Physics 303/573

Charge in a Magnetic Field

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1 Physical setup

A particle with charge q in a magnetic field $\vec{B}$ experiences a force perpendicular to its velocity $\vec{v}$and to the magnetic field:

$$\vec{F} = q\vec{v} \times \vec{B} \tag{1.1}$$

(this is called the Lorentz force law). Using Newton's law, we have

$$\dot{\vec{v}} = \omega\vec{v} \times \hat{B} \tag{1.2}$$

where the *cyclotron frequency* ω is

$$\omega = \frac{qB}{m} \tag{1.3}$$

and $B = |\vec{B}|$. If we take, e.g., $\vec{B} = B\hat{z}$, using $\hat{y} \times \hat{z} = \hat{x}$, $\hat{x} \times \hat{z} = -\hat{y}$, and $\hat{z} \times \hat{z} = 0$, we find

$$\dot{v}_x = \omega v_y \quad , \quad \dot{v}_y = -\omega v_x \quad , \quad \dot{v}_z = 0 \tag{1.4}$$

There are many ways to solve these equations, and we will see several of them later, but now I will discuss the way the book uses.

2 Complex notation

We introduce a complex quantity η:

$$\eta = v_x + iv_y \tag{2.5}$$

where i is $\sqrt{-1}$, that is, $i^2 = -1$. The real part of η is v_x and the imaginary part is v_y; we can draw η in the complex plane as shown in Fig. 2.13 in the book. Now it is easy to see that

$$\dot{\eta} = \dot{v}_x + i\dot{v}_y = \omega v_y - i\omega v_x = -i\omega(v_x + iv_y) = -i\omega\eta \tag{2.6}$$

Thus the two equations have become a single trivial first order differential equation whose solution we can immediately write down:

$$\eta(t) = \eta_0 e^{-i\omega t} \tag{2.7}$$

where $\eta_0 = \eta(0)$ We can easily read off v_x and v_y by taking the real and imaginary parts. As the book discusses, one of the basic deep theorems of complex numbers is

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{2.8}$$

An easy way to prove this is to note that is obviously true for $\theta = 0$, and then compute:

$$\frac{d}{d\theta} e^{i\theta} = ie^{i\theta} \tag{2.9}$$

but

$$\frac{d}{d\theta}(\cos\theta + i\sin\theta) = -\sin\theta + i\cos\theta = i(\cos\theta + i\sin\theta) \tag{2.10}$$

Since the two quantities are equal at one point and obey the same first order differential equation, they must be the same for all θ.

3 Solution

Now we can find v_x, v_y:

$$\begin{aligned} v_x(t) = \mathrm{Re}[\eta(t)] &= \mathrm{Re}[\eta_0 e^{-i\omega t}] \\ &= \mathrm{Re}[(v_x(0) + iv_y(0))(\cos(\omega t) - i\sin(\omega t)] \\ &= v_x(0)\cos(\omega t) + v_y(0)\sin(\omega t) \end{aligned} \tag{3.11}$$

and similarly

$$v_y(t) = \mathrm{Im}[\eta(t)] = v_y(0)\cos(\omega t) - v_x(0)\sin(\omega t) \tag{3.12}$$

Of course, $\dot{v}_z(t) = 0$ implies that the z-component of the velocity is constant.

$$v_z(t) = v_z(0) \tag{3.13}$$

Notice that the magnitude of the velocity is constant; indeed, v_z and $|v_x\hat{x} + v_y\hat{y}|$ are separately constant.

We can easily integrate (3.11,3.12,3.13) to find:

$$\begin{aligned} x(t) &= x_{org} + \frac{1}{\omega}(v_x(0)\sin(\omega t) - v_y(0)\cos(\omega t)) \\ y(t) &= y_{org} + \frac{1}{\omega}(v_y(0)\sin(\omega t) + v_x(0)\cos(\omega t)) \\ z(t) &= z_0 + v_z(0)t \end{aligned} \tag{3.14}$$

where x_{org}, y_{org} are the coordinates at the center of a spiral that the particle traces as it moves up along the z-axis with constant z-velocity. In the x, y plane, it moves along a circle with radius

$$r_P = \frac{|v_R|}{\omega} = \frac{m|v_R|}{qB} = \frac{|p_R|}{qB} \tag{3.15}$$

where $\vec{v}_R = v_x\hat{x} + v_y\hat{y}$ is the component of the velocity in the x, y-plane.

One can also integrate the solution (2.7) as in the book, and find, defining $\xi = x + iy$,

$$\xi(t) = \xi_{org} + i\frac{\eta_0}{\omega}e^{-\omega t} \tag{3.16}$$

Then $x(t)$ and $y(t)$ are found as the real and imaginary parts of $\xi(t)$.