Physics 303/573

Midterm

Wednesday, October 26, 2016

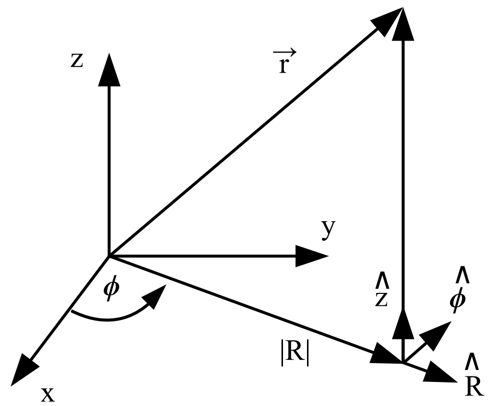
Note: Look over the whole midterm before you start, and do what you find easiest first.

Print name: ____

Formulas

Summation convention: repeated indices are implicit summations. $q_i p_i \equiv \sum_i q_i p_i$, *etc.*

Figure 1: **Cylindrical polar coordinates:**
$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z} = R\hat{R} + z\hat{z}$.
Length: $r^2 = R^2 + z^2$, $R^2 = x^2 + y^2$.
Cartesian: $x = R\cos\phi$, $y = R\sin\phi$, $z = z$.
Kinetic energy in cylindrical polar coordinates: $T = \frac{1}{2}m(\dot{R}^2 + R^2\dot{\phi}^2 + \dot{z}^2)$.
Grad: $\vec{\nabla} f(R, \phi, z) = \hat{R}\frac{\partial}{\partial R}f + \hat{\phi}\frac{1}{R}\frac{\partial}{\partial \phi}f + \hat{z}\frac{\partial}{\partial z}f$
Curl: $\vec{\nabla} \times \vec{v} = \hat{R}\left(\frac{1}{R}\frac{\partial}{\partial \phi}v_z - \frac{\partial}{\partial z}v_\phi\right) + \hat{\phi}\left(\frac{\partial}{\partial z}v_R - \frac{\partial}{\partial R}v_z\right) + \hat{z}\frac{1}{R}\left(\frac{\partial}{\partial R}(Rv_\phi) - \frac{\partial}{\partial \phi}v_R\right)$
Div: $\vec{\nabla} \cdot \vec{v} = \frac{1}{R}\left(\frac{\partial}{\partial R}(Rv_R) + \frac{\partial}{\partial \phi}v_\phi\right) + \frac{\partial}{\partial z}v_z$



Center of mass:

$$\vec{r}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}$$

Damped harmonic oscillator: (damping proportional to the velocity):

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0 \text{ has solutions } x(t) = (x_{max})e^{-\beta t}\cos(\omega t + \varphi) \text{ where } \omega^2 = \omega_0^2 - \beta^2 .$$

The energy is $E(t) = \frac{1}{2}m(\dot{x}^2 + \omega_0^2 x^2)$.

Driven damped harmonic oscillator:

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = \frac{1}{m}F_\omega e^{i\omega_F t} \text{ has special solution } x(t) = A_{\omega_F}e^{i\omega_F t} \text{ where } A_{\omega_F} = \frac{1}{m}\frac{F_\omega}{\omega_0^2 - \omega_F^2 + 2i\beta\omega_F}$$

transient solutions are determined by boundary conditions. System is linear so solutions superpose.

1. Consider a vector field $\vec{v}(R, \phi, z) = (z\cos\phi)\hat{R} + (z\sin\phi)\hat{\phi} + R\hat{z}$ in cylindrical polar coordinates.

(a) What is the length of this vector field?

(b) What is the time derivative of this vector?

(c) What is the divergence of this vector field?

(d) What is this vector field in cartesian coordinates?

2. Consider a mass m moving along the x-axis that feels a frictional force $F = -Av(1 + B|v|)(1 + C|v|)$ where $A, B, C > 0$ are positive constants. Find the time t as a function of the speed v when $v > 0$ if the mass has a speed v_0 at $t = 0$. Hint–you may find the following identity useful:

$$\frac{1}{(x+p)(x+q)} = \frac{1}{q-p}\left(\frac{1}{x+p} - \frac{1}{x+q}\right)$$

3. Consider a solid cube with uniform density, and edge-length ℓ whose center is at the origin. Where is its center of mass? (Hint: you do not need to integrate–use symmetry.) Now imagine that a solid cylindrical plug with radius $\ell/3$ and height $\ell/2$ is drilled out from the top so that the top of the plug is just at the top of the cylinder and and the center of the plug is $\ell/6$ from two of the edges. Where is the plug's outer side? (Sketch the drilled-out cube). Where is the center of mass of the drilled out cube? (Hint: again, you do not need to integrate.)

4. Consider a force (in cylindrical coordinates)

$$\vec{F}_1 = R^a \hat{\phi}$$

where a is a constant.

(a) Find the work this force does on a particle that follows a circular path of radius R_0 from $\phi = 0$ to $\phi = 2\pi$. Is this force conservative for any value of a?

(b) Find the work this force does on a particle that follows a closed path in four sections: First along a straight radial line from $R = R_1$ to $R = R_2$ along $\phi = 0$, then along an arc of radius R_2 from $\phi = 0$ to $\phi = \pi/2$, then back along a radial line from $R = R_2$ to $R = R_1$ along $\phi = \pi/2$, and finally an arc of radius R_1 back from $\phi = \pi/2$ to $\phi = 0$. Is there any value of a for which this vanishes?

(c) What is the curl of this force? Is there any value of a for which it vanishes away from $R = 0$?

5. Consider a mass m attached to a spring with spring constant $k = m\omega^2$ and unstretched length ℓ that slides along a horizontal track; for $x < 5\ell/4$, the air track is working and there is no significant friction, but for $x > 5\ell/4$ the air holes are blocked and there is a coefficient of sliding friction μ. (Recall that this means it feels a frictional force opposing its motion with a magnitude μmg where mg is the normal force due to gravity. Remember that frictional forces cannot make something move from rest–they can only resist motion.)

(a) What is the equation of motion for the mass in the region *without* friction? What is the equation of motion for the mass in the region *with* friction when it is moving in the positive x direction? What is the equation of motion for the mass in the region *with* friction when it is moving in the negative x direction?

(b) Suppose the the mass starts at rest ($v_0 = 0$) with the spring *compressed* to 3/4 its length: $x_0 = 4\ell/4$

(i) What is the solution $x(t)$ and $v(t)$ What is the maximum value of x the mass reaches?

(ii) Suppose the the mass starts at rest with the spring *compressed* to 1/2 its length: $\ell/2$ let the mass start at rest at $x(0) = \ell/2$, and recall that there is friction for $x > 5\ell/4$. find $x(t)$ for $x < 5\ell/4$; find $x(t)$ for $x > 5\ell/4$ as long as the velocity does not change sign (Suppose that the spring constant is large enough so that it overcomes the frictional force). After the velocity changes sign, what will $x(t)$ be? (Hint: the velocity and the position do not jump.)

(c) Sketch $x(t)$ and $v(t)$ for a few cycles, and then sketch the motion in phase space, with $x(t)$ along the horizontal and $\frac{v(t)}{\omega}$ along the vertical. Will the mass move forever, or will it come to rest? If it comes to rest, what determines how many cycles it moves before it comes to rest?