a) Take $|\psi\rangle = c_1|\beta\rangle + c_2|\gamma\rangle$ w/ c_1 and c_2 arbitrary

$$\hat{H}|\psi\rangle = c_1 E_2|\beta\rangle + c_2 E_2|\gamma\rangle$$

$$= E_2(c_1|\beta\rangle + c_2|\gamma\rangle)$$

$$\Rightarrow \boxed{\hat{H}|\psi\rangle = E_2|\psi\rangle} \qquad (1)$$

b) The matrix elements are $\langle\psi_1|H|\psi_2\rangle$. Since $|\alpha\rangle$, $|\beta\rangle$, and $|\gamma\rangle$ are orthogonal and eigenkets of A, the off diagonal terms such as $\langle\alpha|\hat{H}|\gamma\rangle$ will vanish, and we have

$$\boxed{H = \begin{bmatrix} E_1 & 0 & 0 \\ 0 & E_2 & 0 \\ 0 & 0 & E_2 \end{bmatrix}} \qquad (2)$$

c) The matrix for A is found in similar fashion

$$\langle\alpha|\hat{A}|\alpha\rangle = a$$

$$\langle\alpha|\hat{A}|\beta\rangle = \langle\beta|\hat{A}|\alpha\rangle = 0$$

$$\langle\alpha|\hat{A}|\gamma\rangle = \langle\gamma|\hat{A}|\alpha\rangle = 0$$

$$\langle\beta|\hat{A}|\beta\rangle = \langle\beta|(b|\beta\rangle + c|\gamma\rangle) = b$$

$$\langle\beta|\hat{A}|\gamma\rangle = \langle\beta| -ic|\beta\rangle + b|\gamma\rangle) = -c$$

$$\langle\gamma|\hat{A}|\gamma\rangle = \langle\gamma|(-ic|\beta\rangle + b|\gamma\rangle) = b$$

$$\langle\gamma|\hat{A}|\beta\rangle = \langle\gamma|(ic|\beta\rangle + ic|\gamma\rangle) = c$$

Putting it all together

$$\boxed{A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & -ic \\ 0 & ic & b \end{bmatrix}} \qquad (3)$$

We can verify that this is correct by doing $A|\beta\rangle$ in matrix/coloumn vector notation

$$A|\beta\rangle \rightarrow \begin{bmatrix} a & 0 & 0 \\ 0 & b & -ic \\ 0 & ic & b \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ b \\ c \end{bmatrix} \rightarrow b|\beta\rangle + c|\gamma\rangle \qquad (4)$$

✓

Does $\hat{A}$ commute with $\hat{H}$?

$$AH = \begin{bmatrix} a & 0 & 0 \\ 0 & b & -ic \\ 0 & ic & b \end{bmatrix} \begin{bmatrix} E_1 & 0 & 0 \\ 0 & E_2 & 0 \\ 0 & 0 & E_2 \end{bmatrix}$$

$$\begin{bmatrix} aE_1 & 0 & 0 \\ 0 & E_2 b & -icE_2 \\ 0 & iE_2 c & E_2 b \end{bmatrix} \leftarrow AH$$

$$HA = \begin{bmatrix} E_1 & 0 & 0 \\ 0 & E_2 & 0 \\ 0 & 0 & E_2 \end{bmatrix} \begin{bmatrix} a & 0 & 0 \\ 0 & b & ic \\ 0 & ic & b \end{bmatrix} \qquad (5)$$

$$\begin{bmatrix} aE_1 & 0 & 0 \\ 0 & bE_2 & -icE_2 \\ 0 & icE_2 & bE_2 \end{bmatrix} \leftarrow$$ HA is the same as AH

so $[A, H] = 0$

d) Now we can use the fact that commuting observables can have simultaneous eigenkets, so if we diagonalize A and find its eigenvectors, they will be eigenvectors of H.

To find the eigenvalues of A

$$\begin{vmatrix} a-\lambda & 0 & 0 \\ 0 & b-\lambda & -ic \\ 0 & ic & b-\lambda \end{vmatrix} = (a-\lambda)\left((b-\lambda)^2 - c\right) = 0$$

$$\Rightarrow \lambda = a \quad \text{or} \quad \lambda_{\pm} = b \pm c$$

$|\alpha\rangle \leftrightarrow \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is one obvious eigenvector w/ eigenvalue a. The other eigenvectors are found from solving

$$\lambda_+ \Rightarrow \begin{bmatrix} b & -ic \\ ic & b \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = (b+c) \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

$$bV_1 - icV_2 = (b+c)V_1$$

$$icV_1 + bV_2 = (b+c)V_2$$

$$\Rightarrow \boxed{V_1 = -iV_2} \Rightarrow \chi_+ = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ -i \\ 1 \end{bmatrix} \quad (6)$$

Similarly

$$\text{for } \lambda_- \Rightarrow \boxed{V_1 = +iV_2} \Rightarrow \chi_- = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ +i \\ 1 \end{bmatrix} \quad (7)$$

Or in terms of ket notation, we have new simultaneous eigenkets

$$\boxed{\begin{aligned} |E_1, a\rangle &= |\alpha\rangle \\ |E_2, b+c\rangle &= \frac{1}{\sqrt{2}}\left(-i|\beta\rangle + |\gamma\rangle\right) \\ |E_2, b-c\rangle &= \frac{1}{\sqrt{2}}\left(+i|\beta\rangle + |\gamma\rangle\right) \end{aligned}} \qquad (8)$$