Physics 303/573

Homework 7 due Wednesday, November 8.

1. Suppose there is a flat piece of glass whose front surface is at $x = 0$ and whose back surface is at $x = \ell$. Suppose as well that its index of refraction depends only on the height: $n^2(y) = 1 + (n_0^2 - 1)e^{-a|y|}$. (This has index of refraction n_0 at $y = 0$, and tends to 1 as $|y| \to \infty$).

a) Write down the functional that needs to be extremized to use Fermat's principle (see my notes). (Hint–this problem becomes *much* easier if you describe the path by $x(y)$ rather than $y(x)$. Can you say why?) You may restrict yourself to paths that have positive y in the glass. What is the equation that you get?

b) Find the general form of the function that describes the path $x(y)$ of a light beam through the glass that starts at $x = 0, y = 0$ by extremizing the time the light takes. Be careful to find the integration constants: express it in terms of the angle the light makes to the surface at $x = 0$.

(You may find the website http://www.integral-calculator.com/ useful–it not only finds integrals, it explains how it does them).

c) Consider two points, one at $x = -d, y = h$ and the second at $x = d + \ell, y = h$. Calculate the time it would take for light to go between the two points along a straight line, taking into account the time delay in the glass. Now consider a path between the two points that goes a little way higher a point $h + \delta$ to the glass, then straight across the glass, and then back down to the other point, and calculate the time this would take. Is this longer or shorter than the time the straight line takes (you can try an example such as $d = a = h = \ell = 1, n_0 = 2$ and plot the result as a function of δ, or give an analytic argument). Does the answer surprise you?

2. Consider a particle of mass M that is constrained to move on a cone defined by $z^2 = A^2(x^2 + y^2)$ (A is a constant) with no external force other than the constraint on the particle. (take $z > 0$).

a) Write down the Lagrangian in Cartesian coordinates with a Lagrange multiplier that imposes the constraints.

b) Find the Euler-Lagrange equations in Cartesian coordinates.

c) Rewrite the Lagrangian in cylindrical polar coordinates. Solve the constraint for z and substitute the solution into the Lagrangian to get a Lagrangian $L(R, \dot{R}, \dot{\phi})$.

d) Find the Euler-Lagrange equations that follow from $L(R, \dot{R}, \dot{\phi})$ that you found.

e) Find the effective potential for R in terms of L_z, the z-component of the angular momentum.

3. Consider the same system as above, but now turn on a gravitational force.

a) Write the Lagrangian in cylindrical polar coordinates. Solve the constraint for z and substitute the solution into the Lagrangian to get a Lagrangian $L(R, \dot{R}, \dot{\phi})$ (you don't need to bother with Lagrange multipliers in cylindrical polar coordinates).

b) Find the Euler-Lagrange equations that follow from $L(R, \dot{R}, \dot{\phi})$ that you found.

c) Find the effective potential for R in terms of L_z, the z-component of the angular momentum. Sketch the effective potential (recall that $z > 0$).

d) For a given L_z, what is the equilibrium value of R?

e) Is the equilibrium stable or unstable?

4. Consider a bead of mass m constrained to move along a helical wire described in cylindrical coordinates by the equations $R = R_0$, R_0 a constant, and $z = \beta\phi$, where β is a constant and ϕ is the angle around the z-axis. Let gravity act on the bead.

a) Write down the Lagrangian–since the constraints are so simple, there is no need to use Lagrange multipliers, and you can simply use z as your variable. Don't forget the gravitational potential.

b) Write down the Euler-Lagrange equation. Find the general solution for arbitrary initial z_0 and $\dot{z}_0$. How does this differ from a freely falling bead?