Neiman and Gopinath(2014): Notes

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Parameters : Parameters of elasticity

- α: between capital K and labor L
- μ: between capital labor K, L and intermediate input X
- ρ: between domestic inputs Z and foreign inputs M
- θ: same output elasticity. among domestic inputs. among foreign inputs. among final goods.

Other parameters:

- f: $f > 0$ scale fixed cost
- λ: convex parameter

Environment Firm i:

Production function:

$$Y_i = A_i \left(K_i^{\alpha} L_{p,i}^{1-\alpha}\right)^{1-\mu} X_i^{\mu}$$

$L_{p,i}$: production labor(also have fixed cost labor $L_{f,i}$) CES aggregator of intermediates:

$$X_i = (Z_i^{\rho} + M_i^{\rho})^{1/\rho}$$

Domestic input bundle:

$$Z_i = \left(\int_j z_{ij}^{\theta} \, dj\right)^{1/\theta}$$

Imported input bundle:

$$M_i = \left(\int_{k \in \Omega_i} m_{ik}^{\theta} \, dk\right)^{1/\theta}$$

Endogenous import bundle: Ω Resource constraint(assume roundabout production):

$$Y_i = g_i + z_i = g_i + \int_j z_{ji} \, dj$$

Aggregate final good:

$$G = \left(\int_i g_i^{\theta} \, di \right)^{1/\theta}$$

The elasticity of substitution among all kinds of output are the same, θ

Fixed cost of importing:

$$F(|\Omega_i|) = f \, |\Omega_i|^{\lambda}, \quad f > 0, \, \lambda > 0$$

Firm's problem : Given wage w, rental rate r, set of domestic intermediate input price $\{p_j\}$, constant foreign price p_m across all foreign varieties, firm i chooses:

- capital K_i
- production labor $L_{p,i}$
- intermediate inputs from firm j: $\{z_{ij}\}$
- import bundle: Ω
- quantity of each import input k: m_{ik}. Same across all varieties since the foreign price of each variety is fixed

Unit cost function:

$$C_i = \frac{1}{A_i} \left(\frac{P_V}{1-\mu} \right)^{1-\mu} \left(\frac{P_{X_i}}{\mu} \right)^{\mu} = \frac{1}{\mu^{\mu}(1-\mu)^{1-\mu}} \, P_V^{1-\mu} \, P_{X_i}^{\mu} \, / \, A_i$$

, where the variable input price index:

$$P_V = \left(\frac{r}{\alpha} \right)^{\alpha} \left(\frac{w}{1-\alpha} \right)^{1-\alpha} \text{, constant across firms}$$

, the intermediate input

$$P_{X_i} = (P_Z^{\frac{\rho}{\rho-1}} + P_{M_i}^{\frac{\rho}{\rho-1}})^{\frac{\rho-1}{\rho}} \text{, firm unique intermediate input price}$$

, where the domestic input price index:

$$P_Z = \left(\int_i p_i^{\theta/(\theta-1)} \, di \right)^{(\theta-1)/\theta} \text{, Constant across firms}$$

, imported input price index

$$P_{M_i} = \left(\int_{k \in \Omega_i} p_m^{\frac{\theta}{\theta-1}} \, di \right)^{\frac{\theta-1}{\theta}} = p_m \, |\Omega_i|^{\frac{\theta-1}{\theta}} \text{, differ in measure of varieties of each firm}$$

FOC conditions : firm i's FOC:

$$wL_{p,i} = (1-\mu)(1-\alpha)C_i Y_i$$

$$rK_i = (1-\mu)\alpha \, C_i Y_i$$

$$P_{X_i} X_i = \mu \, C_i Y_i$$

$$z_{ij} = \left(\frac{p_j}{P_Z} \right)^{1/(\theta-1)} \left(\frac{P_Z}{P_{X_i}} \right)^{1/(\rho-1)} X_i$$

$$m_i = \left(\frac{p_m}{P_{M_i}} \right)^{1/(\theta-1)} \left(\frac{P_{M_i}}{P_{X_i}} \right)^{1/(\rho-1)} X_i$$

Demand demand for firm i:

$$g_i + \int_j z_{ji}\, dj = \left(\frac{p_i}{P_G}\right)^{1/(\theta-1)} G + \int_j \left(\frac{p_i}{P_Z}\right)^{1/(\theta-1)} \left(\frac{P_Z}{P_{X_j}}\right)^{1/(\rho-1)} X_j\, dj$$

Profit maximization monopolistic competition, so $p_i = \frac{C_i}{\theta}$. Firm i then chooses Ω_i to maximize profits net the fixed cost of importing varieties

$$\Omega_i = \arg\max_{\Omega_i} \{\Pi_i - wF(|\Omega_i|)\}$$

Domestic price index Define the share of domestic inputs in total spending on intermediates, the share if decreasing in technology:

$$\gamma_i \equiv \frac{P_Z Z_i}{P_{X_i} X_i}$$

Then the domestic input price index:

$$P_Z = \frac{\left(r^\alpha w^{1-\alpha}\right)^{\frac{1}{1-\mu}}}{(\epsilon\theta)^{\frac{1}{1-\mu}}} Q_{\gamma\theta\rho}^{-\frac{1}{1-\mu}}$$

, where

$$Q_{\gamma\theta\rho} = \left(\sum_i \gamma_i^{\mu\frac{\rho-1}{\rho}} A_i^{\frac{\theta}{1-\theta}}\right)^{\frac{1-\theta}{\theta}}$$

and,

$$\epsilon = \mu^\mu (1-\mu)^{1-\mu} \left(\alpha^\alpha (1-\alpha)^{1-\alpha}\right)^{1-\mu}$$

Roundabout production Quantity demand of intermediate inputs for firm j from firm i:

$$z_{ji} = \left(\frac{p_i}{P_Z}\right)^{\frac{1}{\theta-1}} Z_j$$

Domestic input share for firm j:

$$\gamma_j = \frac{P_Z Z_j}{P_{X_j} X_j}$$

For this cost-minimization problem, we will apply the Shephard Lemma to obtain the Hicksian Demand here:

- The aggregate unit cost for the intermediate input is:

$$P_{X_j} = (P_Z^{\frac{\rho}{\rho-1}} + P_{M_j}^{\frac{\rho}{\rho-1}})^{\frac{\rho-1}{\rho}}$$

- The cost of X_j units of intermediate good is:

$$X_j P_{X_j}$$

- By Shephard Lemma, the Hicksian Demand for the domestic intermediate goods is:

$$Z_j = \frac{\partial (P_{X_j} X_j)}{\partial P_Z} = X_j \frac{\partial P_{X_j}}{\partial P_Z}$$

$$\frac{Z_j}{X_j} = \frac{\partial P_{X_j}}{\partial P_Z}$$

- Plug in the rewrite the domestic input share for firm j:

$$\gamma_j = \frac{P_Z \, Z_j}{P_{X_j} \, X_j} = \frac{P_Z}{P_{X_j}} \frac{\partial P_{X_j}}{\partial P_Z}$$

- Recall:

$$P_{X_j} = (P_Z^{\frac{\rho}{\rho-1}} + P_{M_j}^{\frac{\rho}{\rho-1}})^{\frac{\rho-1}{\rho}}$$

$$\frac{\partial P_{X_j}}{\partial P_Z} = P_{X_j}^{\frac{\rho}{\rho-1}-1} P_Z^{\frac{\rho}{\rho-1}-1}$$

- Plug back to γ:

$$\gamma_j = \frac{P_Z \, Z_j}{P_{X_j} \, X_j} = \frac{P_Z}{P_{X_j}} \frac{\partial P_{X_j}}{\partial P_Z} = \left(\frac{P_Z}{P_{X_j}} \right)^{\frac{\rho}{\rho-1}}$$

Now we have the γ:

$$\gamma_j = \left(\frac{P_Z}{P_{X_j}} \right)^{\frac{\rho}{\rho-1}}$$

Recall the unit cost function:

$$C_i = \frac{1}{A_i} \left(\frac{P_V}{1-\mu} \right)^{1-\mu} \left(\frac{P_{X_i}}{\mu} \right)^{\mu} = \frac{1}{\mu^{\mu}(1-\mu)^{1-\mu}} P_V^{1-\mu} P_{X_i}^{\mu} \, / \, A_i$$

Then the cost share of domestic intermediate goods is μ:

$$P_{X_j} X_j = \mu C_j y_j$$

Given the monopolistic competition assumption, the optimal pricing rule is:

$$p_j = \frac{1}{\theta} C_j$$

, multiple both sides by y_j, then the total revenue versus total cost is:

$$p_j y_j = \frac{1}{\theta} C_j \, y_j$$

Combine $P_{X_j} X_j = \mu C_j y_j$ and $p_j y_j = \frac{1}{\theta} C_j \, y_j$:

$$P_{X_j} X_j = \mu \theta p_j \, y_j$$

Now we have three main equations:

- Domestic intermediate input demand: $z_{ji} = \left(\frac{p_i}{P_Z} \right) Z_j$
- Share of spending on domestic inputs: $\gamma_j = \frac{P_Z \, Z_j}{P_{X_j} X_j} = \left(\frac{P_Z}{P_{X_j}} \right)^{\frac{\rho}{\rho-1}}$. $Z_j = \frac{\gamma_j P_{X_j} X_j}{P_Z}$
- The share of intermediate inputs to total revenue:$P_{X_j} X_j = \mu \theta p_j \, y_j$

Combine the three equations, we have:

- Apply equation $Z_j = \frac{\gamma_j P_{X_j} X_j}{P_Z Z_j}$:

$$z_{ji} = \left(\frac{p_i}{P_Z}\right)^{\frac{1}{\theta-1}} Z_j = \left(\frac{p_i}{P_Z}\right)^{\frac{1}{\theta-1}} \frac{\gamma_j P_{X_j} X_j}{P_Z}$$

- Apply equation $P_{X_j} X_j = \mu\theta p_j\, y_j$:

$$z_{ji} = \left(\frac{p_i}{P_Z}\right)^{\frac{1}{\theta-1}} \frac{\gamma_j P_{X_j} X_j}{P_Z} = \left(\frac{p_i}{P_Z}\right)^{\frac{1}{\theta-1}} \frac{\gamma_j \mu\theta p_j\, y_j}{P_Z}$$

- Sum over all j:

$$\int_j z_{ji}\, dj = \mu\theta p_i^{\frac{1}{\theta-1}} P_Z^{\frac{\theta}{1-\theta}} \int_j \gamma_j p_j y_j\, dj$$

So, there is fixed point for y_i:

$$y_i = g_i + \int_j z_{ji}\, dj$$

$$y_i = g_i + \mu\theta p_i^{\frac{1}{\theta-1}} P_Z^{\frac{\theta}{1-\theta}} \int_j \gamma_j p_j y_j dj$$

Productivity decomposition : decompose the productivity growth step by step:

Changes in productivity of firm i: Using first-order approxiamation:

$$\Delta \ln PR_i = \frac{(1-\theta)}{\theta(1-\mu)} \left[\Delta \ln V_i + \frac{\mu\theta}{1-\mu\theta} \left(\Delta \ln X_i - \Delta \ln Y_i\right)\right] - \frac{(1-\mu\theta)}{\theta(1-\mu)} s_{L_i}(1-\omega_{L_p,i})\Delta \ln L_{f,i} + \frac{\Delta \ln A_i}{1-\mu}.$$

How to derive:

- Firms are price taker of input price. Given the prices of inputs and constant markup $\frac{1}{\theta}$:

$$p_i \frac{\partial Y_i}{\partial L_{p,i}} = \frac{w}{\theta}, \quad p_i \frac{\partial Y_i}{\partial K_i} = \frac{r}{\theta}, \quad p_i \frac{\partial Y_i}{\partial X_i} = \frac{P_{X_i}}{\theta}.$$

  - LHS: increase in revenue of 1 unit increase of specific inputs
  - RHS: constant markup over the input cost

- Given nominal value-added $VA_i = p_i Y_i - P_{X_i} X_i$, take the total derivatives and and rearrange, we have:

$$\Delta \ln Y_i^{VA} = \frac{\Delta \ln Y_i - s_X^Y \Delta \ln X_i}{1 - s_X^Y} = \Delta \ln Y_i - \frac{s_X^Y}{1 - s_X^Y} \left(\Delta \ln X_i - \Delta \ln Q_i\right),$$

  , where s_X^Y is the *revenue* share of intermediates, $s_X^Y = \frac{P_{X_i} X_i}{P_i Y_i}$., which equals constant $\mu\theta$.(Apply constant markup and Shephard Lemma)

- Apply the first-order Taylor expansion of production function $Y_i = A_i(K_i^{\alpha} L_{p,i}^{1-\alpha})^{1-\mu} X_i^{\mu}$, we have:

$$\Delta Y_i = \frac{\partial Y_i}{\partial K_i}\Delta K_i + \frac{\partial Y_i}{\partial L_{p,i}}\Delta L_i^p + \frac{\partial Y_i}{\partial X_i}\Delta X_i + \frac{\partial Y_i}{\partial A_i}\Delta A_i,$$

  Take the log term and plug in the $p_i \frac{\partial Y_i}{\partial L_{p,i}} = \frac{w}{\theta}, \quad p_i \frac{\partial Y_i}{\partial K_i} = \frac{r}{\theta}, \quad p_i \frac{\partial Y_i}{\partial X_i} = \frac{P_{X_i}}{\theta}.$, we have:

$$\Delta \ln Y_i = \frac{(1-s^Y_{X_i})}{\theta} s_{k,i}\Delta \ln K_i + \frac{(1-s^Y_{X_i})}{\theta}\omega_{Lp} s_{Li}\Delta \ln L_i^p + \frac{s^Y_{X,i}}{\theta}\Delta \ln X_i + \frac{F_{A_i}A_i}{Y_i P_i}\Delta \ln A_i.$$

  , where

  - Capital share: $\frac{rK_i}{rK_i + wL_i} = s_{k,i}$
  - Labor share: $\frac{wL_i}{rK_i + wL_i} = s_{L,i}$
  - Production labor: $\omega_{Lp} \equiv \frac{L_i^p}{L_i}$

- Express value added growth as variable input growth, intermediate intensity, and technology:

$$\Delta \ln Y_i^{VA} = \frac{(1-\mu\theta)}{\theta(1-\mu)}\left[s_{k,i}\Delta \ln K_i + \omega_{Lp} s_{Li}\Delta \ln L_i^p\right] + \frac{\mu\theta}{1-\mu\theta}\left[\frac{(1-\mu\theta)}{\theta(1-\mu)} - 1\right](\Delta \ln X_i - \Delta \ln Y_i) + \frac{F_{A_i}A_i}{(1-\mu)Y_i p_i}\Delta \ln A_i.$$

  - The goal is to find the coefficient of $\Delta \ln X_i - \Delta \ln Y_i$. Others are just rearranging the non-X terms.
  - Start from:

$$\Delta \ln Y_i = \frac{(1-s^Y_{X_i})}{\theta} s_{k,i}\Delta \ln K_i + \frac{(1-s^Y_{X_i})}{\theta}\omega_{Lp} s_{Li}\Delta \ln L_i^p + \frac{s^Y_{X,i}}{\theta}\Delta \ln X_i + \frac{F_{A_i}A_i}{Y_i P_i}\Delta \ln A_i.$$

    , we can express $\Delta \ln Y_i$ as:

$$\Delta \ln Y_i = \frac{s^Y_{X,i}}{\theta}\Delta \ln X_i + \Phi$$

    , where Φ are no-X terms. Specifically,

$$\Phi = \frac{(1-s^Y_{X_i})}{\theta} s_{k,i}\Delta \ln K_i + \frac{(1-s^Y_{X_i})}{\theta}\omega_{Lp} s_{Li}\Delta \ln L_i^p + \frac{F_{A_i}A_i}{Y_i P_i}\Delta \ln A_i$$

    Then we rearrange and have:

$$\Delta \ln Y_i = \frac{s^Y_{X,i}}{\theta}(\Delta \ln X_i - \Delta \ln Y_i + \Delta \ln Y_i) + \Phi$$

$$\left(1 - \frac{s^Y_{X,i}}{\theta}\right)\Delta \ln Y_i = \frac{s^Y_{X,i}}{\theta}(\Delta \ln X_i - \Delta \ln Y_i) + \Phi$$

$$\Delta \ln Y_i = \frac{s^Y_{X,i}}{\theta - s^Y_{X,i}}(\Delta \ln X_i - \Delta \ln Y_i) + \frac{\theta}{\theta - s^Y_{X,i}}\Phi$$

  - Plug the equation above in

$$\Delta \ln Y_i^{VA} = \Delta \ln Y_i - \frac{s^Y_X}{1 - s^Y_X}\left(\Delta \ln X_i - \Delta \ln Y_i\right),$$

$$\Delta \ln Y_i^{VA} = \frac{s^Y_{X,i}}{\theta - s^Y_{X,i}}(\Delta \ln X_i - \Delta \ln Y_i) - \frac{s^Y_{X,i}}{1 - s^Y_{X,i}}\left(\Delta \ln X_i - \Delta \ln Y_i\right) + \frac{\theta}{\theta - s^Y_{X,i}}\Phi$$

, rearrange and plug in $s_{X,i}^{Y} = \mu\theta$, the coefficient of $(\Delta \ln X_i - \Delta \ln Y_i)$ is:

$$\frac{\mu\theta}{1-\mu\theta}\left[\frac{(1-\mu\theta)}{\theta(1-\mu)} - 1\right]$$

, then we get:

$$\Delta \ln Y_i^{VA} = \frac{(1-\mu\theta)}{\theta(1-\mu)}\left[s_{k,i}\Delta \ln K_i + \omega_{Lp} s_{Li} \Delta \ln L_i^p\right] +$$
$$\frac{\mu\theta}{1-\mu\theta}\left[\frac{(1-\mu\theta)}{\theta(1-\mu)} - 1\right](\Delta \ln X_i - \Delta \ln Y_i) + \frac{F_{A_i} A_i}{(1-\mu) Y_i p_i}\Delta \ln A_i$$

- Derive the welfare related productivity:
  - Value-added version of production function:

$$Y_i^{VA} = PR_i \cdot F(K_i, L_i)$$

    , where PR is the welfare related productivity
  - Take the log differentiation:

$$\Delta \ln Y_i^{VA} = \Delta \ln PR_i + \frac{\partial \ln F}{\partial \ln K}\Delta \ln K_i + \frac{\partial \ln F}{\partial \ln L}\Delta \ln L_i$$

    * Firm's cost minimization problem:

$$\min_{K,L} \; rK + wL \quad \text{s.t.} \quad AF(K,L) \geq Y^{VA}$$

    * F.O.C condition:

$$p^{VA} A F_K = r, \qquad p^{VA} A F_L = w$$

    * Multiple both sides by $\frac{K}{p^{VA}Y^{VA}}$, or $\frac{L}{p^{VA}Y^{VA}}$

$$\frac{rK}{p^{VA}Y^{VA}} = \frac{AF_K K}{AF} = \frac{F_K K}{F}, \qquad \frac{wL}{p^{VA}Y^{VA}} = \frac{AF_L L}{AF} = \frac{F_L L}{F}$$

    * Apply the chain rule:

$$\frac{\partial \ln F}{\partial \ln K} = \frac{\partial \ln F}{\partial F} \cdot \frac{\partial F}{\partial K} \cdot \frac{\partial K}{\partial \ln K} = \frac{1}{F} \cdot F_K \cdot K = \frac{F_K K}{F}.$$

    * Then define:

$$s_k \equiv \frac{rK}{p^{VA}Y^{VA}}, \qquad s_l \equiv \frac{wL}{p^{VA}Y^{VA}}.$$

    * Plug back, we have:

$$\Delta \ln Y_i^{VA} = \Delta \ln PR_i + s_k \Delta \ln K_i + s_l \Delta \ln L_i$$

$$\Delta \ln PR_i = \Delta \ln Y_i^{VA} - s_k \Delta \ln K_i - s_l \Delta \ln L_i$$

  - Then we split the production labor and fixed cost labor:

$$\Delta \ln PR_i = \Delta \ln Y_i^{VA} - s_{k,i}\Delta \ln K_i - s_{l,i}\omega_{Lp}\Delta \ln L_i^P - s_{l,i}(1-\omega_{Lp})\Delta \ln L_i^F.$$

– Combine with the

$$\Delta \ln Y_i^{VA} = \frac{(1-\mu\theta)}{\theta(1-\mu)}\left[s_{k,i}\Delta \ln K_i + \omega_{Lp} s_{Li} \Delta \ln L_i^p\right] +$$
$$\frac{\mu\theta}{1-\mu\theta}\left[\frac{(1-\mu\theta)}{\theta(1-\mu)} - 1\right](\Delta \ln X_i - \Delta \ln Y_i) + \frac{F_{A_i} A_i}{(1-\mu) Y_i p_i} \Delta \ln A_i$$

, we have the productivity growth decomposed to variable input growth, relative growth of intermediate inputs to output, fixed cost growth, and technology growth

$$\begin{aligned}\Delta \ln PR_i = &\frac{(1-\theta)}{\theta(1-\mu)}\left[\Delta \ln V_i + \frac{\mu\theta}{1-\mu\theta}(\Delta \ln X_i - \Delta \ln Y_i)\right] \\ &- \frac{(1-\mu\theta)}{\theta(1-\mu)} s_{L_i}\left(1-\omega_{L_{p,i}}\right)\Delta \ln L_{f,i} + \frac{\Delta \ln A_i}{(1-\mu)}.\end{aligned}$$

Changes in industry level productivity: from individual productivity to manufacturing industry level productivity:

$$\begin{aligned}\Delta \ln PR = &\frac{\mu}{1-\mu}\cdot\frac{1-\theta}{\theta\mu}\Delta \ln V \\ &+ \frac{\mu}{1-\mu}\left[\left(\frac{1-\theta}{1-\mu\theta} - \frac{1-\gamma}{1-\mu}\right)\frac{\theta-1}{\theta}\sum_i \omega_i \Delta \ln \omega_i\right] \\ &+ \frac{\mu}{1-\mu}\left[\frac{1-\rho}{\rho}\left(\frac{\theta(1-\mu)}{1-\mu\theta} + \frac{\mu(1-\gamma)}{1-\mu}\right)\sum_i \omega_i \Delta \ln \gamma_i\right] \\ &- \frac{\mu}{1-\mu}(1-\gamma)\Delta \ln p_m\end{aligned}$$

, where ω_i is firm i's share of industry value added.

$$\omega_i \equiv \frac{P_i^{VA} Y_i^{VA}}{P^{VA} Y^{VA}}$$

, recall the firm level productivity:

$$\begin{aligned}\Delta \ln PR_i = &\frac{(1-\theta)}{\theta(1-\mu)}\left[\Delta \ln V_i + \frac{\mu\theta}{1-\mu\theta}(\Delta \ln X_i - \Delta \ln Y_i)\right] \\ &- \frac{(1-\mu\theta)}{\theta(1-\mu)} s_{L_i}\left(1-\omega_{L_{p,i}}\right)\Delta \ln L_{f,i} + \frac{\Delta \ln A_i}{(1-\mu)}.\end{aligned}$$

, the industry level will take

- Change in variable inputs
- Change in intermediate intensity
- Change in labor for fixed cost

into consideration.

Steps to derive industry level productivity:

- Terms: $\Delta \ln X_i - \Delta \ln Y_i$
  - Combine:

$$C_i = \frac{1}{\mu^\mu (1-\mu)^{1-\mu}} \frac{P_V^{1-\mu} P_{X_i}^{\mu}}{A_i},$$
$$P_{X_i} X_i = \mu C_i Y_i$$
$$p_i = \frac{C_i}{\theta}$$

into $\ln X_i - \Delta \ln Y_i$, we have:

$$\Delta \ln X_i - \Delta \ln Y_i = \Delta \ln p_i - \Delta \ln P_{X_i} = (\mu - 1)\Delta \ln P_{X_i}.$$

  - Recall how we derive γ_i:

$$\gamma_i = \frac{P_Z Z_i}{P_{X_i} X_i} = \left(\frac{P_Z}{P_{X_i}}\right)^{\frac{\rho}{\rho-1}}$$

$$P_{X_i} = P_Z \gamma_i^{\frac{1-\rho}{\rho}}.$$

  - Now we derive domestic input price index P_Z:

$$P_Z = \left[\int_i p_i^{\frac{\theta}{\theta-1}}\, di\right]^{\frac{\theta-1}{\theta}} = \left[\int_i \left(\frac{C_i}{\theta}\right)^{\frac{\theta}{\theta-1}}\, di\right]^{\frac{\theta-1}{\theta}}$$

    * The unit cost is:

$$C_i = \frac{1}{\mu^\mu (1-\mu)^{1-\mu}} \frac{P_V^{1-\mu} P_{X_i}^{\mu}}{A_i},$$

    , where the variable input price index is:

$$P_V = \alpha^{-\alpha}(1-\alpha)^{-(1-\alpha)} r^\alpha w^{1-\alpha}.$$

    * Plug back in to P_Z:

$$P_Z = \left[\int_i \left(\frac{\left(r^\alpha w^{1-\alpha}\right)^{1-\mu} P_{X_i}^{\mu}}{\epsilon \theta A_i}\right)^{\frac{\theta}{\theta-1}}\, di\right]^{\frac{\theta-1}{\theta}}.$$

    , where

$$\epsilon = \mu^\mu (1-\mu)^{1-\mu} \left(\alpha^\alpha (1-\alpha)^{1-\alpha}\right)^{1-\mu}.$$

    * Plug in $P_{X_i} = P_Z \gamma_i^{\frac{1-\rho}{\rho}}$, we have:

$$P_Z = \frac{\left(r^\alpha w^{1-\alpha}\right)^{1-\mu}}{\epsilon\theta} \left[\int_i \left(P_Z \gamma_i^{\frac{1-\rho}{\rho}}\right)^{\frac{\mu\theta}{\theta-1}} A_i^{\frac{\theta}{1-\theta}}\, di\right]^{\frac{\theta-1}{\theta}} = \frac{\left(r^\alpha w^{1-\alpha}\right)^{1-\mu} P_Z^{\mu}}{\epsilon\theta} \left[\int_i \gamma_i^{\mu\frac{\rho-1}{\rho}\frac{\theta}{1-\theta}} A_i^{\frac{\theta}{1-\theta}}\, di\right]^{\frac{\theta-1}{\theta}}$$

    * Rearrange and get P_Z and P_X in terms of domestic expenditure ratio and technology:

$$P_Z = \frac{\left(r^\alpha w^{1-\alpha}\right)^{1-\mu}}{(\epsilon\theta)^{\frac{1}{1-\mu}}} Q_{\gamma\theta\rho}^{-\frac{1}{1-\mu}}$$

$$P_{X_i} = \frac{\left(r^\alpha w^{1-\alpha}\right)^{1-\mu}}{(\epsilon\theta)^{\frac{1}{1-\mu}}} Q_{\gamma\theta\rho}^{-\frac{1}{1-\mu}} \gamma_i^{\frac{1-\rho}{\rho}}$$

, where

$$Q_{\gamma\theta\rho} = \left[\int_i \gamma_i^{\mu\frac{\rho-1}{\rho}\frac{\theta}{1-\theta}} A_i^{\frac{\theta}{1-\theta}}\, di\right]^{\frac{1-\theta}{\theta}}$$

, here are some take-away: $Q_{\gamma\theta\rho} = \left[\int_i \gamma_i^{\mu\frac{\rho-1}{\rho}\frac{\theta}{1-\theta}} A_i^{\frac{\theta}{1-\theta}}\, di\right]^{\frac{1-\theta}{\theta}}$ is a index that aggregates firm heterogeneity in technology A_i and sourcing choice γ_i

- Technology A_i: higher overall A, higher Q, lower P_Z and P_X
- Domestic price index γ: higher γ, lower Q, higher P_Z

* Now we have:

$$\Delta \ln P_{X_i} = \frac{1}{\mu-1}\Delta \ln Q_{\gamma\theta\rho} + \frac{1-\rho}{\rho}\Delta \ln \gamma_i.$$

– Plug $\Delta \ln P_{X_i}$ into:

$$\Delta \ln X_i - \Delta \ln Y_i = (\mu-1)\Delta \ln P_{X_i}.$$

$$\Delta \ln X_i - \Delta \ln Y_i = \Delta \ln Q_{\gamma\theta\rho} + \frac{(1-\rho)(\mu-1)}{\rho}\Delta \ln \gamma_i$$

- Allocation of labor between production and fixed cost. Term: $\omega_{L_{P,i}}$:

– Firm i's choice of importing scope solves firm maximization problem:

$$\widetilde{\Pi}_i = \Pi_i - wL^F = (1-\theta)p_i Y_i - wL^F,$$

subject to:

$$Y_i = g_i + \int_j z_{ji}dj = \left(\frac{p_i}{P_G}\right)^{\frac{1}{\theta-1}} G + \int_j \left(\frac{p_i}{P_Z}\right)^{\frac{1}{\theta-1}} \left(\frac{P_Z}{P_{X_j}}\right)^{\frac{1}{\rho-1}} X_j dj.$$

– To simplify, we define:

$$\widetilde{D} \equiv \left(\frac{1}{P_G}\right)^{\frac{1}{\theta-1}} G + \int_j \left(\frac{1}{P_Z}\right)^{\frac{1}{\theta-1}} \left(\frac{P_Z}{P_{X_j}}\right)^{\frac{1}{\rho-1}} X_j dj$$

, then

$$Y_i = p_i^{\frac{1}{\theta-1}}\widetilde{D}$$

$$p_i Y_i = p_i^{\frac{\theta}{\theta-1}}\widetilde{D}$$

, and variable profit as :

$$\Pi_i = (1-\theta)p_i Y_i = (1-\theta)p_i^{\frac{\theta}{\theta-1}}\widetilde{D}.$$

, include the fixed cost term:

$$F(\Omega_i) = f\,\Omega_i^{\lambda}, \quad f>0,\ \lambda>0$$

, rewrite the total profit as:

$$\widetilde{\Pi}_i = (1-\theta)p_i^{\frac{\theta}{\theta-1}}\widetilde{D} - wL^F$$

$$\widetilde{\Pi}_i = (1-\theta)p_i^{\frac{\theta}{\theta-1}}\widetilde{D} - wf\,\Omega_i^{\lambda}$$

– FOC condition:

–
$$\frac{\partial \Pi_i}{\partial \Omega_i} = w\,\frac{\partial L^F}{\partial \Omega_i}$$

* LHS:
$$\ln \Pi_i = \ln(1-\theta) + \frac{\theta}{\theta-1}\ln p_i + \ln \widetilde{D}$$

  · Take FOC derivative:
$$\frac{1}{\Pi_i}\frac{\partial \Pi_i}{\partial \Omega_i} = \frac{\theta}{\theta-1}\frac{\partial \ln p_i}{\partial \Omega_i}$$

  · Combine:
$$C_i = \frac{1}{\mu^{\mu}(1-\mu)^{1-\mu}}\frac{P_V^{1-\mu}P_{X_i}^{\mu}}{A_i},$$
$$P_{X_i}X_i = \mu C_i Y_i$$
$$p_i = \frac{C_i}{\theta}$$

  , we have:

$$\ln p_i = \mu \ln P_{X_i} + \text{others}$$

  , then we have:

$$\frac{1}{\Pi_i}\frac{\partial \Pi_i}{\partial \Omega_i} = \frac{\theta}{\theta-1}\frac{\partial \ln p_i}{\partial \Omega_i} = \frac{\theta}{\theta-1}\mu\frac{\partial \ln P_{X_i}}{\partial \Omega_i}$$

  · Combine
$$P_{X_i} = \left(P_Z^{\frac{\rho}{\rho-1}} + P_{M_i}^{\frac{\rho}{\rho-1}}\right)^{\frac{\rho-1}{\rho}}$$

  ,
$$P_{M_i} = \left(\int_{k\in\Omega_i} p_m^{\frac{\theta}{\theta-1}}\right)^{\frac{\theta-1}{\theta}} = p_m|\Omega_i|^{\frac{\theta-1}{\theta}}$$

  , we have
$$\frac{\partial \ln P_{X_i}}{\partial \Omega_i} = \frac{\theta-1}{\theta}\left(\frac{P_{M_i}}{P_{X_i}}\right)^{\frac{\rho}{\rho-1}}\frac{1}{\Omega}$$

  , plug back, we have:
$$\frac{1}{\Pi_i}\frac{\partial \Pi_i}{\partial \Omega_i} = \frac{\theta}{\theta-1}\frac{\partial \ln p_i}{\partial \Omega_i} = \frac{\theta}{\theta-1}\mu\frac{\partial \ln P_{X_i}}{\partial \Omega_i} = \mu\left(\frac{P_{M_i}}{P_{X_i}}\right)^{\frac{\rho}{\rho-1}}\frac{1}{\Omega}$$

* RHS $w\,\frac{\partial L^F}{\partial \Omega_i}$, fixed cost of importing is:

$$F(|\Omega_i|) = f\,|\Omega_i|^{\lambda}, \quad f>0,\ \lambda>0$$

, take FOC, we have:
$$\text{RHS} = w\lambda^F\frac{1}{\Omega}$$

* Connect LHS and RHS, we have:

$$\frac{\partial \Pi_i}{\partial \Omega_i} = w\,\frac{\partial L^F}{\partial \Omega_i}$$

$$\frac{\partial \Pi_i}{\partial \Omega_i} = \Pi_i \mu \left(\frac{P_{M_i}}{P_{X_i}}\right)^{\frac{\rho}{\rho-1}} \frac{1}{\Omega} = w\lambda L^F \frac{1}{\Omega}$$

- Connect to domestic intermediate input ratio:

$$\gamma_i = \frac{P_Z Z_i}{P_{X_i} X_i} = \left(\frac{P_Z}{P_{X_i}}\right)^{\frac{\rho}{\rho-1}}$$

By Shephard Lemma:

$$1 - \gamma_i = \left(\frac{P_{M_i}}{P_{X_i}}\right)^{\frac{\rho}{\rho-1}}$$

Plug in

$$\frac{\partial \Pi_i}{\partial \Omega_i} = \Pi_i \mu \left(\frac{P_{M_i}}{P_{X_i}}\right)^{\frac{\rho}{\rho-1}} \frac{1}{\Omega} = w\lambda L^F \frac{1}{\Omega}$$

, we have

$$\Pi_i \mu (1-\gamma_i) = w\lambda L^F$$

, express Π_i with price and output

$$(1-\theta) p_i Y_i \mu (1-\gamma_i) = w\lambda L^F$$

- So we can express labor in fixed cost as:

$$w L_i^F = \lambda^{-1}(1-\theta) p_i Y_i \mu (1-\gamma_i)$$

, from Cobb-Douglas function and FOC conditions:

$$\frac{\partial Y_i}{\partial L_{p,i}} L_{p,i} = (1-\mu)(1-\alpha)\, Y_i.$$

$$p_i \frac{\partial Y_i}{\partial L_{p,i}} = \frac{w}{\theta}$$

, so

$$w L_i^P = (1-\mu)(1-\alpha)\theta p_i Y_i$$

-

$$\frac{L_i^P}{L_i^F} = \frac{(1-\mu)(1-\alpha)\theta}{\lambda^{-1}(1-\theta)\mu(1-\gamma_i)}.$$

- So now we get how does the firm allocate the production labor and the labor in fixed cost:

$$1 - \omega_{L_{P,i}} = \frac{L_F}{L} = \frac{w L_i^F}{w L_i^F + w L_i^P} = \frac{\lambda^{-1}(1-\theta)\mu(1-\gamma_i)}{\lambda^{-1}(1-\theta)\mu(1-\gamma_i) + (1-\mu)(1-\alpha)\theta}.$$

- Term: importing fixed cost L^F:

$$L^F \equiv f_v \Omega_i^{\lambda}$$

$$\Delta \ln L^F \equiv \lambda \Delta \ln \Omega_i$$

- Deal with $\Delta \ln \Omega_i$:

$$P_{M_i} = \left(\int_{k \in \Omega_i} p_m^{\frac{\theta}{\theta-1}} \right)^{\frac{\theta-1}{\theta}} = p_m \Omega_i^{\frac{\theta-1}{\theta}}$$

$$\Delta \ln \Omega_i = \frac{\theta}{\theta - 1} (\Delta \ln P_{M_i} - \Delta \ln p_m)$$

So,

$$\Delta \ln L^F \equiv \lambda \Delta \ln \Omega_i = \lambda \frac{\theta}{\theta - 1} (\Delta \ln P_{M_i} - \Delta \ln p_m)$$

- Given:

$$\gamma_i = \frac{P_Z Z_i}{P_{X_i} X_i} = \left(\frac{P_Z}{P_{X_i}} \right)^{\frac{\rho}{\rho-1}}$$

By Shephard Lemma:

$$1 - \gamma_i = \left(\frac{P_{M_i}}{P_{X_i}} \right)^{\frac{\rho}{\rho-1}}$$

So,

$$\ln(1 - \gamma_i) = \frac{\rho}{\rho - 1} \left[\ln P_{M_i} - \ln P_{X_i} \right]$$

$$\ln P_{M_i} = \ln P_{X_i} - \frac{1 - \rho}{\rho} \ln(1 - \gamma_i),$$

$$\Delta \ln P_{M_i} = \Delta \ln P_{X_i} - \frac{1 - \rho}{\rho} \Delta \ln(1 - \gamma_i)$$

- Plug into:

$$\Delta \ln L^F \equiv \lambda \Delta \ln \Omega_i = \lambda \frac{\theta}{\theta - 1} (\Delta \ln P_{M_i} - \Delta \ln p_m)$$

$$= \lambda \frac{\theta}{\theta - 1} (\Delta \ln P_{X_i} - \frac{1 - \rho}{\rho} \Delta \ln(1 - \gamma_i) - \Delta \ln p_m))$$

- Given:

$$w L_i^F = \lambda^{-1} (1 - \theta) p_i Y_i \mu (1 - \gamma_i)$$

$$w L_i^P = (1 - \mu)(1 - \alpha) \theta p_i Y_i$$

$$P_i^{VA} Y_i^{VA} = (1 - \mu\theta) \, P_i Y_i,$$

$$s_{L,i} = \frac{w L_i^P + w L_i^F}{P_i^{VA} Y_i^{VA}} = \frac{\lambda^{-1} (1 - \theta) \mu (1 - \gamma_i) + (1 - \mu)(1 - \alpha)\theta}{1 - \mu\theta},$$

$$s_{L,i} (1 - \omega_{lp}) \, \Delta \ln L^F = \frac{\lambda^{-1} (1 - \theta) \mu (1 - \gamma_i)}{1 - \mu\theta} \, \Delta \ln L^F.$$

$$\frac{(1-\mu\theta)}{\theta(1-\mu)} s_{L,i}(1-\omega_{lp})\Delta \ln L^F = \frac{(1-\mu\theta)}{\theta(1-\mu)} \frac{\lambda^{-1}(1-\theta)\mu(1-\gamma_i)}{(1-\mu\theta)} \Delta \ln L^F$$
$$= -\frac{\mu(1-\gamma_i)}{(1-\mu)} \left[\Delta \ln P_{X_i} - \frac{1-\rho}{\rho} \Delta \ln(1-\gamma_i) - \Delta \ln p_m \right].$$

- Term $Q_{\gamma\theta\rho}$:
  - First get the value-added share:

$$VA_i \equiv p_i y_i - P_{X_i} X_i, \qquad VA \equiv PY - P_X X.$$

$$\frac{P_{X_i} X_i}{p_i y_i} = s^Y_{X,i} = \mu\theta$$
$$VA_i = (1-\mu\theta)\, p_i y_i,$$
$$VA = (1-\mu\theta) \sum_j p_j y_j.$$

$$\omega_i \equiv \frac{VA_i}{VA} = \frac{(1-\mu\theta) p_i y_i}{(1-\mu\theta)\sum_j p_j y_j} = \frac{p_i y_i}{\sum_j p_j y_j}.$$

$$p_i y_i = p_i^{\frac{\theta}{\theta-1}} \tilde{D}, \qquad \sum_j p_j y_j = \tilde{D} \int_j p_j^{\frac{\theta}{\theta-1}} dj = \tilde{D}\, P^{\frac{\theta}{\theta-1}},$$

, where

$$\tilde{D} \equiv Y P^{-\frac{1}{\theta-1}},$$

$$p_i y_i = \tilde{D}\, p_i^{\frac{\theta}{\theta-1}}$$

, so we can write the value-added weight as:

$$\omega_i = \frac{p_i^{\frac{\theta}{\theta-1}} \tilde{D}}{\tilde{D}\, P^{\frac{\theta}{\theta-1}}} = \left(\frac{p_i}{P}\right)^{\frac{\theta}{\theta-1}} = \left(\frac{p_i}{\left(\int_j p_j^{\frac{\theta}{\theta-1}} dj\right)^{\frac{\theta-1}{\theta}}} \right)^{\frac{\theta}{\theta-1}}.$$

, given the unit cost function and constant markup, we know

$$p_i \propto P_V^{1-\mu}\, P_{X_i}^{\mu}\, A_i^{-1},$$

, then given the expression of γ, we have:

$$p_i \propto \gamma_i^{\mu\frac{1-\rho}{\rho}}\, A_i^{-1}.$$

, so we have:

$$\omega_i = \left(\frac{p_i}{\left(\int_j p_j^{\frac{\theta}{\theta-1}} dj\right)^{\frac{\theta-1}{\theta}}} \right)^{\frac{\theta}{\theta-1}} = \left(\frac{(\gamma_i)^{\mu\frac{1-\rho}{\rho}} (A_i)^{-1}}{\left(\int_j (\gamma_j)^{\mu\frac{1-\rho}{\rho}} (A_j)^{-\frac{\theta}{\theta-1}} dj\right)^{\frac{\theta-1}{\theta}}} \right)^{\frac{\theta}{\theta-1}}.$$

Notice that the denominator is $Q_{\gamma\theta\rho}$.

- Now we have:

$$Q_{\gamma\theta\rho} = \left[\int_i \gamma_i^{\mu\frac{\theta}{\theta-1}\frac{1-\rho}{\rho}} (A_i)^{\frac{\theta}{1-\theta}} di\right]^{\frac{1-\theta}{\theta}} = \omega_i^{\frac{\theta-1}{\theta}} A_i \, \gamma_i^{\mu\frac{\rho-1}{\rho}},$$

, so:

$$\Delta \ln Q_{\gamma\theta\rho} = \frac{\theta-1}{\theta} \sum_i \omega_i \, \Delta \ln \omega_i + \mu \frac{\rho-1}{\rho} \sum_i \omega_i \, \Delta \ln \gamma_i, \qquad \text{given } \Delta \ln A_i = 0.$$

- Now we can aggregate the firm i's productivity weighted by value-added share, to get the industry level productivity:

$$\Delta \ln PR_i = \frac{(1-\theta)}{\theta(1-\mu)} \left[\Delta \ln V_i + \frac{\mu\theta}{1-\mu\theta} (\Delta \ln X_i - \Delta \ln Y_i)\right] - \frac{(1-\mu\theta)}{\theta(1-\mu)} s_{L_i} \left(1 - \omega_{L_{p,i}}\right) \Delta \ln L_{f,i} + \frac{\Delta \ln A_i}{(1-\mu)}.$$

  - We apply the approximation in the expression:

$$\Delta \ln(1-\gamma_i) = -\frac{\gamma_i}{1-\gamma_i} \Delta \ln \gamma_i.$$

  - Replace the related terms

- Aggregate, we get the aggregate growth:

$$\Delta \ln PR = \frac{\mu}{1-\mu} \frac{1-\theta}{\theta\mu} \Delta \ln V + \frac{\mu}{1-\mu} \left[\left(\frac{1-\theta}{1-\mu\theta} - \frac{1-\gamma}{1-\mu}\right) \frac{\theta-1}{\theta} \sum_i \omega_i \Delta \ln \omega_i\right]$$

$$+ \frac{\mu}{1-\mu} \left[\frac{1-\rho}{\rho} \left(\frac{\theta(1-\mu)}{1-\mu\theta} + \frac{\mu(1-\gamma)}{1-\mu}\right) \sum_i \omega_i \Delta \ln \gamma_i\right] - \frac{\mu}{1-\mu} (1-\gamma) \Delta \ln p_m.$$