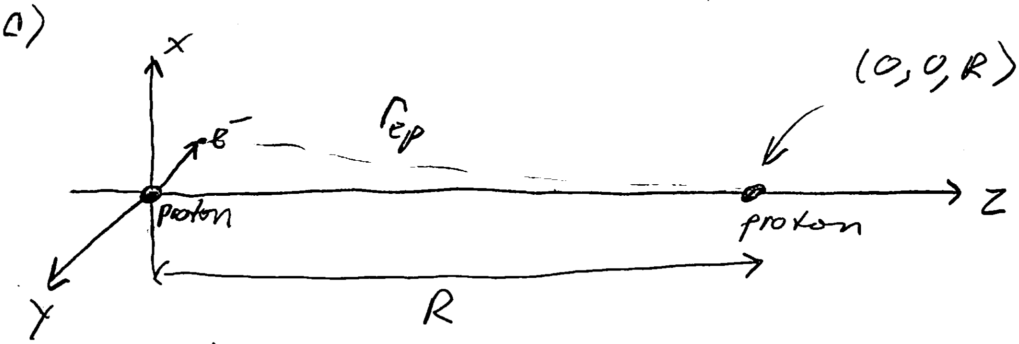
a)



Neglecting Nuclear kinetic energy

$$\hat{H} = \underbrace{\frac{\hat{p}^2}{2m} - \frac{1}{4\pi\varepsilon_0}\frac{e^2}{r}}_{\text{Hydrogen atom}} + \frac{1}{4\pi\varepsilon_0}\frac{e^2}{R} - \frac{1}{4\pi\varepsilon_0}\frac{e^2}{r_{ep}}$$

$$= \vec{H}_0 + \underbrace{\frac{e^2}{4\pi\varepsilon_0}\left(\frac{1}{R} - \frac{1}{r_{ep}}\right)}_{\hat{H}'}$$

$$\vec{r}_{ep} = x\hat{e}_x + y\hat{e}_y + (R-z)\hat{e}_z$$

$$|\vec{r}_{ep}|^2 = x^2 + y^2 + (R-z)^2$$

$$= x^2+y^2+z^2 + R^2 - 2Rz$$

keeping only terms to first order in z,x,y and lower

$$|\vec{r}_{ep}| \approx \sqrt{R^2 - 2Rz} \approx R\left(1 - 2\frac{z}{R}\right)^{1/2}$$

$$\approx R\left(1 - \frac{z}{R}\right)$$

$$\frac{1}{|\vec{r}_{ep}|} \approx \frac{1}{R(1-z/R)} = \left(\frac{1}{R}\right)\left(1 + \frac{z}{R} + \ldots\right)$$

to first order

$$\Rightarrow \hat{H}' = \frac{e^2}{4\pi\varepsilon_0}\left(\frac{1}{R} - \frac{1}{R}\left(1+\frac{z}{R}\right)\right)$$

$$\boxed{\hat{H}' = \frac{e^2}{4\pi\varepsilon_0}\left(\frac{-z}{R^2}\right)}$$

b) Taking $\psi^{(0)}$ = ground state of H-atom $= \psi_{1s}$

$$E^{(1)} = \langle \psi_{1s} | \frac{e^2}{4\pi\varepsilon_0} \frac{-z}{R^2} | \psi_{1s} \rangle$$

$$= \frac{-e^2}{4\pi\varepsilon_0 R^2} \langle \psi^{(0)} | z | \psi^{(0)} \rangle \rightarrow 0 \text{ by symmetry}$$

So we have to go to second order

$$E_1^{(2)} = \sum_{m \neq n} \frac{|\langle \psi_m | \hat{H}' | \psi_n \rangle|^2}{E_1 - E_m}$$

$$= \left(\frac{-e^2}{4\pi\varepsilon_0 R^2}\right)^2 \sum_{m \neq 1} \frac{|\langle \psi_m | z | \psi_1 \rangle|^2}{E_1 - E_m}$$

Now since E_1 is the ground state, all $E_{m\neq 1}$ are higher in energy and thus all terms in the sum are negative and we have

$$\boxed{E^{(2)} = \frac{-C}{R^4}}$$ with $$\boxed{C = \left(\frac{e^2}{4\pi\varepsilon_0}\right)^2 \sum_{m \neq 1} \frac{|\langle \psi_m | z | \psi_1 \rangle|^2}{|E_1 - E_m|}}$$

c) With $E_1 - E_m \approx E_1$ the sum can be evaluated with a trick. Expanding...

$$C = \left(\frac{e^2}{4\pi\varepsilon_0}\right)^2 \frac{1}{|E_1|} \sum_{m\neq 1} \langle\psi_1|z|\psi_m\rangle\langle\psi_m|z|\psi_1\rangle$$

Since $\langle\psi_1|z|\psi_1\rangle = 0$, might as well include $m=1$ in the sum, then you have the identity operator to work with! Viz.

$$\sum_{m=1,2,\ldots} \langle\psi_1|\hat{z}|\psi_m\rangle\langle\psi_m|\hat{z}|\psi_1\rangle$$

$$= \langle\psi_1|\hat{z}\left(\sum_m |\psi_m\rangle\langle\psi_m|\right)\hat{z}|\psi_1\rangle$$

$$= \langle\psi_1|\hat{z}^2|\psi_1\rangle$$

Now we just have one integral to do.

$$\psi_1 = \frac{1}{\sqrt{\pi a_0^3}} e^{-r/a_0}$$

$$z = r\cos\theta \Rightarrow \langle z^2\rangle = \int_0^\pi d\theta \int_0^{2\pi} d\phi \int_0^\infty dr\, r^2 \sin\theta\, r^2\cos^2\theta\, \frac{e^{-2r/a_0}}{\pi a_0^3}$$

$$= \frac{2\cancel{\pi}}{\cancel{\pi} a_0^3} \underbrace{\int_0^\pi d\theta \sin\theta\cos^2\theta}_{I_1} \underbrace{\int_0^\infty dr\, r^4 e^{-2r/a_0}}_{I_2}$$

$$I_1 = -\int_1^{-1} du\, u^2$$

$u = \cos\theta$
$du = -\sin\theta\, d\theta$

$$= \int_{-1}^{1} du\, u^2 = \left.\frac{u^3}{3}\right|_{-1}^{1} = \frac{2}{3}$$

$$I_2 = \int_0^\infty \frac{a_0}{2} dz \left(\frac{a_0}{2} z\right)^4 e^{-z}$$

$z = 2r/a_0$
$dz = \frac{2}{a_0} dr$

$$= \left(\frac{a_0}{2}\right)^5 \underbrace{\int_0^\infty dz\, z^4 e^{-z}}_{\Gamma(5) = 4! = 24}$$

$$= \frac{24 a_0^5}{32} = \frac{3}{4} a_0^5$$

So

$$\langle z^2 \rangle = \frac{\cancel{2}}{a_0^3} \frac{\cancel{2}}{\cancel{3}} \frac{\cancel{3}}{\cancel{4}} a_0^5$$

$$= a_0^2$$

$$\Rightarrow \boxed{C = \left(\frac{e^2}{4\pi\varepsilon_0}\right)^2 a_0^2} = 1 \text{ in atomic units!}$$