Version 2

PHY 308 Midterm 1, Spring 2017

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100 points total. The following information may or may not be useful

Planck's constant: $\hbar = 1.055 \times 10^{-34}$ J-s $= 6.582 \times 10^{-16}$ eV-s
Speed of light: $c = 3 \times 10^8$ m/s
Mass of the electron: $m_e = 9.1 \times 10^{-31}$ kg
The electron volt: 1 eV $= 1.6 \times 10^{-19}$ J
$\hbar^2/2m_e = 3.81$ eV-Å^2.

1 True/False (18 points)

Circle **T** if the statement is always true. Otherwise circle **F** for false. 2 points each

T F The state function of a quantum system is always equal to a function of time multiplied by a function of the coordinates.

T F Every linear combination of eigenfunctions of the Hamiltonian is an eigenfunction of the Hamiltonian.

T F The probability density is independent of time for a stationary state.

T F Eigenkets of a Hermitian operator with different eigenvalues are always orthogonal.

T F If the Hermitian operator $\hat{B}$ corresponds to a physical property of the quantum-mechanical system, the state function $|\Psi\rangle$ must be an eigenfunction of $\hat{B}$.

T F All eigenfunctions of Hermitian operators must be real functions.

T F When the state function $|\Psi\rangle$ is an eigenket of the Hermitian operator $\hat{B}$ with eigenvalue b, we are certain to measure b when the observable corresponding to $\hat{B}$ is measured.

T F The magnitude of the absorption coefficient for microwave radiation by a gas of NO molecules will be similar to that of a gas of O_2 molecules because their moments of inertia are similar.

T F The energy difference between the $n = 1$ and $n = 0$ states of the harmonic oscillator is the same as the energy difference between the $n = 2$ and $n = 1$ states of the harmonic oscillator.

2 Multiple Choice (18 points)

Circle **one** answer for each question. 3 points each.

Consider a particle moving in 1 dimension ~~at~~ described by the wave function $\Psi(x, t_0) = A(x)e^{ikx}$, with A(x) real. The probability current evaluated at t_0 is

at $t = t_0$

a) 0 everywhere.
b) $[\hbar k/m]|\Psi(x, t_0)|^2$
c) $[\hbar k/(2m)]|\Psi(x, t_0)|^2$
d) $-[\hbar k/m]|\Psi(x, t_0)|^2$
e) none of the above.

Consider a quantum bouncing ball, moving in one dimension with the potential $V(y) = mgy$ for $y > 0$ and $V(y) = \infty$ for $y \leq 0$. The boundary conditions on the stationary states $\psi_n(y)$ are

a) $\psi_n(y \leq 0) = 0$, $\psi_n(y = \infty) = 0$
b) $\psi_n(y \leq 0) = 0$, $\psi_n(y = \infty) = 0$, and $\frac{d\psi_n}{dy}$ continuous at $x = 0$
c) $\psi_n(y \leq 0) = 0$, $\psi_n(y = \infty) = 0$, $\frac{d\psi_n}{dy}$ continuous at $x = 0$, and ψ_n must be either an even or odd function of y
d) $\psi_n(y \leq 0) = 0$, $\psi_n(y = \infty) = 0$, and all derivatives of ψ_n continuous at $x = 0$

The simple harmonic oscillator potential, $V(x) = \frac{1}{2}m\omega^2 x^2$, has which of the following properties:

a) The energy levels are evenly spaced and given by $(n + 1/2)\hbar\omega$, with $n = 0, 1, 2, ...$
b) With x the displacement from equilibrium, it is the lowest-order approximation to the potential energy of the nuclei in a diatomic molecule near equilibrium, with m the reduced mass
c) The number of nodes in the n^{th} energy eigenstate wavefunction, with corresponding energy $(n + 1/2)\hbar\omega$, is n.
d) all of the above.
e) a) and c) only.

For a particle moving in 1 dimension with Hamiltonian $\hat{H}(\hat{x}, \hat{p})$, the stationary states are:

a) Solutions to the time-independent Schrödinger equation, $\hat{H}\psi = E\psi$.
b) States with $\langle \hat{p} \rangle = 0$. e^{ikx}
c) Eigenfunctions of the position operator, $\hat{x}$.
d) all of the above
e) a) and c) only.
f) none of the above.

A system is in a quantum state $|\Psi\rangle$ and one makes a measurement of the observable $\hat{O}$. According to quantum theory, the result of the measurement will be

a) the expectation value of $\hat{O}$, given by $\langle \Psi | \hat{O} | \Psi \rangle$.
b) a random value ~~between~~ with probability given by a a normal (Gaussian) ~~probability~~ distribution with mean $\langle \Psi | \hat{O} | \Psi \rangle$ and standard deviation $\sqrt{\langle \Psi | \hat{O}^2 | \Psi \rangle - \langle \Psi | \hat{O} | \Psi \rangle^2}$.
c) that the measurement will collapse the wavefunction to one of the stationary states $|\psi_n\rangle$, such that one will measure $\langle \psi_n | \hat{O} | \psi_n \rangle$.
d) one of the eigenvalues λ_n of $\hat{O}$ with probability $|\langle \lambda_n | \Psi \rangle|^2$, where $|\lambda_n\rangle$ are the normalized eigenkets of $\hat{O}$ with eigenvalues λ_n.
e) none of the above.

When an operator $\hat{A}$ commutes with the Hamiltonian $\hat{H}$ of a quantum system, it means that

a) its expectation value $\langle \hat{A} \rangle$ is a constant of the motion.
b) one can construct simultaneous eigenkets of $\hat{A}$ and $\hat{H}$.
c) The set of expectation values of $\hat{A}$ calculated using the stationary states, $\langle \psi_n | \hat{A} | \psi_n \rangle$, are non-degenerate.
d) all of the above.
e) a) and b) only.
f) b) and c) only.

3 The Classical Limit (12 points)

Explain when it is appropriate to use classical mechanics to describe the motion of a particle, and when quantum mechanics must be used. You are encouraged to use a combination of words, equations, and pictures, as appropriate.

The classical limit occurs when $xp \gg \hbar$, such that the uncertainty principle $\sigma_x \sigma_p \geq \hbar/2$ does not present practical limits. One can also think of CM as the $\hbar \to 0$ limit of QM.

For systems in thermal equilibrium, another useful metric is the probability of finding the system in any one quantum state. If $\frac{\Delta E}{kT} \ll 1$, then this probability is small, and one is probably OK w/ CM.

In terms of wave mechanics $xp \gg \hbar$ corresponds to $\frac{x}{\lambda_{DB}} \gg 2$. If a particle is confined to a dimension x, CM is appropriate when x is much larger than the particle's de Broglie wavelength.

4 Resonant Tunneling Diodes (15 points)

Consider an electron with kinetic energy $E = 0.1$ eV incident from the left ($x < 0$) on the potential shown in figure 1.

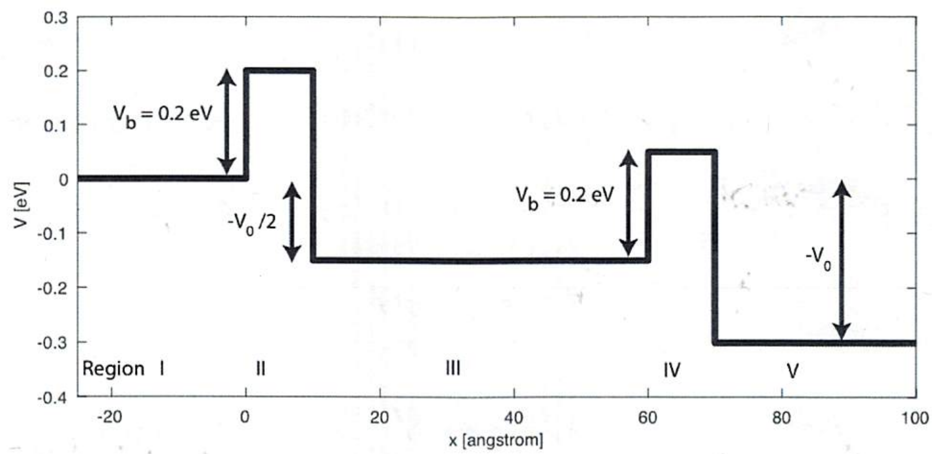


Figure 1: Scattering potential for problem 4. A 0.1 eV electron is incident from the left. The barriers are 0.2 eV high and 10 Å thick. The spacing between the barriers is 50 Å. The wave function has a different form in each labeled region.

a) Write down the general form of the wavefunction in each region of the potential in terms of plane waves or exponential decays. For example, in region I, we have $E > V$, so we can write the wave function as a sum of leftward (incident) and rightward (reflected) going plane waves $\psi(x) = Ae^{ikx} + Be^{-ikx}$, with $k = \sqrt{\frac{2mE}{\hbar^2}}$

b) Apply the boundary conditions at $x = 0$ to your answer to part a) to relate the parameters of the wave function in region I to the parameters of the wave function in region II. Just set up the equations, do not worry about solving them.

c) Why do you get the wrong answer for the transmission coefficient in this problem (and $R + T \neq 1$) if you simply use the square modulus of the amplitude of the wave function in region V to calculate T? How do you properly calculate T

d) As the bias potential is swept from 0 - 2 eV, one observes resonances in the transmission probability (I-V curve) of this device. Why?

e) Can this potential support bound states trapped between the two barriers? Why or why not?

a) I $\psi = Ae^{ik_1x} + Be^{-ik_1x}$ $\qquad k_1 = \sqrt{\frac{2mE}{\hbar^2}}$

II $\psi = Ce^{-K_2x} + De^{+K_2x}$ $\qquad K_2 = \sqrt{\frac{2m(V_b - E)}{\hbar^2}}$

III $\psi = Fe^{ik_3x} + Ge^{-ik_3x}$ $\qquad k_3 = \sqrt{\frac{2m(E + V_0/2)}{\hbar^2}}$

More space for 4

IV $\psi = He^{-K_4 x} + Ie^{+K_4 x}$ $\quad K_4 = \sqrt{\frac{2m(V_b - \frac{V_0}{2} - E)}{\hbar^2}}$

↑ Note that this still works for $E > V_b - V_0/2$ b/c K becomes imaginary.

V $\psi = Le^{ik_5 x}$ w/ $K = \sqrt{\frac{2m(E-V_0)}{\hbar^2}}$

Note that there is no reflected wave

b) $\psi_I(0) = \psi_{II}(0) \Rightarrow$ $\boxed{A + B = C + D}$

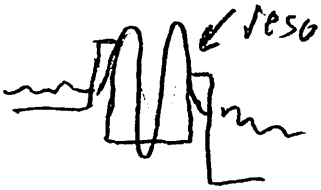
$\psi_I'(0) = \psi_{II}'(0) \Rightarrow$ $\boxed{ik_1 A - ik_1 B = -KC + KD}$

c) $T \neq \frac{|L|^2}{|A|^2}$ b/c phase $k_1 \neq k_5$. Phase velocities are different.

Have to use the probability current.

d) One observes resonances when an integer number of waves fit btwn. the barriers

resonant ψ



e) No bound states b/c they will escape to $x = \infty$ by tunnelling.

5 Half Harmonic Oscillator (12 points)

Consider a particle of mass m moving in one dimension on the potential of half a harmonic oscillator

$$V(x) = \begin{cases} \infty, & x \le 0 \\ m\omega^2 x^2/2, & x > 0 \end{cases}$$

What are the stationary states $\chi_n(x)$ in terms of the stationary state solutions of the full harmonic oscillator with the same vibrational frequency ω: $\psi_n(x)$? What is the energy level spacing? Sketch the ground state wave function of the half harmonic oscillator.

The diff. eqn. is the same as the SHO, just different b.c.'s, so the SHO sol^n's solve the problem, just that only the odd ones meet the b.c. $\psi_n(0) = 0$, $\psi_n(+\infty) = 0$

so

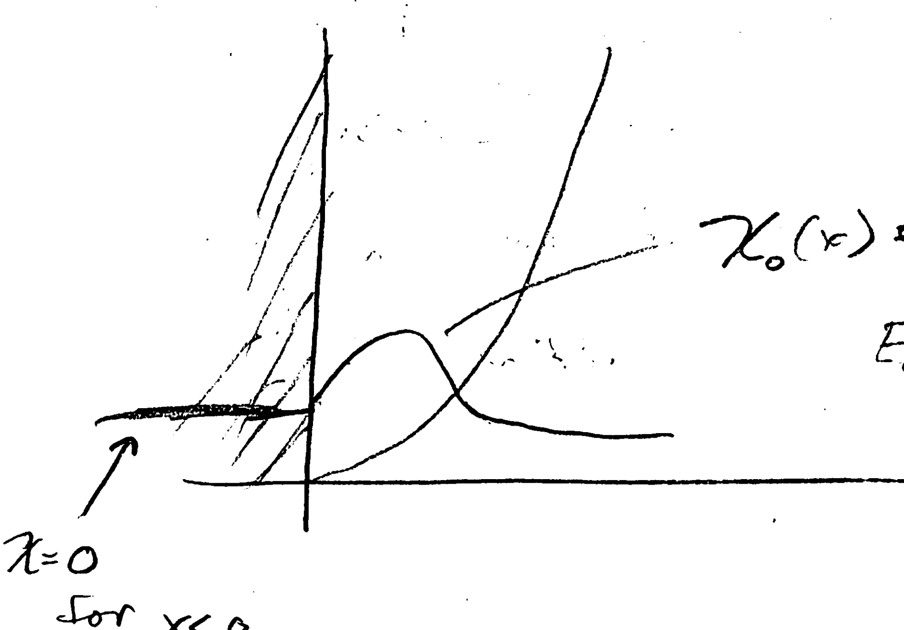
$$\boxed{\chi_m(x) = \begin{cases} \sqrt{2}\,\psi_{2m+1} & x \ge 0 \\ 0 & x < 0 \end{cases}}$$

where ψ_n are the SHO sol^n

$m = 0, 1, 2, \ldots$

$2m+1 = 1, 3, 5, \ldots$

The energy level spacing is now $\boxed{2\hbar\omega}$



$\chi_0(x) \propto x e^{-\frac{m\omega}{2\hbar}x^2}$ for $x > 0$

$E_0 = \frac{3}{2}\hbar\omega$

6 Two-Dimensional Harmonic Oscillator (14 points)

Consider a particle confined to a symmetric two-dimensional harmonic oscillator potential $V(x,y) = \frac{1}{2}m\omega^2(x^2 + y^2)$. The time-independent Schrödinger equation reads

$$\hat{H}\psi(x,y) = -\frac{\hbar^2}{2m}\frac{\partial^2\psi(x,y)}{\partial x^2} + \frac{1}{2}m\omega^2 x^2\psi(x,y) - \frac{\hbar^2}{2m}\frac{\partial^2\psi(x,y)}{\partial y^2} + \frac{1}{2}m\omega^2 y^2\psi(x,y) = E\psi(x,y) \qquad (6.1)$$

a) Solve the Schrödinger equation and provide formulae for the energy eigenvalues and stationary state wavefunctions. You may express your answers in terms of the solutions to the 1D harmonic oscillator (ψ_n), we have studied in class without re-deriving them.

If you can guess the answers without derivation that is fine, just write.

b) What is the degeneracy of the ground state, the first excited state, and the second excited state?

c) Show that the operator $\hat{L}_z = x\hat{p}_y - y\hat{p}_x$, commutes with the Hamiltonian.

d) Find one simultaneous eigenfunction of $\hat{H}$ and $\hat{L}_z$ and determine its eigenvalues.

Hint: For parts of this problem it may be helpful to use polar coordinates (r,ϕ). In ~~cylindrical~~ polar coordinates

$$\cancel{\nabla^2 f(r,\phi) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial\phi^2}}, \qquad \hat{L}_z = -i\hbar\frac{\partial}{\partial\phi}$$

$[\hat{H}_x, \hat{H}_y] = 0 \Rightarrow$ sep. var.

a) TISE $\rightarrow \left(\hat{H}_{SHO,x} + \hat{H}_{SHO,y}\right)\psi_x(x)\,\psi_y(y) = E\psi_x\psi_y$

$\psi_x(x) = \psi_n(x)$ $\rightarrow$ Normal SHO wavefcns.

$\psi_y(y) = \psi_m(y)$

$$\boxed{\psi_{nm}(x,y) = \psi_n(x)\,\psi_m(y)}$$

$E_{nm} = \left(n + m + \frac{1}{2} + \frac{1}{2}\right)\hbar\omega$

$$\boxed{E_{nm} = (n+m+1)\hbar\omega}$$

b) ground state $E_{00} = \hbar\omega \rightarrow$ only one way (nondegenerate) $\boxed{g_0 = 1}$

First excited state $\underbrace{E_{01} = E_{10}}_{\text{two ways}} = 2\hbar\omega$ $\boxed{g_1 = 2}$

Second excited state

$3\hbar\omega = \underbrace{E_{02} = E_{20} = E_{11}}_{\text{three ways}}$ $\boxed{g_2 = 3}$

c) Using the hint $r^2 = x^2+y^2$

$$\hat{H} = -\frac{\hbar^2}{2m}\nabla^2 + \frac{1}{2}m\omega^2 r^2$$

Now one can see that $\left[\hat{H}, \frac{\partial}{\partial\phi}\right] = 0$ or $[\hat{H}, \hat{L}_z] = 0$

This makes sense since the potential is cylindrically symmetric!

d) The ground state $\psi_{00} = \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} e^{-\frac{m\omega}{2\hbar}(x^2+y^2)}$

$$= \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} e^{-\frac{m\omega}{2\hbar}r^2}$$

does <u>not</u> depend on ϕ

so $L_z\psi_{00} = \underset{\substack{\uparrow \\ \text{eigenvalue}}}{0}\,\psi_{00}$ and $\hat{H}\psi_{00} = \hbar\omega\psi_{00}$

so this is a silmutaneous eigenket of $\hat{H}$ and $\hat{L}_z$

For higher n it is more complicated and one must derive the Laguerre Gaussian modes!