PHY 308 Midterm 2, Spring 2017
Stony Brook University
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100 points total. The following information may or may not be useful

Planck's constant: $\hbar = 1.055 \times 10^{-34}$ J-s $= 6.582 \times 10^{-16}$ eV-s
Speed of light: $c = 3 \times 10^8$ m/s
Mass of the electron: $m_e = 9.1 \times 10^{-31}$ kg
The electron volt: 1 eV $= 1.6 \times 10^{-19}$ J
The fine structure constant $\alpha = 1/137$

Table 4.2: The first few spherical harmonics, $Y_l^m(\theta, \phi)$.

$Y_0^0 = \left(\frac{1}{4\pi}\right)^{1/2}$	$Y_2^{\pm2} = \left(\frac{15}{32\pi}\right)^{1/2} \sin^2\theta e^{\pm 2i\phi}$
$Y_1^0 = \left(\frac{3}{4\pi}\right)^{1/2} \cos\theta$	$Y_3^0 = \left(\frac{7}{16\pi}\right)^{1/2} (5\cos^3\theta - 3\cos\theta)$
$Y_1^{\pm1} = \mp\left(\frac{3}{8\pi}\right)^{1/2} \sin\theta e^{\pm i\phi}$	$Y_3^{\pm1} = \mp\left(\frac{21}{64\pi}\right)^{1/2} \sin\theta(5\cos^2\theta - 1)e^{\pm i\phi}$
$Y_2^0 = \left(\frac{5}{16\pi}\right)^{1/2} (3\cos^2\theta - 1)$	$Y_3^{\pm2} = \left(\frac{105}{32\pi}\right)^{1/2} \sin^2\theta\cos\theta e^{\pm 2i\phi}$
$Y_2^{\pm1} = \mp\left(\frac{15}{8\pi}\right)^{1/2} \sin\theta\cos\theta e^{\pm i\phi}$	$Y_3^{\pm3} = \mp\left(\frac{35}{64\pi}\right)^{1/2} \sin^3\theta e^{\pm 3i\phi}$

TABLE 4.7: The first few radial wave functions for hydrogen, $R_{nl}(r)$.

$R_{10} = 2a^{-3/2}\exp(-r/a)$
$R_{20} = \frac{1}{\sqrt{2}}a^{-3/2}\left(1 - \frac{1}{2}\frac{r}{a}\right)\exp(-r/2a)$
$R_{21} = \frac{1}{\sqrt{24}}a^{-3/2}\frac{r}{a}\exp(-r/2a)$
$R_{30} = \frac{2}{\sqrt{27}}a^{-3/2}\left(1 - \frac{2}{3}\frac{r}{a} + \frac{2}{27}\left(\frac{r}{a}\right)^2\right)\exp(-r/3a)$
$R_{31} = \frac{8}{27\sqrt{6}}a^{-3/2}\left(1 - \frac{1}{6}\frac{r}{a}\right)\left(\frac{r}{a}\right)\exp(-r/3a)$
$R_{32} = \frac{4}{81\sqrt{30}}a^{-3/2}\left(\frac{r}{a}\right)^2\exp(-r/3a)$
$R_{40} = \frac{1}{4}a^{-3/2}\left(1 - \frac{3}{4}\frac{r}{a} + \frac{1}{8}\left(\frac{r}{a}\right)^2 - \frac{1}{192}\left(\frac{r}{a}\right)^3\right)\exp(-r/4a)$
$R_{41} = \frac{\sqrt{5}}{16\sqrt{3}}a^{-3/2}\left(1 - \frac{1}{4}\frac{r}{a} + \frac{1}{80}\left(\frac{r}{a}\right)^2\right)\frac{r}{a}\exp(-r/4a)$
$R_{42} = \frac{1}{64\sqrt{5}}a^{-3/2}\left(1 - \frac{1}{12}\frac{r}{a}\right)\left(\frac{r}{a}\right)^2\exp(-r/4a)$
$R_{43} = \frac{1}{768\sqrt{35}}a^{-3/2}\left(\frac{r}{a}\right)^3\exp(-r/4a)$

1 True/False

Circle **T** if the statement is always true. Otherwise circle **F** for false. 2 points each

T F The energy levels of the free rigid rotor are evenly spaced.

T F The energy levels of the free rigid rotor are nondegenerate.

T F Within an accuracy of $\alpha^2 m_e c^2$ (i.e. neglecting fine structure, etc.) the energy levels of the hydrogen atom are degenerate with dengeracy n^2 where n is the principal quantum number.

T F The wave function describing the positions of two Fermions must always be antisymmetric with respect to interchange of the particles' spatial positions and spins.

T F The variational principle states that for any quantum system, the expectation value of the Hamiltonian evaluated using any normalized trial wave function will always be larger than the energy of the system's ground state.

2 Multiple Choice

Circle **one** answer for each question. 3 points each.

Which property applies to identical particles that are labeled "Fermions":
a) they possess integer spin angular momentum
b) their angular momentum cannot be added.
c) they always have $\langle \hat{S}^2 \rangle = 0$
d) they obey the Pauli principle.

The atomic unit of length is (circle the closest answer):
a) 0.529 nm
b) 0.529 Å
c) 0.529 μm
d) 13.6 eV

Which of the following is *not* a valid set of quantum numbers for the orbitals of the hydrogen atom $\psi_{nlm}(r, \theta, \phi) = R_{nl}(r) Y_\ell^m(\theta, \phi)$ with energy -13.6 eV/n^2:
a) $n = 1, \ell = 0, m = 0$
b) $n = 2, \ell = 0, m = 1$
c) $n = 2, \ell = 1, m = -1$
d) $n = 5, \ell = 2, m = 1$
e) none of the above.

If the spin angular momenta of two spin-1 particles are added, the possible m values for the z-component of the total spin angular momentum (such that $\hat{S}_z |\psi\rangle = m\hbar |\psi\rangle$) are
a) m = -1, 0, or 1
b) m = -2, 1, 0, 1, or 2
c) m = 2, 0, or -2
d) m = -2,-3/2, -1, -1/2, 0, 1/2, 1, 3/2, or 2
e) m = 2, 1, or 0
f) none of the above.

3 Spin Statistics

Explain the difference between a Boson and a Fermion. Explain the symmetry requirements on the wave functions of identical Bosons and Fermions and how this is related to the Pauli exclusion principle.

4 The Quantum Bouncing Ball

Consider a particle of mass m in a one dimensional potential given by

$$V(x) = \begin{cases} Ax, & x > 0 \\ \infty, & x \leq 0 \end{cases} \tag{4.1}$$

This problem can be solved exactly using Airy functions but here we will use the variational principle. Consider an un-normalized trial solution of the form

$$\psi(x) = \begin{cases} (\alpha x)e^{-\alpha x}, & x > 0 \\ 0, & x \leq 0 \end{cases} \tag{4.2}$$

Here α is a variational parameter that you can vary to minimize the energy. Use the variational principle to find the optimum value of α and the corresponding minimum energy. You may find the following integral relation useful

$$\int_0^\infty u^n e^{-u} du = n! \tag{4.3}$$

5 A Proton Spin in a Changing Magnetic Field

Consider a proton, with gyromagnetic ratio γ, immersed in a uniform magnetic field B_0 in the z direction. At times $t \leq 0$, the spin is aligned along the magnetic field in the z direction. At t = 0, the magnetic field in the z direction is suddenly turned off and replaced with a magnetic field of strength B_0 in the x-direction. Describe the dynamics, if there are any, of the proton spin for times $t \geq 0$. Half credit for a qualitative only explanation. Full credit for a qualitative explanation plus an expression for the expectation of the spin vector $\langle \vec{I} \rangle (t)$.

6 Rotational Selection Rules

Consider a polar diatomic molecule with permanent dipole moment μ_0 whose rotations are well described by the rigid rotor, such that its rotational wave functions are given by

$$\psi_{\text{rot}} = Y_J^m(\theta, \phi) = i^{m+|m|} N_{Jm} P_J^m(\cos\theta) e^{im\phi} \tag{6.1}$$

where P_J^m is the associated Legendre polynomial and N_{Jm} is a normalization constant that does not concern us here.

a) Consider the interaction of this molecule with microwaves polarized in the z-direction. Write down an integral expression for the dipole matrix element $\langle J', m' | \hat{\mu}_z | J, m \rangle$ in spherical coordinates, where $\hat{\mu}_z = \mu_0 \cos\theta$.

b) Use your expression to derive the rotation selection rules $\Delta m = 0$, and $\Delta J = \pm 1$.

For part b), the following recursion relation may be useful:

$$(2J+1)xP_J^m(x) = (J-m+1)P_{J+1}^m(x) + (J+m)P_{J-1}^m(x) \tag{6.2}$$

along with the orthogonality relation

$$\int_{-1}^{1} dx P_{J'}^m(x) P_J^m(x) = \frac{2(J+m)!}{(2J+1)(J-m)!}\delta_{JJ'} \tag{6.3}$$

Also remember that you are only asked to show when something is non-zero. You do not actually have to produce a numerical result, so you do not need to keep track of all the constants!