Projection onto Basis Soln

$$\psi(x) = \left(\frac{1}{L}\right)^{1/2}\left[\frac{1}{4} - \left(\frac{x}{L} - \frac{1}{2}\right)^2\right]$$

$$f_n(x) = \left(\frac{2}{L}\right)^{1/2} \sin\left(\frac{n\pi x}{L}\right)$$

a) Convert to Dirac notation

$$|\psi\rangle = \sum_{n=1}^{\infty} c_n |f_n\rangle$$

$$c_n = \langle f_n|\psi\rangle$$

b)

$$c_1 = \frac{\sqrt{2}}{L}\int_0^L dx \sin\left(\pi \frac{x}{L}\right)\left[\frac{1}{4} - \left(\frac{x}{L} - \frac{1}{2}\right)^2\right]$$

let $u = x/L - \frac{1}{2}$

$du = \frac{dx}{L}$

$$c_1 = \frac{\sqrt{2}}{\cancel{L}}\cancel{L}\int_{-1/2}^{+1/2} du \sin\left(\pi u + \pi/2\right)\left[\frac{1}{4} - u^2\right]$$

$\sin(t + \pi/2) = \cos t$

$$= \sqrt{2}\left\{\frac{1}{4}\int_{-1/2}^{+1/2} du \cos(\pi u) - \int_{-1/2}^{1/2} du\, u^2 \cos(\pi u)\right\}$$

The first integral is trivial

$$\frac{1}{4}\int_{-1/2}^{1/2} du \cos(\pi u) = \frac{1}{4\pi}\left[\sin(\pi/2) - \sin(-\pi/2)\right] \qquad (1)$$

$$= \frac{1}{2\pi}$$

The second integral can be done quickly w/ the tabular method

u^2		$\cos \pi u$
	$(+)$	
$2u$		$\frac{1}{\pi} \sin \pi u$
	$(-)$	
2		$-\frac{1}{\pi^2} \cos \pi u$
	$(+)$	
0		$-\frac{1}{\pi^3} \sin(\pi u)$

$$\int_{-1/2}^{1/2} du\, u^2 \cos(\pi u) =$$

0 at both bounds

$$\left[\frac{u^2}{\pi} \sin \pi u + \frac{2u}{\pi^2} \cos \pi u - \frac{2}{\pi^3} \sin \pi u\right]_{-1/2}^{1/2} \qquad (2)$$

$$\frac{1}{4\pi} \cdot 2 + 0 - \frac{2}{\pi^3} \cdot 2$$

$$\frac{1}{2\pi} - \frac{4}{\pi^3}$$

$$\Rightarrow \quad c_1 = \sqrt{2}\left[\cancel{\frac{1}{2\pi}} - \left(\cancel{\frac{1}{2\pi}} - \frac{4}{\pi^3}\right)\right]$$

$$\boxed{c_1 = \frac{4\sqrt{2}}{\pi^3}}$$

Now C_2 and C_4 are zero by symmetry b/c

$C_2 = \int$ 

odd even

$$\int \text{odd} = 0$$

and likewise for C_4. So we are left w/ C_3. Using same u-sub as ~~before~~

$$C_3 = \sqrt{2} \int_{-1/2}^{1/2} du \, \sin\left(3\pi u + \tfrac{3\pi}{2}\right) \left[\tfrac{1}{4} - u^2\right]$$

$\vdots$

similar algebra to that done for C_1 gives

$$C_3 = \frac{4\sqrt{2}}{27\pi^3}$$

See MATLAB for plot.