a) $\langle \hat{L}_z \rangle = \langle \Psi | \hat{L}_z | \Psi \rangle = \langle \Psi | \hbar l | \Psi \rangle$

$= \hbar l \langle \Psi | \Psi \rangle$

$$\boxed{\langle \hat{L}_z \rangle = \hbar l}$$

b) The raising and lowering operators

$\hat{L}_\pm = \hat{L}_x \pm i \hat{L}_y$

can be used to show this. Since $\hat{L}^2 |\Psi\rangle = l(l+1)\hbar^2 |\Psi\rangle$, we are on the top rung and

$$\hat{L}_+ |\Psi\rangle = 0$$

so $\langle \Psi | \hat{L}_+ | \Psi \rangle$

$$= \langle \Psi | \hat{L}_x | \Psi \rangle + i \langle \Psi | \hat{L}_y | \Psi \rangle \quad (1)$$

Now, since $\hat{L}_x$, $\hat{L}_y$ are Hermitian, they must have real expectation values. This means that both terms in (1) must be zero independently, or

$$\boxed{\langle \hat{L}_x \rangle = \langle \hat{L}_y \rangle = 0}$$

c) $\langle \hat{L}^2 \rangle - \langle \hat{L}_z^2 \rangle = \langle \hat{L}_x^2 \rangle + \langle \hat{L}_y^2 \rangle$

$\hbar^2 l(l+1) - \hbar^2 l^2 = 2 \langle \hat{L}_x^2 \rangle$

$$\Rightarrow \boxed{\langle \hat{L}_x^2 \rangle = \langle \hat{L}_y^2 \rangle = \frac{\hbar^2 l}{2}}$$

d) $\sigma_{L_x}^2 = \langle \hat{L}_x^2 \rangle - \cancel{\langle \hat{L}_x \rangle^2}^{\,0}$

$$= \frac{\hbar^2 l}{2}$$

$\sigma_{L_y}^2 = \frac{\hbar^2 l}{2}$ by same reasoning

Generalized uncertainty princ.

$[\hat{L}_x, \hat{L}_y] = i\hbar \hat{L}_z$

$$\sigma_{L_x} \sigma_{L_y} \overset{?}{\geq} \frac{1}{2} |\langle [\hat{L}_x, \hat{L}_y] \rangle|$$

$$\frac{\hbar^2 l}{2} \overset{?}{\geq} \frac{1}{2} | i\hbar \langle \hat{L}_z \rangle |$$

$$\frac{\hbar^2 l}{2} \overset{?}{\geq} \frac{1}{2} | i\hbar \, \hbar l |$$

$$\frac{\hbar^2 l}{2} \geq \frac{\hbar^2 l}{2}$$

✓ satisfies gen. unc. princ.