a) The 1s binding energy (and all the other energy scales) of the $+Ze$, e^- system scales as Z^2, so for C^{5+}, we expect

$$IP\ C^{5+} = Z^2 \cdot 13.6\,eV$$
$$= \boxed{489.6\ eV} \quad (1)$$

Carbon K-edge is only at 284.2 eV, substantially lower. The electron is less tightly bound in a real neutral carbon atom than C^{5+} because the other 5 electrons (1) screen the nuclear charge and (2) have repulsive interactions w/ the electron "in" the 1s orbital

b) For Uranium, $Z = 92$ and we expect

$$IP\ U^{91+} = Z^2 \cdot 13.6\,eV$$
$$= (92)^2 \cdot 13.6\,eV$$
$$= \boxed{115\ keV} \quad (2)$$

c) The NIST database lists the binding energies

and $\boxed{\begin{array}{ll} U\ XC\,II & 131.821\ keV \\ C\ VI & 489.9\ eV \end{array}}$ (3)

The Hydrogenic result for carbon is off by

$$\frac{0.3\,eV}{489.9\,eV} = 6\times10^{-4} = 0.06\%$$

For Uranium

$$\frac{17\,keV}{127\,keV} = 13\% \quad \text{much worse}$$

The quantum result is quite a bit off because we are using a result from nonrelativistic quantum mechanics on this very heavy system. The Virial theorem tells us that $\langle T \rangle = -\frac{1}{2}\langle V \rangle$ so

$$\langle T \rangle \approx 66\ keV \approx 0.13\ m_e c^2 \tag{4}$$

and we see error on this level. A relativistic treatment is required either via perturbation theory (see for example Griffiths) or via Dirac and/or Dirac-Fock equations.

d) First convert the proton radius to atomic units

$$r_p = 8.78 \times 10^{-16}\ m = 1.66 \times 10^{-5}\ a.u.$$

So from this we can see this will be a small effect. In atomic units, the 1s orbital is

$$\psi_{100} = \frac{1}{\sqrt{\pi}} e^{-r}$$

The probability that we are inside the proton is

$$P_c = \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\, d\theta \int_0^{r_p} r^2 dr\, |\psi_{100}|^2 \tag{5}$$

since ψ_{100} does not depend on θ or ϕ $\int d\Omega \to 4\pi$

So we have $P_e = 4\int_0^{r_p} r^2 dr e^{-2r}$

Now since $r_p \ll 1$, we can to a good approximation neglect the exponential in the integrand and set $e^{-2r} \approx 1$, so then we get the intuitive result

$$P_e \simeq \frac{4}{3} r_p^3 = 6\times 10^{-15} \qquad (6)$$

e) Now for muonic hydrogen we need to look at the mass dependence of the Bohr radius

$$a_0^{(e^-)} = \frac{4\pi\varepsilon_0 \hbar^2}{m_e e^2}$$

So now the orbital will be rescaled by the factor

$$\gamma = \frac{a_0^{(\mu)}}{a_0^{(e^-)}} = \frac{m_e}{m_\mu} = \frac{1}{207}$$

So since the proton size is the same there will be $(207)^3$ more muon wave function inside of it.

More formally, in atomic units, the muonium 1s wavefunction is

$$\psi_{100}^{(\mu)} = \frac{1}{\sqrt{\pi}\gamma^3} e^{-r/\gamma}$$

and

$$p_{\mu} = \frac{1}{\pi r_B^3} \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta \int_0^{r_p} dr\, r^2 e^{-2r/r_B}$$

$$= \frac{4\pi}{\pi r_B^3} \int_0^{r_p} dr\, r^2 e^{-2r/r_B}$$

$$\approx \frac{4}{3} r_p^3 \left(\frac{1}{r_B}\right)^3 = \boxed{(207)^3 p_e}$$