a) $\Psi(x,t) = Ae^{i(k_0x - \omega t)}$

Insert into 1D TDSE

LHS: $i\hbar \frac{\partial \Psi}{\partial t} = i\hbar(-i\omega)\Psi = \hbar\omega \Psi$

RHS: $\frac{-\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} = \frac{-\hbar^2}{2m}(ik_0)^2 = \frac{\hbar^2 k_0^2}{2m}$

Equating LHS and RHS, we get

$$\cancel{\hbar}\omega = \frac{\hbar^{\cancel{2}} k_0^2}{2m}$$

$$\boxed{\omega = \frac{\hbar k_0^2}{2m}} \quad (1)$$

To get the wave speed, we recognize that

$$\Psi(x,t) = Ae^{ik_0(x - \frac{\omega}{k_0}t)}$$

So the whole pattern moves in the +x direction w/ phase velocity

$$\boxed{v_p = \frac{\omega}{k_0} = \frac{\hbar k_0}{2m}} \quad (2)$$

b) For the wave eqn.

LHS: $\frac{\partial^2 \Psi}{\partial t^2} = (-i\omega)^2 \Psi = -\omega^2 \Psi$

RHS: $c^2 \frac{d^2 \Psi}{dt^2} = c^2 (ik_0)^2 \Psi = -k_0^2 c^2 \Psi$

Equating LHS and RHS, we get

$$\boxed{\omega = ck_0} \quad (3)$$

c) Now for the diffusion eqn. and

$$\Psi = \sqrt{\frac{t_0}{t}} \exp\left(\frac{-x^2}{4Dt}\right)$$

LHS: $$\frac{\partial \Psi}{\partial t} = -\frac{1}{2}\frac{t_0^{1/2}}{t^{3/2}} \exp\left(\frac{-x^2}{4Dt}\right) + \sqrt{\frac{t_0}{t}} \exp\left(\frac{-x^2}{4Dt}\right)\left(\frac{+x^2}{4Dt^2}\right)$$

$$= \sqrt{\frac{t_0}{t}} \exp\left(\frac{-x^2}{4Dt}\right)\left[\frac{x^2}{4Dt^2} - \frac{1}{2t}\right]$$

RHS: $$D\frac{\partial^2 \Psi}{\partial x^2} = D\frac{\partial}{\partial x}\left[\sqrt{\frac{t_0}{t}} \exp\left(\frac{-x^2}{4Dt}\right)\left(\frac{-2x}{4Dt}\right)\right]$$

$$= \cancel{D}\sqrt{\frac{t_0}{t}}\left(\frac{-2}{4\cancel{D}t}\right) \exp\left(\frac{-x^2}{4Dt}\right)\left[1 + x\left(\frac{-2x}{4Dt}\right)\right]$$

$$= \sqrt{\frac{t_0}{t}} \exp\left(\frac{-x^2}{4Dt}\right)\left[\frac{x^2}{4Dt^2} - \frac{1}{2t}\right]$$

The LHS agrees w/ the RHS, so Ψ satisfies the diffusion equation.

The width of Ψ is proportional to the $\sqrt{t}$. You can see this by, for example, looking at the time dependence of the $1/e$ half width x_e, which would be when

$$\frac{x_e^2}{4Dt} = 1$$

$$\Rightarrow \boxed{x_e = \sqrt{4Dt}}$$

(4)

d) Now for 1D TDSE

$$\Psi = \sqrt{\frac{\pi}{\alpha+i\beta t}}\, e^{i(k_0 x - \omega_0 t)} \exp\left(-\frac{(x-V_g t)^2}{4(\alpha+i\beta t)}\right) \quad (5)$$

We need to use the chain rule a lot here, lets make some definitions to save writing

$$f \equiv \sqrt{\frac{\pi}{\alpha+i\beta t}}$$

$$g \equiv e^{i(k_0 x - \omega_0 t)}$$

$$h \equiv \exp\left(\frac{-(x-V_g t)^2}{4(\alpha+i\beta t)}\right)$$

So $\Psi = fgh$

Now

$$\frac{\partial f}{\partial t} = \frac{\left(\frac{-1}{2}\right)\sqrt{\pi}}{(\alpha+i\beta t)^{3/2}}(i\beta)$$

$$= -\frac{1}{2}\frac{i\beta}{\alpha+i\beta t} f$$

$$\frac{\partial f}{\partial x} = 0$$

$$\frac{\partial g}{\partial t} = -i\omega_0 g \qquad \frac{\partial h}{\partial x} = \frac{-2(x-V_g t)}{4(\alpha+i\beta t)} h$$

$$\frac{\partial g}{\partial x} = +ik_0 g$$

$$\frac{\partial h}{\partial t} = \left(\frac{-2(x-V_g t)(-V_g)}{4(\alpha+i\beta t)} + \frac{(x-V_g t)^2}{(4(\alpha+i\beta t))^2} 4i\beta\right) h$$

$$\frac{\partial h}{\partial t} = \frac{x - v_g t}{4(\alpha + i\beta t)}\left(2v_g + i\beta \frac{(x-v_g t)}{4(\alpha+i\beta t)}\right) h$$

So the LHS of the TDSE reads

$$i\hbar \frac{\partial \Psi}{\partial t} = i\hbar\left(\frac{\partial f}{\partial t} gh + f\frac{\partial g}{\partial t} h + fg\frac{\partial h}{\partial t}\right)$$

$$= i\hbar fgh\left(\underbrace{-\frac{1}{2}\frac{i\beta}{\alpha+i\beta t}}_{LHS1} \underbrace{- i\omega_0}_{LHS2} + \underbrace{\frac{2v_g(x-v_g t)}{4(\alpha+i\beta t)}}_{LHS3} \underbrace{+ 4i\beta \frac{(x-v_g t)^2}{(4(\alpha+i\beta t))^2}}_{LHS4}\right)$$

The RHS reads

$$-\frac{\hbar^2}{2m} f \frac{\partial}{\partial x}\left(ik_0 gh - \frac{2(x-v_g t)}{4(\alpha+i\beta t)} gh\right)$$

$$= \frac{-\hbar^2}{2m} f\left(ik_0\left(h\frac{\partial g}{\partial x} + g\frac{\partial h}{\partial x}\right) - \frac{2}{4(\alpha+i\beta t)} gh - \frac{2(x-v_g t)}{4(\alpha+i\beta t)}\left(h\frac{\partial g}{\partial x} + g\frac{\partial h}{\partial x}\right)\right)$$

$$= -\frac{\hbar^2}{2m} fgh\left(\underbrace{(ik_0)^2}_{RHS1} \underbrace{- ik_0 \frac{2(x-v_g t)}{4(\alpha+i\beta t)}}_{RHS2} \underbrace{- \frac{2}{4(\alpha+i\beta t)}}_{RHS3}\right.$$

$$\left. \underbrace{- \frac{2(x-v_g t)}{4(\alpha+i\beta t)}\left(ik_0\right.}_{RHS4} \underbrace{\left.- \frac{2(x-v_g t)}{4(\alpha+i\beta t)}\right)}_{RHS5}\right)$$

If we take $\boxed{\omega_0 = \frac{\hbar k_0^2}{2m}}$ as in a) then

RHS1 = LHS2 and if we take

$\boxed{v_g = \frac{\hbar k_0}{m}}$, then RHS 4 + RHS 2 = LHS 3

and we are left with an equation for β

$$\frac{1}{2}\frac{\hbar\beta}{\alpha+i\beta t} - \hbar\beta\frac{4(x-v_g t)^2}{(4(\alpha+i\beta t))^2} = \frac{\hbar^2}{2m}\left(\frac{2}{4(\alpha+i\beta t)} - 4\frac{(x-v_g t)^2}{[4(\alpha+i\beta t)]^2}\right)$$

$$\hbar\beta\left(\cancel{\frac{1}{2}\frac{1}{\alpha+i\beta t} - 4\frac{(x-v_g t)^2}{[4(\alpha+i\beta t)]^2}}\right) = \frac{\hbar^2}{2m}\left(\cancel{\frac{1}{2}\frac{1}{\alpha+i\beta t} - 4\frac{(x-v_g t)^2}{[4(\alpha+i\beta t)]^2}}\right)$$

$$\Rightarrow \boxed{\beta = \frac{\hbar}{2m}}$$

(e) The center moves at $\boxed{v_g = \frac{\hbar k_0}{m}}$

(f) $|\Psi|^2 \propto \exp\left[-(x-v_g t)^2\left(\frac{1}{4(\alpha+i\beta t)} + \frac{1}{4(\alpha-i\beta t)}\right)\right]$

$$= \exp\left[\frac{-(x-v_g t)^2}{4}\left(\frac{2\alpha}{\alpha^2+\beta^2 t^2}\right)\right]$$

As in part c), the width of the Gaussian is determined by the denominator of the exponent. For $\beta t \gg \alpha$, the width of $|\Psi|^2$ will grow as

$$\text{width} \propto t$$

This is faster than for the diffusion equation result, for which the width was proportional to $\sqrt{t}$.

(g) The plane wave sol^n is not normalizable and does not represent a real particle. Only through a superposition of waves that make up a wavepacket can we accurately describe a free particle. The wavepacket moves at the group velocity which is $\hbar k_0/m$. This result is actually independent of the precise functional form of the wavepacket (see Griffiths' Book).

The result of $\hbar k_0/m = v_g$ makes sense b/c classically, we would expect

$$p = mv \quad \text{and} \quad v = \frac{mv}{m} = \frac{p}{m}$$

The plane wave result give $v = \frac{p}{2m}$ which does not correspond to classical mechanics in the appropriate limit $(\hbar \to 0)$

It is also interesting to note that quantum diffusion coefficient $\beta = \frac{\hbar}{2m}$ goes to zero in the classical limit while the group velocity remains finite at p/m b/c for a particle of energy E

$$E = \frac{\hbar^2 k^2}{2m}$$

as $\hbar \to 0$ $k \to \infty$ in the classical limit so $\hbar k/m = v_g \to p_{classical}/m = v.$

$\beta \to 0$ in the classical limit makes sense b/c baseballs retain their size.