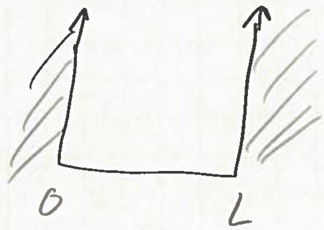
Particle w/ Speed Bump

a)

$$\hat{H}_0 = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} +$$



$$\hat{H}' =$$

$$\psi_n^{(0)} = \begin{cases} \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right) & 0 < x < L \qquad n = 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases}$$

$$E_1^{(1)} = \langle \psi_1^{(0)} | \hat{H}' | \psi_1^{(0)} \rangle$$

$$= \int_{\frac{3}{8}L}^{\frac{5}{8}L} dx \sqrt{\frac{2}{L}}\sin\left(\frac{\pi x}{L}\right) V_0 \sqrt{\frac{2}{L}}\sin\left(\frac{\pi x}{L}\right)$$

$$= \frac{2V_0}{L}\int_{3/8 L}^{5/8 L} dx \sin^2\left(\frac{\pi x}{L}\right) = \frac{2V_0}{L}\int_{3/8 L}^{5/8 L} dx \left(\frac{1-\cos(2\pi x/L)}{2}\right)$$

$$= \frac{V_0}{L}\int_{3/8 L}^{5/8 L} dx - \frac{V_0}{L}\int_{3/8 L}^{5/8 L} dx \cos\left(\frac{2\pi x}{L}\right)$$

$$= \frac{V_0}{4} - \frac{V_0}{L}\frac{L}{2\pi}\left[\sin\left(\frac{2\pi x}{L}\right)\right]_{3/8 L}^{5/8 L}$$

$$= \frac{V_0}{4} - \frac{V_0}{2\pi}\Big[\underbrace{\sin\left(\frac{5\pi}{4}\right)}_{-\frac{\sqrt{2}}{2}} - \underbrace{\sin\left(\frac{3\pi}{4}\right)}_{\frac{\sqrt{2}}{2}}\Big] \qquad (1)$$

$$E_1^{(1)} = \frac{V_0}{4} - \frac{V_0}{2\pi}\left(-\sqrt{2}\right)$$

$$\boxed{E_1^{(1)} = V_0\left(\frac{1}{4} + \frac{\sqrt{2}}{2\pi}\right)}$$

For $n=2$, everything is the same except $\pi \to 2\pi$ in eqn. 1

$$E_2^{(2)} = \frac{V_0}{4} - \frac{V_0}{2\cdot 2\pi}\left[\underbrace{\sin\left(\frac{10\pi}{4}\right)}_{+1} - \underbrace{\sin\left(\frac{6\pi}{4}\right)}_{-1}\right] \quad = 2$$

$$\boxed{E_2^{(2)} = \frac{V_0}{4} - \frac{V_0}{2\pi}}$$

b)

$$|\psi_1^{(1)}\rangle = \sum_{m=2}^{\infty} \frac{\langle \psi_m^{(0)} | \hat{H}' | \psi_1^{(0)} \rangle}{E_1^{(0)} - E_m^{(0)}} |\psi_m^{(0)}\rangle$$

$$\psi_1^{(1)}(x) = \sum_{m=2}^{\infty} \frac{V}{(1^2 - m^2)\frac{m^2h^2}{8mL^2}} \underbrace{\left[\int_{\frac{3}{8}L}^{\frac{5}{8}L} dx \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{\pi x}{L}\right)\right]}_{\downarrow} \psi_m^{(0)}(x)$$

$$\left[\frac{\sin\left(\frac{\pi x}{L}(m-1)\right)}{\frac{2\pi}{L}(m-1)} - \frac{\sin\left(\frac{\pi x}{L}(m+1)\right)}{\frac{2\pi}{L}(m+1)}\right]_{x=3/8 L}^{x=5/8 L}$$

Every other term is zero b/c since the perturbation is even around $x = 1/2$, the product $\psi_m^* \psi_1$ needs to be even as well, which means only even m will contribute.