a) $$\left[\frac{1}{x}, \hat{p}_x\right] = -i\hbar\left[\frac{1}{x}, \frac{\partial}{\partial x}\right]$$

It is often helpful to use a test function to not get mixed up

$$\left[\frac{1}{x}, \frac{\partial}{\partial x}\right] f = \frac{1}{x}\frac{\partial f}{\partial x} - \frac{\partial}{\partial x}\left(\frac{1}{x} f\right)$$

$$= \cancel{\frac{1}{x}\frac{\partial f}{\partial x}} - \left(-\frac{1}{x^2} f + \cancel{\frac{1}{x}\frac{\partial f}{\partial x}}\right)$$

$$= \frac{1}{x^2} f$$

$$\Rightarrow \boxed{\left[\frac{1}{x}, \hat{p}_x\right] = \frac{-i\hbar}{x^2}} \qquad (1)$$

b) $$[V, \hat{p}] f = V\left(-i\hbar\frac{\partial f}{\partial x}\right) - \left(-i\hbar\frac{\partial}{\partial x}\right)(Vf)$$

$$= \cancel{-i\hbar V\frac{\partial f}{\partial x}} + i\hbar f\frac{\partial V}{\partial x} + \cancel{i\hbar V\frac{\partial f}{\partial x}}$$

$$= i\hbar f\frac{\partial V}{\partial x}$$

$$\Rightarrow [V, \hat{p}] = i\hbar\frac{\partial V}{\partial x}$$

c) $$[x\hat{p}_y - y\hat{p}_x, y\hat{p}_z - z\hat{p}_y] = [x\hat{p}_y, y\hat{p}_z - z\hat{p}_y] - [y\hat{p}_x, y\hat{p}_z - z\hat{p}_y]$$

$$= [x\hat{p}_y, y\hat{p}_z] - [x\hat{p}_y, z\hat{p}_y] - [y\hat{p}_x, y\hat{p}_z] + [y\hat{p}_x, z\hat{p}_y] \qquad (2)$$

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Now the 2nd and 3rd commutators here are obviously zero because the derivatives do not involve the coordinates.

The remaining commutators can be rearranged using the identities

$$[AB, C] = A[B, C] + [A, C]B$$

$$[A, BC] = B[A, C] + [A, B]C$$

so $$[x\hat{p}_y, y\hat{p}_z] = x[\hat{p}_y, y\hat{p}_z] + \cancelto{0}{[x, y\hat{p}_z]}\hat{p}_y$$

$$= x\left(y\cancelto{0}{[\hat{p}_y, \hat{p}_z]} + [\hat{p}_y, y]\hat{p}_z\right)$$

$$= -i\hbar\, x\hat{p}_z$$

And $$[y\hat{p}_x, z\hat{p}_y] = \cancelto{0}{x[\hat{p}_x, z\hat{p}_y]} + [y, z\hat{p}_y]\hat{p}_x$$

$$= \left(z[y, \hat{p}_y] + \cancelto{0}{[y, z]}\hat{p}_y\right)\hat{p}_x$$

$$= z\hat{p}_x(i\hbar)$$

So the final result is

$$\boxed{[x\hat{p}_y - y\hat{p}_x, y\hat{p}_z - z\hat{p}_y] = i\hbar(zp_x - xp_z)}$$

If you already know about angular momentum, you can identify this as a commutator relation for orbital angular momentum

$$\boxed{[\hat{L}_z, \hat{L}_x] = i\hbar\hat{L}_y}$$

(d) $$\left[x^2\frac{\partial^2}{\partial y^2}, y\frac{\partial}{\partial x}\right] = x^2\left[\frac{\partial^2}{\partial y^2}, y\frac{\partial}{\partial x}\right] + \left[x^2, y\frac{\partial}{\partial x}\right]\frac{\partial^2}{\partial y^2}$$

$$= -x^2\left(y\cancelto{0}{\left[\frac{\partial}{\partial x}, \frac{\partial^2}{\partial y^2}\right]} + \left[y, \frac{\partial^2}{\partial y^2}\right]\frac{\partial}{\partial x}\right)$$

$$- \left(y\left[\frac{\partial}{\partial x}, x^2\right] + \cancelto{0}{\left[y, x^2\right]}\frac{\partial}{\partial x}\right)\frac{\partial^2}{\partial y^2}$$

$$= +x^2\left[\frac{\partial^2}{\partial y^2}, y\right]\frac{\partial}{\partial x} + y\left[x^2, \frac{\partial}{\partial x}\right]\frac{\partial^2}{\partial y^2}$$

$$= x^2\left(\frac{\partial}{\partial y}\left[\frac{\partial}{\partial y}, y\right] + \left[\frac{\partial}{\partial y}, y\right]\frac{\partial}{\partial y}\right)\frac{\partial}{\partial x}$$

$$+ y\left(x\left[x, \frac{\partial}{\partial x}\right] + \left[x, \frac{\partial}{\partial x}\right]x\right)\frac{\partial^2}{\partial y^2}$$

Now this is broken down into the elementary commutator

$$\left[q, \frac{\partial}{\partial q}\right]f = q\frac{\partial f}{\partial q} - \frac{\partial}{\partial q}(qf)$$

$$= -f$$

$$\left[q, \frac{\partial}{\partial q}\right] = -1$$

So we have

$$\left[x^2\frac{\partial^2}{\partial y^2}, y\frac{\partial}{\partial x}\right] = x^2\left(1 + 1\frac{\partial}{\partial y}\right)\frac{\partial}{\partial x} + y(-x - x)\frac{\partial^2}{\partial y^2}$$

$$= \boxed{2x^2\frac{\partial^2}{\partial x\partial y} - 2xy\frac{\partial^2}{\partial y^2}}$$