiating again, we find[2]

$$\begin{aligned}\vec{a}_0 \equiv \dot{\vec{v}}_0 &= \frac{d}{dt}\left(\vec{v} + \vec{\Omega} \times \vec{r}\right) \\ &= \vec{a} + \dot{\vec{\Omega}} \times \vec{r} + \vec{\Omega} \times \vec{v} + \vec{\Omega} \times \left(\vec{v} + \vec{\Omega} \times \vec{r}\right) \\ &= \vec{a} + \dot{\vec{\Omega}} \times \vec{r} + 2\vec{\Omega} \times \vec{v} + \vec{\Omega} \times (\vec{\Omega} \times \vec{r})\end{aligned} \tag{3.26}$$

[1] I will refer to equations in Lecture 1 by adding a prefix L1.

[2] Because $\vec{\Omega} \times \vec{\Omega} = 0$, if we differentiate $\vec{\Omega}$ either in the rotating or the inertial basis, we get the same answer: $\dot{\vec{\Omega}} = \dot{\Omega}_m\hat{e}_m$.

where $\vec{a}$ is the acceleration in the rotating frame:

$$\vec{a} = \ddot{r}_m \hat{e}_m \tag{3.27}$$

The first term in (3.26), $\dot{\vec{\Omega}} \times \vec{r}$, is the transverse acceleration and is just the expected term from angular acceleration $\dot{\Omega}$; the second term $2\vec{\Omega} \times \vec{v}$ in (3.26) is the Coriolis acceleration, and the third term $\vec{\Omega} \times (\vec{\Omega} \times \vec{r})$ is the familiar centripetal acceleration.

In Newton's second law, these give rise to fictitious forces:

$$\vec{F} = \vec{F}_0 - m \left(\dot{\vec{\Omega}} \times \vec{r} + 2\vec{\Omega} \times \vec{v} + \vec{\Omega} \times (\vec{\Omega} \times \vec{r}) \right) = m\vec{a} \tag{3.28}$$

These forces are illustrated in Figure 2.

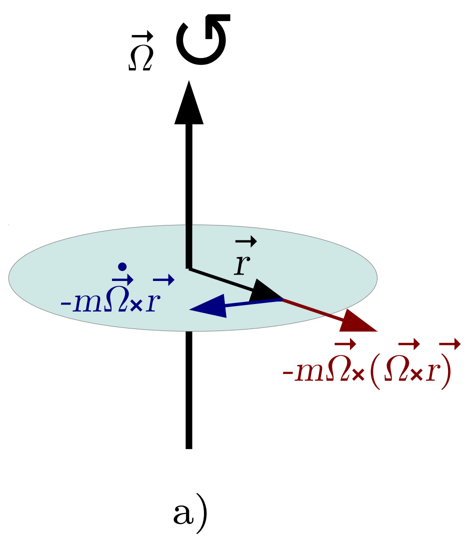

Figure 2: Fictitious forces in a rotating frame: a) The transverse force $-m\dot{\vec{\Omega}} \times \vec{r}$ is tangential and centrifugal force $-m\vec{\Omega} \times (\vec{\Omega} \times \vec{r})$ is radial. b) The coriolis force $-2m\vec{\Omega} \times \vec{v}$ is tangential if the velocity vector $\vec{v}$ is radial and radial if $\vec{v}$ is tangential.

3.1 Hurricanes, cyclones, and typhoons

If we look at the Coriolis force in Figure 2b) from above, as if we are looking down on the northern hemisphere, we see that a mass moving in the rotating disk appears to be pushed in a clockwise direction, whereas if we view the disk from below, the mass would move in a counterclockwise direction. We might expect hurricanes (properly called cyclones) to spin clockwise in the northern hemisphere, and counterclockwise in the southern hemisphere, but this is precisely the opposite of what happens. The reason is that hurricanes have hot low-pressure centers, so air is pulled in towards the center by the pressure gradient. The Coriolis force deflects the radially in-falling air tangentially as would be expected from Figure 2b), but the pressure gradient doesn't let it get out, and the net effect is a counterclockwise rotation. This can be seen in the first panel of Figure 3, which is taken from wikipedia[3] (the caption is adapted from the wikipedia caption).

There are also storms called anti-cyclones with cold high-pressure centers. Then the air is pushed out by the pressure gradient, and all the forces and velocities are reversed– these rotate clockwise in the northern hemisphere and counterclockwise in the southern hemisphere, as in the second panel of Figure 3. For both the hurricane and the anti-cyclone, we can neglect centrifugal forces.

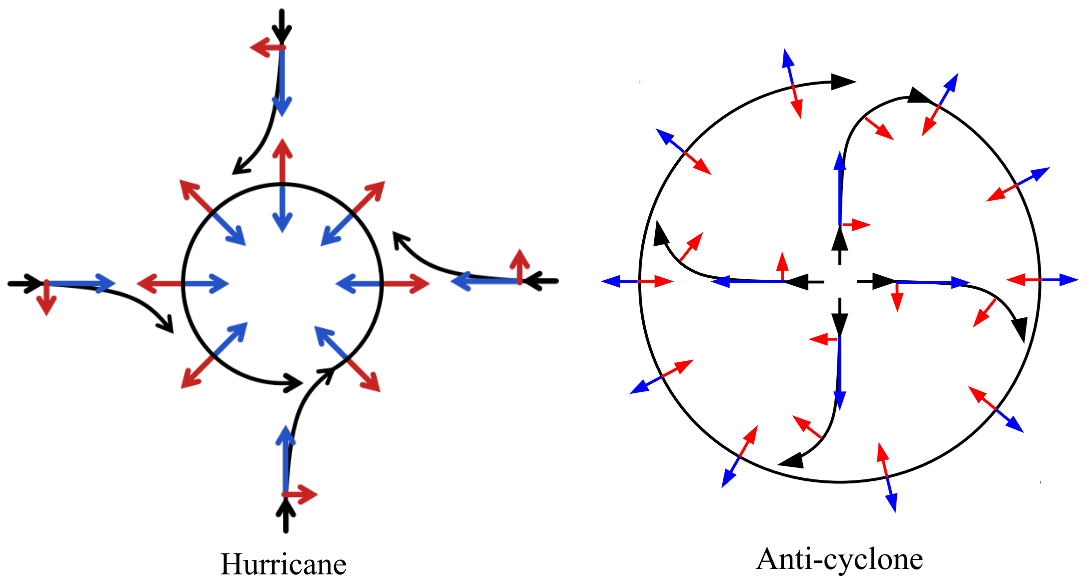

Figure 3: Flow and forces in a hurricane (low pressure) and anti-cyclone (high pressure) in the Northern Hemisphere. The pressure-gradient force is represented by **blue arrows**, the Coriolis force (always perpendicular to the velocity) by **red arrows**.

3.2 Deflection of a projectile

Consider a coordinate system at a latitude λ (the angle from the equator) with the x-axis pointing east, the y-axis pointing north, and the z-axis pointing up. Then the force on a

[3] http://en.wikipedia.org/wiki/File:Coriolis_effect10.svg

mass due to gravity is $\vec{F} = -mg\,\hat{z}$, and the earth's rotation vector is

$$\vec{\Omega} = \Omega(\cos\lambda\,\hat{y} + \sin\lambda\,\hat{z}) \tag{3.29}$$

We will use (3.28); however, we can simplify our calculation by noting that the transverse force vanishes since the the earth's rotation is essentially constant. We can also ignore the centrifugal force because it though it changes the direction and magnitude of the gravitational force slightly, the earth bulges in such a way as to keep the effective gravitational force perpendicular to the surface, and over short distances, the gravitational force is still very close to constant. This leaves only the Coriolis force:

$$-mg\,\hat{z} - 2m\vec{\Omega}\times\vec{v} = m\vec{a} \quad\Rightarrow\quad \vec{a} = -g\,\hat{z} - 2\vec{\Omega}\times\vec{v} \tag{3.30}$$

In this case, the Coriolis term is (using (L1.2.17) from Lecture 1):

$$\begin{aligned} -2\vec{\Omega}\times\vec{v} &= -2\Omega(\cos\lambda\,\hat{y} + \sin\lambda\,\hat{z}) \times (\dot{x}\hat{x} + \dot{y}\hat{y} + \dot{z}\hat{z}) \\ &= -2\Omega[\cos\lambda\,(\dot{z}\hat{x} - \dot{x}\hat{z}) + \sin\lambda\,(\dot{x}\hat{y} - \dot{y}\hat{x})] \end{aligned} \tag{3.31}$$

and hence we have:

$$\ddot{x} = -2\Omega(\cos\lambda\,\dot{z} - \sin\lambda\,\dot{y}) \tag{3.32}$$

$$\ddot{y} = -2\Omega\sin\lambda\,\dot{x} \tag{3.33}$$

$$\ddot{z} = -g + 2\Omega\cos\lambda\,\dot{x} \tag{3.34}$$

We can immediately integrate these once:

$$\dot{x} - v_x(0) = -2\Omega(\cos\lambda\,z - \sin\lambda\,y) \tag{3.35}$$

$$\dot{y} - v_y(0) = -2\Omega\sin\lambda\,x \tag{3.36}$$

$$\dot{z} - v_z(0) = -gt + 2\Omega\cos\lambda\,x \tag{3.37}$$

where we choose our coordinates so that the origin is at the initial position $\vec{r}(0) = 0$. Substituting (3.37) and (3.37) into (3.32) we find

$$\begin{aligned} \ddot{x} &= -2\Omega[\cos\lambda\,(v_z(0) - gt + 2\Omega\cos\lambda\,x) - \sin\lambda\,(v_y(0) - 2\Omega\sin\lambda\,x)] \\ &= -4\Omega^2 x + 2\Omega\cos\lambda\,gt - 2\Omega(\cos\lambda\,v_z(0) - \sin\lambda\,v_y(0)) \end{aligned} \tag{3.38}$$

We can solve this exactly, but there is no point–a projectile will stay in the air a minute or two, and since the earth's rotates once per day, we have

$$\Omega T \sim \frac{1\ min}{1\ day} = \frac{1}{24\cdot 60} \sim 10^{-3} \tag{3.39}$$

and thus the term $-4\Omega^2 x$ can safely be ignored. We have

$$x(t) = v_x(0)t + \frac{1}{3}\Omega\cos\lambda\, gt^3 - \Omega(\cos\lambda\, v_z(0) - \sin\lambda\, v_y(0))t^2 + \mathcal{O}(\Omega^2) \tag{3.40}$$

Substituting back in (3.37) and (3.37), keeping only the leading term, and integrating, we find

$$\begin{aligned} y(t) &= v_y(0)t - \Omega\sin\lambda\, v_x(0)t^2 + \mathcal{O}(\Omega^2) \\ z(t) &= v_z(0)t - \frac{1}{2}gt^2 + \Omega\cos\lambda\, v_x(0)t^2 + \mathcal{O}(\Omega^2) \end{aligned} \tag{3.41}$$

Suppose a projectile drops a distance h; if $\vec{v}(0) = 0$, (3.40) and (3.41) reduce to (ignoring all the $\mathcal{O}(\Omega^2)$ terms)

$$x(t) = \frac{1}{3}\Omega\cos\lambda\, gt^3 \quad , \quad y(t) = 0 \quad , \quad z(t) = -\frac{1}{2}gt^2 \tag{3.42}$$

(recall that we assumed $\vec{r}(0) = 0$). Then the projectile lands in a time

$$t_h = \sqrt{\frac{2h}{g}} \tag{3.43}$$

and projectile, instead of dropping straight down, is displaced to the east by a distance

$$x(t_h) = \cos\lambda\,\frac{2\Omega h}{3}\sqrt{\frac{2h}{g}} \tag{3.44}$$

At a latitude $\lambda = 45°$, for a height of $100m$, this gives a small but nontrivial effect: about $.015m = 1.5cm$. For large enough initial velocities, the effect can be much larger.

3.3 Foucault pendulum

The Foucault pendulum is a long pendulum that leaves a mark on the ground as it goes back and forth. If started in a simple motion in one plane, the plane of the motion slowly changes due to the Coriolis force.

As above, we start with (3.28), and keep only the external force and the Coriolis term.

$$m\vec{a} = -mg\hat{z} + \vec{T} - 2m\vec{\Omega}\times\dot{\vec{r}} \tag{3.45}$$

where $\vec{T}$ is the tension, and is a vector pointing up along the rope. We assume that the length of the rope ℓ is very long and the amplitude of the motion is much small, so we can use the small angle approximation. In this approximation, the pendulum moves only side-to-side, and the z components of the force must vanish; hence we can use $T_z \approx |\vec{T}| \equiv T = mg$. Then we have:

$$-mg\hat{z} + \vec{T} \approx -\frac{T}{\ell}(x\hat{x} + y\hat{y}) \approx -\frac{mg}{\ell}(x\hat{x} + y\hat{y}) \tag{3.46}$$

Dropping the terms proportional to $\dot{z}$ (because in the small angle approximation they are neglible), we get:

$$\ddot{x} = -\frac{g}{\ell}x + 2\Omega\sin\lambda\,\dot{y} \quad , \quad \ddot{y} = -\frac{g}{\ell}y - 2\Omega\sin\lambda\,\dot{x} \tag{3.47}$$

Since only the combination $\Omega\sin\lambda$ appears in these equations, we define $\Omega_\lambda = \Omega\sin\lambda$, and, as usual, the unperturbed frequency of oscillation $\omega_0^2 = \frac{g}{\ell}$ We also introduce the complex variable

$$\eta = x + iy \tag{3.48}$$

Then the equation simplifies to

$$\ddot{\eta} = -\omega_0^2\eta - 2i\Omega_\lambda\dot{\eta} \tag{3.49}$$

This looks like a damped harmonic oscillator with imaginary damping $\beta = i\Omega_\lambda$ (see equation (LHI:2.30))[4] and the frequency of oscillation is

$$\Omega_f^2 = \omega_0^2 + \Omega_\lambda^2 \approx \omega_0^2 \tag{3.50}$$

where the last approximation follows because the frequency of oscillations is much greater than the angular frequency of the earth's rotation: $\omega_0 >> \Omega_\lambda$. The general solution to (3.49) depends on two complex or four real integration constants, which is what we expect for a two dimensional harmonic oscillator (see equation (LHI:2.37)):

$$\eta(t) = e^{-i\Omega_\lambda t}\left(A_0^{(+)}e^{i\omega_0 t} + A_0^{(-)}e^{-i\omega_0 t}\right) \tag{3.51}$$

We want to choose a solution with the initial motion in a plane, *e.g.*, the xz-plane; this means both $y(0) = \dot{y}(0) = 0$ and hence $\eta(0)$ and $\dot{\eta}(0)$ should both be real. For convenience, let's also chose to start the pendulum from rest, so $\dot{\eta}(0) = 0$. Thus we want

$$A_0^{(+)} + A_0^{(-)} = x(0) \quad , \quad A_0^{(+)}(-\Omega_\lambda + \omega_0) + A_0^{(-)}(-\Omega_\lambda - \omega_0) = 0 \tag{3.52}$$

and hence

$$A_0^{(+)} = \frac{1}{2}x(0)\left(1 + \frac{\Omega_\lambda}{\omega_0}\right) \quad , \quad A_0^{(-)} = \frac{1}{2}x(0)\left(1 - \frac{\Omega_\lambda}{\omega_0}\right) \tag{3.53}$$

The solution can be written as

$$x(t) + iy(t) = x(0)e^{-i\Omega_\lambda t}[\cos(\omega_0 t) + i\frac{\Omega_\lambda}{\omega_0}\sin(\omega_0 t)] \tag{3.54}$$

[4] I will refer to equations in the "Notes on the Harmonic Oscillator Part I" by the prefix LHI.

Taking the real and imaginary parts, we see that the direction of the motion rotates in the xy-plane with an angular frequency $\Omega_\lambda = \Omega \sin \lambda$. The last term in (3.54) is of order Ω_λ/ω_0, so it can be dropped, and we find

$$x(t) = x(0) \cos(\Omega_\lambda t) \cos(\omega_0 t) \quad , \quad y(t) = x(0) \sin(\Omega_\lambda t) \cos(\omega_0 t) \tag{3.55}$$

Since $\Omega_\lambda << \omega_0$, we see that this describes a linear oscillation with angular frequency ω_0 in a plane that rotates with angular frequency $\Omega_\lambda = \Omega \sin \lambda$.