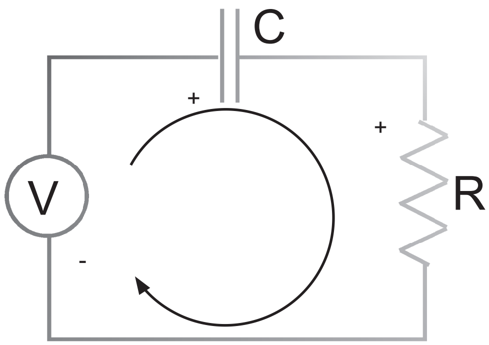
Derivation of Complex Impedence



Using the loop law with the loop drawn above and writing the voltage across the capacitor as $V_C = q/C$ and the voltage across the resistor as $V_R = iR$, one gets the following:

$$\begin{aligned} 0 &= -V(t) + V_C + V_R \\ &= -V(t) + q/C + iR. \end{aligned}$$

Differentiating the above and substituting $i \equiv dq/dt$ gives

$$\begin{aligned} 0 &= \frac{-dV}{dt} + (1/C)\frac{dq}{dt} + R\frac{di}{dt} \\ &= \frac{-dV}{dt} + (1/C)i + R\frac{di}{dt}. \end{aligned}$$

This is the fundamental equation used to determine the current $i(t)$ in the circuit as a function of time. It is a first order, inhomogeneous, linear differential equation with constant coefficients. There are standard techniques for solving these equantions which are not discussed (directly) here.

Frequency Domain

However, it is often useful to understand how a circuit responds to signals of different frequencies. To do this, we use the Fourier transform. The Fourier transform $F(\omega)$ of a function $f(t)$ is given by

$$F(\omega) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt\, f(t)\, \exp^{-j\omega t}$$

Applying this to the current equation gives

$$0 = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt\, f(t)\, \exp^{-j\omega t} \left[\frac{-dV}{dt} + (1/C)i + R\frac{di}{dt}\right]$$

and doing the integrals gives

$$\begin{aligned} 0 &= -j\omega\, V(\omega) + (1/C)\, i(\omega) + j\omega R i(\omega) \\ &= -V(\omega) + \frac{1}{j\omega C} i(\omega) + R i(\omega) \end{aligned}$$

in which $i(\omega)$ is the Fourier transform of $i(t)$ and $V(\omega)$ is the Fourier transform of $V(t)$. If $V(t) = V_0 \cos(\omega t)$, then $V(\omega) = V_0$ and the previous equation becomes

$$0 = -V_0 + \frac{1}{j\omega C} i(\omega) + R i(\omega)$$

Interpretation
The previous equation still is the sum of the voltages around the loop. An advantage of this form is that it is simply an algebraic equation for the current $i(\omega)$, but as a function of frequency ω. The current is then simply

$$i(\omega) = V_0 / \Big[\frac{1}{j\omega C} + R\Big]$$

Furthermore, the voltage across the capacitor is

$$V_C(\omega) = \frac{1}{j\omega C} i(\omega)$$

Like Ohm's law, this is a linear relationship between V and i, so all of the same ideas hold. The only difference is that the "resistance" is a frequency-dependent complex number. However, all of the same calculational methods still apply, and we identify the "complex impedence" as

$$Z_C = \frac{1}{j\omega C}$$

This is the basis of RC circuit analysis resulting in high– and low–pass filters and their combination into band–pass filters.

Decomposition of an arbitrary signal into its frequency components

The frequency domain is of particular interest for two major reasons. The first is because many signals have a particular frequency range (for example, WUSB has a carrier frequency of 91 MHz). The second is that any periodic (under some reasonable assumptions) signal $V(t)$ can be written as a sum of terms each of a specific frequency via a Fourier series. The general statement of a Fourier series for a function $V(t)$ with period T_0 (so that $V(t) = V(t + T_0)$) is

$$\begin{aligned} V(t) &= \Sigma_{n=-\infty}^{\infty} c_n e^{j2\pi nt/T_0} \\ \text{with } \mathrm{c_n} &= \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} V(t)\, e^{-j2\pi t/T_0}\, dt. \end{aligned}$$

For an even function $V(-t) = V(t)$ this reduces to the simpler form

$$V(t) = \frac{a_0}{2} + \Sigma_{n=1}^{\infty} a_n \cos(2\pi nt/T_0) \tag{1}$$

$$a_n = \frac{2}{T_0}\int_{-T_0/2}^{T_0/2} V(t)\, \cos(2\pi nt/T_0)\, dt. \tag{2}$$

As examples, Table 1 gives the coefficients a_n for the three waveforms available from our signal generators, assuming a peak is centered at $t = 0$, and Fig. 1 shows the triangle and squares waves from the series.

Signal	Fourier Coeff, a_n, $n \geq 1$
Cosine	$a_1 = V_0$; $a_n = 0,\ n \neq 1$
Triangle	$a_n = \frac{4V_0((-1)^n - 1)}{(\pi n)^2}$
Square	$a_n = \frac{4V_0 \sin(\pi n/2)}{\pi n}$

Table 1: Fourier coefficients for the signals available from the Phy335 signal generator (SG). The SG voltage amplitude is V_0 and the frequency is f_0. This gives the angular frequency as $\omega_0 = 2\pi f_0$ and to the period as $T_0 = 1/f_0$. The DC offset is assumed to be zero, so $a_0 = 0$.

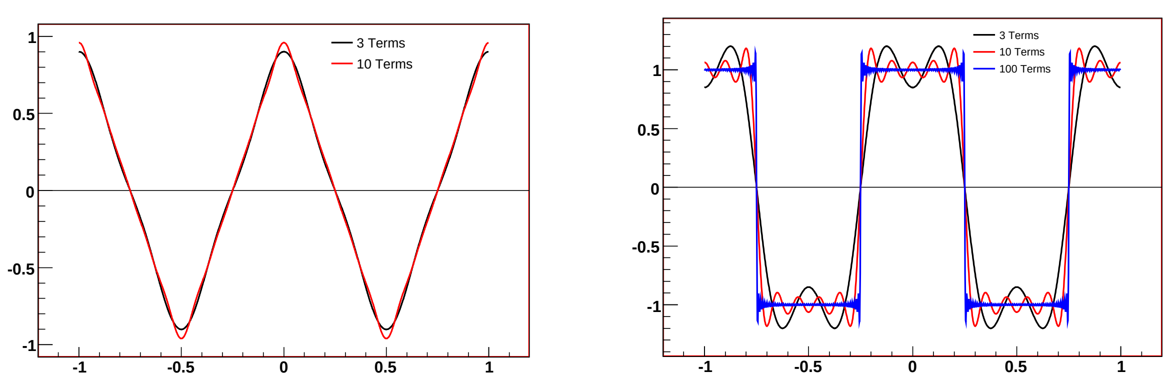


Figure 1: The triangle (left) and square (right) waves resulting from the above Fourier series. The different colors are for different numbers of terms in the sums.