Unit2: Resistor/Capacitor-Filters

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Abstract

Basic *RC*-filters were constructed and properties such as transfer functions were examined for high-pass, low-pass, and bandpass filters. The 3dB point of the high-pass filter was measured to be ???±? Hz while that of the low-pass filter was measured to be 1593± 0.8 Hz. It was demonstrated that low-pass filters have the effect of integrating the input signal, while high-pass filters serve to differentiate the input signal. Furthermore the range over which each type of circuit was determined empirically, giving a working range of approximately 1.2 ± 0.1 kHz and higher for the low pass filter as an integrator and $50 - 600$ Hz for the high-pass filter as a differentiator. Measurements served to confirm the theory for linear *RC* circuits quite well.

1 Introduction

1.1 Ideal Capacitors

Resistors and capacitors constitute two of the most ubiquitous circuit elements used in electronics. A capacitor is a circuit element whose function is to store charge between two conductors, hence storing electrical energy in the form of a field $\mathbf{E}(\mathbf{R}, t)$. This is in contrast to an inductor, which stores energy in the form of a magnetic field $\mathbf{B}(\mathbf{R}, t)$. The process of storing energy in the capacitor is known as "charging", as an equal amount of charges of opposite sign build on each conductor (Rizzoni, Section 4.1, pp. 138-139).[1] A capacitor is defined by its ability to hold charge, which is proportional to the applied voltage,

$$Q = CV, \tag{1}$$

with the proportionality constant C called the **capacitance**. A simple model for a capacitor consists of two parallel conducting plates of cross-sectional area A separated by air or a dielectric. The presence of the dielectric does not allow for the flow of DC current; therefore a capacitor acts as an open circuit in the presence of a DC current. However, if the voltage across the capacitor terminals changes as a function of time, the charge accumulated on the capacitor plates is given by

$$q(t) = Cv(t) \tag{2}$$

Although no current can flow through the capacitor if the voltage across it is constant, a time-varying voltage will cause charge to vary in time. Thus if the charge is changing in time, the current in the circuit is given by

$$i(t) = \frac{dq(t)}{dt} = C\frac{dv(t)}{dt} \tag{3}$$

Eq.(3) is the defining circuit law for a capacitor (Rizzoni, Section 4.1, pg. 139).

1.2 Analysis of Elementary RC-Filters

Simple combinations of passive circuit elements, i.e. resistors and capacitors, can be used to remove an unwanted signal component or enhance a desired one.[2] Frequency filters exploit the frequency dependence of the passive circuit elements and allow only specific frequency ranges to pass from input to output (Mellissinos, Sec. 3.1.5, pg. 103). Where the equations governing voltage and current for purely resistive circuits with DC sources led to

[1] There is usually some form of dielectric, or non-conducting material between the conductors of a capacitor. In addition to air, other dielectrics that are used to separate conductors are plastic films such as mylar.

[2] **Passive** components refer to those that consume energy, but are incapable of power gain. A passive circuit does not require an external power source beyond the signal, and can be constructed from linear circuit elements such as resistors, capacitors, inductors, and transformers.

solving a system of algebraic equations, the analysis of such circuits as the result of applying Kirchhoff's voltage and current laws requires differential equations.

1.2.1 The Low-Pass Filter

Consider the circuit combination of a single resistor and a capacitor, given in **Figure 1**. Let v_{out} be the value at the terminal terminal where the output

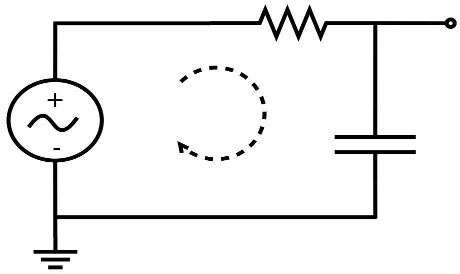

Figure 1: Schematic of low-pass filter

voltage is taken. Kirchhoff's laws allow this problem to be treated in a similar fashion to a voltage divider, where each circuit element has complex impedance Z.[3] The transfer function is defined to be the ratio of response of system output to system input,

$$H(j\omega) = \frac{|v_{out}|}{|v_{in}|} \tag{4}$$

and is found by taking the voltage drops across the capacitor (iZ_C)and dividing it by the total voltage drop across the circuit, $iZ_C + iR$ (Wylie, Section 8.7, pg. 337). This yields

$$\frac{v_{out}}{v_{in}} = \frac{iZ_C}{i(Z_R + Z_C)} = \frac{i\frac{1}{j\omega C}}{i(R + \frac{1}{j\omega C})} = \frac{\frac{-1}{j\omega C}}{R - \frac{j}{\omega C}} = \frac{1}{1 + j\omega RC} \tag{5}$$

[3]The complex impedance, Z is the generalization of resistance for various circuit elements in relation to alternating voltage sources, and is equivalent to stating that capacitors and inductors act as **frequency-dependent resistors**. This leads directly to a generalization of Ohm's law from the familiar $V = IR$ in the case of constant voltage to $v = iZ(\omega)$, where v and i vary in time, and the "resistance" Z is dependent upon frequency. The impedance of a resistor is simply $Z_R = R$, while the impedance for a capacitor is given by $Z_C = 1/(j\omega C) = -j/(\omega C)$, where j is the imaginary unit ($j = \sqrt{-1}$). The impedance of an inductor is proportional to frequency, namely $Z_L = j\omega L$.

The **transfer function**, more commonly called the **gain** is a real quantity and thus cannot be expressed in complex form; however, multiplication of v_{out}/v_{in} by its complex conjugate gives a real number. Therefore,

$$H(j\omega) = \left|\frac{v_{out}}{v_{in}}\right| = \sqrt{\frac{v_{out}}{v_{in}} \cdot \frac{\overline{v_{out}}}{v_{in}}} = \sqrt{\frac{1}{1+j\omega RC} \cdot \frac{1}{1-j\omega RC}} = \frac{1}{\sqrt{1+\omega^2 R^2 C^2}} \tag{6}$$

This is the governing equation for a **low-pass filter**. At low (relative to the -3dB point)[4] frequencies, the transfer function is close to 1, while at high values of frequency, the output voltage is attenuated and will be quite small relative to the input voltage.

1.2.2 High-Pass Filter

In contrast to the circuit given above, consider the circuit given in **Figure 2**. Superficially, the only differences are the locations of the capacitor and

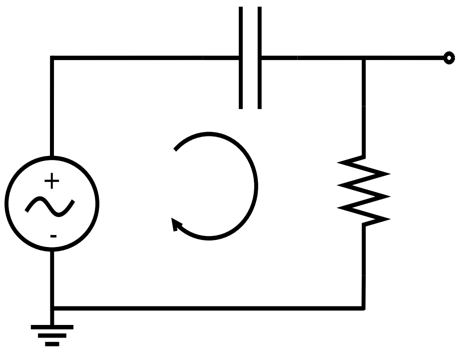

Figure 2: schematic of high-pass filter

resistor relative to the AC voltage source. Applying the same method as above to obtain the relationship between v_{out} and v_{in}, the following transfer

[4]The decibel is a logarithmic unit that expresses the magnitude of a physical quantity as a ratio of some known quantity. For example, in the the case of expressing the ratio of the output voltage amplitude to that of input, the gain in decibels is taken as dB = $20\log(v_{out}/v_{in})$. A gain of 1 corresponds to 0 dB, while -3dB corresponds to a gain of 0.7079, or $1/\sqrt{2}$; **therefore the -3 dB point is the frequency at which the ratio of output voltage to input voltage is equal to $1/\sqrt{2}$ or roughly 70%.**

function is obtained

$$H_{HighPass} = \frac{v_{out}}{v_{in}} = \frac{iR}{i(Z_C + R)} = \frac{R}{\frac{1}{j\omega C} + R} = \frac{1}{1 + \frac{1}{j\omega RC}} = \frac{j\omega RC}{1 + j\omega RC} \tag{7}$$

Again, since this is a complex quantity, the magnitude of this ratio is obtained via a similar procedure, giving

$$H(j\omega) = \left|\frac{v_{out}}{v_{in}}\right| = \sqrt{\frac{v_{out}}{v_{in}} \cdot \frac{\overline{v_{out}}}{v_{in}}} = \sqrt{\frac{j\omega RC}{1 + j\omega RC} \cdot \frac{-j\omega RC}{1 - j\omega RC}} = \frac{\omega RC}{\sqrt{1 + \omega^2 R^2 C^2}} \tag{8}$$

Eq.(7) expresses the gain of a **high-pass** filter in terms of the circuit elements R and C and the driving angular frequency ω. As the input frequency increases, the gain approaches one, since the capacitor acts like a short circuit (essentially becoming a wire) at very high frequencies. The gain will clearly be quite low for small angular frequencies.

These two simple circuits have very important properties that are used in modern electronic devices. Furthermore, these two circuits may be combined to pass only a narrow frequency range, making a **bandpass** filter. This experiment serves to verify the properties of high and low-pass filters, and their respective roles as signal differentiators and integrators.

2 Procedure

Both low and high pass filters were constructed on the job board, with the resistor and capacitor chosen such that the -3dB (read 'negative three D B point') was between 1 and 2 kHz. The frequency of the -3dB point is given by[5]

$$f = \frac{1}{2\pi RC}. \tag{9}$$

To have a circuit that turns on between 1-2 kHz, the time constant RC must be on the order of milliseconds; therefore $R = 1.262 \pm 0.001$ kΩ and $C = 80.6 \pm 0.1$ nF. Measurements of the output voltage were taken over a range of frequencies covering five orders of magnitude (10^1 to 10^5). In addition to voltage, the time difference between two points of constant phase was measured to see the relationship between phase angle ϕ and frequency.

[5]See derivation in laboratory notebook pg.

All AC voltage measurements were taken using a dual-channel Tektronix oscilloscope.

The output of the low-pass filter was taken to be the input of the high-pass filter (see **Figure 3**) to make a bandpass filter. Measurements of output

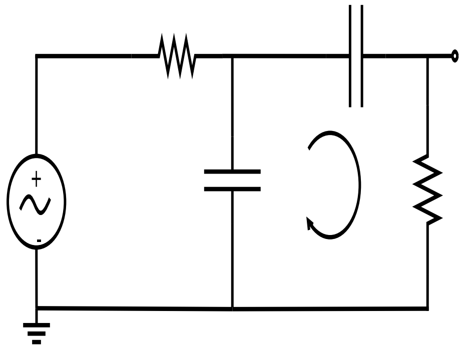

Figure 3: Band-Pass filter

voltage, and time differences were again taken over a frequency range from several Hz, to several hundred thousand Hz.

Finally, various waveforms from the signal generator were applied to both filters individually (uncoupling them from the previous portion of the experiment), and the ranges over which each circuit integrated (low-pass) or differentiated the input signal (high-pass) were examined.

3 Results and Observations with Error Analysis

A plot of the gain as a function of the driving frequency for the low-pass filter is given below in **Figure 4**. All plots were performed with the statistical package ROOT.[6] The graph clearly shows that for low frequency values (relative to the -3dB point), the gain is quite close to 1. There is a sharp exponential drop in the transfer function as the frequency increases, it is therefore helpful and more revealing to view the transfer function against the natural logarithm ($\log_e(f)$ or $\ln(f)$), given below in **Figure 5**.

[6]visit root.cern.ch for documentation and instructions on how to use ROOT

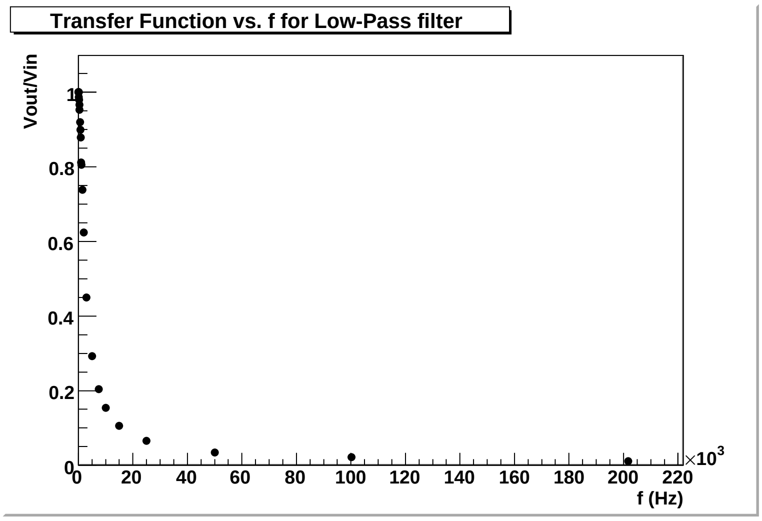

Figure 4: Plot of transfer function vs. f for low pass filter

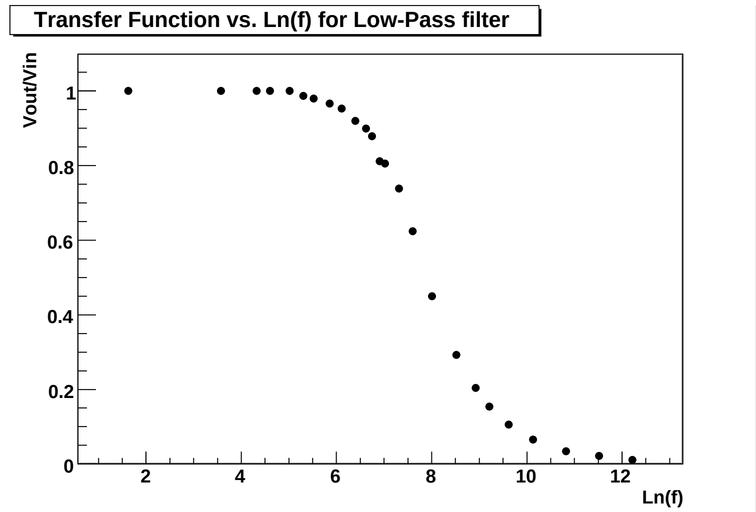

Figure 5: Plot of transfer function vs. $\ln(f)$ for low-pass filter

The transfer function is roughly 1 for low driving frequencies, while it eventually decreases to zero at high frequencies. The measured -3dB point was found to be 1593 $\pm$ 0.8 Hz, which matches reasonably well with the calculated value of 1567 Hz.

Similar plots were constructed for the high-pass filter, given in **Figure 6** and **Figure 7**. The plots show inverse behavior as compared to the low-pass

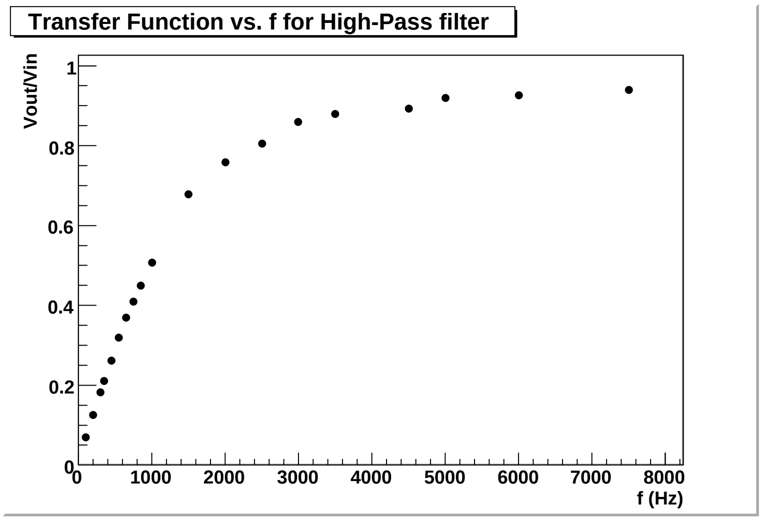

Figure 6: Plot of Transfer Function vs. f for High Pass filter.

filter, namely a gain of zero at low frequencies that approaches one for high frequencies.

A question one might ask is whether or not the output voltage stays in phase with the input over the whole range of function for either filter; in other words, does the output follow the input for all frequencies for all ω. A plot of phase shift against frequency and $\ln(f)$ is given below for each circuit.

The data show that for low frequencies, the low-pass filter yields an output that is in phase with the input voltage. This is in contrast to the high-pass filter, which has a $\pi/2$ phase shift at low frequencies that eventually diminishes. The phase shifts can be understood with the aid of a highly practical tool known as a phasor.

3.1 Phasor Theory

It is a fundamental property of complex numbers that they may be expressed in a form $z = a + jb$, or in a polar form, $z = re^{j\theta}$ (Brown & Churchill, Sec.

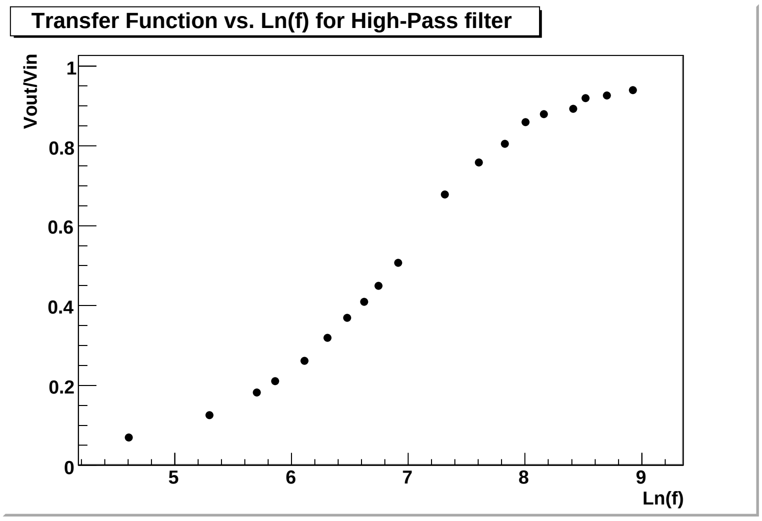

Figure 7: Plot of Transfer function vs. $\ln(f)$ for high-pass filter

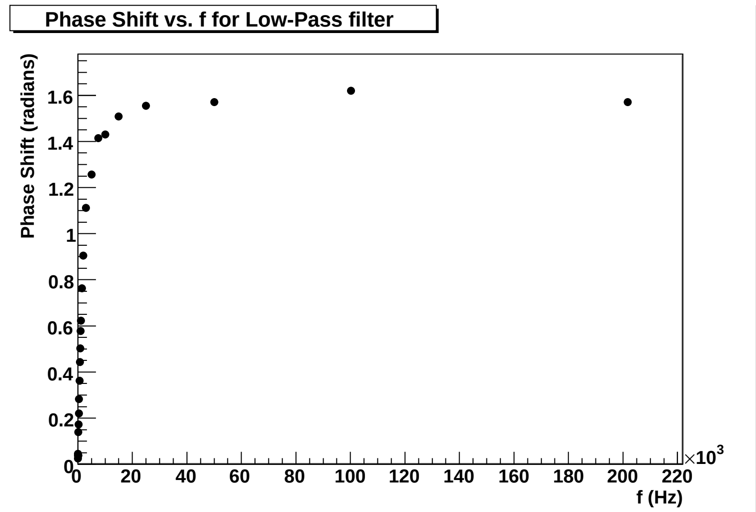

Figure 8: Plot of phase shift vs. frequency for low-pass filter

6, p. 15). When expressed in polar form and placed on coordinate axes, complex numbers may be treated like Cartesian vectors with a magnitude and direction. In this notation r represents the "norm" or magnitude of a

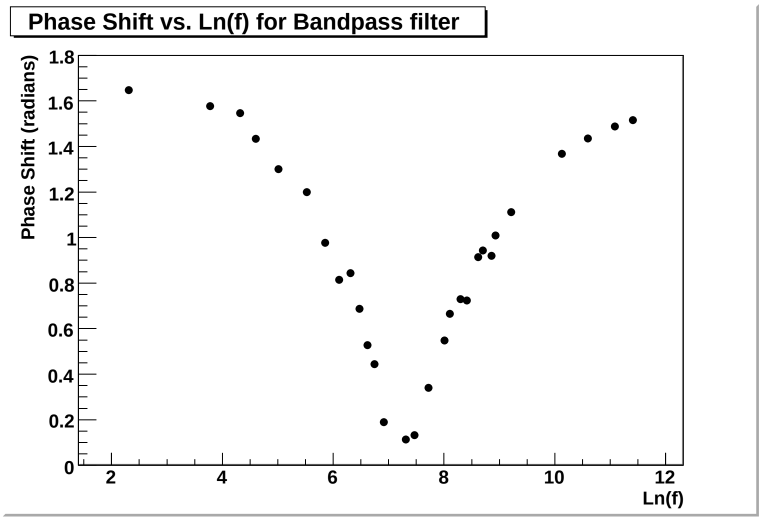

Figure 9: Plot of ϕ vs. $\ln(f)$ for low-pass filter

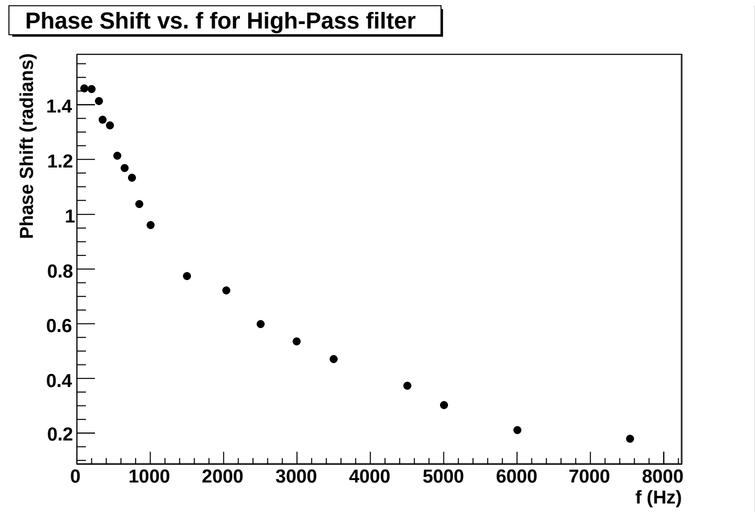

Figure 10: Plot of ϕ vs. frequency for high-pass filter

complex number, while the angle θ, called the argument, or $\mathbf{arg}(z)$ represents some orientation relative to the zero angle. Euler's formula,

$$e^{j\theta} = \cos\theta + j\sin\theta \tag{10}$$

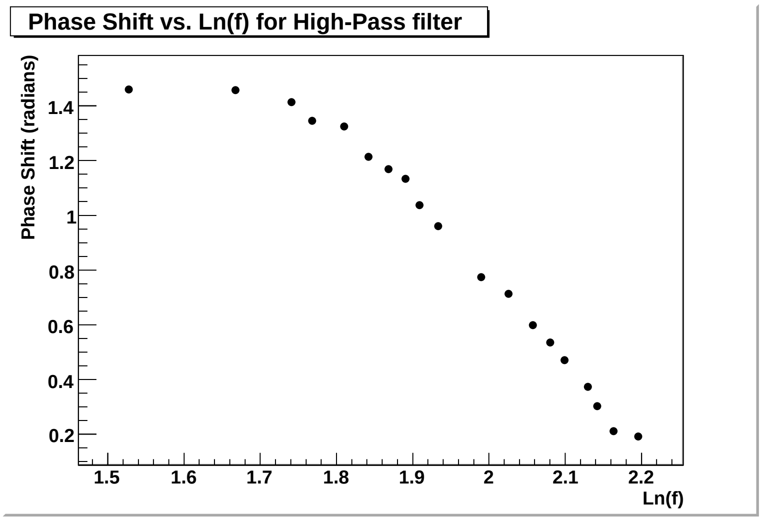

Figure 11: Plot of ϕ vs. $\ln(f)$ for high-pass filter

allows any complex number to be converted essentially from polar form to a sum of sine and cosine terms and vise-versa. These complex numbers expressed in polar form are called **phasors** when plotted on a set of coordinate axes. A phasor is a vector whose length is proportional to the maximum value of the variable it represents (voltage, current, etc), and which rotates counterclockwise at an angular speed equal to that of the driving frequency (Serway & Jewett, Section 33.2, pg. 1036). Thus a sinusoidally varying volt-

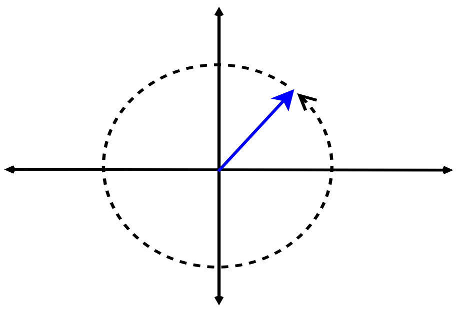

Figure 12: Phasor representation of Voltage

age may be represented graphically as the projection of a vector with length

V_0 that rotates at the driving frequency!

Now consider a sinusoidally varying input voltage of the form $v(t) = V_0 \cos(\omega t)$. This can be written as $v_{in}(t) = V_0 e^{j\omega t}$. As expressed earlier, the relationship between input and output voltage can be put into the form

$$v_{out} = H(j\omega) v_{in} e^{j\phi} \tag{11}$$

Assuming v_{out} is also a complex exponential function, then the output can be re-written as

$$v_{out} = H(j\omega) V_0 e^{j\omega t} e^{j\phi} = H(j\omega) V_0 e^{j(\omega t + \phi)} \tag{12}$$

Therefore, $v_{out} = H(j\omega) V_0 \cos(\omega t + \phi)$ This implies that the output voltage crests at a different time than the input time, and this time is proportional to the phase (Mellissinos, sec 3.1 pg 97). Going back to the complex form of the transfer function for the low-pass filter

$$H(j\omega) = \frac{1}{1 + j\omega RC},$$

and using the definition of the transfer function, Eq.(4), the voltages can be expressed as

$$v_{out} = v_{in} \frac{1}{1 + j\omega RC}. \tag{13}$$

Since the output voltage is a sinusoidally oscillating function, it may be expressed as a complex exponential[7],

$$v_{out} = ||\mathbf{v}_{out}|| e^{j\phi}.$$

The goal is to obtain an analytic expression for the argument ϕ. The argument $\arg(z)$ of a complex number $z = re^{j\theta}$ expressed in polar form may be obtained from its Cartesian form $z = a + jb$ by taking the arctangent of the imaginary component b denoted by $\mathrm{Im}(z)$ divided by the real component a denoted $\mathrm{Re}(z)$ (Brown & Churchill, Section 6, pg. 15), or

$$\theta = \arctan \frac{Im(z)}{Re(z)}. \tag{14}$$

[7]Since the phasor is simply treated as a vector, $||\mathbf{v}||$ represents the norm or magnitude of the vector $\mathbf{v}$.

To obtain the the argument of the transfer function in terms of the driving frequency and the passive circuit elements used, take the imaginary part of Eq.(5) and divide it by the real part of Eq.(5). This gives

$$\phi = \arctan \frac{Im(H)}{Re(H)} = \arctan \frac{\omega RC/(1+\omega^2 R^2 C^2)}{1/(1+\omega^2 R^2 C^2)} \tag{15}$$

The argument ϕ physically represents the phase angle between input and output voltage may be obtained to give

$$\phi = \arctan(\omega RC) \tag{16}$$

Stated another way, the output voltage for a low-pass filter is shifted relative to the input voltage by an amount $\arctan(\omega RC)$. Pictorially this would be represented by two arrows situated on top of each other that eventually separate to give a right angle ($\pi/2$ phase shift). Going back to the graph of the phase shift, it may be concluded that the expression for the phase shift matches the empirical results. Again, for a low pass filter, there is little to no phase shift at low frequencies, but at high frequencies, the capacitor (see Figure 1) acts like a short-circuit and provides a path for the current to flow. The output voltage is then shifted from the input by 90 degrees or $\pi/2$ radians as shown in **Figure 13**.

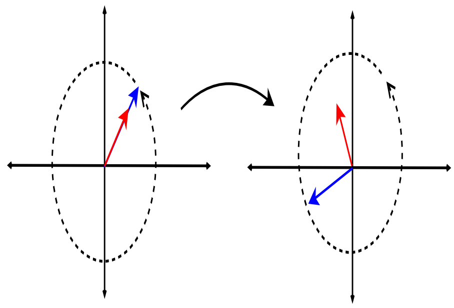

Figure 13: Phasor representation of phase shift

The same procedure may be applied to the high-pass filter using the transfer function given in Eq.(7). Then

$$\phi = \arctan \frac{j\omega RC/(1+\omega^2 R^2 C^2)}{\omega^2 R^2 C^2/(1+\omega^2 R^2 C^2)} = \arctan(1/\omega RC) \tag{17}$$

Again the plot of phase shift versus $\ln(f)$ for the high-pass filter shows good agreement with the theoretical description given in Eq.(16). At low frequencies, the capacitor tries to store charge but is unable to do so. Yet at higher frequencies, the charge on the capacitor tends to accumulate, thus making the capacitor behave more like a wire. Therefore, the output signal will be in phase with the input signal.

3.2 Differentiator and Integrator Theory

Consider again the low-pass filter given in **Figure 1**. For high frequencies such that $\omega >> RC$, the capacitor has insufficient time to charge up, therefore the voltage drop across the capacitor is small. The input voltage may be approximated by the voltage across the resistor.

$$v_{in} \approx iZ_R = IR. \tag{18}$$

Therefore, for large frequencies,

$$v_{out} = v_C = \frac{q}{C} = \int \frac{i}{C} dt \approx \int \frac{v_{in}}{RC} dt. \tag{19}$$

Thus low-pass filters have the property of integrating the applied input signal. This was observed on the oscilloscope for sinusoidal, square, and triangular waveforms. The sinusoidal signal was integrated to give a cosine (another sine wave shifted by $\pi/2$ radians, while the triangular wave was integrated to give a sine wave, and finally the square wave yielded a triangular wave [8]. The working range of the low-pass filter as an integrator was observed to be 1.0 kHz and greater for the triangular wave, 1.5 kHz and greater for the sinusoidal waveform. Lastly, for the square wave pulse, the circuit differentiated signals greater than 5 kHz.

The opposite effect is observed when viewing the output of the high-pass filter. This time only low frequencies are considered such that $\omega << RC$ so that the capacitor has time to charge until its voltage is roughly equal to that of the source. Then $v_{in} \approx iZ_C \approx \frac{i}{\omega C}$ For low frequencies $v_{in} \approx v_c$. Therefore

$$v_{out} = v_R = iR = R\frac{dq}{dt} = R\frac{d(Cv_C)}{dt} = RC\frac{dv_{in}}{dt}. \tag{20}$$

Thus the high-pass filter has the effect of differentiating the input signal. This was also observed for the various waveforms. The sine wave was differentiated

[8] Please refer to lab notebook pg. ... for sketches of the waveforms

to give a cosine wave, the triangular wave was differentiated to give a square wave, and the square wave was differentiated to give a series of high spikes at the edges of the square wave.[9] The high-pass filter worked well as a differentiator in the range of 50 to 750 Hz for the square pulse, 50 to 550 Hz for the triangular pulse and from 50 to 1 kHz for a sinusoidal pulse.

3.3 Band-Pass Filter

The combined filter **(Figure 3)** serves to pass a high output for a select frequency range, while attenuating signals that have too high a frequency or too low a frequency. Bandpass filters are found in radio transmitters and receivers. A plot of the transfer function is given below Because of the large

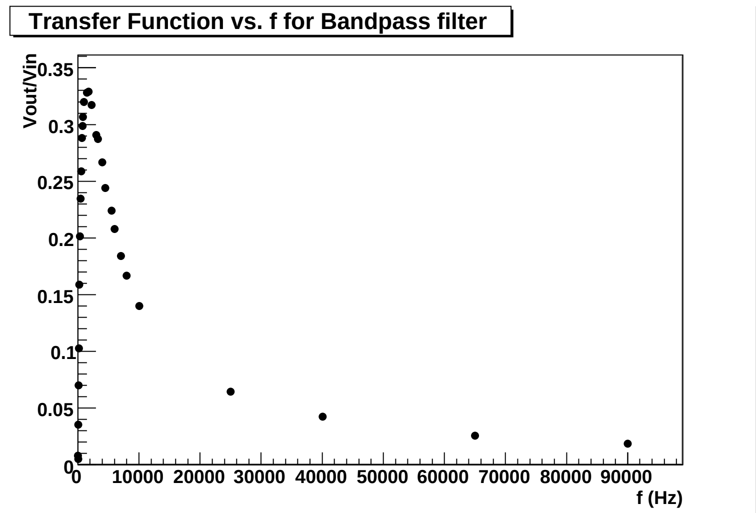

Figure 14: Plot of Transfer function versus f for bandpass filter

range over which the data were taken, it is helpful to look at a plot of the transfer function against the natural logarithm of the frequency, given below in **Figure 15**

The data show a bell-curve characteristic of a bandpass filter. Low frequencies are not passed well while frequencies on the order of several hundred thousand Hertz are not passed either. The bandpass filter exhibits a $\pi/2$

[9]Please reference notebook pg. ... for the sketches of the waveforms

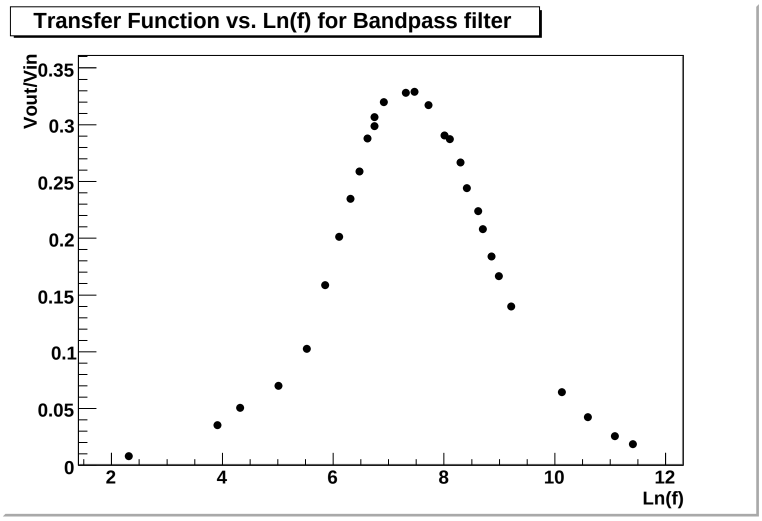

Figure 15: Plot of transfer function vs. $\ln(f)$ for bandpass filter.

phase shift for low frequencies. However, this sharply decreases to a phase shift of approximately zero in the working range the circuit (near 1 to 5 kHz). The phase shift increases to its original value of $\pi/2$ above 50,000 Hz.

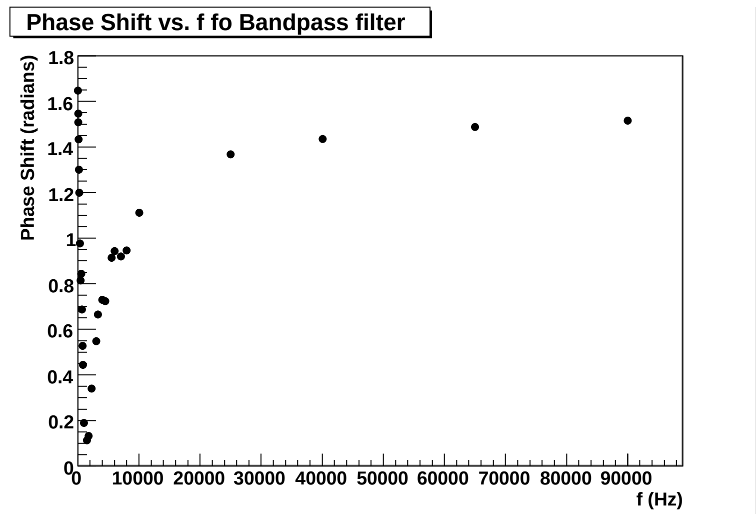

Figure 16: Plot of ϕ vs. f for bandpass filter

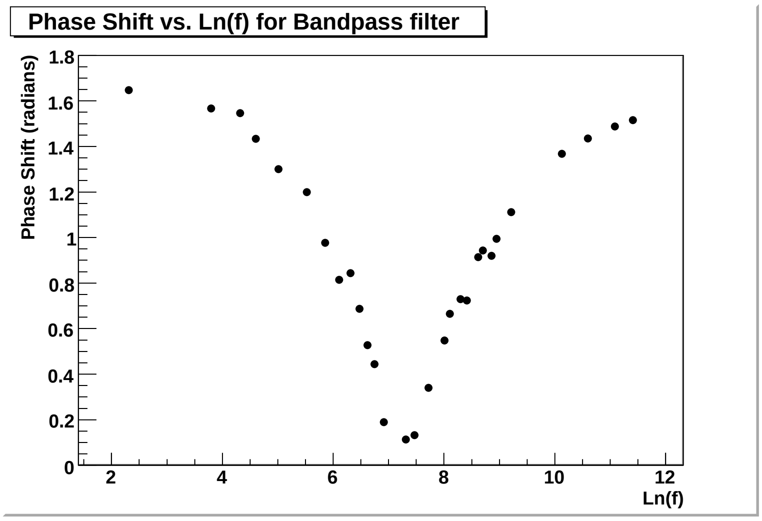

Figure 17: Plot of ϕ vs. $\ln(f)$ for bandpass filter

4 Conclusion

Passive circuit elements such as resistors and capacitors can be used to pass voltages above certain frequency ranges. High-pass filters work well for large frequencies relative to the -3dB point. They also serve to differentiate the input signal. Low-pass filters operate well at low frequencies- as their name implies, and serve to integrate the input signal. The data serve to confirm the behavior of these circuits (i.e. the transfer functions) based upon linear circuit theory (Kirchhoff's and Ohm's Laws).

5 References

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