Momentum Space

a)

$$\phi(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_0^\infty dx\, e^{-ipx/\hbar}\, N x e^{-\alpha x}$$

let $k = p/\hbar$

$$\phi(k) = \frac{N}{\sqrt{2\pi\hbar}} \int_0^\infty dx\, e^{-(\alpha+ik)x}\, x$$

Tabular method

x		$e^{-(\alpha+ik)x}$
	$(+)$	
1		$-\frac{1}{(\alpha+ik)} e^{-(\alpha+ik)x}$
	$(-)$	
0		$\frac{1}{(\alpha+ik)^2} e^{-(\alpha+ik)x}$

$\rightarrow 0$ at both band

$$\Rightarrow \boxed{\phi(k) = \frac{2\alpha^{3/2}}{\sqrt{2\pi\hbar}} \frac{1}{(\alpha+ik)^2}}$$

b) Normalization

$$\int_{-\infty}^{\infty} dp\, |\phi(p)|^2 = \hbar \int_{-\infty}^{\infty} dk\, |\phi(k)|^2$$

$$= \frac{4\alpha^3}{2\pi\cancel{\hbar}} \cancel{\hbar} \underbrace{\int_{-\infty}^{\infty} dk\, \frac{1}{(\alpha-ik)^2} \frac{1}{(\alpha+ik)^2}}_{I_1}$$

Momentum space

This integral can be done in many ways. The fastest way is to rewrite it as a contour integral on the complex plane and use the residue thm. Rewriting the integrand

$$W = \frac{1}{(\alpha - ik)^2} \frac{1}{(\alpha + ik)^2}$$

$$= \frac{1}{(k - i\alpha)^2} \frac{1}{(k + i\alpha)^2}$$

and closing in the upper half plane around $z = i\alpha$ (second order pole), the residue is given by

$$C_{-1} = \lim_{k \to i\alpha} \frac{d}{dk} \left(\frac{1}{k + i\alpha} \right)^2$$

$$= \lim_{k \to i\alpha} \frac{-2}{(k + i\alpha)^3} = \frac{-2}{-8i\alpha^3} = \frac{1}{4i\alpha^3}$$

$$\Rightarrow I_1 \to 2\pi i C_{-1} = 2\pi \cancel{i} \left(\frac{1}{4\cancel{i}\alpha^3} \right)$$

$$= \frac{2\pi}{4\alpha^3}$$

$$\Rightarrow \boxed{\int_{-\infty}^{\infty} dp\, |\phi(p)|^2 = \frac{\cancel{4\alpha^3}}{\cancel{2\pi}} \frac{\cancel{2\pi}}{\cancel{4\alpha^3}} = 1} \checkmark$$

$\phi(p)$ is normalized

c) Again using the residue thm.

$$\langle p \rangle = \frac{4\alpha^3}{2\pi\cancel{\hbar}}\cancel{\hbar}\int_{-\infty}^{\infty} dk \, \frac{1}{(k-i\alpha)^2}\, \hbar k \, \frac{1}{(k+i\alpha)^2}$$

Use 2nd order pole at $k = i\alpha$

$$\lim_{k\to i\alpha} \frac{d}{dk} \frac{\hbar k}{(k+i\alpha)^2} = \lim_{k\to i\alpha} \hbar\left(\frac{1}{(k+i\alpha)^2} - \frac{2k}{(k+i\alpha)^3}\right)$$

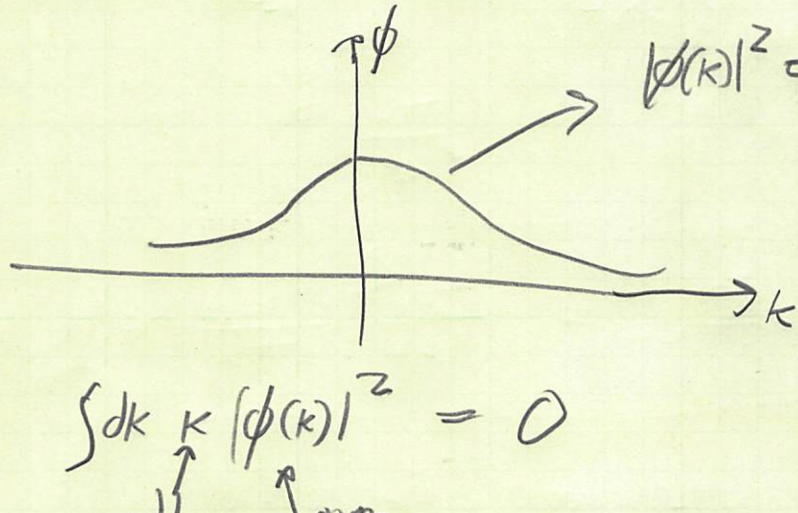
$$= \hbar\left(\frac{-1}{4\alpha^2} - \frac{2i\alpha}{(2i\alpha)^3}\right)$$

$$= \hbar\left(\frac{-1}{4\alpha^2} - \frac{\cancel{2i\alpha}}{\underset{4}{\cancel{-8i\alpha^3}}}\right)$$

$$= \hbar\left(\frac{-1}{4\alpha^2} + \frac{1}{4\alpha^2}\right)$$

$$\boxed{\langle p \rangle = 0}$$

Which could also be deduced from a symmetry argument b/c $|\phi(p)|^2$ is even



$$|\phi(k)|^2 \propto \frac{1}{(k^2+\alpha^2)^2}$$

so that $\int dk \, \underset{\text{odd}}{k} \, \underset{\text{even}}{|\phi(k)|^2} = 0$