PHY 303

Mechanics

Aug 30, 2017

$$\mathcal{L}(\vec{v}) = \mathcal{L}(\sum v^i \hat{e}_i)$$
$$= \sum v^i \mathcal{L}(\hat{e}_i)$$
$$= \sum v^i \mathcal{L}_i^j \hat{e}_j$$

| i Row
| j col

$\boxed{?}$

we use

$$\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}}$$

we chose

$$e_i \cdot e_j = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} = I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

orthonormal basis

$$\vec{v}_i = \vec{v} \cdot \hat{e}_i$$

$$\vec{v} \cdot \vec{w} = \sum v^i \hat{e}_i \cdot w^j \hat{e}_j$$
$$= \sum v^i w^j \underbrace{\hat{e}_i \cdot \hat{e}_j}_{\delta_{ij}}$$
$$= \sum v^i w_i$$

In 3D

$$\hat{e}_x = \hat{x} = \hat{i}$$
$$\hat{e}_y = \hat{y} = \hat{j}$$
$$\hat{e}_z = \hat{z} = \hat{k}$$

$$\mathcal{L}_{ij} = \mathcal{L}(\hat{e}_i) \cdot \hat{e}_j$$

matrix element

Usually are Rotation or Reflection

$$\hat{e}_i \cdot \hat{e}_j = \delta_{ij}$$

orthogonal transformations

A linear transformations that preserve $|\vec{v}|$, $\vec{v} \cdot \vec{w}$ for $\forall\, \vec{v}, \vec{w}$

$|\vec{v}||\vec{w}|\cos\theta$

prime

$$\hat{e}_i' = R(\hat{e}_i) = R_i^j \hat{e}_j$$

$$\hat{e}_i' \cdot \hat{e}_j' = \delta_{ij} \qquad \Rightarrow \hat{e}_i' \cdot \hat{e}_j = R_{ij}$$

$$\Rightarrow R_i^k \hat{e}_k R_j^m \hat{e}_m = \delta_{ij}$$
$$\Rightarrow R_i^k R_j^m \underbrace{\hat{e}_k \cdot \hat{e}_m}_{\delta_{ij}} = R_i^k \cdot R_{jk}$$
$$= R_i^k R^T_{kj} = \delta_{ij}$$
$$\Leftrightarrow R R^T = I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Active

$$R(\vec{v}) = v^i R(\hat{e}_i)$$
$$= v^i R_i^j \hat{e}_j$$

Passive

$\vec{v}$ in a different basis

$$\vec{v} = v^i \hat{e}_i = v^{i'} \hat{e}_i'$$
$$= v^{i'} R_j^i \hat{e}_i$$
$$\Rightarrow v^i = v^{j'} R_j^i = (R^T)^i_j v^{j'}$$
$$v^{j'} = R^j_i v^i \quad (\text{using } R^T R = R R^T = I) \quad [I, \mathbb{1}]$$

3 Dimensions - Cross product

Anti symmetric : $\vec{v} \times \vec{w} = -\vec{w} \times \vec{v}$

linear : $(a\vec{v} + b\vec{w}) \times \vec{u} = a(\vec{v} \times \vec{u}) + b(\vec{w} \times \vec{u})$

Right Hand Rule : $\hat{x} \times \hat{y} = \hat{z}$, $\hat{y} \times \hat{z} = \hat{x}$, $\hat{z} \times \hat{x} = \hat{y}$

Alternating tensor

$$\varepsilon^{ijk} = \begin{cases} +1 & \text{if } ijk = \begin{matrix} xyz \\ yzx \\ zxy \end{matrix} \\ 0 & \text{if any 2 } ijk \text{ are the same} \\ -1 & \text{if } ijk = \begin{matrix} yxz \\ zyx \\ xzy \end{matrix} \end{cases}$$

$$\det(M) = \sum \frac{1}{6} \varepsilon^{ijk} \varepsilon^{abc} M_{ia} M_{jb} M_{kc}$$

$$(M^{-1})^{ck} = \frac{1}{2\det(M)} \varepsilon^{ijk} \varepsilon^{abc} M_{ia} M_{jb}$$

$$\hat{e}_i \times \hat{e}_j = \varepsilon_{ijk} \hat{e}_k \qquad \Rightarrow \hat{e}_k = \frac{1}{2} \varepsilon_{ijk} \hat{e}_i \times \hat{e}_j$$

$\leftarrow$ e.g. $\hat{z} = \frac{1}{2}(\hat{x} \times \hat{y} - \hat{y} \times \hat{x})$

$= \hat{x} \times \hat{y}$

$$\Rightarrow \vec{v} \times \vec{w} = (v^i \hat{e}_i) \times (w^j \hat{e}_j) = v^i w^j \varepsilon_{ijk} \hat{e}_k$$

$$(\hat{e}_i \times \hat{e}_j) \cdot \hat{e}_k = \varepsilon_{ijk}$$

Orthogonal transformation.

$\varepsilon'_{ijk} = (\det R)\,\varepsilon_{ijk}$

$\det(AB) = (\det(A))(\det(B)) \Rightarrow \underbrace{\det(R)\,\det(R^T)}_{(\det(R))^2 \Rightarrow \det(R) = \pm 1} = 1$

$\det(A^T) = \det(A)$

stay same for rotation.

change sign for reflection

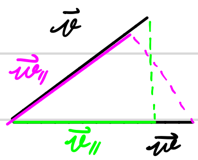
$\vec v \cdot (\vec w \times \vec u) = v^i w^j u^k \varepsilon_{ijk}$

$\varepsilon_{ijk}\,\varepsilon^{ijm} = 2\delta^m_n$

$\varepsilon_{ijk}\,\varepsilon^{imn} = \delta^m_j \delta^n_k - \delta^n_j \delta^m_k$

$\vec v \cdot \vec w = |\vec v||\vec w|\cos\theta = |\vec v|\,|\vec w_\parallel|$

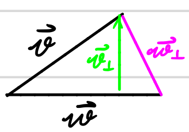
$= |\vec v_\parallel|\,|\vec w|$



$|\vec v \times \vec w| = |\vec v||\vec w|\sin\theta$

$= |\vec v|\,|\vec w_\perp| = |\vec w|\,|\vec v_\perp|$

$= 2$ Area of $\triangle$



2D Cartesian Coordinates

$\vec v = v_x \hat x + v_y \hat y$

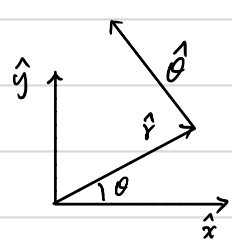
$\vec r = x\hat x + y\hat y$

Polar Coordinates

$\vec r = |r|\hat r$ $\qquad \vec v = v_r \hat r + v_\theta \vec\theta$

$\vec r = \hat x \cos\theta + \hat y \sin\theta$

$\hat\theta = \hat y \cos\theta - \hat x \sin\theta$



$$\begin{pmatrix} \hat{r} \\ \hat{\theta} \end{pmatrix} = R(\theta) \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix}$$

$$\vec{v} = v_x \hat{x} + v_y \hat{y} = v_r \hat{r} + v_\theta \hat{\theta}$$

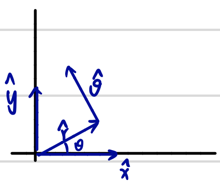
$$\begin{pmatrix} v_r \\ v_\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} v_x \\ v_y \end{pmatrix}$$

2D Polar Coordinates

Sep 6, 2017

$$R(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$\begin{pmatrix} \hat{r} \\ \hat{\theta} \end{pmatrix} = R(\theta) \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} = \begin{pmatrix} \cos\theta\,\hat{x} + \sin\theta\,\hat{y} \\ \cos\theta\,\hat{y} + \sin\theta\,\hat{x} \end{pmatrix}$$

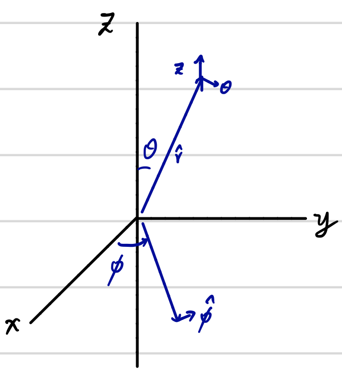


$$\vec{v} = v_x \hat{x} + v_y \hat{y} \equiv v_i \hat{e}_i$$

$$= v_r \hat{r} + r_\theta \hat{\theta}$$

$$\begin{pmatrix} v_r \\ v_\theta \end{pmatrix} = R(\theta) \begin{pmatrix} v_x \\ v_y \end{pmatrix}$$

3D Cylindrical Polar Coordinates



$$\vec{r} = R\hat{R} + z\hat{z}$$

$$|r| = \sqrt{R^2 + z^2} = \sqrt{\vec{r}\cdot\vec{r}}$$

$$R = \sqrt{x^2 + y^2}$$

$$\begin{pmatrix} \hat{R} \\ \hat{\phi} \\ \hat{z} \end{pmatrix} = R(\phi) \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix} \qquad R(\phi) = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

For a general vector, $\vec{v} = v_R \hat{R} + v_\phi \hat{\phi} + v_z \hat{z}$

Spherical Polar Coordinates

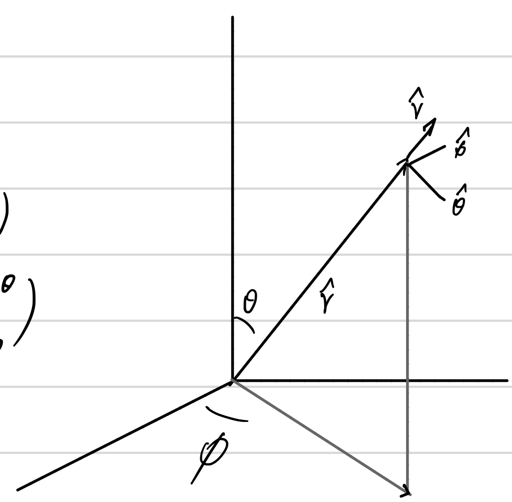
$\vec{r} = r\hat{r}$, $\vec{v} = v_r \hat{r} + v_\theta \hat{\theta} + v_\phi \hat{\phi}$

$$\begin{pmatrix} \hat{\theta} \\ \hat{\phi} \\ \hat{r} \end{pmatrix} = R_y(\theta) R_z(\phi) \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix}$$

$$= \begin{pmatrix} (\hat{x}\cos\phi + \hat{y}\sin\theta)\cos\phi - \hat{z}\sin\theta \\ \hat{y}\cos\phi - \hat{x}\sin\phi \\ (\hat{x}\cos\phi + \hat{y}\sin\phi)\sin\theta + \hat{z}\sin\theta \end{pmatrix}$$

$$R_z(\phi) = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$R_y(\theta) = \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix}$$



$\vec{v} = v_i \, \hat{e}_i$ $\quad\quad \not{\exists}_i \; \hat{e}_i$

$$\frac{d\vec{v}}{dt} = \frac{dv_i}{dt}\hat{e}_i + v_i \frac{d\hat{e}_i}{dt}$$

$$\hat{e}\cdot\hat{e} = 1 \qquad \left(\frac{d\hat{e}}{dt}\right)\cdot \hat{e} = 0$$

Cartesian:

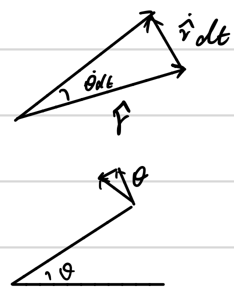
$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$$

$$\vec{v} = \frac{d\vec{r}}{dt} = \dot{x}\hat{x} + \dot{y}\hat{y} + \dot{z}\hat{z} \qquad \dot{x} = \frac{dx}{dt}$$

velocity

$$\vec{a} = \frac{d\vec{v}}{dt} = \ddot{x}\hat{x} + \ddot{y}\hat{y} + \ddot{z}\hat{z}$$

Polar Coordinates.



$$\frac{d}{dt}\hat{r} = \dot{\theta}\hat{\theta}$$

$$\frac{d}{dt}\hat{\theta} = -\dot{\theta}\hat{r}$$

$$\vec{r} = r\hat{r}$$

$$\vec{v} = \frac{d}{dt}\vec{r} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$$

$$\vec{a} = \frac{d}{dt}\vec{v} = \ddot{r}\hat{r} + 2\dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} - r\dot{\theta}^2\hat{r}$$

$$\begin{pmatrix}\hat{r}\\ \hat{\theta}\end{pmatrix} = R(\theta)\begin{pmatrix}\hat{x}\\ \hat{y}\end{pmatrix} \qquad R^{-1} = R^T$$

$$\frac{d}{dt}\begin{pmatrix}\hat{r}\\ \hat{\theta}\end{pmatrix} = \left(\frac{d}{dt}R(\theta)\right)\begin{pmatrix}\hat{x}\\ \hat{y}\end{pmatrix} = \underbrace{\dot{R}(\theta)R^T(\theta)}_{\dot{\theta}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}}\begin{pmatrix}\hat{r}\\ \hat{\theta}\end{pmatrix}$$

$$\dot{R}_z(\phi)R_z^T(\phi) = \dot{\phi}\begin{pmatrix}0 & 1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$$

$$\dot{R}_y(\phi)R_y^T(\theta) = \dot{\theta}\begin{pmatrix}0 & 0 & -1\\ 0 & 0 & 0\\ 1 & 0 & 0\end{pmatrix}$$

$$R_x(\phi)R_x^T(x) = \dot{x}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 1\\ 0 & -1 & 0\end{pmatrix}$$

$$1 = (AB)^{-1}AB = B^{-1}(A^{-1}A)B = B^{-1}B = 1$$

$$(AB)^T = B^T A^T$$

$$(AB)^{-1} = B^{-1}A^{-1}$$

$$\frac{d}{dt}\begin{pmatrix}\hat\theta\\ \hat\phi\\ \hat r\end{pmatrix} = \dot R_y \hat R_z(\phi)\begin{pmatrix}\hat x\\ \hat y\\ \hat z\end{pmatrix} + R_y(\theta)\dot R_z(\phi)\begin{pmatrix}\hat x\\ \hat y\\ \hat z\end{pmatrix}$$

$$= \dot R_y(\theta)\underbrace{R_z(\phi) R_z^T(\phi)}_{1} R_y^T(\theta)\begin{pmatrix}\hat\theta\\ \hat\phi\\ \hat r\end{pmatrix} + R_y(\theta)\dot R_z(\phi) R_z^T(\phi) R_y^T(\theta)\begin{pmatrix}\hat\theta\\ \hat\phi\\ \hat r\end{pmatrix}$$

$$= \dot\theta\begin{pmatrix}0&0&-1\\0&0&0\\1&0&0\end{pmatrix}\begin{pmatrix}\hat\theta\\ \hat\phi\\ \hat r\end{pmatrix} + \dot\phi R_y(\theta)\begin{pmatrix}0&1&0\\-1&0&0\\0&0&0\end{pmatrix} R_y^T(\theta)\begin{pmatrix}\hat\theta\\ \hat\phi\\ \hat r\end{pmatrix}$$

Physics

$$\vec{F} = m\vec{a}$$

$$\vec{F}_g = -mg\hat{z}$$

$$\vec{F}_{fric} = -f(v)\hat{v}$$

T-E for low v → $f(0) + \underbrace{|v|\frac{\partial f}{\partial v}\Big|_0}_{\text{linear drag}} + \underbrace{\frac{1}{2}|v|^2\frac{\partial^2 f}{\partial v^2}\Big|_0}_{\text{quadratic drag}} + \cdots$

$f(0)$: μN for sliding friction; 0 for gas or liquid

$$\vec{F} = m\ddot{\vec{r}} = -mg\hat{z} - \frac{m}{\tau}\dot{\vec{r}} \quad \text{Linear drag}$$

separable in cartesian coordinates.

$$\ddot{\vec{r}} = -\frac{1}{\tau}\dot{\vec{r}} - g\hat{z}$$

$$\dot{\vec{r}} - \vec{v}_0 = -\frac{1}{\tau} = -\frac{1}{\tau}(\vec{r}(t) - r_0) - gt\hat{z} \Rightarrow \dot{\vec{v}} = -\frac{1}{\tau}\vec{v} - g\hat{z}$$

$\dot{\vec{r}} = \vec{v}(t)$

$$\vec{r}(t) = \vec{r}_0 - \tau(\vec{r}(t) - \vec{v}_0 + gt\hat{z})$$

$$\dot{v}_x = \frac{1}{\tau}v_x, \quad \dot{v}_y = -\frac{1}{\tau}v_y, \quad \dot{v}_z = -\frac{1}{\tau}v_z - g.$$

$$v_x(t) = \dot{x}(0)e^{-t/\tau}$$

$$\frac{dv_x}{dt} = -\frac{1}{\tau}v_x$$

$$\ln\frac{v_x}{v_x(0)} = \int\frac{dv_x}{v_x} = -\int\frac{dt}{\tau} = -\frac{t}{\tau}$$

$$v_y(t) = v_y(0)e^{-t/\tau}$$

$$\int_{v_z(0)}^{v_z(t)}\frac{dv_z}{v_z + g\tau} = -\frac{t}{\tau}$$

$$= \ln\left(\frac{v_z(t) + g\tau}{v_z(0) + g\tau}\right)$$

$$v_z(t) = -g\tau + (v_z(0) + g\tau)e^{-t/\tau}$$

$$= v_0 e^{-t/\tau} - g\tau(1 - e^{-t/\tau})$$

$$v_{ter} = -g\tau$$

Force

Sep 10, 2017
Book

Newton II : $\vec{F} = m\vec{a}$

$$\vec{p} = m\vec{v}$$

$$\dot{\vec{p}} = m\dot{\vec{v}} = m\vec{a}$$

$$\vec{F} = \dot{p}$$

$\Rightarrow \ddot{x}(t) = \frac{F_0}{m}$ moving alone x-axis, constant force.

$\dot{x}(t) = \int \ddot{x}(t)dt = v_0 + \frac{F_0}{m}t$

$x(t) = \int \dot{x}(t)dt = x_0 + v_0 t + \frac{F_0}{2m}t^2$

Inertial Frames

An inertial frame is defined as one when the first law holds.

Newton III

$\vec{F}_{21}$: force exerted on 2 by 1

$\vec{F}_{12} = -\vec{F}_{21}$ Newton III

Not hold since $F_{12}(t)$, $F_{21}(t)$ measure at same time t.

magnetic force : not hold

2D ○

$$\begin{cases} x = r\cos\phi \\ y = r\sin\phi \end{cases} \qquad \begin{cases} r = \sqrt{x^2+y^2} \\ \phi = \arctan(y/x) \end{cases} \qquad \begin{aligned} \hat{r} &= \cos\phi\,\hat{x} + \sin\phi\,\hat{y} \\ \hat{\phi} &= -\sin\phi\,\hat{x} + \cos\phi\,\hat{y} \end{aligned}$$

$$\Delta\hat{r} \approx \Delta\phi\,\hat{\phi} \approx \dot{\phi}\,\Delta t\,\hat{\phi}$$

$$\Rightarrow \frac{d\hat{r}}{dt} = \dot{\phi}\,\hat{\phi}$$

Sep 11, 2017

$\vec{F} = -mg\hat{z} - f(v)\hat{v}$

$f(v) = f_0 + f_1|v| + f_2|v|^2 + \cdots$

assume f_0, f_2 negligible

$\Rightarrow \ddot{\vec{r}} = -\frac{1}{\tau}\dot{\vec{r}} - gz$

$\dot{\vec{r}} - \dot{\vec{r}}_0 = -\frac{1}{\tau}(\vec{r} - \vec{r}_0) - gt\hat{z}$

init value

$\dot{\vec{v}} = -\frac{1}{\tau}\vec{v} - g\hat{z}$

$\vec{r} = \vec{r}_0 - \tau(\vec{v} - \vec{v}_0 + gt\hat{z})$

$$\begin{cases} \dot{v}_x = -\frac{1}{\tau}v_x \\ \dot{v}_y = -\frac{1}{\tau}v_y \\ \dot{v}_z = -\frac{1}{\tau}v_z - g \end{cases} \Rightarrow \begin{cases} v_x = v_x(0)\,e^{-t/\tau} \\ v_y = v_y(0)\,e^{-t/\tau} \\ v_z = (v_z(0) + \tau g)\,e^{-t/\tau} - \tau g \\ \quad = v_z(0)\,e^{-t/\tau} - g\tau(1 - e^{-t/\tau}) \end{cases}$$

as $t \to \infty$, $e^{-t/\tau} \to 0$

$\vec{v} = -g\tau\hat{z} = -v_{ter}\hat{z}$

Quadratic Drag

$\vec{F} = -mg\hat{z} - c|v|\vec{v}$

$\sqrt{v_x^2 + v_y^2 + v_z^2}$

$c \propto$ Area, ℓ

$F \sim \frac{[m][\ell]}{[t]^2}$

$cv^2 \sim \frac{[m]}{[\ell]^3}\frac{[\ell]^2[\ell]^2}{[t]^2} \longleftarrow$ agrees.

Let take $v_y = 0$

$\dot{v}_x = -\frac{c}{m}\sqrt{v_x^2 + v_z^2}\,v_x$

$\dot{v}_z = -g - \frac{c}{m}\sqrt{v_x^2 + v_z^2}\,v_z$

$\Rightarrow \dot{v}_z = -g + \boxed{\frac{v_z}{v_x}}\dot{v}_x$

slope

let $w = \frac{v_z}{v_x} \Rightarrow \dot{w} = \frac{\dot{v}_z}{v_x} - \frac{v_z\dot{v}_x}{v_x^2}$

$= \frac{1}{v_x}(\dot{v}_z - \frac{v_z}{v_x}\dot{v}_x)$

$\Rightarrow \dot{w} = -\frac{g}{v_x}$

$$\dot v_x = -\frac{c}{m}\sqrt{1+w^2}\, v_x^2$$

$$\dot w = \frac{g\dot v_x}{v_x^2} = -\frac{cg}{m}\sqrt{1+w^2}$$

$$\Rightarrow \ddot w \dot w = -\frac{cg}{m}\sqrt{1+w^2}\,\dot w$$

$$\frac{1}{2}\frac{d}{dt}(\dot w^2) = \frac{d}{dt}\left(-\frac{cg}{m}\int^{w} d\tilde w \sqrt{1+\tilde w^2}\right)$$

$$\dot w^2 = -\frac{2cg}{m}\int_{w_0}^{w} d\tilde w \sqrt{1+\tilde w^2}$$

$$= -\frac{cg}{m}\left(w\sqrt{1+w^2} + \ln(w+\sqrt{1+w^2}) - A\right)$$

$$A = w_0\sqrt{1+w_0^2} + \ln(w_0+\sqrt{1+w_0^2}) + \frac{mg}{c\dot x_0^2}$$

For any function

$$\dot w^2 = f(w)$$

$$t = \pm\int_{w_0}^{w(t)} \frac{d\tilde w}{\sqrt{f(\tilde w)}}$$

For our case

$$t = -\sqrt{\frac{m}{cg}}\int_{w_0}^{w(t)} \frac{d\tilde w}{\sqrt{A - \tilde w\sqrt{1+\tilde w^2} - \ln(\cdots)}}$$

$x(t) =$

$$v_x = -\frac{g}{\dot w} = -\frac{g}{dw/dt}$$

$x(w)$

$z(w)$ using $v_z = w v_x$

$$z(w) = z_0 - g\int_{w_0}^{w} \frac{\tilde w\, d\tilde w}{\dot{\tilde w}^2}$$

limits

$$\frac{g^2}{v_x^2} = \dot w^2 \approx -\frac{cg}{m}w^2 + \cdots$$

$$= -\frac{cg}{m}\frac{v_z^2}{v_x^2} + \cdots$$

goes more and more $x, y \to 0$

$$\Rightarrow v_z \to \sqrt{\frac{mg}{c}} \quad (v_{ter})$$

at the top, $w = 0$.

$$v_{top} = \frac{v_{ter}}{\sqrt{A}}$$

$x(w)$ approaches ~~aprach~~ asymptote

$$x \approx x_{max} + \frac{m}{cw}$$

$$z \approx -\frac{m}{c}\ln(-w), \quad t \approx \sqrt{\frac{m}{g}}\ln(-w)$$

$$\Rightarrow z = -v_{ter} t, \quad x = x_{max} - \frac{m}{c} e^{\sqrt{\frac{g}{m}}t}$$

General monomial drag

$$\dot{v}_x = -\frac{c}{m}\left(v_x^2 + v_z^2\right)^\alpha v_x$$

$$\dot{v}_z = -g - \frac{c}{m}\left(v_x^2 + v_z^2\right)^\alpha v_z$$

$$= -g + \frac{v_z}{v_x}\dot{v}_x$$

$$\Rightarrow \dot{w} = -\frac{g}{v_x}$$

$$\dot{v}_x = \frac{g\ddot{w}}{\dot{w}^2} = -\frac{c}{m}\left(1+w^2\right)^\alpha v_x^{2\alpha+1}$$

$$= -\frac{c}{m}\left(1+w^2\right)^\alpha \left(-\frac{g}{\dot{w}}\right)^{2\alpha+1}$$

$$\Rightarrow \underbrace{\ddot{w}\left(-\dot{w}\right)^{2\alpha}}_{\frac{1}{2\alpha+1}\frac{d}{dt}\left(-\dot{w}\right)^{2\alpha+1}} = \underbrace{g^{2\alpha}\frac{c}{m}\left(1+w^2\right)^\alpha \dot{w}}_{\frac{d}{dt}\int_{w_0}^{w} g^{2\alpha}\frac{c}{m}\left(\tilde{w}^2+1\right)^\alpha d\tilde{w}}$$

constant drag: $\alpha = -\frac{1}{2}$

Projectile motion 2.30 -> 2.33 Sep 13, 2017

$x = a + bx^2 + cx^3$ x is small

to leading order $b, c \ldots$ not nesseary to be small.

$x = a$

to next order

$x = a + ba^2$

to third order

$x = a + ba^2 + ca^3 + b(a^2 b)$

$= a + ba^2 + (c + 2b^2)\, a^3$

A charge moving in a magnetic field.

$\vec{F} = m\vec{a} = q\,\vec{v} \times \vec{B}$

$\dot{\vec{v}} = \omega \vec{v} \times \hat{B}$ where $\omega = \frac{q}{m}|B|$

e.g. $\vec{B} = B\hat{z}$, $|B| = B$

$(v_x\hat{x} + v_y\hat{y} + v_z\hat{z}) \times \hat{z} = -v_x\hat{y} + v_y\hat{x}$

$\Rightarrow \dot{v}_x = \omega v_y$

$\dot{v}_y = -\omega v_x$

$\dot{v}_z = 0$

$\eta = v_x + i v_y$, $i^2 = -1$

$\dot{\eta} = \dot{v}_x + i\dot{v}_y = \omega v_y - \omega i v_x$

$= -i\omega\eta$

WORK

$\eta = \eta_0 e^{-i\omega t} = (v_x(0) + i v_y(0))(\cos\omega t - i\sin\omega t)$

$\frac{d\eta}{dt} = -i\omega\eta \Rightarrow \int \frac{d\eta}{\eta} = -i\omega \int dt$

$\ln\eta - \ln\eta_0 = -i\omega t$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$v_x(t) = \mathrm{Re}(\eta(t))$$
$$= \mathrm{Re}(\eta_0 e^{-i\omega t}), \quad \eta_0 = v_x(0) + i v_y(0)$$
$$= v_x(0)\cos\omega t + v_y(0)\sin\omega t$$
$$v_y(t) = \mathrm{Im}(\eta(t)) = v_y(0)\cos\omega t - v_x(0)\sin\omega t.$$

$$v_z(t) = v_z(0)$$

$$\Rightarrow \quad x(t) = x_0 + \frac{1}{\omega}(v_x(0)\sin\omega t - v_y(0)\cos\omega t)$$
$$y(t) = y_0 + \frac{1}{\omega}(v_y(0)\sin\omega t + v_x(0)\cos\omega t)$$
$$z(t) = z_0 + v_z(0)\,t$$

radius $\frac{1}{\omega}|v_0| = \frac{m|v_\perp|}{qB} = \frac{|P|}{qB}$

$$\eta = \dot{\xi}, \quad \xi = x + iy \Rightarrow \xi = \xi_{0xy} + \frac{i}{\omega}\eta_0 e^{-i\omega t}$$

$$\vec{r}_{cm} = \sum_i \frac{m_i \vec{r}_i}{M}, \quad M = \sum_i m_i$$

continuous

$$M = \int d^3\vec{r}\,\rho(\vec{r}) \quad \text{e.g. } \rho(x,y,z) \quad \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz\, \rho(x,y,z)$$
$$\rho(R,\phi,z) \quad \int_{-\infty}^{\infty} dz \int_{-\infty}^{+\infty} dR \int_0^{2\pi} R\,d\phi\, \rho(R,\phi,z)$$

$$\vec{r}_{cm} = \frac{\int d^3\vec{r}\,\rho(\vec{r})\,\vec{r}}{M}$$

$$M_1 = \sum_1^k m_i$$
$$M_2 = \sum_{k+1}^n m_i$$

$$\vec{r}_{cm}^{(1)} = \frac{\sum_1^k m_i \vec{r}_i}{M_1} \quad r_{cm}^{(2)} = \frac{\sum_{n+1}^n m_i \vec{r}_i}{M_2}, \quad \vec{r}_{cm} = \frac{M_1 \vec{r}_{cm}^{(1)} + M_2 \vec{r}_{cm}^{(2)}}{M_1 + M_2} = \frac{\sum_1^n m_i \vec{r}_i}{\sum_1^n m_i}$$

$$\vec{v}_{cm} = \frac{\sum_i m_i \vec{v}_i}{M}$$

$$\Rightarrow \vec{P}_{cm} = \sum \vec{p}_i = \vec{P}_{TOT}$$

$$\vec{P}_{cm} = M\vec{v}_{cm}$$

Forces

$$\vec{F}_i = \underset{\uparrow}{\vec{F}_{(ext),i}} + \sum_{j\neq i}^{n} \vec{F}_{ij} \leftarrow$$

external force   force that the j'th exerts on the i^{th}

Newton II

$$\vec{F}_i = \dot{\vec{p}}_i$$

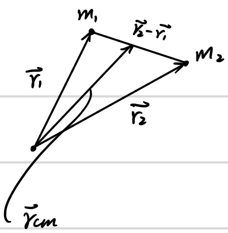
$$\dot{\vec{P}}_{TOT} = \sum_{i=1}^{n} \vec{F}_i = \sum_{i=1}^{n} \vec{F}_{(ext),i} + \underbrace{\sum_i \sum_{j\neq i} \vec{F}_{ij}}_{=0}$$

However: Newton III

$$\Rightarrow F_{ij} = -F_{ji}$$

$$\Rightarrow \dot{\vec{P}}_{TOT} = F_{ext} = \sum_{i=1}^{n} F_{(ext),i}$$

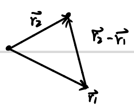
no external force $\Rightarrow$ total $\vec{p}$ conserved.



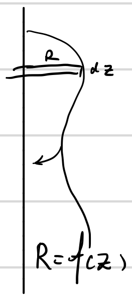
$$\vec r_{cm} = \frac{m_1 \vec r_1 + m_2 \vec r_2}{m_1+m_2} = \frac{(m_1+m_2)\vec r_1 + m_2(\vec r_2 - \vec r_1)}{m_1+m_2}$$

$$= \vec r_1 + \frac{m_2}{m_1+m_2}(\vec r_2 - \vec r_1)$$

$$\alpha = \frac{m_2}{M} \qquad 1-\alpha = \frac{m_1}{M}.$$



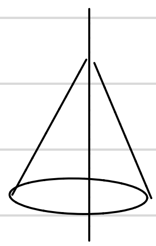
$$\frac{m\vec 0 + m\vec r_1 + m\vec r_2}{3m} = \frac{\vec r_1 + \vec r_2}{3}$$



$$\vec r_{cm} = z_{cm}\hat z$$

$$\int_0^h \rho\pi (f(z))^2 dz = M$$

$$\int_0^h \rho\pi (f(z))^2 z\,dz$$



$$f(z) = R_0(1 - z/h)$$

$$M = \pi\rho R_0^3 \int_0^h (1-z/h)^2 dz$$

$$= \pi\rho R_0^2 \underbrace{\int_0^h \left(1 - \frac{2z}{h} + \frac{z^2}{h^2}\right) dz}_{h - h + \frac{1}{3}h}$$

$$= \frac{\pi}{3}\rho R_0^2 h.$$

$$z_{cm} = \frac{1}{M}\int_0^h \rho\pi (f(z))^2 z\,dz$$

$$= \frac{3}{h}\int_0^h \underbrace{\left(z - \frac{2z^2}{h} + \frac{z^3}{h^2}\right)}_{\frac{1}{2}h^2 - \frac{2}{3}h^2 + \frac{1}{4}h^2 = \frac{1}{12}h^2} dz = \frac{h}{4}$$

$\vec{p}(t) = m(t)\,\vec{v}(t)$

$\vec{F} = \dot{\vec{p}}(t) = \dot{m}(t)\,\vec{v}(t) + m(t)\,\dot{\vec{v}}(t)$ even if $F_{ext} = 0$, $m\dot{\vec{v}} = -\dot{m}\,\vec{v} \neq 0$

for 1-D motion

$m\dot{v} = -\dot{m}v$

$\frac{\dot{v}}{v} = -\frac{\dot{m}}{m}$

$m\dot{v}(t) = -\dot{m}\,v_{exhaust}$

$v(t) = v_0 - v_{ext}\ln\left(\frac{m}{m_0}\right)$

$= v_0 - v_{ext} + \ln\left(m_0/m(t)\right)$

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$P = mv \Rightarrow \frac{dP_{tot}}{dt} = \frac{dm}{dt} V_{ex} + m\frac{dV}{dt}$

$dv = \frac{-V_{ex}}{m} dm \quad , v - v_0 = -V_{ex} (\ln m - \ln m_0)$

$v(t) = v_0 + v_{ex} \ln {}^{m_0}/_{m(t)}$

$\vec{\mathcal{L}}_i = \vec{r}_i \times \vec{p}_i$ angular momentum.

$\vec{\mathcal{L}}_{tot} = \sum_i \vec{r}_i \times \vec{p}_i$

$\dot{\vec{\mathcal{L}}}_{tot} = \sum_i \dot{\vec{r}} \times \vec{p}_i + \sum_i \vec{r}_i \times \dot{\vec{p}}_i$

$\downarrow$

$m_i \dot{\vec{r}}_i$

$\dot{\vec{r}}_i \times \dot{\vec{r}}_i = 0$

$= \sum_i \vec{r}_i \times \vec{F}_i \leftarrow$

$\vec{F}_i = \vec{F}_{i(ext)} + \sum_{j \neq i} F_{ij}$

$\Rightarrow \dot{\vec{\mathcal{L}}}_{tot} = \sum_i \vec{r}_i \times \vec{F}_{i(ext)} + \underbrace{\sum_{i,j \neq i} \vec{r}_i \times F_{ij}}_{\frac{1}{2}\sum_{i,j} \underbrace{(\vec{r}_i - \vec{r}_j)}_{r_{ij}} \times \vec{F}_{ij}}$ since $F_{ji} = -F_{ij}$

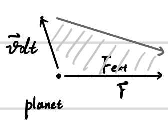
if $\vec{F}_{ij} = f(\vec{r}_{ij})\, \hat{r}_{ij} \Rightarrow \dot{\vec{\mathcal{L}}}_{tot} = \sum_i \vec{r}_i \times F_{i(ext)} = \vec{N} \quad (\vec{\tau})$ torque

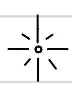
Force

torque, not ΣF.

e.g. Heavy sun

$F_{ext} \parallel \vec{r} \;,\; \vec{N} = 0$

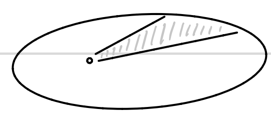




$0 = \dot{\vec{\mathcal{L}}} \Rightarrow \vec{\mathcal{L}}$ constant.

$dA = \frac{1}{2} |\vec{r} \times \vec{v} dt|$

$\frac{dA}{dt} = \frac{1}{2} |\vec{r} \times \vec{v}| = \frac{1}{2m} |\mathcal{L}|$



$\dot{\vec{\mathcal{L}}} = 0$

Area inpln ve.

COM Frame

$\vec{\bar{r}}_i = \vec{r}_i - \vec{r}_{cm}$

$\sum m_i \vec{\bar{r}}_i = 0$

$\vec{\bar{p}}_i = m_i \dot{\vec{r}}_i - m_i \dot{\vec{r}}_{cm}$

$\sum \vec{\bar{p}}_i = \vec{P}_{TOT} - \vec{P}_{cm} = 0 = \sum m_i \dot{\vec{\bar{r}}}_i$

$\Rightarrow$ C.O.M doesn't move in its own frame.

$$\vec{\bar{r}}_i = \vec{r}_i - \vec{r}_{cm} \Leftrightarrow \vec{r}_i = \vec{r}_{cm} + \vec{\bar{r}}_i$$

$$\Rightarrow \vec{\mathcal{L}} = \sum (\vec{r}_{cm} + \vec{r}_i) \times m_i (\dot{\vec{r}}_{cm} + \dot{\vec{\bar{r}}}_i)$$

$$= \underbrace{M \vec{r}_{cm} \times \dot{\vec{r}}_{cm}}_{\vec{r}_{cm} \times \vec{P}_{cm} = \vec{\mathcal{L}}_{orb}} + \underbrace{\vec{r}_{cm} \sum m_i \dot{\vec{\bar{r}}}_i}_{0} + \underbrace{(\sum \vec{\bar{r}}_i m_i)}_{0} \times \dot{\vec{r}}_{cm} + \underbrace{\sum \vec{\bar{r}}_i \times M_i \dot{\vec{\bar{r}}}_i}_{\vec{\mathcal{L}}_{spin}}$$

$$T_{tot} (\overset{\text{notation}}{=} KE) = \sum \frac{1}{2} m_i \vec{v} \cdot \vec{v}_j = \frac{1}{2} \sum m_i (\dot{\vec{r}}_{cm} + \dot{\vec{\bar{r}}}_i) \cdot (\dot{\vec{r}}_{cm} + \dot{\vec{\bar{r}}}_i)$$

$$= \frac{1}{2} M (\dot{\vec{r}}_{cm})^2 + \underbrace{\frac{1}{2} \sum m_i \cdot \dot{\vec{\bar{r}}}_i \cdot \dot{\vec{\bar{r}}}_i}_{\frac{1}{2} \sum m_i (\vec{v}_i)^2} = T_{cm} + T_{int}$$

Taylor

$$f(x+dx) = f(x) + dx\frac{\partial f}{\partial x} + \underbrace{O(dx^2)}_{\text{Higher order way}} \iff \frac{\partial f}{\partial x} = \lim_{dx\to 0}\frac{f(x+dx)-f(x)}{dx}$$

$$f(\vec{r}+d\vec{r}) = f(\vec{r}) + d\vec{r}\cdot\vec{\nabla}f(\vec{r}) + O(|d\vec{r}|^2)$$

def : gradient

e.g. Cartesian Coordinates.

$$d\vec{r} = dx\hat{x} + dy\hat{y} + dz\hat{z}$$

$$f(x+dx,\ y+dy,\ y+dz) = f(x,y,z) + dx\frac{\partial f}{\partial x} + dy\frac{\partial f}{\partial y} + dz\frac{\partial f}{\partial z} + \cdots$$

$$\Rightarrow \vec{\nabla} = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$$

in cylindrical coordinates

$$\vec{r} = R\hat{R} + z\hat{z}$$

$$d\vec{r} = dR\hat{R} + R\,d\phi\,\hat{\phi} + dz\hat{z}$$

$$f(R+dR,\ \phi+d\phi,\ z+dz)$$

$$= f(R,\phi,z) + dR\frac{\partial f}{\partial R} + d\phi\frac{\partial f}{\partial \phi} + dz\frac{\partial f}{\partial z} + \cdots$$

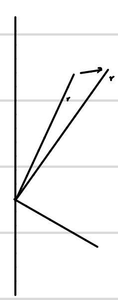
$$\Rightarrow \vec{\nabla} = \hat{R}\frac{\partial}{\partial R} + \frac{1}{R}\hat{\phi}\frac{\partial}{\partial \phi} + \hat{z}\frac{\partial}{\partial z}$$

in

$$\vec{r} = r\hat{r} \qquad d\vec{r} = dr\,\hat{r} + r(d\theta\,\hat{\theta} + d\phi\sin\theta\,\hat{\phi})$$

$$\Rightarrow f(r+dr,\theta+d\theta,\phi+d\phi) = f(r,\theta,\phi) + dr\frac{\partial f}{\partial r} + d\theta\frac{\partial f}{\partial \theta} + d\phi\frac{\partial f}{\partial \phi} + \cdots$$

$$\Rightarrow \vec{\nabla} = \hat{r}\frac{\partial}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial}{\partial \theta} + \frac{\hat{\phi}}{r\sin\theta}\frac{\partial}{\partial \phi}.$$



Line Integral

Vector field $\vec{V}(\vec{r})$

$$\int_{\underset{\text{path}}{\gamma}} d\vec{r}\cdot\vec{V}(\vec{r}) = \lim_{n\to\infty}\sum_{i=1}^{n} d\vec{r}_i\cdot\vec{V}(\vec{r}_0 + \tfrac{1}{2}d\vec{r}_i)$$

$$\int_{\vec{a}}^{\vec{b}} \nabla f(\vec{r}) \cdot dr = f(\vec{b}) - f(\vec{a})$$ independent of the path γ

true for all dim

path from $\vec{a}$ to $\vec{b}$ s.t. $\vec{b} = \vec{a} + \sum d\vec{r}_i$

$$\int_{\vec{a}}^{\vec{a}+d\vec{r}_1} \vec{\nabla} f(r) \cdot d\vec{r} + \int_{\vec{a}+d\vec{r}_1}^{\vec{a}+d\vec{r}_1+d\vec{r}_2} \nabla f(r) \cdot d\vec{r} + \cdots$$

$$d\vec{r}_1 \cdot \vec{\nabla} f(\vec{\phi}) + d\vec{r}\ \vec{\nabla} f(a+d\vec{r}_1) + \ldots + d\vec{r}_{n-1} \cdot \vec{\nabla} f(b - d\vec{r}_n - d\vec{r}_{n-1}) + d\vec{r}_n \cdot \vec{\nabla} f(b - d\vec{r}_n)$$

$$= f(\cancel{\vec{a}+d\vec{r}_1}) \boxed{-f(\vec{a})} + f(\vec{a}+d\vec{r}_1+d\vec{r}_2) - f(\cancel{\vec{a}+d\vec{r}_1}) + \cdots + f(\cancel{\vec{b}-d\vec{r}_n}) - f(\vec{b} - d\vec{r}_n - d\vec{r}_{n-1}) \boxed{+f(\vec{b})} - f(\cancel{b-d\vec{r}_n})$$

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$$f(\vec{r}+d\vec{r}) = f(\vec{r}) + d\vec{r}\cdot\vec{\nabla} f(\vec{r}) + \cdots$$

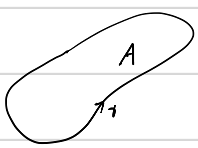
$$\Rightarrow \int_{\vec{a}}^{\vec{b}} \vec{\nabla} f\cdot d\vec{r} = f(\vec{b}) - f(\vec{a})$$, independent from path

$$\Rightarrow \oint \vec{\nabla} f\cdot d\vec{r} = 0$$

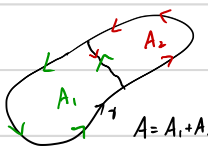
Gradient of a scalar function is a vector field, i.e., a vector associated to points in space.

$\Rightarrow$ such vector field: $\oint F(\vec{r})\, d\vec{r} = 0$.

General vector fields have line integrals that DO depend on the path. $\oint F(\vec{r})\, d\vec{r} \neq 0$

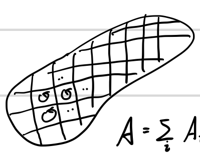


Circulation : $\mathrm{Circ}(\underset{\substack{\uparrow\\ \text{vector field}}}{\vec{V}(\vec{r})}, \underset{\substack{\uparrow\\ \text{path}}}{\gamma}) = \oint_{\gamma=\partial A} \vec{V}\cdot d\vec{r}$



$A = A_1 + A_2$

$$\Rightarrow \mathrm{Circ}(\gamma) = \mathrm{Circ}(\gamma_1) + \mathrm{Circ}(\gamma_2)$$



$A = \sum_i A_i$

$$\Rightarrow \mathrm{Circ}(\gamma) = \sum \mathrm{Circ}(\gamma_i)$$

$$\oint_{\gamma=\partial A} \vec{V}\cdot d\vec{r} = \sum_{i=1}^{n} \oint_{\gamma_i=\partial A_i} \vec{V}\cdot d\vec{r}$$

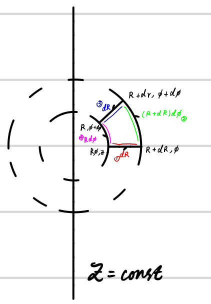
Consider the limit: $\lim_{|A|\to 0} \frac{1}{A} \oint_{\gamma=\partial A} \vec{V}\cdot d\vec{r} \overset{\substack{\text{definition}\\ \downarrow}}{\equiv} (\vec{\nabla}\times\vec{V})\cdot \underset{\substack{\uparrow\\ \hat{n}\perp A}}{\hat{n}}$

Stokes Thorem

$$\iint_A (\vec{\nabla}\times\vec{V})\cdot \hat{n}_A\, dA = \lim_{n\to\infty} \sum_{1}^{n} A_i \left(\frac{1}{A_i} \oint_{\gamma_i=\partial A_i} \vec{V}\cdot d\vec{r}\right) = \oint_{\gamma=\partial A} \vec{V}\cdot d\vec{r}$$

$$\Leftrightarrow \iint_A (\vec{\nabla}\times\vec{V})\cdot d\vec{A} = \oint_{\gamma=\partial A} \vec{V}\cdot d\vec{r}$$

Let's compute the formula for the curl in cylindrical coords.



① $V_R(\bar{R},\phi)\,dR$

② $V_\phi(R+dR,\bar{\phi})(R+dR)\,d\phi$

③ $-V_R(\bar{R},\phi+d\phi)\,dR$

④ $-V_\phi(R,\bar{\phi})\,R\,d\phi$

$z=$ const

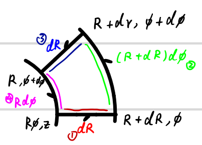
$$= ① + ② + ③ + ④$$

$$\Rightarrow \oint \vec{V}\cdot d\vec{r} = V_R(\bar{R},\phi)\,dR - V_R(\bar{R},\phi+d\phi)\,dR + V_\phi(R+dR,\bar{\phi})(R+dR)\,d\phi - V_\phi(R,\bar{\phi})R\,d\phi$$

$$= V_R(R,\tilde{\phi})\,dR\Big|^{\phi}_{\phi+d\phi} + V_\phi(\tilde{R},\phi)R\,d\phi\Big|^{R+dR}_{R}$$

$$= -\frac{\partial V_R}{\partial \phi}\,d\phi\,dR + \frac{\partial}{\partial R}(R\cdot V_\phi)\,dR\,d\phi$$

Area:



$$A = R\,dR\,d\phi + \cdots \longrightarrow \frac{1}{A}\left(-\frac{\partial V_R}{\partial \phi}\,d\phi\,dR + \frac{\partial}{\partial R}(R\cdot V_\phi)\,dR\,d\phi\right)$$

$$= \frac{1}{R}\left(\frac{\partial}{\partial R}(RV_\phi) - \frac{\partial V_R}{\partial \phi}\right)$$

$$= (\vec{\nabla}\times\vec{V})_z$$

In $\perp$ coor

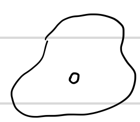
$$\vec{\nabla}\times\vec{V} = \hat{z}\left(\frac{\partial}{\partial x}V_y - \frac{\partial}{\partial y}V_x\right) + \hat{x}\left(\frac{\partial}{\partial y}V_z - \frac{\partial}{\partial z}V_y\right) + \hat{y}\left(\frac{\partial}{\partial z}V_x - \frac{\partial}{\partial x}V_z\right)$$

, $\vec{\nabla}\times(\vec{\nabla}f) = 0$

Subtlety

$$\vec{V} = \frac{\hat{y}x - \hat{x}y}{x^2+y^2} = \vec{\nabla}\tan^{-1}\left(\frac{y}{x}\right) = \vec{\nabla}\theta = \frac{\hat{\theta}}{R}$$

Such surface have to take out the point



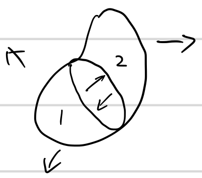
Stokes thm:

$$\iint_A (\vec{\nabla}\times\vec{V})\cdot dA = \oint_{\Gamma=\partial A} \vec{V}\cdot d\vec{r}$$

if $\partial A = 0$ (closed surface, no boundary)

$$\Rightarrow \iint_A (\vec{\nabla}\times\vec{V})\cdot d\vec{A} = 0$$

↳ For general vector $\vec{V}$, $\Phi(A) = \iint_A \vec{V}\cdot \hat{n}_A\,dA = \iint_A \vec{V}\cdot d\vec{A}$



$A = \partial V$

$\Phi(V) = \Phi(V_1) + \Phi(V_2)$

$$\vec{\nabla} \cdot \vec{V} \overset{\text{def.}}{=} \lim_{V \to 0} \frac{1}{|V|} \iint_{A = \partial V} \vec{V} \cdot d\vec{A}$$

divergence.

$$\Phi(A = \partial V) = \int_{A = \partial V} \vec{V} \cdot d\vec{A} = \iiint_V \vec{\nabla} \cdot \vec{V}\, d\vec{V} = \lim_{i \to \infty} \Phi(A_i = \partial V_i)$$

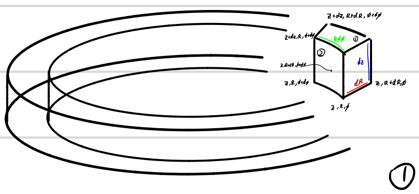
$$= \lim_{n \to \infty} \sum_i^n \Phi(V_i) = \lim_{n \to \infty} \sum_1^n V_i \left(\frac{1}{V_i} \Phi(V_i) \right)$$

$$= \lim_{n \to 0} \sum_1^n V_i (\vec{\nabla} \cdot \vec{V})$$

$$= \iiint_V (\vec{\nabla} \cdot \vec{V})\, d\vec{V}$$

Sep 25, 2017

Diff Operator	Gradient	Curl	Divergence
Def.	$\lim_{\lvert d\vec r\rvert\to 0} f(\vec r + d\vec r) - f(\vec r) \equiv d\vec r \cdot \nabla f(\vec r)$	$\hat n_A \cdot \nabla\times\vec V(\vec r) = \lim_{\lvert A\rvert\to 0} \frac{1}{\lvert A\rvert} \oint_{\gamma=\partial A} \vec V\cdot d\vec r$	$\vec\nabla\cdot\vec V = \lim_{\lvert V\rvert\to 0}\frac{1}{\lvert V\rvert}\oiint_{A=\partial V} \vec V\cdot \underbrace{\hat n_A\, dA}_{d\vec A}$
Int	$f(\vec r_2) - f(\vec r_1) = \int_C \vec\nabla f\cdot d\vec r$	Circulation $\oint_{\gamma=\partial A} V\cdot d\vec r$	Flux $\oint_{A=\partial V} \vec V\cdot d\vec A$
Th	fundamental thm of calculus	Stokes thm $\iint_A (\vec\nabla\times\vec V)\cdot d\vec A = \oint_{\gamma=\partial A}\vec V\cdot d\vec r$	Gauss thm $\iiint_V (\vec\nabla\cdot\vec V)\, dV = \oint_{A=\partial V}\vec V\cdot d\vec A$



$V = R\,dR\,dz\,d\phi$

$A_1 = A_1' = R\,dR\,d\phi$

① $(V_z(z+dz, \bar R, \bar\phi) - V_z(z, \bar R, \bar\phi))\,R\,dR\,d\phi$

② $A_2 = R\,d\phi\,dz$, $A_2' = (R+dR)\,d\phi\,dz$

$V_R(R+dR)\cdot(R+dR)\,d\phi\,dz - V_R(R, \cdots)\,R\,d\phi\,dz.$

③ $A_3 = dR\,dz$, $(V_\phi(\phi+d\phi, \dots) - V_\phi(\phi))\,dR\,dz.$

$$\Rightarrow \left[\frac{\partial V_z}{\partial z} dz\, R\,dR\,d\phi + \frac{\partial R V_R}{\partial R} dR\,d\phi\,dz + \frac{\partial V_\phi}{\partial\phi} d\phi\,dR\,dz\right]\frac{1}{V}$$

$$\vec\nabla\cdot\vec V = \frac{\partial V_z}{\partial z} + \frac{1}{R}\left(\frac{\partial}{\partial R} R V_R + \frac{\partial}{\partial\phi} V_\phi\right)$$

$\vec{F} = m\dot{\vec{v}}$, $KE = T = \frac{1}{2} m\vec{v}\cdot\vec{v} = \frac{1}{2} m|v|^2$

$$\dot{T} = m\dot{\vec{v}}\cdot\vec{v} = \vec{v}\cdot\vec{F}$$

$$\Delta T = T_2 - T_1 = \int_{t_1}^{t_2} \vec{F}\cdot\vec{v}\,dt$$

$$= \int_{t_1 \text{ along path}}^{t_2} \vec{F}\cdot d\vec{r}$$

ΔT the work done by $\vec{F}$.

if we parametrize the path

$$\vec{r} = \vec{r}(s) \quad s.t. \quad \vec{r}_1 = \vec{r}(s_1)$$
$$\vec{r}_2 = \vec{r}(s_2)$$

$$\Rightarrow \int_{s_1}^{s_2} F\cdot\frac{d\vec{r}}{ds}\,ds$$

$\oint \vec{F}\cdot d\vec{r} \neq 0$, $\vec{F}$ is not conservite

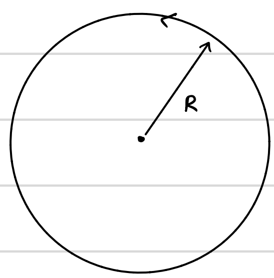
$$\vec{F} = \frac{1}{2}(x\hat{y} - y\hat{x}) = \frac{1}{2}R\hat{\phi}$$

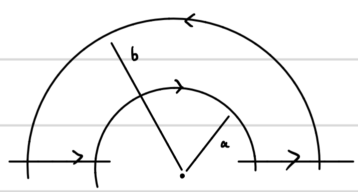
$$(\vec{\nabla}\times\vec{F})_z = \frac{\partial}{\partial x}F_y - \frac{\partial}{\partial y}F_x = 1$$

$$\vec{F}\cdot d\vec{r} = \vec{F}\cdot(dR\hat{R} + R\,d\phi\,\hat{\phi} + dz\hat{z})$$

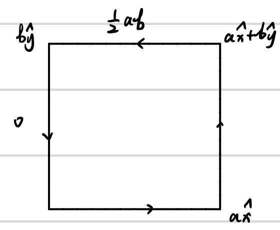
$$= \frac{1}{2}R^2\,d\phi$$

$$\Delta T = \pi R^2$$





$\frac{1}{2}\pi b^2 - \frac{1}{2}\pi a^2$



$$\frac{1}{2}x \Big|_{x=a} \int_a^b dy = \frac{1}{2}ab$$

$$\int \vec{F}\cdot d\vec{r} = -\frac{1}{2}\Big|_{y=0} \int dx = 0$$

$$\vec{r} = f(s)\hat{x} + y(s)\hat{y}$$

$$= F(s)\hat{R} + G(s)\hat{\phi}$$

If $\int_{\vec{r}_1}^{\vec{r}_2} \vec{F}\cdot dr$ does not depend on the path, the Force is conservative and we can write $\vec{F} = -\vec{\nabla} V(\vec{r})$

$$\int_{\vec{r}_1}^{\vec{r}_2} \vec{F}\cdot dr = -\int_{\vec{r}_1}^{\vec{r}_2} \nabla V\cdot d\vec{r} = -(V(\vec{r}_2) - V(\vec{r}_1))$$

$$T(\vec{r}_2) - T(\vec{r}_1) = V(\vec{r}_2) + V(\vec{r}_1)$$

$$E(\vec{r}_2) = T(\vec{r}_2) + V(\vec{r}_1) = T(\vec{r}_1) + V(\vec{r}_1) = E(\vec{r}_1)$$

$$0 = \frac{dE}{dt} = m\vec{v}\cdot\dot{\vec{v}} + \vec{\nabla}V\cdot\vec{v}$$
$$= \dot{\vec{r}}\cdot(m\ddot{\vec{r}} - \vec{F}) = 0$$

1 Dim undamped Harmane ossillator

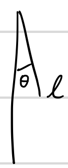
$$F(x) = \cancel{F(0)}^{\,0} + \frac{dF}{dx}x + \cdots = -\underbrace{k}_{-\frac{dF}{dx}}x$$

$$m\ddot{x} = -kx \Rightarrow \ddot{x} = -\omega^2 x, \quad F(x) = -kx = -\frac{\partial}{\partial x}\overbrace{\frac{1}{2}kx^2}^{V(x)}$$

$$\omega = \sqrt{\frac{k}{m}} \qquad E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$$

$$V(\theta) = mgl(1-\cos\theta) \approx mgl\frac{1}{2}\theta^2$$

$1 - \frac{1}{2}\theta^2 + \cdots$



θ l

$$T = \frac{1}{2}m(l\dot{\theta})^2 = \frac{1}{2}ml^2\dot{\theta}^2$$

$$\omega = \sqrt{\frac{mgl}{ml^2}}$$

$m \Leftrightarrow ml^2$

$k \Leftrightarrow mgl$

$$\ddot{\theta} = -\omega^2\sin\theta$$

$m\ddot{x} = -kx$ Sep 27, 2017

$\omega = \sqrt{\frac{k}{m}}$

$m \Leftrightarrow \mu l^2 \qquad T = \frac{1}{2} m\dot{x}^2, \quad U = \frac{1}{2} kx^2$

$\frac{1}{2}\mu(l\dot{\theta})^2 = \frac{1}{2}\mu l^2 \dot{\theta}^2$

$U = \mu g l (1-\cos\theta) \approx \frac{1}{2} \underbrace{\mu g l}_{\Updownarrow\, k} \theta^2 \quad , \quad \omega = \sqrt{\frac{g}{l}}$

$p = m\dot{x}, \quad \dot{p} = -kx$

$$\begin{pmatrix} \dot{x} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} \frac{1}{m} p \\ -kx \end{pmatrix} = \begin{pmatrix} \frac{1}{m} & 0 \\ 0 & -k \end{pmatrix} \begin{pmatrix} x \\ p \end{pmatrix}$$

$a = x + iAp \Leftrightarrow x = a - iAp$

$\dot{a} = \dot{x} + iA\dot{p}$

$= \frac{1}{m} p - iAkx = -iAka + \frac{1}{m} p - A^2 kp$

choose $A^2 k = \frac{1}{m} \Rightarrow A = \frac{1}{\sqrt{mk}} = \frac{1}{m\omega}$

$\Rightarrow \dot{a} = -i\sqrt{\frac{k}{m}}\, a$

$= -i\omega a$

$\Rightarrow a_0 = x_0 + \frac{iP_0}{m\omega} \quad , \quad \frac{P_0}{m} = v_0$

$x(t) = x_0 \cos\omega t + \frac{p}{m\omega} \sin\omega t \qquad = x_{max}\cos(\omega t + \phi)$

$\frac{p(t)}{m\omega} = -x_0 \sin\omega t + \frac{p_0}{m\omega}\cos\omega t$

$v(t) = -x_0 \omega \sin\omega t + v_0 \cos\omega t$

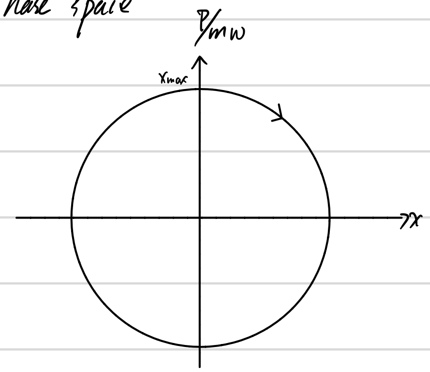
$a(t) = a_0 e^{-i\omega t} = x_{max} e^{-i(\omega t + \phi)}$

ω is the angular frequency.

period $T = \frac{2\pi}{\omega}$. $\quad x(t+T) = x(t)$

frequency $= \frac{1}{T} = \frac{\omega}{2\pi}$

Phase space



Energy: $E = \frac{p^2}{2m} + \frac{1}{2} k x^2$

$= \frac{1}{2} k \left(x^2 + \frac{p^2}{(m\omega)^2}\right) \qquad k = m\omega^2$

$= \frac{1}{2} k\, a^* a \qquad$ Recall: $a = x + \frac{ip}{m\omega}$

3D:

$T = \frac{1}{2} m \dot{\vec{r}} \cdot \dot{\vec{r}} = \frac{1}{2} m \dot{x}_i^2$

$U = \frac{1}{2} \sum k_i x_i^2 \quad , \quad \vec{r} = x_i \hat{e}_i$

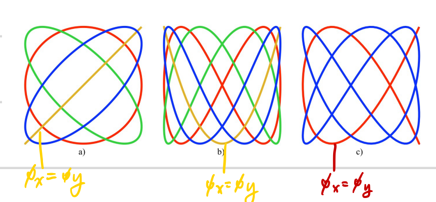
$\ddot{x}_i = -\omega_i^2 x_i \quad \omega_i = \sqrt{\frac{k_i}{m}}$

Consider $\omega_z = 0$, $z = 0$, $\omega_x = \omega_y$

$x(t) = x_{max} \cos(\omega_x t + \phi_x)$

$y(t) = y_{max} \cos(\omega_y t + \phi_y)$

when $\omega_1 = \omega_y \qquad \omega_1 = 2\omega_2 \qquad \omega_1 = 3\omega_3$



Constant external force

$$F = -kx + F_0$$
$$= -k(\underbrace{x - \tfrac{F_0}{k}}_{y})$$

$m\ddot{x} = F$

$\Rightarrow m\ddot{y} = -ky$

$\Rightarrow$ Just shifts the equilibrium point.

Damped harmonic motion.

$$m\ddot{x} + b\dot{x} + kx = 0$$

frictional constant $\frac{[m]}{[t]}$

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 = 0$$

undamped frequency

$$\beta = \frac{b}{2m}, \quad \omega_0^2 = \frac{k}{m}.$$

$\dot{x} = v$,

$\dot{v} = -2\beta v - \omega_0^2 x$

$\alpha = x + iAv \Leftrightarrow x = \alpha - iAv$

$\Rightarrow \dot{\alpha} = \dot{x} + iA\dot{v} = v - iA(2\beta v + \omega_0^2 x)$

$\alpha - iAv$

$= -iA\omega_0^2\alpha + \underline{(1 - 2iA\beta - A^2\omega_0^2)}v.$

choose A s.t. $= 0$

$\Rightarrow \omega_0^2 A^2 + 2iA\beta - 1 = 0$

$$A = \frac{-2i\beta \pm \sqrt{-\beta^2 4 + 4\omega_0^2}}{2\omega_0^2} = \frac{-i\beta \pm \omega}{\omega_0^2} \quad \omega = \sqrt{\omega_0^2 - \beta^2} \qquad [\Rightarrow \dot{\alpha} = -iA_\pm\omega_0^2\alpha]$$

$$\dot{\alpha}_\pm = -iA_\pm\omega_0^2\alpha_\pm$$

$$\begin{cases}\dot{a}_\pm = -iA_\pm \omega_0^2 a_\pm \\ a_\pm = x + iA_\pm v = x + \left(\frac{\beta}{\omega_0^2} \pm \frac{i\omega}{\omega_0^2}\right) v\end{cases}$$

$$\Rightarrow \quad \dot{a}_\pm = \mp(i\omega - \beta)\, a_\pm \qquad \boxed{?}$$

$$a_\pm(t) = |a_{max}|\, e^{-\beta t}\, e^{\mp i(\omega t + \phi)}$$

$$v = \frac{a - x}{iA} = \frac{i(x-a)}{A}$$

$$a_+ - a_- = \frac{2i\omega v}{\omega_0^2}$$

$$v(t) = \frac{\omega_0^2}{\omega}\frac{(a_+(t) - a_-(t))}{i2} = \frac{\omega_0^2}{\omega}|a_{max}|\, e^{-\beta t}\left(-\sin(\omega t + \phi)\right) \qquad \left[\ \frac{e^{i\theta} - e^{-i\theta}}{2i} = \sin\theta\ \right]$$

$$= -\frac{\omega_0^2}{\omega} e^{-\beta t} \sin(\omega t + \phi)$$

$$\Rightarrow x(t) = \frac{1}{2}(a_+ + a_-) - \frac{\beta}{\omega_0^2} v(t)$$

$$= |a_{max}|\, e^{-\beta t}\left[\cos(\omega t + \phi) + \frac{\beta}{\omega}\sin(\omega t + \phi)\right]$$

$$= x_{max}\, e^{-\beta t}\cos(\omega t + \varphi), \quad \varphi = \phi - \psi, \quad \frac{\beta}{\omega} = \tan\psi, \quad x_{max} = \frac{|a_{max}|}{\cos\psi}$$

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$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$$

$$v = \dot{x}$$

$$\dot{v} + 2\beta v + \omega_0^2 x = 0$$

$$a = x + iAx$$

$$\dot{a} = -Ai\omega_0^2 a$$

$$A_\pm = \frac{-i\beta \pm \omega}{\omega_0^2} \quad \text{where } \omega = \sqrt{\omega_0^2 - \beta^2}$$

$$\dot{a} = (-\beta \mp i\omega)a \Rightarrow a_\pm = a_\pm(0)\, e^{-\beta t \mp i\omega t} = |a_{max}|\, e^{-\beta t}\, e^{\mp i(\omega t + \phi)}$$

ω is real, $\omega_0 > \beta \Rightarrow$ under damped

$\omega = 0$ $\quad \omega_0 = \beta \Rightarrow$ critidly damped

ω is imaginary $\omega_0 < \beta \Rightarrow$ over damed.

$\hookrightarrow q = i\omega \Rightarrow e^{-(\beta \pm q)t}$, $q = \sqrt{\beta^2 - \omega_0^2} < \beta$

$$\mathcal{L} = \frac{1}{2} \overset{\text{const}}{\overset{\downarrow}{m}} \dot{q}^2 + U(q)$$

$$\dot{q}^2 = \frac{2}{m}(E - U(q))$$

$$\frac{dq}{dt} = \sqrt{\frac{2}{m}}\sqrt{E - U(q)}$$

$$t = \int_{q_0}^{q(t)} \frac{d\tilde{q}}{\sqrt{E - U(\tilde{q})}}$$

Under dampeed

$$x(t) = x_{max}\, e^{-\beta t} \cos(\omega t + \varphi)$$

$$E = \frac{1}{2} m\dot{x}^2 + \frac{1}{2} m\omega_0^2 x^2$$

$$a_0(t) = x(t) + \frac{iV(t)}{\omega_0} \Rightarrow E = \frac{1}{2} m\omega_0^2\, a(t)\, a_0^*(t)$$

Over Damped

$$a_\pm(t) = x(t) + \frac{\beta \pm q}{\omega_0^2} V$$

$$= a_\pm(0)\, e^{-(\beta \pm q)t}$$

$$v(t) = \omega_0^2 \frac{a_+(t) - a_-(t)}{2q} = \frac{\omega_0^2}{2q}\left(a_+(0)\, e^{-(\beta+q)t} + a_-(0)\, e^{-(\beta - q)t}\right)$$

$$x(t) = \frac{a_+(t) + a_-(t)}{2} - \frac{\beta}{\omega_0^2} V.$$

$\ddot{x} + 2\beta\dot{x} - w_0^2 x = 0$

$D = \frac{d}{dt}$ is a linear operator

$(D^2 + 2\beta D + w_0^2)x = 0$

$\Rightarrow (D+\beta+iw)(D+\beta-iw)x = 0$ under

$(D+\beta)^2 x = 0$ critical

$(D+\beta+q)(D+\beta-q)x = 0$ over.

$(D+A)(D+B)x = 0$

$(D+A)x$ is solved by x_A

general solution:

$(D+B)x = x_A$ $\quad$ $(D+B)x_A = [(D+A)+(B-A)]x_A$

$(D+B)x_B = 0$ $\quad$ $= (B-A)x_A$

$(D+A)(D+B)(x_A+x_B) = (D+A)(D+B)x_A$

$= (D+A)(B-A\ x_A$

for constant $B-A \Rightarrow (B-A)(B+A)x_A = 0$.

$x_A = x_A(0)e^{-At}$, $x_B = x_B(0)e^{-Bt}$

$x(0) = x_A(0) + x_B(0)$

$V(0) = -(Ax_A(0) + Bx_B(0))$

$$x_A(0) = \frac{Bx(0)+V(0)}{B-A}$$

$$x_B(0) = \frac{Ax(0)+V(0)}{A-B}$$

When $A=B$

$$(D+A)^2 x = 0$$

$$(D+A) x_A = 0$$

$$(D+A) x = x_A$$

$$(D+A) t x_A = x_A + \underbrace{t D x_A + A t x_A}_{t(D+A)x_A = 0}$$

$$x(t) = x_A + C t x_A$$

$$x(0) e^{-At}$$

$$x(t) = e^{-At} (x_0 + t(v_0 + A x_0))$$

Under: $A = \beta + i\omega$, $B = \beta - i\omega$

$$x(t) = x_{max} e^{-\beta t} \cos(\omega t + \varphi)$$

$$\Rightarrow x(0) = x_{max} \cos\varphi$$

$$v(t) = -x_{max} e^{-\beta t} (\beta \cos(\omega t + \varphi) + \omega \sin(\omega t + \varphi))$$

$$v(0) = -x_{max} (\beta \cos\varphi + \omega \sin\varphi)$$

Critically

$$x(t) = -e^{\beta t} (x_0 + t(v_0 + \beta x_0))$$

Over

$$e^{-(\beta+q)t} \qquad e^{-(\beta-\rho)t}$$

Q: quality factor

$$Q = \frac{\omega}{2\beta} \to \infty \text{ as } \beta \to 0.$$

$$Q' = \frac{2\pi E_{max}}{\Delta E}, \quad E(0) - E(\underset{T = \frac{2\pi}{\omega}}{\overset{\uparrow}{T}}) \quad \text{energy lost in one cy}$$

$$x(t) = e^{-\beta t} f(t), \quad f(t+T) = f(t)$$

$$v(t) = e^{-\beta t} g(t), \quad \underset{\text{not } d}{\overset{}{d'}}(t+T) = f'(t)$$

$$\mathcal{E} = \tfrac{1}{2} m v(t)^2 + \tfrac{1}{2} k x(t)^2$$

$$= e^{-2\beta t} g(t), \quad g(t + \tfrac{2\pi}{\omega}) = g(t)$$

$$Q' = \frac{2\pi E(0)}{E(0) - e^{-\frac{4\pi\beta}{\omega}} E(0)} = \frac{2\pi}{1 - e^{-\frac{4\pi\beta}{\omega}}}$$

when $\beta \to 0$, $e^{-4\pi\beta/\omega} \to 1 - \frac{4\pi\beta}{\omega}$

$$Q' = \frac{2\pi}{1 - 1 + \frac{4\pi\beta}{\omega}} = \frac{\omega}{2\beta} = Q$$

N cycles in time T_N

$$\Rightarrow T = \frac{T_N}{N}$$

$$w = \frac{2\pi}{T} = \frac{2\pi N}{T_N}$$

$$e^{-\beta M T} = \frac{1}{e} = e^{-1}$$

$$\beta M T = 1 \Rightarrow \beta = \frac{1}{MT} = \frac{N}{MT_N}$$

$$w_1 = \sqrt{w^2 + \beta^2} = \frac{N}{T_N}\sqrt{4\pi^2 + \frac{1}{M^2}}$$

$$Q = \frac{w}{2\beta} = \frac{2\pi N}{2T_N N} T_N M = \pi M$$

If we hadn't Taylor expanded the potential, we would have found:

$$\ddot{\theta} = -\frac{g}{\ell}\sin\theta \tag{1.12}$$

This illustrates both how we can Taylor expand the potential (or equivalently the force), and how we can use any variable to measure the deviation from equilibrium.

1.3 Momentum

We can rewrite the equation (1.2) if we introduce the momentum $p = m\dot{x}$ as a system of first order differential equations:

$$p = m\dot{x} \quad , \quad \dot{p} = -kx \quad \Rightarrow \quad \begin{pmatrix} \dot{x} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} \frac{1}{m}p \\ -kx \end{pmatrix} = \begin{pmatrix} \frac{1}{m} & 0 \\ 0 & -k \end{pmatrix}\begin{pmatrix} x \\ p \end{pmatrix} \tag{1.13}$$

Recall that it is convenient to rewrite this in terms of a single complex variable. I'll do it in a slightly different way than before; let's write

$$x = a - iAp \Leftarrow \quad a = x + Aip \tag{1.14}$$

where A is a constant that we need to find. Then

$$\dot{a} = \dot{x} + Ai\dot{p} = \frac{1}{m}p - Aikx = \frac{1}{m}p - Aik(a - Aip) = -Aika + \left(\frac{1}{m} - A^2k\right)p \tag{1.15}$$

$$= -iAka + \frac{1}{m}p - A^2kp$$

which gives us an equation for a if we choose $A^2k = \frac{1}{m}$ or:

$$A = \sqrt{\frac{1}{km}} = \frac{1}{m\omega} = \pm\frac{1}{\sqrt{mk}} \tag{1.16}$$

$$\Rightarrow \dot{a} = \mp i\sqrt{\frac{k}{m}}\,a = \mp i\omega a$$

and hence we have

$$a = x + \frac{i}{m\omega}p = x + \frac{i}{\omega}\dot{x} \tag{1.17}$$

and, since $k(\frac{1}{m\omega}) = \omega$

$$\dot{a} = -i\omega a \quad \Rightarrow a(t) = a_0 e^{-i\omega t} \Rightarrow a_0 = x_0 + i\frac{p_0}{m} \tag{1.18}$$

$$\Rightarrow x(t) = x_0 \cos\omega t + \frac{p_0}{m\omega}\sin\omega t$$

Notice that (1.17) makes sense dimensionally: p has dimensions $\frac{[M][L]}{[T]}$, and ω has dimensions $\frac{1}{[T]}$, so the ratio $\frac{p}{m\omega}$ has dimensions of length. The equation (1.18) is the easiest differential equation there is; its solution is

$$a = a_0\, e^{-i\omega(t-t_0)} = x_{max}\, e^{-i(\omega t+\phi)} \tag{1.19}$$

where x_{max} is the maximum displacement and ϕ is an arbitrary phase angle. Using one of the many amazing formulae that Euler discovered, (following, in this case, the work of

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$\ddot{x}_1 + 2\beta\dot{x}_1 + \omega_0^2 x_1 = \frac{1}{m}F_1(t)$ $x_1(t)$ solve this equation (1)

$\ddot{x}_2 + 2\beta\dot{x}_2 + \omega_0^2 x_2 = \frac{1}{m}F_2(t)$ $x_2(t)$ " " " (2)

$\Rightarrow a x_1 + b x_2 = x_{lc}$

$\Rightarrow \ddot{x}_{lc} + 2\beta\dot{x}_{lc} + \omega_0^2 x_{lc} = a\frac{1}{m}F_1(t) + b\frac{1}{m}F_2(t)$ since $\frac{d}{dt}$ is linear.

Let $\mathcal{L}$ is a linear op, and

$\mathcal{L}x_1 = f_1(t)$, $\mathcal{L}x_2 = f_2(t)$

where $\mathcal{O} = D^2 + 2\beta D + \omega_0^2$

$\Rightarrow \mathcal{O}(ax_1 + bx_2) = a\mathcal{O}x_1 + b\mathcal{O}x_2 = af_1(t) + bf_2(t)$

Suppose $F_2 = 0$, $a = 1$

$\Rightarrow x = x_1 + \underset{\substack{\uparrow \\ \text{undriven case}}}{b x_2}$ solves (1)

If $\mathcal{L}x = \frac{1}{m}F(t)$, F is complex

$\Rightarrow \mathcal{L}(\mathrm{Re}\,x) = \frac{1}{m}\mathrm{Re}\,F(t)$

We will study

$$F(t) = F_\omega e^{i\omega_F t}$$
$$= |F_\omega| e^{i(\omega_F t + \varphi_F)}$$

$$\mathcal{L}A_{\omega_F} e^{i\omega_F t} = (-\omega_F^2 + 2\beta i\omega_F + \omega_0^2)A_{\omega_F} e^{i\omega_F t}$$

$$\mathcal{L}x = \frac{1}{m}F_\omega e^{i\omega_F t} \Rightarrow A_{\omega_F} = \frac{1}{m}\frac{F_\omega}{\omega_0^2 - \omega_F^2 + 2i\beta\omega_F}$$

$$x(t) = \frac{1}{m}\frac{F_\omega}{\omega_0^2 - \omega_F^2 + 2i\beta\omega_F}e^{i\omega_F t}$$
$$= \frac{1}{m}F_\omega\frac{\boxed{\omega_0^2 - \omega_F^2}\boxed{-2i\beta\omega_F}}{\boxed{(\omega_0^2 - \omega_F^2)^2 + 4\beta^2\omega_F^2}}e^{i\omega_F t}$$

When $\beta = 0$, $A_\omega = \frac{F_\omega}{m}\frac{1}{\omega_0^2 - \omega_F^2}$

$\beta \neq 0$, $|A_\omega| = \frac{|F_\omega|}{m}\frac{1}{\sqrt{(\omega_0^2 - \omega_F^2)^2 + 4\beta^2\omega_F^2}}$ $\quad \frac{A_\omega}{F_\omega} = \left|\frac{A_\omega}{F_\omega}\right|e^{-i\phi} = \frac{1}{m}(\boxed{\cos\phi} - \boxed{i\sin\phi}) \Rightarrow \tan\phi = \frac{2\beta\omega_F}{\omega_0^2 - \omega_F^2}$

General real solution for $F(t) = \mathrm{Re}\, F_w e^{i\omega t}$

constants determined by initial conditions

$$x(t) = \mathrm{Re}\,(A_w e^{i\omega_F t}) + x_{max} e^{-\beta t} \cos(\underbrace{\omega}_{\sqrt{\omega_0^2-\beta^2}} t + \varphi)$$

$$= |A_w| \cos(\omega_F t + \phi_F) + x_{max} e^{-\beta t} \cos(\omega t + \varphi)$$

determined by special solution.

e.g. $x(0) = v(0) = 0$

$x(0)=0$ $\quad |A_w| \cos \phi_F + \underline{x_{max}} \cos \underline{\varphi} = 0$

$v(0)=0$ $\quad -|A_w|\, \omega_F \sin\phi_F - \underline{x_{max}}(\beta \cos \underline{\varphi} + \omega \sin \underline{\varphi}) = 0$ $\qquad$ unknow

$\vec{v} = v_x \hat{x} + v_y \hat{y} + v_z \hat{z}$ Oct 9, 2017

$$\int_{\gamma \sim \hat{x}} \vec{v} \cdot d\vec{r} = \int v_x dx \qquad \text{e.g. } \gamma = x \hat{e}_x \Big|_0^s \Rightarrow \int_\gamma v \cdot dr = \int_0^s v_x(x, y=0, z=0)\, dx.$$

$$\int_\gamma \vec{v} \cdot d\vec{r} \qquad \gamma = f_x(t)\hat{x} + f_y(t)\hat{y} + f_z(t)\hat{z}$$

$$\Rightarrow dr = (\dot{f}_x \hat{x} + \dot{f}_y \hat{y} + \dot{f}_z \hat{z})\, dt$$

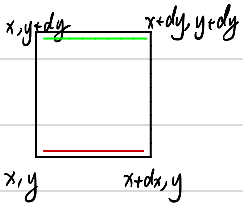
$$\int_0^s (\dot{f}_x v_x(f_x(t), f_y(t), f_z(t)) + \dot{f}_y v_y + \dot{f}_z v_z)\, dt$$

e.g. $\gamma = x\hat{x} \Rightarrow d\vec{r} = dx\, \hat{x}$

$$\Rightarrow f_x(t) = x, \quad f_y(t) = f_z(t) = 0$$

$$\Rightarrow \int_0^s v_x(x, 0, 0)\, dx$$

Curl



- $\int_x^{x+dx} d\tilde{x}\, v_x(\tilde{x}, y)$
$= dx\, v_x(x,y)$

$$\lim_{dx \to 0} \int_x^{x+dx} f(\tilde{x})\, d\tilde{x} = f(x)\, dx \quad ?$$

- $\int_{x+dx}^{x} v_x(\tilde{x}, y+dy)\, d\tilde{x}$
$= -dx\, v_x(x, y+dx)$

$$\Rightarrow -\frac{\partial v_x}{\partial y} dx\, dy.$$

$$v_x(\tilde{x}, y+dy) = v_x(x, y+dy) + (\tilde{x} - x)\frac{\partial v_x}{\partial v_y} \cdots \quad ?$$

$$\frac{1}{2}\tilde{x}^2 - \tilde{x}x$$

$F(t+T) = F(t) \leftarrow$ linear

$$T = \frac{2\pi}{\omega_F} \Leftrightarrow \omega_F = \frac{2\pi}{T}$$

$$F(t) = \frac{1}{2}\sum_{-\infty}^{\infty} \tilde{F}_n e^{in\omega_F t} = \frac{1}{2}\tilde{F}_0 + \frac{1}{2}\sum_{-\infty}^{\infty}{}' \underbrace{\tilde{F}_n}_{\mathrm{Re}\tilde{F}_n + i\,\mathrm{Im}\tilde{F}_n}(\cos(n\omega_F t) + i\sin(n\omega_F t)) \qquad F_{-n} = F_n^* \text{ for real } F(t)$$

$$= \frac{1}{2}\tilde{F}_0 + \sum_1^{\infty} \mathrm{Re}(\tilde{F}_n)\cos(n\omega_F t) - \mathrm{Im}(\tilde{F}_n)\sin(n\omega_F t)$$

$$= \frac{1}{2}\tilde{F}_0 + \frac{1}{2}\sum_1^{\infty}(F_n + F_{-n})\cos(n\omega_F t) + i(F_n - F_n)\sin(n\omega_F t)$$

$$= \frac{1}{2}\tilde{F}_0 + \sum_1^{\infty}\left[(\mathrm{Re}\, F_n)\cos(n\omega_F t) - \mathrm{Im}(F_n)\sin(n\omega_F t)\right]$$

$$\frac{1}{T}\int_0^T dt\, e^{inw_F t} = \begin{cases} 1 & \text{if } n=0 \\ \frac{1}{inw_F T}\left(e^{inw_F T}-1\right) = \frac{1}{inw_F T}\left(e^{2\pi i n}-1\right)=0 & \text{if } n\neq 0 \end{cases}$$

$$\vec{v} = v_i \hat{e}_i \Rightarrow v_i = \hat{e}_i \cdot \vec{v}$$

Def: $f_1(t)\cdot f_2(t) = \frac{1}{T}\int_0^T f_1^*(t)\, f_2(t)\, dt$

eng $f_1 = e^{inw_F t}$, $f_2 = e^{imw_F t}$

$\Rightarrow f_1 \cdot f_2 = \frac{1}{T}\int_0^T e^{-inw_F t}\, e^{imw_F t}\, dt$

$= 0$ if $m \neq n$

1 if $m = n$

$\lceil F_n \to \tilde{F}_n \rfloor$

$$\tilde{F}_n = \frac{2}{T}\int_0^T F(t)\, e^{-inw_F t} = \frac{w_F}{\pi}\int_0^{\frac{w_F}{2\pi}} F(t)\, e^{-inw_F t}\, dt$$

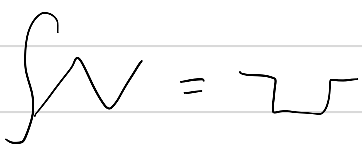
$$\ddot{x} + 2\beta\dot{x} + w_0^2 x = \frac{1}{m}F(t) = \frac{1}{2m}\sum_{-\infty}^{\infty}\tilde{F}_n e^{inw_F t}$$

$$x_n = A_{nw_F}\, e^{inw_F t} = \frac{\tilde{F}_n}{2m} = \frac{e^{inw_F t}}{(w_0^2 - (nw_F^2)) + 2in\beta w_F}$$

$$x = \sum_{-\infty}^{+\infty} x_n$$

$\tilde{F}_n = |F_n| e^{-i\varphi_n}$ $\quad F_n e^{inw_F t} = |F_n|\, e^{i(nw_F t - \varphi_n)}$

$$= \frac{1}{2}\tilde{F}_0 + \sum_1^{\infty} |F_n| \cos(nw_F t - \varphi_n)$$



Fourier Transform

$T \to \infty$

$$\frac{1}{T}\int_{-T/2}^{T/2} dt\, e^{inw_F t} = \begin{cases} \frac{1}{inw_F T}\left(e^{inw_F T/2} - e^{inw_F T/2}\right) = \frac{1}{inw_F T}\left(e^{\pi i n} - e^{-\pi i n}\right) = 0 \\ 1 \quad n=0 \end{cases}$$

$$\mathcal{F}_n = \frac{T}{2}\tilde{F}_n = \int_{-T/2}^{T/2} F(t)\, e^{-in\frac{2\pi}{T}t}$$

$$F(t) = \sum_{-\infty}^{\infty} \frac{\mathcal{F}_n}{T}\, e^{in\frac{2\pi}{T}t}$$

Fourier Transform

$T \to \infty$

$$\frac{1}{T}\int_{-T/2}^{T/2} dt\, e^{in\omega_T t} = \begin{cases} \frac{1}{in\omega_T T}\left(e^{in\omega_T T/2} - e^{in\omega_T T/2}\right) = \frac{1}{in\omega_T T}\left(e^{\pi i n} - e^{-\pi i n}\right) = 0 \\ 1 \quad n=0 \end{cases}$$

$$\mathcal{F}_n = \frac{T}{2}\tilde{F}_n = \int_{-T/2}^{T/2} F(t) e^{-in\frac{2\pi}{T}t}$$

$$F(t) = \sum_{-\infty}^{\infty} \frac{\mathcal{F}_n}{T} e^{in\frac{2\pi}{T}t}$$

def : $S = \frac{n}{T}$ as $T \to$, S is continue's variable

$$\mathcal{F}_n = \frac{T}{2}\tilde{F}_n = \int_{-T/2}^{T/2} F(t) e^{-in\frac{2\pi}{T}t} = \int_{-T/2}^{T/2} F(t) e^{-i2\pi St} = \int_{-T/2}^{T/2} F(t)\, e^{-i\omega t}$$

$$F(t) = \sum_{-\infty}^{\infty} \frac{\mathcal{F}_n}{T} e^{in\frac{2\pi}{T}t} = \sum_{-\infty}^{\infty} \frac{\mathcal{F}_n}{T} e^{i2\pi St} = \sum_{-\infty}^{\infty} \frac{\mathcal{F}_n}{T} e^{i\omega t}$$

$n = \frac{T\omega}{2\pi}$

let $2\pi S = \omega$

$$= \int_{-\infty}^{\infty} \mathcal{F}(S)\, e^{i2\pi St}$$

$$dS\,\mathcal{F}(S) = \frac{\mathcal{F}_{n=TS}}{T} \qquad \omega = 2\pi S,\quad \tilde{F}(\omega) = \mathcal{F}(S)$$

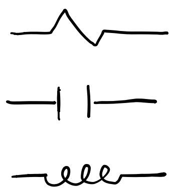
$$F(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{F}(\omega) e^{i\omega t} d\omega \qquad A(\omega) = \frac{1}{m}\tilde{F}(\omega_{div}) \frac{1}{(\omega_0 - \omega_{div})^2 + 2i\beta\omega_{div}}$$

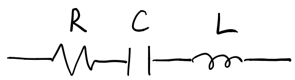
$$\tilde{F}(\omega) = \int_{-\infty}^{\infty} F(t)\, e^{-i\omega t} dt \qquad x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{x}(\omega_{div})\, e^{i\omega_{div} t} d\omega_{div}$$

$V = IR = \dot{Q}R$ Ohm's Law

$Q = CV$

$V = L\dot{I} = L\ddot{Q}$





$V_R + V_C + V_L = V_T \equiv V$

$$V(t) = R\dot{Q} + \frac{Q}{C} + L\ddot{Q}$$
$$= L\ddot{Q} + R\dot{Q} + \frac{1}{C}Q$$

$Q \leftrightarrow x$

$L \leftrightarrow m$ $\quad\left(\beta = \frac{b}{2m} = \frac{R}{2L}\right)$

$\frac{1}{C} \leftrightarrow k$ $\quad\left(\omega_0^2 = \frac{k}{m} = \frac{1}{LC}\right)$

$R \leftrightarrow b$

$$\Rightarrow V(t) = \ddot{Q} + 2\beta\dot{Q} + \omega_0^2 Q \quad \text{where } \beta = \frac{R}{2L},\ \omega_0^2 = \frac{1}{LC},\ \omega^2 = \omega_0^2 - \beta^2 = \frac{1}{LC} - \frac{R^2}{4L^2}$$

Quality $\rightarrow \mathcal{Q} = \frac{\omega_0}{2\beta} = \sqrt{\frac{1}{LC}} \cdot \frac{L}{R} = \frac{1}{R}\sqrt{\frac{L}{C}}$

$$Q(t) = A_{\omega_F} e^{i\omega_F t}$$

$$A_{\omega_F} = \frac{F}{m}\frac{1}{\omega_0^2 - \omega_F^2 + 2i\beta\omega_F}$$
$$= \frac{V}{L}\frac{1}{\frac{1}{LC} - \omega_F^2 + \frac{iR}{L}\omega_F}$$
$$= V\frac{1}{\frac{1}{C} - L\omega_F^2 + 2iR\omega_F}$$

$\left(\text{for } V(t) = Ve^{i\omega_F t}\right)$

$$I(t) = I_{\omega_F} e^{i\omega_F t} = V\frac{\omega_F}{\frac{1}{C} - L\omega_F^2 + 2iR\omega_F}$$

$$|I_{\omega_F}| = \frac{\omega_F}{\left[\left(\frac{1}{C} - L\omega_F^2\right)^2 + R^2\omega_F^2\right]^{1/2}}$$

$$\frac{\partial |I_{\omega_F}|}{\partial \omega_F} = \frac{1}{[\ \ ]} - \frac{1}{2}\frac{\omega_F}{[\ \ ]^{3/2}}\left(2R^2\omega_F - 4\left(\frac{1}{C} - L\omega_F^2\right)L\omega_F\right) = 0$$

$$\left(\frac{1}{C} - L\omega_F^2\right)^2 + \cancel{R^2\omega_F^2} - \cancel{R^2\omega_F^2} - \underbrace{2\left(\frac{1}{C} - L\omega_F^2\right)}L\omega_F^2 = 0$$

$$\frac{1}{C} - L\omega_F^2 = 0$$

$$\Rightarrow \omega_F^2 = \frac{1}{LC} = \omega_0^2$$

WKB

Slowing Varying in time

$$\ddot{x} + \omega_0^2(t)x = 0$$

for const ω

$$i\omega_0 t = i\int_0^t \omega_0 dt$$

[for QM:

$$\frac{\partial^2\psi(x)}{\partial x^2} + f(x)\psi(x) = 0$$]

might exp $x(t) \simeq C_+(t)e^{i\int \omega_0(t)dt} + C_-(t)\, e^{-i\int \omega_0(t)dt}$

$$x = e^{\lambda(t)} = \underbrace{e^{\alpha(t)}}_{A(t)}\, e^{i\theta(t)}$$

$$\dot{x} = x(\dot{\alpha} + i\dot{\theta}), \quad \ddot{x} = x(\dot{\alpha} + i\dot{\theta}) + x(\ddot{\alpha} + i\ddot{\theta})$$

$$\begin{cases} \ddot{\alpha} + \dot{\alpha}^2 - \dot{\theta}^2 + \omega_0^2 = 0 \\ 2\dot{\alpha}\dot{\theta} + \ddot{\theta} = 0 \end{cases} \Rightarrow \dot{\alpha} = -\frac{\ddot{\theta}}{2\dot{\theta}} \overset{\text{chain rule}}{=} -\frac{1}{2}\frac{d}{dt}\ln\dot{\theta}$$

$$\alpha(t) = c - \frac{1}{2}\ln\dot{\theta}$$

$$A(t) = e^{\alpha(t)} = \frac{\mathbb{C}}{\sqrt{\dot{\theta}}} \quad \text{where } \mathbb{C} = e^c$$

Approximation

assume $\dot{\alpha}^2, \ddot{\alpha} \ll \omega_0^2, \dot{\theta}^2$

$$\dot{\theta} = \pm\omega_0(t)$$

$$\theta = \pm\int_0^t \omega_0$$

$$\Rightarrow \frac{\mathbb{C}_+}{\sqrt{\omega_0(t)}}\, e^{i\int^t \omega_0} + \frac{\mathbb{C}_-}{\sqrt{\omega_0(t)}}\, e^{-i\int^t \omega_0}$$

$$m\ddot{x} = F(x) = m\left(-\omega_0^2 x + \Delta f(x)\right)$$

$$\Delta f(x) = \sum_{n=2}^{\infty} f_n \lambda^{n-1} x^n$$

$$x(t) = \sum_{0}^{\infty} x_n(t)\lambda^n$$

$$\left\ulcorner m \sum \frac{f_n}{n+1} \lambda^{n-1} x^{n+1} + \frac{1}{2} m \omega_0^2 x^2 = U(x) \right\lrcorner$$

$$\ddot{x} + \omega_0^2 x = \Delta f(x)$$

$$\underbrace{\ddot{x}_0 + \lambda \ddot{x}_1 + \lambda^2 \ddot{x}_2 + \cdots}_{\ddot{x}} + \underbrace{\omega_0^2 x_0 + \omega_0^2 \lambda x_1 + \omega_0^2 \lambda^2 x_2 + \cdots}_{\omega_0^2 x} = 0 + \lambda f_2 x^2 + \lambda f_3 x^3 + \cdots$$

$$= \lambda f_2 x_0^2 + 2\lambda^2 f_2 x_0 x_1 + \lambda^3 f_3 x_0^2 + \cdots$$

$O(\lambda^0)$: $\ddot{x}_0 + \omega_0^2 x_0 = 0$ ← solve

$O(\lambda^1)$: $\ddot{x}_1 + \omega_0^2 x_1 = f_2 x_0^2$ ← get effective driving force

$O(\lambda^2)$: $\ddot{x}_2 + \omega_0^2 x_2 = 2 f_2 x_0 x_1 + f_3 x_0^3$ ←

$$\left\ulcorner \cos^2 \omega_0 = \frac{1+\cos 2\omega_0}{2} \right\lrcorner$$

$$\left\ulcorner \frac{e^{-i\omega t} + e^{i\omega t}}{2} = \frac{e^{3i\omega t} + 3e^{i\omega t} + 3e^{-i\omega t} + e^{-3i\omega t}}{9} \right.$$

$$\left. = \frac{1}{4}\cos \omega t + \frac{3}{4}\cos \omega t. \right\lrcorner$$

$$\ddot{x} + \omega_0^2 x = \lambda x^2$$

$$x(t) = \bar{x} + A\cos(\omega t) \qquad \bar{x} \propto \lambda + \cdots$$

$$(\omega_0^2 - \omega^2) A \boxed{\cos(\omega t)} + \boxed{\omega_0^2 \bar{x}} \qquad \omega = \omega_0 + \lambda + \cdots$$

$$= \lambda\left(\boxed{\bar{x}^2} + 2\bar{x} A \boxed{\cos \omega t} + A^2 \underbrace{\cos^2 \omega t}_{\frac{\cos(2\omega t) + \boxed{1}}{2}}\right)$$

$$\Rightarrow \omega_0^2 \bar{x} = \boxed{\lambda \bar{x}^2} + \frac{\lambda}{2} A^2$$

$$\bar{x} \approx \frac{\lambda}{2\omega_0^2} A^2 \quad \left(\bar{x} = \frac{\lambda}{2\omega_0^2} A^2 + O(\lambda^3)\right)$$

$$\Rightarrow \omega_0^2 - \omega^2 = \lambda \bar{x} 2$$

$$\omega = \sqrt{\omega_0^2 - \lambda \bar{x} 2}$$

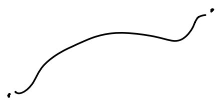
$$= \sqrt{\omega_0^2 - \frac{\lambda^2 A^2}{\omega_0}} = \omega_0 \sqrt{1 - \frac{\lambda^2 A^2}{\omega_0^3}} \approx \omega_0\left(1 - \frac{\lambda^2 A^2}{2\omega_0^3}\right)$$

E.O.C.

Oct 16, 2017

Shortest path between two point.

$y(x)$



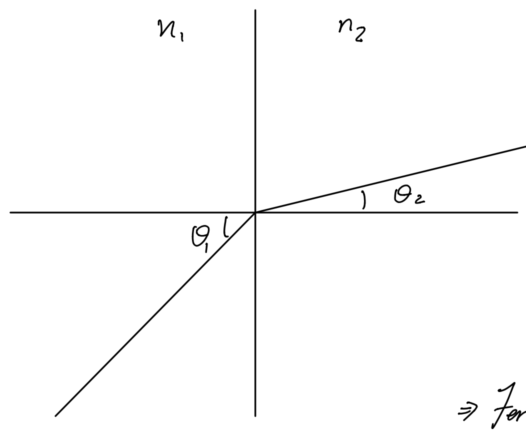
$$\sqrt{dx^2+dy^2} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$l[y] = \int_{x_1}^{x_2} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

find $y(x)$ s.t. $l[y]$ is minimized.

$$\vec{r} = y(t)\hat{y} + x(t)\hat{x}, \quad l[\vec{r}] = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$\sqrt{d\vec{r}\cdot d\vec{r}} = \sqrt{\frac{d\vec{r}}{dt}\cdot\frac{d\vec{r}}{dt}}\, dt.$$



Snell's law

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

$n = \frac{c}{v}$ ← speed of light in vacuum / " in medium

Alhazon[1021]: light moves at different speeds in different materials

⇒ Fermat[1662] proposed light moves to <u>minimized</u> the time it travels between 2 points.

$$t[y(x)] = \int_{x_1}^{x_2} dx\, \frac{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}{v(x,y)}$$

$$n = \frac{c}{v}$$

$$= \frac{1}{c}\int_{x_1}^{x_2} dx\; n(x,y)\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

General form

$$S[y(x)] = \int_{x_1}^{x_2} \mathcal{L}\left(y, \frac{dy}{dx}, x\right)$$

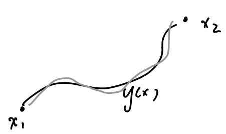
analog of setting $\frac{df}{dx} = 0$?

$$f(x+\delta) = f(x) + \delta \cancelto{0}{\frac{df}{dx}} + \dots$$

at extremum

At extremum, function doesn't chang when we change the argument

Want to find $y(x)$ s.t. $S[y(x)]$ doesn't chang when we change $y(x)$ a little.



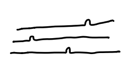
Independent but small.

• $y(x) + \underbrace{\delta y(x)}_{?}$ $\qquad \underbrace{\delta y(x_1) = \delta y(x_2) = 0}_{?}$

$$\to f(x+\delta) - f(x) \approx 0$$

$$\underset{\substack{\uparrow \\ \text{action}}}{S}[y(x) + \delta y(x)] - S[y(x)] = \int_{x_1}^{x_2} dx \Big[\underbrace{\mathcal{L}\left(y + \delta y, \frac{d}{dx}(y+\delta y), x\right) - \mathcal{L}\left(y, \frac{dy}{dx}, x\right)}_{\frac{\partial L}{\partial y}\delta y + \underbrace{\frac{\partial L}{\partial \frac{dy}{dx}} \frac{d}{dx}\delta y}_{\text{int by part}} + \dots} \Big]$$

$$= \int_{x_1}^{x_2} dx \left(\frac{\partial L}{\partial y}\delta y - \left(\frac{d}{dx}\frac{\partial \mathcal{L}}{\partial (\frac{dy}{dx})}\right)\delta y \right) = \int_{x_1}^{x_2} dx\, \delta y(x) \left(\frac{\partial L}{\partial y} - \frac{d}{dx}\left(\frac{\partial L}{\partial \frac{dy}{dx}}\right)\right)$$

δy can look like  , can be anything

$$\Rightarrow \frac{\partial L}{\partial y} - \frac{d}{dx}\left(\frac{\partial L}{\partial \frac{dy}{dx}}\right) = 0$$

For $l[y] = \int_{x_1}^{x_2} \underbrace{\sqrt{1+(\frac{dy}{dx})^2}}_{\mathcal{L}(y,\frac{dy}{dx},x)}\, dx$

$$\Rightarrow 0 - \frac{d}{dx}\left(\frac{\frac{dy}{dx}}{\sqrt{1+(\frac{dy}{dx})^2}}\right) = 0 \quad \boxed{?}$$

$$\Rightarrow \frac{\frac{dy}{dx}}{\sqrt{1+(\frac{dy}{dx})^2}} = const \Rightarrow \frac{dy}{dx} = const\sqrt{1+(\frac{dy}{dx})^2}$$

$$(\frac{dy}{dx})^2 = const^2\,(1+(\frac{dy}{dx})^2)$$

$$(\frac{dy}{dx})^2 = \frac{c^2}{1-c^2} = a^2$$

$$\frac{dy}{dx} = a$$

$$y = ax+b$$

For $t[y(x)]$:

$$\frac{\partial n}{\partial y}\sqrt{1+(\frac{dy}{dx})^2} - \frac{\frac{d}{dx}\, n(x,y)\frac{dy}{dx}}{\sqrt{1+(\frac{dy}{dx})^2}} = 0$$

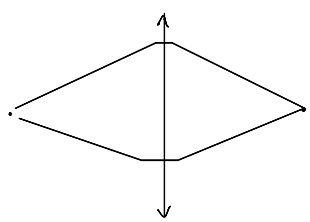
if $n(x,y) = n(x)$, then

$$n(x)\frac{dy}{dx} = \overset{const}{\overset{\downarrow}{a}}\sqrt{1+(\frac{dy}{dx})^2}$$

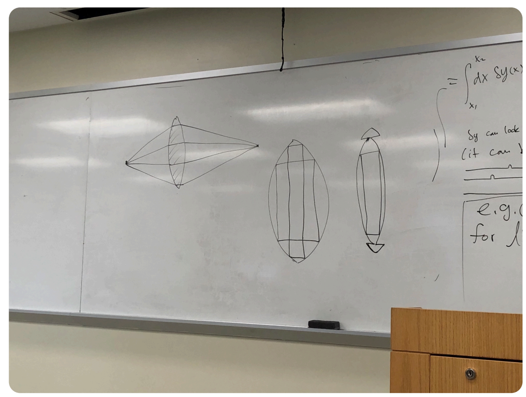
$$\frac{dy}{dx} = \frac{a}{\sqrt{n^2-a^2}} = \tan\theta$$

$\Rightarrow n(x)\sin\theta = a \qquad (\sin\theta = \frac{a}{n}\cos\theta = \sqrt{1-\frac{a^2}{n^2}} = \frac{1}{n}\sqrt{n^2-a^2}$

$\Rightarrow n(x)\sin\theta$ is a const.



total time is same for all path



$\boxed{?}$

$q_i(t)$, $\dot{q}_i(t)$ many functions.

$$S[\{q_i(t)\}] = \int_{t_1}^{t_2} dt\, \mathcal{L}(q_i(t), \dot{q}_i(t), t)$$

independent variations $\delta q_i(t)$ s.t. $\delta q_i(t_1) = \delta q_i(t_2) = 0$

$$\delta S = \int_{t_1}^{t_2} dt\left[\mathcal{L}(q_i+\delta q_i, \dot{q}_i + \frac{d}{dt}\delta q_i, t) - \mathcal{L}(q_i, \dot{q}_i, t)\right]$$

$$= \int_{t_1}^{t_2} dt\left(\sum_i \delta q_i \frac{\partial L}{\partial q_i} + (\frac{d}{dt}\delta q_i)\frac{\partial L}{\partial \dot{q}_i}\right)$$

$$= \int_{t_1}^{t_2} dt \sum_i \delta q_i(t)\left(\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}\right)$$

$$\Rightarrow \frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = 0$$

Euler-Lagrange Equations

for $i = 1 \dots n$, there are n diff-eqn's e.g. 1D Newton's Law

$$\mathcal{L} = \frac{1}{2} m \dot{q}^2 - U(q)$$

$$\frac{\partial L}{\partial \dot{q}} = m\dot{q} \Rightarrow -\frac{\partial U}{\partial q} - \frac{d}{dt} m\dot{q} = 0$$

$$\Rightarrow m\ddot{q} = -\frac{\partial U}{\partial q}$$

e.g. $\mathcal{L} = \frac{1}{2} m \sum \dot{q}_i^2 - U(\{q_i\})$

$$m\ddot{q}_i = -\frac{\partial}{\partial q} U$$

if $\vec{r} = \hat{e}_i q_i$ for $\hat{e}_i$ cartesian basis

$$\Rightarrow m\ddot{\vec{r}} = -\vec{\nabla} U$$

$\frac{1}{2} m(\underset{\dot{q}_1^2}{\underset{\uparrow}{\dot{x}^2}} + \underset{\dot{q}_2^2}{\underset{\uparrow}{\dot{y}^2}})$ let $x = r\cos\theta$, $x = r\sin\theta$, $\dot{x} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta$, $\dot{y} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta$

$$\dot{x}^2 + \dot{y}^2 = \dot{r}^2 + r^2\dot{\theta}^2$$

$$= \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2)$$

$$\mathcal{L} = \frac{1}{2} m(\overbrace{\dot{r}^2 + r^2\dot{\theta}^2}^{\text{How come in the first place}}) - U(r,\theta)$$

$$\frac{\partial L}{\partial r} = \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} \Rightarrow -\frac{\partial U}{\partial r} + mr\dot{\theta}^2 = m\ddot{r}$$

$$\frac{\partial L}{\partial \theta} = \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} \Rightarrow -\frac{\partial U}{\partial \theta} = \frac{d}{dt}(mr^2\dot{\theta})$$

Notice if $U = U(r) \Leftrightarrow \frac{dU}{d\theta} = 0$

$$\Rightarrow \frac{d}{dt}(mr^2\dot{\theta}) = 0$$

$$mr^2\dot{\theta} = \underset{\text{const}}{\underset{\uparrow}{\mathcal{L}_z}}$$ $\boxed{?}$ How do you know it → $\mathcal{L}_z$?

$$\dot{\theta} = \frac{\mathcal{L}_z}{mr^2}$$

$$\Rightarrow m\ddot{r} = -\frac{\partial U}{\partial r} + \frac{\mathcal{L}_z^2}{mr^3}$$

$$= -\frac{\partial}{\partial r}\left(U + \frac{L_z^2}{2mr^2}\right)$$

$$S[q^i(t)] = \int dt\, \mathcal{L}(q^i(t), \dot{q}^i(t), t)$$

$$\delta S = S[q^i(t) + \delta q^i(t)] - S[q^i(t)] = 0 \quad \Leftrightarrow \quad \frac{\partial L}{\partial q^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} = 0$$

Hamilton's principle

if $\mathcal{L} = T - U$

$$\Rightarrow \frac{\partial L}{\partial q^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} = 0 \Leftrightarrow \text{Newton's Laws}$$

Particles move along path $q^i(t)$ that extremize $S[q_i(t)]$

e.g. $T = \frac{1}{2}\sum_i m_i \dot{q}_i^2$, $U = U(q_i)$

$$\mathcal{L} = \frac{1}{2}\sum_i m_i \dot{q}_i^2 - U(q^i)$$

$$\Rightarrow \frac{\partial \mathcal{L}}{\partial q^i} = -\frac{\partial U}{\partial q^i}, \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} = \frac{d}{dt}(\cancel{\Sigma}\, m_i \dot{q}_i) = \cancel{\Sigma}\, m_i \ddot{q}^i$$

$$\cancel{\Sigma}\, m_i \ddot{q}_i = -\frac{\partial U}{\partial q^i}$$

$$T = \frac{1}{2}\sum_{ij} m_{ij}\dot{q}^i \dot{q}^j$$

$$\Rightarrow \sum_j m_{ij}\ddot{q}^j = -\frac{\partial U}{\partial q^i}$$

Nice Use : Polar Coordinates.

$$\vec{x} = R\hat{R} + z\hat{z}$$

$$\dot{\vec{x}} = \dot{R}\hat{R} + R\dot{\theta}\hat{\theta} + \dot{z}\hat{z}$$

$$\dot{\vec{x}}\cdot\dot{\vec{x}} = \dot{R}^2 + R^2\dot{\theta}^2 + \dot{z}^2$$

$$\Rightarrow T = \frac{1}{2}m(\dot{R}^2 + R^2\dot{\theta}^2 + \dot{z}^2)$$

$U(R, \theta, z)$

$R \quad \frac{d}{dt}(m\dot{R}) = mR\dot{\theta}^2 - \frac{\partial U}{\partial R} \quad \Leftarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i}$

$\theta \quad \frac{d}{dt}(mR^2\dot{\theta}) = -\frac{\partial U}{\partial \theta} \quad 2mR\dot{R}\dot{\theta} + mR^2\ddot{\theta} = mR(2\dot{R}\dot{\theta} + R\ddot{\theta})$

$z \quad \frac{d}{dt}(m\dot{z}) = -\frac{\partial U}{\partial z}$

$$\vec{F} = m\ddot{\vec{r}} = \frac{d}{dt}(\dot{R}\hat{R} + R\dot{\theta}\hat{\theta} + \dot{z}\hat{z}) = (\ddot{R}\hat{R} + 2\dot{R}\dot{\theta}\hat{\theta} - R\dot{\theta}^2\hat{R} + \ddot{z}\hat{z} + R\ddot{\theta}\hat{\theta})$$

$$\frac{d}{dt}\hat{R} = \dot{\theta}\hat{\theta}$$

$$\frac{d}{dt}\hat{\theta} = -\dot{\theta}\hat{R}$$

$$-\vec{\nabla}U = -(\hat{R}\frac{\partial}{\partial R}U + \frac{1}{R}\hat{\theta}\frac{\partial U}{\partial \theta} + \hat{z}\frac{\partial}{\partial z}U)$$

$$m(\ddot{R} - R\dot{\theta}^2) = -\frac{\partial U}{\partial R}$$

$$m(2\dot{R}\dot{\theta} + R\ddot{\theta}) = -\frac{1}{R}\frac{\partial U}{\partial \theta}$$

$$m\ddot{z} = -\frac{\partial U}{\partial z}$$

- $\frac{d}{dt}(\underbrace{mR^2\dot{\theta}}_{L_z}) = -\frac{\partial U}{\partial \theta} = R F_\theta = \overset{\text{torq}}{\Gamma_z}$

$\Rightarrow \dot{L}_z = \Gamma_z$

$\Rightarrow \dot{\theta} = \frac{L_z}{mR^2}$

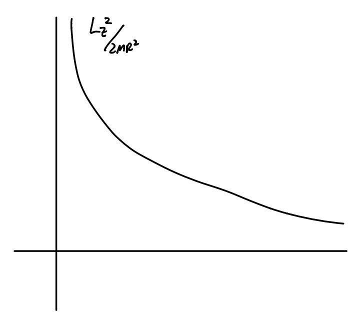
$m\ddot{R} = \frac{L_z^2}{mR^3} - \frac{\partial U}{\partial R}$

W $\Gamma_z = 0 \Rightarrow \dot{L}_z = 0 \Leftrightarrow L_z$ is conserved

$\Rightarrow$ effective potential

$m\ddot{R} = -\frac{\partial U_{eff}}{\partial R}$, $U_{eff} = U + \frac{L_z^2}{2mR^2}$

$$\frac{\partial}{\partial R}\left(\frac{L_z^2}{2mR^2}\right) = \frac{L_z^2}{2m}\left(-\frac{2\cdot 1}{R^3}\right) = -\frac{L_z^2}{mR^3}$$



$\Gamma_z = 0 \Leftrightarrow \frac{\partial U}{\partial \theta} = 0 \Rightarrow U(R, z)$, L is independent of θ (depends only on $\dot{\theta}$)

When L depends only on $\dot{q}^i$ for some particular i, then $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = 0$ for that i $\Rightarrow \underbrace{\frac{\partial L}{\partial \dot{q}_i}}_{p_i}$ is conserved.

L doesn't change if we shift the particular $q^i \to q^i + c$

$\Rightarrow$ symmetry!

if for some q^i, $\mathcal{L}$ doesn't depend on $\dot{q}_i$, we get

$\frac{\partial L}{\partial \dot{q}_i} = 0$ ⇐ a constraint on this q's and $\dot{q}$'s

e.g. If we want to impose a const $f(q^i) = 0$

add a coordinate λ and λ term $\quad \mathcal{L} \to \mathcal{L} - \lambda f(q^i)$

Lagrange multiplier

E-L EQ for λ is $f(q^i) = 0$ imposes the const

$$\frac{d}{dt}\frac{dL}{\partial \dot{q}_i} = \frac{\partial L_0}{\partial q^i} - \lambda \frac{\partial f}{\partial q_i}$$

e.g. $x^2 + y^2 = R^2$ R const

$y = \sqrt{R^2 - x^2}$

$\frac{1}{2} m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2} m\left(\dot{x}^2 + \left(\frac{x\dot{x}}{\sqrt{R^2 - x^2}}\right)^2\right)$

$U(x,y) = U(x, \sqrt{R^2 - x^2})$

$$\Rightarrow m\frac{d}{dt}\left(\dot{x} + \frac{x^2}{R^2 - x^2}\dot{x}\right) = -\frac{\partial U}{\partial x} - \frac{\partial U}{\partial y}\left(\frac{-x}{\sqrt{R^2 - x^2}}\right) + \frac{\partial}{\partial x}\left(\frac{1}{2} m\frac{x^2 \dot{x}^2}{R^2 - x^2}\right)$$

L-M: $\frac{1}{2} m(\dot{x}^2 + \dot{y}^2) - U(x,y) - \lambda(x^2 + y^2 - R^2)$

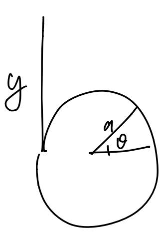
$\lambda: \quad x^2 + y^2 = R^2$

$m\ddot{x} = -\frac{\partial U}{\partial x} - 2\lambda x$

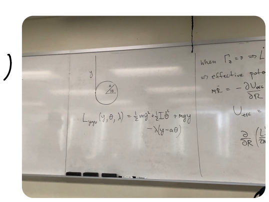
$m\ddot{y} = -\frac{\partial U}{\partial y} - 2\lambda y$

$\Rightarrow$ $m(y\ddot{x} + x\ddot{y}) = -y\frac{\partial U}{\partial x} + x\frac{\partial U}{\partial y}$

$m\left(\ddot{x} - \frac{x}{y}\ddot{y}\right) = -\frac{\partial U}{\partial x} + \frac{x}{y}\frac{\partial U}{\partial y}$



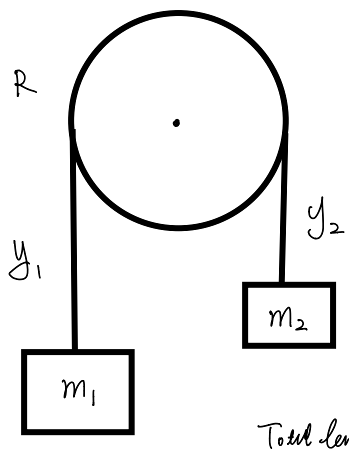
$\mathcal{L}_{yoyo}(y, \theta, \lambda) = \frac{1}{2} m\dot{y}^2 = \frac{1}{2} I\dot{\theta}^2 + mgy - \lambda(y - a\theta)$



y	$m\ddot{y} = mg - \lambda$	$m\ddot{y} + \frac{I}{a}\ddot{\theta} = mg$
θ	$I\ddot{\theta} = \lambda a$	$\ddot{\theta} = \frac{\ddot{y}}{a}$
λ	$y = a\theta$	$\Rightarrow \ddot{y}\left(1 + \frac{I}{ma^2}\right) = g, \quad \ddot{y} = \frac{g}{1 + \frac{I}{ma^2}}$

$g \geq g_{eff} \geq \frac{1}{2} g$

Atwood Machine



Total length of String
ℓ

$$\frac{1}{2} I \left(\frac{\dot{y}_1}{a}\right)^2$$

K ⇒ $\mathcal{L} = \frac{1}{2} m_1 \dot{y}_1^2 + \frac{1}{2} m_2 \dot{y}_2^2 + \frac{1}{2} I \dot{\theta}^2$

U ⇒ $+ m_1 g y_1 + m_2 g y_2$

λ { $-\lambda (y_1 + y_2 + \pi a - \ell)$ $\boxed{?}$

Turns without slipping means

$a\dot{\theta} = \dot{y}_1$

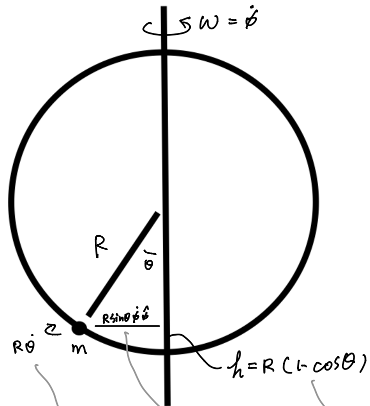
$$\frac{\partial L}{\partial \lambda} = 0 \Rightarrow y_1 + y_2 + \pi a - \ell = 0 \Rightarrow y_2 = \ell - \pi a - y_1$$

$$\Rightarrow \mathcal{L}(y_1, \dot{y}_1) = \frac{1}{2}\left(m_1 + m_2 + \frac{I}{a^2}\right)\dot{y}_1^2 + m g y_1 + \underbrace{m_2 g (\ell - \pi a}_{\text{const}} - y_1)$$

$$= \frac{1}{2}\left(m_1 + m_2 + \frac{I}{a^2}\right)\dot{y}_1^2 + (m_1 - m_2) g y_1 + C$$

$$\left(m_1 + m_2 + I/a^2\right)\ddot{y}_1 = (m_1 - m_2) g$$

$$\ddot{y}_1 = \frac{m_1 - m_2}{m_1 + m_2 \, I/a^2} g$$



$\vec r = R\dot\theta\hat\theta + R\sin\theta\,\dot\phi\,\hat\phi$

$\mathcal{L}(\theta,\dot\theta) = \frac{1}{2}m\left(R^2\dot\theta^2 + R^2\sin^2\theta\,\omega^2\right) - mgR(1-\cos\theta)$

$\frac{d}{dt}\frac{\partial L}{\partial\dot\theta} = \frac{\partial L}{\partial\theta} \Rightarrow mR\ddot\theta = mR^2\sin\theta\cos\theta\,\omega^2 - mgR\sin\theta$

$\ddot\theta = (\omega^2\cos\theta - \frac{g}{R})\sin\theta$

Possible eq point: $\sin\theta = 0$ or, $\cos\theta = \frac{g}{\omega^2 R}$

$\Downarrow$

① $\theta_{eq} = 0$ $\quad \cos\theta_{eq} = \frac{g}{\omega^2 R}$ only exist if $\omega^2 R \geq g$ ③

② or $\theta_{eq} = \pi$

①: $\theta = \theta_{eq} + \varepsilon = \varepsilon$

$\sin\theta \approx \varepsilon,\ \cos\theta \approx 1$

$\ddot\varepsilon = \underbrace{(\omega^2 - \frac{g}{R})}_{-\Omega^2}\varepsilon \Rightarrow \Omega$ is real when $\omega^2 < \frac{g}{R} \Rightarrow \omega^2 R < g \Rightarrow$ ③

$\Omega = \sqrt{\frac{g}{R} - \omega^2}$

Stable when $\omega^2 R < g$

not stable when $\omega^2 R > g$

② $\theta = \pi + \varepsilon$

$\sin(\pi + \varepsilon) \approx -\varepsilon$

$\cos(\pi + \varepsilon) \approx -1$

$\Rightarrow \ddot\varepsilon = (-\omega^2 - \frac{g}{R})(-\varepsilon)$

$= \underbrace{(\omega^2 + \frac{g}{R})}_{\Omega^2}\varepsilon$

Never Stable!

$\ddot q + \omega^2 q = 0$

③ $\theta = \theta_{eq} + \varepsilon$

$\cos\theta_{eq} = \frac{g}{\omega^2 R}$ $\sin\theta_{eq} = \sqrt{1 - \frac{g^2}{\omega^4 R^2}}$

$\sin\theta = \cos\theta_{eq}\sin\varepsilon + \cos\varepsilon\sin\theta_{eq}$ $\cos\theta = \cos\theta_{eq}\cos\varepsilon - \sin\theta_{eq}\sin\varepsilon$

$= \frac{g\varepsilon}{\omega^2 R} + \sqrt{1 - \frac{g^2}{\omega^4 R^2}}$ $\approx \frac{g}{\omega^2 R} - \sqrt{1 - \frac{g^2}{\omega^4 R^2}}\,\varepsilon$

$$\ddot{\varepsilon} = \left(\cancel{\frac{g}{R}} - \omega^2\sqrt{1 - \frac{g^2}{\omega^4 R^2}}\,\varepsilon - \cancel{\frac{g}{R}}\right)\sqrt{1 - \frac{g^2}{R^2\omega^4}}$$

$$= -\omega^2\left(1 - \frac{g^2}{R^2\omega^4}\right)\varepsilon$$

$$= -\underbrace{\left(\omega^2 - \frac{g^2}{R^2\omega^2}\right)}_{\Omega^2}\varepsilon$$

$\Rightarrow$

$\varepsilon(t) = \varepsilon_{max}\cos(\Omega_3 t + \varphi)$

$\theta(t) = \theta_{eq} + \varepsilon_{max}\cos(\Omega_3 t + \varphi)$

Recall

if for some q_0 $\mathcal{L}(\{q_i\},\{\dot{q}_i\}) = \mathcal{L}(\{q_a\},\{\dot{q}_0,\dot{q}_a\})$

does not depend on q_0, only on $\dot{q}_0$

$$\Rightarrow q_0 \text{ EL. } \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_0} = 0 \Rightarrow \frac{\partial L}{\partial \dot{q}_0} \text{ is constant in time (i.e., conserved)}$$

Suppose we chose new coordinates

$Q_i(q_0, q_a)$

Up $L \Rightarrow L_0$

$$\mathcal{L}_\Delta = \mathcal{L}_0 + \frac{d}{dt}\Delta(q_0, q_a) = \frac{\partial \Delta}{\partial q_0}\dot{q}_0 + \frac{\partial \Delta}{\partial q_a}\dot{q}_a$$

$$S_\Delta = \int_{t_1}^{t_2}(\mathcal{L}_0 + \dot{\Delta})\,dt = S_0 + \Delta\Big|_{t_1}^{t_2}$$

$$\frac{d}{dt}\frac{\partial L_\Delta}{\partial \dot{q}_0} = \frac{\partial L_\Delta}{\partial q_0} = \frac{d}{dt}\frac{\partial L_0}{\partial \dot{q}_0} + \underbrace{\frac{d}{dt}\frac{\partial \Delta}{\partial q_0}}_{} = \underbrace{\frac{\partial^2}{\partial q_0^2}\Delta\,\dot{q}_0 + \left(\frac{\partial^2}{\partial q_0 \partial q_a}\Delta\right)\dot{q}^a}_{=}$$

$$\Rightarrow \frac{d}{dt}\frac{\partial L_0}{\partial \dot{q}^0} = 0$$

$$\delta q = \alpha,\ \delta \dot{q}_0 = 0 \Rightarrow \delta L = 0$$

Infinitesimal

Symmetry transformation:

$$\delta q_i = \alpha R^i(q_i, \dot{q}_i, t)$$

$(q_i \to q_i + \delta q_i)$ is a symmetry if $\delta L = \alpha \dot{K}$

$$\delta L = \left(\frac{\partial L}{\partial \dot{q}^i}\frac{d}{dt}R^i + \frac{\partial L}{\partial q^i}R^i\right) \Rightarrow \underbrace{\delta L - \alpha \dot{K}}_{=0} = \alpha\left(\frac{\partial L}{\partial \dot{q}^i}\frac{d}{dt}R^i + \underbrace{\frac{\partial L}{\partial q^i}}_{}R^i - \dot{K}\right)$$

E-L: $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i}$

$$0 = \alpha\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}R_i - K\right)$$

$\Rightarrow \underline{\frac{\partial L}{\partial \dot{q}_i}R^i - K}$ is conserved

Q

e.g. $\delta q^0 = \alpha \Rightarrow R^a = 0,\ R^0 = 1$

$\delta L = 0 \Rightarrow K = 0$

$\Rightarrow Q = \frac{\partial L}{\partial \dot{q}_0} = P_0$ e.g. $L = \frac{1}{2}m\dot{q}_0^2 \Rightarrow P_0 = m\dot{q}$

$L = \frac{1}{2}m(\dot{R}^2 + R^2\dot{\theta}^2 + \dot{z}^2) - U(R, z)$

$\Rightarrow P_\theta = mR^2\dot{\theta} = L_z$ is conserved

$\delta \vec{r} = \vec{\alpha} \times \vec{r}$

$(\delta x = \alpha_y z = -\alpha_z y$, 1tz

3-symmetries with 3 parameters $\alpha_x, \alpha_y, \alpha_z \Leftrightarrow \vec{\alpha}$

$\Rightarrow$ 3 conserved Noether charges $\vec{Q}$

$R^x_y = z$

$R^x_z = -y \ldots$

$L = \frac{1}{2}m\dot{\vec{r}}\cdot\dot{\vec{r}} - U(|\vec{r}|)$ $\delta L = 0 \Rightarrow K = 0$

$Q = \frac{\partial L}{\partial \dot{\vec{r}}} \times \vec{r} = m\dot{\vec{r}} \times \vec{r} = -\vec{L}$ angular momentum

Noether's Theorem

If for $\delta q_i = \alpha R_i(q,\dot q)$

$$\delta L = \frac{\partial L}{\partial q_i}\delta q_i + \frac{\partial L}{\partial \dot q_i}\frac{d}{dt}\delta q_i$$

$$= \alpha \dot{k}(q,\dot q,t)$$

$\Rightarrow Q = \frac{\partial L}{\partial \dot q_i}R_i - K$ is conserved. (Noether charge)

e.g. if $\mathcal{L}$ depends only on $\dot q$, not on q

$\Rightarrow \delta q = \alpha \Rightarrow \delta L = 0$

$\Rightarrow R = 1,\ k = 0$

$\Rightarrow Q = \frac{\partial L}{\partial \dot q} = P$

Same in diff Coor

e.g. $\delta\varphi = \alpha$, $\mathcal{L} = \frac{1}{2}m(\dot R^2 + R^2\dot\varphi^2 + \dot z^2) + U(R,z)$

$\Rightarrow R=1,\ K=0,\ \delta L = 0 \Rightarrow \frac{\partial L}{\partial \dot\varphi} = mR^2\dot\varphi = \mathcal{L}_z$

e.g. $\delta x = \alpha y$, $\delta y = -\alpha x$ (Rotation around z)

$\Rightarrow R_x = y,\ R_y = -x$

Assume $\delta\mathcal{L} = 0$, $k=0$

$\Rightarrow \mathcal{L} = \frac{1}{2}m(\dot x^2 + \dot y^2 + \dot z^2) - U(x^2+y^2, z)$

$\Rightarrow Q = \frac{\partial L}{\partial \dot x}\underset{\uparrow \atop y}{R_x} + \frac{\partial}{\partial \dot y}(-x) = m(\dot x y - \dot y x) = m[(x\hat x + y\hat y)\times(\dot x\hat x + \dot y\hat y)]\cdot\hat z = \mathcal{L}_z$

$U(|r|)$, $\delta\vec r = \vec\alpha\times\vec r \Rightarrow \vec Q = \vec r\times\vec p = \vec{\mathcal{L}}$

e.g. $\mathcal{L} = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) + s(x\dot{y} - y\dot{x})$ [partial in $\vec{B}$]

$\delta x = \alpha_x \qquad \delta \mathcal{L} = s(\alpha_x \dot{y} - \alpha_y \dot{x})$

$\delta y = \alpha_y$

$\Rightarrow x: R = 1,\ k = s y \qquad Q_x = \frac{\partial L}{\partial \dot{x}} \cdot \overset{1}{R} - s y = m\dot{x} - 2 s y$

$y: R = 1,\ k = -s x \qquad Q_y = \underbrace{\frac{\partial L}{\partial \dot{y}}}_{PR} + \underbrace{s x}_{k} = m\dot{y} + 2 s x$

$\dot{Q}_x = m\ddot{x} - 2 s \dot{y} = 0$

$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x}$

$\frac{d}{dt}(m\dot{x} - s y) = s\dot{y}$

Time translation is a symmetry when $\mathcal{L}$ does not depend on t explicitly.

if $\mathcal{L}(q, \dot{q})$, $\delta q = \alpha \dot{q}$

$\Rightarrow \delta L = \left(\frac{\partial L}{\partial q} \dot{q} + \frac{\partial L}{\partial \dot{q}} \ddot{q}\right) \alpha = \alpha \frac{d}{dt} \mathcal{L}$

$\Rightarrow K = \mathcal{L},\ R = \dot{q}$

$E \qquad Q = H = \frac{\partial L}{\partial \dot{q}} \dot{q} - \mathcal{L} = p\dot{q} - \mathcal{L}$

e.g. $\mathcal{L} = \frac{1}{2} m \dot{q}^2 - U(q)$

$H = m\dot{q}^2 - \frac{1}{2} m \dot{q}^2 + U$

$E \qquad = \frac{1}{2} m \dot{q}^2 + U = E$

Generalized Coordinate $\frac{\partial L}{\partial \dot{q}_i} = P_i$ Generalized momentum

$\Rightarrow$ E-L: $\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i}$, $\dot{P}_i = F_i$ where $F_i = \frac{\partial L}{\partial q_i}$ Generalized force.

$$H(p_i, q_i) = p_i \dot{q}_i - \mathcal{L}(q_i, \dot{q}_i)$$

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \Rightarrow \dot{q}_i = f_i(p_i, q_i)$$

$\mathcal{L}$agrangian $\quad \mathcal{L}(q, \dot{q})$

Hamiltonian $\quad H(q, p)$

$$\frac{\partial H}{\partial q_i} = \underbrace{p_j \frac{\partial \dot{q}_j}{\partial q_i}}_{\times} - \frac{\partial L}{\partial q_i} - \underbrace{\frac{\partial L}{\partial \dot{q}_j} \frac{\partial \dot{q}_j}{\partial q_i}}_{\times} = -\frac{\partial L}{\partial q_i} = -\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} = -\dot{p}_i$$

$$p_i = \frac{\partial L}{\partial \dot{q}_i}$$

$$\Rightarrow \frac{\partial H}{\partial q_i} = -\dot{p}_i$$

$$\frac{\partial H}{\partial p_i} = \dot{q}_i + \underbrace{p_j \frac{\partial \dot{q}_j}{\partial p_i} - \frac{\partial L}{\partial \dot{q}_j} \cdot \frac{\partial \dot{q}_j}{\partial p_i}}_{\times} = \dot{q}_i$$

$$\Rightarrow \frac{\partial H}{\partial p_i} = \dot{q}_i$$

$$\begin{cases} \dfrac{\partial H}{\partial q_i} = -\dot{p}_i \\ \dfrac{\partial H}{\partial p_i} = \dot{q}_i \end{cases}$$

$$\mathcal{L} = \frac{1}{2} \sum m \dot{q}_i^2 - U(q_i)$$

$$H = \sum \frac{p_i^2}{2m_i} + U(q_i) = E$$

Suppose we have some $A(p,q)$. Want to find $\dot{A}$
Observable

$$\dot{A} = \frac{\partial A}{\partial p_i}\dot{p}_i + \frac{\partial A}{\partial q_i}\dot{q}_i = -\frac{\partial A}{\partial p_i}\frac{\partial H}{\partial q_i} + \frac{\partial A}{\partial q_i}\frac{\partial H}{\partial p_i}$$

Poisson Bracket

$$\frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q^i}$$

$$[A, B]_{PB} = \frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i} - \frac{\partial B}{\partial q_i}\frac{\partial A}{\partial p_i}$$

$$\dot{A} = [A, H]_{PB}$$

$$\dot{q} = [q, H]_{PB} = \frac{\partial H}{\partial p}$$

$$\dot{p} = -\frac{\partial H}{\partial q}$$

Properties of PB's

$$\underset{\underset{[AB,H]}{\parallel}}{\frac{d}{dt}(AB)} = \dot{A}B + A\dot{B} = [A,H]B + A[B,H]$$

$$[AB, C] = [A,C]B + A[B,C]$$

$$[A,B] = -[B,A]$$

$$\underset{\underset{[[A,B],H]}{\parallel}}{\frac{d}{dt}[A,B]} = [\dot{A},B] + [A,\dot{B}]$$

$$= [[A,H],B] + [A,[B,H]]$$

$$[[A,B],C] + [[C,A]B] + [[B,C],A] = 0$$

Noether's Theory revisited

$$0 = \dot{Q} = [Q, H] = \frac{\partial Q}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial Q}{\partial p_i}\frac{\partial H}{\partial q_i}$$

$$\Rightarrow \exists \text{ symetry } \delta p_i = \alpha \frac{\partial Q}{\partial q_i}, \quad \delta q_i = -\alpha \frac{\partial Q}{\partial p_i}$$

$$\Rightarrow \delta H = [Q, H] \qquad = \alpha [Q, p_i] \qquad = \alpha [Q, q_i]$$

$$\dot{Q} = 0 \Leftrightarrow [Q, H] = 0 \Leftrightarrow \delta H = 0$$

$$\delta A = \alpha [Q, A] = \frac{\partial A}{\partial q_i}\delta q_i + \frac{\partial A}{\partial p_i}\delta p_i$$

Is this consitent?

since Q conserve

$$\delta \dot{q}_i \overset{?}{=} \frac{d}{dt}\delta q_i = [\delta q_i, H] = \alpha [[Q, q_i], H] = \alpha \left\{ \overbrace{[[Q,H], q_i]}^{0} + [Q, [q_i, H]] \right\}$$

$$\delta \dot{q}_i = \alpha [Q, [q_i, H]] \xleftarrow{=}$$

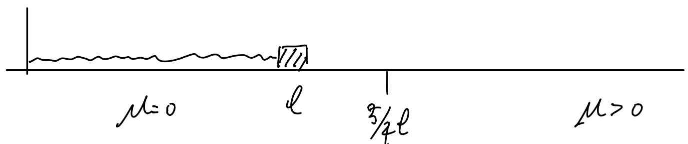
e.g. $Q = m(x\dot{y} - y\dot{x})$

$$= x p_y - y p_x$$

$$\delta x = \alpha [Q, x] = \left(\underbrace{\frac{\partial Q}{\partial x}}_{0} \frac{\partial x}{\partial p_x} - \frac{\partial Q}{\partial p_x} \overbrace{\frac{\partial x}{\partial x}}^{1} \right) \alpha = -\alpha y$$

$$\delta y = \alpha x$$

1. m, $k = m\omega^2$, l



$\beta = 0$, No "damping."

F_f depend on $\hat{v}$, $F_f = -\mu m g \frac{\dot{x}}{|\dot{x}|}$

$\mu = 0$ $\quad m\ddot{x} = -k(x-l) \Rightarrow \ddot{x} = -\omega^2(x-l)$

$\mu > 0$ $\quad m\ddot{x} = -k(x-l) - \mu m g \Rightarrow \ddot{x} = -\omega^2(x-l) - \mu g$

$$= -\omega^2(x-(l-\Delta)) \quad \Delta = \frac{\mu g}{\omega^2}$$

$\mu > 0$, $\dot{x} < 0$, $F_f = +\mu m g$

$m\ddot{x} = -k(x-l) + \mu m g$, $\ddot{x} = -\omega^2(x-(l+\Delta))$

b) $x_0 = \frac{3}{4}l$, $v_0 = 0$, $\quad x - l = -\frac{l}{4}\cos\omega t$

or $\quad x - l = A\cos\omega t$, $\quad \frac{3}{4}l - l = A$

$$x = l - \frac{l}{4}\cos\omega t$$

$$v = \dot{x} = \frac{l}{4}\omega\sin\omega t$$

max: $\cos = -1$, $t = \frac{\pi}{\omega}$

$$x = \frac{5}{4}l$$

c) $x_0 = \frac{l}{2}, \ v_0 = 0$

$$\frac{l}{2} \leq x_0 \leq \frac{5l}{4}$$

$$\frac{l}{2} - l = A \Rightarrow A = -\frac{l}{2}$$

$$x(t) = l - \frac{l}{2}\cos\omega t,$$
$$v(t) = \frac{l\omega}{2}\sin\omega t;$$

When do we reach friction

$$x(t) = \frac{5}{4}l$$
$$\Rightarrow \frac{5}{4}l = l - \frac{l}{2}\cos\omega t$$
$$\cos\omega t = -\frac{1}{2}, \quad \sin\omega t = \sqrt{1-\cos^2\omega t} = \frac{\sqrt{3}}{2}$$

$$x(t_1) = \frac{5}{4}l_1, \quad v(t_1) = \frac{\sqrt{3}}{4}l\omega$$
$$x(t) = l - \Delta + A_1\cos\omega t + B_1\sin\omega t$$
$$v(t) = -A_1\omega\sin\omega t + B_1\omega\cos\omega t$$

at t_1 $\quad \frac{5}{4}l = l - \Delta - \frac{A_1}{2} + \frac{\sqrt{3}}{2}B_1$

$$\frac{\sqrt{5}}{4}l\omega = -A_1\omega\frac{\sqrt{3}}{2} - \frac{B_1\omega}{2}$$

When does v change sign?

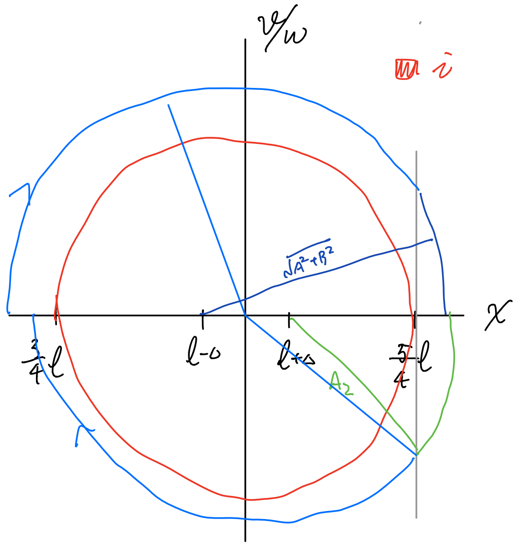
$$v(t_2) = 0$$
$$-\frac{V(t_2)}{\omega} = A_1\sin\omega t_2 + B_1\cos\omega t_2 = 0 \Rightarrow \tan\omega t_2 = +\frac{B_1}{A_1}$$
$$x(t_2) = l - \Delta + A_1\cos\omega t_2 + B_1\sin\omega t_2$$
$$x(t) = l\Delta + A_2\cos(\omega(t-t_2))$$
$$v(t) = -A_2\omega\sin(t-t_2)$$

$$A_2 + l + \Delta = l - \Delta + A_1 \cos\omega t_2 + B_1 \sin\omega t_2$$

$$= l - \Delta + \sqrt{A_1^2 + B_1^2}$$

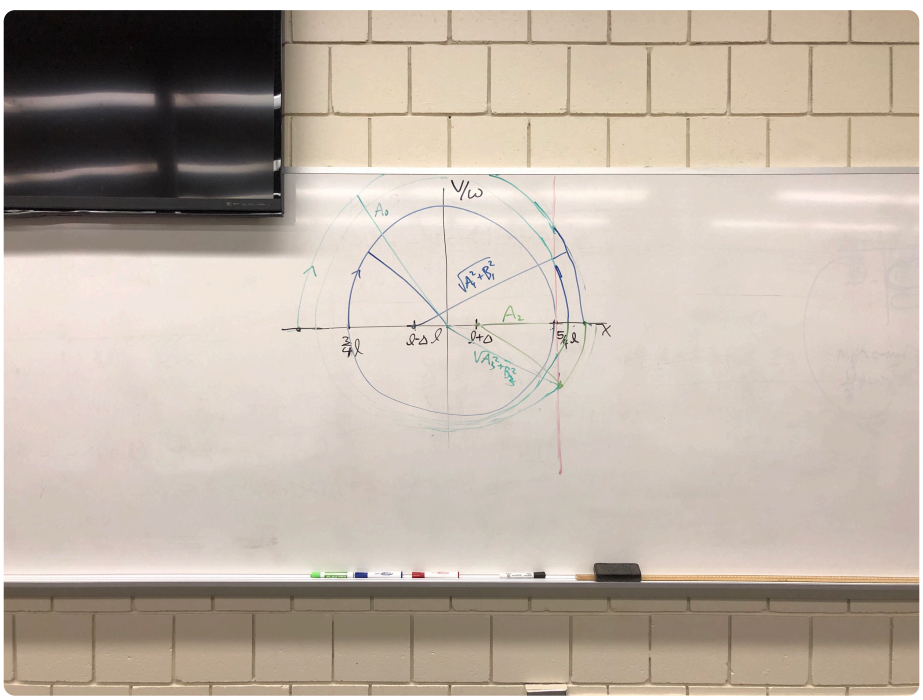
$$\Rightarrow A_2 = \sqrt{A_1^2 + B_1^2} - 2\Delta$$

$$\Gamma \frac{y}{x} \cdot \frac{y/x}{\sqrt{1 + y^2/x^2}} \quad \frac{1}{\sqrt{1^2 + y^2/x^2}}$$



$$\sin\omega t_2 = \frac{B_1}{\sqrt{A_1^2 + B_1^2}}$$

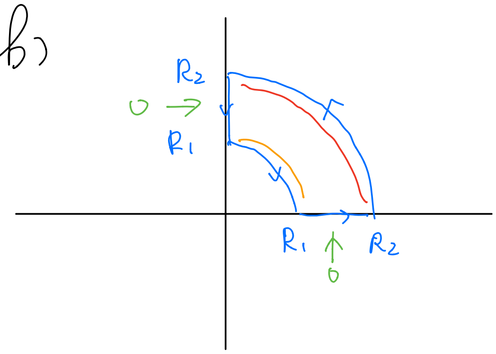
$$\cos\omega t_2 = \frac{A_1}{\sqrt{A_1^2 + B_1^2}}$$



4. $\vec{F} = R^{a}\hat{\phi}$, $W = \int \vec{F}\cdot d\vec{r}$, $\vec{r} = R\hat{R} + z\hat{z}$, $d\vec{r} = dR\hat{R} + Rd\phi\hat{\phi} + dz\hat{z}$

$$\vec{F}\cdot d\vec{r} = R^{a} R\, d\phi \quad , \quad R = R_0 \quad \int_0^{2\pi} R^{a+1} d\phi = 2\pi R_0^{a+1}$$

b)



$$\int_{\frac{\pi}{2}}^{0} R_1^{a+1} d\phi = -\frac{\pi}{2} R_1^{a+1}$$

$$\int_0^{\pi/2} R_2^{a+1} d\phi = \frac{\pi}{2} R_2^{a+1}$$

$$\Rightarrow \frac{\pi}{2}\left(R_2^{a+1} - R_1^{a+1}\right)$$

it is 0 when $a = -1$

$$\vec{\nabla}\times\vec{V} = \hat{R}\left(\frac{1}{r}\frac{\partial}{\partial\phi}V_z - \frac{\partial}{\partial z}V_\phi\right) + \hat{\phi}\left(\frac{1}{R}\frac{\partial}{\partial z}V_R - \frac{\partial}{\partial R}V_z\right) + \hat{z}\frac{1}{R}\left(\underbrace{\frac{\partial}{\partial R}(RV_\phi)}_{\frac{\partial}{\partial R}(R^{a+1})} - \frac{\partial}{\partial\phi}V_R\right)$$

$$= \hat{z}(a+1)R^{a-1}$$

$\uparrow$

0 when $a = -1$

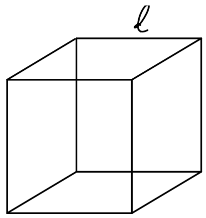
$(R \neq 0)$

for $\vec{F} = \frac{\hat{\phi}}{R}$, almost conservative

$\oint = \pi \neq$ winding around the z-axis.

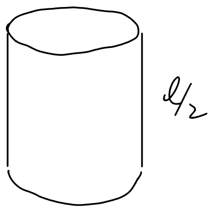
$$\frac{\hat{\phi}}{R} = \vec{\nabla}\phi$$

3

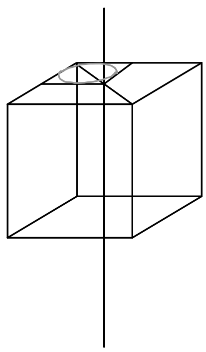


$\vec{r}_{cm} = 0$

$M = \rho \ell^3$



$M = -\rho h \pi r^2 = -\frac{\ell}{2}\pi\left(\frac{\ell}{6}\right)^2 \rho = -\rho\, \ell^3 \frac{\pi}{72}$



$\vec{r}_{cm} = \frac{\ell}{4}\hat{z} + \frac{\ell}{6}(\hat{x}+\hat{y})$

$$\vec{r}_{cm} = \frac{M_{cube}\,\vec{r}^{\,cube}_{cm} + M_{cyl}\,\vec{r}^{\,cyl}_{cm}}{M_{cube}+M_{cyl}}$$

$$= \frac{-\rho \ell^3 \frac{\pi}{72}\ell\left(\frac{1}{4}\hat{z}+\frac{1}{6}(\hat{x}+\hat{y})\right)}{\rho\ell^3 - \rho\ell^3\frac{\pi}{72}}$$

$$= \frac{-\pi\ell\left(\frac{1}{4}\hat{z}+\frac{1}{6}(\hat{x}+\hat{y})\right)}{72-\pi}$$

2 $F = f(v)$

$m\dot{v} = f(v) = -Av(1+B|v|)(1+C|v|)$

$= \frac{1}{Av(1+Bv)(1+Cv)}$

$m\int_{v_i}^{v}\frac{dv}{f(v)} = t$

$= \frac{1}{A}\left(\frac{\alpha}{v} + \frac{\beta}{1+Bv} + \frac{\gamma}{1+Cv}\right)$

$-\frac{m}{A}\alpha\ln\left(\frac{v}{v_0}\right) + \frac{\beta}{B}\ln\left(\frac{1+Bv}{1+Bv_0}\right) + \frac{\gamma}{C}\ln\left(\frac{1+Cv}{1+Cv_0}\right)$

1. a) $\vec{V} = z(\cos\phi\hat{R} + \sin\phi\hat{\phi}) + R\hat{z}$

$|v|^2 = \vec{V}\cdot\vec{V} = z^2\cos^2\phi + z^2\sin^2\phi + R^2$

$= z^2 + R^2$

b) $\dot{\vec{V}} = \dot{z}(\cos\phi\hat{R} + \sin\phi\hat{\phi}) + \dot{R}\hat{z}$

$= \dot{z}(\dot{\phi}(-\sin\phi)\hat{R} + \cos\phi\dot{\phi}\hat{\phi}$

$+ \dot{\phi}\cos\phi\hat{\phi} + \sin\phi(-\dot{\phi}\hat{R}))$

$= (\dot{z}\cos\phi - 2z\dot{\phi}\sin\phi)\hat{R} +$

$(\dot{z}\sin\phi + 2z\cos\phi\dot{\phi})\hat{\phi} + \dot{R}\hat{z}$

c) $\nabla\cdot V = \frac{1}{R}\frac{\partial}{\partial R}(R\underset{z\cos\phi}{V_R}) + \frac{1}{R}\frac{\partial}{\partial\phi}\underset{z\sin\phi}{V_\phi} + \frac{\partial}{\partial z}\underset{R}{V_z}$

$= \frac{z\cos\phi}{R} + \frac{z}{R}\cos\phi + 0$

$= 2\frac{z\cos\phi}{R}$

d) $\hat{R} = \hat{x}\cos\phi + \hat{y}\sin\phi$

$\hat{\phi} = \hat{y}\cos\phi - \hat{y}\sin\phi$

$\vec{V} = z(\cos\phi(\cos\phi\hat{x} + \sin\phi\hat{y}) + \sin\phi(\cos\hat{y} - \sin\phi\hat{x})$

$+ \sqrt{x^2+y^2}\hat{z}$

$= z[\underset{\frac{x^2-y^2}{x^2+y^2}}{(\cos^2\phi - \sin^2\phi)}\hat{x} + \underset{\frac{xy}{x^2+y^2}}{2\cos\phi\sin\phi}\hat{y}] + \sqrt{x^2+y^2}\hat{z}$

$= \frac{z}{x^2+y^2}((x^2-y^2)\hat{x} + 2xy\hat{y}) + \sqrt{x^2+y^2}\hat{z}$

$$\mathcal{L} = \frac{1}{2} m_1 \dot{\vec{r}}_1 \cdot \dot{\vec{r}}_1 + \frac{1}{2} m_2 \dot{\vec{r}}_2 \cdot \dot{\vec{r}}_2 - U(\vec{r}_1, \vec{r}_2)$$

No external force $\Rightarrow U(\vec{r}_2 - \vec{r}_1) \Leftrightarrow$ Symmetry : $\delta \vec{r}_1 = \vec{c}$, $\delta \vec{r}_2 = \vec{c} \Rightarrow \delta(\vec{r}_1 - \vec{r}_2) = 0$

$\boxed{?}$

$$r_{cm} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2}, \quad \vec{r} = \vec{r}_1 - \vec{r}_2, \quad m_1 + m_2 = M \quad \Leftrightarrow \quad r_1 = r_{cm} + \frac{m_2 \vec{r}}{M}, \quad r_2 = r_{cm} - \frac{m_1 \vec{r}}{M}$$

$$\Rightarrow \mathcal{L} = \frac{1}{2} m_1 \left(\dot{\vec{r}}_{cm} + \frac{m_2 \dot{\vec{r}}}{M}\right)^2 + \frac{1}{2} m_2 \left(\dot{\vec{r}}_{cm} - \frac{m_1 \dot{\vec{r}}}{M}\right)^2 - U(\vec{r})$$

$$= \frac{1}{2} M \dot{\vec{r}}_{cm} \cdot \dot{\vec{r}}_{cm} + \underbrace{\frac{m_1 m_2^2 + m_2 m_1}{2M^2}}_{\frac{m_1 m_2}{2M} =: \frac{1}{2}\mu \text{ reduced mass}} \dot{\vec{r}} \cdot \dot{\vec{r}} - U(r)$$

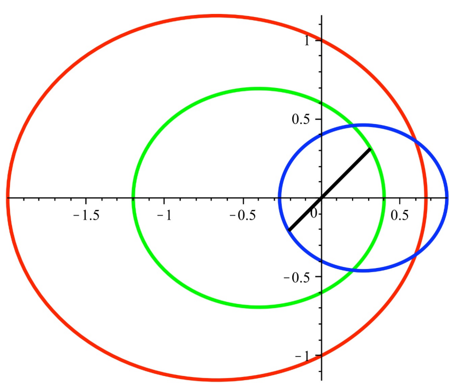
$$\mu = \frac{m_1 m_2}{M} = \frac{m_1 m_2}{m_1 + m_2} \Leftrightarrow \frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}$$

$$= \underbrace{\frac{1}{2} M \dot{\vec{r}}_{cm} \cdot \dot{\vec{r}}_{cm}}_{\mathcal{L}(\dot{\vec{r}}_{cm})} + \underbrace{\frac{1}{2}\mu \dot{\vec{r}} \cdot \dot{\vec{r}} - U(\vec{r})}_{\mathcal{L}_{red}(\vec{r}, \dot{\vec{r}})}$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\vec{r}}_{cm}} = 0 = \dot{\vec{P}}_{cm} \Rightarrow \vec{P}_{cm} = const = m \vec{v}_{cm}(0)$$

$$\left(\vec{P}_{cm} = \vec{P}_{tot}\right) \qquad \vec{r}_{cm} = r_{cm}(0) + \vec{v}_c t$$

When $m_1 >> m_2$, $M \approx m_1$



Central Force : $U(|\vec{r}_1 - \vec{r}_2|)$

New symmetry: Rotational symmetry $\Rightarrow$ conservation of angular momutum.

central potential

$$\Rightarrow \mathcal{L} = \underbrace{\frac{1}{2} M \dot{\vec{r}}_{cm} \cdot \dot{\vec{r}}_{cm}}_{\mathcal{L}(\dot{\vec{r}}_{cm})} + \underbrace{\frac{1}{2}\mu \dot{\vec{r}} \cdot \dot{\vec{r}} - U(|\vec{r}|)}_{\mathcal{L}_{red}(|\vec{r}|, \dot{\vec{r}})}$$

Symmetries: rotation of $\vec{r}_{cm}$ rotations of $\vec{r}$

$$\vec{L} = \vec{r} \times \vec{p}$$

centeral potential $\Rightarrow$ planar motion in c.o.m. frame.

Planar motion : Use polar coor.

$$\mathcal{L} = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) - U(r)$$

$(\vec{r} = r\hat{r} \,,\ \dot{\vec{r}} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta})$

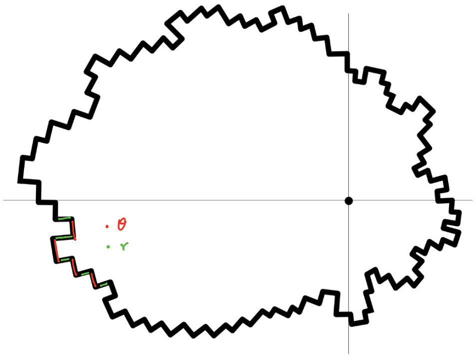
symmetry : $\delta\theta = \alpha$

E-L: $\frac{d}{dt}(\underbrace{\mu r^2 \dot{\theta}}) = 0$, $\mu \ddot{r} = \mu r \dot{\theta}^2 \overbrace{- \frac{\partial U}{\partial r}}^{F_r}$

L_z is conserved.

$$\mu r^2 \dot{\theta} = L_z \Rightarrow \dot{\theta} = \frac{L_z}{\mu r^2} , \ \dot{\theta}^2 = \frac{L_z^2}{\mu^4}$$

$$\mu \ddot{r} = \frac{L_z^2}{\mu r^3} - \frac{\partial U}{\partial r} = -\frac{\partial}{\partial r}\overbrace{\left(\frac{L_z^2}{2\mu r^2} + U\right)}^{U_{eff}}$$



$$\vec{F} = -\frac{\partial U}{\partial r}\hat{r}$$

no work for tangential segments ($\vec{F}\cdot\hat{\theta} = 0$)

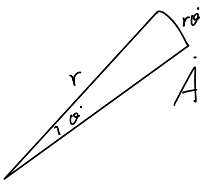
radial: work does not depend θ, only on Δr

$\Rightarrow$ total $\Delta r = 0$ $\Rightarrow$ no total work on r

$\Rightarrow$ central force is conservative.

No explicit time dependent

$\Rightarrow$ $H = T + U$ conserved.



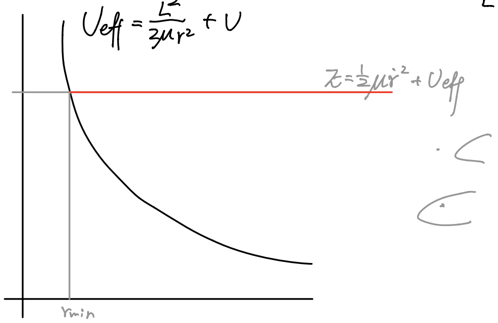
$\dot{A} = \frac{1}{2} r^2 \dot{\theta} = \frac{L_z}{2\mu}$ conserved

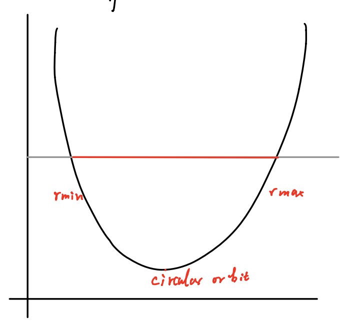
$$T = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2)$$

$$= \frac{1}{2}\mu\dot{r}^2 + \frac{L_z}{2\mu r^2}$$

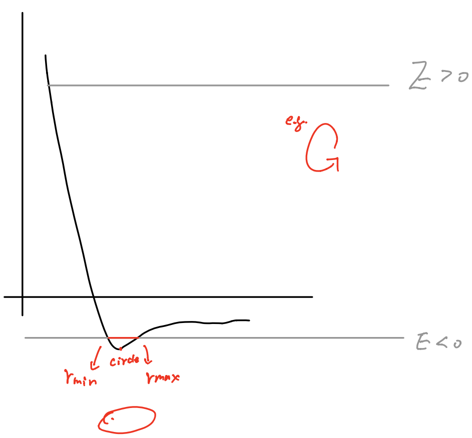
$$E = \frac{1}{2}\mu\dot{r}^2 + \underbrace{\frac{L_z^2}{2\mu r^2} + U(r)}_{U_{eff}(\vec{r})}$$

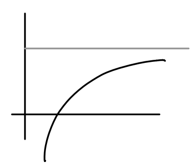
$U_{eff} = \frac{L^2}{2\mu r^2} + U$

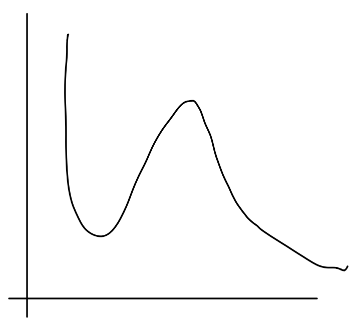




e.g. k ~







$$\vec{F}_{ij} = -\frac{G m_i m_j}{|r_{ij}|^2}\hat{r} = -\vec{F}_{ji} \qquad (r_{ij} = \vec{r_i} - \vec{r_j} \,, \ |r_{ij}|^2 = \vec{r}_{ij} \cdot \vec{r}_{ij}$$

Gravititational mass

$F = ma \leftarrow$ inertial mass

$T = \frac{1}{2} m\dot{x}^2 \Leftrightarrow \vec{E} \cdot \vec{E}$, $E = \vec{\nabla}\varphi - \vec{A}$ $(\vec{\nabla} \cdot \vec{E} = 0)$ $\frac{d}{dt}E = \vec{\nabla} \times \vec{B}$

$U = U(x) \Leftrightarrow \vec{B} \cdot \vec{B}$, $B = \vec{\nabla} \times \vec{A} \Rightarrow \vec{\nabla} \cdot \vec{B} = 0$

$L = T - U \Leftrightarrow E^2 - B^2$

$x \Leftrightarrow \vec{A}$

Principle of Equivalence Eötvös Experiment

$F_{12} = -G \frac{m_{1g} M_{2g}}{r_{12}^2} \hat{r}_{12}$

on earth $M_\otimes = m_g g \Rightarrow g = \frac{GM_\otimes}{R_\otimes^2}$

$\vec{F}_m = -m_g g = m_{int} \ddot{q}$ for pendulum : $\omega^2 = \frac{g}{\ell} \underbrace{\boxed{\frac{m_g}{m_{int}}}}$ ← const, chose to be 1

?

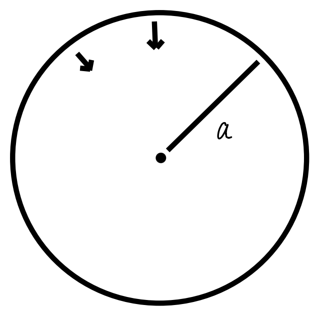
(Chose G to make it 1)

Two Particle

$\vec{F} = -\frac{GM\mu}{r^2} \hat{r}$ $(\overset{m_1+m_2}{M} \mu = m_1 m_2)$

$\vec{g} = -\frac{GM}{r^2} \hat{r}$

$\Phi_g = -\frac{GM}{a^2} 4\pi a^2 = -4\pi GM$



$$\mathcal{L} = \frac{1}{2}\mu \dot{\vec{r}}\cdot\dot{\vec{r}} + G\frac{M\mu}{r} = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{GM\mu}{r}$$

$$\vec{F} = -\frac{GM\mu}{r^2}\hat{r} = -\vec{\nabla}\left(-\frac{GM\mu}{r}\right)$$

$$\Rightarrow \delta\theta: \frac{d}{dt}(\mu r^2\dot{\theta}) = 0 \Rightarrow \mu r^2 \dot{\theta} = L_z \Rightarrow \dot{\theta} = \frac{L_z}{M r^2} \qquad \left(\int r^2(\theta)d\theta = \frac{L_z t}{\mu}\right)$$

$$\delta r: \frac{d}{dt}(\mu\dot{r}) = \mu r\dot{\theta}^2 - \frac{GM\mu}{r^2} = \frac{L_z^2}{\mu r^3} - \frac{GM\mu}{r^2} = -\frac{\partial}{\partial r}\overbrace{\left(\frac{L_z^2}{2\mu r^2} - \frac{GM\mu}{r}\right)}^{U_{eff}}$$

$$\ddot{r} = \frac{L_z^2}{\mu^2 r^3} - \frac{GM}{r^2} \qquad \left(\frac{L_z^2}{\mu^2} = \ell^2\right)$$

orbit is $r(\theta)$ not $r(t)$, $\theta(t)$

$$\dot{r} = \frac{dr}{d\theta}\frac{d\theta}{dt} = \frac{dr}{d\theta}\dot{\theta} = \frac{dr}{d\theta}\frac{\ell}{r^2} \overset{\text{chain}}{=} -\ell\frac{d}{d\theta}\frac{1}{r}$$

let $u(\theta) = \frac{1}{r(\theta)}$ $\Rightarrow \dot{r} = -\ell\frac{du}{d\theta}$ $\qquad \frac{du}{d\theta} = u'$

$$\ddot{r} = \frac{d}{dt}\dot{r} = \dot{\theta}\frac{d}{d\theta}(-\ell u') = \dot{\theta}(-\ell u'') = \ell u^2(-\ell u'') = -\ell^2 u^2 u''$$

$$\Rightarrow -\ell^2 u^2 u'' = \ell^2 u^3 - GMu^2$$

$$\Rightarrow \underbrace{u'' = -u}_{w=1} + \frac{GM}{\ell^2}$$

$$\Rightarrow u(\theta) = A\cos\theta + \frac{GM}{\ell^2}$$

$$\Rightarrow r(\theta) = \frac{1}{A\cos\theta + \frac{GM}{\ell^2}} = \frac{\alpha}{1+\varepsilon\cos\theta}, \quad \alpha = \frac{\ell^2}{GM} = \frac{L_z^2}{GM\mu^2}, \quad \varepsilon = \frac{A\ell^2}{GM}$$

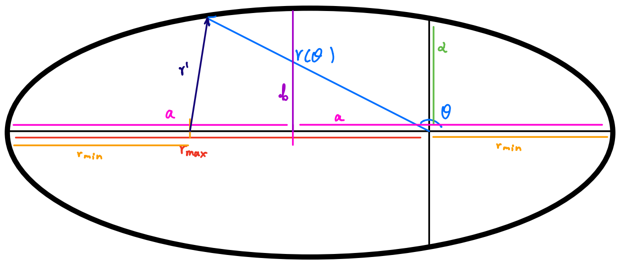
scale of orbit (α) ↓ eccentricity (ε) ↓

e.g. $\varepsilon = 0 \Rightarrow$ circle, $r = \text{const}$

$0 < \varepsilon < 1$ ellipse $\rightarrow r_{min} = r(0) = \frac{\alpha}{1+\varepsilon}$, $r_{max} = r(\pi) = \frac{\alpha}{1-\varepsilon}$, $r(\theta = \frac{\pi}{2}) = \alpha$

$\varepsilon = 1$ parabola

$\varepsilon > 1$ hyperbola



$$a = \frac{r_{max} + r_{min}}{2} = \text{semimajor axis}$$

$$= \frac{1}{2}\left(\frac{\alpha}{1+\varepsilon} + \frac{\alpha}{1-\varepsilon}\right) = \frac{\alpha}{1-\varepsilon^2}$$

$$r_{min} = a(1-\varepsilon), \quad r_{max} = a(1+\varepsilon)$$

$r + r' = \text{const}$ for ellipse

$$r' = \sqrt{r^2 + 4a^2\varepsilon^2 - 2\cdot 2a\varepsilon r(-\cos\theta)}$$

$$r(\theta) = \frac{\alpha}{1+\varepsilon\cos\theta} \Rightarrow \varepsilon\cos\theta = \frac{\alpha}{r} - 1 \Rightarrow r\varepsilon\cos\theta = \alpha - r = a(1-\varepsilon^2) -$$

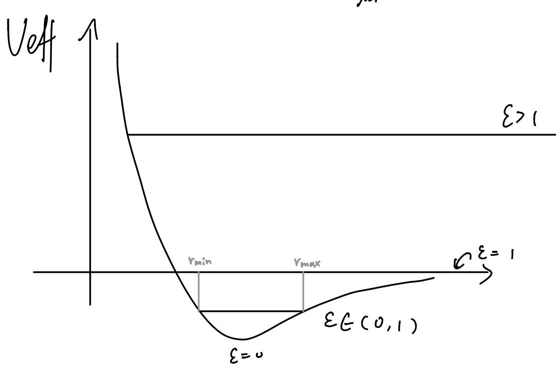
$$r' = \sqrt{r^2 + \cancel{4a^2\varepsilon^2} + 4a(a(1\cancel{-\varepsilon^2}) - r)}$$

$$= \sqrt{r^2 + 4a^2 - 4ar}$$

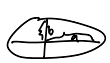
$$= \pm(r - 2a) \Rightarrow r' + r = 2a \Rightarrow \text{ellipse}$$

$$= 2a - r$$

$$\delta r: \ \frac{d}{dt}(\mu\dot{r}) = \mu r\dot{\theta}^2 - \frac{GM\mu}{r^2} = \frac{L_z^2}{\mu r^3} - \frac{GM\mu}{r^2} = -\frac{\partial}{\partial r}\overbrace{\left(\frac{L_z^2}{2\mu r^2} - \frac{GM\mu}{r}\right)}^{V_{eff}}$$



Kepler 3

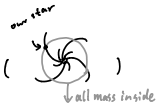
A for  $A_r = \pi a b$ $\left(\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1\right)$, $A = \int dy \int dx\, 1$

$$A = \int_0^T \dot{A}\,dt = \frac{L_z T}{2\mu} = \pi a b$$

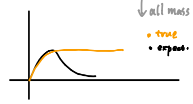
$$\dot{A} = \frac{L_z}{2\mu} \qquad \alpha = \frac{l^2}{GM} = \frac{L_z^2}{GM\mu^2}$$

$$T = \frac{2\mu\pi a b}{L_z} = 2\mu\pi a^{3/2}\frac{\sqrt{\alpha}}{L_z} = \frac{2\mu\pi a^{3/2}}{L_z}\frac{L_z}{\sqrt{GM}\,\mu} = \frac{2\pi a^{3/2}}{\sqrt{GM}}$$

$$T = \frac{2\pi\, a^{3/2}}{\sqrt{GM}} = \frac{2\pi a}{v(a)}$$

$$\Rightarrow M(a) = \frac{a v^2(a)}{G} = 4\pi \int_0^a \rho(r) r^2 dr$$ ()

$$\Rightarrow v(a) = \sqrt{\frac{GM(a)}{a}}$$



if $\rho = c r^{-\alpha}$

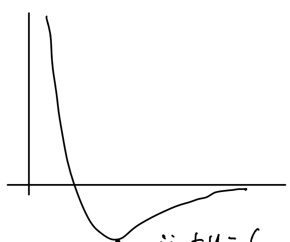
$r = \frac{\alpha}{1+\varepsilon\cos\varphi}$ for $E = \frac{1}{2}(\dot r^2 + r^2\dot\theta^2) - \frac{GMm}{r} = \frac{L_z^2}{2\mu r_{min}^2} - \frac{GM\mu}{r_{min}} = \frac{L^2(1+\varepsilon)^2}{2\mu\alpha^2} - \frac{GM\mu(1+\alpha)}{\alpha}$

$= \frac{G^2M^2\mu^3(\varepsilon^2-1)}{2L_z^2}$

$\mu r^2\dot\theta = L_z$; $\dot r = 0$ at r_{min}, r_{max}

$r_{min} = \frac{\alpha}{1+\varepsilon}$; $\alpha = \frac{L_z^2}{GM\mu^2}$ $\qquad \alpha = a(1-\varepsilon^2)$ $\qquad = -\frac{GM\mu}{2a}$

$$\Rightarrow \frac{G^2M^2\mu^3(\varepsilon^2-1)}{2L^2} \Rightarrow \varepsilon = \sqrt{1 + \frac{2L_z^2 E}{G^2M^2\mu^3}}$$



$\ddot u + u = C$

Circular orbit: $\dot r = \ddot r = 0$, $r = a$

$\Rightarrow F(a) = m f(a)$

$-L^2$

$\frac{1}{2}\mu(\dot r^2 + r^2\dot\theta^2) - U(r)$, $L_z = m r^2\dot\theta$

$-\frac{l^2\mu}{a^3} = f(a) \Leftarrow -\frac{L_z^2}{\mu a^2} = f(a)$

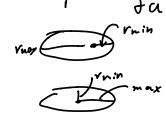
$r = a + \delta(t) \Rightarrow \ddot r = f(r) + \frac{L^2}{\mu r^3} \Leftrightarrow \frac{d^2u}{d\theta^2} + u = -\frac{1}{\mu l^2 u^2} f\left(\frac{1}{u}\right)$

$u = \frac{1}{r} = \frac{1}{a+\delta} = \frac{1}{a}\frac{1}{1+\delta/a} \approx \frac{1}{a}\left(1 - \frac{\delta}{a}\right)$

$= \frac{1}{a} - \frac{\delta}{a^2}$

$$\Rightarrow -\frac{1}{a^2}\frac{\partial^2\delta}{\partial\theta^2} + \cancel{\frac{1}{a}} - \frac{\delta}{a^2} = -\frac{1}{\mu l^2}\underbrace{(a+\delta)^2 f(a+\delta)}_{\cancel{a^2 f(a)} + \underbrace{2\delta f(a) + a^2\delta\frac{df}{da}}_{\delta a\left(2f + a\frac{df}{da}\right)}}$$

$$\Rightarrow \frac{\partial^2\delta}{\partial\theta^2} + \overbrace{\left(3 + \frac{a}{f(a)}\frac{\partial f}{\partial a}\right)}^{\omega_0^2}\delta = 0$$



as long as $\omega_0^2 > 0$, we get stable obart

ellipse $\Rightarrow \omega_0^2 = 1 \Rightarrow f \propto \frac{1}{r^2}$, $\omega_0^2 = 4 \Rightarrow f \propto r$

e.g. take $f = -c\,a^{n} \Rightarrow \frac{\partial f}{\partial a} = -c n a^{n-1}$

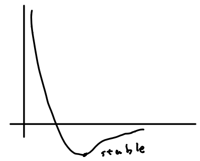
$$a f' = -c n a^{n}$$

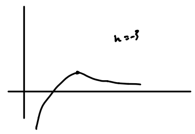
$$\Rightarrow \frac{a f'}{f} = n$$

$$\Rightarrow \omega_0^2 = 3 + n$$

stable for $n > -3$

for $\begin{cases} n = -2, & \omega_0^2 = 1 \\ = 1, & \omega_0^2 = 4 \end{cases}$





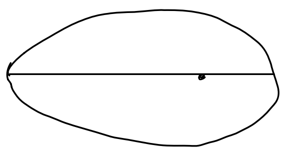
$U(r) = \frac{c}{n+1} r^{n+1} \quad (n \neq -1)$

$c \ln r \quad (n = -1)$

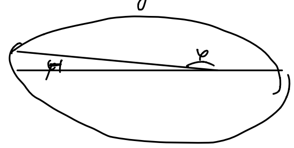
$n+1 = 2 \quad \frac{1}{2} k r^2$

$n+1 = -1 \quad -\frac{GMm}{r}$

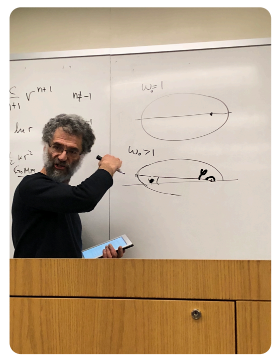
early



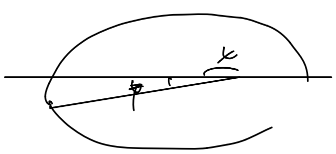
$\omega_0 = 1$

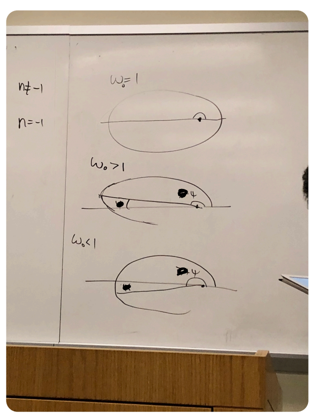


$\omega_0 > 1$



$\omega_0 < 1$





$$\omega_0^2 = 3 + \frac{1}{f(a)} a \frac{\partial f}{\partial a}$$

$$f(r) = -\frac{GM\mu}{r^2} + \beta r^n$$

$$r \frac{df}{dr} = \frac{2GM\mu}{r^2} + \beta n r^n$$

$$\frac{1}{f(a)} a \frac{df}{da} = \frac{\frac{2GM\mu}{a^2} + \beta n a^n}{-\frac{GM\mu}{a^2} + \beta a^n} \Rightarrow = \frac{-2 - \frac{a^2}{GM\mu}\beta n a^n}{1 - \frac{a^2}{GM\mu}\beta a^n}$$

$$\simeq -\left(2 + \frac{a^{n+1}\beta n}{GM\mu}\right)\left(1 + \frac{a^{n+2}}{GM\mu}\beta\right)$$

$$\approx -2 - \frac{(n+2)\beta a^{n+2}}{GM\mu} + O(\beta^2)$$

$$\omega_0^2 = 1 - \frac{(n+2)}{GM\mu}\beta a^{n+2}$$

$\psi < \pi$ when $n+2<0$

$\psi > \pi$ $\quad n+2>0$

$$\psi = \frac{\pi}{\omega_0} = \frac{\pi}{\sqrt{\omega_0^2}} \approx \frac{\pi}{\sqrt{1 - \cdots}}$$

$$\approx \pi\left(1 + \frac{1}{2}\frac{(n+2)\beta a^{n+2}}{GM\mu}\right)$$

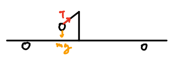
$$\vec{r}_0 = \vec{r} + \vec{R}$$

inertial, accelerated frame, position of origin

$$\vec{v}_0 = \vec{v} + \vec{V}, \quad \vec{a}_0 = \vec{a} + \vec{A}$$

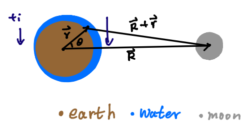
$$\vec{F}_0 = m\vec{a}_0 = m\vec{a} + m\vec{A}$$

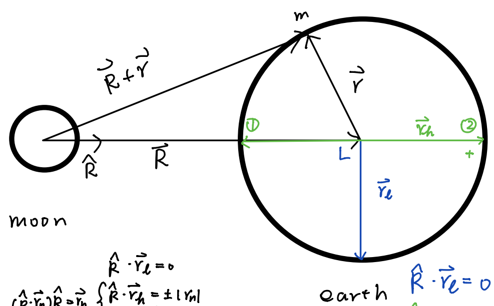
$$m\vec{a} = \vec{F} - m\vec{A} = \vec{F}_{eff}$$



$$\vec{g}_{eff} = \vec{g} - \vec{A}$$

$$\Omega = \sqrt{\frac{g_{eff}}{\ell}} = \sqrt{\frac{\sqrt{g^2 + A^2}}{\ell}}$$





$\hat{R}\cdot\vec{r}_l = 0$

$(\hat{R}\cdot\vec{r}_h)\hat{R} = \vec{r}_h \quad \begin{cases} \hat{R}\cdot\vec{r}_h = \pm|r_h| \\ \hat{R} = \pm\hat{r}_h \end{cases}$

$\hat{R}\cdot\vec{r}_l = 0$

$\hat{R}\cdot\vec{r}_h = \pm|r_h|$

① $-$ ② $+$

$$\vec{F}_{moon} = +\frac{GM_e M_\oplus}{R^2}\hat{R}$$

$$\Rightarrow \vec{F}_{earth} = \frac{-GM_m M_\oplus}{R^2}\hat{R}$$

$$\Rightarrow \vec{A}_\oplus = -\frac{GM_m\hat{R}}{R^2} = -\frac{GM_m\vec{R}}{R^3}$$

$$ma - F_{eff} = \vec{F}_0 - m\vec{A}_\oplus$$

$$= -mg\hat{r} - \frac{GM_m m(\vec{R}+\vec{r})}{|\vec{R}+\vec{r}|^3} \text{ (Force by moon)}$$

$F_{tidal}(\vec{R},\vec{r})$ $\quad -m\left(-\frac{GM_m\vec{R}}{R^3}\right)$ $(\vec{A}\,m)$

$$+\vec{F}_{other}$$

$$F_{tidal} = GM_m\, m\left(\frac{\vec{R}}{R^3} - \frac{\vec{R}+\vec{r}}{|\vec{R}+\vec{r}|^3}\right)$$

if $|\vec{R}| >> |\vec{r}|$

$$= \frac{GM_m m}{R^2}\left(\frac{\vec{R}}{|R|} - \frac{\frac{\vec{R}+\vec{r}}{|R|}}{\left|\frac{\vec{R}+\vec{r}}{|R|}\right|^3}\right)$$

$$= \frac{GM_m m}{R^2}\left(\hat{R} - \frac{\hat{R}+\frac{\vec{r}}{|R|}}{\left|\hat{R}+\frac{\vec{r}}{R}\right|^3}\right)$$

$$\frac{1}{\left|\hat{R}+\frac{\vec{r}}{|R|}\right|^3} = \left(\frac{1}{(\hat{R}+\frac{\vec{r}}{|R|})\cdot(\hat{R}+\frac{\vec{r}}{|R|})}\right)^{3/2}$$

$$\doteq \frac{1}{\left(1+2\frac{\hat{R}\cdot\vec{r}}{|R|}+\frac{|\vec{r}|^2}{|R|^2}\right)^{3/2}}$$

$$\doteq \frac{1}{1+\frac{3\hat{R}\cdot\vec{r}}{|R|}} + \mathcal{O}\left(\frac{|r|^2}{|R|^2}\right)$$

$$\doteq 1 - \frac{3\hat{R}\cdot\vec{r}}{|R|} + \mathcal{O}\left(\frac{|r|^2}{|R|^2}\right)$$

$\left[\ (1+x)^n = 1+nx+\mathcal{O}(x^2)\right.$

$\left.\ \hat{R}\cdot\vec{r} \leq |\vec{r}|\right]$

$$= \frac{GM_m m}{|\vec{R}|^2}\left(\hat{R} - \underbrace{\left(\hat{R}+\frac{\vec{r}}{|R|}\right)\left(1-\frac{3\hat{R}\cdot\vec{r}}{|R|}\right.}_{\mathcal{O}\left(\frac{|r|^2}{|R|^2}\right)} + \left.\mathcal{O}\left(\frac{|r|^2}{|R|^2}\right)\right)\right)$$

$$= \frac{GM_m m}{|R|^2}\left(\frac{3\hat{R}\,\hat{R}\cdot\vec{r}}{|R|} - \frac{\vec{r}}{|R|}\right)$$

$$= \frac{GM_m m}{|R|^3}\left(3\hat{R}(\hat{R}\cdot\vec{r}) - \vec{r}\right)$$

$$F_t(\vec{r},\vec{R}) = \frac{GM_m m}{|R|^3}\left(3\hat{R}(\hat{R}\cdot\vec{r}) - \vec{r}\right)$$

$$F_t(\vec{r}_l) = -\frac{GM_m m\,\vec{r}_l}{|R|^3}, \quad F_t(\vec{r}_h) \approx \frac{2GM_m m\,\vec{r}_h}{|\vec{R}|^3}$$

$$\Rightarrow \vec{F}_t = -\vec{\nabla}_{\vec{r}}\, U_{ti}$$

$$U_{ti} = -\frac{GM_m m}{2|\vec{R}|^3}\left(3(\hat{R}\cdot\vec{r})^2 - |\vec{r}|^2\right)$$

$$\frac{GM_\oplus}{|r_\oplus|^2}$$

$$\Delta U_t \equiv U(\vec{r}_h) - U(\vec{r}_\ell) = -mg(|r_h| - |r_\ell|)$$

$$= -\frac{GM_m m}{2|R|^3}\left(2|r_h|^2 + |r_\ell|^2\right)$$

$$\Delta U_{tot} = \Delta U_{ti} + mgh = 0 \qquad r_h \approx r_\oplus \approx r_\ell \quad (r_h + r_\ell \ll r_\oplus)$$

$$h = |r_h| - |r_\ell| \approx \frac{3M_m}{2|R|^3}\,\frac{|r_\oplus|^4}{M_\oplus} \approx .5\,m$$

$$\vec{r}_0 = r_{(0)i}\,\hat{e}_{i(0)} = r_i\,\hat{e}_i = \vec{r}$$

$$\hat{e}_i = \underset{\uparrow\,[\text{Rotation}]}{R_{ij}}\,\hat{e}_{(0)j} \;;\; r_i = R_{ij}\,r_{(0)i}$$

$$R^T = R^{-1}, \quad R_{ij}R^T_{jk} = R_{ij}R_{kj} = \delta_{ik}$$

$$\vec{v}_0 = \dot{\vec{r}} = \dot{r}_i\hat{e}_i + r_i\dot{\hat{e}}_i = \underbrace{\dot{r}_i\,\hat{e}_i}_{\vec{v}} + r_i\,\dot{R}_{ij}\,\underbrace{R_{kj}\,\hat{e}_k}_{\hat{e}_{(0)j}}$$

$$= \dot{\vec{r}}_0$$

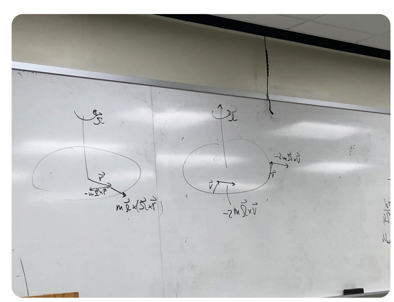
$$\vec{\Omega} = \Omega_m\,\hat{e}_m$$

points along axis of rotation, magnitude is how fast the rotating

$$\dot{R}_{ij}\,R_{kj} = \varepsilon_{ikm}\,\Omega_m$$

$$\Omega_m = \tfrac{1}{2}\,\varepsilon_{mij}\,\dot{R}_{ik}\,R_{jk}$$

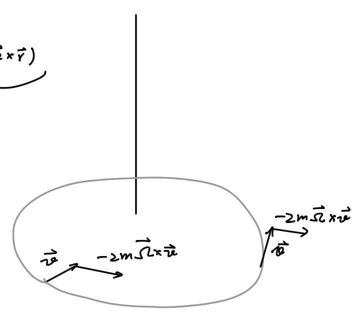
$$\varepsilon_{ijk} = \begin{cases} 1 & ijk \text{ even} \\ -1 & \text{odd} \\ 0 & 2 \text{ are eq.} \end{cases}$$

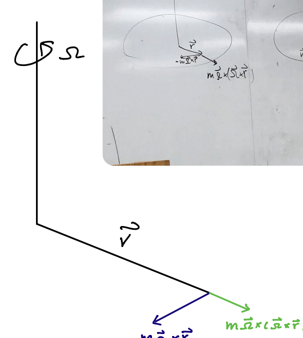


$$\vec{v}_0 = \vec{v} + \vec{\Omega}\times\vec{r} \qquad (\dot{\vec{\Omega}}_0 = \dot{\vec{\Omega}})$$

$$\vec{a}_0 = \dot{\vec{v}}_0 = \underset{\underset{\ddot{r}_i\hat{e}_i}{\parallel}}{\vec{a}} + \underbrace{\vec{\Omega}\times\vec{v} + \dot{\vec{\Omega}}\times\vec{r} + \vec{\Omega}\times(\vec{v} + \vec{\Omega}\times\vec{r})}_{\text{fictitious force}}$$

$$\Rightarrow \vec{a}_0 = \underset{\substack{\uparrow\\ \text{acceleration}\\ \text{in Rotation frame}}}{\vec{a}} + \underset{\substack{\uparrow\\ \text{Coriolis}\\ \text{accel.}}}{2\vec{\Omega}\times\vec{v}} + \underset{\substack{\uparrow\\ \text{transverse}\\ \text{accel.}}}{\dot{\vec{\Omega}}\times\vec{r}} + \underset{\substack{\uparrow\\ \text{Centripetal}\\ \text{accel}}}{\vec{\Omega}\times(\vec{\Omega}\times\vec{r})}$$

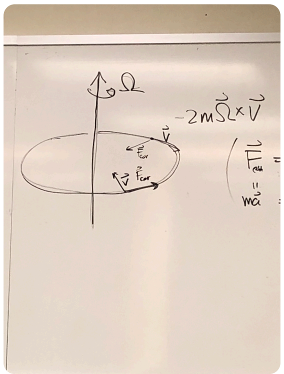




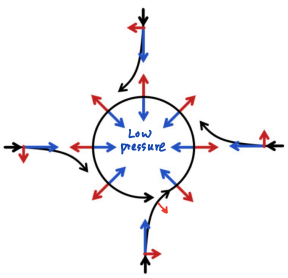
$$m\vec{a} = \vec{F}_{eff} = \vec{F}_0 - 2m\vec{\Omega}\times\vec{v} - m\dot{\Omega}\times\vec{r} - m\vec{\Omega}\times(\vec{\Omega}\times\vec{r})$$

$-2m\vec\Omega\times\vec v$

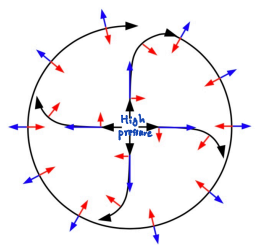
$$\underset{m\vec a}{\vec F_{eff}} = \underset{m\vec a_0}{\vec F_0} - \vec F_{fictions}$$



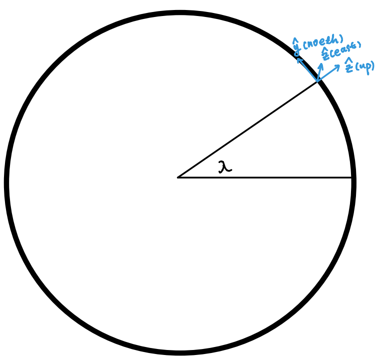
Coriolis force in northern hemisphere pushes projectiles clockwise



Hurricane



Anti-cyclone



$$\vec F_g = -mg\hat z$$

$$\vec\Omega = \Omega(\cos\lambda\,\hat y + \sin\lambda\,\hat z)$$

$$\vec F = \underbrace{\vec F_0}_{-mg\hat z} - m(\underbrace{\dot{\vec\Omega}\times\vec r}_{0} + 2\vec\Omega\times\vec v + \underbrace{\vec\Omega\times(\vec\Omega\times\vec r)}_{\text{inside } mg\hat z})$$

$$\Rightarrow m\vec a = -mg\hat z - 2m\vec\Omega\times\vec v$$

$$\vec a = -g\hat z - 2\vec\Omega\times\vec v$$

$$= -g\hat z - 2\vec\Omega\times(\dot x\hat x + \dot y\hat y + \dot z\hat z)$$

$$\vec a = -g\hat z - \Omega(\cos\lambda(\dot z\hat x - \dot x\hat z) + \sin\lambda(\dot x\hat y - \dot y\hat x))$$

$\Rightarrow \ddot x = -2\Omega(\cos\lambda\,\dot z - \sin\lambda\,\dot y)$ $\qquad \dot x - v_x(0) = -2\Omega(\cos\lambda(z-z_0) - \sin\lambda(y-y_0))$

$\ddot y = -2\Omega\sin\lambda\,\dot x$ $\qquad \dot y - v_y(0) = -2\Omega\sin\lambda(x-x_0)$

$\ddot z = -g + 2\Omega\cos\lambda\,\dot x$ $\qquad \dot z - v_z(0) = -gt + 2\Omega(\cos\lambda(x-x_0))$

Chose $r(0)=0$

$\dot{x} - v_x(0) = -2\Omega(\cos\lambda\ z - \sin\lambda\ y)$ $\qquad$ $\ddot{x} = -2\Omega\left(\cos\lambda\,(v_z(0) - gt + 2\Omega\cos\lambda\, x) - \sin\lambda\,(v_y(0) - 2\Omega\sin\lambda\, x)\right)$

$\dot{y} - v_y(0) = -2\Omega\sin\lambda\ x$

$\dot{z} - v_z(0) = -gt + 2\Omega\cos\lambda\ x$

$T_{earth} = 1$ day $\qquad \Omega T_{proj} = \frac{1\,\text{min}}{1\,\text{day}} \sim 10^{-3}$

$T_{projectile} \sim 1$ min

$\ddot{x} = \sim \Rightarrow \ddot{x} \doteq -2\Omega(\cos\lambda\,(v_z(0) - gt) - \sin\lambda\, v_y(0))$

$\dot{x} \approx \qquad t \quad t^2 \qquad t$

$$x(t) = v_x(0)t + \frac{1}{3}\Omega\cos\lambda\, g t^3 - \Omega(\cos\lambda\, v_z(0) - \sin\lambda\, v_y(0))t^2 + \mathcal{O}(\Omega^2)$$

$$y(t) = v_y(0)t - \Omega\sin\lambda\, v_x(0)t^2 + \mathcal{O}(\Omega^2)$$

$$z(t) = -\frac{1}{2}gt^2 + v_z(0)t + \Omega\cos\lambda\, v_x(0)t^2 + \mathcal{O}(\Omega^2)$$

e.g. $\vec{v}(0) = 0$, we let the object fall a hight h

$-h = \frac{1}{2}gt_h^2 \Rightarrow t_h = \sqrt{\frac{2h}{g}}$

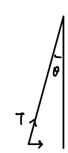
$y(t_h) = 0$, $x(t_h) = \frac{1}{2}\Omega\cos\lambda\, g\, t_h^3 = \frac{2}{3}\Omega\cos\lambda\, h\sqrt{\frac{2h}{g}} = \frac{2}{3}h\cos\lambda\,\Omega t_h$

We find, if we drop an object, it is displaced east.

Foucault Pendulum

$\vec{F} = -mg\hat{z} - 2m\vec{\Omega}\times\vec{v} + \vec{T}$

$T_z = mg$, $\vec{T} - mg\hat{z} \doteq \frac{T}{\ell}(x\hat{x} + y\hat{y}) \approx \frac{mg}{\ell}(x\hat{x} + y\hat{y})$



$\Rightarrow \begin{cases} \ddot{x} = -\frac{g}{\ell}x + 2\Omega\sin\lambda\,\dot{y} \\ \ddot{y} = -\frac{g}{\ell}y - 2\Omega\sin\lambda\,\dot{x} \end{cases}$

$\eta = x + iy \Rightarrow \ddot{\eta} = -\frac{g}{\ell}\eta - 2i\Omega_\lambda\dot{\eta}$

$\Omega_\lambda = \Omega\sin\lambda$ damped HO with $\beta = i\Omega_\lambda$, $\omega_0 = \sqrt{\frac{g}{\ell}}$, $\omega = \sqrt{\omega_0^2 + \underset{\text{small}}{\Omega_\lambda^2}} = \omega_0$

$$\eta(t) = e^{-i\Omega_\lambda t}\left(A_+(0)\,e^{i\omega_0 t} + A_-(0)\,e^{i\omega_0 t}\right)$$

Init Cond:

$y(0)=0 \quad \dot{y}(0)=0 \qquad \dot{x}(0)=0$

$\Rightarrow$

$\mathrm{Im}(A_+(0)+A_-(0)) = 0$

$-\Omega_\lambda(A_+(0)+A_-(0)) + \omega_0(A_+(0)-A_-(0)) = 0$

$A_+(0) + A_-(0) = x(0)$

$\Rightarrow A_\pm(0) = \frac{x_0}{2}\left(1 \pm \frac{\Omega_\lambda}{\omega_0}\right) \qquad \cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}, \quad i\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2}$

$$x(t)+y(t) = x_0 e^{-i\Omega_\lambda t}\Big(\cos\omega_0 t + \underbrace{i\frac{\Omega_\lambda}{\omega_0}\sin\omega_0 t}_{\text{Small}}\Big)$$

$\Rightarrow x(t) \simeq x_0 \cos\Omega_\lambda t \cos\omega t$

$y(t) \simeq -x_0 \sin\Omega_\lambda t \cos\omega_0 t$

plane of oscilation rotates with a angular frequency

$\Omega_\lambda = \Omega \sin\lambda$

$$\vec r_{cm} = \frac{\sum m_i \vec r_i}{\sum m_i}, \quad \sum m_i = M$$

$$\rho(\vec r) = |V(\vec r)|^{-1} \sum_{i \in V(\vec r)} m_i \qquad \sigma(\vec r) = |A(\vec r)|^{-1} \sum_{i \in A(\vec r)} m_i$$

$$\vec r_{cm} = \frac{\iiint \vec r \rho(\vec r)\, dV}{\iiint \rho(\vec r)\, dV} = \frac{1}{M} \iiint \vec r \rho(\vec r)\, dV \qquad r_{cm} = \frac{\iint \vec r \sigma(\vec r)\, dA}{\iint \sigma(\vec r)\, dA} = \frac{1}{M} \iint \vec r \sigma(\vec r)\, dA$$

Symmetry:

e.g. m_i is at $\vec r_i$. $i = 1 \cdots n$

$m_{i+n} = m_i$ at $-\vec r_i$

$$\Rightarrow \vec r_{cm} = \frac{\sum_1^{2n} m_i r_i}{\sum_1^{2n} m_i} = \frac{\sum_1^n m_i \vec r_i + \sum_1^n m_i (-\vec r_i)}{\sum m_i} = \vec 0$$

$$\vec v_{cm} = \dot{\vec r}_{cm} = \frac{\sum m_i \vec v_i}{M} = \frac{1}{M} \iiint \rho \vec v_i \, dV$$

$$\vec P_{tot} = \sum \vec p_i = \sum m_i \vec v_i = M \vec v_{cm} = \vec P_{cm}$$

$$\vec F_i = \dot{\vec p}_i, \quad \dot{\vec P}_{cm} = \sum \vec F_i, \quad \vec F_i = \underset{\substack{\uparrow \\ \text{external force}}}{\vec F_{ext\, i}} + \underset{\substack{\uparrow \\ \text{force due to } j \text{ on } i}}{\vec F_{ij}}, \quad \vec F_{ij} = -\vec F_{ji} \Rightarrow \dot{\vec P}_{cm} = \sum \vec F_{ext\, i}$$

$$\vec L = \vec r \times \vec p, \quad \vec L_{tot} = \sum \vec r_i \times \vec p_i, \quad \dot{\vec L}_{tot} = \sum \vec r_i \times \dot{\vec p}_i = \sum \vec r_i \times \vec F_i = \sum_i \vec r_i \times F_{ext\, i} + \sum_i \sum_j \vec r_i \times \vec F_{ij}$$

$$= \sum_i \vec r_i \times \vec F_{(ext)\, i} + \frac{1}{2} \sum_{i,j} \vec r_{ij} \times \vec F_{ij} \qquad (\vec r_{ij} = \vec r_i - \vec r_j)$$

for central force

$$\vec F_{ij} \times \vec r_{ij} = 0 \qquad \Rightarrow \dot{\vec L}_{tot} = \vec N_{(ext)} = \sum \vec r_i \times \vec F_{ext\, i}$$

$$\vec{r}'_i = \vec r_i - \vec r_{cm} \Rightarrow \vec v' = \vec v_i - \vec v_{cm} \Rightarrow \vec P'_{tot} = \vec P_{tot} - \vec P_{cm} = 0$$

$$\vec L_{tot} = \sum (\vec r'_i + \vec r_{cm}) \times (\vec p'_i + \vec P_{cm}) = \sum \vec r'_i \times \vec p'_i + \sum \vec r'_i \times P_{cM} + \vec r_{cm} \times \sum \vec p'_i + \vec r_{cm} \times \vec P_{cm}$$

$\sum m_i \dot{\vec r}_i$; 0

$$\sum m_i \vec r'_i = \underset{M r_{cm}}{\left(\sum m_i \vec r_i\right)} - \sum m_i \vec r_{cm}$$

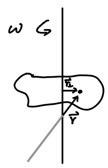
$$\vec L_{tot} = \sum_i (\vec r'_i + \vec r_{cm}) \times m_i (\dot{\vec r}'_i + \dot{\vec r}_{cm})$$

$$= \sum_i \vec r'_i \times m_i \dot{\vec r}'_i + \vec r_{cm} \times \sum m_i \dot{\vec r}_{cm} + \vec r_{cm} \times \underset{0}{\boxed{\sum m_i \dot{\vec r}'_i}} + \left(\underset{0}{\boxed{\sum \vec r'_i m_i}}\right) \times \dot{\vec r}_{cm}$$

$$= \sum_i \vec r'_i \times \vec p'_i + \vec r_{cm} \times P_{cm} = \vec L_{spin} + \vec L_{orbital}$$

$$T = \frac{1}{2} \sum m_i \dot{\vec r}_i \cdot \dot{\vec r}_i = \frac{1}{2} \sum m_i (\dot{\vec r}' + \dot{\vec r}_{cm}) \cdot (\dot{\vec r}' + \dot{\vec r}_{cm}) = \underset{\text{internal } T}{\frac{1}{2} \sum m_i \dot{\vec r}' \cdot \dot{\vec r}'} + \underset{\text{C.O.M. } T}{\frac{1}{2} M \dot{\vec r}_{cm} \cdot \dot{\vec r}_{cm}}$$

Rigid Body : $|\vec r_{ij}|$ is constant , $\vec r_{ij}\cdot\vec r_{kl}$ = constant .



$|\vec v_i| = |r_\perp|\,\omega = |\vec r\times\vec\omega|$ $\quad \vec v_i = \vec\omega\times\vec r \quad (\dot{\vec r}_0 = \dot r_i\hat e_i + \vec\omega\times\vec r)$

$T = \sum \frac12 m_i|v_i|^2 = \sum_i \frac12 m_i\,|r_{i\perp}|^2\omega^2 := \frac12 I_\omega\,\omega^2$, $I_\omega = \sum_i m_i(r_{i\perp})^2 = \iiint\rho(\vec r)\,|\vec r_\perp|^2\,dV$

$\vec L_{tot} = \sum m_i\vec r_i\times\dot{\vec r}_i = \sum m_i\vec r_i\times(\vec\omega\times\vec r_i) = \sum m_i(|\vec r_i|^2\vec\omega - (\vec\omega\cdot\vec r_i)\,\vec r_i)$

$\hookrightarrow \vec r\times(\vec\omega\times\vec r) = \hat e_a\,\varepsilon^{abc}r_b\,(\varepsilon_{cde}\,\omega^d r^e)$

$L_\omega = \hat\omega\cdot\vec L$

$= \omega\sum_i m_i(|\vec r_i|^2 - (\hat\omega\cdot\vec r_i)^2)$

$= \omega\sum_i m_i\,(r_{i\perp})^2$

$= I_\omega\,\omega$

$\vec v\times\vec w = \hat e_a\,\varepsilon^{abc}v_b w_c$, $\vec v = v_a\hat e_b$

$= \hat e_a(\delta^a_d\delta^b_e - \delta^a_e\delta^b_d)\,r_b\,r_e\,\omega^d$

$= \hat e_a\omega_a r_b r_b - \hat e_a r_a(r_b\omega_b) = \vec\omega\,r^2 - \vec r\,(\vec r\cdot\vec\omega)$

Linear motion		**Rotational motion**	
Position	x	Angle	θ
Velocity	$v = \dot x$	Angular velocity	$\omega = \dot\theta$
Mass	m	Moment of inertia	I
Linear Momentum	$p = mv$	Angular momentum	$L = I\omega$
Force	$F = m\dot v$	Torque	$N = I\dot\omega$
Kinetic energy	$T = \frac12 mv^2$	Kinetic energy	$T = \frac12 I\omega^2$

Table 1: One-dimensional linear motion compared to rotational motion about a fixed axis

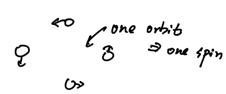
Parallel axis th: I about an axis parallel to an axis through the CM

$I = I_{cm} + m(r_{\perp cm})^2$

$L = I\omega = I_{cm}\underbrace{\omega_{spin} + m(r_{\perp cm})^2\omega_{orbit}}_{same}$

$\parallel$

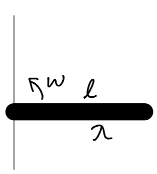
$L_{spin} + L_{orbit}$



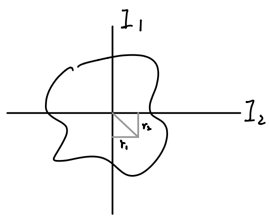
$\Rightarrow I_\omega = I_{cm}\,\omega + m(r_{\perp cm})^2\,\omega$

$m = \lambda l$, $I_0(l) = I_{cm} + m\left(\frac l2\right)^2 \quad I_{cm} = 2\,I_0\left(\frac l2,\frac m2\right) = \frac14 I_0(l,m)$

$I_0 = \frac14 I_0 + \frac14 ml^2$, $I_0 = \frac13 ml$



$\perp$ Axis : Lamina (thin flat thing)



$I_\perp = I_1 + I_2$

$L_a \hat{e}_a$

$\Downarrow$

$\vec{L} = \mathbb{I} \vec{\omega}$

$\Uparrow$

moment of inertia tensor

$L_a = \sum_b \mathbb{I}_{ab} \omega_b$

$v_a' = R_{ab} v_b \quad \mathbb{I}'_{ab} = R_{ac} R_{bd} \mathbb{I}_{cd} \quad , \quad \underset{\sim}{\mathbb{I}}' = R \underset{\sim}{\mathbb{I}} R^T$

$$I_w = \sum_i m_i (\vec{r}_{i\perp})^2$$ $i = 1 \cdots n$, # of particles $\delta_{ab} = \begin{cases} 1 & \text{if } a=b \\ 0 & \text{if } a \neq b \end{cases}$

$$L_w = I_w w$$ $a, b.. = 1 \cdots 3$, dim

$$\vec{L} = \underset{\sim}{\mathbb{I}} \vec{w} \quad (L_a = I_{ab} w_b)$$

$$I_{ab} = \sum_{i=1}^{n} m_i \left((r_i)^2 \delta_{ab} - (r_i)_a (r_i)_b \right) \qquad \cancel{\sum_i} \sum_b (r_i^2 \delta_{ab} - r_{ia} r_{ib}) r_{ib} = 0 \qquad \vec{r_i} = r_{ia} \hat{e}_a$$

$$\underset{\substack{\uparrow \\ \sum_b r_{ib}^2}}{r_i^2} r_{ia} - r_{ia} r_i^2 = 0$$

$$I_{ab} = \sum_i^n m_i \begin{pmatrix} r_i^2 - x_i^2 & -x_i y_i & -x_i z_i \\ -y_i x_i & r_i^2 - y_i^2 & -y_i z_i \\ -z_i x_i & -z_i y_i & r_i^2 - z_i^2 \end{pmatrix}$$

$$\hat{w} \cdot \vec{L} = \hat{w}_a I_{ab} w_b = |w| (\hat{w}_a I_{ab} \hat{w}_b) = w \sum_i \hat{w}_a I_{ab}^i \hat{w}_b$$

$$\vec{w} = w_1 \hat{r}_i + w_2 \hat{r}_{i\perp}$$

by rotating our basis, we can write

$$I'_{ab} = \begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \lambda_2 & \vdots \\ 0 & \cdots & \lambda_3 \end{pmatrix}$$ principle moments

$$I_{ab} = R_{ac} I'_{cd} R_{bd}$$

$$T = \underbrace{\frac{1}{2} M \vec{v}_{cm} \cdot \vec{v}_{cm}}_{\text{KE of COM}} + \underbrace{\frac{1}{2} \sum m_i \vec{v}_i \cdot \vec{v}_i}_{\text{spin KE}}$$

$$T_{spin} = \frac{1}{2} I_w w^2 = \frac{1}{2} \vec{w} \cdot \underset{\sim}{\mathbb{I}} \vec{w} = \frac{1}{2} w_a I_{ab} w_b = \frac{1}{2} (\lambda_1 w_1^2 + \lambda_2 w_2^2 + \lambda_3 w_3^2)$$

e.g. 4 partical mass m

$$\vec{r_1} = \hat{x}, \ \vec{r_2} = \hat{y}, \ \vec{r_3} = \hat{z}, \ \vec{r_4} = -\hat{x} - \hat{y} - \hat{z}$$

$$r_{cm} = 0$$

$$I_{ab} = \begin{pmatrix} 4 & -1 & -1 \\ -1 & 4 & -1 \\ -1 & -1 & 4 \end{pmatrix}$$

$$I \vec{v} = \lambda \vec{v} \qquad R_{ab} = \hat{e}_a \cdot \hat{r}_b = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \end{pmatrix}$$

$$\hat{e}_1 = \frac{1}{\sqrt{3}} (\hat{x} + \hat{y} + \hat{z}) \quad \lambda_1 = 2 \qquad R_{ab} I_{bc} \underset{\substack{\uparrow \\ R_{cd}^T}}{R_{dc}} = m \begin{pmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

$$\hat{e}_2 = \frac{1}{\sqrt{2}} (\hat{x} - \hat{y}) \quad \lambda_2 = 3$$

$$\hat{e}_3 = \frac{1}{\sqrt{6}} (\hat{x} + \hat{y} - 2\hat{z}) \quad \lambda_3 = 5$$

$$\frac{d\vec{v}}{dt} = \dot{\vec{v}} + \vec{w} \times \vec{v} \qquad \frac{d\vec{L}}{dt} = \vec{N} \Rightarrow \dot{\vec{L}} + \vec{w} \times \vec{L} = \vec{N}$$

$$\frac{d}{dt}(v_a \hat{e}_a) \quad \dot{v}_a \hat{e}_a \qquad \vec{L} = L_a \hat{e}_a$$

$$\dot{\hat{e}}_a = \vec{w} \times \hat{e}_a$$

take $\hat{e}_a$ along the principle axis

$\vec{L} = \mathbb{I}\vec{w} \quad L_a = \lambda_a w_a \not\Sigma_a$

$$\lambda_1 \dot{w}_1 + \underbrace{(w_2 L_3 - w_3 L_2)}_{(\lambda_3-\lambda_2) w_2 w_3} = N_1$$

$$\lambda_2 \dot{w}_2 + (\lambda_1 - \lambda_3) w_1 w_3 = N_2$$

$$\lambda_3 \dot{w}_3 + (\lambda_2 - \lambda_1) w_1 w_2 = N_3$$

When $\vec{N} = 0$, choose $\lambda_3 > \lambda_2 > \lambda_1$

$\lambda_3 - \lambda_2 > 0$, $\lambda_2 - \lambda_1 > 0$, But $\lambda_1 - \lambda_3 < 0$

3 simple solution for $N=0$, if 2 w's $= 0$ and 1 is constant

but non-zero $\dot{w}_a = 0$, and $w_a w_b = 0$ when $a \neq b$

Stability: e.g. constant part of w_1 is big, $w_2 + w_3$ are small.

$\lambda_1 \dot{w}_1 \approx 0 \quad \Rightarrow w_1$ still const

$$\lambda_2 \dot{w}_2 + (\lambda_1 - \lambda_3) w_1 w_3 \approx 0 \Rightarrow \lambda_2 \dot{w}_2 + \underbrace{\boxed{\frac{(\lambda_3 - \lambda_1) w_1^2 (\lambda_2 - \lambda_2)}{\lambda_3} w_2}}_{\lambda_2 w_{eff}^2} \approx 0 \Rightarrow \text{stable !}$$

$$\lambda_3 \dot{w}_2 + (\lambda_2 - \lambda_1) w_1 w_2 \approx 0$$

w_3 is big

$$w_{eff}^2 = \frac{(\lambda_2 - \lambda_3)(\lambda_1 - \lambda_3)}{\lambda_2 \lambda_1} > 0$$

w_2 is big

$$w_{eff}^2 = \frac{(\lambda_2 - \lambda_3)(\lambda_2 - \lambda_1)}{\lambda_1 \lambda_3} < 0$$

$$R(\psi,\theta,\phi) = R_z(\psi)\, R_y(\theta)\, R_z(\phi)$$

$$R_z(\psi) = \begin{pmatrix} c\psi & s\psi & 0 \\ -s\psi & c\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad R_y(\theta) = \begin{pmatrix} c\theta & 0 & -s\theta \\ 0 & 1 & 0 \\ s\theta & 0 & c\theta \end{pmatrix}$$

$$\hat{e}_a(\psi,\theta,\phi) = R_{ab}\, \hat{e}_{0a} \leftarrow \text{inertial basis vector.}$$

$\uparrow$ basis vector along principle axes

$$\dot{\hat{e}}_a = \vec{w} \times \hat{e}_a \Longleftrightarrow w_a = \frac{1}{2} \varepsilon_{abc} \underbrace{\dot{R}_{bd} R_{cd}}_{(\dot{R} R^T)_{bc}}$$

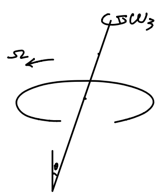
$$\dot{R} R^T = \underline{\dot{R}_1 R_1^T} + R_1 \underline{\dot{R}_2 R_2^T} R_1^T + R_1 R_2 \underline{\dot{R}_3 R_3^T} R_2^T R_1^T$$

$$\dot{R}_1 R^T = \dot{\psi} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_1 = \lambda_2$$

$$T_{spin} = \frac{1}{2}\left(\lambda_1 (\dot{\theta}^2 + (\dot{\phi}\sin\theta)^2) + \lambda_3 (\dot{\psi} + \dot{\phi}\cos\theta)^2\right)$$

$$= \frac{1}{2} \vec{w} \cdot \underset{\approx}{I}\, \vec{w}$$

$$L = T - U = \frac{1}{2}\lambda_1 (\dot{\theta}^2 + (\dot{\phi}\sin\theta)^2) + \frac{1}{2}\lambda_3 (\dot{\psi} + \dot{\phi}\cos\theta)^2 - mg\, R\cos\theta$$



奇雪李

Xue

雪

李

Li

Shan

Snow

qi

奇

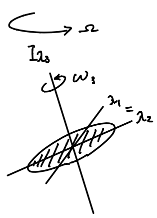
Odd

Stm

Miracle

Spinning top

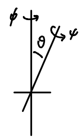
$\vec{L} = \underset{\sim}{I}\,\vec{\omega}$ along principal axes $\underset{\sim}{I} = \begin{pmatrix}\lambda_1&0&0\\0&\lambda_2&0\\0&0&\lambda_3\end{pmatrix}$, $\lambda_2=\lambda_3$ by sym.



$T = \frac{1}{2}\vec{\omega}\cdot\underset{\sim}{I}\,\vec{\omega}$, $\dot{\hat{e}}_a = \vec{\omega}\times\hat{e}_a$

$\mathcal{L} = T - U$

$= \frac{1}{2}\left(\lambda_1(\dot\theta + (\dot\phi\sin\theta)^2) + \lambda_3(\dot\psi + \dot\phi\cos\theta)^2\right) - MgR\cos\theta$



ψ : $\frac{d}{dt}\frac{\partial L}{\partial\dot\psi} = \frac{\partial L}{\partial\psi} = 0 \Rightarrow \lambda_3(\dot\psi + \dot\phi\cos\theta) = P_\psi$, const.

ϕ : $\frac{d}{dt}\frac{\partial L}{\partial\dot\phi} = 0 \quad \Rightarrow \lambda_1\dot\phi\sin^2\theta + \lambda_3\cos\theta\underbrace{(\dot\psi + \dot\phi\cos\theta)}_{P_\psi} = P_\phi$ const.

θ : $\frac{d}{dt}\frac{\partial L}{\partial\dot\theta} = \frac{\partial L}{\partial\theta}$

$\Rightarrow \lambda_1\ddot\theta = \lambda_1\dot\phi^2\cos\theta\sin\theta + \lambda_3(\dot\psi + \dot\phi\cos\theta)\dot\phi(-\sin\theta) + MgR\sin\theta$

$= (\lambda_1 - \lambda_3)\dot\phi^2\cos\theta\sin\theta - \lambda_3\dot\psi\dot\phi\sin\theta + MgR\sin\theta$

$\mathcal{L} = \frac{1}{2}me^{-\lambda t}\dot x^2$

$\frac{d}{dt}(me^{-\lambda t}\dot x) = 0$

$me^{-\lambda t}\ddot x - m\lambda e^{\lambda t}\dot x = 0$

$\ddot x - \lambda\dot x = 0$

Steady precession : $\dot\theta = \ddot\theta = 0$

$\dot\psi$, $\dot\phi$ are const. $\Omega = \dot\phi$, $\boxed{?}$ $\omega_3 = \frac{P_\psi}{\lambda_3} = \dot\psi + \Omega\cos\theta$

$0 = \lambda_1\Omega^2\cos\theta + \lambda_3\Omega\omega_3 + MgR$

$$\Omega_\pm = \frac{\lambda_3\omega_3 \pm \sqrt{\lambda_3\omega_3^2 - 4MgR\lambda_1\cos\theta}}{2\lambda_1\cos\theta} = \frac{\lambda\omega_3}{2\lambda_1\cos\theta}\left(1 \pm \sqrt{1 - \frac{4MgR\lambda_1\cos\theta}{\lambda_3^2\omega_3^2}}\right) \approx \begin{cases} \frac{\lambda_3\omega_3}{2\lambda_1\cos\theta} \\ \frac{\lambda_3\omega_3}{2\lambda_1\cos\theta}\left(\frac{2MgR\lambda_1\cos\theta}{\lambda_3^2\omega_3^2}\right) = \frac{MgR}{\lambda_3\omega_3} \end{cases}$$

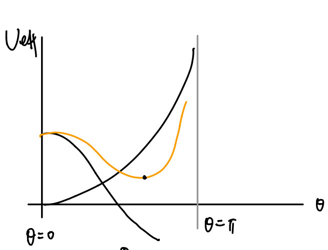
$\mathcal{H} = P_\theta\dot\theta + P_\psi\dot\psi + P_\phi\dot\phi - \mathcal{L}$ in θ, ψ, ϕ and P_θ, P_ψ, P_ϕ

$\mathcal{H} = \frac{P_\theta^2}{2\lambda_1} + \frac{P_\psi^2}{2\lambda_3} + \frac{(P_\phi - P_\psi\cos\theta)^2}{2\lambda_1\sin^2\theta} + MgR\cos\theta$

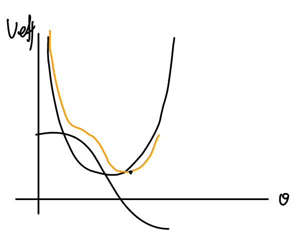
$U_{eff} = \frac{(P_\phi - P_\psi\cos\theta)^2}{2\lambda_1\sin^2\theta} + MgR\cos\theta$

When $P_\phi = P_\psi$ $\quad \frac{P_\phi^2(1-\cos\theta)^2}{2\lambda_1\sin^2\theta} = \frac{P_\phi(1-\cos\theta)^2}{2\lambda(1-\cos^2\theta)} = \frac{P_\phi^2}{2\lambda_1}\frac{1-\cos\theta}{1+\cos\theta} < \infty$ except $\theta = \pi$

$P_\phi \neq P_\psi$ $\quad \frac{(P_\phi - P_\psi\cos\theta)^2}{2\lambda_1\sin^2\theta} \to \infty$ at $\theta = 0, \pi$



$P_\phi = P_\psi$

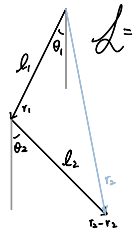


$\neq$

$$\dot{\phi} = \frac{1}{\lambda_3 \sin^2\theta}(P_\phi - \cos\phi \, P_\psi) \quad P_\phi > P_\psi \text{ or } P_\phi < P_\psi$$

Double pendulum

$$\frac{1}{2} m_{ij} \dot{q}^i \dot{q}^j + k_{ij} q^i q^j$$



$$\mathcal{L} = \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2) - m_1 g y_1 - m_2 g y_2 + \lambda_1 (x_1^2 + y_1^2 - l_1^2) + \lambda_2 ((x_2 - x_1)^2 + (y_2 - y_1)^2 - l_2^2)$$

$$\vec{r}_1 = l_1 \hat{r}(\theta_1) = l_1 (\hat{x} \cos\theta_1 - \hat{y} \sin\theta_1)$$

$$\vec{r}_2 - \vec{r}_1 = l_2 (\hat{x} \cos\theta_2 - \hat{y} \sin\theta_2)$$

$$\dot{\vec{r}}_1 = -l_1 \dot{\theta}_1 (\hat{x} \sin\theta_1 + \hat{y} \cos\theta_1) \qquad \dot{\vec{r}}_2 = \dot{\vec{r}} - l_2 \dot{\theta}_2 (\hat{x} \sin\theta_2 + \hat{y} \cos\theta_2)$$

$$\mathcal{L} = \frac{1}{2} m_1 l_1^2 \dot{\theta}_1 + \frac{1}{2} m_2 l_2^2 \dot{\theta}_2 + \frac{1}{2} m_2 l_1^2 \dot{\theta}_1 + m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 (\sin\theta_1 \sin\theta_2 + \cos\theta_1 \cos\theta_2) - m_1 g l_1 (1 - \cos\theta_1) + m_2 g (l_1 (1 - \cos\theta_1) + l_2 (1 - \cos\theta_2))$$

Small angle

$$= \frac{1}{2} m_1 l_1^2 \dot{\theta}_1 + \frac{1}{2} m_2 l_2^2 \dot{\theta}_2^2 + \frac{1}{2} m_2 l_1^2 \dot{\theta}_1^2 + m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 - \frac{m_1}{2} g l_1 \theta_1^2 - \frac{m_2}{2} g (l_1 \theta_1^2 + l_2 \theta_2^2)$$

$$z = f(x, y)$$

$$\frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 + \left(\frac{\partial f}{\partial x} \dot{x} + \frac{\partial f}{\partial y} \dot{y}\right)^2\right)$$

$-mg f(x, y) \rightarrow$ if equilibrium is at $x = y = 0$

$$f(x, y) = \frac{1}{2} x^2 \frac{\partial^2 f}{\partial x^2}\bigg|_{x, y = 0} + \frac{1}{2} y^2 \frac{\partial^2 f}{\partial y^2}\bigg|_{x, y = 0} + xy \frac{\partial^2 f}{\partial x \partial y}\bigg|_{x, y = 0} + \cdots$$

$$m_{ij} = m \begin{pmatrix} 1 + (\frac{\partial f}{\partial x})^2 & \frac{\partial f}{\partial x} \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial x} \frac{\partial f}{\partial y} & 1 + (\frac{\partial f}{\partial y})^2 \end{pmatrix}$$

$$k_{ij} = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}$$

$$m_{ij} = \begin{pmatrix} m_1 l_1^2 + m_2 l_1^2 & m_2 l_1 l_2 \\ m_2 l_1 l_2 & m_2 l_2^2 \end{pmatrix}$$

$$m_{ij} \ddot{q}^j + k_{ij} q^j = 0$$

$$= \ddot{q}^i + (m^{-1} k)^{ij} q^i = 0$$

10. $\mathcal{L} = \frac{1}{2}\mu(\dot{R}^2 + R^2\dot{\theta}^2) + \frac{GM\mu}{R}$, $\mu \approx m$

$$\mathcal{H}(p,q) = p_i \dot{q}^i - \mathcal{L}(q_i, \dot{q}_i)$$

$$p_i = \frac{\partial L}{\partial \dot{q}^i} \Rightarrow \dot{q}^j(p,q)$$

$$\{q^i\} = \{\theta, R\}$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = \mu R^2 \dot{\theta} \Rightarrow \dot{\theta} = \frac{p_\theta}{\mu R^2}$$

$$p_R = \frac{\partial L}{\partial \dot{R}} = \mu \dot{R} \Rightarrow \dot{R} = \frac{p_R}{\mu}$$

$$H = p_\theta \dot{\theta} + p_R \dot{R} - \mathcal{L}$$

$$= \frac{p_\theta^2}{\mu R^2} + \frac{p_R^2}{\mu} - \frac{1}{2}\mu\left[\left(\frac{p_R}{\mu}\right)^2 + R^2\left(\frac{p_\theta}{\mu R^2}\right)^2\right] - \frac{GM\mu}{R}$$

$$= \frac{p_\theta^2}{2\mu R^2} + \frac{p_R^2}{2\mu} - \frac{GM\mu}{R}$$

$$U_{eff} = \frac{\mathcal{L}_z^2}{2\mu R^2} - \frac{GM\mu}{R}$$

$$R_{eq} = R_{circ} = \frac{\partial U_{eff}}{\partial R} = 0 \qquad (R^{n\prime} = nR^{n-1})$$

$$\Rightarrow -\frac{\mathcal{L}_z^2}{\mu R^3} + \frac{GM\mu}{R^2} = 0 \quad , R \neq 0$$

$$R = \frac{\mathcal{L}_z^2}{\mu^2 GM}$$

for $R > R_{DM} \Rightarrow M \to \frac{4}{3}\pi\rho R_{DM}^3 + M$

$R < R_{DM} \Rightarrow M \to \frac{4}{3}\pi\rho R^3 + M \qquad U_{eff} \to \frac{\mathcal{L}_z^2}{2\mu R^2} - \frac{GM\mu}{R} - \frac{4}{3}\pi\rho G\mu R^2$

$$-\frac{\mathcal{L}_z^2}{\mu R^3} + \frac{GM\mu}{R^2} \underbrace{- \frac{8}{3}\pi\rho G\mu R}_{\delta} \Rightarrow R_{eq} = \frac{L_z^2}{\mu^2 GM} + \mathcal{O}(\delta)$$

$$-\frac{L_z^2}{\mu^2 GM} + R_{eq} - \frac{\delta}{GM\mu}R_{eq}^4 = 0 \quad , R_{eq} = \frac{L_z^2}{\mu^2 GM} + \frac{\delta}{GM\mu}\left(\frac{L_z^2}{\mu^2 GM}\right)^4 + \mathcal{O}(\delta^2)$$

$$\omega^2 = \frac{1}{\mu}\left.\frac{\partial^2 U_{eff}}{\partial R^2}\right|_{R_{eq}} \qquad \Big(\mu\ddot{R} + \frac{\partial U_{eff}}{\partial R} = 0 , \quad R = R\delta \Rightarrow \mu\ddot{\delta} + \underbrace{\left.\frac{\partial^2 U_{eff}}{\partial R^2}\right|_{R_{eq}}}_{\mu\omega^2} \delta = 0 \Big) \qquad (\ddot{q} + \omega^2 q = 0)$$

$$\Rightarrow +\frac{3L_z}{\mu^2 R^4} - \frac{2GM}{R^3} - \left.\frac{\delta}{\mu}\right|_{R_{eq}} \to \frac{3L_z^2}{\mu^2}\left(\frac{1}{\frac{L_z^2}{\mu GM} + \delta\alpha}\right)^4 - 2GM\left(\frac{1}{\frac{L_z^2}{\mu GM} + \delta\alpha}\right)^3 - \delta_\mu \ , \quad \alpha = \frac{L_z^8}{(GM\mu)^5\mu^4} \ , \Rightarrow \left(\frac{1}{R_0(1+\frac{\delta\alpha}{R_0})}\right)^4 \approx \frac{1}{R_0^4}\left(1 - 4\frac{\delta\alpha}{R_0}\right)$$

$$\omega^2 = \frac{\mu^6 (GM)^4}{L_z^6} + \frac{7\delta}{\mu} \quad (\ddot{R} + \omega^2 R = 0 \text{ , real want } R'' + \omega_\theta^2 R = 0)$$

$$\dot{\theta} \frac{\partial}{\partial \theta} = \frac{\partial}{\partial t} \quad , \quad \ddot{R}$$

$$\overset{\shortparallel}{\frac{L_z}{\mu R_{eq}^2}} \quad \omega_\theta = \frac{1}{\dot{\theta}} \omega \quad \frac{\partial}{\partial t} \cos \omega t = -\omega \sin \omega t = \dot{\theta} \frac{\partial}{\partial \theta} \cos \omega_\theta \theta(t)$$

$$\Rightarrow \omega_\theta^2 = 1 + \frac{3 L_z^6 \delta}{\mu^7 (GM)^4}$$

9. $\ddot{x} + 2\beta \dot{x} + \omega_0^2 x = \frac{F}{m} = \sin(2\omega_0 t) + \frac{1}{2}\cos(\frac{1}{2}\omega_0 t)$

$$\mathrm{Im}\, e^{2i\omega_0 t} \quad \mathrm{Re}\, \frac{1}{2} e^{\frac{1}{2} i \omega_0 t}$$

$$x(t) = \mathrm{Im} \frac{e^{2i\omega_0 t}}{\omega_0^2 - (2\omega_0)^2 + 4i\beta\omega_0} + \frac{1}{2} \mathrm{Re} \frac{e^{\frac{1}{2} i \omega_0 t}}{\omega_0^2 - (\frac{\omega_0}{2})^2 + i\beta\omega_0}$$

$$(\omega_F^2 = 2i\beta\omega_F)$$

$$= \mathrm{Im} \frac{e^{2i\omega_0 t}}{-3\omega_0^2 + 4i\beta\omega_0} + \frac{1}{2} \mathrm{Re} \frac{e^{\frac{1}{2} i \omega t}}{\frac{3}{4}\omega_0^2 + i\beta\omega_0}$$

$$x_p(0) = \mathrm{Im} \frac{1}{-3\omega_0^2 + 4i\beta\omega_0} + \frac{1}{2} \mathrm{Re} \frac{1}{\frac{3}{4}\omega_0^2 + i\beta\omega_0}$$

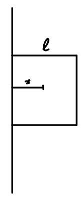
$$\dot{x}_p(0) = \mathrm{Im} \frac{2i\omega_0}{-3\omega_0^2 + 4i\beta\omega_0} + \frac{1}{2} \mathrm{Re} \frac{\frac{1}{2} i\omega_0}{\frac{3}{4}\omega_0^2 + i\beta\omega_0}$$

$$x_H(t) = e^{-\beta t} (A \cos \omega t + B \sin \omega t) \quad , \quad \omega = \sqrt{\omega_0^2 - \beta^2}$$

$$\dot{x}_H(t) = e^{-\beta t} \left((A(-\beta) + B\omega) \cos \omega t + (A(-\omega) - \beta B) \sin \omega t \right)$$

$$x = x_p + x_H \quad , \quad x(0) = \dot{x}(0) = 0$$

$$\Rightarrow x_p(0) = -A \quad , \quad \dot{x}_p(0) = B\omega - \beta A$$

8. 

$M_{sq} = \sigma l^2$, $dM = \sigma l dy$

$\underset{dm}{I} = \frac{1}{3}\sigma l dy l^2$, $I_{rod} = \frac{1}{3}\sigma l^4 = \frac{1}{3}M_{sq} l^2$ $\left(= l^2 \int_0^l \frac{1}{3}\sigma l dy \right.$

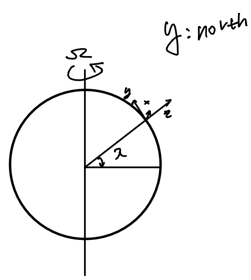
$\sum M r_i^2 \to \int_0^l \int_0^l dx dy \ \sigma x^2 = l\sigma \int_0^l x^2 dx = \frac{1}{3} l \sigma l^3 = \frac{1}{3}\sigma l^{4} = \frac{1}{3}Ml^2$

$I_{edge} = \frac{1}{3}Ml^2$, $I_{edge} = I_{cm} + (\frac{l}{2})^2 M$

$\frac{1}{3}Ml^2 = I_{cm} + \frac{1}{4}Ml^2$

$\Rightarrow I_{cm} = \frac{1}{12}Ml^2.$

$I_{\perp} = \frac{1}{6}Ml^2$

7 

$\frac{d}{dt}\hat{e}_i = \vec{\Omega}\times\hat{e}_i$

$\Rightarrow \vec{v}_0 = \vec{v} + \vec{\Omega}\times\vec{r}$ $\left(= \dot{r}_i\hat{e}_i + (\vec{\Omega}\times\hat{e}_i) r_i \right)$

$\vec{r} = \vec{r}_0 = r_i\hat{e}_i = r_{0i}\hat{e}_{i0}$

$\vec{v}_0 \equiv \dot{r}_{i(n)}\hat{e}_{i(0)} = \dot{r}_i\hat{e}_i + r_i\dot{\hat{e}}_i$

$\vec{a}_0 = \vec{a} + 2\vec{\Omega}\times\vec{v} + \underbrace{\dot{\vec{\Omega}}\times\vec{r}}_{\vec{\Omega}\text{ is const}} + \vec{\Omega}\times(\vec{\Omega}\times\vec{r}) = \frac{\vec{F}_0}{m}$

$\vec{F}_0 \underbrace{- 2m\vec{\Omega}\times\vec{v}}_{\text{cor}} \underbrace{- m\vec{\Omega}\times(\vec{\Omega}\times\vec{r})}_{\text{centf..}} = \vec{F}_{eff} = m\vec{a}$

$\vec{\Omega} = \left(\cos\lambda\,\hat{y} + \sin\lambda\,\hat{e}\right)\Omega$

$F_{cor}(\lambda) = -2m\Omega\left(-\cos\lambda\, v_x\hat{z} - \sin\lambda\, v_y\hat{x} + \sin\lambda\, v_x\hat{y} + \cos\lambda\, v_z\hat{x}\right)$

$\left(\vec{v} = \sum_i v_i\hat{i}\right)$

6.

$$\mathcal{L} = \frac{1}{2}m_1(\dot{\vec{r}}_1)^2 + \frac{1}{2}m_2(\dot{\vec{r}}_2)^2 - m_1 g z_1 - m_2 g z_2 - \frac{1}{2}k(\|\vec{r}_1-\vec{r}_2\| - \ell)^2$$

$$-\lambda_1(z_1 - f(x_1,y_1)) + \lambda_2(z_2 - f(x_2,y_2)) \qquad \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = -\frac{\partial}{\partial r}U\right)$$

$$\left(\frac{d}{dx}x^2 = 2x, \quad \frac{d}{dx}(-x)^2 = -2x(-1) = 2x\right)$$

$$f = 0 \Rightarrow \mathcal{L} = \frac{1}{2}m_1(\dot{\vec{r}}_1)^2 + \frac{1}{2}m_2(\dot{\vec{r}}_2)^2 - \frac{1}{2}k(\|\vec{r}_1-\vec{r}_2\| - \ell)^2$$

$$= \frac{1}{2}m_1(\dot{x}_1^2+\dot{y}_1^2) + \frac{1}{2}m_2(\dot{x}_2^2+\dot{y}_2^2) - k\left(((x_1-x_2)^2+(y_1-y_2)^2)^{\frac{1}{2}} - \ell\right)^2$$

$$= \frac{1}{2}M(\dot{x}_{cm}^2 + \dot{y}_{cm}^2) + \frac{1}{2}\mu(\dot{x}^2+\dot{y}^2) - \frac{1}{2}k(\sqrt{x^2+y^2} - \ell)^2$$

$$M\ddot{x}_{cm} = 0, \quad M\ddot{y}_{cm} = 0, \quad \rightarrow x_{cm}(t) = x_{cm}(0) + \dot{x}_{cm}(0)t, \quad y_{cm}(t) = y_{cm}(0) + \dot{y}_{cm}(0)t.$$

$$\frac{1}{2}\mu(\dot{x}^2+\dot{y}^2) - \frac{1}{2}k(\sqrt{x^2+y^2} - \ell)^2$$

$\delta x = \alpha y, \delta y = -\alpha x$ rotations in x-y plane

$$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} 1 & \alpha \\ -\alpha & 1 \end{pmatrix}$$

$$Q = \frac{\partial L}{\partial \dot{x}}R_x + \frac{\partial L}{\partial \dot{y}}R_y, \quad R_x = y, \quad R_y = -x.$$

$$= (\mu\dot{x}y - \dot{y}x)\mu$$

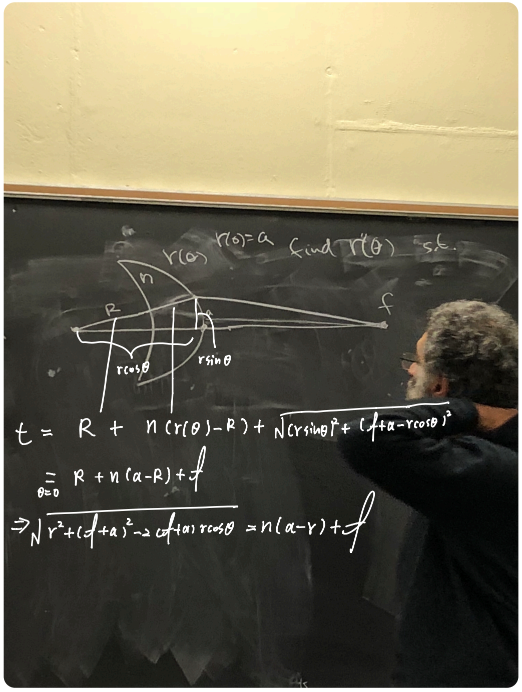
$$\frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{1}{2}k(r-\ell)^2, \quad L_z = \mu r^2\dot{\theta}, \quad \mu\ddot{r} = \mu r\dot{\theta}^2 - k(r-\ell) = \frac{L_z^2}{\mu r^3} - k(r-\ell),$$

$$\frac{L_z^2}{\mu^2} - \frac{k}{\mu}(r_{eq}^4 - r_{eq}^3\ell) = 0, \quad -\frac{3L_z^2}{\mu^2 r_{eq}^4} - \frac{k}{\mu} = -\omega^2$$

$$\ell = 0, \quad x = x_{max}(\cos(\omega t + \varphi_x)) \qquad \omega = \sqrt{\frac{k}{\mu}}$$

$$y = y_{max}(\cos(\omega t + \varphi_y))$$

5.



All same time

4.

$$\mathcal{L} = \frac{1}{2}\left(\dot R^2 + R^2 \overset{\omega^2}{\overset{\downarrow}{\dot\theta^2}} + \dot z^2\right) - mgz + \lambda(z - f(R)) \quad , \dot\theta = \omega$$

$\Rightarrow$ $m\ddot z = -mg + \lambda$

$m\ddot R = mR\omega^2 - \lambda \frac{\partial f}{\partial R}$

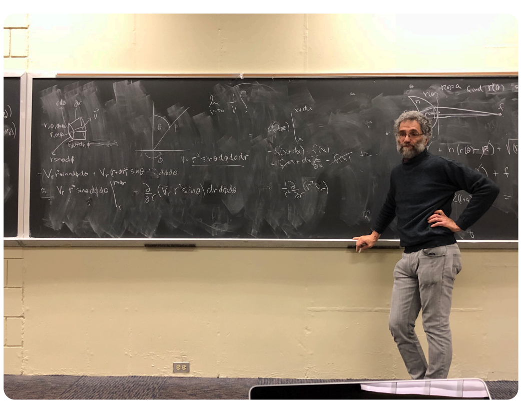
$z = f(R)$, $\dot z = \frac{\partial f}{\partial R}\dot R$

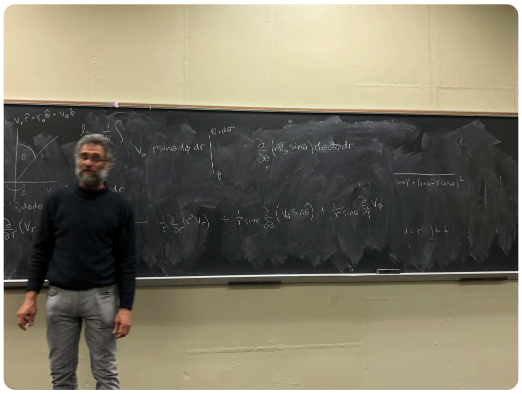
$\Rightarrow \frac{1}{2}m\left(1 + \left(\frac{\partial f}{\partial R}\right)^2\right)\dot R^2 - mg f(R) + \frac{1}{2}mR^2\omega^2$

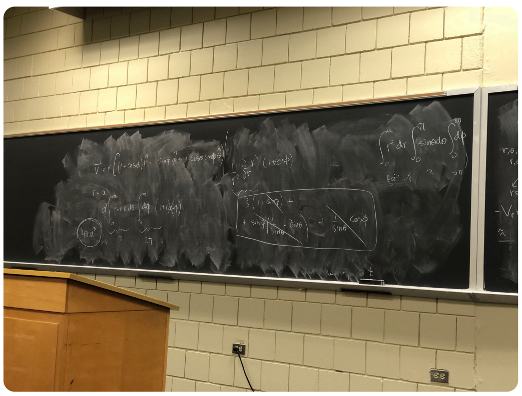
$$\frac{d}{dt}\left[m\left(1 + \left(\frac{\partial f}{\partial R}\right)^2\right)\dot R\right] = -mg\frac{\partial f}{\partial R} + m\dot R^2\left(\frac{\partial f}{\partial R}\right)\frac{\partial^2 f}{\partial R^2} + mR\omega^2$$

$0 = -mg\frac{\partial f}{\partial R} + mR\omega^2$, $\frac{\partial f}{\partial R} = \frac{\omega^2}{g}R \Rightarrow f = f_0 + \frac{\omega^2}{2g}R^2$

2.

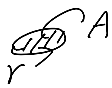






3a) $\vec{\Omega} = \Omega \hat{z}$

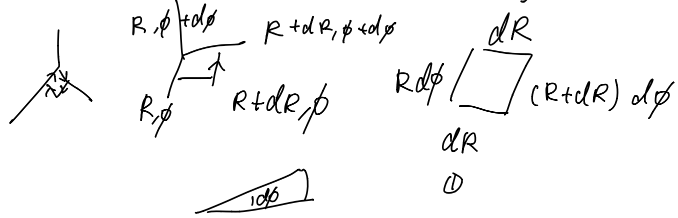
Circulation $\oint \vec{v}\cdot d\vec{s}$



curl : $\nabla \times$

$$\hat{n}_A \cdot (\vec{\nabla}\times\vec{V}) = \lim_{A\to 0} \frac{1}{A} \oint_{\gamma=\partial A} \vec{V}\cdot d\vec{s}$$

$$\hat{z} \cdot (\vec{\nabla}\times\vec{V}) = \lim_{A\to \infty} \frac{1}{A} \oint$$



$$V_R\, dR\Big|^{\phi}_{\phi+d\phi} = -\frac{\partial V_R}{\partial \phi}\, d\phi\, dR = (V_R\, dR)(\phi) - \underbrace{(V_R\, dR)(\phi+d\phi)}$$

$$-\left[(V_R\, dR)\phi + \frac{\partial}{\partial\phi}(V_R\, dR)\, d\phi + O\right]$$

$$\doteq -\left(\frac{\partial}{\partial\phi} V_R\right) dR\, d\phi$$

$A = R\, d\phi\, dR$

① + ③ $\Rightarrow$ $-\frac{1}{R}\frac{\partial}{\partial\phi} V_R$

$$V_\phi R\, d\phi\,\Big|_{R}^{R+dR} = \frac{\partial}{\partial R}(V_\phi R)\, d\phi\, dR$$

$$\Rightarrow \hat{z}\cdot(\vec{\nabla}\times\vec{V}) = \frac{1}{R}\left(\frac{\partial}{\partial R}(V_\phi R) - \frac{\partial}{\partial\phi} V_R\right)$$

$$\vec{V} = (R-z)\cos\phi\,\hat{R} + Rz\,\hat{\phi} + (R+z)\sin\phi\,\hat{z}$$

$R=a$, $z=h$ ↻ $\oint_0^{2\pi} R V_\phi\, d\phi = a^2 h \int_0^{2\pi} d\phi$

$= 2\pi a^2 h$

$$\hat{z}\cdot(\vec{\nabla}\times\vec{V}) = \frac{1}{R}\left(\frac{\partial}{\partial R}(R^2 z) - \frac{\partial}{\partial\phi}(R-z)\cos\phi\right)$$

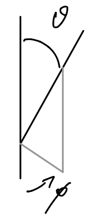
$$= 2z + \left(1-\frac{z}{R}\right)\sin\phi$$

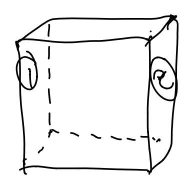
$$\int_0^a \int_0^{2\pi} R\,d\phi\,dR\left(2z + \underbrace{\left(1-\frac{z}{R}\right)\sin\phi}_{0}\right) = 4\pi h\int_0^a R\,dR = 4\pi\frac{a^2}{2}h = 2\pi a^2 h$$

flux $\oint_{A=\partial V} \vec{V}\cdot\underbrace{\hat{n}_A\, dA}_{\vec{dA}}$

divergent

$$\vec{\nabla}\cdot\vec{V} = \lim_{V\to 0}\frac{1}{V}\oint_{A=\partial V}\vec{V}\cdot\hat{n}_A\, dA$$





"Cube" with volume $dr \times r\,d\theta \times r\sin\theta\, d\varphi$

$\hat{r}$, $\hat{\theta}$, $\hat{\varphi}$

What's changing as we go from one face to another? ① + ②: $\left(V_\theta\, r\sin\theta\, d\varphi\, dr\right)\Big|_\theta^{\theta+d\theta}$ r θ,φ r $\theta+d\theta,\varphi$

$$\frac{\partial}{\partial\theta}\left(V_\theta\sin\theta\right) r\,d\varphi\, dr\, d\theta$$

We keep doing this for parallel faces

$\hat{r}$: $V_r\, r^2\sin\theta\, d\varphi\, d\theta\Big|_r^{r+dr} = \frac{\partial}{\partial r}\left(V_r r^2\right)\sin\theta\, d\varphi\, d\theta\, dr$

$\hat{\theta}$: $V_\theta\, r\sin\theta\, d\varphi\, dr\Big|_\theta^{\theta+d\theta} = \frac{\partial}{\partial\theta}\left(V_\theta\sin\theta\right) r\,d\varphi\, dr\, d\theta$

$\hat{\varphi}$: $v_\varphi \, r\,dr\,d\theta \Big|_{\varphi}^{\varphi+d\varphi} = \frac{\partial}{\partial\varphi}(v_\varphi)\, r\,dr\,d\theta\,d\varphi$

b) Flux through sphere $\Rightarrow$ $\vec{r}$ is perpendicular to surface $(r=a)$

$$\Phi = \oiint \vec{v}\cdot d\vec{Area} = \int_0^{\pi}\int_0^{2\pi} a(1+\cos\varphi)\cdot a^2\sin\theta\,d\varphi\,d\theta$$

$$= 2\pi a^3 \int_0^{\pi} \sin\theta\,d\theta = 4\pi a^3$$

c) Compute divergence (just plug in)

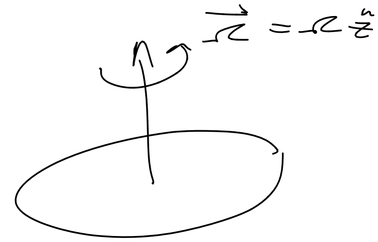
$$\vec{\nabla}\cdot\vec{v} = 3(1+\cos\varphi) + \frac{1}{\sin\theta}\left(r\cos^2\theta\sin\varphi - r\sin^2\theta\cos\varphi\right) + r\cos\varphi$$

d) $$\Phi = \int_0^{\pi}\int_0^{a}\int_0^{2\pi}\left[3 + 3\cos\varphi + (\cdots\cdots)\right] r^2\sin\theta\,d\varphi\,dr\,d\theta$$

Drops out, all $\sin\varphi$ or $\cos\varphi$ integrated $0\to 2\pi$, so they're all zero

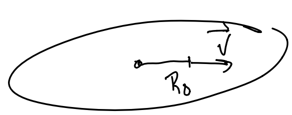
$$\Phi = \int_0^{\pi}\int_0^{a} 3r^2\sin\theta\,dr\,d\theta = 4\pi a^3$$

3) Ant on turntable



a) See diagram

b)



$\hat{z}$ is <u>fixed</u>

Deriving Coriolis force:

$$\vec r_0 = \vec r = R\hat R + z\hat z$$

$$\dot{\vec r}_0 = \vec V_0 = \dot{\vec r} = \underbrace{\dot R\hat R + \dot z\hat z}_{\vec V} + R\dot{\hat R} \qquad \dot{\hat R} = \dot\theta\hat\theta$$

$$= \vec V + \vec\Omega\times\vec r$$

$$\dot{\hat e}_i = \vec\Omega\times\hat e_i$$

$$\frac{\vec F_0}{m} = \vec a_0 = \frac{d}{dt}\left(\vec V + \vec\Omega\times\vec r\right) = \vec a + 2\vec\Omega\times\vec V + \dot{\vec\Omega}\times\vec r + \cancelto{0}{\vec\Omega\times\vec\Omega}\times\vec r + \vec\Omega\times(\vec\Omega\times\vec r)$$

$$m\vec a = \vec F_{eff} = \vec F_0 - m\left(2\vec\Omega\times\vec V + \dot{\vec\Omega}\times\vec r + \vec\Omega\times(\vec\Omega\times\vec r)\right)$$

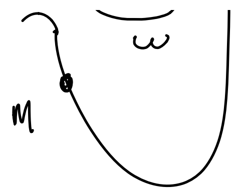
Differentiating Basis Vectors

Back to the ant:

$$-2m\vec\Omega\times\hat r|v| = -2m|v|\hat\varphi\,\Omega$$, vector describing Coriolis force

c) $-2m\vec\Omega\times v\hat\varphi = 2m\Omega v\hat R$ (ant moving tangentially)

d) 0

4) 

$z = f(R)$ $\quad (R, \varphi, z) \quad \dot{\varphi} = \omega$

a) $L = \frac{1}{2} m\left(\dot{R}^2 + R^2 \overset{\omega^2}{\cancel{\dot{\varphi}^2}} + \dot{z}^2\right) - mgz + \lambda\left(z - f(R)\right)$

$\underline{R:}$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{R}} = \frac{\partial L}{\partial R} \implies m\ddot{R} = mR\omega^2 - \lambda \frac{\partial f}{\partial R}$$

$\underline{\lambda:}$ $\quad z - f(R) = 0$

$\underline{z:}$ $\quad m\ddot{z} = -mg + \lambda$

b) Plug in $f(R)$ into Lagrangian, $\dot{z} = \frac{\partial f}{\partial R} \cdot \dot{R}$

(Supposedly easier this way)

$$L(R) = \frac{1}{2} m\left(\dot{R}^2 + R^2\omega^2 + \left(\frac{\partial f}{\partial R}\right)^2 \dot{R}^2\right) - mg\frac{\partial f}{\partial R}$$

$$\implies m\frac{d}{dt}\left[\left(1 + \left(\frac{\partial f}{\partial R}\right)^2\right)\dot{R}^2\right] = mR\omega^2 - mg\frac{\partial f}{\partial R} + m\frac{\partial^2 f}{\partial R^2}\frac{\partial f}{\partial R}\dot{R}^2$$

What $f(R)$ implies $\ddot{R} = 0$ when $\dot{R} = 0$?

$$\frac{R\omega^2}{g} = \frac{\partial f}{\partial R}, \text{ so } f(R) = \frac{1}{2}\frac{\omega^2 R^2}{g}$$

$\underline{\text{My method:}}$ $\quad m\ddot{z} = 0 = -mg + \lambda$, so $\lambda = mg$

$$m\ddot{R} = 0 = Rm\omega^2 - \lambda\frac{\partial f}{\partial R}$$

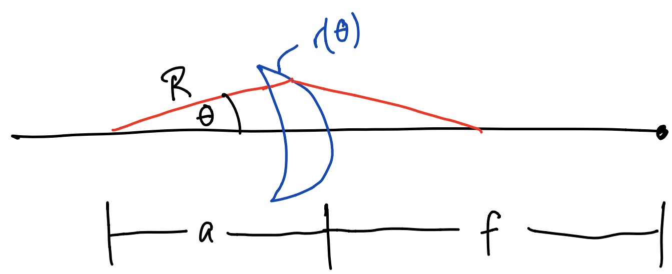
$$R\cancel{m}\omega^2 = \cancel{m}g\,\frac{\partial f}{\partial R}$$

$$\frac{R^2}{2}\frac{\omega^2}{g} = f(R)$$

d) Bucket of water will settle into parabola

5)

a)



At 90^0 to surface, goes straight through

b) All paths are extremal, so all take same time.
Said in weird way, listen at ~ 1:30:00 on recording

c)

$\theta = 0$: $ct = R + n(a-R) + f$

$\theta \neq 0$: $ct = R + n(r(\theta) - R) + \sqrt{r^2\sin^2\theta + (f+a-r\cos\theta)^2}$

Set equal to each other since all paths take same time, get

$$(n(a-r) - f)^2 = r^2 + (f+a)^2 - 2(f+a)r\cos\theta$$

6) a)

$$L = \frac{1}{2} m_1 |\dot{\vec{r}}_1|^2 + \frac{1}{2} m_2 |\dot{\vec{r}}_2|^2 - \frac{1}{2} k \left(\sqrt{|\vec{r}_1 - \vec{r}_2|^2} - \ell \right)^2 - m_1 g z_1 - m_2 g z_2$$
$$+ \lambda_1 (z_1 - f(x_1, y_1)) + \lambda_2 (z_2 - f(x_2, y_2))$$

same bowl, same constraint

b) $f = 0$ for rest of problem

$$L = \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{x}_2^2) + \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2) - \frac{1}{2} k \left(\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} - \ell \right)^2$$

c) $$L = \frac{1}{2} M (\dot{x}_{cm}^2 + \dot{y}_{cm}^2) + \frac{1}{2} \mu (\dot{x}^2 + \dot{y}^2)$$
$$- \frac{1}{2} k \left(\sqrt{x^2 + y^2} - \ell \right)^2$$

(from $T = \frac{1}{2}\left(M|\vec{v}_{cm}|^2 + \mu |\vec{v}|^2\right)$ on formula sheet)

d) $M \ddot{\vec{r}}_{cm} = 0$; $\vec{r}_{cm}(t) = \vec{r}_{cm}(0) + \vec{V}_{cm}(0)\, t$

e) $$L(\vec{r}) = \frac{1}{2} \mu (\dot{x}^2 + \dot{y}^2) - \frac{1}{2} k \left(\sqrt{x^2 + y^2} - \ell \right)^2$$

$\delta x = \alpha y$
$\delta y = -\alpha x$

Symmetric under rotation

$\delta L = 0$ Invariant

$$\delta L = \mu\left(\cancel{\dot{x}\alpha y - \dot{x}\alpha y}\right)^{\to 0} - k\left(\sqrt{(\cdots)} - \ell\right)\left(\cancel{\frac{x\alpha y - y\alpha x}{\sqrt{(\cdots)}}}\right)^{\to 0} = 0$$

$$Q = \frac{\partial L}{\partial \dot{x}} R_x + \frac{\partial L}{\partial \dot{y}} R_y = \mu \dot{x} y - \mu \dot{y} x = \mu(\dot{x}y - \dot{y}x)$$

f) Rewriting in polar coordinates

$$L = \frac{1}{2}\mu\left(\dot{R}^2 + R^2\dot{\varphi}^2\right) - \frac{1}{2}k\left(R - \ell\right)^2$$

$$\mu R^2 \dot{\varphi} = L_z \longrightarrow$$

$$x = R\cos\varphi$$
$$y = R\sin\varphi$$
$$\dot{x} = \dot{R}\cos\varphi - R\dot{\varphi}\sin\varphi$$
$$\dot{y} = \dot{R}\sin\varphi + R\dot{\varphi}\cos\varphi$$
$$\mu(\dot{x}y - y\dot{x}) = -\mu R^2\varphi$$

$$\mu\ddot{R} = -k(R-\ell) + \frac{L_z^2}{\mu R^3}$$

$$= -\frac{\partial}{\partial R}\underbrace{\left(\frac{1}{2}k(R-\ell)^2 + \frac{L_z^2}{2\mu R^2}\right)}_{U_{eff}}$$

$$k(R-\ell) - \frac{L_z^2}{\mu R^3} = 0$$

$$\mu\omega^2 = k + \frac{3L_z^2}{\mu R^4} = k + 3k\left(1 - \ell/R\right)$$

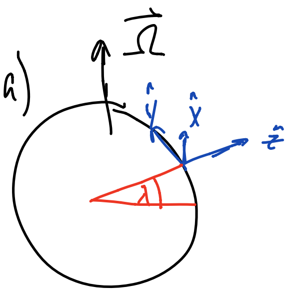
$$\omega^2 = \frac{k}{\mu}\left(4 - \frac{3\ell}{R_{eff}}\right)$$

g) $L = \frac{1}{2}\mu\left(\dot{x}^2 + \dot{y}^2\right) - \frac{1}{2}k\left(x^2+y^2\right)$

$x(t) = x_{MAX}\cos(\omega t + \varphi_x)$ $\quad \omega = \sqrt{\frac{k}{m}}$

$y(t) = y_{MAX}\cos(\omega t + \varphi_y)$

7) a)



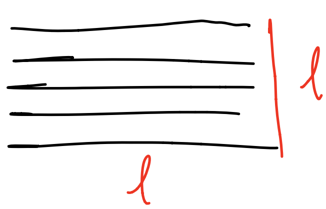
$\vec{\Omega} = \Omega\left(\cos\lambda\,\hat{y} + \sin\lambda\,\hat{z}\right)$

$\dot{\hat{x}} = \vec{\Omega}\times\hat{x} = \Omega\left(-\cos\lambda\,\hat{z} + \sin\lambda\,\hat{y}\right)$

Did derivation in Problem 3

b)

8)



a) $I = \frac{1}{3}m\ell^2$

b) Parallel Axis: $I = I_{cm} + m\left(\frac{\ell}{2}\right)^2$

$$\frac{1}{3}ml^2 = I_{cm} + \frac{1}{4}ml^2$$
$$\Rightarrow I_{cm} = \frac{ml^2}{12}$$

Perpendicular Axis: $I_\perp = I_1 + I_2 = \cancel{2}\cdot\frac{1}{\cancel{12}_{6}}ml^2$

9) a) See Formula sheet

$$\ddot{x} + 2\beta x + \omega_0^2 x = \sin(2\omega_0 t) + \frac{1}{2}\cos\left(\frac{1}{2}\omega_0 t\right)$$

Linearity: $\sin(2\omega_0 t) = I_m\, e^{2i\omega_0 t}$

$\cos\left(\frac{1}{2}\omega_0 t\right) = R_e\, e^{i\omega_0 t/2}$

$$x(t) = I_m \frac{e^{2i\omega_0 t}}{\omega_0^2 - 4\omega_0^2 + 4i\beta\omega_0} + \frac{1}{2} R_e \frac{e^{\frac{1}{2}i\omega_0 t}}{\omega_0^2 - \frac{1}{4}\omega_0^2 + i\beta\omega_0}$$

b) $x_{trans} = e^{-\beta t}\left(A\cos(\omega t) + B\sin(\omega t)\right)$ $\quad \omega^2 = \omega_0^2 - \beta^2$

$$x(0) = I_m \frac{1}{-3\omega_0^2 + 4i\beta\omega_0} + 2 R_e \frac{1}{3\omega_0^2 + 4i\beta\omega_0}$$

$x_{trans}(0) = A$

$V_{trans}(0) = -\beta A + B\omega$

$$V(0) = I_m\left(\frac{2i\omega_0}{-3\omega_0^2 + 4i\beta\omega_0}\right) + R_e\left(\frac{i\omega_0}{3\omega_0^2 + 4i\beta\omega_0}\right)$$

$x(t) + x_{trans}(t) = 0$ at $t = 0$

$\dot{x}(t) + \dot{x}_{trans}(t) = 0$ at $t = 0$

c) Yes, as long as there's damping

d) No

e) No f) NO

10) a) Rotational, leads to conservation of angular momentum

$T - U$

b) $L = \frac{1}{2} m \left(\dot{R}^2 + R^2 \dot{\varphi}^2 \right) + \frac{GMm}{R}$ $\qquad T - \left(\frac{-GMm}{R} \right) = +$

c) $L(q^i, \dot{q}^i)$ $\qquad \dot{q}^i = \frac{\partial H}{\partial p^i}$

$H(q^i, p_i)$ $\qquad p_i = \frac{\partial L}{\partial \dot{q}_i}$

$$H = p_i \dot{q}^i - L = p_R \dot{R} + p_\varphi \dot{\varphi} - L$$

$p, R, \varphi \qquad p, R, \varphi$

$$p_\theta = mR^2 \dot{\theta} \Rightarrow \dot{\theta} = p_\theta / mR^2$$

$$p_R = m\dot{R} \Rightarrow \dot{R} = \frac{p_R}{m}$$

$$H = \frac{p_R^2}{m} + \frac{p_\varphi^2}{mR^2} - \frac{1}{2} m\left(\frac{p_R^2}{m^2} + R^2 \frac{p_\varphi^2}{m^2R^4}\right) - \frac{GMm}{R}$$

$$= \frac{1}{2m}\left(p_R^2 + \frac{p_\varphi^2}{R^2}\right) - \frac{GMm}{R}$$

$$\dot{p}_\varphi = -\frac{\partial H}{\partial \varphi} = 0$$

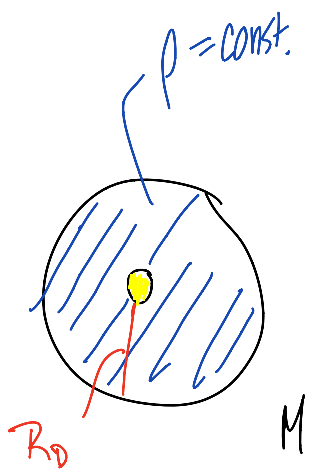
$p_\varphi = L_z$

$$U_{eff} = \frac{L_z^2}{2mR^2} - \frac{GMm}{R}$$

d) $U'_{eff} = 0 \Rightarrow \quad \frac{L_z^2}{mR^3} = \frac{GMm}{R^2}$

$$R_{eq} = \frac{L_z^2}{GMm^2}$$

e)



$$M \longrightarrow M + \frac{4}{3}\pi\rho R^3$$

$$M \longrightarrow M + \frac{4}{3}\pi\rho R_D^3$$

$$U = \frac{L_z^2}{2mR^2} - \frac{GMm}{R} - Gm\,\frac{4}{3}\pi\rho R^2$$

$- L_z^2$

$$U' = 0 = \frac{\quad_z}{M} + GMmR - \frac{8}{3} G m \pi \rho R^4$$

$$R_{eq} = \frac{L_z^2}{GMm^2} + \delta R \qquad \delta R = \frac{8}{3}\frac{\pi\rho}{M}\frac{L_z^8}{(GMm^2)^4}$$

f) $$\omega^2 = U''_{eff} = \frac{1}{m}\left(\frac{3L_z^2}{mR^4} - \frac{2GMm}{R^3} - \frac{8}{3} G m \pi \rho\right)$$

$r(\theta)$

$$\dot{r} = \frac{\partial r}{\partial \theta}\dot{\theta}$$

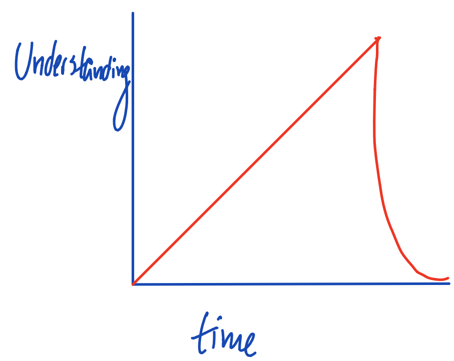
$$r = f(\cos \omega t)$$

$$r = f(\cos \Omega_{orbital}\, \theta)$$

$$\left(\frac{1}{\dot{\theta}}\right)^2 \omega^2 = \Omega_{orb}^2 \qquad \dot{\theta} = \frac{L_z}{MR^2}$$

Rocek not sure what this is...

$$\frac{1}{\dot{\theta}^2} = \frac{M^2R^4}{L_z^2} = \frac{m^2 R_{eq}^4}{L_z^2}$$



20 10-pt parts on the Final