PHY 308 Appetizer for first Midterm, Spring 2017
Stony Brook University
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100 points total. The following information may or may not be useful

Planck's constant: $\hbar = 1.055 \times 10^{-34}$ J-s $= 6.582 \times 10^{-16}$ eV-s
Speed of light: $c = 3 \times 10^{8}$ m/s
Mass of the electron: $m_{\text{electron}} = 9.1 \times 10^{-31}$ kg
The electron volt: 1 eV $= 1.6 \times 10^{-19}$ J

1 True/False (18 points)

Circle **T** if the statement is always true. Otherwise circle **F** for false. 2 points each

T F The time independent Schrödinger equation can be derived from the time dependent Schrödinger equation.

T F If an operator commutes with the Hamiltonian, it must be Hermitian.

T F The dimension of a vector space is equal to the number of linearly independent vectors that can be constructed in the space

T F Linearly independent eigenkets of a Hermitian operator are always orthogonal.

T F The state function of a quantum system is always equal to a function of time multiplied by a function of the coordinates.

T F When the state function $|\Psi\rangle$ is initially an eigenket of the Hermitian operator $\hat{B}$ with eigenvalue b, and $\hat{B}$ does not commute with the Hamiltonian, we are certain to measure b when the observable corresponding to $\hat{B}$ is measured for all later times.

T F The ground state wavefunction of the simple harmonic oscillator is of the form of a Gaussian, $\psi \propto \exp(-\alpha x^2)$, where α is a constant and x is the displacement of the harmonic oscillator from equilibrium.

T F The energy levels for a particle trapped in a one-dimensional box are evenly spaced.

T F For $m = 0$, the spherical harmonics, $Y_J^m(\theta, \phi)$, are constant on the surface of a sphere on centered at the origin.

2 Multiple Choice (18 points)

Circle **one** answer for each question. 3 points each.

Consider a particle moving in 1 dimension described by the wave function $\Psi(x,t)$, the probability current $J = -\frac{i\hbar}{2m}\left[\Psi^* \frac{\partial \Psi}{\partial x} - \Psi \frac{\partial \Psi^*}{\partial x}\right]$ at a given point $x = x_0$ describes

a) the probability of finding the particle at x_0.
b) the time derivative of the probability of finding the particle at x_0.
c) the time derivative of the expectation value of x_0.
d) the time derivative of finding the particle at all positions less than x_0.
e) none of the above.

Consider the time-independent Schrödinger equation (TISE) for a rectangular potential well of finite depth V_0 and width $2a$, where V_0 and a are constants. Explicitly, $V(x) = 0$ for $|x| < a$ and $V(x) = -V_0$ for $|x| \leq a$. Which of the following statements are true:

a) There is always at least one bound state if the product aV_0 is nonzero.
b) Normalized free state solutions to the TISE with $E > 0$ can be constructed.
c) Some energy levels are degenerate, but the degeneracy can be broken by considering parity.
d) all of the above.

The simple harmonic oscillator potential, $V(x) = \frac{1}{2}m\omega^2 x^2$, has which of the following properties:

a) The energy levels are evenly spaced and given by $(n + 1/2)\hbar\omega$, with $n = 0, 1, 2, ...$
b) With x the displacement from equilibrium, it is the lowest-order approximation to the potential energy of the nuclei in a diatomic molecule near equilibrium.
c) The number of nodes in the n^{th} energy eigenstate wavefunction, with corresponding energy $(n + 1/2)\hbar\omega$, is n.
d) all of the above.
e) a) and c) only.

The raising and lowering operators for the simple harmonic oscillator, $\hat{a} = \frac{1}{\sqrt{2m\hbar\omega}}(m\omega\hat{x} + i\hat{p})$ and $\hat{a}^\dagger = \frac{1}{\sqrt{2m\hbar\omega}}(m\omega\hat{x} - i\hat{p})$, have which of the following properties:

a) They are Hermitian operators.
b) They commute with each other.
c) Given a solution of the harmonic oscillator problem, they can be used to generate new solutions.
d) all of the above
e) a) and c) only.
f) none of the above.

A system is in a quantum state $|\Psi\rangle$ and one makes a measurement of the observable $\hat{O}$. According to quantum theory, the result of the measurement will be

a) the expectation value of $\hat{O}$, given by $\langle\Psi|\hat{O}|\Psi\rangle$.
b) a random value between given by a a normal (Gaussian) probability distribution with mean $\langle\Psi|\hat{O}|\Psi\rangle$ and standard deviation $\sqrt{\langle\Psi|\hat{O}^2|\Psi\rangle - \langle\Psi|\hat{O}|\Psi\rangle^2}$.
c) one of the eigenvalues λ_n of $\hat{O}$ with probability $|\langle\lambda_n|\Psi\rangle|^2$, where $|\lambda_n\rangle$ are the normalized eigenkets of $\hat{O}$ with eigenvalues λ_n.
d) that the measurement will collapse the wavefunction to one of the stationary states $|\psi_n\rangle$, such that one will measure $\langle\psi_n|\hat{O}|\psi_n\rangle$.
e) none of the above.

When two operators, $\hat{A}$ and $\hat{B}$, commute, it means that

a) it does not matter in what order you act them on a ket, vector, or function, *viz.* $\hat{A}\hat{B}|\psi\rangle = \hat{B}\hat{A}|\psi\rangle$.
b) they belong to the same symmetry group.
c) one can construct simultaneous eigenkets of $\hat{A}$, $\hat{B}$, and the Hamiltonian.
d) all of the above.

3 Dispersion Relations (12 points)

Explain what a dispersion relation is and how one uses it to calculate the group and phase velocities of a wave packet. Explain the physical meaning of the group and phase velocities. You are encouraged to use a combination of words, equations, and pictures, as appropriate.

4 The Generalized Uncertainty Principle (13 points)

The generalized uncertainty principle states that

$$\sigma_A^2 \sigma_B^2 \geq \frac{1}{4} \left| \langle [A, B] \rangle \right|^2 \tag{4.1}$$

where σ_A and σ_B are the uncertainties in the observables A and B associated with the operators $\hat{A}$ and $\hat{B}$, and as usual the $\langle\rangle$ notation indicates the expectation value. Use equation (4.1) to prove the uncertainty relation

$$\sigma_x \sigma_H \geq \frac{\hbar}{2m} \left| \langle \hat{p} \rangle \right| \tag{4.2}$$

for a 1D particle with Hamiltonian $\hat{H} = \hat{p}^2/2m + V(x)$. For stationary states, this doesn't tell you much – why?

5 Projection Onto a Basis (12 points)

Consider a particle that half fills a 1D box of length L, such that it's wavefunction is

$$\phi(x) = \begin{cases} 0, & 0 < x < L/2 \\ \sqrt{2/L}, & L/2 < x < L \\ 0, & \text{outside the box}, x < 0 \text{ or } x > L \end{cases} \tag{5.1}$$

This is not an eigenstate of the "particle in the box" (PIB) Hamiltonian so calculating things like the energy and the time evolution of this particle's wavefunction would be very complicated. However, things can be made significantly simpler if we represent the wavefunction as a sum of normalized, orthogonal, PIB eigenstates:

$$\phi(x) = \sum_{n=1}^{\infty} c_n \psi_n(x) \tag{5.2}$$

$$\psi_n(x) = \left(\frac{2}{L}\right)^{1/2} \sin\left(\frac{n\pi x}{L}\right) \tag{5.3}$$

a) Derive an integral expression for the coefficients. You can use Dirac notation to save writing if desired, but give the final result as a integral over x. If you know the answer, but don't know how to derive it, just write it down for partial credit.

b) Are any of the coefficients zero based on symmetry considerations? Why or why not?

c) Evaluate the integral from part a) to derive a general formula for the coefficients c_n.

6 Particle *Moving* in a box (14 points)

A "wave-packet" can be formed by a coherent superposition of stationary states. Unlike stationary states, wave packets move. Consider the superposition of particle in the box stationary states.

$$\Psi(x, t = 0) = \frac{1}{\sqrt{2}}\psi_1(x) + \frac{1}{\sqrt{2}}\psi_2(x) = \left(\frac{1}{L}\right)^{1/2}\left[\sin\left(\frac{\pi x}{L}\right) + \sin\left(\frac{2\pi x}{L}\right)\right] \tag{6.1}$$

for $0 < x < L$ and $\Psi = 0$ outside of this interval.

a) Write out $\Psi(x, t)$.

b) What is the expectation value of the system's total energy $\langle\hat{H}\rangle$? Does this change with time?

c) Neglecting the discontinuity of $\frac{d\psi}{dx}$ at $x = 0$ and $x = L$, derive an expression for the time dependent expectation value $\langle\hat{p}\rangle(t)$. *Hint: The integrals might be easier if you write the sine and cosine functions as* $\sin(kx) = (e^{ikx} - e^{-ikx})/2i$ *and* $\cos(kx) = (e^{ikx} + e^{-ikx})/2$

d) Based on the hint above, one might think that if a measurement of the particle's momentum is made, there are four possible outcomes with equal probabilities, $\pm\hbar(\pi/L)$ and $\pm(\hbar 2\pi/L)$. Why is this wrong? Make a rough sketch of the momentum space wave function. You may find the following fact insightful:

$$\frac{1}{\sqrt{2\pi}}\int_{-L/2}^{L/2} dx e^{ikx} = \sqrt{\frac{2}{\pi}}\left[\frac{\sin(kL/2)}{k}\right] \tag{6.2}$$

More space for 6

7 Degenerate Perturbation Theory (13 points)

Consider a quantum system with a two-fold degenerate energy level, with orthonormal energy eigenkets $|\alpha\rangle$ and $|\beta\rangle$, such that initially

$$\hat{H}_0 |\alpha\rangle = E^{(0)} |\alpha\rangle \tag{7.1}$$

$$\hat{H}_0 |\beta\rangle = E^{(0)} |\beta\rangle \tag{7.2}$$

Here $\hat{H}_0$ is the systems initial hamiltonian. Now consider the case where a perturbation is added to the system such that the Hamiltonian becomes $\hat{H} = \hat{H}_0 + \hat{H}'$. The matrix elements of $\hat{H}'$ in the $|\alpha\rangle$, $|\beta\rangle$ basis are

$$\langle\alpha|\hat{H}'|\alpha\rangle = \langle\beta|\hat{H}'|\beta\rangle = 0 \tag{7.3}$$

$$\langle\alpha|\hat{H}'|\beta\rangle = \langle\beta|\hat{H}'|\alpha\rangle = V \tag{7.4}$$

where V is a real positive number. Find the new energy eigenvalues and corresponding eigenvectors of the perturbed system. Express the eigenvectors in terms of $|\alpha\rangle$ and $|\beta\rangle$. Are the energy levels still degenerate?