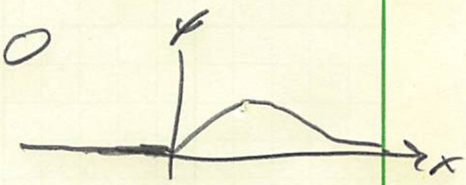
$$\psi(x) = \begin{cases} Nxe^{-\alpha x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$



1) Normalization

$$1 = \int dx\, |\psi|^2$$

$$= |N|^2 \int_0^\infty dx\, x^2 e^{-2\alpha x}$$

Tabular integration

$$\begin{array}{ccc} x^2 & & e^{-2\alpha x} \\ & (+) & \\ 2x & & -\frac{1}{2\alpha} e^{-2\alpha x} \\ & (-) & \\ 2 & & \frac{1}{4\alpha^2} e^{-2\alpha x} \\ & (+) & \\ 0 & & -\frac{1}{8\alpha^3} e^{-2\alpha x} \end{array}$$

These two terms are zero at both bounds

$$-\frac{1}{2\alpha} e^{-2\alpha x} x^2$$

$$-\frac{1}{2\alpha^2} e^{-2\alpha x} x$$

$$-\frac{1}{4\alpha^3} e^{-2\alpha x}$$

$$1 = |N|^2 \left[-\frac{1}{4\alpha^3} e^{-2\alpha x} \right]_0^\infty$$

$$= |N|^2 \left(\frac{1}{4\alpha^3} \right) \Rightarrow |N| = 2\alpha^{3/2}$$

Since the problem says N is real $\boxed{N = \pm 2\alpha^{3/2}}$

Wave fcn Pushups

b) Looking for maximum of $|\psi|^2$

$$\frac{d}{dx}|\psi|^2 = 0$$

$$|N|^2 \frac{d}{dx}\left(x^2 e^{-2\alpha x}\right) = \cancel{|N|^2}\left(2x\cancel{e^{-2\alpha x}} + x^2(-2\alpha)\cancel{e^{-2\alpha x}}\right) = 0$$

$$\cancel{2}\cancel{x} - \cancel{2}\alpha x^{\cancel{2}} = 0$$

$$1 - \alpha x = 0$$

$$\boxed{x_{max} = \frac{1}{\alpha}}$$

c)

$$\langle x\rangle = \langle\psi|\hat{x}|\psi\rangle = \int_{-\infty}^{\infty} dx\, \psi^*(x)\, x\, \psi(x)$$

$$= |N|^2 \int_0^{\infty} dx\; xe^{-\alpha x}\;\; x\;\; xe^{-\alpha x}$$

$$= |N|^2 \int_0^{\infty} dx\;\; x^3 e^{-2\alpha x}$$

The procedure here is the same as a), or we can re-arrange to make it look like the Γ-fcn.

$$u = 2\alpha x$$
$$du = 2\alpha\, dx \qquad \Rightarrow \langle x\rangle = \left(\frac{1}{2\alpha}\right)^4 |N|^2 \underbrace{\int_0^{\infty} du\, u^{4-1} e^{-u}}_{\Gamma(4) = 3! = 6}$$

Wave fcn pushups

$$\langle x \rangle = \left(\frac{1}{2\alpha}\right)^{\cancel{4}\,2} \frac{\cancel{(2\alpha)^3}}{\underset{\substack{\uparrow \\ |N|^2}}{2}}\; 6$$

$$\boxed{\langle x \rangle = \frac{3}{2}\frac{1}{\alpha}} \qquad (1)$$

d) Similarly

$$\langle x^2 \rangle = \left(\frac{1}{2\alpha}\right)^{\cancel{5}\,2} \left(\frac{\cancel{(2\alpha)^3}}{2}\right) \Gamma(5)$$

$$= \left(\frac{1}{2\alpha}\right)^2 \frac{24}{2} = \frac{12}{4\alpha^2} = \boxed{\frac{3}{\alpha^2} = \langle x^2 \rangle}$$

e)

$$\sigma_x^2 = \langle x^2 \rangle - \langle x \rangle^2 = \frac{3}{\alpha^2} - \frac{9}{4}\frac{1}{\alpha^2}$$

$$\boxed{\sigma_x^2 = \frac{3}{4}\frac{1}{\alpha^2}}$$

f) For $\hat{p}$:

$$\langle \psi | \hat{p} | \psi \rangle = \langle \hat{p} \rangle = |N|^2 \int_0^\infty dx\; x e^{-\alpha x}\left(-i\hbar \frac{\partial}{\partial x}\right) x e^{-\alpha x}$$

$$= -i\hbar |N|^2 \int_0^\infty dx\; x e^{-\alpha x}\left(e^{-\alpha x} + x e^{-\alpha x}(-\alpha)\right)$$

$$= -i\hbar |N|^2 \int_0^\infty dx\, (x - \alpha x^2) e^{-2\alpha x}$$

Wave fcn

Pushups

$u = 2\alpha x$

$du = 2\alpha dx$

$$\langle\hat{p}\rangle = -i\hbar|N|^2\left(\frac{1}{2\alpha}\right)\int_0^\infty du\left(\frac{u}{2\alpha} - \frac{u}{4\alpha^2}\alpha\right)e^{-u}$$

$$= -\frac{i\hbar|N|^2}{2\alpha}\left(\cancel{\frac{1}{2\alpha}\Gamma(2) - \frac{1}{2\cdot 2\alpha}\Gamma(3)}\right)$$

$$\boxed{\langle\hat{p}\rangle = 0}$$

g)

$$\langle\hat{p}^2\rangle = \int_{-\infty}^{\infty} dx\ \psi^*(x)\left(-i\hbar\frac{d}{dx}\right)^2\psi(x)$$

$$= -\hbar^2|N|^2\int_0^\infty dx\ xe^{-\alpha x}\frac{d^2}{dx^2}\left(xe^{-\alpha x}\right)$$

$$= -\hbar^2|N|^2\int_0^\infty dx\ xe^{-\alpha x}\frac{d}{dx}\left(e^{-\alpha x} - \alpha xe^{-\alpha x}\right)$$

$$= -\hbar^2|N|^2\int_0^\infty dx\ xe^{-\alpha x}\left(-\alpha e^{-\alpha x} - \alpha e^{-\alpha x} - \alpha xe^{-\alpha x}(-\alpha)\right)$$

$$= -\hbar^2|N|^2\int_0^\infty dx\left(\alpha^2x^2 - 2\alpha x\right)e^{-2\alpha x}$$

$$= -\hbar^2|N|^2\left(\alpha^2\underbrace{\int_0^\infty dx\ x^2e^{-2\alpha x}}_{1/|N|^2 = \frac{1}{4\alpha^3}} - 2\alpha\underbrace{\int_0^\infty dx\ xe^{-2\alpha x}}_{\frac{1}{(2\alpha)^2}\Gamma(2)}\right)$$

$$= -\hbar^2\, 4\alpha^3\left(\frac{1}{4\alpha} - \frac{2\alpha}{2\alpha}\right)$$

$$= \boxed{\hbar^2\alpha^2 = \langle\hat{p}^2\rangle}$$

h)

$$\sigma_p^2 = \langle p^2 \rangle - \langle p \rangle^2$$

$$= \hbar^2 \alpha^2 - 0$$

$$\boxed{\sigma_p^2 = \hbar^2 \alpha^2}$$

i)

$$\sigma_x \sigma_p = \sqrt{\frac{3}{4}} \frac{1}{\cancel{\alpha}} \times \hbar \cancel{\alpha}$$

$$= \frac{\sqrt{3}}{2} \hbar > \frac{\hbar}{2}$$

Heisenberg!

✓