a)

$$\psi = e^{-\alpha x^2}$$

Using the integral tables in the back of Levine

$$\langle\psi|\psi\rangle = \int_{-\infty}^{\infty} dx\, e^{-2\alpha x^2} \qquad u = \sqrt{2\alpha}\, x \qquad du = \sqrt{2\alpha}\, dx$$

$$= \frac{1}{\sqrt{2\alpha}} \underbrace{\int_{-\infty}^{\infty} du\, e^{-u^2}}_{\sqrt{\pi}}$$

$$\boxed{\langle\psi|\psi\rangle = \frac{\sqrt{\pi}}{\sqrt{2\alpha}}} \qquad (1)$$

$$\langle\psi|\hat{H}|\psi\rangle = \frac{-\hbar^2}{2m}\int_{-\infty}^{\infty} dx\, \psi \frac{d^2\psi}{dx^2} + \frac{1}{2}m\omega^2 \int_{-\infty}^{\infty} dx\, \psi x^2 \psi$$

$$\frac{d^2\psi}{dx^2} = \frac{d}{dx}\left(-2\alpha x\, e^{-\alpha x^2}\right)$$

$$= -2\alpha\left(e^{-\alpha x^2} + (-2\alpha x) x\, e^{-\alpha x^2}\right)$$

$$= -2\alpha\, e^{-\alpha x^2}\left(1 - 2\alpha x^2 e^{-\alpha x^2}\right)$$

So

$$\langle\psi|\hat{H}|\psi\rangle = \frac{2\hbar^2\alpha}{2m}\int_{-\infty}^{\infty} dx\, e^{-2\alpha x^2}$$

$$+ \left(\frac{1}{2}m\omega^2 - \frac{2\hbar^2\alpha^2}{m}\right)\int_{-\infty}^{\infty} dX\, X^2 e^{-2\alpha x^2}$$

↓ Integral Table

$$= \frac{\hbar^2\alpha}{m}\sqrt{\frac{\pi}{2\alpha}} + \left(\frac{1}{2}m\omega^2 - \frac{2\hbar^2\alpha^2}{m}\right) \cancel{2}\left(\frac{\cancel{2!}}{2^3\, 1!}\right)\left(\frac{\pi}{(2\alpha)^3}\right)^{1/2}$$

$$\frac{\hbar^2}{m}\sqrt{\frac{\pi}{2}}\,\alpha^{+1/2} + \left(\frac{1}{2}m\omega^2 - \frac{2\hbar^2}{m}\alpha^2\right)\left(\frac{\pi^{1/2}}{2^{5/2}}\right)\alpha^{-3/2}$$

$$\left(\frac{\hbar^2}{m}\sqrt{\frac{\pi}{2}} - \frac{\hbar^2}{m}\frac{\pi^{1/2}}{2^{3/2}}\right)\alpha^{+1/2} + \frac{1}{2}m\omega^2\frac{\pi^{1/2}}{2^{5/2}}\alpha^{-3/2}$$

$$\boxed{\frac{\hbar^2}{2m}\sqrt{\frac{\pi}{2}}\,\alpha^{1/2} + m\omega^2\frac{\pi^{1/2}}{2^{7/2}}\alpha^{-3/2}} \tag{2}$$

So

$$\frac{\langle\psi|\hat{H}|\psi\rangle}{\langle\psi|\psi\rangle} = \frac{\frac{\hbar^2}{2m}\cancel{\sqrt{\frac{\pi}{2}}}\,\alpha^{\cancel{1/2}\,1} + m\omega^2\frac{\cancel{\sqrt{\pi}}}{2^{\cancel{7/2}\,3}}\alpha^{\cancel{-3/2}\,-1}}{\cancel{\sqrt{\frac{\pi}{2}}}\,\cancel{\alpha^{-1/2}}}$$

$$\boxed{\frac{\langle\psi|\hat{H}|\psi\rangle}{\langle\psi|\psi\rangle} = \frac{\hbar^2}{2m}\alpha + \frac{m\omega^2}{8}\alpha^{-1}} \tag{3}$$

So now we minimize this w/ respect to α by setting the derivative of eqn (3) to zero

$$\frac{\partial}{\partial\alpha}(3) \to 0 \Rightarrow \frac{\hbar^2}{2m} - \frac{m\omega^2}{8}\alpha^{-2} = 0$$

$$\alpha^{-2} = \frac{8\hbar^2}{2m^2\omega^2} = \frac{4\hbar^2}{m^2\omega^2} \tag{4}$$

$$\Rightarrow \boxed{\alpha = \frac{m\omega}{2\hbar}}$$

The (−) solution to the quadratic eqn (4) is nonsense because then ψ would not be normalizable

This is the exact result because the basis function chosen is of the same functional form as the Harmonic oscillator ground state. Usually we are not so lucky.

b) SEE MATLAB CODE