SECOND EDITION

ADVANCED ENGINEERING ELECTROMAGNETICS

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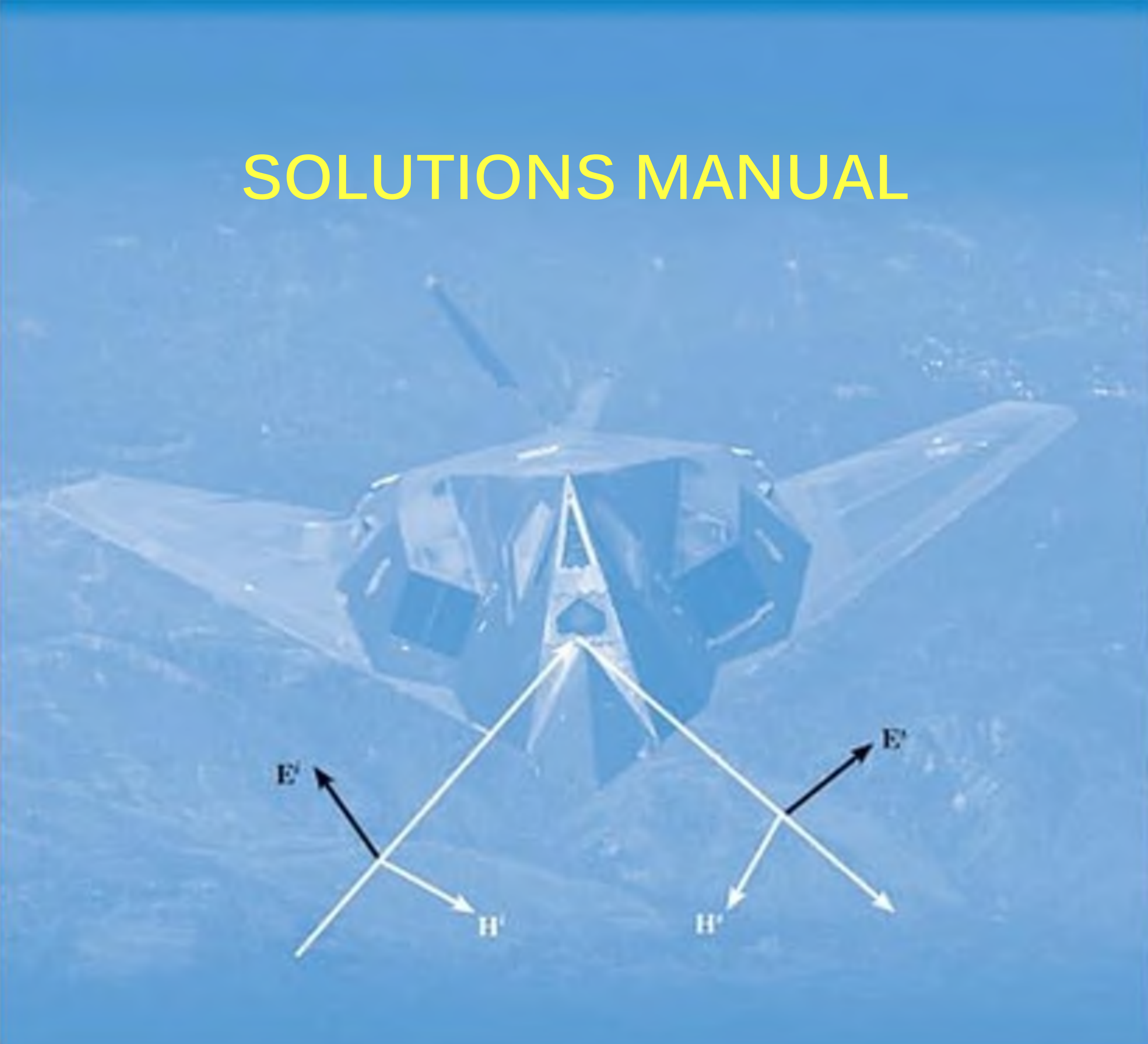
SOLUTIONS MANUAL



CHAPTER 1

1.1 $\nabla \times \underline{H} = \underline{J}_{ic} + \frac{\partial \underline{D}}{\partial t}$

Taking the divergence of both sides

$$\nabla \cdot (\nabla \times \underline{H}) = \nabla \cdot \underline{J}_{ic} + \nabla \cdot \frac{\partial \underline{D}}{\partial t} = \nabla \cdot \underline{J}_{ic} + \frac{\partial}{\partial t} \nabla \cdot \underline{D}$$

Using the vector identity of

$\nabla \cdot (\nabla \times \underline{A} = 0)$ and (1-3) we can write that

$$0 = \nabla \cdot \underline{J}_{ic} + \frac{\partial}{\partial t}(q_{ve}) \Rightarrow \boxed{\nabla \cdot \underline{J}_{ic} = -\frac{\partial q_{ve}}{\partial t}}$$

1.2 $\nabla \times \underline{E} = -\underline{M}_i - \frac{\partial \underline{B}}{\partial t}$

Taking a surface integral of both sides, we can write that

$$\iint_S (\nabla \times \underline{E}) \cdot d\underline{s} = -\iint_S \underline{M}_i \cdot d\underline{s} - \frac{\partial}{\partial t} \iint_S \underline{B} \cdot d\underline{s}$$

Applying Stokes' theorem of (1-7) to the left side of the equation above leads to

$$\oint_C \underline{E} \cdot d\underline{\ell} = -\iint_S \underline{M}_i \cdot d\underline{s} - \frac{\partial}{\partial t} \iint_S \underline{B} \cdot d\underline{s}$$

Using the same procedure, we can write that

$$\oint_C \underline{H} \cdot d\underline{\ell} = \iint_S \underline{J}_i \cdot d\underline{s} + \iint_S \underline{J}_c \cdot d\underline{s} + \frac{\partial}{\partial t} \iint_S \underline{D} \cdot d\underline{s}$$

For the remaining three equations of Table 1-1 we proceed as follows:

$$\nabla \cdot \underline{D} = q_{ve}$$

Taking a volume integral of both sides, we can write that

$$\iiint_V \nabla \cdot \underline{D} \, dv = \iiint_V q_{ve} ds = Q_e$$

1.2 cont'd

Applying the divergence theorem of (1-8) on the left side of the equation above leads to

$$\oiint_S \underline{D} \cdot d\underline{s} = Q_e$$

Using the same procedure, we can write that

$$\oiint_S \underline{B} \cdot d\underline{s} = Q_m$$

and

$$\oiint_S \underline{J}_{ic} \cdot d\underline{s} = -\frac{\partial}{\partial t}\iiint_V q_{ve}\, dv = -\frac{\partial Q_e}{\partial t}$$

1.3

(a) $\underline{D} = \hat{a}_x(3+x)$

$$Q_e = \oiint_S \underline{D} \cdot d\underline{s} = \int_0^1\int_0^1 \hat{a}_x(3+x)\Big|_{x=0} \cdot (-\hat{a}_x\, dy\, dz) + \int_0^1\int_0^1 \hat{a}_x(3+x)\Big|_{x=1} \cdot dy\, dz = -3+4 = 1$$

(b) $\underline{D} = \hat{a}_y(4+y^2)$

$$Q_e = \oiint_S \underline{D} \cdot d\underline{s} = \int_0^1\int_0^1 \hat{a}_y(4+y^2)\Big|_{y=0} \cdot (-\hat{a}_y\, dx\, dz) + \int_0^1\int_0^1 \hat{a}_y(4+y^2)\Big|_{y=1} \cdot \hat{a}_y\, dx\, dz = -4+5 = 1$$

1.4 $\underline{D}_2 = 6\hat{a}_x + 3\hat{a}_z$, $\chi_{e2} = \varepsilon_{sr2} - 1 = 2.56 - 1 = 1.56$

(a) $\underline{E}_2 = \dfrac{\underline{D}_2}{\varepsilon_{s2}} = \dfrac{1}{2.56\varepsilon_0}(6\hat{a}_x + 3\hat{a}_z) = \dfrac{1}{\varepsilon_0}\left(\dfrac{6}{2.56}\hat{a}_x + \dfrac{3}{2.56}\hat{a}_z\right)$

$$\underline{E}_2 = \frac{1}{\varepsilon_0}(2.34\hat{a}_x + 1.1718\hat{a}_z)$$

(b) $\underline{P}_2 = \varepsilon_0 \chi_e \underline{E}_2 = \varepsilon_0\left[1.56\dfrac{1}{\varepsilon_0}(2.34\hat{a}_x + 1.1718\hat{a}_z)\right]$

$$\underline{P}_2 = 3.65\hat{a}_x + 1.828\hat{a}_z$$

(c) $E_{1x} = E_{2x}$ Continuity of tangential components of $\underline{E}$-field

$$\frac{D_{1x}}{\varepsilon_0} = \frac{D_{2x}}{\varepsilon_{r2}\varepsilon_0} \Rightarrow D_{1x} = \frac{D_{2x}}{\varepsilon_{r2}} = \frac{6}{2.56} = 2.344$$

$\hat{n} \cdot (\underline{D}_2 - \underline{D}_1) = q_{es}$ Discontinuity of normal components of $\underline{D}$ density.

$\hat{n} = \hat{a}_z$: $D_{2z} - D_{1z} = q_{es} = 0.2 \Rightarrow D_{1z} = D_{2z} - q_{es} = 3 - 0.2 = 2.8$

$$D_{1z} = 2.8$$

$$\underline{D}_1 = 2.344\hat{a}_x + 2.8\hat{a}_z$$

Cont'd

$\boxed{1.4 \text{ cont'd}}$

(d) $\varepsilon_1 \underline{E}_1 = \underline{D}_1 \Rightarrow \varepsilon_0 \underline{E}_1 = \underline{D}_1 \Rightarrow \underline{E}_1 = \frac{1}{\varepsilon_0}\underline{D}_1 = \frac{1}{\varepsilon_0}\left(2.34\hat{a}_x + 2.8\hat{a}_z\right)$

$$\underline{E}_1 = \frac{1}{\varepsilon_0}\left(2.34\hat{a}_x + 2.8\hat{a}_z\right)$$

(e) $\chi_{e1} = \varepsilon_{sr1} - 1 = 1 - 1 = 0$

$$\underline{P}_1 = \varepsilon_0 \chi_{e1} \underline{E}_1 = 0$$

$\boxed{1.5}$ $\underline{H}_1 = 3\hat{a}_x + \hat{a}_z 9, \quad \mu_2 = 4\mu_0$

(a) $\underline{B}_1 = \mu_0 \underline{H}_1 = \mu_0 (3\hat{a}_x + 9\hat{a}_z)$

(b) $\underline{M}_1 = \chi_{m1} \underline{H}_1, \quad \chi_{m1} = \mu_{sr} - 1 = 1 - 1 = 0$

$$\underline{M}_1 = 0$$

(c) $H_{2x} = H_{1x} = 3$ Continuity of tangential $\underline{H}$-field

Continuity of normal $\underline{B}$ density

$$B_{2z} = B_{1z} = 9\mu_0 \Rightarrow \mu_2 H_{2z} = 9\mu_0 \Rightarrow 4\mu_0 H_{2z} = 9\mu_0$$

$$H_{2z} = \frac{9}{4} = 2.25$$

$$\underline{H}_2 = 3\hat{a}_x + 2.25\hat{a}_z$$

(d) $\underline{B}_2 = \mu_2 H_2 = 4\mu_0\left(3\hat{a}_x + \frac{9}{4}\hat{a}_z\right) = \mu_0\left(12\hat{a}_x + 9\hat{a}_z\right)$

(e) $\underline{M}_2 = \chi_{m2} \underline{H}_2 \qquad \chi_{m2} = \mu_{2r} - 1 = 4 - 1 = 3$

$$\underline{M}_2 = 3\left(3\hat{a}_x + 2.25\hat{a}_z\right) = 9\hat{a}_x + 6.75\hat{a}_z$$

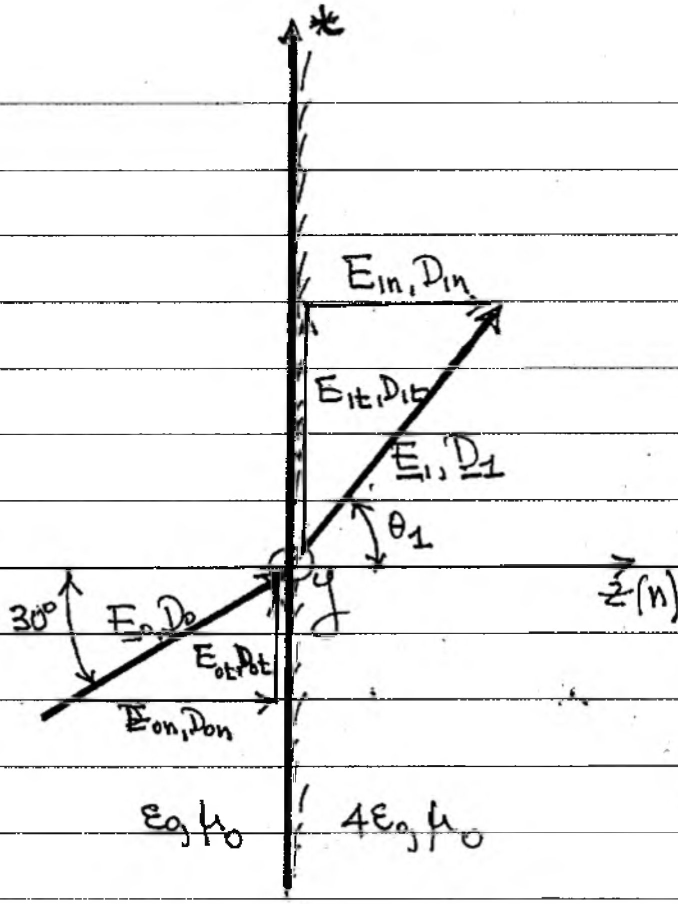
$\boxed{1.6}$ $D_0 = \varepsilon_0 E_0$

(a) $E_{on} = E_0 \cos 30^0 = 0.866 E_0$

$E_{ot} = E_0 \sin 30^0 = 0.5 E_0$

$D_{on} = \varepsilon_0 E_{on} = 0.866 \varepsilon_0 E_0$

$D_{ot} = \varepsilon_0 E_{ot} = 0.5 \varepsilon_0 E_0$



From B.Cs.

$E_{1t} = E_{ot} = 0.5 E_0$

$D_{1t} = \varepsilon_1 E_{1t} = 0.5(4)\varepsilon_0 E_0$

$D_{1t} = 2\varepsilon_0 E_0$

$D_{1n} = D_{on} = 0.866 \varepsilon_0 E_0$

$E_{1n} = \dfrac{D_{1n}}{\varepsilon_1} = \dfrac{0.866 \varepsilon_0 E_0}{4\varepsilon_0}$

$E_{1n} = 0.2165 E_0$

$\boxed{E_1 = (0.5\hat{a}_t + 0.2165\hat{a}_n) E_0}$

$E_1 = \sqrt{(E_{1t})^2 + (E_{1n})^2} = \sqrt{(0.5)^2 + (0.2165)^2} E_0 = \sqrt{0.25 + 0.046875} E_0$

$E_1 = \sqrt{0.296875} E_0 = 0.54486 E_0$

$\boxed{E_1 = 0.54486 E_0}$

$\boxed{D_1 = (2\hat{a}_t + 0.866\hat{a}_n)\varepsilon_0 E_0}$

$D_1 = \sqrt{(D_{1t})^2 + (D_{1n})^2} = \sqrt{(2)^2 + (0.866)^2}\, \varepsilon_0 E_0 = \sqrt{4 + 0.749956}\, \varepsilon_0 E_0$

$D_1 = \sqrt{4.749956}\, \varepsilon_0 E_0 = 2.17944 \varepsilon_0 E_0 = 0.54486(4\varepsilon_0) E_0$

$\boxed{D_1 = 2.17944 \varepsilon_0 E_0 = 0.54486(4\varepsilon_0) E_0}$

(b) $\theta_1 = \tan^{-1}\left(\dfrac{E_{1t}}{E_{1n}}\right) = \tan^{-1}\left(\dfrac{0.5 E_0}{0.2165 E_0}\right) = \tan^{-1}(2.309) = \boxed{66.587^0}$

$\theta_1 = \tan^{-1}\left(\dfrac{D_{1t}}{D_{1n}}\right) = \tan^{-1}\left(\dfrac{2\varepsilon_0 E_0}{0.866\varepsilon_0 E_0}\right) = \tan^{-1}\left(\dfrac{2}{0.866}\right) = \tan^{-1}(2.308) = 66.571^0$

$\boxed{\theta_1 = 66.571^0}$

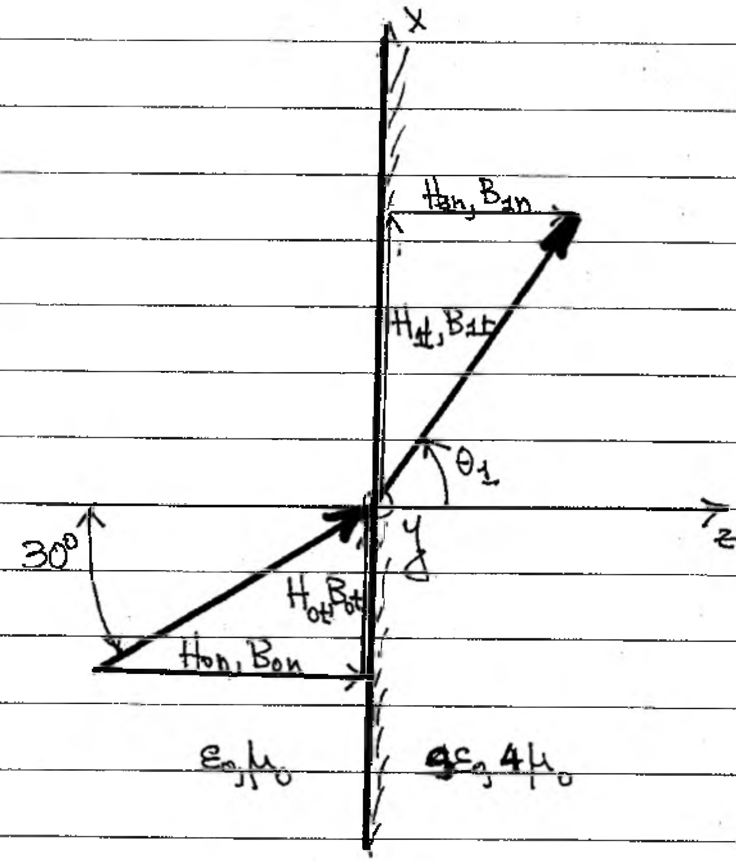
1.7 $B_0 = \mu_0 H_0$

$H_{on} = H_0 \cos 30^\circ = 0.866 H_0$

$H_{ot} = H_0 \sin 30^\circ = 0.5 H_0$

$B_{on} = \mu_0 H_{on} = 0.866 \mu_0 H_0$

$B_{ot} = \mu_0 H_{ot} = 0.5 \mu_0 H_0$



From B.Cs.

$H_{1t} = H_{ot} = 0.5 H_0$

$B_{1t} = \mu_1 H_{1t} = 0.5\, 4\mu_0 H_0$

$B_{1t} = 2\mu_0 H_0$

$B_{1n} = B_{on} = 0.866 \mu_0 H_0$

$H_{1n} = \dfrac{B_{1n}}{\mu_1} = \dfrac{0.866 \mu_0 H_0}{4\mu_0}$

$H_{1n} = 0.2165 H_0$

$\boxed{H_1 = (0.5\hat{a}_t + 0.2165\hat{a}_n) H_0}$

$H_1 = \sqrt{(H_{1t})^2 + (H_{1n})^2} = \sqrt{(0.5)^2 + (0.2165)^2}\, H_0 = \sqrt{0.296875}\, H_0$

$\boxed{H_1 = 0.54486 H_0}$

$\underline{B_1} = \boxed{(2\hat{a}_t + 0.866\hat{a}_n) \mu_0 H_0}$

$B_1 = \sqrt{(B_{1t})^2 + (B_{1n})^2} = \sqrt{(2)^2 + (0.866)^2}\, \mu_0 H_0 = \sqrt{4.751}\, \mu_0 H_0$

$\boxed{B_1 = 2.18 \mu_0 H_0 = 0.545 (4\mu_0) H_0}$

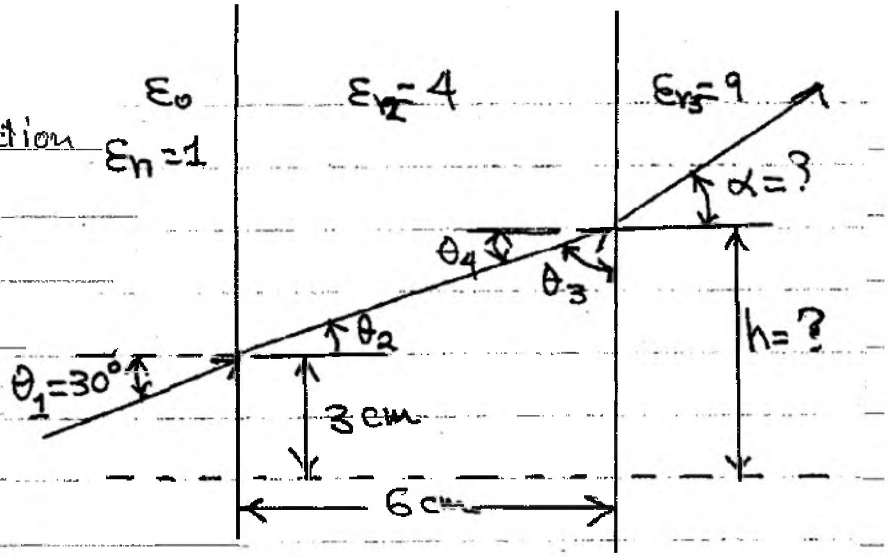
b. $\theta_1 = \tan^{-1}\left(\dfrac{H_{1t}}{H_{1n}}\right) = \tan^{-1}\left(\dfrac{0.5 H_0}{0.2165 H_0}\right) = \tan^{-1}(2.309) = 66.587^\circ$

$\boxed{\theta_1 = 66.587^\circ}$

$\theta_1 = \tan^{-1}\left(\dfrac{B_{1t}}{B_{1n}}\right) = \tan^{-1}\left(\dfrac{2\mu_0 H_0}{0.8666 \mu_0 H_0}\right) = \tan^{-1}(2.308) = 66.571^\circ$

$\boxed{\theta_1 = 66.571^\circ}$

1.8 Snell's Law of Refraction



(a) $\beta_1 \sin\theta_1 = \beta_2 \sin\theta_2$

$\omega\sqrt{\mu_1\varepsilon_1}\sin\theta_1 = \omega\sqrt{\mu_2\varepsilon_2}\sin\theta_2$

Since $\mu_1=\mu_2=\mu_0$

$\sqrt{\varepsilon_1}\sin\theta_1 = \sqrt{\varepsilon_2}\sin\theta_2$

$\sin\theta_2 = \sqrt{\frac{\varepsilon_1}{\varepsilon_2}}\sin\theta_1$

$= \sqrt{\frac{1}{\varepsilon_2/\varepsilon_1}}\sin\theta_1$

$\sin\theta_2 = \sqrt{\frac{1}{4}}\sin\theta_1 = \frac{1}{2}\sin(30^\circ) = \frac{1}{2}\left(\frac{1}{2}\right) = \frac{1}{4}$

$\theta_2 = \sin^{-1}\left(\frac{1}{4}\right) = 14.4775^\circ = \theta_4$

$\beta_2 \sin\theta_4 = \beta_3 \sin\alpha \Rightarrow \omega\sqrt{\mu_2\varepsilon_2}\sin\theta_4 = \omega\sqrt{\mu_3\varepsilon_3}\sin\alpha$

Since $\mu_2=\mu_3=\mu_0$:

$\sqrt{\varepsilon_3}\sin\alpha = \sqrt{\varepsilon_2}\sin\theta_4$

$\alpha = \sin^{-1}\left[\sqrt{\frac{\varepsilon_2}{\varepsilon_3}}\sin\theta_4\right] = \sin^{-1}\left(\sqrt{\frac{4}{9}}\sin\theta_4\right) = \sin^{-1}\left[\frac{2}{3}\sin(14.4775^\circ)\right]$

$\alpha = \sin^{-1}\left[\frac{2}{3}(0.25)\right] = \sin^{-1}(0.1667)$

$$\boxed{\alpha = 9.594^\circ}$$

(b) $\tan\theta_2 = \left(\frac{h-3}{6}\right) = \tan(14.4775^\circ)$

$\frac{h-3}{6} = (0.2582) \Rightarrow h-3 = 6(0.2582)$

$h = 3 + 6(0.2582) = 4.5492\text{ cm}$

$$\boxed{h = 4.5492\text{ cm}}$$

1.9 $\underline{D} = \varepsilon_0 \underline{E}$, $Q_e = \oint\!\!\oint_S \underline{D}\cdot d\underline{s}$

$$Q_e = \oint\!\!\oint_S \underline{D}\cdot d\underline{s} = \varepsilon_0 \iint_S \hat{a}_z E_z\Big|_{z=0}\cdot(-\hat{a}_z ds) + \varepsilon_0\iint_S \hat{a}_z E_z\Big|_{z=h}\cdot\hat{a}_z ds$$

$$Q_e = -\varepsilon_0 \iint_S \left[-\frac{c}{h} - \frac{bh^2}{6\varepsilon_0}\right] ds + \varepsilon_0 \iint_S \left[-\frac{c}{h} + 2\frac{bh^2}{6\varepsilon_0}\right] ds = \frac{3h^2 b}{6}(\pi a^2) = \frac{\pi}{2} b(ha)^2$$

1.10 $\varepsilon_r = 4,\ \mu_r = 9,\ a = 4\text{ cm}$

$$\underline{H} = 3\hat{a}_\rho + 6\hat{a}_\phi + 8\hat{a}_z$$

(a) $\underline{B} = \mu\underline{H} = \mu_r\mu_o\underline{H} = 9\mu_o(3\hat{a}_\rho + 6\hat{a}_\phi + 8\hat{a}_z) = \mu_o(27\hat{a}_\rho + 54\hat{a}_\phi + 72\hat{a}_z)$

$$\underline{B} = = \mu_o(27\hat{a}_\rho + 54\hat{a}_\phi + 27\hat{a}_z)$$

(b) $\underline{H}_o = \hat{a}_\rho H_{\rho o} + \hat{a}_\phi H_{\phi o} + \hat{a}_z H_{zo}$

$$H_{\phi_o} = H_\phi = 6$$

$$H_{z_o} = H_z = 8$$

$$B_{\rho o} = B_\rho = \mu_o H_{\rho o} = \mu H_\rho = \mu_r\mu_o H_\rho$$

$$H_{\rho o} = \mu_r H_\rho = 9(3) = 27$$

$$\underline{H}_o = (27\hat{a}_\rho + 6\hat{a}_\phi + 8\hat{a}_z)$$

(c)

$$\underline{B}_o = \mu_o\underline{H}_o = \mu_o(27\hat{a}_\rho + 6\hat{a}_\phi + 8\hat{a}_z)$$

1.11 $\nabla\cdot\underline{E} = 0$ for a source-free and homogeneous medium. Thus

$$\nabla\cdot\underline{E} = \left[\hat{a}_x\frac{\partial}{\partial x} + \hat{a}_y\frac{\partial}{\partial y} + \hat{a}_z\frac{\partial}{\partial z}\right]\cdot\left[\hat{a}_x A(x+y) + \hat{a}_y B(x-y)\right]\cos(\omega t) = 0$$

$$A\frac{\partial}{\partial x}(x+y) + B\frac{\partial}{\partial y}(x-y) = A(1) + B(-1) = 0 \Rightarrow A = B$$

Also

$$\underline{E} = \hat{a}_x A(x+y) + \hat{a}_y B(x-y)$$

$$\nabla\times\underline{E} = -j\omega\mu\underline{H} \Rightarrow \underline{H} = -\frac{1}{j\omega\mu}\begin{vmatrix}\hat{a}_x & \hat{a}_y & \hat{a}_z\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ A(x+y) & B(x-y) & 0\end{vmatrix}$$

$$\underline{H} = -\frac{1}{j\omega\mu}\left[\hat{a}_x(0) + \hat{a}_y(0) + \hat{a}_z(B-A)\right]$$

$$\nabla\times\underline{H} = j\omega\varepsilon\underline{E} \Rightarrow \underline{E} = 0 \Rightarrow A = B = 0$$

$\boxed{1.12}$

$$\underline{\mathcal{B}} = \hat{a}_z \frac{10^{-12}}{1+25\rho}\cos(1500\pi t)$$

a. $$\Psi_m = \iint_{S_0} \underline{\mathcal{B}}\cdot d\underline{s} = \int_0^{2\pi}\int_0^{a} \hat{a}_z\mathcal{B}_z\cdot\hat{a}_z\,\rho\,d\rho\,d\phi = \int_0^{2\pi}\int_0^{a}\mathcal{B}_z\,\rho\,d\rho\,d\phi$$

$$= 2\pi\int_0^a \mathcal{B}_z\,\rho\,d\rho = 2\pi\times10^{-12}\cos(1500\pi t)\int_0^a \frac{\rho}{1+25\rho}\,d\rho$$

Using the integral

$$\int \frac{u}{a+bu}\,du = \frac{1}{b^2}\left[a+bu-a\ln(a+bu)\right]$$

we can write the flux as

$$\Psi_m = 2\pi\times10^{-12}\cos(1500\pi t)\left\{\frac{1}{(25)^2}\left[1+25\rho-\ln(1+25\rho)\right]_0^a\right\}$$

$$= 2\pi\times10^{-12}\cos(1500\pi t)\left\{\frac{1}{625}\left[25a-\ln(1+25a)\right]\right\}$$

$$\Psi_m = 2\pi\times10^{-12}\cos(1500\pi t)\left\{\frac{1}{625}\left[2.5-\ln(3.5)\right]\right\} = 1.2539\times10^{-14}\cos(1500\pi t)$$

b. $$\oint_C \underline{\mathcal{E}}\cdot d\underline{\ell} = -\frac{\partial\Psi}{\partial t}$$

$$\int_0^{2\pi}(\hat{a}_\phi\mathcal{E}_\phi)\cdot\hat{a}_\phi\,\rho\,d\phi = -\frac{\partial}{\partial t}\int_0^{2\pi}\int_0^{\rho}\underline{\mathcal{B}}\cdot d\underline{s} = -\frac{\partial}{\partial t}\int_0^{2\pi}\int_0^{\rho}(\hat{a}_z\mathcal{B}_z)\cdot\hat{a}_z\,\rho\,d\rho\,d\phi$$

$$2\pi\rho\,\mathcal{E}_\phi = -2\pi\frac{\partial}{\partial t}\int_0^{\rho}\mathcal{B}_z\,\rho\,d\rho = 2\pi\times10^{-12}(1500\pi)\sin(1500\pi t)\int_0^{\rho}\frac{\rho}{1+25\rho}\,d\rho$$

$$\rho\,\mathcal{E}_\phi = 1500\pi\times10^{-12}\sin(1500\pi t)\left\{\frac{1}{(25)^2}\left[1+25\rho-\ln(1+25\rho)\right]_0^{\rho}\right\}$$

$$\mathcal{E}_\phi = \frac{7.5398\times10^{-12}}{\rho}\left[25\rho-\ln(1+25\rho)\right]\sin(1500\pi t)$$

<u>To check:</u> Use Maxwell's equation $\nabla\times\underline{\mathcal{E}} = -\frac{\partial\underline{\mathcal{B}}}{\partial t}$

$$\hat{a}\;\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho\,\mathcal{E}_\phi) = -\frac{\partial}{\partial t}\left[\hat{a}_z\frac{10^{-12}}{1+25\rho}\cos(1500\pi t)\right]$$

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left\{7.5398\times10^{-12}\left[25\rho-\ln(1+25\rho)\right]\sin(1500\pi t)\right\} = 1500\pi\times10^{-12}\sin(1500\pi t)\frac{1}{1+25\rho}$$

$$\frac{4.712\times10^{-9}}{1+25\rho}\sin(1500\pi t) = \frac{4.712\times10^{-9}}{1+25\rho}\sin(1500\pi t)$$

QED

$\boxed{1.13}$ $\underline{B} = \hat{a}_x B_x \cos(2y)\sin(\omega t - \pi z) + \hat{a}_y B_y \cos(2x)\cos(\omega t - \pi z)$

$$\nabla \times \underline{H} = \cancel{\underline{J}_i}^{\,0} + \cancel{\underline{J}_c}^{\,0} + \underline{J}_d = \underline{J}_d = \frac{\partial \underline{D}}{\partial t}$$

$$\underline{J}_d = \nabla \times \underline{H} = \frac{1}{\mu_0}\nabla \times \underline{B} = \frac{1}{\mu_0}\left\{-\hat{a}_x \frac{\partial B_y}{\partial z} + \hat{a}_y \frac{\partial B_x}{\partial z} + \hat{a}_z\left[\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right]\right\}$$

$$= \frac{1}{\mu_0}\left\{-\hat{a}_x \pi B_y \cos(2x)\sin(\omega t - \pi z) - \hat{a}_y \pi B_x \cos(2y)\cos(\omega t - \pi z) + \hat{a}_z\left[-2B_y \sin(2x)\cos(\omega t - \pi z) + 2B_x \sin(2y)\sin(\omega t - \pi z)\right]\right\}$$

$$\underline{J}_d = -\hat{a}_x 2.5\times10^6 B_y \cos(2x)\sin(\omega t - \pi z) - \hat{a}_y 2.5\times10^6 B_x \cos(2y)\cos(\omega t - \pi z) + \hat{a}_z\left[-1.59\times10^6 B_y \sin(2x)\cos(\omega t - \pi z) + 1.59\times10^6 B_x \sin(2y)\sin(\omega t - \pi z)\right]$$

$\boxed{1.14}$ $\underline{J}_d = \hat{a}_x yz + \hat{a}_y y^2 + \hat{a}_z xyz$, $\quad I_d = \oiint_S \underline{J}_d \cdot d\underline{s}$

$$I_d = \int_{-1/2}^{1/2}\int_{-1/2}^{1/2} (\hat{a}_x yz)\Big|_{x=-1/2} \cdot (-\hat{a}_x dy dz) + \int_{-1/2}^{1/2}\int_{-1/2}^{1/2} (\hat{a}_x yz)\Big|_{x=1/2} \cdot (\hat{a}_x dy dz)$$

$$+ \int_{-1/2}^{1/2}\int_{-1/2}^{1/2} (\hat{a}_y y^2)\Big|_{y=-1/2} \cdot (-\hat{a}_y dx dz) + \int_{-1/2}^{1/2}\int_{-1/2}^{1/2} (\hat{a}_y y^2)\Big|_{y=1/2} \cdot (\hat{a}_y dx dz)$$

$$+ \int_{-1/2}^{1/2}\int_{-1/2}^{1/2} (\hat{a}_z xyz)\Big|_{z=-1/2} \cdot (-\hat{a}_z dx dy) + \int_{-1/2}^{1/2}\int_{1/2}^{1/2} (\hat{a}_z xyz)\Big|_{z=1/2} \cdot (\hat{a}_z dx dy)$$

$$I_d = 0 + 0 + 0 = 0$$

$\boxed{1.15}$ $\underline{\mathcal{D}} = \hat{a}_r \frac{10^{-9}}{4\pi}\frac{1}{r^2}\cos(\omega t - \beta r) = \mathrm{Re}\left[\hat{a}_r \frac{10^{-9}}{4\pi r^2} e^{j(\omega t - \beta r)}\right] = \mathrm{Re}\left[\underline{D} e^{j\omega t}\right]$

where $\underline{D} = \hat{a}_r \frac{10^{-9}}{4\pi r^2} e^{-j\beta r}$, $\quad \beta = \omega\sqrt{\mu_0 \varepsilon_0}$

$$Q_e = \oiint_S \underline{D} \cdot d\underline{s} = \int_0^{2\pi}\int_0^{\pi} (\hat{a}_r D_r) \cdot \hat{a}_r r^2 \sin\theta \, d\theta \, d\phi = \frac{10^{-9}}{4\pi} e^{-j\beta r} \int_0^{2\pi}\int_0^{\pi} \sin\theta \, d\theta \, d\phi$$

$$Q_e = \frac{10^{-9}}{4\pi} e^{-j\beta r} \, 2\pi \left(-\cos\theta\right)_0^{\pi} = 10^{-9} e^{-j\beta r}$$

$|Q_e| = 10^{-9}$ Coulombs

$\boxed{1.16}$ $\underline{\mathcal{E}} = Re\left[\underline{E}(r,\theta)e^{j\omega t}\right] = Re\left[\hat{a}_\phi E_o \sin\theta \frac{e^{-j\beta_o r}}{r} e^{j\omega t}\right] = \hat{a}_\phi E_o \sin\theta \frac{\cos(\omega t - \beta_o r)}{r}$

where $\underline{E}(r,\theta) = \hat{a}_\phi E_o \sin\theta \frac{e^{-j\beta_o r}}{r}$

Using Maxwell's equation

$$\underline{H} = -\frac{1}{j\omega\mu_o}\nabla\times\underline{E} = -\frac{1}{j\omega\mu_o}\left[\hat{a}_r \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(E_\phi \sin\theta\right) - \hat{a}_\theta \frac{1}{r}\frac{\partial}{\partial r}\left(rE_\phi\right)\right]$$

$$= -\hat{a}_r \frac{2E_o}{j\omega\mu_o}\cos\theta \frac{e^{-j\beta_o r}}{r^2} - \hat{a}_\theta\sqrt{\frac{\varepsilon_o}{\mu_o}}E_o \sin\theta \frac{e^{-j\beta_o r}}{r}$$

$$\underline{H} \simeq -\hat{a}_\theta\sqrt{\frac{\varepsilon_o}{\mu_o}}E_o\sin\theta \frac{e^{-j\beta_o r}}{r} \Rightarrow \underline{\mathcal{H}} = Re\left[\underline{H}e^{j\omega t}\right] \simeq -\hat{a}_\theta\sqrt{\frac{\varepsilon_o}{\mu_o}}E_o \sin\theta \frac{\cos(\omega t - \beta_o r)}{r}$$

$\boxed{1.17}$ $v(t) = 10\cos(\omega t)$, $\underline{\mathcal{J}}_c = \sigma\underline{\mathcal{E}}$, $\underline{\mathcal{J}}_d = \frac{\partial \underline{\mathcal{D}}}{\partial t} = \varepsilon\frac{\partial\underline{\mathcal{E}}}{\partial t}$

$\underline{\mathcal{E}}(t) = \frac{10}{2\times10^{-2}}\cos(\omega t) = 500\cos(\omega t)$, $\frac{\partial\underline{\mathcal{E}}}{\partial t} = -500\omega\sin(\omega t)$

a. $f = 1$ MHz

$|\underline{\mathcal{J}}_c|_{max} = |\sigma\underline{\mathcal{E}}|_{max} = (3.7\times10^{-4})500 = 0.185$

$|\underline{\mathcal{J}}_d|_{max} = \left|\varepsilon\frac{\partial\underline{\mathcal{E}}}{\partial t}\right|_{max} = |-500\omega\varepsilon\sin(\omega t)|_{max} = \frac{2.56}{36} = 0.07111$

b. $f = 100$ MHz

$|\underline{\mathcal{J}}_c|_{max} = |\sigma\underline{\mathcal{E}}|_{max} = (3.7\times10^{-4})500 = 0.185$

$|\underline{\mathcal{J}}_d|_{max} = \left|\varepsilon\frac{\partial\underline{\mathcal{E}}}{\partial t}\right|_{max} = \frac{2.56(100)}{36} = 7.111$

$\boxed{1.18}$ $\underline{\mathcal{E}} = \left[\hat{a}_y 5 + \hat{a}_z 10\right]\cos(\omega t - \beta x) = Re\left[(\hat{a}_y 5 + \hat{a}_z 10)e^{j(\omega t - \beta x)}\right] = Re\left[\underline{E}e^{j\omega t}\right]$

where $\underline{E} = (\hat{a}_y 5 + \hat{a}_z 10)e^{-j\beta x}$

a. $\underline{H} = -\frac{1}{j\omega\mu_o}\nabla\times\underline{E} = \hat{a}_y\left(-\frac{\partial E_z}{\partial x}\right) + \hat{a}_z\left(\frac{\partial E_y}{\partial x}\right) = 4.244\times10^{-3}\left(-\hat{a}_y 10 + \hat{a}_z 5\right)e^{-j\beta x}$

$\underline{\mathcal{H}} = Re\left[\underline{H}e^{j\omega t}\right] = 4.244\times10^{-3}\left(-\hat{a}_y 10 + \hat{a}_z 5\right)\cos(\omega t - \beta x)$

(continued)

$\boxed{1.18 \text{ cont'd.}}$ The tangential components of the electric and magnetic fields must be continuous across the boundaries.

The normal components of the electric and magnetic fields must be discontinuous across the boundaries.

b. $\mathcal{E}_z^o(y=h^+) = 10\cos(\omega t-\beta x)$; $\mathcal{E}_y^o(y=h^+) = \frac{\varepsilon}{\varepsilon_0}(5)\cos(\omega t-\beta x) = 12.8\cos(\omega t-\beta x)$

$$\underline{\mathcal{E}}^o(y=h^+) = (\hat{a}_y 12.8 + \hat{a}_z 10)\cos(\omega t-\beta x)$$

$\mathcal{H}_z^o(y=h^+) = 4.244\times10^{-3}(5)\cos(\omega t-\beta x)$; $\mathcal{H}_y^o(y=h^+) = 4.244\times10^{-3}(-10)\cos(\omega t-\beta x)$

$$\underline{\mathcal{H}}(y=h^+) = 4.244\times10^{-3}(-\hat{a}_y 10 + \hat{a}_z 5)\cos(\omega t-\beta x)$$

In a similar manner

$$\underline{\mathcal{E}}^o(y=-h^+) = (\hat{a}_y 12.8 + \hat{a}_z 10)\cos(\omega t-\beta x)$$

$$\underline{\mathcal{H}}(y=-h^-) = 4.244\times10^{-3}(-\hat{a}_y 10 + \hat{a}_z 5)\cos(\omega t-\beta x)$$

$\boxed{1.19}$

$$\underline{\mathcal{J}} = \hat{a}_z 10^4 e^{-10^6 y}\cos(2\pi\times10^9 t)$$

$$\underline{\mathcal{J}}(y=0.25\times10^{-3}) = \hat{a}_z 10^4 e^{-10^6(2.5\times10^{-4})}\cos(2\pi\times10^9 t) = \hat{a}_z 10^4 e^{-250}\cos(2\pi\times10^9 t) \simeq 0$$

$$I = \iint_S \underline{\mathcal{J}}\cdot d\underline{s} \simeq 2\int_0^{2.5\times10^{-4}}\int_0^{5\times10^{-3}}\left[\hat{a}_z 10^4 e^{-10^6 y}\cos(2\pi\times10^9 t)\right]\cdot\hat{a}_z\, dx\, dy$$

$$I \simeq 2(5\times10^{-3})(10^4)\cos(2\pi\times10^9 t)\int_0^{2.5\times10^{-4}} e^{-10^6 y}\, dy = 10^{-4}\cos(2\pi\times10^9 t)$$

$\boxed{1.20}$ a. $\underline{\mathcal{E}} = \hat{a}_y E_0 \sin\left(\frac{\pi}{a}x\right)\cos(\omega t-\beta_z z) = \mathrm{Re}\left[\hat{a}_y E_0 \sin\left(\frac{\pi}{a}x\right)e^{j(\omega t-\beta_z z)}\right] = \mathrm{Re}\left[\underline{E}\, e^{j\omega t}\right]$

where $\underline{E} = \hat{a}_y E_0 \sin\left(\frac{\pi}{a}x\right)e^{-j\beta_z z}$

$$\underline{H} = -\frac{1}{j\omega\mu_0}\nabla\times\underline{E} = \hat{a}_x\frac{1}{j\omega\mu_0}\frac{\partial E_y}{\partial z} - \hat{a}_z\frac{1}{j\omega\mu_0}\frac{\partial E_y}{\partial x}$$

$$= -\hat{a}_x\frac{\beta_z}{\omega\mu_0}E_0\sin\left(\frac{\pi}{a}x\right)e^{-j\beta_z z} + \hat{a}_z j\frac{E_0}{\omega\mu_0}\left(\frac{\pi}{a}\right)\cos\left(\frac{\pi}{a}x\right)e^{-j\beta_z z}$$

$$\underline{\mathcal{H}} = \mathrm{Re}\left[\underline{H}\, e^{j\omega t}\right] = -\hat{a}_x\frac{\beta_z}{\omega\mu_0}E_0\sin\left(\frac{\pi}{a}x\right)\cos(\omega t-\beta_z z) + \hat{a}_z\frac{E_0}{\omega\mu_0}\left(\frac{\pi}{a}\right)\cos\left(\frac{\pi}{a}x\right)\cos\left(\omega t+\frac{\pi}{2}-\beta_z z\right)$$

b. Using $\nabla\times\underline{H} = j\omega\varepsilon_0\underline{E} \Rightarrow \hat{a}_y\left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right) = j\omega\varepsilon_0 E_0\sin\left(\frac{\pi}{a}x\right)e^{-j\beta_z z}$

$$\hat{a}_y j\frac{E_0}{\omega\mu_0}\left[\beta_z^2 + \left(\frac{\pi}{a}\right)^2\right]\sin\left(\frac{\pi}{a}x\right)e^{-j\beta_z z} = j\omega\varepsilon_0 E_0\sin\left(\frac{\pi}{a}x\right)e^{-j\beta_z z}$$

$$\omega\varepsilon_0 = \frac{1}{\omega\mu_0}\left[\beta_z^2 + \left(\frac{\pi}{a}\right)^2\right] \Rightarrow \beta_z = \pm\sqrt{\omega^2\mu_0\varepsilon_0 - \left(\frac{\pi}{a}\right)^2}$$

$\boxed{1.21}$ $\mathcal{E} = \hat{a}_\rho\left(\frac{100}{\rho}\right)\cos(10^8 t - \beta z) = \text{Re}\left[\hat{a}_\rho \frac{100}{\rho} e^{j(10^8 t - \beta z)}\right] = \text{Re}\left[\underline{E}\, e^{j\omega t}\right]$

where $\underline{E} = \hat{a}_\rho \frac{100}{\rho} e^{-j\beta z}$

a. $\underline{H} = -\frac{1}{j\omega\mu_0}\nabla\times\underline{E} = -\frac{1}{j\omega\mu_0}\left(\hat{a}_\phi \frac{\partial E_\rho}{\partial z}\right) = \hat{a}_\phi \frac{\beta}{\omega\mu} \frac{100}{\rho} e^{-j\beta z}$

$\mathcal{H} = \text{Re}\left[\underline{H} e^{j\omega t}\right] = \text{Re}\left[\underline{H} e^{j10^8 t}\right] = \hat{a}_\phi \frac{\beta}{\omega\mu}\left(\frac{100}{\rho}\right)\cos(10^8 t - \beta z)$

b. $\nabla\times\underline{H} = j\omega\varepsilon\,\underline{E} \Rightarrow -\hat{a}_\rho \frac{\partial H_\phi}{\partial z} = \hat{a}_\rho j\omega\varepsilon \frac{100}{\rho} e^{-j\beta z}$

$$\hat{a}_\rho j \frac{\beta^2}{\omega\mu_0}\left(\frac{100}{\rho}\right) e^{-j\beta z} = \hat{a}_\rho j\omega\varepsilon \left(\frac{100}{\rho}\right) e^{-j\beta z}$$

$$\omega\varepsilon = \frac{\beta^2}{\omega\mu_0} \Rightarrow \beta^2 = \omega^2\mu_0\varepsilon$$

c. $\mathcal{J}_d = \varepsilon\frac{\partial\mathcal{E}}{\partial t} = \varepsilon\frac{\partial}{\partial t}\left[\hat{a}_\rho \frac{100}{\rho}\cos(10^8 t - \beta z)\right] = -\hat{a}_\rho \frac{10^{10}\varepsilon}{\rho}\sin(10^8 t - \beta z)$

$= -\hat{a}_\rho \varepsilon_r \frac{8.854\times10^{-2}}{\rho}\sin(10^8 t - \beta z) = -\hat{a}_\rho \frac{2.25(8.854\times10^{-2})}{\rho}\sin(10^8 t - \beta z)$

$\mathcal{J}_d = -\hat{a}_\rho \frac{0.1992}{\rho}\sin(10^8 t - \beta z)$

$\boxed{1.22}$ $\mathcal{H} = \hat{a}_\phi \frac{2}{\rho}\cos\left(\frac{\pi}{\ell}z\right)\cos(4\pi\times10^8 t) = \text{Re}\left[\hat{a}_\phi \frac{2}{\rho}\cos\left(\frac{\pi}{\ell}z\right)e^{j4\pi\times10^8 t}\right] = \text{Re}\left[\underline{H} e^{j4\pi\times10^8 t}\right]$

where $\underline{H} = \hat{a}_\phi \frac{2}{\rho}\cos\left(\frac{\pi}{\ell}z\right)$, $\omega = 4\pi\times10^8$

a. $\underline{E} = \frac{1}{j\omega\varepsilon_0}\nabla\times\underline{H} = -\hat{a}_\rho \frac{1}{j\omega\varepsilon_0}\frac{\partial H_\phi}{\partial z} = -\hat{a}_\rho j \frac{1}{\omega\varepsilon_0}\left(\frac{\pi}{\ell}\right)\frac{2}{\rho}\sin\left(\frac{\pi}{\ell}z\right)$

$\mathcal{E} = \text{Re}\left[\underline{E} e^{j\omega t}\right] = \hat{a}_\rho \frac{1}{\omega\varepsilon_0}\left(\frac{\pi}{\ell}\right)\frac{2}{\rho}\sin\left(\frac{\pi}{\ell}z\right)\sin(4\pi\times10^8 t)$

b. $\mathcal{J}_s = \hat{n}\times(\mathcal{H}_2 - \mathcal{H}_1)$: At $\rho = a$: $\mathcal{J}_s = \hat{a}_\rho\times(\hat{a}_\phi\mathcal{H}_\phi)\Big|_{\rho=a} = \hat{a}_z \frac{2}{a}\cos\left(\frac{\pi}{\ell}z\right)\cos(4\pi\times10^8 t)$

At $\rho = b$: $\mathcal{J}_s = -\hat{a}_\rho\times(\hat{a}_\phi\mathcal{H}_\phi)\Big|_{\rho=b} = -\hat{a}_z \frac{2}{b}\cos\left(\frac{\pi}{\ell}z\right)\cos(4\pi\times10^8 t)$

c. $\mathcal{J}_d = \varepsilon_0\frac{\partial\mathcal{E}_0}{\partial t} = \hat{a}_\rho \frac{4\pi\times10^8}{\omega}\left(\frac{\pi}{\ell}\right)\frac{2}{\rho}\sin\left(\frac{\pi}{\ell}z\right)\cos(4\pi\times10^8 t) = \hat{a}_\rho \frac{\pi}{\ell}\left(\frac{2}{\rho}\right)\sin\left(\frac{\pi}{\ell}z\right)\cos(4\pi\times10^8 t)$

d. $I_d = \int_0^\ell\int_0^{2\pi}\mathcal{J}_d\cdot d\underline{s} = \int_0^\ell\int_0^{2\pi}(\hat{a}_\rho\mathcal{J}_d)\cdot\hat{a}_\rho\,\rho\,d\phi\,dz = \frac{\pi}{\ell}(4\pi)\left[-\frac{\ell}{\pi}\cos\left(\frac{\pi}{\ell}z\right)\right]_0^\ell\cos(4\pi\times10^8 t)$

$I_d = 8\pi\cos(4\pi\times10^8 t)$

$\boxed{1.23}$ $\quad \nabla \times \underline{\mathcal{E}} = -\underline{\mathcal{M}}_i - \frac{\partial \underline{\mathcal{B}}}{\partial t}$

Defining

$$\underline{\mathcal{E}} = \mathrm{Re}\left[\underline{E}\, e^{j\omega t}\right]$$

$$\underline{\mathcal{M}}_i = \mathrm{Re}\left[\underline{M}_i\, e^{j\omega t}\right]$$

$$\underline{\mathcal{B}} = \mathrm{Re}\left[\underline{B}\, e^{j\omega t}\right]$$

Thus

$$\nabla \times \left(\mathrm{Re}\left[\underline{E}\, e^{j\omega t}\right]\right) = -\mathrm{Re}\left[\underline{M}_i e^{j\omega t}\right] - \frac{\partial}{\partial t}\left(\mathrm{Re}\left[\underline{B}\, e^{j\omega t}\right]\right)$$

Interchanging differentiation with the Real part leads to

$$\mathrm{Re}\left[\nabla \times \left(\underline{E}\, e^{j\omega t}\right)\right] = \mathrm{Re}\left[-\underline{M}_i\, e^{j\omega t}\right] + \mathrm{Re}\left[-\frac{\partial}{\partial t}\left(\underline{B}\, e^{j\omega t}\right)\right]$$

or

$$\mathrm{Re}\left[\left(\nabla \times \underline{E}\right) e^{j\omega t}\right] = \mathrm{Re}\left[-\underline{M}_i\, e^{j\omega t}\right] + \mathrm{Re}\left[-j\omega \underline{B}\, e^{j\omega t}\right]$$

__Lemma:__ If $\underline{A}$ and $\underline{B}$ are complex quantities and

$$\mathrm{Re}\left[\underline{A}\, e^{j\omega t}\right] = \mathrm{Re}\left[\underline{B}\, e^{j\omega t}\right] \quad \text{for all } t$$

then

$$\underline{A} = \underline{B}$$

Using this lemma, we can write that

$$\nabla \times \underline{E} = -\underline{M}_i - j\omega \underline{B}$$

The same procedure can be used for all the other differential form equations.

For the integral form

$$\oint_C \underline{\mathcal{E}} \cdot d\underline{\ell} = -\iint_S \underline{\mathcal{M}}_i \cdot d\underline{s} - \frac{\partial}{\partial t}\iint_S \underline{\mathcal{B}} \cdot d\underline{s}$$

Using the above definitions, we can write the integral form as

$$\oint_C \mathrm{Re}\left[\underline{E}\, e^{j\omega t}\right] \cdot d\underline{\ell} = -\iint_S \mathrm{Re}\left[\underline{M}_i e^{j\omega t}\right] \cdot d\underline{s} - \frac{\partial}{\partial t}\iint_S \mathrm{Re}\left[\underline{B}\, e^{j\omega t}\right] \cdot d\underline{s}$$

$$\mathrm{Re}\left\{\left[\oint \underline{E} \cdot d\underline{\ell}\right] e^{j\omega t}\right\} = \mathrm{Re}\left\{\left[-\iint_S \underline{M}_i \cdot d\underline{s}\right] e^{j\omega t}\right\} + \mathrm{Re}\left\{\left[-j\omega \iint_S \underline{B} \cdot d\underline{s}\right] e^{j\omega t}\right\}$$

Using the above lemma leads to

$$\oint_C \underline{E} \cdot d\underline{\ell} = -\iint_S \underline{M}_i \cdot d\underline{s} - j\omega \iint_S \underline{B} \cdot d\underline{s}$$

The same procedure can be used for all the other integral form equations.

$\boxed{1.24}$

$$\underline{\mathcal{E}} = \mathrm{Re}\left[\underline{E}\,e^{j\omega t}\right] = \mathrm{Re}\left[(\underline{E}_R + j\underline{E}_x)(\cos\omega t + j\sin\omega t)\right]$$

$$= \mathrm{Re}\left[(\underline{E}_R\cos\omega t - \underline{E}_x\sin\omega t) + j(\underline{E}_R\sin\omega t + \underline{E}_x\cos\omega t)\right]$$

$$\underline{\mathcal{E}} = (\underline{E}_R\cos\omega t - \underline{E}_x\sin\omega t)$$

Similarly

$$\underline{\mathcal{E}} = \frac{1}{2}\left[\underline{E}\,e^{j\omega t} + (\underline{E}\,e^{j\omega t})^*\right] = \frac{1}{2}\left[\underline{E}\,e^{j\omega t} + \underline{E}^* e^{-j\omega t}\right]$$

$$= \frac{1}{2}\left[(\underline{E}_R + j\underline{E}_x)(\cos\omega t + j\sin\omega t) + (\underline{E}_R - j\underline{E}_x)(\cos\omega t - j\sin\omega t)\right]$$

$$\underline{\mathcal{E}} = \frac{1}{2}\left[2(\underline{E}_R\cos\omega t - \underline{E}_x\sin\omega t)\right] = (\underline{E}_R\cos\omega t - \underline{E}_x\sin\omega t)$$

$\boxed{1.25}$

(a) $$\underline{\mathcal{E}}(z,t) = \hat{a}_x E_0 \sin\left[(\omega t - \beta_0 z) - \frac{\pi}{2}\right]$$

$$= \hat{a}_x E_0\left[\sin(\omega t - \beta_0 z)\underbrace{\cos\left(-\frac{\pi}{2}\right)}_{0} + \cos(\omega t - \beta_0 z)\underbrace{\sin\left(-\frac{\pi}{2}\right)}_{-1}\right]$$

$$= -\hat{a}_x E_0 \cos(\omega t - \beta_0 z) = -\hat{a}_x E_0 \,\mathrm{Re}\left[e^{j(\omega t - \beta_0 z)}\right]$$

$$\underline{\mathcal{E}}(z,t) = -\hat{a}_x E_0\,\mathrm{Re}\left[e^{j\omega t}e^{-j\beta_0 z}\right] = \mathrm{Re}\Big[\underbrace{-\hat{a}_x E_0 e^{-j\beta_0 z}}_{E_x(z)} e^{j\omega t}\Big]$$

$$\underline{E}(z) = -\hat{a}_x E_0 e^{-j\beta_0 z}$$

(b) $$\nabla\times\underline{E} = -j\omega\mu_0\underline{H} \Rightarrow \underline{H} = -\frac{1}{j\omega\mu_0}\nabla\times\underline{E} = -\frac{1}{j\omega\mu_0}\nabla\times\left[-\hat{a}_x E_0 e^{-j\beta_0 z}\right]$$

$$\underline{H} = -\frac{1}{j\omega\mu_0}\begin{vmatrix}\hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & 0 & 0\end{vmatrix} = -\frac{1}{j\omega\mu_0}\left[\hat{a}_x(0) + \hat{a}_y\left(\frac{\partial E_x}{\partial z}\right) + \hat{a}_z\left(-\cancelto{0}{\frac{\partial E_x}{\partial y}}\right)\right]$$

$$\underline{H} = -\frac{1}{j\omega\mu_0}\left[\hat{a}_y\left(\frac{\partial E_x}{\partial z}\right)\right] = -\hat{a}_y\frac{1}{j\omega\mu_0}\frac{\partial}{\partial z}\left[-E_0 e^{-j\beta_0 z}\right] = -\hat{a}_y E_0\frac{\beta_0}{\omega\mu_0}e^{-j\beta_0 z}$$

$$\underline{H} = -\hat{a}_y E_0\frac{\omega\sqrt{\mu_0\varepsilon_0}}{\omega\mu_0}e^{-j\beta_0 z} = -\hat{a}_y E_0\sqrt{\frac{\varepsilon_0}{\mu_0}}e^{-j\beta_0 z} = -\hat{a}_y\frac{E_0}{\eta_0}e^{-j\beta_0 z}, \quad \eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}}$$

(c) $$\underline{S}_{ave} = \frac{1}{2}\mathrm{Re}\left(\underline{E}\times\underline{H}^*\right) = \frac{1}{2}\mathrm{Re}\left[\left(-\hat{a}_x E_0 e^{-j\beta_0 z}\right)\times\left(-\hat{a}_y E_0\sqrt{\frac{\varepsilon_0}{\mu_0}}e^{-j\beta_0 z}\right)^*\right]$$

$$\underline{S}_{ave} = \hat{a}_z\frac{1}{2}|E_0|^2\sqrt{\frac{\varepsilon_0}{\mu_0}} = \hat{a}_z\frac{1}{2\eta_0}|E_0|^2$$

$\boxed{1.26}$

$$\underline{H} = \hat{a}_\phi H_0 \frac{e^{-j\beta_0\rho}}{\sqrt{\rho}}$$

$$\underline{E} = \frac{1}{j\omega\varepsilon_0}\nabla\times\underline{H} = \frac{1}{j\omega\varepsilon_0}\left[\hat{a}_z \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho H_\phi)\right] = \frac{H_0}{j\omega\varepsilon_0}\frac{\hat{a}_z}{\rho}\frac{\partial}{\partial\rho}\left[\rho^{1/2}e^{-j\beta_0\rho}\right]$$

$$= \hat{a}_z \frac{H_0}{j\omega\varepsilon_0}\frac{1}{\rho}\left[\rho^{1/2}(-j\beta_0 e^{-j\beta_0\rho}) + \frac{1}{2}\frac{1}{\rho^{1/2}}e^{-j\beta_0\rho}\right]$$

$$= \hat{a}_z H_0\left[-\frac{\beta_0}{\omega\varepsilon_0}\frac{e^{-j\beta_0\rho}}{\sqrt{\rho}} + \frac{e^{-j\beta_0\rho}}{j2\omega\varepsilon_0(\rho)^{3/2}}\right] \xrightarrow{\rho\to\text{large}} -\hat{a}_z H_0 \frac{\beta_0}{\omega\varepsilon_0}\frac{e^{-j\beta_0\rho}}{\sqrt{\rho}}$$

$$\underline{E} = -\hat{a}_z H_0 \frac{\omega\sqrt{\mu_0\varepsilon_0}}{\omega\varepsilon_0}\frac{e^{-j\beta_0\rho}}{\sqrt{\rho}} = -\hat{a}_z H_0\sqrt{\frac{\mu_0}{\varepsilon_0}}\frac{e^{-j\beta_0\rho}}{\sqrt{\rho}}$$

$\boxed{1.27}$

$$\underline{E} = \hat{a}_r E_r(r,\theta) + \hat{a}_\theta E_\theta(r,\theta),\ E_r = E_0\frac{\cos\theta}{r^2}\left(1+\frac{1}{j\beta_0 r}\right)e^{-j\beta_0 r},\ E_\theta = jE_0\frac{\beta_0\sin\theta}{2r}\left[1+\frac{1}{j\beta_0 r}-\frac{1}{(\beta_0 r)^2}\right]e^{-j\beta_0 r}$$

$$\underline{H} = -\frac{1}{j\omega\mu_0}\nabla\times\underline{E} = -\frac{1}{j\omega\mu_0}\left\{\frac{\hat{a}_\phi}{r}\left[\frac{\partial}{\partial r}(rE_\theta) - \frac{\partial E_r}{\partial\theta}\right]\right\}$$

Expanding using the above electric field components leads to

$$\underline{H} = \hat{a}_\phi j\sqrt{\frac{\varepsilon_0}{\mu_0}}\frac{\beta_0\sin\theta}{2r}\left(1+\frac{1}{j\beta_0 r}\right)e^{-j\beta_0 r}$$

or

$$H_r = H_\theta = 0$$

$$H_\phi = j\sqrt{\frac{\varepsilon_0}{\mu_0}}\frac{\beta_0\sin\theta}{2r}\left(1+\frac{1}{j\beta_0 r}\right)e^{-j\beta_0 r}$$

$\boxed{1.28}$ $\underline{E} = \hat{a}_\phi E_0 \frac{\sin\theta}{r}\left(1 + \frac{1}{j\beta_0 r}\right)e^{-j\beta_0 r} = \hat{a}_\phi E_\phi(r,\theta)$

$$\underline{H} = -\frac{1}{j\omega\mu_0}\nabla\times\underline{E} = -\frac{1}{j\omega\mu_0}\left\{\frac{\hat{a}_r}{r\sin\theta}\frac{\partial}{\partial\theta}\left(E_\phi\sin\theta\right) + \frac{\hat{a}_\theta}{r}\left[-\frac{\partial}{\partial r}(rE_\phi)\right]\right\}$$

Using the above electric field component leads to

$\underline{H} = \hat{a}_r H_r + \hat{a}_\theta H_\theta$ where $H_r = j\,\frac{2E_0\cos\theta}{\omega\mu_0 r^2}\left(1 + \frac{1}{j\beta r}\right)e^{-j\beta r}$

$$H_\theta = -\frac{E_0\sin\theta}{\eta r}\left[1 + \frac{1}{j\beta_0 r} - \frac{1}{(\beta_0 r)^2}\right]e^{-j\beta_0 r}$$

$\boxed{1.29}$ $\underline{E} = \hat{a}_z(1+j)\sin\left(\frac{\pi}{a}x\right)\sin\left(\frac{\pi}{b}y\right)$

Using Maxwell's equation of $\nabla\times\underline{E} = -j\omega\mu_0\underline{H}$ leads to

$$\underline{H} = -\frac{1}{j\omega\mu_0}\left[\hat{a}_x\frac{\partial E_z}{\partial y} - \hat{a}_y\frac{\partial E_z}{\partial x}\right] = -\frac{(1+j)}{j\omega\mu_0}\left[\hat{a}_x\left(\frac{\pi}{b}\right)\cos\left(\frac{\pi}{b}y\right)\sin\left(\frac{\pi}{a}x\right) - \hat{a}_y\left(\frac{\pi}{a}\right)\sin\left(\frac{\pi}{b}y\right)\cos\left(\frac{\pi}{a}x\right)\right]$$

Now using Maxwell's equation of $\nabla\times\underline{H} = \underline{J}_c + j\omega\varepsilon\,\underline{E} = \sigma\underline{E} + j\omega\varepsilon\,\underline{E}$ leads to

$$\nabla\times\underline{H} = -\hat{a}_z\frac{(1+j)}{j\omega\mu_0}\left[\left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2\right]\sin\left(\frac{\pi}{b}y\right)\sin\left(\frac{\pi}{a}x\right) = \hat{a}_z(\sigma + j\omega\varepsilon)(1+j)\sin\left(\frac{\pi}{b}y\right)\sin\left(\frac{\pi}{a}x\right)$$

Equating both sides, we can write that

$$(\sigma + j\omega\varepsilon) = -\frac{1}{j\omega\mu_0}\left[\left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2\right] \Rightarrow \quad \begin{aligned}&\sigma = 0\\ &\varepsilon_r = \frac{\varepsilon}{\varepsilon_0} = \frac{1}{\omega^2\mu_0\varepsilon_0}\left[\left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2\right]\end{aligned}$$

$\boxed{1.30}$ $\underline{E}^i = \hat{a}_x e^{-j\beta_0 z} \Rightarrow \underline{H}^i = -\frac{1}{j\omega\mu_0}\nabla\times\underline{E}^i = -\hat{a}_y\frac{1}{j\omega\mu_0}\frac{\partial E_x^i}{\partial z} = \hat{a}_y\sqrt{\frac{\varepsilon_0}{\mu_0}}\,e^{-j\beta_0 z}$

$\underline{E}^r = -\hat{a}_x e^{+j\beta_0 z} \Rightarrow \underline{H}^r = -\hat{a}_y\frac{1}{j\omega\mu_0}\frac{\partial E_x^r}{\partial z} = \hat{a}_y\sqrt{\frac{\varepsilon_0}{\mu_0}}\,e^{+j\beta_0 z}$

$$\underline{J}_s = \hat{n}\times(\underline{H}_2 - \underline{H}_1) = \hat{n}\times\underline{H}_2 = -\hat{a}_z\times\hat{a}_y(H^i + H^r)\Big|_{z=0} = \hat{a}_x(H^i + H^r)\Big|_{z=0} = \hat{a}_x\sqrt{\frac{\varepsilon_0}{\mu_0}}\left(e^{-j\beta_0 z} + e^{+j\beta_0 z}\right)\Big|_{z=0}$$

$$= \hat{a}_x 2\sqrt{\frac{\varepsilon_0}{\mu_0}} = \hat{a}_x\frac{2}{377} = \hat{a}_x\, 5.3\times10^{-3}\ \text{A/m}$$

1.31 $\underline{E}^i = \hat{a}_y E_0 e^{-j\beta_0(x\sin\theta_i + z\cos\theta_i)}$, $\underline{E}^r = \hat{a}_y E_0 \Gamma_h e^{-j\beta_0(x\sin\theta_i - z\cos\theta_i)}$

Along the interface $(z=0)$ $\left(\underline{E}^i + \underline{E}^r\right)^{tan}_{z=0} = 0 = \hat{a}_y E_0 (1+\Gamma_h) e^{-j\beta_0 x \sin\theta_i}$

which is satisfied provided $(1+\Gamma_h)=0 \Rightarrow \Gamma_h = -1$

1.32 Using Maxwell's equation of $\nabla\times\underline{E} = -j\omega\mu_0\underline{H}$, the magnetic

a. field components corresponding to the electric fields of Problem 1.31 can be written as

$$\underline{H}^i = \frac{E_0}{\sqrt{\mu_0/\varepsilon_0}}\left(-\hat{a}_x\cos\theta_i + \hat{a}_z\sin\theta_i\right) e^{-j\beta_0(x\sin\theta_i + z\cos\theta_i)}$$

$$\underline{H}^r = -\frac{E_0}{\sqrt{\mu_0/\varepsilon_0}}\left(\hat{a}_x\cos\theta_i + \hat{a}_z\sin\theta_i\right) e^{-j\beta_0(x\sin\theta_i - z\cos\theta_i)}$$

b.

$$\underline{J}_s = \hat{n}\times\left(\underline{H}_2 - \underline{H}_1\right)\Big|_{z=0} = -\hat{a}_z\times\left(\underline{H}^i + \underline{H}^r\right)\Big|_{z=0} = -\hat{a}_z\times\left[\hat{a}_x(H^i+H^r) + \hat{a}_z(H^i+H^r)\right]_{z=0}$$

$$\underline{J}_s = -\hat{a}_z\times\hat{a}_x(H^i+H^r) = \hat{a}_y\frac{2E_0}{\sqrt{\mu_0/\varepsilon_0}}\cos\theta_i\, e^{-j\beta_0 x\sin\theta_i}$$

1.33 $\underline{E}^i = \left(\hat{a}_x\cos\theta_i - \hat{a}_z\sin\theta_i\right)E_0 e^{-j\beta_0(x\sin\theta_i + z\cos\theta_i)}$

$\underline{E}^r = \left(\hat{a}_x\cos\theta_i + \hat{a}_z\sin\theta_i\right)\Gamma_e e^{-j\beta_0(x\sin\theta_i - z\cos\theta_i)}$

Along the interface $(z=0)$ $\left(\underline{E}^i + \underline{E}^r\right)^{tan}_{z=0} = 0 = \hat{a}_x\cos\theta_i E_0(1+\Gamma_e)e^{-j\beta_0 x\sin\theta_i}$

which is satisfied provided $(1+\Gamma_e)=0 \Rightarrow \Gamma_e = -1$

1.34 Along the interface the normal components of the electric flux density must be continuous; that is

$$\hat{n}\cdot\left(\underline{D}_2 - \underline{D}_1\right) = \hat{n}\cdot\underline{D}_2 = -\hat{n}\cdot\hat{a}_z\varepsilon_0 E_0\sin\theta_i(1+\Gamma_e)e^{-j\beta_0 x\sin\theta_i} = 0$$

$$= -\hat{a}_z\cdot\hat{a}_z\varepsilon_0 E_0\sin\theta_i(1+\Gamma_e)e^{-j\beta_0 x\sin\theta_i} = 0$$

which is satisfied provided

$$(1+\Gamma_e)=0 \Rightarrow \Gamma_e = -1$$

1.35

a. Given the electric fields of Problem 1.33, the corresponding magnetic field components can be found using Maxwell's equation of

$$\nabla\times\underline{E} = -j\omega\mu_0\underline{H} \Rightarrow \underline{H} = -\frac{1}{j\omega\mu_0}\nabla\times\underline{E} = -\frac{1}{j\omega\mu_0}\left[\hat{a}_x\frac{\partial E_z}{\partial y}+\hat{a}_y\left(\frac{\partial E_x}{\partial z}-\frac{\partial E_z}{\partial x}\right)+\hat{a}_z\frac{\partial E_x}{\partial y}\right]$$

For the incident field

$$E_x^i = \cos\theta_i\, E_0 e^{-j\beta_0(x\sin\theta_i+z\cos\theta_i)}$$

$$E_y^i = 0$$

$$E_z^i = -\sin\theta_i\, E_0 e^{-j\beta_0(x\sin\theta_i+z\cos\theta_i)}$$

Using these

$$\frac{\partial E_x^i}{\partial y} = 0 \ , \ \frac{\partial E_z^i}{\partial y} = 0$$

However

$$\frac{\partial E_x^i}{\partial z} = -j\beta_0\cos^2\theta_i\, E_0 e^{-j\beta_0(x\sin\theta_i+z\cos\theta_i)}$$

$$\frac{\partial E_z^i}{\partial x} = +j\beta_0\sin^2\theta_i\, E_0 e^{-j\beta_0(x\sin\theta_i+z\cos\theta_i)}$$

Thus we can write the incident magnetic field as

$$\underline{H}^i = -\frac{E_0}{j\omega\mu_0}\hat{a}_y(-j\beta_0)\left(\cos^2\theta_i+\sin^2\theta_i\right)e^{-j\beta_0(x\sin\theta_i+z\cos\theta_i)}$$

$$= \hat{a}_y\frac{\beta_0}{\omega\mu_0}E_0 e^{-j\beta_0(x\sin\theta_i+z\cos\theta_i)} = \hat{a}_y\sqrt{\frac{\varepsilon_0}{\mu_0}}E_0 e^{-j\beta_0(x\sin\theta_i+z\cos\theta_i)}$$

Using the same procedure we can write the reflected magnetic field as

$$\underline{H}^r = -\hat{a}_y\sqrt{\frac{\varepsilon_0}{\mu_0}}E_0\Gamma_e e^{-j\beta_0(x\sin\theta_i-z\cos\theta_i)} = \hat{a}_y\sqrt{\frac{\varepsilon_0}{\mu_0}}E_0 e^{-j\beta_0(x\sin\theta_i-z\cos\theta_i)}$$

b. $\underline{J}_s = \hat{n}\times(\underline{H}_2-\underline{H}_1) = \hat{n}\times\underline{H}_2 = -\hat{a}_z\times\hat{a}_y(H^i+H^r)\big|_{z=0} = \hat{a}_x 2E_0\sqrt{\frac{\varepsilon_0}{\mu_0}}e^{-j\beta_0 x\sin\theta_i}$

1.36 To determine the coefficients Γ_0 and T_0 we apply the boundary conditions along the interface at $z=0$. To do this we first find the corresponding magnetic field components. This is accomplished using Maxwell's equation of $\nabla\times\underline{E} = -j\omega\mu_0\underline{H}_0 \Rightarrow \underline{H}_0 = -\frac{1}{j\omega\mu_0}\nabla\times\underline{E}$. Doing

(continued)

$\boxed{1.36 \text{ cont'd}}$ this for each component leads to

$$\underline{H}^i = \hat{a}_y\sqrt{\frac{\varepsilon_0}{\mu_0}}\,E_0 e^{-j\beta_0 z}, \quad \underline{H}^r = -\hat{a}_y\sqrt{\frac{\varepsilon_0}{\mu_0}}\,E_0\Gamma_0 e^{+j\beta_0 z}, \quad \underline{H}^t = \hat{a}_y\sqrt{\frac{\varepsilon}{\mu_0}}\,E_0 T_0 e^{-j\beta z}$$

Applying the boundary conditions on the continuity of the tangential electric and magnetic fields along the interface at $z=0$ leads to

$1+\Gamma_0 = T_0$ from continuity of the electric fields

$1-\Gamma_0 = \sqrt{\varepsilon_r}\,T_0 = \sqrt{81}\,T_0$ from continuity of the magnetic fields

Solving these two equations, we find that

$$T_0 = \frac{2}{1+\sqrt{\varepsilon_r}} = \frac{2}{1+\sqrt{81}} = \frac{2}{1+9} = \frac{1}{5} = 0.2$$

$$\Gamma_0 = T_0 - 1 = \frac{1}{5} - 1 = -\frac{4}{5} = -0.8$$

$\boxed{1.37}$ The boundary conditions require continuity of the tangential components of the electric and magnetic fields.

<u>Electric Fields</u>: $(\underline{E}^i + \underline{E}^r)^{tan}_{z=0} = (\underline{E}^t)^{tan}_{z=0}$

$$1 + \Gamma_h = T_h$$

<u>Magnetic Fields</u>; $(\underline{H}^i + \underline{H}^r)^{tan}_{z=0} = (\underline{H}^t)^{tan}_{z=0}$

$$\cos\theta_i(-1+\Gamma_h)\sqrt{\varepsilon_0} = -\sqrt{1-\frac{\varepsilon_0}{\varepsilon}\sin^2\theta_i}\,\sqrt{\varepsilon}\;T_h$$

Solving these two equations leads to

$$\Gamma_h = \frac{\cos\theta_i - \sqrt{\frac{\varepsilon}{\varepsilon_0}}\sqrt{1-\frac{\varepsilon_0}{\varepsilon}\sin^2\theta_i}}{\cos\theta_i + \sqrt{\frac{\varepsilon}{\varepsilon_0}}\sqrt{1-\frac{\varepsilon_0}{\varepsilon}\sin^2\theta_i}}$$

$$T_h = \frac{2\cos\theta_i}{\cos\theta_i + \sqrt{\frac{\varepsilon}{\varepsilon_0}}\sqrt{1-\frac{\varepsilon_0}{\varepsilon}\sin^2\theta_i}}$$

1.38 The boundary conditions require continuity of the normal components of the electric flux density and magnetic flux density.

Electric Flux Density: $(\underline{D}^i + \underline{D}^r)^{nor}_{z=0} = (\underline{D}^t)^{nor}_{z=0}$

Since there are no normal components, this boundary condition is automatically satisfied.

Magnetic Flux Density: $(\underline{B}^i + \underline{B}^r)^{nor}_{z=0} = (\underline{B}^t)^{nor}_{z=0}$

$$\sin\theta_i\, \mu_o \sqrt{\varepsilon_o}\,(1 + \Gamma_h) = \sqrt{\frac{\varepsilon_o}{\varepsilon}}\, \sin\theta_i\, \mu_o \sqrt{\varepsilon}\, T_h$$

or

$$1 + \Gamma_h = T_h$$

This is identical to one of the equations for the solution of Problem 1.37 However we do not have another equation from the normal components of the electric field. Therefore we can not solve for Γ_h and T_h using only the normal components of Problem 1.37.

1.39 The boundary conditions require continuity of the tangential components of the electric and magnetic fields.

Electric Fields: $(\underline{E}^i + \underline{E}^r)^{tan}_{z=0} = (\underline{E}^t)^{tan}_{z=0}$

$$\cos\theta_i\,(1 + \Gamma_e) = \sqrt{1 - \frac{\varepsilon_o}{\varepsilon}\sin^2\theta_i}\; T_e$$

Magnetic Fields: $(\underline{H}^i + \underline{H}^r)^{tan}_{z=0} = (\underline{H}^t)^{tan}_{z=0}$

$$\sqrt{\varepsilon_o}\,(1 - \Gamma_e) = \sqrt{\varepsilon}\; T_e$$

Solving these two equations leads to

$$\Gamma_e = \frac{-\cos\theta_i + \sqrt{\frac{\varepsilon_o}{\varepsilon}}\sqrt{1 - \frac{\varepsilon_o}{\varepsilon}\sin^2\theta_i}}{\cos\theta_i + \sqrt{\frac{\varepsilon_o}{\varepsilon}}\sqrt{1 - \frac{\varepsilon_o}{\varepsilon}\sin^2\theta_i}}$$

$$T_e = \frac{2\sqrt{\frac{\varepsilon_o}{\varepsilon}}\cos\theta_i}{\cos\theta_i + \sqrt{\frac{\varepsilon_o}{\varepsilon}}\sqrt{1 - \frac{\varepsilon_o}{\varepsilon}\sin^2\theta_i}}$$

$\boxed{1.40}$ The boundary conditions require continuity of the normal components of the electric flux density and magnetic flux density.

Electric Flux Density: $(\underline{D}^i + \underline{D}^r)^{nor}_{z=0} = (\underline{D}^t)^{nor}_{z=0}$

$$\sin\theta_i\, \varepsilon_0 (-1 + \Gamma_e) = -\sqrt{\frac{\varepsilon_0}{\varepsilon}}\, \varepsilon \sin\theta_i\, T_e$$

or

$$\sqrt{\varepsilon_0}\,(1 - \Gamma_e) = \sqrt{\varepsilon}\, T_e$$: This equation is identical to one of the equations for the solution of Problem 1.37.

Magnetic Flux Density: $(\underline{B}^i + \underline{B}^r)^{nor}_{z=0} = (\underline{D}^t)^{nor}_{z=0}$

Since there are no normal components, this boundary condition is automatically satisfied. However we only have one equation and two unknowns; therefore we can not solve for Γ_e and T_e using only the normal components of Problem 1.37.

$\boxed{1.41}$ From Problem 1.16 and at large distances

$$\underline{\mathcal{E}} = \hat{a}_\phi E_0 \sin\theta \frac{\cos(\omega t - \beta_0 r)}{r},\quad \underline{\mathcal{E}} = \mathrm{Re}[\underline{E} e^{j\omega t}] \Rightarrow \underline{E} = E_0 \sin\theta \frac{e^{-j\beta_0 r}}{r}\hat{a}_\phi$$

$$\underline{\mathcal{H}} = -\hat{a}_\theta \sqrt{\frac{\varepsilon_0}{\mu_0}} E_0 \sin\theta \frac{\cos(\omega t - \beta_0 r)}{r},\quad \underline{\mathcal{H}} = \mathrm{Re}[\underline{H} e^{j\omega t}] \Rightarrow \underline{H} = -\hat{a}_\theta \sqrt{\frac{\varepsilon_0}{\mu_0}} E_0 \sin\theta \frac{e^{-j\beta_0 r}}{r}$$

a. $$\underline{S}_{av} = \underline{S} = \frac{1}{2}\mathrm{Re}(\underline{E} \times \underline{H}^*) = \frac{1}{2}\mathrm{Re}(\hat{a}_\phi E_\phi \times \hat{a}_\theta H_\theta^*) = \frac{1}{2}\mathrm{Re}(\hat{a}_r E_\phi H_\theta^*)$$

$$= \hat{a}_r \frac{|E_0|^2}{2}\sqrt{\frac{\varepsilon_0}{\mu_0}}\, \frac{\sin^2\theta}{r^2}$$

b. $$P_{av} = \oiint_{S_a} \underline{S}_{av} \cdot d\underline{s} = \oiint_{S_a} \underline{S}_{av} \cdot \hat{a}_r r^2 \sin\theta\, d\theta\, d\phi = \frac{|E_0|^2}{2}\sqrt{\frac{\varepsilon_0}{\mu_0}} \int_0^{2\pi}\int_0^{\pi} \sin^3\theta\, d\theta\, d\phi = \frac{|E_0|^2}{2}\sqrt{\frac{\varepsilon_0}{\mu_0}}(2\pi)\left(\frac{4}{3}\right)$$

$$= |E_0|^2 \frac{4\pi}{3}\sqrt{\frac{\varepsilon_0}{\mu_0}}$$

$\boxed{1.42}$ $$\underline{H}^{inc} = \frac{1}{377}(-\hat{a}_x \cos\theta_i + \hat{a}_z \sin\theta_i),\quad \underline{H}^{ref} = \frac{1}{377}(-\hat{a}_x \cos\theta_i - \hat{a}_z \sin\theta_i)$$

$$\underline{H}^{total} = \underline{H}^{inc} + \underline{H}^{ref} = -\hat{a}_x \frac{2}{377}\cos\theta_i = -\hat{a}_x 5.31\times 10^{-3}\cos\theta_i$$

$$\underline{J}_s = \hat{n} \times \underline{H}^{total} = \hat{a}_y \times (-\hat{a}_x 5.31\times 10^{-3}\cos\theta_i) = \hat{a}_z\, 5.31\times 10^{-3}\cos\theta_i$$

$\boxed{1.43}$ $$\underline{E} = \hat{a}_y E_0 \sin\left(\frac{\pi}{a}x\right) e^{-j\beta_z z}$$

a. $$\underline{H} = -\frac{1}{j\omega\mu_0}\nabla \times \underline{E} = -\frac{1}{j\omega\mu_0}\left(-\hat{a}_x \frac{\partial E_y}{\partial z} + \hat{a}_z \frac{\partial E_y}{\partial x}\right) = -\hat{a}_x \frac{\beta_z}{\omega\mu_0} E_0 \sin\left(\frac{\pi}{a}x\right)e^{-j\beta_z z} - \hat{a}_z \frac{E_0}{j\omega\mu_0}\left(\frac{\pi}{a}\right)\cos\left(\frac{\pi}{a}x\right)e^{-j\beta_z z}$$

b. $$P_s = -\frac{1}{2}\iiint_V (\underline{H}^* \cdot \cancel{\underline{M}_i}^{\,0} + \underline{E} \cdot \cancel{\underline{J}_i^*}^{\,0})\, dv = 0$$

c. $$P_e = \oiint_S \left(\frac{1}{2}\underline{E} \times \underline{H}^*\right) \cdot d\underline{s}$$

$\boxed{\text{Cont'd}}$

1.43 cont'd $\frac{1}{2}\underline{E}\times\underline{H}^{*} = \frac{1}{2}\hat{a}_y E_y \times\left(-\hat{a}_x H_x^{*} - \hat{a}_z H_z^{*}\right) = \frac{1}{2}\left(\hat{a}_z E_y H_x^{*} - \hat{a}_x E_y H_z^{*}\right)$

$$\underline{S} = \frac{1}{2}\underline{E}\times\underline{H}^{*} = \hat{a}_z \frac{\beta_z}{2\omega\mu_0}|E_0|^2 \sin^2\left(\frac{\pi}{a}x\right) + \hat{a}_x \frac{|E_0|^2}{j2\omega\mu_0}\left(\frac{\pi}{a}\right)\sin\left(\frac{\pi}{a}x\right)\cos\left(\frac{\pi}{a}x\right)$$

<u>From the two side walls at x=0 and x=a:</u>

$$\int_0^1\int_0^b \left(\hat{a}_x S_x\right)\Big|_{x=0}\cdot\left(-\hat{a}_x\,dy\,dz\right) + \int_0^1\int_0^b \left(\hat{a}_x S_x\right)\Big|_{x=a}\cdot\left(\hat{a}_x\,dy\,dz\right) = 0 + 0 = 0$$

<u>From the front and back cross sections at z=0 and z=1:</u>

$$\int_0^b\int_0^a \left(\hat{a}_z S_z\right)\Big|_{z=0}\cdot\left(+\hat{a}_z\,dx\,dy\right) = \frac{\beta_z}{2\omega\mu_0}|E_0|^2\int_0^b\int_0^a \sin^2\left(\frac{\pi}{a}x\right)dx\,dy = \frac{\beta_z}{2\omega\mu_0}\frac{ab}{2}|E_0|^2$$

$$\int_0^b\int_0^a \left(\hat{a}_z S_z\right)\Big|_{z=1}\cdot\left(-\hat{a}_z\,dx\,dy\right) = -\frac{\beta_z}{2\omega\mu_0}|E_0|^2\int_0^b\int_0^a \sin^2\left(\frac{\pi}{a}x\right)dx\,dy = -\frac{\beta_z}{2\omega\mu_0}\frac{ab}{2}|E_0|^2$$

<u>From the top and bottom walls at y=0 and y=b:</u>

Since there are no y components of the power density, there is no contribution from the top and bottom walls.

Therefore

$$P_e = 0 + \frac{\beta_z}{2\omega\mu_0}\frac{ab}{2}|E_0|^2 - \frac{\beta_z}{2\omega\mu_0}\frac{ab}{2}|E_0|^2 = 0$$

d. $P_d = \frac{1}{2}\iiint_V \sigma|\underline{E}|^2 dv = 0$

e. $$\overline{W}_m = \iiint_V \frac{1}{4}\mu_0|\underline{H}|^2 dv = \frac{\mu_0}{4}|E_0|^2\left\{\int_0^1\int_0^b\int_0^a\left(\frac{\beta_z}{\omega\mu_0}\right)^2\sin^2\left(\frac{\pi}{a}x\right)dx\,dy\,dz + \left(\frac{1}{\omega\mu_0}\right)^2\left(\frac{\pi}{a}\right)^2\int_0^1\int_0^b\int_0^a\cos^2\left(\frac{\pi}{a}x\right)dx\,dy\,dz\right\}$$

$$= \frac{\mu_0}{4}|E_0|^2\left\{\left(\frac{\beta_z}{\omega\mu_0}\right)^2\frac{ab}{2} + \left(\frac{1}{\omega\mu_0}\right)^2\left(\frac{\pi}{a}\right)^2\left(\frac{ab}{2}\right)\right\} = \frac{\mu_0}{4}|E_0|^2\left\{\frac{\varepsilon_0}{\mu_0}\frac{ab}{2} - \left(\frac{1}{\omega\mu_0}\right)^2\left(\frac{\pi}{a}\right)^2\frac{ab}{2} + \left(\frac{1}{\omega\mu_0}\right)^2\left(\frac{\pi}{a}\right)^2\frac{ab}{2}\right\}$$

$$\overline{W}_m = \frac{\varepsilon_0}{8}ab|E_0|^2$$

f. $$\overline{W}_e = \iiint_V \frac{1}{4}\varepsilon_0|\underline{E}|^2 dv = \frac{\varepsilon_0}{4}|E_0|^2\int_0^1\int_0^b\int_0^a \sin^2\left(\frac{\pi}{a}x\right)dx\,dy\,dz = \frac{\varepsilon_0}{4}|E_0|^2\frac{ab}{2} = \frac{\varepsilon_0}{8}ab|E_0|^2$$

$$\overline{W}_e = \frac{\varepsilon_0}{8}ab|E_0|^2$$

Ultimately $P_s = P_e + P_d + j2\omega\left(\overline{W}_m - \overline{W}_e\right)$

$0 = 0 + 0 + j2\omega\left(\frac{\varepsilon_0}{8}ab|E_0|^2 - \frac{\varepsilon_0}{8}ab|E_0|^2\right) = 0 + 0 + 0 = 0$

$0 = 0$ QED

$\boxed{1.44}$ $$-\nabla\cdot\left(\frac{1}{2}\underline{E}\times\underline{H}^*\right)=\frac{1}{2}\underline{H}^*\cdot\cancel{\underline{M}_i}^{\,0}+\frac{1}{2}\underline{E}\cdot\cancel{\underline{J}_i^*}^{\,0}+j2\omega\left(\frac{1}{4}\mu_o|\underline{H}|^2-\frac{1}{4}\varepsilon_o|\underline{E}|^2\right)$$

$$\frac{1}{2}\left(\underline{E}\times\underline{H}^*\right)=\hat{a}_x\frac{|E_o|^2}{j2\omega\mu_o}\left(\frac{\pi}{a}\right)\sin\left(\frac{\pi}{a}x\right)\cos\left(\frac{\pi}{a}x\right)+\hat{a}_z\frac{\beta_z}{2\omega\mu_o}|E_o|^2\sin^2\left(\frac{\pi}{a}x\right)$$

$$=\hat{a}_x\frac{|E_o|^2}{j4\omega\mu_o}\left(\frac{\pi}{a}\right)\sin\left(\frac{2\pi}{a}x\right)+\hat{a}_z\frac{\beta_z}{2\omega\mu_o}|E_o|^2\sin^2\left(\frac{\pi}{a}x\right)$$

$$-\nabla\cdot\left(\frac{1}{2}\underline{E}\times\underline{H}^*\right)=-\frac{|E_o|^2}{j4\omega\mu_o}\left(\frac{\pi}{a}\right)\left(\frac{2\pi}{a}\right)\cos\left(\frac{2\pi}{a}x\right)=j\frac{|E_o|^2}{2\omega\mu_o}\left(\frac{\pi}{a}\right)^2\cos\left(\frac{2\pi}{a}x\right)$$

$$\frac{1}{4}\mu_o|\underline{H}|^2=\frac{\mu_o}{4}\frac{|E_o|^2}{(\omega\mu_o)^2}\left[\beta_z^2\sin^2\left(\frac{\pi}{a}x\right)+\left(\frac{\pi}{a}\right)^2\cos^2\left(\frac{\pi}{a}x\right)\right]$$

$$=\frac{\mu_o}{4}\frac{|E_o|^2}{(\omega\mu_o)^2}\left[\beta_o^2\sin^2\left(\frac{\pi}{a}x\right)+\left(\frac{\pi}{a}\right)^2\left(\cos^2\frac{\pi}{a}x-\sin^2\frac{\pi}{a}x\right)\right]$$

$$=\frac{\mu_o}{4}\frac{|E_o|^2}{(\omega\mu_o)^2}\left[\beta_o^2\sin^2\left(\frac{\pi}{a}x\right)+\left(\frac{\pi}{a}\right)^2\left(\frac{1+\cos\left(\frac{2\pi}{a}x\right)}{2}-\frac{1-\cos\left(\frac{2\pi}{a}x\right)}{2}\right)\right]$$

$$\frac{1}{4}\mu_o|\underline{H}|^2=\frac{|E_o|^2}{4}\varepsilon_o\sin^2\left(\frac{\pi}{a}x\right)+\frac{|E_o|^2}{4}\frac{1}{\omega^2\mu_o}\left(\frac{\pi}{a}\right)^2\cos\left(\frac{2\pi}{a}x\right)$$

$$\frac{1}{4}\varepsilon_o|\underline{E}|^2=\frac{|E_o|^2}{4}\varepsilon_o\sin^2\left(\frac{\pi}{a}x\right)$$

Therefore the conservation of energy equation in differential form can be written as

$$j\frac{|E_o|^2}{2\omega\mu_o}\left(\frac{\pi}{a}\right)^2\cos\left(\frac{2\pi}{a}x\right)=0+0+j2\omega\left[\frac{|E_o|^2}{4}\varepsilon_o\sin^2\left(\frac{\pi}{a}x\right)+\frac{|E_o|^2}{4}\frac{1}{\omega^2\mu_o}\left(\frac{\pi}{a}\right)^2\cos\left(\frac{2\pi}{a}x\right)-\frac{|E_o|^2}{4}\varepsilon_o\sin^2\left(\frac{\pi}{a}x\right)\right]$$

$$j\frac{|E_o|^2}{2\omega\mu_o}\left(\frac{\pi}{a}\right)^2\cos\left(\frac{2\pi}{a}x\right)=j\frac{|E_o|^2}{2\omega\mu_o}\left(\frac{\pi}{a}\right)^2\cos\left(\frac{2\pi}{a}x\right)\qquad \text{QED}$$

$\boxed{1.45}$ Boundary Conditions on PEC:

(a) $E_x(0\le x\le a,\ y=0,\ 0\le z\le c)=E_x(\text{bottom wall})=0$

$$E_x=\cos(\beta_x x)\sin(\beta_y y)\sin(\beta_z z)\big|_{y=0}=0$$

$E_x(0\le x\le a,\ y=b,\ 0\le z\le c)=E_x(\text{top wall})=0$

$$E_x=\cos(\beta_x x)\sin(\beta_y b)\sin(\beta_z z)=0$$

$$\sin(\beta_y b)=0\Rightarrow\beta_y b=\sin^{-1}(0)=n\pi,\quad n=0,\pm1,\pm2,\ldots$$

$$\boxed{\beta_y=(n\pi/b),\quad n=0,\pm1,\pm2,\ldots}$$

$\boxed{\text{Cont'd}}$

$\boxed{1.45 \text{ (con't)}}$

(b) $E_y(x=0, 0\le y\le b, 0\le z\le c) = E_y(\text{left wall}) = 0$

$$E_y = \sin(0)\cos(\beta_y y)\sin(\beta_z z) = 0$$

$E_y(x=a, 0\le y\le b, 0\le z\le c) = E_y(\text{right wall}) = 0$

$$E_y = \sin(\beta_x a)\cos(\beta_y y)\sin(\beta_z z) = 0$$

$$\sin(\beta_x a) = 0 \Rightarrow \beta_x a = \sin^{-1}(0) = m\pi,\ m = 0, \pm 1, \pm 2, \ldots$$

$$\boxed{\beta_x = (m\pi/a),\ m = 0, \pm 1, \pm 2, \ldots}$$

($m = n \neq 0$ simultaneously; $m = n = 0$ trivial solution; E-fields vanish)

c. $E_y(0\le x\le a, 0\le y\le b, z=0) = E_y(\text{front wall}) = 0$

$$E_y = \sin(\beta_x x)\cos(\beta_y y)\sin(0) = 0$$

$E_y(0\le x\le a, 0\le y\le b, z=c) = E_y(\text{back wall}) = 0$

$$E_y = \sin\beta_x x\cos(\beta_y y)\sin(\beta_z c) = 0$$

$$\sin(\beta_z c) = 0 \Rightarrow \beta_z c = \sin^{-1}(0) = p\pi,\ p = \cancel{0}, \pm 1, \pm 2, \ldots$$

$$\boxed{\beta_z = (p\pi/c) = (p\pi/c) \quad p = \pm 1, \pm 2, \ldots}$$

$p \neq 0$ because $p = 0$ leads to trivial solution; E-fields vanish

$\boxed{1.46}$ $\underline{E} = \hat{a}_y E_0 \sin\left(\frac{\pi}{a}x\right)\sin\left(\frac{\pi}{c}z\right)$, $\omega_r = \omega = \frac{1}{\sqrt{\mu_0\varepsilon_0}}\sqrt{\left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{c}\right)^2}$

a. $$\underline{H} = -\frac{1}{j\omega\mu_0}\nabla\times\underline{E} = -\frac{1}{j\omega\mu_0}\left[-\hat{a}_x\frac{\partial E_y}{\partial z} + \hat{a}_z\frac{\partial E_y}{\partial x}\right]$$

$$= \hat{a}_x\frac{E_0}{j\omega\mu_0}\left(\frac{\pi}{c}\right)\sin\left(\frac{\pi}{a}x\right)\cos\left(\frac{\pi}{c}z\right) - \hat{a}_z\frac{E_0}{j\omega\mu_0}\left(\frac{\pi}{a}\right)\cos\left(\frac{\pi}{a}x\right)\sin\left(\frac{\pi}{c}z\right)$$

b. $$P_s = -\frac{1}{2}\iiint_V\left[\underline{H}^*\cdot\underline{M}_i + \underline{E}\cdot\underline{J}_i^*\right]dv = 0$$

(continued)

1.46 cont'd.

c. $P_e = \oint\!\!\oint_{S_a} \left(\frac{1}{2}\underline{E}\times\underline{H}^*\right)\cdot d\underline{s} = \oint\!\!\oint_{S_a} \underline{S}\cdot d\underline{s}$, $\underline{S} = \frac{1}{2}\underline{E}\times\underline{H}^*$

$$\underline{S} = \frac{1}{2}\underline{E}\times\underline{H}^* = \frac{1}{2}\hat{a}_y E_y \times \left(\hat{a}_x H_x^* - \hat{a}_z H_z^*\right) = \frac{1}{2}\left(-\hat{a}_z E_y H_x^* - \hat{a}_x E_y H_z^*\right)$$

$$= \frac{1}{2}\left[\hat{a}_x \frac{E_o}{j\omega\mu_o}\left(\frac{\pi}{a}\right)\sin\left(\frac{\pi}{a}x\right)\cos\left(\frac{\pi}{a}x\right)\sin^2\left(\frac{\pi}{c}z\right) + \hat{a}_z \frac{|E_o|^2}{j\omega\mu_o}\left(\frac{\pi}{c}\right)\sin^2\left(\frac{\pi}{a}x\right)\sin\left(\frac{\pi}{c}z\right)\cos\left(\frac{\pi}{c}z\right)\right]$$

Contributions to P_e from the different walls:

Left and right walls:

$$\int_0^c\int_0^b (\hat{a}_x S_x)\Big|_{x=0}\cdot(-\hat{a}_x\,dy\,dz) = 0\ ;\quad \int_0^c\int_0^b (\hat{a}_x S_x)\Big|_{x=0}\cdot(\hat{a}_x\,dy\,dz) = 0$$

Back and front walls:

$$\int_0^b\int_0^a (\hat{a}_z S_z)\Big|_{z=0}\cdot(-\hat{a}_z\,dx\,dy) = 0\ ;\quad \int_0^b\int_0^a (\hat{a}_z S_z)\Big|_{z=0}\cdot(\hat{a}_z\,dx\,dy) = 0$$

Top and bottom walls:

Since there are no y components of the power density, there are no contributions from the top and bottom walls.

Therefore $P_e = 0+0+0+0+0 = 0$

d. $P_d = \frac{1}{2}\iiint_V \sigma|\underline{E}|^2 dv = 0$

e. $$\overline{W}_m = \frac{\mu_o}{4}\iiint_V |\underline{H}|^2 dv = |E_o|^2\frac{\mu_o}{4}\frac{1}{(\omega\mu_o)^2}\left[\left(\frac{\pi}{c}\right)^2\int_0^c\int_0^b\int_0^a \sin^2\left(\frac{\pi}{a}x\right)\cos^2\left(\frac{\pi}{c}z\right)dx\,dy\,dz + \left(\frac{\pi}{a}\right)^2\int_0^c\int_0^b\int_0^a \cos^2\left(\frac{\pi}{a}x\right)\sin^2\left(\frac{\pi}{c}z\right)dx\,dy\,dz\right]$$

$$\overline{W}_m = |E_o|^2\frac{\mu_o}{4}\frac{1}{(\omega\mu_o)^2}\left[\left(\frac{\pi}{c}\right)^2\frac{abc}{4} + \left(\frac{\pi}{a}\right)^2\frac{abc}{4}\right] = |E_o|^2\frac{abc}{16}\frac{\varepsilon_o}{\omega^2\mu_o\varepsilon_o}\left[\left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{c}\right)^2\right] = |E_o|^2\frac{abc}{16}\varepsilon_o$$

f. $$\overline{W}_e = \frac{\varepsilon_o}{4}\iiint_V |\underline{E}|^2 dv = |E_o|^2\frac{\varepsilon_o}{4}\int_0^c\int_0^b\int_0^a \sin^2\left(\frac{\pi}{a}x\right)\sin^2\left(\frac{\pi}{c}z\right)dx\,dy\,dz = |E_o|^2\frac{\varepsilon_o}{4}\,\frac{abc}{4} = |E_o|^2\frac{abc}{16}\varepsilon_o$$

Ultimately
$$P_s = P_e + P_d + j2\omega\left(\overline{W}_m - \overline{W}_e\right)$$
$$0 = 0 + 0 + j2\omega\left(|E_o|^2\frac{abc}{16}\varepsilon_o - |E_o|^2\frac{abc}{16}\varepsilon_o\right)$$
$$0 = 0 \qquad \text{QED}$$

CHAPTER 2

2.1 $\underline{P} = \hat{a}_y\, 2.762 \times 10^{-11}$ C/m², $\underline{E} = \hat{a}_y\, 2$ V/m

a. $q_{sb} = |\underline{P}| = 2.762 \times 10^{-11}$ C/m²

$q_{sb} = +2.762 \times 10^{-11}$ C/m² at $y = 2$ cm face

$q_{sb} = -2.762 \times 10^{-11}$ C/m² at $y = 0$ cm face

b. $$Q_p = \oiint_S \underline{P} \cdot d\underline{s} = \underbrace{\int_0^{.06}\int_0^{.04} \hat{a}_y P_y \cdot \hat{a}_y\, dx\, dz}_{y=2\text{ cm face}} + \underbrace{\int_0^{.06}\int_0^{.04} \hat{a}_y P_y \cdot (-\hat{a}_y\, dx\, dz)}_{y=0\text{ cm face}} = \int_0^{.06}\int_0^{.04} P_y\, dx\, dz - \int_0^{.06}\int_0^{.04} P_y\, dx\, dz = 0$$

c. $$\oiint_S \underline{P} \cdot d\underline{s} = \iiint_V \nabla \cdot \underline{P}\, dv = \iiint_V q_{vb}\, dv = Q_b$$

$$q_{vb} = \nabla \cdot \underline{P} = \frac{\partial P_y}{\partial y} = 0$$

d. $$\varepsilon_r = \frac{\varepsilon}{\varepsilon_o} = \frac{(1+\chi_e)\varepsilon_o}{\varepsilon_o} = 1 + \chi_e \,; \qquad \underline{D} = \varepsilon_o \underline{E}_a + \underline{P} = (1+\chi_e)\varepsilon_o \underline{E}_a$$

$$\chi_e = \frac{P}{\varepsilon_o E_a} = \frac{2.762 \times 10^{-11}}{8.854 \times 10^{-12}(2)} = \frac{2.762}{1.7708} = 1.5597$$

$$\varepsilon_r = 1 + 1.5597 = 2.5597 \simeq 2.56$$

2.2 $\underline{P} = \hat{a}_\rho \frac{2}{\rho} \times 10^{-10}$ C/m², $\underline{E} = \hat{a}_\rho \frac{7.53}{\rho}$ V/m; $\qquad a \le \rho \le b$

a. $$q_{sb}\Big|_{\rho=a} = -|P| = -\frac{2 \times 10^{-10}}{2 \times 10^{-2}} = -10^{-8} \text{ C/m}^2$$

$$q_{sb}\Big|_{\rho=b} = +|P| = \frac{2 \times 10^{-10}}{6 \times 10^{-2}} = \frac{1}{3} \times 10^{-8} \text{ C/m}^2$$

$q_{sb} = 0$ elsewhere

b. $$Q_p\Big|_{\rho=a} = q_{sb} A_a = -\frac{2}{\rho} \times 10^{-10}(2\pi\rho\ell) = -4\pi\ell \times 10^{-10} = -1.256 \times 10^{-10}$$

$$Q_p\Big|_{\rho=b} = q_{sb} A_b = \frac{2}{\rho} \times 10^{-10}(2\pi\rho\ell) = 4\pi\ell \times 10^{-10} = +1.256 \times 10^{-10}$$

$$Q_{total} = Q_p\Big|_{\rho=a} + Q_p\Big|_{\rho=b} = 0$$

(continued)

2.2 cont'd.

c. $Q_p = \oiint_S \underline{P}\cdot d\underline{s} = \iiint_V \nabla\cdot\underline{P}\,dv = \iiint_V q_{vb}\,dv$

$$q_{vb} = \nabla\cdot\underline{P} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho P_\rho) = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{2}{\rho}\times10^{-10}\right) = 0$$

d. $\underline{D} = \varepsilon\underline{E}_a = \varepsilon_0\underline{E}_a + \underline{P} = \varepsilon_0\underline{E}_a + \chi_e\varepsilon_0\underline{E}_a = \varepsilon_0(1+\chi_e)\underline{E}_a$

$$\varepsilon_r = \frac{\varepsilon}{\varepsilon_0} = (1+\chi_e)$$

$$\chi_e = \frac{P}{\varepsilon_0 E_a} = \frac{\frac{2}{\rho}\times10^{-10}}{\varepsilon_0\frac{7.53}{\rho}} = \frac{2\times10^{-10}}{7.53\,\varepsilon_0} = 2.9998$$

$$\varepsilon_r = 1+\chi_e = 1+2.9998 = 3.9998 \simeq 4$$

2.3 $\underline{P} = \hat{a}_r\frac{31.87}{r^2}\times10^{-12}$ C/m^2, $\underline{E} = \hat{a}_r\frac{0.45}{r^2}$ V/m ; $a\le r\le b$

a. $q_{sb}\Big|_{r=a} = -|\underline{P}| = -\frac{31.87\times10^{-12}}{(2\times10^{-2})^2} = -\frac{31.87\times10^{-12}}{4\times10^{-4}} = -7.9675\times10^{-8}\ C/m^2$

$$q_{sb}\Big|_{r=b} = +|\underline{P}| = \frac{31.87\times10^{-12}}{(4\times10^{-2})^2} = \frac{31.87\times10^{-12}}{16\times10^{-4}} = 1.9919\times10^{-8}\ C/m^2$$

b. $Q_p\Big|_{r=a} = q_{sb}A_a = -\frac{31.87\times10^{-12}}{r^2}(4\pi r^2) = -31.87(4\pi\times10^{-12}) = -4.005\times10^{-10}$

$$Q_p\Big|_{r=b} = q_{sb}A_b = \frac{31.87\times10^{-12}}{r^2}(4\pi r^2) = 31.87(4\pi\times10^{-12}) = 4.005\times10^{-10}$$

$$Q_{total} = Q_p\Big|_{r=a} + Q_p\Big|_{r=b} = 0$$

c. $Q_p = \oiint_S \underline{P}\cdot d\underline{s} = \iiint_V \nabla\cdot\underline{P}\,dv = \iiint_V q_{vb}\,dv$

$$q_{vb} = \nabla\cdot\underline{P} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2P_r) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{31.87}{r^2}\times10^{-12}\right) = 0$$

d. $\chi_e = \frac{P}{\varepsilon_0 E_a} = \frac{\frac{31.87\times10^{-12}}{r^2}}{\varepsilon_0\frac{0.45}{r^2}} = \frac{31.87\times10^{-12}}{0.45\,\varepsilon_0} = 7.9989$

$$\varepsilon_r = 1+\chi_e = 1+7.9989 = 8.9989 \simeq 9$$

$\boxed{2.4}$

a. $E_o(0.25\times10^{-3}) + E_1(1\times10^{-3}) = 100$ Volts

Also $\varepsilon_o E_o = \varepsilon_1 E_1 \Rightarrow E_1 = \frac{\varepsilon_o}{\varepsilon_1}E_o$

Thus $E_o(0.25\times10^{-3}) + E_o\left(\frac{10^{-3}}{5}\right) = E_o(0.25+0.2)\times10^{-3} = 0.45\times10^{-3}E_o = 100$

$$E_o = \frac{100\times10^{3}}{0.45} = 2.222\times10^{5} \text{ V/m}$$

$$E_1 = \frac{\varepsilon_o}{\varepsilon_1}E_o = \frac{2.222\times10^{5}}{5} = 0.4444\times10^{5} \text{ V/m}$$

b. $D_o = \varepsilon_o E_o = 8.854\times10^{-12}(2.222\times10^{5}) = 19.6755\times10^{-7} = 1.96755\times10^{-6}$ C/m²

$D_1 = \varepsilon_1 E_1 = 5(8.854\times10^{-12})(0.4444\times10^{5}) = 1.96755\times10^{-6} = D_o$ C/m²

c. $q_{s} = 1.96755\times10^{-6}$ C/m² in upper plate where voltage is positive

$q_s = -1.96755\times10^{-6}$ C/m² in lower plate where voltage is negative

d. $Q = q_s A = 1.96755\times10^{-6}(2\times10^{-2}) = 3.9351\times10^{-8}$ C in upper plate

$Q = q_s A = -1.96755\times10^{-6}(2\times10^{-2}) = -3.9351\times10^{-8}$ C in lower plate

e. $C_o = \frac{Q}{V_o}$, $V_o = E_o(0.25\times10^{-3}) = 2.222\times10^{5}(0.25\times10^{-3}) = 55.5556$ Volts

$$C_o = \frac{3.9351\times10^{-8}}{55.5556} = 7.0832\times10^{-10}$$

$C_1 = \frac{Q}{V_1}$, $V_1 = E_1(1\times10^{-3}) = 44.4444$ Volts

$$C_1 = \frac{3.9351\times10^{-8}}{44.4444} = 8.854\times10^{-10}$$

$$C_t = \frac{Q}{V} = \frac{3.9351\times10^{-8}}{100} = 3.9351\times10^{-10} = \frac{C_oC_1}{C_o+C_1} = \frac{7.0775(8.8628)}{7.0775+8.8628}\times10^{-10}$$

f. $W_{eo} = \frac{1}{2}C_oV_o^2 = \frac{1}{2}(7.0832\times10^{-10})(55.5556)^2 = 1.0931\times10^{-6}$ Joules

$W_{e1} = \frac{1}{2}C_1V_1^2 = \frac{1}{2}(8.854\times10^{-10})(44.4444)^2 = 0.87445\times10^{-6}$ Joules

$W_{et} = W_{eo} + W_{e1} = 1.96755\times10^{-6}$ Joules

Also $W_{et} = \frac{1}{2}C_tV^2 = \frac{1}{2}(3.9351\times10^{-10})(100)^2 = 1.96755\times10^{-6}$ Joules

2.5 Before the insertion of the slab the electric flux density between the plates was equal to

$$D = \varepsilon_o E_o = 8.854\times10^{-12}\left(\frac{100}{1.25\times10^{-3}}\right) = 7.0832\times10^{-7}\ C/m^2$$

Thus the surface charge density was equal to $|q_s| = D = 7.0832\times10^{-7}\ C/m^2$

a,b. After the removal of the battery and the insertion of the dielectric sheet, the charge density on the plates remains the same as before the insertion of the dielectric sheet. Thus

$q_s = +7.0832\times10^{-7}\ C/m^2$ on the upper plate where the voltage is positive.

$$Q = q_s A = 7.0832\times10^{-7}(2\times10^{-2}) = 14.1664\times10^{-9}\ C$$

$q_s = -7.0832\times10^{-7}\ C/m^2$ on the lower plate where the voltage is negative.

$$Q = q_s A = -7.0832\times10^{-7}(2\times10^{-2}) = -14.1664\times10^{-9}\ C$$

c. Using the boundary conditions on the normal components of the electric flux density

$$\hat{n}\cdot(\underline{D}_2 - \underline{D}_1) = q_s \quad \text{or} \quad D_{2n} - \cancelto{0}{D_{1n}} = D_{2n} = q_s$$

Thus the electric flux density in the dielectric slab and free space is equal to

$$D_o = D_1 = 7.0832\times10^{-7}\ C/m^2$$

d. $$E_o = \frac{D_o}{\varepsilon_o} = \frac{7.0832\times10^{-7}}{8.854\times10^{-12}} = 80\times10^{3}\ V/m$$

$$E_1 = \frac{D_1}{\varepsilon_1} = \frac{7.0832\times10^{-7}}{5\,\varepsilon_o} = 16\times10^{3}\ V/m$$

e. $$V_o = E_o(0.25\times10^{-3}) = 20\ \text{Volts}$$

$$V_1 = E_1(1\times10^{-3}) = 16\ \text{Volts}$$

$$V_t = V_o + V_1 = 20 + 16 = 36\ \text{Volts}$$

f. $$C_o = \frac{Q}{V_o} = \frac{14.1664\times10^{-9}}{20} = 0.70832\times10^{-9} = 7.0832\times10^{-10}$$

$$C_1 = \frac{Q}{V_1} = \frac{14.1664\times10^{-9}}{16} = 0.8854\times10^{-9} = 8.854\times10^{-10}$$

$$C_t = \frac{Q}{V_t} = \frac{14.1664\times10^{-9}}{36} = 0.39351\times10^{-9} = 3.9351\times10^{-10} = \frac{C_o C_1}{C_o + C_1} = \frac{7.0832(8.854)\times10^{-10}}{7.0832+8.854}$$

g. $$W_{eo} = \tfrac{1}{2}C_o V_o^2 = \tfrac{1}{2}(7.0832\times10^{-10})(20)^2 = 0.14166\times10^{-6}\ \text{Joules}$$

$$W_{e1} = \tfrac{1}{2}C_1 V_1^2 = \tfrac{1}{2}(8.854\times10^{-10})(16)^2 = 0.11333\times10^{-6}\ \text{Joules}$$

$$W_{et} = \tfrac{1}{2}C_t V_t^2 = \tfrac{1}{2}(3.9351\times10^{-10})(36)^2 = 0.25499\times10^{-6} = W_{eo} + W_{e1}\ \text{Joules}$$

$\boxed{2.6}$ Before the insertion of the slab

$$E = \frac{V}{4\times10^{-2}} = \frac{8}{4\times10^{-2}} = 200 \text{ V/m}$$

$$D = \varepsilon_0 E = 8.854\times10^{-12}(200) = 17.708\times10^{-10} = 1.7708\times10^{-9} \text{ C/m}^2$$

a. $q_s = \pm D = \pm 1.7708\times10^{-9}$ C/m²

$Q = \pm q_s A = \pm 1.7708\times10^{-9}(64\times10^{-4}) = \pm 113.3312\times10^{-13} = \pm 1.133312\times10^{-11}$ C

b. $E = 200$ V/m

c. $D = 1.7708\times10^{-9}$ C/m²

d. $C = \varepsilon_0 \dfrac{A}{d} = 8.854\times10^{-12}\dfrac{64\times10^{-4}}{4\times10^{-2}} = 141.664\times10^{-14} = 1.41664\times10^{-12}$

e. $W_e = \frac{1}{2}CV^2 = \frac{1}{2}(1.41664\times10^{-12})(8)^2 = 45.3326\times10^{-12}$ Joules

After the insertion of the slab the electric field intensity remained the same; that is $E_0 = E_1 = 200$ Volts/m

$$D_0 = \varepsilon_0 E_0 = 8.854\times10^{-12}(200) = 1.7708\times10^{-9} \Rightarrow q_{s0} = 1.7708\times10^{-9}$$

$$D_1 = \varepsilon_1 E_1 = 2.56 D_0 = 2.56(1.7708\times10^{-9}) = 4.5332\times10^{-9} \Rightarrow q_{s1} = 4.5332\times10^{-9}$$

f. $Q_0 = \pm q_{s0} A = \pm 1.7708\times10^{-9}(32\times10^{-4}) = \pm 5.66656\times10^{-12}$ C (in free space)

$Q_1 = \pm q_{s1} A = \pm 4.5332\times10^{-9}(32\times10^{-4}) = \pm 14.50639\times10^{-12}$ C (in dielectric)

g. $E_0 = E_1 = 200$ V/m

h. $D_0 = 1.7708\times10^{-9}$ C/m² (in free space)

$D_1 = 4.5332\times10^{-9}$ C/m² (in dielectric)

i. $C_0 = \dfrac{Q_0}{V} = \dfrac{5.66656\times10^{-12}}{8} = 0.70832\times10^{-12}$ (in free space)

$C_1 = \dfrac{Q_1}{V} = \dfrac{14.50639\times10^{-12}}{8} = 1.813299\times10^{-12}$ (in dielectric)

j. $C_t = \dfrac{Q_t}{V} = \dfrac{(5.66656+14.50639)\times10^{-12}}{8} = \dfrac{20.17295\times10^{-12}}{8} = 2.521619\times10^{-12} = C_0 + C_1$

k. $W_{e0} = \frac{1}{2}C_0V^2 = \frac{1}{2}(0.70832\times10^{-12})(8)^2 = 22.66624\times10^{-12}$ (in free space)

$W_{e1} = \frac{1}{2}C_1V^2 = \frac{1}{2}(1.813299\times10^{-12})(8)^2 = 58.02528\times10^{-12}$ (in dielectric)

l. $W_{et} = \frac{1}{2}C_tV^2 = \frac{1}{2}(2.521619\times10^{-12})(8)^2 = 80.6918\times10^{-12}$ Joules

$W_{et} = W_{e0} + W_{e1}$

$\boxed{2.7}$ Parts (a)-(e) are the same as in the solution of Problem 2.6.

f. After the removal of the voltage source and the insertion of the slab, the total charge on the plates stays the same as before the removal of the source. Therefore from part a of Problem 2.6

$$Q_0 + Q_1 = Q_t = 1.133312\times10^{-11}$$

Also along the interface between the free space and the dielectric slab the tangential components of the electric field must be continuous. Therefore

$$E_0 = E_1 \Rightarrow \frac{D_0}{\varepsilon_0} = \frac{D_1}{\varepsilon_1} \Rightarrow \frac{q_{s0}}{\varepsilon_0} = \frac{q_{s1}}{\varepsilon_1} \Rightarrow \frac{Q_0/A_0}{\varepsilon_0} = \frac{Q_1/A_1}{\varepsilon_1} \Rightarrow Q_1 = \frac{\varepsilon_1 A_1}{\varepsilon_0 A_0} Q_0$$

Since the two areas are the same $A_0 = A_1$. Thus

$$Q_1 = \frac{\varepsilon_1}{\varepsilon_0} Q_0 = \varepsilon_r Q_0 \quad \text{where } \varepsilon_r = \frac{\varepsilon_1}{\varepsilon_0}$$

Using these two equations

$$Q_0 + Q_1 = Q_t$$

$$Q_1 = \varepsilon_r Q_0$$

we can write that

$$Q_0 = \frac{Q_t}{1+\varepsilon_r} = \frac{1.133312\times10^{-11}}{1+2.56} = 0.31835\times10^{-11} = 3.1835\times10^{-12}\ \text{C} \quad \text{(in free space)}$$

$$Q_1 = \varepsilon_r Q_0 = 2.56\,(3.1835\times10^{-1}) = 8.14966\times10^{-12}\ \text{C} \quad \text{(in dielectric)}$$

g. $$q_{s0} = \frac{Q_0}{A_0} = \frac{3.1835\times10^{-12}}{32\times10^{-4}} = 0.9948\times10^{-9} = D_0 \Rightarrow E_0 = \frac{0.9948\times10^{-9}}{8.854\times10^{-12}} = 112.36\ \text{V/m} \quad \text{(in free space)}$$

$$q_{s1} = \frac{Q_1}{A_1} = \frac{8.14966\times10^{-12}}{32\times10^{-4}} = 2.5468\times10^{-9} = D_1 \Rightarrow E_1 = \frac{2.5468\times10^{-9}}{2.56(8.854\times10^{-12})} = 112.36\ \text{V/m} \quad \text{(in dielectric)}$$

h. $D_0 = 0.9948\times10^{-9}\ \text{C/m}^2$ (in free space); $D_1 = 2.5468\times10^{-9}\ \text{C/m}^2$ (in dielectric)

i. $$C_0 = \frac{Q_0}{V_0} = \frac{3.1835\times10^{-12}}{112.36(4\times10^{-2})} = 0.708326\times10^{-12}\ \text{farads} = \varepsilon_0\frac{A_0}{d}$$

$$C_1 = \frac{Q_1}{V_1} = \frac{8.14966\times10^{-12}}{112.36(4\times10^{-2})} = 1.81329\times10^{-12}\ \text{farads} = \varepsilon_1\frac{A_1}{d}$$

j. $C_t = C_0 + C_1 = 2.521616\times10^{-12}$ farads

k. $W_0 = \frac{1}{2}C_0V_0^2 = \frac{1}{2}(0.708326\times10^{-12})[112.36(4\times10^{-2})]^2 = 7.15396\times10^{-12}$ Joules

$W_1 = \frac{1}{2}C_1V_1^2 = \frac{1}{2}(1.81329\times10^{-12})[112.36(4\times10^{-2})]^2 = 18.31389$ Joules

l. $W_t = W_0 + W_1 = 25.46785\times10^{-12}$ Joules

$\boxed{2.8}$ $V = 100$ Volts, $A = 2cm^2 = 2\,cm^2\left(\frac{m}{100\,cm}\right)^2 = 2\times10^{-4}$

(a) Before the insertion of the slab, the electric flux density between the plates is equal to

$$D = \varepsilon_0 E_0 = 8.854\times10^{-12}\left(\frac{100}{1.25\times10^{-3}}\right) = 7.0832\times10^{-7}\ C/m^2$$

Thus the surface charge density is equal to $|q_s| = D = 7.0832\times10^{-7}\ C/m^2$

After removal of the battery and the insertion of the dielectric slab, the charge density remains the same. Thus,

$$Q_u = Q(\text{upper}) = q_s(\text{upper})A = +7.0832\times10^{-7}(2\times10^{-4}) = +14.1664\times10^{-11}\ C$$
$$Q_l = Q(\text{lower}) = q_s(\text{lower})A = -7.0832\times10^{-7}(2\times10^{-4}) = -14.1664\times10^{-11}\ C$$

(b) $q_s(\text{upper}) = +7.0832\times10^{-7}\ C/m^2$

$q_s(\text{lower}) = -7.032\times10^{-7}\ C/m^2$

(c) The electric flux density in the dielectric slab and free space is

$$\hat{n}\cdot(D_2 - D_1) = q_s \Rightarrow D_{2n} - \cancelto{0}{D_{1n}} = D_{2n} = q_s$$
$$D_d = 7.0832\times10^{-7}\ C/m^2$$
$$D_0 = 7.0832\times10^{-7}\ C/m^2$$

(d) $\varepsilon_d E_d = D_d \Rightarrow E_d = \frac{D_d}{\varepsilon_d} = \frac{7.0832\times10^{-7}}{5(8.854\times10^{-12})} = \frac{0.8\times10^{5}}{5} = 16\times10^{+3}\ V/m$

$\varepsilon_0 E_0 = D_0 \Rightarrow E_0 = \frac{D_0}{\varepsilon_0} = \frac{7.0832\times10^{-7}}{8.854\times10^{-12}} = 0.8\times10^{5} = 80\times10^{3}\ V/m$

(e) $V_d = E_d d_d = 16\times10^{+3}(1\times10^{-3}) = 16$ Volts

$V_0 = E_0 d_0 = 80\times10^{+3}(0.25\times10^{-3}) = 20$ Volts

$V_t = V_0 + V_d = 20 + 16 = 36$ Volts

(f) $C_d = \frac{Q}{V_d} = \frac{14.1664\times10^{-11}}{16} = 0.8854\times10^{-11} = 8.854\times10^{-12}$ farads

$C_0 = \frac{Q}{20} = \frac{14.1664\times10^{-11}}{20} = 0.70832\times10^{-11} = 7.0832\times10^{-12}$ farads

$$C_t = \frac{Q}{V_t} = \frac{14.1664\times10^{-11}}{36} = 3.9351\times10^{-12} = \frac{C_0 C_1}{C_0 + C_1} = \frac{7.0832(8.854)\times10^{-24}}{(7.0832+8.854)\times10^{-12}}$$

$\boxed{2.9}$ $A = 1\ cm^2$, $V = 10$ volt, $\varepsilon_{r1} = 2$, $\varepsilon_{r2} = 6$, $d_1 = d_2 = 1\ cm$

(a) $E_1 d_1 + E_2 d_2 = 10 \Rightarrow E_1 + E_2 = \frac{10}{10^{-2}} = 10^3\ V/m$

$D_1 = D_2 \Rightarrow \varepsilon_1 E_1 = \varepsilon_2 E_2 \Rightarrow E_1 = \frac{E_2}{\varepsilon_1}(\varepsilon_2) = \frac{\varepsilon_2}{\varepsilon_1} E_2 = \frac{6}{2} E_2 = 3E_2$

$E_1 + E_2 = 3E_2 + E_2 = 4E_2 = 10^3 \Rightarrow E_2 = 0.25 \times 10^3\ V/m$, $E_1 = 3E_2 = 0.75 \times 10^3\ V/m$

$E_1 = 0.75 \times 10^3\ V/m$, $E_2 = 0.25 \times 10^3\ V/m$

(b) $D_1 = \varepsilon_1 E_1 = 2\varepsilon_0 (0.75 \times 10^3) = 1.5\varepsilon_0 \times 10^3\ c/m^2$

$D_2 = \varepsilon_2 E_2 = 6\varepsilon_0 (0.25 \times 10^3) = 1.5\varepsilon_0 \times 10^3\ c/m^2$

(c) $Q_{upper} = A_1 q_{s1} = 10^{-4}\varepsilon_0 (1.5 \times 10^3) = 0.15\varepsilon_0\ C$

$Q_{lower} = A_2 q_{s2} = 10^{-4}\varepsilon_0 (1.5 \times 10^3) = 0.15\varepsilon_0\ C$

(d) $C_1 = \frac{Q}{V_1} = \frac{Q_1}{E_1 d_1} = \frac{0.15\varepsilon_0}{(0.75 \times 10^3)(10^{-2})} = 0.02\varepsilon_0\ F$

$C_2 = \frac{Q_2}{V_2} = \frac{0.15\varepsilon_0}{E_2 d_2} = \frac{0.15\varepsilon_0}{(0.25 \times 10^3)(10^{-2})} = 0.06\varepsilon_0\ F$

$C_T = \frac{Q}{V} = \frac{0.15\varepsilon_0}{10} = 0.015\varepsilon_0\ F$

(e)

$C_1 = \varepsilon_1 \frac{A_1}{d_1} = 2\varepsilon_0 \frac{10^{-4}}{10^{-2}} = 0.02\varepsilon_0$

$C_2 = \varepsilon_2 \frac{A_2}{d_2} = 6\varepsilon_0 \frac{10^{-4}}{10^{-2}} = 0.06\varepsilon_0$

$C_T = \frac{C_1 C_2}{C_1 + C_2} = \frac{(0.02\varepsilon_0)(0.06\varepsilon_0)}{(0.02\varepsilon_0 + 0.06\varepsilon_0)} = \frac{0.02(0.06)}{0.08}\varepsilon_0$

$C_T = \frac{2(6)}{8} \times 10^{-2}\varepsilon_0 = 0.015\varepsilon_0$

(f) They are the same, as they should be.

$\boxed{2.10}$

(a) $E_1 = \frac{10}{1\text{ cm}}\left(\frac{100\text{ cm}}{1\text{ m}}\right) = \boxed{10^{+3}\ \text{V/m}}$

$E_2 = \frac{10}{1\text{ cm}}\left(\frac{100\text{ cm}}{1\text{ m}}\right) = \boxed{10^{+3}\ \text{V/m}}$

(b) $D_1 = E_1\varepsilon_1 = 10^{+3}\left(2\times 8.854\times 10^{-12}\right) = \boxed{17.708\times 10^{-9}\ \text{C/m}^2}$

$D_2 = E_2\varepsilon_2 = 10^{+3}\left(6\times 8.854\times 10^{-12}\right) = \boxed{53.124\times 10^{-9}\ \text{C/m}^2}$

(c) $q_{s1} = \boxed{17.708\times 10^{-9}\ \text{C/m}^2}$

$q_2 = \boxed{53.124\times 10^{-9}\ \text{C/m}^2}$

(d) $Q_1 = q_{s1}A = 17.708\times 10^{-9}\ \frac{\text{C}}{\text{m}^2}\left(1\times 10^{-4}\ \text{m}^2\right) = 17.708\times 10^{-13} = \boxed{1.7708\times 10^{-12}\ \text{C}}$

$Q_2 = q_2 A = 53.124\times 10^{-9}\ \frac{\text{C}}{\text{m}^2}\left(1\times 10^{-4}\ \text{m}^2\right) = 53.124\times 10^{-13} = \boxed{5.3124\times 10^{-12}\ \text{C}}$

(e) $C_1 = \frac{Q_1}{V_1} = \frac{1.7708\times 10^{-12}}{10} = \boxed{1.7708\times 10^{-13}\ \text{farads}}$

$C_2 = \frac{Q_2}{V_2} = \frac{5.3124\times 10^{-12}}{10} = \boxed{5.3124\times 10^{-13}\ \text{farads}}$

(f) $C_T = C_1 + C_2 = (1.7708 + 5.3124)\times 10^{-13} = 7.0832\times 10^{-13} = \boxed{0.70832\times 10^{-12}\ \text{f}}$

(g) $C_1 = \varepsilon_1 \frac{A_1}{S_1} = 2\left(8.854\times 10^{-12}\right)\frac{10^{-4}}{10^{-2}} = 17.708\times 10^{-14} = \boxed{1.7708\times 10^{-13}\ \text{farads}}$

$C_2 = \varepsilon_2 \frac{A_2}{S_2} = 6\left(8.854\times 10^{-12}\right)\frac{\cancel{10^{-4}}}{10^{-2}} = 53.124\times 10^{-14} = \boxed{5.3124\times 10^{-13}\ \text{farad}}$

(h) $C_T = C_1 + C_2 = (1.7708 + 5.3124)\times 10^{-13} = 7.0824\times 10^{-13} = \boxed{0.70824\times 10^{-12}\ \text{farads}}$

(i) They are the same as they should. Both are derived based on basic and related principles.

$\boxed{2.11}$

Before the insertion of the slabs and before the power supply is disconnected

$$E_1 = E_2 = \frac{10}{1\times10^{-2}} = 1{,}000 \text{ V/m} = 10^3 \text{ V/m}$$

$$D_1 = D_2 = \varepsilon_0 E_1 = \varepsilon_0 E_2 = 8.854\times10^{-12}(10^3) = 8.854\times10^{-9} \text{ C/m}^2$$

$$q_{s1} = q_{s2} = D_1 = D_2 = 8.854\times10^{-9} \text{ C/m}^2$$

$$Q_1 = Q_2 = q_{s1}A_1 = q_{s2}A_2 = 8.854\times10^{-9}(10^{-4}) = 8.854\times10^{-13}$$

After the removal of the power supply, the charge density (C/m²) in each plate remains in the presence of the slabs is the same before the insertion of the slabs.

(a) $q_{s1} = q_{s2} = 8.854\times10^{-9} \text{ C/m}^2$

(b) $D_1 = D_2 = |q_{s1}| = |q_{s2}| = 8.854\times10^{-9} \quad \text{C/m}^2$

(c) $Q_1 = -Q_2 = q_{s1}A_1 = q_{s2}A_2 = 8.854\times10^{-13} \quad \text{C}$

(d) $D_1 = \varepsilon_1 E_1 \Rightarrow E_1 = \frac{D_1}{\varepsilon_1} = \frac{8.854\times10^{-9}}{2(8.854\times10^{-12})} = \frac{10^3}{2} = 500 \text{ V/m}$

$D_2 = \varepsilon_2 E_2 \Rightarrow E_2 = \frac{D_2}{\varepsilon_2} = \frac{8.854\times10^{-9}}{6(8.854\times10^{-12})} = \frac{10^3}{6} = 166.67 \text{ V/m}$

(e) $V_1 = E_1 d_1 = 500(10^{-2}) = 5 \text{ Volts}$

$V_2 = E_2 d_2 = 166.67(10^{-2}) = 1.6667 \text{ Volts}$

(f) $C_1 = \varepsilon_1 \frac{A_1}{d_1} = 2(8.854\times10^{-12})\frac{10^{-4}}{10^{-2}} = 17.708\times10^{-14} \text{ f}$

$C_2 = \varepsilon_2 \frac{A_2}{d_2} = 6(8.854\times10^{-12})\frac{10^{-4}}{10^{-2}} = 53.124\times10^{-14} \text{ f}$

(g) $C_1 = \frac{Q_1}{V_1} = \frac{8.854\times10^{-13}}{5} = 1.7708\times10^{-13} = 17.708\times10^{-14} \text{ f}$

$C_2 = \frac{Q_2}{V_2} = \frac{8.854\times10^{-13}}{1.6667} = 5.3124\times10^{-13} = 53.708\times10^{-14} \text{ f}$

(h) They should be the same. Same definition

2.12 Using Coulomb's Law

a. $\oint\!\!\oint_S \underline{D}\cdot d\underline{s} = \int_0^{\ell}\int_0^{2\pi} (\hat{a}_\rho D_\rho)\cdot \hat{a}_\rho \rho\, d\phi\, dz = \int_0^{\ell}\int_0^{2\pi} \varepsilon_0 E_\rho \rho\, d\phi\, dz = 2\pi\varepsilon_0 \ell \rho E_\rho = Q$

or

$$E_\rho = \frac{Q}{2\pi\varepsilon_0 \ell}\frac{1}{\rho} = \frac{Q}{\rho}(3\times10^{11}) = 3\times10^{11}\frac{Q}{\rho}$$

Since the voltage is 10 Volts, then

$$V = \int_a^b \underline{E}\cdot d\underline{\ell} = \int_a^b (\hat{a}_\rho E_\rho)\cdot\hat{a}_\rho d\rho = 3\times10^{11} Q \int_a^b \frac{d\rho}{\rho} = 3\times10^{11} Q \ln(\rho)\Big|_a^b$$

$$V = 10 = 3\times10^{11} Q \ln\left(\frac{b}{a}\right) = 3\times10^{11} Q \ln(2) = 3\times10^{11} Q (0.69315)$$

$$Q = \frac{10}{3\times10^{11}(0.69315)} = 48.0898\times10^{-12} \text{ C}$$

$$E_\rho = 3\times10^{11}\frac{Q}{\rho} = 3\times10^{11}\frac{48.0898\times10^{-12}}{\rho} = \frac{14.42695}{\rho}$$

b. $Q = 48.0898\times10^{-12}$ C

c. $$q_{sa} = -\frac{Q}{A_a} = -\frac{Q}{2\pi a \ell} = -\frac{48.0898\times10^{-12}}{2\pi(2\times10^{-2})(6\times10^{-2})} = -6.378\times10^{-9} \text{ C/m}^2$$

$$q_{sb} = \frac{Q}{A_b} = \frac{Q}{2\pi b \ell} = \frac{48.0898\times10^{-12}}{2\pi(4\times10^{-2})(6\times10^{-2})} = 3.189\times10^{-9} \text{ C/m}^2$$

d. $$D = \varepsilon_0 E_\rho = 8.854\times10^{-12}\left(\frac{14.42695}{\rho}\right) = \frac{1.27736}{\rho}\times10^{-10} \text{ C/m}^2$$

e. $$C = \frac{Q}{V} = \frac{48.0898\times10^{-12}}{10} = 4.80898\times10^{-12} \text{ farads}$$

f. $$W_e = \frac{1}{2}CV^2 = \frac{1}{2}(4.80898\times10^{-12})(100) = 2.4045\times10^{-10} \text{ Joules}$$

2.13 After the removal of the voltage source, the total charge stays the same. Using this we can answer the questions.

a. Same as in part b of Problem 2.12

$Q = 48.0898\times10^{-12}$ C

b. Same as in part c of Problem 2.12

$q_{sa} = -6.378\times10^{-9}$ C/m²

$q_{sb} = 3.189\times10^{-9}$ C/m² (continued)

$\boxed{2.13 \text{ cont'd}}$ Same as in part d of Problem 2.12 (in free space and dielectric)

c. $D_{\rho o} = D_{\rho 1} = \dfrac{1.27736\times10^{-10}}{\rho}$

d.
$$E_{\rho o} = \frac{D_{\rho o}}{\varepsilon_o} = \frac{1.27736\times10^{-10}}{8.854\times10^{-12}\,\rho} = \frac{14.4269}{\rho}$$
$$E_{\rho 1} = \frac{D_{\rho 1}}{\varepsilon_1} = \frac{1.27736\times10^{-10}}{2.56(8.854\times10^{-12})\,\rho} = \frac{5.6355}{\rho}$$

e.
$$V = \int_a^c \underline{E}_1\cdot d\underline{\ell} + \int_c^b \underline{E}_o\cdot d\underline{\ell} = 5.6355\int_a^c \frac{d\rho}{\rho} + 14.4269\int_c^b \frac{d\rho}{\rho} = 5.6355\ln\left(\frac{c}{a}\right) + 14.4269\ln\left(\frac{b}{c}\right)$$
$$= 5.6355\ln\left(\frac{3}{2}\right) + 14.4269\ln\left(\frac{4}{3}\right) = 5.6355(0.405465) + 14.4269(0.28768)$$
$$V = 2.285 + 4.1504 = 6.4354 \text{ Volts}$$

f. $C = \dfrac{Q}{V} = \dfrac{48.0898\times10^{-12}}{6.4354} = 7.472698\times10^{-12}$ farads

g. $W_e = \frac{1}{2}CV^2 = \frac{1}{2}(7.472698)(6.4354)^2\times10^{-12} = 1.54738\times10^{-10}$ Joules

$\boxed{2.14}$ Since the voltage source is connected at all times, the ~~total~~ voltage is maintained at 10 Volts. Also the normal components of the electric flux density is continuous along the interface.

a.
$$V = \int_a^c \underline{E}_1\cdot d\underline{\ell} + \int_c^b \underline{E}_o\cdot d\underline{\ell} = 10 \text{ Volts} = \int_a^c \hat{a}_\rho E_{\rho 1}\cdot\hat{a}_\rho\, d\rho + \int_c^b \hat{a}_\rho E_{\rho o}\cdot\hat{a}_\rho\, d\rho$$
$$V = \int_a^c E_{\rho 1}\,d\rho + \int_c^b E_{\rho o}\,d\rho = \int_a^c \frac{D_{\rho 1}}{\varepsilon_1}\,d\rho + \int_c^b \frac{D_{\rho o}}{\varepsilon_o}\,d\rho = \frac{1}{\varepsilon_1}\int_a^c D_{\rho 1}\,d\rho + \int_c^b D_{\rho o}\,d\rho$$

However
$$D_{\rho 1} = D_{\rho o} = \frac{Q}{2\pi\ell}\frac{1}{\rho}$$

Thus
$$V = \frac{1}{\varepsilon_1}\frac{Q}{2\pi\ell}\ln\left(\frac{c}{a}\right) + \frac{1}{\varepsilon_o}\frac{Q}{2\pi\ell}\ln\left(\frac{b}{c}\right) = \frac{Q}{\varepsilon_o(2\pi\ell)}\left[\frac{\ln(c/a)}{\varepsilon_r} + \ln(b/c)\right] = 10$$
$$Q = \frac{10(2\pi\varepsilon_o\ell)}{\frac{\ln(3/2)}{2.56} + \ln(4/3)} = \frac{10^{-10}}{3\left[\frac{0.405465}{2.56} + 0.28768\right]} = 7.47276\times10^{-9}\ \text{C}$$
$$E_{\rho o} = \frac{Q}{2\pi\varepsilon_o\ell}\frac{1}{\rho} = \frac{7.47276\times10^{-9}}{2\pi\varepsilon_o(6\times10^{-2})\,\rho} = \frac{2{,}241.827}{\rho} \quad \text{(in free space)}$$

(continued)

$\boxed{\text{2.14 cont'd.}}$ $E_{\rho 1} = \dfrac{Q}{2\pi\varepsilon_1 \ell}\dfrac{1}{\rho} = \dfrac{E_{\rho o}}{\varepsilon_r} = \dfrac{875.7136}{\rho}$ (in dielectric)

b. $D_{\rho o} = \varepsilon_o E_{\rho o} = \dfrac{19.849\times10^{-9}}{\rho}$ C/m²

$D_{\rho 1} = \varepsilon_1 E_{\rho 1} = \dfrac{19.849\times10^{-9}}{\rho}$ C/m²

c. $q_{sa} = D_{\rho 1}\Big|_{\rho=2\times10^{-2}} = \dfrac{19.849\times10^{-9}}{2\times10^{-2}} = 0.99246\times10^{-6}$ C/m²

$q_{sb} = D_{\rho o}\Big|_{\rho=4\times10^{-2}} = \dfrac{q_{sa}}{2} = 0.49623\times10^{-6}$ C/m²

d. $Q_a = q_{sa}(2\pi a\ell) = 0.99246\times10^{-6}(2\pi\times2\times10^{-2}\times6\times10^{-2}) = 74.8295\times10^{-10}$ C

$Q_b = q_{sb}(2\pi b\ell) = 0.49623\times10^{-6}(2\pi\times4\times10^{-2}\times6\times10^{-2}) = 74.8295\times10^{-10}$ C

e. $C = \dfrac{Q}{V} = \dfrac{74.8295\times10^{-10}}{10} = 7.48295\times10^{-10} = 0.748295\times10^{-9}$ farads

f. $W_e = \frac{1}{2}CV^2 = \frac{1}{2}(0.748295\times10^{-9})(100) = 37.41475\times10^{-9}$ Joules

$\boxed{2.15}$ The voltage source is connected across at all times.

a. $E_o = \dfrac{100}{4\times10^{-2}} = 2{,}500$ V/m $= E_1$

b. $D_o = \varepsilon_o E_o = 22.135\times10^{-9}$ C/m² (in free space)

$D_1 = \varepsilon_1 E_1 = 2.56 D_o = 56.6656\times10^{-9}$ C/m² (in dielectric)

c. $Q_o = 22.135\times10^{-9}(2\times10^{-2}) = 0.4427\times10^{-9}$ C (in free space)

$Q_1 = 56.6656\times10^{-9}(2\times10^{-2}) = 1.13331\times10^{-9}$ C (in dielectric)

d. $W_{eo} = \frac{1}{2}C_o V_o^2 = \frac{1}{2}\left(\frac{Q_o}{V_o}\right)V_o^2 = \frac{1}{2}Q_o V_o = \frac{1}{2}(0.4427\times10^{-9})(100) = 22.135\times10^{-9}$ J (free space)

$W_{e1} = \frac{1}{2}C_1 V_1^2 = \frac{1}{2}\left(\frac{Q_1}{V_1}\right)V_1^2 = \frac{1}{2}Q_1 V_1 = \frac{1}{2}(1.13331\times10^{-9})(100) = 56.6655\times10^{-9}$ J (dielectric)

e. $Q_t = Q_o + Q_1 = 1.57601\times10^{-9}$

f. $C_t = \dfrac{Q_t}{V} = \dfrac{1.57601\times10^{-9}}{100} = 1.57601\times10^{-11} = 15.7601\times10^{-12} = C_o + C_1$ farads

g. $W_{et} = \frac{1}{2}C_t V^2 = \frac{1}{2}(15.7601\times10^{-12})(100)^2 = 78.8005\times10^{-9}$ J $= W_{eo} + W_{e1}$

2.16

(a) $\oiint_S \underline{D}\cdot d\underline{s} = \int_0^{\ell}\int_0^{\pi} \hat{a}_\rho D_{\rho 0}\cdot \hat{a}_\rho\, dz\rho d\phi + \int_0^{\ell}\int_{\pi}^{2\pi} \hat{a}_\rho D_{\rho 1}\cdot \hat{a}_\rho\, dz\rho d\phi$

$\oiint_S \underline{D}\cdot d\underline{s} = \pi\ell(D_{\rho 0}+D_{\rho 1})\rho = Q_t \Rightarrow D_{\rho 0}+D_{\rho 1} = \frac{Q_t}{2\pi\ell}\left(\frac{1}{\rho}\right)$

Also, because the tangential electric field components ar continuous along the radial interfaces between the dielectric free space

$$E_{\rho 0} = E_{\rho 1} \Rightarrow \frac{D_{\rho 0}}{\varepsilon_0} = \frac{D_{\rho 1}}{\varepsilon_1} \Rightarrow D_{\rho 1} = \frac{\varepsilon_1}{\varepsilon_0}D_{\rho 0} = \varepsilon_r D_{\rho 0},\ \varepsilon_r = \frac{\varepsilon_1}{\varepsilon_0}$$

Thus

$$D_{\rho 0}(1+\varepsilon_r) = \frac{Q_t}{2\pi\ell}\frac{1}{\rho} \Rightarrow D_{\rho 0} = \frac{Q_t}{2\pi\ell(1+\varepsilon_r)}\frac{1}{\rho},\ D_{\rho 1} = \frac{Q_t}{2\pi\ell}\frac{\varepsilon_r}{1+\varepsilon_r}\frac{1}{\rho}$$

$$E_{\rho 0} = \frac{D_{\rho 0}}{\varepsilon_0} = \frac{Q_t}{\pi\varepsilon_0\ell(1+\varepsilon_r)}\frac{1}{\rho},\ \text{in free space}$$

$$E_{\rho 1} = \frac{D_{\rho 1}}{\varepsilon_1} = \frac{Q_t}{\pi\varepsilon_1\ell}\frac{\varepsilon_r}{1+\varepsilon_r}\frac{1}{\rho},\ \text{in dielectric}$$

(b) $\int_a^b \underline{E}\cdot d\underline{\ell} = \int_a^b \hat{a}_\rho E_{\rho 0}\cdot\hat{a}_\rho d\rho = \int_a^b E_{\rho 0}d\rho = \frac{Q_t \ln(b/a)}{2\pi\varepsilon_0\ell(1+\varepsilon_r)} = V = 10$

$$Q_t = \frac{10(\pi\varepsilon_0\ell)(1+\varepsilon_r)}{\ln(2)} = \frac{(1+2.56)\times10^{-10}}{6\ln(2)} = 0.856\times10^{-10}\ \text{C}$$

(c,d) $D_{\rho 0} = \varepsilon_0 E_{\rho 0} = \frac{Q_t}{\pi\ell(1+\varepsilon_r)}\frac{1}{\rho} = \frac{0.856\times10^{-10}}{2\pi(6\times10^{-2})(1+2.56)}\frac{1}{\rho} = \frac{1.2756\times10^{-10}}{\rho}\ \text{C/m}^2$

$$D_{\rho 0}\big|_{\rho=a} = \frac{1.2756\times10^{-10}}{2\times10^{-2}} = 0.6378\times10^{-8}\ \text{C/m}^2 \Rightarrow q_{s0}\big|_{\rho=a} = 0.6378\times10^{-8}\ \text{C/m}^2$$

$$D_{\rho 0}\big|_{\rho=b} = \frac{1.2756\times10^{-10}}{4\times10^{-2}} = 0.3189\times10^{-8}\ \text{C/m}^2 \Rightarrow q_{s0}\big|_{\rho=b} = 0.3189\times10^{-8}\ \text{C/m}^2$$

$$D_{\rho 1} = \varepsilon_1 E_{\rho 1} = \frac{Q_t}{\pi\ell}\frac{\varepsilon_r}{1+\varepsilon_r}\frac{1}{\rho} = \varepsilon_r D_{\rho 0} = \frac{3.2655\times10^{-10}}{\rho}\ \text{C/m}^2$$

$$D_{\rho 1}\big|_{\rho=a} = \frac{3.2655\times10^{-10}}{2\times10^{-2}} = 1.63277\times10^{-8}\ \text{C/m}^2 \Rightarrow q_{s1}\big|_{\rho=a} = 1.63277\times10^{-8}\ \text{C/m}^2$$

$$D_{\rho 1}\big|_{\rho=b} = \frac{3.2655\times10^{-10}}{4\times10^{-2}} = 0.81638\times10^{-8}\ \text{C/m}^2 \Rightarrow q_{s1}\big|_{\rho=b} = 0.81638\times10^{-8}\ \text{C/m}^2$$

(e)

$C_0 = \frac{Q_0}{V} = \frac{0.6378\times10^{-8}(\pi\times2\times10^{-2}\times6\times10^{-2})}{10} = 2.40445\times10^{-12}$ farads (in free space)

$C_1 = \frac{Q_1}{V} = \frac{1.63277\times10^{-8}(\pi\times2\times10^{-2}\times6\times10^{-2})}{10} = 6.155398\times10^{-12}$ farads (in dielectric)

$C_t = C_0 + C_1 = 8.559847\times10^{-12}$ farads

(f) $W_{e0} = \frac{1}{2}C_0V^2 = \frac{1}{2}(2.40445\times10^{-12})(100) = 1.2022\times10^{-10}$ Joules (in free space)

$W_{e1} = \frac{1}{2}C_1V^2 = \frac{1}{2}(6.155398\times10^{-12})(100) = 3.077699\times10^{-10}$ Joules (in dielectric)

$W_t = W_{e0} + W_{e1} = 4.279949\times10^{-10}$ Joules (total)

$\boxed{2.17}$ From Problem 2.16 $E_\rho = \frac{Q}{2\pi\varepsilon\ell\rho}$ where Q is the total charge.

a. Since the charge stays the same at all times and is the same as that of Problem 2.16, or $Q_t = 85.5985\times10^{-12}$ C. Also to satisfy the boundary conditions along the interface between the two media

$$E_{\rho 0} = E_{\rho 1}$$

b. $D_{\rho 0} = \varepsilon_0 E_{\rho 0} = \frac{1.2756\times10^{-10}}{\rho}$ C/m²

$D_{\rho 1} = \varepsilon_1 E_{\rho 1} = \varepsilon_r E_{\rho 0} = \frac{3.2655\times10^{-10}}{\rho}$ C/m²

c. $q_{s0}|_{\rho=a} = -D_{\rho 0}|_{\rho=a} = -\frac{1.2756\times10^{-10}}{2\times10^{-2}} = -0.6378\times10^{-8}$ C/m²

$q_{s0}|_{\rho=b} = D_{\rho 0}|_{\rho=b} = \frac{1.2756\times10^{-10}}{4\times10^{-2}} = 0.3189\times10^{-8}$ C/m²

$q_{s1}|_{\rho=a} = -D_{\rho 1}|_{\rho=a} = -\frac{3.2655\times10^{-10}}{2\times10^{-2}} = -1.63277\times10^{-8}$ C/m²

$q_{s1}|_{\rho=b} = D_{\rho 1}|_{\rho=b} = \frac{3.2756\times10^{-10}}{4\times10^{-2}} = 0.81638\times10^{-8}$ C/m²

d. $Q_0|_{\rho=a} = q_{s0}|_{\rho=a}(\pi a\ell) = -0.6378\times10^{-8}(\pi\times2\times10^{-2}\times6\times10^{-2}) = -24.0445\times10^{-12}$ C

$Q_0|_{\rho=b} = q_{s0}|_{\rho=b}(\pi b\ell) = 0.3189\times10^{-8}(\pi\times4\times10^{-2}\times6\times10^{-2}) = 24.0445\times10^{-12}$ C

$Q_1|_{\rho=a} = q_{s1}|_{\rho=a}(\pi a\ell) = -1.63277\times10^{-8}(\pi\times2\times10^{-2}\times6\times10^{-2}) = -61.5540\times10^{-12}$ C

$Q_1|_{\rho=b} = q_{s1}|_{\rho=b}(\pi b\ell) = 1.63277\times10^{-8}(\pi\times4\times10^{-2}\times6\times10^{-2}) = 61.5540\times10^{-12}$ C

e. $C_0 = \frac{Q_0}{V_0} = \frac{24.0445\times10^{-12}}{10} = 2.40445\times10^{-12}$ farads in free space

$C_1 = \frac{Q_1}{V_1} = \frac{61.5540\times10^{-12}}{10} = 6.15540\times10^{-12}$ farads in polystyrene

$C_t = C_0 + C_1 = (2.40445 + 6.15540)\times10^{-12} = 8.55985\times10^{-12}$

f. $W_{e0} = \frac{1}{2}C_0V_0^2 = \frac{1}{2}(2.40445\times10^{-12})(100) = 1.2022\times10^{-10}$ Joules (in free space)

$W_{e1} = \frac{1}{2}C_1V_1^2 = \frac{1}{2}(6.15540\times10^{-12})(100) = 3.0777\times10^{-10}$ Joules (in polystyrene)

$W_{et} = \frac{1}{2}C_tV_0^2 = \frac{1}{2}C_tV_1^2 = \frac{1}{2}(8.55985\times10^{-12})(100) = 4.2799\times10^{-10}$ Joules

$= W_{e0} + W_{e1}$

2.18

$\underline{\mathcal{E}} = \hat{a}_z 10^{-3} \sin(2\pi\times10^7 t), \quad \varepsilon_{sr} = 2.56$

(a) $1+\chi_e = \frac{\varepsilon_s}{\varepsilon_0} = \varepsilon_{sr} \Rightarrow \chi_e = \varepsilon_{sr} - 1 = 2.56-1 = 1.56$

(b) $\underline{\mathcal{D}} = \varepsilon_s \underline{\mathcal{E}} = \hat{a}_z 2.56(\varepsilon_0) 10^{-3} \sin(2\pi\times10^7 t) = \hat{a}_z 2.56\times10^{-3}\varepsilon_0 \sin(2\pi 10^7 t)$

$\underline{\mathcal{D}} = \hat{a}_z 2.56\times10^{-3} \varepsilon_0 \sin(2\pi\times10^7 t)$

(c) $\underline{\mathcal{P}} = \varepsilon_0 \chi_e \underline{\mathcal{E}} = \hat{a}_z \varepsilon_0 (1.56\times10^{-3} \varepsilon_0) \sin(2\pi\times10^7 t) = \hat{a}_z 1.56\times10^{-3} \varepsilon_0 \sin(2\pi\times10^7 t)$

$\underline{\mathcal{P}} = \hat{a}_z 1.56\times10^{-3} \varepsilon_0 \sin(2\pi\times10^7 t)$

(d) $\underline{\mathcal{J}}_d = \frac{\partial \underline{\mathcal{D}}}{\partial t} = \hat{a}_z 2.56\times10^{-3} \varepsilon_0 \frac{\partial}{\partial t} \sin(2\pi\times10^7 t)$

$= \hat{a}_z (2.56\times10^{-3}\times2\pi\times10^7) \varepsilon_0 \cos(2\pi\times10^7 t)$

$\underline{\mathcal{J}}_d = \hat{a}_z 16.085\times10^4 \varepsilon_0 \cos(2\pi\times10^7 t)$

(e) $\underline{\mathcal{J}}_p = \frac{\partial \underline{\mathcal{P}}}{\partial t} = \hat{a}_z \frac{\partial}{\partial t}\left[1.56\times10^{-3} \varepsilon_0 \sin(2\pi\times10^7 t)\right]$

$= \hat{a}_z 1.56\times10^{-3} (2\pi\times10^7) \varepsilon_0 \cos(2\pi\times10^7 t)$

$\underline{\mathcal{J}}_p = \hat{a}_z 9.802\times10^4 \varepsilon_0 \cos(2\pi\times10^7 t)$

$\boxed{2.19}$ $\underline{M} = \hat{a}_z 1.245\times10^6$ A/m , $\underline{H} = \hat{a}_z 5\times10^3$ A/m

a. $\underline{J}_{ms} = \underline{M}\times\hat{n}$

$y=0:$ $\quad \underline{J}_{ms} = \hat{a}_z M_z \times(-\hat{a}_y) = \hat{a}_x M_z = \hat{a}_x 1.245\times10^6$

$x=4\text{cm}:$ $\quad \underline{J}_{ms} = \hat{a}_z M_z \times \hat{a}_x = \hat{a}_y M_z = \hat{a}_y 1.245\times10^6$

$y=6\text{ cm}:$ $\quad \underline{J}_{ms} = \hat{a}_z M_z \times \hat{a}_y = -\hat{a}_x M_z = -\hat{a}_x 1.245\times10^6$

$x=0:$ $\quad \underline{J}_{ms} = \hat{a}_z M_z \times(-\hat{a}_x) = -\hat{a}_y M_z = -\hat{a}_y 1.245\times10^6$

$z=0:$ $\quad \underline{J}_{ms} = \hat{a}_z M_z \times(-\hat{a}_z) = 0$

$z=1\text{ cm}:$ $\quad \underline{J}_{ms} = \hat{a}_z M_z \times \hat{a}_z = 0$

b. $\underline{J}_m = \nabla\times\underline{M} = 0$

c. $I_m = \oiint_S \underline{J}_m\cdot d\underline{s} = \iiint_V (\nabla\cdot\underline{J}_m)\, dv = 0$

d. $\chi_m = \dfrac{M}{H} = \dfrac{1.245\times10^6}{5\times10^3} = 249$

$\mu_r = 1+\chi_m = 1+249 = 250$

$\boxed{2.20}$ $\underline{H} = \hat{a}_\phi \dfrac{0.3183}{\rho}$ A/m , $\underline{M} = \hat{a}_\phi \dfrac{190.67}{\rho}$ A/m

a. $\underline{J}_{ms} = \underline{M}\times\hat{n}$

$\rho=a:$ $\quad \underline{J}_{ms} = \hat{a}_\phi M_\phi \times(-\hat{a}_\rho)\Big|_{\rho=a} = \hat{a}_z M_\phi\Big|_{\rho=a} = \hat{a}_z \dfrac{190.67}{1\times10^{-2}} = \hat{a}_z 19.067\times10^3$

$\rho=b:$ $\quad \underline{J}_{ms} = \hat{a}_\phi M_\phi \times \hat{a}_\rho\Big|_{\rho=b} = -\hat{a}_z M_\phi\Big|_{\rho=b} = -\hat{a}_z \dfrac{190.67}{3\times10^{-2}} = -\hat{a}_z 6.3557\times10^3$

b. $\underline{J}_m = \nabla\times\underline{M} = \hat{a}_z \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho M_\phi) = \hat{a}_z\frac{1}{\rho}\frac{\partial}{\partial\rho}(190.67) = 0$

c. $I_m = \oiint_S \underline{J}_m\cdot d\underline{s} = \iiint_V (\nabla\cdot\underline{J}_m)\, dv = 0$

d. $\chi_m = \dfrac{M}{H} = \dfrac{190.67/\rho}{0.3183/\rho} = \dfrac{190.67}{0.3183} = 599.03 \simeq 599$

$\mu_r = 1+\chi_m = 1+599 = 600$

$z=0:$ $\underline{J}_{ms} = \hat{a}_\phi M_\phi\times\hat{a}_z = \hat{a}_\rho M_\phi$

$\underline{J}_{ms} = \hat{a}_\rho \dfrac{160.67}{\rho}$

$z=-\ell:$ $\underline{J}_{ms} = \hat{a}_\phi M_\phi\times(-\hat{a}_z)$

$= -\hat{a}_\rho M_\phi$

$\underline{J}_{ms} = -\hat{a}_\rho \dfrac{160.67}{\rho}$

$\boxed{2.21}$ $\underline{M} = \hat{a}_\phi\, 10 \quad \frac{A}{m}$

a. $\underline{J}_{ms} = \underline{M} \times \hat{n}\Big|_{\rho=1} = (\hat{a}_\phi M_\phi \times \hat{a}_\rho)\Big|_{\rho=1} = -\hat{a}_z M_\phi\Big|_{\rho=1} = -\hat{a}_z 10 \quad A/m$

b. $\underline{J}_m = \nabla \times \underline{M} = \hat{a}_z \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho M_\phi) = \hat{a}_z \frac{1}{\rho}\frac{\partial}{\partial\rho}(10\rho) = \hat{a}_z \frac{10}{\rho} \quad A/m^2$

c. $I_m = \iint_S \underline{J}_m \cdot d\underline{s} = \int_0^{2\pi}\int_0^{1} \hat{a}_z J_{mz} \cdot \hat{a}_z \rho\, d\rho\, d\phi = \int_0^{2\pi}\int_0^{1} 10\, d\rho\, d\phi = 2\pi(10) = 62.83185\ A$

$\boxed{2.22}$ $\underline{J} \simeq \hat{a}_z J_0\, e^{-10^2[(a-|x|)+(a-|y|)]} \quad A/m^2$

$$I = \iint_{S_0} \underline{J} \cdot d\underline{s} = 4\int_0^a\int_0^a \hat{a}_z J_z \cdot \hat{a}_z\, dx\, dy = 4J_0\left[\int_0^a e^{-100(a-x)}dx \int_0^a e^{-100(a-y)}dy\right]$$

$$= 4J_0\left[\int_0^a e^{-100(a-x)}dx\right]^2 = 4J_0\left\{\left[\frac{1}{100}e^{-100(a-x)}\right]_0^a\right\}^2 = 4J_0\left\{\frac{1}{100}\left[1-e^{-100a}\right]\right\}^2$$

$$I = 4J_0\left\{\frac{1}{100}\left[1-e^{-1}\right]\right\}^2 = 4J_0\left[\frac{1}{100}(1-0.36788)\right]^2 = 2(7.9915\times10^{-5})J_0 \quad A$$
$$= 15.983\times10^{-5} J_0 \quad A$$

$\boxed{2.23}$ $\underline{J} = \hat{a}_z J_0\, e^{-10^4(a-\rho)} \quad A/m^2$

a. $I = \iint_{S_0} \underline{J} \cdot \hat{a}_z\, d\underline{s} = \int_0^{2\pi}\int_0^a \hat{a}_z J_z \cdot \hat{a}_z \rho\, d\rho\, d\phi = \int_0^{2\pi}\int_0^a J_0 e^{-10^4(a-\rho)}\rho\, d\rho\, d\phi$

$$= 2\pi J_0 \int_0^a e^{-10^4(a-\rho)}\rho\, d\rho = 2\pi J_0 \int_0^a e^{-10^4 a} e^{10^4\rho}\rho\, d\rho$$

Let $u = 10^4\rho \Rightarrow du = 10^4 d\rho \Rightarrow d\rho = 10^{-4}du,\ \rho = 10^{-4}u$

$$I = 2\pi\times10^{-8} J_0 \int_0^{10^4 a} e^{-10^4 a} e^{u} u\, du = 2\pi\times10^{-8} J_0\, e^{-10^4 a}\int_0^{c} e^{u} u\, du$$

Using the integral $\int_0^c e^u u\, du = e^u(u-1)\Big|_0^c$ we can write that

$$I = 2\pi J_0\times10^{-8} e^{-10^4 a}\left[e^{10^4 a}(10^4 a-1) - e^0(0-1)\right]_{a=0.01} = 0.622\times10^{-5} J_0 = 10\ A$$

$I = 10 \simeq 0.622\times10^{-5} J_0 \Rightarrow J_0 = 10/0.622\times10^{-5} = 1.6077\times10^{6}\ A/m^2$

b. $J = J_0\, e^{-10^4(a-\rho)} = e^{-1} J_0 = 0.368\, J_0 \Rightarrow 10^4(a-\rho) = 1$

$$a-\rho = 10^{-4} \Rightarrow \rho = a - 10^{-4}$$

Distance from surface $= 10^{-4}$ m (very small; almost on the surface)

$\boxed{2.24}$ $\sigma = 5.76\times10^{7}$ S/m , $\varepsilon = 8.854\times10^{-12}$ f/m

$$T_r = \frac{\varepsilon}{\sigma} = \frac{8.854\times10^{-12}}{5.76\times10^{7}} = 1.5372\times10^{-19} \text{ secs.}$$

In the microwave region ($f = 1$-10 GHz) the period is

$$f = 1 \text{ GHz}: \quad T = \frac{1}{f} = \frac{1}{10^9} = 10^{-9} \gg T_r = 1.5372\times10^{-19}$$

$$f = 2 \text{ GHz}: \quad T = \frac{1}{f} = \frac{1}{10^{10}} = 10^{-10} \gg T_r = 1.5372\times10^{-19}$$

For x-rays $[\lambda = (1-10)\times10^{-8} \text{ cm}]$ the period is

$$\lambda = 1\times10^{-8}\text{ cm}: \quad f = \frac{v}{\lambda} = \frac{3\times10^{10}}{10^{-8}} = 3\times10^{+18}$$

$$T = \frac{1}{f} = \frac{1}{3\times10^{18}} = 0.333\times10^{-18} = 3.33\times10^{-19}$$

$$\lambda = 10\times10^{-8}\text{ cm}: \quad f = \frac{v}{\lambda} = \frac{3\times10^{10}}{10\times10^{-8}} = 3\times10^{+17}$$

$$T = \frac{1}{f} = \frac{1}{3\times10^{17}} = 0.333\times10^{-17} = 3.33\times10^{-18}$$

which are comparable to the relaxation time constant.

$\boxed{2.25}$ $\sigma = 3.96\times10^{7}$ S/m, $\mu_e = 2.2\times10^{-3}$ m²/(V-s), $\underline{E} = \hat{a}_x 2$ V/m

a. $\sigma = -\mu_e q_{ve} \Rightarrow q_{ve} = -\dfrac{\sigma}{\mu_e} = -\dfrac{3.96\times10^{7}}{2.2\times10^{-3}} = -1.8\times10^{10}$ C/m³

b. $\underline{v}_e = -\mu_e \underline{E} = -2.2\times10^{-3}(\hat{a}_x E_x) = -\hat{a}_x(2.2\times2\times10^{-3}) = -\hat{a}_x 4.4\times10^{-3}$ m/sec.

c. $\underline{J}_c = \sigma\underline{E} = 3.96\times10^{+7}(\hat{a}_x 2) = \hat{a}_x 7.92\times10^{7}$ A/m²

d. $I = \iint_{S_o} \underline{J}_c \cdot d\underline{s} = \iint_{S_o} \hat{a}_x J_{cx} \cdot \hat{a}_x\, ds = \iint_{S_o} J_{cx}\, ds = 7.92\times10^{7}(S_o) = 7.92\times10^{7}(10^{-3})$

$I = 7.92\times10^{4}$ A

e. $N_e = \dfrac{q_{ve}}{q_e} = \dfrac{-1.8\times10^{10}}{-1.602\times10^{-19}} = 1.1236\times10^{29}$ $\dfrac{\text{electrons}}{\text{m}^3}$

Chapter 3

3.1 $\nabla\times\underline{E} = -\underline{M}_i - j\omega\mu\underline{H}$, $\nabla\times\underline{H} = \underline{J}_i + \sigma\underline{E} + j\omega\varepsilon\underline{E}$

$$\nabla\times\nabla\times\underline{E} = -\nabla\times\underline{M}_i - j\omega\mu\nabla\times\underline{H}$$

$$\nabla\times\nabla\times\underline{H} = \nabla\times\underline{J}_i + \sigma\nabla\times\underline{E} + j\omega\varepsilon\nabla\times\underline{E}$$

Using Maxwell's equations from above and the vector identify of

$$\nabla\times\nabla\times\underline{F} = \nabla(\nabla\cdot\underline{F}) - \nabla^2\underline{F}$$

we can write

$$\nabla(\nabla\cdot\underline{E}) - \nabla^2\underline{E} = -\nabla\times\underline{M}_i - j\omega\mu\left[\underline{J}_i + \sigma\underline{E} + j\omega\varepsilon\underline{E}\right]$$
$$= -\nabla\times\underline{M}_i - j\omega\mu\underline{J}_i - j\omega\mu\sigma\underline{E} + \omega^2\mu\varepsilon\underline{E}$$

Since $\nabla\cdot\underline{D} = \nabla\cdot(\varepsilon\underline{E}) = \varepsilon\nabla\cdot\underline{E} = q_{ve} \Rightarrow \nabla\cdot\underline{E} = \frac{q_{ve}}{\varepsilon}$

Now we can write that

$$\nabla^2\underline{E} = \nabla\times\underline{M}_i + j\omega\mu\underline{J}_i + \frac{1}{\varepsilon}\nabla q_{ve} + j\omega\mu\sigma\underline{E} - \omega^2\mu\varepsilon\underline{E}$$

which is an uncoupled second-order differential equation.

Using the equation for the magnetic field from above along with Maxwell's equations and the vector identity, we can write

$$\nabla(\nabla\cdot\underline{H}) - \nabla^2\underline{H} = \nabla\times\underline{J}_i + (\sigma + j\omega\varepsilon)\nabla\times\underline{E} = \nabla\times\underline{J}_i + (\sigma + j\omega\varepsilon)(-\underline{M}_i - j\omega\mu\underline{H})$$

Since $\nabla\cdot\underline{B} = \nabla\cdot(\mu\underline{H}) = \mu\nabla\cdot\underline{H} = q_{vm} \Rightarrow \nabla\cdot\underline{H} = \frac{1}{\mu}q_{vm}$

then

$$\nabla^2\underline{H} = -\nabla\times\underline{J}_i + \sigma\underline{M}_i + \frac{1}{\mu}\nabla q_{vm} + j\omega\varepsilon\underline{M}_i + j\omega\mu\sigma\underline{H} - \omega^2\mu\varepsilon\underline{H}$$

which also is an uncoupled second order differential equation.

3.2 $\frac{d^2f}{dx^2} = -\beta_x^2 f$, $f = f_1 = A_1 e^{-j\beta_x x} + B_1 e^{+j\beta_x x}$

Using $f = f_1 = A_1 e^{-j\beta_x x}$, then

$$(-j\beta_x)^2 A_1 e^{-j\beta_x x} = -\beta_x^2 A_1 e^{-j\beta_x x} = -\beta_x A_1 e^{-j\beta_x x} \qquad \text{Q.E.D.}$$

(continued)

$\boxed{3.2 \text{ cont'd.}}$ The same can be shown by letting $f = f_1 = B_1 e^{+j\beta_x x}$

Now let us try the cosinusoidal solutions.

Let $f = f_2 = C_1 \cos(\beta_x x)$

Substituting this into the differential equation leads to

$$-\beta_x^2 C_1 \cos(\beta_x) = -\beta_x^2 C_1 \cos(\beta_x x) \qquad \text{Q.E.D.}$$

The same can be shown by letting $f = f_2 = D_1 \sin(\beta_x x)$

$\boxed{3.3}$ $E_x^-(x,y,z;t) = [C_1 \cos(\beta_x x) + D_1 \sin(\beta_x x)][C_2 \cos(\beta_y y) + D_2 \sin(\beta_y y)] B_3 \cos(\omega t + \beta_z z)$

To follow a point z_p for different values of t we must maintain constant the amplitude of the cosine term. This is accomplished by letting

$$\omega t + \beta_z z = C_1 = \text{constant}$$

Taking a derivative of both sides with respect to time, we can write

$$\omega(1) + \beta_z \frac{dz}{dt} = 0 \Rightarrow \omega + \beta_z v_p = 0 \Rightarrow v_p = -\frac{\omega}{\beta_z}$$

which indicates that the wave is moving in the $-z$ direction.

$\boxed{3.4}$ $\nabla^2 E_x - \gamma^2 E_x = 0 = \dfrac{d^2 E_x}{dx^2} + \dfrac{d^2 E_x}{dy^2} + \dfrac{d^2 E_x}{dz^2} - \gamma^2 E_x$

Letting $E_x(x,y,z) = f(x)\, g(y)\, h(z)$ and substituting above leads to

$$gh \frac{d^2 f}{dx^2} + fh \frac{d^2 g}{dy^2} + fg \frac{d^2 h}{dz^2} - \gamma^2 fgh = 0$$

Diving both sides by fgh, we can write

$$\frac{1}{f}\frac{d^2 f}{dx^2} + \frac{1}{g}\frac{d^2 g}{dy^2} + \frac{1}{h}\frac{d^2 h}{dz^2} = \gamma^2$$

By letting each term on the left equal to a constant leads to

$$\frac{1}{f}\frac{d^2 f}{dx^2} = \gamma_x^2 \Rightarrow f = f_1 = A_1 e^{-\gamma_x x} + B_1 e^{+\gamma_x x};\quad f = f_2 = C_1 \cosh(\gamma_x x) + D_1 \sinh(\gamma_x x)$$

$$\frac{1}{g}\frac{d^2 g}{dy^2} = \gamma_y^2 \Rightarrow g = g_1 = A_2 e^{-\gamma_y y} + B_2 e^{+\gamma_y y};\quad g = g_2 = C_2 \cosh(\gamma_y y) + D_2 \sinh(\gamma_y y)$$

$$\frac{1}{h}\frac{d^2 h}{dz^2} = -\gamma_z^2 \Rightarrow h = h_1 = A_3 e^{-\gamma_z z} + B_3 e^{+\gamma_z z};\quad h = h_2 = C_3 \cosh(\gamma_z z) + D_3 \sinh(\gamma_z z)$$

provided that $\quad \gamma_x^2 + \gamma_y^2 + \gamma_z^2 = \gamma^2$

$\boxed{3.5}$ $\nabla(\nabla\cdot\underline{E}) - \nabla\times\nabla\times\underline{E} = -\beta^2\underline{E}$, $\underline{E} = \hat{a}_\rho E_\rho + \hat{a}_\phi E_\phi + \hat{a}_z E_z$

Using cylindrical coordinates

$$\nabla\cdot\underline{E} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho E_\rho) + \frac{1}{\rho}\frac{\partial E_\phi}{\partial\phi} + \frac{\partial E_z}{\partial z}$$

$$\nabla(\nabla\cdot\underline{E}) = \nabla\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho E_\rho) + \frac{1}{\rho}\frac{\partial E_\phi}{\partial\phi} + \frac{\partial E_z}{\partial z}\right]$$

$$= \hat{a}_\rho\left\{\frac{\partial}{\partial\rho}\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho E_\rho) + \frac{1}{\rho}\frac{\partial E_\phi}{\partial\phi} + \frac{\partial E_z}{\partial z}\right]\right\} + \hat{a}_\phi\frac{1}{\rho}\left\{\frac{\partial}{\partial\phi}\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho E_\rho) + \frac{1}{\rho}\frac{\partial E_\phi}{\partial\phi} + \frac{\partial E_z}{\partial z}\right]\right\}$$

$$+\hat{a}_z\left\{\frac{\partial}{\partial z}\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho E_\rho) + \frac{1}{\rho}\frac{\partial E_\phi}{\partial\phi} + \frac{\partial E_z}{\partial z}\right]\right\}$$

$$\nabla(\nabla\cdot\underline{E}) = \hat{a}_\rho\left[\frac{\partial^2 E_\rho}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial E_\rho}{\partial\rho} - \frac{E_\rho}{\rho^2} + \frac{1}{\rho}\frac{\partial^2 E_\phi}{\partial\rho\partial\phi} - \frac{1}{\rho^2}\frac{\partial E_\phi}{\partial\phi} + \frac{\partial^2 E_z}{\partial\rho\partial z}\right]$$

$$+\hat{a}_\phi\left[\frac{1}{\rho}\frac{\partial^2 E_\rho}{\partial\rho\partial\phi} + \frac{1}{\rho^2}\frac{\partial E_\rho}{\partial\phi} + \frac{1}{\rho^2}\frac{\partial^2 E_\phi}{\partial\phi^2} + \frac{1}{\rho}\frac{\partial^2 E_z}{\partial\phi\partial z}\right] + \hat{a}_z\left[\frac{\partial^2 E_\rho}{\partial\rho\partial z} + \frac{1}{\rho}\frac{\partial E_\rho}{\partial z} + \frac{1}{\rho}\frac{\partial^2 E_\phi}{\partial\phi\partial z} + \frac{\partial^2 E_z}{\partial z^2}\right]$$

$$\nabla\times\underline{E} = \hat{a}_\rho\left[\frac{1}{\rho}\frac{\partial E_z}{\partial\phi} - \frac{\partial E_\phi}{\partial z}\right] + \hat{a}_\phi\left[\frac{\partial E_\rho}{\partial z} - \frac{\partial E_z}{\partial\rho}\right] + \hat{a}_z\left[\frac{\partial E_\phi}{\partial\rho} + \frac{E_\phi}{\rho} - \frac{1}{\rho}\frac{\partial E_\rho}{\partial\phi}\right]$$

$$\nabla\times\nabla\times\underline{E} = \hat{a}_\rho\left\{\frac{1}{\rho}\frac{\partial^2 E_\phi}{\partial\rho\partial\phi} + \frac{1}{\rho^2}\frac{\partial E_\phi}{\partial\phi} - \frac{1}{\rho^2}\frac{\partial^2 E_\rho}{\partial\phi^2} - \frac{\partial^2 E_\rho}{\partial z^2} + \frac{\partial^2 E_z}{\partial\rho\partial z}\right\}$$

$$+\hat{a}_\phi\left\{\frac{1}{\rho}\frac{\partial^2 E_z}{\partial\phi\partial z} - \frac{\partial^2 E_\phi}{\partial z^2} - \frac{\partial^2 E_\phi}{\partial\rho^2} - \frac{1}{\rho}\frac{\partial E_\phi}{\partial\rho} + \frac{E_\phi}{\rho^2} + \frac{1}{\rho}\frac{\partial^2 E_\rho}{\partial\rho\partial\phi} - \frac{1}{\rho^2}\frac{\partial E_\rho}{\partial\phi}\right\}$$

$$+\hat{a}_z\left\{\frac{\partial^2 E_\rho}{\partial\rho\partial z} + \frac{1}{\rho}\frac{\partial E_\rho}{\partial z} - \frac{\partial^2 E_z}{\partial\rho^2} - \frac{1}{\rho}\frac{\partial E_z}{\partial\rho} - \frac{1}{\rho^2}\frac{\partial^2 E_z}{\partial\phi^2} + \frac{1}{\rho}\frac{\partial^2 E_\phi}{\partial\phi\partial z}\right\}$$

Substituting all these into the wave equation and equating identical components from the left and right sides, it leads to

ρ component:

$$\left[\frac{\partial^2 E_\rho}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial E_\rho}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2 E_\rho}{\partial\phi^2} + \frac{\partial^2 E_\rho}{\partial z^2}\right] - \frac{E_\rho}{\rho^2} - \frac{2}{\rho^2}\frac{\partial E_\phi}{\partial\phi} = -\beta^2 E_\rho$$

or $\nabla^2 E_\rho + \left(-\frac{E_\rho}{\rho^2} - \frac{2}{\rho^2}\frac{\partial E_\phi}{\partial\phi}\right) = -\beta^2 E_\rho$

ϕ component:

$$\left[\frac{\partial^2 E_\phi}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial E_\phi}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2 E_\phi}{\partial\phi^2} + \frac{\partial^2 E_\phi}{\partial z^2}\right] - \frac{1}{\rho^2} + \frac{2}{\rho^2}\frac{\partial E_\rho}{\partial\phi} = -\beta^2 E_\phi$$

or $\nabla^2 E_\phi + \left[-\frac{E_\phi}{\rho^2} + \frac{2}{\rho^2}\frac{\partial E_\rho}{\partial\phi}\right] = -\beta^2 E_\phi$

z component:

$$\left[\frac{\partial^2 E_z}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial E_z}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2 E_z}{\partial\phi^2} + \frac{\partial^2 E_z}{\partial z^2}\right] = -\beta^2 E_z$$

or $\nabla^2 E_z = -\beta^2 E_z$

$\boxed{3.6}$ $\quad \underline{E} = \hat{a}_\rho E_\rho(\rho,\phi,z) + \hat{a}_\phi E_\phi(\rho,\phi,z) + \hat{a}_z E_z$

Using the rectangular to cylindrical coordinate transformation, we can write

$$\hat{a}_\rho = \hat{a}_x \cos\phi + \hat{a}_y \sin\phi$$

$$\hat{a}_\phi = -\hat{a}_x \sin\phi + \hat{a}_y \cos\phi$$

Therefore

$$\frac{\partial \hat{a}_\rho}{\partial \rho} = 0\ ;\quad \frac{\partial \hat{a}_\phi}{\partial \rho} = 0\ ;\quad \frac{\partial \hat{a}_z}{\partial \rho} = 0$$

$$\frac{\partial \hat{a}_\rho}{\partial \phi} = -\hat{a}_x \sin\phi + \hat{a}_y \cos\phi = \hat{a}_\phi\ ;\quad \frac{\partial^2 \hat{a}_\rho}{\partial \phi^2} = -\hat{a}_x \cos\phi - \hat{a}_y \sin\phi = -\hat{a}_\rho$$

$$\frac{\partial \hat{a}_\phi}{\partial \phi} = -\hat{a}_x \cos\phi - \hat{a}_y \sin\phi = -\hat{a}_\rho\ ;\quad \frac{\partial^2 \hat{a}_\phi}{\partial \phi^2} = \hat{a}_x \sin\phi - \hat{a}_y \cos\phi = -\hat{a}_\phi$$

$$\frac{\partial \hat{a}_z}{\partial \phi} = 0$$

$$\frac{\partial \hat{a}_\rho}{\partial z} = 0\ ;\quad \frac{\partial \hat{a}_\phi}{\partial z} = 0\ ;\quad \frac{\partial \hat{a}_z}{\partial z} = 0$$

$$\nabla^2 \underline{E} = \nabla^2(\hat{a}_\rho E_\rho) + \nabla^2(\hat{a}_\phi E_\phi) + \nabla^2(\hat{a}_z E_z) = \nabla^2(\hat{a}_\rho E_\rho) + \nabla^2(\hat{a}_\phi E_\phi) + \hat{a}_z \nabla^2 E_z$$

$$\nabla^2(\hat{a}_\rho E_\rho) = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left[\rho \frac{\partial}{\partial \rho}(\hat{a}_\rho E_\rho)\right] + \frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2}(\hat{a}_\rho E_\rho) + \frac{\partial^2(\hat{a}_\rho E_\rho)}{\partial z^2}$$

$$\frac{\partial^2}{\partial \phi^2}(\hat{a}_\rho E_\rho) = \frac{\partial}{\partial \phi}\left[\frac{\partial}{\partial \phi}(\hat{a}_\rho E_\rho)\right] = \frac{\partial}{\partial \phi}\left[\hat{a}_\rho \frac{\partial E_\rho}{\partial \phi} + E_\rho \frac{\partial \hat{a}_\rho}{\partial \phi}\right] = \frac{\partial}{\partial \phi}\left[\hat{a}_\rho \frac{\partial E_\rho}{\partial \phi} + \hat{a}_\phi E_\rho\right]$$

$$= \hat{a}_\rho \frac{\partial^2 E_\rho}{\partial \phi^2} + \frac{\partial E_\rho}{\partial \phi}\frac{\partial \hat{a}_\rho}{\partial \phi} + \hat{a}_\phi \frac{\partial E_\rho}{\partial \phi} + E_\rho \frac{\partial \hat{a}_\phi}{\partial \phi}$$

$$= \hat{a}_\rho \frac{\partial^2 E_\rho}{\partial \phi^2} + \hat{a}_\phi \frac{\partial E_\rho}{\partial \phi} + \hat{a}_\phi \frac{\partial E_\rho}{\partial \phi} - \hat{a}_\rho E_\rho = -\hat{a}_\rho E_\rho + \hat{a}_\rho \frac{\partial^2 E_\rho}{\partial \phi^2} + \hat{a}_\phi 2 \frac{\partial E_\rho}{\partial \phi}$$

$$\frac{\partial^2}{\partial z^2}(\hat{a}_\rho E_\rho) = \frac{\partial}{\partial z}\left[\frac{\partial}{\partial z}(\hat{a}_\rho E_\rho)\right] = \frac{\partial}{\partial z}\left[\hat{a}_\rho \frac{\partial E_\rho}{\partial z} + E_\rho \cancelto{0}{\frac{\partial \hat{a}_\rho}{\partial z}}\right] = \hat{a}_\rho \frac{\partial^2 E_\rho}{\partial z^2} + \frac{\partial E_\rho}{\partial z}\cancelto{0}{\frac{\partial \hat{a}_\rho}{\partial z}}$$

Therefore

$$\nabla^2(\hat{a}_\rho E_\rho) = \hat{a}_\rho\left[\frac{\partial^2 E_\rho}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial E_\rho}{\partial \rho} + \frac{1}{\rho^2}\left(\frac{\partial^2 E_\rho}{\partial \phi^2} - E_\rho\right) + \frac{\partial^2 E_\rho}{\partial z^2}\right] + \hat{a}_\phi \frac{2}{\rho^2}\frac{\partial E_\rho}{\partial \phi}$$

Similarly

$$\nabla^2(\hat{a}_\phi E_\phi) = \frac{\partial^2}{\partial \rho^2}(\hat{a}_\phi E_\phi) + \frac{1}{\rho}\frac{\partial}{\partial \rho}(\hat{a}_\phi E_\phi) + \frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2}(\hat{a}_\phi E_\phi) + \hat{a}_\phi \frac{\partial^2 E_\phi}{\partial z^2}$$

$$= -\hat{a}_\rho \frac{2}{\rho^2}\frac{\partial E_\phi}{\partial \phi} + \hat{a}_\phi\left[\frac{\partial^2 E_\phi}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial E_\phi}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 E_\phi}{\partial \phi^2} - \frac{E_\phi}{\rho^2} + \frac{\partial^2 E_\phi}{\partial z^2}\right]$$

$$\nabla^2(\hat{a}_z E_z) = \hat{a}_z \nabla^2 E_z = \hat{a}_z\left[\frac{\partial^2 E_z}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial E_z}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 E_z}{\partial \phi^2} + \frac{\partial^2 E_z}{\partial z^2}\right]$$

(continued)

3.6 cont'd. Substituting all these terms into the vector wave equation and collecting terms with the same direction vectors leads to

ρ component:

$$\left[\frac{\partial^2 E_\rho}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial E_\rho}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2 E_\rho}{\partial\phi^2}+\frac{\partial^2 E_\rho}{\partial z^2}\right]-\frac{E_\rho}{\rho^2}-\frac{2}{\rho^2}\frac{\partial E_\phi}{\partial\phi}=-\beta^2 E_\rho$$

or $$\nabla^2 E_\rho+\left[-\frac{E_\rho}{\rho^2}-\frac{2}{\rho^2}\frac{\partial E_\phi}{\partial\phi}\right]=-\beta^2 E_\rho$$

ϕ component:

$$\left[\frac{\partial^2 E_\phi}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial E_\phi}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2 E_\phi}{\partial\phi^2}+\frac{\partial^2 E_\phi}{\partial z^2}\right]-\frac{E_\phi}{\rho^2}+\frac{2}{\rho^2}\frac{\partial E_\rho}{\partial\phi}=-\beta^2 E_\phi$$

or $$\nabla^2 E_\phi+\left[-\frac{E_\phi}{\rho^2}+\frac{2}{\rho^2}\frac{\partial E_\rho}{\partial\phi}\right]=-\beta^2 E_\phi$$

z component:

$$\left[\frac{\partial^2 E_z}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial E_z}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2 E_z}{\partial\phi^2}+\frac{\partial^2 E_z}{\partial z^2}\right]=-\beta^2 E_z$$

or $$\nabla^2 E_z=-\beta^2 E_z$$

3.7 For large arguments the Bessel function of the first kind and the Hankel function of the second kind can be written according to (IV-13) and (IV-17) as

$$J_m(\beta\rho)\overset{\beta\rho\to\text{large}}{\simeq}\sqrt{\frac{2}{\pi\beta\rho}}\cos\left(\beta\rho-\frac{\pi}{4}-\frac{m\pi}{2}\right)$$

$$H_m^{(2)}(\beta\rho)\overset{\beta\rho\to\text{large}}{\simeq}\sqrt{\frac{2}{\pi\beta\rho}}\,e^{-j\left[\beta\rho-\frac{m\pi}{2}-\frac{\pi}{4}\right]}$$

which are recognized as decaying cosine and complex exponential functions. Thus they represent, respectively, standing and traveling waves.

3.8 For large argument the Hankel functions of the first and second kind can be written according to (IV-16) and (IV-17) as

$$H_m^{(1)}(\beta\rho)\overset{\beta\rho\to\text{large}}{\simeq}\sqrt{\frac{2}{\pi\beta\rho}}\,e^{j\left[\beta\rho-\frac{m\pi}{2}-\frac{\pi}{4}\right]}$$

$$H_m^{(2)}(\beta\rho)\overset{\beta\rho\to\text{large}}{\simeq}\sqrt{\frac{2}{\pi\beta\rho}}\,e^{-j\left[\beta\rho-\frac{m\pi}{2}-\frac{\pi}{4}\right]}$$

(continued)

$\boxed{3.8 \text{ cont'd.}}$ For an $e^{j\omega t}$ time convention the time varying field represented by the Hankel functions can be written as

$$\mathcal{H}_m^{(1)}(\beta\rho;t) = \text{Re}\left[H_m^{(1)}(\beta\rho)e^{j\omega t}\right] = \sqrt{\frac{2}{\pi\beta\rho}}\cos\left[\omega t + \beta\rho - \frac{m\pi}{2} - \frac{\pi}{4}\right]$$

$$\mathcal{H}_m^{(2)}(\beta\rho;t) = \text{Re}\left[H_m^{(2)}(\beta\rho)e^{j\omega t}\right] = \sqrt{\frac{2}{\pi\beta\rho}}\cos\left[\omega t - \beta\rho + \frac{m\pi}{2} + \frac{\pi}{4}\right]$$

For a constant wavefront

$$\omega t + \beta\rho - \frac{m\pi}{2} - \frac{\pi}{4} = C_1 \Rightarrow \omega(1) + \beta\frac{d\rho}{dt} = 0 \Rightarrow \omega + \beta v_p = 0 \Rightarrow v_p = -\frac{\omega}{\beta}$$

$$\omega t - \beta\rho + \frac{m\pi}{2} + \frac{\pi}{4} = C_2 \Rightarrow \omega(1) - \beta\frac{d\rho}{dt} = 0 \Rightarrow \omega - \beta v_p = 0 \Rightarrow v_p = +\frac{\omega}{\beta}$$

Therefore $H_m^{(1)}(\beta\rho)$ represents for an $e^{j\omega t}$ time convention waves traveling in the $-\rho$ direction while $H_m^{(2)}(\beta\rho)$ represents waves traveling in the $+\rho$ direction. It is obvious, by following a similar procedure, that the opposite would be true for an $e^{-j\omega t}$ time convention.

$\boxed{3.9}$ For large argument the Bessel function of the first kind can be written according to (IV-13) as

$$J_m(\gamma\rho) \underset{\gamma\rho\to\text{large}}{\simeq} \sqrt{\frac{2}{\pi\gamma\rho}}\cos\left[\gamma\rho - \frac{\pi}{4} - \frac{m\pi}{2}\right] = \sqrt{\frac{2}{\pi\gamma\rho}}\cos\left\{\left[\alpha\rho - \frac{\pi}{4} - \frac{m\pi}{2}\right] + j\beta\rho\right]$$

$$\underset{\gamma\rho\to\text{large}}{\simeq} \sqrt{\frac{2}{\pi\gamma\rho}}\cos(\alpha_e\rho + j\beta\rho) = \sqrt{\frac{2}{\pi\gamma\rho}}\left\{\cos(\alpha_e\rho)\cos(j\beta\rho) - \sin(\alpha_e\rho)\sin(j\beta\rho)\right\}$$

$$\underset{\gamma\rho\to\text{large}}{\simeq} \sqrt{\frac{2}{\pi\gamma\rho}}\left\{\cos(\alpha_e\rho)\cosh(\beta\rho) - j\sin(\alpha_e\rho)\sinh(\beta\rho)\right\}$$

$$\underset{\gamma\rho\to\text{large}}{\simeq} \sqrt{\frac{2}{\pi\gamma\rho}}\left\{\cos(\alpha_e\rho)\left(\frac{e^{\beta\rho}+e^{-\beta\rho}}{2}\right) - j\sin(\alpha_e\rho)\left(\frac{e^{\beta\rho}-e^{-\beta\rho}}{2}\right)\right\}$$

$$J_m(\gamma\rho) \underset{\gamma\rho\to\text{large}}{\simeq} \sqrt{\frac{2}{\pi\gamma\rho}}\left\{\frac{e^{\beta\rho}}{2}\left[\cos(\alpha_e\rho) - j\sin(\alpha_e\rho)\right] + \frac{e^{-\beta\rho}}{2}\left[\cos(\alpha_e\rho) + j\sin(\alpha_e\rho)\right]\right\}$$

where $\gamma = \alpha + j\beta$ and $\alpha_e\rho = \alpha\rho - \frac{\pi}{4} - \frac{m\pi}{2}$

It is apparent from the above that the Bessel functions of complex argument represent attenuating standing waves.

$\boxed{3.10}$ For large argument the Hankel functions of the first and second kind can be written according to (IV-16) and (IV-17) as

$$H_m^{(1)}(\gamma\rho) \overset{\gamma\rho\to\text{large}}{\simeq} \sqrt{\frac{2}{j\pi\gamma\rho}}\, e^{j\left[-j\gamma\rho-\frac{m\pi}{2}-\frac{\pi}{4}\right]} = \sqrt{\frac{2}{j\pi\gamma\rho}}\, e^{j\left[-j(\alpha+j\beta)\rho-\frac{m\pi}{2}-\frac{\pi}{4}\right]}$$

$$\overset{\gamma\rho\to\text{large}}{\simeq} \sqrt{\frac{2}{j\pi\gamma\rho}}\, e^{+\alpha\rho}\, e^{-j\left(\beta\rho-\frac{m\pi}{2}-\frac{\pi}{4}\right)} = \sqrt{\frac{2}{\pi\gamma\rho}}\, e^{\alpha\rho}\, e^{+j\left(\beta\rho-\frac{m\pi}{2}\right)}$$

which represents an attenuating traveling wave in the $-\rho$ direction because of the

$e^{\alpha\rho}$ and $e^{j\beta\rho}$ factors.

$$H_m^{(2)}(\gamma\rho) \overset{\gamma\rho\to\text{large}}{\simeq} \sqrt{\frac{2}{-j\pi\gamma\rho}}\, e^{-j\left[-j\gamma\rho-\frac{m\pi}{2}-\frac{\pi}{4}\right]} = \sqrt{\frac{2}{-j\pi\gamma\rho}}\, e^{+j\left[j\gamma\rho+\frac{m\pi}{2}+\frac{\pi}{4}\right]}$$

$$\overset{\gamma\rho\to\text{large}}{\simeq} \sqrt{\frac{2}{\pi\gamma\rho}}\, e^{-\alpha\rho}\, e^{-j\left(\beta\rho-\frac{m\pi}{2}-\frac{\pi}{2}\right)}$$

which represents an attenuating traveling wave in the $+\rho$ direction because of the

$e^{-\alpha\rho}$ and $e^{-j\beta\rho}$ factors.

$\boxed{3.11}$

$$\underline{E} = \hat{a}_r E_r(r,\theta,\phi) + \hat{a}_\theta E_\theta(r,\theta,\phi) + \hat{a}_\phi E_\phi(r,\theta,\phi)$$

$$\nabla(\nabla\cdot\underline{E}) - \nabla\times\nabla\times\underline{E} = -\beta^2\underline{E}$$

Using spherical coordinates

$$\nabla\cdot\underline{E} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2E_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, E_\theta) + \frac{1}{r\sin\theta}\frac{\partial E_\phi}{\partial\phi}$$

$$\nabla(\nabla\cdot\underline{E}) = \hat{a}_r\frac{\partial}{\partial r}[\nabla\cdot\underline{E}] + \hat{a}_\theta\frac{1}{r}\frac{\partial}{\partial\theta}[\nabla\cdot\underline{E}] + \hat{a}_\phi\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}[\nabla\cdot\underline{E}]$$

$$= \hat{a}_r\frac{\partial}{\partial r}\left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2E_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, E_\theta) + \frac{1}{r\sin\theta}\frac{\partial E_\phi}{\partial\phi}\right]$$

$$+ \hat{a}_\theta\frac{1}{r}\frac{\partial}{\partial\theta}\left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2E_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, E_\theta) + \frac{1}{r\sin\theta}\frac{\partial E_\phi}{\partial\phi}\right]$$

$$+ \hat{a}_\phi\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2E_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, E_\theta) + \frac{1}{r\sin\theta}\frac{\partial E_\phi}{\partial\phi}\right]$$

(continued)

3.11 cont'd.

$$\nabla\times\underline{E} = \frac{\hat{a}_r}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(E_\phi\sin\theta)-\frac{\partial E_\theta}{\partial\phi}\right]+\frac{\hat{a}_\theta}{r}\left[\frac{1}{\sin\theta}\frac{\partial E_r}{\partial\phi}-\frac{\partial}{\partial r}(rE_\phi)\right]+\frac{\hat{a}_\phi}{r}\left[\frac{\partial}{\partial r}(rE_\theta)-\frac{\partial E_r}{\partial\theta}\right]$$

$$= \hat{a}_r\, ER + \hat{a}_\theta ET + \hat{a}_\phi EP$$

where
$$ER = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(E_\phi\sin\theta)-\frac{\partial E_\phi}{\partial\phi}\right]$$

$$ET = \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial E_r}{\partial\phi}-\frac{\partial}{\partial r}(rE_\phi)\right]$$

$$EP = \frac{1}{r}\left[\frac{\partial}{\partial r}(rE_\phi)-\frac{\partial E_r}{\partial\theta}\right]$$

$$\nabla\times\nabla\times E = \hat{a}_r\,\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(EP\sin\theta)-\frac{\partial}{\partial\phi}(ET)\right]$$
$$+\hat{a}_\theta\,\frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}(ER)-\frac{\partial}{\partial r}(rEP)\right]$$
$$+\hat{a}_\phi\,\frac{1}{r}\left[\frac{\partial}{\partial r}(rEP)-\frac{\partial}{\partial\theta}(ER)\right]$$

Using all these it can be shown that $\nabla(\nabla\cdot\underline{E})-\nabla\times\nabla\times\underline{E} = -\beta^2\underline{E}$ reduces by equating identical components from the left and right sides to

<u>r component:</u>

$$\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2E_r\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial E_r}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2E_r}{\partial\phi^2}\right]-\frac{2}{r^2}\left[E_r+E_\theta\cot\theta+\csc\theta\frac{\partial E_\phi}{\partial\phi}+\frac{\partial E_\theta}{\partial\theta}\right]=-\beta^2E_r$$

or
$$\nabla^2E_r-\frac{2}{r^2}\left[E_r+E_\theta\cot\theta+\csc\theta\frac{\partial E_\phi}{\partial\phi}+\frac{\partial E_\theta}{\partial\theta}\right]=-\beta^2E_r$$

<u>θ component:</u>

$$\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2E_\theta\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial E_\theta}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2E_\theta}{\partial\phi^2}\right]-\frac{1}{r^2}\left[E_\theta\csc^2\theta-2\frac{\partial E_r}{\partial\theta}+2\cot\theta\csc\theta\frac{\partial E_\phi}{\partial\phi}\right]=-\beta^2E_\theta$$

or
$$\nabla^2E_\theta-\frac{1}{r^2}\left[E_\theta\csc^2\theta-2\frac{\partial E_r}{\partial\theta}+2\cot\theta\csc\theta\frac{\partial E_\phi}{\partial\phi}\right]=-\beta^2E_\theta$$

<u>φ component:</u>

$$\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2E_\phi\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial E_\phi}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2E_\phi}{\partial\phi^2}\right]-\frac{1}{r^2}\left[E_\phi\csc^2\theta-2\csc\theta\frac{\partial E_r}{\partial\phi}-2\cot\theta\csc\theta\frac{\partial E_\theta}{\partial\phi}\right]=-\beta^2E_\phi$$

or
$$\nabla^2E_\phi-\frac{1}{r^2}\left[E_\phi\csc^2\theta-2\csc\theta\frac{\partial E_r}{\partial\phi}-2\cot\theta\csc\theta\frac{\partial E_\theta}{\partial\phi}\right]=-\beta^2E_\phi$$

$\boxed{3.12}$ $\underline{E} = \hat{a}_r E_r(r,\theta,\phi) + \hat{a}_\theta E_\theta(r,\theta,\phi) + \hat{a}_\phi E_\phi(r,\theta,\phi)$

Using the rectangular to spherical coordinate transformation, we can write

$$\hat{a}_r = \hat{a}_x \sin\theta\cos\phi + \hat{a}_y \sin\theta\sin\phi + \hat{a}_z \cos\theta$$

$$\hat{a}_\theta = \hat{a}_x \cos\theta\cos\phi + \hat{a}_y \cos\theta\sin\phi - \hat{a}_z \sin\theta$$

$$\hat{a}_\phi = -\hat{a}_x \sin\phi + \hat{a}_y \cos\phi$$

Therefore

$$\frac{\partial \hat{a}_r}{\partial r} = 0\ ;\quad \frac{\partial \hat{a}_\theta}{\partial r} = 0\ ;\quad \frac{\partial \hat{a}_\phi}{\partial r} = 0$$

$$\frac{\partial \hat{a}_r}{\partial \theta} = \hat{a}_x \cos\theta\cos\phi + \hat{a}_y \cos\theta\sin\phi - \hat{a}_z \sin\theta = +\hat{a}_\theta$$

$$\frac{\partial \hat{a}_\theta}{\partial \theta} = -\hat{a}_x \sin\theta\cos\phi - \hat{a}_y \sin\theta\sin\phi - \hat{a}_z \cos\theta = -\hat{a}_r$$

$$\frac{\partial \hat{a}_\phi}{\partial \theta} = 0$$

$$\frac{\partial \hat{a}_r}{\partial \phi} = -\hat{a}_x \sin\theta\sin\phi + \hat{a}_y \sin\theta\cos\phi = \sin\theta\left[-\hat{a}_x\sin\phi + \hat{a}_y\cos\phi\right] = \sin\theta\,\hat{a}_\phi$$

$$\frac{\partial \hat{a}_\theta}{\partial \phi} = -\hat{a}_x \cos\theta\sin\phi + \hat{a}_y \cos\theta\cos\phi = \cos\theta\left[-\hat{a}_x\sin\phi + \hat{a}_y\cos\phi\right] = \cos\theta\,\hat{a}_\phi$$

$$\frac{\partial \hat{a}_\phi}{\partial \phi} = -\hat{a}_x\cos\phi - \hat{a}_y\sin\phi = -\sin\theta\,\hat{a}_r - \cos\theta\,\hat{a}_\theta = -\hat{a}_x \sin^2\theta\cos\phi - \hat{a}_y\sin^2\theta\sin\phi - \hat{a}_z\sin\theta\cos\theta$$
$$- \hat{a}_x \cos^2\theta\cos\phi - \hat{a}_y\cos^2\theta\sin\phi + \hat{a}_z\sin\theta\cos\theta$$
$$= -\hat{a}_x\cos\phi\left(\sin^2\theta + \cos^2\theta\right) - \hat{a}_y\sin\phi\left(\sin^2\theta+\cos^2\theta\right)$$

$$\frac{\partial \hat{a}_\phi}{\partial \phi} = -\hat{a}_x\cos\phi - \hat{a}_y\sin\phi$$

$$\nabla^2 \underline{E} = \overbrace{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \underline{E}}{\partial r}\right)}^{\text{I}} + \overbrace{\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial \underline{E}}{\partial\theta}\right)}^{\text{II}} + \overbrace{\frac{1}{r^2\sin^2\theta}\frac{\partial^2 \underline{E}}{\partial\phi^2}}^{\text{III}}$$

I. $$\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial(\hat{a}_r E_r)}{\partial r}\right] + \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial(\hat{a}_\theta E_\theta)}{\partial r}\right] + \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial(\hat{a}_\phi E_\phi)}{\partial r}\right] = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \underline{E}}{\partial r}\right)$$

$$\hat{a}_r\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial E_r}{\partial r}\right] + \hat{a}_\theta\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial E_\theta}{\partial r}\right] + \hat{a}_\phi\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial E_\phi}{\partial r}\right] = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \underline{E}}{\partial r}\right)$$

II. $$\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\underline{E}}{\partial\theta}\right) = \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial(\hat{a}_r E_r)}{\partial\theta}\right] + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial(\hat{a}_\theta E_\theta)}{\partial\theta}\right] + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial(\hat{a}_\phi E_\phi)}{\partial\theta}\right]$$

$$\frac{\partial}{\partial\theta}(\underline{E}) = \frac{\partial}{\partial\theta}\left(\hat{a}\,E_r + \hat{a}_\theta E_\theta + \hat{a}_\phi E_\phi\right) = \hat{a}_r\left(\frac{\partial E_r}{\partial\theta} - E_\theta\right) + \hat{a}_\theta\left(\frac{\partial E_\theta}{\partial\theta} + E_r\right) + \hat{a}_\phi\frac{\partial E_\phi}{\partial\theta}$$

(continued)

3.12 cont'd.

$$\frac{\partial^2 \mathbf{E}}{\partial\theta^2} = \hat{a}_r\left[\frac{\partial^2 E_r}{\partial\theta^2} - \frac{\partial E_\theta}{\partial\theta} - \left(\frac{\partial E_\theta}{\partial\theta} + E_r\right)\right] + \hat{a}_\theta\left[\frac{\partial^2 E_\theta}{\partial\theta^2} + \frac{\partial E_r}{\partial\theta} + \left(\frac{\partial E_r}{\partial\theta} - E_\theta\right)\right] + \hat{a}_\phi\frac{\partial^2 E_\phi}{\partial\theta^2}$$

$$= \hat{a}_r\left(\frac{\partial^2 E_r}{\partial\theta^2} - 2\frac{\partial E_\theta}{\partial\theta} - E_r\right) + \hat{a}_\theta\left(\frac{\partial^2 E_\theta}{\partial\theta^2} + 2\frac{\partial E_r}{\partial\theta} - E_\theta\right) + \hat{a}_\phi\frac{\partial^2 E_\phi}{\partial\theta^2}$$

Therefore

$$\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\mathbf{E}}{\partial\theta}\right) = \frac{\cos\theta}{r^2\sin\theta}\frac{\partial\mathbf{E}}{\partial\theta} + \frac{1}{r^2}\frac{\partial^2\mathbf{E}}{\partial\theta^2}$$

$$= \hat{a}_r\frac{1}{r^2}\left[\frac{\partial^2 E_r}{\partial\theta^2} + \frac{\cos\theta}{\sin\theta}\frac{\partial E_r}{\partial\theta} - 2\frac{\partial E_\theta}{\partial\theta} - E_r - E_\theta\frac{\cos\theta}{\sin\theta}\right]$$

$$+ \hat{a}_\theta\frac{1}{r^2}\left[\frac{\partial^2 E_\theta}{\partial\theta^2} + \frac{\cos\theta}{\sin\theta}\frac{\partial E_\theta}{\partial\theta} + 2\frac{\partial E_r}{\partial\theta} + E_r\frac{\cos\theta}{\sin\theta} - E_\theta\right]$$

$$+ \hat{a}_\phi\frac{1}{r^2}\left[\frac{\partial^2 E_\phi}{\partial\theta^2} + \frac{\cos\theta}{\sin\theta}\frac{\partial E_\phi}{\partial\theta}\right]$$

or

$$\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\mathbf{E}}{\partial\theta}\right) = \frac{1}{r^2}\left\{\hat{a}_r\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial E_r}{\partial\theta}\right) - 2\frac{\partial E_\theta}{\partial\theta} - E_\theta\frac{\cos\theta}{\sin\theta} - E_r\right]\right.$$

$$+ \hat{a}_\theta\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial E_\theta}{\partial\theta}\right) + 2\frac{\partial E_r}{\partial\theta} + E_r\frac{\cos\theta}{\sin\theta} - E_\theta\right]$$

$$\left. + \hat{a}_\phi\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial E_\phi}{\partial\theta}\right)\right]\right\}$$

III. $$\frac{1}{r^2\sin^2\theta}\frac{\partial^2\mathbf{E}}{\partial\phi^2} = \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\left[\hat{a}_r E_r + \hat{a}_\theta E_\theta + \hat{a}_\phi E_\phi\right]$$

$$\frac{\partial\mathbf{E}}{\partial\phi} = \hat{a}_r\left(\frac{\partial E_r}{\partial\phi} - E_\phi\sin\theta\right) + \hat{a}_\theta\left(\frac{\partial E_\theta}{\partial\phi} - E_\phi\cos\theta\right) + \hat{a}_\phi\left(\frac{\partial E_\phi}{\partial\phi} + E_r\sin\theta + E_\theta\cos\theta\right)$$

$$\frac{\partial^2\mathbf{E}}{\partial\phi^2} = \hat{a}_r\left[\frac{\partial^2 E_r}{\partial\phi^2} - \frac{\partial E_\phi}{\partial\phi}\sin\theta - \left(\frac{\partial E_\phi}{\partial\phi} + E_r\sin\theta + E_\theta\cos\theta\right)\sin\theta\right]$$

$$+ \hat{a}_\theta\left[\frac{\partial^2 E_\theta}{\partial\phi^2} - \frac{\partial E_\phi}{\partial\phi}\cos\theta - \left(\frac{\partial E_\phi}{\partial\phi} + E_r\sin\theta + E_\theta\cos\theta\right)\cos\theta\right]$$

$$+ \hat{a}_\phi\left[\frac{\partial^2 E_\phi}{\partial\phi^2} + \frac{\partial E_r}{\partial\phi}\sin\theta + \frac{\partial E_\theta}{\partial\phi}\cos\theta + \left(\frac{\partial E_r}{\partial\phi} - E_\phi\sin\theta\right)\sin\theta + \left(\frac{\partial E_\theta}{\partial\phi} - E_\phi\cos\theta\right)\cos\theta\right]$$

Therefore

$$\frac{1}{r^2\sin^2\theta}\frac{\partial^2\mathbf{E}}{\partial\phi^2} = \frac{1}{r^2\sin^2\theta}\left\{\hat{a}_r\left[\frac{\partial^2 E_r}{\partial\phi^2} - 2\frac{\partial E_\phi}{\partial\phi}\sin\theta - E_r\sin^2\theta - E_\theta\sin\theta\cos\theta\right]\right.$$

$$+ \hat{a}_\theta\left[\frac{\partial^2 E_\theta}{\partial\phi^2} - 2\frac{\partial E_\phi}{\partial\phi}\cos\theta - E_r\sin\theta\cos\theta - E_\theta\cos^2\theta\right]$$

$$\left. + \hat{a}_\phi\left[\frac{\partial^2 E_\phi}{\partial\phi^2} + 2\frac{\partial E_r}{\partial\phi}\sin\theta + 2\frac{\partial E_\theta}{\partial\phi}\cos\theta - E_\phi\right]\right\}$$

(continued)

3.12 cont'd. Finally using I, II and III, it can be shown that

$$\nabla^2 \underline{E} = \nabla^2(\hat{a}_r E_r + \hat{a}_\theta E_\theta + \hat{a}_\phi E_\phi) = -\beta^2 \underline{E} = -\beta^2(\hat{a}_r E_r + \hat{a}_\theta E_\theta + \hat{a}_\phi E_\phi)$$

reduces by equating identical components from the left and right sides to

$$\nabla^2 E_r - \frac{2}{r^2}\left[E_r + E_\theta \cot\theta + \csc\theta \frac{\partial E_\phi}{\partial \phi} + \frac{\partial E_\theta}{\partial \theta}\right] = -\beta^2 E_r$$

$$\nabla^2 E_\theta - \frac{1}{r^2}\left[E_\theta \csc^2\theta - 2\frac{\partial E_r}{\partial \theta} + 2\cot\theta \csc\theta \frac{\partial E_\phi}{\partial \phi}\right] = -\beta^2 E_\theta$$

$$\nabla^2 E_\phi - \frac{1}{r^2}\left[E_\phi \csc^2\theta - 2\csc\theta \frac{\partial E_r}{\partial \phi} - 2\cot\theta \csc\theta \frac{\partial E_\theta}{\partial \phi}\right] = -\beta^2 E_\phi$$

where $\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\psi) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial \psi}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}$

3.13 According to (3-90a)

$$j_n(\beta r) = \sqrt{\frac{\pi}{2\beta r}}\, J_{n+\frac{1}{2}}(\beta r) \underset{\sim}{\xrightarrow{\beta r \to large}} \sqrt{\frac{\pi}{2\beta r}}\sqrt{\frac{2}{\pi \beta r}} \cos\left(\beta r - \frac{\pi}{2} - \frac{n\pi}{2}\right)$$

$$j_n(\beta r) \underset{\sim}{\xrightarrow{\beta r \to large}} \frac{1}{\beta r} \cos\left(\beta r - \frac{\pi}{2} - \frac{n\pi}{2}\right)$$

which represents a standing wave in the r direction. The same can be shown for the spherical Bessel function of (3-90b).

3.14 According to (3-91)

$$h_n^{(2)}(\beta r) = \sqrt{\frac{\pi}{2\beta r}}\, H^{(2)}_{n+\frac{1}{2}}(\beta r) \underset{\sim}{\xrightarrow{\beta r \to large}} \sqrt{\frac{\pi}{2\beta r}}\sqrt{\frac{2}{\pi\beta r}}\, e^{-j\left[\beta r - \frac{n\pi}{2} - \frac{\pi}{2}\right]}$$

$$h_n^{(2)}(\beta r) \underset{\sim}{\xrightarrow{\beta r \to large}} \frac{1}{\beta r} e^{-j\left[\beta r - \frac{n\pi}{2} - \frac{\pi}{2}\right]}$$

which represents a traveling wave in the +r direction (for $e^{+j\omega t}$).

Similarly

$$h_n^{(1)}(\beta r) \underset{\sim}{\xrightarrow{\beta r \to large}} \frac{1}{\beta r} e^{j\left[\beta r - \frac{n\pi}{2} - \frac{\pi}{2}\right]}$$

which represents a traveling wave in the −r direction (for $e^{+j\omega t}$).

$\boxed{3.15}$ According to (V-9) the Legendre polynomials are expressed in terms of cosine functions of the angle θ. Since cosine functions are used to represent standing wave functions, then the Legendre polynomials, and in turn the Legendre functions of (V-19a), are used to represent standing waves.

$\boxed{3.16}$ According to (3-90a)

$$j_n(\beta r) \equiv \sqrt{\frac{\pi}{2\beta r}}\, J_{n+\frac{1}{2}}(\beta r)$$

Since

$$\hat{J}_n(\beta r) \equiv \beta r\, j_n(\beta r)$$

then

$$\hat{J}_n(\beta r) = \beta r\, j_n(\beta r) = \beta r \sqrt{\frac{\pi}{2\beta r}}\, J_{n+\frac{1}{2}}(\beta r) = \sqrt{\frac{\pi\beta r}{2}}\, J_{n+\frac{1}{2}}(\beta r)$$

According to

$$h_n^{(2)}(\beta r) \equiv \sqrt{\frac{\pi}{2\beta r}}\, H_{n+\frac{1}{2}}^{(2)}(\beta r)$$

Since

$$\hat{H}_n^{(2)}(\beta r) \equiv \beta r\, h_n^{(2)}(\beta r)$$

then

$$\hat{H}_n^{(2)}(\beta r) = \beta r\, h_n^{(2)}(\beta r) = \beta r \sqrt{\frac{\pi}{2\beta r}}\, H_{n+\frac{1}{2}}^{(2)}(\beta r) = \sqrt{\frac{\pi\beta r}{2}}\, H_{n+\frac{1}{2}}^{(2)}(\beta r)$$

CHAPTER 4

4.1 According to Maxwell's equation of $\nabla \times \underline{H} = j\omega\varepsilon \underline{E}$

$$\hat{a}_x\left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right) + \hat{a}_y\left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right) + \hat{a}_z\left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right) = j\omega\varepsilon\left(\hat{a}_x E_x + \hat{a}_y E_y + \hat{a}_z E_z\right)$$

For a uniform plane wave traveling in the z direction the only component of $\underline{H}$ that contributes, assuming an x component of $\underline{E}$, is

$$-\hat{a}_x \frac{\partial H_y}{\partial z} = \hat{a}_x\, j\omega\varepsilon E_x \Rightarrow \frac{\partial H_y}{\partial z} = -j\omega\varepsilon E_x$$

Similarly using Maxwell's equation of $\nabla \times \underline{E} = -j\omega\mu \underline{H}$

$$\hat{a}_x\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right) + \hat{a}_y\left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right) + \hat{a}_z\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right) = -j\omega\mu\left(\hat{a}_x H_x + \hat{a}_y H_y + \hat{a}_z H_z\right)$$

the only components that contribute to the plane wave traveling in the x direction are

$$\hat{a}_y \frac{\partial E_x}{\partial z} = -\hat{a}_y\, j\omega\mu H_y \Rightarrow \frac{\partial E_x}{\partial z} = -j\omega\mu H_y \Rightarrow H_y = -\frac{1}{j\omega\mu}\frac{\partial E_x}{\partial z}$$

Substituting this equation into the one above, we can write

$$\frac{\partial H_y}{\partial z} = \frac{\partial}{\partial z}\left[-\frac{1}{j\omega\mu}\frac{\partial E_x}{\partial z}\right] = -\frac{1}{j\omega\mu}\frac{\partial^2 E}{\partial z^2} = -j\omega\varepsilon E_x \Rightarrow \frac{\partial^2 E_x}{\partial z^2} = -\omega^2\mu\varepsilon E_x$$

Since the field is only a function of z, then

$$\frac{d^2 E_x}{dz^2} = -\omega^2\mu\varepsilon E_x = -\beta^2 E_x \Rightarrow E_x = E_0^+ e^{-j\beta z} + E_0^- e^{+j\beta z},\quad E_x^+ = E_0^+ e^{-j\beta z}$$

Then

$$H_y^+ = -\frac{1}{j\omega\mu}\frac{\partial E_x^+}{\partial z} = -\frac{1}{j\omega\mu}\frac{\partial}{\partial z}\left[E_0^+ e^{-j\beta z}\right] = \frac{1}{\omega\mu}\beta E_0^+ e^{-j\beta z}$$

$$= \frac{\beta}{\omega\mu} E_0^+ e^{-j\beta z} = \frac{\omega\sqrt{\mu\varepsilon}}{\omega\mu} E_0^+ e^{-j\beta z} = \frac{1}{\sqrt{\mu/\varepsilon}} E_0^+ e^{-j\beta z}$$

$$H_y^+ = \frac{1}{\sqrt{\mu/\varepsilon}} E_x^+$$

4.2 $\underline{E} = \hat{a}_y (E_o^+ e^{-j\beta z} + E_o^- e^{+j\beta z})$

Using Maxwell's equation of

$$\nabla \times \underline{E} = -j\omega\mu \underline{H} \Rightarrow \underline{H} = -\frac{1}{j\omega\mu} \nabla \times \underline{E} = -\frac{1}{j\omega\mu}\left[\hat{a}_x\left(-\frac{\partial E_y}{\partial z}\right)\right] = \hat{a}_x \frac{1}{j\omega\mu}\frac{\partial E_y}{\partial z}$$

$$H_x = \frac{1}{j\omega\mu}\frac{\partial E_y}{\partial z} = \frac{\beta}{\omega\mu}\left(-E_o^+ e^{-j\beta z} + E_o^- e^{+j\beta z}\right) = \frac{\omega\sqrt{\mu\varepsilon}}{\omega\mu}\left(-E_o^+ e^{-j\beta z} + E_o^- e^{+j\beta z}\right)$$

$$H_x = \sqrt{\frac{\varepsilon}{\mu}}\left(-E_o^+ e^{-j\beta z} + E_o^- e^{+j\beta z}\right) = \frac{1}{\eta}\left(-E_o^+ e^{-j\beta z} + E_o^- e^{+j\beta z}\right)$$

or

$$\underline{H} = \hat{a}_x \frac{1}{\eta}\left(-E_o^+ e^{-j\beta z} + E_o^- e^{+j\beta z}\right)$$

4.3 $\underline{H} = \frac{1}{120\pi}(\hat{a}_x - 2\hat{a}_y) e^{-j\beta_o z}$

a. $\underline{E} = \frac{120\pi}{120\pi}(-\hat{a}_y - 2\hat{a}_x) e^{-j\beta_o z} = -(2\hat{a}_x + \hat{a}_y) e^{-j\beta_o z}$

b. $\underline{\mathcal{E}} = \mathrm{Re}\left[\underline{E} e^{j\omega t}\right] = -(2\hat{a}_x + \hat{a}_y)\cos(\omega t - \beta_o z)$

$\underline{\mathcal{H}} = \mathrm{Re}\left[\underline{H} e^{j\omega t}\right] = \frac{1}{120\pi}(\hat{a}_x - 2\hat{a}_y)\cos(\omega t - \beta_o z)$

$\underline{\mathcal{S}} = \underline{\mathcal{E}} \times \underline{\mathcal{H}} = -\frac{1}{120\pi}(2\hat{a}_x + \hat{a}_y) \times (\hat{a}_x - 2\hat{a}_y)\cos^2(\omega t - \beta_o z) = \hat{a}_z \frac{5}{120\pi}\cos^2(\omega t - \beta_o z)$

$= \hat{a}_z\, 1.326 \times 10^{-2} \cos^2(\omega t - \beta z)$

c. $\underline{S}_{ave} = \frac{1}{2}\mathrm{Re}(\underline{E} \times \underline{H}^*) = \frac{1}{2}\mathrm{Re}\left[-(2\hat{a}_x + \hat{a}_y) e^{-j\beta_o z} \times \frac{1}{120\pi}(\hat{a}_x - 2\hat{a}_y) e^{+j\beta_o z}\right]$

$= \hat{a}_z \frac{5}{240\pi} = \hat{a}_z\, 6.63 \times 10^{-3}$ W/m^2

4.4 $\underline{E} = (\hat{a}_x + j\hat{a}_z) e^{-j\beta_o y} + (2\hat{a}_x - j\hat{a}_z) e^{+j\beta_o y}$

a. $\underline{H} = (j\hat{a}_x - \hat{a}_z)\frac{1}{\eta_o} e^{-j\beta_o y} + (j\hat{a}_x + 2\hat{a}_z)\frac{1}{\eta_o} e^{+j\beta_o y}$

b. $\underline{S}_{av}^+ = \frac{1}{2}\mathrm{Re}\left[\underline{E}^+ \times \underline{H}^{+*}\right] = \frac{1}{2}\mathrm{Re}\left[(\hat{a}_x + j\hat{a}_z) e^{-j\beta_o y} \times (-j\hat{a}_x - \hat{a}_z)\frac{1}{\eta_o} e^{+j\beta_o y}\right] = \hat{a}_y \frac{1}{\eta_o} = \hat{a}_y\, 2.6525 \times 10^{-3}$

c. $\underline{S}_{av}^- = \frac{1}{2}\mathrm{Re}\left[\underline{E}^- \times \underline{H}^{-*}\right] = \frac{1}{2}\mathrm{Re}\left[(2\hat{a}_x - j\hat{a}_z) e^{+j\beta_o y} \times (-j\hat{a}_x + 2\hat{a}_z)\frac{1}{\eta_o} e^{-j\beta z}\right] = -\hat{a}_y \frac{5}{2\eta_o} = -\hat{a}_y \frac{1.326 \times 10^{-2}}{2}$

$\underline{S}_{av}^- = -\hat{a}_y\, 0.6625 \times 10^{-2}$

$\boxed{4.5}$ $\quad \underline{H} = 10^{-6}[-\hat{a}_x(2+j) + \hat{a}_z(1+j3)]e^{+j\beta y}$

a. $\underline{E} = \eta_o \times 10^{-6}[\hat{a}_x(1+j3) + \hat{a}_z(2+j)]e^{+j\beta y}$

b. $\underline{S}_{av} = \frac{1}{2}\,\mathrm{Re}[\underline{E}\times\underline{H}^*] = \frac{\eta_o}{2}\times 10^{-12}\,\mathrm{Re}\left\{[\hat{a}_x(1+j3)+\hat{a}_z(2+j)]e^{j\beta y}\times[-\hat{a}_x(2-j)+\hat{a}_z(1-j3)]e^{-j\beta y}\right\}$

$= -\frac{15}{2}\eta_o \times 10^{-12}\,\hat{a}_y = -\hat{a}_y\, 2.8275\times 10^{-9}$

$\boxed{4.6}$ $\quad \underline{E} = 10^{-3}(\hat{a}_x + j\hat{a}_y)\sin(\beta_o z)$

a. Standing wave because of $\sin(\beta_o z)$ wave function.

b. $\underline{E} = 10^{-3}(\hat{a}_x + j\hat{a}_y)\sin(\beta_o z) = 10^{-3}(\hat{a}_x + j\hat{a}_y)\left(\frac{e^{j\beta_o z} - e^{-j\beta_o z}}{2j}\right) = \frac{10^{-3}}{2}(-j\hat{a}_x + \hat{a}_y)e^{+j\beta_o z} - \frac{10^{-3}}{2}(-j\hat{a}_x+\hat{a}_y)e^{-j\beta_o z}$

$= \underbrace{\frac{10^{-3}}{2}(-j\hat{a}_x + \hat{a}_y)\,e^{+j\beta_o z}}_{-z \text{ direction}} - \underbrace{\frac{10^{-3}}{2}(-j\hat{a}_x + \hat{a}_y)e^{-j\beta_o z}}_{+z \text{ direction}}$

c. $\underline{H} = -\frac{1}{j\omega\mu_o}\nabla\times\underline{E} = -\frac{1}{j\omega\mu_o}\left[-\hat{a}_x\frac{\partial E_y}{\partial z} + \hat{a}_y\frac{\partial E_x}{\partial z}\right] = 2.6525\times 10^{-6}(\hat{a}_x + j\hat{a}_y)\cos(\beta_o z) = \frac{10^{-3}}{377}(\hat{a}_x + j\hat{a}_y)\cos(\beta_o z)$

d. $\underline{S}_{av} = \frac{1}{2}\mathrm{Re}(\underline{E}\times\underline{H}^*) = \frac{1}{2}2.6525\times 10^{-9}\,\mathrm{Re}[(\hat{a}_x + j\hat{a}_y)\sin(\beta_o z)\times(\hat{a}_x + j\hat{a}_y)\cos(\beta_o z)] = 0$

$\boxed{4.7}$ $\quad \underline{H} = 10^{-6}(\hat{a}_y + j\hat{a}_z)\cos(\beta_o x)$

a. Standing wave because of the $\cos(\beta_o x)$ function.

b. $\underline{H} = 10^{-6}(\hat{a}_y + j\hat{a}_z)\cos(\beta_o x) = 10^{-6}(\hat{a}_y + j\hat{a}_z)\left(\frac{e^{j\beta_o x} + e^{-j\beta_o x}}{2}\right) = \frac{10^{-6}}{2}(\hat{a}_y + j\hat{a}_z)e^{j\beta_o x} + \frac{10^{-6}}{2}(\hat{a}_y + j\hat{a}_z)e^{-j\beta_o x}$

$= \underbrace{\frac{10^{-6}}{2}(\hat{a}_y + j\hat{a}_z)e^{+j\beta_o x}}_{-x \text{ direction}} + \underbrace{\frac{10^{-6}}{2}(\hat{a}_y + j\hat{a}_z)e^{-j\beta_o x}}_{+x \text{ direction}}$

c. $\underline{E} = \frac{1}{j\omega\varepsilon}\nabla\times\underline{H} = \frac{\beta\times 10^{-6}}{j\omega\varepsilon}[j\hat{a}_y \sin\beta_o x - \hat{a}_z \sin\beta_o x] = \{\eta[\hat{a}_y + j\hat{a}_z]\sin\beta_o x\}10^{-6}$

d. $\underline{S}_{av} = \frac{1}{2}\mathrm{Re}(\underline{E}\times\underline{H}^*) = \frac{10^{-12}}{2}\mathrm{Re}[\eta(\hat{a}_y + j\hat{a}_z)\sin\beta_o x \;\times(\hat{a}_y - j\hat{a}_z)\cos\beta_o x] = 0$

$\boxed{4.8}$ $\underline{E} = \hat{a}_x 4\times10^{-3} e^{+j\beta_0 z}$, $\beta_0 = \omega\sqrt{\mu_0\varepsilon_0} = 2\pi\times3\times10^8 \frac{1}{3\times10^8} = 2\pi$ rad/m

a. $\underline{H} = -\hat{a}_y \frac{4\times10^{-3}}{\eta_0} e^{+j\beta_0 z} = -\hat{a}_y 10.61\times10^{-6} e^{+j\beta_0 z}$

b. $\mathcal{E} = \mathrm{Re}\left[\underline{E} e^{j\omega t}\right] = \hat{a}_x 4\times10^{-3}\cos(\omega t + \beta_0 z)$

$\mathcal{H} = \mathrm{Re}\left[\underline{H} e^{j\omega t}\right] = -\hat{a}_y 10.61\times10^{-6}\cos(\omega t + \beta_0 z)$

c. $\underline{S}_{av} = \frac{1}{2}\mathrm{Re}\left(\underline{E}\times\underline{H}^*\right) = -\hat{a}_z 21.22\times10^{-9}$

$\underline{\mathcal{S}} = \underline{\mathcal{E}}\times\underline{\mathcal{H}} = -\hat{a}_z 42.44\times10^{-9}\cos^2(\omega t + \beta_0 z)$

d. $\bar{w}_e = \frac{1}{4}\varepsilon_0|E|^2 = \frac{1}{4}(8.854\times10^{-12})16\times10^{-6} = 35.416\times10^{-18}$ J/m^3

$\bar{w}_m = \frac{1}{4}\mu_0|H|^2 = \frac{1}{4}(4\pi\times10^{-7})(10.61\times10^{-6})^2 = 353.656\times10^{-19} = 35.3656\times10^{-18}$ J/m^3

$w_e = \frac{1}{2}\varepsilon_0\mathcal{E}^2 = \frac{1}{2}(8.854\times10^{-12})(16\times10^{-6})\cos^2(\omega t + \beta_0 z) = 70.832\times10^{-18}\cos^2(\omega t - \beta_0 z)$

$w_m = \frac{1}{2}\mu_0\mathcal{H}^2 = \frac{1}{2}(4\pi\times10^{-7})(10.61\times10^{-6})^2 = 70.731\times10^{-18}\cos^2(\omega t - \beta_0 z)$

$\boxed{4.9}$

a. $H_0 = \frac{E_0}{\eta_0} = \frac{1\times10^{-3}}{377} = 2.6525\times10^{-6}$ A/m

b. $\underline{S}_{av} = \frac{1}{2}\mathrm{Re}\left[\underline{E}\times\underline{H}^*\right] = \hat{a}_z\frac{|\underline{E}|^2}{2\eta_0} = \hat{a}_z\frac{(10^{-3})^2}{2(377)} = \hat{a}_z 1.3263\times10^{-9}$

c. $\left.\begin{aligned}\underline{\mathcal{E}} &= \mathrm{Re}\left[\underline{E}e^{j\omega t}\right] = \hat{a}_x E_0\cos(\omega t - \beta_0 z)\\ \underline{\mathcal{H}} &= \mathrm{Re}\left[\underline{H}e^{j\omega t}\right] = \hat{a}_y H_0\cos(\omega t - \beta_0 z)\end{aligned}\right\}\begin{aligned}w_e &= \frac{1}{2}\varepsilon_0\mathcal{E}^2 = 4.427\times10^{-18}\\ w_m &= \frac{1}{2}\mu_0\mathcal{H}^2 = \frac{1}{2}(4\pi\times10^{-7})(2.6525\times10^{-6})^2 = 4.4207\times10^{-18}\end{aligned}$

$\boxed{4.10}$ a. $\underline{E} = \hat{a}_y 10^{-3}e^{-j2\pi z} \Rightarrow \beta = \frac{\omega}{v} \Rightarrow v = \frac{\omega}{\beta} = \frac{2\pi\times10^8}{2\pi} = 10^8$ m/sec.

b. $v = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{1}{\sqrt{\mu_0\varepsilon_r\varepsilon_0}} = \frac{1}{\sqrt{\varepsilon_r}\sqrt{\mu_0\varepsilon_0}} = \frac{v_0}{\sqrt{\varepsilon_r}} \Rightarrow \varepsilon_r = \left(\frac{v_0}{v}\right)^2 = \left(\frac{3\times10^8}{10^8}\right)^2 = 9$

c. $\beta = \frac{2\pi}{\lambda} \Rightarrow \lambda = \frac{2\pi}{\beta} = \frac{2\pi}{2\pi} = 1$ meters

d. $\underline{E} = \hat{a}_y 10^{-3}e^{-j2\pi z}$, $\underline{H} = -\hat{a}_x\left[10^{-3}/\left(\frac{377}{3}\right)\right]e^{-j2\pi z} = -\hat{a}_x 7.9576\times10^{-6}e^{-j2\pi z}$

$\underline{S}_{av} = \frac{1}{2}\mathrm{Re}\left(\underline{E}\times\underline{H}^*\right) = \frac{\hat{a}_z}{2}\frac{|\underline{E}|^2}{\eta} = \frac{\hat{a}_z}{2}\frac{10^{-6}}{125.667} = \hat{a}_z 3.978\times10^{-9}$

e. $\left.\begin{aligned}\bar{w}_e &= \frac{1}{4}\varepsilon|\underline{E}|^2 = \frac{9}{4}(8.854\times10^{-12})(10^{-3})^2 = 19.9215\times10^{-18}\\ \bar{w}_m &= \frac{1}{4}\mu_0|\underline{H}|^2 = \frac{1}{4}(4\pi\times10^{-7})(7.9576\times10^{-6})^2 = 19.8936\times10^{-18}\end{aligned}\right\}\bar{w} = \bar{w}_e + w_m = 39.8151\times10^{-18}$

$\boxed{4.11}$ $\underline{E} = \hat{a}_z 10^{-3}(1+j)e^{-j(2/3)\pi x}$

a. $\beta = \frac{2\pi}{\lambda} = \frac{2\pi}{3} \Rightarrow \lambda = 3$ meters

b. $f = \frac{\upsilon}{\lambda} = \frac{3\times10^8}{3} = 10^8$ m/sec

c. $\underline{H} = -\hat{a}_y \frac{10^{-3}}{377}(1+j)e^{-j(2/3)\pi x} = -\hat{a}_y 2.6525\times10^{-6}(1+j)e^{-j(2/3)\pi x}$

$\boxed{4.12}$

a. $\upsilon = \frac{1}{\sqrt{\mu_o \varepsilon}} = \frac{1}{\sqrt{\mu_o \varepsilon_o \varepsilon_r}} = \frac{3\times10^8}{3} = 10^8$ m/sec.

b. $Z_o = \sqrt{\frac{\mu_o}{\varepsilon}} = \sqrt{\frac{\mu_o}{\varepsilon_o \varepsilon_r}} = \frac{377}{3} = 125.67$

c. $\eta_o = Z_o = \frac{377}{3} = 125.67$

d. $\lambda = \frac{\upsilon}{f} = \frac{10^8}{10^6} = 100$ meters

$\boxed{4.13}$ $P_r = 50\times10^{-3}$

a. $(S_o)_{av} = \frac{P_r}{4\pi R^2} = \frac{50\times10^{-3}}{4\pi(3\times10^3)^2} = \frac{50\times10^{-3}}{36\pi\times10^6} = 0.4421\times10^{-9}$ W/m²

b. $(S_o)_{av} = \frac{1}{2}\frac{|E_{peak}|^2}{\eta_o} \Rightarrow |E_{peak}| = \sqrt{2\eta_o (S_o)_{av}} = \sqrt{2(377)(4.421\times10^{-10})} = 5.7736\times10^{-4}$

$|E_{rms}| = \frac{1}{\sqrt{2}}|E_{peak}| = \frac{5.7736}{\sqrt{2}}\times10^{-4}$ V/m $= 4.0826\times10^{-4}$ V/m

$|H_{rms}| = \frac{|E_{rms}|}{377} = \frac{4.0826}{377}\times10^{-6}$ A/m $= 1.0829\times10^{-6}$ V/m

c. $\bar{w}_e = \frac{1}{4}\varepsilon_o|E_{peak}|^2 = \frac{1}{2}\varepsilon_o|E_{rms}|^2 = \frac{1}{2}(8.854\times10^{-12})|4.0826\times10^{-4}|^2 = 0.7379\times10^{-18}$ J/m³

$\bar{w}_m = \frac{1}{4}\mu_o|H_{peak}|^2 = \frac{1}{2}\mu_o|H_{rms}|^2 = \frac{1}{2}(4\pi\times10^{-7})|1.0829\times10^{-6}|^2 = 0.7368\times10^{-18}$ J/m³

$\bar{w} = \bar{w}_e + \bar{w}_m = 1.4744\times10^{-18}$ J/m³

$\boxed{4.14}$ $\underline{E} = \hat{a}_x 10^{-4}(1+j)e^{-j\beta_o z} \Rightarrow \underline{S}_{ave} = \frac{1}{2}\text{Re}(\underline{E}\times\underline{H}^*) = \hat{a}_z\frac{|\underline{E}|^2}{2\eta_o} = \hat{a}_z\frac{10^{-8}}{377} = 2.6525\times10^{-11}\hat{a}_z$

$P_{av} = \underline{S}_{av}\cdot\hat{a}_z A = 2.6525\times10^{-11}(20\times10^{-4}) = 53.05\times10^{-15} = 5.305\times10^{-14}$ W

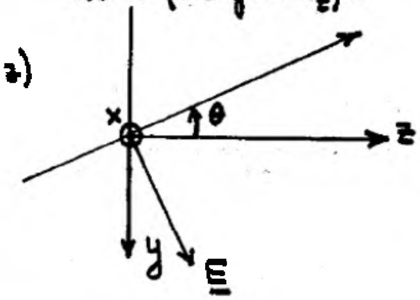
$\boxed{4.15}$ $\underline{E} = 5\times10^{-3}(4\hat{a}_y + 3\hat{a}_z)e^{j(6y-8z)}$

$$\underline{E} = 5\times10^{-3}(5)\left(\frac{4}{5}\hat{a}_y + \frac{3}{5}\hat{a}_z\right)e^{j10(0.6y-0.8z)} = 25\times10^{-3}(0.8\hat{a}_y + 0.6\hat{a}_z)e^{j10(0.6y-0.8z)}$$

a.
$$\underline{E} = 25\times10^{-3}(\cos\theta\,\hat{a}_y + \sin\theta\,\hat{a}_z)e^{j\beta_0(\sin\theta y - \cos\theta z)}$$

$$\cos\theta = 0.8 \Rightarrow \theta = \cos^{-1}(0.8) = 36.87^\circ$$

$$\sin\theta = 0.6 \Rightarrow \theta = \sin^{-1}(0.6) = 36.87^\circ$$



b.
$$\underline{E} = 25\times10^{-3}(\cos\theta\,\hat{a}_y + \sin\theta\,\hat{a}_z)e^{j(\beta_y y - \beta_z z)}$$

$\beta_r = 10$ rad/m

$\beta_y = 6$ rad/m

$\beta_z = 8$ rad/m

c. $\lambda_r = \frac{2\pi}{\beta_r} = \frac{2\pi}{10} = 0.6283$ m ; $\lambda_y = \frac{2\pi}{\beta_y} = \frac{2\pi}{6} = 1.047$ m ; $\lambda_z = \frac{2\pi}{\beta_z} = \frac{2\pi}{8} = 0.7854$ m

d. $v_{pr} = \frac{1}{\sqrt{\mu_0\varepsilon_0}} = 3\times10^8$ m/sec.

$$\cos\theta = \frac{v_r}{v_z} \Rightarrow v_z = \frac{v_r}{\cos\theta} = \frac{3\times10^8}{0.8} = 3.75\times10^8 \text{ m/sec.}$$

$$\cos\left(\frac{\pi}{2}-\theta\right) = \frac{v_r}{v_y} \Rightarrow v_y = \frac{v_r}{\cos(\frac{\pi}{2}-\theta)} = \frac{v_r}{\sin\theta} = \frac{3\times10^8}{0.6} = 5\times10^8 \text{ m/sec}$$

e. $v_{er} = 3\times10^8$ m/sec

$$v_{ez} = v\cos\theta = 3\times10^8(0.8) = 2.4\times10^8 \text{ m/sec.}$$

$$v_{ey} = v\sin\theta = 3\times10^8(0.6) = 1.8\times10^8 \text{ m/sec.}$$

f. $\beta_0 = \omega\sqrt{\mu_0\varepsilon_0} = 2\pi f\sqrt{\mu_0\varepsilon_0} = 10 \Rightarrow f = \frac{10}{2\pi\sqrt{\mu_0\varepsilon_0}} = \frac{15\times10^8}{\pi} = 4.7746\times10^8 = 0.47746\times10^9$

g. $\underline{H} = -\hat{a}_x\frac{25\times10^{-3}}{377}e^{j10(0.6y-0.8z)} = -\hat{a}_x 66.313\times10^{-6}e^{j(6y-8z)}$

$\boxed{4.16}$ $\underline{E} = \underline{E}^+ + \underline{E}^- = E_0^+(\hat{a}_x\cos\theta_i - \hat{a}_z\sin\theta_i)e^{-j\beta(x\sin\theta_i + z\cos\theta_i)} + E_0^-(\hat{a}_x\cos\theta_i - \hat{a}_z\sin\theta_i)e^{+j\beta(x\sin\theta_i + z\cos\theta_i)}$

$$\underline{H}^+ = -\frac{1}{j\omega\mu}\nabla\times\underline{E}^+ = -\frac{1}{j\omega\mu}\left[\hat{a}_x\cancelto{0}{\frac{\partial E_z^+}{\partial y}} + \hat{a}_y\left(\frac{\partial E_x^+}{\partial z} - \frac{\partial E_z^+}{\partial x}\right) + \hat{a}_z\cancelto{0}{\frac{\partial E_x^+}{\partial y}}\right] = \hat{a}_y\frac{1}{j\omega\mu}\left(-\frac{\partial E_x^+}{\partial z} + \frac{\partial E_z^+}{\partial x}\right)$$

$$\frac{\partial E_x^+}{\partial z} = -j\beta\cos^2\theta_i\,E_0^+ e^{-j\beta(x\sin\theta_i + z\cos\theta_i)}$$

$$\frac{\partial E_z^+}{\partial x} = +j\beta\sin^2\theta_i\,E_0^+ e^{-j\beta(x\sin\theta_i + z\cos\theta_i)}$$

$\underline{H}^+ = \hat{a}_y\frac{\beta E_0^+}{\omega\mu}(\cos^2\theta_i + \sin^2\theta_i)E_0^+e^{-j\beta(x\sin\theta_i + z\cos\theta_i)} = \hat{a}_y\frac{E_0}{\eta}E_0^+e^{-j\beta(x\sin\theta_i + z\cos\theta_i)}$. Similarly for $\underline{H}^-$.

$\boxed{4.17}$ $\underline{E} = \underline{E}^+ + \underline{E}^- = \hat{a}_y\left[E_0^+ e^{-j\beta(x\sin\theta_i + z\cos\theta_i)} + E_0^- e^{+j\beta(x\sin\theta_i + z\cos\theta_i)}\right]$

$$\underline{H}^+ = -\frac{1}{j\omega\mu}\nabla\times\underline{E}^+ = -\frac{1}{j\omega\mu}\left[\hat{a}_x\left(-\frac{\partial E_y^+}{\partial z}\right) + \hat{a}_z\frac{\partial E_y^+}{\partial x}\right]$$

$$\frac{\partial E_y^+}{\partial z} = -j\beta E_0^+ e^{-j\beta(x\sin\theta_i + z\cos\theta_i)}\cos\theta_i$$

$$\frac{\partial E_y^+}{\partial x} = -j\beta E_0^+ e^{-j\beta(x\sin\theta_i + z\cos\theta_i)}\sin\theta_i$$

$$\underline{H}^+ = \frac{\beta E_0^+}{\omega\mu}\left[-\hat{a}_x\cos\theta_i + \hat{a}_z\sin\theta_i\right]e^{-j\beta(x\sin\theta_i + z\cos\theta_i)} = \frac{E_0^+}{\eta}\left(-\hat{a}_x\cos\theta_i + \hat{a}_z\sin\theta_i\right)e^{-j\beta(x\sin\theta_i + z\cos\theta_i)}$$

Similarly for $\underline{H}^- = \frac{E_0^-}{\eta}\left(\hat{a}_x\cos\theta_i - \hat{a}_z\sin\theta_i\right)e^{+j\beta(x\sin\theta_i + z\cos\theta_i)}$

$\boxed{4.18}$ $\underline{\beta}^+\cdot\underline{r} - \omega t = \beta(x\sin\theta_i + z\cos\theta_i) - \omega t = C_0^+$

$$\frac{\partial}{\partial t}\left[\underline{\beta}^+\cdot\underline{r} - \omega t\right] = \beta\cos\theta_i\frac{\partial z}{\partial t} - \omega(1) = 0 \Rightarrow \frac{\partial z}{\partial t} = v_{pz} = \frac{\omega}{\beta\cos\theta_i} = \frac{v}{\cos\theta_i} = \frac{1}{\sqrt{\mu\varepsilon}\cos\theta_i}$$

$$\underline{\beta}^+\cdot\underline{r} - \omega t = \hat{a}_r\cdot\hat{a}_r r - \omega t = \beta r - \omega t = C_0^+$$

$$\frac{\partial}{\partial t}\left[\beta r - \omega t\right] = \beta\frac{\partial r}{\partial t} - \omega(1) = 0 \Rightarrow \frac{\partial r}{\partial t} = v_{pr} = \frac{\omega}{\beta} = \frac{1}{\sqrt{\mu\varepsilon}}$$

$\boxed{4.19}$ $\underline{E}^+ = E_0^+\left(\hat{a}_x\cos\theta_i - \hat{a}_z\sin\theta_i\right)e^{-j\beta(x\sin\theta_i + z\cos\theta_i)}$

$$\underline{H}^+ = \hat{a}_y\frac{E_0^+}{\eta}e^{-j\beta(x\sin\theta_i + z\cos\theta_i)}$$

$$\underline{\mathcal{E}}^+ = \mathrm{Re}\left[\underline{E}^+ e^{j\omega t}\right] = E_0^+\left(\hat{a}_x\cos\theta_i - \hat{a}_z\sin\theta_i\right)\cos\left[\omega t - \beta(x\sin\theta_i + z\cos\theta_i)\right]$$

$$\underline{\mathcal{H}}^+ = \mathrm{Re}\left[\underline{H}^+ e^{j\omega t}\right] = \hat{a}_y\frac{E_0^+}{\eta}\cos\left[\omega t - \beta(x\sin\theta_i + z\cos\theta_i)\right]$$

$$\underline{\mathcal{S}}^+ = \underline{\mathcal{E}}^+\times\underline{\mathcal{H}}^+ = \frac{(E_0^+)^2}{\eta}\left(\hat{a}_z\cos\theta_i + \hat{a}_x\sin\theta_i\right)\cos^2\left[\omega t - \beta(x\sin\theta_i + z\cos\theta_i)\right]$$

$$S_z^+ = \frac{\left[E_0^+\right]^2}{\eta}\cos^2\left[\omega t - \beta(x\sin\theta_i + z\cos\theta_i)\right]\cos\theta_i = \frac{(E_0^+)^2}{\eta}\cos\theta_i\cos^2\left[\omega t - \beta(x\sin\theta_i + z\cos\theta_i)\right]$$

$$w_e = \frac{1}{2}\varepsilon\left|\underline{\mathcal{E}}\right|^2 = \frac{(E_0)^2}{2}\varepsilon\cos^2\left[\omega t - \beta(x\sin\theta_i + z\cos\theta_i)\right]$$

$$w_m = \frac{1}{2}\mu\left|\underline{\mathcal{H}}\right|^2 = \frac{\mu}{2\eta^2}(E_0)^2\cos^2\left[\omega t - \beta(x\sin\theta_i + z\cos\theta_i)\right] = \frac{(E_0)^2}{2}\varepsilon\cos^2\left[\omega t - \beta(x\sin\theta_i + z\cos\theta_i)\right]$$

$$v_{ez}^+ = \frac{S_z^+}{w_e + w_m} = \frac{\left[(E_0^+)^2/\eta\right]\cos\theta_i}{\frac{(E_0^+)^2}{2}\varepsilon + \frac{(E_0^+)^2}{2}\varepsilon} = \frac{\cos\theta_i}{\eta\varepsilon} = \frac{1}{\sqrt{\mu\varepsilon}}\cos\theta_i = v\cos\theta_i \le v$$

4.20 $f = 3$ GHz, $\sigma = 5.76\times10^7$ S/m, $\varepsilon = \varepsilon_0, \mu = \mu_0$

a. Since $\frac{\sigma}{\omega\varepsilon_0} = \frac{5.76\times10^7}{2\pi(3\times10^9)\left(\frac{10^{-9}}{36\pi}\right)} = 6(5.76)\times10^7 \gg 1$

then $\eta \simeq \sqrt{\frac{\omega\mu}{2\sigma}}(1+j) = \sqrt{\frac{\omega\mu}{\sigma}}\angle 45^\circ = \sqrt{\frac{2\pi(3\times10^9)(4\pi\times10^{-7})}{2(5.76\times10^7)}}(1+j) = 2\pi\sqrt{\frac{30}{5.76}}\times10^{-3}(1+j)$

$\simeq 14.3393(1+j)\times10^{-3} = 14.3393\times10^{-3}(1+j)$

b. $\delta = \frac{1}{\alpha} \simeq \sqrt{\frac{2}{\omega\mu\sigma}} = \sqrt{\frac{2}{2\pi(3\times10^9)4\pi\times10^{-7}(5.76\times10^7)}} = \frac{1\times10^{-4}}{2\pi\sqrt{30(5.76)}} = 1.2107\times10^{-6}$ meters

4.21 $\underline{H} = (\hat{a}_y + j2\hat{a}_z)H_0 e^{-\alpha x}e^{-j\beta x}$, $H_0 = 1\ \mu$A/m, $\sigma = 10^{-4}$ S/m, $\varepsilon_r = 9$, $f = 10^9$ Hz

a. $\underline{E} = \eta H_0(j2\hat{a}_y - \hat{a}_z)e^{-\alpha x}e^{-j\beta x}$, $\frac{\sigma}{\omega\varepsilon} = \frac{10^{-4}}{2\pi\times10^9\,\frac{9\times10^{-9}}{36\pi}} = 2\times10^{-4} \ll 1$

$\eta \simeq \sqrt{\frac{\mu}{\varepsilon}} = \sqrt{\frac{\mu_0}{\varepsilon_r\varepsilon_0}} = \frac{377}{3} = 125.67$, $\alpha \simeq \frac{\sigma}{2}\sqrt{\frac{\mu_0}{\varepsilon}} = \frac{10^{-4}}{2}\sqrt{\frac{\mu_0}{9\varepsilon_0}} = \frac{377\times10^{-4}}{2(3)} = 62.83\times10^{-4}$ N/m

b. $\underline{S}_{av} = \frac{1}{2}\mathrm{Re}[\underline{E}\times\underline{H}^*] \simeq \frac{1}{2\eta}|\underline{E}|^2 \simeq \frac{\eta}{2}|H_0|^2 5 e^{-2\alpha x} = \hat{a}_x 314.167 e^{-2\alpha x}\times10^{-12}$

c. $\beta \simeq \omega\sqrt{\mu\varepsilon} = 2\pi\times10^9\frac{(3)}{3}\times10^{-8} = 2\pi\times10 = 62.83$ rad/m

d. $\beta = \frac{\omega}{v} \Rightarrow v = \frac{\omega}{\beta} \simeq \frac{1}{\sqrt{\mu\varepsilon}} = 10^8$ m/sec

e. $\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{20\pi} = \frac{1}{10} = 0.1$ meters

f. $\alpha \simeq \frac{\sigma}{2}\sqrt{\frac{\mu_0}{\varepsilon}} = \frac{10^{-4}}{2}\sqrt{\frac{\mu_0}{9\varepsilon_0}} = 62.83\times10^{-4}$ N/m

g. $\delta = \frac{1}{\alpha} \simeq \frac{1}{62.83\times10^{-4}} = 159.1596$ m

4.22 $\sigma = 4$ S/m, $\varepsilon_r = 81$, $\mu_r = 1$, $f = 10^4$ Hz

$\frac{\sigma}{\omega\varepsilon} = \frac{4}{2\pi\times10^4\,\frac{81\times10^{-9}}{36\pi}} = \frac{4(18)}{81}\times10^5 = \frac{8}{9}\times10^5 \gg 1$

a. $\alpha \simeq \beta \simeq \sqrt{\frac{\omega\mu\sigma}{2}} = \sqrt{\frac{2\pi\times10^4(4\pi\times10^{-7})4}{2}} = 4\pi\times10^{-2}\sqrt{10} = 0.39738 \Rightarrow \gamma = \alpha + j\beta = 0.39738(1+j)$

b. $\beta = \frac{\omega}{v} \Rightarrow v = \frac{\omega}{\beta} = \frac{2\pi\times10^4}{0.39738} = 15.811\times10^4 = 1.5811\times10^5 = \sqrt{\frac{2\omega}{\mu\sigma}}$

c. $\lambda = \frac{2\pi}{\beta} \simeq \frac{2\pi}{0.39738} = 15.811$ meters

d. $\alpha \simeq 0.39738$ N/m

e. $\delta = \frac{1}{\alpha} \simeq \frac{1}{0.39738} = 2.5165$ meters

$\boxed{4.23}$ $\sigma = 10^{-1}$ S/m, $\varepsilon_r = 4$, $\mu_r = 1$, $f = 10^6$ Hz

$$\frac{\sigma}{\omega\varepsilon} = \frac{10^{-1}}{2\pi\times 10^6\left(\frac{4\times 10^{-9}}{36\pi}\right)} = \frac{9}{2}\times 10^2 = 450 \gg 1$$

a. $\dfrac{1.104\times 10^{-2}}{3\times 10^{-2}} = 0.368$

$$\delta = \sqrt{\frac{2}{\omega\mu\sigma}} = \sqrt{\frac{2}{2\pi\times 10^6(4\pi\times 10^{-7})10^{-1}}} = \frac{1}{2\pi}\times 10 = 1.5915 \text{ m}$$

b. $\alpha = \dfrac{1}{\delta} = \dfrac{\pi}{5} = 0.6283$ N/m $= 0.6283(8.68)\dfrac{\text{dB}}{\text{m}} = 5.4538\dfrac{\text{dB}}{\text{m}}$

Total attenuation $= \alpha\delta = 1 = 8.68$ dB

c. $\lambda = \dfrac{2\pi}{\beta} = \dfrac{2\pi}{\alpha} = \dfrac{2\pi}{0.6283} = \dfrac{2\pi}{\pi/5} = 10$ meters, since $\beta \simeq \alpha$ when $\dfrac{\sigma}{\omega\varepsilon} \gg 1$

d. $v = \dfrac{\omega}{\beta} = \lambda f = 10\times 10^6 = 10^7$ m/sec.

e. $\eta \simeq \sqrt{\dfrac{\omega\mu}{2\sigma}}(1+j) = \sqrt{\dfrac{2\pi\times 10^6(4\pi\times 10^{-7})}{2\times 10^{-1}}}(1+j) = 2\pi(1+j) = 6.283(1+j)$

$\boxed{4.24}$ $\underline{E} = 10^{-2}\left[\hat{a}_x\sqrt{2} + \hat{a}_z(1+j)e^{j\pi/4}\right]e^{-j\beta y} = 10^{-2}\left(\hat{a}_x\sqrt{2} + \hat{a}_z\sqrt{2}\,e^{j\pi/2}\right)e^{-j\beta y}$

a. Circular polarization.

b. Clockwise because the z component leads the x component.

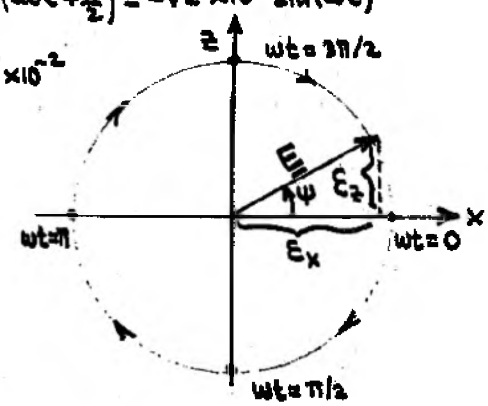
c. $\mathcal{E}_x = \sqrt{2}\times 10^{-2}\cos(\omega t - \beta y) \overset{y=0}{\Longrightarrow} \mathcal{E}_x = \sqrt{2}\times 10^{-2}\cos(\omega t)$

$\mathcal{E}_z = \sqrt{2}\times 10^{-2}\cos(\omega t + \frac{\pi}{2} - \beta y) \overset{y=0}{\Longrightarrow} \mathcal{E}_z = \sqrt{2}\times 10^{-2}\cos(\omega t + \frac{\pi}{2}) = -\sqrt{2}\times 10^{-2}\sin(\omega t)$

$|\mathcal{E}| = \sqrt{\mathcal{E}_x^2 + \mathcal{E}_z^2} = \sqrt{2}\times 10^{-2}\sqrt{\cos^2\omega t + \sin^2\omega t} = \sqrt{2}\times 10^{-2}$

$\psi = \tan^{-1}\left(\dfrac{\mathcal{E}_z}{\mathcal{E}_x}\right) = \tan^{-1}\left[\dfrac{-\sin\omega t}{\cos\omega t}\right]$

$\psi = \tan^{-1}(-\tan\omega t) = -\omega t$



$\boxed{4.25}$ $\underline{H} = \dfrac{10^{-3}}{120\pi}(\hat{a}_x - j\hat{a}_z)e^{+j\beta y}$

a. Circular polarization

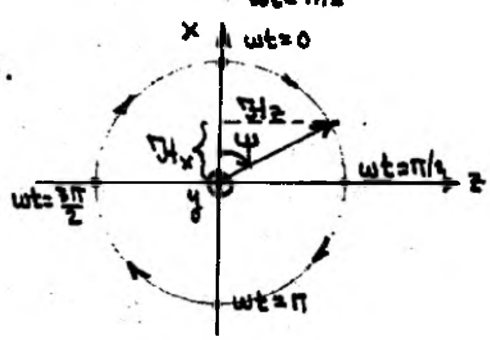
b. Clockwise because the x component leads the z.

c. $\mathcal{H}_x = \dfrac{10^{-3}}{120\pi}\cos(\omega t + \beta y) \overset{y=0}{\Longrightarrow} \mathcal{H}_x = \dfrac{10^{-3}}{120\pi}\cos\omega t$

$\mathcal{H}_z = \dfrac{10^{-3}}{120\pi}\cos(\omega t - \frac{\pi}{2} + \beta y) \overset{y=0}{\Longrightarrow} \mathcal{H}_z = \dfrac{10^{-3}}{120\pi}\sin\omega t$

$|\mathcal{H}| = \sqrt{\mathcal{H}_x^2 + \mathcal{H}_y^2} = 10^{-3}/120\pi$

$\psi = \tan^{-1}(\mathcal{H}_z/\mathcal{H}_x) = \tan^{-1}(\tan\omega t) = \omega t$



4.26 $\underline{H} = \left[\hat{a}_x(1+j) + \hat{a}_z\sqrt{2}\, e^{j\pi/4}\right]\frac{E_0}{\eta_0}e^{-j\beta_0 y} = \left[\hat{a}_x\sqrt{2}e^{j\pi/4} + \hat{a}_z\sqrt{2}\, e^{j\pi/4}\right]\frac{E_0}{\eta_0}e^{-j\beta_0 y}$

(a) Linear polarization because two components of the same time phase.

(b) No rotation.

(c) $\underline{E} = \sqrt{2}e^{j\pi/4}(-\hat{a}_x+\hat{a}_z)E_0 e^{-j\beta_0 y} = \sqrt{2}(-\hat{a}_x+\hat{a}_z)E_0 e^{j(\frac{\pi}{4}-\beta y)}$

4.27 Let $\underline{E} = \hat{a}_x E_0 e^{-j\beta z}$ which is linearly polarized. Then

$$\underline{E} = \hat{a}_x E_0 e^{-j\beta z} = \left[\hat{a}_x \frac{E_0}{2}e^{-j\beta z} + j\hat{a}_y\frac{E_0}{2}e^{-j\beta z}\right] + \left[\hat{a}_x\frac{E_0}{2}e^{-j\beta z} - j\hat{a}_y\frac{E_0}{2}e^{-j\beta z}\right]$$

$$= \underbrace{\frac{E_0}{2}(\hat{a}_x + j\hat{a}_y)e^{-j\beta z}}_{\text{CIRCULAR: CCW}} + \underbrace{\frac{E_0}{2}(\hat{a}_x - j\hat{a}_y)e^{-j\beta z}}_{\text{CIRCULAR: CW}}$$

The same can be shown for any other linearly polarized field.

4.28

$$\underline{E} = (\hat{a}_x + j2\hat{a}_y)e^{-j600\pi z} = (\hat{a}_x + j2\hat{a}_y)e^{-j\beta z}$$

(a) $\beta = \frac{2\pi}{\lambda} = 600\pi \Rightarrow \lambda = \frac{2\pi}{600\pi} = \frac{1}{300} = \boxed{3.333\times10^{-3} \text{ meters}}$

(b) $v = \lambda f = \frac{1}{300}(10\times10^{9}) = \boxed{3.333\times10^{7} \text{ m/sec.}}$

(c) $\beta = \omega\sqrt{\mu\varepsilon} = 2\pi f\sqrt{\mu_0\varepsilon_r\varepsilon_0} = 2\pi f\sqrt{\mu_0\varepsilon_0}\sqrt{\varepsilon_r} = \frac{2\pi f}{v_0}\sqrt{\varepsilon_r} = \beta$

$\sqrt{\varepsilon_r} = \frac{\beta v_0}{2\pi f} = \frac{600\pi(3\times10^{8})}{2\pi(10\times10^{9})} = 9$

$\boxed{\varepsilon_r = (9)^2 = 81}$

(d) $\eta = \sqrt{\frac{\mu}{\varepsilon}} = \sqrt{\frac{\mu_0}{\varepsilon_r\varepsilon_0}} = \frac{1}{\sqrt{\varepsilon_r}}\sqrt{\frac{\mu_0}{\varepsilon_0}} = \frac{377}{\sqrt{81}} = \frac{377}{9} = \boxed{41.89}$

(e) For a uniform plane wave

$$Z_w = \eta = \sqrt{\frac{\mu}{\varepsilon}} = \boxed{41.89}$$

cont'd

$\boxed{4.28 \text{ cont'd}}$

(f) $\underline{H} = \frac{1}{\eta}(\hat{a}_y - j2\hat{a}_x)e^{-j\overbrace{600\pi}^{\beta} z} = \frac{1}{41.89}(-j2\hat{a}_x + \hat{a}_y)e^{-j\overbrace{600\pi}^{\beta} z}$

$$\underline{H} = (-j\,47.74\hat{a}_x + 23.87\hat{a}_y)e^{-j600\pi z}$$ (Using Right Rule between $\underline{E}$ and $\underline{H}$ fields)

Alternate: You can also use Maxwell's equation to find the $\underline{H}$-field.

$$\underline{H} = +j\frac{1}{\omega\mu}\nabla\times\underline{E} = j\frac{1}{\omega\mu}\begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & 0 \end{vmatrix} = j\frac{1}{\omega\mu}\left[\hat{a}_x\left(-\frac{\partial E_y}{\partial z}\right) + \hat{a}_y\left(\frac{\partial E_x}{\partial z}\right) + \hat{a}_z(0)\right]$$

$$= j\frac{1}{\omega\mu}\left[\hat{a}_x\left(-\frac{\partial E_y}{\partial z}\right) + \hat{a}_y\left(\frac{\partial E_x}{\partial z}\right)\right] = j\frac{1}{\omega\mu}\left[-\hat{a}_x(2j)(-j\overbrace{600\pi}^{\beta}) + \hat{a}_y(-j\underbrace{600\pi}_{\beta})\right]e^{-j600\pi z}$$

$$\underline{H} = j\frac{(-j\beta)}{\omega\mu}\left[-\hat{a}_x(2j) + \hat{a}_y\right]e^{-j600\pi z} = \sqrt{\frac{\varepsilon}{\mu}}(-j2\hat{a}_x + \hat{a}_y)e^{-j600\pi}$$

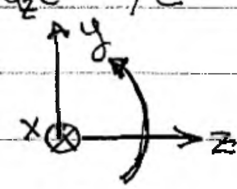
$$\underline{H} = \frac{1}{\eta}(-j2\hat{a}_x + \hat{a}_y)e^{-j600\pi} = (-j\,47.74\hat{a}_x + 23.87\hat{a}_y)e^{-j600\pi z}$$

which is the same as above (and as it should be).

$\boxed{4.29}$ $\underline{E} = \hat{a}_y e^{-j(\beta_o x - \pi/4)} + \hat{a}_z e^{-j(\beta_o x - \pi/2)}$

$\underline{E} = \hat{a}_y e^{+j\pi/4}e^{-j\beta_o x} + \hat{a}_z e^{+j\pi/2}e^{-j\beta_o x} = (\hat{a}_y e^{+j\pi/4} + \hat{a}_z e^{+j\pi/2})e^{-j\beta_o x}$

(a) Wave is traveling in the $+x$ direction



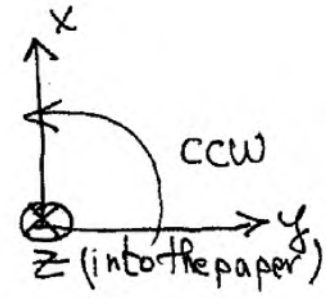
(b) Polarization is elliptical because:

- 2 orthogonal components not of the same magnitude
- Phase difference is 45°.

(c) The rotation is CCW (LH) because:

- z component is leading the y component in phase.
- Rotate the leading component (z component) toward the lagging component (y component) thru smaller angular rotation; thus CCW (LH).

$\boxed{4.30}$



$$\underline{E} = \hat{a}_x 2 e^{-j(\beta_0 z - \frac{\pi}{4})} + \hat{a}_y e^{-j(\beta_0 z - \frac{3\pi}{4})}$$

$$\underline{E} = \hat{a}_x 2 e^{+j\frac{\pi}{4}} e^{-j\beta_0 z} + \hat{a}_y e^{+j\frac{3\pi}{4}} e^{-j\beta_0 z}$$

$$|E_{x0}^+| = 2, \ |E_{y0}^+| = 1, \ \phi_x = \frac{\pi}{4}, \ \phi_y = \frac{3\pi}{4}$$

Wave travels in $+z$ direction

Using (4-50a) as a reference

$$\underline{\mathcal{E}} = \hat{a}_x E_x^+ e^{j(\omega t - \beta z)} + \hat{a}_y E_y^+ e^{j(\omega t - \beta z)}$$

$$\underline{\mathcal{E}} = \left[\hat{a}_x |E_{x0}^+| e^{j\phi_x} e^{-j\beta z} + \hat{a}_y |E_{y0}^+| e^{j\phi_y} e^{-j\beta z}\right] e^{j\omega t}$$

$$\underline{E} = \left[\hat{a}_x |E_{x0}^+| e^{j\phi_x} + \hat{a}_y |E_{y0}^+| e^{j\phi_y}\right] e^{-j\beta z}$$

And using (4-58a)

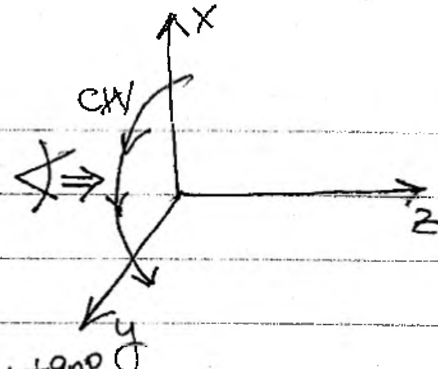
$$\left\{\begin{array}{l} \gamma = \tan^{-1}\left(\frac{|E_{y0}^+|}{|E_{x0}^+|}\right) = \tan^{-1}\left(\frac{1}{2}\right) = 26.565^\circ \\ \delta = \phi_y - \phi_x = \frac{3\pi}{4} - \frac{\pi}{4} = \frac{\pi}{2} = 90^\circ \end{array}\right\} \begin{array}{l} \gamma = 26.565^\circ \\ \delta = 90^\circ \end{array}$$

$$\left.\begin{array}{l} 2\varepsilon = \sin^{-1}\left[\sin(2\gamma)\sin(\delta)\right] = \sin^{-1}\left[\sin(2\gamma)(1)\right] = 2\gamma \\ \varepsilon = \gamma = 26.565^\circ \\ 2\tau = \tan^{-1}\left[\tan(2\gamma)\cos\delta\right] = \tan^{-1}(0) = 0^\circ \\ \tau = 0^\circ \end{array}\right\} \begin{array}{l} \varepsilon_r = 26.565^\circ \\ \tau = 0^\circ \end{array}$$

- $2\gamma = 53.13^\circ$, $\delta = 90^\circ \Rightarrow$ elliptical (along principal xy-plane)
- $2\varepsilon = 2\gamma = 53.13^\circ$, $\tau = 0^\circ \Rightarrow$ elliptical (along principal xy-plane)
- <u>CCW (LH)</u> because <u>upper</u> hemisphere of Poincaré Sphere.
- $AR = \cot(\varepsilon) = \frac{1}{\tan\varepsilon} = \frac{1}{0.5} = 2$

4.31

$E(z) = (j2\hat{a}_x + 5\hat{a}_y)E_o e^{-j\beta z}$

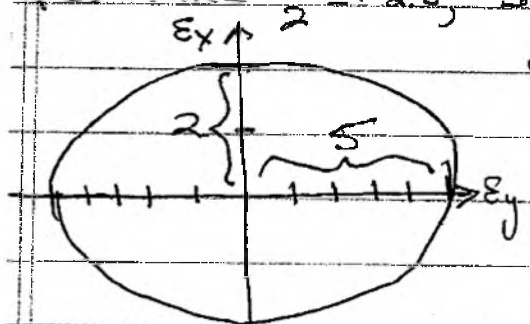


(a) Elliptical because

- 2 components, not of same magnitude and 90° time-phase difference

(b) CW. x-component leads y-component by $+90^\circ$. Rotate x-component towards y-component.

(c) $AR = -\frac{5}{2} = -2.5$. Since it is elliptical with 90° phase difference between the 2 components, the major and minor axes of the ellipse align with the principal axes.



(d) $\gamma = \tan^{-1}\left(\frac{E_{yo}}{E_{xo}}\right) = \tan^{-1}\left(\frac{5}{2}\right) = 68.2^\circ$ or $\gamma = \tan^{-1}\left(\frac{E_{xo}}{E_{yo}}\right) = \tan^{-1}\left(\frac{2}{5}\right)$

$\gamma = 21.8^\circ$

$\delta = \phi_y - \phi_x = 0 - 90 = -90^\circ$

Option I

$\gamma = 68.2^\circ, \delta = -90^\circ$

$2\varepsilon = \sin^{-1}[\sin(2\gamma)\sin\delta] = \sin^{-1}[\sin(2\gamma)(-1)]$

$= \sin^{-1}[-\sin 2\gamma] = -\sin^{-1}(\sin 2\gamma)$

$2\varepsilon = -136.4 \Rightarrow \boxed{\varepsilon = -68.2^\circ}$

$2\tau = \tan^{-1}[\tan 2\gamma \cos\delta] = \tan^{-1}(\tan 2\gamma \cdot 0)$

$2\tau = \tan^{-1}(0) = 0 \Rightarrow \boxed{\tau = 0}$

Option II

$\gamma = 21.8^\circ, \delta = -90^\circ$

$2\varepsilon = \sin^{-1}[\sin(2\gamma)\sin\delta] = \sin^{-1}[\sin 2\gamma(-1)]$

$= \sin^{-1}(-\sin 2\gamma) = -\sin^{-1}(\sin 2\gamma)$

$2\varepsilon = -2\gamma = -43.6^\circ \Rightarrow \boxed{\varepsilon = -21.8^\circ}$

$2\tau = \tan^{-1}[\tan 2\gamma \cos\delta] = \tan^{-1}[\tan 2\gamma(0)]$

$2\tau = \tan^{-1}(0) = 0 \Rightarrow \boxed{\tau = 0}$

X

$\gamma = 68.2^\circ, \delta = -90^\circ$

$\varepsilon = -68.2^\circ, \tau = 0^\circ$

$AR = \cot(\varepsilon) = \cot(-68.2)$

$AR = -0.4$

X

$\gamma = 21.8^\circ, \delta = -90^\circ$

$\varepsilon = -21.8^\circ, \tau = 0^\circ$

$AR = \cot(\varepsilon) = \cot(-21.8^\circ)$

$AR = -2.5$

Cont'd.

431 cont'd Option III

$$\gamma = 68.2^\circ,\ \delta = -90^\circ$$

$$2\varepsilon = \sin^{-1}[\sin(2\gamma)\sin(\delta)] = \sin^{-1}(-\sin 2\gamma) = \begin{cases} \sin^{-1}\left[\dfrac{-\sin(2\gamma)}{1}\right] \\ \sin^{-1}\left[\dfrac{\sin(2\gamma)}{-1}\right] \end{cases}$$

Taking the first one:

$$2\varepsilon = \sin^{-1}\left[\frac{-\sin(2\gamma)}{1}\right]_{\gamma=68.2^\circ} = \sin^{-1}\left[\frac{-0.690}{1}\right] = -43.6^\circ \text{ or } 223.6^\circ$$

$$\varepsilon = -21.8^\circ,\ \text{or } 111.8^\circ$$

Since $-45^\circ \le \varepsilon \le +45^\circ \Rightarrow \boxed{\varepsilon = -21.8^\circ}$ *

$$2\tau = \tan^{-1}[\tan(2\gamma)\cos\delta]_{\delta=-90^\circ} = \tan^{-1}(0) = 0^\circ \text{ or } 180^\circ$$

$$\Rightarrow \tau = 0^\circ \text{ or } 90^\circ$$

$\gamma = 68.2^\circ,\ \delta = -90^\circ,\ \varepsilon = -21.8^\circ,\ \tau = 0^\circ$
$\gamma = 68.2,\ \delta = -90^\circ,\ \varepsilon = -21.8^\circ,\ \tau = 90^\circ$

Taking the second one:

$$2\varepsilon = \sin^{-1}\left[\frac{\sin(2\gamma)}{-1}\right]$$

This is not valid because it indicates a negative hypotenuse!

So, In Summary, the options are:

Options		
1st	X	$\gamma = 68.2^\circ,\ \delta = -90^\circ,\ \varepsilon = 68.2^\circ,\ \tau = 0^\circ \Rightarrow AR = -0.4$
2nd	X	$\gamma = 21.8^\circ,\ \delta = -90^\circ,\ \varepsilon = -21.8^\circ,\ \tau = 0^\circ \Rightarrow AR = -2.5$
3rd	X	$\gamma = 68.2^\circ,\ \delta = -90^\circ,\ \varepsilon = -21.8^\circ\ \tau = 0^\circ \Rightarrow AR = -2.5$
4th	✓	$\gamma = 68.2^\circ,\ \delta = -90^\circ\ \varepsilon = -21.8^\circ,\ \tau = 90^\circ \Rightarrow AR = -2.5$

(e) $AR = \cot(\varepsilon) = \cot(-21.8^\circ) = -2.5$ $\boxed{AR = -2.5}$

** Using Figure 4-19, τ is the tilt angle from the x-component toward the major axis. For our expression, $E = (j2\hat{a}_x + 5\hat{a}_y)e^{-j\beta z}$, the y-component represents the major axis. Therefore, according to Figure 4-19, $\tau = 90^\circ$. Therefore the 4th option is the correct one since it gives the correct AR and tilt angle τ.

4.32 $\underline{H} = j(\hat{a}_y - j\hat{a}_z)\frac{E_o}{\eta_o}e^{+j\beta_o x}$

a. Circular polarization because of two equal components and 90° time phase difference.

b. CCW because the y component leads the z component.

c. $\underline{E} = E_o(-\hat{a}_y + j\hat{a}_z)e^{+j\beta_o x}$

$$\underline{S}_{av} = \frac{1}{2}Re(\underline{E}\times\underline{H}^*) = -\hat{a}_x\frac{1}{2\eta_o}|\underline{E}|^2 = -\hat{a}_x\frac{2|E_o|^2}{2(377)} = -\hat{a}_x\, 2.6525\times10^{-3}$$

d. $\gamma = \tan^{-1}\left[\frac{1}{1}\right] = \tan^{-1}(1) = \pi/4 \Rightarrow 2\gamma = 90°$
$\delta = \phi_y - \phi_z = -180° - 90° = -270° = +90°$ } North pole

4.33 $\underline{E} = 2\times10^{-3}(\hat{a}_x + \hat{a}_y)e^{-j2z}$

a. Linear polarization because of 2 components with no time phase difference.

b. $\gamma = \tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4}$, $\delta = \phi_y - \phi_x = 0°$

$2\epsilon = \sin^{-1}[\sin(2\gamma)\sin\delta] = \sin^{-1}(0) = 0°$

$2\tau = \tan^{-1}[\tan(2\gamma)\cos\delta] = \tan^{-1}[\infty(1)] = 90° \Rightarrow \tau = 45°$

c. The polarization point is located on the equator at the y axis. It represents linear polarization at a tilt angle of 45°.

4.34 $\underline{E} = E_o(\hat{a}_x - j2\hat{a}_y)e^{-j\beta z}$, $AR = -2$

a. $\delta = \phi_y - \phi_x = -\frac{\pi}{2} - 0 = -90°$

$\epsilon = \cot^{-1}(AR) = \cot^{-1}(-2) = -26.56°$

$\gamma = \frac{1}{2}\sin^{-1}\left[\frac{\sin(2\epsilon)}{\sin\delta}\right] = \frac{1}{2}\sin^{-1}\left[\frac{\sin(-53.13°)}{\sin(-90°)}\right] = 26.56°$

$\tau = \frac{1}{2}\tan^{-1}[\tan(2\gamma)\cos\delta] = \frac{1}{2}[\tan(53.13°)\cos(-90°)] = 0°$

b. Elliptical polarization, CW with an axial ratio of −2.

c. $2\gamma = 53.13°$

Angle between $P_w P_a = 90° - 53.13° = 36.8699°$

$V = \cos\left(\frac{P_w P_a}{2}\right) = \cos\left(\frac{36.8699}{2}\right) = 0.9487 = 20\log_{10}(0.9487) = -0.4575$ dB

4.35 $AR = \infty$, $2\gamma = 109.47° \Rightarrow \gamma = 54.735°$

a. $\gamma = \tan^{-1}\left(\frac{E_{yo}}{E_{xo}}\right) = 54.735° \Rightarrow \frac{E_{yo}}{E_{xo}} = \tan(54.735°) = \sqrt{2} \Rightarrow E_{yo} = \sqrt{2}E_{xo}$

E_y is more dominant ($E_y = \sqrt{2}E_x$)

b. $\epsilon = \cot^{-1}(\infty) = 0°$

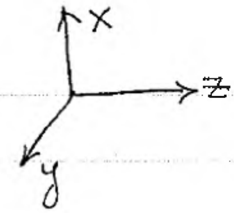
$\cos(2\gamma) = \cos(2\epsilon)\cos(2\tau)\Big|_{\epsilon=0} = \cos(2\tau) \Rightarrow 2\tau = 2\gamma \Rightarrow \tau = 54.735°$

$\tan\delta = \frac{\tan(2\epsilon)}{\sin(2\tau)}\Big|_{\epsilon=0,\ \tau=54.735°} = \frac{0}{0.9428} = 0 \Rightarrow \delta = 0°$

} $\gamma = 54.735°$, $\epsilon = 0°$, $\tau = 54.735°$, $\delta = 0°$

c. Linear because $\delta = 0°$; also because $AR = \infty$

$\boxed{4.36}$ $E_w = (\hat{a}_x + j\hat{a}_y)e^{-j\beta z}E_0$



(a) $E_{aa} = (\hat{a}_x + j\hat{a}_y)e^{+j\beta z}E_0$

(b) $E_{ab} = (\hat{a}_x - j\hat{a}_y)e^{+j\beta z}E_0$

1. Wave: Circular, CCW, AR=1 (North pole: $\delta=90°$, $2\gamma=90°$)
2. Antenna(aa): Circular, CW, AR=−1 (South pole: $\delta=-90°$, $2\gamma=90°$)
3. Antenna(ab): Circular, CCW, AR=+1 (North pole: $\delta=90°$, $2\gamma=90°$)
4. <u>w:aa</u>

$$V = \cos\left(\frac{P_wP_a}{2}\right) = \cos\left(\frac{180°}{2}\right) = \cos(90°) = 0 = -\infty \text{ dB}$$

5. <u>w:ab</u>

$$V = \cos\left(\frac{P_wP_a}{2}\right) = \cos\left(\frac{0°}{2}\right) = \cos(0°) = 1 = 0 \text{ dB}$$

$\boxed{437}$ $E_{x0}^{+} = E_{y0}^{+}$, AR $= \infty$

a.

$$\gamma = \tan^{-1}\left(\frac{E_{y0}^{+}}{E_{x0}^{+}}\right) = 45°$$

$$\varepsilon = \cot^{-1}(\infty) = 0°$$

$$\cos(2\gamma) = \cos(2\varepsilon)\cos(2\tau)\Big|_{\varepsilon=0} = \cos(2\tau) \Rightarrow 2\gamma = 2\tau \Rightarrow \tau = \gamma = 45°$$

$$\tan\delta = \frac{\tan(2\varepsilon)}{\sin(2\tau)}\Bigg|_{\varepsilon=0°,\ \tau=45°} = 0 \Rightarrow \delta = 0°$$

b.

$$V = \cos\left[\frac{2\gamma - 2\gamma_a}{2}\right] = \cos(\gamma - \gamma_a) = \cos(54.735° - 45°) = \cos(9.735°) = 0.9856$$

$$= 20\log_{10}(0.9856) = -0.1259 \text{ dB}$$

Chapter 5

5.1 $\underline{E}^i = \hat{a}_y\, 2\times10^{-3}\, e^{-j\beta z}$

a. $\underline{H}^i = -\hat{a}_x \dfrac{2\times10^{-3}}{\eta} e^{-j\beta z} = -\hat{a}_x \dfrac{2\times10^{-3}}{376.73/2} e^{-j\beta z} = -\hat{a}_x \dfrac{2\times10^{-3}}{188.37} e^{-j\beta z} = -\hat{a}_x\, 1.06\times10^{-5} e^{-j\beta z}$

b. $\eta_2 = \eta_0 = \sqrt{\dfrac{\mu_0}{\varepsilon_0}} = 376.73$, $\eta = \sqrt{\dfrac{\mu_0}{\varepsilon}} = \sqrt{\dfrac{\mu_0}{4\varepsilon_0}} = \dfrac{376.73}{2} = 188.37$

$$\Gamma^b = \frac{\eta_0 - \eta}{\eta_0 + \eta} = \frac{\sqrt{\frac{\mu_0}{\varepsilon_0}} - \frac{1}{2}\sqrt{\frac{\mu_0}{\varepsilon_0}}}{\sqrt{\frac{\mu_0}{\varepsilon_0}} + \frac{1}{2}\sqrt{\frac{\mu_0}{\varepsilon_0}}} = \frac{2-1}{2+1} = \frac{1}{3}$$

$$T^b = \frac{2\eta_0}{\eta_0 + \eta} = \frac{2\sqrt{\mu_0/\varepsilon_0}}{\sqrt{\frac{\mu_0}{\varepsilon_0}} + \frac{1}{2}\sqrt{\frac{\mu_0}{\varepsilon_0}}} = \frac{2}{\frac{3}{2}} = \frac{4}{3} = 1.333$$

c. $\underline{E}^r = \hat{a}_y \Gamma^b E_0 e^{+j\beta z} = \hat{a}_y \dfrac{2\times10^{-3}}{3} e^{+j\beta z} = \hat{a}_y\, 0.667\times10^{-3} e^{j\beta z}$

$\underline{H}^r = \hat{a}_x \dfrac{\Gamma^b E_0}{\eta} e^{+j\beta z} = \hat{a}_x \dfrac{2\times10^{-3}}{3(188.37)} e^{+j\beta z} = \hat{a}_x\, 3.54\times10^{-6} e^{j\beta z}$

$\underline{E}^t = \hat{a}_y T^b E_0 e^{-j\beta_0 z} = \hat{a}_y \dfrac{4}{3}(2\times10^{-3}) e^{-j\beta_0 z} = \hat{a}_y\, 2.667\times10^{-3} e^{-j\beta_0 z}$

$\underline{H}^t = -\hat{a}_x \dfrac{T^b E_0 e^{-j\beta_0 z}}{\eta_0} = -\hat{a}_x \dfrac{2.667\times10^{-3}}{376.73} e^{-j\beta_0 z} = -\hat{a}_x\, 7.08\times10^{-6} e^{-j\beta_0 z}$

d. $\underline{S}^i_{av} = \frac{1}{2}\mathrm{Re}(\underline{E}^i\times\underline{H}^{i*}) = \hat{a}_z \dfrac{|E^i|^2}{2\eta} = \hat{a}_z \dfrac{|2\times10^{-3}|^2}{2(188.37)} = \hat{a}_z\, 1.06\times10^{-8}$ W/m²

$\underline{S}^r_{av} = \frac{1}{2}\mathrm{Re}(\underline{E}^r\times\underline{H}^{r*}) = -\hat{a}_z|\Gamma^b|^2 S^i_{av} = -\hat{a}_z\left(\frac{1}{3}\right)^2(1.06\times10^{-8}) = -\hat{a}_z\, 1.18\times10^{-9}$ W/m²

$\underline{S}^t_{av} = \frac{1}{2}\mathrm{Re}(\underline{E}^t\times\underline{H}^{t*}) = \hat{a}_z(S^i - S^r_{av}) = \hat{a}_z(1-|\Gamma^b|^2)S^i_{av} = \hat{a}_z|T^b|^2\dfrac{\eta}{\eta_0}S^i_{av}$

$= \hat{a}_z\, 9.44\times10^{-9}$ W/m²

5.2 $$\Gamma^b = \frac{\eta_1 - \eta_0}{\eta_1 + \eta_0} = \frac{\sqrt{\frac{\mu_0}{81\varepsilon_0}} - \sqrt{\frac{\mu_0}{\varepsilon_0}}}{\sqrt{\frac{\mu_0}{81\varepsilon_0}} + \sqrt{\frac{\mu_0}{\varepsilon_0}}} = \frac{1-9}{1+9} = \frac{-8}{10} = -0.8$$

$S^{ref}_{av} = |\Gamma^b|^2 S^i_{av}$

$\dfrac{S^{ref}_{av}}{S^i_{av}} = |\Gamma^b|^2 = |-0.8|^2 = 0.64$ or 64 %

$S^t_{av} = (1-|\Gamma^b|^2)S^i_{av}$

$\dfrac{S^t_{av}}{S^i_{av}} = (1-|\Gamma^b|^2) = 1-0.64 = 0.36$ or 36%

$\boxed{5.3}$ a. $\Gamma^b = \dfrac{\eta_2-\eta_1}{\eta_2+\eta_1} = \dfrac{\frac{1}{3}\sqrt{\mu_0/\varepsilon_0} - \frac{1}{2}\sqrt{\mu_0/\varepsilon_0}}{\frac{1}{3}\sqrt{\mu_0/\varepsilon_0} - \frac{1}{2}\sqrt{\mu_0/\varepsilon_0}} = \dfrac{\frac{1}{3}-\frac{1}{2}}{\frac{1}{3}+\frac{1}{2}} = \dfrac{2-3}{2+3} = -\dfrac{1}{5} = -0.2$

$$T^b = \frac{2\eta_2}{\eta_2+\eta_1} = \frac{\frac{2}{3}\sqrt{\mu_0/\varepsilon_0}}{\frac{1}{3}\sqrt{\frac{\mu_0}{\varepsilon_0}}+\frac{1}{2}\sqrt{\frac{\mu_0}{\varepsilon_0}}} = \frac{\frac{2}{3}}{\frac{1}{3}+\frac{1}{2}} = \frac{\frac{4}{6}}{\frac{5}{6}} = \frac{4}{5} = 0.8$$

b. $\dfrac{S_{av}^{ref}}{S_{av}^{i}} = |\Gamma^b|^2 = |-0.2|^2 = 0.04$ or 4%

$\dfrac{S_{av}^{t}}{S_{av}^{i}} = \left(1-|\Gamma^b|^2\right) = \left(1-|0.2|^2\right) = 0.96$ or 96%

$\boxed{5.4}$ $f = 3$ GHz, $\eta_1 = \eta_0$, $\eta_2 = \frac{1}{2}\eta_0$

a. $\Gamma^b = \dfrac{\eta_2-\eta_1}{\eta_2+\eta_1} = \dfrac{\frac{1}{2}\eta_0-\eta_0}{\frac{1}{2}\eta_0+\eta_0} = \dfrac{\frac{1}{2}-1}{\frac{1}{2}+1} = \dfrac{-\frac{1}{2}}{\frac{3}{2}} = -\dfrac{1}{3} = -0.3333$

b. $\text{SWR} = \dfrac{1+|\Gamma^b|}{1-|\Gamma^b|} = \dfrac{1+\frac{1}{3}}{1-\frac{1}{3}} = \dfrac{\frac{4}{3}}{\frac{2}{3}} = 2$

c. $E^{total} = E_0 e^{-j\beta_0 z} + \Gamma^b E_0 e^{+j\beta_0 z} = E_0 e^{-j\beta_0 z}\left[1+\Gamma^b e^{-j2\beta_0 z}\right] = 2\times10^{-3} e^{-j\beta_0 z}\left[1+\Gamma^b e^{-j2\beta_0 z}\right]$

$|E^{total}| = 2\times10^{-3}\left|1-\frac{1}{3}e^{-j2\beta_0 z}\right|$

$|E^{total}|_{max} = 2\times10^{-3}\left|1+\frac{1}{3}\right| = 2\times10^{-3}\left(\frac{4}{3}\right) = \frac{8\times10^{-3}}{3} = 2.667\times10^{-3}$ when $2\beta_0 z_{max} = \pi, 3\pi, 5\pi, \ldots$

$z_{max} = \dfrac{m\pi}{2\beta_0} = \dfrac{m\pi}{2\left(\frac{2\pi}{\lambda_0}\right)} = \dfrac{m}{4}\lambda_0 = \dfrac{m}{4}\left(\dfrac{3\times10^8}{3\times10^9}\right) = m\left(\dfrac{0.1}{4}\right) = 0.025m,\ m = 1,3,5,$

$z_{max} = 0.025m$ meters where $m = 1, 3, 5, \ldots$

$|E^{total}|_{min} = 2\times10^{-3}\left|1-\frac{1}{3}\right| = 2\times10^{-3}\left(\frac{2}{3}\right) = \frac{4\times10^{-3}}{3} = 1.333\times10^{-3}$ when $2\beta_0 z_{min} = 0, 2\pi, 4\pi, \ldots$

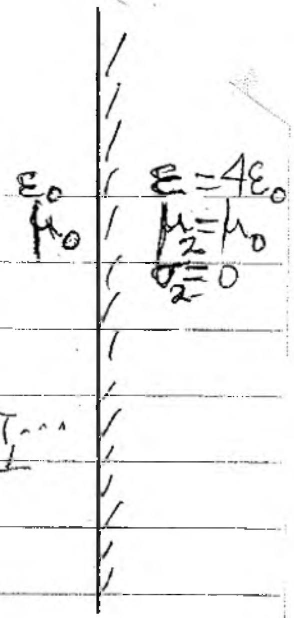
$z_{min} = \dfrac{2n\pi}{2\beta_0} = \dfrac{2\pi n}{2\left(\frac{2\pi}{\lambda_0}\right)} = \dfrac{n}{2}\lambda_0 = \dfrac{n}{2}\left(\dfrac{3\times10^8}{3\times10^9}\right) = n\left(\dfrac{0.1}{2}\right) = 0.05n,\ n = 0, 1, 2, \ldots$

$z_{min} = 0.05n$ meters where $n = 0, 1, 2, \ldots$

d. $|E^{total}|_{max} = 2\times10^{-3}\left|1+\frac{1}{3}\right| = 2\times10^{-3}\left(\frac{4}{3}\right) = 2.667\times10^{-3}$ V/m

$|E^{total}|_{min} = 2\times10^{-3}\left|1-\frac{1}{3}\right| = 2\times10^{-3}\left(\frac{2}{3}\right) = 1.333\times10^{-3}$ V/m

$\boxed{5.5}$



(a) $\Gamma^b = \frac{\eta-\eta_0}{\eta+\eta_0} = \frac{\sqrt{\frac{\mu_0}{4\varepsilon_0}}-\sqrt{\frac{\mu_0}{\varepsilon_0}}}{\sqrt{\frac{\mu_0}{4\varepsilon_0}}+\sqrt{\frac{\mu_0}{\varepsilon_0}}} = \frac{\frac{1}{2}-1}{\frac{1}{2}+1} = \frac{-\frac{1}{2}}{\frac{3}{2}} = -\frac{1}{3}$

$|E(\text{Standing Wave})|_{max} = \left|1+\Gamma^b e^{+j2\beta_0 z}\right|_{\substack{max\\ z=-\ell}} = \left|1-\frac{1}{3}e^{-j2\beta_0\ell}\right|_{max}$

$|E(SW)|_{max} = 4/3$ when $-2\beta_0\ell = m\pi$, $m = \pi, 3\pi, 5\pi, \ldots$

(b) $2\beta_0\ell = \pi \Rightarrow \ell = \pi/2\beta_0 = \pi/2(2\pi/\lambda_0) = \frac{\pi\lambda_0}{4\pi} = \lambda_0/4$

$\boxed{\ell = \lambda/4}$

(c) $|E(SW)|_{min} = \left|1+\Gamma^b e^{+j2\beta_0 z}\right|_{\substack{min\\ z=-\ell}} = \left|1-\frac{1}{3}e^{-j2\beta_0\ell}\right|_{min}$

$2\beta_0\ell = 0 \Rightarrow \ell = 0 \qquad |E|_{min} = 1-\frac{1}{3} = \frac{2}{3}$

(d) $2\beta_0\ell = 0 \Rightarrow \boxed{\ell = 0}$ Also $2\beta_0\ell = 2\pi \Rightarrow \boxed{\ell = \frac{2\pi}{2\left(\frac{2\pi}{\lambda_0}\right)} = \frac{\lambda_0}{2}}$

(e) $SWR = \frac{E_{max}}{E_{min}} = \frac{4/3}{2/3} = \boxed{2}$

(f) $Z_{in}(\ell = \lambda_0/4)\Big|_{E_{max}} = \eta_0\left[\frac{1+\Gamma_{in}(\ell_0=\lambda/4)}{1-\Gamma_{in}(\ell_0=\lambda/4)}\right] = \eta_0\left[\frac{1+\left(-\frac{1}{3}\right)e^{-j\frac{2\pi}{\lambda_0}(2)\frac{\lambda_0}{4}}}{1-\left(-\frac{1}{3}\right)e^{-j\frac{2\pi}{\lambda_0}(2)\frac{\lambda_0}{4}}}\right]$

$= \eta_0\left[\frac{1+\frac{1}{3}(+1)}{1-\frac{1}{3}(+1)}\right] = \eta_0\left[\frac{4/3}{2/3}\right] = \eta_0\left(\frac{2}{1}\right) = 2\eta_0$

$Z_{in}(\ell = \lambda_0/4)\Big|_{max} = 2(\eta_0) = 2(377) = \boxed{754}$

(g) $Z_{in}(\ell=0)\Big|_{E_{min}} = \eta_0\left[\frac{1+\Gamma_{in}(\ell=0)}{1-\Gamma_{in}(\ell=0)}\right] = \eta_0\left[\frac{1-\frac{1}{3}}{1-\left(-\frac{1}{3}\right)}\right] = \eta_0\left[\frac{\frac{2}{3}}{\frac{4}{3}}\right] = \frac{\eta_0}{2}$

$Z_{in}(\ell=0)\Big|_{min} = \frac{\eta_0}{2} = \frac{377}{2} = \boxed{188.5}$

$\boxed{5.6}$

$\underline{E} = (\hat{a}_x - j\hat{a}_y)\, e^{-j6\pi z}$, z = meters, $f = 10^8$ Hz = 100 MHz

(a) $\beta = \omega\sqrt{\mu\varepsilon} = 2\pi f\sqrt{\mu_0\varepsilon_0\varepsilon_r} = 2\pi(10^8)\sqrt{\mu_0\varepsilon_0\varepsilon_r} = 6\pi$

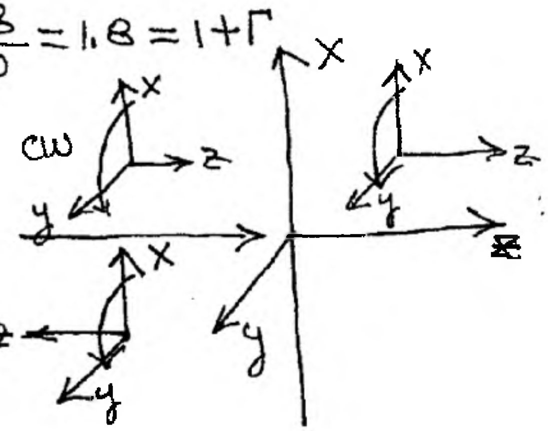
$$\frac{10^8\sqrt{\varepsilon_r}}{3\times 10^8} = 3 \Rightarrow \frac{\sqrt{\varepsilon_r}}{3} = 3 \Rightarrow \sqrt{\varepsilon_r} = 9 \Rightarrow \underline{\varepsilon_r = 81}$$

(b) $\Gamma_x = \Gamma_y = \dfrac{\eta_0 - \eta}{\eta_0 + \eta} = \dfrac{\sqrt{\frac{\mu_0}{\varepsilon_0}} - \sqrt{\frac{\mu_0}{\varepsilon_r\varepsilon_0}}}{\sqrt{\frac{\mu_0}{\varepsilon_0}} + \sqrt{\frac{\mu_0}{\varepsilon_r\varepsilon_0}}} = \dfrac{1 - \frac{1}{\sqrt{\varepsilon_r}}}{1 + \frac{1}{\sqrt{\varepsilon_r}}} = \dfrac{\sqrt{\varepsilon_r} - 1}{\sqrt{\varepsilon_r} + 1} = \dfrac{9-1}{9+1} = \dfrac{8}{10}$

$\Gamma_x = \Gamma_y = 0.8$

(c) $T_x = T_y = \dfrac{2\eta_0}{\eta_1 + \eta_0} = 2\dfrac{\sqrt{\mu_0/\varepsilon_0}}{\sqrt{\frac{\mu_0}{\varepsilon_r\varepsilon_0}} + \sqrt{\frac{\mu_0}{\varepsilon_0}}} = 2\dfrac{1}{\frac{1}{\sqrt{\varepsilon_r}} + 1} = 2\dfrac{\sqrt{\varepsilon_r}}{1 + \sqrt{\varepsilon_r}}$

$T_x = T_y = 2\dfrac{9}{1+9} = \dfrac{18}{10} = 1.8 = 1 + \Gamma$



(d) $E^r_x = \Gamma_x E^i_x = 0.8\, e^{+j6\pi z}$

$E^r_y = \Gamma_y E^i_y = -j0.8\, e^{+j6\pi z}$

$\underline{\text{Circular}}$

(e) $\underline{\text{CCW}}$

(f) $E^t_x = T_x E^i_x = 1.8\, e^{-j\beta_0 z}$

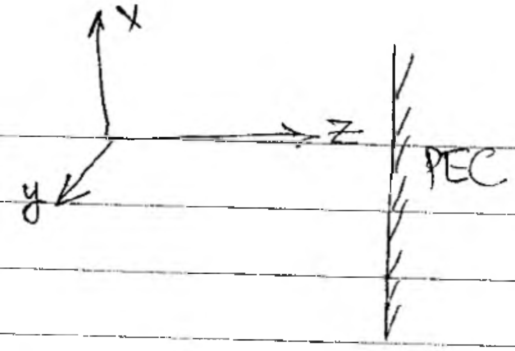
$E^t_y = T_y E^i_y = -j1.8\, e^{-j\beta_0 z}$

$\underline{\text{Circular}}$

(g) $\underline{\text{CW}}$

$\boxed{5.7}$

$$\underline{E}^i = (j\hat{a}_x + \hat{a}_y) E_o e^{-j\beta z}$$



(a)

$$\underline{E}^r = (-j\hat{a}_x - \hat{a}_y) e^{+j\beta z} = -(j\hat{a}_x + \hat{a}_y) e^{+j\beta z}$$

2 components, same magnitude, 90° out of phase ⇒ CP
Since the x component leads the y component ⇒ CCW(LH)
∴ Reflected field is **LHC**

(b)

On Poincaré sphere incident field is on South pole.
On Poincaré sphere reflected field is on North pole.
Therefore the great circle angle between the two is

$$P_w P_a = 180°$$

The normalized voltage output is

$$V = C\cos\left(\frac{P_w P_a}{2}\right) = C\cos\left(\frac{180°}{2}\right) = C\cos(90°) = 0$$

$$V = 0 = -\infty \text{ dB}$$

The polarization of the antenna and the incident wave are crosspolarized (mismatched in polarization).
Therefore output is $0 = -\infty$ dB.

$\boxed{5.8}$ $\underline{E}^i = \hat{a}_x E_0 e^{-j\beta_0 z}$, $\Gamma^b = \frac{\eta_2 - \eta_0}{\eta_2 + \eta_0} = \frac{0-\eta_0}{0+\eta_0} = -1$

a. $\underline{E}^r = \hat{a}_x \Gamma^b E_0 e^{+j\beta_0 z} = -\hat{a}_x E_0 e^{+j\beta_0 z}$

b. $\underline{H}^i = \hat{a}_y \frac{E_0}{\eta_0} e^{-j\beta_0 z} = \hat{a}_y \frac{E_0}{377} e^{-j\beta_0 z} = \hat{a}_y\, 2.6525\times10^{-3} E_0 e^{-j\beta_0 z}$

$\underline{H}^r = -\hat{a}_y \frac{\Gamma^b E_0}{\eta_0} e^{+j\beta_0 z} = \hat{a}_y \frac{E_0}{377} e^{+j\beta_0 z} = \hat{a}_y\, 2.6525\times10^{-3} E_0 e^{+j\beta_0 z}$

c. $\underline{J}_s = \hat{n}\times(\underline{H}^i + \underline{H}^r)\Big|_{z=0} = \hat{n}\times(2\underline{H}^i)\Big|_{z=0} = -2\hat{a}_z\times\hat{a}_y\, 2.6525\times10^{-3} E_0 = \hat{a}_x\, 5.3\times10^{-3}$ A/m

$\boxed{5.9}$ $\eta_1 = \sqrt{\eta_0\eta_2} = \left\{\sqrt{\frac{\mu_0}{\varepsilon_0}}\,\frac{1}{2}\sqrt{\frac{\mu_0}{\varepsilon_0}}\right\}^{1/2} = \frac{1}{\sqrt{2}}\sqrt{\frac{\mu_0}{\varepsilon_0}} = \frac{1}{2}\eta_0 = \sqrt{\frac{\mu_0}{2\varepsilon_0}} = \sqrt{\frac{\mu_0}{\varepsilon_r\varepsilon_0}} \Rightarrow \varepsilon_r = 2$

$\boxed{5.10}$ In the three different regions, we can write the corresponding fields as:

assume the electric fields are x-polarized while the magnetic fields are y-polarized.

$$\left.\begin{aligned} E_1 &= E_1^+ e^{-j\beta_0 z} + E_1^- e^{+j\beta_0 z} \\ H_1 &= \frac{E_1^+}{\eta_0} e^{-j\beta_0 z} - \frac{E_1^-}{\eta_0} e^{+j\beta_0 z} \end{aligned}\right\}\quad z \le 0$$

$$\left.\begin{aligned} E_2 &= E_2^+ e^{-j\beta_1 z} + E_2^- e^{+j\beta_1 z} \\ H_2 &= \frac{E_2^+}{\eta_1} e^{-j\beta_1 z} - \frac{E_2^-}{\eta_1} e^{+j\beta_1 z} \end{aligned}\right\}\quad 0 \le z \le t$$

$$\left.\begin{aligned} E_3 &= E_3^+ e^{-j\beta_0(z-t)} \\ H_3 &= \frac{E_3^+}{\eta_0} e^{-j\beta_0(z-t)} \end{aligned}\right\}\quad t \le z$$

Applying the continuity of the tangential electric and magnetic fields at the interface at $z=0$, we have

$$E_1^+ + E_1^- = E_2^+ + E_2^- \Rightarrow E_1^+ + E_1^- = E_2^+ + E_2^- \quad (1)$$

$$\frac{E_1^+}{\eta_0} - \frac{E_1^-}{\eta_0} = \frac{E_2^+}{\eta_1} - \frac{E_2^-}{\eta_1} \Rightarrow E_1^+ - E_1^- = \frac{\eta_0}{\eta_1}(E_2^+ - E_2^-) \quad (2)$$

Solving (1) and (2), we get

$$E_1^+ = \frac{E_2^+}{2}\left(1+\frac{\eta_0}{\eta_1}\right) + \frac{E_2^-}{2}\left(1-\frac{\eta_0}{\eta_1}\right) = E_2^+\left(\frac{\eta_1+\eta_0}{2\eta_1}\right) + E_2^-\left(\frac{\eta_1-\eta_0}{2\eta_1}\right) \quad (3)$$

cont'd.

5.10 cont'd.

$$E_1^- = E_2^+\left(\frac{\eta_1-\eta_0}{2\eta_1}\right) + E_2^-\left(\frac{\eta_1+\eta_0}{2\eta_1}\right) \qquad (4)$$

Now apply the boundary conditions at $z=t$, and we get the following:

$$E_2^+ e^{-j\beta_1 t} + E_2^- e^{+j\beta_1 t} = E_3^+ \Rightarrow E_2^+ e^{-j\beta_1 t} + E_2^- e^{+j\beta_1 t} = E_3^+ \qquad (5)$$

$$\frac{E_2^+}{\eta_1}e^{-j\beta_1 t} - \frac{E_2^-}{\eta_1}e^{+j\beta_1 t} = \frac{E_3^+}{\eta_0} \Rightarrow E_2^+ e^{-j\beta_1 t} - E_2^- e^{+j\beta_1 t} = \frac{\eta_1}{\eta_0}E_3^+ \qquad (6)$$

Solving (5) and (6) for E_2^+ and E_2^- leads to

$$E_2^+ = \frac{E_3^+ e^{+j\beta_1 t}}{2}\left(\frac{\eta_1+\eta_0}{\eta_0}\right) = E_3^+ e^{+j\beta_1 t}\left(\frac{\eta_1+\eta_0}{2\eta_0}\right) = E_3^+ e^{+j\beta_1 t}\left(\frac{\eta_0+\eta_1}{2\eta_0}\right) \qquad (7)$$

$$E_2^- = E_3^+ e^{-j\beta_1 t}\left(\frac{\eta_0-\eta_1}{2\eta_0}\right) = -E_3^+ e^{-j\beta_1 t}\left(\frac{\eta_1-\eta_0}{2\eta_0}\right) \qquad (8)$$

Substituting (7) and (8) into (3) we can write that

$$E_1^+ = E_3^+\frac{e^{j\beta_1 t}}{\eta_0\eta_1}\left(\frac{\eta_0+\eta_1}{2}\right)^2 - E_3^+\frac{e^{-j\beta_1 t}}{\eta_0\eta_1}\left(\frac{\eta_1-\eta_0}{2}\right)^2 = \frac{E_3^+}{4\eta_0\eta_1}\left[e^{+j\beta_1 t}(\eta_0+\eta_1)^2 - e^{-j\beta_1 t}(\eta_1-\eta_0)^2\right]$$

or

$$\boxed{\frac{E_3^+}{E_1^+} \equiv T^b = \frac{4\eta_0\eta_1}{(\eta_1+\eta_0)^2 e^{+j\beta_1 t} - (\eta_1-\eta_0)^2 e^{-j\beta_1 t}} = \frac{4\eta_0\eta_1 e^{-j\beta_1 t}}{(\eta_1+\eta_0)^2 - (\eta_1-\eta_0)^2 e^{-j2\beta_1 t}}}$$

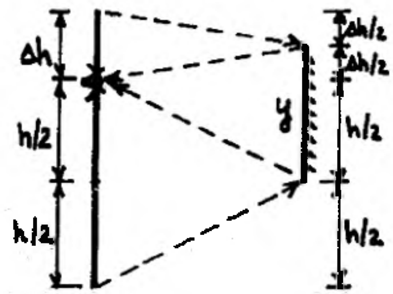
To find the reflection coefficient, we substitute (7) and (8) into (3) and (4) and then take the ratio of (4) and (3). This leads to

$$\frac{E_1^-}{E_1^+} \equiv \Gamma^b = \frac{\frac{e^{j\beta_1 t}}{4\eta_0\eta_1}(\eta_1-\eta_0)(\eta_1+\eta_0) - \frac{e^{-j\beta_1 t}}{4\eta_0\eta_1}(\eta_1+\eta_0)(\eta_1-\eta_0)}{\frac{e^{j\beta_1 t}}{4\eta_0\eta_1}(\eta_1+\eta_0)(\eta_1+\eta_0) - \frac{e^{-j\beta_1 t}}{4\eta_0\eta_1}(\eta_1-\eta_0)(\eta_1-\eta_0)}$$

$$\boxed{\frac{E_1^-}{E_1^+} \equiv \Gamma = \frac{(\eta_1+\eta_0)(\eta_1-\eta_0)\left(1-e^{-j2\beta_1 t}\right)}{(\eta_1+\eta_0)^2 - (\eta_1-\eta_0)^2 e^{-j2\beta_1 t}}}$$

5.11 As can be seen from the figure the mirror dimensions are equal to

$$y = \frac{h}{2} + \frac{\Delta h}{2} = \frac{1}{2}(h + \Delta h)$$



5.12 For the prism with a dielectric constant of 2.25, the critical angle is

$$\theta_c = \sin^{-1}\sqrt{\frac{\varepsilon_0}{\varepsilon_r \varepsilon_0}} = \sin^{-1}\sqrt{\frac{1}{2.25}} = \sin^{-1}\sqrt{0.4444} = \sin^{-1}(0.6666) = 41.81^\circ$$

Therefore at the hypotenuse the reflection coefficient $|\Gamma| = 1$ since the incident angle of 45° is greater than the critical angle of 41.81°.

$$\Gamma_{20} = \frac{\eta_2 - \eta_0}{\eta_2 + \eta_0} = \frac{\frac{1}{1.5}\eta_0 - \eta_0}{\frac{1}{1.5}\eta_0 + \eta_0} = \frac{1 - 1.5}{1 + 1.5} = \frac{-0.5}{2.5} = -\frac{1}{5} = -0.2$$

$$\Gamma_{02} = -\Gamma_{20} = 0.2$$

$$\frac{S_{av}^{ex}}{S_{av}^{i}} = (1 - |\Gamma_{20}|^2)(1^2)(1 - |\Gamma_{02}|^2) = (1 - |\Gamma_{20}|^2)^2 = [1 - (0.2)^2]^2 = (0.96)^2 = 0.9216 \text{ or } 92.16\%$$

5.13 $\beta_0 \sin\theta_i = \beta_2 \sin\theta_2 \Rightarrow \theta_2 = \sin^{-1}\left(\frac{\beta_0}{\beta_2}\sin\theta_i\right) = \sin^{-1}\left(\frac{\beta_0}{2\beta_0}\sin 30^\circ\right) = \sin^{-1}\left(\frac{1}{4}\right) = 14.48^\circ$

$\beta_0 \sin(30^\circ) = \beta_2 \sin\theta_2 = \beta_0 \sin\theta_3 \Rightarrow \theta_3 = 30^\circ$

5.14 $$|\Gamma_\perp^b|^2 = \left|\frac{\cos\theta_i - \sqrt{\frac{\varepsilon_2}{\varepsilon_1}}\sqrt{1 - \left(\frac{\varepsilon_1}{\varepsilon_2}\right)\sin^2\theta_i}}{\cos\theta_i + \sqrt{\frac{\varepsilon_2}{\varepsilon_1}}\sqrt{1 - \frac{\varepsilon_1}{\varepsilon_2}\sin^2\theta_i}}\right|^2 = \left|\frac{\cos\theta_i - 2\sqrt{1 - \frac{1}{4}\sin^2\theta_i}}{\cos\theta_i + 2\sqrt{1 - \frac{1}{4}\sin^2\theta_i}}\right|^2 = \frac{1}{4}$$

When this equation is solved iteratively, it is found that $\theta_i \simeq 52^\circ$.

5.15 $$|\Gamma_\parallel|^2 = \left|\frac{-\cos\theta_i + \sqrt{\frac{\varepsilon_1}{\varepsilon_2}}\sqrt{1 - \frac{\varepsilon_1}{\varepsilon_2}\sin^2\theta_i}}{\cos\theta_i + \sqrt{\frac{\varepsilon_1}{\varepsilon_2}}\sqrt{1 - \frac{\varepsilon_1}{\varepsilon_2}\sin^2\theta_i}}\right|^2 = \left|\frac{-\cos\theta_i + \frac{1}{2}\sqrt{1 - \frac{1}{4}\sin^2\theta_i}}{\cos\theta_i + \frac{1}{2}\sqrt{1 - \frac{1}{4}\sin^2\theta_i}}\right|^2 = \frac{1}{4}$$

When this equation is solved iteratively, it is found that $\theta_i \simeq 81.65^\circ$.

5.16 $\theta_B = \tan^{-1}\sqrt{\frac{\varepsilon_2}{\varepsilon_1}}$

$\varepsilon_r = 2.56: \ \theta_B = \tan^{-1}(1.6) = 57.99^\circ$

$\varepsilon_r = 4: \ \theta_B = \tan^{-1}(2.0) = 63.435^\circ$

$\varepsilon_r = 9: \ \theta_B = \tan^{-1}(3.0) = 71.565^\circ$

$\varepsilon_r = 16: \ \theta_B = \tan^{-1}(4) = 75.964^\circ$

$\varepsilon_r = 25: \ \theta_B = \tan^{-1}(5) = 78.69^\circ$

$\varepsilon_r = 81: \ \theta_B = \tan^{-1}(9) = 83.66^\circ$

$\boxed{5.17}$ The easiest way to solve this problem is to use the same procedure as used for the solution of Problem 5.10 The end result for the reflection and transmission coefficients is the same as that of Problem 5.10 except that instead of using the intrinsic impedances we must use the wave impedances in the z-direction, and we must use the phase constant along the z-direction.

Thus
$$\Gamma_{\parallel}^{b} = \frac{(Z_{1z}+Z_{0z})(Z_{1z}-Z_{0z})\left[1-e^{-j2\beta_{1z}t}\right]}{(Z_{1z}+Z_{0z})^2-(Z_{1z}-Z_{0z})^2 e^{-j2\beta_{1z}t}}$$

$$T_{\parallel}^{b} = \frac{4Z_{0z}Z_{1z}\,e^{-j\beta_{1z}t}}{(Z_{1z}+Z_{0z})^2-(Z_{1z}-Z_{0z})^2 e^{-j2\beta_{1z}t}}$$

where according to (4-20b) and Figure 4-6

$$\left.\begin{aligned} Z_{0z} &= \eta_0\cos\theta_i \\ Z_{1z} &= \eta_1\cos\theta_1 \\ \beta_{1z} &= \beta_1\cos\theta_1 \end{aligned}\right\} \quad \begin{aligned} &\beta_0\sin\theta_i = \beta_1\sin\theta_1 \\ &\sin\theta_1 = \frac{\beta_0}{\beta_1}\sin\theta_i = \sqrt{\frac{\epsilon_0}{\epsilon_1}}\sin\theta_i \\ &\cos\theta_1 = \sqrt{1-\sin^2\theta_1} = \sqrt{1-\frac{\epsilon_0}{\epsilon_1}\sin^2\theta_i} \end{aligned}$$

$$\eta_0 = \sqrt{\frac{\mu_0}{\epsilon_0}},\quad \eta_1 = \sqrt{\frac{\mu_1}{\epsilon_1}}$$

$$\beta_1 = \omega\sqrt{\mu_1\epsilon_1}$$

$\boxed{5.18}$ As for Problem 5.17, the easiest way to solve this problem is to use the same procedure as used for the solution of Problem 5.10. The end result for the reflection and transmission coefficients is the same as that of Problem 5.10 except that instead of using the intrinsic impedances we must use the wave impedances in the z-direction, and we must use the phase constant along the z-direction.

Thus
$$\Gamma_{\perp}^{b} = \frac{(Z_{1z}+Z_{0z})(Z_{1z}-Z_{0z})\left[1-e^{-j2\beta_{1z}t}\right]}{(Z_{1z}+Z_{0z})^2-(Z_{1z}-Z_{0z})^2 e^{-j2\beta_{1z}t}}$$

$$T_{\perp} = \frac{4Z_{0z}Z_{1z}\,e^{-j\beta_{1z}t}}{(Z_{1z}+Z_{0z})^2-(Z_{1z}-Z_{0z})^2 e^{-j2\beta_{1z}t}}$$

where according to (4-21b) and Figure 4-6

$$\left.\begin{aligned} Z_{0z} &= \frac{\eta_0}{\cos\theta_i} \qquad \beta_{1z} = \beta_1\cos\theta_1 \\ Z_{1z} &= \frac{\eta_1}{\cos\theta_1} \qquad \beta_1 = \omega\sqrt{\mu_1\epsilon_1} \end{aligned}\right\} \quad \begin{aligned} &\beta_0\sin\theta_i = \beta_1\sin\theta_1 \\ &\sin\theta_1 = \frac{\beta_0}{\beta_1}\sin\theta_i = \sqrt{\frac{\epsilon_0}{\epsilon_1}}\sin\theta_i \\ &\cos\theta_1 = \sqrt{1-\sin^2\theta_1} = \sqrt{1-\frac{\epsilon_0}{\epsilon_1}\sin^2\theta_i} \end{aligned}$$

$\boxed{5.19}$ $\theta_c = \sin^{-1}\sqrt{\frac{\epsilon_2}{\epsilon_1}} = \sin^{-1}\left(\sqrt{\frac{4}{9}}\right) = \sin^{-1}\left(\frac{2}{3}\right) = \sin^{-1}(0.667) = 41.8°$

$\boxed{5.20}$ $\theta_B = \tan^{-1}\left(\sqrt{\frac{\epsilon_2}{\epsilon_1}}\right)$, $\theta_c = \sin^{-1}\left(\sqrt{\frac{\epsilon_2}{\epsilon_1}}\right)$

(a) $\theta_B = \tan^{-1}\left(\sqrt{\frac{1}{81}}\right) = \tan^{-1}\left(\frac{1}{9}\right) = 6.34°$ $\qquad \theta_c = \sin^{-1}\left(\sqrt{\frac{1}{81}}\right) = \sin^{-1}\left(\frac{1}{9}\right) = 6.379°$

(b) $\theta_B = \tan^{-1}\left(\sqrt{81}\right) = \tan^{-1}(9) = 83.64°$ $\qquad \theta_c = \sin^{-1}(\sqrt{81}) = \sin^{-1}(9) =$ does not exist

(c) $\theta_B = \tan^{-1}\left(\sqrt{\frac{1}{9}}\right) = \tan^{-1}\left(\frac{1}{3}\right) = 18.436°$ $\qquad \theta_c = \sin^{-1}\left(\sqrt{\frac{1}{9}}\right) = \sin^{-1}\left(\frac{1}{3}\right) = 19.47°$

$\boxed{5.21}$ $\theta_c = \sin^{-1}\left(\sqrt{\frac{\epsilon_2}{\epsilon_1}}\right)$

a. $\theta_c = \sin^{-1}\left(\frac{1}{1.6}\right) = 38.68°$ $\qquad \theta_c = \sin^{-1}\left(\frac{1}{4}\right) = 14.478°$

$\theta_c = \sin^{-1}\left(\frac{1}{2}\right) = 30°$ $\qquad \theta_c = \sin^{-1}\left(\frac{1}{5}\right) = 11.54°$

$\theta_c = \sin^{-1}\left(\frac{1}{3}\right) = 19.47°$ $\qquad \theta_c = \sin^{-1}\left(\frac{1}{9}\right) = 6.379°$

b. $\theta_B = \tan^{-1}\left(\sqrt{\frac{\epsilon_2}{\epsilon_1}}\right)$

$\theta_B = \tan^{-1}\left(\frac{1}{1.6}\right) = 32.0°$ $\qquad \theta_B = \tan^{-1}\left(\frac{1}{4}\right) = 14.036°$

$\theta_B = \tan^{-1}\left(\frac{1}{2}\right) = 26.565°$ $\qquad \theta_B = \tan^{-1}\left(\frac{1}{5}\right) = 11.31°$

$\theta_B = \tan^{-1}\left(\frac{1}{3}\right) = 18.435°$ $\qquad \theta_B = \tan^{-1}\left(\frac{1}{9}\right) = 6.34°$

c. As the dielectric constant gets larger or the difference between the two media gets greater, the critical angle and Brewster angle are approaching the same value.

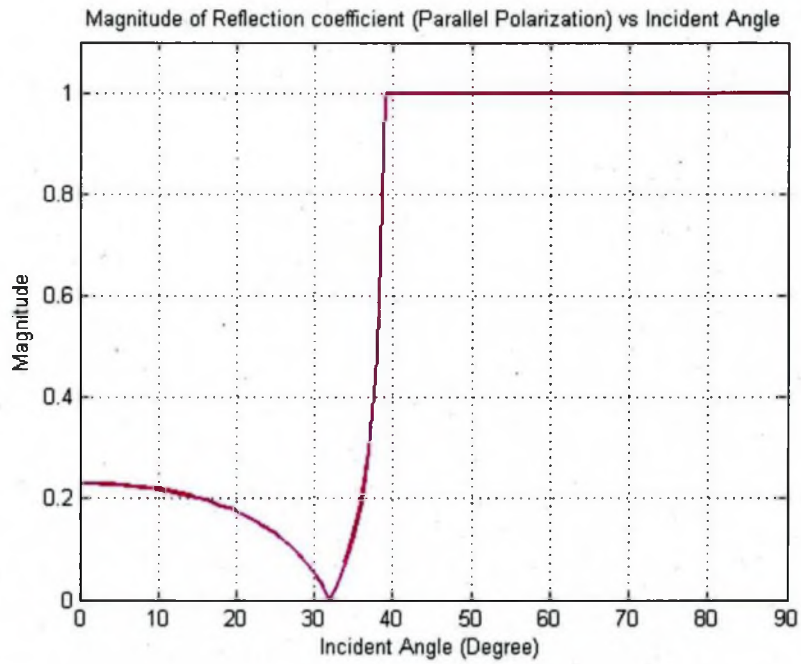
d. The plots are shown in the pages to follow.

e. The plots are shown in the pages that follow.

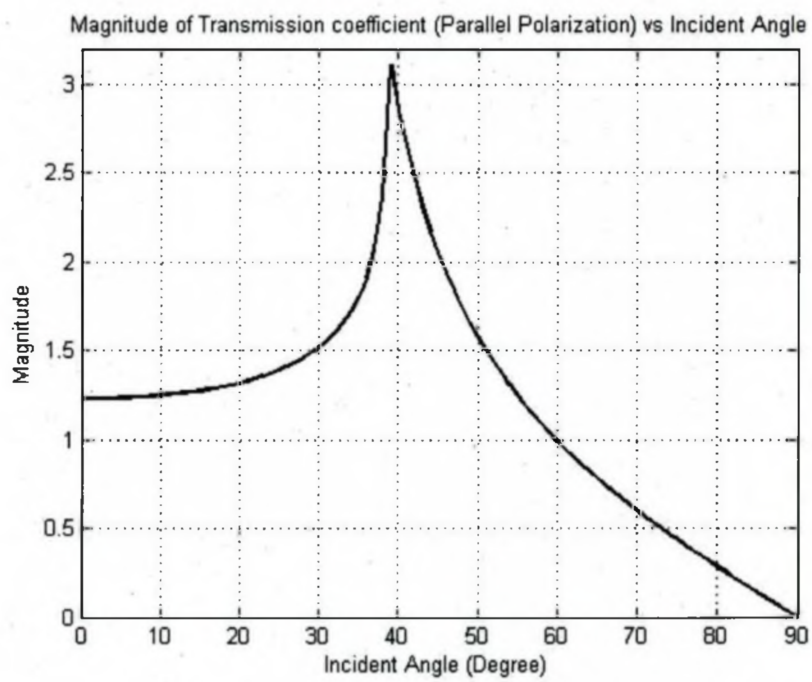
Cont'd.

5.21 Cont'd

$|\Gamma_{\parallel}|$

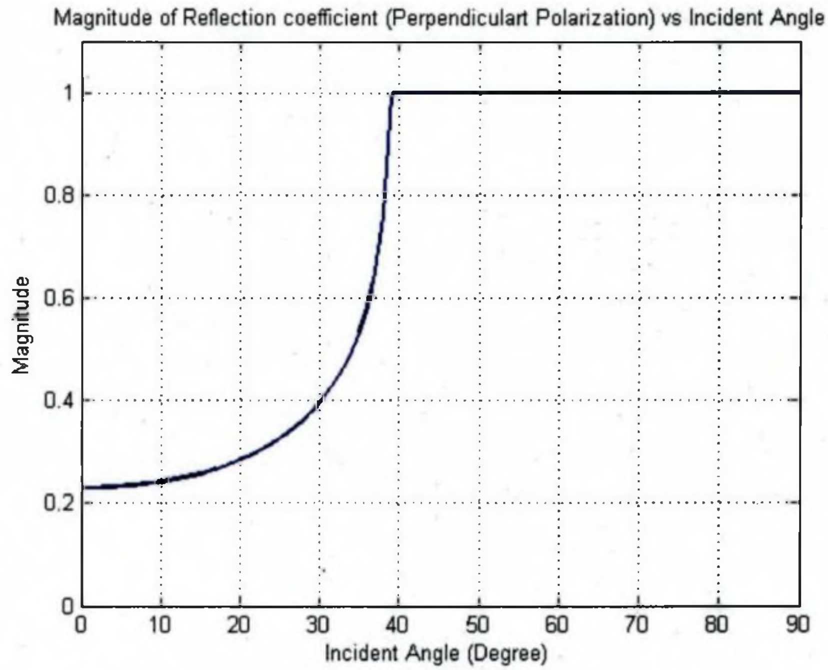


$|T_{\parallel}|$

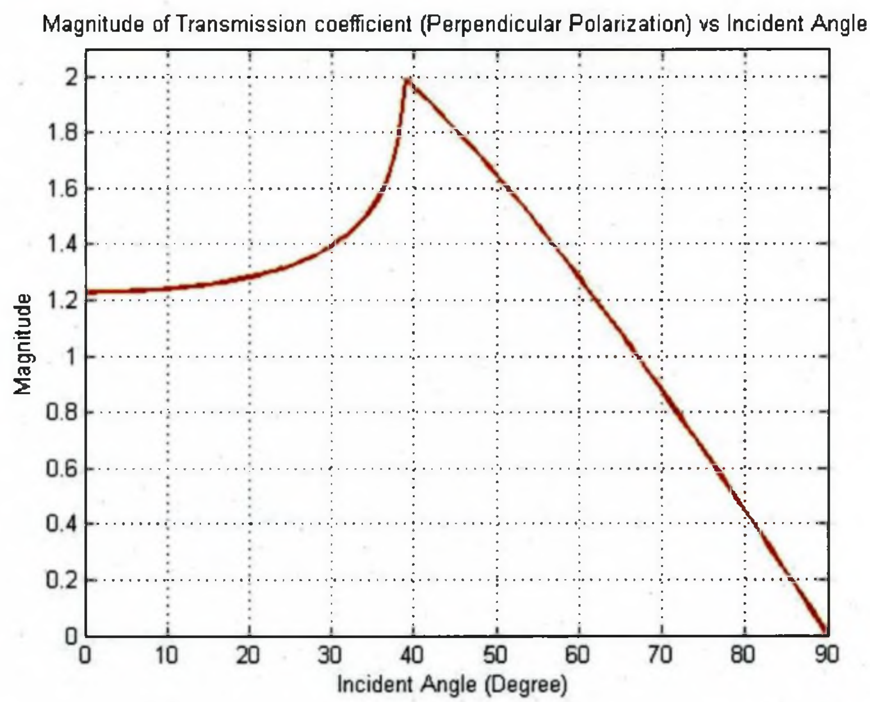


5.21

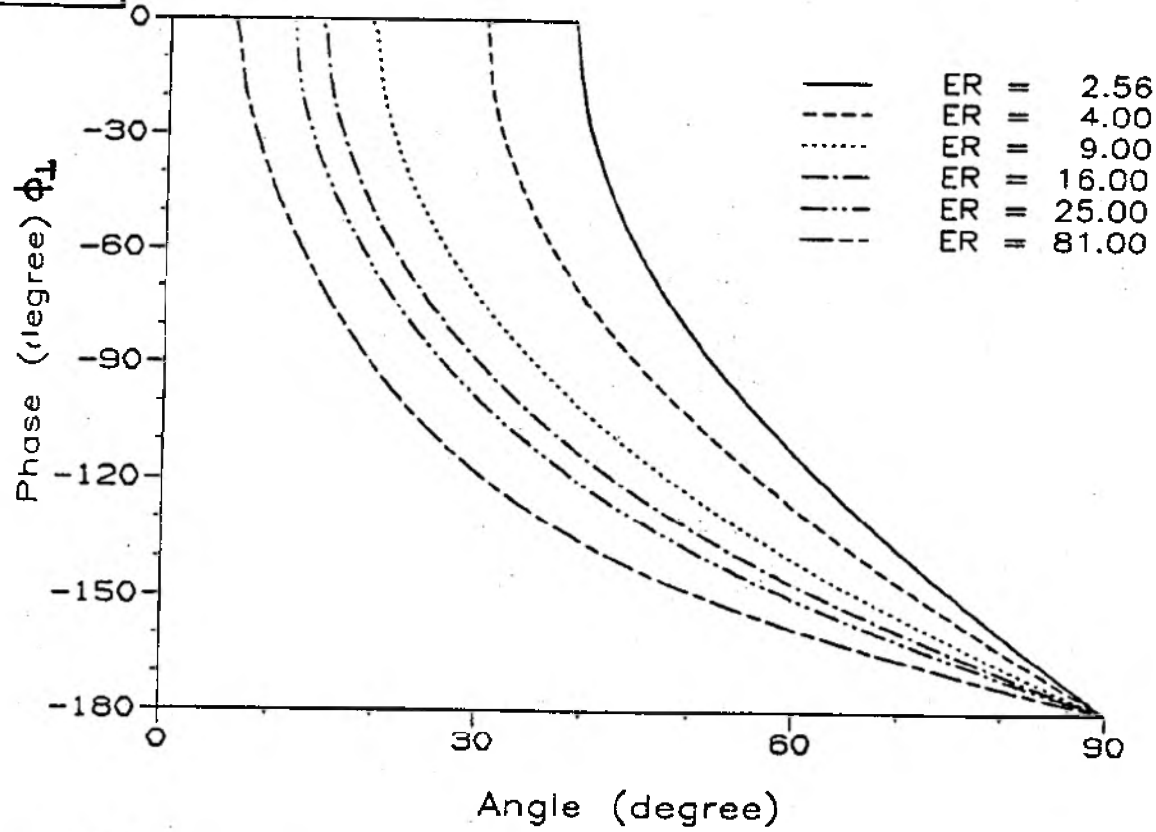
$|\Gamma_{\perp}|$



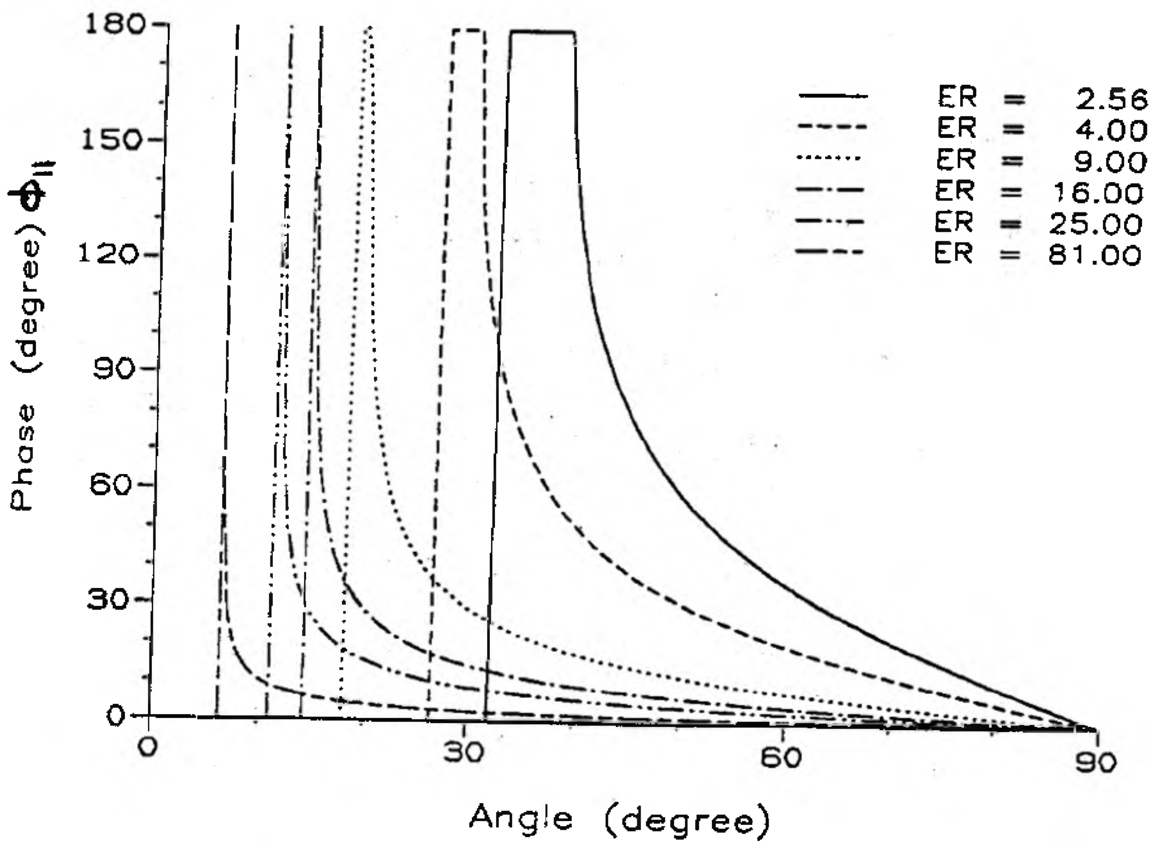
$|T_{\perp}|$



5.21 Cont'd.

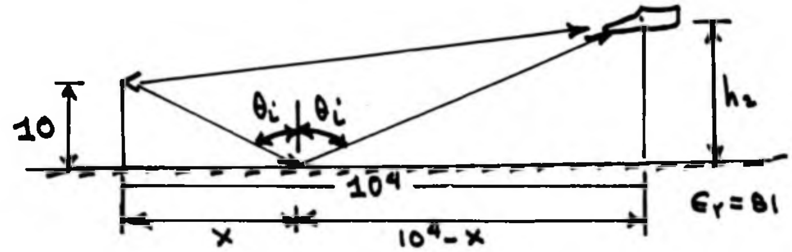


ER = 2.56
ER = 4.00
ER = 9.00
ER = 16.00
ER = 25.00
ER = 81.00
Phase (degree) $\phi_\perp$
0
-30
-60
-90
-120
-150
-180
0
30
60
90
Angle (degree)



ER = 2.56
ER = 4.00
ER = 9.00
ER = 16.00
ER = 25.00
ER = 81.00
Phase (degree) $\phi_\parallel$
180
150
120
90
60
30
0
0
30
60
90
Angle (degree)

$\boxed{5.22}$ $\theta_B = \tan^{-1}\left(\sqrt{\frac{\epsilon_2}{\epsilon_1}}\right)$



For the reflected wave not to possess a parallel polarized component, then the incidence angle must be equal to the Brewster angle. Thus

$$\theta_i = \theta_B = \tan^{-1}\left(\frac{x}{10}\right) = \tan^{-1}\left(\sqrt{\frac{\epsilon_2}{\epsilon_1}}\right) = \tan^{-1}\left(\sqrt{81}\right) = \tan^{-1}(9) = 83.66°$$

$$x = 90 \text{ meter}$$

Thus

$$\tan^{-1}\left(\frac{10^4-90}{h_2}\right) = \tan^{-1}\left(\sqrt{81}\right) = \tan^{-1}(9) \Rightarrow \frac{10^4-90}{h_2} = 9 \Rightarrow h_2 = \frac{10^4-90}{9} = 1{,}101.11\,\text{m}$$

$\boxed{5.23}$ For a height of 1,101.11m the incident angle is equal to 83.66° which is equal to the Brewster angle (see the solution to Problem 5.22). The refraction angle θ_t measured from the vertical is equal to

$$\beta_0 \sin\theta_i = \beta \sin\theta_t \Rightarrow \theta_t = \sin^{-1}\left(\frac{\beta_0}{\beta}\sin\theta_i\right) = \sin^{-1}\left[\frac{1}{9}\sin(83.66°)\right] = 6.34°$$

Therefore

$$\Gamma_{\parallel} = \frac{\eta\cos\theta_t - \eta_0\cos\theta_i}{\eta\cos\theta_t + \eta_0\cos\theta_i} = \frac{\cos\theta_t - \sqrt{\epsilon_r}\cos\theta_i}{\cos\theta_t + \sqrt{\epsilon_r}\cos\theta_i} = \frac{\cos(6.34°) - 9\cos(83.66°)}{\cos(6.34°) + 9\cos(83.66°)} = 0 = |\Gamma_{\parallel}|\underline{/\zeta^r_{\parallel}} \quad \begin{matrix} |\Gamma_{\parallel}| = 0 \\ |\zeta^r_{\parallel}| = 0° \end{matrix}$$

$$T_{\parallel} = \frac{\cos\theta_i}{\cos\theta_t}(1+\Gamma_{\parallel}) = \frac{\cos\theta_i}{\cos\theta_t} = \frac{\cos(83.66°)}{\cos(6.34°)} = 0.111\underline{/0°} = |T_{\parallel}|\underline{/\xi^t_{\parallel}} \Rightarrow \begin{matrix} |T_{\parallel}| = 0.111 \\ \xi^t_{\parallel} = 0° \end{matrix}$$

$$\Gamma_{\perp} = \frac{\eta\cos\theta_i - \eta_0\cos\theta_t}{\eta\cos\theta_i + \eta_0\cos\theta_t} = \frac{\cos\theta_i - \sqrt{\epsilon_r}\cos\theta_t}{\cos\theta_i + \sqrt{\epsilon_r}\cos\theta_t} = -0.9756 = |\Gamma_{\perp}|\underline{/\zeta^r_{\perp}} \Rightarrow \begin{matrix} |\Gamma_{\perp}| = 0.9756 \\ \zeta^r_{\perp} = 180° \end{matrix}$$

$$T_{\perp} = 1 + \Gamma_{\perp} = 1 - 0.9756 = 0.0244 = |T_{\perp}|\underline{/\xi^t_{\perp}} \Rightarrow \begin{matrix} |T_{\perp}| = 0.0244 \\ |\xi^t_{\perp}| = 0° \end{matrix}$$

$\delta_i = -90°;\ \gamma_i = \tan^{-1}\left(\frac{1}{1}\right) = 45°$

a. $\left.\begin{matrix} \delta_r = \delta_i - \pi + (\zeta^r_{\perp} - \zeta^r_{\parallel}) = -90° - 180° + (180° - 0) = -90° \\ \gamma_r = \tan^{-1}\left(\frac{|\Gamma_{\parallel}|}{|\Gamma_{\perp}|}\tan\gamma_i\right) = \tan^{-1}\left(\frac{0}{0.9756}(1)\right) = 0° \end{matrix}\right\}$ The reflected field is linearly polarized.

cont'd.

5.23 cont'd.

b. $\delta_t = \delta_i + (\xi^t_{\perp} - \xi^t_{\parallel}) = -90 + (0-0) = -90°$

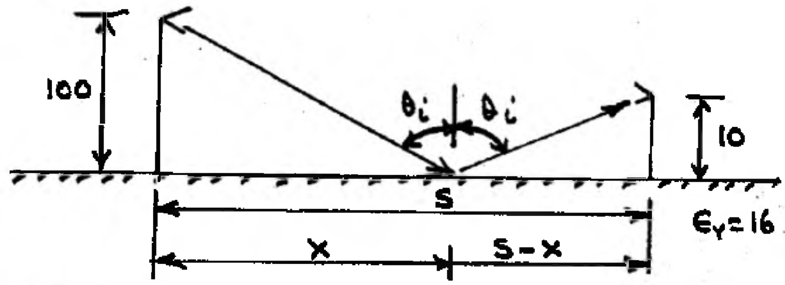
$\gamma_t = \tan^{-1}\left(\frac{|T^b_{\perp}|}{|T^b_{\parallel}|}\tan\gamma_i\right) = \tan^{-1}\left[\frac{0.0244}{0.111}(1)\right] = \tan^{-1}(0.2198) = 12.398°$

Therefore the transmitted wave is elliptically, right-hand polarized.

5.24

$\tan(90-\theta_i) = \frac{100}{x}$ (1)

$\tan(90-\theta_i) = \frac{10}{s-x}$ (2)



Therefore

$\frac{100}{x} = \frac{10}{s-x} \Rightarrow x = \frac{100}{110}s = \frac{10}{11}s \Rightarrow s = \frac{11}{10}x$

To eliminate reflections for the parallel-polarized component the incidence angle must be equal to the Brewster angle. Thus

$\theta_i = \tan^{-1}\left(\sqrt{\frac{\epsilon_2}{\epsilon_0}}\right) = \tan^{-1}(\sqrt{16}) = 75.96°$

Therefore from (1)

$x = \frac{100}{\tan(90° - 75.96°)} = \frac{100}{\tan(14.04°)} = \frac{100}{0.25} = 400 \text{ m} \Rightarrow s = \frac{11}{10}x = \frac{11}{10}(400) = 440 \text{ m}$

5.25 $\theta_c = \sin^{-1}\left(\sqrt{\frac{1}{81}}\right) = \sin^{-1}\left(\frac{1}{9}\right) = 6.379°$

$\tan\theta_c = \frac{x_1}{d} \Rightarrow x_1 = d\tan\theta_c = d\tan(6.379°) = 0.1118d$

Therefore $|x| \le 0.1118d$

5.26 a. $\theta_r = \theta_c = \sin^{-1}\left(\sqrt{\frac{\epsilon_0}{\epsilon}}\right) = \sin^{-1}\left(\frac{1}{\sqrt{\epsilon_r}}\right) = 30° \Rightarrow \frac{1}{\sqrt{\epsilon_r}} = \sin(30°) = \frac{1}{2} \Rightarrow \epsilon_r = 4$

Since $\theta_c \ge \theta_r \Rightarrow \epsilon_r \le 4$

b. $\beta\sin\theta_i = \beta_0\sin\theta_t = \beta_0\sin\theta_e$

$\theta_e = \sin^{-1}\left(\frac{\beta}{\beta_0}\sin\theta_i\right) = \sin^{-1}\left(\sqrt{\epsilon_r}\sin 30°\right) = \sin^{-1}\left(2\frac{1}{2}\right) = \sin^{-1}(1) = 90°$

5.27

a. $\theta_B = \tan^{-1}\left(\sqrt{\frac{\epsilon_0}{4\epsilon_0}}\right) = \tan^{-1}\left(\frac{1}{2}\right) = 26.565°$

b. $\theta_c = \sin^{-1}\left(\sqrt{\frac{\epsilon_0}{4\epsilon_0}}\right) = \sin^{-1}\left(\frac{1}{2}\right) = 30°$

5.28

a. $\beta_0 \sin\theta_i = \beta \sin\theta_t = \beta \sin\theta_c \Rightarrow \sin\theta_c = \frac{\beta_0}{\beta}\sin\theta_i$

$\theta_c = \sin^{-1}\left[\frac{1}{\sqrt{\epsilon_r}}\sin(90°)\right] = \sin^{-1}\left(\frac{1}{9}\right) = 6.379° \Rightarrow 2\theta_c = 12.758°$

b. $\tan\theta_c = \frac{D/2}{h} \Rightarrow h = \frac{D/2}{\tan\theta_c} = \frac{D/2}{\tan(6.379°)} = 4.472D$

5.29 a. To make the brightness temperature equal to the thermal temperature, the reflection coefficient must be zero. Therefore the incidence angle must be the Brewster angle and the polarization must be <u>parallel</u>.

b. $|\Gamma_{\parallel}| = 0 \Rightarrow \theta_i = \theta_B = \tan^{-1}\left(\sqrt{\frac{\epsilon}{\epsilon_0}}\right) = \tan^{-1}\left(\sqrt{81}\right) = \tan^{-1}(9) = 83.66°$

5.30 Since the electric field must be reduced to 0.368 of its original value, it must travel a distance equal to its skin depth. Since

$$\frac{\sigma}{\omega\epsilon} = \frac{3}{2\pi(10^4)\left(\frac{10^{-9}}{36\pi}\right)81} = \frac{3(18)}{81}\times10^5 = \frac{2}{3}\times10^5 >> 1$$

then

$$\delta = \sqrt{\frac{2}{\omega\mu\sigma}} = \sqrt{\frac{1}{f\pi\mu\sigma}} = \sqrt{\frac{1}{\pi(4\pi\times10^{-7})(10^4)3}} = \frac{1}{2\pi}\sqrt{\frac{1000}{3}} = 2.9057 \text{ meters}$$

5.31

a. $$\frac{\sigma}{\omega\epsilon} = \frac{1}{2\pi(10^6)\left(\frac{81\times10^{-9}}{36\pi}\right)} = \frac{18}{81}\times10^3 = \frac{2}{9}\times10^3 >> 1 \Rightarrow \text{Good conductor}$$

$$\eta = \sqrt{\frac{\omega\mu}{2\sigma}}(1+j) = \sqrt{\frac{2\pi\times10^6(4\pi\times10^{-7})}{2(1)}}(1+j) = 1.987(1+j) = 2.810\underline{\angle 45°}$$

$$\Gamma = \frac{\eta-\eta_0}{\eta+\eta_0} = \frac{1.987(1+j)-377}{1.987(1+j)+377} = \frac{-375.013+j1.987}{378.987+j1.987} = 0.9895\underline{\angle 179.4°}$$

$$E^r = |\Gamma|E^i = 0.9895(1\times10^{-3}) = 0.9895\times10^{-3} \text{ V/m}$$

b. $$SWR = \frac{1+|\Gamma|}{1-|\Gamma|} = \frac{1+0.9895}{1-0.9895} = 189.476$$

c. $$S^i = \frac{1}{2\eta_0}|E^i|^2 = \frac{1}{2(377)}(10^{-3})^2 = 1.326\times10^{-9} \text{ W/m}^2$$

$$S^r = \frac{1}{2\eta_0}|E^r|^2 = \frac{1}{2\eta_0}|\Gamma E^i|^2 = |\Gamma|^2 S^i = (0.9895)^2(1.326\times10^{-9}) = 1.2983\times10^{-9} \text{ W/m}^2$$

cont'd.

$\boxed{5.31 \text{ cont'd.}}$

d. $T = \frac{2\eta}{\eta+\eta_0} = \frac{2(2.810\angle 45^\circ)}{378.987+j1.987} = \frac{2(2.810\angle 45^\circ)}{378.992\angle 0.3} = 0.0148\angle 44.7^\circ$

$E^t = |T|E^i = 0.0148\times10^{-3}$ V/m

e. $S_L^t = S^i - S^r = S^i - |\Gamma|^2 S^i = (1-|\Gamma|^2)S^i = (1-|0.9895|^2)1.326\times10^{-9} = 0.0277\times10^{-9}$ W/m²

f. $\delta = \sqrt{\frac{2}{\omega\mu\sigma}} = \sqrt{\frac{1}{\pi f\mu\sigma}} = \sqrt{\frac{1}{\pi(10^6)(4\pi\times10^{-7})(1)}} = \frac{1}{2\pi}\sqrt{10} = 0.5033$ m

g. $\lambda = 2\pi\sqrt{\frac{2}{\omega\mu\sigma}} = 2\pi\delta = 2\pi(0.5033) = 3.1623$ m

$d = 20\lambda = 2(3.1623) = 63.2455$ m

h. $v = \sqrt{\frac{2\omega}{\mu\sigma}} = \sqrt{\frac{2(2\pi\times10^6)}{4\pi\times10^{-7}(1)}} = \sqrt{10}(10^6) \simeq 3.1623\times10^6$ m/sec.

$t = \frac{d}{v} = \frac{100}{3.1623\times10^6} = 31.623\times10^{-6}$ sec.

i. $\frac{v}{v_0} = \frac{3.1623\times10^6}{3\times10^8} = 1.054\times10^{-2}$

$\boxed{5.32}$ $\eta_2 \simeq \sqrt{\frac{\omega\mu_0}{2\sigma_2}}(1+j)$, $\eta_1 \simeq \sqrt{\frac{\omega\mu_0}{2\sigma_1}}(1+j)$

$$|\Gamma| = \left|\frac{\eta_2-\eta_1}{\eta_2+\eta_1}\right| = \left|\frac{\sqrt{\frac{1}{\sigma_2}}-\sqrt{\frac{1}{\sigma_1}}}{\sqrt{\frac{1}{\sigma_2}}+\sqrt{\frac{1}{\sigma_1}}}\right| = \left|\frac{\sqrt{\sigma_1}-\sqrt{\sigma_2}}{\sqrt{\sigma_1}+\sqrt{\sigma_2}}\right| = \frac{\sqrt{\sigma_1}-\sqrt{\sigma_2}}{\sqrt{\sigma_1}+\sqrt{\sigma_2}}$$

$$SWR = \frac{1+|\Gamma|}{1-|\Gamma|} \Rightarrow |\Gamma| = \frac{SWR-1}{SWR+1} = \frac{1.5-1}{1.5+1} = \frac{0.5}{2.5} = \frac{1}{5}$$

$$|\Gamma| = \frac{1}{5} = \frac{\sqrt{\sigma_1}-\sqrt{\sigma_2}}{\sqrt{\sigma_1}+\sqrt{\sigma_2}} \Rightarrow \sqrt{\sigma_1}+\sqrt{\sigma_2} = 5\sqrt{\sigma_1}-5\sqrt{\sigma_2} \Rightarrow 4\sqrt{\sigma_1} = 6\sqrt{\sigma_2} \Rightarrow \sqrt{\frac{\sigma_1}{\sigma_2}} = 1.5 \Rightarrow \frac{\sigma_1}{\sigma_2} = (1.5)^2 = 2.25$$

$\boxed{5.33}$ a. For right hand circular $\psi = 90^\circ$

b. $\underline{H}^i = (j\hat{a}_y - \hat{a}_z)\frac{E_0}{\eta_0}e^{+j\beta_0 x}$

c. $\frac{\sigma}{\omega\varepsilon} = \frac{10^{-1}}{2\pi(10^9)\left(\frac{81\times10^{-9}}{36\pi}\right)} = \frac{18}{81}\times10^{-1} \ll 1 \Rightarrow$ Good dielectric

$$\Gamma = \frac{\eta-\eta_0}{\eta+\eta_0} \simeq \frac{\sqrt{\mu_0/\varepsilon}-\sqrt{\mu_0/\varepsilon_0}}{\sqrt{\mu_0/\varepsilon}+\sqrt{\mu_0/\varepsilon_0}} = \frac{\sqrt{\varepsilon_0}-\sqrt{\varepsilon}}{\sqrt{\varepsilon_0}+\sqrt{\varepsilon}} = \frac{1-\sqrt{\varepsilon_r}}{1+\sqrt{\varepsilon_r}} = \frac{1-\sqrt{81}}{1+\sqrt{81}} = \frac{1-9}{1+9} = -0.8$$

$\underline{E}^r = -0.8(\hat{a}_y + j\hat{a}_z)E_0 e^{-j\beta_0 x}$

$\underline{H}^r = -\frac{0.8}{377}(\hat{a}_z - j\hat{a}_y)E_0 e^{-j\beta_0 x} = 2.122\times10^{-3}(j\hat{a}_y - \hat{a}_z)E_0 e^{-j\beta_0 x}$

cont'd.

5.33 Cont'd.

d. Circular, CCW

e. $T = \dfrac{2\eta}{\eta+\eta_0} \simeq \dfrac{2\sqrt{\frac{\mu_0}{\varepsilon}}}{\sqrt{\frac{\mu_0}{\varepsilon}}-\sqrt{\frac{\mu_0}{\varepsilon_0}}} = 2\dfrac{\frac{1}{\sqrt{\varepsilon}}}{\frac{1}{\sqrt{\varepsilon}}-\frac{1}{\sqrt{\varepsilon_0}}} = 2\dfrac{\sqrt{\varepsilon_0}}{\sqrt{\varepsilon_0}-\sqrt{\varepsilon}} = \dfrac{2}{1+\sqrt{\varepsilon_r}} = \dfrac{2}{1+9} = 0.2$

$\underline{E}^t = 0.2(\hat{a}_y + j\hat{a}_z)e^{+\gamma x} = 0.2(\hat{a}_y+j\hat{a}_z)e^{+(\alpha+j\beta)x}$

$\underline{H}^t = \dfrac{0.2}{\frac{377}{\sqrt{81}}}(j\hat{a}_y-\hat{a}_z)E_0e^{+(\alpha+j\beta)x} = (8)5.3\times10^{+4}(j\hat{a}_y-\hat{a}_z)E_0e^{+(\alpha+j\beta)x}$
$= 47.7454\times10^{-4}(j\hat{a}_y-\hat{a}_z)E_0e^{+(\alpha+j\beta)x}$

$\alpha \simeq \dfrac{\sigma}{2}\sqrt{\dfrac{\mu_0}{\varepsilon}} = \dfrac{10^{-1}}{2}\left(\dfrac{377}{9}\right) = 2.0944$ Np/m

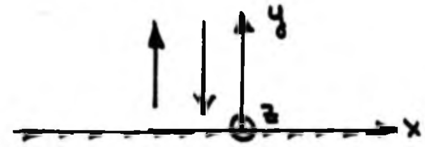
$\beta \simeq \omega\sqrt{\mu_0\varepsilon} = \beta_0\sqrt{81} = 9\beta_0 = 9\omega\sqrt{\mu_0\varepsilon_0} = \dfrac{9(2\pi\times10^9)}{3\times10^8} = 9(20.944) = 188.5$ rad/m

f. Circular, CW

g. $\dfrac{S^r}{S^i} = |\Gamma|^2 = |-0.8|^2 = 0.64 = 64\%$

$\dfrac{S^t}{S^i} = 1-|\Gamma|^2 = (1-|-0.8|^2) = 0.36 = 36\%$

5.34 a. Because the reflection coefficient is -1, the polarization is also left-hand circularly polarized.



b. $\underline{E}^i = (\hat{a}_x - j\hat{a}_z)e^{+j\beta y}$ (CW/RH)

$\underline{E}^r = -(\hat{a}_x - j\hat{a}_z)e^{-j\beta y}$ (CCW/LH)

$V = \cos\left(\dfrac{P_w P_a}{2}\right) = \cos\left(\dfrac{180}{2}\right) = \cos(90^\circ) = 0$ One @ north pole, the other @ south pole $\Rightarrow 2\gamma = 180^\circ$

c. $V = \cos\left(\dfrac{0^\circ}{2}\right) = \cos(0^\circ) = \cos(0^\circ) = 1$ Both @ north pole $\Rightarrow 2\gamma = 0^\circ$

5.35 $\dfrac{\sigma}{\omega\varepsilon} = \dfrac{4}{2\pi(10^7)\left(\frac{81\times10^{-9}}{36\pi}\right)} = \dfrac{4(18)}{81}\times10^2 = \dfrac{8}{9}\times10^2 \gg 1 \Rightarrow$ Good conductor

$\eta \simeq \sqrt{\dfrac{\omega\mu}{2\sigma}}(1+j) = \sqrt{\dfrac{\pi\times10^7(4\pi\times10^{-7})}{4}}(1+j) = \pi(1+j) = \sqrt{2}\pi\angle 45^\circ$

$\Gamma = \dfrac{\eta-\eta_0}{\eta+\eta_0} \simeq \dfrac{\pi+}{\pi+j\pi+377} = \dfrac{-373.858+j\pi}{380.142+j\pi} = \dfrac{373.871\angle 179.5185}{380.155\angle 0.4735^\circ} = 0.98345\angle 179.045^\circ$

Because the intrinsic impedance is very small and the reflection coefficient is almost -1, the results of this problem are almost identical to those of problem 5.34.

5.35 According to (5-20a)-(5-22d)

$$\underline{E}_{\parallel}^{i}=(\hat{a}_x\cos\theta_i-\hat{a}_z\sin\theta_i)E_0 e^{-j\beta_1(x\sin\theta_i+z\cos\theta_i)}$$

$$\underline{H}_{\parallel}^{i}=\hat{a}_y\frac{E_0}{\eta_1}e^{-j\beta_1(x\sin\theta_i+z\cos\theta_i)}$$

$$\underline{E}_{\parallel}^{r}=(\hat{a}_x\cos\theta_r+\hat{a}_z\sin\theta_r)\Gamma_{\parallel}^{b}E_0 e^{-j\beta_1(x\sin\theta_r-z\cos\theta_r)}$$

$$\underline{H}_{\parallel}^{r}=-\hat{a}_y\frac{\Gamma_{\parallel}^{b}E_0}{\eta_1}e^{-j\beta_1(x\sin\theta_r-z\cos\theta_r)}$$

$$\underline{E}_{\parallel}^{t}=(\hat{a}_x\cos\theta_t-\hat{a}_z\sin\theta_t)T_{\parallel}^{b}E_0 e^{-j\beta_2(x\sin\theta_t+z\cos\theta_t)}$$

$$\underline{H}_{\parallel}^{t}=\hat{a}_y\frac{T_{\parallel}^{b}E_0}{\eta_2}e^{-j\beta_2(x\sin\theta_t+z\cos\theta_t)}$$

When the incidence angle is greater than the critical angle, then the exponentials for the transmitted fields can be written as those given by (5-43a)-(5-43b) for the perpendicular polarization. Thus the transmitted fields take the form of

$$\underline{E}_{\parallel}^{t}=(\hat{a}_x\cos\theta_t-\hat{a}_z\sin\theta_t)T_{\parallel}^{b}E_0 e^{-\alpha_e z}e^{-j\beta_e x}$$

$$\underline{H}_{\parallel}^{t}=\hat{a}_y\frac{T_{\parallel}^{b}E_0}{\eta_2}e^{-\alpha_e z}e^{-j\beta_e x}$$

where $\alpha_e=\omega\sqrt{\mu_1\varepsilon_1\sin^2\theta_i-\mu_2\varepsilon_2}\Big|_{\theta_i>\theta_c}$

$\beta_e=\omega\sqrt{\mu_1\varepsilon_1}\sin\theta_i\Big|_{\theta_i>\theta_c}$

The reflection and transmission coefficients ($\Gamma_{\parallel}^{b}$ and $T_{\parallel}^{b}$) of (5-24c) and (5-24d) reduce, respectively, to

$$\Gamma_{\parallel}^{b}\Big|_{\theta_i>\theta_c}=\frac{-\sqrt{\frac{\mu_1}{\varepsilon_1}}\cos\theta_i+\sqrt{\frac{\mu_2}{\varepsilon_2}}\cos\theta_t}{\sqrt{\frac{\mu_1}{\varepsilon_1}}\cos\theta_i+\sqrt{\frac{\mu_2}{\varepsilon_2}}\cos\theta_t}=\frac{-\sqrt{\frac{\mu_1}{\varepsilon_1}}\cos\theta_i+j\sqrt{\frac{\mu_2}{\varepsilon_2}}\sqrt{\frac{\mu_1\varepsilon_1}{\mu_2\varepsilon_2}\sin^2\theta_i-1}}{\sqrt{\frac{\mu_1}{\varepsilon_1}}\cos\theta_i+j\sqrt{\frac{\mu_2}{\varepsilon_2}}\sqrt{\frac{\mu_1\varepsilon_1}{\mu_2\varepsilon_2}\sin^2\theta_i-1}}$$

$$=|\Gamma_{\parallel}^{b}|e^{j\psi_{\parallel}}$$

where $|\Gamma_{\parallel}^{b}|=1$

$$\psi_{\parallel}=\pi-\psi_0=\pi-\psi_0,\ \psi_0=\tan^{-1}\left(\frac{X_{\parallel}}{R_{\parallel}}\right),\ X_{\parallel}=\sqrt{\frac{\mu_2}{\varepsilon_2}}\sqrt{\frac{\mu_1\varepsilon_1}{\mu_2\varepsilon_2}\sin^2\theta_i-1},\ R_{\parallel}=\sqrt{\frac{\mu_1}{\varepsilon_1}}\cos\theta_i$$

cont'd.

5.36 Cont'd.

$$T_{11}^{b}=\frac{2\sqrt{\frac{\mu_2}{\epsilon_2}}\cos\theta_i}{\sqrt{\frac{\mu_1}{\epsilon_1}}\cos\theta_i+\sqrt{\frac{\mu_2}{\epsilon_2}}\cos\theta_t}=\frac{2\sqrt{\frac{\mu_2}{\epsilon_2}}\cos\theta_i}{\sqrt{\frac{\mu_1}{\epsilon_1}}\cos\theta_i+j\sqrt{\frac{\mu_2}{\epsilon_2}}\sqrt{\frac{\mu_1\epsilon_1}{\mu_2\epsilon_2}\sin^2\theta_i-1}}=|T_{11}^{b}|e^{-j\psi_0}$$

$$|T_{11}^{b}|=\frac{2R_{11}}{\sqrt{R_{11}^2+X_{11}^2}}$$

$$\underline{\underline{S}}_{av}^{i}=\frac{1}{2}\text{Re}\left(\underline{E}^{i}\times\underline{H}^{i*}\right)=\frac{1}{2}\text{Re}\left[\left(\hat{a}_xE_x^{i}+\hat{a}_zE_z^{i}\right)\times\hat{a}_yH_y^{*}\right]=\frac{1}{2}\text{Re}\left[\hat{a}_zE_x^{i}H_y^{i*}-\hat{a}_xE_z^{i}H_y^{i*}\right]$$

$$=\frac{1}{2}\left\{\hat{a}_z\frac{|E_0|^2}{\eta_1}\cos\theta_i+\hat{a}_x\frac{|E_0|^2}{\eta_1}\sin\theta_i\right\}=\frac{|E_0|^2}{2\eta_1}\left\{\hat{a}_z\cos\theta_i+\hat{a}_x\sin\theta_i\right\}$$

$$\underline{\underline{S}}_{av}^{r}=\frac{1}{2}\text{Re}\left(\underline{E}^{r}\times\underline{H}^{r}\right)=\frac{1}{2}\text{Re}\left[\left(\hat{a}_xE_x^{r}+\hat{a}_zE_z^{r}\right)\times\left(\hat{a}_yH_y^{r*}\right)\right]=\frac{1}{2}\text{Re}\left[\hat{a}_zE_x^{r}H_y^{r*}-\hat{a}_xE_z^{r}H_y^{r*}\right]$$

$$=\frac{1}{2}\left\{-\hat{a}_z\frac{|E_0|^2|\Gamma_{11}^{b}|^2}{\eta_1}\cos\theta_r+\hat{a}_x\frac{|E_0|^2|\Gamma_{11}^{b}|^2}{\eta_1}\sin\theta_r\right\}=\frac{|\Gamma_{11}^{b}|^2|E_0|^2}{2\eta_1}\left\{-\hat{a}_z\cos\theta_r+\hat{a}_x\sin\theta_r\right\}$$

$$\underline{\underline{S}}_{av}^{t}=\frac{1}{2}\text{Re}\left\{\underline{E}^{t}\times\underline{H}^{t*}\right\}=\frac{1}{2}\text{Re}\left\{\left(\hat{a}_xE_x^{t}+\hat{a}_zE_z^{t}\right)\times\hat{a}_yH_y^{t*}\right\}=\frac{1}{2}\text{Re}\left[\hat{a}_zE_x^{t}H_y^{t*}-\hat{a}_xE_z^{t}H_y^{t*}\right]$$

$$=\frac{\text{Re}}{2}\left\{\hat{a}_z\frac{|T_{11}^{b}|^2|E_0|^2}{\eta_2}\cos\theta_t+\hat{a}_x\frac{|T_{11}^{b}|^2|E_0|^2}{\eta_2}\sin\theta_t\right\}e^{-2\alpha_e z}=\frac{|T_{11}^{b}|^2|E_0|^2}{2\eta_2}\text{Re}\left[\hat{a}_z\cos\theta_t+\hat{a}_x\sin\theta_t\right]e^{-2\alpha_e z}$$

Since $\cos\theta_t=\pm j\sqrt{\frac{\mu_1\epsilon_1}{\mu_2\epsilon_2}\sin^2\theta_i-1}\Big|_{\theta_i>\theta_c}=\text{imaginary}$ [see (5-42b)]

$$\sin\theta_t=\sqrt{\frac{\mu_1\epsilon_1}{\mu_2\epsilon_2}}\sin\theta_i\Big|_{\theta_i>\theta_c}$$

then

$$\underline{\underline{S}}_{av}^{t}=\hat{a}_x\sqrt{\frac{\mu_1\epsilon_1}{\mu_2\epsilon_2}}\sin\theta_i\frac{|T_{11}^{b}|^2|E_0|^2}{2\eta_2}e^{-2\alpha_e z}\Big|_{\theta_i>\theta_c}$$

$\boxed{5.37}$

$\varepsilon_{r2} = 4,\ \theta_i = 60^\circ$

(a) $Z_w^i = \eta_0 = 377$ ohms because plane wave

$Z_w^r = \eta_0 = 377$ ohms because plane wave

$Z_w^t = \eta_2 = \dfrac{\eta_0}{\sqrt{\varepsilon_r}} = \dfrac{377}{2} = 188.5$ because plane wave

(b) $\boxed{Z_{z0}^{i+}(\text{directional})} = \dfrac{\eta_0}{\cos\theta_i} = \dfrac{377}{\cos(60^\circ)} = \dfrac{377}{0.5} = \boxed{754 \text{ ohms}}$

$Z_{z2}^{t+}(\text{directional}) = \dfrac{\eta_2}{\cos\theta_t}$

$\beta_0 \sin\theta_i = \beta_2 \sin\theta_t \Rightarrow \theta_t = \sin^{-1}\left[\dfrac{\beta_0}{\beta_2}\sin\theta_i\right] = \sin^{-1}\left[\dfrac{1}{\sqrt{\varepsilon_r}}\sin\theta_i\right]$

$= \sin^{-1}\left(\dfrac{1}{2}\sin(60^\circ)\right) = \sin^{-1}\left[\dfrac{0.866}{2}\right]$

$\theta_t = \sin^{-1}(0.433) = 25.669^\circ$

$\boxed{Z_{z2}^{t+}} = \dfrac{\eta_2}{\cos\theta_t} = \dfrac{\eta_0/\sqrt{\varepsilon_r}}{\cos(25.669^\circ)} = \dfrac{188.5}{0.9014} = \boxed{209.12 \text{ ohms}}$

(c) $\Gamma_{in}^{+z} = \dfrac{Z_{z2}^{t+} - Z_{z0}^{i+}}{Z_{z2}^{t+} + Z_{z0}^{i+}} = \dfrac{209.12 - 754}{209.12 + 754} = -\dfrac{544.88}{963.12} = -0.5657$

$\boxed{|\Gamma_{in}^{+z}| = 0.5657,\ \angle\Gamma_{in}^{+z} = 180^\circ}$

OR

$\Gamma_\perp = \dfrac{\cos\theta_i - \sqrt{\varepsilon_r}\sqrt{1 - \frac{1}{\varepsilon_r}\sin^2\theta_i}}{\cos\theta_i + \sqrt{\varepsilon_r}\sqrt{1 - \frac{1}{\varepsilon_r}\sin^2\theta_i}} = \dfrac{\cos(60^\circ) - \sqrt{4}\sqrt{1 - \frac{1}{4}\sin^2(60^\circ)}}{\cos(60^\circ) + \sqrt{4}\sqrt{1 - \frac{1}{4}\sin^2(60^\circ)}}$

$= \dfrac{0.5 - 2\sqrt{1 - 0.75/4}}{0.5 + 2\sqrt{1 - 0.75/4}} = \dfrac{0.5 - 2(0.9011388)}{0.5 + 2(0.9011388)} = \dfrac{0.5 - 1.802776}{0.5 + 1.802776}$

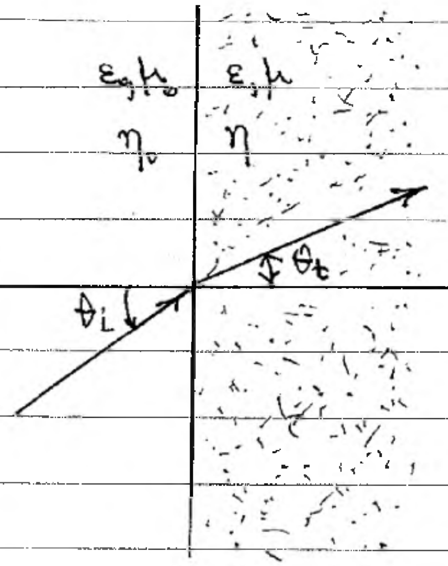
$\Gamma_\perp^{in} = -\dfrac{1.302766}{2.302766} = -0.5657 \Rightarrow \boxed{|\Gamma_\perp| = 0.5657,\ \angle\Gamma_\perp = 180^\circ}$

(d) $SWR = \dfrac{1 + |\Gamma_{in}|}{1 - |\Gamma_{in}|} = \dfrac{1 + 0.5657}{1 - 0.5657} = \dfrac{1.5657}{0.4343} = \boxed{3.605}$

5.38

(a) Snell's Law of Refraction

$$\beta_0 \sin\theta_i = \beta \sin\theta_t$$



$$\sin\theta_t = \frac{\beta_0}{\beta}\sin\theta_0 = \frac{\omega\sqrt{\mu_0\varepsilon_0}}{\omega\sqrt{\varepsilon\mu}}\sin\theta_0$$

$$= \frac{\sqrt{\mu_0\varepsilon_0}}{\sqrt{\mu_0\varepsilon_0\varepsilon_r\mu_r}}\sin\theta_0$$

$$\sin\theta_t = \frac{1}{\sqrt{\varepsilon_r\mu_r}}\sin\theta_0 \overset{\mu_r\varepsilon_r \gg 1}{\simeq} 0$$

$\theta_t \simeq \sin^{-1}(0) \simeq 0°$ for both $\perp$ and $\parallel$ polarizations

(b)

$$\Gamma_\perp = \frac{\eta\cos\theta_i - \eta_0\cos\theta_t}{\eta\cos\theta_i + \eta_0\cos\theta_t} \overset{\theta_t=0}{=} \frac{\eta\cos\theta_i - \eta_0}{\eta\cos\theta_i + \eta_0} = 0 \Rightarrow \eta\cos\theta_i - \eta_0 = 0$$

$$\theta_i = \cos^{-1}\left(\frac{\eta_0}{\eta_1}\right) = \cos^{-1}\left(\frac{\eta_0}{\eta_0\sqrt{\frac{\mu_r}{\varepsilon_r}}}\right) = \cos^{-1}\left(\sqrt{\frac{\varepsilon_r}{\mu_r}}\right)$$

exists if $\mu_r \geq \varepsilon_r$

$$\Gamma_\parallel = \left.\frac{-\eta_0\cos\theta_i + \eta\cos\theta_t}{\eta_0\cos\theta_i + \eta\cos\theta_t}\right| \overset{\theta_t=0}{=} \frac{-\eta_0\cos\theta_i + \eta}{\eta_0\cos\theta_i + \eta} = 0$$

$$-\eta_0\cos\theta_i + \eta = 0 \Rightarrow \cos\theta_i = \frac{\eta}{\eta_0}$$

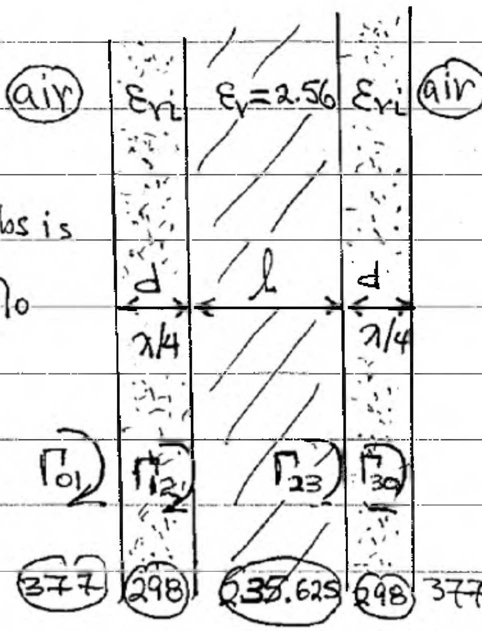
$$\theta_i = \cos^{-1}\left(\frac{\eta}{\eta_0}\right) = \cos^{-1}\left(\frac{\eta_0\sqrt{\frac{\mu_r}{\varepsilon_r}}}{\eta_0}\right) = \cos^{-1}\left(\sqrt{\frac{\mu_r}{\varepsilon_r}}\right)$$

exists if $\varepsilon_r \geq \mu_r$

$\boxed{5.39}$ Due to the symmetry of the problem, both dielectric slabs have the same thickness and dielectric constant.

(a) To match the slab on each side, we need $\lambda_i/4$ impedance transformers.

Thus $d = \lambda_i/4$



(d) The instrinsic impedance of the $\lambda/4$ slabs is

$$\eta_i = \sqrt{\eta\eta_o} = \sqrt{\frac{\eta_o}{\sqrt{2.56}}\eta_o} = \frac{\eta_o}{\sqrt{1.6}} = 0.7906\eta_o$$

$$\eta_i = 298 \text{ ohms}$$

(c) $$\eta_i = \sqrt{\frac{\mu_o}{\varepsilon_{ri}\varepsilon_o}} = \frac{\sqrt{\mu_o/\varepsilon_o}}{\sqrt{\varepsilon_{ri}}} = \frac{377}{\sqrt{\varepsilon_{ri}}} = 298$$

$$\sqrt{\varepsilon_{ri}} = 377/298 = 1.2651$$

$$\varepsilon_{ri} = 1.6$$

(b) The wavelength in the slabs is

$$\lambda = \frac{\lambda_o}{\sqrt{\varepsilon_{ri}}} = \frac{30\times10^9/10\times10^9}{\sqrt{1.6}} = \frac{3}{\sqrt{1.6}} = \frac{3}{1.2649} = 2.3717 \text{ cm}$$

Therefore the thickness of each slab is

$$d = \frac{\lambda}{4} = \frac{2.3717}{4} = 0.5929 \text{ cm}$$

(d) From above

$$\eta_i = 298 \text{ ohms}$$

(e) Since the $\lambda/4$, the total input reflection coefficient is zero thus the VSWR = 1

$$\Gamma_{01} = \frac{\eta_i - \eta_o}{\eta_i + \eta_o} = \frac{298-377}{298+377} = -0.117; \quad \Gamma_{12} = \frac{\eta - \eta_i}{\eta + \eta_i} = \frac{235.625-298}{235.625+298} = -0.1169$$

$$\Gamma_{23} = -\Gamma_{12} = +0.1169, \quad \Gamma_{30} = -\Gamma_{01} = 0.117$$

5.40

$$\left| \frac{\Gamma_{12}\left(1+e^{-j\pi x}\right)}{1+\left(\Gamma_{12}\right)^2 e^{-j\pi x}} \right| = \rho_m$$

$x = f / f_o; \quad f_o = 10$ GHz

$$\Gamma_{12} = \frac{1-\sqrt{2}}{1+\sqrt{2}} = \frac{-0.41421}{2.41421} = -0.1716$$

Using an iterative program or solving it otherwise, the following solutions are obtained.

....

1. $\rho_m = 0.05, \quad \Gamma_{12} = -0.1726,$

The lower (f_l) and upper (f_u) frequencies, and bandwidth (Δf), due to frequency response symmetry about f_o, are:

$f_l = 9.095$ GHz; $\quad f_u = 10.905$ GHz

$\Delta f = \left(f_u - f_l\right) = (10.905 - 9.095)$ GHz $= 1.81$ GHz

....

2. $\rho_m = 0.10, \quad \Gamma_{12} = -0.1726$

The lower (f_l) and upper (f_u) frequencies, and bandwidth (Δf), due to frequency response symmetry about f_o, are:

$f_l = 8.165$ GHz; $f_u = 11.835$ GHz

$\Delta f = \left(f_u - f_l\right) = (11.835 - 8.165)$ GHz $= 3.67$ GHz

$\boxed{5.41}$

Exact Transmission Line Model:

$$\Gamma_{in}\left(z=-d^{-}\right)=\frac{Z_{in}\left(z=-d^{+}\right)-\eta_1}{Z_{in}\left(z=-d^{+}\right)+\eta_1}=\frac{N}{D}$$

$$N=\eta_2\left[\left(\eta_3+\eta_2\right)+\left(\eta_3-\eta_2\right)e^{-j2\beta_2 d}\right]$$
$$-\eta_1\left[\left(\eta_3+\eta_2\right)-\left(\eta_3-\eta_2\right)e^{-j2\beta_2 d}\right]$$

$$D=\eta_2\left[\left(\eta_3+\eta_2\right)+\left(\eta_3-\eta_2\right)e^{-j2\beta_2 d}\right]$$
$$+\eta_1\left[\left(\eta_3+\eta_2\right)-\left(\eta_3-\eta_2\right)e^{-j2\beta_2 d}\right]$$

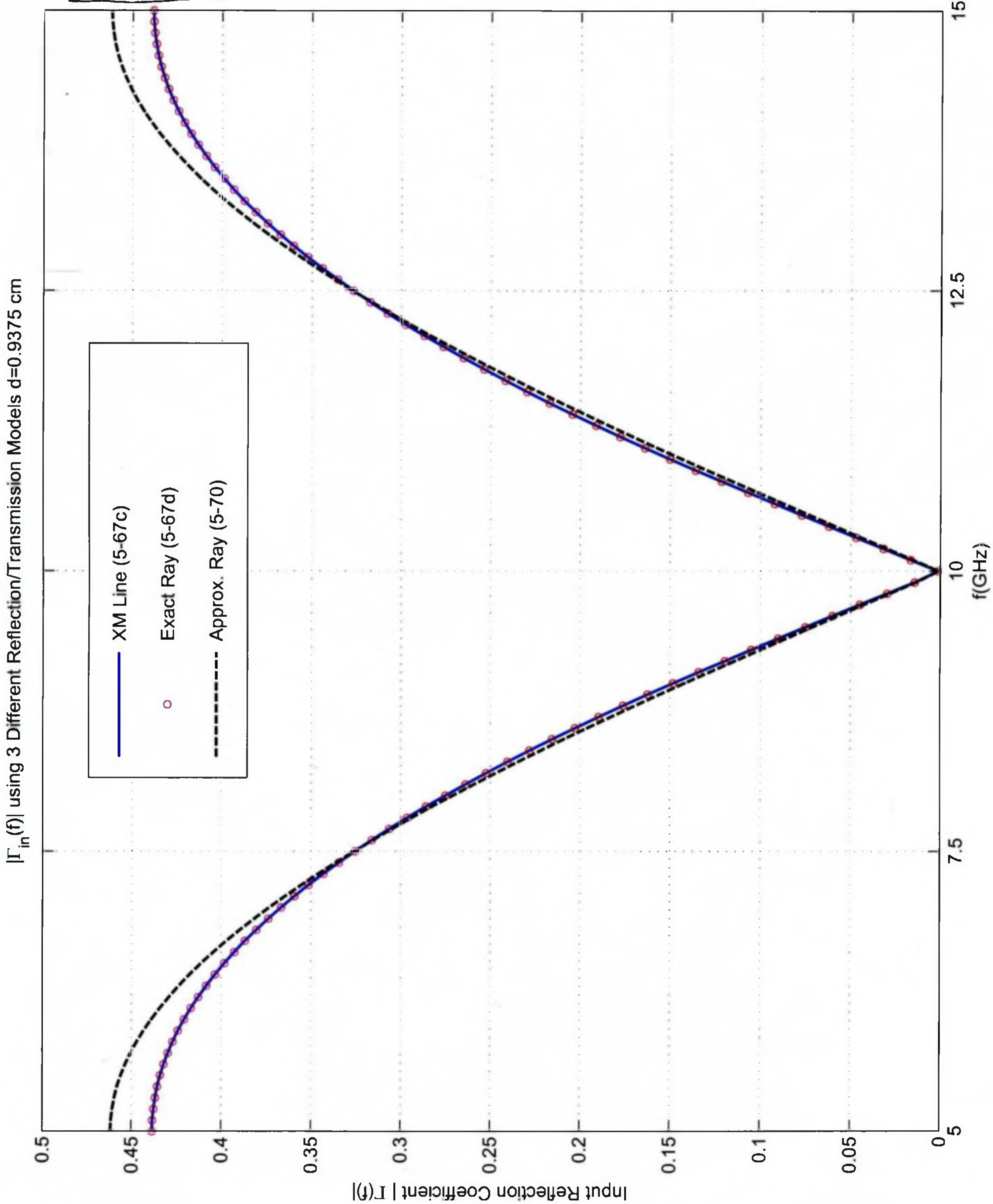
Exact Ray-Tracing Model:

$$\Gamma_{in}(z=-d^{-})=\frac{\Gamma_{12}+\Gamma_{23}e^{-j2\beta_2 d}}{1+\Gamma_{12}\Gamma_{23}e^{-j2\beta_2 d}}$$

Approximate Ray-Tracing Model:

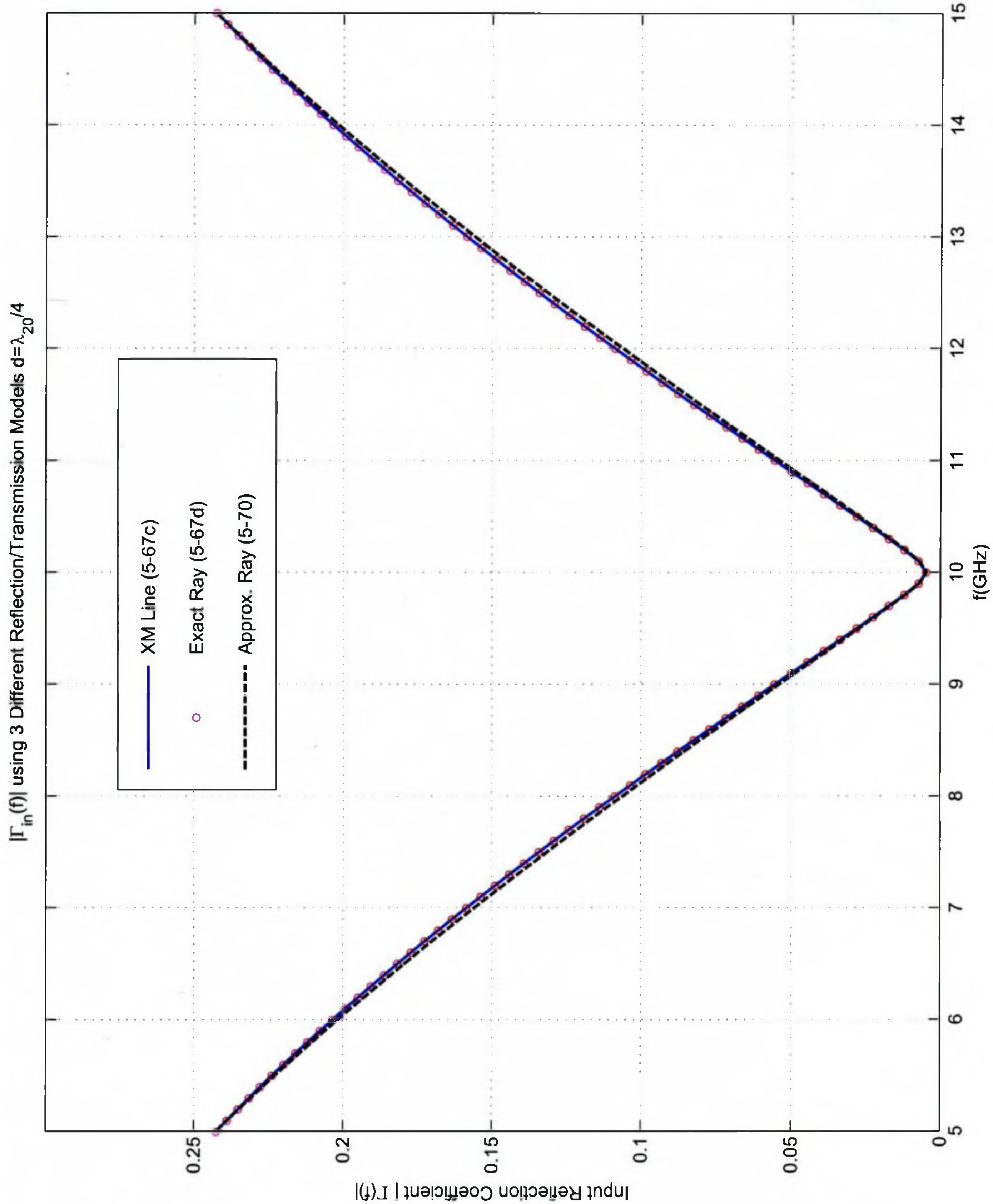
$$\Gamma_{in}(z=-d^{-})\underset{|\Gamma_{23}|<<1}{\overset{|\Gamma_{12}|<<1}{\simeq}}\Gamma_{12}+\Gamma_{23}e^{-j2\beta_2 d}$$

5.41 cont'd



|Γin(f)| using 3 Different Reflection/Transmission Models d=0.9375 cm
XM Line (5-67c)
Exact Ray (5-67d)
Approx. Ray (5-70)
Input Reflection Coefficient | Γ(f)|
f(GHz)
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5
5
7.5
10
12.5
15

5.41 cont'd



|Γin(f)| using 3 Different Reflection/Transmission Models d=λ20/4
XM Line (5-67c)
Exact Ray (5-67d)
Approx. Ray (5-70)
Input Reflection Coefficient | Γ(f)|
f(GHz)
0
0.05
0.1
0.15
0.2
0.25
5
6
7
8
9
10
11
12
13
14
15

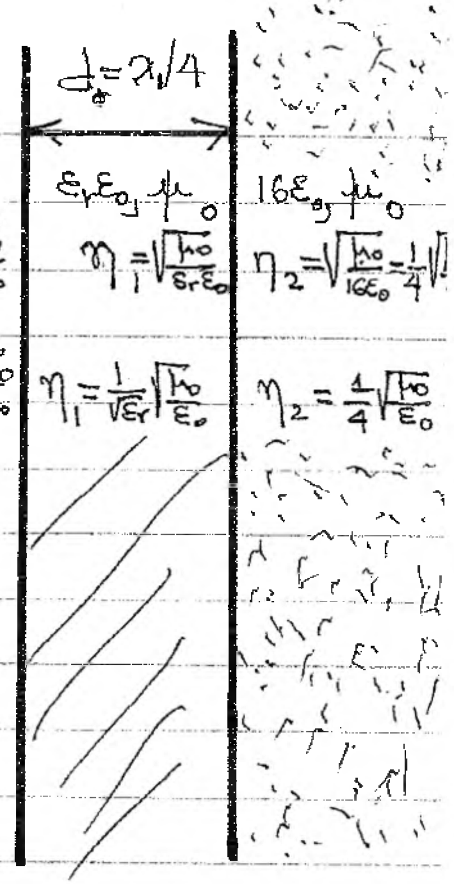
5.42

(a) $d = \lambda_1/4$

(b) $\eta_1 = \sqrt{\eta_0 \eta_2} = \left\{ \sqrt{\frac{\mu_0}{\varepsilon_0}} \left(\frac{1}{4} \sqrt{\frac{\mu_0}{\varepsilon_0}} \right) \right\}^{1/2}$

$$\eta_1 = \frac{1}{2}\sqrt{\frac{\mu_0}{\varepsilon_0}} = \sqrt{\frac{\mu_0}{\varepsilon_0}}\frac{1}{\sqrt{\varepsilon_r}}$$

$$\frac{1}{\sqrt{\varepsilon_r}} = \frac{1}{2} \Rightarrow \boxed{\varepsilon_r = 4}$$



(c) $\lambda_1 = \frac{v}{f} = \frac{v_0/\sqrt{\varepsilon_r}}{1\times10^9} = \frac{3\times10^{10}/\sqrt{4}}{1\times10^9}$

$$\lambda_1 = \frac{3}{2}\times10 = \frac{30}{2} = 15\text{cm}$$

$$d = \frac{\lambda_1}{4} = \frac{15}{4} = \boxed{3.75\text{ cm}}$$

5.43

a. $\Gamma_{01} = \frac{\eta_1 - \eta_0}{\eta_1 + \eta_0} = \frac{\frac{\eta_0}{2} - \eta_0}{\frac{\eta_0}{2} + \eta_0} = \frac{\frac{1}{2} - 1}{\frac{1}{2} + 1} = -\frac{1}{3}$

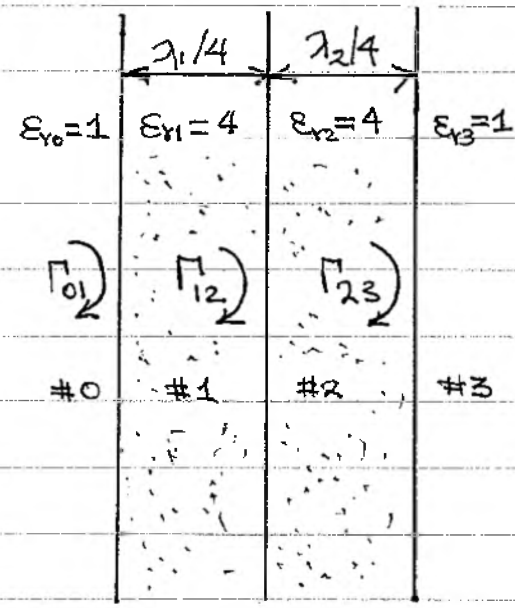
$$\boxed{\Gamma_{01} = -\frac{1}{3}}$$

$$\Gamma_{12} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{\frac{\eta_0}{2} - \frac{\eta_0}{2}}{\frac{\eta_0}{2} + \frac{\eta_0}{2}} = 0$$

$$\boxed{\Gamma_{12} = 0}$$

$$\Gamma_{23} = \frac{\eta_3 - \eta_2}{\eta_3 + \eta_2} = \frac{\eta_0 - \eta_0/2}{\eta_0 + \eta_0/2} = \frac{1 - 1/2}{1 + 1/2} = \frac{1}{3}$$

$$\boxed{\Gamma_{23} = \frac{1}{3}}$$



(b)

$$\Gamma_{in} = \Gamma_{01} + \Gamma_{12}e^{-j2\beta_1 d_1} + \Gamma_{23}e^{-j(2\beta_1 d_1 + 2\beta_2 d_2)} = \Gamma_{01} + \Gamma_{12}e^{-j2\left(\frac{2\pi}{\lambda_1}\right)\left(\frac{\lambda_1}{4}\right)} + \Gamma_{23}e^{-j2\left(\frac{2\pi}{\lambda_1}\frac{\lambda_1}{4} + \frac{2\pi}{\lambda_2}\frac{\lambda_2}{4}\right)}$$

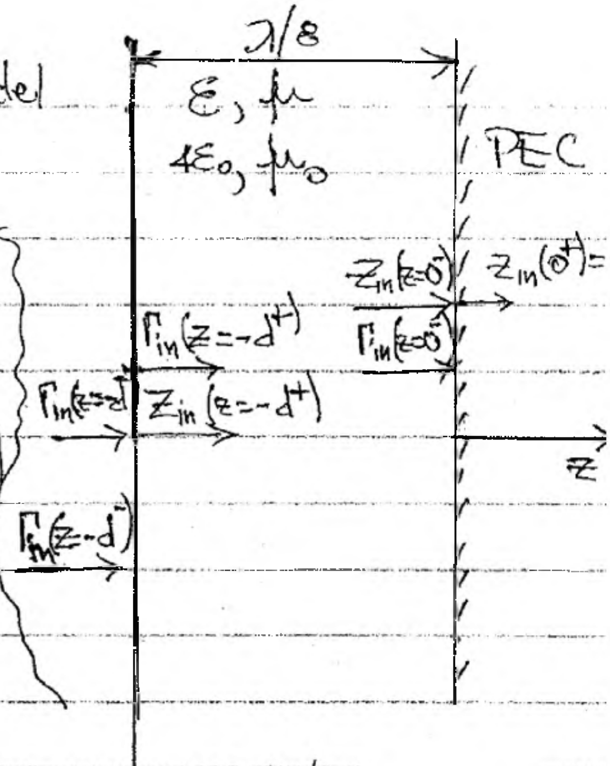
$$= \Gamma_{01} + \Gamma_{12}e^{-j\pi} + \Gamma_{23}e^{-j2\left(\frac{\pi}{2}+\frac{\pi}{2}\right)} = \Gamma_{01} + \Gamma_{12}e^{-j\pi} + \Gamma_{23}e^{-j2\pi}$$

$$\Gamma_{in} = -\frac{1}{3} + 0 + \frac{1}{3} = 0$$

$$\boxed{\Gamma_{in} = 0}$$

5.44

Using one of the transmission line model



(a) $Z_{in}(z=0^+)=0$ PEC

$$\Gamma_{in}(z=0^-)=\frac{Z_{in}(z=0^+)-\eta}{Z_{in}(z=0^+)+\eta}=\frac{0-\eta}{0+\eta}=-1$$

$$\Gamma_{in}(z=-d^+)=\Gamma_{in}(z=0^-)e^{-j2\beta d}=-1e^{-j2\left(\frac{2\pi}{\lambda}\right)\left(\frac{\lambda}{8}\right)}$$

$$=-1e^{-j\pi/2}=-1(-j)=+j$$

$$Z_{in}(z=-d^+)=\eta\left[\frac{1+\Gamma_{in}(z=-d^+)}{1-\Gamma_{in}(z=-d^+)}\right]$$

$$=\eta\left[\frac{1+j}{1-j}\right]=\eta\left[\frac{\sqrt{2}\,\angle 45^\circ}{\sqrt{2}\,\angle -45^\circ}\right]=\eta\angle{+90}=\eta e^{j\pi/2}=j\eta$$

$$Z_{in}(z=-d^+)=j\eta=j\frac{1}{2}\eta_0$$

$$\Gamma_{in}(z=-d^-)=\Gamma_{in}(\text{total})=\frac{Z_{in}(z=-d^+)-\eta_0}{Z_{in}(z=-d^+)+\eta_0}=\frac{j\frac{1}{2}\eta_0-\eta_0}{j\frac{1}{2}\eta_0+\eta_0}=\frac{j\frac{1}{2}-1}{j\frac{1}{2}+1}$$

$$=-\frac{1-j\frac{1}{2}}{1+j\frac{1}{2}}=-\frac{2-j}{2+j}=-\frac{\sqrt{5}\,\angle -26.565^\circ}{\sqrt{5}\,\angle +26.565^\circ}=-1\angle -53.13$$

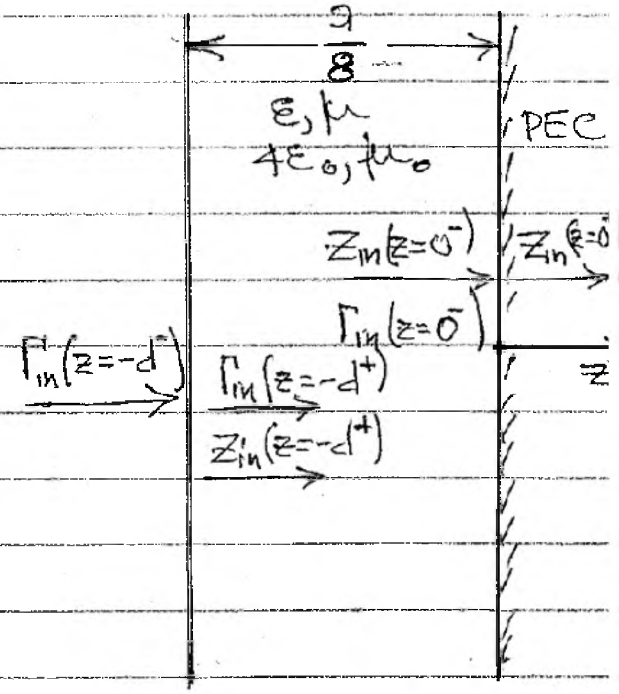
$$=-1(0.6-j0.8)=-0.6+j0.8=1\angle 180^\circ-53.13^\circ$$

$$\Gamma_{in}(\text{total})=1\angle 126.87^\circ \Rightarrow |\Gamma_{in}(\text{total})|=1$$

(b) $$\text{SWR}=\frac{1+|\Gamma_{in}(\text{total})|}{1-|\Gamma_{in}(\text{total})|}=\frac{1+1}{1-1}=\infty$$

$\boxed{\text{alternate}}$

(a) Using the other transmission line model or ray-tracing model



$$\Gamma(z=-d^-) = \frac{\Gamma_{12}+\Gamma_{23}e^{-j2\beta_2 d}}{1+\Gamma_{12}\Gamma_{23}e^{-j2\beta_2 d}}$$

$$\Gamma_{12} = \frac{\eta_2-\eta_1}{\eta_2+\eta_1} = \frac{\frac{1}{2}\eta_0-\eta_0}{\frac{1}{2}\eta_0+\eta_0} = \frac{+\frac{1}{2}-1}{\frac{1}{2}+1}$$

$$\Gamma_{12} = \frac{-\frac{1}{2}}{\frac{3}{2}} = -\frac{1}{3}$$

$$\Gamma_{23} = \frac{\eta_3-\eta_2}{\eta_3+\eta_2} = \frac{0-\eta_2}{0+\eta_2} = -1$$

$$2\beta_2 d = 2\left(\frac{2\pi}{\lambda_2}\right)\left(\frac{\lambda_2}{8}\right) = \frac{\pi}{2}$$

$$\Gamma_{in}(z=-d^-) = \frac{-\frac{1}{3}-1e^{-j\pi/2}}{1+\left(-\frac{1}{3}\right)(-1)e^{-j\pi/2}} = -\frac{\frac{1}{3}+e^{-j\pi/2}}{1+\frac{1}{3}e^{-j\pi/2}} = -\frac{\frac{1}{3}-j}{1-\frac{1}{3}j}$$

$$= -\frac{1-j3}{3-j} = -\frac{\sqrt{10}\,\angle -71.565}{\sqrt{10}\,\angle -18.43495} = -1\angle -53.13^\circ$$

$$\Gamma_{in}(z=-d^-) = -(0.6-j0.8) = -0.6+j0.8 = 1\angle 180^\circ-53.13^\circ$$

$$\Gamma_{in}(\text{total}) = 1\angle 126.87^\circ \Rightarrow |\Gamma_{in}(\text{total})| = 1, \quad \angle\Gamma_{in}(\text{total}) = 126.87^\circ$$

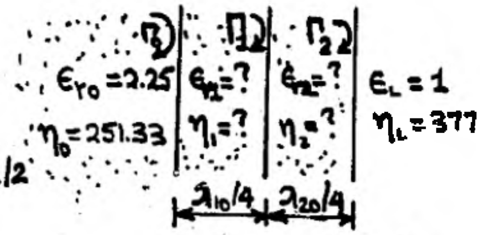
(b) $$SWR = \frac{1+|\Gamma_{in}(\text{total})|}{1-|\Gamma_{in}(\text{total})|} = \frac{1+1}{1-1} = \infty$$

$\boxed{5.45}$

$f_0 = 2$ GHz $N = 2$

$\frac{\Delta f}{f_0} = 0.5$ $\eta_L = 377$

$\eta_0 = 251.33$



a. $\frac{\Delta f}{f_0} = 0.5 = 2 - \frac{4}{\pi}\cos^{-1}\left[\frac{\Gamma_m}{(\eta_L-\eta_0)/(\eta_L+\eta_0)}\right]^{1/2}$

$$\Gamma_m = \frac{\eta_L-\eta_0}{\eta_L+\eta_0}\left\{\cos\left[\frac{\pi}{4}\left(2-\frac{\Delta f}{f_0}\right)\right]\right\}^2 = \frac{377-251.33}{377+251.33}\left\{\cos\left[\frac{\pi}{4}(2-0.5)\right]\right\}^2 = 0.0293$$

b. $\Gamma_n = 2^{-N}\frac{\eta_L-\eta_0}{\eta_L+\eta_0}\frac{N!}{(N-n)!n!}$

$n=0$: $\Gamma_0 = \frac{1}{4}\left(\frac{377-251.33}{377+251.33}\right)\frac{2!}{2!0!} = \frac{1}{4}(0.2) = 0.05$

$n=1$: $\Gamma_1 = \frac{1}{4}(0.2)\frac{2!}{1!1!} = 2(0.05) = 0.1$

$n=2$: $\Gamma_2 = \frac{1}{4}(0.2)\frac{2!}{(2-2)!2!} = 0.05$

c. $\Gamma_0 = 0.05 = \frac{\eta_1-\eta_0}{\eta_1+\eta_0} \Rightarrow \eta_1 = \eta_0\frac{1+\Gamma_0}{1-\Gamma_0} = 251.33\,\frac{1+0.05}{1-0.05} = 277.786 \Rightarrow \epsilon_{r1} = 1.842$

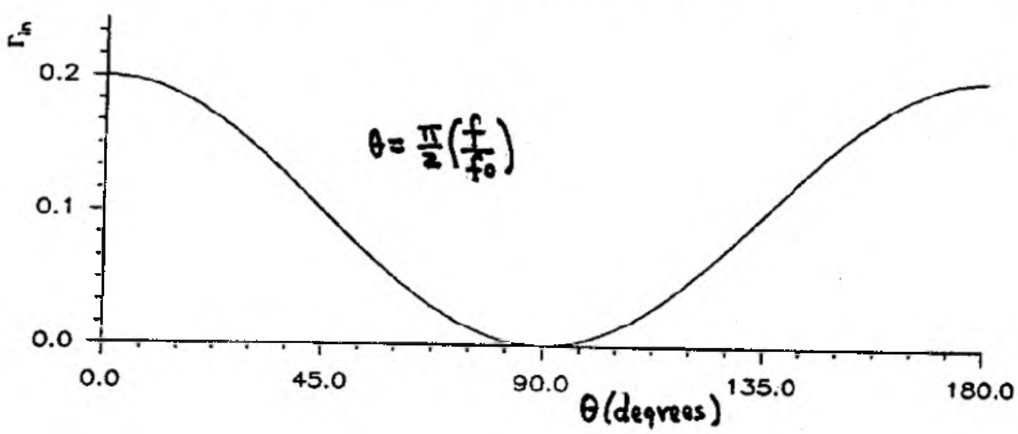
$$d_1 = \frac{\lambda_{10}}{4} = \frac{30\times10^9}{4(2\times10^9)\sqrt{1.842}} = 2.763 \text{ cm}$$

$\Gamma_1 = 0.1 = \frac{\eta_2-\eta_1}{\eta_2+\eta_1} \Rightarrow \eta_2 = \eta_1\frac{1+\Gamma_1}{1-\Gamma_1} = 277.786\,\frac{1+0.1}{1-0.1} = 339.516 \Rightarrow \epsilon_{r2} = 1.233$

$$d_2 = \frac{\lambda_{20}}{4} = \frac{30\times10^9}{4(2\times10^9)\sqrt{1.233}} = 3.377 \text{ cm}$$

d. $\frac{\Delta f}{f_0} = \frac{\Delta f}{2\text{ GHz}} = 0.5 \Rightarrow \Delta f = 1$ GHz $\Rightarrow f_\ell = f_0 - \frac{\Delta f}{2} = 1.5$ GHz, $f_u = f_0 + \frac{\Delta f}{2} = 2.5$ GHz

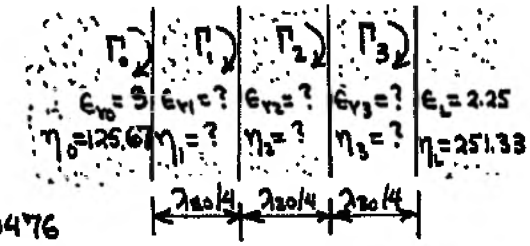
e. $|\Gamma_{in}(f)| = \frac{\eta_L-\eta_0}{\eta_L+\eta_0}\cos^2(\theta) = \frac{377-251.33}{377+251.33}\cos^2\left[\frac{\pi}{2}\frac{f}{f_0}\right] = 0.2\cos^2\left(\frac{\pi}{2}\frac{f}{f_0}\right)$



$\boxed{5.46}$ $f_0 = 3$ GHz $N=3$

$SWR|_{max} = 1.1$ $\eta_L = 251.33$

$\eta_0 = 125.67$



a. $SWR_m = \dfrac{1+\Gamma_m}{1-\Gamma_m} \Rightarrow \Gamma_m = \dfrac{SWR_m - 1}{SWR_m+1} = \dfrac{1.1-1}{1.1+1} = 0.0476$

$$\frac{\Delta f}{f_0} = 2 - \frac{4}{\pi}\cos^{-1}\left[\frac{\Gamma_m}{(\eta_L-\eta_0)/(\eta_L+\eta_0)}\right]^{1/3} = 2 - \frac{4}{\pi}\cos^{-1}\left[\frac{0.0476}{(251.33-125.67)/(251.33+125.67)}\right]^{1/3} =$$

$$\frac{\Delta f}{f_0} = 2 - \frac{4}{\pi}\cos^{-1}(0.1428)^{1/3} = 2 - \frac{4}{\pi}\cos^{-1}(0.5227) = 2 - 1.3 = 0.7 \Rightarrow \Delta f = 2.1 \text{ GHz}$$

$$f_l = f_0 - \frac{\Delta f}{2} = (3-1.05)\text{ GHz} = 1.95 \text{ GHz},\quad f_u = f_0 + \frac{\Delta f}{2} = (3+1.05)\text{ GHz} = 4.05 \text{ GHz}$$

b. $\Gamma_n = 2^{-N}\dfrac{\eta_L-\eta_0}{\eta_L+\eta_0}\dfrac{N!}{(N-n)!\,n!} = 2^{-3}\dfrac{251.33-125.67}{251.33+125.67}\dfrac{N!}{(N-n)!\,n!} = \dfrac{1}{24}\dfrac{N!}{(N-n)!\,n!}$

$n=0:\ \Gamma_0 = \dfrac{1}{24}\dfrac{3!}{(3-0)!\,0!} = \dfrac{1}{24}$

$n=1:\ \Gamma_1 = \dfrac{1}{24}\dfrac{3!}{(3-1)!\,1!} = \dfrac{1}{24}(3) = \dfrac{1}{8}$

$n=2:\ \Gamma_2 = \dfrac{1}{24}\dfrac{3!}{(3-2)!\,2!} = \dfrac{1}{24}(3) = \dfrac{1}{8}$

$n=3:\ \Gamma_3 = \dfrac{1}{24}\dfrac{3!}{(3-3)!\,3!} = \dfrac{1}{24}$

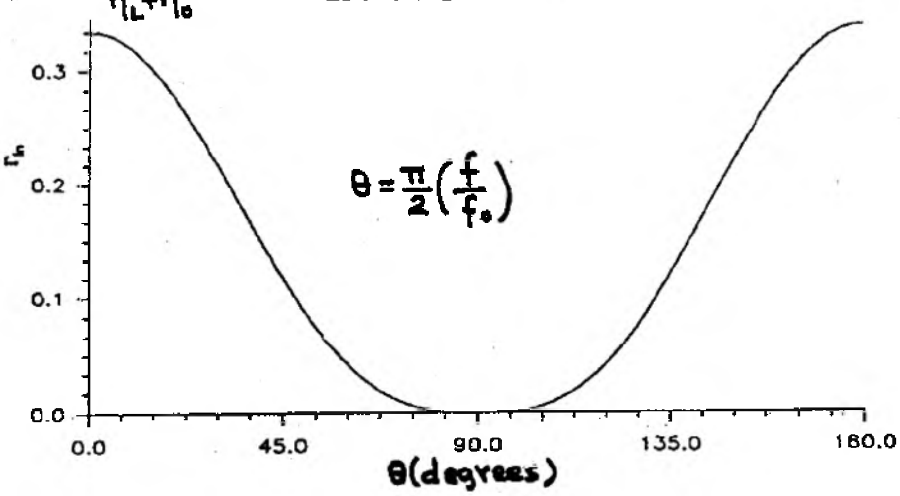
c. From Part a, $\Gamma_m = 0.0476$

d. $\Gamma_0 = \dfrac{1}{24} = \dfrac{\eta_1-\eta_0}{\eta_1+\eta_0} \Rightarrow \eta_1 = \eta_0\dfrac{1+\Gamma_0}{1-\Gamma_0} = 125.67\dfrac{1+\frac{1}{24}}{1-\frac{1}{24}} = 136.598,\ \epsilon_{r1} = 7.617,\ d_1 = 0.906$ cm

$\Gamma_1 = \dfrac{1}{8} = \dfrac{\eta_2-\eta_1}{\eta_2+\eta_1} \Rightarrow \eta_2 = \eta_1\dfrac{1+\Gamma_1}{1-\Gamma_1} = 136.598\dfrac{1+\frac{1}{8}}{1-\frac{1}{8}} = 175.626,\ \epsilon_{r2} = 4.608,\ d_2 = 1.165$ cm

$\Gamma_2 = \dfrac{1}{8} = \dfrac{\eta_3-\eta_2}{\eta_3+\eta_2} \Rightarrow \eta_3 = \eta_2\dfrac{1+\Gamma_2}{1-\Gamma_2} = 175.626\dfrac{1+1/8}{1-1/8} = 225.804,\ \epsilon_{r3} = 2.788,\ d_3 = 1.497$ cm

e. $|\Gamma_{in}(f)| = \dfrac{\eta_L-\eta_0}{\eta_L+\eta_0}\cos^3(\theta) = \dfrac{251.33-125.67}{251.33+125.67}\cos^3\left(\dfrac{\pi}{2}\dfrac{f}{f_0}\right) = \dfrac{1}{3}\cos^3\left(\dfrac{\pi}{2}\dfrac{f}{f_0}\right)$



$\boxed{5.47}$ $f_0 = 10$ GHz $\quad N=2$

$\frac{\Delta f}{f_0} = 0.375 \quad \eta_L = 188.5$

$\eta_0 = 377$

a. Using (5-75) for $N=2$

$$\Gamma_{in}(f) = 2e^{-j2\theta}\left[\Gamma_0 \cos(2\theta) + \tfrac{1}{2}\Gamma_1\right] \qquad (1)$$

which when equated to (5-81) leads to

$$2e^{-j2\theta}\left[\Gamma_0 \cos(2\theta) + \tfrac{1}{2}\Gamma_1\right] = e^{-j2\theta}\,\frac{\eta_L-\eta_0}{\eta_L+\eta_0}\,\frac{T_2(\sec\theta_m\cos\theta)}{T_2(\sec\theta_m)} \qquad (2)$$

or

$$2\Gamma_0\cos(2\theta) + \Gamma_1 = \frac{\eta_L-\eta_0}{\eta_L+\eta_0}\left(\frac{1}{T_2(\sec\theta_m)}\right)T_2(\sec\theta_m\cos\theta) \qquad (3)$$

Since the fractional bandwidth is 0.375, then

$$\Delta f = 2(f_0 - f_m)$$

$$\theta_m = \beta_m \ell = \frac{2\pi}{\lambda_m}\left(\frac{\lambda_0}{4}\right) = \frac{\pi}{2}\left(\frac{\lambda_0}{\lambda_m}\right) = \frac{\pi}{2}\left(\frac{f_m}{f_0}\right) \Rightarrow f_m = \frac{2}{\pi}f_0\theta_m$$

$$\Delta f = 2\left[f_0 - \frac{2}{\pi}f_0\theta_m\right] = 2f_0\left(1-\frac{2}{\pi}\theta_m\right) \Rightarrow \frac{\Delta f}{f_0} = 2\left(1-\frac{2}{\pi}\theta_m\right) = 2-\frac{4}{\pi}\theta_m$$

$$\theta_m = \frac{\pi}{4}\left(2-\frac{\Delta f}{f_0}\right) = \frac{\pi}{4}(2-0.375) = \frac{13}{32}\pi = 0.40625\pi \quad = 1.2763 = 73.125^\circ$$

Thus $\sec\theta_m = \sec(73.125^\circ) = 3.445$

$$T_2(\sec\theta_m) = 2\sec^2\theta_m - 1 = 2(3.445)^2 - 1 = 22.736 \qquad (4)$$

Therefore we can write (3) using (5-89) for $T_2(\sec\theta_m\cos\theta)$ as

$$2\Gamma_0\cos(2\theta) + \Gamma_1 = \frac{-377+188.5}{377+188.5}\,\frac{1}{22.736}\left[\sec^2\theta_m\cos(2\theta) + (\sec^2\theta_m - 1)\right]$$

$$= -\frac{1}{3}\left(\frac{1}{22.736}\right)\left[\sec^2\theta_m\cos(2\theta) + (\sec^2\theta_m-1)\right]$$

$$= -0.01466\left[\sec^2\theta_m\cos(2\theta) + (\sec^2\theta_m-1)\right]$$

$$= -0.01466(\sec^2\theta_m)\cos 2\theta - 0.01466(\sec^2\theta_m - 1)$$

$$= -0.01466(3.445)^2\cos(2\theta) - 0.01466\left[(3.445)^2-1\right]$$

$$2\Gamma_0\cos(2\theta) + \Gamma_1 = -0.174\cos(2\theta) - 0.1593$$

From these $\quad \Gamma_1 = -0.1593$

$$2\Gamma_0 = -0.174 \Rightarrow \Gamma_0 = -0.087$$

Cont'd.

5.47 cont'd.

$$n=0:\ \Gamma_0 = 0.087 = \frac{\eta_1-\eta_0}{\eta_1+\eta_0} \Rightarrow \eta_1 = \eta_0 \frac{1+\Gamma_0}{1-\Gamma_0} = 377\,\frac{1-0.087}{1+0.087} = 316.65$$

$$n=1:\ \Gamma_1 = -0.1593 = \frac{\eta_2-\eta_1}{\eta_2+\eta_1} \Rightarrow \eta_2 = \eta_1 \frac{1+\Gamma_1}{1-\Gamma_1} = 316.65\,\frac{1-0.1593}{1+0.1593} = 229.63$$

$$n=2:\ \Gamma_2 = \Gamma_0 = -0.087 = \frac{\eta_3-\eta_2}{\eta_3+\eta_2}$$

$$\epsilon_{r1} = \left(\frac{377}{316.65}\right)^2 = 1.4175 \Rightarrow d_1 = \frac{\lambda_{10}}{4} = \frac{30\times10^9}{4(10\times10^9)\sqrt{1.4175}} = 0.63 \text{ cm}$$

$$\epsilon_{r2} = \left(\frac{377}{229.63}\right)^2 = 2.695 \Rightarrow d_2 = \frac{\lambda_{20}}{4} = \frac{30\times10^9}{4(10\times10^9)\sqrt{2.695}} = 0.457 \text{ cm}$$

b. Within the bandwidth, the maximum value of (5-81) occurs when $\theta=\theta_m$. Thus

$$|\Gamma_{in}|_{max} = \Gamma_m = \left|\frac{\eta_L-\eta_0}{\eta_L+\eta_0}\right| \frac{T_2(\sec\theta_m\cos\theta_m)}{T_2(\sec\theta_m)} = \left|\frac{\eta_L-\eta_0}{\eta_L+\eta_0}\right| \frac{T_2(1)}{T_2(\sec\theta_m)} = \left|\frac{\eta_L-\eta_0}{\eta_L+\eta_0}\right| \frac{1}{T_2(\sec\theta_m)}$$

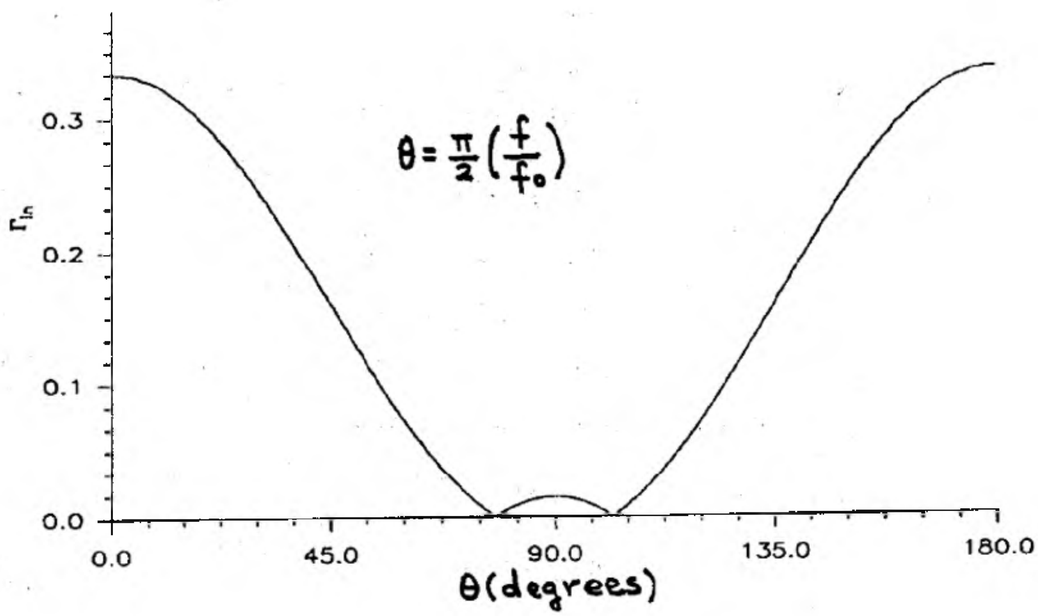
Since $T_2(1)=1$

Thus $$\Gamma_m = \frac{1}{3}\left(\frac{1}{22.736}\right) = 0.01466$$

$$SWR_m = \frac{1+\Gamma_m}{1-\Gamma_m} = \frac{1+0.01466}{1-0.01466} = 1.03$$

c. $$|\Gamma_{in}(f)| = \left|\frac{\eta_L-\eta_0}{\eta_L+\eta_0}\right| \frac{1}{T_2(\sec\theta_m)} T_2(\cos\theta\sec\theta_m) = \Gamma_m T_2(\sec\theta_m\cos\theta) = 0.174\cos(2\theta)+0.1593$$

$$|\Gamma_{in}(f)| = 0.174\cos(2\theta)+0.1593, \quad \theta = \frac{\pi}{2}\left(\frac{f}{f_0}\right)$$



$\boxed{5.48}$ $f_0 = 2$ GHz $N=2$

$\frac{\Delta f}{f_0} = 0.5$ $\eta_L = 377$

$\eta_0 = 251.33$

a. Using (5-78) and (5-79)

$$\frac{\Delta f}{f_0} = 2 - \frac{4}{\pi}\theta_m \Rightarrow \theta_m = \frac{\pi}{4}\left(2 - \frac{\Delta f}{f_0}\right) = \frac{\pi}{4}(2-0.5) = \frac{3\pi}{8} = 67.5^\circ \Rightarrow \begin{cases} \sec\theta_m = 2.613 \\ \sec^2\theta_m = 6.828 \\ \sec^2\theta_m - 1 = 5.828 \end{cases}$$

$$T_2(\sec\theta_m) = 2\sec^2\theta_m - 1 = 2(6.828) - 1 = 12.656$$

Within the bandwidth the maximum value of (5-81) occurs when $\theta = \theta_m$. Then $T_2(\sec\theta_m \cos\theta_m) = T_2(1) = 1$. Thus

$$|\Gamma_{in}|_{max} = \Gamma_m = \left|\frac{\eta_L - \eta_0}{\eta_L + \eta_0}\right| \frac{1}{T_2(\sec\theta_m)} = \frac{0.2}{12.656} = 0.0158$$

b. Using (5-75) for $N=2$ and equating it to (5-81) leads to

$$\Gamma_{in}(f) = 2e^{-j2\theta}\left[\Gamma_0 \cos(2\theta) + \frac{1}{2}\Gamma_1\right] = e^{-j2\theta}\frac{\eta_L - \eta_0}{\eta_L + \eta_0}\frac{1}{T_2(\sec\theta_m)} T_2(\sec\theta_m \cos\theta)$$

$$2\Gamma_0 \cos(2\theta) + \Gamma_1 = \Gamma_m T_2(\sec\theta_m \cos\theta)$$

By using (5-83) for $T_2(\sec\theta_m\cos\theta)$

$$\begin{aligned} 2\Gamma_0\cos(2\theta) + \Gamma_1 &= \Gamma_m\left[\sec^2\theta_m \cos(2\theta) + (\sec^2\theta_m - 1)\right] \\ &= 0.0158\left[6.828\cos(2\theta) + 5.828\right] \\ 2\Gamma_0\cos(2\theta) + \Gamma_1 &= 0.108\cos(2\theta) + 0.092 \end{aligned}$$

From these

$$2\Gamma_0 = 0.108 \Rightarrow \Gamma_0 = 0.054$$

$$\Gamma_1 = 0.092$$

c. $n=0$: $\Gamma_0 = 0.054 = \frac{\eta_1 - \eta_0}{\eta_1 + \eta_0} \Rightarrow \eta_1 = \eta_0 \frac{1+\Gamma_0}{1-\Gamma_0} = 251.33 \frac{1+0.054}{1-0.054} = 280.023$

$$\epsilon_{r1} = \left(\frac{377.0}{280.023}\right)^2 = 1.812, \quad d_1 = \frac{\lambda_{10}}{4} = \frac{30\times10^9}{4(2\times10^9)\sqrt{1.812}} = 2.785 \text{ cm}$$

$n=1$: $\Gamma_1 = 0.092 = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} \Rightarrow \eta_2 = \eta_1 \frac{1+\Gamma_1}{1-\Gamma_1} = 280.023 \frac{1+0.092}{1-0.092} = 336.768$

$$\epsilon_{r2} = \left(\frac{377}{336.768}\right)^2 = 1.253 \Rightarrow d_2 = \frac{\lambda_{20}}{4} = \frac{30\times10^9}{4(2\times10^9)\sqrt{1.253}} = 3.35 \text{ cm}$$

d. From the solution of Problem 5.45

$$f_l = f_0 - \frac{\Delta f}{2} = \left(2 - \frac{1}{2}\right)\text{GHz} = 1.5 \text{ GHz}, \quad f_h = f_0 + \frac{\Delta f}{2} = \left(2 + \frac{1}{2}\right)\text{GHz} = 2.5 \text{ GHz}$$

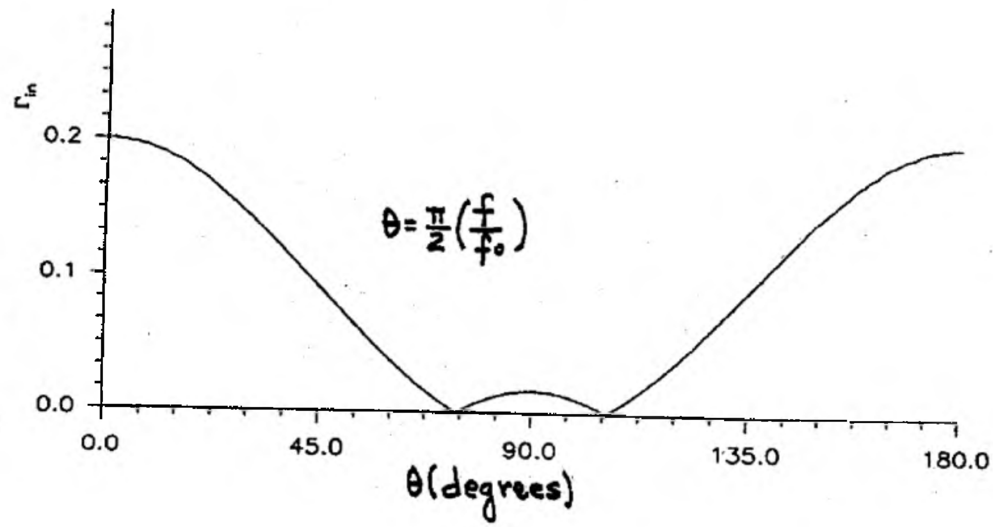
Cont'd.

5.48 Cont'd.

e. $$|\Gamma_{in}(f)| = \left|\frac{\eta_L-\eta_0}{\eta_L+\eta_0}\right| \frac{1}{T_2(\sec\theta_m)} T_2(\cos\theta\sec\theta_m) = \Gamma_m T_2(\sec\theta_m\cos\theta)$$

$$= 0.0158\, T_2(\sec\theta_m\cos\theta) = 0.108\cos(2\theta)+0.092$$

$$|\Gamma_{in}(f)| = 0.108\cos(2\theta)+0.092, \quad \theta = \frac{\pi}{2}\left(\frac{f}{f_0}\right)$$



$\boxed{5.49}$ $f_0 = 3$ GHz $N=3$
$SWR_{max} = 1.1$ $\eta_L = 251.33$
$\eta_0 = 125.67$

a. $SWR_m = \dfrac{1+\Gamma_m}{1-\Gamma_m} \Rightarrow \Gamma_m = \dfrac{SWR_m - 1}{SWR_m + 1} = \dfrac{1.1-1}{1.1+1} = 0.0476$

Within the bandwidth the maximum value of (5-81) occurs when $\theta = \theta_m$.
Then $T_3(\sec\theta_m \cos\theta_m) = T_3(1) = 1$. Thus

$$|\Gamma_{in}(f)|_{max} = \Gamma_m = \left|\frac{\eta_L - \eta_0}{\eta_L + \eta_0}\right| \frac{1}{T_3(\sec\theta_m)} = \frac{1}{3}\,\frac{1}{T_3(\sec\theta_m)} \Rightarrow T_3(\sec\theta_m) = \frac{1}{3\Gamma_m} = 7$$

From (5-83)

$$T_3(\sec\theta_m = 4(\sec\theta_m)^3 - 3\sec\theta = 7 = \sec\theta_m\left[4\sec^2\theta_m - 3\right]$$

By iteration $\theta_m = 44.87° \Rightarrow \begin{cases} \sec\theta_m = 1.411 \\ \sec^2\theta_m = 1.991 \\ \sec^3\theta_m = 2.809 \end{cases}$
$\theta_m = 0.783$ rad.

Therefore

$$\frac{\Delta f}{f_0} = 2\left(1 - \frac{2}{\pi}\theta_m\right) = 1.003 \simeq 1$$

b. Using (5-75) for $N=3$ and equating it to (5-81) for $N=3$, lead to

$$\Gamma_{in}(f) = 2e^{-j3\theta}\left[\Gamma_0\cos(3\theta) + \Gamma_1\cos(\theta)\right] = e^{-j3\theta}\frac{\eta_L - \eta_0}{\eta_L + \eta_0}\,\frac{T_3(\sec\theta_m\cos\theta)}{T_3(\sec\theta_m)}$$

$$2\left[\Gamma_0\cos(3\theta) + \Gamma_1\cos(\theta)\right] = \frac{\eta_L - \eta_0}{\eta_L + \eta_0}\,\frac{1}{T_3(\sec\theta_m)}T_3(\sec\theta_m\cos\theta) = \Gamma_m T_3(\sec\theta_m\cos\theta)$$

$$2\left[\Gamma_0\cos(3\theta) + \Gamma_1\cos(\theta)\right] = \Gamma_m T_3(\sec\theta_m\cos\theta)$$

Using the expansion of (5-89) to represent $T_3(\sec\theta_m\cos\theta)$, we can write that

$$2\left[\Gamma_0\cos(3\theta) + \Gamma_1\cos(\theta)\right] = 0.0476\left[\sec^3\theta_m\cos(3\theta) + 3(\sec^3\theta_m - \sec\theta_m)\cos\theta\right]$$

$$= 0.0476(2.809)\cos(3\theta) + 0.0476(3)(2.809 - 1.411)\cos\theta$$

$$2\left[\Gamma_0\cos(3\theta) + \Gamma_1\cos(\theta)\right] = 0.1337\cos(3\theta) + 0.1996\cos(\theta)$$

Matching terms

$$2\Gamma_0 = 0.1337 \Rightarrow \Gamma_0 = 0.06685$$
$$2\Gamma_1 = 0.1996 \Rightarrow \Gamma_1 = 0.0998$$

From symmetry
$$\Gamma_2 = \Gamma_1 = 0.0998$$
$$\Gamma_3 = \Gamma_0 = 0.06685$$

cont'd.

$\boxed{5.49 \text{ cont'd.}}$

c. From Part a, $\Gamma_m = 0.0476$

d. $\Gamma_0 = 0.06685 = \dfrac{\eta_1 - \eta_0}{\eta_1 + \eta_0} \Rightarrow \eta_1 = \eta_0 \dfrac{1+\Gamma_0}{1-\Gamma_0} = 125.67\, \dfrac{1+0.06685}{1-0.06685} = 143.67$

$\Gamma_1 = 0.0998 = \dfrac{\eta_2 - \eta_1}{\eta_2 + \eta_1} \Rightarrow \eta_2 = \eta_1 \dfrac{1+\Gamma_1}{1-\Gamma_1} = 143.67\, \dfrac{1+0.0998}{1-0.0998} = 175.533$

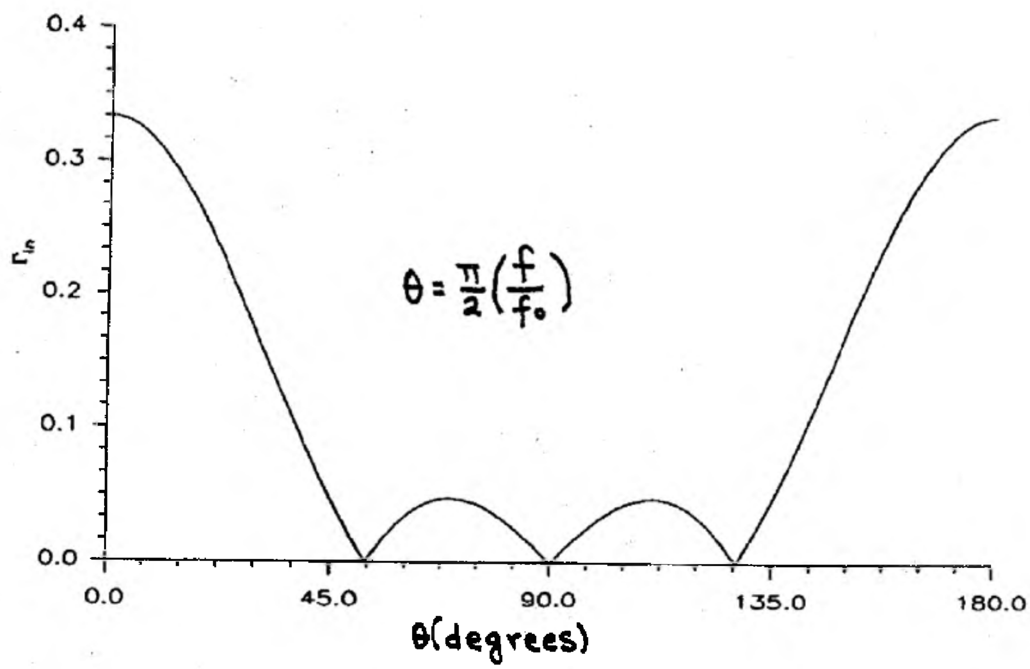
$\Gamma_2 = 0.0998 = \dfrac{\eta_3 - \eta_2}{\eta_3 + \eta_2} \Rightarrow \eta_3 = \eta_2 \dfrac{1+\Gamma_2}{1-\Gamma_2} = 175.533\, \dfrac{1+0.0998}{1-0.0998} = 214.453$

$\epsilon_{r1} = \left(\dfrac{377}{143.67}\right)^2 = 6.886 \Rightarrow d_1 = \dfrac{30\times10^9}{4(3\times10^9)\sqrt{6.886}} = 0.953 \text{ cm}$

$\epsilon_{r2} = \left(\dfrac{377}{175.533}\right)^2 = 4.613 \Rightarrow d_2 = \dfrac{30\times10^9}{4(3\times10^9)\sqrt{4.613}} = 1.164 \text{ cm}$

$\epsilon_{r3} = \left(\dfrac{377}{214.453}\right)^2 = 3.090 \Rightarrow d_3 = \dfrac{30\times10^9}{4(3\times10^9)\sqrt{3.090}} = 1.422 \text{ cm}$

e. $\Gamma_{in}(f) = 0.1337\cos(3\theta) + 0.1996\cos(\theta)$ where $\theta = \dfrac{\pi}{2}\left(\dfrac{f}{f_0}\right)$



5.50 $\theta_i = 30^\circ,\ AR = -2$

$$\gamma_i = \tan^{-1}\left(\frac{E_\perp^0}{E_\parallel^0}\right) = \tan^{-1}\left(\frac{1}{2}\right) = 26.565^\circ$$

$$\delta_i = \phi_\perp^i - \phi_\parallel^i = -90^\circ$$

$$\epsilon^i = \cot^{-1}(AR^i) = \cot^{-1}(-2) = -26.565^\circ$$

$$\gamma^i = \frac{1}{2}\sin^{-1}\left[\frac{\sin(2\epsilon^i)}{\sin\delta^i}\right] = \frac{1}{2}\sin^{-1}\left[\frac{\sin(-53.13^\circ)}{\sin(-90^\circ)}\right] = 26.565^\circ = 0.4636 \text{ rad.}$$

$$\tan\gamma^i = \tan(26.565^\circ) = 0.5$$

$$\Gamma_\parallel^b = -1 \Rightarrow |\Gamma_\parallel^b| = 1,\ \zeta_\parallel^r = 180^\circ$$

$$\Gamma_\perp^b = -1 \Rightarrow |\Gamma_\perp^b| = 1,\ \zeta_\perp^r = 180^\circ$$

$$\gamma^r = \tan^{-1}\left[\frac{|\Gamma_\perp^b|}{|\Gamma_\parallel^b|}\tan\gamma^i\right] = \tan^{-1}\left(\frac{1}{2}\right) = 26.565^\circ$$

$$\delta^r = (\delta^i - \pi) + (\zeta_\perp^r - \zeta_\parallel^r) = -\frac{\pi}{2} - \pi = -\frac{3\pi}{2} = \frac{\pi}{2}$$

$$2\epsilon^r = \sin^{-1}[\sin(2\gamma^r)\sin(\delta^r)] = \sin^{-1}[\sin(53.13^\circ)\sin(90^\circ)] = \sin^{-1}(0.8)$$

$$2\epsilon^r = 53.13$$

$$\epsilon^r = 26.565^\circ$$

$$AR^r = \cot(26.665^\circ) = 2$$

Therefore the reflected field is elliptically polarized with CCW rotation and an axial ratio of $AR = 2$

5.51 $\theta_i = 30^\circ,\ AR = -2$

$$\delta_i = \phi_\perp^i - \phi_\parallel^i = -90^\circ$$

$$\epsilon^i = \cot^{-1}(AR^i) = \cot^{-1}(-2) = -26.565^\circ$$

$$\gamma^i = \frac{1}{2}\sin^{-1}\left[\frac{\sin 2\epsilon^i}{\sin\delta^i}\right] = \frac{1}{2}\sin^{-1}\left[\frac{\sin(-53.13^\circ)}{\sin(-90^\circ)}\right] = 26.565^\circ$$

$$\tan\gamma_i = 0.5$$

From Example 5.13

$$\Gamma_\perp^b = -0.824 \Rightarrow |\Gamma_\perp^b| = 0.824,\ \zeta_\perp^r = 180^\circ;\quad \Gamma_\parallel^b = -0.773 \Rightarrow |\Gamma_\parallel^b| = 0.773,\ \zeta_\parallel^r = 180^\circ$$

$$\gamma^r = \tan^{-1}\left[\frac{|\Gamma_\perp^b|}{|\Gamma_\parallel^b|}\tan\gamma^i\right] = \tan^{-1}\left(\frac{0.824}{0.773}\frac{1}{2}\right) = 28.06^\circ$$

$$\delta^r = (\delta^i - \pi) + (\zeta_\perp^r - \zeta_\parallel^r) = -90^\circ - 180^\circ + (180^\circ - 180^\circ) = -270^\circ = +90^\circ$$

$\delta^r = +90^\circ$ so that $-180^\circ < \delta^r < 180^\circ$

$$2\epsilon^r = \sin^{-1}[\sin(2\gamma^r)\sin\delta^r] = \sin^{-1}[\sin(56.114^\circ)\sin(+90^\circ)] = 56.114^\circ$$

$$\epsilon_r = 28.057^\circ = 0.4897 \text{ rad.}$$

$$AR^r = \cot(\epsilon^r) = 1.8762$$

Therefore the reflected field is elliptically polarized with counter-clockwise rotation (CCW) and with an axial ratio of $AR = 1.8762$.

From Example 5.13

$$T_\perp^b = 0.1758 \Rightarrow |T_\perp^b| = 0.1758,\ \xi_\perp^t = 0^\circ$$

$$T_\parallel^b = 0.197 \Rightarrow |T_\parallel^b| = 0.197,\ \xi_\parallel^t = 0^\circ$$

$$\gamma^t = \tan^{-1}\left[\frac{|T_\perp^b|}{|T_\parallel^b|}\tan\gamma^i\right] = \tan^{-1}\left[\frac{0.1758}{0.197}\frac{1}{2}\right] = \tan^{-1}(0.4462) = 24.0464^\circ$$

$$\delta^t = \delta^i + (\xi_\perp^t - \xi_\parallel^t) = -90^\circ + (0^\circ - 0^\circ) = -90^\circ$$

$$2\epsilon^t = \sin^{-1}[\sin(2\gamma^t)\sin(\delta^t)] = \sin^{-1}[\sin(48.093^\circ)\sin(-90^\circ)]$$

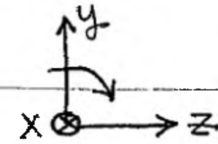
$$2\epsilon^t = \sin^{-1}(-0.7442) = -48.0929^\circ$$

$$\epsilon^t = -24.0464^\circ = -0.4197 \text{ rad.}$$

$$AR^b = \cot(-24.0464^\circ) = -2.2411$$

Therefore the transmitted field is elliptically polarized with clockwise (CW) rotation and with an axial ratio of $AR = -2.2411$.

$\boxed{5.52}$ $E^i(z) = (\hat{a}_z + j2\hat{a}_y)e^{-j\beta_o x}$

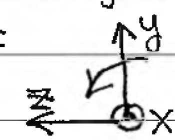


(a) $\underline{E}^r(z) = -(\hat{a}_z + j2\hat{a}_y)e^{+j\beta_o x}$

(b)
- Elliptical: 2 components; not of the same magnitude; 90° phase diff
- CW: y component is leading the z component
- Rotate y component to z component thru smaller angular range (CW)
- $AR = \frac{2}{1} = -2$ since it is elliptical with 90° phase difference; The major and minor axes of the ellipse align with principal axes.

(c) $\underline{E}^r(z) = -(\hat{a}_z + j2\hat{a}_y)e^{+j\beta_o x}$

- Elliptical: 2 components; not of the same magnitude; 90° phase diff.
- CCW: y component is leading the z component



  Rotate y component to z component thru smaller angular range (CCW)
- $AR = \frac{2}{1} = 2$ since it is elliptical with 90° phase difference; the major and minor axes of the ellipse align with principal axes.

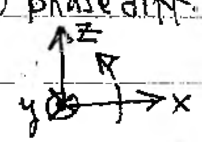
$\boxed{5.53}$ $E^i = (2\hat{a}_x - j\hat{a}_z)E_o e^{-j\beta_o y}$

(a) Since reflecting surface is PMC $\Rightarrow \Gamma = +1$. Therefore

$$\underline{E}^r = (2\hat{a}_x - j\hat{a}_y)E_o e^{+j\beta_o y}$$

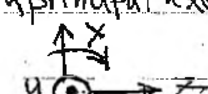
(b) $\underline{E}^i = (2\hat{a}_x - j\hat{a}_z)E_o e^{-j\beta_o y}$

- Elliptical: 2 components; not of the same magnitude; 90° phase diff.
- CCW: x component is leading the z component



  Rotate the x-component to z component thru smaller angular range (CCW)
- $AR = \frac{2}{1} = 2$ (major and minor axes of ellipse align with principal axes)

(c) $\underline{E}^r = (2\hat{a}_x - j\hat{a}_z)E_o e^{+j\beta_o y}$



- Elliptical: 2 components; not of the same magnitude; 90° phase diff.
- CW: x component is leading the z component
  Rotate the x-component to the z component thru smaller range.
- $AR = -\frac{2}{1} = -2$ (major and minor axes of ellipse align with principal axes)

5.54 $\varepsilon = 9\varepsilon_0,\ \mu = \mu_0,\ \sigma = 0$; CCW (LH) CP, $\theta_i = 18.43495°$

$$\theta_c = \sin^{-1}\left(\sqrt{\frac{\varepsilon_2}{\varepsilon_1}}\right) = \sin^{-1}\left(\sqrt{\frac{1}{\varepsilon_{r1}}}\right) = \sin^{-1}\left(\sqrt{\frac{1}{9}}\right) = 19.4712°$$

$$\theta_B = \tan^{-1}\left(\sqrt{\frac{\varepsilon_2}{\varepsilon_1}}\right) = \tan^{-1}\left(\sqrt{\frac{1}{\varepsilon_{r1}}}\right) = \tan^{-1}\left(\sqrt{\frac{1}{9}}\right) = 18.4349°$$

(a) Since one of the components is totally transmitted, because the incident angle is equal to the Brewster angle, the reflected field has only one component. Therefore the reflected field is <u>linearly</u> polarized.

(b) <u>No rotation</u> because it is linearly polarized.

(c,d)
$$T_\perp^b = \frac{2\cos\theta_i}{\cos\theta_i + \sqrt{\frac{\varepsilon_2}{\varepsilon_1}}\sqrt{1-\frac{\varepsilon_1}{\varepsilon_2}\sin^2\theta_i}} = \frac{2\cos(18.43495)}{\cos(18.43495) + \sqrt{\frac{1}{9}}\sqrt{1-9\sin^2(18.43495)}}$$

$$= \frac{2(0.9487)}{0.9487 + \frac{1}{3}(0.3162)} = 1.8$$

$$T_\parallel^b = \frac{2\sqrt{\frac{\varepsilon_1}{\varepsilon_2}}\cos\theta_i}{\cos\theta_i + \sqrt{\frac{\varepsilon_1}{\varepsilon_2}}\sqrt{1-\frac{\varepsilon_1}{\varepsilon_2}\sin^2\theta_i}} = \frac{2(3)\cos(18.43495)}{\cos(18.43495) + \sqrt{9}\sqrt{1-9\sin^2(18.43495)}}$$

$$= \frac{2(3)0.9487}{0.9487 + 3(0.3162)} = 3$$

1. Since the transmission coefficients are not identical, the transmitted field components will not be identical.
2. Also both transmission coefficients are real positive (zero phase angle); the transmitted field components will maintain the same phase as the incident field components; i.e. will have the same sense of rotation.

Thus the polarization of the transmitted wave is:

(c) <u>Elliptically polarized</u> because the 2 field components are not equal.

(d) The <u>rotation is CCW(LH)</u> because the transmitted field components maintain the same phase as the incident components.

$\boxed{5.55}$ $\varepsilon = 9\varepsilon_0,\ \mu = \mu_0 = \sigma = 0$; CW(RH) CP, $\theta_i = 18.43495°$

$$\theta_c = \sin^{-1}\left(\sqrt{\frac{\varepsilon_2}{\varepsilon_1}}\right) = \sin^{-1}\left(\sqrt{\frac{1}{\varepsilon_{r1}}}\right) = \sin^{-1}\left(\sqrt{\frac{1}{9}}\right) = 19.4712°$$

$$\theta_B = \tan^{-1}\left(\sqrt{\frac{\varepsilon_2}{\varepsilon_1}}\right) = \tan^{-1}\left(\sqrt{\frac{1}{\varepsilon_{r1}}}\right) = \tan^{-1}\left(\sqrt{\frac{1}{9}}\right) = 18.4349$$

Based on the solution of Problem 5.54:

(a) The reflected field is linearly polarized.

(b) No rotation because the reflected wave is linearly polarized.

(c) $T_{\perp}^{b}$ (based on the solution of Problem 5.54) $= 1.8$

$T_{\parallel}^{b}$ (based on the solution of Problem 5.54) $= 3$

1. Since the transmission coefficients are not identical, the transmitted field components will not be identical.
2. Also both transmission coefficients are real positive (zero phase angle); therefore the transmitted field components will maintain the same phase as the incident field components; i.e., will have the same sense of rotation.

Thus the polarization of the transmitted wave is:

(c) Elliptically polarized because the 2 field components are not equal.

(d) The rotation is CW (RH) because the transmitted field components maintain the same phase as the incident field components; thus have the same sense of rotation.

$\boxed{5.56}$ Based on the geometry and ray-tracing method illustrated in Figure 5-11, we can write the transmitted wave as a summation of the following terms:

$$T = T_{32}\left(T_{21}e^{-j\theta}\right) + T_{32}\Gamma_{21}\Gamma_{23}T_{21}e^{-j3\theta} + T_{32}\Gamma_{21}^2\Gamma_{23}^2T_{21}e^{-j5\theta} + \cdots$$

Factoring the common terms $T_{32}T_{21}e^{-j\theta}$ that appear in each of the terms, we can write the above equation as

$$T = T_{32}T_{21}e^{-j\theta}\left[1 + \left(\Gamma_{21}\Gamma_{23}e^{-j2\theta}\right) + \left(\Gamma_{21}\Gamma_{23}e^{-j2\theta}\right)^2 + \cdots\right]$$

which can be written in compact form, since the terms within the bracket form a geometric series, as

$$T = T_{32}T_{21}e^{-j\theta}\left[\frac{1}{1-\Gamma_{21}\Gamma_{23}e^{-j2\theta}}\right] \underset{\Gamma_{21}=-\Gamma_{12}}{=} \frac{T_{32}T_{21}e^{-j\theta}}{1+\Gamma_{12}\Gamma_{23}e^{-j2\theta}}$$

Since

$$T_{32} = \frac{2\eta_3}{\eta_3+\eta_2},\ T_{21} = \frac{2\eta_2}{\eta_2+\eta_1} \text{ and } \eta_3 = \eta_1$$

we can reduce the above to

$$T = \frac{4\eta_1\eta_2e^{-j\theta}}{(\eta_1+\eta_2)^2}\left[\frac{1}{1+\Gamma_{12}\Gamma_{23}e^{-j2\theta}}\right] \underset{\Gamma_{23}=-\Gamma_{12}}{=} \frac{4\eta_1\eta_2e^{-j\theta}}{(\eta_1+\eta_2)^2}\left[\frac{1}{1-(\Gamma_{12})^2e^{-j2\theta}}\right]$$

Since $\theta = |\beta_2|d$, the transmission coefficient can be written as

$$T = \frac{4\eta_1\eta_2e^{-j|\beta_2|d}}{(\eta_1+\eta_2)^2}\left[\frac{1}{1-(\Gamma_{12})^2e^{-j2|\beta_2|d}}\right]$$

*In the first printing of the book, there is an extra factor of 2 in the exponential at the numerator; this is a typographical error.

$\boxed{5\text{-}57}$ For parallel polarization, the wavenumbers are the same as those of the perpendicular polarization and given by (5-110a)–(5-110c), or

$$\underline{\beta}_i = \eta_1 \frac{\omega}{v_0}(\hat{a}_x \sin\theta_i + \hat{a}_z \cos\theta_i) \quad (1)$$
$$\underline{\beta}_r = \eta_1 \frac{\omega}{v_0}(\hat{a}_x \sin\theta_i - \hat{a}_z \cos\theta_i) \quad (2)$$
$$\underline{\beta}_t = |\eta_2| \frac{\omega}{v_0}(\hat{a}_x \sin|\theta_t| - \hat{a}_z \cos|\theta_t|) \quad (3)$$

The electric and magnetic fields for parallel polarization, based on the geometries of Figure 5-4 and 5-29 (but with the transmitted fields of Figure 5-4 in the 4th quadrant), we can write them as

$$\underline{E}^i_{\parallel} = (\hat{a}_x \cos\theta_i - \hat{a}_z \sin\theta_i) E_0 e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}$$
$$\underline{H}^i_{\parallel} = \hat{a}_y \frac{E_0}{\eta_1} e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}$$

$$\underline{E}^r_{\parallel} = (\hat{a}_x \cos\theta_r + \hat{a}_z \sin\theta_r) \Gamma^b_{\parallel} E_0 e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}$$
$$\underline{H}^r_{\parallel} = -\hat{a}_y \frac{\Gamma^b_{\parallel} E_0}{\eta_1} e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}$$

$$\underline{E}^t_{\parallel} = (+\hat{a}_x \cos\theta_t + \hat{a}_z \sin\theta_t) T^b_{\parallel} E_0 e^{-j|\beta_2|(+x\sin|\theta_t| - z\cos|\theta_t|)}$$
$$\underline{H}^t_{\parallel} = \hat{a}_y \frac{T^b_{\parallel} E_0}{|\eta_2|} e^{-j|\beta_2|(x\sin|\theta_t| - z\cos|\theta_t|)}$$

The corresponding Poynting vectors can be written, using the above equations for the incident, reflected and transmitted electric and magnetic fields, as

$$\underline{S}^i_{\parallel} = \frac{1}{2}\mathrm{Re}(\underline{E}^i_{\parallel} \times \underline{H}^{i*}_{\parallel}) = \frac{|E_0|^2}{2\eta_1}(\hat{a}_x \sin\theta_i + \hat{a}_z \cos\theta_i) \quad (4)$$

$$\underline{S}^r_{\parallel} = \frac{1}{2}\mathrm{Re}(\underline{E}^r_{\parallel} \times \underline{H}^{r*}_{\parallel}) = \frac{|E_0\Gamma_{\parallel}|^2}{2\eta_1}(\hat{a}_x \sin\theta_r - \hat{a}_z \cos\theta_r) \quad (5)$$

$$\underline{S}^t_{\parallel} = \frac{1}{2}\mathrm{Re}(\underline{E}^t_{\parallel} \times \underline{H}^{t*}_{\parallel}) = \frac{|E_0 T_{\parallel}|^2}{2|\eta_2|}(-\hat{a}_x \sin|\theta_t| + \hat{a}_z \cos|\theta_t|) \quad (6)$$

$\boxed{\text{cont'd}}$

5-57 (cont'd)

It is apparent by examining (1)-(3) and (4)-(6) that the wavenumbers and Poynting vector for parallel polarization:

- Incident Fields

  The wavenumber and Poynting vector are both parallel to each other and both point inward toward the origin.

- Reflected Fields

  The wave number and Poynting vector are both parallel to each other and both point outward from the origin.

- Transmitted Fields

  The wave number and Poynting vector are antiparallel to each other, with the wave number pointing inward toward the origin while the Poynting vector pointing outward from the origin.

These are the same observations made for the perpendicular polarization as represented, respectively, by (5-110a)-(5-110c) and (5-111a)-(5-111c) and illustrated graphically in Figure 5-29.

CHAPTER 6

$\boxed{6.1}$ If $\underline{H}_e = j\omega\varepsilon \nabla \times \underline{\Pi}_e$

Maxwell's curl equation $\nabla \times \underline{E}_e = -j\omega\mu \underline{H}_e$ can be written as

$$\nabla \times \underline{E}_e = -j\omega\mu \underline{H}_e = -j\omega\mu (j\omega\varepsilon \nabla \times \underline{\Pi}_e) = \omega^2\mu\varepsilon \nabla \times \underline{\Pi}_e \tag{1}$$

or

$$\nabla \times (\underline{E}_e - \omega^2\mu\varepsilon \underline{\Pi}_e) = \nabla \times (\underline{E}_e - \beta^2 \underline{\Pi}_e) = 0 \quad \text{where } \beta^2 = \omega^2\mu\varepsilon.$$

Letting

$$\underline{E}_e - \beta^2 \underline{\Pi}_e = -\nabla \phi_e \Rightarrow \underline{E}_e = -\nabla \phi_e + \beta^2 \underline{\Pi}_e \tag{2}$$

Taking the curl of (1) and using the vector identity of

$$\nabla \times \nabla \times \underline{A} = \nabla(\nabla \cdot \underline{A}) - \nabla^2 \underline{A}$$

leads to

$$\nabla \times \underline{H}_e = j\omega\varepsilon \nabla \times \nabla \times \underline{\Pi}_e = j\omega\varepsilon \left[\nabla(\nabla \cdot \underline{\Pi}_e) - \nabla^2 \underline{\Pi}_e\right]$$

Using Maxwell's equation of

$$\nabla \times \underline{H}_e = \underline{J} + j\omega\varepsilon \underline{E}_e \tag{3}$$

reduces (3) to

$$\underline{J} + j\omega\varepsilon \underline{E}_e = j\omega\varepsilon \left[\nabla(\nabla \cdot \underline{\Pi}_e) - \nabla^2 \underline{\Pi}_e\right] \tag{4}$$

Substituting (2) into (4) reduces to

$$\nabla^2 \underline{\Pi}_e + \beta^2 \underline{\Pi}_e = j \frac{\underline{J}}{\omega\varepsilon} + \left[\nabla(\nabla \cdot \underline{\Pi}_e) + \nabla \phi_e\right] \tag{5}$$

Letting $\phi_e = -\nabla \cdot \underline{\Pi}_e$ simplifies (5) to

$$\nabla^2 \underline{\Pi}_e + \beta^2 \underline{\Pi}_e = j \frac{\underline{J}}{\omega\varepsilon} \tag{6}$$

and (2) to

$$\underline{E}_e = \nabla(\nabla \cdot \underline{\Pi}_e) + \beta^2 \underline{\Pi}_e \tag{7}$$

Comparing (6) with (6-16) leads to the relation

$$\underline{\Pi}_e = -j \frac{1}{\omega\mu\varepsilon} \underline{A} \tag{8}$$

$\boxed{6.2}$ If $\underline{E}_m = -j\omega\mu\nabla\times\underline{\Pi}_m$

Maxwell's curl equation $\nabla\times\underline{H}_m = j\omega\varepsilon\,\underline{E}_m$ can be written as

$$\nabla\times\underline{H}_m = j\omega\varepsilon(-j\omega\mu\nabla\times\underline{\Pi}_m) = +\omega^2\mu\varepsilon\nabla\times\underline{\Pi}_m \quad (1)$$

or

$$\nabla\times(\underline{H}_m - \omega^2\mu\varepsilon\underline{\Pi}_m) = \nabla\times(\underline{H}_m - \beta^2\underline{\Pi}_m) = 0 \text{ where } \beta^2 = \omega^2\mu\varepsilon.$$

Letting

$$\underline{H}_m - \beta^2\underline{\Pi}_m = -\nabla\phi_m \Rightarrow \underline{H}_m = -\nabla\phi_m + \beta^2\underline{\Pi}_m \quad (2)$$

Taking the curl of (1) and using the vector identity of

$$\nabla\times\nabla\times\underline{F} = \nabla(\nabla\cdot\underline{F}) - \nabla^2\underline{F}$$

leads to

$$\nabla\times\underline{E}_m = -j\omega\mu\nabla\times\nabla\times\underline{\Pi}_m = -j\omega\mu\left[\nabla(\nabla\cdot\underline{\Pi}_m) - \nabla^2\underline{\Pi}_m\right] \quad (3)$$

Using Maxwell's equation of

$$\nabla\times\underline{E}_m = -\underline{M} - j\omega\mu\underline{H}_m$$

reduces (3) to

$$-\underline{M} - j\omega\mu\underline{H}_m = -j\omega\mu\left[\nabla(\nabla\cdot\underline{\Pi}_m) - \nabla^2\underline{\Pi}_m\right] \quad (4)$$

Substituting (2) into (4) reduces to

$$\nabla^2\underline{\Pi}_m + \beta^2\underline{\Pi}_m = j\frac{\underline{M}}{\omega\mu} + \left[\nabla(\nabla\cdot\underline{\Pi}_m) + \nabla\phi_m\right] \quad (5)$$

Letting

$$\phi_m = -\nabla\cdot\underline{\Pi}_m$$

simplifies (5) to

$$\nabla^2\underline{\Pi}_m + \beta^2\underline{\Pi}_m = j\frac{\underline{M}}{\omega\mu} \quad (6)$$

and (2) to

$$\underline{H}_m = \nabla(\nabla\cdot\underline{\Pi}_m) + \beta^2\underline{\Pi}_m \quad (7)$$

Comparing (6) with (6-28) leads to the relation

$$\underline{\Pi}_m = -j\frac{1}{\omega\mu\varepsilon}\underline{F} \quad (8)$$

6.3 $\underline{H}_A = \frac{1}{\mu}\nabla\times\underline{A}$, $\underline{A} = \frac{\mu}{4\pi}\iiint_V \underline{J}(r')\frac{e^{-j\beta R}}{R}dv'$, $\underline{E}_A = \frac{1}{j\omega\varepsilon}\nabla\times\underline{H}_A$

$$\underline{H}_A = \frac{1}{\mu}\nabla\times\underline{A} = \frac{1}{\mu}\nabla\times\left[\frac{\mu}{4\pi}\iiint_V \underline{J}(r')\frac{e^{-j\beta R}}{R}dv'\right] = \frac{1}{4\pi}\nabla\times\iiint_V \underline{J}(r')\frac{e^{-j\beta R}}{R}dv'$$

$$\underline{H}_A = \frac{1}{4\pi}\iiint_V \nabla\times\left[\underline{J}(r')\frac{e^{-j\beta R}}{R}\right]dv' \qquad (1)$$

Using the identity

$$\nabla\times(\alpha\underline{B}) = \alpha\,\nabla\times\underline{B} - \underline{B}\times\nabla\alpha \qquad (2)$$

we can write (1), remembering that $\underline{J}(r')$ is not a function of r, as

$$\underline{H}_A = -\frac{1}{4\pi}\iiint_V \left[\underline{J}(r')\times\nabla\left(\frac{e^{-j\beta R}}{R}\right)\right]dv' \qquad (3)$$

Now using (3) we can write the electric field as

$$\underline{E}_A = \frac{1}{j\omega\varepsilon}\nabla\times\underline{H}_A = -\frac{1}{j\omega\varepsilon}\nabla\times\left\{\frac{1}{4\pi}\iiint_V\left[\underline{J}(r')\times\nabla\left(\frac{e^{-j\beta R}}{R}\right)\right]dv'\right\}$$

$$\underline{E}_A = -\frac{1}{j4\pi\omega\varepsilon}\iiint_V \nabla\times\left[\underline{J}(r')\times\nabla\left(\frac{e^{-j\beta R}}{R}\right)\right]dv' \qquad (4)$$

Now using the vector identity

$$\nabla\times(\underline{B}\times\underline{C}) = \underline{B}\,\nabla\cdot\underline{C} - \underline{C}\,\nabla\cdot\underline{B} + (\underline{C}\cdot\nabla)\underline{B} - (\underline{B}\cdot\nabla)\underline{C} \qquad (5)$$

we can write (4) as

$$\underline{E}_A = -\frac{1}{j4\pi\omega\varepsilon}\iiint_V\left\{\underline{J}\,\nabla\cdot\nabla\left(\frac{e^{-j\beta R}}{R}\right) - \nabla\left(\frac{e^{-j\beta R}}{R}\right)[\nabla\cdot\underline{J}] + \left[\left(\nabla\frac{e^{-j\beta R}}{R}\right)\cdot\nabla\right]\underline{J} - (\underline{J}\cdot\nabla)\nabla\left(\frac{e^{-j\beta R}}{R}\right)\right\}dv' \qquad (6)$$

Again, since $\underline{J}(r')$ is not a function of r, the second and third terms are equal to zero. Therefore (6) reduces to

$$\underline{E}_A = -\frac{1}{j4\pi\omega\varepsilon}\iiint_V\left\{\underline{J}(r')\nabla^2\left(\frac{e^{-j\beta R}}{R}\right) - [\underline{J}(r')\cdot\nabla]\nabla\left(\frac{e^{-j\beta R}}{R}\right)\right\}dv' \qquad (7)$$

Since $\nabla^2\left(\frac{e^{-j\beta R}}{R}\right) = \frac{1}{R^2}\frac{\partial}{\partial R}\left[R^2\frac{\partial}{\partial R}\left(\frac{e^{-j\beta R}}{R}\right)\right] = -\beta^2\frac{e^{-j\beta R}}{R}$, then (7) reduces to

$$\underline{E}_A = -j\frac{1}{4\pi\omega\varepsilon}\iiint_V\left\{\left[\beta^2\underline{J}(r') + (\underline{J}(r')\cdot\nabla)\nabla\right]\frac{e^{-j\beta R}}{R}\right\}dv' \qquad (8)$$

$\boxed{6.4}$ $\underline{E}_F = -\frac{1}{\varepsilon}\nabla\times\underline{F}$, $\underline{F} = \frac{\varepsilon}{4\pi}\iiint_V \underline{M}(r')\frac{e^{-j\beta R}}{R}dv'$, $\underline{H}_F = -\frac{1}{j\omega\mu}\nabla\times\underline{E}_F$

$$\underline{E}_F = -\frac{1}{\varepsilon}\nabla\times\underline{F} = -\frac{1}{\varepsilon}\nabla\times\left[\frac{\varepsilon}{4\pi}\iiint_V \underline{M}(r')\frac{e^{-j\beta R}}{R}dv'\right] = -\frac{1}{4\pi}\nabla\times\iiint_V \underline{M}\frac{e^{-j\beta R}}{R}dv'$$

$$\underline{E}_F = -\frac{1}{4\pi}\iiint_V \nabla\times\left[\underline{M}(r')\frac{e^{-j\beta R}}{R}\right]dv' \qquad (1)$$

Using the identity $\quad \nabla\times(\alpha\underline{B}) = \alpha\nabla\times\underline{B} - \underline{B}\times\nabla\alpha \qquad (2)$

we can write (1), rembering that $\underline{M}(r')$ is not a function of r, as

$$\underline{E}_F = +\frac{1}{4\pi}\iiint_V \left[\underline{M}(r')\times\nabla\left(\frac{e^{-j\beta R}}{R}\right)\right]dv' \qquad (3)$$

Now using (3) we can write the magnetic field as

$$\underline{H}_F = -\frac{1}{j\omega\mu}\nabla\times\underline{E}_F = -\frac{1}{j\omega\mu}\nabla\times\left\{\frac{1}{4\pi}\iiint_V\left[\underline{M}(r')\times\nabla\left(\frac{e^{-j\beta R}}{R}\right)\right]dv'\right.$$

$$\underline{H}_F = -\frac{1}{j4\pi\omega\mu}\iiint_V \nabla\times\left[\underline{M}(r')\times\nabla\left(\frac{e^{-j\beta R}}{R}\right)\right]dv' \qquad (4)$$

Now using the vector identity

$$\nabla\times(\underline{B}\times\underline{C}) = \underline{B}\,\nabla\cdot\underline{C} - \underline{C}\,\nabla\cdot\underline{B} + (C\cdot\nabla)\underline{B} - (\underline{B}\cdot\nabla)\underline{C} \qquad (5)$$

we can write (4) as

$$\underline{H}_F = -\frac{1}{j4\pi\omega\mu}\iiint_V\left\{\underline{M}\nabla\cdot\nabla\left(\frac{e^{-j\beta R}}{R}\right) - \nabla\left(\frac{e^{-j\beta R}}{R}\right)[\nabla\cdot\underline{M}] + \left[\left(\nabla\frac{e^{-j\beta R}}{R}\right)\cdot\nabla\right]\underline{M} - (\underline{M}\cdot\nabla)\nabla\left(\frac{e^{-j\beta R}}{R}\right)\right\}dv' \qquad (6)$$

Again, since $\underline{M}(r')$ is not a function of r, the second and third terms are equal to zero. Therefore (6) reduces to

$$\underline{H}_F = -\frac{1}{j4\pi\omega\mu}\iiint_V\left\{\underline{M}(r')\nabla^2\left(\frac{e^{-j\beta R}}{R}\right) - \left[\underline{M}(r')\cdot\nabla\right]\nabla\left(\frac{e^{-j\beta R}}{R}\right)\right\}dv' \qquad (7)$$

Since

$$\nabla^2\left(\frac{e^{-j\beta R}}{R}\right) = \frac{1}{R^2}\frac{\partial}{\partial R}\left[R^2\frac{\partial}{\partial R}\left(\frac{e^{-j\beta R}}{R}\right)\right] = -\beta^2\frac{e^{-j\beta R}}{R}$$

then (7) reduces to

$$\underline{H}_F = -j\frac{1}{4\pi\omega\mu}\iiint_V\left\{\left[\beta^2\underline{M}(r') + (\underline{M}(r')\cdot\nabla)\nabla\right]\frac{e^{-j\beta R}}{R}\right\}dv' \qquad (8)$$

6.5 A. $TEM^x \Rightarrow E_x = H_x = 0$

It is apparent that by examining (6-41) and (6-43) that $E_x = H_x = 0$ can be obtained by any of the following three combinations.

1. $A_y = A_z = F_y = F_z = 0, \ A_x \neq 0, \ F_x \neq 0, \ \partial/\partial y \neq 0, \ \partial/\partial z \neq 0$

For this combination, according to (6-41)

$$E_x = -j\omega A_x - j\frac{1}{\omega\mu\varepsilon}\frac{\partial^2 A_x}{\partial x^2} = -j\frac{1}{\omega\mu\varepsilon}\left(\frac{\partial^2}{\partial x^2} + \omega^2\mu\varepsilon\right)A_x = 0$$

provided that

$$A_x(x,y,z) = A_x^+(y,z)e^{-j\beta x} + A_x^-(y,z)e^{+j\beta x}$$

Similarly according to (6-43)

$$H_x = -j\omega F_x - j\frac{1}{\omega\mu\varepsilon}\frac{\partial^2 F_x}{\partial x^2} = -j\frac{1}{\omega\mu\varepsilon}\left(\frac{\partial^2}{\partial x^2} + \omega^2\mu\varepsilon\right)F_x = 0$$

provided that

$$F_x(x,y,z) = F_x^+(y,z)e^{-j\beta x} + F_x^-(y,z)e^{+j\beta x}$$

Also according to (6-41) and (6-43)

$$E_y = \left(-\frac{1}{\sqrt{\mu\varepsilon}}\frac{\partial A_x^+}{\partial y} - \frac{1}{\varepsilon}\frac{\partial F_x^+}{\partial z}\right)e^{-j\beta x} + \left(\frac{1}{\sqrt{\mu\varepsilon}}\frac{\partial A_x^-}{\partial y} - \frac{1}{\varepsilon}\frac{\partial F_x^-}{\partial z}\right)e^{+j\beta x} = E_y^+ + E_y^-$$

$$E_z = \left(-\frac{1}{\sqrt{\mu\varepsilon}}\frac{\partial A_x^+}{\partial z} + \frac{1}{\varepsilon}\frac{\partial F_x^+}{\partial y}\right)e^{-j\beta x} + \left(\frac{1}{\sqrt{\mu\varepsilon}}\frac{\partial A_x^-}{\partial z} + \frac{1}{\varepsilon}\frac{\partial F_x^-}{\partial y}\right)e^{+j\beta x} = E_z^+ + E_z^-$$

$$H_y = \left(\frac{1}{\mu}\frac{\partial A_x^+}{\partial z} - \frac{1}{\sqrt{\mu\varepsilon}}\frac{\partial F_x^+}{\partial y}\right)e^{-j\beta x} + \left(\frac{1}{\mu}\frac{\partial A_x^-}{\partial z} + \frac{1}{\sqrt{\mu\varepsilon}}\frac{\partial F_x^-}{\partial y}\right)e^{+j\beta x}$$

$$= \sqrt{\frac{\varepsilon}{\mu}}\left(\frac{1}{\sqrt{\mu\varepsilon}}\frac{\partial A_x^+}{\partial z} - \frac{1}{\varepsilon}\frac{\partial F_x^+}{\partial y}\right)e^{-j\beta x} + \sqrt{\frac{\varepsilon}{\mu}}\left(\frac{1}{\sqrt{\mu\varepsilon}}\frac{\partial A_x^-}{\partial z} + \frac{1}{\varepsilon}\frac{\partial F_x^-}{\partial y}\right)$$

$$H_y = \sqrt{\frac{\varepsilon}{\mu}}\left(-E_z^+\right) + \sqrt{\frac{\varepsilon}{\mu}}\left(E_z^-\right) = H_y^+ + H_y^-$$

$$H_z = \left(-\frac{1}{\mu}\frac{\partial A_x^+}{\partial y} - \frac{1}{\sqrt{\mu\varepsilon}}\frac{\partial F_x^+}{\partial z}\right)e^{-j\beta x} + \left(\frac{1}{\mu}\frac{\partial A_x^-}{\partial y} + \frac{1}{\sqrt{\mu\varepsilon}}\frac{\partial F_x^-}{\partial z}\right)e^{+j\beta x}$$

$$= \sqrt{\frac{\varepsilon}{\mu}}\left(-\frac{1}{\sqrt{\mu\varepsilon}}\frac{\partial A_x^+}{\partial y} - \frac{1}{\varepsilon}\frac{\partial F_x^+}{\partial z}\right)e^{-j\beta x} + \sqrt{\frac{\varepsilon}{\mu}}\left(\frac{1}{\sqrt{\mu\varepsilon}}\frac{\partial A_x^-}{\partial y} - \frac{1}{\varepsilon}\frac{\partial F_x^-}{\partial z}\right)e^{+j\beta x}$$

$$= \sqrt{\frac{\varepsilon}{\mu}}\left(E_y^+\right) + \sqrt{\frac{\varepsilon}{\mu}}\left(-E_y^-\right) = H_z^+ + H_z^-$$

cont'd.

$\boxed{6.5 \text{ Cont'd.}}$

2. $\underline{A_x = A_y = A_z = F_y = F_z = 0,\ F_x \neq 0,\ \partial/\partial y \neq 0,\ \partial/\partial z \neq 0}$

For this combination

$$E_x = 0$$

$$H_x = -j\omega F_x - j\frac{1}{\omega\mu\varepsilon}\frac{\partial^2 F_x}{\partial x^2} = -\frac{1}{\omega\mu\varepsilon}\left(\frac{\partial^2 F_x}{\partial x^2} + \omega^2\mu\varepsilon F_x\right) = -j\frac{1}{\omega\mu\varepsilon}\left(\frac{\partial^2}{\partial x^2} + \omega^2\mu\varepsilon\right)F_x = 0$$

provided that

$$F_x(x,y,z) = F_x^+(y,z)e^{-j\beta x} + F_x^-(y,z)e^{+j\beta x}$$

Also according to (6-41) and (6-43)

$$E_y = -\frac{1}{\varepsilon}\frac{\partial F_x^+}{\partial z}e^{-j\beta x} - \frac{1}{\varepsilon}\frac{\partial F_x^-}{\partial z}e^{+j\beta x} = E_y^+ + E_y^-$$

$$E_z = \frac{1}{\varepsilon}\frac{\partial F_x^+}{\partial y}e^{-j\beta x} + \frac{1}{\varepsilon}\frac{\partial F_x^-}{\partial y}e^{+j\beta x} = E_z^+ + E_z^-$$

$$H_y = \sqrt{\frac{\varepsilon}{\mu}}\left(-\frac{1}{\varepsilon}\frac{\partial F_x^+}{\partial y}\right)e^{-j\beta x} + \sqrt{\frac{\varepsilon}{\mu}}\left(\frac{1}{\varepsilon}\frac{\partial F_x^-}{\partial y}\right)e^{+j\beta x} = \sqrt{\frac{\varepsilon}{\mu}}\left(-E_z^+\right) + \sqrt{\frac{\varepsilon}{\mu}}\left(E_z^-\right) = H_y^+ + H_y^-$$

$$H_z = \sqrt{\frac{\varepsilon}{\mu}}\left(-\frac{1}{\varepsilon}\frac{\partial F_x^+}{\partial z}\right)e^{-j\beta x} - \sqrt{\frac{\varepsilon}{\mu}}\left(-\frac{1}{\varepsilon}\frac{\partial F_x^-}{\partial z}\right)e^{+j\beta x} = \sqrt{\frac{\varepsilon}{\mu}}\left(E_y^+\right) + \sqrt{\frac{\varepsilon}{\mu}}\left(-E_y^-\right) = H_z^+ + H_z^-$$

3. $\underline{A_y = A_z = F_x = F_y = F_z = 0,\ A_x \neq 0,\ \partial/\partial y \neq 0,\ \partial/\partial z \neq 0}$

For this combination

$$E_x = -j\omega A_x - j\frac{1}{\omega\mu\varepsilon}\frac{\partial^2 A_x}{\partial x^2} = -j\frac{1}{\omega\mu\varepsilon}\left(\frac{\partial^2}{\partial x^2} + \omega^2\mu\varepsilon\right)A_x = 0$$

$$H_x = 0$$

provided that

$$A_x(x,y,z) = A_x^+(y,z)e^{-j\beta x} + A_x^-(y,z)e^{+j\beta x}$$

Also according to (6-41) and (6-43)

$$E_y = \left(-\frac{1}{\sqrt{\mu\varepsilon}}\frac{\partial A_x^+}{\partial y}\right)e^{-j\beta x} + \left(\frac{1}{\sqrt{\mu\varepsilon}}\frac{\partial A_x^-}{\partial y}\right)e^{+j\beta x} = E_y^+ + E_y^-$$

$$E_z = \left(-\frac{1}{\sqrt{\mu\varepsilon}}\frac{\partial A_x^+}{\partial z}\right)e^{-j\beta x} + \left(\frac{1}{\sqrt{\mu\varepsilon}}\frac{\partial A_x^-}{\partial z}\right)e^{+j\beta x} = E_z^+ + E_z^-$$

$$H_y = \sqrt{\frac{\varepsilon}{\mu}}\left(\frac{1}{\sqrt{\mu\varepsilon}}\frac{\partial A_x^+}{\partial z}\right) + \sqrt{\frac{\varepsilon}{\mu}}\left(\frac{1}{\sqrt{\mu\varepsilon}}\frac{\partial A_x^-}{\partial z}\right)e^{+j\beta x} = \sqrt{\frac{\varepsilon}{\mu}}\left(-E_z^+\right) + \sqrt{\frac{\varepsilon}{\mu}}\left(E_z^-\right) = H_y^+ + H_y^-$$

$$H_z = \sqrt{\frac{\varepsilon}{\mu}}\left(-\frac{1}{\sqrt{\mu\varepsilon}}\frac{\partial A_x^+}{\partial y}\right) - \sqrt{\frac{\varepsilon}{\mu}}\left(\frac{1}{\sqrt{\mu\varepsilon}}\frac{\partial A_x^-}{\partial y}\right) = \sqrt{\frac{\varepsilon}{\mu}}\left(E_y^+\right) + \sqrt{\frac{\varepsilon}{\mu}}\left(-E_y^-\right) = H_z^+ + H_z^-$$

Cont'd.

$\boxed{\text{6.5 cont'd.}}$ **B. $TEM^y \Rightarrow E_y = H_y = 0$**

It is apparent that by examining (6-41) and (6-43) that $E_y = H_y = 0$ can be obtained by any of the following three combinations.

1. $A_x = A_z = F_x = F_z = 0,\ A_y \neq 0,\ F_y \neq 0,\ \partial/\partial x \neq 0,\ \partial/\partial z \neq 0$

For this combination, according to (6-41)

$$E_y = -j\omega A_y - j\frac{1}{\omega\mu\varepsilon}\frac{\partial^2 A_y}{\partial y^2} = -j\frac{1}{\omega\mu\varepsilon}\left(\frac{\partial^2}{\partial y^2} + \omega^2\mu\varepsilon\right)A_y = 0$$

provided that

$$A_y(x,y,z) = A_y^+(x,z)e^{-j\beta y} + A_y^-(x,z)e^{+j\beta y}$$

Similarly according to (6-43)

$$H_y = -j\omega F_y - j\frac{1}{\omega\mu\varepsilon}\frac{\partial^2 F_y}{\partial y^2} = -j\frac{1}{\omega\mu\varepsilon}\left(\frac{\partial^2}{\partial y^2} + \omega^2\mu\varepsilon\right)F_y = 0$$

provided that

$$F_y(x,y,z) = F_y^+(x,z)e^{-j\beta y} + F_y^-(x,z)e^{+j\beta y}$$

Also according to (6-41) and (6-43)

$$E_x = \left(-\frac{1}{\sqrt{\mu\varepsilon}}\frac{\partial A_y^+}{\partial x} + \frac{1}{\varepsilon}\frac{\partial F_y^+}{\partial z}\right)e^{-j\beta y} + \left(\frac{1}{\sqrt{\mu\varepsilon}}\frac{\partial A_y^-}{\partial x} + \frac{1}{\varepsilon}\frac{\partial F_y^-}{\partial z}\right)e^{+j\beta y} = E_x^+ + E_x^-$$

$$E_z = \left(-\frac{1}{\sqrt{\mu\varepsilon}}\frac{\partial A_y^+}{\partial z} - \frac{1}{\varepsilon}\frac{\partial F_y^+}{\partial x}\right)e^{-j\beta y} + \left(\frac{1}{\sqrt{\mu\varepsilon}}\frac{\partial A_y^-}{\partial z} - \frac{1}{\varepsilon}\frac{\partial F_y^-}{\partial x}\right)e^{+j\beta y} = E_z^+ + E_z^-$$

$$H_x = \left(-\frac{1}{\mu}\frac{\partial A_y^+}{\partial z} - \frac{1}{\sqrt{\mu\varepsilon}}\frac{\partial F_y^+}{\partial x}\right)e^{-j\beta y} + \left(-\frac{1}{\mu}\frac{\partial A_y^-}{\partial z} + \frac{1}{\sqrt{\mu\varepsilon}}\frac{\partial F_y^-}{\partial x}\right)e^{-j\beta y}$$

$$= \sqrt{\frac{\varepsilon}{\mu}}\left(\frac{1}{\sqrt{\mu\varepsilon}}\frac{\partial A_y^+}{\partial z} - \frac{1}{\varepsilon}\frac{\partial F_y^+}{\partial x}\right)e^{-j\beta y} + \sqrt{\frac{\varepsilon}{\mu}}\left(-\frac{1}{\sqrt{\mu\varepsilon}}\frac{\partial A_y^-}{\partial z} + \frac{1}{\varepsilon}\frac{\partial F_y^-}{\partial x}\right)e^{-j\beta y}$$

$$H_x = \sqrt{\frac{\varepsilon}{\mu}}\left(E_z^+\right) + \sqrt{\frac{\varepsilon}{\mu}}\left(-E_z^-\right) = H_x^+ + H_x^-$$

$$H_z = \left(\frac{1}{\mu}\frac{\partial A_y^+}{\partial x} - \frac{1}{\sqrt{\mu\varepsilon}}\frac{\partial F_y^+}{\partial z}\right)e^{-j\beta y} + \left(\frac{1}{\mu}\frac{\partial A_y^-}{\partial x} + \frac{1}{\sqrt{\mu\varepsilon}}\frac{\partial F_y^-}{\partial z}\right)e^{+j\beta y}$$

$$= \sqrt{\frac{\varepsilon}{\mu}}\left(\frac{1}{\sqrt{\mu\varepsilon}}\frac{\partial A_y^+}{\partial x} - \frac{1}{\varepsilon}\frac{\partial F_y^+}{\partial z}\right)e^{-j\beta y} + \sqrt{\frac{\varepsilon}{\mu}}\left(\frac{1}{\sqrt{\mu\varepsilon}}\frac{\partial A_y^-}{\partial x} + \frac{1}{\varepsilon}\frac{\partial F_y^-}{\partial z}\right)e^{+j\beta y}$$

$$H_z = \sqrt{\frac{\varepsilon}{\mu}}\left(-E_x^+\right) + \sqrt{\frac{\varepsilon}{\mu}}\left(E_x^-\right) = H_z^+ + H_z^-$$

cont'd.

6.5 cont'd.

2. $A_x = A_y = A_z = F_x = F_z = 0,\ F_y \neq 0,\ \partial/\partial x \neq 0,\ \partial/\partial z \neq 0$

For this combination

$$E_y = 0$$

$$H_y = -j\omega F_y - j\frac{1}{\omega\mu\varepsilon}\frac{\partial^2 F_y}{\partial y^2} = -j\frac{1}{\omega\mu\varepsilon}\left(\frac{\partial^2}{\partial y^2} + \omega^2\mu\varepsilon\right)F_y = 0$$

provided that

$$F_y(x,y,z) = F_y^+(x,z)e^{-j\beta y} + F_y^-(x,z)e^{+j\beta y}$$

Also according to (6-41) and (6-43)

$$E_x = \frac{1}{\varepsilon}\frac{\partial F_y^+}{\partial z}e^{-j\beta y} + \frac{1}{\varepsilon}\frac{\partial F_y^-}{\partial z}e^{+j\beta y} = E_x^+ + E_x^-$$

$$E_z = -\frac{1}{\varepsilon}\frac{\partial F_y^+}{\partial x}e^{-j\beta y} - \frac{1}{\varepsilon}\frac{\partial F_y^-}{\partial x}e^{+j\beta y} = E_z^+ + E_z^-$$

$$H_x = \sqrt{\frac{\varepsilon}{\mu}}\left(-\frac{1}{\varepsilon}\frac{\partial F_y^+}{\partial x}\right)e^{-j\beta y} + \sqrt{\frac{\varepsilon}{\mu}}\left(\frac{1}{\varepsilon}\frac{\partial F_y^-}{\partial x}\right)e^{+j\beta y} = \sqrt{\frac{\varepsilon}{\mu}}(E_z^+) + \sqrt{\frac{\varepsilon}{\mu}}(-E_z^-) = H_x^+ + H_x^-$$

$$H_z = \sqrt{\frac{\varepsilon}{\mu}}\left(-\frac{1}{\varepsilon}\frac{\partial F_y^+}{\partial z}\right)e^{-j\beta y} + \sqrt{\frac{\varepsilon}{\mu}}\left(\frac{1}{\varepsilon}\frac{\partial F_y^-}{\partial z}\right)e^{+j\beta y} = \sqrt{\frac{\varepsilon}{\mu}}(-E_x^+) + \sqrt{\frac{\varepsilon}{\mu}}(E_x^-) = H_z^+ + H_z^-$$

3. $A_x = A_z = F_x = F_y = F_z = 0,\ A_y \neq 0,\ \partial/\partial x \neq 0,\ \partial/\partial z \neq 0$

For this combination

$$E_y = -j\omega A_y - j\frac{1}{\omega\mu\varepsilon}\frac{\partial^2 A_y}{\partial y^2} = -j\frac{1}{\omega\mu\varepsilon}\left(\frac{\partial^2}{\partial y^2} + \omega^2\mu\varepsilon\right)A_y = 0$$

$$H_y = 0$$

provided that

$$A_y(x,y,z) = A_y^+(x,z)e^{-j\beta y} + A_y^-(x,z)e^{+j\beta y}$$

Also according to (6-41) and (6-43)

$$E_x = \left(-\frac{1}{\sqrt{\mu\varepsilon}}\frac{\partial A_y^+}{\partial x}\right)e^{-j\beta y} + \left(\frac{1}{\sqrt{\mu\varepsilon}}\frac{\partial A_y^-}{\partial x}\right)e^{+j\beta y} = E_x^+ + E_x^-$$

$$E_z = \left(-\frac{1}{\sqrt{\mu\varepsilon}}\frac{\partial A_y^+}{\partial z}\right)e^{-j\beta y} + \left(\frac{1}{\sqrt{\mu\varepsilon}}\frac{\partial A_y^-}{\partial z}\right)e^{+j\beta y} = E_z^+ + E_z^-$$

$$H_x = \sqrt{\frac{\varepsilon}{\mu}}\left(-\frac{1}{\sqrt{\mu\varepsilon}}\frac{\partial A_y^+}{\partial z}\right)e^{-j\beta y} + \sqrt{\frac{\varepsilon}{\mu}}\left(-\frac{1}{\sqrt{\mu\varepsilon}}\frac{\partial A_y^-}{\partial z}\right)e^{+j\beta y} = \sqrt{\frac{\varepsilon}{\mu}}(E_z^+) + \sqrt{\frac{\varepsilon}{\mu}}(-E_z^-) = H_x^+ + H_x^-$$

$$H_z = \sqrt{\frac{\varepsilon}{\mu}}\left(\frac{1}{\sqrt{\mu\varepsilon}}\frac{\partial A_y^+}{\partial x}\right)e^{-j\beta y} + \sqrt{\frac{\varepsilon}{\mu}}\left(\frac{1}{\sqrt{\mu\varepsilon}}\frac{\partial A_y^-}{\partial x}\right)e^{+j\beta y} = \sqrt{\frac{\varepsilon}{\mu}}(-E_x^+) + \sqrt{\frac{\varepsilon}{\mu}}(E_x^-) = H_z^+ + H_z^-$$

6.6

A. $TEM^{\phi} \Rightarrow E_\phi = H_\phi = 0$

It is apparent that by examining (6-50) and (6-51) that $E_\phi = H_\phi = 0$ can be obtained by any of the following three combinations.

1. $A_\rho = A_z = F_\rho = F_z = 0,\quad A_\phi \neq 0,\ F_\phi \neq 0,\ \partial/\partial\rho \neq 0,\ \partial/\partial z \neq 0$

For this combination, according to (6-50) and (6-51)

$$E_\phi = -j\omega A_\phi - j\frac{1}{\omega\mu\varepsilon}\frac{1}{\rho}\frac{\partial}{\partial\phi}\left(\frac{1}{\rho}\frac{\partial A_\phi}{\partial\phi}\right) = -j\omega A_\phi - j\frac{1}{\omega\mu\varepsilon}\frac{1}{\rho^2}\frac{\partial^2 A_\phi}{\partial\phi^2} = 0$$

$$\frac{\partial^2 A_\phi}{\partial\phi^2} + \omega^2\mu\varepsilon\rho^2 A_\phi = 0$$

$$H_\phi = -j\omega F_\phi - j\frac{1}{\omega\mu\varepsilon}\frac{1}{\rho}\frac{\partial}{\partial\phi}\left(\frac{1}{\rho}\frac{\partial F_\phi}{\partial\phi}\right) = -j\omega F_\phi - j\frac{1}{\omega\mu\varepsilon}\frac{1}{\rho^2}\frac{\partial^2 F_\phi}{\partial\phi^2} = 0$$

$$\frac{\partial^2 F_\phi}{\partial\phi^2} + \omega^2\mu\varepsilon\rho^2 F_\phi = 0$$

$$E_\rho = -j\frac{1}{\omega\mu\varepsilon}\frac{\partial}{\partial\rho}\left(\frac{1}{\rho}\frac{\partial A_\phi}{\partial\phi}\right) + \frac{1}{\varepsilon}\frac{\partial F_\phi}{\partial z}$$

$$E_z = -j\frac{1}{\omega\mu\varepsilon}\frac{\partial}{\partial z}\left(\frac{1}{\rho}\frac{\partial A_\phi}{\partial\phi}\right) - \frac{1}{\varepsilon}\frac{1}{\rho}\left[\frac{\partial}{\partial\rho}(\rho F_\phi)\right] = -j\frac{1}{\omega\mu\varepsilon}\frac{1}{\rho}\frac{\partial^2 A_\phi}{\partial\phi\,\partial z} - \frac{1}{\varepsilon}\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho F_\phi)$$

$$H_\rho = -j\frac{1}{\omega\mu\varepsilon}\frac{\partial}{\partial\rho}\left(\frac{1}{\rho}\frac{\partial F_\phi}{\partial\phi}\right) - \frac{1}{\mu}\frac{\partial A_\phi}{\partial z}$$

$$H_z = -j\frac{1}{\omega\mu\varepsilon}\frac{\partial}{\partial z}\left(\frac{1}{\rho}\frac{\partial F_\phi}{\partial\phi}\right) + \frac{1}{\mu}\frac{1}{\rho}\left[\frac{\partial}{\partial\rho}(\rho A_\phi)\right] = -j\frac{1}{\omega\mu\varepsilon}\frac{1}{\rho}\frac{\partial^2 F_\phi}{\partial\phi\,\partial z} + \frac{1}{\mu}\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_\phi)$$

2. $A_\rho = A_\phi = A_z = F_\rho = F_z = 0,\ F_\phi \neq 0,\ \partial/\partial\rho \neq 0,\ \partial/\partial z \neq 0$

For this combination, according to (6-50) and (6-51)

$E_\phi = 0$

$$H_\phi = -j\omega F_\phi - j\frac{1}{\omega\mu\varepsilon}\frac{1}{\rho^2}\frac{\partial^2 F_\phi}{\partial\phi^2} = 0 \Rightarrow \frac{\partial^2 F_\phi}{\partial\phi^2} + \omega^2\mu\varepsilon\rho^2 F_\phi = 0$$

$$E_\rho = \frac{1}{\varepsilon}\frac{\partial F_\phi}{\partial z}$$

$$H_\rho = -j\frac{1}{\omega\mu\varepsilon}\frac{\partial}{\partial\rho}\left(\frac{1}{\rho}\frac{\partial F_\phi}{\partial\phi}\right)$$

$$E_z = -\frac{1}{\varepsilon}\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho F_\phi)$$

$$H_z = -j\frac{1}{\omega\mu\varepsilon}\frac{1}{\rho}\frac{\partial^2 F_\phi}{\partial\phi\,\partial z}$$

3. $A_\rho = A_z = F_\rho = F_\phi = F_z = 0,\ A_\phi \neq 0,\ \partial/\partial\rho \neq 0,\ \partial/\partial z \neq 0$

For this combination, according to (6-50) and (6-51)

$$E_\phi = -j\omega A_\phi - j\frac{1}{\omega\mu\varepsilon}\frac{1}{\rho^2}\frac{\partial^2 A_\phi}{\partial\phi^2} = 0 \Rightarrow \frac{\partial^2 A_\phi}{\partial\phi^2} + \omega^2\mu\varepsilon\rho^2 A_\phi = 0 \qquad H_\phi = 0$$

$$E_\rho = -j\frac{1}{\omega\mu\varepsilon}\frac{\partial}{\partial\rho}\left(\frac{1}{\rho}\frac{\partial A_\phi}{\partial\phi}\right) \qquad H_\rho = -\frac{1}{\mu}\frac{\partial A_\phi}{\partial z}$$

$$E_z = -j\frac{1}{\omega\mu\varepsilon}\frac{1}{\rho}\frac{\partial^2 A_\phi}{\partial\phi\,\partial z} \qquad H_z = \frac{1}{\mu}\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_\phi)$$

Cont'd.

6.6 Cont'd. **B. $TEM^z \Rightarrow E_z = H_z = 0$**

1. $A_\rho = A_\phi = F_\rho = F_\phi = 0, \quad A_z \neq 0, F_z \neq 0, \quad \partial/\partial\rho \neq 0, \partial/\partial\phi \neq 0$

For this combination, according to (6-50) and (6-51)

$$E_z = -j\omega A_z - j\frac{1}{\omega\mu\varepsilon}\frac{\partial^2 A_z}{\partial z^2} = 0 \Rightarrow \frac{\partial^2 A_z}{\partial z^2} + \omega^2\mu\varepsilon A_z = 0$$

$$H_z = -j\omega F_z - j\frac{1}{\omega\mu\varepsilon}\frac{\partial^2 F_z}{\partial z^2} = 0 \Rightarrow \frac{\partial^2 F_z}{\partial z^2} + \omega^2\mu\varepsilon F_z = 0$$

$$E_\rho = -j\frac{1}{\omega\mu\varepsilon}\frac{\partial^2 A_z}{\partial\rho\partial z} - \frac{1}{\varepsilon\rho}\frac{\partial F_z}{\partial\phi}$$

$$E_\phi = -j\frac{1}{\omega\mu\varepsilon}\frac{1}{\rho}\frac{\partial^2 A_z}{\partial\phi\partial z} + \frac{1}{\varepsilon}\frac{\partial F_z}{\partial\rho}$$

$$H_\rho = -j\frac{1}{\omega\mu\varepsilon}\frac{\partial^2 F_z}{\partial\rho\partial z} + \frac{1}{\mu\rho}\frac{\partial A_z}{\partial\phi}$$

$$H_\phi = -j\frac{1}{\omega\mu\varepsilon}\frac{1}{\rho}\frac{\partial^2 F_z}{\partial\phi\partial z} - \frac{1}{\mu}\frac{\partial A_z}{\partial\rho}$$

2. $A_\rho = A_\phi = A_z = F_\rho = F_\phi = 0, \quad F_z \neq 0, \quad \partial/\partial\rho \neq 0, \partial/\partial\phi \neq 0$

For this combination, according to (6-50) and (6-51)

$$E_z = 0$$

$$H_z = -j\omega F_z - j\frac{1}{\omega\mu\varepsilon}\frac{\partial^2 F_z}{\partial z^2} = 0 \Rightarrow \frac{\partial^2 F_z}{\partial z^2} + \omega^2\mu\varepsilon F_z = 0$$

$$E_\rho = -\frac{1}{\varepsilon\rho}\frac{\partial F_z}{\partial\phi}$$

$$E_\phi = \frac{1}{\varepsilon}\frac{\partial F_z}{\partial\rho}$$

$$H_\rho = -j\frac{1}{\omega\mu\varepsilon}\frac{\partial^2 F_z}{\partial\rho\partial z}$$

$$H_\phi = -j\frac{1}{\omega\mu\varepsilon}\frac{1}{\rho}\frac{\partial^2 F_z}{\partial\phi\partial z}$$

3. $A_\rho = A_\phi = F_\rho = F_\phi = F_z = 0, \quad A_z \neq 0, \quad \partial/\partial\rho \neq 0, \partial/\partial\phi \neq 0$

For this combination, according to (6-50) and (6-51)

$$E_z = -j\omega A_z - j\frac{1}{\omega\mu\varepsilon}\frac{\partial^2 A_z}{\partial z^2} = 0 \Rightarrow \frac{\partial^2 A_z}{\partial z^2} + \omega^2\mu\varepsilon A_z = 0$$

$$H_z = 0$$

$$E_\rho = -j\frac{1}{\omega\mu\varepsilon}\frac{\partial^2 A_z}{\partial\rho\partial z}$$

$$E_\phi = -j\frac{1}{\omega\mu\varepsilon}\frac{1}{\rho}\frac{\partial^2 A_z}{\partial\phi\partial z}$$

$$H_\rho = \frac{1}{\mu\rho}\frac{\partial A_z}{\partial\phi}$$

$$H_\phi = -\frac{1}{\mu}\frac{\partial A_z}{\partial\rho}$$

$\boxed{6.7}$

A. $TM^x \Rightarrow H_x = 0$

It is apparent by examining (6-41) and (6-43) that $H_x=0$ can be obtained by letting

$$A_y = A_z = F_x = F_y = F_z = 0 \text{ and } A_x \neq 0.$$

When these are substituted into (6-41) and (6-43), we get (6-61).

B. $TM^y \Rightarrow H_y = 0$

It is apparent by examining (6-41) and (6-43) that $H_y=0$ can be obtained by letting

$$A_x = A_z = F_x = F_y = F_z = 0$$

When these are substituted into (6-41) and (6-43), we get (6-64).

$\boxed{6.8}$

A. $TM^x \Rightarrow H_x = 0$

Let $E_x \neq 0$ so that $\nabla^2 E_x + \beta^2 E_x = 0$. Solve this for E_x.

According to (6-61) E_x is proportional to A_x. Therefore

$$A_x = \frac{E_x}{-j\frac{1}{\omega\mu\varepsilon}\left(\frac{\partial^2}{\partial x^2}+\beta^2\right)E_x} E_x = C_x E_x \quad \text{where } C_x = j\frac{E_x}{\frac{1}{\omega\mu\varepsilon}\left(\frac{\partial^2}{\partial x^2}+\beta^2\right)E_x}$$

Thus we can write the remaining $\underline{E}$-and $\underline{H}$-field components as

$$E_y = -j\frac{C_x}{\omega\mu\varepsilon}\frac{\partial^2 E_x}{\partial x \partial y} \qquad H_y = \frac{C_x}{\mu}\frac{\partial E_x}{\partial z}$$

$$E_z = -j\frac{C_x}{\omega\mu\varepsilon}\frac{\partial^2 E_x}{\partial x \partial z} \qquad H_z = -\frac{C_x}{\mu}\frac{\partial E_x}{\partial y}$$

B. $TM^y \Rightarrow H_y = 0$

Let $E_y \neq 0$ so that $\nabla^2 E_y + \beta^2 E_y = 0$. Solve this for E_y.

According to (6-64) E_y is proportional to A_y. Therefore

$$A_y = \frac{E_y}{-j\frac{1}{\omega\mu\varepsilon}\left(\frac{\partial^2}{\partial y^2}+\beta^2\right)E_y} E_y = C_y E_y \quad \text{where } C_y = +j\frac{E_y}{\frac{1}{\omega\mu\varepsilon}\left(\frac{\partial^2}{\partial y^2}+\beta^2\right)E_y}$$

Thus we can write the remaining $\underline{E}$-and $\underline{H}$-field components as

$$E_x = -j\frac{C_y}{\omega\mu\varepsilon}\frac{\partial^2 E_y}{\partial x \partial y} \qquad H_x = -\frac{C_y}{\mu}\frac{\partial E_y}{\partial z}$$

$$E_z = -j\frac{C_y}{\omega\mu\varepsilon}\frac{\partial^2 E_y}{\partial y \partial z} \qquad H_z = \frac{C_y}{\mu}\frac{\partial E_y}{\partial x}$$

cont'd.

6.8 Cont'd.

C. $TM^z \Rightarrow H_z = 0$

Let $E_z \neq 0$ so that $\nabla^2 E_z + \beta^2 E_z = 0$. Solve this for E_z.

According to (6-59) E_z is proportional to A_z. Therefore

$$A_z = \frac{E_z}{-j\frac{1}{\omega\mu\varepsilon}\left(\frac{\partial^2}{\partial z^2}+\beta^2\right)E_z} E_z = C_z E_z \quad \text{where} \quad C_z = +j\,\frac{E_z}{\frac{1}{\omega\mu\varepsilon}\left(\frac{\partial^2}{\partial z^2}+\beta^2\right)E_z}$$

Thus we can write the remaining $\underline{E}$- and $\underline{H}$-field components as

$$E_x = -j\frac{C_z}{\omega\mu\varepsilon}\frac{\partial^2 E_z}{\partial x \partial z} \qquad H_x = \frac{C_z}{\mu}\frac{\partial E_z}{\partial y}$$

$$E_y = -j\frac{C_z}{\omega\mu\varepsilon}\frac{\partial^2 E_z}{\partial y \partial z} \qquad H_y = -\frac{C_z}{\mu}\frac{\partial E_z}{\partial x}$$

$\boxed{6.9}$

TE^z: $E_z = 0$, $H_z(x,y,z) = f(x)g(y)h(z) = f(x)g(y)e^{-j\beta_z z}$

$$\nabla\times\underline{E} = -j\omega\mu\underline{H} \Rightarrow \underline{H} = -\frac{1}{j\omega\mu}\nabla\times\underline{E} = -\frac{1}{j\omega\mu}\begin{vmatrix}\hat{a}_x & \hat{a}_y & \hat{a}_z\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ E_x & E_y & 0\end{vmatrix}$$

$$\underline{H} = -\frac{1}{j\omega\mu}\left[\hat{a}_x\frac{\partial E_y}{\partial z} + \hat{a}_y\frac{\partial E_x}{\partial z} + \hat{a}_z\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)\right]$$

$$H_x = \frac{1}{j\omega\mu}\frac{\partial E_y}{\partial z} = \frac{-j\beta_z}{j\omega\mu}E_y = -\frac{\beta_z}{\omega\mu}E_y \qquad (1)$$

$$H_y = -\frac{1}{j\omega\mu}\frac{\partial E_x}{\partial z} = -\frac{-j\beta_z}{j\omega\mu}E_x = +\frac{\beta_z}{\omega\mu}E_x \qquad (2)$$

$$H_z = \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)\frac{1}{j\omega\mu} \qquad (3)$$

$$\nabla\times\underline{H} = +j\omega\varepsilon\underline{E} \Rightarrow \underline{E} = \frac{1}{j\omega\varepsilon}\nabla\times\underline{H} = \frac{1}{j\omega\varepsilon}\begin{vmatrix}\hat{a}_x & \hat{a}_y & \hat{a}_z\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ H_x & H_y & H_z\end{vmatrix}$$

$$\underline{E} = \frac{1}{j\omega\varepsilon}\left[\hat{a}_x\left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right) + \hat{a}_y\left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right) + \hat{a}_z\left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)\right]$$

$$E_x = \frac{1}{j\omega\varepsilon}\left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right) = \frac{1}{j\omega\varepsilon}\left(\frac{\partial H_z}{\partial y} + j\beta_z H_y\right) \qquad (4)$$

$$E_y = \frac{1}{j\omega\varepsilon}\left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right) = \frac{1}{j\omega\varepsilon}\left(-j\beta_z H_x - \frac{\partial H_z}{\partial x}\right) \qquad (5)$$

$$E_z = \frac{1}{j\omega\varepsilon}\left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right) = 0 \qquad (6)$$

Cont'd.

$\boxed{6.9 \text{ cont'd}}$

Substitute (4) into (2) and using that $\boxed{\beta^2 = \omega^2\mu\varepsilon = \beta_c^2 + \beta_z^2 \Rightarrow \beta^2 - \beta_z^2 = \beta_c^2}$

$$H_y = +\frac{\beta_z}{\omega\mu} E_x = \frac{\beta_z}{\omega\mu}\left[\frac{1}{j\omega\varepsilon}\left(\frac{\partial H_z}{\partial y} + j\beta_z H_y\right)\right] = -j\frac{\beta_z}{\omega^2\mu\varepsilon}\left(\frac{\partial H_z}{\partial y} + j\beta_z H_y\right)$$

$$= -j\frac{\beta_z}{\omega^2\mu\varepsilon}\frac{\partial H_z}{\partial y} + \frac{\beta_z^2}{\omega^2\mu\varepsilon} H_y = -j\frac{\beta_z}{\beta^2}\frac{\partial H_z}{\partial y} + \frac{\beta_z^2}{\beta^2} H_y$$

$$H_y\left(1 - \frac{\beta_z^2}{\beta^2}\right) = H_y\left(\frac{\beta^2 - \beta_z^2}{\beta^2}\right) = H_y\left(\frac{\beta_c^2}{\beta^2}\right) = -j\frac{\beta_z}{\beta^2}\frac{\partial H_z}{\partial y}$$

$$H_y = \frac{\beta^2}{\beta_c^2}\left(-j\frac{\beta_z}{\beta^2}\right)\frac{\partial H_z}{\partial y} = -j\frac{\beta_z}{\beta_c^2}\frac{\partial H_z}{\partial y}$$

$$\boxed{H_y = -j\frac{\beta_z}{\beta_c^2}\frac{\partial H_z}{\partial y}} \qquad (7)$$

Substitude (5) into (1), and using that $\boxed{\beta^2 = \omega^2\mu\varepsilon = \beta_c^2 + \beta_z^2 \Rightarrow \beta^2 - \beta_z^2 = \beta_c^2}$

$$H_x = -\frac{\beta_z}{\omega\mu} E_y = -\frac{\beta_z}{\omega\mu}\left[\frac{1}{j\omega\varepsilon}\left(-j\beta_z H_x - \frac{\partial H_z}{\partial x}\right)\right] = +j\frac{\beta_z}{\omega^2\mu\varepsilon}\left(-j\beta_z H_x - \frac{\partial H_z}{\partial x}\right)$$

$$= \frac{\beta_z^2}{\omega^2\mu\varepsilon} H_x - j\frac{\beta_z}{\omega^2\mu\varepsilon}\frac{\partial H_z}{\partial x} = \frac{\beta_z^2}{\beta^2} H_x - j\frac{\beta_z}{\beta^2}\frac{\partial H_z}{\partial x}$$

$$H_x\left(1 - \frac{\beta_z^2}{\beta^2}\right) = H_x\left(\frac{\beta^2 - \beta_z^2}{\beta^2}\right) = H_x\left(\frac{\beta_c^2}{\beta^2}\right) = -j\frac{\beta_z}{\beta^2}\frac{\partial H_z}{\partial x}$$

$$H_x = \frac{\beta^2}{\beta_c^2}\left(-j\frac{\beta_z}{\beta^2}\right)\frac{\partial H_z}{\partial x} = -j\frac{\beta_z}{\beta_c^2}\frac{\partial H_z}{\partial x}$$

$$\boxed{H_x = -j\frac{\beta_z}{\beta_c^2}\frac{\partial H_z}{\partial x}} \qquad (8)$$

Cont'd.

6.9 cont'd

Substitute (7) into (4)

$$E_x = \frac{1}{j\omega\varepsilon}\left(\frac{\partial H_z}{\partial y} + j\beta_z H_y\right) = \frac{1}{j\omega\varepsilon}\left[\frac{\partial H_z}{\partial y} + j\beta_z\left(-j\frac{\beta_z}{\beta_c^2}\right)\frac{\partial H_z}{\partial y}\right]$$

$$= \frac{1}{j\omega\varepsilon}\left[\frac{\partial H_z}{\partial y} + \frac{\beta_z^2}{\beta_c^2}\frac{\partial H_z}{\partial y}\right] = \frac{1}{j\omega\varepsilon}\left(1 + \frac{\beta_z^2}{\beta_c^2}\right)\frac{\partial H_z}{\partial y} = \frac{1}{j\omega\varepsilon}\left(\frac{\beta_c^2+\beta_z^2}{\beta_c^2}\right)\frac{\partial H_z}{\partial y}$$

$$E_x = \frac{1}{j\omega\varepsilon}\frac{\beta^2}{\beta_c^2}\frac{\partial H_z}{\partial y} = \frac{\omega^2\mu\varepsilon}{j\omega\varepsilon}\frac{1}{\beta_c^2}\frac{\partial H_z}{\partial y} = -j\frac{\omega\mu}{\beta_c^2}\frac{\partial H_z}{\partial y}$$

$$\boxed{E_x = -j\frac{\omega\mu}{\beta_c^2}\frac{\partial H_z}{\partial y}} \qquad (9)$$

Substitute (8) into (5)

$$E_y = \frac{1}{j\omega\varepsilon}\left(-j\beta_z H_x - \frac{\partial H_z}{\partial x}\right) = \frac{1}{j\omega\varepsilon}\left[-j\beta_z\left(-j\frac{\beta_z}{\beta_c^2}\frac{\partial H_z}{\partial x}\right) - \frac{\partial H_z}{\partial x}\right]$$

$$= \frac{1}{j\omega\varepsilon}\left[-\frac{\beta_z^2}{\beta_c^2}\frac{\partial H_z}{\partial x} - \frac{\partial H_z}{\partial x}\right] = -\frac{1}{j\omega\varepsilon}\left(\frac{\beta_z^2}{\beta_c^2}+1\right)\frac{\partial H_z}{\partial x} = -\frac{1}{j\omega\varepsilon}\left(\frac{\beta_z^2+\beta_c^2}{\beta_c^2}\right)\frac{\partial H_z}{\partial x}$$

$$E_y = -\frac{1}{j\omega\varepsilon}\frac{\beta^2}{\beta_c^2}\frac{\partial H_z}{\partial x} = -\frac{1}{j\omega\varepsilon}\frac{\omega^2\mu\varepsilon}{\beta_c^2}\frac{\partial H_z}{\partial x} = +j\frac{\omega\mu}{\beta_c^2}\frac{\partial H_z}{\partial x}$$

$$\boxed{E_y = +j\frac{\omega\mu}{\beta_c^2}\frac{\partial H_z}{\partial x}} \qquad (10)$$

cont'd

6.9 cont'd

SUMMARY

TE^z:

$$E_x = -j\frac{\omega\mu}{\beta_c^2}\frac{\partial H_z}{\partial y}$$

$$E_y = +j\frac{\omega\mu}{\beta_c^2}\frac{\partial H_z}{\partial x}$$

$$E_z = 0 = \frac{1}{j\omega\varepsilon}\left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right) \Rightarrow \frac{\partial H_x}{\partial y} = \frac{\partial H_y}{\partial x}$$

$$H_x = -j\frac{\beta_z}{\beta_c^2}\frac{\partial H_z}{\partial x}$$

$$H_y = -j\frac{\beta_z}{\beta_c^2}\frac{\partial H_z}{\partial y}$$

$$H_x = h_z(x,y)e^{-j\beta_z z} = f(x)g(y)e^{-j\beta_z z}$$

$$\boxed{\nabla^2 H_z + \beta^2 H_z = (\nabla^2 + \beta^2)H_z(x,y,z) = 0}$$

6.10

TM^z: It can be shown that for TM^z $\left[H_z = 0,\ E_z(x,y,z) = e(x,y)e^{-j\beta_z z}\right]$

$$E_x = -j\frac{\beta_z}{\beta_c^2}\frac{\partial E_z}{\partial x},\quad E_y = -j\frac{\beta_z}{\beta_c^2}\frac{\partial E_z}{\partial y}$$

$$H_x = j\frac{\omega\varepsilon}{\beta_c^2}\frac{\partial E_z}{\partial y},\quad H_y = -j\frac{\omega\varepsilon}{\beta_c^2}\frac{\partial E_z}{\partial x}$$

$$E_z(x,y,z) = e(x,y)e^{-j\beta_z z} = f(x)g(y)e^{-j\beta_z z}$$

$$\boxed{\nabla^2 E_z + \beta^2 E_z = (\nabla^2 + \beta^2)E_z(x,y,z) = 0}$$

$\boxed{6.11}$ $\underline{TM^z \Rightarrow H_z = 0}$

Let $E_z \neq 0$ so that $\nabla^2 E_z + \beta^2 E_z = 0$. Solve this for E_z. Then according to (6-70) E_z is proportional to A_z. Therefore

$$A_z = \frac{E_z}{-j\frac{1}{\omega\mu\varepsilon}\left(\frac{\partial^2}{\partial z^2} + \beta^2\right)E_z} E_z = C_z E_z \quad \text{where} \quad C_z = j\frac{E_z}{\frac{1}{\omega\mu\varepsilon}\left(\frac{\partial^2}{\partial z^2}+\beta^2\right)E_z}$$

Thus we can write the remaining $\underline{E}$- and $\underline{H}$-field components in terms of E_z as

$$E_\rho = -j\frac{C_z}{\omega\mu\varepsilon}\frac{\partial^2 E_z}{\partial\rho\,\partial z} \qquad H_\rho = \frac{1}{\mu}\frac{C_z}{\rho}\frac{\partial E_z}{\partial\phi}$$

$$E_\phi = -j\frac{C_z}{\omega\mu\varepsilon}\frac{1}{\rho}\frac{\partial^2 E_z}{\partial\phi\,\partial z} \qquad H_\phi = -\frac{C_z}{\mu}\frac{\partial E_z}{\partial\rho}$$

$\boxed{6.12}$ A. $\underline{TE^z \Rightarrow E_z = 0}$

It is apparent by examining (6-41) and (6-43) that $E_z = 0$ can be obtained by letting

$A_x = A_y = A_z = F_x = F_y = 0, \; F_z \neq 0$

When these are substituted into into (6-41) and (6-43), we get (6-72).

B. $\underline{TE^x \Rightarrow E_x = 0}$

It is apparent by examining (6-41) and (6-43) that $E_x = 0$ can be obtained by letting

$A_x = A_y = A_z = F_y = F_z = 0, \; F_x \neq 0$

When these are substituted into (6-41) and (6-43), we get (6-74).

C. $\underline{TE^y \Rightarrow E_y = 0}$

It is apparent by examining (6-41) and (6-43) that $E_y = 0$ can be obtained by letting

$A_x = A_y = A_z = F_x = F_z = 0, \; F_y \neq 0$

When these are substituted into (6-41) and (6-43), we get (6-77).

$\boxed{6.13}$ $TE^z \Rightarrow E_z = 0$

It is apparent by examining (6-50) and (6-51) that $E_z=0$ can be obtained by letting

$$A_x = A_y = A_z = F_x = F_y = 0, \quad F_z \neq 0$$

When these are substituted into (6-50) and (6-51), we get (6-80).

$\boxed{6.14}$ A. $TE^x \Rightarrow E_x = 0$

Let $H_x \neq 0$ so that $\nabla^2 H_x + \beta^2 H_x = 0$. Solve this for H_x. Then according to (6-74) H_x is proportional to F_x. Therefore

$$F_x = \frac{H_x}{-j\frac{1}{\omega\mu\varepsilon}\left(\frac{\partial^2}{\partial x^2}+\beta^2\right)H_x} H_x = D_x H_x \quad \text{where } D_x = j\frac{H_x}{\frac{1}{\omega\mu\varepsilon}\left(\frac{\partial^2}{\partial x^2}+\beta^2\right)H_x}$$

Thus we can write the remaining $\underline{E}$- and $\underline{H}$-field components in terms of H_x as

$$E_y = -\frac{D_x}{\varepsilon}\frac{\partial H_x}{\partial z} \qquad H_y = -j\frac{D_x}{\omega\mu\varepsilon}\frac{\partial^2 H_x}{\partial x \partial y}$$

$$E_z = \frac{D_x}{\varepsilon}\frac{\partial H_x}{\partial y} \qquad H_z = -j\frac{D_x}{\omega\mu\varepsilon}\frac{\partial^2 H_x}{\partial x \partial z}$$

B. $TE^y \Rightarrow E_y = 0$

Let $H_y \neq 0$ so that $\nabla^2 H_y + \beta^2 H_y = 0$. Solve this for H_y. Then according to (6-77) H_y is proportional to F_y. Therefore

$$F_y = \frac{H_y}{-j\frac{1}{\omega\mu\varepsilon}\left(\frac{\partial^2}{\partial y^2}+\beta^2\right)H_y} H_y = D_y H_y \quad \text{where } D_y = j\frac{H_y}{\frac{1}{\omega\mu\varepsilon}\left(\frac{\partial^2}{\partial y^2}+\beta^2\right)H_y}$$

Thus we can write the remaining $\underline{E}$ and $\underline{H}$-field components in terms of H_y as

$$E_x = \frac{D_x}{\varepsilon}\frac{\partial H_y}{\partial z} \qquad H_x = -j\frac{D_x}{\omega\mu\varepsilon}\frac{\partial^2 H_y}{\partial x \partial y}$$

$$E_z = -\frac{D_x}{\varepsilon}\frac{\partial H_y}{\partial x} \qquad H_z = -j\frac{1}{\omega\mu\varepsilon}\frac{\partial^2 H_y}{\partial y \partial z}$$

Cont'd.

6.14 Cont'd.

C. $TE^z \Rightarrow E_z = 0$

Let $H_z \neq 0$ so that $\nabla^2 H_z + \beta^2 H_z = 0$. Solve this for H_z. Then according to (6-72) H_z is proportional to F_z. Therefore

$$F_z = \frac{H_z}{-j\frac{1}{\omega\mu\varepsilon}\left(\frac{\partial^2}{\partial z^2}+\beta^2\right)H_z} H_z = D_z H_z \quad \text{where } D_z = j\frac{H_z}{\frac{1}{\omega\mu\varepsilon}\left(\frac{\partial^2}{\partial z^2}+\beta^2\right)H_z}$$

Thus we can write the remaining $\underline{E}$- and $\underline{H}$-field components in terms of H_z as

$$E_x = -\frac{D_z}{\varepsilon}\frac{\partial H_z}{\partial y} \qquad H_x = -j\frac{D_z}{\omega\mu\varepsilon}\frac{\partial^2 H_z}{\partial x \partial z}$$

$$E_y = \frac{D_z}{\varepsilon}\frac{\partial H_z}{\partial x} \qquad H_y = -j\frac{D_z}{\omega\mu\varepsilon}\frac{\partial^2 H_z}{\partial y \partial z}$$

6.15 $TE^z \Rightarrow E_z = 0$

Let $H_z \neq 0$ so that $\nabla^2 H_z + \beta^2 H_z = 0$. Solve this for H_z. Then according to (6-80) H_z is proportional to F_z. Therefore

$$F_z = \frac{H_z}{-j\frac{1}{\omega\mu\varepsilon}\left(\frac{\partial^2}{\partial z^2}+\beta^2\right)H_z} H_z = D_z H_z \quad \text{where } D_z = j\frac{H_z}{\frac{1}{\omega\mu\varepsilon}\left(\frac{\partial^2}{\partial z^2}+\beta^2\right)H_z}$$

Thus we can write the remaining $\underline{E}$- and $\underline{H}$-field components in terms of H_z as

$$E_\rho = -\frac{D_z}{\varepsilon\rho}\frac{\partial H_z}{\partial \phi} \qquad H_\rho = -j\frac{D_z}{\omega\mu\varepsilon}\frac{\partial^2 H_z}{\partial \rho \partial z}$$

$$E_\phi = \frac{D_z}{\varepsilon}\frac{\partial H_z}{\partial \rho} \qquad H_\phi = -j\frac{D_z}{\omega\mu\varepsilon}\frac{1}{\rho}\frac{\partial^2 H_z}{\partial \phi \partial z}$$

$\boxed{6.16}$ $A_{z1} = C_1 \dfrac{e^{-j\beta r}}{r}$, $A_{z2} = C_2 \dfrac{e^{+j\beta r}}{r}$

$$\frac{dA_{z1}}{dr} = C_1\left[-j\beta\frac{e^{-j\beta r}}{r} - \frac{e^{-j\beta r}}{r^2}\right], \quad \frac{dA_{z2}}{dr} = C_2\left[j\beta\frac{e^{+j\beta r}}{r} - \frac{e^{+j\beta r}}{r^2}\right]$$

$$\frac{d^2A_{z1}}{dr^2} = C_1\left[-\beta^2\frac{e^{-j\beta r}}{r} + j\beta\frac{e^{-j\beta r}}{r^2} + j\beta\frac{e^{-j\beta r}}{r^2} + 2\frac{e^{-j\beta r}}{r^3}\right]$$

$$= C_1\left[-\beta^2\frac{e^{-j\beta r}}{r} + j2\beta\frac{e^{-j\beta r}}{r^2} + 2\frac{e^{-j\beta r}}{r^3}\right]$$

$$\frac{d^2A_{z2}}{dr^2} = C_2\left[-\beta^2\frac{e^{+j\beta r}}{r} - j\beta\frac{e^{+j\beta r}}{r^2} - j\beta\frac{e^{+j\beta r}}{r^2} + 2\frac{e^{+j\beta r}}{r^3}\right]$$

$$= C_2\left[-\beta^2\frac{e^{+j\beta r}}{r} - j2\beta\frac{e^{+j\beta r}}{r^2} + 2\frac{e^{-j\beta r}}{r^3}\right]$$

Substituting each of these into (6-84a) balances the equation.

$\boxed{6.17}$

a. $A_z = \dfrac{\mu}{4\pi}\displaystyle\iiint_V \frac{J_z}{r}\,dv'$, $\nabla^2 A_z = \dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2\dfrac{\partial A_z}{\partial r}\right)$

When $r \neq 0$ and $\underline{r} \notin V'$, then

$$\begin{cases} \dfrac{\partial A_z}{\partial r} = -\dfrac{\mu}{4\pi}\displaystyle\iiint_V \frac{J_z}{r^2}\,dv' \\ r^2\dfrac{\partial A_z}{\partial r} = -\dfrac{\mu}{4\pi}\displaystyle\iiint_V J_z\,dv' \\ \dfrac{\partial}{\partial r}\left(r^2\dfrac{\partial A_z}{\partial r}\right) = 0 \end{cases}$$

Therefore when $r \neq 0\ (\underline{r} \notin V')$ $\nabla^2 A_z = 0$. However when $\underline{r} \to \underline{r}'$

$$\nabla^2 A_z(x,y,z) = \frac{\mu}{4\pi}\iiint_V \nabla\cdot\left(\nabla\frac{1}{r}\right)J_z(x',y',z')\,dv' = \frac{\mu}{4\pi}\iiint_{\substack{r\in S_\varepsilon \\ \varepsilon\to 0}} \nabla\cdot\left(-\frac{\underline{r}}{r^3}\right)J_z(x',y',z')\,dx'dy'dz'$$

where $S_\varepsilon = \{\underline{r}' : |\underline{r}' - \underline{r}| \le \varepsilon\}$

Thus

$$\nabla^2 A_z = -\frac{\mu}{4\pi}\oiint_{\substack{S \\ \varepsilon\to 0}} \frac{\underline{r}}{r^3}\cdot\hat{n}\, J_z(x',y',z')\, r^2\sin\theta\, d\theta\, d\phi = -\frac{\mu}{4\pi}\int_0^{2\pi}\int_0^{\pi} J_z(x,y,z)\sin\theta\, d\theta\, d\phi$$

$$\nabla^2 A_z = -\frac{\mu}{4\pi}J_z(2)(2\pi) = -\mu J_z \Rightarrow \nabla^2 A_z = -\mu J_z$$

cont'd.

$\boxed{\text{6.17 cont'd.}}$

b. $A_z = \frac{\mu}{4\pi}\iiint_V J_z \frac{e^{-j\beta r}}{r}\,dv'$, $\nabla^2 A_z + \beta^2 A_z = -\mu J_z$

When $r \neq 0$ and $\underline{r} \neq v'$.

$$(\nabla^2+\beta^2)\frac{e^{-j\beta r}}{r} = \beta^2\frac{e^{-j\beta r}}{r} + \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial}{\partial r}\left(\frac{e^{-j\beta r}}{r}\right)\right]$$

$$= \beta^2\frac{e^{-j\beta r}}{r} + \frac{1}{r^2}\frac{\partial}{\partial r}\left[-(1+j\beta r)\,e^{-j\beta r}\right]$$

$$(\nabla^2+\beta^2)\frac{e^{-j\beta r}}{r} = \beta^2\frac{e^{-j\beta r}}{r} + \frac{1}{r^2}\left[-j\beta e^{-j\beta r} + j\beta e^{-j\beta r} - \beta^2 r\,e^{-j\beta r}\right] = 0$$

Thus when $r \neq 0$ and $\underline{r} \neq v'$

$$\nabla^2 A_z + \beta^2 A_z = \frac{\mu}{4\pi}(\nabla^2+\beta^2)\iiint_V J_z\frac{e^{-j\beta r}}{r}\,dv' = \frac{\mu}{4\pi}\iiint_V J_z\left[(\nabla^2+\beta^2)\frac{e^{-j\beta r}}{r}\right]dv' = 0$$

When $\underline{r} \to \underline{r}'$

$$\nabla^2 A_z + \beta^2 A_z = \lim_{\varepsilon\to 0}\left\{\frac{\mu}{4\pi}\iiint_{S_\varepsilon}(\nabla^2+\beta^2)\frac{e^{-j\beta r}}{r}J_z(x',y',z')\,dv'\right\}$$

$$= \lim_{\substack{\varepsilon\to 0\\ r\to 0}}\iiint_{S_\varepsilon}\nabla\cdot\left(-\frac{\underline{r}}{r^3}\right)e^{-j\beta r}J_z(x,y,z)\,r^2\sin\theta\,d\theta\,d\phi$$

$$\nabla^2 A_z + \beta^2 A_z = -\frac{\mu}{4\pi}\int_0^{2\pi}\int_0^{\pi}J_z(x,y,z)\sin\theta\,d\theta\,d\phi = -\frac{\mu}{4\pi}J_z(x,y,z)(2)(2\pi)$$

$$\nabla^2 A_z + \beta^2 A_z = -\frac{\mu}{4\pi}J_z(4\pi) = -\mu J_z$$

Thus

$$\nabla^2 A_z + \beta^2 A_z = -\mu J_z$$

$\boxed{6.18}$ $H_r = H_\theta = 0$, $\quad H_\phi = j\,\dfrac{\beta I_e \ell \sin\theta}{4\pi r}\left(1+\dfrac{1}{j\beta r}\right)e^{-j\beta r}$

$$\nabla\times\underline{H} = \frac{\hat{a}_r}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\left(H_\phi \sin\theta\right)\right] - \frac{\hat{a}_\theta}{r}\left[\frac{\partial}{\partial r}\left(r H_\phi\right)\right]$$

$$\frac{\partial}{\partial\theta}\left(H_\phi \sin\theta\right) = 2\sin\theta\cos\theta\, H_\phi$$

$$\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(H_\phi\sin\theta\right) = j\,\frac{\beta I_e \ell \cos\theta}{2\pi r^2}\left(1+\frac{1}{j\beta r}\right)e^{-j\beta r}$$

$$r H_\phi = j\,\frac{\beta I_e \ell \sin\theta}{4\pi}\left(1+\frac{1}{j\beta r}\right)e^{-j\beta r}$$

$$\frac{\partial}{\partial r}\left(r H_\phi\right) = j\,\frac{\beta I_e \ell \sin\theta}{4\pi}\left\{-j\beta\left(1+\frac{1}{j\beta r}\right)e^{-j\beta r} + \left(-\frac{1}{j\beta r^2}\right)e^{-j\beta r}\right\}$$

$$= j\,\frac{\beta I_e \ell \sin\theta}{4\pi}(-j\beta)\left\{1+\frac{1}{j\beta r}-\frac{1}{(\beta r)^2}\right\}e^{-j\beta r}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r H_\phi\right) = \frac{\beta^2 I_e \ell \sin\theta}{4\pi r}\left[1+\frac{1}{j\beta r}-\frac{1}{(\beta r)^2}\right]e^{-j\beta r}$$

Using the above equations we can write that

$$\underline{E} = \frac{1}{j\omega\varepsilon}\nabla\times\underline{H} = \frac{1}{j\omega\varepsilon}\left\{\frac{\hat{a}_r}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\left(H_\phi\sin\theta\right)\right] - \frac{\hat{a}_\theta}{r}\left[\frac{\partial}{\partial r}\left(r H_\phi\right)\right]\right\}$$

is equal to

$$E_r = \eta\,\frac{I_e \ell \cos\theta}{2\pi r^2}\left[1+\frac{1}{j\beta r}\right]e^{-j\beta r}$$

$$E_\theta = j\eta\,\frac{\beta I_e \ell \sin\theta}{4\pi r}\left[1+\frac{1}{j\beta r}-\frac{1}{(\beta r)^2}\right]e^{-j\beta r}, \quad E_\phi = 0$$

$\boxed{6.19}$ When $\beta r \gg 1$ the most dominant term in each of the above expressions is the first one. Also since the most dominant term for E_r varies as $1/r^2$ while for E_θ varies as $1/r$, the E_r component is very small compared to E_θ so that is set approximately equal to zero. Therefore

$E_r \simeq 0$, $E_\phi = H_r = H_\theta = 0$

$E_\theta \simeq j\eta\,\dfrac{\beta I_e \ell e^{-j\beta r}}{4\pi r}\sin\theta$, $H_\phi \simeq E_\theta/\eta$

6.20

- $\underline{S}_{ave} = \frac{1}{2}\text{Re}(\underline{E}\times\underline{H}^*) = \frac{1}{2}\text{Re}\left[\hat{a}_\theta E_\theta \times \left(\hat{a}_\phi \frac{E_\theta}{\eta}\right)^*\right] = \hat{a}_r \frac{1}{2\eta}\left|E_\theta\right|^2$

$$= \frac{\hat{a}_r}{2\eta}\left|j\eta \frac{\beta I_0 \ell e^{-j\beta r}}{4\pi r}\sin\theta\right|^2 = \hat{a}_r \frac{\eta\beta^2|I_0\ell|^2}{32\pi^2 r^2}\sin^2\theta = \hat{a}_r\eta\frac{(4\pi^2/\lambda^2)|I_0\ell|^2}{32\pi^2r^2}\sin^2\theta$$

$$\underline{S}_{ave} = \hat{a}_r \frac{\eta}{8}\left|\frac{I_0\ell}{\lambda}\right|^2 \frac{\sin^2\theta}{r^2}$$

- $U = r^2 S_{ave} = \frac{\eta}{8}\left|\frac{I_0\ell}{\lambda}\right|^2 \sin^2\theta$

- $P_{rad} = \int_0^{2\pi}\int_0^{\pi} U(\theta,\phi)\sin\theta\, d\theta\, d\phi = \frac{\eta}{8}\left|\frac{I_0\ell}{\lambda}\right|^2 \int_0^{2\pi} d\phi \int_0^{\pi}\sin^3\theta\, d\theta = \frac{\eta}{8}\left|\frac{I_0\ell}{\lambda}\right|^2 (2\pi)\left(\frac{4}{3}\right)$

$$P_{rad} = \eta\left(\frac{\pi}{3}\right)\left|\frac{I_0\ell}{\lambda}\right|^2$$

- $U = \frac{\eta}{8}\left|\frac{I_0\ell}{\lambda}\right|^2 \sin^2\theta \Rightarrow U_{max}\Big|_{\theta=90^\circ} = \frac{\eta}{8}\left|\frac{I_0\ell}{\lambda}\right|^2$

$$D_0 = \frac{4\pi U_{max}}{P_{rad}} = \frac{4\pi\left[\frac{\eta}{8}\left|\frac{I_0\ell}{\lambda}\right|^2\right]}{\eta\left(\frac{\pi}{3}\right)\left|\frac{I_0\ell}{\lambda}\right|^2} = \frac{3}{2}$$

$$D_0 = \frac{3}{2}(\text{dimensionless}) = 10\log_{10}\left(\frac{3}{2}\right) = 1.761\text{ dB}$$

- $R_r = \frac{2P_{rad}}{|I_0|^2} = 2\frac{\eta\left(\frac{\pi}{3}\right)\left|\frac{I_0\ell}{\lambda}\right|^2}{|I_0|^2} = \eta\left(\frac{2\pi}{3}\right)\left|\frac{I_0\ell}{\lambda}\right|^2 \frac{1}{|I_0|^2}$

$$R_r = 120\pi\left(\frac{2\pi}{3}\right)\left(\frac{\ell}{\lambda}\right)^2 = 80\pi^2\left(\frac{\ell}{\lambda}\right)^2$$

$$R_r = 80\pi^2\left(\frac{\ell}{\lambda}\right)^2$$

$\boxed{6.21}$

$$\underline{A} = \frac{\mu}{4\pi}\int \underline{I}_e \frac{e^{-j\beta R}}{R}\, dl' = \frac{\mu}{4\pi}\int_{-l/2}^{+l/2} \hat{a}_x I_0 \frac{e^{-j\beta r}}{r}\, dx' = \hat{a}_x \frac{I_0 \mu}{4\pi}\frac{e^{-jkr}}{r}\int_{-l/2}^{+l/2} dx'$$

$$\underline{A} = \hat{a}_x \frac{l\mu I_0 e^{-j\beta r}}{4\pi r} = \hat{a}_x A_x \Rightarrow A_x = \frac{\mu I_0 l}{4\pi r} e^{-j\beta r}$$

$$\begin{pmatrix} A_r \\ A_\theta \\ A_\phi \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{pmatrix} \begin{pmatrix} A_x \\ 0 \\ 0 \end{pmatrix}$$

$$A_r = A_x \sin\theta\cos\phi = \frac{\mu I_0 l e^{-j\beta r}}{4\pi r}\sin\theta\cos\phi$$

$$A_\theta = A_x \cos\theta\cos\phi = \frac{\mu I_0 l e^{-j\beta r}}{4\pi r}\cos\theta\cos\phi$$

$$A_\phi = -A_x \sin\phi = -\frac{\mu I_0 l e^{-j\beta r}}{4\pi r}\sin\phi$$

- **In far-field:**

$$\left.\begin{aligned} E_r &\simeq 0 \\ E_\theta &\simeq -j\omega A_\theta \\ E_\phi &\simeq -j\omega A_\phi \end{aligned}\right\} \begin{matrix}(3\text{-}58a) \\ (3\text{-}58b)\end{matrix} \Rightarrow \left.\begin{aligned} E_r &\simeq 0 \\ E_\theta &= -j\frac{\omega\mu I_0 l e^{-j\beta r}}{4\pi r}\cos\theta\cos\phi \\ E_\phi &= -j\frac{\omega\mu I_0 l e^{-j\beta r}}{4\pi r}\sin\phi \end{aligned}\right\} \begin{aligned} H_r &\simeq 0 \\ H_\phi &= \frac{E_\theta}{\eta} \\ H_\theta &= -\frac{E_\phi}{\eta} \end{aligned}$$

- $\underline{S}_{ave} = \frac{1}{2}\text{Re}\left[\underline{E}\times\underline{H}^*\right] = \frac{1}{2}\text{Re}\left[(\hat{a}_\theta E_\theta + \hat{a}_\phi E_\phi)\times(\hat{a}_\theta H_\theta + \hat{a}_\phi H_\phi)^*\right]$

$$\underline{S}_{ave} = \frac{1}{2\eta}\left\{|E_\theta|^2 + |E_\phi|^2\right\} = \hat{a}_r \frac{\eta}{8}\left|\frac{I_0 l}{\lambda}\right|^2 \frac{\sin^2\theta}{r^2} = \hat{a}_r W_{ave} = \hat{a}_r W_r$$

- $U = \frac{r^2}{2\eta}\left[|E_\theta|^2 + |E_\phi|^2\right]$ (2-12a)

$$U = \left(\frac{\omega\mu I_0 l}{4\pi}\right)^2 \frac{1}{2\eta}\left[\cos^2\theta\cos^2\phi + \sin^2\phi\right]$$

$$= B_0\left[\cos^2\theta\cos^2\phi + \sin^2\phi\right]$$

$$B_0 = \frac{1}{2\eta}\left(\frac{\omega\mu I_0 l}{4\pi}\right)^2 = \frac{1}{2\eta}\left(\frac{\eta\omega\mu I_0 l}{\eta 4\pi}\right)^2 = \frac{1}{2\eta}\left[\frac{\eta\omega\mu I_0 l}{4\pi\sqrt{\mu/\varepsilon}}\right]^2$$

$$= \frac{1}{2\eta}\left[\frac{\eta\omega\sqrt{\mu\varepsilon}}{4\pi} I_0 l\right] = \frac{1}{2\eta}\left[\frac{\eta\beta I_0 l}{4\pi}\right] = \frac{\eta^2}{2\eta}\left(\frac{\beta I_0 l}{4\pi}\right)^2 = \frac{\eta}{2}\left(\frac{\beta I_0 l}{4\pi}\right)^2$$

$$B_0 = \frac{\eta}{2}\left(\frac{\beta I_0 l}{4\pi}\right)^2$$

$$U = B_0(\cos^2\theta\cos^2\phi + \sin^2\phi) \Rightarrow U_{max} = B_0 \text{ when } \phi = 90°, 270°$$

$$0 \leqslant \theta \leqslant 180°$$

Cont'd

6.21 cont'd

- $$P_{\text{rad}} = \int_0^{2\pi}\int_0^{\pi} U \sin\theta \, d\theta \, d\phi$$
$$= B_0 \left\{ \int_0^{2\pi}\int_0^{\pi} \underbrace{\cos^2\theta\cos^2\phi\sin\theta\,d\theta\,d\phi}_{I_1} + \int_0^{2\pi}\int_0^{\pi} \underbrace{\sin^2\phi\sin\theta\,d\theta\,d\phi}_{I_2} \right\}$$
$$I_1 = \int_0^{2\pi} \cos^2\phi\,d\phi \int_0^{\pi} \cos^2\theta\sin\theta\,d\theta = \int_0^{2\pi}\cos^2\phi\,d\phi\int_0^{\pi}\cos^2\theta\,d(-\cos\theta)$$
$$= -\int_0^{2\pi}\left(\frac{1+\cos(2\phi)}{2}\right)d\phi\int_0^{\pi}(\cos\theta)^2\,d(\cos\theta)$$
$$= -\tfrac{1}{2}\left[(\phi + \tfrac{1}{2}\sin 2\phi)_0^{2\pi}\right]\left[\frac{\cos^3\theta}{3}\right]_0^{\pi}$$
$$I_1 = -\tfrac{1}{2}[(2\pi)]\left(-\frac{1}{3}-\frac{1}{3}\right) = \tfrac{1}{2}(2\pi)\left(\frac{2}{3}\right) = \frac{2\pi}{3}$$
$$I_2 = \int_0^{2\pi}\int_0^{\pi}\sin^2\phi\sin\theta\,d\theta\,d\phi = \int_0^{2\pi}\sin^2\phi\,d\phi\int_0^{\pi}\sin\theta\,d\theta$$
$$= \int_0^{2\pi}\left(\frac{1-\cos(2\phi)}{2}\right)d\phi\int_0^{\pi}\sin\theta\,d\theta$$

$$I_2 = \tfrac{1}{2}\left[\phi + \tfrac{1}{2}\sin 2\phi\right]_0^{2\pi}(-\cos\theta)_0^{\pi} = \tfrac{1}{2}(2\pi)[-(-1)+1] = 2\pi$$
$$I_1 + I_2 = \frac{2\pi}{3} + 2\pi = \frac{8\pi}{3}$$
$$P_{\text{rad}} = B_0(I_1+I_2) = B_0\left(\frac{8\pi}{3}\right) = \left(\frac{8\pi}{3}\right)\left[\frac{\eta}{2}\left(\frac{\beta I_0 \ell}{4\pi}\right)^2\right] = \frac{8\pi}{3}\left[\frac{\eta}{2}\left(\frac{\beta\ell}{4\pi}\right)^2|I_0|^2\right]$$
$$D_0 = \frac{4\pi U_{\max}}{P_{\text{rad}}} = \frac{4\pi(B_0)}{\frac{8\pi}{3}(B_0)} = \frac{3}{2} = 1.761 \text{ dB}$$
$$\boxed{D_0 = 1.5 = +1.761 \text{ dB}}$$

- $$R_r = \frac{2P_{\text{rad}}}{|I_0|^2} = 2\left(\frac{8\pi}{3}\right)\left[\frac{\eta}{2}\left(\frac{\beta\ell}{4\pi}\right)^2\right] = 8\pi^2\left(\frac{\ell}{\lambda}\right)^2$$

$\boxed{6.22}$ $\underline{I}_e = \hat{a}_y I_o$

- $\underline{A} = \frac{\mu}{4\pi}\int \underline{I}_e \frac{e^{-j\beta r}}{r} dy' = \hat{a}_y I_o \frac{\mu e^{-j\beta r}}{4\pi r}\int_{-\ell/2}^{+\ell/2} dy' = \hat{a}_y I_o \frac{\ell \mu e^{-j\beta r}}{4\pi r}, \quad A_y = \frac{\mu I_o \ell}{4\pi r} e^{-j\beta r}$

$A_r = A_y \sin\theta \sin\phi, \quad A_\theta = A_y \cos\theta \sin\phi, \quad A_\phi = A_y \cos\phi$

$A_r = \frac{\ell \mu I_o e^{-j\beta r}}{4\pi r} \sin\theta \sin\phi, \quad A_\theta = \frac{\ell \mu I_o e^{-j\beta r}}{4\pi r} \cos\theta \sin\phi, \quad A_\phi = \frac{\ell \mu I_o e^{-j\beta r}}{4\pi r} \cos\phi$

- $$E_r \simeq 0 \qquad H_r = 0$$
$$E_\theta \simeq -j\omega A_\theta = -j\frac{\omega\mu I_0 l e^{-j\beta r}}{4\pi r}\cos\theta\sin\phi \qquad H_\theta = -E_\phi/\eta$$
$$E_\phi \simeq -j\omega A_\phi = -j\frac{\omega\mu I_0 l e^{-j\beta r}}{4\pi r}\cos\phi \qquad H_\phi = E_\theta/\eta$$

- $\underline{S}_{ave} = \frac{1}{2}\mathrm{Re}\left[\underline{E}\times\underline{H}^*\right] = \frac{1}{2}\mathrm{Re}\left[(\hat{a}_\theta E_\theta + \hat{a}_\phi E_\phi)\times(\hat{a}_\theta H_\theta + \hat{a}_\phi H_\phi)^*\right]$

$\underline{S}_{ave} = \frac{1}{2\eta}\left\{|E_\theta|^2 + |E_\phi|^2\right\} = \hat{a}_r \frac{\eta}{8}\left|\frac{I_o \ell}{\lambda}\right|^2 \frac{\sin^2\theta}{r^2} = \hat{a}_r W_{ave} = \hat{a}_r W_r$

- $$U(\theta,\phi) = \frac{r^2}{2\eta}[|E_\theta|^2 + |E_\phi|^2] = \frac{1}{2\eta}\left(\frac{\omega\mu I_0 l}{4\pi}\right)^2[\cos^2\theta\sin^2\phi + \cos^2\phi]$$
$$U(\theta,\phi) = \frac{1}{2\eta}\left(\frac{\eta\omega\mu I_0 l}{4\pi\sqrt{\mu/\varepsilon}}\right)^2[\cos^2\theta\sin^2\phi + \cos^2\phi] = B_0(\cos^2\theta\sin^2\phi + \cos^2\phi)$$
$$B_0 = \frac{1}{2\eta}\left(\frac{\eta\omega\sqrt{\mu\varepsilon}I_0 l}{4\pi}\right)^2 = \frac{\eta}{2}\left(\frac{\beta I_0 l}{4\pi}\right)^2$$
$$U(\theta,\phi) = B_0(\cos^2\theta\sin^2\phi + \cos^2\phi) \Rightarrow U_{max} = B_0 @ \phi = 0^\circ, 180^\circ$$
$$P_{rad} = \int_0^{2\pi}\int_0^{\pi} U(\theta,\phi)\sin\theta\, d\theta\, d\phi = B_0\int_0^{2\pi}\int_0^{\pi}(\cos^2\theta\sin^2\phi + \cos^2\phi)\sin\theta\, d\theta\, d\phi$$
$$= B_0\left\{\underbrace{\int_0^{2\pi}\int_0^{\pi}\cos^2\theta\sin^2\phi\sin\theta\, d\theta\, d\phi}_{I_1} + \underbrace{\int_0^{2\pi}\int_0^{\pi}\cos^2\phi\sin\theta\, d\theta\, d\phi}_{I_2}\right\}$$
$$I_1 = \int_0^{2\pi}\sin^2\phi\, d\phi\int_0^{\pi}\cos^2\theta\sin\theta\, d\theta$$
$$= \int_0^{2\pi}\left(\frac{1-\cos 2\phi}{2}\right)d\phi\int_0^{\pi}\cos^2\theta\, d(-\cos\theta)$$
$$= -\tfrac{1}{2}\left[\phi - \tfrac{1}{2}\sin 2\phi\right]_0^{2\pi}\left(\frac{\cos^3\theta}{3}\right)_0^{\pi} = -\tfrac{1}{2}(2\pi)\left(\frac{-1-1}{3}\right) = \frac{2\pi}{3}$$
$$I_2 = \int_0^{2\pi}\int_0^{\pi}\cos^2\phi\sin\theta\, d\theta\, d\phi = \int_0^{2\pi}\left(\frac{1+\cos 2\phi}{2}\right)d\phi\int_0^{\pi}\sin\theta\, d\theta$$
$$= \tfrac{1}{2}\left[\phi + \tfrac{1}{2}\sin 2\phi\right]_0^{2\pi}(-\cos\theta)_0^{\pi} = \tfrac{1}{2}(2\pi)(2) = 2\pi$$
$$P_{rad} = B_0(I_1 + I_2) = B_0\left(\frac{2\pi}{3} + 2\pi\right) = B_0\left(\frac{8\pi}{3}\right), \quad B_o = \frac{\eta}{2}\left(\frac{\beta I_o \ell}{4\pi}\right)^2 = \frac{\eta}{2}\left(\frac{\beta\ell}{4\pi}\right)^2|I_o|^2$$

- $$D_0 = \frac{4\pi U_{max}}{P_{rad}} = \frac{4\pi B_0}{\frac{8\pi}{3}B_0} = \frac{3}{2} \text{ (same as in Problem 6.21)}$$

- $R_r = \frac{2P_{rad}}{|I_o|^2} = 2\left(\frac{8\pi}{3}\right)\left[\frac{\eta}{2}\left(\frac{\beta\ell}{4\pi}\right)^2\right] = 80\pi^2\left(\frac{\ell}{\lambda}\right)^2$

$\boxed{6.23}$ $\underline{E}_A = -j\omega \underline{A} - j\frac{1}{\omega\mu\varepsilon}\nabla(\nabla\cdot\underline{A})$

$$\underline{A} \simeq \left[\hat{a}_r A_r'(\theta,\phi) + \hat{a}_\theta A_\theta'(\theta,\phi) + \hat{a}_\phi A_\phi'(\theta,\phi)\right]\frac{e^{-j\beta r}}{r}$$

$$\psi = \nabla\cdot\underline{A} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_r\right) - \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(A_\theta \sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$$

$$= -j\beta\frac{e^{-j\beta r}}{r} + \frac{e^{-j\beta r}}{r^2}\left[\cdots\cdots\right] + \frac{e^{-j\beta r}}{r^3}\left[\cdots\cdots\right] + \cdots\cdots$$

$$\nabla(\nabla\cdot\underline{A}) = \nabla\psi = \hat{a}_r\frac{\partial\psi}{\partial r} + \hat{a}_\theta\frac{1}{r}\frac{\partial\psi}{\partial\theta} + \hat{a}_\phi\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\theta}$$

$$= \hat{a}_r\left\{\frac{1}{r}\left[-\omega^2\mu\varepsilon\, e^{-j\beta r}A_r'(\theta,\phi)\right] + \frac{1}{r^2}\left[\cdots\right] + \frac{1}{r^3}\left[\cdots\right] + \cdots\right\}$$

$$+ \hat{a}_\theta\left\{\frac{1}{r}(0) + \frac{1}{r^2}\left[\cdots\right] + \frac{1}{r^3}\left[\cdots\right] + \cdots\right\}$$

$$+ \hat{a}_\phi\left\{\frac{1}{r}(0) + \frac{1}{r^2}\left[\cdots\right] + \frac{1}{r^3}\left[\cdots\right] + \cdots\right\}$$

Therefore

$$\underline{E}_A = -j\omega\underline{A} - j\frac{1}{\omega\mu\varepsilon}\nabla(\nabla\cdot\underline{A})$$

$$\underline{E}_A \simeq -j\omega\left[\hat{a}_r A_r' + \hat{a}_\theta A_\theta' + \hat{a}_\phi A_\phi'\right]\frac{e^{-j\beta r}}{r}$$

$$- j\frac{1}{\omega\mu\varepsilon}\left\{\hat{a}_r\left[\omega^2\mu\varepsilon\frac{e^{-j\beta r}}{r} + \frac{1}{r^2}(\cdots) + \frac{1}{r^3}(\cdots) + \cdots\right]\right.$$

$$+ \hat{a}_\theta\left[\frac{1}{r}(0) + \frac{1}{r^2}(\cdots) + \frac{1}{r^3}(\cdots) + \cdots\right]$$

$$\left. + \hat{a}_\phi\left[\frac{1}{r}(0) + \frac{1}{r^2}(\cdots) + \frac{1}{r^3}(\cdots) + \cdots\right]\right\}$$

or

$$\underline{E}_A \simeq \frac{1}{r}\left\{-j\omega e^{-j\beta r}\left[\hat{a}_r(0) + \hat{a}_\theta A_\theta' + \hat{a}_\phi A_\phi'\right] + \frac{1}{r^2}\left[\cdots\right] + \cdots\right\}$$

In a similar manner, it can be shown that

$$\underline{H} = \frac{1}{\mu}\nabla\times\underline{A}$$

$$\simeq \frac{1}{r}\left\{j\frac{\omega}{\eta}e^{-j\beta r}\left[\hat{a}_r(0) + \hat{a}_\theta A_\phi' - \hat{a}_\phi A_\theta'\right]\right\} + \frac{1}{r^2}\left[\cdots\right] + \frac{1}{r^3}\left[\cdots\right] + \cdots\}$$

$\boxed{6.24}$ $R = \left[r^2 + (r')^2 - 2rr'\cos\psi\right]^{1/2} = r\left[1 + \left(\frac{r'}{r}\right)^2 - \frac{2r'}{r}\cos\psi\right]^{1/2}$

Using the binomial expansion of

$$(a+b)^n = \frac{a^n b^0}{0!} + \frac{n a^{n-1} b^1}{1!} + \frac{n(n-1) a^{n-2} b^2}{2!} + \frac{n(n-1)(n-2) a^{n-3} b^3}{3!} + \cdots\cdots$$

it can be shown by letting

$a = r^2$

$b = (r')^2 - 2rr'\cos\psi$

$n = 1/2$

that

$$R = r - r'\cos\psi + \frac{1}{r}\left(\frac{r'^2}{2}\sin^2\psi\right) + \frac{1}{r^2}\left[\frac{(r')^3}{2}\cos\psi\sin\psi\right] + \frac{1}{r^3}\left[\frac{(r')^4}{8}\left(-1 + 6\cos^2\psi - 5\cos^4\psi\right)\right] + \cdots$$

By keeping the first two terms, the most significant neglected term is the third whose maximum value is

$$\left.\frac{1}{r}\left(\frac{r'^2}{2}\sin\psi\right)\right|_{max} = \frac{(r')^2}{2r} \quad \text{when } \psi = \frac{\pi}{2}$$

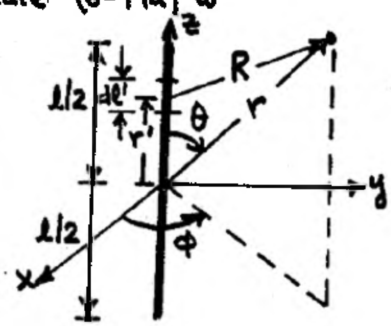
Using the $\pi/8$ rad (or $22.5°$) as the maximum phase error, then

$$\frac{\beta(r')^2}{2r} \leq \frac{\pi}{8}$$

If the maximum value of r' is $D/2$, the

$$\frac{\pi}{\lambda}\frac{1}{r}\left(\frac{D^2}{4}\right) \leq \frac{\pi}{8} \Rightarrow r \geq \frac{2D^2}{\lambda}$$

$\boxed{6.25}$ Using (6-112a) and (6-112b) we can reduce (6-97a) to



$$\underline{A} \simeq \frac{\mu}{4\pi}\frac{e^{-j\beta r}}{r}\int_{-l/2}^{+l/2} \underline{I}_e\, e^{+j\beta r'\cos\psi}\, dl'$$

$$\underline{I}_e = \hat{a}_z I_z(z') = \hat{a}_z I_0 \cdot \begin{cases} \sin\left[\beta\left(\frac{l}{2} - z'\right)\right], & 0 \leq z' \leq l/2 \\ \sin\left[\beta\left(\frac{l}{2} + z'\right)\right], & -l/2 \leq z' \leq 0 \end{cases}$$

$$\underline{A} \simeq \hat{a}_z \frac{\mu I_0}{4\pi}\frac{e^{-j\beta r}}{r}\int_{-l/2}^{+l/2} I_z(z')\, e^{j\beta z'\cos\theta}\, dz' = \hat{a}_z A_z$$

$$\left.\begin{aligned} \underline{E} &\simeq -j\omega \underline{A} \\ E_\theta &\simeq -j\omega A_\theta \\ E_\phi &\simeq -j\omega A_\phi \end{aligned}\right\} \quad \begin{aligned} A_\theta &= -A_z \sin\theta \\ A_\phi &= 0 \end{aligned}$$

cont'd.

6.25 cont'd.

$$A_\theta \simeq -\frac{\mu I_0 \sin\theta}{4\pi r} e^{-j\beta r} \int_{-\ell/2}^{+\ell/2} I_z(z') e^{j\beta z'\cos\theta} dz'$$

$$E_\theta = +j \frac{\omega\mu I_0 \sin\theta\, e^{-j\beta r}}{4\pi r} \int_{-\ell/2}^{\ell/2} I_z(z') e^{j\beta z'\cos\theta} dz'$$

Using the sinusoidal current distribution and the integral of

$$\int e^{\alpha x} \sin(\beta x+\gamma)\, dx = \frac{e^{\alpha x}}{\alpha^2+\beta^2}\left[\alpha \sin(\beta x+\gamma) - \beta\cos(\beta x+\gamma)\right]$$

we can write that

$$E_\theta = j\eta \frac{\beta I_0 \sin\theta\, e^{-j\beta r}}{4\pi r}\left\{ \int_{-\ell/2}^{0} \sin\left[\beta\left(\frac{\ell}{2}+z'\right)\right] e^{j\beta z'\cos\theta} dz' + \int_{0}^{\ell/2} \sin\left[\beta\left(\frac{\ell}{2}-z'\right)\right] e^{j\beta z'\cos\theta} dz'\right\}$$

After some mathematical manipulations, we can reduce the above equation to

$$E_\theta \simeq j\eta \frac{I_0 e^{-j\beta r}}{2\pi r}\left[\frac{\cos\left(\frac{\beta\ell}{2}\cos\theta\right) - \cos\left(\frac{\beta\ell}{2}\right)}{\sin\theta}\right], \quad H_\phi \simeq +\frac{E_\theta}{\eta}$$

6.26 When $\ell = \lambda/2$ we can write the far-zone electric and magnetic fields of Problem 6.25 when $\beta\ell/2 = \frac{2\pi}{\lambda}\left(\frac{\lambda}{4}\right) = \frac{\pi}{2}$

$$E_\theta \simeq j\eta \frac{I_0 e^{-j\beta r}}{2\pi r}\left[\frac{\cos\left(\frac{\pi}{2}\cos\theta\right) - \cos\left(\frac{\pi}{2}\right)}{\sin\left(\frac{\pi}{2}\right)}\right]$$

$$E_\theta \simeq j\eta \frac{I_0 e^{-j\beta r}}{2\pi r}\left[\frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}\right]$$

$$H_\phi \simeq \frac{E_\theta}{\eta}$$

$$E_r \simeq E_\phi \simeq H_r \simeq H_\theta \simeq 0$$

$\boxed{6.27}$ Since the operator ∇ in the solution of Problem 6.3 acts on the coordinates of the sources while the observation point is maintained fixed, then

$$\nabla\left(\frac{e^{-j\beta R}}{R}\right) = -\hat{R}\left(j\beta + \frac{1}{R}\right)\frac{e^{-j\beta R}}{R} \tag{1}$$

$$(\underline{J}\cdot\nabla)\nabla\left(\frac{e^{-j\beta R}}{R}\right) = \left[\beta^2(\underline{J}\cdot\hat{R})\hat{R} + \frac{3}{R}\left(j\beta + \frac{1}{R}\right)(\underline{J}\cdot\hat{R})\hat{R} - \frac{\underline{J}}{R}\left(j\beta + \frac{1}{R}\right)\right]\frac{e^{-j\beta R}}{R} \tag{2}$$

Thus for first-order variations in R, (1) and (2) reduce to

$$\nabla\left(\frac{e^{-j\beta R}}{R}\right) \simeq -\hat{R}\left(j\beta\,\frac{e^{-j\beta R}}{R}\right) \tag{3}$$

$$(\underline{J}\cdot\nabla)\nabla\left(\frac{e^{-j\beta R}}{R}\right) \simeq -\hat{R}\,\beta^2(\underline{J}\cdot\hat{R})\,\frac{e^{-j\beta R}}{R} \tag{4}$$

Using (3) and (4) we can reduce (3) and (8) of the solution of Problem 6.3 to

$$\underline{H}_A \simeq -\frac{1}{4\pi}\iiint_V\left[\underline{J}\times\left(-j\beta\hat{R}\,\frac{e^{-j\beta R}}{R}\right)\right]dv' = +j\frac{\beta}{4\pi}\iiint_V\left[(\underline{J}\times\hat{R})\,\frac{e^{-j\beta R}}{R}\right]dv' \tag{5}$$

$$\underline{E}_A \simeq -j\,\frac{1}{4\pi\omega\varepsilon}\iiint_V\left[\beta^2\underline{J} - \hat{R}\beta^2(\underline{J}\cdot\hat{R})\right]\frac{e^{-j\beta R}}{R}\,dv' = -j\eta\,\frac{\beta}{4\pi}\iiint_V\left[\underline{J} - \hat{R}(\underline{J}\cdot\hat{R})\right]\frac{e^{-j\beta R}}{R}\,dv' \tag{6}$$

Using the geometry of Figure 6-3 we can write for far-field observations

$$R \simeq r - \underline{r}'\cdot\hat{a}_r \quad \text{for phase variations} \tag{7a}$$

$$R \simeq r \quad \text{for amplitude variations} \tag{7b}$$

Thus (5) and (6) of above can be reduced to

$$\underline{H}_A \simeq j\,\frac{\beta e^{-j\beta}}{4\pi r}\iiint_V(\underline{J}\times\hat{a}_r)\,e^{j\beta\underline{r}'\cdot\hat{a}_r}\,dv' \tag{8}$$

$$\underline{E}_A \simeq -j\eta\,\frac{\beta e^{-j\beta r}}{4\pi r}\iiint_V\left[\underline{J} - \hat{a}_r(\underline{J}\cdot\hat{a}_r)\right]e^{j\beta\underline{r}'\cdot\hat{a}_r}\,dv' \tag{9}$$

$\boxed{6.28}$ Using the same procedure as in Problem 6.27 solution, we can reduce (3) and (4) of the solution of Problem 6.4 for far-field observations to

$$\underline{E}_F \simeq -j\,\frac{\beta e^{-j\beta r}}{4\pi r}\iiint_V(\underline{M}\times\hat{a}_r)\,e^{+j\beta\underline{r}'\cdot\hat{a}_r}\,dv'$$

$$\underline{H}_F \simeq -j\,\frac{\beta e^{-j\beta r}}{4\pi\eta r}\iiint_V\left[\underline{M} - \hat{a}_r(\underline{M}\cdot\hat{a}_r)\right]e^{j\beta\underline{r}'\cdot\hat{a}_r}\,dv'$$

$\boxed{6.29}$ $\underline{E}_a = \hat{a}_z E_o$, $\underline{M}_s = -2\hat{n}\times\underline{E}_a = -2\hat{a}_x\times\hat{a}_z E_o = \hat{a}_y 2E_o$

Thus $M_y = 2E_o$, $M_x = M_z = J_x = J_y = J_z = 0$

$N_\theta = N_\phi = 0$

$$L_\theta = \iint_{S_a}\left[\cancel{M_x}^{0}\cos\theta\cos\phi + M_y\cos\theta\sin\phi - \cancel{M_z}^{0}\sin\theta\right]e^{j\beta(y'\sin\theta\sin\phi + z'\cos\theta)}dy'dz'$$

$$= 2E_o\cos\theta\sin\phi\int_{-a/2}^{a/2} e^{j\beta y'\sin\theta\sin\phi}dy'\int_{-b/2}^{b/2} e^{j\beta z'\cos\theta}dz'$$

$$L_\theta = 2E_o ab\left[\cos\theta\sin\phi\frac{\sin Y}{Y}\frac{\sin Z}{Z}\right],\quad Y = \frac{\beta a}{2}\sin\theta\sin\phi,\quad Z = \frac{\beta b}{2}\cos\theta$$

$$L_\phi = \iint_{S_a}\left[-\cancel{M_x}^{0}\sin\phi + M_y\cos\phi\right]e^{j\beta(y'\sin\theta\sin\phi + z'\cos\theta)}dy'dz'$$

$$= 2E_o ab\left[\cos\phi\frac{\sin Y}{Y}\frac{\sin Z}{Z}\right]$$

Using (6-122a)-(6-122f), we have $E_r \simeq H_r \simeq 0$ and

$$E_\theta \simeq -\frac{j\beta e^{-j\beta r}}{4\pi r}\left[L_\phi + \eta\cancel{N_\theta}^{0}\right] = -j\frac{ab\beta E_o e^{-j\beta r}}{2\pi r}\left[\cos\phi\frac{\sin Y}{Y}\frac{\sin Z}{Z}\right]$$

$$E_\phi \simeq \frac{j\beta e^{-j\beta r}}{4\pi r}\left[L_\theta - \eta\cancel{N_\phi}^{0}\right] = -j\frac{ab\beta E_o e^{-j\beta r}}{2\pi r}\left[-\cos\theta\sin\phi\frac{\sin Y}{Y}\frac{\sin Z}{Z}\right]$$

$$H_\theta \simeq -\frac{E_\phi}{\eta},\quad H_\phi \simeq +\frac{E_\theta}{\eta}$$

$\boxed{6.30}$ $\underline{E}_a = \hat{a}_z E_o$, $\underline{H}_a = -\hat{a}_y\frac{E_o}{\eta}$

$\underline{M}_s = -\hat{n}\times\underline{E}_a$, $\underline{J}_s = \hat{n}\times\underline{H}_a$

$$\underline{M}_s = -\hat{a}_x\times(\hat{a}_z E_o) = \hat{a}_y E_o \Rightarrow M_y = E_o$$

$$\underline{J}_s = \hat{a}_x\times\left(-\hat{a}_y\frac{E_o}{\eta}\right) = -\hat{a}_z\frac{E_o}{\eta} \Rightarrow J_z = -\frac{E_o}{\eta}$$

$$N_\theta = \iint\left(\cancel{J_x}^{0}\cos\theta\cos\phi + \cancel{J_y}^{0}\cos\theta\sin\phi - J_z\sin\theta\right)e^{+j\beta r'\cos\psi}ds'$$

$$= -\iint J_z\sin\theta e^{j\beta r'\cos\psi}ds' = +\frac{E_o}{\eta}\sin\theta\int_{-a/2}^{+a/2} e^{j\beta y'\sin\theta\sin\phi}dy'\int_{-b/2}^{+b/2} e^{j\beta z'\cos\theta}dz'$$

$$N_\theta = \frac{E_o ab}{\eta}\sin\theta\left[\left(\frac{\sin Y}{Y}\right)\left(\frac{\sin Z}{Z}\right)\right]$$

$Y = \frac{\beta a}{2}\sin\theta\sin\phi$

$Z = \frac{\beta b}{2}\cos\theta$

$$N_\phi = \iint_S\left(-\cancel{J_x}^{0}\sin\phi + \cancel{J_y}^{0}\cos\phi\right)e^{j\beta r'\cos\psi}ds' = 0$$

$\boxed{\text{cont'd}}$

6.30 cont'd

$$L_\theta = \iint_S \left(\cancel{M_x}^{\,0}\cos\theta\cos\phi + M_y\cos\theta\sin\phi - \cancel{M_z}^{\,0}\sin\theta\right)e^{j\beta r'\cos\psi}\,ds' = \iint_S M_y\cos\theta\sin\phi\, e^{j\beta r'\cos\psi}\,ds'$$

$$= E_o\cos\theta\sin\phi\int_{-a/2}^{+a/2} e^{j\beta y'\sin\theta\sin\phi}\,dy'\int_{-b/2}^{+b/2} e^{j\beta z'\cos\theta}\,dz' = E_o ab\cos\theta\sin\phi\left[\frac{\sin Y}{Y}\,\frac{\sin Z}{Z}\right]$$

$$L_\phi = \iint_S \left(-\cancel{M_x}^{\,0}\sin\phi + M_y\cos\phi\right)e^{j\beta r'\cos\psi}\,dx = \iint_S M_y\cos\phi\, e^{j\beta r'\cos\psi}\,ds'$$

$$= E_o\cos\phi\int_{-a/2}^{+a/2} e^{j\beta y'\sin\theta\sin\phi}\,dy'\int_{-b/2}^{b/2} e^{j\beta z'\cos\theta}\,dz' = E_o(ab)\cos\phi\left[\frac{\sin Y}{Y}\,\frac{\sin Z}{Z}\right]$$

$$E_r \simeq 0,\quad E_\theta \simeq -j\frac{\beta e^{-j\beta r}}{4\pi r}\left(L_\phi + \eta N_\theta\right) = -jE_o\frac{ab\beta e^{-j\beta r}}{4\pi r}\left\{(\cos\phi + \sin\theta)\frac{\sin Y}{Y}\,\frac{\sin Z}{Z}\right\}$$

$$E_\phi \simeq +j\frac{\beta e^{-j\beta r}}{4\pi r}\left(L_\theta - \eta N_\phi\right) = +jE_o\frac{ab\beta e^{-j\beta r}}{4\pi r}\left\{\cos\theta\sin\phi\,\frac{\sin Y}{Y}\,\frac{\sin Z}{Z}\right\}$$

$$H_r \simeq 0,\quad H_\theta = -E_\phi/\eta,\quad H_\phi = E_\theta/\eta$$

6.31 $\underline{E}_a = \hat{a}_x E_o,\ \underline{M}_s = -2\hat{n}\times\underline{E}_a = -2\hat{a}_y\times\hat{a}_x E_o = \hat{a}_z 2E_o$

Thus $M_z = 2E_o,\ M_x = M_y = J_x = J_y = J_z = 0$

$N_\theta = N_\phi = 0$

$$L_\theta = \iint_{S_a}\left[\cancel{M_x}^{\,0}\cos\theta\cos\phi + M_y\cos\theta\sin\phi - M_z\sin\theta\right]e^{j\beta(x'\sin\theta\cos\phi + z'\cos\theta)}\,dx'dz'$$

$$= -2E_o\sin\theta\int_{-a/2}^{a/2} e^{j\beta z'\cos\theta}\,dz'\int_{-b/2}^{b/2} e^{j\beta x'\sin\theta\cos\phi}\,dx'$$

$$L_\theta = -2E_o ab\left[\sin\theta\,\frac{\sin X}{X}\,\frac{\sin Z}{Z}\right],\quad X = \frac{\beta b}{2}\sin\theta\cos\phi,\ Z = \frac{\beta a}{2}\cos\theta$$

cont'd

$\boxed{6.31 \text{ Cont'd.}}$

$$L_\phi = \iint_{S_a} \left[-\cancel{M_x}^{\,0}\sin\phi + \cancel{M_y}^{\,0}\cos\phi\right] e^{j\beta(x'\sin\theta\cos\phi + z'\cos\theta)}\,dx'dz' = 0$$

Using (6-122a)-(6-122f), we have

$$E_r \simeq H_r \simeq 0$$

$$E_\theta \simeq -\frac{j\beta e^{-j\beta r}}{4\pi r}\left[\cancel{L_\phi}^{\,0} + \eta \cancel{N_\theta}^{\,0}\right] = 0$$

$$E_\phi \simeq \frac{j\beta e^{-j\beta r}}{4\pi r}\left[L_\theta - \eta N_\phi\right] = -j\,\frac{ab\beta e^{-j\beta r}}{2\pi r}E_0\left[\sin\theta\,\frac{\sin X}{X}\,\frac{\sin Z}{Z}\right]$$

$$H_\theta \simeq -\frac{E_\phi}{\eta}\,, \quad H_\phi \simeq \frac{E_\theta}{\eta} = 0$$

$\boxed{6.32}$ $\underline{E}_a = \hat{a}_y E_0$, $\underline{M}_s = -2\hat{n}\times\underline{E}_a = -2\hat{a}_z\times\hat{a}_y E_0 = \hat{a}_x 2E_0$

Thus $M_x = 2E_0$, $M_y = M_z = J_x = J_y = J_z = 0$

$$N_\theta = N_\phi = 0$$

$$L_\theta = \iint_{S_a}\left[M_x\cos\theta\cos\phi + \cancel{M_y}^{\,0}\cos\theta\sin\phi - \cancel{M_z}^{\,0}\sin\theta\right] e^{j\beta(x'\sin\theta\cos\phi + y'\sin\theta\sin\phi)}\,dx'dy'$$

$$= 2E_0\cos\theta\cos\phi\int_{-b/2}^{b/2} e^{j\beta y'\sin\theta\sin\phi}\,dy'\int_{-a/2}^{a/2} e^{j\beta x'\sin\theta\cos\phi}\,dx'$$

$$L_\theta = 2abE_0\left[\cos\theta\cos\phi\,\frac{\sin X}{X}\,\frac{\sin Y}{Y}\right], \quad X = \frac{\beta a}{2}\sin\theta\cos\phi,\; Y = \frac{\beta b}{2}\sin\theta\sin\phi$$

$$L_\phi = \iint_{S_a}\left[-M_x\sin\phi + \cancel{M_y}^{\,0}\cos\phi\right] e^{j\beta(x'\sin\theta\cos\phi + y'\sin\theta\sin\phi)}\,dx'dy'$$

$$= -2abE_0\left[\sin\phi\,\frac{\sin X}{X}\,\frac{\sin Y}{Y}\right]$$

Using (6-122a)-(6-122f), we have $E_r \simeq H_r \simeq 0$

$$E_\theta \simeq -\frac{j\beta e^{-j\beta r}}{4\pi r}\left[L_\phi + \eta\cancel{N_\theta}^{\,0}\right] = j\,\frac{ab\beta e^{-j\beta r}}{2\pi r}E_0\left[\sin\phi\,\frac{\sin X}{X}\,\frac{\sin Y}{Y}\right]$$

$$E_\phi \simeq \frac{j\beta e^{-j\beta r}}{4\pi r}\left[L_\theta - \eta\cancel{N_\phi}^{\,0}\right] = j\,\frac{ab\beta e^{-j\beta r}}{2\pi r}E_0\left[\cos\theta\cos\phi\,\frac{\sin X}{X}\,\frac{\sin Y}{Y}\right]$$

$$H_\theta \simeq -\frac{E_\phi}{\eta}\,, \quad H_\phi \simeq \frac{E_\theta}{\eta}$$

$\boxed{6.33}$ $\underline{E}_a = \hat{a}_z E_0 \cos\left(\frac{\pi}{a}y'\right)$

The only difference between this problem and 6.29 is that for the y variations the integral reduces to

$$\int_{-a/2}^{a/2} \cos\left(\frac{\pi}{a}y'\right) e^{j\beta y'\sin\theta\sin\phi}\,dy' = -\left(\frac{\pi a}{2}\right)\frac{\cos\left(\frac{\beta a}{2}\sin\theta\sin\phi\right)}{\left(\frac{\beta a}{2}\sin\theta\sin\phi\right)^2-\left(\frac{\pi}{2}\right)^2}$$

Thus $\quad E_r \simeq H_r \simeq 0$

$$E_\theta \simeq -\frac{j\beta e^{-j\beta r}}{4\pi r}L_\phi = j\,\frac{ab\beta E_0 e^{-j\beta r}}{4r}\left[\cos\phi\,\frac{\cos Y}{(Y)^2-(\pi/2)^2}\,\frac{\sin Z}{Z}\right],\quad Y=\frac{\beta a}{2}\sin\theta\sin\phi$$

$$E_\phi \simeq \frac{j\beta e^{-j\beta r}}{4\pi r}L_\theta = j\,\frac{ab\beta E_0 e^{-j\beta r}}{4r}\left[-\cos\theta\sin\phi\,\frac{\cos Y}{(Y)^2-(\pi/2)^2}\,\frac{\sin Z}{Z}\right],\quad Z=\frac{\beta b}{2}\cos\theta$$

$$H_\theta \simeq -\frac{E_\phi}{\eta},\quad H_\phi \simeq +\frac{E_\theta}{\eta}$$

$\boxed{6.34}$ $\underline{E}_a = \hat{a}_x E_0 \cos\left(\frac{\pi}{a}z'\right)$

The only difference between this problem and 6.31 is that for the z variations the integral reduces to

$$\int_{-a/2}^{a/2} \cos\left(\frac{\pi}{a}z'\right) e^{j\beta z'\cos\theta}\,dz' = -\left(\frac{\pi a}{2}\right)^2\frac{\cos\left(\frac{\beta a}{2}\cos\theta\right)}{\left(\frac{\beta a}{2}\cos\theta\right)^2-\left(\frac{\pi}{2}\right)^2}$$

Thus $E_r \simeq H_r \simeq 0 \simeq E_\theta$

$$E_\phi \simeq j\frac{\beta e^{-j\beta r}}{4\pi r}L_\theta = j\,\frac{ab\beta E_0 e^{-j\beta r}}{4r}\left[\sin\theta\,\frac{\sin X}{X}\,\frac{\cos Z}{(Z)^2-(\pi/2)^2}\right],\quad X=\frac{\beta b}{2}\sin\theta\cos\phi$$

$$H_\theta \simeq -\frac{E_\phi}{\eta},\quad H_\phi \simeq \frac{E_\theta}{\eta}=0$$

$\boxed{6.35}$ $\underline{E}_a = \hat{a}_y E_0 \cos\left(\frac{\pi}{a}x'\right)$

The only difference between this problem and 6.32 is that for the x variations the integral reduces to

$$\int_{-a/2}^{a/2} \cos\left(\frac{\pi}{a}x'\right) e^{j\beta x'\sin\theta\cos\phi}\,dx' = -\left(\frac{\pi a}{2}\right)^2\frac{\cos\left(\frac{\beta a}{2}\sin\theta\cos\phi\right)}{\left(\frac{\beta a}{2}\sin\theta\cos\phi\right)^2-\left(\frac{\pi}{2}\right)^2}$$

Thus $\quad E_r \simeq H_r \simeq 0$

$$E_\theta \simeq -\frac{j\beta e^{-j\beta r}}{4\pi r}L_\phi = -j\,\frac{ab\beta E_0 e^{-j\beta r}}{4r}\left[\sin\phi\,\frac{\cos X}{(X)^2-(\pi/2)^2}\,\frac{\sin Y}{Y}\right],\quad X=\frac{\beta a}{2}\sin\theta\cos\phi$$

$$E_\phi \simeq \frac{j\beta e^{-j\beta r}}{4\pi r}L_\theta \simeq -j\,\frac{ab\beta E_0 e^{-j\beta r}}{4r}\left[\cos\theta\cos\phi\,\frac{\cos X}{(X)^2-(\pi/2)^2}\,\frac{\sin Y}{Y}\right],\quad Y=\frac{\beta b}{2}\sin\theta\sin\phi$$

$$H_\theta = -E_\phi/\eta,\quad H_\phi \simeq E_\theta/\eta$$

$\boxed{6.36}$

(a) In the E-plane $E_r \simeq E_\phi \simeq 0$ and the normalized E_θ

$$E_{\theta n} = \left(\frac{1+\cos\theta}{2}\right)\frac{\sin\left(\frac{\beta b}{2}\sin\theta\right)}{\frac{\beta b}{2}\sin\theta} \underset{\sim}{\overset{b\to \text{large}}{}} \frac{\sin\left(\frac{\beta b}{2}\sin\theta\right)}{\frac{\beta b}{2}\sin\theta}$$

The 0.707 point of the sinc function occurs when its argument is 1.391. Thus

$$\frac{\beta b}{2}\sin\theta_h = 1.391 \Rightarrow \theta_h = \sin^{-1}\left(\frac{2.782}{\beta b}\right) = \sin^{-1}\left(\frac{0.443\lambda}{b}\right)$$

Therefore the half-power beamwidth Θ_h is given by

$$\Theta_h = 2\theta_h = 2\sin^{-1}\left(\frac{0.443\lambda}{b}\right)\Bigg|_{b=3\lambda} = 2\sin^{-1}\left(\frac{0.443}{3}\right) = 2(8.49) = 16.98^\circ$$

(b) In the H-plane $E_r \simeq E_\theta \simeq 0$ and the normalized E_ϕ

$$E_{\phi n} = \left(\frac{1+\cos\theta}{2}\right)\frac{\sin\left(\frac{\beta a}{2}\sin\theta\right)}{\frac{\beta a}{2}\sin\theta} \underset{\sim}{\overset{a\to \text{large}}{}} \frac{\sin\left(\frac{\beta a}{2}\sin\theta\right)}{\frac{\beta a}{2}\sin\theta}$$

Therefore

$$\frac{\beta a}{2}\sin\theta_h = 1.391 \Rightarrow \theta_h = \sin^{-1}\left(\frac{0.443\lambda}{a}\right)$$

$$\Theta_h = 2\theta_h = 2\sin^{-1}\left(\frac{0.443\lambda}{a}\right)\Bigg|_{a=4\lambda} = 2\sin^{-1}\left(\frac{0.443}{4}\right) = 2(6.36) = 12.72^\circ$$

$\boxed{6.37}$ (a) $\underline{E}_a = \hat{a}_y E_o \left[1-\left(\frac{\rho'}{a}\right)^2\right], \ \rho' \le a$

(b) $\underline{E}_a = \hat{a}_y E_o \left[1-\left(\frac{\rho'}{a}\right)^2\right]^2, \ \rho' \le a$

(a) $\underline{E}_a = \hat{a}_y E_o \left[1-\left(\frac{\rho'}{a}\right)^2\right], \ \rho \le a$

We start by writing the magnetic current from the expression of the electric field given in the exercise

$$\mathbf{M}_s = -2\hat{n} \times \mathbf{E}_a = \hat{a}_x 2E_0 \left[1-(\rho'/a)^2\right].$$

Since the electric currents are zero then N_θ and N_ϕ will be equal to zero. We note that the aperture fields are given in terms of their cartesian components, so we use

$$\begin{aligned} L_\theta &= \int_S M_x \cos\theta\cos\phi\, e^{j\beta r'\cos\psi} ds' \\ &= 2E_0\cos\theta\cos\phi \int_0^a \int_0^{2\pi} \rho'[1-(\rho'/a)^2] e^{j\beta\rho'\sin\theta\cos(\phi-\phi')} d\rho' d\phi'. \end{aligned}$$

We observe that

$$\int_0^{2\pi} e^{j\beta\rho'\sin\theta\cos(\phi-\phi')} d\phi' = 2\pi J_0(\beta\rho'\sin\theta),$$

which yields

$$L_\theta = 4\pi E_0\cos\theta\cos\phi \int_0^a J_0(\beta\rho'\sin\theta)[1-(\rho'/a)^2]\rho' d\rho'.$$

By making the substitution

$$t = \beta\rho'\sin\theta, \rho' = \frac{t}{\beta sin\theta}, \Rightarrow dt = \beta\rho'\sin\theta d\rho', d\rho' = \frac{dt}{\beta sin\theta}$$

the previous integral becomes

$$L_\theta = 4\pi E_0\cos\theta\cos\phi \left[\frac{1}{(\beta\sin\theta)^2}\int_0^{\beta a\sin\theta} tJ_0(t)\,dt - \frac{1}{a^2(\beta\sin\theta)^4}\int_0^{\beta a\sin\theta} t^3 J_0(t)dt\right].$$

Then we use the Bessel function indefinite integrals

$$\int x^{p+1}J_p(\alpha x)dx = \frac{1}{\alpha}x^{p+1}J_{p+1}(\alpha x) + C \Rightarrow \int tJ_0(t)dt = tJ_1(t) + C$$

$$\int t^3 J_0(t)dt = t^3 J_1((t) - 2t^2 J_2(t) + C$$

and we obtain

$$\begin{aligned} L_\theta &= 4\pi E_0\cos\theta\cos\phi \left[\frac{a}{\beta\sin\theta}J_1(\beta a\sin\theta) - \frac{a}{\beta\sin\theta}J_1(\beta a\sin\theta) + \frac{2}{(\beta\sin\theta)^2}J_2(\beta a\sin\theta)\right] \\ &= 8\pi E_0 \frac{\cos\theta\cos\phi}{(\beta\sin\theta)^2} J_2(\beta a\sin\theta). \end{aligned}$$

Cont'd

The procedure to find the expression for L_ϕ is very similar and yields

$$
\begin{aligned}
L_\phi &= \int_s -M_x \sin\phi e^{j\beta r' \cos\psi}\, ds' \\
&= -2E_0 \sin\phi \int_0^a \rho' \left[1-(\rho'/a)^2\right] \int^2 \pi_0 e^{j\beta\rho' \sin\theta\cos(\phi-\phi')} d\phi' d\rho' \\
&= -2E_0 \sin\phi \int_0^a \rho' \left[1-(\rho'/a)^2\right] 2\pi J_0\left(\beta\rho'\sin\theta\right) d\rho' \\
&= -4\pi E_0 \sin\phi \int_0^a \left[\rho' J_0\left(\beta\rho'\sin\theta\right) - \frac{\rho'^3}{a^2} J_0\left(\beta\rho'\sin\theta\right)\right] d\rho' \\
&= -8\pi E_0 \sin\phi \frac{J_2(\beta a\sin\theta)}{(\beta\sin\theta)^2}
\end{aligned}
$$

It is now possible to write the expressions of the electric and magnetic fields. The result is

$$
\begin{aligned}
E_r &\simeq 0 \\
E_\theta &\simeq -\frac{j\beta e^{-j\beta r}}{4\pi r} L_\phi = j2\beta E_0 \frac{e^{-j\beta r}}{r} \sin\phi \frac{J_2(\beta a\sin\theta)}{(\beta\sin\theta)^2} \\
E_\phi &\simeq +\frac{j\beta e^{-j\beta r}}{4\pi r} L_\theta = j2\beta E_0 \frac{e^{-j\beta r}}{r} \cos\theta\cos\phi \frac{J_2(\beta a\sin\theta)}{(\beta\sin\theta)^2} \\
H_r &\simeq 0 \\
H_\theta &\simeq -\frac{1}{\eta}\frac{j\beta e^{-j\beta r}}{4\pi r} L_\theta = -E_\phi/\eta = j2\frac{\beta E_0}{\eta}\frac{e^{-j\beta r}}{r} \cos\theta\cos\phi \frac{J_2(\beta a\sin\theta)}{(\beta\sin\theta)^2} \\
H_\phi &\simeq -\frac{1}{\eta}\frac{j\beta e^{-j\beta r}}{4\pi r} L_\phi = \quad E_\theta/\eta = j2\frac{\beta E_0}{\eta}\frac{e^{-j\beta r}}{r} \sin\phi \frac{J_2(\beta a\sin\theta)}{(\beta\sin\theta)^2}
\end{aligned}
$$

(b) $\quad \underline{E}_a = \hat{a}_y E_0\left[1-(\rho'/a)^2\right]^2, \ \rho' \leq a$

This case is very similar to previous one. The magnetic current is

$$
\mathbf{M}_s = -2\hat{n}\times\mathbf{E}_a = \hat{a}_x 2E_0\left[1-(\rho'/a)^2\right]^2.
$$

and

$$
N_\theta = 0, N_\phi = 0.
$$

We have

$$
\begin{aligned}
L_\theta &= \int_s M_x \cos\theta\cos\phi e^{j\beta r'\cos\psi} ds' \\
&= \int_s \left(2E_0\left[1-(\rho'/a)^2\right]^2 \cos\theta\cos\phi\right) e^{j\beta\rho' sin\theta \cos(\phi-\phi')} \rho' d\phi' d\rho' \\
&= 2E_0\cos\theta\cos\phi \int_0^a \rho'\left[1-(\rho'/a)^2\right]^2 \int_0^{2\pi} e^{j\beta\rho'\sin\theta\cos(\phi-\phi')} d\phi' d\rho' \\
&= 2E_0\cos\theta\cos\phi \int_0^a \rho'\left[1-(\rho'/a)^2\right]^2 2\pi J_0\left(\beta\rho'\sin\theta\right) d\rho' \\
&= 4\pi E_0\cos\theta\cos\phi \int_0^a \left[\rho' J_0\left(\beta\rho'\sin\theta\right) - 2\frac{\rho'^3}{a^2} J_0\left(\beta\rho' sin\theta\right) + \frac{\rho'^5}{a^4} J_0\left(\beta\rho'\sin\theta\right)\right] d\rho'
\end{aligned}
$$

$\boxed{\text{Cont'd}}$

Using the substitution, we obtain

$$L_\theta = 4\pi E_0 \cos\theta \cos\phi \int_0^{\beta a \sin\theta} \left[\frac{t}{(\beta \sin\theta)^2} J_0(t) - 2\frac{t^3}{(\beta\sin\theta)^4 a^2} J_0(t) + \frac{t^5}{(\beta\sin\theta)^6 a^4} J_0(t) \right] dt.$$

We use again the Bessel indefinite integrals, and

$$\int t^m J_n(t)dt = t^m J_{n+1}(t) - (m-n-1)\int t^{m-1} J_{n+1}(t)dt$$
$$\Rightarrow \int t^5 J_0(t)dt = t^5 J_1(t) - 4\left[\int t^4 J_1(t)dt\right] = t^5 J_1(t) - 4\left[t^4 J_2(t) - 2\int t^3 J_2(t)dt\right]$$
$$= t^5 J_1(t) - 4\left[t^4 J_2(t) - 2t^3 J_3(t)\right] + C$$

Hence,

$$\begin{aligned} L_\theta &= 4\pi E_0 \cos\theta\cos\phi \left[\frac{\beta a \sin\theta}{(\beta\sin\theta)^2} J_1(\beta a \sin\theta) - \frac{2(\beta a \sin\theta)^3}{(\beta\sin\theta)^4 a^2} J_1(\beta a\sin\theta) + \frac{4(\beta a\sin\theta)^2}{(\beta\sin\theta)^4 a^2} J_2(\beta a\sin\theta) \right] \\ &+ 4\pi E_0\cos\theta\cos\phi \left[\frac{(\beta a\sin\theta)^5}{(\beta\sin\theta)^6 a^4} J_1(\beta a\sin\theta) - 4\frac{(\beta a\sin\theta)^4}{(\beta\sin\theta)^6 a^4} J_2(\beta a\sin\theta) + 8\frac{(\beta a\sin\theta)^3}{(\beta\sin\theta)^6 a^4} J_3(\beta a\sin\theta) \right] \\ &= 32\pi E_0\cos\theta\cos\phi \frac{J_3(\beta a\sin\theta)}{a(\beta\sin\theta)^3} \end{aligned}$$

and, similarly,

$$L_\phi = -32\pi E_0 \sin\phi \frac{1}{a(\beta\sin\theta)^3} J_3(a\beta\sin\theta)$$

The expressions of the fields are again obtained from

$$\begin{aligned} E_r &\simeq 0 \\ E_\theta &\simeq -\frac{j\beta e^{-j\beta r}}{4\pi r} L_\phi = j32\beta E_0 \frac{e^{-j\beta r}}{r} \sin\phi \frac{J_3(\beta a\sin\theta)}{a(\beta\sin\theta)^3} \\ E_\phi &\simeq +\frac{j\beta e^{-j\beta r}}{4\pi r} L_\theta = j32\beta E_0 \frac{e^{-j\beta r}}{r} \cos\theta\cos\phi \frac{J_3(\beta a\sin\theta)}{a(\beta\sin\theta)^3} \\ H_r &\simeq 0 \\ H_\theta &\simeq -\frac{1}{\eta}\frac{j\beta e^{-j\beta r}}{4\pi r} L_\theta = -E_\phi/\eta = j32\frac{\beta E_0}{\eta}\frac{e^{-j\beta r}}{r}\cos\theta\cos\phi\frac{J_3(\beta a\sin\theta)}{a(\beta\sin\theta)^3} \\ H_\phi &\simeq -\frac{1}{\eta}\frac{j\beta e^{-j\beta r}}{4\pi r} L_\phi = \quad E_\theta/\eta = j32\frac{\beta E_0}{\eta}\frac{e^{-j\beta r}}{r}\sin\phi\frac{J_3(\beta a\sin\theta)}{a(\beta\sin\theta)^3} \end{aligned}$$

$\boxed{6.38}$ $\underline{E}_a = -\hat{a}_\rho \dfrac{V}{\rho\ln(b/a)}\dfrac{1}{\rho'} = -\hat{a}_\rho C\,\dfrac{1}{\rho'}$ where $C = \dfrac{V}{\ell n(b/a)}$

$\underline{M}_s = -2\hat{n}\times\underline{E}_a = -2\hat{a}_z\times\left(-\hat{a}_\rho\dfrac{C}{\rho'}\right) = \hat{a}_\phi\, 2\dfrac{C}{\rho'} \Rightarrow M_\phi = \dfrac{2C}{\rho'}$

Thus using (6-131c)

$$L_\theta = \iint_{S_a}\left[\cancelto{0}{M_\rho}\cos\theta\cos(\phi-\phi') + M_\phi\cos\theta\sin(\phi-\phi') - \cancelto{0}{M_z}\sin\theta\right]e^{j\beta r'\cos\psi}\,ds'$$

$$= 2C\cos\theta\int_a^b\left[\int_0^{2\pi}\sin(\phi-\phi')\,e^{j\beta\rho'\sin\theta\cos(\phi-\phi')}\,d\phi'\right]d\rho'$$

Because of azimuthal symmetry, the field is not a function of ϕ. Choosing $\phi=0$

$$L_\theta = 2C\cos\theta\int_a^b\left[-\int_0^{2\pi}\sin\phi'\,e^{j\beta\rho'\sin\theta\cos\phi'}\,d\phi'\right]d\rho' = 2C\cos\theta\int_a^b[0]\,d\rho' = 0$$

$$L_\phi = \iint_{S_a}\left[-\cancelto{0}{M_\rho}\sin(\phi-\phi') + M_\phi\cos(\phi-\phi')\right]e^{j\beta r'\cos\psi}\,ds' = 2C\int_a^b\left[\int_0^{2\pi}\cos\phi'\,e^{j\beta\rho'\sin\theta\cos\phi'}\,d\phi'\right]d\rho'$$

$$= 2C\int_a^b\left[\int_0^{\pi}\cos\phi'\,e^{j\beta\rho'\sin\theta\cos\phi'}\,d\phi' + \underbrace{\int_\pi^{2\pi}\cos\phi'\,e^{j\beta\rho'\sin\theta\cos\phi'}\,d\phi'}_{\text{Let } \phi''=\phi'-\pi \Rightarrow d\phi''=d\phi'}\right]d\rho'$$

$$= 2C\int_a^b\left[\int_0^{\pi}\cos\phi'\,e^{j\beta\rho'\sin\theta\cos\phi'}\,d\phi' - \int_0^{\pi}\cos\phi''\,e^{-j\beta\rho'\sin\theta\cos\phi''}\,d\phi''\right]$$

$$L_\phi = 2C\int_a^b\left[J_1(\beta\rho'\sin\theta) - J_1(-\beta\rho'\sin\theta)\right]d\rho' = 4C\int_a^b J_1(\beta\rho'\sin\theta)\,d\rho'$$

Using $\int x^{1-p}J_p(\alpha x)\,dx = -\dfrac{1}{\alpha}x^{1-p}J_{p-1}(\alpha x)$

$$L_\phi = -4C\left[\frac{1}{\beta\sin\theta}\left(J_0(\beta b\sin\theta) - J_0(\beta a\sin\theta)\right)\right]$$

If however the slot is very thin, L_ϕ can be approximated by

$$L_\phi = 4C\int_a^b J_1(\beta\rho'\sin\theta)\,d\rho' \simeq 4C\,J_1(\beta a'\sin\theta)\int_a^b d\rho' = 4C(b-a)J_1(\beta a'\sin\theta),\quad a' = \frac{a+b}{2}$$

Using (6-122a)-(6-122f), we can write the radiated fields as

$E_r \simeq H_r \simeq 0$

$$E_\theta \simeq -\frac{j\beta e^{-j\beta r}}{4\pi r}\left[L_\phi + \eta\cancelto{0}{N_\theta}\right] = j\frac{Ce^{-j\beta r}}{\pi r}\left[\frac{J_0(\beta b\sin\theta) - J_0(\beta a\sin\theta)}{\sin\theta}\right] = j\frac{(a-b)C\beta e^{-j\beta r}}{\pi r}J_1(\beta a'\sin\theta)$$

$$E_\phi \simeq \frac{j\beta e^{-j\beta r}}{4\pi r}\left[L_\theta - \eta N_\phi\right] = 0,\quad H_\theta = -\frac{E_\phi}{\eta},\quad H_\phi = \frac{E_\theta}{\eta}$$

6.39

(a) For the E-plane the normalized electric field is

$$E_{\theta n} = \frac{J_1(\beta a \sin\theta)}{\beta a \sin\theta}$$

whose maximum value is 0.5 when $\beta a \sin\theta = 0$. Its 0.707 value of its maximum value, or its 0.3535 value, occurs according to the table on page 943 when $\beta a \sin\theta_h \simeq 1.616 \Rightarrow \theta_h = \sin^{-1}\left(\frac{1.616}{\beta a}\right) = \sin^{-1}\left(\frac{0.2572\lambda}{a}\right)$

Thus the half-power beamwidth is equal to

$$\Theta_h = 2\theta_h = 2\sin^{-1}\left(\frac{0.2572\lambda}{a}\right)$$

When $a = 3\lambda$

$$\Theta_h = 2\sin^{-1}\left(\frac{0.2572}{3}\right) = 2\sin^{-1}(0.08573) = 2(4.918^\circ) = 9.836^\circ$$

(b) In the H-plane the normalized electric field is

$$E_{\phi n} = \cos\theta \, \frac{J_1(\beta a \sin\theta)}{\beta a \sin\theta} \overset{a \to \text{large}}{\simeq} \frac{J_1(\beta a \sin\theta)}{\beta a \sin\theta}$$

since the $J_1(x)/x$ term varies much faster than the $\cos\theta$ term. Therefore the half-power beamwidth in the H-plane is approximately the same as that in the E-plane, or

$$\Theta_h \simeq 9.836^\circ$$

CHAPTER 7

7.1 a. $H_\phi = \dfrac{E_\theta}{\eta} = j\,\dfrac{\beta I_0 \ell e^{-j\beta r}}{4\pi r}\sin\theta\left[2\cos(\beta h\cos\theta)\right]$

b. $\underline{S}_{av} = \frac{1}{2}\mathrm{Re}\left[\underline{E}\times\underline{H}^*\right] = \frac{1}{2}\mathrm{Re}\left[\hat{a}_\theta E_\theta \times \hat{a}_\phi H_\phi^*\right] = \frac{\hat{a}_r}{2}\mathrm{Re}\left[E_\theta H_\phi^*\right]$

$= \hat{a}_r \frac{1}{2}\mathrm{Re}\left[E_\theta \frac{E_\theta^*}{\eta}\right] = \hat{a}_r \frac{1}{2}\frac{|E_\theta|^2}{\eta}$

$\underline{S}_{av} = \hat{a}_r\,\frac{\eta}{2}\left|\frac{\beta I_0 \ell}{4\pi r}\right|^2 \sin^2\theta\left|2\cos(\beta h\cos\theta)\right|^2 = \hat{a}_r 2\eta\left|\frac{\beta I_0 \ell}{4\pi r}\right|^2 \sin^2\theta\cos^2(\beta h\cos\theta)$

c. $P_{av} = \oiint_S \underline{S}_{av}\cdot d\underline{s} = \int_0^{2\pi}\int_0^{\pi/2} \hat{a}_r S_{av}\cdot\hat{a}_r r^2\sin\theta\,d\theta\,d\phi = \int_0^{2\pi}\int_0^{\pi/2} S_{av} r^2 \sin\theta\,d\theta\,d\phi$

$P_{av} = 4\pi\eta\left|\frac{\beta I_0 \ell}{4\pi}\right|^2 \int_0^{\pi/2}\sin^3\theta\cos^2(\beta h\cos\theta)\,d\theta = \pi\eta\left|\frac{I_0\ell}{\lambda}\right|^2\int_0^{\pi/2}\sin^3\theta\cos^2(\beta h\cos\theta)\,d\theta$

$P_{rad} = P_{av} = \pi\eta\left|\frac{I_0\ell}{\lambda}\right|^2 I$

where $I = I_1 + I_2 = \int_0^{\pi/2}\sin^3\theta\cos^2(\beta h\cos\theta)\,d\theta = \int_0^{\pi/2}\sin^3\theta\left[\frac{1+\cos(2\beta h\cos\theta)}{2}\right]d\theta$

$I = I_1 + I_2 = \frac{1}{2}\int_0^{\pi/2}\sin^3\theta\,d\theta + \frac{1}{2}\int_0^{\pi/2}\sin^3\theta\cos(2\beta h\cos\theta)\,d\theta$

$I_1 = \frac{1}{2}\int_0^{\pi/2}\sin^3\theta\,d\theta = -\frac{1}{6}\cos\theta\left(\sin^2\theta+2\right)\Big|_0^{\pi/2} = \frac{1}{3}$

$I_2 = \frac{1}{2}\int_0^{\pi/2}\sin^3\theta\cos(2\beta h\cos\theta)\,d\theta = \frac{1}{2}\int_0^{\pi/2}\sin^2\theta\cos(2\beta h\cos\theta)\sin\theta\,d\theta$

Let $u = \sin^2\theta$, $du = 2\sin\theta\cos\theta\,d\theta$; $dv = -\frac{\cos(2\beta h\cos\theta)}{2\beta h}\,d(2\beta h\cos\theta)$, $v = -\frac{1}{2\beta h}\sin(2\beta h\cos\theta)$

Thus

$I_2 = \underbrace{-\frac{\sin^2\theta}{4\beta h}\sin(2\beta h\cos\theta)\Big|_0^{\pi/2}}_{=0} + \frac{1}{2\beta h}\int_0^{\pi/2}\cos\theta\sin(2\beta h\cos\theta)\sin\theta\,d\theta$

Let $u = \cos\theta$, $du = -\sin\theta\,d\theta$; $dv = -\frac{1}{2\beta h}\sin(2\beta h\cos\theta)\,d(2\beta h\cos\theta)$, $v = \frac{1}{2\beta h}\cos(2\beta h\cos\theta)$

$I_2 = 0 + \frac{1}{2\beta h}\left\{\frac{\cos\theta}{2\beta h}\cos(2\beta h\cos\theta)\Big|_0^{\pi/2} + \frac{1}{2\beta h}\int_0^{\pi/2}\cos(2\beta h\cos\theta)\sin\theta\,d\theta\right\}$

$I_2 = \frac{1}{2\beta h}\left\{-\frac{1}{2\beta h}\cos(2\beta h) - \frac{1}{(2\beta h)^2}\sin(2\beta h\cos\theta)\Big|_0^{\pi/2}\right\} = \left\{-\frac{\cos(2\beta h)}{(2\beta h)^2} + \frac{\sin(2\beta h)}{(2\beta h)^3}\right\}$

Therefore

$P_{rad} = \pi\eta\left|\frac{I_0\ell}{\lambda}\right|^2\left[\frac{1}{3} - \frac{\cos(2\beta h)}{(2\beta h)^2} + \frac{\sin(2\beta h)}{(2\beta h)^3}\right]$

$\boxed{7.2}$ a. From Problem 7.1, part b

$$S_{av} = 2\eta\left|\frac{\beta I_0 \ell}{4\pi r}\right|^2 \sin^2\theta \cos^2(\beta h\cos\theta) = \frac{\eta}{2}\left|\frac{I_0\ell}{2r}\right|^2 \sin^2\theta\cos^2(\beta h\cos\theta)$$

$$U = r^2 S_{av} = \frac{\eta}{2}\left|\frac{I_0\ell}{\lambda}\right|^2 \sin^2\theta\cos^2(\beta h\cos\theta)$$

b. $$D_0 = \frac{4\pi U_{max}}{P_{rad}}, \quad U_{max} = U|_{max} = \frac{\eta}{2}\left|\frac{I_0\ell}{\lambda}\right|^2$$

From Problem 7.1, part c

$$P_{rad} = \pi\eta\left|\frac{I_0\ell}{\lambda}\right|^2\left[\frac{1}{3} - \frac{\cos(2\beta h)}{(2\beta h)^2} + \frac{\sin(2\beta h)}{(2\beta h)^3}\right] = \pi\eta\left|\frac{I_0\ell}{\lambda}\right|^2 F(\beta h)$$

where $F(\beta h) = \left[\frac{1}{3} - \frac{\cos(2\beta h)}{(2\beta h)^2} + \frac{\sin(2\beta h)}{(2\beta h)^3}\right]$

Thus

$$D_0 = \frac{4\pi\,\frac{\eta}{2}\left|\frac{I_0\ell}{\lambda}\right|^2}{\pi\eta\left|\frac{I_0\ell}{\lambda}\right|^2 F(\beta h)} = \frac{2}{F(\beta h)}$$

c. $$R_r = \frac{2P_{rad}}{|I_0|^2} = 2\pi\eta\left|\frac{\ell}{\lambda}\right|^2 F(\beta h)$$

$\boxed{7.3}$ The radiation characteristics of a vertical <u>magnetic</u> dipole can be obtained easily by referring to the solution of the radiation characteristics of a vertical <u>electric</u> dipole and using duality.

Based on the solution of Example 6-3, and duality, we can write:

- $F_r = \dfrac{I_m \ell e^{-j\beta r}}{4\pi\varepsilon r}\cos\theta, \quad F_\theta = -\dfrac{I_m\ell e^{-j\beta r}}{4\pi\varepsilon r}\sin\theta, \quad F_\phi = 0$
- $H_r = \dfrac{I_m\ell\cos\theta}{2\pi\eta r^2}\left(1 + \dfrac{1}{j\beta r}\right)e^{-j\beta r}$

  $H_\theta = j\dfrac{\beta I_m \ell\sin\theta}{4\pi\eta r}\left[1 + \dfrac{1}{j\beta r} - \dfrac{1}{(\beta r)^2}\right]e^{-j\beta r}$

  $H_\phi = 0$
- $E_r = E_\theta = 0$
- $E_\phi = -j\dfrac{\beta I_m\ell\sin\theta}{4\pi r}\left(1 + \dfrac{1}{j\beta r}\right)e^{-j\beta r}$

$\boxed{\text{Cont'd}}$

$\boxed{7.3 \text{ cont'd}}$

- $\underline{S}_{ave} = \frac{1}{2}\mathrm{Re}[\underline{E}\times\underline{H}^*] = \hat{a}_r \frac{1}{2\eta}\left(\frac{\beta l}{4\pi}\right)^2 \frac{|I_m|^2}{r^2}\sin^2\theta = \hat{a}_r S_r = \hat{a}_r S_{ave}$

- $U \simeq r^2 S_{ave} = \frac{1}{2\eta}\left(\frac{\beta l}{4\pi}\right)^2 |I_m|^2 \sin^2\theta \Rightarrow U_{max} = \frac{1}{2\eta}\left(\frac{\beta l}{4\pi}\right)^2 |I_m|^2$

- $P_{rad} = \int_0^{2\pi}\int_0^{\pi} U(\theta,\phi)\sin\theta\, d\theta\, d\phi = \frac{1}{2\eta}\left(\frac{\beta l}{4\pi}\right)^2 \frac{4}{3}(2\pi) = \frac{8\pi}{2(3\eta)}\left(\frac{\beta l}{4\pi}\right)^2 |I_m|^2$

- $D_0 = \frac{4\pi U_{max}}{P_{rad}} = \frac{4\pi\left[\frac{1}{2\eta}\left(\frac{\beta l}{4\pi}\right)^2\right]}{\left(\frac{1}{2}\right)\frac{8\pi}{3\eta}\left(\frac{\beta l}{4\pi}\right)^2} = \frac{3}{2}$

- $R_r = \frac{2P_{rad}}{|I_m|^2} = \frac{2}{2}\left[\frac{8\pi}{3\eta}\left(\frac{\beta l}{4\pi}\right)^2\right] = \frac{8\pi}{12\eta}\left(\frac{l}{\lambda}\right)^2 = \frac{8}{12(120)}\left(\frac{l}{\lambda}\right)^2$

$R_r = \frac{1}{180}\left(\frac{l}{\lambda}\right)^2$

$\boxed{7.4}$ $E_r = E_\theta = H_\phi = 0$, $F_r = \frac{I_m l e^{-j\beta r}}{4\pi\varepsilon r}\cos\theta$

$E_\phi = -j\frac{\beta I_m l e^{-j\beta r}}{4\pi r}\sin\theta$, $F_\theta = -\frac{I_m l e^{-j\beta r}}{4\pi\varepsilon r}\sin\theta$

$H_\theta = j\frac{\beta I_m l e^{-j\beta r}}{4\pi\eta r}\sin\theta$, $F_\phi = 0$

$\boxed{7.5}$ $F = \frac{\varepsilon}{4\pi}\int_{-l/2}^{+l/2} I_m \frac{e^{-j\beta R}}{4\pi R}\, dl \simeq \hat{a}_x \frac{\varepsilon I_m e^{-j\beta r}}{4\pi r}\int_{-l/2}^{+l/2} dx' = \hat{a}_x \frac{\varepsilon l I_m e^{-j\beta r}}{4\pi r}$

- Using (II-12a)

$F_r = F_x \sin\theta\cos\phi$, $F_\theta = F_x\cos\theta\cos\phi$, $F_\phi = -F_x\sin\phi$

$F_r = \frac{\varepsilon l I_m e^{-j\beta r}}{4\pi r}\sin\theta\cos\phi$, $F_\theta = \frac{\varepsilon l I_m e^{-j\beta r}}{4\pi r}\cos\theta\cos\phi$, $F_\phi = -\frac{\varepsilon l I_m e^{-j\beta r}}{4\pi r}\sin\phi$

- Using (6-119a)-(6-119d)

$E_\theta = -j\omega\eta F_\phi = j\frac{\omega\eta\varepsilon l I_m e^{-j\beta r}}{4\pi r}\sin\phi = j\frac{\beta l I_m e^{-j\beta r}}{4\pi r}\sin\phi$

$E_\phi = +j\omega\eta F_\theta = j\frac{\omega\eta\varepsilon l I_m e^{-j\beta r}}{4\pi r}\cos\theta\cos\phi = j\frac{\beta l I_m e^{-j\beta r}}{4\pi r}\cos\theta\cos\phi$

$\boxed{\text{cont'd}}$

7.5 cont'd

$$H_\theta = -j\omega F_\theta = -j\frac{\omega\varepsilon \ell I_m e^{-j\beta r}}{4\pi r}\cos\theta\cos\phi = -E_\phi/\eta$$
$$H_\phi = -j\omega F_\phi = j\frac{\omega\varepsilon \ell I_m e^{-j\beta r}}{4\pi r}\sin\phi = E_\theta/\eta$$

- $\underline{S}_{ave} = \frac{1}{2}\mathrm{Re}[\underline{E}\times\underline{H}^*] = \hat{a}_r\frac{1}{2\eta}\left[|E_\theta|^2 + |E_\phi|^2\right] = \hat{a}_r\frac{1}{2\eta}\left[\frac{\beta\ell I_m}{4\pi r}\right]^2\left[\sin^2\phi + \cos^2\theta\cos^2\phi\right]$

$$\underline{S}_{ave} = \hat{a}_r S_r = \hat{a}_r\frac{|I_m|^2}{2\eta r^2}\left[\frac{\beta\ell}{4\pi}\right]^2\left[\sin^2\phi + \cos^2\theta\cos^2\phi\right]$$

- $\mathcal{U} = r^2 S_r = \frac{|I_m|^2}{2\eta}\left(\frac{\beta\ell}{4\pi}\right)^2\left[\sin^2\phi + \cos^2\theta\cos^2\phi\right]$
- $P_{rad} = \int_0^{2\pi}\int_0^{\pi}\mathcal{U}\sin\theta\, d\theta\, d\phi = \frac{|I_m|^2}{2\eta}\left(\frac{\beta\ell}{4\pi}\right)^2\int_0^{2\pi}\int_0^{\pi}\left[\sin^2\phi + \cos^2\theta\cos^2\phi\right]\sin\theta\, d\theta\, d\phi$

Using a similar integration procedure as in solution to Problem 6.21:

$$P_{rad} = \frac{|I_m|^2}{2\eta}(2\pi)\left(\frac{4}{3}\right)\left(\frac{\beta\ell}{4\pi}\right)^2 = \frac{|I_m|^2}{2\eta}\left(\frac{8\pi}{3}\right)\left(\frac{\beta\ell}{4\pi}\right)^2$$

$$\mathcal{U}_{max} = \frac{|I_m|^2}{2\eta}\left(\frac{\beta\ell}{4\pi}\right)^2\left[\sin^2\phi + \cos^2\theta\cos^2\phi\right]_{\substack{\theta=0,\ \phi=0 \\ \text{or } \theta=90°,\ \phi=90°}} = \frac{|I_m|^2}{2\eta}\left(\frac{\beta\ell}{4\pi}\right)^2$$

$$D_0 = \frac{4\pi\,\mathcal{U}_{max}}{P_{rad}} = \frac{4\pi}{(8\pi)/3} = \left(\frac{3}{2}\right)$$

- $R_r = \frac{2P_{rad}}{|I_m|^2} = 2\,\frac{1}{2\eta}\left(\frac{8\pi}{3}\right)\left(\frac{\beta\ell}{4\pi}\right)^2 = \frac{1}{\eta}\left(\frac{8\pi}{3}\right)\left[\frac{2\pi\ell}{\lambda(4\pi)}\right]^2 = \frac{1}{\eta}\left(\frac{8\pi}{4}\right)\left(\frac{\ell}{\lambda}\right)^2$

$$R_r = \frac{8\pi}{120\pi(4)3}\left(\frac{\ell}{\lambda}\right)^2 = \frac{1}{3(60)}\left(\frac{\ell}{\lambda}\right)^2 = \frac{1}{180}\left(\frac{\ell}{\lambda}\right)^2$$

7.6 $\underline{F} = \frac{\varepsilon}{4\pi}\int_{-\ell/2}^{+\ell/2}\underline{I}_m\frac{e^{-j\beta R}}{R}\,d\ell \ \underset{\sim}{\overset{R\simeq r}{=}}\ \hat{a}_y\frac{\varepsilon I_m e^{-j\beta r}}{4\pi r}\int_{-\ell/2}^{+\ell/2}dy'$

$$\underline{F} = \hat{a}_y\frac{\varepsilon\ell I_m e^{-j\beta r}}{4\pi r} = \hat{a}_y F_y,\quad F_y = \frac{\varepsilon\ell I_m e^{-j\beta r}}{4\pi r}$$

$$F_r = F_y\sin\theta\sin\phi = \frac{\varepsilon\ell I_m e^{-j\beta r}}{4\pi r}\sin\theta\sin\phi$$
$$F_\theta = F_y\cos\theta\sin\phi = \frac{\varepsilon\ell I_m e^{-j\beta r}}{4\pi r}\cos\theta\sin\phi$$
$$F_\phi = F_y\cos\phi = \frac{\varepsilon\ell I_m e^{-j\beta r}}{4\pi r}\cos\phi$$

cont'd

7.6 **Cont'd** Using (6-119a)-(6-119d)

$$E_\theta = -j\omega\eta F_\phi = -j\,\frac{\omega\eta\varepsilon \ell I_m e^{-j\beta r}}{4\pi r}\cos\phi = -j\,\frac{\beta \ell I_m e^{-j\beta r}}{4\pi r}\cos\phi$$

$$E_\phi = j\omega\eta F_\theta = j\,\frac{\omega\eta\varepsilon \ell I_m e^{-j\beta r}}{4\pi r}\cos\theta\sin\phi = j\,\frac{\beta \ell I_m e^{-j\beta r}}{4\pi r}\cos\theta\sin\phi$$

$$H_\theta = -j\omega F_\theta = -j\,\frac{\omega\varepsilon \ell I_m e^{-j\beta r}}{4\pi r}\cos\theta\sin\phi = -j\,\frac{\beta \ell I_m e^{-j\beta r}}{4\pi\eta r}\cos\theta\sin\phi = -\frac{E_\phi}{\eta}$$

$$H_\phi = -j\omega F_\phi = -j\,\frac{\omega\varepsilon \ell I_m e^{-j\beta r}}{4\pi r}\cos\phi = -j\,\frac{\beta \ell I_m e^{-j\beta r}}{\eta 4\pi r}\cos\phi = \frac{E_\theta}{\eta}$$

7.7 For a horizontal electric dipole, directed along the y-direction, the far-zone electric and magnetic fields are given by (see solution of Problem 6.22) by

$$A_r = \frac{\ell\mu I_0 e^{-j\beta r}}{4\pi r}\sin\theta\sin\phi,\quad A_\theta = \frac{\ell\mu I_0 e^{-j\beta r}}{4\pi r}\cos\theta\sin\phi,\quad A_\phi = \frac{\ell\mu I_0 e^{-j\beta r}}{4\pi r}\cos\phi$$

$E_r \simeq H_r \simeq 0$, $E_\theta \simeq$

$$E_\theta \simeq -j\,\frac{\omega\mu I_0 \ell e^{-j\beta r}}{4\pi r}\cos\theta\sin\phi \quad , \; H_\theta \simeq -E_\phi/\eta$$

$$E_\phi \simeq -j\,\frac{\omega\mu I_0 \ell e^{-j\beta r}}{4\pi r}\cos\phi \quad , \; H_\phi \simeq E_\theta/\eta$$

When the infinitesimal electric dipole is placed a height h above a PEC ground plane, its image is, according to Figures 7-2 and 7-5, in the $-y$ direction (same magnitude, 180° phase difference). Therefore, according to (7-17) its array factor is

$$AF = 2j\sin(\beta h\cos\theta)$$

and the total fields are now given by

$$E_\theta \simeq -j\,\frac{\omega\mu I_0 \ell e^{-j\beta r}}{4\pi r}\left[2j\sin(\beta h\cos\theta)\right]\cos\theta\sin\phi$$

$$E_\phi \simeq -j\,\frac{\omega\mu I_0 \ell e^{-j\beta r}}{4\pi r}\left[2j\sin(\beta h\cos\theta)\right]\cos\phi$$

Cont'd

7.7 (cont'd) The radiation intensity is given

- $$U = \frac{1}{2\eta} r^2 \left[|E_\theta|^2 + |E_\phi|^2\right] = \frac{|I_0|^2}{2\eta}\left[\left(\frac{\omega\mu\ell}{4\pi}\right)^2\left\{\cos^2\theta\sin^2\phi + \cos^2\phi\right\}\right]\left[2\sin(\beta h\cos\theta)\right]^2$$

which reduces to

$$U = \frac{\eta}{2}|I_0|^2\left(\frac{\ell}{\lambda}\right)^2\left[\cos^2\theta\sin^2\phi + \cos^2\phi\right]\sin^2(\beta h\cos\theta)$$

- The power radiated is

$$P_{rad} = \int_0^{2\pi}\int_0^{\pi/2} U(\theta,\phi)\sin\theta\, d\theta\, d\phi = \frac{\eta}{2}|I_0|^2\left(\frac{\ell}{\lambda}\right)^2\int_0^{2\pi}\int_0^{\pi/2}\left[\cos^2\theta\sin^2\phi + \cos^2\phi\right]\sin^2(\beta h\cos\theta)\sin\theta\, d\theta\, d\phi$$

Using an integration procedure similar to that indicated in the solution of Problem 7.1, it can be shown that the above reduces to

$$P_{rad} = \eta\left(\frac{\pi}{2}\right)|I_0|^2\left(\frac{\ell}{\lambda}\right)^2\left[\frac{2}{3} - \frac{\sin(2\beta h)}{2\beta h} - \frac{\cos(2\beta h)}{(2\beta h)^2} + \frac{\sin(2\beta h)}{(2\beta h)^3}\right]$$

- Using the maximum value of $U_{max} = U(\theta,\phi)$ from above, it can be shown that the directivity can be written as

$$D_0 = \frac{4\pi U_{max}}{P_{rad}} = \begin{cases} \dfrac{4\sin^2(\beta h)}{R(\beta h)} & \beta h \le \pi/2 \ (h \le \lambda/4) \\[2ex] \dfrac{4}{R(\beta h)} & \beta h > \pi/2 \ (h > \lambda/4) \end{cases}$$

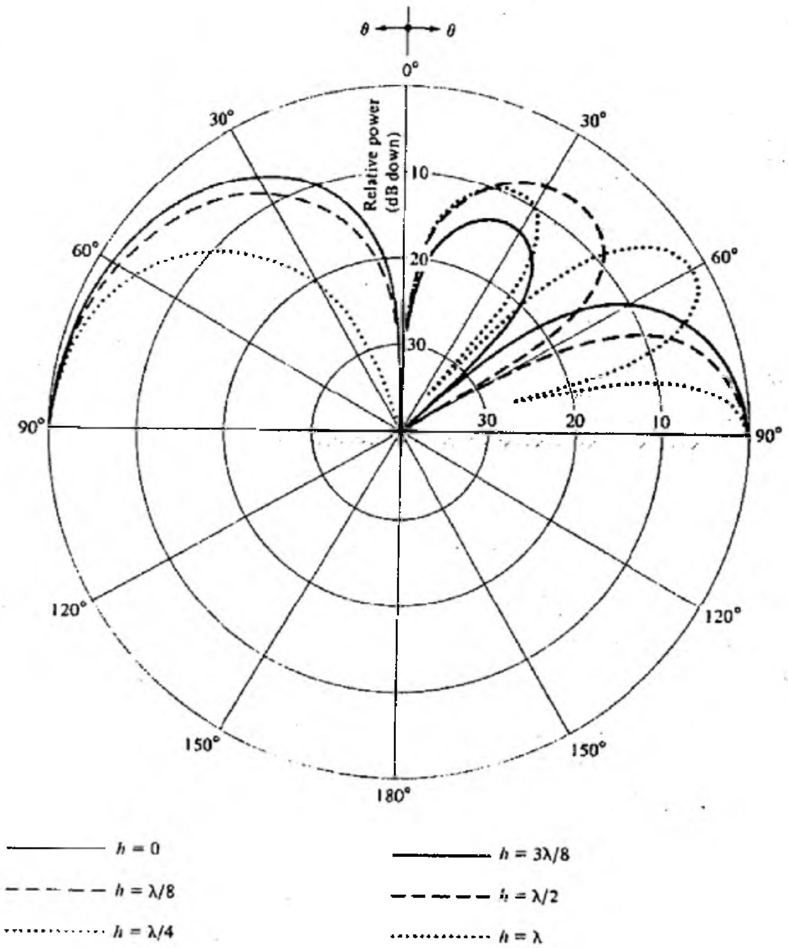
$$R(\beta h) = \left[\frac{2}{3} - \frac{\sin(2\beta h)}{2\beta h} - \frac{\cos(2\beta h)}{(2\beta h)^2} + \frac{\sin(2\beta h)}{(2\beta h)^3}\right]$$

- The radiation resistance, according to the definition of

$$R_r = \frac{2P_{rad}}{|I_0|^2} = 2\eta\left(\frac{\pi}{2}\right)\left(\frac{\ell}{\lambda}\right)^2\left[\frac{2}{3} - \frac{\sin(2\beta h)}{2\beta h} - \frac{\cos(2\beta h)}{(2\beta h)^2} + \frac{\sin(2\beta h)}{(2\beta h)^3}\right]$$

$$= \eta\pi\left(\frac{\ell}{\lambda}\right)^2\left[\frac{2}{3} - \frac{\sin(2\beta h)}{2\beta h} - \frac{\cos(2\beta h)}{(2\beta h)^2} + \frac{\sin(2\beta h)}{(2\beta h)^3}\right]$$

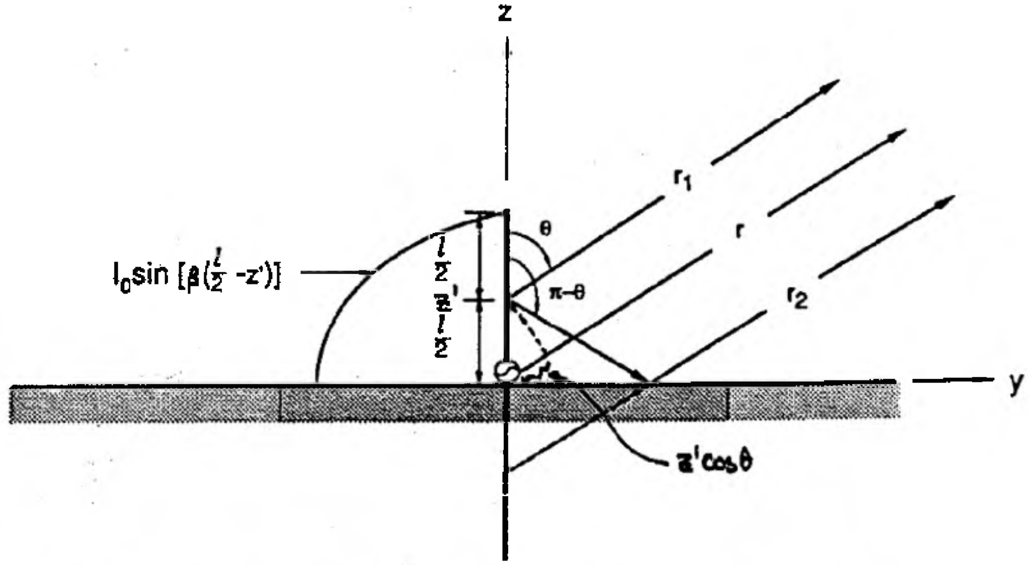
7.8



θ θ
0°
30°
30°
60°
60°
90°
90°
120°
120°
150°
150°
180°
Relative power (dB down)
10
20
30
30
20
10

$h = 0$
$h = \lambda/8$
$h = \lambda/4$
$h = 3\lambda/8$
$h = \lambda/2$
$h = \lambda$

$\boxed{7.9}$



The direct far-zone field is obtained using (7-9) modified as

$$E_\theta^i = j\eta \frac{\beta e^{-j\beta r_1}}{4\pi r_1}\sin\theta \int_{-\ell/2}^{\ell/2} I_z(z')\, e^{j\beta z'\sin\theta}\, dz' = j\eta \frac{\beta e^{-j\beta r_1} I_0}{4\pi r_1}\sin\theta \int_{-\ell/2}^{\ell/2} \sin\left[\beta\left(\frac{\ell}{2}-z'\right)\right] e^{j\beta z'\cos\theta}\, dz' \qquad (1)$$

The integral can be integrated as

$$\int_{-\ell/2}^{\ell/2} \sin\left[\beta\left(\frac{\ell}{2}-z'\right)\right] e^{j\beta z'\cos\theta}\, dz' = \frac{e^{j\beta z'\cos\theta}}{(j\beta\cos\theta)^2+\beta^2}\left\{ j\beta\cos\theta \sin\left[\beta\left(\frac{\ell}{2}-z'\right)\right] + \beta\cos\left[\beta(\ell-z')\right]\right\}\Bigg|_{-\ell/2}^{\ell/2}$$

$$= \frac{1}{\beta^2\sin^2\theta}\left\{\beta e^{+j\frac{\beta\ell}{2}\cos\theta} - e^{-j\frac{\beta\ell}{2}\cos\theta}\left[j\beta\cos\theta\sin(\beta\ell)+\beta\cos(\beta\ell)\right]\right\}$$

$$\int_{-\ell/2}^{\ell/2} \sin\left[\beta\left(\frac{\ell}{2}-z'\right)\right] e^{j\beta z'\cos\theta}\, dz' = \frac{1}{\beta\sin^2\theta}\left\{e^{j\frac{\beta\ell}{2}\cos\theta} - e^{-j\frac{\beta\ell}{2}\cos\theta}\left[j\cos\theta\sin(\beta\ell)+\cos(\beta\ell)\right]\right\} \qquad (2)$$

Substituting (2) into (1) we can write the direct far-zone field as

$$E_\theta^i = j\eta \frac{I_0 e^{-j\beta r_1}}{4\pi r_1}\left\{\frac{e^{j\frac{\beta\ell}{2}\cos\theta} - e^{-j\frac{\beta\ell}{2}\cos\theta}\left[j\cos\theta\sin(\beta\ell)+\cos(\beta\ell)\right]}{\sin\theta}\right\} \qquad (3)$$

Using the far zone approximations of

$$\begin{array}{ll} r_1 \simeq r - \frac{\ell}{2}\cos\theta & \text{for phase terms} \\ r_1 \simeq r & \text{for amplitude terms} \end{array} \qquad (4)$$

cont'd.

7.9 Cont'd. we can reduce (3) to

$$E_\theta^i = j\eta \frac{I_0 e^{-j\beta r}}{4\pi r}\left\{\frac{e^{j\beta \ell \cos\theta} - j\cos\theta \sin(\beta\ell) - \cos(\beta\ell)}{\sin\theta}\right\} \tag{5}$$

By referring to the geometry of the figure it is evident that the reflected field can be obtained from the direct field by replacing θ by $\pi-\theta$ in (3). Doing this, the reflected field can be written by referring to (3) as

$$E_\theta^r = j\eta \frac{I_0 e^{-j\beta r_2}}{4\pi r_2}\left\{\frac{e^{j\frac{\beta\ell}{2}\cos(\pi-\theta)} - e^{-j\frac{\beta\ell}{2}\cos(\pi-\theta)}\left[j\cos(\pi-\theta)\sin(\beta\ell)+\cos(\beta\ell)\right]}{\sin(\pi-\theta)}\right\}$$

$$= j\eta \frac{I_0 e^{-j\beta r_2}}{4\pi r_2}\left\{\frac{e^{-j\frac{\beta\ell}{2}\cos\theta} - e^{j\frac{\beta\ell}{2}\cos\theta}\left[-j\cos\theta\sin(\beta\ell)+\cos(\beta\ell)\right]}{\sin\theta}\right\} \tag{6}$$

which by using the far zone approximations of

$$\begin{aligned} r_2 &\simeq r + \frac{\ell}{2}\cos\theta && \text{for phase terms} \\ r_2 &\simeq r && \text{for amplitude terms} \end{aligned} \tag{7}$$

reduces to

$$E_\theta^r = j\eta \frac{I_0 e^{-j\beta r}}{4\pi r}\left\{\frac{e^{-j\beta\ell\cos\theta} + j\cos\theta\sin(\beta\ell) - \cos(\beta\ell)}{\sin\theta}\right\} \tag{8}$$

Combining (5) and (8) leads to

$$E_\theta^t = E_\theta^i + E_\theta^r = j\eta \frac{I_0 e^{-j\beta r}}{2\pi r}\left\{\frac{\cos(\beta\ell\cos\theta) - \cos(\beta\ell)}{\sin\theta}\right\} \tag{9}$$

When $\ell = \lambda/4$, (9) reduces to

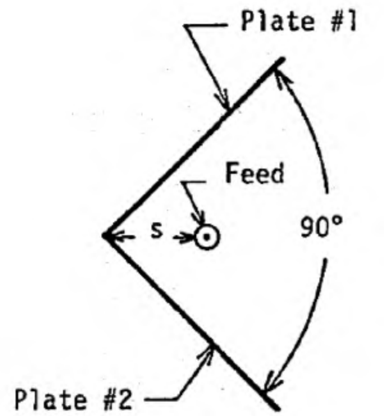
$$E_\theta^t = j\eta \frac{I_0 e^{-j\beta r}}{2\pi r}\left\{\frac{\cos(\frac{\pi}{2}\cos\theta)}{\sin\theta}\right\} \tag{10}$$

The other electric and magnetic field components are

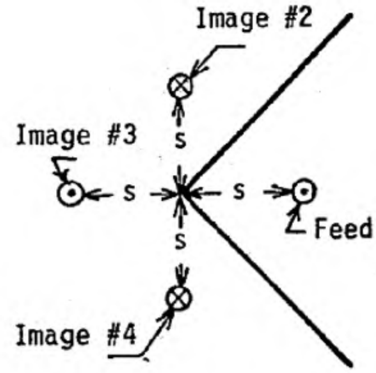
$$E_r \simeq E_\phi \simeq H_r \simeq H_\theta \simeq 0$$

$$H_\phi^t = \frac{E_\theta^t}{\eta}$$

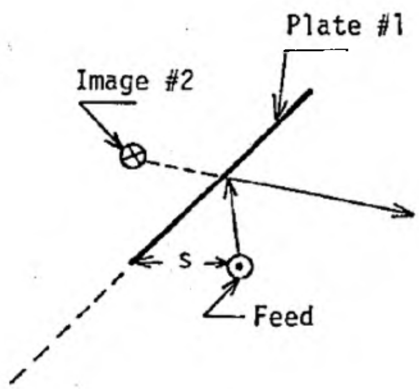
$\boxed{7.10}$ a. The number of images (3 of them), their polarizations, and their positions are shown below.



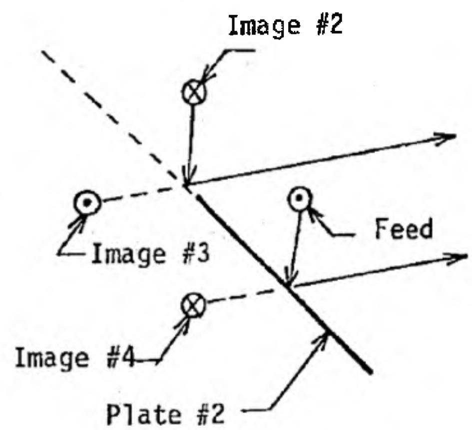
(a) 90° corner reflector



(b) images for 90° corner reflector



(c) placement of image #2 (due to feed)



(d) placement of images #4 and #3 (due to feed and image #2)

cont'd.

7.10 cont'd. b. The total field is derived by summing the contributions from the feed and its images. Thus

$$E^t(r,\theta,\phi) = E_1(r_1,\theta,\phi) + E_2(r_2,\theta,\phi) + E_3(r_3,\theta,\phi) + E_4(r_4,\theta,\phi) \qquad (1)$$

In the far zone, the normalized field can be written as

$$E^t(r,\theta,\phi) = f(\theta,\phi)\frac{e^{-j\beta r_1}}{r_1} - f(\theta,\phi)\frac{e^{-j\beta r_2}}{r_2} + f(\theta,\phi)\frac{e^{-j\beta r_3}}{r_3} - f(\theta,\phi)\frac{e^{-j\beta r_4}}{r_4}$$

$$= \left[e^{j\beta s\cos\psi_1} - e^{j\beta s\cos\psi_2} + e^{j\beta s\cos\psi_3} - e^{j\beta s\cos\psi_3}\right] f(\theta,\phi)\frac{e^{-j\beta r}}{r} \qquad (2)$$

where

$$\begin{aligned} \cos\psi_1 &= \hat{a}_x \cdot \hat{a}_r = \sin\theta\cos\phi \\ \cos\psi_2 &= \hat{a}_y \cdot \hat{a}_r = \sin\theta\sin\phi \\ \cos\psi_3 &= -\hat{a}_x \cdot \hat{a}_r = -\sin\theta\cos\phi \\ \cos\psi_3 &= -\hat{a}_y \cdot \hat{a}_r = -\sin\theta\sin\phi \end{aligned} \qquad (3)$$

since

$$\hat{a}_r = \hat{a}_x \sin\theta\cos\phi + \hat{a}_y \sin\theta\sin\phi + \hat{a}_z\cos\theta \qquad (3a)$$

Substituting (3) into (2) we can write it as

$$E^t(r,\theta,\phi) = 2\left[\cos(\beta s\sin\theta\cos\phi) - \cos(\beta s\sin\theta\sin\phi)\right] f(\theta,\phi)\frac{e^{-j\beta r}}{r} \qquad (4)$$

By letting the field of the isolated element (in this case the dipole) to be

$$E_\theta^0 = f(\theta,\phi)\frac{e^{-j\beta r}}{r} = j\eta\frac{\beta I_0 \ell e^{-j\beta r}}{4\pi r}\sin\theta \qquad (5)$$

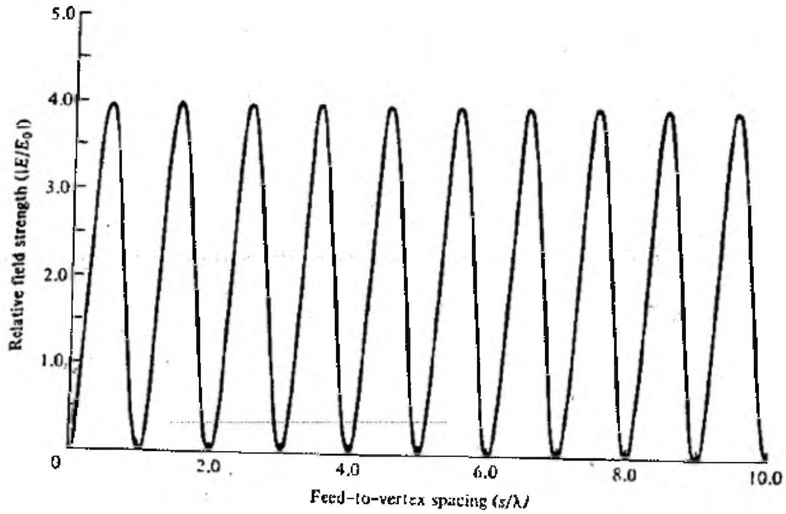
then (4) can be written as

$$E^t(r,\theta,\phi) = E_\theta^0 \; F(\beta s)$$

where

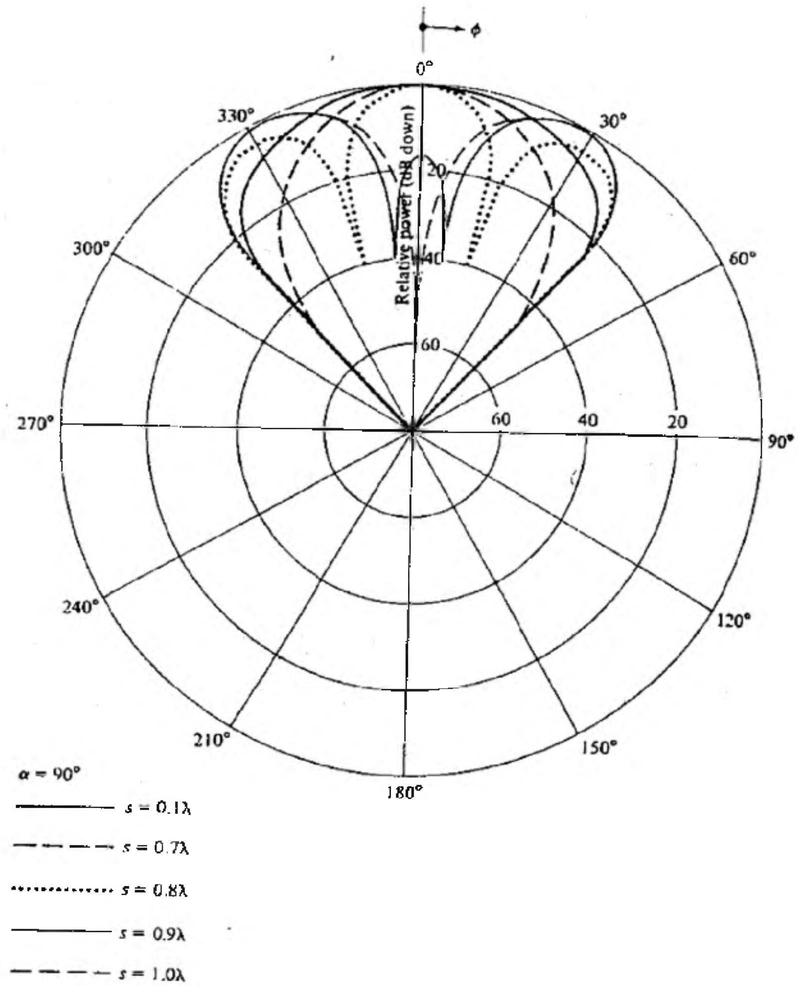
$$F(\beta s) = 2\left[\cos(\beta s\sin\theta\cos\phi) - \cos(\beta s\sin\theta\sin\phi)\right]$$

7.11

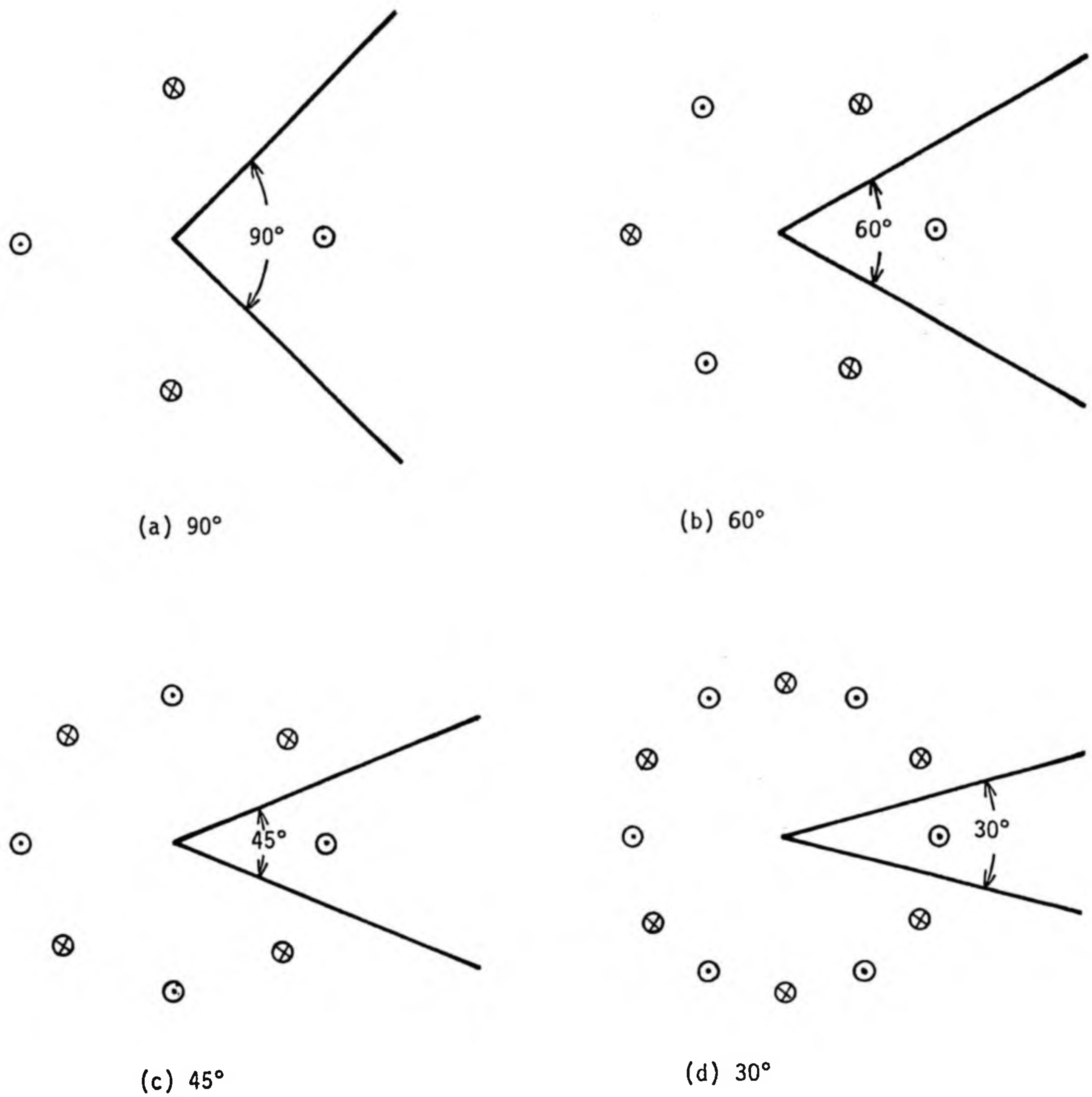


The maximum value of $F(\beta s)$ is 4. The function is periodic with a period of λ.

7.12



7.13 For all three corner reflectors ($\alpha = 60°, 45°, 30°$), the geometrical coordinate system is that shown in Problem 7.10. The sources will be numbered so that the feed will be #1. The images will be designated as #2, #3, ... (in a clockwise rotation) as shown in the figure of the solution of Problem 7.10 for the 90° corner reflector. The image arrangement for each geometry is shown below.



cont'd.

$\underline{\alpha = 60^\circ}$

Using the arrangement of Figure b of the previous page

$$E(r,\theta,\phi) = \left(\frac{e^{-j\beta r_1}}{r_1} + \frac{e^{-j\beta r_2}}{r_2} + \frac{e^{-j\beta r_3}}{r_3} + \frac{e^{-j\beta r_4}}{r_4} + \frac{e^{-j\beta r_5}}{r_5} + \frac{e^{-j\beta r_6}}{r_6} \right) f(\theta,\phi)$$

for the main source (#1) and the five images. For far-field observations

For Phase Terms

$r_1 \simeq r - s\cos\psi_1 = r - s(\hat{a}_x \cdot \hat{a}_r) = r - s\sin\theta\cos\phi$

$r_2 \simeq r - s\cos\psi_2 = r - s\left[(0.5\hat{a}_x + 0.866\hat{a}_y)\cdot\hat{a}_r\right] = r - s(0.5\sin\theta\cos\phi + 0.866\sin\theta\sin\phi)$

$r_3 \simeq r - s\cos\psi_3 = r - s\left[(-0.5\hat{a}_x + 0.866\hat{a}_y)\cdot\hat{a}_r\right] = r - s(-0.5\sin\theta\cos\phi + 0.866\sin\theta\sin\phi)$

$r_4 \simeq r - s\cos\psi_4 = r - s(-\hat{a}_x \cdot \hat{a}_r) = r + s\sin\theta\cos\phi$

$r_5 \simeq r - s\cos\psi_5 = r - s\left[(-0.5\hat{a}_x - 0.866\hat{a}_y)\cdot\hat{a}_r\right] = r + s(0.5\sin\theta\cos\phi + 0.866\sin\theta\sin\phi)$

$r_6 \simeq r - s\cos\psi_6 = r - s\left[(0.5\hat{a}_x - 0.866\hat{a}_y)\cdot\hat{a}_r\right] = r + s(-0.5\sin\theta\cos\phi + 0.866\sin\theta\sin\phi)$

where $\hat{a}_r = \hat{a}_x\sin\theta\cos\phi + \hat{a}_y\sin\theta\sin\phi + \hat{a}_z\cos\theta$

For Amplitude Terms

$r_1 \simeq r_2 \simeq r_3 \simeq r_4 \simeq r_5 \simeq r_6 \simeq r$

Making these substitutions and combining terms (first with fourth, second with fifth, third with sixth), we can write that

$$E(r,\theta,\phi) = f(\theta,\phi)\frac{e^{-j\beta r}}{r} 2\Big\{\sin(\beta s\sin\theta\cos\phi) - \sin\left[\beta s(0.5\sin\theta\cos\phi + 0.866\sin\theta\sin\phi)\right] + \sin\left[\beta s(-0.5\sin\theta\cos\phi + 0.866\sin\theta\sin\phi)\right]$$

Using the identities of

$$\sin(x \pm y) = \sin(x)\cos(y) \pm \cos(x)\sin(y)$$

reduces the field to

$$\frac{E}{E_0} = 2\left[\sin(\beta s\sin\theta\cos\phi) - 2\sin\left(\frac{\beta s}{2}\sin\theta\cos\phi\right)\cos\left(\frac{\sqrt{3}}{2}\beta s\sin\theta\sin\phi\right)\right] \quad \text{where } E_0 = f(\theta,\phi)\frac{e^{-j\beta r}}{r}$$

Utilizing the identity of

$$\sin(2x) = 2\sin(x)\cos(x)$$

we can write in the final form that

$$\frac{E}{E_0} = 4\sin\left(\frac{\beta s}{2}\sin\theta\cos\phi\right)\left[\cos\left(\frac{\beta s}{2}\sin\theta\cos\phi\right) - \cos\left(\frac{\sqrt{3}}{2}\beta s\sin\theta\sin\phi\right)\right]$$

Cont'd.

7.13 cont'd.

$\alpha = 45°$

Using the source arrangement of Figure c

$$E(r,\theta,\phi) = \left(\frac{e^{-j\beta r_1}}{r_1} + \frac{e^{-j\beta r_2}}{r_2} + \frac{e^{-j\beta r_3}}{r_3} + \frac{e^{-j\beta r_4}}{r_4} + \frac{e^{-j\beta r_5}}{r_5} + \frac{e^{-j\beta r_6}}{r_6} + \frac{e^{-j\beta r_7}}{r_7} + \frac{e^{-j\beta r_8}}{r_8}\right) f(\theta,\phi)$$

For far-field observations

For Phase Terms

$$r_1 \simeq r - s\cos\psi_1 = r - s(\hat{a}_x \cdot \hat{a}_r) = r - s\sin\theta\cos\phi$$
$$r_2 \simeq r - s\cos\psi_2 = r - s\left(\frac{1}{\sqrt{2}}(\hat{a}_x + \hat{a}_y)\cdot\hat{a}_r\right) = r - \frac{s}{\sqrt{2}}(\sin\theta\cos\phi + \sin\theta\sin\phi)$$
$$r_3 \simeq r - s\cos\psi_3 = r - s(\hat{a}_y \cdot \hat{a}_r) = r - s\sin\theta\sin\phi$$
$$r_4 \simeq r - s\cos\psi_4 = r - s\left[\frac{1}{\sqrt{2}}(-\hat{a}_x + \hat{a}_y)\cdot\hat{a}_r\right] = r + \frac{s}{\sqrt{2}}(\sin\theta\cos\phi - \sin\theta\sin\phi)$$
$$r_5 \simeq r - s\cos\psi_5 = r - s(-\hat{a}_x \cdot \hat{a}_r) = r + s\sin\theta\cos\phi$$
$$r_6 \simeq r - s\cos\psi_6 = r - s\left[\frac{1}{\sqrt{2}}(-\hat{a}_x - \hat{a}_y)\cdot\hat{a}_r\right] = r + \frac{s}{\sqrt{2}}(\sin\theta\cos\phi + \sin\theta\sin\phi)$$
$$r_7 \simeq r - s\cos\psi_7 = r - s(-\hat{a}_y \cdot \hat{a}_r) = r + s\sin\theta\sin\phi$$
$$r_8 \simeq r - s\cos\psi_8 = r - s\left[\frac{1}{\sqrt{2}}(+\hat{a}_x - \hat{a}_y)\cdot\hat{a}_r\right] = r - \frac{s}{\sqrt{2}}(\sin\theta\cos\phi - \sin\theta\sin\phi)$$

For Amplitude Terms

$$r_1 \simeq r_2 \simeq r_3 \simeq r_4 \simeq r_5 \simeq r_6 \simeq r_7 \simeq r_8 \simeq r$$

Making these substitutions and combining terms (first with fifth, second with sixth, third with seventh, fourth with eighth), we can write

$$\frac{E}{E_0} = 2\left[\cos(\beta s\sin\theta\cos\phi) + \cos(\beta s\sin\theta\sin\phi) - 2\cos\left(\frac{\beta s}{\sqrt{2}}\sin\theta\cos\phi\right)\cos\left(\frac{\beta s}{\sqrt{2}}\sin\theta\sin\phi\right)\right]$$

where $E_0 = f(\theta,\phi)\dfrac{e^{-j\beta r}}{r}$

$\alpha = 30°$

The procedure for this reflector follows those of the others. Using the geometry of Figure d

$$E = \left(\frac{e^{-j\beta r_1}}{r_1} + \frac{e^{-j\beta r_2}}{r_2} + \frac{e^{-j\beta r_3}}{r_3} + \frac{e^{-j\beta r_4}}{r_4} + \frac{e^{-j\beta r_5}}{r_5} + \frac{e^{-j\beta r_6}}{r_6} + \frac{e^{-j\beta r_7}}{r_7} + \frac{e^{-j\beta r_8}}{r_8} + \frac{e^{-j\beta r_9}}{r_9} + \frac{e^{-j\beta r_{10}}}{r_{10}} + \frac{e^{-j\beta r_{11}}}{r_{11}} + \frac{e^{-j\beta r_{12}}}{r_{12}}\right) f(\theta,\phi)$$

For far-field observations

For Phase Terms

$$r_1 \simeq r - s\cos\psi_1 = r - s(\hat{a}_x \cdot \hat{a}_r) = r - s\sin\theta\cos\phi$$
$$r_2 \simeq r - s\cos\psi_2 = r - s\left[\tfrac{1}{2}(\sqrt{3}\,\hat{a}_x + \hat{a}_y)\cdot\hat{a}_r\right] = r - \frac{s}{2}(\sqrt{3}\sin\theta\cos\phi + \sin\theta\sin\phi)$$
$$r_3 \simeq r - s\cos\psi_3 = r - s\left[\tfrac{1}{2}(\hat{a}_x + \sqrt{3}\,\hat{a}_y)\cdot\hat{a}_r\right] = r - \frac{s}{2}(\sin\theta\cos\phi + \sqrt{3}\sin\theta\sin\phi)$$

Cont'd.

7.13 cont'd.

$$r_4 \simeq r - s\cos\psi_4 = r - s(\hat{a}_y\cdot\hat{a}_r) = r - s\sin\theta\sin\phi$$

$$r_5 \simeq r - s\cos\psi_5 = r - s\left[\tfrac{1}{2}(-\hat{a}_x+\sqrt{3}\,\hat{a}_y)\cdot\hat{a}_r\right] = r - \tfrac{s}{2}(-\sin\theta\cos\phi+\sqrt{3}\,\sin\theta\sin\phi)$$

$$r_6 \simeq r - s\cos\psi_6 = r - s\left[\tfrac{1}{2}(-\sqrt{3}\,\hat{a}_x+\hat{a}_y)\cdot\hat{a}_r\right] = r - \tfrac{s}{2}(-\sqrt{3}\,\sin\theta\cos\phi+\sin\theta\sin\phi)$$

$$r_7 \simeq r - s\cos\psi_7 = r - s(-\hat{a}_x\cdot\hat{a}_r) = r + s\sin\theta\cos\phi$$

$$r_8 \simeq r - s\cos\psi_8 = r - s\left[\tfrac{1}{2}(-\sqrt{3}\,\hat{a}_x-\hat{a}_y)\cdot\hat{a}_r\right] = r + \tfrac{s}{2}(\sqrt{3}\,\sin\theta\cos\phi+\sin\theta\sin\phi)$$

$$r_9 \simeq r - s\cos\psi_9 = r - s\left[\tfrac{1}{2}(-\hat{a}_x-\sqrt{3}\,\hat{a}_y)\cdot\hat{a}_r\right] = r + \tfrac{s}{2}(\sin\theta\cos\phi+\sqrt{3}\,\sin\theta\sin\phi)$$

$$r_{10} \simeq r - s\cos\psi_{10} = r - s(-\hat{a}_y\cdot\hat{a}_r) = r + s\sin\theta\sin\phi$$

$$r_{11} \simeq r - s\cos\psi_{11} = r - s\left[\tfrac{1}{2}(\hat{a}_x-\sqrt{3}\,\hat{a}_y)\cdot\hat{a}_r\right] = r + \tfrac{s}{2}(-\sin\theta\cos\phi+\sqrt{3}\,\sin\theta\sin\phi)$$

$$r_{12} \simeq r - s\cos\psi_{12} = r - s\left[\tfrac{1}{2}(\sqrt{3}\,\hat{a}_x-\hat{a}_y)\cdot\hat{a}_r\right] = r + \tfrac{s}{2}(-\sqrt{3}\,\sin\theta\cos\phi+\sin\theta\sin\phi)$$

For Amplitude Terms

$$r_1 \simeq r_2 \simeq r_3 \simeq r_4 \simeq r_5 \simeq r_6 \simeq r_7 \simeq r_8 \simeq r_9 \simeq r_{10} \simeq r_{11} \simeq r_{12} \simeq r$$

Making these substitutions and combining terms (first with seventh, second with eighth, third with nineth, fourth with tenth, fifth with eleventh, sixth with twelveth), we can write

$$\frac{E}{E_0} = 2\left[\cos(\beta s\sin\theta\cos\phi) - 2\cos\left(\frac{\sqrt{3}}{2}\beta s\sin\theta\cos\phi\right)\cos\left(\frac{\beta s}{2}\sin\theta\sin\phi\right)\right.$$
$$\left. -\cos(\beta s\sin\theta\sin\phi + 2\cos\left(\frac{\beta s}{2}\sin\theta\cos\phi\right)\cos\left(\frac{\sqrt{3}}{2}\beta s\sin\theta\sin\phi\right)\right]$$

where $E_0 = f(\theta,\phi)\dfrac{e^{-j\beta r}}{r}$

7.14 The period of each is

$\alpha = 60^\circ$: $\Delta s = 2.0\lambda$

$\alpha = 45^\circ$: $\Delta s = 16.69\lambda$

$\alpha = 30^\circ$: $\Delta s = 30.0\lambda$

while the peak values are

$\alpha = 60^\circ$: $|E/E_0|_{max} = 5.2$

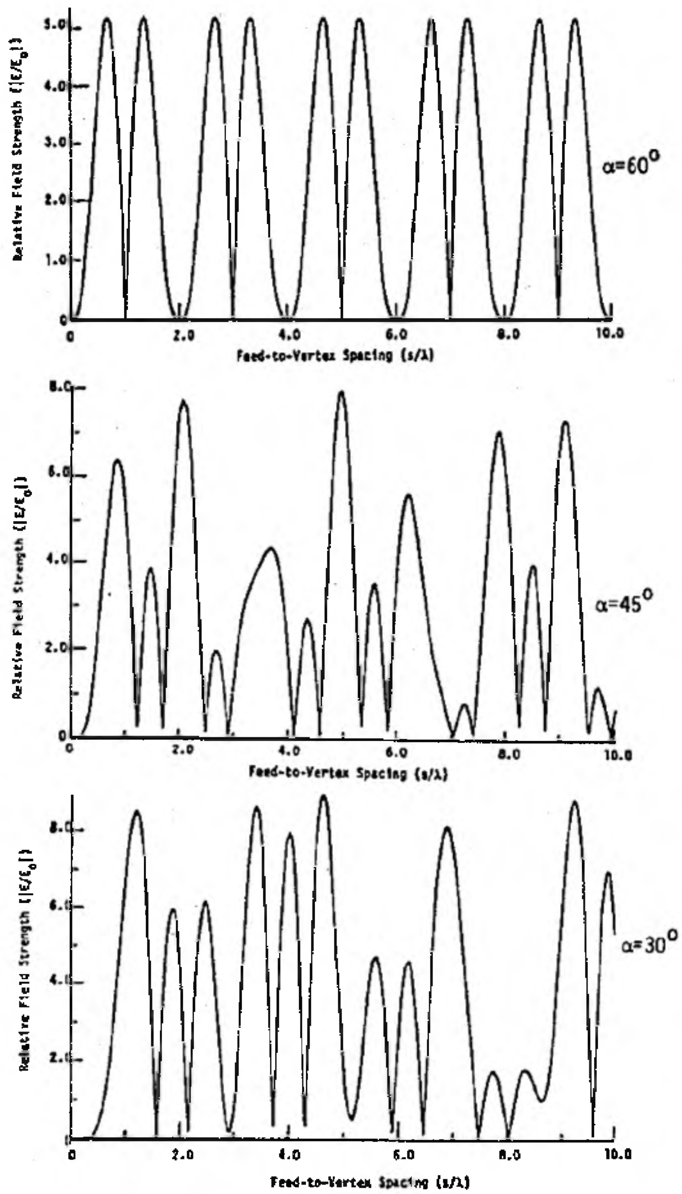
$\alpha = 45^\circ$: $|E/E_0|_{max} = 8.0$

$\alpha = 30^\circ$: $|E/E_0|_{max} = 9.0$

The plots of $|E/E_0|$ as a function of s in the range $0 \le s \le 10\lambda$ are shown in the next page

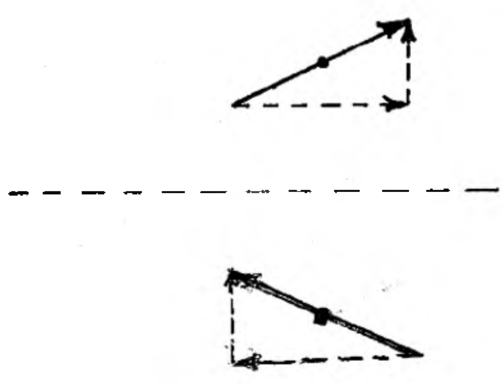
cont'd.

7.14 cont'd.

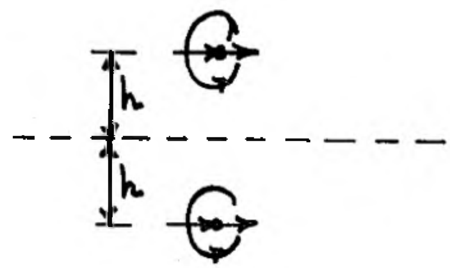


Relative field strengths along the axis ($\theta=90^\circ$, $\phi=0^\circ$) for $\alpha=60^\circ$, 45°, 30° corner reflectors as a function of feed-to-vertex spacing.

7.15



7.16 Since a small loop can be modeled as a linear magnetic dipole, as shown in the figure, then using the sources and images of Figure 7-2(a) we choose the image to be another magnetic horizontal linear dipole as shown in the figure. The corresponding small loop to represent the image is also shown.



7.17

a. The equivalent of a small circular loop is a linear magnetic dipole perpendicular to the plane of the loop ($\perp$ to the x-y plane) and pointed in the +z direction.

The image of the magnetic dipole above a PMC is a linear magnetic dipole in the same direction (+z) as the actual source.

Therefore the image is also a small electric loop with the same magnitude and phase as the actual source.

b. In the same direction as the actual source; CCW.

7.18

- To maintain the maximum along the z-axis, the field must add (in phase) with that of the actual dipole. Therefore the smallest phase (excluding zero) of the additional round trip of the reflected field must be 2π rads or $360°$. Thus

$$2\beta h = 2\pi \Rightarrow h = \frac{2\pi}{2\beta} = \frac{2\pi}{2(2\pi/\lambda)} = \frac{\lambda}{2}$$

$$\boxed{h = \lambda/2}$$

- $\lambda = 30\times10^9/10\times10^9 = 3\text{ cm} \Rightarrow \boxed{h = 1.5\text{ cm}}$

$\boxed{7.19}$ • To maintain the maximum along the z-axis, the field must add (in phase) with that of the actual dipole. Since the image is in the opposite direction of the actual source (dipole), the smallest phase of the round trip of the reflected field must be π rads or $180°$. Thus

$$2\beta h = \pi \Rightarrow h = \pi/2\beta = \pi/2(2\pi/\lambda) = \boxed{\lambda/4}$$

• $\lambda = 30\times10^9/(10\times10^9) = 3\text{ cm} \Rightarrow \boxed{h = 0.75\text{ cm}}$

$\boxed{7.20}$ • Since the source is an electric dipole, the image is in the opposite direction. Therefore smallest phase of the round trip of the reflected field must be π rads or $180°$. Thus

$$2\beta h = \pi \Rightarrow h = \pi/(2\beta) = \pi/[2(2\pi/\lambda)] = \boxed{\lambda/4}$$

• $\lambda = 30\times10^9/(10\times10^9) = 3\text{ cm} \Rightarrow \boxed{h = 0.75\text{ cm}}$

$\boxed{7.21}$ • Since the source is an electric dipole and the ground plane is a PMC, the image is in the same direction as the actual element. Therefore the smallest phase of the round trip of the reflected field must (excluding zero) 2π rads ($360°$). Thus

$$2\beta h = 2\pi \Rightarrow h = 2\pi/2\beta = 2\pi/[2(2\pi/\lambda)] = \boxed{\lambda/2}$$

• $\lambda = 30\times10^9/(10\times10^9) = 3\text{ cm} \Rightarrow \boxed{h = 1.5\text{ cm}}$

$\boxed{7.22}$

$$\underline{E}^i = \hat{a}_x E_o e^{-j\beta z} \qquad \underline{E}^r = \hat{a}_x \Gamma E_o e^{+j\beta z} \qquad \underline{E}^t = \hat{a}_x T E_o e^{-j\beta z}$$

$$\underline{H}^i = \hat{a}_y \frac{E_o}{\eta} e^{-j\beta z} \qquad \underline{H}^r = -\hat{a}_y \frac{\Gamma E_o}{\eta_o} e^{+j\beta z} \qquad \underline{H}^t = \frac{\hat{a}_y}{\eta} T E_o e^{-j\beta z}$$

$$\underline{J}_{eq} = j\omega(\varepsilon-\varepsilon_o)\underline{E}^t = \hat{a}_x j\omega(\varepsilon-\varepsilon_o) T E_o e^{-j\beta z}$$

$$\underline{M}_{eq} = j\omega(\mu-\mu_o)\underline{H}^t = \hat{a}_y j\omega(\mu-\mu_o)\frac{T E_o}{\eta} e^{-j\beta z}$$

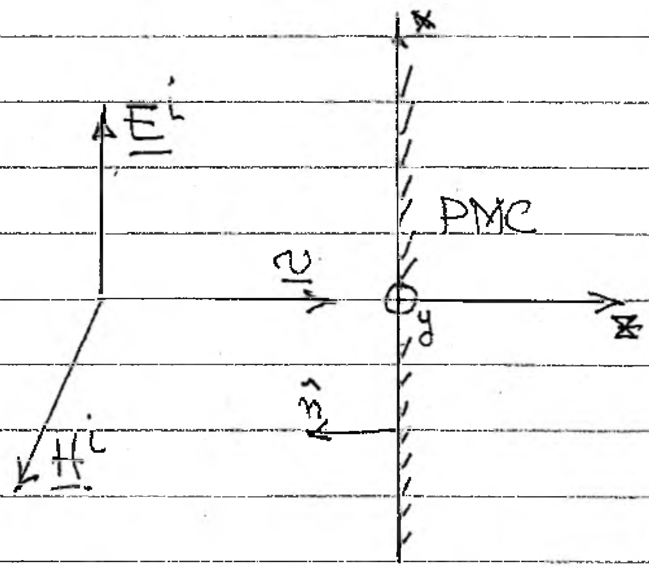
$$\Gamma = \frac{\eta-\eta_o}{\eta+\eta_o} \qquad T = \frac{2\eta}{\eta+\eta_o}$$

7.23

$\underline{E}_1^i = \hat{a}_x E_0 e^{-j\beta_0 z}$

a. $\underline{H}_1^i = \hat{a}_y \dfrac{E_0}{\eta_0} e^{-j\beta_0 z} = \hat{a}_y \dfrac{E_0}{377} e^{-j\beta_0 z}$

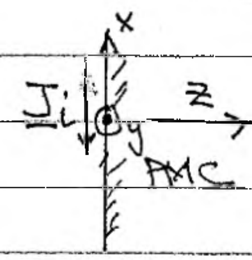
$\underline{H}_1^i = \hat{a}_y\, 2.6525 \times 10^{-3} E_0 e^{-j\beta_0 z}$



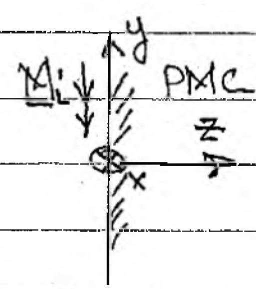
b. Using Figure 7-12(a), but with PMC instead of PEC, the current densities that you need are, according to Figure 7-12(b) with the PMC instead of PEC, are

$$\underline{J}_i = -\hat{n} \times \underline{H}_1 \Big|_{z=0} = (-\hat{a}_z) \times \left(\hat{a}_y \frac{E_0}{\eta_0} e^{-j\beta_0 z}\right)\Big|_{z=0} = \hat{a}_z \times \hat{a}_y \frac{E_0}{\eta_0} = -\hat{a}_x \frac{E_0}{\eta_0}$$

$$\underline{M}_i = \hat{n} \times \underline{E}_1 \Big|_{z=0} = -\hat{a}_z \times \hat{a}_x E_0 e^{-j\beta_0 z}\Big|_{z=0} = -\hat{a}_y E_0$$

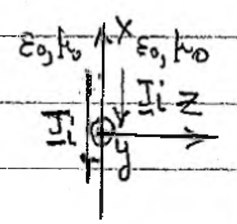


However since both of them are next (parallel) to a PMC, then $\underline{M}_i$ is shorted out by the PMC. The $\underline{J}_i$, which is also parallel to the PMC (flat and infinite in extent), using image theory of Figure 7-2(b), the image is the same as the actual source (magnitude and phase). Thus



$$\underline{J}_i \Big|_{z=0} = -2\hat{a}_x \frac{E_0}{\eta_0} = -\hat{a}_x 2 \left(2.6525 \times 10^{-3} E_0\right)$$

$$\underline{J}_i = -\hat{a}_x\, 5.305 \times 10^{-3} E_0$$



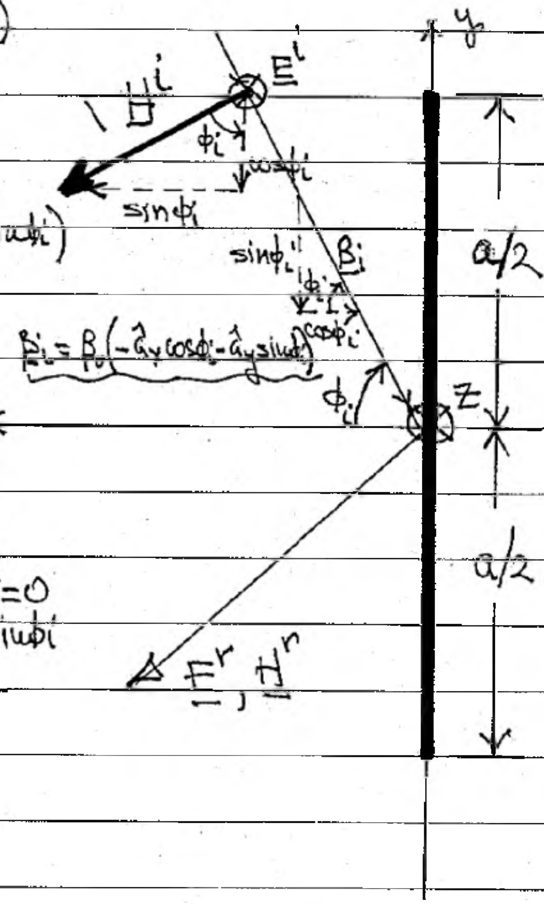
$\boxed{7.24}$ $\underline{E}^i = \hat{a}_z E_0 e^{-j\underline{\beta}_i \cdot \underline{r}}$

$= \hat{a}_z E_0 e^{-j[\beta(-x\cos\phi_i - y\sin\phi_i)(\hat{a}_x x' + \hat{a}_y y' + \hat{a}_z z')]}$

$\underline{E}^i = \hat{a}_z E_0 e^{+j\beta(x'\cos\phi_i + y'\sin\phi_i)}$

$\underline{H}^i = H_0(\hat{a}_x \sin\phi_i - \hat{a}_y \cos\phi_i) e^{+j\beta(x'\cos\phi_i + y'\sin\phi_i)}$

$H_0 = E_0/\eta$



(a) Induction Equivalent

$\underline{M}_i = 2\hat{n} \times \underline{E}^i \Big|_{x'=0} = 2\hat{a}_x \times (\hat{a}_z E^i)\Big|_{x'=0} = +2\hat{a}_y E^i \Big|_{x'=0}$

$\underline{M}_i = +\hat{a}_y 2E_0 e^{j\beta y'\sin\phi_i} \Rightarrow M_y = +2E_0 e^{j\beta y'\sin\phi_i}$

$M_x = M_z = J_x = J_y = J_z = 0$

$M_y = +2E_0 e^{j\beta y'\sin\phi_i}$

According to (6-125a)-(6-125e)

$N_\theta = N_\phi = 0$

$$L_\theta = \iint_{S'} [M_x \cos\theta_s \cos\phi_s + M_y \cos\theta_s \sin\phi_s - M_z \sin\theta_s] e^{j\beta r'\cos\psi} ds'$$

$$L_\phi = \iint_{S'} [-M_x \sin\phi_s + M_y \cos\phi_s] e^{j\beta r'\cos\psi} ds'$$

According to (6-125a)-(6-128d)

$$N_\theta = N_\phi = 0$$

$$L_\theta = \iint \left(\cancel{M_x} \cos\theta_s \cos\phi_s + M_y \cos\theta_s \sin\phi_s - \cancel{M_z \sin\theta_s} \right) e^{j\beta r' \cos\psi}\, ds'$$

$$= \int_{-a/2}^{a/2}\int_{-b/2}^{b/2} M_y \cos\theta_s \sin\phi_s\, e^{j\beta(y' \sin\theta_s \sin\phi_s + z' \cos\theta_s)}\, dy'\, dz'$$

$$L_\theta = +2E_0 \cos\theta_s \sin\phi_s \int_{-b/2}^{+b/2} e^{j\beta z' \cos\theta_s}\, dz' \int_{-a/2}^{+a/2} e^{j\beta(\sin\theta_s \sin\phi_s + \sin\phi_i) y'}\, dy'$$

According to
$$\int_{-c/2}^{c/2} e^{j\alpha z}\, dz = c\left[\frac{\sin\left(\frac{\alpha}{2}c\right)}{\frac{\alpha}{2}c}\right]$$

thus

$$L_\theta = +2E_0 \cos\theta_s \sin\phi_s \left[b\, \frac{\sin\left(\frac{\beta b}{2}\cos\theta_s\right)}{\frac{\beta b}{2}\cos\theta_s} \right] \left[a\, \frac{\sin\left[\frac{\beta a}{2}(\sin\theta_s \sin\phi_s + \sin\phi_i)\right]}{\frac{\beta a}{2}(\sin\theta_s \sin\phi_s + \sin\phi_i)} \right]$$

$$L_\theta = +2E_0\, ab \cos\theta_s \sin\phi_s \left[\frac{\sin(Y)}{Y}\right]\left[\frac{\sin(Z)}{Z}\right]$$

$$Y = \frac{\beta a}{2}(\sin\theta_s \sin\phi_s + \sin\phi_i)$$

$$Z = \frac{\beta b}{2}\cos\theta_s$$

$$L_\phi = \iint_S \left(-\cancelto{0}{M_x} \sin\phi_s + M_y \cos\phi_s\right) e^{j\beta r' \cos\psi}\, ds'$$

$$L_\phi = +2E_0 ab \cos\phi_s \left[\frac{\sin Y}{Y}\right]\left[\frac{\sin Z}{Z}\right]$$

$$E_\theta \simeq -j\frac{\beta e^{-j\beta r}}{4\pi r}\left(L_\phi + \eta \cancelto{0}{N_\theta}\right) = -j\frac{\beta e^{-j\beta r}}{4\pi r} L_\phi$$

$$E_\phi \simeq +j\frac{\beta e^{-j\beta r}}{4\pi r}\left(L_\theta - \eta \cancelto{0}{N_\phi}\right) = +j\frac{\beta e^{-j\beta r}}{2\pi r} L_\theta$$

$$E_\theta \simeq -j\frac{\beta e^{-j\beta r}}{4\pi r}\left[+2E_0 ab \cos\phi_s\right]\left[\frac{\sin(Y)}{Y}\right]\left[\frac{\sin(Z)}{Z}\right]$$

$$\boxed{E_\theta \simeq -j\frac{abE_0 \beta e^{-j\beta r}}{2\pi r}\left[\cos\phi_s \frac{\sin Y}{Y}\frac{\sin Z}{Z}\right]}$$

$$E_\phi \simeq +j\frac{\beta e^{-j\beta r}}{4\pi r} L_\theta = +j\frac{\beta e^{-j\beta r}}{4\pi r}\left[+2E_0 ab \cos\theta_s \sin\phi_s\right]\left[\frac{\sin Y}{Y}\right]\left[\frac{\sin Z}{Z}\right]$$

$$\boxed{E_\phi \simeq +j\frac{abE_0 \beta e^{-j\beta r}}{2\pi r}\left[\cos\theta_s \sin\phi_s \frac{\sin Y}{Y}\frac{\sin Z}{Z}\right]}$$

$$Y = \frac{\beta a}{2}\left[\sin\theta_s \sin\phi_s + \sin\phi_i\right]$$

$$Z = \frac{\beta b}{2}\cos\theta_s$$

For backscattering: $\theta_s = \pi/2$, $\phi_s = \phi_i$:

$$E_\theta \simeq j\frac{abE_0 \beta e^{-j\beta r}}{2\pi r}\left[\cos\phi_i \frac{\sin(\beta a \sin\phi_i)}{\beta a \sin\phi_i}\right]$$

$$E_\phi \simeq +j\frac{abE_0 \beta e^{-j\beta r}}{2\pi r}\left[0\right] = 0$$

(b) Physical Equivalent:

$$\underline{J}_p = 2\hat{n} \times \underline{H}^i \Big|_{x'=0}$$

$$\underline{H}^i = \frac{E_o}{\eta}\left(\hat{a}_x \sin\phi_i - \hat{a}_y \cos\phi_i\right) e^{j\beta(x'\cos\phi_i + y'\sin\phi_i)}$$

$$\underline{J}_p = 2\hat{a}_x \times \left(\frac{E_o}{\eta}\right)\left(\hat{a}_x \sin\phi_i - \hat{a}_y \cos\phi_i\right) e^{j\beta(x'\cos\phi_i + y'\sin\phi_i)}\Big|_{x'=0}$$

$$\underline{J}_p = -\hat{a}_z 2\frac{E_o}{\eta}\cos\phi_i\, e^{j\beta y'\sin\phi_i}$$

$$J_z = -2\frac{E_o}{\eta}\cos\phi_i\, e^{j\beta y'\sin\phi_i}, \quad J_x = J_y = 0$$

$$M_x = M_y = M_z = 0$$

Using (6-125a)-(6-128c)

$$L_\theta = L_\phi = 0$$

$$N_\theta = \iint_S \left[\cancel{J_x}^{\,0} \cos\theta_s \cos\phi_s + \cancel{J_y}^{\,0} \cos\theta_s \sin\phi_s - J_z \sin\theta_s\right] e^{j\beta r'\cos\psi}\, ds'$$

$$N_\phi = \iint_S \left[-\cancel{J_x}^{\,0} \sin\phi_s + \cancel{J_y}^{\,0} \cos\phi_s\right] e^{j\beta r'\cos\psi}\, ds' = 0$$

$$N_\theta = \int_{-b/2}^{b/2}\int_{-a/2}^{a/2} -J_z \sin\theta_s\, e^{j\beta r'\cos\psi}\, ds' = +2\frac{E_o}{\eta}\cos\phi_i \sin\theta_s \int_{-b/2}^{+b/2} e^{j\beta z'\cos\theta_s}\, dz' \int_{-a/2}^{+a/2} e^{j\beta(\sin\theta_s \sin\phi_s + \sin\phi_i)y'}\, dy'$$

$$N_\theta = +2\frac{E_o}{\eta}\cos\phi_i \sin\theta_s \left[b\,\frac{\sin\left(\frac{\beta b}{2}\cos\theta_s\right)}{\frac{\beta b}{2}\cos\theta_s}\right]\left[a\,\frac{\sin\left[\frac{\beta a}{2}(\sin\theta_s \sin\phi_s + \sin\phi_i)\right]}{\frac{\beta a}{2}(\sin\theta_s \sin\phi_s + \sin\phi_i)}\right]$$

$$N_\theta = +2\frac{E_o ab}{\eta}\cos\phi_i \sin\theta_s \left[\frac{\sin Y}{Y}\right]\left[\frac{\sin Z}{Z}\right]$$

$$E_\theta \simeq -j\frac{\beta e^{-j\beta r}}{4\pi r}\left[\cancel{L_\phi}^{\,0}+\eta N_\theta\right] = -j\eta\frac{\beta e^{-j\beta r}}{4\pi r}N_\theta$$

$$\simeq -j\eta\frac{\beta e^{-j\beta r}}{4\pi r}\left[+2\frac{E_0 ab}{\eta}\cos\phi_i\sin\theta_s\left[\frac{\sin Y}{Y}\right]\left[\frac{\sin Z}{Z}\right]\right]$$

$$\boxed{E_\theta \simeq -j\frac{abE_0\beta}{2\pi r}\left[\cos\phi_i\sin\theta_s\left[\frac{\sin Y}{Y}\right]\left[\frac{\sin Z}{Z}\right]\right] \qquad E_\phi \simeq +j\frac{\beta e^{-j\beta r}}{4\pi r}\left[\cancel{L_\theta}^{\,0}-\eta\cancel{N_\phi}^{\,0}\right] = 0}$$

$$Y = \frac{\beta a}{2}\left[\sin\theta_s\sin\phi_s + \sin\phi_i\right]$$

$$Z = \frac{\beta b}{2}\cos\theta_s$$

For backscattering: $\theta_s = \pi/2,\ \phi_s = \phi_i$

$$E_\theta \simeq -j\frac{abE_0\beta}{2\pi r}\left[\cos\phi_i\,\frac{\sin(\beta a\sin\phi_i)}{\beta a\sin\phi_i}\right]$$

(which, for backscattering is the same as for the Induction Equivalent. The solutions are not the same in other directions.)

7.25 $\underline{E}_a = \hat{a}_z E_0, \underline{M}_s = -2\hat{n} \times \underline{E}_a = -2\hat{a}_x \times \hat{a}_z E_0 = \hat{a}_y 2E_0$ Thus
$M_y = 2E_0, M_x = M_z = J_x = J_y = J_z = 0$
$N_\theta = N_\phi = 0$

$$L_\theta = \iint_{Sa} [\cancel{M_x}^0 \cos\theta\cos\phi + M_y \cos\theta \sin\phi - \cancel{M_z}^0 \sin\theta] e^{j\beta(y'\sin\theta\sin\phi + z'\cos\theta)}\, dy'\, dz'$$

$$= 2E_0 \cos\theta \cdot \sin\phi \int_{-a/2}^{a/2} e^{j\beta y'\sin\theta\sin\phi}\, dy' \int_{-b/2}^{b/2} e^{j\beta z'\cos\theta}\, dz'$$

$$L_\theta = 2E_0 ab \left[\cos\theta \sin\phi \frac{\sin Y}{Y}\frac{\sin Z}{Z}\right], Y = \frac{\beta a}{2}\sin\theta \cdot \sin\phi, Z = \frac{\beta b}{2}\cos\theta$$

$$L_\phi = \iint_{Sa} [-\cancel{M_x}^0 \sin\phi + M_y \cos\phi] e^{j\beta(y'\sin\theta\cdot\sin\phi + z'\cos\theta)}\, dy'\, dz'$$

$$= 2E_0 \cdot a \cdot b \left[\cos\phi \cdot \frac{\sin Y}{Y} \cdot \frac{\sin Z}{Z}\right]$$

$\vdots$

$E_r \simeq 0$

$$E_\theta \simeq -j\frac{\beta e^{-j\beta r}}{4\pi r}[L_\phi + \eta \cancel{N_\theta}^0] = -j\frac{ab\beta E_0 e^{-jkr}}{2\pi r}\left[\cos\phi \cdot \frac{\sin Y}{Y}\frac{\sin Z}{Z}\right]$$

$$E_\phi \simeq j\frac{\beta e^{-j\beta r}}{4\pi r}[L_\theta - \eta \cancel{N_\phi}^0] = +j\frac{ab\beta E_0 e^{-j\beta r}}{2\pi r}\left[+\cos\theta \cdot \sin\phi \frac{\sin Y}{Y} \cdot \frac{\sin Z}{Z}\right]$$

$$H_r \simeq 0, \quad H_\theta \simeq -\frac{E_\phi}{\eta}, \quad H_\phi \simeq +\frac{E_\theta}{\eta}$$

7.26 Assuming the magnetic field at the aperture is related to the electric field by the wave impedance Z_a

$\underline{H}_a = -\hat{a}_y \frac{E_0}{Z_a}$, and using the equivalent model of 7.8(c) and an equivalent procedure as that outlined in Figure 7.10 but using a PMC ground plane (instead of a PEC), the equivalent current densities to represent the aperture and PMC ground plane are

$$\underline{J}_s = 2\hat{n}\times\underline{H}_a = 2\hat{a}_x \times \left(-\hat{a}_y \frac{E_0}{Z_a}\right) = -\hat{a}_z \frac{2E_0}{Z_a} \Rightarrow J_z = -\frac{2E_0}{Z_a}, \quad J_x = J_y = 0$$

$\underline{M}_s = 0$

$$N_\theta = \iint_S -J_z \sin\theta e^{j\beta r'\cos\psi}\, ds' = -\frac{2E_0 ab}{Z_a}\left[\sin\theta \frac{\sin Y}{Y}\frac{\sin Z}{Z}\right] \quad Y = \frac{\beta a}{2}\sin\theta\sin\phi, \; Z = \frac{\beta b}{2}\cos\theta$$

$N_\phi = L_\theta = L_\phi = 0$

$$E_\theta = j\frac{ab\beta e^{-j\beta r}}{2\pi r}\left(\frac{\eta}{Z_a}\right)\left[\sin\theta \frac{\sin Y}{Y}\frac{\sin Z}{Z}\right], \quad E_\phi = 0$$

$$H_\theta = \frac{-E_\phi}{0} = 0, \quad H_\phi = \frac{E_\theta}{\eta}$$

$\boxed{7.27}$ $\underline{E}_a = \hat{a}_z E_0, \underline{M}_s = -\hat{n} \times \underline{E}_a = \hat{a}_y E_0 \Rightarrow M_x = M_z = 0, M_y = E_0$

$$\underline{H}_a = -\hat{a}_y \frac{E_0}{Z_a}, \underline{J}_s = \hat{n} \times \underline{H}_a = \hat{a}_x \times \left(-\hat{a}_y \frac{E_0}{Z_a}\right) = -\hat{a}_z \frac{E_0}{Z_a}$$

$$\Rightarrow J_x = J_y = 0, J_z = -E_0/Z_a$$

From Problem 7.25, without the factor of 2

$$L_\theta = E_0 \cdot a \cdot b \left[\cos\theta \sin\phi \cdot \frac{\sin Y}{Y} \cdot \frac{\sin Z}{Z}\right], Y = \frac{\beta a}{2} \sin\theta \sin\phi, Z = \frac{\beta b}{2} \cos\theta$$

$$L_\phi = E_0 \cdot ab \left[\cos\phi \cdot \frac{\sin Y}{Y} \cdot \frac{\sin Z}{Z}\right]$$

$$N_\theta = \iint_{Sa} [\cancel{J_x}^0 \cos\theta\cos\phi + \cancel{J_y}^0 \cos\theta\sin\phi - J_z \sin\theta] e^{j\beta(y'\sin\theta\sin\phi + z'\cos\theta)}\, dy'\, dz'$$

$$= +\frac{E_0 ab}{Z_a}\left[\sin\theta \cdot \frac{\sin Y}{Y} \cdot \frac{\sin Z}{Z}\right]$$

$$N_\phi = \iint_{Sa} [-\cancel{J_x}^0 \sin\phi + \cancel{J_y}^0 \cos\phi] e^{j\beta(y'\sin\theta\sin\phi + z'\cos\theta)}\, dy'\, dz' = 0$$

$$E_r \simeq 0$$

$$E_\theta \simeq -j\frac{\beta e^{-j\beta r}}{4\pi r}[L_\phi + \eta N_\theta] = -j\frac{\beta ab E_0 e^{-j\beta r}}{4\pi r}\left[\left(\cos\phi + \left(\frac{\eta}{Z_a}\right)\sin\theta\right)\frac{\sin Y}{Y} \cdot \frac{\sin Z}{Z}\right]$$

$$E_\phi \simeq j\frac{\beta e^{-j\beta r}}{4\pi r}[L_\theta - \eta \cancel{N_\phi}^0] = -j\frac{\beta ab E_0 e^{-j\beta r}}{4\pi r}\left[-\cos\theta\sin\phi \cdot \frac{\sin Y}{Y} \cdot \frac{\sin Z}{Z}\right]$$

$$H_r \simeq 0$$

$$H_\theta \simeq -E_\phi/\eta$$

$$H_\phi \simeq E_\theta/\eta$$

$\boxed{7.28}$

The only difference between this problem and 7.25 is that for the y variations the integral reduces to

$$\int_{-a/2}^{a/2} \cos\left(\frac{\pi}{a}y'\right) e^{j\beta y'\sin\theta\sin\phi} dy' = -\left(\frac{\pi a}{2}\right) \frac{\cos\left(\frac{\beta a}{2}\sin\theta\sin\phi\right)}{\left(\frac{\beta a}{2}\sin\theta\sin\phi\right)^2 - \left(\frac{\pi}{2}\right)^2}$$

Thus

$$E_r \simeq 0$$

$$E_\theta \simeq -j\frac{\beta e^{-j\beta r}}{4\pi r} L_\phi = j\frac{ab\beta E_0 e^{-j\beta r}}{4r}\left[\cos\phi \cdot \frac{\cos Y}{(Y)^2 - (\pi/2)^2} \cdot \frac{\sin Z}{Z}\right], Y = \frac{\beta a}{2}\sin\theta\sin\phi$$

$$E_\phi \simeq j\frac{\beta e^{-j\beta r}}{4\pi r} \cdot L_\theta = +j\frac{ab\beta E_0 \cdot e^{-j\beta r}}{4r}\left[-\cos\theta \cdot \sin\phi \frac{\cos Y}{(Y)^2 - (\pi/2)^2} \cdot \frac{\sin Z}{Z}\right],$$

$$Z = \frac{\beta b}{2}\cos\theta$$

$$H_r \simeq 0, \quad H_\theta \simeq -E_\phi/\eta, \quad H_\phi \simeq E_\theta/\eta$$

$\boxed{7.29}$ Assuming the magnetic field at the aperture is related to the electric field by the wave impedance Z_a

$$\underline{H}_a = -\hat{a}_y \frac{E_o}{Z_a} \cos\left(\frac{\pi}{a} y'\right)$$

and using the equivalent model of 7.8(c) and an equivalent procedure as that outlined in Figure 7.10 but using a PMC ground plane (instead of a PEC), the equivalent current densities to represent the aperture and PMC ground plane are

$$\underline{J}_s = 2\hat{n} \times \underline{H}_a = 2\hat{a}_x \times \left[-\hat{a}_y \frac{E_o}{Z_a} \cos\left(\frac{\pi}{a} y'\right)\right] = -\hat{a}_z \frac{2E_o}{Z_a} \cos\left(\frac{\pi}{a} y'\right)$$

$$\underline{J}_z = -\frac{2E_o}{Z_a} \cos\left(\frac{\pi}{a} y'\right), \quad J_x = J_y = \underline{M}_s = 0$$

Using the integration with respect to y' from the solution of Problem 7.28

$$\int_{-a/2}^{+a/2} \cos\left(\frac{\pi}{a} y'\right) e^{j\beta y' \sin\theta \sin\phi} \, dy' = -\left(\frac{\pi a}{2}\right) \frac{\cos\left(\frac{\beta a}{2} \sin\theta \sin\phi\right)}{\left(\frac{\beta a}{2} \sin\theta \sin\phi\right)^2 - \left(\frac{\pi}{2}\right)^2}$$

$$N_\theta = \iint_S -J_z \sin\theta \, e^{j\beta r' \cos\psi} \, ds' = \frac{2E_o ab\pi}{2Z_a} \left[\sin\theta \frac{\cos(Y)}{(Y)^2 - \left(\frac{\pi}{2}\right)^2} \frac{\sin Z}{Z}\right]$$

$$N_\phi = L_\theta = L_\phi = 0, \quad Y = \frac{\beta a}{2} \sin\theta \sin\phi, \quad Z = \frac{\beta b}{2} \cos\theta$$

$$E_\theta = -j \frac{ab\beta e^{-j\beta r}}{2 \, Y} \left(\frac{\eta}{Z_a}\right) \left[\sin\theta \frac{\cos(Y)}{(Y)^2 - (\pi/2)^2} \frac{\sin(Z)}{Z}\right], \quad E_\phi = 0, \; H_\phi = E_\theta/\eta, \; H_\theta = -\frac{E_\phi}{\eta} = 0$$

$\boxed{7.30}$ Since the aperture is not mounted on a ground plane, then both electric ($\underline{J}_s$) and magnetic ($\underline{M}_s$) current densities must be used, as in Problem 7.27. Using the solution of Problem 7.27 and integration of the solutions of Problems 7.28 and 7.29 we can write (with $Z_a = \eta$)

$$E_\theta \simeq j \frac{ab\beta E_0 e^{-j\beta r}}{8r} \left[(\cos\phi + \sin\theta) \frac{\cos Y}{(Y)^2 - (\pi/2)^2} \frac{\sin Z}{Z}\right]$$

$$E_\phi \simeq -j \frac{ab\beta E_0 e^{-j\beta r}}{8r} \left[+\cos\theta \cdot \sin\phi \cdot \frac{\cos Y}{(Y)^2 - \left(\frac{\pi}{2}\right)^2}\right]$$

$$Y = \frac{\beta a}{2} \sin\theta \sin\phi, \; Z = \frac{\beta b}{2} \cos\theta, \; H_r \simeq 0, \; H_\theta \simeq -E_\phi/\eta, \; H_\phi \simeq E_\theta/\eta$$

7.31 $\underline{E}_a = \hat{a}_x E_0,\ M_s = -2\hat{n}\times\underline{E}_a = -2\hat{a}_y\times\hat{a}_x E_0 = \hat{a}_z 2E_0$

Thus $M_z = 2E_0,\ M_x = M_y = J_x = J_y = J_z = 0$

$N_\theta = N_\phi = 0$

$$L_\theta = \iint_{Sa} [\cancel{M_x}^0 \cos\theta\cdot\cos\phi + \cancel{M_y}^0 \cos\theta\sin\phi - M_z\cdot\sin\theta] e^{j\beta(x'\sin\theta\cos\phi + z'\cos\theta)}\, dx'\, dz'$$

$$= -2E_0\sin\theta\int_{-a/2}^{a/2} e^{j\beta z'\cos\theta}\, dz' \int_{-b/2}^{b/2} e^{j\beta x'\sin\theta\cos\phi}\, dx'$$

$$L_\theta = -2E_0 ab\left[\sin\theta\cdot\frac{\sin x}{x}\cdot\frac{\sin Z}{Z}\right], X = \frac{\beta b}{2}\sin\theta\cos\phi, Z = \frac{\beta a}{2}\cos\theta$$

$$L_\phi = \iint_{Sa} [-\cancel{M_x}^0\sin\phi + \cancel{M_y}^0\cos\phi] e^{j\beta(x'\sin\theta\cos\phi + z'\cos\theta)}\, dx'\, dz' = 0$$

$$E_r \simeq 0,\quad E_\theta \simeq -\frac{j\beta e^{-j\beta r}}{4\pi r}[\cancel{L_\phi}^0 + \eta\cancel{N_\theta}^0] = 0$$

$$E_\phi \simeq j\frac{\beta e^{-j\beta r}}{4\pi r}[L_\theta - \eta\cancel{N_\phi}^0] = -j\frac{ab\beta E_0 e^{-j\beta r}}{2\pi r}\left[\sin\theta\cdot\frac{\sin X}{X}\cdot\frac{\sin Z}{Z}\right]$$

$$H_r \simeq 0,\quad H_\theta \simeq -\frac{E_\phi}{\eta},\quad H_\phi \simeq \frac{E_\theta}{\eta} = 0$$

7.32 Assuming the magnetic field at the aperture is related to the electric field by the wave impedance Z_a $\underline{H}_a = -\hat{a}_z\frac{E_0}{Z_a}$, and using the equivalent model of 7.8(c) and an equivalent procedure as that outlined in Figure 7.10 but using a PMC ground plane (instead of a PEC), the equivalent current densities to represent the aperture and PMC ground plane are

$\underline{J}_s = 2\hat{n}\times\underline{H}_a = 2\hat{a}_y\times\left(-\hat{a}_z\frac{E_0}{Z_a}\right) = -\hat{a}_x\frac{2E_0}{Z_a},\ J_y = J_z = 0$

$\underline{M}_s = 0$

$$N_\theta = \iint_S J_x\cos\theta\cos\phi\, e^{j\beta r'\cos\psi}\, ds' = -\frac{2E_0}{Z_a}\cos\theta\cos\phi\,\frac{\sin X}{X}\,\frac{\sin Z}{Z}$$

$$N_\phi = \iint_S -J_x\sin\phi\, e^{j\beta r'\cos\psi}\, ds' = +\frac{2E_0}{Z_a}\sin\phi\,\frac{\sin X}{X}\,\frac{\sin Z}{Z}$$

$$E_\theta = -\frac{j\beta e^{-j\beta r}}{2\pi r}\left(\frac{\eta}{Z_a}\right)\left[\cos\theta\cos\phi\,\frac{\sin X}{X}\,\frac{\sin Z}{Z}\right],\quad X = \frac{\beta b}{2}\sin\theta\cos\phi,\ Z = \frac{\beta a}{2}\cos\theta$$

$H_\theta = -E_\phi/\eta,\ H_\phi = E_\theta/\eta$

$\boxed{7.33}$

$$\mathbf{E}_a = \hat{a}_x E_0, \quad \underline{M}_s = -\hat{n} \times \underline{E}_a = \hat{a}_z E_0 \Rightarrow M_x = M_y = 0, \quad M_z = E_0$$

$$\underline{H}_a = -\hat{a}_z \frac{E_0}{Z_a}, \quad \underline{J}_s = \hat{n} \times \underline{H}_a = \hat{a}_y \times \left(-\hat{a}_z \frac{E_0}{Z_a}\right) = -\hat{a}_x \frac{E_0}{Z_a} \Rightarrow J_y = J_z = 0,$$

$$J_x = -\frac{E_0}{Z_a}$$

From Problem 7.31, without the factor of 2

$$L_\theta = -E_0 ab \left[\sin\theta \cdot \frac{\sin X}{X} \cdot \frac{\sin Z}{Z}\right]$$

$$L_\phi = 0$$

$$X = \frac{\beta b}{2}\sin\theta\cos\phi, Z = \frac{\beta a}{2}\cos\theta$$

$$N_\theta = \iint_{Sa} [J_x \cos\theta\cos\phi + \cancel{J_y}^0 \cos\theta\sin\phi - \cancel{J_z}^0 \sin\theta] e^{j\beta(x'\sin\theta\cos\phi + z'\cos\theta)}\, dx'\, dz'$$

$$= -\frac{E_0}{Z_a} ab \left[\cos\theta\cos\phi \cdot \frac{\sin X}{X} \cdot \frac{\sin Z}{Z}\right]$$

$$N_\phi = \iint_{Sa} [-J_x \sin\phi + J_y\cos\phi] e^{j\beta(x'\sin\theta\cos\phi + z'\cos\theta)}\, dx'\, dz'$$

$$= \frac{E_0}{Z_a} ab \left[\sin\phi \cdot \frac{\sin X}{X} \cdot \frac{\sin Z}{Z}\right]$$

$$E_r \simeq 0$$

$$E_\theta \simeq -j\frac{\beta e^{-j\beta r}}{4\pi r}[\cancel{L_\phi}^0 + \eta N_\theta] = j\frac{\beta ab E_0 e^{-j\beta r}}{4\pi r}\left[\cos\theta\cos\phi \frac{\sin X}{X}\frac{\sin Z}{Z}\right]\left(\frac{\eta}{Z_a}\right)$$

$$E_\phi \simeq j\frac{\beta e^{-j\beta r}}{4\pi r}[L_\theta - \eta N_\phi] = j\frac{\beta ab E_0 e^{-j\beta r}}{4\pi r}\left[-\left(\sin\theta + \left(\frac{\eta}{Z_a}\right)\sin\phi\right) \cdot \frac{\sin X}{X} \cdot \frac{\sin Z}{Z}\right]$$

$$H_r \simeq 0, H_\theta \simeq -E_\phi/\eta, H_\phi \simeq E_\theta/\eta$$

$\boxed{7.34}$

The only difference between this problem and 7.31 is that for the z variations the integral reduces to

$$\int_{-a/2}^{a/2} \cos\left(\frac{\pi}{a}z'\right) e^{j\beta z'\cos\theta} dz' = -\left(\frac{\pi a}{2}\right) \frac{\cos\left(\frac{\beta a}{2}\cos\theta\right)}{\left(\frac{\beta a}{2}\cos\theta\right)^2 - \left(\frac{\pi}{2}\right)^2}$$

$$E_r \simeq 0$$

$$E_\theta \simeq 0$$

$$E_\phi \simeq j\frac{\beta e^{-j\beta r}}{4\pi r}L_\theta = j\frac{ab\beta E_0 e^{-j\beta r}}{4r}\left[\sin\theta \frac{\sin X}{X}\frac{\cos Z}{(Z)^2 - \left(\frac{\pi}{2}\right)^2}\right]$$

$$X = \frac{\beta b}{2}\sin\theta\cos\phi, \quad Z = \frac{\beta a}{2}\cos\theta$$

$$H_r \simeq 0, \quad H_\theta \simeq -E_\phi/\eta, \quad H_\phi \simeq E_\theta/\eta = 0$$

$\boxed{7.35}$ Assuming the magnetic field at the aperture is related to the electric field by the wave impedance Z_a

$$\underline{H}_a = -\hat{a}_x \frac{E_0}{Z_a}\cos\left(\frac{\pi}{a}z'\right)$$

and using the equivalent model of 7.8(c) and an equivalent procedure as that outlined in Figure 7.10 but using a PMC ground plane (instead of a PEC), the equivalent current densities to represent the aperture and the PMC ground plane are

$$\underline{J}_s = 2\hat{n}\times\underline{H}_a = 2\hat{a}_y \times\left[-\hat{a}_x\frac{E_0}{Z_a}\cos\left(\frac{\pi}{a}z'\right)\right] = \hat{a}_z\frac{2E_0}{Z_a}\cos\left(\frac{\pi}{a}z'\right)$$

$$J_z = \frac{2E_0}{Z_a}\cos\left(\frac{\pi}{a}z'\right),\quad J_x = J_y = \underline{M}_s = 0$$

Using the integration with respect to z' from the solution of Problem 7.34

$$\int_{-a/2}^{+a/2}\cos\left(\frac{\pi}{a}z'\right)e^{j\beta z'\cos\theta}\,dz' = -\left(\frac{\pi a}{2}\right)\frac{\cos\left(\frac{\beta a}{2}\cos\theta\right)}{\left(\frac{\beta a}{2}\cos\theta\right)^2-\left(\frac{\pi}{2}\right)^2}$$

$$N_\theta = \iint -J_z\sin\theta\, e^{j\beta r'\cos\psi}\,ds' = \frac{2ab\pi E_0}{2Z_a}\left[\sin\theta\,\frac{\sin X}{X}\,\frac{\cos Z}{(Z)^2-(\pi/2)^2}\right]$$

$$N_\phi = L_\theta = L_\phi = 0,\quad X = \left(\frac{\beta b}{2}\sin\theta\cos\phi\right),\quad Z = \frac{\beta a}{2}\cos\theta$$

$$E_\theta = -j\frac{\beta abE_0e^{-j\beta r}}{4r}\left[\sin\theta\,\frac{\sin X}{X}\,\frac{\cos(Z)}{(Z)^2-(\pi/2)^2}\right]$$

$$E_\phi = 0,\quad H_\theta = -E_\phi/\eta = 0,\quad H_\phi = E_\theta/\eta$$

$\boxed{7.36}$ Since the aperture is not mounted on a ground plane, then both the electric ($\underline{J}_s$) and magnetic ($\underline{M}_s$) current densities must be used, as in Problem 7.33. Using the solution of Problem 7.33 and integration of the solutions of Problems 7.33 and 7.34, we can write (with $Z_a = \eta$)

$$E_\theta \simeq j\frac{kabE_0e^{-jkr}}{8r}\left[\cos\theta\cos\phi\cdot\frac{\sin X}{X}\cdot\frac{\cos Z}{(Z)^2-(\pi/2)^2}\right]$$

$$E_\phi \simeq j\frac{kabE_0e^{-jkr}}{8r}\left[-(\sin\theta+\sin\phi)\cdot\frac{\sin X}{X}\cdot\frac{\cos Z}{(Z)^2-(\pi/2)^2}\right]$$

$$H_\theta \simeq -E_\phi/\eta$$

$$H_\phi \simeq E_\theta/\eta$$

$$X = \frac{kb}{2}\sin\theta\cdot\cos\phi,\quad Z = \frac{ka}{2}\cos\theta$$

7.37 $\quad \underline{E}_a = \hat{a}_y E_0$

(a) $\underline{M}_s = 2\hat{n}\times\underline{E}_a = 2\hat{a}_z \times (\hat{a}_y E_0) = +\hat{a}_x 2E_0 \Rightarrow M_x = +2E_0$

$\underline{J}_s = 0$

(b) $N_\theta = N_\phi = 0$

$$L_\theta = \iint_S M_x \cos\theta\cos\phi\, e^{j\beta r'\cos\psi}\, ds' = +2E_0\cos\theta\cos\phi \int_{-a/2}^{+a/2} e^{j\beta x'\sin\theta\cos\phi}\,dx' \int_{-b/2}^{+b/2} e^{j\beta y'\sin\theta\sin\phi}\,dy'$$

$$L_\theta = +2E_0 ab\left[\cos\theta\cos\phi\,\frac{\sin(X)}{X}\,\frac{\sin(Y)}{Y}\right],\quad X = \frac{\beta a}{2}\sin\theta\cos\phi,\ Y = \frac{\beta b}{2}\sin\theta\sin\phi$$

$$L_\phi = \iint_S -M_x\sin\phi = -2E_0 ab\left[\sin\phi\,\frac{\sin X}{X}\,\frac{\sin Y}{Y}\right]$$

$$E_\theta = +\frac{j\beta abE_0 e^{-j\beta r}}{2\pi r}\left[\sin\phi\,\frac{\sin X}{X}\,\frac{\sin Y}{Y}\right]$$

$$E_\phi = +j\frac{\beta abE_0 e^{-j\beta r}}{2\pi r}\left[\cos\theta\cos\phi\,\frac{\sin X}{X}\,\frac{\sin Y}{Y}\right]$$

7.38 Assuming the magnetic field at the aperture is related to the electric field by the wave impedance Z_a, $\underline{H}_a = -\hat{a}_x \frac{E_0}{Z_a}$, and using the equivalent model of 7.8(c) and an equivalent procedure as that outlined in Figure 7.10 but using a PMC ground plane (instead of a PEC), the equivalent current densities to represent the aperture and PMC ground plane are

$$\underline{J}_s = 2\hat{n}\times\underline{H}_a = 2\hat{a}_z\times\left(-\hat{a}_x\frac{E_0}{Z_a}\right) = -\hat{a}_y\frac{2E_0}{Z_a} \Rightarrow J_y = -\frac{2E_0}{Z_a},\ J_x = J_z = 0$$

$\underline{M}_s = 0$

$$N_\theta = \iint_S J_y\cos\theta\sin\phi\, e^{j\beta r'\cos\psi}\,ds' = -\frac{2E_0}{Z_a}\cos\theta\sin\phi\int_{-a/2}^{+a/2} e^{j\beta x'\sin\theta\cos\phi}\,dx'\int_{-b/2}^{+b/2} e^{j\beta y'\sin\theta\sin\phi}\,dy'$$

$$= -\frac{2abE_0}{Z_a}\left[\cos\theta\sin\phi\,\frac{\sin X}{X}\,\frac{\sin Y}{Y}\right],\quad X = \frac{\beta a}{2}\sin\theta\cos\phi,\ Y = \frac{\beta b}{2}\sin\theta\sin\phi$$

$$N_\phi = \iint_S J_y\cos\phi\, e^{j\beta r'\cos\psi}\,ds' = -\frac{(ab)2E_0}{Z_a}\left[\cos\phi\,\frac{\sin X}{X}\,\frac{\sin Y}{Y}\right],\quad L_\theta = L_\phi = 0$$

$$E_\theta = +\frac{j\beta abE_0 e^{-j\beta r}}{2\pi r}\,\frac{\eta}{Z_a}\left[\cos\theta\sin\phi\,\frac{\sin X}{X}\,\frac{\sin Y}{Y}\right]$$

$$E_\phi = j\frac{\beta abE_0 e^{-j\beta r}}{2\pi r}\,\frac{\eta}{Z_a}\left[\cos\phi\,\frac{\sin X}{X}\,\frac{\sin Y}{Y}\right]$$

$\boxed{7.39}$ When the aperture is not mounted on a ground plane, we need both $\underline{J}_s$ and $\underline{M}_s$ ($\underline{J}_s = \hat{n}\times \underline{H}_a$, $\underline{M}_s = -\hat{n}\times\underline{E}_a$).

From the solutions of 7.37 and 7.38 (with $Z_a = \eta$)

$$\underline{M}_s = \hat{a}_x E_o, \quad \underline{J}_s = -\hat{a}_y \frac{E_o}{\eta}$$

$$L_\theta = E_o ab\left[\cos\theta\cos\phi \frac{\sin X}{X}\frac{\sin Y}{Y}\right], \quad L_\phi = -E_o ab\left[\sin\phi \frac{\sin X}{X}\frac{\sin Y}{Y}\right]$$

$$N_\theta = -\frac{E_o ab}{\eta}\left[\cos\theta\sin\phi \frac{\sin X}{X}\frac{\sin Y}{Y}\right], \quad N_\phi = -\frac{E_o ab}{\eta}\left[\cos\phi\left(\frac{\sin X}{X}\right)\frac{\sin Y}{Y}\right]$$

$$E_\theta = j\frac{\beta ab E_o e^{-j\beta r}}{4\pi r}\left[\sin\phi(1+\cos\theta)\frac{\sin X}{X}\frac{\sin Y}{Y}\right], \quad X = \frac{\beta a}{2}\sin\theta\cos\phi, \quad Y = \frac{\beta b}{2}\sin\theta\sin\phi$$

$$E_\phi = j\frac{\beta ab E_o e^{-j\beta r}}{4\pi r}\left[\cos\phi(1+\cos\theta)\frac{\sin X}{X}\frac{\sin Y}{Y}\right]$$

$\boxed{7.40}$

The only difference between this problem and 7.37 is that for the x variations the integral reduces to

$$\int_{-a/2}^{+a/2}\cos\left(\frac{\pi}{a}x'\right) e^{j\beta x'\sin\theta\cos\phi}\,dx' = -\left(\frac{\pi a}{2}\right)\frac{\cos\left(\frac{\beta a}{2}\sin\theta\cos\phi\right)}{\left(\frac{\beta a}{2}\sin\theta\cos\phi\right)^2-\left(\frac{\pi}{2}\right)^2}$$

Thus, from the solution of 7.37 $\left[X = \frac{\beta a}{2}\sin\theta\cos\phi,\ Y = \frac{\beta b}{2}\sin\theta\sin\phi\right]$

$$L_\theta = -E_o\pi ab\left[\cos\theta\cos\phi\frac{\cos(X)}{(X)^2-(\pi/2)^2}\frac{\sin Y}{Y}\right], \quad L_\phi = E_o\pi ab\left[\sin\phi\frac{\cos X}{(X)^2-(\pi/2)^2}\frac{\sin Y}{Y}\right]$$

$$E_\theta = -j\frac{\beta ab E_o e^{-j\beta r}}{4r}\left[\sin\phi\frac{\cos X}{(X)^2-(\pi/2)^2}\frac{\sin Y}{Y}\right]$$

$$E_\phi = -j\frac{\beta ab E_o e^{-j\beta r}}{4r}\left[\cos\theta\cos\phi\frac{\cos(X)}{(X)^2-(\pi/2)^2}\frac{\sin Y}{Y}\right]$$

$$H_\theta = -E_\phi/\eta, \quad H_\phi = E_\theta/\eta$$

$\boxed{7.41}$

Assuming the magnetic field at the aperture is related to the electric field by the wave impedance Z_a

$$\underline{H}_a = -\hat{a}_x \frac{E_0}{Z_a} \cos\left(\frac{\pi}{a}x'\right)$$

and using the equivalent model of 7.8(c) and an equivalent procedure as that outlined in Figure 7.10 but using a PMC ground plane (instead of a PEC) the equivalent current densities to represent the aperture and PMC ground plane are.

$$\underline{J}_s = -\hat{a}_y \frac{2E_0}{Z_a} \cos\left(\frac{\pi}{a}x'\right) \Rightarrow J_y = -\frac{2E_0}{Z_a}\cos\left(\frac{\pi}{a}x'\right),\ J_x = J_z = 0$$

$$\underline{M}_s = 0$$

Using the integral of $\int_{-a/2}^{+a/2} \cos\left(\frac{\pi}{a}x'\right) dx' = -\left(\frac{\pi a}{2}\right)\frac{\cos(X)}{(X)^2-(\pi/2)^2}$, $X = \frac{\beta a}{2}\sin\theta\cos\phi$, $Y = \frac{\beta b}{2}\sin\theta\sin\phi$

$$N_\theta = -\frac{2E_0 ab}{Z_a}\left(-\frac{\pi}{2}\right)\left[\cos\theta\sin\phi \frac{\cos X}{(X)^2-(\pi/2)^2}\frac{\sin Y}{Y}\right],\ N_\phi = -\frac{2E_0 ab}{Z_a}\left(-\frac{\pi}{2}\right)\left[\cos\phi\frac{\cos X}{(X)^2-(\frac{\pi}{2})^2}\frac{\sin Y}{Y}\right]$$

$$E_\theta = -j\frac{\beta ab E_0}{4r}\left(\frac{\eta}{Z_a}\right)\left[\cos\theta\sin\phi\frac{\cos X}{(X)^2-(\pi/2)^2}\frac{\sin Y}{Y}\right],\ E_\phi = -j\frac{\beta ab E_0}{4r}\left(\frac{\eta}{Z_a}\right)\left[\cos\phi\frac{\cos X}{(X)^2-(\pi/2)^2}\frac{\sin Y}{Y}\right]$$

$$H_\theta = -E_\phi/\eta,\ H_\phi = E_\theta/\eta$$

$\boxed{7.42}$

$$\underline{E}_a = \hat{a}_y E_0 \cos\left(\frac{\pi}{a}x'\right) \Rightarrow \underline{M}_s = -\hat{n}\times\underline{E}_a = \hat{a}_x E_0\cos\left(\frac{\pi}{a}x'\right)$$

Thus $M_x = E_0\cos\left(\frac{\pi}{a}x'\right)$, $M_y = M_z = 0$

$$L_\theta = \int_{-b/2}^{b/2}\int_{-a/2}^{a/2} M_x \cos\theta\cos\phi e^{j\beta(x'\sin\theta\cos\phi + y'\sin\theta\sin\phi)}\, dx'\, dy'$$

$$= E_0\cos\theta\cos\phi\int_{-a/2}^{a/2}\cos\left(\frac{\pi}{a}x'\right)e^{j\beta x'\sin\theta\cos\phi}\,dx'\int_{-b/2}^{b/2}e^{j\beta y'\sin\theta\sin\phi}\,dy'$$

Since

$$\int_{-a/2}^{a/2}\cos\left(\frac{\pi}{a}x'\right)e^{j\beta x'\sin\theta\cos\phi}\,dx' = -\left(\frac{\pi a}{2}\right)\frac{\cos(X)}{(X)^2-\left(\frac{\pi}{2}\right)^2}$$

$$\int_{-b/2}^{b/2}e^{j\beta y'\sin\theta\sin\phi}\,dy' = b\frac{\sin Y}{Y}, \quad X = \frac{\beta a}{2}\sin\theta\cos\phi, Y = \frac{\beta b}{2}\sin\theta\sin\phi$$

then

$$L_\theta = -\frac{\pi ab}{2}E_0\left[\cos\theta\cdot\cos\phi\frac{\cos X}{(X)^2-\left(\frac{\pi}{2}\right)^2}\cdot\frac{\sin Y}{Y}\right]$$

Similarly

$$L_\phi = \int_{-b/2}^{b/2} \int_{-a/2}^{a/2} -M_x \sin\phi e^{j\beta(x'\sin\theta\cos\phi + y'\sin\theta\sin\phi)}\, dx'\, dy'$$

$$= -E_0 \sin\phi \int_{-a/2}^{a/2} \cos\left(\frac{\pi}{a}x'\right) e^{j\beta x'\sin\theta\cos\phi}\, dx' \int_{-b/2}^{b/2} e^{j\beta y'\sin\theta\sin\phi}\, dy'$$

$$L_\phi = +\frac{\pi ab}{2} E_0 \left[\sin\phi \frac{\cos X}{(X)^2 - \left(\frac{\pi}{2}\right)^2} \cdot \frac{\sin Y}{Y}.\right]$$

$$\underline{H}_a \simeq -\hat{a}_x \frac{E_a}{\eta} \Rightarrow \underline{J}_s = \hat{n} \times \underline{H}_a = \hat{a}_z \times \left(-\hat{a}_x \frac{E_a}{\eta}\right) = -\hat{a}_y \frac{E_a}{\eta} = -\hat{a}_y \frac{E_0}{\eta} \cos\left(\frac{\pi}{a}x'\right)$$

$$J_x = J_z = 0, J_y = -\frac{E_0}{\eta} \cos\left(\frac{\pi}{a}x'\right)$$

$$N_\theta = \int_{-b/2}^{b/2} \int_{-a/2}^{a/2} J_y \cos\theta \sin\phi e^{j\beta(x'\sin\theta\cos\phi + y'\sin\theta\sin\phi)}\, dx'\, dy'$$

$$= -\frac{E_0}{\eta} \cos\theta \sin\phi \int_{-a/2}^{a/2} \cos\left(\frac{\pi}{a}x'\right) e^{j\beta x'\sin\theta\cos\phi}\, dx' \int_{-b/2}^{b/2} e^{j\beta y'\sin\theta\sin\phi}\, dy'$$

$$= +\frac{\pi ab E_0}{2\eta} \left[\cos\theta \sin\phi \cdot \frac{\cos X}{(X)^2 - (\pi/2)^2} \cdot \frac{\sin Y}{Y}\right]$$

$$N_\phi = \int_{-b/2}^{b/2} \int_{-a/2}^{a/2} J_y \cos\phi e^{j\beta(x'\cos\phi\sin\theta + y'\sin\theta\sin\phi)}\, dx'\, dy'$$

$$= +\frac{\pi ab E_0}{2\eta} \left[\cos\phi \cdot \frac{\cos X}{(X)^2 - \left(\frac{\pi}{2}\right)^2} \cdot \frac{\sin Y}{Y}\right]$$

$$E_r \simeq 0$$

$$E_\theta \simeq -j \frac{\beta e^{-j\beta r}}{4\pi r} [L_\phi + \eta N_\theta] = -j \frac{ab\beta E_0 e^{-j\beta r}}{8r} \left[\sin\phi(1+\cos\theta) \frac{\cos X}{(X)^2 - \left(\frac{\pi}{2}\right)^2} \cdot \frac{\sin Y}{Y}\right]$$

$$E_\phi \simeq j \frac{\beta e^{-j\beta r}}{4\pi r} [L_\theta - \eta N_\phi] = -j \frac{ab\beta E_0 e^{-j\beta r}}{8r} \left[\cos\phi(1+\cos\theta) \frac{\cos X}{(X)^2 - \left(\frac{\pi}{2}\right)^2} \cdot \frac{\sin Y}{Y}\right]$$

$$H_r \simeq 0, \; H_\theta = -E_\phi/\eta, \; H_\phi = E_\theta/\eta$$

$\boxed{7.43}$ Since $\underline{E}_a = \hat{a}_\rho E_\rho + \hat{a}_\phi E_\phi$ exists only over the circular aperture where $\rho' \le a$, and it is zero elsewhere, then over the aperture and outside it along a flat plane coincident with the circular aperture we can form the electric and magnetic equivalent current, as shown below

$\underline{M}_a = \hat{n} \times \underline{E}_a$ $\quad \underline{M}_s = 0,\ \underline{J}_s \ne 0 \quad \underline{J}_s, \underline{M}_a \uparrow \hat{n} \quad \underline{M}_s = 0,\ \underline{J}_s \ne 0 \quad \epsilon_0, \mu_0$

$\leftarrow a \rightarrow \leftarrow a \rightarrow$

Now if we move a perfect electric conductor to coincide with the interface we have the following equivalent.

$\underline{M}_s = \underline{J}_s = 0 \quad \underline{J}_s = 0, \underline{M}_a \uparrow \hat{n} \quad \underline{M}_s = \underline{J}_s = 0 \quad \epsilon_0, \mu_0$

$\sigma = \infty$

which is equivalent to

$\underline{M}_s = 2\hat{n} \times \underline{E}_a \quad \underline{J}_s = \underline{M}_s = 0 \quad \underline{J}_s = 0, \underline{M}_s = 2\underline{M}_a \quad \underline{J}_s = \underline{M}_s = 0 \quad \epsilon_0, \mu_0$

$\underline{M}_s = 2\hat{a}_z \times (\hat{a}_\rho E_\rho + \hat{a}_\phi E_\phi) = 2\hat{a}_\phi E_\rho - 2\hat{a}_\rho E_\phi \quad \leftarrow a \rightarrow \leftarrow a \rightarrow \quad \epsilon_0, \mu_0$

$\boxed{7.44}$ $\underline{E}^i = \hat{a}_x E_o e^{-j\beta_o(y\sin\theta_i - z\cos\theta_i)}$

$\underline{H}^i = \frac{E_o}{\eta}\left(-\hat{a}_y\cos\theta_i - \hat{a}_z\sin\theta_i\right)e^{-j\beta_o(y\sin\theta_i - z\cos\theta_i)}$

a. <u>Induction Equivalent</u>

$$\underline{M}_i = 2\hat{n}\times\underline{E}^i\Big|_{\substack{z=0\\y=y'}} = 2\hat{a}_z\times\hat{a}_x E_x\Big|_{\substack{z=0\\y=y'}} = \hat{a}_y 2E_o e^{-j\beta_o y\sin\theta_i}$$

$$M_x = M_z = 0,\quad M_y = 2E_o e^{-j\beta_o y'\sin\theta_i}$$

Using (6-125a)-(6-125d)

$$N_\theta = N_\phi = 0$$

$$L_\theta = \iint_{S_a} M_y\cos\theta_s\sin\phi_s e^{j\beta r'\cos\psi}\,ds' = 2E_o\cos\theta\sin\phi\iint_{S_a} e^{-j\beta_o y'\sin\theta_i}e^{j\beta r'\cos\psi}\,ds'$$

Using (6-130b), (6-132a) and (6-132b) we can write

$$r'\cos\psi = x'\sin\theta_s\cos\phi_s + y'\sin\theta_s\sin\phi_s$$

$$ds' = dx'dy' = \rho'd\rho'd\phi'$$

Thus

$$L_\theta = 2E_o\cos\theta_s\sin\phi_s\iint_{S_a} e^{j\beta_o\left[x'\sin\theta_s\cos\phi_s + y'(\sin\theta_s\sin\phi_s - \sin\theta_i)\right]}\rho'd\rho'd\phi'$$

Since $x' = \rho'\cos\phi'$
$y' = \rho'\sin\phi'$

then the exponent of the exponential can be written as

$$x'\sin\theta_s\cos\phi_s + y'(\sin\theta_s\sin\phi_s - \sin\theta_i) = \rho'\left[\underbrace{\sin\theta_s\cos\phi_s}_{A}\cos\phi' + \sin\phi'\underbrace{(\sin\theta_s\sin\phi_s - \sin\theta_i)}_{B}\right]$$

$$= \rho'\left(A\cos\phi' + B\sin\phi'\right)$$

$$= \rho'\sqrt{A^2+B^2}\left(\frac{A}{\sqrt{A^2+B^2}}\cos\phi' + \frac{B}{\sqrt{A^2+B^2}}\sin\phi'\right)$$

$$= \rho'\sqrt{A^2+B^2}\left(\cos\phi_o\cos\phi' + \sin\phi_o\sin\phi'\right)$$

$$x'\sin\theta_s\cos\phi_s + y'(\sin\theta_s\sin\phi_s - \sin\theta_i) = \sqrt{A^2+B^2}\,\rho'\cos(\phi' - \phi_o)$$

where $A = \sin\theta_s\cos\phi_s$

$B = \sin\theta_s\sin\phi_s - \sin\theta_i$

$\phi_o = \tan^{-1}\left(\frac{B}{A}\right)$

cont'd.

7.14 Cont'd. Thus we can write L_θ as

$$L_\theta = 2E_0 \cos\theta_s \sin\phi_s \int_0^a \left\{ \int_0^{2\pi} e^{j\beta_0\sqrt{A^2+B^2}\,\rho'\cos(\phi'-\phi_0)}\, d\phi' \right\} \rho' d\rho'$$

$$= 2E_0 \cos\theta_s \sin\phi_s \int_0^a \left[2\pi J_0\left(\beta_0\sqrt{A^2+B^2}\,\rho'\right)\right] \rho' d\rho' = 4\pi E_0 \cos\theta_s \sin\phi_s \int_0^a J_0\left(\beta_0\sqrt{A^2+B^2}\,\rho'\right)\rho' d\rho'$$

$$L_\theta = 4\pi E_0 \cos\theta_s \sin\phi_s \left[a^2\, \frac{J_1\left(\beta_0 a\sqrt{A^2+B^2}\right)}{\beta_0 a\sqrt{A^2+B^2}} \right] = 4\pi a^2 E_0 \cos\theta_s \sin\phi_s\, \frac{J_1\left(\beta_0 a\sqrt{A^2+B^2}\right)}{\beta_0 a\sqrt{A^2+B^2}}$$

Using (6-125d)

$$L_\phi = \iint_{S_a} M_y \cos\phi_s\, e^{j\beta_0 r'\cos\psi}\, ds' = 4\pi a^2 E_0 \cos\phi_s\, \frac{J_1\left(\beta_0 a\sqrt{A^2+B^2}\right)}{\beta_0 a\sqrt{A^2+B^2}}$$

The electric and magnetic scattered fields can be written using (6-122a) - (6-122f) as

$$E_r^s \simeq H_r^s = 0$$

$$E_\theta^s \simeq -j\frac{\beta_0 e^{-j\beta_0 r}}{4\pi r}\left(L_\phi + \eta \overset{0}{\cancel{N_\theta}}\right) = -j\,\frac{\beta_0 a^2 E_0 e^{-j\beta_0 r}}{r} \cos\phi_s\, \frac{J_1\left(\beta_0 a\sqrt{A^2+B^2}\right)}{\beta_0 a\sqrt{A^2+B^2}}$$

$$E_\phi^s \simeq j\,\frac{\beta_0 e^{-j\beta_0 r}}{4\pi r}\left(L_\theta - \eta \overset{0}{\cancel{N_\phi}}\right) = j\,\frac{\beta_0 a^2 E_0 e^{-j\beta_0 r}}{r} \cos\theta_s \sin\phi_s\, \frac{J_1\left(\beta_0 a\sqrt{A^2+B^2}\right)}{\beta_0 a\sqrt{A^2+B^2}}$$

$$H_\theta^s \simeq -\frac{E_\phi^s}{\eta}, \quad H_\phi^s = \frac{E_\theta^s}{\eta}$$

For backscatter observations

$$\theta_s = \theta_i, \quad \phi_s = 3\pi/2$$

$$A = \sin\theta_s \cos\phi_s = 0$$

$$B = \sin\theta_s \sin\phi_s - \sin\theta_i = -\sin\theta_i - \sin\theta_i = -2\sin\theta_i \quad \Big\} \sqrt{A^2+B^2} = 2\sin\theta_i$$

Thus

$$E_\theta^s \simeq 0$$

$$E_\phi^s \simeq -j\,\frac{\beta_0 a^2 E_0 e^{-j\beta_0 r}}{r}\cos\theta_i\, \frac{J_1(2\beta_0 a\sin\theta_i)}{2\beta_0 a \sin\theta_i}$$

$$H_\theta^s \simeq -\frac{E_\phi^s}{\eta}$$

$$H_\phi^s \simeq \frac{E_\theta^s}{\eta}$$

cont'd.

7.44 Cont'd.

b. Physical Equivalent

$$\underline{J}_p = 2\hat{n}\times\underline{H}^i\Big|_{\substack{z=0\\y=y'}} = 2\hat{a}_z\times(\hat{a}_y H_y^i + \hat{a}_z H_z^i)\Big|_{\substack{z=0\\y=y'}} = -\hat{a}_x 2H_y^i\Big|_{\substack{z=0\\y=y'}} = \hat{a}_x 2\frac{E_0}{\eta}\cos\theta_i e^{-j\beta_0 y'\sin\theta_i}$$

$$J_y = J_z = 0, \quad J_x = \frac{2E_0}{\eta}\cos\theta_i e^{-j\beta_0 y\sin\theta_i}$$

Using (6-125a)-(6-125d) and the results of the induction equivalent, we can write

$$N_\theta = \iint_{S_a} J_x\cos\theta_s\cos\phi_s e^{j\beta_0 r'\cos\psi}\,ds' = \frac{4\pi a^2 E_0}{\eta}\cos\theta_i\cos\theta_s\cos\phi_s\;\frac{J_1(\beta_0 a\sqrt{A^2+B^2})}{\beta_0 a\sqrt{A^2+B^2}}$$

$$N_\phi = \iint_{S_a} -J_x\sin\phi_s e^{j\beta_0 r'\cos\psi}\,ds' = -\frac{4\pi a^2 E_0}{\eta}\cos\theta_i\sin\phi_s\;\frac{J_1(\beta_0 a\sqrt{A^2+B^2})}{\beta_0 a\sqrt{A^2+B^2}}$$

$$A = \sin\theta_s\cos\phi_s, \quad B = \sin\theta_s\sin\phi_s - \sin\theta_i$$

Now using (6-122a)-(6-122f) we can write the scattered electric and magnetic fields as

$$E_r^s \simeq H_r^s \simeq 0$$

$$E_\theta^s \simeq -j\frac{\beta_0 e^{-j\beta_0 r}}{4\pi r}(\cancelto{0}{L_\phi} + \eta N_\theta) = -j\frac{\beta_0 a^2 E_0 e^{-j\beta_0 r}}{r}\cos\theta_i\cos\theta_s\cos\phi_s\;\frac{J_1(\beta_0 a\sqrt{A^2+B^2})}{\beta_0 a\sqrt{A^2+B^2}}$$

$$E_\phi^s \simeq j\frac{\beta_0 e^{-j\beta_0 r}}{4\pi r}(\cancelto{0}{L_\theta} - \eta N_\phi) = j\frac{\beta_0 a^2 E_0 e^{-j\beta_0 r}}{r}\cos\theta_i\sin\phi_s\;\frac{J_1(\beta_0 a\sqrt{A^2+B^2})}{\beta_0 a\sqrt{A^2+B^2}}$$

$$H_\theta^s \simeq -\frac{E_\phi^s}{\eta}, \quad H_\phi^s \simeq \frac{E_\theta^s}{\eta}$$

For backscatter observations

$$\theta_s = \theta_i, \quad \phi_s = 3\pi/2$$

$$\left.\begin{array}{l} A = 0 \\ B = -2\sin\theta_i \end{array}\right\}\sqrt{A^2+B^2} = 2\sin\theta_i$$

$$E_\theta^s \simeq 0$$

$$E_\phi^s \simeq -j\frac{\beta_0 a^2 E_0 e^{-j\beta_0 r}}{r}\cos\theta_i\;\frac{J_1(2\beta_0 a\sin\theta_i)}{2\beta_0 a\sin\theta_i}$$

$$H_\theta^s \simeq -\frac{E_\phi^s}{\eta}$$

$$H_\phi^s \simeq \frac{E_\theta^s}{\eta}$$

$\boxed{7.45}$ $\underline{H}^i = \hat{a}_x H_0 e^{-j\beta_0(y\sin\theta_i - z\cos\theta_i)}$

$\underline{E}^i = \eta H_0(\hat{a}_y \cos\theta_i + \hat{a}_z \sin\theta_i)e^{-j\beta_0(y\sin\theta_i - z\cos\theta_i)}$

a. <u>Induction Equivalent</u>

$$\underline{M}_i = 2\hat{n}\times\underline{E}^i\Big|_{\substack{z=0\\y=y'}} = 2\hat{a}_z\times(\hat{a}_y E_y + \hat{a}_z E_z)\Big|_{\substack{z=0\\y=y'}} = -\hat{a}_x 2E_y\Big|_{\substack{z=0\\y=y'}} = -\hat{a}_x 2\eta H_0\cos\theta_i e^{-j\beta_0 y'\sin\theta_i}$$

$M_y = M_z = 0,$ $M_x = -2\eta H_0\cos\theta_i e^{-j\beta_0 y'\sin\theta_i}$

Using (6-125a)-(6-125d) and the solution of Problem 7.14, we can write

$$L_\theta = \iint_{S_a} M_x \cos\theta_s\cos\phi_s e^{j\beta_0 r'\cos\psi}\,ds' = -4\pi a^2\eta H_0\cos\theta_i\cos\theta_s\cos\phi_s\,\frac{J_1(\beta_0 a\sqrt{A^2+B^2})}{\beta_0 a\sqrt{A^2+B^2}}$$

$$L_\phi = \iint_{S_a} -M_x\sin\phi_s e^{j\beta_0 r'\cos\psi}\,ds' = +4\pi a^2\eta H_0\cos\theta_i\sin\phi_s\,\frac{J_1(\beta_0 a\sqrt{A^2+B^2})}{\beta_0 a\sqrt{A^2+B^2}}$$

$A = \sin\theta_s\cos\phi_s,\quad B = \sin\theta_s\sin\phi_s - \sin\theta_i$

Now using (6-122a)-(6-122f) we can write the scattered electric and magnetic fields as

$E_r^s \simeq H_r^s \simeq 0$

$$E_\theta^s \simeq -\frac{j\beta_0 e^{-j\beta_0 r}}{4\pi r}(L_\phi + \cancel{\eta N_\theta}^{\,0}) = -j\frac{\beta_0 a^2\eta H_0 e^{-j\beta_0 r}}{r}\cos\theta_i\sin\phi_s\,\frac{J_1(\beta_0 a\sqrt{A^2+B^2})}{\beta_0 a\sqrt{A^2+B^2}}$$

$$E_\phi^s \simeq j\frac{\beta_0 e^{-j\beta_0 r}}{4\pi r}(L_\theta - \cancel{\eta N_\phi}^{\,0}) = -j\frac{\beta_0 a^2\eta H_0 e^{-j\beta_0 r}}{r}\cos\theta_i\cos\theta_s\cos\phi_s\,\frac{J_1(\beta_0 a\sqrt{A^2+B^2})}{\beta_0 a\sqrt{A^2+B^2}}$$

$H_\theta^s \simeq -\dfrac{E_\phi^s}{\eta},\quad H_\phi^s \simeq \dfrac{E_\theta^s}{\eta}$

<u>For backscatter observations</u>

$\theta_s = \theta_i,\ \phi_s = 3\pi/2$

$$\left.\begin{aligned}A &= 0\\ B &= -2\sin\theta_i\end{aligned}\right\}\sqrt{A^2+B^2} = 2\sin\theta_i$$

$$E_\theta^s \simeq +j\frac{\beta_0 a^2\eta H_0 e^{-j\beta_0 r}}{r}\cos\theta_i\,\frac{J_1(2\beta_0 a\sin\theta_i)}{2\beta_0 a\sin\theta_i}$$

$E_\phi^s \simeq 0$

$H_\theta^s \simeq -\dfrac{E_\phi^s}{\eta}$

$H_\phi^s \simeq E_\theta^s/\eta$

cont'd.

7.45 cont'd.

b. Physical Equivalent

$$\underline{J}_P = 2\hat{n}\times\underline{H}^i\Big|_{\substack{z=0\\y=y'}} = 2\hat{a}_z\times\hat{a}_x H_x = \hat{a}_y 2H_0 e^{-j\beta_0 y'\sin\theta_i}$$

$$J_x = J_z = 0, \qquad J_y = 2H_0 e^{-j\beta_0 y'\sin\theta_i}$$

Using (6-125a)-(6-125d) and the solution of Problem 7.44, we can write

$$\left.\begin{aligned} N_\theta &= \iint_{S_a} J_y \cos\theta_s \sin\phi_s e^{j\beta_0 r'\cos\psi}\, ds' = 4\pi a^2 H_0 \cos\theta_s \sin\phi_s \frac{J_1(\beta_0 a\sqrt{A^2+B^2})}{\beta_0 a\sqrt{A^2+B^2}} \\ N_\phi &= \iint_{S_a} J_y \cos\phi_s e^{j\beta_0 r'\cos\psi}\, ds' = 4\pi a^2 H_0 \cos\phi_s \frac{J_1(\beta_0 a\sqrt{A^2+B^2})}{\beta_0 a\sqrt{A^2+B^2}} \end{aligned}\right\}$$

$$A = \sin\theta_s\cos\phi_s,\quad B = \sin\theta_s\sin\phi_s - \sin\theta_i$$

Now using (6-122a)-(6-122f) we can write the scattered electric and magnetic fields as

$$E_\theta^s \simeq -j\frac{\beta_0 e^{-j\beta_0 r}}{4\pi r}\left(\cancelto{0}{L_\phi} + \eta N_\theta\right) = -j\frac{\beta_0 a^2\eta H_0 e^{-j\beta_0 r}}{r}\cos\theta_s\sin\phi_s \frac{J_1(\beta_0 a\sqrt{A^2+B^2})}{\beta_0 a\sqrt{A^2+B^2}}$$

$$E_\phi^s \simeq j\frac{\beta_0 e^{-j\beta_0 r}}{4\pi r}\left(\cancelto{0}{L_\theta} - \eta N_\phi\right) = -j\frac{\beta_0 a^2\eta H_0 e^{-j\beta_0 r}}{r}\cos\phi_s \frac{J_1(\beta_0 a\sqrt{A^2+B^2})}{\beta_0 a\sqrt{A^2+B^2}}$$

$$H_\theta^s \simeq -\frac{E_\phi^s}{\eta}$$

$$H_\phi^s \simeq \frac{E_\theta^s}{\eta}$$

For backscatter directions

$$\theta_s = \theta_i,\ \phi_s = 3\pi/2$$

$$\left.\begin{aligned} A &= 0 \\ B &= -2\sin\theta_i \end{aligned}\right\}\sqrt{A^2+B^2} = 2\sin\theta_i$$

$$E_\theta^s \simeq j\frac{\beta_0 a^2\eta H_0 e^{-j\beta_0 r}}{r}\cos\theta_i \frac{J_1(2\beta_0 a\sin\theta_i)}{2\beta_0 a\sin\theta_i}$$

$$E_\phi^s \simeq 0$$

$$H_\theta^s \simeq -\frac{E_\phi^s}{\eta}$$

$$H_\phi^s \simeq \frac{E_\theta^s}{\eta}$$

Chapter 8

8.1 $v_g = v\cos\theta$, $\cos\theta = \frac{\beta_z}{\beta} = \sqrt{1-(f_c/f)^2}$

$$(f_c)_{10} = \frac{1}{2a\sqrt{\mu_0\varepsilon_0}} = \frac{3\times10^{10}}{2(2.54)(0.9)} = 6.56\times10^9, \quad \frac{(f_c)_{10}}{f} = \frac{6.56}{10} = 0.656$$

$$\cos\theta = \sqrt{1-(0.656)^2} = \sqrt{1-0.43} = \sqrt{0.57} = 0.755$$

$$v_g = 3\times10^8(0.755) = 2.265\times10^8 \text{ m/sec.}$$

$$d = \text{delay}\times\text{velocity} = (2\times10^{-6})(2.265\times10^8) = 4.53\times10^2 = 453 \text{ meters}$$

8.2 a. $$(f_c)_{10} = \frac{1}{2a\sqrt{\mu_0\varepsilon}} = \frac{1}{2a\sqrt{\mu_0\varepsilon_0}\sqrt{\varepsilon_r}} = \frac{v_0}{2a\sqrt{\varepsilon_r}} = \frac{30\times10^9}{2(2.286)\sqrt{2.56}} = 4.101 \text{ GHz}$$

b. $$\lambda_g = \frac{\lambda}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}} = \frac{\lambda_0/\sqrt{\varepsilon_r}}{\sqrt{1-\left(\frac{4.101}{10}\right)^2}} = \frac{30\times10^9/(10\times10^9\sqrt{2.56})}{\sqrt{1-(0.4101)^2}} = \frac{1.875}{0.912} = 2.056 \text{ cm}$$

c. $$Z_w = \frac{\eta}{\sqrt{1-(f_c/f)^2}} = \frac{377/\sqrt{2.56}}{0.912} = 258.36 \text{ Ohms}$$

d. $$v_p = \frac{v}{\sqrt{1-(f_c/f)^2}} = \frac{v_0/\sqrt{\varepsilon_r}}{0.912} = \frac{3\times10^8/\sqrt{2.56}}{0.912} = \frac{1.875\times10^8}{0.912} = 2.056\times10^8 \text{ m/sec.}$$

e. $$v_g = v\sqrt{1-(f_c/f)^2} = \frac{v_0}{\sqrt{\varepsilon_r}}\sqrt{1-(f_c/f)^2} = \frac{3\times10^8}{\sqrt{2.56}}(0.912) = 1.71\times10^8 \text{ m/sec.}$$

8.3 TE_{10} mode; $a = 0.622$ in $= 1.57988$ cm; $b = 0.311$ in $= 0.78944$ cm

a. $$f_c = \frac{1}{2\pi\sqrt{\mu_0\varepsilon_0}}\left(\frac{\pi}{a}\right) = \frac{1}{2a\sqrt{\mu_0\varepsilon_0}} = \frac{30\times10^9}{2(1.57988)} = 9.49439\times10^9$$

$$\beta_z = \beta\sqrt{1-\left(\frac{f_c}{f}\right)^2} = \omega\sqrt{\mu_0\varepsilon_0}\sqrt{1-\left(\frac{9.49439}{15}\right)^2} = \frac{2\pi(15\times10^9)}{30\times10^9}\sqrt{1-(0.63296)^2}$$

$$\beta_z = \pi(0.77418) = 139.35329 \text{ degrees/cm}$$

$$139.35329(\Delta z) = 300^\circ \Rightarrow \Delta z = 300/139.35329 = 2.1528 \text{ cm}$$

$$\boxed{\Delta z = 2.1528 \text{ cm}}$$

(Cont'd)

$\boxed{8.3 \text{ cont'd}}$

b. $(f_c)\Big|_{\varepsilon_r=4} = \dfrac{1}{2\pi\sqrt{\mu_0\varepsilon_r\varepsilon_0}}\left(\dfrac{\pi}{a}\right) = \dfrac{1}{2}(f_c)_{\varepsilon_r=1} = \dfrac{9.49439}{2}\times10^9 = 4.7472\times10^9$

$$\beta_z = 2\pi f\sqrt{\mu_0\varepsilon_0\varepsilon_r}\sqrt{1-\left(\frac{f_c}{f}\right)^2} = \frac{2\pi(15\times10^9)}{30\times10^9}\sqrt{4}\sqrt{1-\left(\frac{4.7472}{15}\right)^2}$$

$$= 2\pi\sqrt{1-(0.31648)^2} = 2\pi\sqrt{1-0.1} = 2\pi(0.9486) = 5.96023 \text{ rad/cm}$$

$$\beta_z = 5.96023 \text{ rad/cm} = 341.496 \text{ degrees/cm}$$

$$\Delta\phi = \beta_z\Delta z = 341.496(2.1528) = 735.173^\circ = 12.83118 \text{ rads}$$

Additional phase: $\Delta\psi = (735.173-720) = 15.173^\circ$

$$\Delta\psi = (12.83118 - 12.5664) = 0.26481 \text{ rads}$$

$\boxed{8.4}$ $a = 2.286$ cm, $b = 1.016$ cm

$$(f_c)_{10} = \frac{1}{2a\sqrt{\mu_0\varepsilon_r\varepsilon_0}} = \frac{30\times10^9}{2(2.286)\sqrt{2.25}} = \frac{30\times10^9}{2(2.286)(1.5)} = 4.3745 \text{ GHz}$$

$$\cos\theta = \sqrt{1-(f_c/f)^2} = \sqrt{1-(4.3745/10)^2} = \sqrt{1-(0.43745)^2} = 0.8992$$

$$v_g = v\cos\theta = \frac{30\times10^9}{\sqrt{2.25}}(0.8992) = 17.9849\times10^9 \text{ cm/sec}$$
$$= 0.179849\times10^8 \text{ m/sec}$$

$$(v_g/d) = 1/2\times10^{-6} = 0.179849\left(\frac{1}{d}\right)\times10^8 = 0.5\times10^{+6}$$

$$d = (2\times10^{-6})\times(0.179849\times10^8) = 35.9698 \text{ meters}$$

$\boxed{8.5}$ a. For the empty $(f_c)_{10} = 6.56\times10^9 \Rightarrow Z_{w0} = \dfrac{\eta_0}{\sqrt{1-(f_c/f)^2}} = \dfrac{377}{\sqrt{1-\left(\frac{6.56}{10}\right)^2}} = \dfrac{377}{0.7547} = 499.5$

For the filled waveguide, from Problem 8.2

$$(f_c)_{10} = 4.101\times10^9 \Rightarrow Z_{w2} = 258.36$$

Therefore for the intermediate waveguide ($\lambda/4$ section)

$$Z_{w1} = \sqrt{Z_{w0}Z_{w2}} = \sqrt{499.5(258.36)} = 359.235 \text{ Ohms}$$

b.

$$Z_{w1} = 359.235 = \frac{377/\sqrt{\varepsilon_{r1}}}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}} = \frac{377/\sqrt{\varepsilon_{r1}}}{\sqrt{1-\left(\frac{f_{co}/\sqrt{\varepsilon_{r1}}}{f}\right)^2}} = \frac{377/\sqrt{\varepsilon_r}}{\sqrt{1-\varepsilon_{r1}\left(\frac{6.56}{10}\right)^2}} = \frac{377/\sqrt{\varepsilon_{r1}}}{\sqrt{1-0.43/\varepsilon_{r1}}}$$

$\varepsilon_{r1} = 1.531$

c. $$\ell = \frac{\lambda_{g1}}{4} = \frac{\lambda_0/\sqrt{\varepsilon_{r1}}}{4\sqrt{1-\left(\frac{f_{c1}}{f}\right)^2}} = \frac{30\times10^9/(10\times10^9\sqrt{1.531})}{4\sqrt{1-\left(\frac{f_{co}/\sqrt{\varepsilon_{r1}}}{f}\right)^2}} = \frac{3/\sqrt{1.531}}{4\sqrt{1-\left(\frac{6.56}{\sqrt{1.531}(10)}\right)^2}} = 0.7149 \text{ cm}$$

$\boxed{8.6}$ a. For the empty waveguide, from Problem 8.5 $\Rightarrow Z_{w0} = 499.5$ Ohms

For filled waveguide with polystyrene, from Problem 8.2 $\Rightarrow Z_{wL} = 258.36$ Ohms

$N = 2$

Using (5-76a) and (5-77)

$$\Gamma_n = 2^{-N}\frac{Z_{wL}-Z_{w0}}{Z_{wL}+Z_{w0}}C_n^N = 2^{-N}\frac{Z_{wL}-Z_{w0}}{Z_{wL}+Z_{w0}}\frac{N!}{(N-n)!\,n!}$$

$\underline{n=0}$: $$\Gamma_0 = 2^{-2}\frac{258.36-499.5}{258.36+499.5}\frac{2!}{2!\,0!} = \frac{1}{4}(-0.318) = -0.0795 = \frac{Z_{w1}-Z_{w0}}{Z_{w1}+Z_{w0}}$$

$$\Rightarrow Z_{w1} = Z_{w0}\left(\frac{1-0.0795}{0+0.0795}\right) = 499.5(0.8527) = \underline{425.928}$$

$\underline{n=1}$: $$\Gamma_1 = 2^{-2}\frac{258.36-499.5}{258.36+499.5}\frac{2!}{1!\,1!} = -0.159 = \frac{Z_{w2}-Z_{w1}}{Z_{w2}+Z_{w1}}$$

$$\Rightarrow Z_{w2} = Z_{w1}\frac{1-0.159}{1+0.159} = 425.928(0.7255) = \underline{309.03}$$

b. $$Z_{w1} = \frac{377/\sqrt{\varepsilon_{r1}}}{\sqrt{1-\left(\frac{6.56}{10\sqrt{\varepsilon_{r1}}}\right)^2}} \Rightarrow \varepsilon_{r1} = 0.43 + \left(\frac{377}{Z_{w1}}\right)^2 = 1.213$$

$$Z_{w2} = \frac{377/\sqrt{\varepsilon_{r2}}}{\sqrt{1-\left(\frac{6.56}{10\sqrt{\varepsilon_{r2}}}\right)^2}} \Rightarrow \varepsilon_{r2} = 0.43 + \left(\frac{377}{Z_{w2}}\right)^2 = 1.918$$

c. $$L_1 = \frac{\lambda_{g1}}{4} = \frac{\lambda_0/\sqrt{\varepsilon_{r1}}}{4\sqrt{1-\left(\frac{6.56}{10\sqrt{\varepsilon_{r1}}}\right)^2}} = \frac{3/\sqrt{1.213}}{4\sqrt{1-0.43/\varepsilon_{r1}}} = 0.8476 \text{ cm}$$

$$L_2 = \frac{\lambda_{g2}}{4} = \frac{\lambda_0/\sqrt{\varepsilon_{r2}}}{4\sqrt{1-\left(\frac{6.56}{10\sqrt{\varepsilon_{r2}}}\right)^2}} = \frac{3/\sqrt{1.918}}{4\sqrt{1-\frac{0.43}{\varepsilon_{r2}}}} = 0.6148 \text{ cm}$$

8.7 The expressions for α_c of the TE^z_{mn} and TM^z_{mn} modes are derived using (8-64a), or

$$\alpha_c = \frac{P_c/\ell}{2P_{mn}} \qquad (1)$$

The procedure follows the derivation of (8-69a) for $\alpha_c^{TE_{10}}$.

a. $TE^z_{mn}(E_z^+=0)$: $E_x^+ = A_{mn}\frac{n\pi}{b\varepsilon}\cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)e^{-j\beta_z z}$

$$E_y^+ = -A_{mn}\frac{m\pi}{a\varepsilon}\sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)e^{-j\beta_z z}$$

$$H_x^+ = A_{mn}\frac{m\pi\beta_z}{a\omega\mu\varepsilon}\sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)e^{-j\beta_z z}$$

$$H_y^+ = A_{mn}\frac{n\pi\beta_z}{b\omega\mu\varepsilon}\cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)e^{-j\beta_z z}$$

$$H_z^+ = -jA_{mn}\frac{\beta_c^2}{\omega\mu\varepsilon}\cos\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)e^{-j\beta_z z}$$

$$P_{mn} = \int_0^b\int_0^a \frac{1}{2}\mathrm{Re}(\underline{E}\times\underline{H}^*)\cdot\hat{a}_z\,dx\,dy = \frac{1}{2}\mathrm{Re}\int_0^b\int_0^a \hat{a}_z(E_xH_y^* - E_yH_x^*)\cdot\hat{a}_z\,dx\,dy$$

$$P_{mn} = \frac{1}{2}\mathrm{Re}\int_0^b\int_0^a (E_xH_y^* - E_yH_x^*)\,dx\,dy$$

$$E_xH_y^* = |A_{mn}|^2\frac{\beta_z}{\omega\mu\varepsilon^2}\left(\frac{n\pi}{b}\right)^2\cos^2\left(\frac{m\pi}{a}x\right)\sin^2\left(\frac{n\pi}{b}y\right)$$

$$E_yH_x^* = -|A_{mn}|^2\frac{\beta_z}{\omega\mu\varepsilon^2}\left(\frac{m\pi}{a}\right)^2\sin^2\left(\frac{m\pi}{a}x\right)\cos^2\left(\frac{n\pi}{b}y\right)$$

$$P_{mn} = \frac{|A_{mn}|^2}{2}\left(\frac{\beta_z}{\omega\mu\varepsilon^2}\right)\left\{\left(\frac{n\pi}{b}\right)^2\frac{ab}{4} + \left(\frac{m\pi}{a}\right)^2\frac{ab}{4}\right\}\varepsilon_m\varepsilon_n = \frac{|A_{mn}|^2}{2}\frac{ab}{4}\left(\frac{\beta_z}{\omega\mu\varepsilon^2}\right)\left[\left(\frac{n\pi}{b}\right)^2 + \left(\frac{m\pi}{a}\right)^2\right]\varepsilon_m\varepsilon_n$$

$$P_c = \frac{R_s}{2}\iint(\underline{J}_s\cdot\underline{J}_s)\,ds = 2\left\{\frac{R_s}{2}\int_0^\ell\int_0^a\left(|H_x|^2+|H_z|^2\right)\Big|_{y=0}dx\,dz + \frac{R_s}{2}\int_0^\ell\int_0^b\left(|H_y|^2+|H_z|^2\right)\Big|_{x=0}dy\,dz\right\}$$

$$P_c = R_s\ell|A_{mn}|^2\left\{\left[\left(\frac{m\pi\beta_z}{a\omega\mu\varepsilon}\right)^2\int_0^a\sin^2\left(\frac{m\pi}{a}x\right)dx + \left(\frac{\beta_c^2}{\omega\mu\varepsilon}\right)^2\int_0^a\cos^2\left(\frac{m\pi}{a}x\right)dx\right]\right.$$
$$\left.+\left[\left(\frac{n\pi\beta_z}{b\omega\mu\varepsilon}\right)^2\int_0^b\sin^2\left(\frac{n\pi}{b}y\right)dy + \left(\frac{\beta_c^2}{\omega\mu\varepsilon}\right)^2\int_0^b\cos^2\left(\frac{n\pi}{b}y\right)dy\right]\right\}$$

$$P_c = \frac{R_s\ell|A_{mn}|^2}{(\omega\mu\varepsilon)^2}\left\{(\beta_z)^2\left[\left(\frac{m\pi}{a}\right)^2\left(\frac{a}{2}\right) + \left(\frac{n\pi}{b}\right)^2\frac{b}{2}\right] + (\beta_c^2)^2\left[\frac{a}{2}\epsilon_m + \frac{b}{2}\epsilon_n\right]\right\},\quad \epsilon_p = \begin{cases}2, & p=0\\ 1, & p\neq 0\end{cases}$$

$$\alpha_c = \frac{4R_s}{ab(\omega\mu\varepsilon)^2}\left(\frac{\omega\mu\varepsilon^2}{\beta_z}\right)\left\{\frac{(\beta_z)^2\left[\left(\frac{m\pi}{a}\right)^2\frac{a}{2} + \left(\frac{n\pi}{b}\right)^2\frac{b}{2}\right]}{\left(\frac{n\pi}{b}\right)^2 + \left(\frac{m\pi}{a}\right)^2} + \frac{\beta_c^4\left(\frac{a}{2}\epsilon_m + \frac{b}{2}\epsilon_n\right)}{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}\right\}\frac{1}{\varepsilon_m\varepsilon_n}$$

cont'd.

$\boxed{\text{8.7 cont'd.}}$ Since $\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2=\beta_c^2$, then

$$\alpha_c^{TE}=\frac{2R_s}{\omega\mu\epsilon_m\epsilon_n}\left\{\frac{\beta_z\left[\frac{m^2}{a}+\frac{n^2}{b}\right)}{ab\left(\frac{m}{a}\right)^2+\left(\frac{n}{b}\right)^2}+\frac{\beta_c^2(a\epsilon_m+b\epsilon_n)}{ab\beta_z}\right\}$$

$$=\frac{2R_s}{\eta\beta\epsilon_m\epsilon_n}\left\{\beta\sqrt{1-\left(\frac{f_c}{f}\right)^2}\frac{(m^2b+n^2a)}{(mb)^2+(na)^2}+\frac{\beta_c^2}{\beta\sqrt{1-\left(\frac{f_c}{f}\right)^2}}\left(\frac{a\epsilon_m+b\epsilon_n}{ab}\right)\right\}$$

$$=\frac{2R_s}{\eta\epsilon_m\epsilon_n}\left\{\sqrt{1-\left(\frac{f_c}{f}\right)^2}\frac{(m^2b+n^2a)}{(mb)^2+(na)^2}+\frac{(\beta_c/\beta)^2}{\sqrt{1-(f_c/f)^2}}\left(\frac{a\epsilon_m+b\epsilon_n}{ab}\right)\right.$$

$$\alpha_c^{TE}=\frac{2R_s}{\eta}\left\{\sqrt{1-\left(\frac{f_c}{f}\right)^2}\frac{m^2b+n^2a}{(mb)^2+(na)^2}+\frac{(f_c/f)^2}{\sqrt{1-(f_c/f)^2}}\left(\frac{a\epsilon_m+b\epsilon_n}{ab}\right)\right\}$$

which can also be written as

$$\alpha_c^{TE}=\frac{2R_s}{\epsilon_m\epsilon_n b\eta\sqrt{1-\left(\frac{f_c}{f}\right)^2}}\left\{\left(\epsilon_m+\frac{b}{a}\epsilon_n\right)\left(\frac{f_c}{f}\right)^2+\frac{b}{a}\left[1-\left(\frac{f_c}{f}\right)^2\right]\frac{m^2ab+(na)^2}{(mb)^2+(na)^2}\right\}$$

$$\epsilon_p=\begin{cases}2, & p=0\\ 1, & p\neq 0\end{cases}$$

b. $\underline{\underline{TM_{mn}^z(H_z^+=0)}}$:

$$E_x^+=B_{mn}\frac{m\pi\beta_z}{a\omega\mu\epsilon}\cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)e^{-j\beta_z z}$$

$$E_y^+=-B_{mn}\frac{n\pi\beta_z}{b\omega\mu\epsilon}\sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)e^{-j\beta_z z}$$

$$E_z^+=-jB_{mn}\frac{\beta_c^2}{\omega\mu\epsilon}\sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)e^{-j\beta_z z}$$

$$H_x^+=B_{mn}\frac{n\pi}{b\mu}\sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)e^{-j\beta_z z}$$

$$H_y^+=-B_{mn}\frac{m\pi}{a\mu}\cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)e^{-j\beta_z z}$$

$$P_{mn}=\int_0^b\int_0^a\frac{1}{2}\mathrm{Re}\left[\underline{E}\times\underline{H}^*\right]\cdot\hat{a}_z\,dx\,dy=\frac{1}{2}\mathrm{Re}\int_0^b\int_0^a\hat{a}_z\left(E_xH_y^*-E_yH_x^*\right)\cdot\hat{a}_z\,dx\,dy$$

$$=\frac{1}{2}\mathrm{Re}\int_0^b\int_0^a\left(E_xH_y^*-E_yH_x^*\right)dx\,dy$$

cont'd.

8.7 cont'd.

$$E_x H_y^* = |B_{mn}|^2 \left(\frac{m\pi}{a}\right)^2 \frac{\beta_z}{\omega\mu^2\varepsilon} \cos^2\left(\frac{m\pi}{a}x\right) \sin^2\left(\frac{n\pi}{b}y\right)$$

$$E_y H_x^* = |B_{mn}|^2 \left(\frac{n\pi}{b}\right)^2 \frac{\beta_z}{\omega\mu^2\varepsilon} \sin^2\left(\frac{m\pi}{a}x\right) \cos^2\left(\frac{n\pi}{b}y\right)$$

$$P_{mn} = |B_{mn}|^2 \frac{\beta_z}{\omega\mu^2\varepsilon}\left\{\left(\frac{m\pi}{a}\right)^2\left(\frac{a}{2}\right)\left(\frac{b}{2}\right) + \left(\frac{n\pi}{b}\right)^2\left(\frac{a}{2}\right)\left(\frac{b}{2}\right)\right\} = |B_{mn}|^2 \frac{ab}{4}\frac{\beta_z}{\omega\mu^2\varepsilon}\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]$$

$$P_c = \frac{R_s}{2}\iint \underline{J}_s \cdot \underline{J}_s \, ds = 2\left\{\frac{R_s}{2}\int_0^\ell \int_0^a |H_x^+|^2_{y=0} \, dx\,dz + \frac{R_s}{2}\int_0^\ell \int_0^b |H_y^+|_{x=0} \, dy\,dz\right\}$$

$$= R_s \ell \left\{\int_0^a |H_x^+|^2_{y=0} \, dx + \int_0^b |H_y^+|^2_{x=0} \, dy\right\}$$

$$P_c = R_s \ell \frac{|B_{mn}|^2}{\mu^2}\left\{\left(\frac{n\pi}{b}\right)^2\frac{a}{2} + \left(\frac{m\pi}{a}\right)^2\frac{b}{2}\right\}$$

Thus

$$\alpha_c^{TM} = \frac{R_s}{\mu^2}\frac{4}{ab}\frac{\omega\mu^2\varepsilon}{\beta_z}\frac{\left[\left(\frac{n\pi}{b}\right)^2\frac{a}{2} + \left(\frac{m\pi}{a}\right)^2\frac{b}{2}\right]}{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} = \frac{2R_s\omega\varepsilon}{\beta_z ab}\frac{\left(\frac{n}{b}\right)^2 a + \left(\frac{m}{a}\right)^2 b}{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$$

$$= \frac{2R_s\omega\varepsilon}{ab\omega\sqrt{\mu\varepsilon}\sqrt{1-\left(\frac{f_c}{f}\right)^2}} \frac{\frac{n^2a^3 + m^2b^3}{a^2b^2}}{\frac{(mb)^2 + (na)^2}{a^2b^2}} = \frac{2R_s}{ab\eta\sqrt{1-\left(\frac{f_c}{f}\right)^2}}\frac{m^2b^3 + n^2a^3}{(mb)^2 + (na)^2}$$

$$\alpha_c^{TM} = \frac{2R_s}{ab\eta\sqrt{1-\left(\frac{f_c}{f}\right)^2}}\frac{m^2b^3 + n^2a^3}{(mb)^2 + (na)^2}$$

$\boxed{8.8}$ $E_x = E_0 \sin(\beta_y y) e^{-\gamma z} \Rightarrow \beta_x = 0,\ \beta_y = \beta_y,\ \beta_z = -j\gamma$

a. $\beta_x^2 + \beta_y^2 + \beta_z^2 = \beta^2 \Rightarrow \beta_y^2 - \gamma^2 = \beta^2 \Rightarrow \gamma^2 = \beta_y^2 - \beta^2$

$$\nabla^2 E_x + \beta^2 E_x = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) E_x + \beta^2 E_x = -\beta_y^2 + \gamma^2 + \beta^2 = 0$$

b. $E_x(y=0) = E_0(0) e^{-\gamma z} = 0$

$E_x(y=b) = E_0 \sin(\beta_y b) e^{-\gamma z} = 0 \Rightarrow \sin(\beta_y b) = 0 \Rightarrow \beta_y b = \sin^{-1}(0) = n\pi,\ n = 1,2,3,\ldots$

$\beta_y = \left(\frac{n\pi}{b}\right),\ n = 1,2,3,4,\ \ldots$

$\gamma = \sqrt{\beta_y^2 - \beta^2} = \sqrt{\left(\frac{n\pi}{b}\right)^2 - \beta^2} = j\sqrt{\beta^2 - \left(\frac{n\pi}{b}\right)^2}$ for $f > f_c$

$\beta_c = \omega_c\sqrt{\mu\varepsilon} = \beta_y = \frac{n\pi}{b} \Rightarrow \omega_c = \frac{1}{\sqrt{\mu\varepsilon}}\left(\frac{n\pi}{b}\right) = 2\pi f_c \Rightarrow f_c = \frac{n}{2b\sqrt{\mu\varepsilon}},\ n = 1,2,3,\ldots$

$$\underline{H} = -\frac{1}{j\omega\mu}\nabla\times\underline{E} = -\frac{1}{j\omega\mu}\left[\hat{a}_x(0) + \hat{a}_y\frac{\partial E_x}{\partial z} - \hat{a}_z\frac{\partial E_x}{\partial y}\right]$$

$$H_y = -\frac{1}{j\omega\mu}\frac{\partial E_x}{\partial z} = \frac{\gamma}{j\omega\mu}E_0\sin(\beta_y y)e^{-\gamma z}\Big|_{\gamma = j\beta_z} = \frac{\beta_z}{\omega\mu}E_0\sin(\beta_y y)e^{-j\beta_z z}$$

$$P_n = \frac{1}{2}\iint \mathrm{Re}\left(\underline{E}\times\underline{H}^*\right)\cdot d\underline{s} = \frac{1}{2}\iint \mathrm{Re}\left[\hat{a}_x E_x \times \left(\hat{a}_y H_y^* + \hat{a}_z H_z^*\right)\right]\cdot\hat{a}_z\,dx\,dy$$

$$= \frac{1}{2}\int_0^b\int_0^\ell \mathrm{Re}\left(\hat{a}_z E_x H_y^* - \hat{a}_y E_x H_z^*\right)\cdot\hat{a}_z\,dx\,dy = \mathrm{Re}\left\{\frac{1}{2}\int_0^b\int_0^\ell E_x H_y^*\,dx\,dy\right\}$$

$$= \mathrm{Re}\left\{\frac{1}{2}\int_0^w\int_0^\ell |E_0|^2\sin^2(\beta_y y)\frac{\beta_z}{\omega\mu}\,dx\,dy\right\} = \frac{|E_0|^2}{4}\frac{\beta_z\ell}{\omega\mu}\int_0^b\left[1 - \cos(2\beta_y y)\right]dy$$

$$P_n = \frac{|E_0|^2}{4}\frac{\beta_z\ell}{\omega\mu}\left[y - \frac{1}{2\beta_y}\sin(2\beta_y y)\right]_0^b = \frac{|E_0|^2}{4}\frac{b\ell\beta_z}{\omega\mu}$$

$$\frac{P_n}{\ell} = \frac{|E_0|^2}{4}\frac{b\beta}{\omega\mu}\sqrt{1 - \left(\frac{f_c}{f}\right)^2}$$

$\boxed{8.9}$ a. $(f_c)_{10} = \frac{1}{2a\sqrt{\mu_0\varepsilon_0}} = \frac{30\times10^9}{2(2.25)} = 6.667\times10^9$

b. $(f_c')_{10} = \frac{1}{3}(f_c)_{10} = \frac{1}{3(2a\sqrt{\mu_0\varepsilon_0})} = \frac{1}{2a\sqrt{\mu_0\varepsilon_0\varepsilon_r}} \Rightarrow \varepsilon_r = 9$

$\varepsilon_r = 9$

8.10 a. $H_z(x=0)=H_z(x=a)=0$
$H_z(y=0)=H_z(y=b)=0$ } These are sufficient to derive the appropriate expressions and evaluate the constants.

b. $F_z=[C_1\cos(\beta_x x)+D_1\sin(\beta_x x)][C_2\cos(\beta_y y)+D_2\sin(\beta_y y)]e^{-j\beta_z z}$

$$E_x=-\frac{1}{\varepsilon}\frac{\partial F_z}{\partial y} \qquad H_x=-j\frac{1}{\omega\mu\varepsilon}\frac{\partial^2 F_z}{\partial x\partial z}$$
$$E_y=\frac{1}{\varepsilon}\frac{\partial F_z}{\partial x} \qquad H_y=-j\frac{1}{\omega\mu\varepsilon}\frac{\partial^2 F_z}{\partial y\partial z}$$
$$E_z=0 \qquad H_z=-j\frac{1}{\omega\mu\varepsilon}\left(\frac{\partial^2}{\partial z^2}+\beta^2\right)F_z$$

$$H_z=-j\frac{1}{\omega\mu\varepsilon}(-\beta_z^2+\beta^2)F_z=-j\frac{\beta_c^2}{\omega\mu\varepsilon}F_z=-j\frac{\beta_c^2}{\omega\mu\varepsilon}[C_1\cos\beta_x x+D_1\sin\beta_x x][C_2\cos\beta_y y+D_2\sin\beta_y y]e^{-j\beta_z z}$$

$$H_z(x=0)=-j\frac{\beta_c^2}{\omega\mu\varepsilon}[C_1(1)+D_1(0)][\quad]e^{-j\beta_z z}=0\Rightarrow C_1=0$$

$$H_z(x=a)=-j\frac{\beta_c^2}{\omega\mu\varepsilon}[D_1\sin(\beta_x a)][\quad]e^{-j\beta_z z}=0\Rightarrow\beta_x a=\sin^{-1}(0)=m\pi\Rightarrow\beta_x=\frac{m\pi}{a},\ m=1,2.$$

$$H_z(y=0)=-j\frac{\beta_c^2}{\omega\mu\varepsilon}D_1\sin(\beta_x x)[C_2(1)+D_2(0)]e^{-j\beta_z z}=0\Rightarrow C_2=0$$

$$H_z(y=b)=-j\frac{\beta_c^2}{\omega\mu\varepsilon}D_1\sin(\beta_x x)D_2\sin(\beta_y b)e^{-j\beta_z z}=0\Rightarrow\beta_y b=\sin^{-1}(0)=n\pi\Rightarrow\beta_y=\frac{n\pi}{b},\ n=1,2,\ldots$$

$$F_z=A_{mn}\sin(\beta_x x)\sin(\beta_y y)e^{-j\beta_z z},\quad \beta_x=\frac{m\pi}{a},\ m=1,2,\ldots\quad \beta_y=\frac{n\pi}{b},\ n=1,2,3,\ldots$$

$$E_x=-A_{mn}\frac{\beta_y}{\varepsilon}\sin(\beta_x x)\cos(\beta_y y)e^{-j\beta_z z},\quad H_x=-A_{mn}\frac{\beta_x\beta_z}{\omega\mu\varepsilon}\cos(\beta_x x)\sin(\beta_y y)e^{-j\beta_z z}$$

$$E_y=+A_{mn}\frac{\beta_x}{\varepsilon}\cos(\beta_x x)\sin(\beta_y y)e^{-j\beta_z z},\quad H_y=-A_{mn}\frac{\beta_y\beta_z}{\omega\mu\varepsilon}\sin(\beta_x x)\cos(\beta_y y)e^{-j\beta_z z}$$

$$E_z=0,\quad H_z=-jA_{mn}\frac{\beta_c^2}{\omega\mu\varepsilon}\sin(\beta_x x)\sin(\beta_y y)e^{-j\beta_z z}$$

$$\beta_c=\sqrt{\beta_x^2+\beta_y^2}=2\pi f_c\sqrt{\mu\varepsilon}=\sqrt{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2}$$

$$(f_c)_{mn}=\frac{1}{2\pi\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2},\quad \begin{array}{l}m=1,2,3,\ldots\\ n=1,2,3,\ldots\end{array}$$

c. If $a>b$ the dominant mode is the TE_{11}. Thus

$$(f_c)_{11}=\frac{1}{2\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{1}{a}\right)^2+\left(\frac{1}{b}\right)^2}$$

$\boxed{8.11}$ a. $H_x(y=0) = H_x(y=b) = 0$ } These are sufficient to derive the appropriate
$H_y(x=0) = H_y(x=a) = 0$ } expressions and evaluate the constants.

b. $A_z = [C_1 \cos\beta_x x + D_1 \sin\beta_x x][C_2 \cos\beta_y y + D_2 \sin\beta_y y]e^{-j\beta_z z}$

$E_x = -j\frac{1}{\omega\mu\varepsilon}\frac{\partial^2 A_z}{\partial x \partial z}$ $\quad H_x = \frac{1}{\mu}\frac{\partial A_z}{\partial y} = \frac{\beta_y}{\mu}[C_1\cos\beta_x x + D_1 \sin\beta_x x][-C_2\sin\beta_y y + D_2\cos\beta_y y]e^{-j\beta_z z}$

$E_y = -j\frac{1}{\omega\mu\varepsilon}\frac{\partial^2 A_z}{\partial y \partial z}$ $\quad H_y = -\frac{1}{\mu}\frac{\partial A_z}{\partial x} = -\frac{\beta_x}{\mu}[-C_1\sin\beta_x x + D_1\cos\beta_x x][C_2\cos\beta_y y + D_2 \sin\beta_y y]e^{-j\beta_z z}$

$E_z = -j\frac{1}{\omega\mu\varepsilon}\left(\frac{\partial^2}{\partial z^2} + \beta^2\right)A_z$ $\quad H_z = 0$

$H_y(x=0) = -\frac{\beta_x}{\mu}[C_1(0) + D_1(1)][+C_2\cos\beta_y y + D_2\sin\beta_y y]e^{-j\beta_z z} = 0 \Rightarrow D_1 = 0$

$H_y(x=a) = -\frac{\beta_x}{\mu}[-C_1\sin(\beta_x a)][\quad]e^{-j\beta_z z} = 0 \Rightarrow \beta_x a = \sin^{-1}(0) = m\pi \Rightarrow \beta_x = \left(\frac{m\pi}{a}\right),\ m = 0,1,2\ldots$

$H_x(y=0) = \frac{\beta_y}{\mu}[\quad][-C_2(0) + D_2(1)]e^{-j\beta_z z} = 0 \Rightarrow D_2 = 0$

$H_x(y=b) = \frac{\beta_y}{\mu}[\quad][-C_2\sin(\beta_y b)]e^{-j\beta_z z} = 0 \Rightarrow \beta_y b = \sin^{-1}(0) = n\pi \Rightarrow \beta_y = \left(\frac{n\pi}{b}\right),\ n = 0,1,2,\ldots$

$m = n \neq 0$

$A_z = B_{mn}\cos(\beta_x x)\cos(\beta_y y)e^{-j\beta_z z}$, $\beta_x = \frac{m\pi}{a}$, $m = 0,1,2,\ldots$; $\beta_y = \frac{n\pi}{b}$, $n = 0,1,2,\ldots$
$m = n \neq 0$

$E_x = B_{mn}\frac{\beta_x\beta_z}{\omega\mu\varepsilon}\sin(\beta_x x)\cos(\beta_y y)e^{-j\beta_z z}$ $\quad H_x = -A_{mn}\frac{\beta_y}{\mu}\cos(\beta_x x)\sin(\beta_y y)e^{-j\beta_z z}$

$E_y = B_{mn}\frac{\beta_y\beta_z}{\omega\mu\varepsilon}\cos(\beta_x x)\sin(\beta_y y)e^{-j\beta_z z}$ $\quad H_y = A_{mn}\frac{\beta_x}{\mu}\sin(\beta_x x)\cos(\beta_y y)e^{-j\beta_z z}$

$E_z = -jA_{mn}\frac{\beta_c^2}{\omega\mu\varepsilon}\cos(\beta_x x)\cos(\beta_y y)e^{-j\beta_z z}$ $\quad H_z = 0$

$$\beta_c = 2\pi f_c\sqrt{\mu\varepsilon} = \sqrt{\beta_x^2 + \beta_y^2} = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$$

$$(f_c)_{mn} = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2},\quad \left.\begin{matrix} m = 0,1,2,\ldots \\ n = 0,1,2,\ldots \end{matrix}\right\} m = n \neq 0$$

c. If $a > b$ the dominant mode is the TM^z_{10}. Thus

$$(f_c)^{TM^z}_{10} = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\left(\frac{\pi}{a}\right) = \frac{1}{2a\sqrt{\mu\varepsilon}}$$

8.12

TE^z

B.C.'s: $E_x(y=0)=E_x(y=b)=0$, $H_y(x=0)=H_y(x=a)=0$ $\Big\}$ $F_z=[C_1\cos\beta_x x+D_1\sin\beta_x x][C_2\cos\beta_y y+D_2\sin\beta_y y]e^{-j\beta_z z}$

$$E_x=-\frac{1}{\varepsilon}\frac{\partial F_z}{\partial y}=-\frac{\beta_y}{\varepsilon}[C_1\cos\beta_x x+D_1\sin\beta_x x][-C_2\sin\beta_y y+D_2\cos\beta_y y]e^{-j\beta_z z}$$

$$H_y=-j\frac{1}{\omega\mu\varepsilon}\frac{\partial^2 F_z}{\partial y\partial z}=-\frac{\beta_y\beta_z}{\omega\mu\varepsilon}[C_1\cos\beta_x x+D_1\sin\beta_x x][-C_2\sin(\beta_y y)+D_2\cos\beta_y y]e^{-j\beta_z z}$$

$E_x(y=0)=0 \Rightarrow D_2=0$

$E_x(y=b)=0 \Rightarrow \sin(\beta_y b)=0 \Rightarrow \beta_y b=\sin^{-1}(0)=n\pi \Rightarrow \beta_y=\frac{n\pi}{b},\ n=0,1,2,\cdots$

$H_y(x=0)=0 \Rightarrow C_1=0$

$H_y(x=a)=0 \Rightarrow \sin(\beta_x a)=0 \Rightarrow \beta_x a=\sin^{-1}(0)=m\pi \Rightarrow \beta_x=\frac{m\pi}{a},\ m=1,2,3,\cdots$

$$F_z=A_{mn}\sin(\beta_x x)\cos(\beta_y y)e^{-j\beta_z z},\ \beta_x=\left(\frac{m\pi}{a}\right), m=1,2,3,\cdots;\ \beta_y=\frac{n\pi}{b}, n=0,1,2,\cdots$$

$E_x=-\frac{1}{\varepsilon}\frac{\partial F_z}{\partial y}=A_{mn}\frac{\beta_y}{\varepsilon}\sin(\beta_x x)\sin(\beta_y y)e^{-j\beta_z z}$, $H_x=-j\frac{1}{\omega\mu\varepsilon}\frac{\partial^2 F_z}{\partial x\partial z}=-A_{mn}\frac{\beta_x\beta_z}{\omega\mu\varepsilon}\cos(\beta_x x)\cos(\beta_y y)e^{-j\beta_z z}$

$E_y=\frac{1}{\varepsilon}\frac{\partial F_z}{\partial x}=A_{mn}\frac{\beta_x}{\varepsilon}\cos(\beta_x x)\cos(\beta_y y)e^{-j\beta_z z}$, $H_y=-j\frac{1}{\omega\mu\varepsilon}\frac{\partial^2 F_z}{\partial y\partial z}=+A_{mn}\frac{\beta_y\beta_z}{\omega\mu\varepsilon}\sin(\beta_x x)\sin(\beta_y y)e^{-j\beta_z z}$

$E_z=0$, $H_z=-j\frac{1}{\omega\mu\varepsilon}\left(\frac{\partial^2}{\partial z^2}+\beta^2\right)F_z=-j\frac{\beta_c^2}{\omega\mu\varepsilon}\sin(\beta_x x)\cos(\beta_y y)e^{-j\beta_z z}$

$(f_c)_{mn}=\frac{1}{2\pi\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2}$, $\lambda_g=\frac{\lambda}{\sqrt{1-(f_c/f)^2}}$, $Z_w=\frac{E_x}{H_y}=-\frac{E_y}{H_x}=\frac{\omega\mu}{\beta_z}$

$$\beta_z=\pm\sqrt{\beta^2-\beta_c^2}=\begin{cases}\beta\sqrt{1-(f_c/f)^2} & f>f_c\\ 0 & f=f_c\\ -j\beta\sqrt{(f_c/f)^2-1} & f<f_c\end{cases}$$

Dominant Mode is the TE_{10}: $f_c^{10}=\frac{1}{2a\sqrt{\mu\varepsilon}}$

TM^z

$\Rightarrow A_z=[C_1\cos(\beta_x x)+D_1\sin(\beta_x x)][C_2\cos\beta_y y+D_2\sin\beta_y y]e^{-j\beta_z z}$

$$E_x=-j\frac{1}{\omega\mu\varepsilon}\frac{\partial^2 A_z}{\partial x\partial z}=-\frac{\beta_x\beta_z}{\omega\mu\varepsilon}[-C_1\sin(\beta_x x)+D_1\cos(\beta_x x)][C_2\cos\beta_y y+D_2\sin\beta_y y]e^{-j\beta_z z}$$

$$H_y=-\frac{1}{\mu}\frac{\partial A_z}{\partial x}=-\frac{\beta_x}{\mu}[-C_1\sin(\beta_x x)+D_1\cos(\beta_x x)][C_2\cos\beta_y y+D_2\sin\beta_y y]e^{-j\beta_z z}$$

$E_x(y=0)=0 \Rightarrow C_2=0$; $E_x(y=b)=\sin(\beta_y b)=0 \Rightarrow \beta_y=\left(\frac{n\pi}{b}\right),\ n=1,2,3,\cdots$

$H_y(x=0)=0 \Rightarrow D_1=0$; $H_y(x=a)=\sin(\beta_x a)=0 \Rightarrow \beta_x=\left(\frac{m\pi}{a}\right),\ m=0,1,2,\cdots$

$$A_z=B_{mn}\cos(\beta_x x)\sin(\beta_y y)e^{-j\beta_z z},\ \beta_x=\frac{m\pi}{a}, m=0,1,2,\cdots;\ \beta_y=\frac{n\pi}{b}, n=1,2,3,\cdots$$

$E_x=-j\frac{1}{\omega\mu\varepsilon}\frac{\partial^2 A_z}{\partial x\partial z}=B_{mn}\frac{\beta_x\beta_z}{\omega\mu\varepsilon}\sin(\beta_x x)\sin(\beta_y y)e^{-j\beta_z z}$, $H_x=\frac{1}{\mu}\frac{\partial A_z}{\partial y}=B_{mn}\frac{\beta_y}{\mu}\cos\beta_x x\cos\beta_y y\,e^{-j\beta_z z}$

$E_y=-j\frac{1}{\omega\mu\varepsilon}\frac{\partial^2 A_z}{\partial y\partial z}=-B_{mn}\frac{\beta_y\beta_z}{\omega\mu\varepsilon}\cos(\beta_x x)\cos(\beta_y y)e^{-j\beta_z z}$, $H_y=-\frac{1}{\mu}\frac{\partial A_z}{\partial x}=B_{mn}\frac{\beta_x}{\mu}\sin(\beta_x x)\sin(\beta_y y)e^{-j\beta_z z}$

$E_z=-j\frac{1}{\omega\mu\varepsilon}\left(\frac{\partial^2}{\partial z^2}+\beta^2\right)A_z=-j\frac{\beta_c^2 B_{mn}}{\omega\mu\varepsilon}\cos(\beta_x x)\sin(\beta_y y)e^{-j\beta_z z}$, $H_z=0$

$Z_w=\frac{E_x}{H_y}=-\frac{E_y}{H_x}=\frac{\beta_z}{\omega\varepsilon}$ $\Big\}$ Dominant Mode is the TM_{01} $\left(f_c=\frac{1}{2b\sqrt{\mu\varepsilon}}\right)$

If $a>b$: Lowest Order Mode is TE_{10}; If $a<b$, Lowest Order Mode is TM_{01}

$\boxed{8.13}$ B.C.'s: $\left.\begin{array}{l} H_x(y=0)=H_x(y=b)=0 \\ E_y(x=0)=E_y(x=a)=0 \end{array}\right\}$ Necessary and sufficient boundary conditions

TE^z: $F_z=[C_1\cos(\beta_x x)+D_1\sin(\beta_x x)][C_2\cos(\beta_y y)+D_2\sin(\beta_y y)]e^{-j\beta_z z}$

$H_x=-j\frac{1}{\omega\mu\varepsilon}\frac{\partial^2 F_z}{\partial x\partial z}=-\frac{\beta_x\beta_z}{\omega\mu\varepsilon}[-C_1\sin\beta_x x+D_1\cos\beta_x x][C_2\cos(\beta_y y)+D_2\sin(\beta_y y)]e^{-j\beta_z z}$

$E_y=\frac{1}{\varepsilon}\frac{\partial F_z}{\partial x}=\frac{\beta_x}{\varepsilon}[-C_1\sin(\beta_x x)+D_1\cos(\beta_x x)][C_2\cos(\beta_y y)+D_2\sin(\beta_y y)]e^{-j\beta_z z}$

$H_x(y=0)=0\Rightarrow C_2=0$; $H_x(y=b)=0\Rightarrow\sin(\beta_y b)=0\Rightarrow\beta_y=\left(\frac{n\pi}{b}\right), n=1,2,3,\ldots$

$E_y(x=0)=0\Rightarrow D_1=0$; $E_y(x=a)=0\Rightarrow\sin(\beta_x a)=0\Rightarrow\beta_x=\left(\frac{m\pi}{a}\right), m=0,1,2,\ldots$

$F_z=A_{mn}\cos(\beta_x x)\sin(\beta_y y)e^{-j\beta_z z}$, $\beta_x=\left(\frac{m\pi}{a}\right), m=0,1,2,\ldots$; $\beta_y=\left(\frac{n\pi}{b}\right), n=1,2,3,\ldots$

$E_x=-\frac{1}{\varepsilon}\frac{\partial F_z}{\partial y}=-A_{mn}\frac{\beta_y}{\varepsilon}\cos(\beta_x x)\cos(\beta_y y)e^{-j\beta_z z}$, $H_x=-j\frac{1}{\omega\mu\varepsilon}\frac{\partial^2 F_z}{\partial x\partial z}=A_{mn}\frac{\beta_x\beta_z}{\omega\mu\varepsilon}\sin\beta_x x\sin\beta_y y\, e^{-j\beta_z z}$

$E_y=\frac{1}{\varepsilon}\frac{\partial F_z}{\partial x}=-A_{mn}\frac{\beta_x}{\varepsilon}\sin(\beta_x x)\sin(\beta_y y)e^{-j\beta_z z}$, $H_y=-j\frac{1}{\omega\mu\varepsilon}\frac{\partial^2 F_z}{\partial y\partial z}=-A_{mn}\frac{\beta_y\beta_z}{\omega\mu\varepsilon}\cos\beta_x x\cos\beta_y y\, e^{-j\beta_z z}$

$E_z=0$, $H_z=-j\frac{1}{\omega\mu\varepsilon}\left(\frac{\partial^2}{\partial z^2}+\beta^2\right)F_z=-jA_{mn}\frac{\beta_c^2}{\omega\mu\varepsilon}\cos(\beta_x x)\sin\beta_y y\, e^{-j\beta_z z}$

$(f_c)_{mn}=\frac{1}{2\pi\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2}$, $\lambda_g=\frac{\lambda}{\sqrt{1-(f_c/f)^2}}$, $Z_w=\frac{E_x}{H_y}=-\frac{E_y}{H_x}=\frac{\omega\mu}{\beta_z}$

$\beta_z=\pm\sqrt{\beta^2-\beta_c^2}=\begin{cases}\beta\sqrt{1-\left(\frac{f_c}{f}\right)^2} & f>f_c\\ 0 & f=f_c\\ -j\beta\sqrt{\left(\frac{f_c}{f}\right)^2-1} & f<f_c\end{cases}$ Dominant Mode is the TE_{01} $(f_c)_{01}=\frac{1}{2b\sqrt{\mu\varepsilon}}$

TM^z: $A_z=[C_1\cos(\beta_x x)+D_1\sin(\beta_x x)][C_2\cos(\beta_y y)+D_2\sin(\beta_y y)]e^{-j\beta_z z}$

$H_x=+\frac{1}{\mu}\frac{\partial A_z}{\partial y}=\frac{\beta_y}{\mu}[C_1\cos\beta_x x+D_1\sin\beta_x x][-C_2\sin\beta_y y+D_2\cos\beta_y y]e^{-j\beta_z z}$

$E_y=-j\frac{1}{\omega\mu\varepsilon}\frac{\partial^2 A_z}{\partial y\partial z}=-\frac{\beta_y\beta_z}{\omega\mu\varepsilon}[C_1\cos(\beta_x x)+D_1\sin(\beta_x x)][-C_2\sin(\beta_y y)+D_2\cos(\beta_y y)]e^{-j\beta_z z}$

$H_x(y=0)=0\Rightarrow D_2=0$; $H_x(y=b)=0\Rightarrow\sin(\beta_y b)=0\Rightarrow\beta_y=\left(\frac{n\pi}{b}\right), n=0,1,2,\ldots$

$E_y(x=0)=0\Rightarrow C_1=0$; $E_y(x=a)=0\Rightarrow\sin(\beta_x a)=0\Rightarrow\beta_x=\left(\frac{m\pi}{a}\right), m=1,2,3,\ldots$

$A_z=B_{mn}\sin(\beta_x x)\cos(\beta_y y)e^{-j\beta_z z}$, $\beta_x=\frac{m\pi}{a}, m=1,2,3,\ldots$; $\beta_y=\frac{n\pi}{b}, n=0,1,2,\ldots$

$E_x=-j\frac{1}{\omega\mu\varepsilon}\frac{\partial^2 A_z}{\partial x\partial z}=-B_{mn}\frac{\beta_x\beta_z}{\omega\mu\varepsilon}\cos\beta_x x\cos\beta_y y\, e^{-j\beta_z z}$, $H_x=\frac{1}{\mu}\frac{\partial A_z}{\partial y}=-B_{mn}\frac{\beta_y}{\mu}\sin\beta_x x\sin\beta_y y\, e^{-j\beta_z z}$

$E_y=-j\frac{1}{\omega\mu\varepsilon}\frac{\partial^2 A_z}{\partial y\partial z}=B_{mn}\frac{\beta_y\beta_z}{\omega\mu\varepsilon}\sin\beta_x x\sin\beta_y y\, e^{-j\beta_z z}$, $H_y=-\frac{1}{\mu}\frac{\partial A_z}{\partial x}=-B_{mn}\frac{\beta_x}{\mu}\cos\beta_x x\cos\beta_y y\, e^{-j\beta_z z}$

$E_z=-j\frac{1}{\omega\mu\varepsilon}\left(\frac{\partial^2}{\partial z^2}+\beta^2\right)A_z=-j\frac{\beta_c^2}{\omega\mu\varepsilon}\sin\beta_x x\cos\beta_y y\, e^{-j\beta_z z}$, $H_z=0$

$Z_w=\frac{E_x}{H_y}=-\frac{E_y}{H_x}=\frac{\beta_z}{\omega\varepsilon}$ $\Big\}$ $(f_c)_{mn}$, λ_g and β_z are the same as above.

The dominant mode is the TM_{10} mode. $(f_c)_{10}=\frac{1}{2a\sqrt{\mu\varepsilon}}$

If $a>b$: Lowest order mode is TM_{10}.

If $a<b$: Lowest order mode is TE_{01}.

8.14

$$TE^z: \ F_z^+(x,y,z) = \left[C_1\cos(\beta_x x) + D_1\sin(\beta_x x)\right]\left[C_2\cos(\beta_y y) + D_2\sin(\beta_y y)\right] A_3 e^{-j\beta_z z} \quad (8\text{-}7)$$

$$\beta_x^2 + \beta_y^2 + \beta_z^2 = \beta^2 \quad (8\text{-}7a)$$

$$E_x = -\frac{1}{\varepsilon}\frac{\partial F_z}{\partial y} \qquad H_x = -j\frac{1}{\omega\mu\varepsilon}\frac{\partial^2 F_z}{\partial x\partial z}$$

$$E_y = \frac{1}{\varepsilon}\frac{\partial F_z}{\partial x} \qquad H_y = -j\frac{1}{\omega\mu\varepsilon}\frac{\partial^2 F_z}{\partial y\partial z}$$

$$E_z = 0 \qquad H_z = -j\frac{1}{\omega\mu\varepsilon}\left(\frac{\partial^2}{\partial z^2}+\beta^2\right)F_z$$

B.C.s: $E_x^+(0\le x\le a, y=0, z) = 0 = E_x^+(0\le x\le a, y=b, z)$

$H_y^+(x=0, 0\le y\le b, z) = 0 = H_y^+(x=a, 0\le y\le b, z)$

$$E_x^+ = -\frac{1}{\varepsilon}\frac{\partial F_z^+}{\partial y} = -\frac{\beta_y}{\varepsilon}\left[C_1\cos(\beta_x x) + D_1\sin(\beta_x x)\right]\left[-C_2\sin(\beta_y y) + D_2\cos(\beta_y y)\right] A_3 e^{-j\beta_z z}$$

B.C. $E_x^+(y=0) = -\frac{\beta_y}{\varepsilon}\left[\quad\right]\left[-C_2(0) + D_2(1)\right] A_3 e^{-j\beta_z z} = 0 \Rightarrow \boxed{D_2 = 0}$

$E_x^+(y=b) = -\frac{\beta_y}{\varepsilon}\left[\quad\right]\left[-C_2(\sin\beta_y b)\right] A_3 e^{-j\beta_z z} = 0$

$$\sin(\beta_y b) = 0 \Rightarrow \beta_y b = \sin^{-1}(0) \Rightarrow \beta_y b = n\pi, \ n = 0, 1, 2, 3, \ldots$$

$$\boxed{\beta_y = \left(\frac{n\pi}{b}\right), \ n = 0, 1, 2, 3, \ldots}$$

$$F_z^+(x,y,z) = C_2\left[C_1\cos(\beta_x x) + D_1\sin(\beta_x x)\right]\cos(\beta_y y) A_3 e^{-j\beta_z z}$$

$$H_y^+ = -j\frac{1}{\omega\mu\varepsilon}\frac{\partial^2 F_z^+}{\partial y\partial z} = +j\frac{\beta_y(-j\beta_z)}{\omega\mu\varepsilon} C_2\left[C_1\cos(\beta_x x) + D_1\sin(\beta_x x)\right]\sin(\beta_y y) A_3 e^{-j\beta_z z}$$

BC. $H_y^+(x=0) = j\frac{\beta_y(-j\beta_z)}{\omega\mu\varepsilon} C_2\left[C_1(1) + D_1(0)\right]\sin(\beta_y y) A_3 e^{-j\beta_z z} = 0 \Rightarrow \boxed{C_1 = 0}$

$H_y^+(x=a) = j\frac{\beta_y(-j\beta_z)}{\omega\mu\varepsilon} C_2\left[D_1\sin(\beta_x a)\right]\sin(\beta_y y) A_3 e^{-j\beta_z z} = 0$

$$\sin(\beta_x a) = 0 \Rightarrow \beta_x a = \sin^{-1}(0) = m\pi$$

$$\boxed{\beta_x = \left(\frac{m\pi}{a}\right), \ m = \cancel{0}, 1, 2, \ldots}$$

$$F_z^+(x,y,z) = A_{mn}\sin(\beta_x x)\cos(\beta_y y) e^{-j\beta_z z} \quad \left\{ \begin{array}{l} \beta_x = \left(\frac{m\pi}{a}\right), \ m = 1, 2, 3, \ldots \\ \beta_y = \left(\frac{n\pi}{b}\right), \ n = 0, 1, 2, \ldots \end{array} \right.$$

TE_{10}^z Dominant Mode:

$$(f_c)_{mn} = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$$

$$(f_c)_{10} = \frac{1}{2\pi\sqrt{\mu_r\mu_0\varepsilon_r\varepsilon_0}}\left(\frac{\pi}{a}\right) = \frac{1}{2a\sqrt{\mu_0\varepsilon_0}\sqrt{\mu_r\varepsilon_r}} = \frac{30\times10^9}{2(2.286)\sqrt{1(81)}}$$

$$(f_c)_{10} = \frac{30\times10^9}{2(2.286)(9)} = \boxed{0.729 \text{ GHz}}$$

8.15 Since dominant mode is TE_{10} :

a. $$(\alpha_c)^{10} = \frac{R_s}{\eta b}\frac{1+2\left(\frac{b}{a}\right)\left(\frac{f_c}{f}\right)^2}{\sqrt{1-(f_c/f)^2}}, \quad R_s = \sqrt{\frac{\omega\mu}{2\sigma}}$$

$$(f_c)^{10} = \frac{1}{2a\sqrt{\mu\varepsilon}} = \frac{30\times10^9}{2(2.286)\sqrt{2.56}} = 4.1\times10^9$$

$$(\alpha_c)^{10} = \frac{1}{b\sqrt{\frac{\mu}{\varepsilon}}}\sqrt{\frac{\omega\mu}{2\sigma}}\frac{1+2\left(\frac{b}{a}\right)\left(\frac{f_c}{f}\right)^2}{\sqrt{1-(f_c/f)^2}} = \frac{1}{b}\sqrt{\frac{\omega\varepsilon}{2\sigma}}\frac{1+2\left(\frac{b}{a}\right)\left(\frac{f_c}{f}\right)^2}{\sqrt{1-(f_c/f)^2}}$$

$b = 1.016\times10^{-2}$ m
$f = 6.15\times10^9$
$f_c = 4.1\times10^9$
$\sigma = 5.76\times10^7$ S/m

$$(\alpha_c)^{10} = 1.6054\times10^{-2}\ \text{Np/m} = 8.68(1.6054\times10^{-2})$$
$$= 1.393\times10^{-1}\ \text{dB/m}$$

b. $$\alpha_d = \frac{\varepsilon_e''}{\varepsilon_e'}\frac{\pi}{\lambda}\left(\frac{\lambda_g}{\lambda}\right) = \tan\delta_e\,\frac{\pi}{\lambda}\frac{\lambda_g}{\lambda}$$

$$\lambda = \frac{\lambda_0}{\sqrt{\varepsilon_r}} = \frac{30\times10^9/6.15\times10^9}{\sqrt{2.56}} = \frac{4.8747}{1.6} = 3.0467\ \text{cm} = 3.0467\times10^{-2}\ \text{m}$$

$$\lambda_g = \frac{\lambda}{\sqrt{1-(f_c/f)^2}} = \frac{3.0467\times10^{-2}}{\sqrt{1-\left(\frac{4.1}{6.15}\right)^2}} = 4.0862\times10^{-2}\ \text{m}$$

$$\alpha_d = 4\times10^{-4}\left(\frac{\pi}{3.0467\times10^{-2}}\right)\left(\frac{4.0862}{3.0467}\right) = 5.5318\times10^{-2}\ \text{Np/m} = 4.802?\ \text{dB/m}$$

8.16 a. From Problem 8.15 $(f_c)^{10} = 4.1\times10^9$

b. $$\cos\theta = \frac{\beta_z}{\beta} = \sqrt{1-\left(\frac{f_c}{f}\right)^2} \Rightarrow \left(\frac{f_c}{f}\right)^2 = 1-\cos^2\theta \Rightarrow f = \frac{f_c}{\sqrt{1-\cos^2\theta}} = \frac{4.1\times10^9}{\sqrt{1-0.5}} = 5.798\times10^9$$

c. $$\lambda_g = \frac{\lambda}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}} = \frac{30\times10^9/(\sqrt{2.56}\times5.798\times10^9)}{\sqrt{1-\left(\frac{4.1}{5.798}\right)^2}} = \frac{3.2338}{0.707} = 4.574\ \text{cm}$$

d. $$\beta_z = \beta\sqrt{1-(f_c/f)^2} = \frac{2\pi}{\lambda_g}$$

$$\beta_z d = \frac{2\pi}{\lambda_g}d = 2\pi \Rightarrow d = \lambda_g = 4.574\ \text{cm}$$

$\boxed{8.17}$ $a = 0.9$ in. $(2.286$ cm$)$, $b = 0.4$ in. $(1.016$ cm$)$, $f = 10$ GHz

a. $$Z_w = \frac{\eta}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}} = \frac{377}{\sqrt{1-\left(\frac{6.56}{10}\right)^2}} = 499.5$$

$$\Gamma_L = \frac{\eta_0 - Z_w}{\eta_0 + Z_w} = \frac{377-499.5}{377+499.5} = -0.1398 = 0.1398\angle 180^\circ$$

b. $$SWR = \frac{1+|\Gamma_L|}{1-|\Gamma_L|} = \frac{1+0.1398}{1-0.1398} = 1.325$$

c. $$\Gamma_{in} = \Gamma_L e^{-2\gamma z} = \Gamma_L e^{-2\alpha_c z} e^{-j2\beta z} \Rightarrow |\Gamma_{in}| = |\Gamma_L| e^{-2\alpha_c z}$$

$$\alpha_c = \frac{R_s}{b\eta}\frac{\left[1+2\left(\frac{b}{a}\right)\left(\frac{f_c}{f}\right)^2\right]}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}} = \frac{1}{b}\frac{\sqrt{\frac{\omega\mu}{2\sigma}}}{\sqrt{\frac{\mu}{\varepsilon}}}\frac{\left[1+2\left(\frac{b}{a}\right)\left(\frac{f_c}{f}\right)^2\right]}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}} = \frac{1}{b}\sqrt{\frac{\omega\varepsilon}{2\sigma}}\frac{\left[1+2\left(\frac{b}{a}\right)\left(\frac{f_c}{f}\right)^2\right]}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}}$$

$(f_c)_{10} = 6.56\times10^9$, $\sigma = 5.76\times10^7$ S/m, $f = 10$ GHz, $\varepsilon = \varepsilon_0$

$$\alpha_c = 1.25217\times10^{-2}\ \text{Np/m} = 1.08813\times10^{-1}\ \text{dB/m}$$

$$\lambda_g = \frac{\lambda}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}} = \frac{\lambda_0/\sqrt{\varepsilon_r}}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}} = \frac{30\times10^9/\left(10\times10^9\sqrt{1}\right)}{\sqrt{1-\left(\frac{6.56}{10}\right)^2}} = \frac{3}{0.7548} = 3.975\ \text{cm} = 3.975\times10^{-2}\ \text{m}$$

$\underline{z=0}$: $|\Gamma_{in}| = |\Gamma_L| = 0.1398$, $SWR = 1.325$

$\underline{z=\lambda_g/4}$: $$|\Gamma_{in}| = |\Gamma_L| e^{-2\alpha z} = 0.1398\, e^{-2(1.25217\times10^{-2})(3.795\times10^{-2}/4)} = 0.1398\, e^{-0.000238}$$

$$|\Gamma_{in}| = 0.1398(0.9998) = 0.1398 \Rightarrow SWR = \frac{1+0.1398}{1-0.1398} = 1.325$$

$\underline{z=\lambda_g/2}$: $$|\Gamma_{in}| = |\Gamma_L| e^{-2\alpha z} = 0.1398\, e^{-2(1.25217\times10^{-2})(3.795\times10^{-2}/2)} = 0.1398\, e^{-0.000475}$$

$$|\Gamma_{in}| = 0.1398(0.9995) = 0.1397 \Rightarrow SWR = \frac{1+0.1397}{1-0.1397} = 1.325$$

8.18

$\varepsilon_r = 81$, $a = 2.286$ cm, $b = 1.016$ cm, $\mu_r = 1$

a. Since the walls of the waveguide can be modeled as PMC, the expressions for the cutoff frequencies are the same as for the PEC except reversed. Therefore for the PMC, the expressions for the cutoff frequencies are:

$$TE^z(PMC) = TM^z(PEC) = \boxed{\frac{1}{2\pi\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2} \quad \begin{matrix} m=1,2,3,\ldots \\ n=1,2,3,\ldots \end{matrix}}$$

$$TM^z(PMC) = TE^z(PEC) = \boxed{\frac{1}{2\pi\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2} \quad \left.\begin{matrix} m=0,1,2,\ldots \\ n=0,1,2,\ldots \end{matrix}\right\} m=n\neq 0}$$

b. For $a>b$, the dominant mode is the $\boxed{TM^z_{10}(PMC)}$

c. $f_c(TM^z_{10}) = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\left(\frac{\pi}{a}\right) = \frac{1}{2a\sqrt{\mu\varepsilon}}\Big|_{\varepsilon=81\varepsilon_0,\ \mu=\mu_0}$

$$= \frac{1}{2a}\frac{1}{\sqrt{\varepsilon_r\varepsilon_0\mu_0}} = \frac{1}{2a}\frac{v_0}{\sqrt{\varepsilon_r}} = \frac{30\times10^9}{2(2.286)\sqrt{81}}$$

$$f_c(TM^z_{10}) = 0.7291\times10^9 = \boxed{0.7291 \text{ GHz}}$$

d. $\beta_z = \beta\sqrt{1-(f_c/f)^2} = \omega\sqrt{\mu\varepsilon}\sqrt{1-\left(\frac{f_c}{2f_c}\right)^2} = 2\pi f\sqrt{\mu\varepsilon}\sqrt{1-\left(\frac{1}{2}\right)^2}$

$$= 2\pi f\frac{\sqrt{\varepsilon_r}}{v_0}\sqrt{1-\left(\frac{1}{4}\right)} = 2\pi\frac{(2\times0.7291\times10^9)}{30\times10^9}\sqrt{\frac{3}{4}}$$

$$\beta_z = 2.3803 \text{ rad/cm} = 2.3803\left(\frac{180}{\pi}\right) = 136.382 \text{ degrees/cm}$$

$$\beta_z d = 136.382(d) = 360 \Rightarrow d = 360/136.382 = 2.6396$$

$$\boxed{d = 2.6396 \text{ cm}}$$

$\boxed{8.19}$ $(f_r)_{101} = \frac{1}{2\pi\sqrt{\mu_0\varepsilon_0}}\sqrt{\left(\frac{\pi}{a}\right)^2+\left(\frac{\pi}{c}\right)^2} = \frac{1}{2\sqrt{\mu_0\varepsilon_0}}\sqrt{\left(\frac{1}{a}\right)^2+\left(\frac{1}{c}\right)^2} = \frac{v_0}{2}\sqrt{\left(\frac{1}{a}\right)^2+\left(\frac{1}{c}\right)^2}$

$$(f_r)^2 = \frac{v_0^2}{4}\left[\frac{1}{a^2}+\frac{1}{c^2}\right] \Rightarrow c = \frac{1}{\sqrt{\left(\frac{2f_r}{v_0}\right)^2-\left(\frac{1}{a}\right)^2}} = \frac{1}{\sqrt{\left[\frac{2(10)}{30}\right]^2-\left(\frac{1}{2}\right)^2}} = 2.2678\text{ cm}$$

$\boxed{8.20}$ $(f_r)_{101} = \frac{1}{2\sqrt{\mu\varepsilon}}\sqrt{\frac{1}{a^2}+\frac{1}{c^2}} \overset{c=a}{=} \frac{1}{\sqrt{2}\sqrt{\mu\varepsilon}\,a} = \frac{v}{\sqrt{2}\,a} \Rightarrow a = \frac{v}{\sqrt{2}(f_r)_{101}}$

a. $a = \frac{v_0/\sqrt{\varepsilon_r}}{\sqrt{2}(10^9)} = \frac{30\times10^9}{\sqrt{2}\times10^9} = 21.216\text{ cm} = 0.21216\text{ m} = c,\ b = a/2 = 0.10608\text{ m}$

b. $a = \frac{v_0/\sqrt{\varepsilon_r}}{\sqrt{2}(10^9)} = \frac{30\times10^9/1.6}{\sqrt{2}(10^9)} = 13.26\text{ cm} = 0.1326\text{ m} = c,\ b = a/2 = 0.0663\text{ m}$

$\boxed{8.21}$ For a PMC resonator the:

TE^z modes are analogous to the TM^z modes of a PEC resonator

TM^z modes are analogous to the TE^z modes of a PEC resonator

a. TE^z PMC (From Section 8.3.2): $\beta_x = \left(\frac{m\pi}{a}\right),\ \beta_y = \left(\frac{n\pi}{b}\right),\ \beta_z = \left(\frac{p\pi}{c}\right)$ $\quad m = 1,2,3,\ldots;\ n = 1,2,3,\ldots;\ p = 0,1,2,\ldots$

b. TM^z PMC (From Section 8.3.1): $\beta_x = \left(\frac{m\pi}{a}\right),\ \beta_y = \left(\frac{n\pi}{b}\right),\ \beta_z = \left(\frac{p\pi}{c}\right)$ $\quad m = 0,1,2,\ldots;\ n = 0,1,2,\ldots;\ p = 1,2,3,\ldots;\ m = n \neq 0$

c. TE^z PMC: $(f_r)^{TE}_{mnp} = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2+\left(\frac{p\pi}{c}\right)^2}$ $\quad m = 1,2,3,\ldots;\ n = 1,2,3,\ldots;\ p = 0,1,2,\ldots$

TM^z PMC: $(f_r)^{TM}_{mnp} = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2+\left(\frac{p\pi}{c}\right)^2}$ $\quad m = 0,1,2,\ldots;\ n = 0,1,2,\ldots;\ p = 1,2,3,\ldots;\ m = n \neq 0$

d. For $a/b = 2$, $c/b = 4$:

Dominant Mode: $TM_{101} \Rightarrow (f_r)^{TM}_{101} = \frac{1}{2\pi\sqrt{\mu_0\varepsilon_0\varepsilon_r}}\sqrt{\left(\frac{\pi}{a}\right)^2+\left(\frac{\pi}{c}\right)^2}$

$$(f_r)^{TM}_{101} = \frac{30\times10^9}{2\sqrt{81}}\sqrt{\left(\frac{1}{1}\right)^2+\left(\frac{1}{2}\right)^2} = \frac{30\times10^9}{2(9)}(1.118) = \boxed{1.86339\text{ GHz}}$$

Next Mode: TM_{102}

$$(f_r)^{TM}_{102} = \frac{30\times10^9}{2\sqrt{81}}\sqrt{\left(\frac{1}{a}\right)^2+\left(\frac{2}{c}\right)^2} = \frac{30\times10^9}{2(9)}\sqrt{\left(\frac{1}{1}\right)^2+\left(\frac{2}{2}\right)^2} = \frac{30\times10^9}{2(9)}\sqrt{2}$$

$$\boxed{(f_r)^{TM}_{102} = 2.357\text{ GHz}}$$

8.22 a. Since there are no x or y variations, then the electric field must be represented as a standing wave. Thus

$$E_y = \left[C_1 \cos(\beta_z z) + D_1 \sin(\beta_z z)\right]$$

$$E_y(z=0) = 0 = C_1(1) + D_1(0) \Rightarrow C_1 = 0$$

$$E_y(z=d) = 0 = D_1 \sin(\beta_z d) = 0 \Rightarrow \beta_z d = \sin^{-1}(0) = n\pi, \quad \beta_z = \frac{n\pi}{d}, \; n = 1,2,3,\ldots$$

Thus $E_y = D_1 \sin(\beta_z z) = E_o \sin(\beta_z z)$, $\cancel{\beta_x^2}^{\,0} + \cancel{\beta_y^2}^{\,0} + \beta_z^2 = \beta^2 \Rightarrow \beta_z = \beta = \omega\sqrt{\mu\varepsilon}$

$$E_y = E_o \sin(\beta z) \Rightarrow \underline{H} = \frac{1}{-j\omega\mu}\nabla\times\underline{E} = -\frac{1}{j\omega\mu}\left[-\hat{a}_x \frac{\partial E_y}{\partial z}\right] = -j\frac{\beta E_o}{\omega\mu}\hat{a}_x \cos(\beta z)$$

$$H_x = -j\frac{\omega\sqrt{\mu\varepsilon}\,E_o}{\omega\mu}\cos(\beta z) = -j\sqrt{\frac{\varepsilon}{\mu}}\cos(\beta z)$$

b. From Part a

$$\beta_z = \frac{n\pi}{d} \Rightarrow d = \frac{n\pi}{\beta_z} = \frac{n\pi}{\beta} = \frac{n\pi}{\frac{2\pi}{\lambda}} = \frac{n\lambda}{2}, \quad n = 1,2,3,\ldots$$

c. $$W = 2W_e = 2\left[\frac{\varepsilon_o}{4}\iiint_V |E|^2 dv\right] = \frac{\varepsilon_o}{2}\int_0^a\int_0^b\int_0^d |E_o|^2 \sin^2(\beta z)\,dz\,dy\,dx$$

$$= \frac{\varepsilon_o}{2}ab\int_0^d |E_o|^2 \sin^2(\beta z)\,dz = \frac{ab\varepsilon_o|E_o|^2}{4}\int_0^d\left[1-\cos(2\beta z)\right]dz$$

$$W = 2W_e = \frac{ab\varepsilon_o|E_o|^2}{4}\left[z - \frac{1}{2\beta}\sin(2\beta z)\right]_0^d = \frac{ab\varepsilon_o d}{4}|E_o|^2 = \frac{\varepsilon_o}{4}abd|E_o|^2$$

$$P_d = 2\left\{\frac{R_m}{2}\int_0^b\int_0^a \underline{J}_s\cdot\underline{J}_s^* \,dx\,dy\right\}, \quad \underline{J}_s = \hat{n}\times\underline{H}\Big|_{z=0} = \hat{a}_z\times\left[-\hat{a}_x j\frac{E_o}{\eta}\cos(\beta z)\right]_{z=0}$$

$$\underline{J}_s = -j\hat{a}_y\frac{E_o}{\eta}$$

$$P_d = R_m\frac{|E_o|^2}{\eta^2}\int_0^b\int_0^a dx\,dy = R_m ab\frac{|E_o|^2}{\eta^2}$$

$$Q = \frac{\omega W}{P_d} = \frac{\omega\left[\frac{\varepsilon_o}{4}abd|E_o|^2\right]}{R_m ab\frac{|E_o|^2}{\eta^2}} = \frac{\omega\varepsilon_o}{4R_m}d\eta^2$$

d. $$R_m = \sqrt{\frac{\omega\mu_o}{2\sigma}} = \sqrt{\frac{2\pi\times 60\times 10^9 (4\pi\times 10^{-7})}{2(5.76\times 10^7)}} = 6.4127\times 10^{-2}$$

$\underline{\underline{d = 5\lambda}} = 5\left(\frac{30\times 10^9}{60\times 10^9}\right) = 2.5\text{ cm} = 2.5\times 10^{-2}\text{ m}$

$$Q = \frac{\omega\varepsilon_o}{4R_m}d\eta^2 = \frac{2\pi\times 60\times 10^9(10^{-9}/36\pi)}{4(6.4127\times 10^{-2})}(2.5\times 10^{-2})(377)^2 = 46{,}237.296$$

$$Q \simeq 46{,}237$$

$\underline{d = 10\lambda}$:

$$Q = 2(46{,}237.296) = 92{,}474.592 \simeq 92{,}475$$

$\boxed{8.23}$ a. $\beta_{xo} = \beta_{xd} = \left(\frac{m\pi}{a}\right)\Big|_{m=1} = \frac{\pi}{a} = \frac{\pi}{2.286} = 1.3743$ Rad/cm

b. Since $\epsilon_d \simeq \epsilon_o$ and $\mu_d = \mu_o$, then according to (8-133)

$\beta_{yo} \simeq \beta_{yd} \simeq 0$

c. According to:

$(f_c)_{10}^{o} = \frac{1}{2a\sqrt{\mu_o \epsilon_o}} = \frac{30\times10^9}{2(2.286)} = 6.56\times10^9$ If filled with air

$(f_c)_{10}^{d} = \frac{1}{2a\sqrt{\mu_o \epsilon_d}} = \frac{6.56}{\sqrt{1.1}} = 6.25\times10^9$ If filled with styrofoam

Therefore the cutoff frequency of the partially filled should be

$6.25\times10^9 < (f_c)_{10} < 6.56\times10^9$

According to (8-135)

$$(f_c)_{10} = \frac{1}{2a\sqrt{\epsilon_r \mu_o \epsilon_o}}\sqrt{\frac{h + \epsilon_r(b-h)}{(b-h)+\mu_r h}} = \frac{30\times10^9}{2(2.286)\sqrt{1.1}}\sqrt{\frac{0.254+1.1(1.016-0.254)}{(1.016-0.254)+0.254}}$$

$$(f_c)_{10} = 6.2563\,(1.0368)\times10^9 = 6.487\times10^9$$

d. $\beta_{xo}^2 + \cancel{\beta_{yo}^2}^{\,0} + \beta_z^2 = \beta_o^2 \Rightarrow \beta_z = \sqrt{\beta_o^2 - \beta_{xo}^2} = \sqrt{\left[\frac{2\pi(1.25)(6.487\times10^9)}{30\times10^9}\right]^2 - (1.3743)^2}$

$= \sqrt{(1.6982)^2 - (1.3743)^2} = 0.9976$ Rad/cm

$\boxed{8.24}$ $\underline{\underline{TE^x(LSE^x)}}$

$F_x = [C_1\cos(\beta_x x) + D_1\sin(\beta_x x)][C_2\cos(\beta_y y) + D_2\sin(\beta_y y)]e^{-j\beta_z z}$

$E_x = 0$ $\qquad H_x = -j\frac{1}{\omega\mu\epsilon}\left(\frac{\partial^2}{\partial x^2} + \beta^2\right)F_x$

$E_y = -\frac{1}{\epsilon}\frac{\partial F_x}{\partial z}$ $\qquad H_y = -j\frac{1}{\omega\mu\epsilon}\frac{\partial^2 F_x}{\partial x \partial y}$

$E_z = \frac{1}{\epsilon}\frac{\partial F_x}{\partial y}$ $\qquad H_z = -j\frac{1}{\omega\mu\epsilon}\frac{\partial^2 F_x}{\partial x \partial z}$

B.C.'s

$E_z(x=0) = E_z(x=a) = 0$

$E_z(y=0) = E\ (y=b) = 0$

$E_z = \frac{\beta_y}{\epsilon}[C_1\cos\beta_x x + D_1\sin\beta_x x][-C_2\sin\beta_y y + D_2\cos\beta_y y]e^{-j\beta_z z}$

$E_z(x=0) = 0 \Rightarrow C_1 = 0$; $E_x(x=a) = 0 \Rightarrow \sin(\beta_x a) = 0 \Rightarrow \beta_x = \left(\frac{m\pi}{a}\right)$, $m = 1,2,3\ldots$

$E_z(y=0) = 0 \Rightarrow D_2 = 0$; $E_z(y=b) = 0 \Rightarrow \sin(\beta_y b) = 0 \Rightarrow \beta_y = \left(\frac{n\pi}{b}\right)$, $n = 0,1,2,\ldots$

$F_x = A_{mn}\sin(\beta_x x)\cos(\beta_y y)e^{-j\beta_z z}$, $\beta_x = \frac{m\pi}{a}$, $m = 1,2,3,\ldots$; $\beta_y = \frac{n\pi}{b}$, $n = 0,1,2,\ldots$

Cont'd.

$\boxed{\text{8.24 Cont'd.}}$ $(f_c)_{mn} = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2}$, $\quad m=1,2,3,\ldots \quad n=0,1,2,\ldots$

$$E_x = 0 \qquad H_x = -jA_{mn}\frac{\beta^2-\beta_x^2}{\omega\mu\varepsilon}\sin(\beta_x x)\cos(\beta_y y)e^{-j\beta_z z}$$

$$E_y = jA_{mn}\frac{\beta_z}{\varepsilon}\sin(\beta_x x)\cos(\beta_y y)e^{-j\beta_z z} \qquad H_y = jA_{mn}\frac{\beta_x\beta_y}{\omega\mu\varepsilon}\cos(\beta_x x)\sin(\beta_y y)e^{-j\beta_z z}$$

$$E_z = -A_{mn}\frac{\beta_y}{\varepsilon}\sin(\beta_x x)\sin(\beta_y y)e^{-j\beta_z z} \qquad H_z = -A_{mn}\frac{\beta_x\beta_z}{\omega\mu\varepsilon}\cos(\beta_x x)\cos(\beta_y y)e^{-j\beta_z z}$$

Dominant mode is the TE^x_{10} $\left(f_{c10} = \frac{1}{2a\sqrt{\mu\varepsilon}}\right)$

$TM^x(LSM^x)$

$$A_x = [C_1\cos(\beta_x x)+D_1\sin(\beta_x x)][C_2\cos(\beta_y y)+D_2\sin(\beta_y y)]e^{-j\beta_z z}$$

$$E_x = -j\frac{1}{\omega\mu\varepsilon}\left(\frac{\partial^2}{\partial x^2}+\beta^2\right)A_x \qquad H_x = 0$$

$$E_y = -j\frac{1}{\omega\mu\varepsilon}\frac{\partial^2 A_x}{\partial x\partial y} \qquad H_y = \frac{1}{\mu}\frac{\partial A_x}{\partial z}$$

$$E_z = -j\frac{1}{\omega\mu\varepsilon}\frac{\partial^2 A_x}{\partial x\partial z} \qquad H_z = -\frac{1}{\mu}\frac{\partial A_x}{\partial y}$$

B.C.'s

$E_z(x=0) = E_z(x=a) = 0$

$E_z(y=0) = E_z(y=b) = 0$

$$E_z = -\frac{\beta_x\beta_z}{\omega\mu\varepsilon}[-C_1\sin(\beta_x x)+D_1\cos(\beta_x x)][C_2\cos(\beta_y y)+D_2\sin(\beta_y y)]e^{-j\beta_z z}$$

$E_z(x=0)=0 \Rightarrow D_1=0$; $E_z(x=a)=0 \Rightarrow \sin(\beta_x a)=0 \Rightarrow \beta_x = \frac{m\pi}{a}$, $m=0,1,2,\ldots$

$E_z(y=0)=0 \Rightarrow C_2=0$; $E_z(y=b)=0 \Rightarrow \sin(\beta_y b)=0 \Rightarrow \beta_y = \frac{n\pi}{b}$, $n=1,2,3,\ldots$

$$A_x = B_{mn}\cos(\beta_x x)\sin(\beta_y y)e^{-j\beta_z z}, \quad \beta_x = \frac{m\pi}{a}, m=0,1,2,\ldots; \quad \beta_y = \frac{n\pi}{b}, n=1,2,3,\ldots$$

$$(f_c)_{mn} = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2}, \quad m=0,1,2,\ldots \quad n=1,2,3,\ldots$$

$$E_x = -jB_{mn}\frac{\beta^2-\beta_x^2}{\omega\mu\varepsilon}\cos(\beta_x x)\sin(\beta_y y)e^{-j\beta_z z}, \quad H_x = 0$$

$$E_y = +jB_{mn}\frac{\beta_x\beta_y}{\omega\mu\varepsilon}\sin(\beta_x x)\cos(\beta_y y)e^{-j\beta_z z}, \quad H_y = -jB_{mn}\frac{\beta_z}{\mu}\cos(\beta_x x)\sin(\beta_y y)e^{-j\beta_z z}$$

$$E_z = B_{mn}\frac{\beta_x\beta_z}{\omega\mu\varepsilon}\sin(\beta_x x)\sin(\beta_y y)e^{-j\beta_z z}, \quad H_z = -B_{mn}\frac{\beta_y}{\mu}\cos(\beta_x x)\cos(\beta_y y)e^{-j\beta_z z}$$

The dominant mode is the TM^x_{01} $\left(f_{c01} = \frac{1}{2b\sqrt{\mu\varepsilon}}\right)$

$\boxed{8.25}$ **B.C.'s:**

$$E_z^d(0\le x\le w, y=0, z)=E_z^d(0\le x\le w, y=b, z)=0$$
$$E_z^o(x=a, 0\le y\le b, z)=0$$
$$E_z^d(x=w, 0\le y\le b, z)=E_z^o(x=w, 0\le y\le b, z)$$
$$E_z^o(w\le x\le a, y=0, z)=E_z^o(w\le x\le a, y=b, z)=0$$
$$E_z^d(x=0, 0\le y\le b, z)=0$$

TE^x:

$$H_z^d(x=w, 0\le y\le b, z)=H_z^o(x=w, 0\le y\le b, z)$$

$$F_x^d=\left[C_1^d\cos(\beta_{xd}x)+D_1^d\sin(\beta_{xd}x)\right]\left[C_2^d\cos(\beta_{yd}y)+D_2^d\sin(\beta_{yd}y)\right]A_3^d e^{-j\beta_z z} \quad (1)$$
$$\beta_{xd}^2+\beta_{yd}^2+\beta_z^2=\beta_d^2=\omega^2\mu_d\varepsilon_d$$

$$F_x^o=\left\{C_1^o\cos[\beta_{xo}(a-x)]+D_1^o\sin[\beta_{xo}(a-x)]\right\}\left\{C_2^o\cos(\beta_{yo}y)+D_2^o\sin(\beta_{yo}y)\right\}A_3^o e^{-j\beta_z z} \quad (2)$$
$$\beta_{xo}^2+\beta_{yo}^2+\beta_z^2=\beta_o^2=\omega^2\mu_o\varepsilon_o$$

By applying the boundary conditions we reduce (1) and (2) to

$$F_x^d=A_{mn}^d\sin(\beta_{xd}x)\cos(\beta_{yd}y)e^{-j\beta_z z}$$
$$\beta_{yd}=\frac{n\pi}{b},\ n=0,1,2,\dots$$
$$\beta_{xd}^2+\beta_{yd}^2+\beta_z^2=\beta_{xd}^2+\left(\frac{n\pi}{b}\right)^2+\beta_z^2=\beta_d^2=\omega^2\mu_d\varepsilon_d$$

$$F_x^o=A_{mn}^o\sin[\beta_{xo}(a-x)]\cos(\beta_{yo}y)e^{-j\beta_z z}$$
$$\beta_{yo}=\frac{n\pi}{b},\ n=0,1,2,\dots$$
$$\beta_{xo}^2+\beta_{yo}^2+\beta_z^2=\beta_{xo}^2+\left(\frac{n\pi}{b}\right)^2+\beta_z^2=\beta_o^2=\omega^2\mu_o\varepsilon_o$$

Now apply boundary conditions along the interface. Doing this leads to

$$\frac{1}{\varepsilon_o}A_{mn}^o\sin[\beta_{xo}(a-w)]=\frac{1}{\varepsilon_d}A_{mn}^d\sin(\beta_{xd}w) \quad (3)$$

$$\frac{\beta_{xo}}{\mu_o\varepsilon_o}A_{mn}^o\cos[\beta_{xo}(a-w)]=-\frac{\beta_{xd}}{\mu_d\varepsilon_d}A_{mn}^d\cos(\beta_{xd}w) \quad (4)$$

Dividing (4) by (3) leads to

$$\frac{\beta_{xo}}{\mu_o}\cot[\beta_{xo}(a-w)]=-\frac{\beta_{xd}}{\mu_d}\cot(\beta_{xd}w)$$
$$\left.\begin{aligned}\beta_{xo}^2+\beta_{yo}^2+\beta_z^2&=\beta_{xo}^2+\left(\frac{n\pi}{b}\right)^2+\beta_z^2=\beta_o^2=\omega^2\mu_o\varepsilon_o\\ \beta_{xd}^2+\beta_{yd}^2+\beta_z^2&=\beta_{xd}^2+\left(\frac{n\pi}{b}\right)^2+\beta_z^2=\beta_d^2=\omega^2\mu_d\varepsilon_d\end{aligned}\right\}\ n=0,1,2,\dots$$

The dominant mode is the TE_{10}^x with a cutoff frequency value of

$$\frac{1}{2a\sqrt{\mu_d\varepsilon_d}}\le(f_c)_{10}^{TE^x}\le\frac{1}{2a\sqrt{\mu_o\varepsilon_o}}$$

cont'd.

8.25 Cont'd.

$\underline{TM^x}$:

$$A_x^d = [C_1^d \cos(\beta_{xd}x) + D_1^d \sin(\beta_{xd}x)][C_2^d \cos(\beta_{yd}y) + D_2^d \sin(\beta_{yd}y)] A_3^d e^{-j\beta_z z} \quad (1)$$

$$\beta_{xd}^2 + \beta_{yd}^2 + \beta_z^2 = \beta_d^2 = \omega^2\mu_d\varepsilon_d$$

$$A_x^o = \{C_1^o \cos[\beta_{xo}(a-x)] + D_1^o \sin[\beta_{xo}(a-x)]\}\{C_2^o \cos(\beta_{yo}y) + D_2^o \sin(\beta_{yo}y)\} A_3^o e^{-j\beta_z z} \quad (2)$$

$$\beta_{xo}^2 + \beta_{yo}^2 + \beta_z^2 = \beta_o^2 = \omega^2\mu_o\varepsilon_o$$

By applying the boundary conditions along the walls of the waveguide we reduce (1) and (2) to

$$A_x^d = A_{mn}^d \cos(\beta_{xd}x)\sin(\beta_{yd}y)e^{-j\beta_z z}$$

$$\beta_{yd} = \frac{n\pi}{b}, \ n = 1,2,3,\ldots\ldots$$

$$\beta_{xd}^2 + \beta_{yd}^2 + \beta_z^2 = \beta_{xd}^2 + \left(\frac{n\pi}{b}\right)^2 + \beta_z^2 = \beta_d^2 = \omega^2\mu_d\varepsilon_d$$

$$A_x^o = A_{mn}^o \cos[\beta_{xo}(a-x)]\sin(\beta_{yo}y)e^{-j\beta_z z}$$

$$\beta_{yo} = \frac{n\pi}{b}, \ n = 1,2,3,\ldots\ldots$$

$$\beta_{xo}^2 + \beta_{yo}^2 + \beta_z^2 = \beta_{xo}^2 + \left(\frac{n\pi}{b}\right)^2 + \beta_z^2 = \beta_o^2 = \omega^2\mu_o\varepsilon_o$$

Now applying the boundary conditions along the interface leads to

$$\frac{\beta_{xd}}{\mu_d\varepsilon_d} A_{mn}^d \sin(\beta_{xd}w) = -\frac{\beta_{xo}}{\mu_o\varepsilon_o} A_{mn}^o \sin[\beta_{xo}(a-w)] \quad (3)$$

$$\frac{1}{\mu_d} A_{mn}^d \cos(\beta_{xd}w) = \frac{1}{\mu_o} A_{mn}^o \cos[\beta_{xo}(a-w)] \quad (4)$$

Dividing (3) by (4) leads to

$$\left.\begin{array}{l} \dfrac{\beta_{xo}}{\varepsilon_o}\tan[\beta_{xo}(a-w)] = -\dfrac{\beta_{xd}}{\varepsilon_d}\tan(\beta_{xd}w) \\ \beta_{xo}^2 + \beta_{yo}^2 + \beta_z^2 = \beta_{xo}^2 + \left(\frac{n\pi}{b}\right)^2 + \beta_z^2 = \beta_o^2 = \omega^2\mu_o\varepsilon_o \\ \beta_{xd}^2 + \beta_{yd}^2 + \beta_z^2 = \beta_{xd}^2 + \left(\frac{n\pi}{b}\right)^2 + \beta_z^2 = \beta_d^2 = \omega^2\mu_d\varepsilon_d \end{array}\right\} n = 1,2,3,\ldots$$

The dominant mode is the TM_{01}^x with a cutoff frequency value of

$$\frac{1}{2b\sqrt{\mu_d\varepsilon_d}} \le (f_c)_{01}^{TM^x} \le \frac{1}{2b\sqrt{\mu_o\varepsilon_o}}$$

$\boxed{8.26}$

1. $$(f_c)_{mn} = \frac{1}{2\pi\sqrt{\mu\epsilon}}\sqrt{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2} \left\{ \begin{array}{l} TE^z \begin{cases} m=0,1,2,\ldots \\ n=0,1,2,\ldots \end{cases} \quad m=n\neq 0 \\ TM^z \begin{cases} m=1,2,3,\ldots \\ n=1,2,3,\ldots \end{cases} \end{array} \right.$$

a. For $b>a$ $(m=0, n=1)$

$$f_c(TE^z_{01}) = \frac{1}{2\pi\sqrt{\mu_0\varepsilon_0}}\left(\frac{\pi}{b}\right) = \frac{1}{2b\sqrt{\mu_0\varepsilon_0}} = \frac{30\times10^9}{2(2.286)} = \boxed{6.5617\ \text{GHz}}$$

b. For $b>a$ $(m=0, n=1)$

$$f_c(TE^y_{01}) = \frac{1}{2\pi\sqrt{\mu_0\varepsilon_0}}\left(\frac{\pi}{b}\right) = \frac{1}{2b\sqrt{\mu_0\varepsilon_0}} \left\{ \begin{array}{l} TE^y \begin{cases} m=0,1,2,\ldots \\ n=1,2,3,\ldots \end{cases} \\ TM^y \begin{cases} m=1,2,3,\ldots \\ n=0,1,2,\ldots \end{cases} \end{array} \right.$$

$$f_c(TE^y_{01}) = \frac{30\times10^9}{2(2.286)} = \boxed{6.5617\ \text{GHz}}$$

c. For $b>a$ $(m=0, n=2)$

$$f_c(TE^z_{02}) = \frac{1}{2\pi\sqrt{\mu_0\varepsilon_0}}\left(\frac{2\pi}{b}\right) = \frac{1}{b\sqrt{\mu_0\varepsilon_0}} = 2(6.5617)\ \text{GHz}$$

$$f_c(TE^z_{02}) = \boxed{13.1234\ \text{GHz}}$$

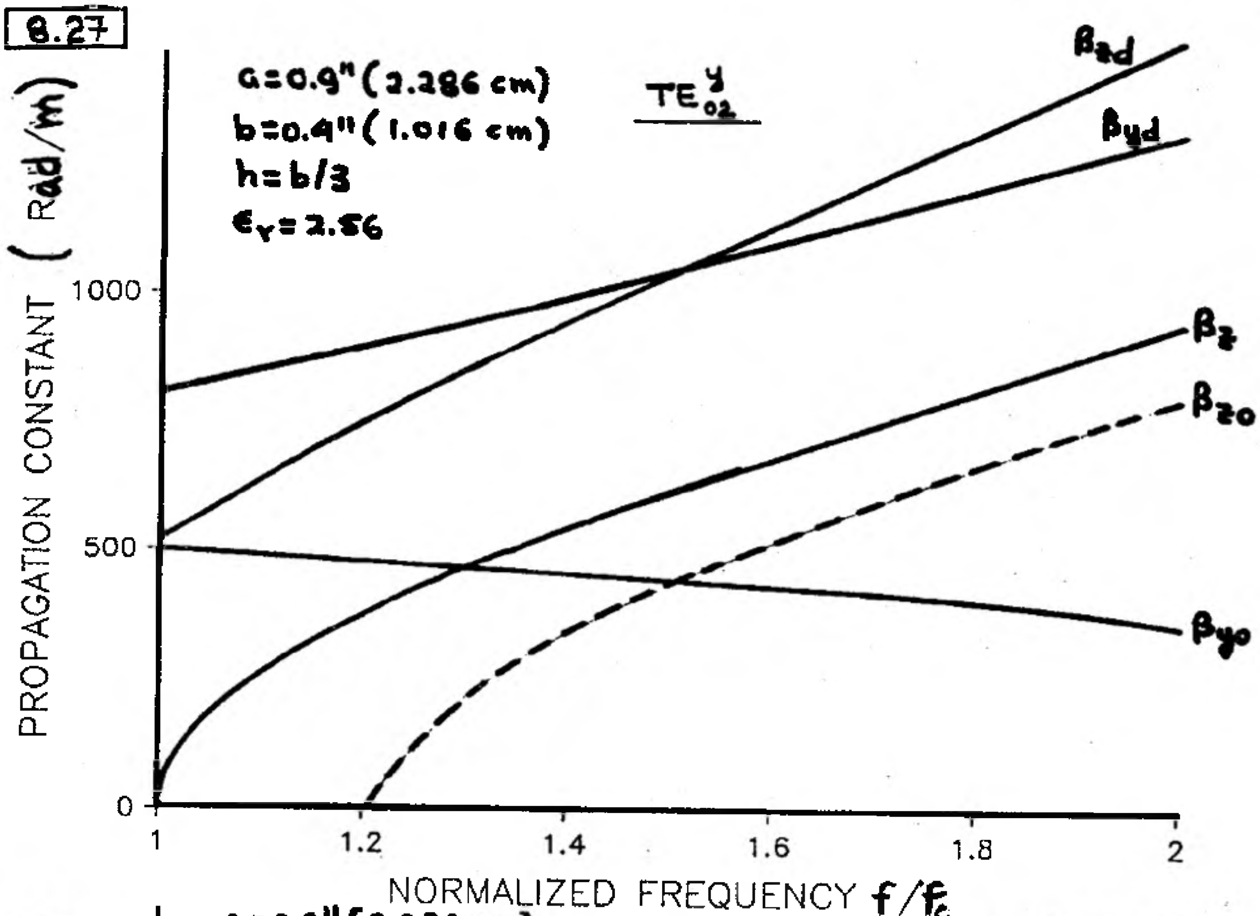
d. $$(f_c)^{TE^z}_{01} = \frac{1}{2\pi\sqrt{\mu_0\varepsilon_r\varepsilon_0}}\left(\frac{\pi}{b}\right) = \frac{1}{2b\sqrt{\mu_0\varepsilon_0}\sqrt{\varepsilon_r}} = 4\times10^9$$

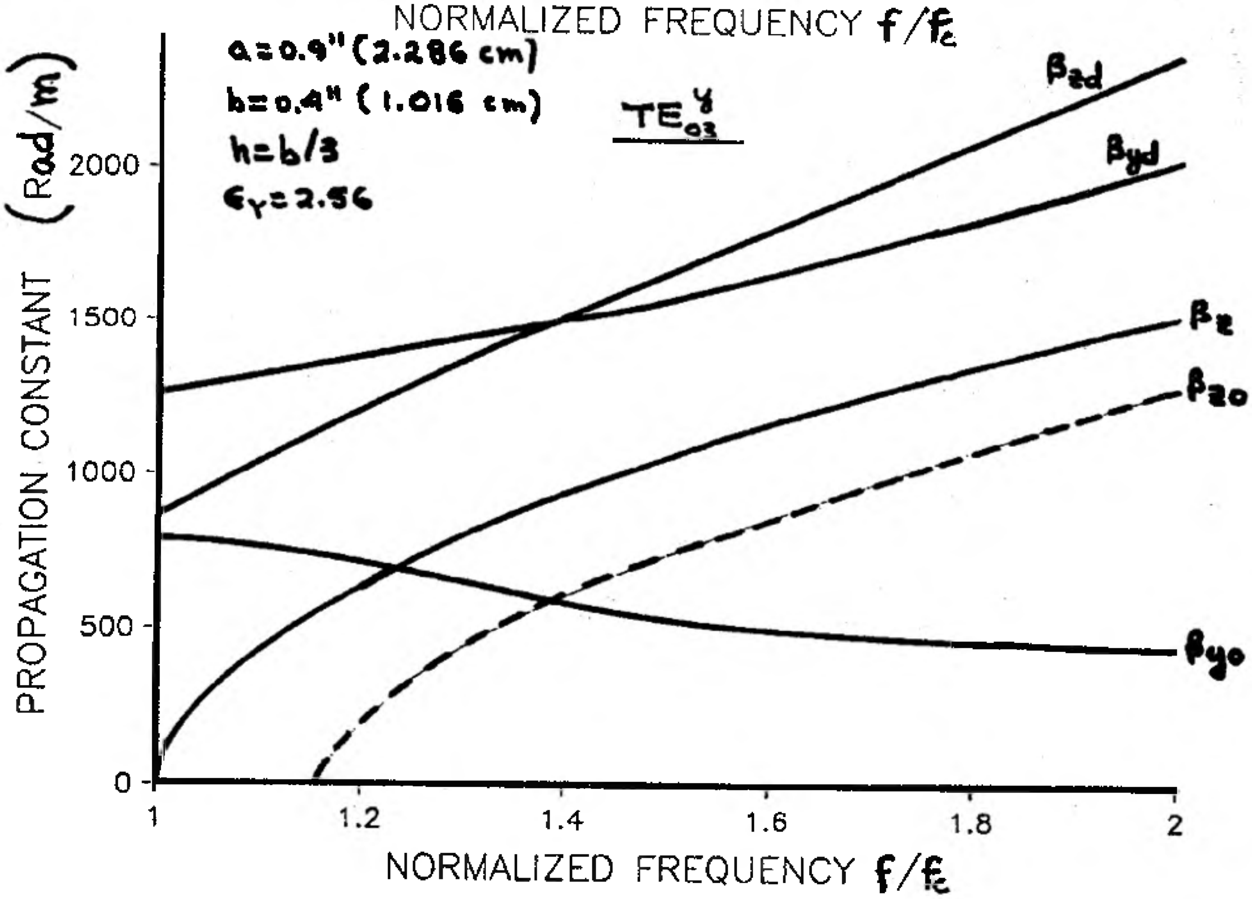
$$\frac{6.5617\times10^9}{\sqrt{\varepsilon_r}} = 4\times10^9$$

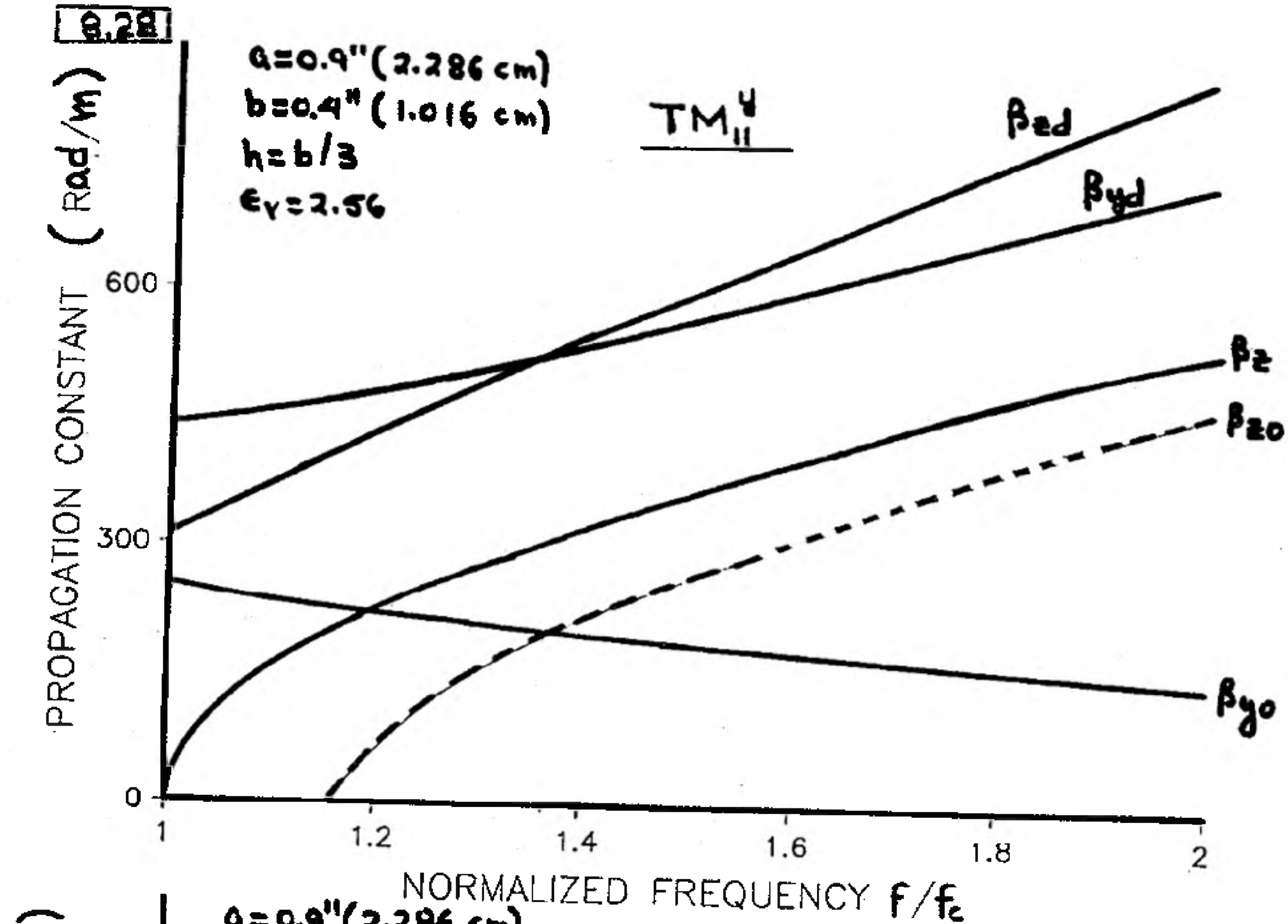
$$\sqrt{\varepsilon_r} = \frac{6.5617}{4} = 1.64$$

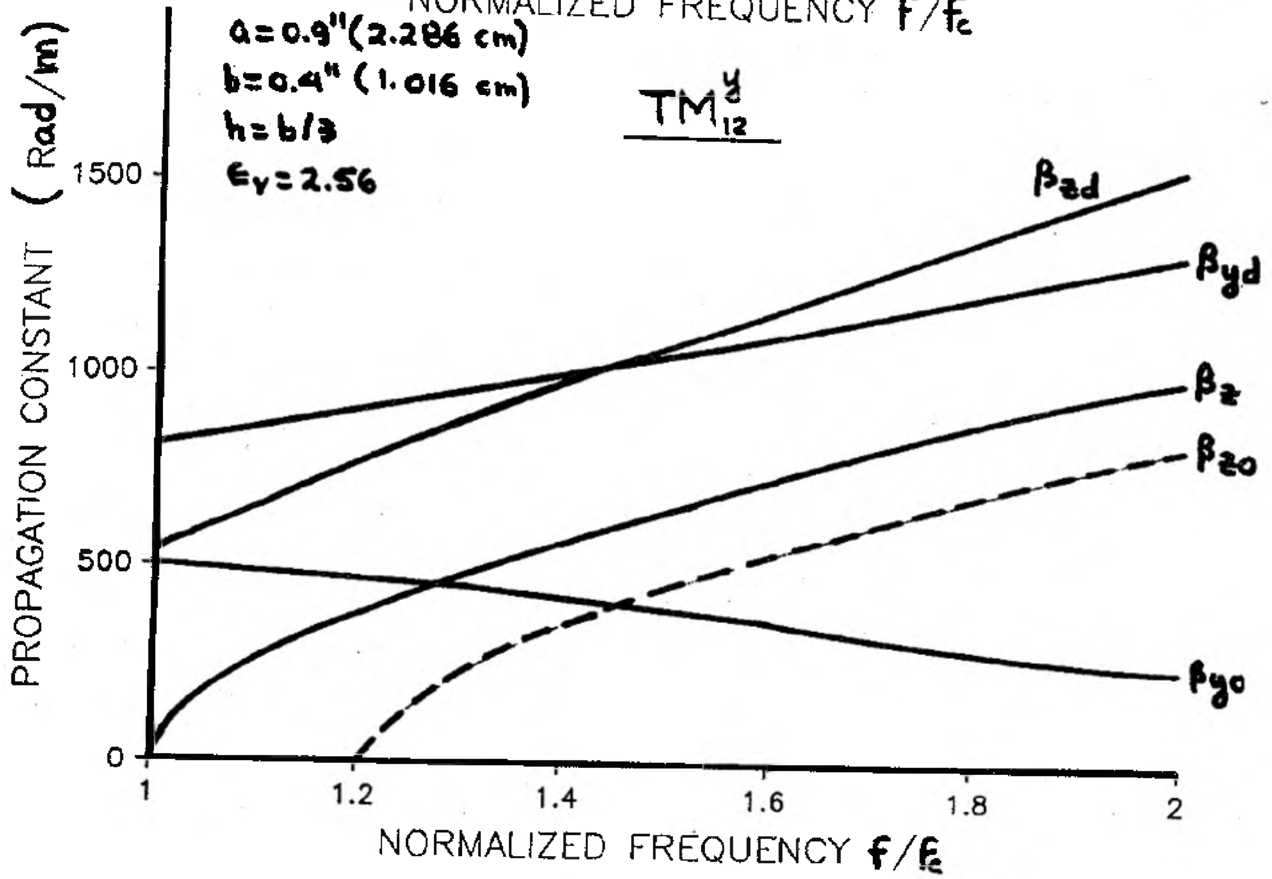
$$\boxed{\varepsilon_r = 2.691}$$

8.27


a=0.9" (2.286 cm)
b=0.4" (1.016 cm)
h=b/3
ϵ_r=2.56
TE^y_{02}
PROPAGATION CONSTANT (Rad/m)
NORMALIZED FREQUENCY f/fc
β_{zd}
β_{yd}
β_z
β_{zo}
β_{yo}
0
500
1000
1
1.2
1.4
1.6
1.8
2


a=0.9" (2.286 cm)
b=0.4" (1.016 cm)
h=b/3
ϵ_r=2.56
TE^y_{03}
PROPAGATION CONSTANT (Rad/m)
NORMALIZED FREQUENCY f/fc
β_{zd}
β_{yd}
β_z
β_{zo}
β_{yo}
0
500
1000
1500
2000
1
1.2
1.4
1.6
1.8
2


8.28
a=0.9"(2.286 cm)
b=0.4" (1.016 cm)
h=b/3
ϵ_r=2.56
TM^y_{11}
β_{zd}
β_{yd}
β_z
β_{zo}
β_{yo}
PROPAGATION CONSTANT (Rad/m)
600
300
0
1
1.2
1.4
1.6
1.8
2
NORMALIZED FREQUENCY f/fc


a=0.9"(2.286 cm)
b=0.4" (1.016 cm)
h=b/3
ϵ_r=2.56
TM^y_{12}
β_{zd}
β_{yd}
β_z
β_{zo}
β_{yo}
PROPAGATION CONSTANT (Rad/m)
1500
1000
500
0
1
1.2
1.4
1.6
1.8
2
NORMALIZED FREQUENCY f/fc

$\boxed{8.29}$

$TE^y(H)$ [Figure 8.14 (b)]

Using the procedure outlined in Section 8.6.1 for Figure 8.13(a), we can write for Figure 8.14(b):

$$Z_{cd} = Z_d^{h} = \frac{\omega \mu_d}{\beta_{xd}}, \quad Z_{c0} = Z_0^{h} = \frac{\omega \mu_0}{\beta_{x0}}.$$

$$\beta_{td} = \beta_{xd}, \quad \beta_{t0} = \beta_{x0}$$

$$\boxed{\frac{\beta_{x0}}{\mu_0} \cot\left[\beta_{x0}(a-w)\right] = -\frac{\beta_{xd}}{\mu_d} \cot(\beta_{xd} w)}$$

$TM^y(E)$ [Figure 8.14(b)]

Using the procedure outlined in Section 8.6.2 for Figure 8.13(a), we can write for Figure 8.14(b):

$$Z_{cd} = Z_d^{e} = \frac{\beta_{xd}}{\omega \varepsilon_d}, \quad Z_{c0} = Z_0^{e} = \frac{\beta_{x0}}{\omega \varepsilon_0}$$

$$\beta_{td} = \beta_{xd}, \quad \beta_{t0} = \beta_{x0}$$

$$\boxed{\frac{\beta_{x0}}{\varepsilon_0} \tan\left[\beta_{x0}(a-w)\right] = -\frac{\beta_{xd}}{\varepsilon_d} \tan(\beta_{xd} h)}$$

Cont'd

8.29 cont'd

$TE^y(H)$ [Figure 8.14(b)]

Using the procedure outlined in Section 8.6.1 for Figure 8.14(a), we can write for Figure 8.14(b):

$$Z_{cd} = Z_d^h = \frac{\omega\mu_d}{\beta_{xd}}, \quad Z_{c0} = Z_0^h = \frac{\omega\mu_0}{\beta_{x0}}.$$

$$\beta_{td} = \beta_{xd}, \quad \beta_{t0} = \beta_{x0}$$

$$\boxed{\frac{\beta_{x0}}{\mu_0}\cot\left[\beta_{x0}(a-w)\right] = -\frac{\beta_{xd}}{\mu_d}\cot(\beta_{xd}w)}$$

$TM^y(E)$ [Figure 8.14(b)]

Using the procedure outlined in Section 8.6.2 for Figure 8.14(a), we can write for Figure 8.14(b):

$$Z_{cd} = Z_d^e = \frac{\beta_{xd}}{\omega\varepsilon_d}, \quad Z_{c0} = Z_0^e = \frac{\beta_{x0}}{\omega\varepsilon_0}$$

$$\beta_{td} = \beta_{xd}, \quad \beta_{t0} = \beta_{x0}$$

$$\boxed{\frac{\beta_{x0}}{\varepsilon_0}\tan\left[\beta_{x0}(a-w)\right] = -\frac{\beta_{xd}}{\varepsilon_d}\tan(\beta_{xd}h)}$$

8.30

$$\left(f_c\right)_{mn}^{TE^y} = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2} \quad \left\{\begin{array}{l} m=0,1,2,\dots \\ n=1,2,3,\dots \end{array}\right. \qquad (8\text{-}100a)$$

a. TE_{01}^y Mode

b. 1. TE_{01}^y: Air-filled: $\left(f_c\right)_{01}^{TE^y} = \frac{1}{2\pi\sqrt{\mu_0\varepsilon_0}}\left(\frac{\pi}{b}\right) = \frac{1}{2b\sqrt{\mu_0\varepsilon_0}}$

2. TE_{01}^y: Dielectric-filled: $\left(f_c\right)_{01}^{TE^y} = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\left(\frac{\pi}{b}\right) = \frac{1}{2b\sqrt{\mu\varepsilon}}$

3. TE_{01}^y: Partially-filled:

$$\sqrt{\frac{\varepsilon_0}{\mu_0}}\cot\left[\omega_c\sqrt{\mu_0\varepsilon_0}(b-h)\right] = -\sqrt{\frac{\varepsilon_d}{\mu_d}}\cot\left[\omega_c\sqrt{\mu_d\varepsilon_d}\,h\right] \qquad (8\text{-}118)$$

$$\sqrt{\frac{\varepsilon_0}{\mu_0}}\cot\left[\omega_c\sqrt{\mu_0\varepsilon_0}(b-h)\right] = -\sqrt{\frac{4\varepsilon_0}{4\mu_0}}\cot\left[\omega_c\sqrt{\mu_d\varepsilon_d}\,h\right] = -\sqrt{\frac{\varepsilon_0}{\mu_0}}\cot\left(\omega_c\sqrt{\mu_d\varepsilon_d}\,h\right)$$

$$\cot\left[\omega_c\sqrt{\mu_0\varepsilon_0}(b-h)\right] = -\cot\left(\omega_c\sqrt{4\mu_0 4\varepsilon_0}\,h\right) = \cot\left[\pi - 4h\omega_c\sqrt{\mu_0\varepsilon_0}\right]$$

$$\omega_c\sqrt{\mu_0\varepsilon_0}(b-h) = \pi - \omega_c 4h\sqrt{\mu_0\varepsilon_0}$$

$$\omega_c\sqrt{\mu_0\varepsilon_0}(b-h+4h) = \omega_c\sqrt{\mu_0\varepsilon_0}(b+3h) = \pi$$

$$2\pi f_c\sqrt{\mu_0\varepsilon_0}(b+3h) = 2\pi f_c\sqrt{\mu_0\varepsilon_0}(2b) = \pi$$

$$h = b/3$$

$$\boxed{\left(f_c\right) = \frac{1}{4b\sqrt{\mu_0\varepsilon_0}} = \frac{1}{2(b+3h)\sqrt{\mu_0\varepsilon_0}}}$$

c. 1. TE_{01}^y (air-filled): $f_c = \frac{1}{2b\sqrt{\mu_0\varepsilon_0}} = \frac{30\times10^9}{2(1.016)} = 14.764$ GHz

2. TE_{01}^y (dielectric-filled): $f_c = \frac{1}{2b\sqrt{\mu_0\varepsilon_0}(4)} = \frac{14.764\text{ GHz}}{4} = 3.691$ GHz

3. TE_{01}^y (partially-filled): $f_c = \frac{1}{4b\sqrt{\mu_0\varepsilon_0}} = \frac{30\times10^9}{4(1.016)} = 7.382$ GHz

d. 3.691 GHz (dielectric-filled) $<$ f_c (partially-filled) = 7.382 GHz $<$ 14.764 GHz (air-filled)

3.691 GHz $<$ 7.382 GHz $<$ 14.764 GHz

as it should be.

8.31

TM^x: $a = 2.286$ cm, $b = 1.1016$ cm, $\varepsilon_r = 1.1$, $\mu_r = 1$, $w = a/2$

From Problem 8.24(b)

$$(f_c)_{mn}(TM^x - LSM^x) = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} \qquad \begin{matrix} m = 0,1,2,\dots \\ n = 1,2,3,\dots \end{matrix}$$

$$\beta_{x0} = \left(\frac{m\pi}{a}\right), \quad \beta_{xd} = \left(\frac{m\pi}{a}\right) \qquad m = 0,1,2,\dots$$

The dominant mode is the TM^x_{01} $(m=0, n=1)$ with

$$(f_c)_{01}^{TM^x} = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\left(\frac{\pi}{b}\right) = \frac{1}{2b\sqrt{\mu\varepsilon}}$$

a. Free-Space: $f_{01}^{TM^x} = \dfrac{30\times 10^9}{2(1.106)} = 13.617$ GHz

b. Dielectric Filled: $f_{01}^{TM^x} = \dfrac{30\times 10^9}{2(1.106)\sqrt{1.1}} = 12.983$ GHz

c. Based on Figure 8-15(b), the pertinent equations for TM^x are:

$$\frac{\beta_{x0}}{\varepsilon}\tan[\beta_{x0}(a-w)] = -\frac{\beta_{xd}}{\varepsilon_d}\tan(\beta_{xd}w)$$

$$\left.\begin{aligned} \beta_{x0}^2 + \beta_{y0}^2 + \beta_z^2 &= \beta_{x0}^2 + \left(\frac{n\pi}{b}\right)^2 + \beta_z^2 = \beta_0^2 = \omega^2\mu_0\varepsilon_0 \\ \beta_{xd}^2 + \beta_{yd}^2 + \beta_z^2 &= \beta_{xd}^2 + \left(\frac{n\pi}{b}\right)^2 + \beta_z^2 = \beta_d^2 = \omega^2\mu_d\varepsilon_d \end{aligned}\right\} \quad n = 1,2,3,\dots$$

The dominant mode is the TM^x_{01} $(m=0, n=1)$ whose cutoff frequency is

$$\frac{1}{2b\sqrt{\mu_d\varepsilon_d}} \le (f_c)_{01}^{TM^x} \le \frac{1}{2b\sqrt{\mu_0\varepsilon_0}}$$

Because $\varepsilon_r = 1.1 \Rightarrow \varepsilon \simeq \varepsilon_0 \Rightarrow \beta_{x0} \simeq \beta_{xd} \simeq$ small $\Rightarrow 0$

Therefore the transendental equation reduces, as shown similarly on Page 401 of the book for the TM^y, to (8-135) which for TM^x mode can be written as

$$(\omega_c)_{01}^{TM^x} \simeq \frac{\pi}{b\sqrt{\varepsilon_r\varepsilon_0\mu_0}}\sqrt{\frac{w + \varepsilon_r(a-w)}{(a-w) + \mu_r w}} \qquad \begin{matrix} a = 2.286 & \varepsilon_r = 1.1 \\ w = 1.143 & \mu_r = 1 \end{matrix}$$

$$(f_c)_{01}^{TM^x} = \frac{30\times 10^9}{2(1.1016)\sqrt{1.1}}\sqrt{\frac{1.143 + 1.1(1.143)}{1.143 + 1.143}} = \boxed{13.3035 \text{ GHz}}$$

d. $\boxed{12.983\times 10^9 < 13.3035\times 10^9 < 13.617\times 10^9}$

$\boxed{8.32}$ $2h = 1$ cm, $\epsilon_r = 5$, $\mu_r = 1$

a. $(f_c)_m = \dfrac{m}{4h\sqrt{\mu_d\epsilon_d - \mu_0\epsilon_0}}$ $\left.\begin{matrix} m = 0,2,4,\ldots\ldots & TM_m^z\text{(odd) and } TE_m^z\text{(odd)} \\ m = 1,3,5,\ldots\ldots, & TM_m^z\text{(even) and } TE_m^z\text{(even)}\end{matrix}\right.$

$m=0$: $(f_c)_0 = 0$

$m=1$: $(f_c)_1 = \dfrac{30\times10^9}{4(0.5\times10^{-2})\sqrt{5-1}} = 7.4949\times10^9$

$\left.\begin{matrix} TM_0^z\text{(odd)} \\ TE_0^z\text{(odd)}\end{matrix}\right\} (f_c)_0 = 0$ $\left.\begin{matrix} TM_1^z\text{(even)} \\ TE_1^z\text{(even)}\end{matrix}\right\} (f_c)_1 = 7.4949\times10^9$

b. Using a graphical solution similar to that of Figure 8-22 it leads to the following answers (radius of circle is $a = 1.6767$)

<u>See graph next page</u>

<u>TM_0^z(odd)</u>

$\alpha_{y0}h \simeq 1.04$ $\beta_{yd}h \simeq 1.325$

$\alpha_{y0} \simeq 208$ Np/m $\beta_{yd} \simeq 265$ Rad/m

<u>TM_1^z(even)</u>

$\alpha_{y0}h \simeq 0.04$ $\beta_{yd}h \simeq 1.675$

$\alpha_{y0} \simeq 8$ Np/m $\beta_{yd} \simeq 335$ Rad/m

<u>TE_0^z(odd)</u> <u>See graph next page</u>

$\alpha_{y0}h \simeq 1.39$ $\beta_{yd}h \simeq 0.956$

$\alpha_{y0} \simeq 278$ Np/m $\beta_{yd} \simeq 191.2$ Rad/m

<u>TE_1^z(even)</u>

$\alpha_{y0}h \simeq 0.180$ $\beta_{yd}h \simeq 1.663$

$\alpha_{y0} \simeq 36$ Np/m $\beta_{yd} \simeq 332.6$ Rad/m

c. Using a graphical solution similar to that of Figure 8-30 it leads to the following answers:

TM_0^z(odd): $\theta_1 \simeq 45.2°$ TE_0^z(odd): $\theta_1 = 59.2°$

TM_1^z(even): $\theta_1 \simeq 26.7°$ TE_1^z(even): $\theta_1 = 27.2°$

<u>See Graph. It follows.</u>

b. <u>Alternate</u> values obtained analytically using iteratively techniques:

<u>TM_0^z(odd)</u>
$\alpha_{y0} = 206.3335$ Np./m
$\beta_{yd} = 264.0467$ Rad/m

<u>TM_1^z(even)</u>
$\alpha_{y0} = 7.0177$ Np/m
$\beta_{yd} = 335.0297$ Rad/m

<u>TE_0^z(odd)</u>
$y_0 = 274.59$ Np/m
$\beta_{yd} = 192.0994$ Rad/m

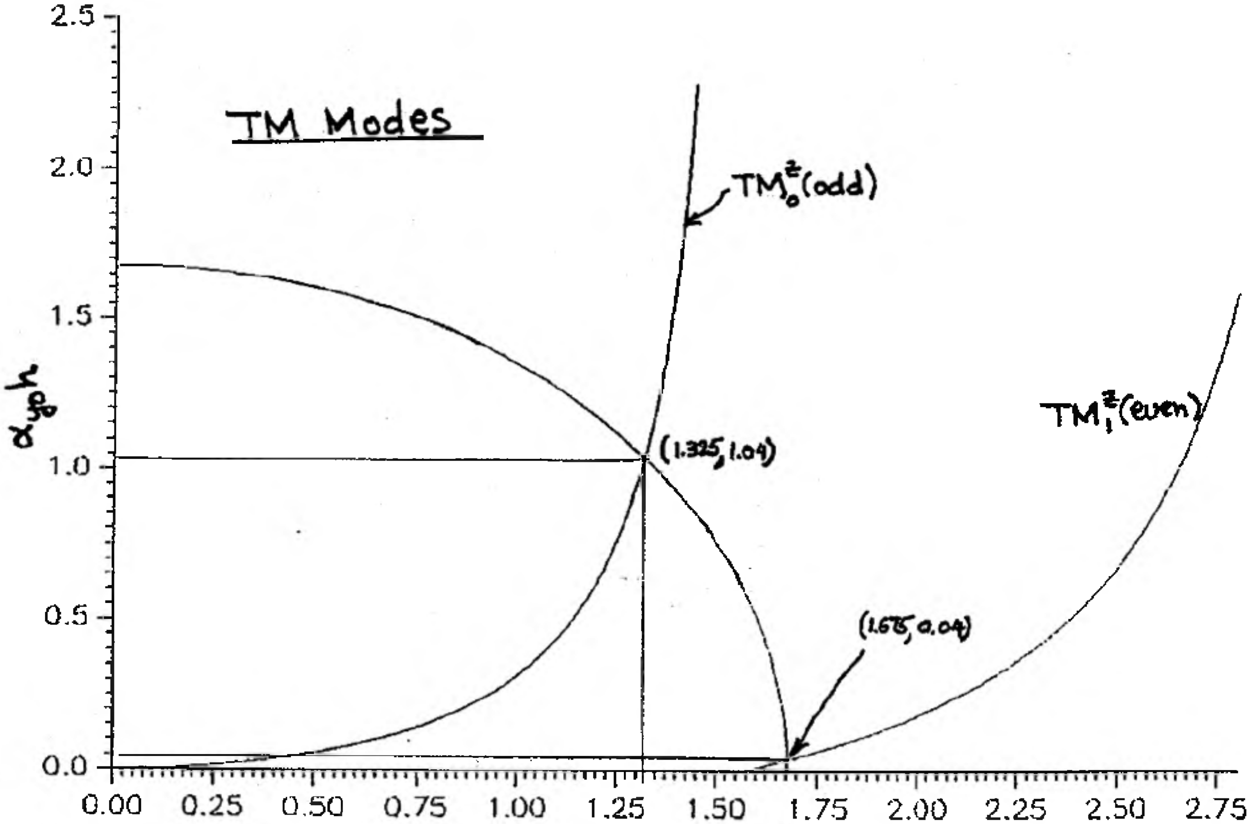
<u>TE_1^z(even)</u>
$\alpha_{y0} = 32.4091$ Np/m
$\beta_{yd} = 333.5323$ Rad/m

c. <u>Alternate</u> values obtained analytically using iteratively techniques:

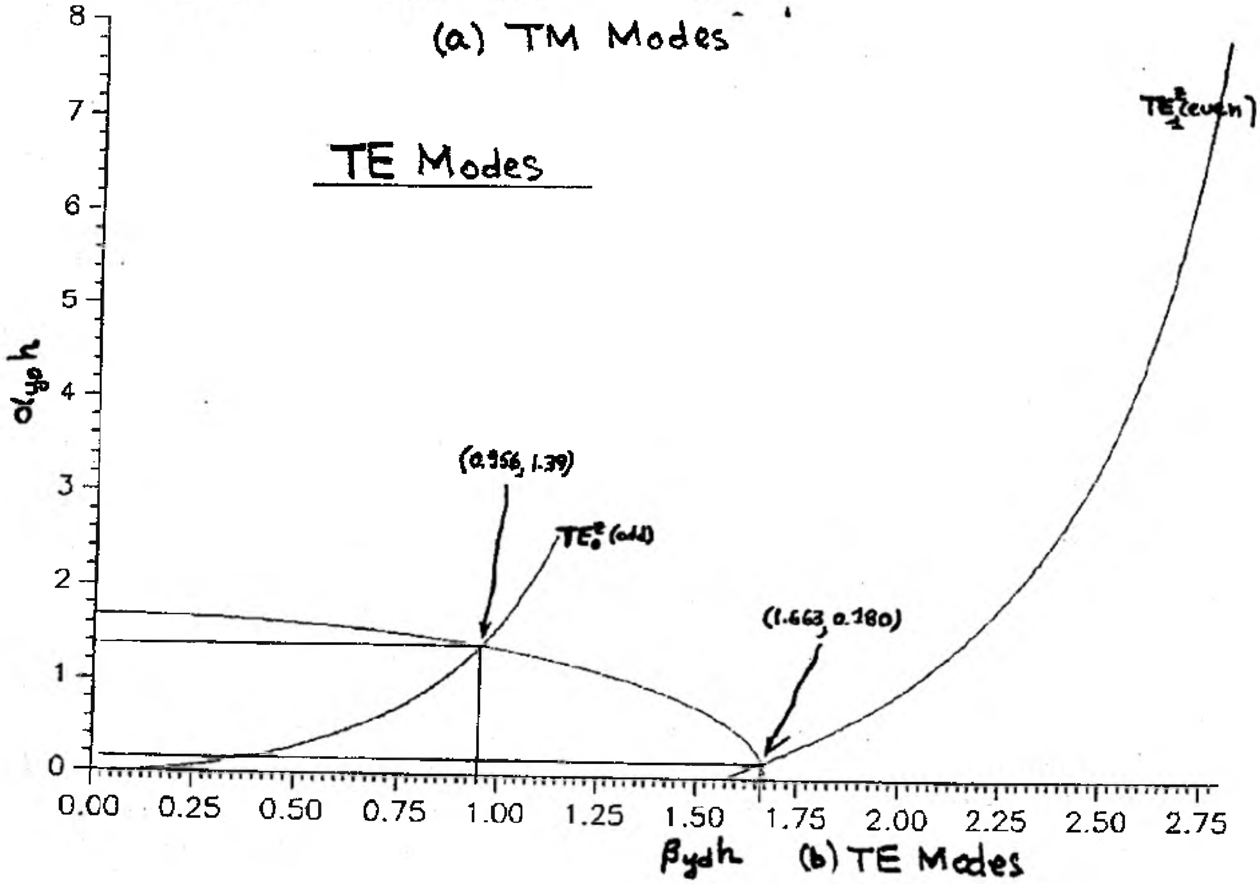
TM_0^z(odd): $\theta_1 = 45.1871°$ TE_0^z(odd): $\theta_1 = 59.1563°$

TM_1^z(even): $\theta_1 = 26.5901°$ TE_1^z(even): $\theta_1 = 27.0944°$

8.32 cont'd.

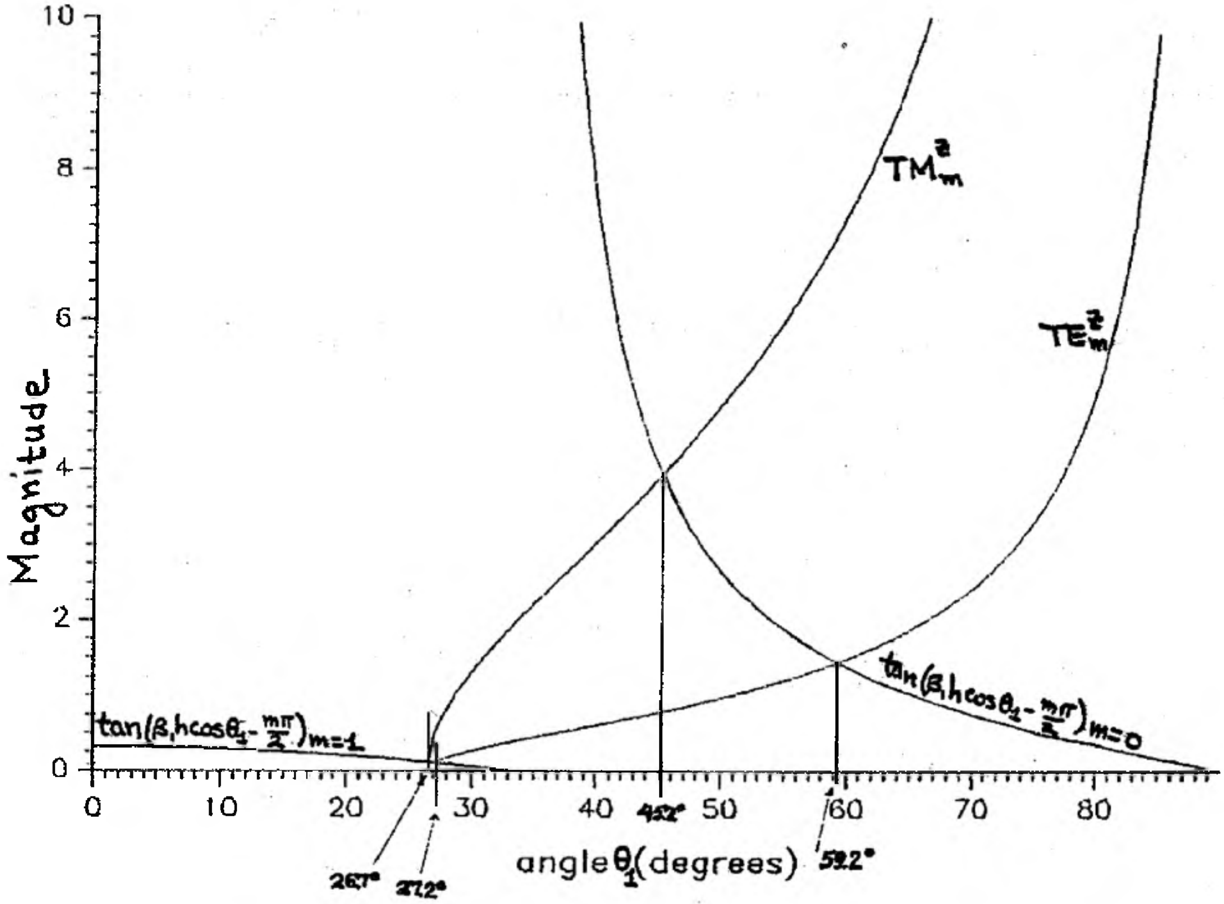


(a) TM Modes



(b) TE Modes

8.32 cont'd.



(c) Incidence angles for TM and TE modes

8.33

a. TM_0 (odd): $f_c = 0$

TE_1 (even):

$$f_c = \frac{1}{4h\sqrt{\mu_0\varepsilon_0}\sqrt{\varepsilon_r - 1}} = \frac{30\times10^9}{4(1.25)\sqrt{4-1}} = 3.464\times10^9$$

b. $\theta_c = \sin^{-1}\sqrt{\frac{1}{4}} = \sin^{-1}(0.5) = 30^\circ \begin{cases} \cos\theta_c = 0.867 \\ \sin\theta_c = 0.5 \end{cases}$

TM_0 (m=0): $\tan\left(\beta_1 h\cos\theta_1 - \frac{m\pi}{2}\right) = \frac{\sqrt{\varepsilon_r}\sqrt{\varepsilon_r\sin^2\theta_1 - 1}}{\cos\theta_1}$, $m = 0, 1, 2, \ldots$

m=0: $\tan(\beta_1 h\cos\theta_1) = \frac{\sqrt{\varepsilon_r}\sqrt{\varepsilon_r\sin^2\theta_1 - 1}}{\cos\theta_1} \Big\{ \theta_1 = 60^\circ$ $\cos\theta_1 = 0.5$, $\sin\theta_1 = 0.867$

$$\beta_1 = \frac{1}{h\cos\theta_1}\tan^{-1}\left[\frac{\sqrt{\varepsilon_r}\sqrt{\varepsilon_r\sin^2\theta_1 - 1}}{\cos\theta_1}\right] = \frac{1}{1.25(0.5)}\tan^{-1}\left\{\frac{\sqrt{4}\sqrt{4(0.867)^2 - 1}}{0.5}\right\}$$

$$\beta_1 = \frac{1}{0.625}\tan^{-1}(5.6664) = \frac{79.9916^\circ(1.3961\,\text{rads})}{0.625} = \frac{1.3961}{0.625}$$

$$\beta_1 = 2.2338 \text{ rads/cm}$$

$$\beta_{yd} = \beta_1\cos\theta_1 = 2.2338(0.5) = 1.1169 \text{ rads/cm}$$

TE_1 (m=1): $\tan\left(\beta_1 h\cos\theta_1 - \frac{m\pi}{2}\right) = \frac{\sqrt{\varepsilon_r\sin^2\theta_1 - 1}}{\sqrt{\varepsilon_r}\cos\theta_1}$

$$\beta_1 = \frac{1}{h\cos\theta_1}\left\{\frac{\pi}{2} + \tan^{-1}\frac{\sqrt{\varepsilon_r\sin^2\theta_1 - 1}}{\sqrt{\varepsilon_r}\cos\theta_1}\right\} = \frac{1}{1.25(0.5)}\left\{\frac{\pi}{2} + \tan^{-1}\left[\frac{\sqrt{4(0.867)^2 - 1}}{\sqrt{4}(0.5)}\right]\right\}$$

$$\beta_1 = \frac{1}{0.625}\left\{\frac{\pi}{2} + 0.9561\right\} = \frac{2.5269}{0.625} = 4.043 \text{ rads/cm}$$

$$\beta_1 = 4.043 \text{ rads/cm}$$

$$\beta_{yd} = \beta_1\cos\theta_1 = 4.043(0.5) = 2.0215 \text{ rads/cm}$$

$$\beta_{yd} = 2.0215 \text{ rads/cm}$$

$\boxed{8.34}$ $(f_c)_m = \dfrac{m}{4h\sqrt{\mu_d\varepsilon_d - \mu_0\varepsilon_0}}$

$$f = (f_c)_1 + 0.1\,(f_c)_1 = 1.1(f_c)_1 = \frac{1.1}{4h\sqrt{\mu_0\varepsilon_0}\sqrt{\mu_r\varepsilon_r - 1}} = \frac{1.1(30\times10^9)}{4(0.635)\sqrt{\varepsilon_r - 1}} = 10\times10^9$$

$$\sqrt{\varepsilon_r - 1} = \frac{1.1(3)}{4(0.635)} = 1.299 \Rightarrow \varepsilon_r - 1 = (1.299)^2 = 1.688$$

$$\varepsilon_r = 2.688$$

At cutoff $\beta_z = \beta_0$.

$$\beta_{yd}\Big|_{\beta_z=\beta_0} = \sqrt{\beta_d^2 - \beta_z^2}\,\Big|_{\beta_z=\beta_0} = \sqrt{\beta_d^2 - \beta_0^2} = \beta_0\sqrt{\varepsilon_r - 1} = \beta_{c0}\sqrt{\varepsilon_r - 1}$$

$$\beta_{yd} = 2\pi f_c\sqrt{\mu_0\varepsilon_0}\sqrt{\varepsilon_r - 1}$$

$$f_c = \frac{10\times10^9}{1.1} = 9.091 \text{ GHz}$$

$$\beta_{yd} = 2\pi\,\frac{9.091\times10^9}{3\times10^8}\sqrt{2.688-1} = 247.375 \text{ Rad/m} = 2.47375 \text{ Rad/cm}$$

$$\alpha_{y0}\Big|_{\beta_z=\beta_0} = \sqrt{\beta_z^2 - \beta_0^2}\,\Big|_{\beta_z=\beta_0} = 0$$

8.35

Given: $\varepsilon_r = 5, \quad \mu_r = 1\ (assumed), \quad thickness = h = 5.625\,\text{cm}$

$$(f_c)_m = \frac{m}{4h\sqrt{\mu_d\varepsilon_d - \mu_0\varepsilon_0}} = \frac{m}{4h\sqrt{\mu_0\varepsilon_0}\sqrt{\mu_r\varepsilon_r - 1}} = \frac{m\cdot c}{4h\sqrt{\mu_r\varepsilon_r - 1}}$$

$c = 30\times10^9$ cm/s
$m = 1,3,5,\ldots,\ \mathrm{TE}^z$ (even)
$m = 0,2,4,\ldots,\ \mathrm{TE}^z$ (odd)

$$(f_c)_0 = \frac{(0)\cdot(30\times10^9\ \text{cm/s})}{(4)\cdot(5.625\,\text{cm})\sqrt{(1)\cdot(5)-1}} = 0\ \text{Hz} \Rightarrow (f_c)_0 = 0\ \text{Hz}\ \left[\mathrm{TM}_0^z\ (\text{odd})\right]$$

$$(f_c)_1 = \frac{(1)\cdot(30\times10^9\ \text{cm/s})}{(4)\cdot(5.625\,\text{cm})\sqrt{(1)\cdot(5)-1}} = 6.67\times10^8\ \text{Hz} \Rightarrow (f_c)_1 = 0.667\ \text{GHz}\ \left[\mathrm{TE}_1^z\ (\text{even})\right]$$

$$(f_c)_2 = \frac{(2)\cdot(30\times10^9\ \text{cm/s})}{(4)\cdot(5.625\,\text{cm})\sqrt{(1)\cdot(5)-1}} = 1.33\times10^9\ \text{Hz} \Rightarrow (f_c)_2 = 1.33\ \text{GHz}\ \left[\mathrm{TM}_2^z\ (\text{odd})\right]$$

$$(f_c)_3 = \frac{(3)\cdot(30\times10^9\ \text{cm/s})}{(4)\cdot(5.625\,\text{cm})\sqrt{(1)\cdot(5)-1}} = 2.00\times10^8\ \text{Hz} \Rightarrow (f_c)_3 = 2.00\ \text{GHz}\ \left[\mathrm{TE}_2^z\ (\text{even})\right]$$

For an operating frequency of 1 GHz, the <u>ONLY</u> the TE^z mode that can propagate inside the slab unattenuated is the TE_1^z mode, whose cutoff frequency is 0.667 GHz (< 1 GHz).

$$\boxed{\mathrm{TE}_1^z}$$

For the TE_1^z mode, find the propagation constant β_z.
Use the Graphical Method:

Plot:
$$-\frac{\mu_0}{\mu}(\beta_{yd}\cdot h)\cot(\beta_{yd}\cdot h) = \alpha_{y0}\cdot h$$
$$(\alpha_{y0}\cdot h)^2 + (\beta_{yd}\cdot h)^2 = a^2$$

$\alpha_{y0}\cdot h$ (ordinate) <u>versus</u> $\beta_{yd}\cdot h$ (abscissa) where,

$$\boxed{a = \omega h\sqrt{\mu_0\varepsilon_0}\sqrt{\mu_r\varepsilon_r - 1} = \frac{\omega h}{c}\sqrt{\mu_r\varepsilon_r - 1} = \beta_0 h\sqrt{\mu_r\varepsilon_r - 1}}$$

and

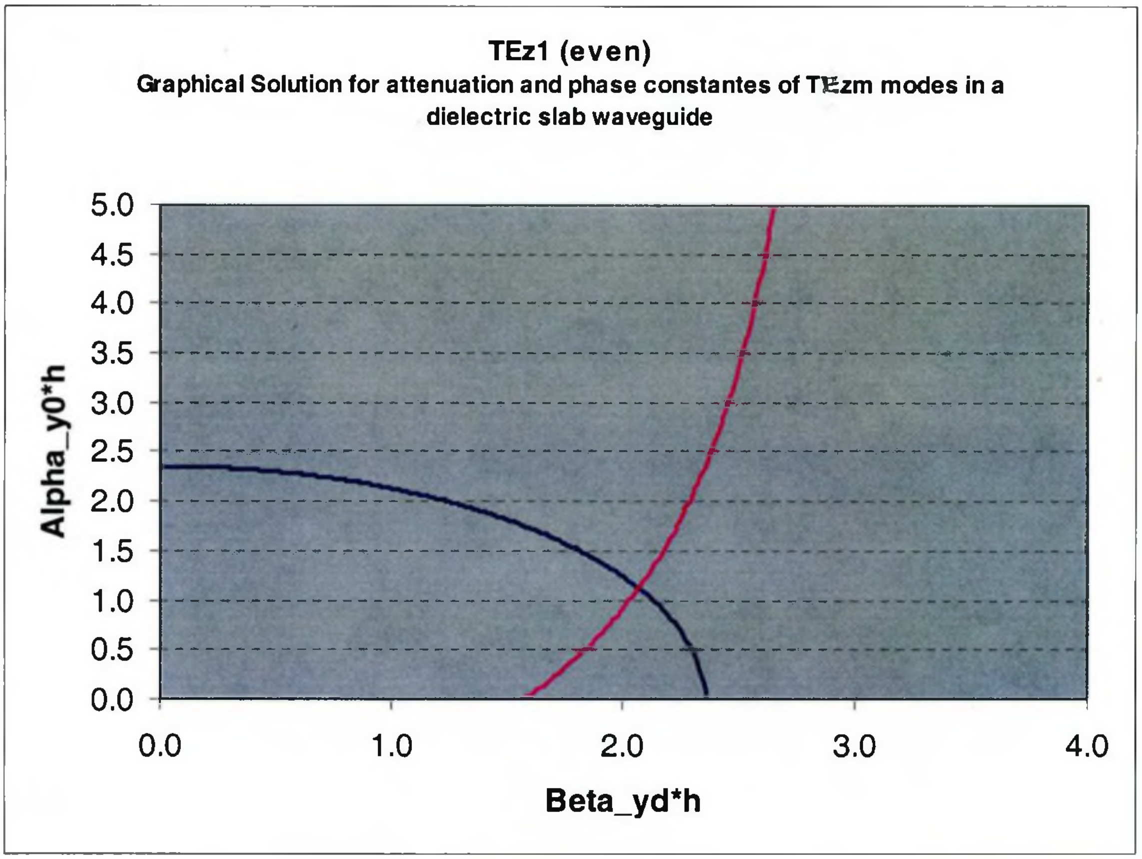
$$\beta_{yd}^2 = \beta_d^2 - \beta_z^2 = \omega^2\mu_d\varepsilon_d - \beta_z^2$$
$$\alpha_{y0}^2 = \beta_z^2 - \beta_0^2 = \beta_z^2 - \omega^2\mu_0\varepsilon_0$$

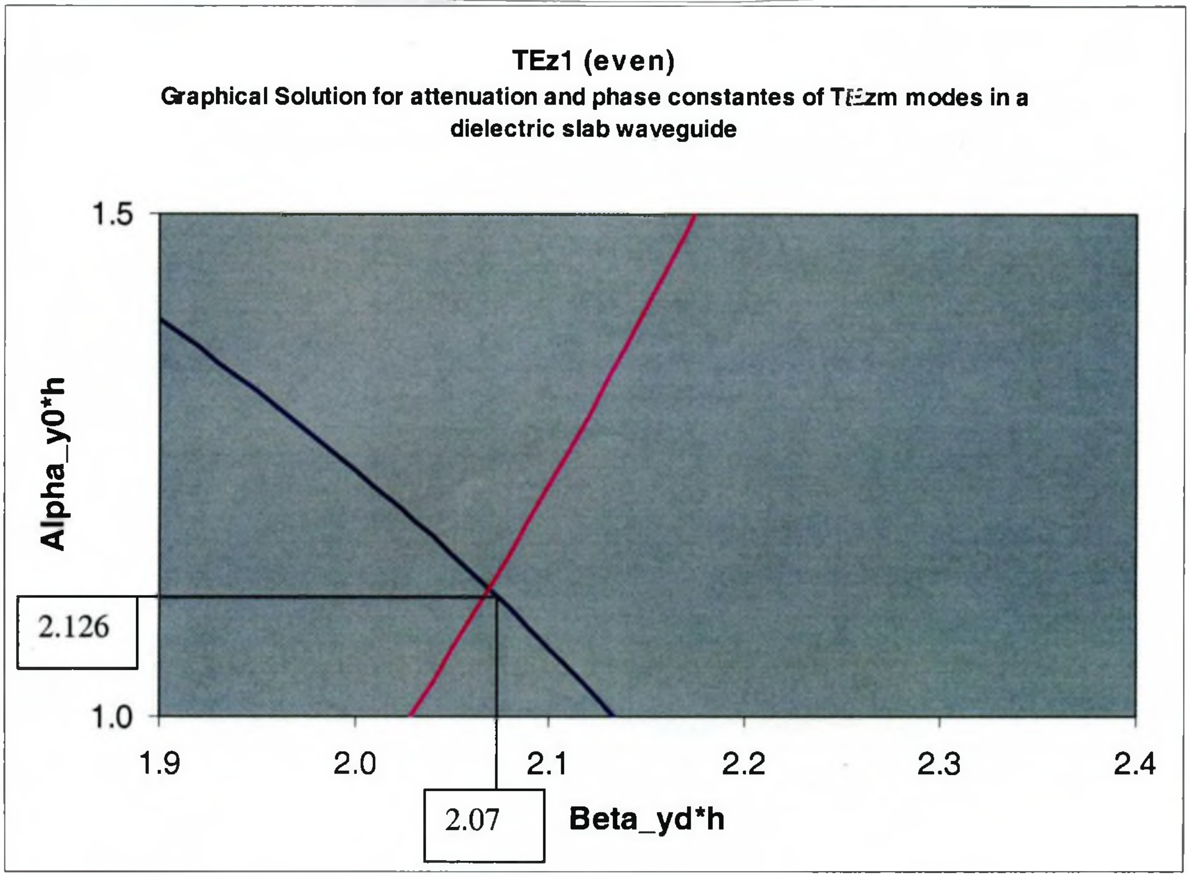
Solve for β_z,

$$\beta_z^2 = \beta_d^2 - \beta_{yd}^2 = \omega^2\mu_d\varepsilon_d - \beta_{yd}^2$$

or

$$\boxed{\begin{aligned}&\beta_z^2 = \alpha_{y0}^2 + \beta_0^2 = \alpha_{y0}^2 + \omega^2\mu_0\varepsilon_0\\ &\beta_z = \pm\sqrt{\alpha_{y0}^2 + \beta_d^2} = \pm\sqrt{\alpha_{y0}^2 + \omega^2\mu_0\varepsilon_0} = \pm\sqrt{\alpha_{y0}^2 + \frac{\omega^2}{c^2}}\end{aligned}}$$





$$\beta_z^2 = \alpha_{y0}^2 + \beta_0^2 = \alpha_{y0}^2 + \omega^2 \mu_0 \varepsilon_0$$

$$\beta_z = \pm\sqrt{\alpha_{y0}^2 + \beta_d^2} = \pm\sqrt{\alpha_{y0}^2 + \omega^2 \mu_0 \varepsilon_0} = \pm\sqrt{\alpha_{y0}^2 + \frac{\omega^2}{c^2}}$$

$$\beta_{yd} \cdot h = 2.07 \text{ rad} \qquad \alpha_{y0} \cdot h = 1.126 \text{ Np}$$

$$\beta_{yd} = \frac{2.07 \text{ rad}}{5.625 \text{ cm}} = 0.368 \text{ rad/cm} = 36.8 \text{ rad/m} \qquad \alpha_{y0} = \frac{1.126 \text{ Np}}{5.625 \text{ cm}} = 0.200 \text{ Np/cm} \simeq 20 \text{ Np/m}$$

$$\beta_z = \pm\sqrt{(0.200)^2 + \frac{\left(2\cdot\pi\cdot 1\times 10^9 \text{ Hz}\right)^2}{\left(30\times 10^9 \text{ cm/s}\right)^2}} = 0.290 \text{ rad/cm}$$

$$\boxed{\begin{array}{c}\beta_z = 0.290 \text{ rad/cm} \\ \beta_z = 29 \text{ rad/m}\end{array}}$$

8.36 $(f_c)_m = \dfrac{m}{4h\sqrt{\mu_d\varepsilon_d - \mu_0\varepsilon_0}} = \dfrac{m}{4h\sqrt{\mu_0\varepsilon_0}\sqrt{\varepsilon_r - 1}}$ $TM^z(\text{odd}): m = 0, 2, 4, \ldots$ $TE^z(\text{even}): m = 1, 3, 5, \ldots$

(a) $m=0$, $TM_0^z(\text{odd})$: $f_c = 0$

(b) $m=1$, $TE_1^z(\text{even})$: $(f_c)_1^{TE} = \dfrac{1}{4h\sqrt{\mu_0\varepsilon_0}\sqrt{\varepsilon_r-1}} = \dfrac{30\times10^9}{4(0.113)\sqrt{\varepsilon_r-1}} = 20\times10^9$

$$\sqrt{\varepsilon_r - 1} = \frac{30}{4(0.113)(20)} = 3.3186 \Rightarrow \varepsilon_r = 1 + (3.3186)^2 = 12.013$$

(c) $TE_1^z(\text{even})$: $(f_c) = 20$ GHz

(d) $\beta_{yd}^2 + \beta_z^2 = \beta_d^2 = \omega^2\mu_0\varepsilon_0\varepsilon_r$

At $f = 20$ GHz, the TE_1^z mode is at cutoff $\Rightarrow \beta_z = \beta_0 = \omega\sqrt{\mu_0\varepsilon_0}$

$$\beta_{yd}^2 = \beta_d^2 - \beta_z^2\Big|_{\beta_z=\beta_0} = \beta_d^2 - \beta_0^2 = (\omega\sqrt{\mu_0\varepsilon_0\varepsilon_r})^2 - (\omega\sqrt{\mu_0\varepsilon_0})^2 = (\omega\sqrt{\mu_0\varepsilon_0})^2(\varepsilon_r - 1)$$

$$= \left[\frac{2\pi(20\times10^9)}{30\times10^9}\right]^2(12.013-1) = 193.2337$$

$\beta_{yd}^2 = 193.2337 \Rightarrow \beta_{yd} = 13.901$ rad/cm $= 796.460$ degrees/cm

$\beta_{yd} = 796.460$ degrees/cm

$$-\alpha_{y0}^2 + \beta_z^2 = \beta_0^2 \Rightarrow \alpha_{y0}^2 = \beta_z^2 - \beta_0^2\Big|_{\substack{\beta_z=\beta_0 \\ @\text{cutoff}}} = 0$$

At cutoff: $\alpha_{y0} = 0$ dB/cm

(e) At 25 GHz $\Rightarrow$ For TE_1^z (even), what is β_{yd} and α_{y0}?

Using the computer program at the end of Chapter 8

$\boxed{\text{Ground\_TE\_TM\_graph}}$

$\beta_{yd} = 16.5909$ rad/cm $= 950.5885$ degrees/cm

$\alpha_{y0} = 5.2039$ Nepers/cm $= 45.1699$ dB/cm

$\boxed{8.37}$

a. $(f_c) = \dfrac{m}{4h\sqrt{\mu_1\varepsilon_1 - \mu_2\varepsilon_2}} = \dfrac{m}{4h\sqrt{\mu_1\varepsilon_1}\sqrt{1-\frac{\mu_2\varepsilon_2}{\mu_1\varepsilon_1}}} = \dfrac{m}{4h\sqrt{\varepsilon_1\mu_0}\sqrt{1-\frac{\varepsilon_2}{\varepsilon_1}}}$

$(f_c)_m = \dfrac{m}{4h\sqrt{\varepsilon_1\mu_0}\sqrt{1-\frac{\varepsilon_{r2}}{\varepsilon_{r1}}}} = \dfrac{m}{4h\sqrt{\varepsilon_{r1}}\sqrt{\mu_0\varepsilon_0}\sqrt{1-\frac{\varepsilon_{r2}}{\varepsilon_{r1}}}}$, $m=0,2,4$ TM^z_m (odd), $m=1,3,5,$ TE^z_m (even)

a. $\boxed{m=0:\ TM^z_0\ (\text{odd})}$

b. $\boxed{m=1:\ TE^z_1\ (\text{even})}$

c. $(f_c)_1 - (f_c)_0 = (3.1-0)\times10^9 = 3.1\times10^9 = \dfrac{1}{4h\sqrt{\varepsilon_{r1}}\sqrt{\mu_0\varepsilon_0}\sqrt{1-\frac{\varepsilon_{r2}}{\varepsilon_{r1}}}}$

$3.1\times10^9 = \dfrac{30\times10^9}{4(2)\sqrt{2.56}}\dfrac{1}{\sqrt{1-\varepsilon_{r2}/\varepsilon_{r1}}} \Rightarrow \sqrt{1-\frac{\varepsilon_{r2}}{\varepsilon_{r1}}} = \dfrac{30}{4(2)(1.6)3.1}$

$\sqrt{1-\frac{\varepsilon_{r2}}{\varepsilon_{r1}}} = 0.756 \Rightarrow 1-\frac{\varepsilon_{r2}}{\varepsilon_{r1}} = (0.756)^2 = 0.5716$

$\frac{\varepsilon_{r2}}{\varepsilon_{r1}} = 1-0.5716 = 0.42839 \Rightarrow \varepsilon_{r2} = 0.42839\,\varepsilon_{r1} = 0.42839(2.56)$

$\varepsilon_{r2} = 0.42839(2.56) = 1.09668 \simeq 1.1$

$\boxed{\varepsilon_{r2} = 1.09668 \simeq 1.1}$

d. $\beta_{yd}^2 + \beta_z^2 = \beta_{d1}^2 = \omega^2\mu_{d1}\varepsilon_{d1} = \omega^2\mu_0\varepsilon_{r1}\varepsilon_0$

At $f=3.1$ GHz the TE^z_1 mode is @ cutoff $\Rightarrow \beta_z = \beta_2 = \omega\sqrt{\mu_0\varepsilon_2}$

$\beta_{yd}^2 = \beta_{d1}^2 - \beta_z^2 = \beta_{d1}^2 - \beta_{d2}^2 = \omega^2\mu_1\varepsilon_1 - \omega^2\mu_2\varepsilon_2 = \omega^2\mu_0\varepsilon_{r1}\varepsilon_0 - \omega^2\mu_0\varepsilon_{r2}\varepsilon_0$

$\beta_{yd}^2 = \omega^2\mu_0\varepsilon_0(\varepsilon_{r1}-\varepsilon_{r2}) = \omega^2\mu_0\varepsilon_0\varepsilon_{r1}\left(1-\frac{\varepsilon_{r2}}{\varepsilon_{r1}}\right)$

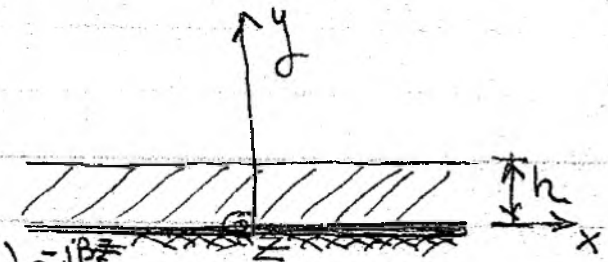
$\beta_{yd} = \omega\sqrt{\mu_0\varepsilon_0}\sqrt{\varepsilon_{r1}}\left(1-\frac{\varepsilon_{r2}}{\varepsilon_{r1}}\right)^{1/2} = 2\pi\left(\frac{3.1\times10^9}{30\times10^9}\right)\sqrt{2.56}\sqrt{1-\frac{\varepsilon_{r2}}{\varepsilon_{r1}}}$ 0.756

$\boxed{\beta_{yd}} = \dfrac{2\pi(3.1)(1.6)(0.756)}{30} = \boxed{0.7854 \text{ rad/cm} = 44.997 \text{ deg/cm}}$

$-\alpha_{y2}^2 + \beta_z^2 = \beta_2^2 \Rightarrow \alpha_{y2}^2 = \beta_z^2 - \beta_2^2\Big|_{\substack{\beta_z=\beta_2 \\ \text{@ cutoff}}} = 0$

$\boxed{\alpha_{y2} = 0}$

$\boxed{8.38}$



$\underline{TM^z(\text{even})}$: $A^d_{ze} = A^d_{me}\cos(\beta_{yd}y)e^{-j\beta_z z}$

$$\left.\begin{aligned} H^d_{xe} &= \frac{1}{\mu_d}\frac{\partial A^d_{ze}}{\partial y} = -A_{me}\frac{\beta_{yd}}{\mu_d}\sin(\beta_{yd}y)e^{-j\beta_z z} \\ H^d_{ye} &= -\frac{1}{\mu_d}\frac{\partial A^d_{ze}}{\partial x} = 0 \\ H^d_{ze} &= 0 \end{aligned}\right\} \quad (6-59)$$

$$H^d_{xe}(y=0) = -A_{me}\frac{\beta_{yd}}{\mu_d}\underbrace{\cancel{\sin(\beta_{yd}y)}}_{0}\Big|_{y=0} e^{-j\beta_z z} = 0$$

$\therefore$ $\underline{TM^z(\text{even})}$ satisfy the additional B.C. $[H^d_{tan}(y=0)=0]$.
Therefore they $\underline{\text{can}}$ propagate as guided waves.

$\underline{TM^z(\text{odd})}$: $A^d_{mo} = A^d_{mo}\sin(\beta_{yd}y)e^{-j\beta_z z}$

$$H^d_{xo} = \frac{1}{\mu_d}\frac{\partial A^d_{zo}}{\partial y} = +A^d_{mo}\frac{\beta_{yd}}{\mu_d}\cos(\beta_{yd}y)e^{-j\beta_z z}$$

$$H^d_{yo} = -\frac{1}{\mu_d}\frac{\partial A^d_{zo}}{\partial x} = 0$$

$$H^d_{zo} = 0$$

$$H^d_{xo}(y=0) = +A^d_{mo}\frac{\beta_{yd}}{\mu_d}\underbrace{\cancel{\cos(\beta_{yd}y)}}_{1}\Big|_{y=0} e^{-j\beta_z z} = A_{mo}\frac{\beta_{yd}}{\mu_d}(1)e^{-j\beta_z z} \neq 0$$

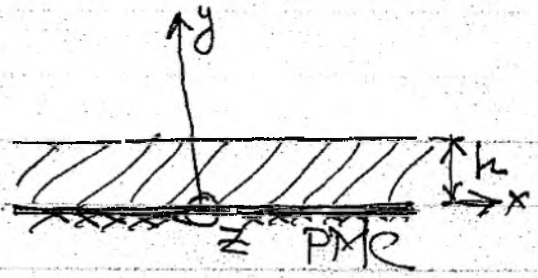
$\therefore$ $\underline{TM^z(\text{odd})}$ do NOT satisfy the additional B.C. $[H^d_{tan}(y=0)=0]$.
Therefore they $\underline{\text{cannot}}$ propagate as guided waves.

(b) $(f_c)_m(\text{even}) = \dfrac{m}{4h\sqrt{\mu_d\varepsilon_d-\mu_0\varepsilon_0}}$, $m = 1, 3, 5, \ldots$

(c) $(f_c)_1(\text{even})\Big|_{m=1} = \dfrac{1}{4h\sqrt{\mu_d\varepsilon_d-\mu_0\varepsilon_0}} = \dfrac{1}{4h\sqrt{\mu_0\varepsilon_0}\sqrt{\mu_r\varepsilon_r-1}} = \dfrac{30\times10^9}{4(0.125)\sqrt{4-1}} = \dfrac{30\times10^9}{0.5(\sqrt{3})} = 34.64\text{ GHz}$

$\boxed{8.39}$

(a) $\underline{TE^z \text{ (even)}}$: $F_{ze}^d = B_{me}^d \cos(\beta_{yd} y) e^{-j\beta_z z}$



$$H_{xe}^d = -j\frac{1}{\omega\mu_d\varepsilon_d}\frac{\partial^2 F_{ze}^d}{\partial x \partial z} = 0$$

$$H_{ye}^d = -j\frac{1}{\omega\mu_d\varepsilon_d}\frac{\partial^2 F_z^d}{\partial y \partial z} = -j\frac{(-\beta_{yd})(-j\beta_z)}{\omega\mu_d\varepsilon_d} B_{me}^d \sin(\beta_{yd} y) e^{-j\beta_z z}$$

$$H_{ze}^d = -j\frac{1}{\omega\mu_d\varepsilon_d}\left(\frac{\partial^2}{\partial z^2}+\beta_d^2\right)F_{zd}^e = -j\frac{1}{\omega\mu_d\varepsilon_d}\left[(-j\beta_z)^2+\beta_d^2\right]B_{me}^d \cos(\beta_{yd} y) e^{-j\beta_z z}$$

$$H_{ze}^d(y=0) = -j\frac{B_{me}^d}{\omega\mu_d\varepsilon_d}\left[-\beta_z^2+\beta_d^2\right]\overset{1}{\cancel{\cos(\beta_{yd} y)}}\Big|_{y=0} e^{-j\beta_z z} \neq 0$$

$\therefore$ TE^z (even) do $\underline{\text{NOT}}$ satisfy the additional B.C. $\left[H_{tan}^d(y=0)=0\right]$. Therefore they $\underline{\text{cannot}}$ propagate as guided waves.

$\underline{TE^z \text{ (odd)}}$: $F_{zo}^d = B_{mo}^d \sin(\beta_{yd} y) e^{-j\beta_z z}$

$$H_{xo}^d = -j\frac{1}{\omega\mu_d\varepsilon_d}\frac{\partial^2 F_{zo}^d}{\partial x \partial z} = 0$$

$$H_{ye}^d = -j\frac{1}{\omega\mu_d\varepsilon_d}\frac{\partial^2 F_z^d}{\partial y \partial z} = -j\frac{(+\beta_{yd})(-j\beta_z)}{\omega\mu_d\varepsilon_d} B_{mo}^d \cos(\beta_{yd} y) e^{-j\beta_z z}$$

$$H_{ze}^d = -j\frac{1}{\omega\mu_d\varepsilon_d}\left(\frac{\partial^2}{\partial z^2}+\beta_d^2\right)F_{ze}^d = -j\frac{B_{mo}^d}{\omega\mu_d\varepsilon_d}\left[(-j\beta_z)^2+\beta_d^2\right]\sin(\beta_{yd} y) e^{-j\beta_z z}$$

$$H_{ze}^d(y=0) = -j\frac{B_{mo}^d}{\omega\mu_d\varepsilon_d}\left(-\beta_z^2+\beta_d^2\right)\overset{0}{\cancel{\sin(\beta_{yd} y)}}\Big|_{y=0} e^{-j\beta_z z} = 0$$

$\therefore$ TE^z (odd) do satisfy the additional B.C. $\left[H_{tan}^d(y=0)=0\right]$. Therefore fore they $\underline{\text{can}}$ propagate as guided waves.

(b) $(f_c)_m^{(odd)} = \dfrac{m}{4h\sqrt{\mu_d\varepsilon_d-\mu_0\varepsilon_0}}$, $m=0,2,4,\ldots$ (8-169f)

(c) $(f_c)_{m=0}^{(odd)} = 0$

$\boxed{8.40}$ $\varepsilon_r = 2.20$

a. $\underline{w/b = 1}$: For zero thickness center conductor

$$\frac{C_f}{\varepsilon} \simeq \frac{1}{\pi}\left[2\ln(2)\right] = 0.4413$$

$$Z_c\sqrt{\varepsilon_r} = \frac{30\pi}{\frac{w/b}{1-t/b} + \frac{C_f}{\varepsilon}} = \frac{30\pi}{w/b + \frac{C_f}{\varepsilon}} = \frac{30\pi}{1+0.4413} = 65.39$$

$$Z_c = \frac{65.39}{\sqrt{2.2}} = 44.086 \text{ Vs. } 44.09 \text{ from Example 8-17.}$$

Good agreement between the two because large w/b.

b. $\underline{w/b = 0.1}$:

$$Z_c\sqrt{\varepsilon_r} = \frac{30\pi}{\frac{w/b}{1-t/b} + \frac{C_f}{\varepsilon}} = \frac{30\pi}{w/b + \frac{C_f}{\varepsilon}} = \frac{30\pi}{0.1+0.4413} = 174.114$$

$$Z_c = \frac{174.114}{\sqrt{2.2}} = 117.387 \text{ Vs. } 131.04 \text{ from Example 8-17.}$$

Poor agreement between the two because of small w/b which leads to greater fringing. Equations (8-200a)-(8-201a) are not as accurate for small values of w/b.

$\boxed{8.41}$

a. $$Z_c\sqrt{\varepsilon_r} = \left.\frac{30\pi}{\frac{w/b}{1-t/b} + \frac{C_f}{\varepsilon}}\right|_{t/b=0} = \frac{30\pi}{\frac{w}{b} + \frac{C_f}{\varepsilon}} = \frac{30\pi}{\frac{w}{b} + 0.4413}$$

$$\frac{w}{b} = -0.4413 + \frac{30\pi}{Z_c\sqrt{\varepsilon_r}} = -0.4413 + \frac{30\pi}{30\sqrt{4}} = -0.4413 + 1.570796$$

$$\boxed{\frac{w}{b} = 1.1295}$$

b. $$k = \tanh\left(\frac{\pi}{2}\frac{w}{b}\right) = \tanh\left[\frac{\pi}{2}(1.1295)\right] = \tanh(1.7742) = 0.944$$

$$k = 0.944 < 1$$

$$\frac{K(k)}{K(k')} \simeq \frac{1}{\pi}\ln\left[2\frac{1+\sqrt{k}}{1-\sqrt{k}}\right]_{k=0.944} = \frac{1}{\pi}\ln\left[2\frac{1+\sqrt{0.944}}{1-\sqrt{0.944}}\right]$$

$$\simeq \frac{1}{\pi}\ln\left[2\frac{1.9716}{0.028365}\right] = \frac{1}{\pi}\ln(139.0159) = \frac{1}{\pi}(4.9346) = 1.5707$$

$$Z_c = \frac{1}{\sqrt{\varepsilon_r}}\frac{30\pi}{K(k)/K(k')} = \frac{1}{\sqrt{4}}\frac{30\pi}{1.5707} = \frac{60}{2} = 30$$

$$\boxed{Z_c = 30 \text{ ohms !!}}$$

$\boxed{8.42}$ $\underline{E} = \hat{a}_y E_o e^{-j\beta z} \Rightarrow \underline{H} = -\hat{a}_x \frac{E_o}{\eta} e^{-j\beta z}$

$$\alpha_c = \frac{P_c/\ell}{2P_d}$$

$$P_d = \int_0^h \int_0^w \frac{1}{2} \mathrm{Re}(\underline{E} \times \underline{H}^*) \cdot \hat{a}_z \, dx\,dy = \frac{1}{2} \mathrm{Re} \int_0^h \int_0^w [\hat{a}_y E_y \times (-\hat{a}_x H_x^*)] \cdot \hat{a}_z \, dx\,dy$$

$$= \frac{1}{2} \mathrm{Re} \int_0^h \int_0^w (\hat{a}_z E_y H_x^*) \cdot \hat{a}_z \, dx\,dy = \frac{1}{2} \mathrm{Re} \int_0^h \int_0^w E_y H_x^* \, dx\,dy$$

$$P_d = \frac{1}{2} \int_0^h \int_0^w \frac{|E_o|^2}{\eta} dx\,dy = \frac{|E_o|^2}{2\eta} wh$$

$$P_c = 2\left\{ \frac{R_s}{2} \int_0^w \int_0^\ell \underline{J}_s \cdot \underline{J}_s^* \, dz\,dx \right\} = R_s \int_0^w \int_0^\ell |\underline{J}_s|^2 dz\,dx$$

$$\underline{J}_s = \hat{n} \times \underline{H} \Big|_{y=0} = \hat{a}_y \times \left(-\hat{a}_x \frac{E_o}{\eta} e^{-j\beta z}\right) = \hat{a}_z \frac{E_o}{\eta} e^{-j\beta z}$$

$$P_c = R_s \frac{|E_o|^2}{\eta^2} \int_0^w \int_0^\ell dz\,dx = R_s \frac{|E_o|^2}{\eta^2} w\ell$$

$$\frac{P_c}{\ell} = R_s \frac{|E_o|^2}{\eta^2} w$$

Therefore

$$\alpha_c = \frac{P_c/\ell}{2P_d} = \frac{R_s \frac{|E_o|^2}{\eta^2} w}{2\frac{|E_o|^2}{2\eta} wh} = R_s \frac{1}{\eta h} = \frac{R_s}{\eta h}$$

$\boxed{8.43}$ The capacitance of a wire of radius a_e and length ℓ located parallel at a height h above a grounded conducting plane is given by

$$C = \frac{2\pi\varepsilon_o \ell}{\cosh^{-1}(h/a_e)} \overset{h \gg a_e}{\simeq} \frac{2\pi\varepsilon_o \ell}{\ln(2h/a_e)} \Rightarrow \frac{C}{\ell} = \frac{2\pi\varepsilon_o}{\cosh^{-1}(h/a_e)} \simeq \frac{2\pi\varepsilon_o}{\ln(2h/a_e)}$$

Using (8-198) we can write the characteristic impedance as

$$Z_c = \frac{\sqrt{\mu_o\varepsilon_o}}{C/\ell} = \frac{\sqrt{\mu_o\varepsilon_o}}{2\pi\varepsilon_o} \cosh^{-1}(h/a_e) \simeq \frac{1}{2\pi}\sqrt{\frac{\mu_o}{\varepsilon_o}} \ln(2h/a_e) = \frac{\eta}{2\pi} \ln\left(\frac{2h}{0.25W}\right)$$

8.44 $w_1 = 1.505$ mm, $h = 1$ mm, $w_3 = 0.549$ mm, $\varepsilon_{r1} = \varepsilon_{r3} = 6$

(a) $w_1/h = 1.505/1 = 1.505$, $w_3/h = 0.549/1 = 0.549$

$$\varepsilon_{r1,eff} = \frac{\varepsilon_{r1}+1}{2} + \frac{\varepsilon_{r1}-1}{2}\left[1+12\left(\frac{h}{w}\right)\right]^{-1/2} = \frac{6+1}{2} + \frac{6-1}{2}\left[\frac{1}{\sqrt{1+\frac{12}{1.505}}}\right]$$

$$= 3.5 + \frac{2.5}{2.9956} = 3.5 + 0.8346 = 4.3346$$

$$\varepsilon_{r3,eff} = 3.5 + 2.5\left\{\frac{1}{\sqrt{1+\frac{12}{0.549}}} + 0.04\left[1-0.549\right]^2\right\} = 3.5+2.5\left[0.2092+0.0081\right]$$

$$= 3.5+2.5(0.2173) = 3.5+0.5433 = 4.0433$$

(b) Using the expressions of (8-205a) and (8-204a) for the characteristic impedances, respectively, for $w/h>1$ and $w/h<1$

$$Z_{c1} = \frac{120\pi/\sqrt{\varepsilon_{r1,eff}}}{\frac{w}{h}+1.393+0.667\ln\left(\frac{w}{h}+1.444\right)} = \frac{120\pi/\sqrt{4.3346}}{1.505+1.393+0.667\ln(1.505+1.444)}$$

$$= \frac{181.0742}{1.505+1.393+0.667(1.0815)} = \frac{181.0742}{3.6193} = 50.03$$

$$Z_{c3} = \frac{60}{\sqrt{\varepsilon_{r3}}}\ln\left[\frac{8}{w/h} + \frac{w/h}{4}\right] = \frac{60}{\sqrt{4.0433}}\ln\left[\frac{8}{0.549} + \frac{0.549}{4}\right]$$

$$= \frac{60}{2.0108}\ln(14.5719+0.1373) = \frac{60}{2.0108}\ln(14.7092) = \frac{60(2.6885)}{2.0108}$$

$$Z_{c3} = 80.22 \text{ ohms}$$

(c) $$Z_{c2} = \sqrt{Z_{c1}Z_{c3}} = \sqrt{50.03(80.22)} = 63.35 \text{ ohms}$$

(d) Assuming the effective dielectric constant of the line between the input and output lines is approximately the average of the two. $\varepsilon_{r2,eff} = \frac{4.3346+4.0433}{2} = 4.189$

the the wavelength @ 2GHz is

$$\lambda_2 = \frac{\lambda_0}{\sqrt{\varepsilon_{r2,eff}}} = \frac{30\times10^9}{\sqrt{4.189}} = 14.6578 \text{ cm}$$

Therefore the length of the $\lambda/4$ transformer is $\frac{14.6578}{4} = 3.6644$ cm

8.45

TE^z (H^z) Modes

For the waveguide geometry of Figure 8-3 the TE^z mode fields can be found using the potential functions

$$\underline{F}^i = \hat{a}_z F_z^i(x,y,z) = \hat{a}_z \psi_h^i(x,y) e^{-j\beta_z z} \qquad (1a)$$

$$\underline{A}^i = 0 \qquad (1b)$$

where i refers to the two media (air and dielectric; i=0 for air and i=d for the dielectric). The normalized field components (normalized with respect to $-\beta_z/\omega\mu\varepsilon$)

$$E_{hx}^i = -\frac{\omega\mu_i\varepsilon_i}{\beta_z}\left[-\frac{1}{\varepsilon}\frac{\partial\psi_h^i(x,y)e^{-j\beta_z z}}{\partial y}\right] = \frac{\omega\mu_i}{\beta_z}\frac{\partial\psi_h}{\partial y}e^{-j\beta_z z} \qquad (2a)$$

$$E_{hy}^i = -\frac{\omega\mu_i\varepsilon_i}{\beta_z}\left[\frac{1}{\varepsilon}\frac{\partial\psi_h^i(x,y)e^{-j\beta_z z}}{\partial x}\right] = -\frac{\omega\mu}{\beta_z}\frac{\partial\psi_h}{\partial x}e^{-j\beta_z z} \qquad (2b)$$

$$E_{hz}^i = 0 \qquad (2c)$$

$$H_{hx}^i = -\frac{\omega\mu_i\varepsilon_i}{\beta_z}\left[-\frac{j}{\omega\mu_i\varepsilon_i}\frac{\partial^2\psi_h^i(x,y)e^{-j\beta_z z}}{\partial x\partial z}\right] = \frac{\partial\psi_h^i}{\partial x}e^{-j\beta_z z} \qquad (2d)$$

8.45 Cont'd.

$$H^i_{hy} = -\frac{\omega\mu_i\varepsilon_i}{\beta_z}\left[-\frac{j}{\omega\mu_i\varepsilon_i}\frac{\partial^2\psi^i_h(x,y)e^{-j\beta_z z}}{\partial y\partial z}\right] = \frac{\partial\psi^i_h}{\partial y}e^{-j\beta_z z} \quad (2e)$$

$$H^i_{hz} = -\frac{\omega\mu_i\varepsilon_i}{\beta_z}\left[-\frac{j}{\omega\mu_i\varepsilon_i}\left(\frac{\partial^2}{\partial z^2}+\beta_i^2\right)\psi^i_h(x,y)e^{-j\beta_z z}\right] = j\left(\frac{\beta_i^2-\beta_z^2}{\beta_z}\right)\psi^i_h e^{-j\beta_z z} \quad (2f)$$

where ψ^i_h satisfies the transverse wave equation of

$$\left[\frac{\partial^2\psi^i_h}{\partial x^2}+\frac{\partial^2\psi^i_h}{\partial y^2}\right] + (\beta_i^2-\beta_z^2)\psi^i_h = \nabla_t^2\psi^i_h + (\beta_i^2-\beta_z^2)\psi^i_h = 0 \quad (3)$$

For the symmetrical modes which require the tangential magnetic field components to vanish along the symmetry plane of the waveguide ($x=0$, $0\le y\le b$), ψ^i_h takes the form of

$$\psi^d_h = \sum_{n=1}^{\infty}\left[A^d_{hn}\cos(\beta^d_{xn}x)+B^d_{hn}\sin(\beta^d_{xn}x)\right]\left[C^d_{hn}\cos(\beta^d_{yn}y)+D^d_{hn}\sin(\beta^d_{yn}y)\right] \quad (4a)$$

$$\psi^0_h = \sum_{n=1}^{\infty}\left[A^0_{hn}\cos(\beta^0_{xn}x)+B^0_{hn}\sin(\beta^0_{xn}x)\right]\left[C^0_{hn}\cos[\beta^0_{yn}(b-y)]+D^0_{hn}\sin[\beta^0_{yn}(b-y)]\right] \quad (4b)$$

For the TE^z modes, the independent boundary conditions on the metallic walls of the waveguide are

$$E^d_{hx}(-a/2\le x\le a/2,\ y=0,\ z) = E^0_{hx}(-a/2\le x\le a/2,\ y=b,\ z) = 0 \quad (5a)$$

$$E^d_{hy}(x=\pm a/2,\ 0\le y\le h,\ z) = E^0_{hy}(x=\pm a/2,\ h\le y\le b,\ z) = 0 \quad (5b)$$

Enforcing the first boundary condition of (5a) using and (4a), leads to

$$E^d_{hx}(-a/2\le x\le a/2,\ y=0,\ z) = \frac{\omega\mu_d\beta^d_{yn}}{\beta_z}\sum_{n=1}^{\infty}\left[A^d_{hn}\cos(\beta^d_{xn}x)+B^d_{hn}\sin(\beta^d_{xn}x)\right]$$

$$\cdot\left[-C^d_{hn}(0)+D^d_{hn}(1)\right]e^{-j\beta_z z} = 0 \Rightarrow D^d_{hn}=0 \quad (6)$$

Applying the first boundary condition of (5b) using (2b),

cont'd.

$\boxed{\text{8.45 Cont'd.}}$ (4a), and (6) leads to

$$E^d_{hy}(x=\pm a/2,\ 0\le y\le h,\ z) = -\frac{\omega\mu_d\beta^d_{xn}}{\beta_z}\sum_{n=1}^{\infty}\left[-A^d_{hn}\sin(\pm\frac{\beta^d_{xn}a}{2})+B^d_{hn}\cos(\pm\frac{\beta^d_{xn}a}{2})\right]$$

$$\cdot\left[C^d_{hn}\cos(\beta^d_{yn}y)\right]e^{-j\beta_z z} = 0 \qquad (7)$$

which is satisfied when

$$-A^d_{hn}\sin(\frac{\beta^d_{xn}a}{2}) + B^d_{hn}\cos(\frac{\beta^d_{xn}a}{2}) = 0 \qquad (7a)$$

$$A^d_{hn}\sin(\frac{\beta^d_{xn}a}{2}) + B^d_{hn}\cos(\frac{\beta^d_{xn}a}{2}) = 0 \qquad (7b)$$

The above two equation are satisfied when

$$A^d_{hn} = 0 \qquad (7c)$$

and

$$\cos(\frac{\beta^d_{xn}a}{2}) = 0 \Rightarrow \frac{\beta^d_{xn}a}{2} = \cos^{-1}(0) = (n-\tfrac{1}{2})\pi$$

$$\boxed{\beta^d_{xn} = \frac{2\pi}{a}(n-\tfrac{1}{2}), \qquad n=1,2,3,\ldots} \qquad (7d)$$

Thus (4a) reduces to

$$\psi^d_h = \sum_{n=1}^{\infty} B^d_{hn}C^d_{hn}\sin(\beta^d_{xn}x)\cos(\beta^d_{yn}y) \qquad (8)$$

where

$$(\beta^d_{xn})^2+(\beta^d_{yn})^2+\beta_z^2 = (\beta_d)^2 \Rightarrow (\beta^d_{yn})^2 = (\beta_d)^2-\left[(\beta^d_{xn})^2+\beta_z^2\right] \qquad (8a)$$

or

$$\beta^d_{yn} = \sqrt{(\beta_d)^2-\left[(\beta^d_{xn})^2+\beta_z^2\right]} = j\sqrt{\left[(\beta^d_{xn})^2+\beta_z^2\right]-(\beta_d)^2} = j\alpha^d_n \qquad (8b)$$

with

cont'd.

8.45 cont'd.

$$\boxed{\alpha_n^d = \sqrt{\left[(\beta_{xn}^d)^2+\beta_z^2\right]-(\beta_d)^2}} \quad \text{(8c)}$$

Therefore (8) can ultimately be written as

$$\psi_h^d = \sum_{n=1}^{\infty} B_{hn}^d C_{hn}^d \sin(\beta_{xn}^d x)\cos(\beta_{yn}^d y) = \sum_{n=1}^{\infty} B_{hn}^d C_{hn}^d \sin(\beta_{xn}^d x)\cos(j\alpha_n^d y)$$

$$\boxed{\psi_h^d = \sum_{n=1}^{\infty} A_{hn}^d \sin(\beta_{xn}^d x)\cosh(\alpha_n^d y)} \quad \text{(9)}$$

where β_{xn}^d and α_n^d are given, respectively, by (7d) and (8c). If the α_n^d is imaginary, then the hyperbolic cosine must be replaced by a trigonometric cosine.

Applying the second boundary condition of (5a) using (2a) and (4b) leads to

$$E_{hx}^0(-a/2\le x\le a/2,\ y=b,\ z) = \frac{\omega\mu_0\beta_{yn}^0}{\beta_z}\sum_{n=1}^{\infty}\left[A_{hn}^0\cos(\beta_{xn}^0 x)+B_{hn}^0\sin(\beta_{xn}^0 x)\right]$$

$$\cdot\left[-C_{hn}^0(0)+D_{hn}^0(1)\right]e^{-j\beta_z z} = 0 \Rightarrow D_{hn}^0=0 \quad \text{(10)}.$$

Enforcing the second boundary condition of (5b) using (2b) (4b), and (10) leads to

$$E_{hy}^0(x=\pm a/2,\ h\le y\le b,\ z) = -\frac{\omega\mu_0\beta_{xn}^0}{\beta_z}\sum_{n=1}^{\infty}\left[-A_{hn}^0\sin(\pm\frac{\beta_{xn}^0 a}{2})+B_{hn}^0\cos(\pm\frac{\beta_{xn}^0 a}{2})\right]$$

$$\cdot\left[-C_{hn}^0\cos[\beta_{yn}^0(h-y)]\right]e^{-j\beta_z z} = 0 \quad \text{(11)}$$

which is satisfied when

$$-A_{hn}^0\sin(\frac{\beta_{xn}^0 a}{2}) + B_{hn}^0\cos(\frac{\beta_{xn}^0 a}{2}) = 0 \quad \text{(11a)}$$

cont'd.

8.45 cont'd.

$$A^0_{hn}\sin\left(\frac{\beta^0_{xn}a}{2}\right) + B^0_{hn}\cos\left(\frac{\beta^0_{xn}a}{2}\right) = 0 \qquad \text{(11b)}$$

The above two equation are satisfied when

$$A^0_{hn} = 0 \qquad \text{(11c)}$$

and

$$\cos\left(\frac{\beta^0_{xn}a}{2}\right) = 0 \Rightarrow \frac{\beta^0_{xn}a}{2} = \cos^{-1}(0) = \left(n - \frac{1}{2}\right)\pi$$

$$\boxed{\beta^0_{xn} = \frac{2\pi}{a}\left(n - \frac{1}{2}\right) = \beta^d_{xn} = \beta_{xn}, \quad n=1,2,3,\ldots} \qquad \text{(11d)}$$

Thus (4b) reduces to

$$\psi^0_h = \sum_{n=1}^{\infty} B^0_{hn}C^0_{hn}\sin(\beta^0_{xn}x)\cos[\beta^0_{yn}(b-y)] \qquad \text{(12)}$$

where

$$(\beta^0_{xn})^2+(\beta^0_{yn})^2+\beta_z^2 = (\beta_0)^2 \Rightarrow (\beta^0_{yn})^2 = (\beta_0)^2-\left[(\beta^0_{xn})^2+\beta_z^2\right] \qquad \text{(12a)}$$

or

$$\beta^0_{yn} = \sqrt{(\beta_0)^2-\left[(\beta^0_{xn})^2+\beta_z^2\right]} = j\sqrt{\left[(\beta^0_{xn})^2+\beta_z^2\right]-(\beta_0)^2} = j\alpha^0_n \qquad \text{(12b)}$$

with

$$\boxed{\alpha^0_n = \sqrt{\left[(\beta^0_{xn})^2+\beta_z^2\right]-(\beta_0)^2}} \qquad \text{(12c)}$$

Therefore (12) can ultimately be written as

$$\psi^0_h = \sum_{n=1}^{\infty} B^0_{hn}C^0_{hn}\sin(\beta^0_{xn}x)\cos[\beta^0_{yn}(b-y)] = \sum_{n=1}^{\infty} B^0_{hn}C^0_{hn}\sin(\beta^0_{xn}x)\cos[j\alpha^0_n(b-y)]$$

$$\boxed{\psi^0_h = \sum_{n=1}^{\infty} B^0_{hn}\sin(\beta^0_{xn}x)\cosh[\alpha^d_n(b-y)} \qquad \text{(13)}$$

cont'd.

B45 cont'd. where β_{xn}^0 and α_n^0 are given, respectively, by (11d) and (12c).

In summary then for the TE^z modes

$\psi_h^d = \sum_{n=1}^{\infty} A_{hn}^d \sin(\beta_{xn}x)\cosh(\alpha_n^d y)$
$\psi_h^0 = \sum_{n=1}^{\infty} B_{hn}^0 \sin(\beta_{xn}x)\cosh[\alpha_n^0(b-y)]$
$\beta_{xn} = \frac{2\pi}{a}(n - \frac{1}{2}), \quad n=1,2,3,\ldots$
$\alpha_n^d = \sqrt{\left[(\beta_{xn})^2+\beta_z^2\right]-(\beta_d)^2}$
$\alpha_n^0 = \sqrt{\left[(\beta_{xn})^2+\beta_z^2\right]-(\beta_o)^2}$

(14a)
(14b)
(14c)
(14d)
(14e)

TM^z (E^z) Modes

Following a similar procedure as for the TE^z modes it can be shown that the fields of the TM^z modes can be derived using the potential functions

$$\underline{A}^i = \hat{a}_z A_z^i(x,y,z) = \hat{a}_z \psi_e^i(x,y) e^{-j\beta_z z} \qquad (15a)$$

$$\underline{F}^i = 0 \qquad (15b)$$

Using the normalized field components (normalized with respect to $-\beta_z/\omega\mu_e\varepsilon_i$) the electric and magnetic fields can be written as

$$E_{ex}^i = \frac{\partial\psi_e}{\partial x} e^{-j\beta_z z} \qquad (16a)$$

$$E_{ey}^i = \frac{\partial\psi_e^i}{\partial y} e^{-j\beta_z z} \qquad (16b)$$

$$E_{ez}^i = j\frac{(\beta_i^2-\beta_z^2)}{\beta_z} \psi_e^i e^{-j\beta_z z} \qquad (16c)$$

cont'd.

8.45 cont'd.

$$H^i_{ex} = -\frac{\omega\varepsilon_i}{\beta_z}\frac{\partial\psi^i_e}{\partial y}\, e^{-j\beta_z z} \tag{16d}$$

$$H^i_{ey} = \frac{\omega\varepsilon_i}{\beta_z}\frac{\partial\psi^i_e}{\partial x}\, e^{-j\beta_z z} \tag{16e}$$

$$H^i_{ez} = 0 \tag{16f}$$

where ψ^i_e satisfies the transverse wave equation of

$$\left(\frac{\partial^2\psi^i_e}{\partial x^2} + \frac{\partial^2\psi^i_e}{\partial y^2}\right) + (\beta_i^2-\beta_z^2)\psi^i_e = \nabla_t^2\psi^i_e + (\beta_i^2-\beta_z^2)\psi^i_e = 0 \tag{17}$$

For the symmetrical modes which require the tangential magnetic field components to vanish along the symmetry plane of the waveguide ($x=0$, $0\le y\le b$), ψ^i_e takes the form of

$$\psi^d_e = \sum_{n=1}^{\infty}\left[A^d_{en}\cos(\beta^d_{xn}x)+B^d_{en}\sin(\beta^d_{xn}x)\right]\left[C^d_{en}\cos(\beta^d_{yn}y)+D^d_{en}\sin(\beta^d_{yn}y)\right] \tag{18a}$$

$$\psi^0_e = \sum_{n=1}^{\infty}\left[A^0_{en}\cos(\beta^0_{xn}x)+B^0_{en}\sin(\beta^0_{xn}x)\right]\left[C^0_{en}\cos[\beta^0_{yn}(h-y)]+D^0_{en}\sin[\beta^0_{yn}(h-y)]\right] \tag{18b}$$

Applying the appropriate boundary conditions along the metallic walls of the waveguide reduces the above scalar potentials to

$$\psi^d_e = \sum_{n=1}^{\infty} A^d_{en}\cos(\beta_{xn}x)\sinh(\alpha^d_n y) \tag{19a}$$

$$\psi^0_e = \sum_{n=1}^{\infty} B^0_{en}\cos(\beta_{xn}x)\sinh[\alpha^0_n(b-y)] \tag{19b}$$

$$\beta_{xn} = \frac{2\pi}{a}\left(n-\frac{1}{2}\right),\quad n=1,2,3,\ldots \tag{19c}$$

$$\alpha^d_n = \sqrt{\left[(\beta_{xn})^2+\beta_z^2\right]-(\beta_d)^2} \tag{19d}$$

$$\alpha^0_n = \sqrt{\left[(\beta_{xn})^2+\beta_z^2\right]-(\beta_o)^2} \tag{19e}$$

cont'd.

845 cont'd. The coefficients A^d_{hn}, B^0_{hn}, A^d_{en} and B^0_{en} are yet unknowns, and their relations will be found by applying the boundary conditions along the air-dielectric interface.

The next step is to apply the appropriate boundary conditions along the air-dielectric interface including the center conductor. It should be stated again that the superposition of the TE^z and TM^z modes must satisfy the boundary along this interface. The appropriate boundary condition along the interface for x>0 that must be enforced are the following (the same apply for x<0):

$$E^d_{ez}(w/2\le x\le a/2,\ y=h,\ z) = E^0_{ez}(w/2\le x\le a/2,\ y=h,\ z) \quad (20a)$$

$$E^d_{hx}(w/2\le x\le a/2,\ y=h,\ z) + E^d_{ex}(w/2\le x\le a/2,\ y=h,\ z)$$
$$= E^0_{hx}(w/2\le x\le a/2,\ y=h,\ z) + E^0_{ex}(w/2\le x\le a/2,\ y=h,\ z) \quad (20b)$$

$$E^d_{ez}(0\le x\le w/2,\ y=h,\ z) = E^0_{ez}(0\le x\le w/2,\ y=h,\ z) \quad (20c)$$

$$H^d_{hx}(w/2\le x\le a/2,\ y=h,\ z) + H^d_{ex}(w/2\le x\le a/2,\ y=h,\ z)$$
$$= H^0_{hx}(w/2\le x\le a/2,\ y=h,\ z) + H^0_{ex}(w/2\le x\le a/2,\ y=h,\ z) \quad (20d)$$

$$E^d_{hx}(0\le x\le w/2,\ y=h,\ z) + E^d_{ex}(0\le x\le w/2,\ y=h,\ z)$$
$$= E^0_{hx}(0\le x\le w/2,\ y=h,\ z) + E^0_{ex}(0\le x\le w/2,\ y=h,\ z) \quad (20e)$$

$$H^d_{hz}(w/2\le x\le a/2,\ y=h,\ z) = H^0_{hz}(w/2\le x\le a/2,\ y=h,\ z) \quad (20f)$$

The boundary conditions of (20a) and (20b) are used, in conjunction with (14a), (14b), (19a) and (19b), to express B^0_{en} and B^0_{hn} each in terms of A^d_{en} and A^d_{hn}.

Applying (20a) leads to

cont'd.

8.45 cont'd.

$$\boxed{B_{en}^{0} = A_{en}^{d} \frac{\beta_d^2-\beta_z^2}{\beta_0^2-\beta_z^2} \frac{\sinh(\alpha_n^d h)}{\sinh[\alpha_n^0 (b-h)]}} \tag{21}$$

while (20b) reduces for $\mu_d=\mu_0$ to

$$\boxed{B_{hn}^{0} = \left[A_{en}^{d} \frac{\beta_{xn}\beta_z}{\omega\mu_0\alpha_n^0} \left\{1 - \frac{\beta_d^2-\beta_z^2}{\beta_0^2-\beta_z^2}\right\} - A_{hn}^{d} \frac{\alpha_n^d}{\alpha_n^0}\right] \frac{\sinh(\alpha_n^d h)}{\sinh[\alpha_n^0 (b-h)]}} \tag{22}$$

Applying (20c) leads to

$$\boxed{\begin{gathered}\sum_{n=1}^{\infty} \bar{A}_{en}^{d} \cos(\beta_{xn} x) = 0 \\ \text{when } 0\le x\le w/2\end{gathered}} \tag{23}$$

where

$$\boxed{\bar{A}_{en}^{d} = A_{en}^{d} \sinh(\alpha_n^d h)} \tag{23a}$$

while (20d) reduces to

$$\boxed{\begin{gathered}\sum_{n=1}^{\infty} \bar{A}_{en}^{d} \beta_{xn} P_n(\beta_z)\cos(\beta_{xn} x) - \sum_{n=1}^{\infty} \bar{A}_{hn}^{d} \beta_{xn} T_n(\beta_z)\cos(\beta_{xn} x) = 0 \\ \text{when } \quad w/2\le x\le a/2\end{gathered}} \tag{24}$$

where

$$\boxed{\bar{A}_{hn}^{d} = A_{hn}^{d} \frac{\omega\mu_0}{\beta_z} \sinh(\alpha_n^d h)} \tag{24a}$$

cont'd.

8.45 cont'd.

$$P_n(\beta_z) = \varepsilon_r \frac{\alpha_n^d}{\beta_{xn}} \coth(\alpha_n^d h) + \frac{\varepsilon_r - \bar{\beta}_z^2}{1-\bar{\beta}_z^2} \frac{\alpha_n^0}{\beta_{xn}} \coth[\alpha_n^0(b-h)] + \bar{\beta}_z^2 \frac{\beta_{xn}}{\alpha_n^d} \frac{1-\varepsilon_r}{1-\bar{\beta}_z^2} \coth[\alpha_n^0(b-h)] \tag{24b}$$

$$T_n(\beta_z) = \bar{\beta}_z^2 \left[\frac{\beta_{xn}}{\alpha_n^d} \coth(\alpha_n^d h) + \frac{\beta_{xn}}{\alpha_n^0} \cot[\alpha_n^0(b-h)]\right] \tag{24c}$$

$$\bar{\beta}_z = \frac{\beta_z}{\beta_0} = \frac{\beta_z}{\omega\sqrt{\mu_0\varepsilon_0}} = \frac{\lambda_0}{\lambda_g} = \sqrt{\varepsilon_{r_{eff}}} = \frac{\nu_p}{\nu_0} \tag{24d}$$

Applying (8-214e) leads to

$$\sum_{n=1}^{\infty} \bar{A}_{en}^d \beta_{xn} \sin(\beta_{xn} x) - \sum_{n=1}^{\infty} \bar{A}_{hn}^d \beta_{xn} \sin(\beta_{xn} x) = 0 \quad \text{when} \quad 0 \le x \le w/2 \tag{25}$$

while (20f) reduces to

$$\sum_{n=1}^{\infty} \bar{A}_{en}^d Q_n(\beta_z) \sin(\beta_{xn} x) - \sum_{n=1}^{\infty} \bar{A}_{hn}^d W_n(\beta_z) \sin(\beta_{xn} x) = 0 \quad \text{when} \quad w/2 \le x \le a/2 \tag{26}$$

where

$$Q_n(\beta_z) = \frac{\beta_{xn}}{\alpha_n^0} \frac{1-\varepsilon_r}{1-\bar{\beta}_z^2} \coth[\alpha_n^0(b-h)] \tag{26a}$$

$$W_n(\beta_z) = \frac{\varepsilon_r - \bar{\beta}_z^2}{1-\bar{\beta}_z^2} \frac{\beta_{xn}}{\alpha_n^d} \coth(\alpha_n^d h) + \frac{\beta_{xn}}{\alpha_n^0} \cot[\alpha_n^0(b-h)] \tag{26b}$$

The objective now is to use (23)-(26b) to solve for

cont'd.

8.45 cont'd.

$\bar{\beta}_z = \beta_z/\beta_0 = \sqrt{\varepsilon_{r_{eff}}}$ of (24d). This can be accomplished by transforming (23)-(26) into an infinite set of homogeneous simultaneous equations for $\bar{A}^d_{en}$ and $\bar{A}^d_{hn}$ by using a conventional procedure such as the singular integral equation of taking a scalar product with a complete set of functions which are appropriate for the various regions. By solving for the zeros of the determinant associated with the above matrix equation, the values of $\bar{\beta}_z$ can be determined.

8.46

Given: $\varepsilon_r = 6.8,\quad \mu_r = 1\ (assumed),\quad w/h = 1.5,\quad t/h = 0.01$

(a) Effective width-to-height ratio at zero frequency.

Since $w/h = 1.5 > \dfrac{1}{2\pi} = 0.159$,

$$\frac{w_{eff}(f=0)}{h} = \frac{w}{h} + \frac{1.25}{\pi}\cdot\frac{t}{h}\left[1+\ln\left(\frac{2h}{t}\right)\right]$$

$$\frac{w_{eff}(f=0)}{h} = 1.5 + \frac{1.25}{\pi}\cdot(0.01)\cdot\left[1+\ln\left(2\cdot\frac{1}{0.01}\right)\right] = 1.525 \quad \Rightarrow \quad \boxed{\frac{w_{eff}(f=0)}{h} = 1.525}$$

(b) Effective dielectric constant at zero frequency.

$$\frac{w_{eff}(f=0)}{h} = 1.525 > 1$$

$$\varepsilon_{r,eff}(f=0) = \frac{\varepsilon_r+1}{2} + \frac{\varepsilon_r-1}{2}\left[1+12\frac{h}{\omega_{eff}(f=0)}\right]^{-\frac{1}{2}}$$

$$\varepsilon_{r,eff}(f=0) = \frac{6.8+1}{2} + \frac{6.8-1}{2}\left[1+12\cdot\left(\frac{1}{1.525}\right)\right]^{-\frac{1}{2}} = 4.874 \quad \Rightarrow \quad \boxed{\varepsilon_{r,eff}(f=0) = 4.874}$$

(c) Characteristic impedance at zero frequency.

$$Z_c(f=0) = \frac{\dfrac{120\pi}{\sqrt{\varepsilon_{r,eff}(f=0)}}}{\dfrac{\omega_{eff}(f=0)}{h} + 1.393 + 0.667\cdot\ln\left[\dfrac{\omega_{eff}(f=0)}{h} + 1.444\right]}$$

$$Z_c(f=0) = \frac{\dfrac{120\pi}{\sqrt{4.874}}}{1.525+1.393+0.667\cdot\ln[1.525+1.444]} = 46.861\ \text{Ohms} \quad \Rightarrow \quad \boxed{Z_c(f=0) = 46.861\ \text{Ohms}}$$

(Problem 8.46 cont.)

(d) Approximate frequency where dispersion will begin when $h = 0.05$

$$f_c \geq 0.3\sqrt{\frac{Z_c(0)}{h}\cdot\frac{1}{\sqrt{\varepsilon_r - 1}}}\times 10^9 = 0.3\sqrt{\frac{46.861\ \text{Ohms}}{0.05\ \text{cm}}\cdot\frac{1}{\sqrt{6.8-1}}}\times 10^9 = 5.918\times 10^9\ \text{Hz}$$

Approximate frequency where dispersion will begin for h = 0.05cm is $\Rightarrow$ $\boxed{f_c \geq 5.918\ \text{GHz}}$

(e) Effective dielectric constant at 15 GHz

$$\varepsilon_{r,eff}(f) = \varepsilon_r - \left[\frac{\varepsilon_r - \varepsilon_{r,eff}(0)}{1 + \frac{\varepsilon_{r,eff}(0)}{\varepsilon_r}\left(\frac{f}{f_t}\right)^2}\right] \Rightarrow f_t = \frac{Z_c(0)}{2\mu_0 h} = \frac{46.861\ \text{Ohms}}{2\cdot(4\pi\times 10^{-7})(0.0005\ \text{cm})} = 37.29\times 10^9\ \text{Hz}$$

$$\varepsilon_{r,eff}(15\ \text{GHz}) = 6.8 - \left[\frac{6.8 - 4.874}{1 + \frac{4.874}{6.8}\left(\frac{15\ \text{GHz}}{37.29\ \text{GHz}}\right)^2}\right] = 5.074 \Rightarrow \boxed{\varepsilon_{r,eff}(15\ \text{GHz}) = 5.074}$$

(f) Characteristic impedance at 15 GHz. Compare with the value if dispersion is neglected.

$$Z_c(f) = Z_c(0)\sqrt{\frac{\varepsilon_{r,eff}(0)}{\varepsilon_{r,eff}(f)}}$$

$$Z_c(f = 15\ \text{GHz}) = (46.861\ \text{Ohms})\sqrt{\frac{4.874}{5.074}} = 45.928\ \text{Ohms} \Rightarrow \boxed{Z_c(f = 15\ \text{GHz}) = 45.928\ \text{Ohms}}$$

If dispersion is neglected, $\boxed{Z_c(f = 15\ \text{GHz}) = Z_c(f = 0\ \text{Hz}) = 46.861\ \text{Ohms}}$.

The characteristic impedance calculation shows no frequency dependence if dispersion neglected.

(g) Phase velocity at 15 GHz. Compare with the value if dispersion is neglected

$$v_p(f) = \frac{c}{\sqrt{\mu_r\cdot\varepsilon_{r,eff}(f)}}$$

$$v_p(f = 15\ \text{GHz}) = \frac{3\times 10^8\ \text{m/s}}{\sqrt{(1)\cdot(5.074)}} = 1.332\times 10^8\ \text{m/s} \Rightarrow \boxed{v_p(f = 15\ \text{GHz}) = 1.332\times 10^8\ \text{m/s}}$$

If dispersion is neglected, then $\varepsilon_{r,eff}(f = 15\ \text{GHz}) = \varepsilon_{r,eff}(f = 0\ \text{Hz}) = 4.874$

(Problem 8.46 cont.)

$$v_p(f=0\,\text{GHz})=\frac{3\times10^8\ \text{m/s}}{\sqrt{(1)\cdot(4.874)}}=1.359\times10^8\ \text{m/s}\Rightarrow \boxed{v_p(f=15\,\text{GHz})=1.359\times10^8\ \text{m/s}}$$

Dispersion Neglected

$$v_p=\frac{3\times10^8\ \text{m/s}}{\sqrt{(1)\cdot(6.8)}}=1.15\times10^8\ \text{m/s}\Rightarrow \quad v_p=1.15\times10^8\ \text{m/s}\quad \text{(no fringing)}$$

(h) Guide wavelength at 15 GHz.

$$\lambda_g(f)=\frac{c}{f\sqrt{\mu_r\cdot\varepsilon_{r,eff}(f)}}$$

$$\lambda_g(f=15\,\text{GHz})=\frac{3\times10^8\ \text{m/s}}{15\times10^9\ \text{GHz}\sqrt{(1)\cdot(5.074)}}=8.879\times10^{-3}\,\text{meters}$$

$$\boxed{\lambda_g(f=15\,\text{GHz})=0.8879\ \text{cm}}$$

CHAPTER 9

9.1

Given: $a = 1.12\,\text{cm},\ \varepsilon_r = ?,\ \mu_r = 1,\ BW = 1.5\,\text{GHz}$ (over a single dominant mode)

Dominant mode for a circular waveguide: TE^z_{11} with $\chi'_{11} = 1.8412$

Next higher-order mode: TM^z_{01} with $\chi_{01} = 2.4049$

BW of the TE^z_{11} mode is: $(f_c)^{\text{TM}^z}_{01} - (f_c)^{\text{TE}^z}_{11}$

$$BW = (f_c)^{\text{TM}^z}_{01} - (f_c)^{\text{TE}^z}_{11} = \frac{\chi_{01}}{2\pi \cdot a\sqrt{\mu\varepsilon}} - \frac{\chi'_{11}}{2\pi \cdot a\sqrt{\mu\varepsilon}} = \frac{\chi_{01} - \chi'_{11}}{2\pi \cdot a\sqrt{\mu\varepsilon}} = \frac{(\chi_{01} - \chi'_{11})\cdot c}{2\pi \cdot a\sqrt{\mu_r\varepsilon_r}}$$

$$\varepsilon_r = \left[\frac{(\chi_{01} - \chi'_{11})\cdot c}{2\pi \cdot a \cdot BW\sqrt{\mu_r}}\right]^2 = \left[\frac{(2.4049 - 1.8412)\cdot 30\times10^9\ \text{cm/s}}{2\pi \cdot (1.12\,\text{cm})\cdot 1.5\times10^9\ \text{Hz}\sqrt{1}}\right]^2 = 2.57$$

$$\boxed{\varepsilon_r = 2.57}$$

Lower Frequency for the TE^z_{11} mode:

$$f_l = (f_c)^{\text{TE}^z}_{11} = \frac{\chi'_{11}}{2\pi \cdot a\sqrt{\mu\varepsilon}} = \frac{\chi'_{11}\cdot c}{2\pi \cdot a\sqrt{\mu_r\varepsilon_r}} = \frac{(1.8412)\cdot(30\times10^9\ \text{cm/s})}{2\pi \cdot (1.12\,\text{cm})\sqrt{(1)\cdot(2.57)}} = 4.90\,\text{GHz}$$

Upper Frequency for the TE^z_{11} mode:

$$f_u = (f_c)^{\text{TM}^z}_{01} = \frac{\chi_{01}\cdot c}{2\pi \cdot a\sqrt{\mu_r\varepsilon_r}} = \frac{(2.4049)\cdot(30\times10^9\ \text{cm/s})}{2\pi \cdot (1.12\,\text{cm})\sqrt{(1)\cdot(2.57)}} = 6.40\,\text{GHz}$$

or

$$f_u = f_l + 1.5\,\text{GHz} = (6.21 + 1.5)\,\text{GHz} = 6.40\,\text{GHz}$$

$$\boxed{\begin{aligned} f_l &= 4.90\,\text{GHz} \\ f_u &= 6.40\,\text{GHz} \end{aligned}}$$

$\boxed{9.2}$ $a = 2\text{ cm},\ L = 5\text{ cm},\ f = 6\text{ GHz}$

$$\beta_z = \beta\sqrt{1-\left(\frac{f_c}{f}\right)^2},\quad f_c = \frac{\chi'_{11}}{2\pi a\sqrt{\mu\varepsilon}} = \frac{1.8412}{2\pi a\sqrt{\mu\varepsilon}}$$

$$(f_{c1}) = \frac{1.8412}{2\pi a\sqrt{\mu_0\varepsilon_0\varepsilon_{r1}}} = \frac{1.8412\times 30\times 10^9}{2\pi(2)\sqrt{2.56}} = \frac{1.8412(30)\times 10^9}{4\pi(1.6)} = 2.7472\times 10^9$$

$$(f_{c0}) = \frac{1.8412}{2\pi a\sqrt{\mu_0\varepsilon_0}} = \frac{1.8412\times 30\times 10^9}{4\pi} = 4.3955\times 10^9$$

$$\beta_{z1} = \beta_1\sqrt{1-\left(\frac{f_{c1}}{f}\right)^2} = \omega\sqrt{\mu_0\varepsilon_0\varepsilon_{r1}}\sqrt{1-\left(\frac{f_{c1}}{f}\right)^2} = \beta_0\sqrt{\varepsilon_{r1}}\sqrt{1-\left(\frac{f_{c1}}{f}\right)^2}$$

$$\beta_{z1} = \frac{2\pi}{\lambda_0}\sqrt{\varepsilon_{r1}}\sqrt{1-\left(\frac{2.7472}{6}\right)^2} = \frac{2\pi}{\lambda_0}(1.6)\sqrt{1-(0.45787)^2} = \frac{2\pi}{\lambda_0}(1.6)(0.88902)$$

$$\beta_{z0} = \beta_0\sqrt{1-\left(\frac{f_{c0}}{f}\right)^2} = \frac{2\pi}{\lambda_0}\sqrt{1-\left(\frac{4.3955}{6}\right)^2} = \frac{2\pi}{\lambda_0}\sqrt{1-(0.73258)^2} = \frac{2\pi}{\lambda_0}(0.68068)$$

$$\Delta\beta = \beta_{z1}-\beta_0 = \frac{2\pi}{\lambda_0}(1.6)(0.88902) - \frac{2\pi}{\lambda_0}(0.68068) = \frac{2\pi}{\lambda_0}\left[1.42243-0.68068\right]$$
$$= \frac{2\pi}{5}(1.42243-0.68068) = 1.78748-0.85537 = 0.93211\ \text{rad/cm}$$

$$\Delta\beta = \frac{2\pi}{\lambda_0}(0.74175),\quad \lambda_0 = 30\times10^9/6\times10^9 = 5\text{ cm}$$

$$\Delta\phi = L\Delta\beta = (5)\frac{2\pi}{\lambda_0}(0.74175) = (5)\frac{2\pi}{5}(0.74175) = 4.66058\ \text{radians}$$

$$\Delta\phi = \Delta\phi_1 - \Delta\phi_0 = (8.93739-4.27684)\ \text{rads} = 512.075^\circ - 245.0448^\circ$$

$$\boxed{\Delta\phi = 267.03167^\circ = 4.66058\ (\text{radians})}$$

$$\boxed{\Delta\phi = \Delta\phi_1 - \Delta\phi_0 = 512.075^\circ - 245.0448^\circ}$$

9.3

$a = 2$ cm, $\varepsilon_r = 2.25$, $f = 3.5$ GHz, delay $= 2$ microseconds

Dominant mode: $TE_{11} \Rightarrow (f_c)_{11}^{TE} = \dfrac{\chi'_{mn}}{2\pi a\sqrt{\mu\varepsilon}} = \dfrac{\chi'_{11}}{2\pi a\sqrt{\mu_0\varepsilon_r\varepsilon_0}}$

From Table 9-1 $\Rightarrow \chi'_{11} = 1.8412$

$$(f_c)_{11}^{TE} = \frac{1.8412(30\times10^9)}{2\pi(2)\sqrt{2.25}} = 2.93084 \text{ GHz}$$

$$v_g v_p = v^2 \Rightarrow v_g = \frac{v^2}{v_p}$$

$$\beta_z = \frac{2\pi}{\lambda_g} \Rightarrow \lambda_g = \frac{2\pi}{\beta_z} \Rightarrow v_p = \lambda_g f = \frac{\lambda f}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}} = \frac{v}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}}$$

$$v_g = \frac{v^2}{v_p} = \frac{v^2}{v}\sqrt{1-\left(\frac{f_c}{f}\right)^2} = v\sqrt{1-\left(\frac{f_c}{f}\right)^2}$$

$$v = \frac{v_0}{\sqrt{\varepsilon_r}} = \frac{3\times10^8}{\sqrt{2.25}} = 2\times10^8 \text{ m/sec}$$

$$v_g = 2\times10^8\sqrt{1-\left(\frac{2.93084}{3.5}\right)^2} = 2\times10^8\sqrt{1-(0.83738)^2} = 2\times(0.54662\times10^8)$$

$$v_g = 1.09324\times10^8 \text{ m/sec}$$

$$\ell = v_g t_d = (1.09324\times10^8)\,2\times10^{-6} = 2.18647\times10^2$$

$$\boxed{\ell = 218.647 \text{ meters}}$$

9.4

$$\Delta\phi = \beta_z \Delta l \Rightarrow \Delta l = \Delta\phi / \beta_z \,, \quad \beta_z = \beta\sqrt{1-\left(\frac{f_c}{f}\right)^2}$$

$$\left(f_c\right)_{11}^{TE} = \frac{\chi'_{11}}{2\pi a\sqrt{\mu_0\varepsilon_0}\sqrt{\varepsilon_r}} = \frac{1.841(30\times10^9)/\sqrt{2.25}}{2\pi(2)} = 2.93 \text{ GHz}$$

$$\beta_z = \beta\sqrt{1-\left(\frac{f_c}{f}\right)^2} = \frac{2\pi}{\lambda}\sqrt{1-\left(\frac{f_c}{f}\right)^2} = \frac{2\pi\sqrt{\varepsilon_r}}{\lambda_0}\sqrt{1-\left(\frac{f_c}{f}\right)^2} = \frac{2\pi\sqrt{2.25}}{5}\sqrt{1-\left(\frac{2.93}{6}\right)^2}$$

$$= \frac{2\pi(1.5)}{5}(0.8727) = 1.6449 \text{ rads/cm} = 94.247^\circ/\text{cm}$$

$$\Delta l = \frac{\Delta\phi}{\beta_z} = \frac{180}{94.247} = 1.9099 \text{ cm}$$

9.5

Vector Potential Component:

For a Circular Waveguide,

$$F_z(\rho,\phi,z) = \left[A_1 J_m(\beta_\rho \rho) + B_1 Y_m(\beta_\rho \rho)\right]\cdot\left[C_2\cos(m\phi) + D_2\sin(m\phi)\right]\cdot\left[A_3 e^{-j\beta_z z} + B_3 e^{+j\beta_z z}\right]$$

$$\beta_\rho^2 + \beta_z^2 = \beta^2$$

$f_1(\rho) = A_1 J_m(\beta_\rho \rho) + B_1 Y_m(\beta_\rho \rho)$ represent standing waves in the $\pm\rho$ directions.
$g_2(\phi) = C_2\cos(m\phi) + D_2\sin(m\phi)$ represent periodic waves in the $\pm\phi$ directions.
$h_3(z) = A_3 e^{-j\beta_z z} + B_3 e^{+j\beta_z a}$ represent traveling waves in the $\pm z$ directions.

Find the constants, $A_1, B_1, C_2, D_2, A_3, B_3, m, \beta_\rho, and \beta_z$ using the following conditions...

1) $E_\phi(\rho = a, \phi, z) = 0$: Tangential component of the Electric field at the waveguide wall must be zero. $E_\rho(\rho, \phi = 2\pi, z)$
2) $E_\rho(\rho, \phi = 0, z) = 0$: Tangential component of the Electric field at the conducting baffle must be zero.
3) The fields must be finite everywhere.

[redacted].

According to 3), $B_1 = 0$ since $Y_m[\rho = 0] = -\infty$.
According to 4), $m = 1, 2, 3, \ldots$
Consider only waves that propagate in the + z direction. Then $B_3 = 0$.

Prob 9-5 Cont.

$$F_z^+(\rho,\phi,z) = A_1 A_3 J_m(\beta_\rho \rho)\cdot[C_2\cos(m\phi) + D_2\sin(m\phi)]\cdot e^{-j\beta_z z}, \quad \text{with } \beta_\rho^2 + \beta_z^2 = \beta^2$$

Apply BC # 1) $E_\phi(\rho = a, \phi, z) = 0$:

$$E_\phi^+(\rho,\phi,z) = \frac{1}{\varepsilon}\frac{\partial}{\partial\rho}F_z^+(\rho,\phi,z) = \beta_\rho \frac{A_1 A_3}{\varepsilon} J_m'(\beta_\rho\rho)\cdot[C_2\cos(m\phi) + D_2\sin(m\phi)]\cdot e^{-j\beta_z z}$$

$$E_\phi^+(\rho = a,\phi,z) = \beta_\rho \frac{A_1 A_3}{\varepsilon} J_m'(\beta_\rho a)\cdot[C_2\cos(m\phi) + D_2\sin(m\phi)]\cdot e^{-j\beta_z z} = 0$$

This is true only for $J_m'(\beta_\rho a) = 0 \quad \Rightarrow \quad \beta_\rho a = \chi_{mn}' \quad \Rightarrow \quad \boxed{\beta_\rho = \frac{\chi_{mn}'}{a}}$

where, χ_{mn}' represents the nth zero (n = 1, 2, 3, … of the derivative of the Bessel function J_m of the first kind of order m (m = 0, 1, 2, 3, …).

Apply BC # 2) $E_\rho(\rho, \phi = 0, z) = 0$:

$$E_\rho^+(\rho,\phi,z) = -\frac{1}{\varepsilon\rho}\frac{\partial}{\partial\phi}F_z^+(\rho,\phi,z) = -m\frac{A_1 A_3}{\varepsilon\rho} J_m(\beta_\rho\rho)\cdot[-C_2\sin(m\phi) + D_2\cos(m\phi)]\cdot e^{-j\beta_z z}$$

$$E_\rho^+(\rho,\phi = 0,z) = -m\frac{A_1 A_3}{\varepsilon\rho} J_m(\beta_\rho\rho)\cdot[-C_2\sin(0) + D_2\cos(0)]\cdot e^{-j\beta_z z} = 0$$

$$E_\rho^+(\rho,\phi = 0,z) = m\frac{A_1 A_3}{\varepsilon\rho} J_m(\beta_\rho\rho)\cdot[D_2]\cdot e^{-j\beta_z z} = 0 \qquad \therefore \qquad \boxed{D_2 = 0}$$

$E_\rho^+(\rho,\phi=2\pi,z) = m\frac{A_1A_3}{\varepsilon\rho}J_m(\beta_\rho\rho)\left[-C_2\sin(2m\pi)\right] = \sin(2m\pi)=0 \Rightarrow 2m\pi = \sin^{-1}(0) = p\pi$

$m = \frac{p\pi}{2}, \ p = 0, 1, 2, 3$

$\boxed{m = 0, \frac{1}{2}, 1, \frac{3}{2} \quad n = 1, 2, 3, 4, \ldots}$

Finally, the simplified expression for the Electric Vector Potential…

$$F_z^+(\rho,\phi,z) = A_1 A_3 C_2 \cdot J_m(\beta_\rho\rho)\cdot[\cos(m\phi)]\cdot e^{-j\beta_z z}$$

$$\boxed{F_z^+(\rho,\phi,z) = A_{mn} J_m(\beta_\rho\rho)\cdot[\cos(m\phi)]\cdot e^{-j\beta_z z}}$$

with $\beta_\rho^2 + \beta_z^2 = \beta^2$ and $\beta_\rho = \frac{\chi_{mn}'}{a}$, $\quad m = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$

Prob 9-5 Cont

$$F_z^+(\rho,\phi,z) = A_{mn} J_m(\beta_\rho \rho)\cdot[\cos(m\phi)]\cdot e^{-j\beta_z z}$$

$$E_\rho^+(\rho,\phi,z) = -\frac{1}{\varepsilon\rho}\frac{\partial F_z^+}{\partial\phi} = A_{mn}\frac{m}{\varepsilon\rho} J_m(\beta_\rho\rho)\cdot[\sin(m\phi)]\cdot e^{-j\beta_z z}$$

$$E_\phi^+(\rho,\phi,z) = \frac{1}{\varepsilon}\frac{\partial F_z^+}{\partial\rho} = A_{mn}\frac{\beta_\rho}{\varepsilon} J_m'(\beta_\rho\rho)\cdot[\cos(m\phi)]\cdot e^{-j\beta_z z}$$

$$E_z^+(\rho,\phi,z) = 0$$

$$H_\rho^+(\rho,\phi,z) = -j\frac{1}{\omega\mu\varepsilon}\frac{\partial^2 F_z^+}{\partial\rho\cdot\partial z} = -A_{mn}\frac{\beta_\rho\beta_z}{\omega\mu\varepsilon} J_m'(\beta_\rho\rho)\cdot[\cos(m\phi)]\cdot e^{-j\beta_z z}$$

$$H_\phi^+(\rho,\phi,z) = -j\frac{1}{\omega\mu\varepsilon}\frac{1}{\rho}\frac{\partial^2 F_z^+}{\partial\phi\cdot\partial z} = A_{mn}\frac{m\beta_z}{\omega\mu\varepsilon}\frac{1}{\rho} J_m(\beta_\rho\rho)\cdot[\sin(m\phi)]\cdot e^{-j\beta_z z}$$

$$H_z^+(\rho,\phi,z) = -j\frac{1}{\omega\mu\varepsilon}\left(\frac{\partial^2}{\partial z^2}+\beta^2\right)F_z^+ = -jA_{mn}\frac{\beta_\rho^2}{\omega\mu\varepsilon} J_m(\beta_\rho\rho)\cdot[\cos(m\phi)]\cdot e^{-j\beta_z z}$$

<u>Cutoff Frequency</u>

$$\beta_\rho^2 + \beta_z^2 = \beta^2 \qquad \Leftarrow \qquad \beta_\rho = \frac{\chi_{mn}'}{a}$$

$$\beta_\rho^2 + 0 = \beta_c^2 \Rightarrow \beta_c = \omega_c\sqrt{\mu\varepsilon} = 2\pi\cdot f_c\sqrt{\mu\varepsilon} = \beta_\rho = \frac{\chi_{mn}'}{a}$$

$$f_c = \frac{\chi_{mn}'}{2\pi\cdot a\cdot\sqrt{\mu\varepsilon}} = \frac{\chi_{mn}'}{2\pi\cdot a\cdot\sqrt{\mu_0\varepsilon_0}\sqrt{\mu_r\varepsilon_r}} = \frac{\chi_{mn}'\cdot c}{2\pi\cdot a\cdot\sqrt{\mu_r\varepsilon_r}}$$

(a) The cutoff frequencies of the three lowest-order propagating modes in order of ascending cutoff frequency when the radius of the cylinder is 1 cm are:

$$(\mathrm{TE}^z)_{\frac{1}{2}1}: \quad (f_c)_{\frac{1}{2}1} = \frac{\chi_{\frac{1}{2}1}'\cdot c}{2\pi\cdot a\cdot\sqrt{\mu_r\varepsilon_r}} = \frac{(1.1655)\cdot(30\times10^9\ \mathrm{cm/s})}{2\pi\cdot(1\,\mathrm{cm})\cdot\sqrt{(1)\cdot(1)}} = 5.565\,\mathrm{GHz}$$

$$(\mathrm{TE}^z)_{11}: \quad (f_c)_{11} = \frac{\chi_{11}'\cdot c}{2\pi\cdot a\cdot\sqrt{\mu_r\varepsilon_r}} = \frac{(1.8412)\cdot(30\times10^9\ \mathrm{cm/s})}{2\pi\cdot(1\,\mathrm{cm})\cdot\sqrt{(1)\cdot(1)}} = 8.791\,\mathrm{GHz}$$

$$(\mathrm{TE}^z)_{\frac{3}{2}1}: \quad (f_c)_{\frac{3}{2}1} = \frac{\chi_{\frac{3}{2}1}'\cdot c}{2\pi\cdot a\cdot\sqrt{\mu_r\varepsilon_r}} = \frac{(2.4605)\cdot(30\times10^9\ \mathrm{cm/s})}{2\pi\cdot(1\,\mathrm{cm})\cdot\sqrt{(1)\cdot(1)}} = 11.748\,\mathrm{GHz}$$

Prob 9-5 Cont

$$\boxed{\begin{aligned}(f_c)_{\frac{1}{2}1} &= 5.565\,\text{GHz}\\ (f_c)_{1y} &= 8.791\,\text{GHz}\\ (f_c)_{\frac{3}{2}1} &= 11.748\,\text{GHz}\end{aligned}}$$

Phase Constant along the axis of the guide:

$$\beta_\rho^2 + \beta_z^2 = \beta^2 \quad \Rightarrow \quad \beta_z^2 = \beta^2 - \beta_\rho^2 = \beta^2 - \left(\frac{\chi'_{mn}}{a}\right)^2 = \beta^2\left[1-\left(\frac{\chi'_{mn}}{a\cdot\beta}\right)^2\right]$$

$$f_c = \frac{\chi'_{mn}}{2\pi\cdot a\cdot\sqrt{\mu\varepsilon}} = \frac{\chi'_{mn}\cdot v}{2\pi\cdot a} = \frac{\chi'_{mn}}{2\pi\cdot a}\cdot\lambda f = \frac{\chi'_{mn}}{2\pi\cdot a}\cdot\left(\frac{2\pi}{\beta}\right)f \quad \Rightarrow \quad \frac{f_c}{f} = \frac{\chi'_{mn}}{\beta\cdot a}$$

$$(\beta_z)_{mn} = \beta\sqrt{1-\left(\frac{\chi'_{mn}}{a\cdot\beta}\right)^2} \quad \Rightarrow \quad \boxed{\beta_z = \beta\sqrt{1-\left(\frac{f_c}{f}\right)^2}} \qquad f > f_c = (f_c)_{mn}$$

Guide Wavelength:

$$(\lambda_g)_{mn} = \frac{2\pi}{(\beta_z)_{mn}} = \frac{2\pi}{\beta\sqrt{1-\left(\frac{f_c}{f}\right)^2}} \quad \Rightarrow \quad \boxed{(\lambda_g)_{mn} = \frac{\lambda}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}}} \qquad f > f_c = (f_c)_{mn}$$

where, λ is the wavelength of the wave in an infinite medium which exists inside the waveguide.

Guide Wavelength:

$$(Z_w^{+z})_{mn}^{\text{TE}} = \frac{E_\rho^+}{H_\phi^+} = \frac{\omega\mu}{(\beta_z)_{mn}} = \frac{\omega\mu}{\beta\sqrt{1-\left(\frac{f_c}{f}\right)^2}} = \frac{\omega\mu}{\omega\sqrt{\mu\varepsilon}\sqrt{1-\left(\frac{f_c}{f}\right)^2}} = \frac{\sqrt{\frac{\mu}{\varepsilon}}}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}}$$

$$\boxed{(Z_w^{+z})_{mn}^{\text{TE}} = \frac{\eta}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}} = \frac{\eta_0\sqrt{\frac{\mu_r}{\varepsilon_r}}}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}}} \qquad f > f_c = (f_c)_{mn}$$

Prob 9-5 Cont

(b) The wave impedance and guide wavelength (in cm) for the lowest-order mode at $f = 1.5\, f_c$ where f is the cutoff frequency of the lowest-order mode.

$$(\lambda_g)_{mn} = \frac{\lambda}{\sqrt{1-\left(\dfrac{f_c}{f}\right)^2}} \qquad \lambda = \frac{v}{f} = \frac{1}{\sqrt{\mu\varepsilon}}\cdot\frac{1}{1.5 f_c} = \frac{c}{\sqrt{\mu_r\varepsilon_r}}\cdot\frac{1}{1.5 f_c}$$

$$(\lambda_g)_{\frac{1}{2}1} = \frac{\dfrac{c}{\sqrt{\mu_r\varepsilon_r}}\cdot\dfrac{1}{1.5 f_c}}{\sqrt{1-\left(\dfrac{f_c}{f}\right)^2}} = \frac{30\times10^9 \text{ cm/s}}{(1.5)\cdot(5.565\times10^9 \text{ Hz})\sqrt{1-\left(\dfrac{1}{1.5}\right)^2}} = 4.822\,\text{cm}$$

$$\boxed{(\lambda_g)_{\frac{1}{2}1} = 4.822\,\text{cm}}$$

$$(Z_w^{+z})^{TE}_{\frac{1}{2}1} = \frac{(120\cdot\pi)\cdot\sqrt{\dfrac{1}{1}}}{\sqrt{1-\left(\dfrac{1}{1.5}\right)^2}} = 505.77\,\Omega$$

$$\boxed{(Z_w^{+z})^{TE}_{\frac{1}{2}1} = 505.77\,\Omega}$$

9.6

$$A_z = [C_1 J_m(\beta_\rho\rho) + \cancel{D_1} Y_m(\beta_\rho\rho)][C_2\cos(m\phi) + D_2\sin(m\phi)]e^{-j\beta_z z}$$

$$A_z = C_1 J_m(\beta_\rho\rho)[C_2\cos(m\phi) + D_2\sin(m\phi)]e^{-j\beta_z z}$$

B.C.'s: $E_z(\phi=0) = E_z(\phi=2\pi) = 0$

$E_z(\rho=a) = 0$

$$E_z = -j\frac{1}{\omega\mu\varepsilon}\left(\frac{\partial^2}{\partial z^2}+\beta^2\right)A_z = -j\frac{C_1}{\omega\mu\varepsilon}(\beta^2-\beta_z^2)J_m(\beta_\rho\rho)[C_2\cos(m\phi)+D_2\sin(m\phi)]e^{-j\beta_z z}$$

$$E_z(\phi=0) = -j\frac{C_1}{\omega\mu\varepsilon}(\beta^2-\beta_z^2)J_m(\beta_\rho\rho)[C_2(1)+D_2(0)]e^{-j\beta_z z} = 0 \Rightarrow C_2 = 0$$

$$E_z(\phi=2\pi) = -j\frac{C_1}{\omega\mu\varepsilon}(\beta^2-\beta_z^2)J_m(\beta_\rho\rho)D_2\sin(2m\pi) = 0 \Rightarrow 2m\pi = \sin^{-1}(0) = p\pi,\; p=1,2,3,\dots$$

$$m = \frac{p}{2},\; p = 1,2,3,\dots \Rightarrow m = \frac{1}{2}, 1, \frac{3}{2}, 2, \dots$$

$$A_z = B_{mn} J_m(\beta_\rho\rho)\sin(m\phi)e^{-j\beta_z z},\quad m = \frac{1}{2}, 1, \frac{3}{2}, 2, \dots$$

cont'd.

$\boxed{\text{9.6 cont'd.}}$ $E_z(\rho=a) = -j B_{mn} \frac{\beta^2-\beta_\rho^2}{\omega\mu\varepsilon} J_m(\beta_\rho \rho) e^{-j\beta_z z} \sin(m\phi) \Big|_{\rho=a}$

$$E_z(\rho=a) = -j B_{mn} \frac{\beta^2-\beta_\rho^2}{\omega\mu\varepsilon} J_m(\beta_\rho a) \sin(m\phi) e^{-j\beta_z z} = 0 \Rightarrow J_m(\beta_\rho a) = 0$$

$$\beta_\rho a = \chi_{mn} \Rightarrow \beta_\rho = \frac{\chi_{mn}}{a}, \quad m = 1/2, 1, 3/2, 2, \ldots \quad n = 1, 2, 3, \ldots$$

$$E_\rho = -j\frac{1}{\omega\mu\varepsilon}\frac{\partial^2 A_z}{\partial\rho\,\partial z} = -B_{mn}\frac{\beta_z\beta_\rho}{\omega\mu\varepsilon} J_m'(\beta_\rho\rho)\sin(m\phi)e^{-j\beta_z z}, \quad H_\rho = \frac{1}{\mu\rho}\frac{\partial A_z}{\partial\phi} = B_{mn}\frac{m}{\mu\rho}J_m(\beta_\rho\rho)\cos(m\phi)e^{-j\beta_z z}$$

$$E_\phi = -j\frac{1}{\omega\mu\varepsilon}\frac{1}{\rho}\frac{\partial^2 A_z}{\partial\phi\,\partial z} = -B_{mn}\frac{m\beta_z}{\omega\mu\varepsilon}\frac{1}{\rho}J_m(\beta_\rho\rho)\cos(m\phi)e^{-j\beta_z z}, \quad H_\phi = -\frac{1}{\mu}\frac{\partial A_z}{\partial\rho} = -B_{mn}\frac{\beta_\rho}{\mu}J_m'(\beta_\rho\rho)\sin(m\phi)e^{-j\beta_z z}$$

$$E_z = -j\frac{1}{\omega\mu\varepsilon}\left(\frac{\partial^2}{\partial z^2}+\beta^2\right)A_z = -j B_{mn}\frac{\beta_\rho^2}{\omega\mu\varepsilon}J_m(\beta_\rho\rho)\sin(m\phi)e^{-j\beta_z z}, \quad H_z = 0$$

$$\beta_z = \begin{cases} \sqrt{\beta^2-\beta_\rho^2} = \sqrt{\beta^2-\left(\frac{\chi_{mn}}{a}\right)^2}, & \beta > \beta_\rho = \chi_{mn}/a \\ 0, & \beta = \beta_\rho \\ -j\sqrt{\beta_\rho^2-\beta^2} = -j\sqrt{\left(\frac{\chi_{mn}}{a}\right)^2-\beta^2}, & \beta < \beta_\rho = \chi_{mn}/a \end{cases}$$

$$(f_c)_{mn} = \frac{\chi_{mn}}{2\pi\sqrt{\mu\varepsilon}}, \quad m = 1/2, 1, 3/2, 2, \ldots \quad n = 1, 2, 3, \ldots$$

$$\beta_z = \frac{2\pi}{\lambda_g} \Rightarrow \lambda_g = \frac{2\pi}{\beta_z} = \frac{2\pi}{\sqrt{\beta^2-\beta_c^2}} = \frac{2\pi}{\beta\sqrt{1-\left(\frac{f_c}{f}\right)^2}} = \frac{\lambda}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}}$$

$$Z_w = \frac{E_\rho}{H_\phi} = -\frac{E_\phi}{H_\rho} = \frac{\beta_z}{\omega\varepsilon} = \frac{\omega\sqrt{\mu\varepsilon}}{\omega\varepsilon}\sqrt{1-\left(\frac{f_c}{f}\right)^2} = \sqrt{\frac{\mu}{\varepsilon}}\sqrt{1-\left(\frac{f_c}{f}\right)^2}$$

a.

$m = 1/2, n = 1$: $\chi_{\frac{1}{2},1} = 3.1416 \Rightarrow (f_c)_{\frac{1}{2},1} = \frac{3.1416(30)}{2\pi(1)} \times 10^9 = 15\times10^9 = 15$ GHz

$m = 1, n = 1$: $\chi_{1,1} = 3.8318 \Rightarrow (f_c)_{11} = \frac{3.8318(30)}{2\pi(1)} \times 10^9 = 18.295\times10^9 = 18.295$ GHz

$m = 3/2, n = 1$: $\chi_{3/2,1} = 4.4934 \Rightarrow (f_c)_{3/2,1} = \frac{4.4934(30)}{2\pi(1)} \times 10^9 = 21.4544\times10^9 = 21.4544$ GHz

b. $\sqrt{1-\left(\frac{f_c}{f}\right)^2} = \sqrt{1-\left(\frac{f_c}{1.5 f_c}\right)^2} = 0.7453$

$$Z_w = \eta\sqrt{1-\left(\frac{f_c}{f}\right)^2} = 377\,(0.7453) =$$

$$\lambda_g = \frac{\lambda}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}} = \frac{30\times10^9/(1.5\times15\times10^9)}{0.7453} = 1.789 \text{ cm}$$

$\boxed{9.7}$

TE^z

$$F_z = \left[A_1 J_m(\beta_\rho \rho) + \cancelto{0}{B_1} Y_m(\beta_\rho \rho)\right]\left[C_2 \cos(m\phi) + D_2 \sin(m\phi)\right] e^{-j\beta_z z}$$

$$E_\rho = -\frac{1}{\varepsilon}\frac{1}{\rho}\frac{\partial F_z}{\partial \phi} = -\frac{A_1}{\varepsilon}\frac{m}{\rho} J_m(\beta_\rho \rho)\left[-C_2 \sin(m\phi) + D_2 \cos(m\phi)\right] e^{-j\beta_z z}$$

$$E_\rho(\phi=0) = -\frac{A_1}{\varepsilon}\frac{m}{\rho} J_m(\beta_\rho \rho)\left[-C_2(0) + D_2(1)\right] e^{-j\beta_z z} = 0 \Rightarrow D_2 = 0$$

$$E_\rho(\phi=\pi) = -\frac{A_1}{\varepsilon}\frac{m}{\rho} J_m(\beta_\rho \rho)\left[-C_2 \sin(m\pi)\right] e^{-j\beta_z z} = 0 \Rightarrow \sin(m\ \ ) = 0 \Rightarrow m\pi = \sin^{-1}(0) = p\pi$$

$$m = p \qquad p = 0,1,2,3,\ldots \Rightarrow m = 0,1,2,3,\ldots$$

$$E_\phi = \frac{1}{\varepsilon}\frac{\partial F_z}{\partial \rho} = A_{mn}\frac{\beta_\rho}{\varepsilon} J_m'(\beta_\rho \rho)\cos(m\phi) e^{-j\beta_z z}$$

$$E_\phi(\rho=a) = A_{mn}\frac{\beta_\rho}{\varepsilon} J_m'(\beta_\rho a)\cos(m\phi) e^{-j\beta_z z} = 0 \Rightarrow \beta_\rho a = \chi_{mn}' \Rightarrow \beta_\rho = \left(\frac{\chi_{mn}'}{a}\right)$$

$$F_z = A_{mn} J_m(\beta_\rho \rho)\cos(m\phi) e^{-j\beta_z z}, \quad \beta_\rho = \frac{\chi_{mn}'}{a}, \quad \begin{matrix} m = 0,1,2,\ldots \\ n = 1,2,3,\ldots \end{matrix}$$

$$(f_c)_{mn} = \frac{\chi_{mn}'}{2\pi a\sqrt{\mu\varepsilon}},$$

$$E_\rho = -\frac{1}{\varepsilon}\frac{1}{\rho}\frac{\partial F_z}{\partial \phi} = +A_{mn}\frac{1}{\varepsilon}\frac{m}{\rho} J_m(\beta_\rho \rho)\sin(m\phi) e^{-j\beta_z z}, \quad H_\rho = -j\frac{1}{\omega\mu\varepsilon}\frac{\partial^2 F_z}{\partial\rho\partial z} = -A_{mn}\frac{\beta_\rho\beta_z}{\omega\mu\varepsilon} J_m'(\beta_\rho \rho)\cos(m\phi) e^{-j\beta_z z}$$

$$E_\phi = \frac{1}{\varepsilon}\frac{\partial F_z}{\partial \rho} = A_{mn}\frac{\beta_\rho}{\varepsilon} J_m'(\beta_\rho \rho)\cos(m\phi) e^{-j\beta_z z}, \quad H_\phi = -j\frac{1}{\omega\mu\varepsilon\rho}\frac{\partial^2 F_z}{\partial\phi\partial z} = A_{mn}\frac{m\beta_z}{\omega\mu\varepsilon\rho} J_m(\beta_\rho \rho)\sin(m\phi) e^{-j\beta_z z}$$

$$E_z = 0, \quad H_z = -j\frac{1}{\omega\mu\varepsilon}\left(\frac{\partial^2}{\partial z^2} + \beta^2\right)F_z = -j\frac{A_{mn}}{\omega\mu\varepsilon}\beta_\rho^2 J_m(\beta_\rho \rho)\cos(m\phi) e^{-j\beta_z z}$$

TM^z

$$A_z = \left[A_1 J_m(\beta_\rho \rho) + \cancelto{0}{B_1} Y_m(\beta_\rho \rho)\right]\left[C_2 \cos(m\phi) + D_2 \sin(m\phi)\right] e^{-j\beta_z z}$$

$$E_z = -j\frac{1}{\omega\mu\varepsilon}\left(\frac{\partial^2}{\partial z^2} + \beta^2\right)A_z = -j A_1 \frac{\beta_\rho^2}{\omega\mu\varepsilon} J_m(\beta_\rho \rho)\left[C_2 \cos(m\phi) + D_2 \sin(m\phi)\right] e^{-j\beta_z z}$$

$$E_z(\phi=0) = -j A_1 \frac{\beta_\rho^2}{\omega\mu\varepsilon} J_m(\beta_\rho \rho)\left[C_2(1) + D_2(0)\right] e^{-j\beta_z z} = 0 \Rightarrow C_2 = 0$$

$$E_z(\phi=\pi) = -j A_1 \frac{\beta_\rho^2}{\omega\mu\varepsilon} J_m(\beta_\rho \rho)\left[D_2 \sin(m\pi)\right] e^{-j\beta_z z} = 0 \Rightarrow m\pi = \sin^{-1}(0) = p\pi, \quad m = p = 1,2,3,\ldots$$

$$E_z(\rho=a) = -j A_1 \frac{\beta_\rho^2}{\omega\mu\varepsilon} J_m(\beta_\rho a)\left[D_2 \sin m\phi\right] e^{-j\beta_z z} = 0 \Rightarrow \beta_\rho a = \chi_{mn} \Rightarrow \beta_\rho = \chi_{mn}/a, \quad n = 1,2,3,\ldots$$

$$A_z = B_{mn} J_m(\beta_\rho \rho)\sin(m\phi) e^{-j\beta_z z}, \quad \beta_\rho = \chi_{mn}/a, \quad m = 1,2,3,\ldots, \quad n = 1,2,3,\ldots$$

$$(f_c)_{mn} = \frac{\chi_{mn}}{2\pi a\sqrt{\mu\varepsilon}}$$

$$E_\rho = -j\frac{1}{\omega\mu\varepsilon}\frac{\partial^2 A_z}{\partial\rho\partial z} = -B_{mn}\frac{\beta_\rho\beta_z}{\omega\mu\varepsilon} J_m'(\beta_\rho \rho)\sin(m\phi) e^{-j\beta_z z}, \quad H_\rho = \frac{1}{\mu\rho}\frac{\partial A_z}{\partial \phi} = B_{mn}\frac{m}{\mu\rho} J_m(\beta_\rho \rho)\cos(m\phi) e^{-j\beta_z z}$$

$$E_\phi = -j\frac{1}{\omega\mu\varepsilon}\frac{1}{\rho}\frac{\partial^2 A_z}{\partial\phi\partial z} = -B_{mn}\frac{m}{\omega\mu\varepsilon}\frac{\beta_z}{\rho} J_m(\beta_\rho \rho)\cos(m\phi) e^{-j\beta_z z}, \quad H_\phi = -\frac{1}{\mu}\frac{\partial A_z}{\partial \rho} = -B_{mn}\frac{\beta_\rho}{\mu} J_m'(\beta_\rho \rho)\sin(m\phi) e^{-j\beta_z z}$$

$$E_z = -j\frac{1}{\omega\mu\varepsilon}\left(\frac{\partial^2}{\partial z^2} + \beta^2\right)A_z = -j\frac{A_{mn}}{\omega\mu\varepsilon}\beta_\rho^2 J_m(\beta_\rho \rho)\sin(m\phi) e^{-j\beta_z z}, \quad H_z = 0$$

$\boxed{9.8}$ The solution to this problem is identical to that of Problem 9.7 with the following exceptions:

$\underline{TE^z}$

$$E_\rho(\phi=\pi/2) = -\frac{A_1}{\epsilon}\frac{m}{\rho}J_m(\beta_\rho\rho)\left[-C_2\sin\left(\frac{m\pi}{2}\right)\right]e^{-j\beta_z z}=0 \Rightarrow \frac{m\pi}{2}=\sin^{-1}(0)=p\pi \Rightarrow m=2p,\ p=0,1,2,\ldots$$

$$m=0,2,4,\ldots$$

$\underline{TM^z}$

$$E_z(\phi=\pi/2) = -jA_1\frac{\beta_\rho^2}{\omega\mu\epsilon}J_m(\beta_\rho\rho)\left[D_2\sin(m\pi/2)\right]e^{-j\beta_z z}=0 \Rightarrow \frac{m\pi}{2}=\sin^{-1}(0)=p\pi,\ p=1,2,3\ldots$$

$$m=2,4,6,\ldots$$

$\boxed{9.9}$ The solution to this problem is identical to that of Problem 9.7 with the following exceptions:

$\underline{TE^z}$

$$E_\rho(\phi=\phi_0) = -\frac{A_1}{\epsilon}\frac{m}{\rho}J_m(\beta_\rho\rho)\left[-C_2\sin(m\phi_0)\right]e^{-j\beta_z z}=0 \Rightarrow m\phi_0=\sin^{-1}(0)=p\pi \Rightarrow m=\frac{p\pi}{\phi_0},\ p=0,1,2,3,\ldots$$

$\underline{TM^z}$

$$E_z(\phi=\phi_0) = -jA_1\frac{\beta_\rho^2}{\omega\mu\epsilon}J_m(\beta_\rho\rho)\left[D_2\sin(m\phi_0)\right]e^{-j\beta_z z}=0 \Rightarrow m\phi_0=\sin^{-1}(0)=p\pi \Rightarrow m=\frac{p\pi}{\phi_0},\ p=1,2,3,\ldots$$

$\boxed{9.10}$ $F_z = [A_1J_m(\beta_\rho\rho)+B_1Y_m(\beta_\rho\rho)][C_2\cos(m\phi)+D_2\sin(m\phi)]e^{-j\beta_z z}$, $m=0,1,2,\ldots$ for 2π periodicity in ϕ.

$$E_\phi = \frac{1}{\epsilon}\frac{\partial F_z}{\partial\rho} = \frac{1}{\epsilon}\beta_\rho[A_1J_m'(\beta_\rho\rho)+B_1Y_m'(\beta_\rho\rho)][C_2\cos(m\phi)+D_2\sin(m\phi)]e^{-j\beta_z z}$$

$$E_\phi(\rho=a) = \frac{1}{\epsilon}\beta_\rho[A_1J_m'(\beta_\rho a)+B_1Y_m'(\beta_\rho a)][C_2\cos(m\phi)+D_2\sin(m\phi)]e^{-j\beta_z z}=0$$

$$A_1J_m'(\beta_\rho a)+B_1Y_m'(\beta_\rho a)=0$$

$$E_\phi(\rho=b) = \frac{1}{\epsilon}\beta_\rho[A_1J_m'(\beta_\rho b)+B_1Y_m'(\beta_\rho b)][C_2\cos(m\phi)+D_2\sin(m\phi)]e^{-j\beta_z z}=0$$

$$A_1J_m'(\beta_\rho b)+B_1Y_m'(\beta_\rho b)=0$$

$$\begin{bmatrix} J_m'(\beta_\rho a) & Y_m'(\beta_\rho a) \\ J_m'(\beta_\rho b) & Y_m'(\beta_\rho b)\end{bmatrix}\begin{bmatrix}A_1\\B_1\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix} \Rightarrow J_m'(\beta_\rho a)Y_m'(\beta_\rho b)-J_m'(\beta_\rho b)Y_m'(\beta_\rho a)=0$$

$$E_\rho = -\frac{1}{\epsilon}\frac{1}{\rho}\frac{\partial F_z}{\partial\phi} = -\frac{m}{\epsilon}\frac{1}{\rho}[A_1J_m(\beta_\rho\rho)+B_1Y_m(\beta_\rho\rho)][-C_2\sin(m\phi)+D_2\cos(m\phi)]e^{-j\beta_z z}$$

$$E_\phi = \frac{1}{\epsilon}\frac{\partial F_z}{\partial\rho} = \frac{\beta_\rho}{\epsilon}[A_1J_m'(\beta_\rho\rho)+B_1Y_m'(\beta_\rho\rho)][C_2\cos(m\phi)+D_2\sin(m\phi)]e^{-j\beta_z z}$$

$$E_z=0$$

$$H_\rho = -j\frac{1}{\omega\mu\epsilon}\frac{\partial^2F_z}{\partial\rho\partial z} = -\frac{\beta_\rho\beta_z}{\omega\mu\epsilon}[A_1J_m'(\beta_\rho\rho)+B_1Y_m'(\beta_\rho\rho)][C_2\cos(m\phi)+D_2\sin(m\phi)]e^{-j\beta_z z}$$

$$H_\phi = -j\frac{1}{\omega\mu\epsilon\rho}\frac{\partial^2F_z}{\partial\phi\partial z} = -\frac{m\beta_z}{\omega\mu\epsilon\rho}[A_1J_m(\beta_\rho\rho)+B_1Y_m(\beta_\rho\rho)][C_2\sin(m\phi)+D_2\cos(m\phi)]e^{-j\beta_z z}$$

$$H_z = -j\frac{1}{\omega\mu\epsilon}\left(\frac{\partial^2}{\partial z^2}+\beta^2\right)F_z = -j\frac{\beta_\rho^2}{\omega\mu\epsilon}[A_1J_m(\beta_\rho\rho)+B_1Y_m(\beta_\rho\rho)][C_2\cos(m\phi)+D_2\sin(m\phi)]e^{-j\beta_z z}$$

$\boxed{9.11}$ $A_z = \left[A_1 J_m(\beta_\rho \rho) + B_1 Y_m(\beta_\rho \rho)\right]\left[C_2 \cos(m\phi) + D_2 \sin(m\phi)\right] e^{-j\beta_z z}$

$$E_z = -j\frac{1}{\omega\mu\varepsilon}\left(\frac{\partial^2}{\partial z^2}+\beta^2\right)A_z = -j\frac{\beta_\rho^2}{\omega\mu\varepsilon}\left[A_1 J_m(\beta_\rho\rho)+B_1 Y_m(\beta_\rho\rho)\right]\left[C_2\cos(m\phi)+D_2\sin(m\phi)\right]e^{-j\beta_z z}$$

$$E_z(\rho=a) = -j\frac{\beta_\rho^2}{\omega\mu\varepsilon}\left[A_1 J_m(\beta_\rho a)+B_1 Y_m(\beta_\rho a)\right]\left[C_2\cos(m\phi)+D_2\sin(m\phi)\right]e^{-j\beta_z z} = 0$$

$$A_1 J_m(\beta_\rho a)+B_1 Y_m(\beta_\rho a) = 0$$

$$E_z(\rho=b) = -j\frac{\beta_\rho^2}{\omega\mu\varepsilon}\left[A_1 J_m(\beta_\rho b)+B_1 Y_m(\beta_\rho b)\right]\left[C_2\cos(m\phi)+D_2\sin(m\phi)\right]e^{-j\beta_z z} = 0$$

$$A_1 J_m(\beta_\rho b)+B_1 Y_m(\beta_\rho b) = 0$$

$$\begin{bmatrix} J_m(\beta_\rho a) & Y_m(\beta_\rho a) \\ J_m(\beta_\rho b) & Y_m(\beta_\rho b)\end{bmatrix}\begin{bmatrix} A_1 \\ B_1\end{bmatrix} = \begin{bmatrix} 0 \\ 0\end{bmatrix} \Rightarrow J_m(\beta_\rho a)Y_m(\beta_\rho b) - J_m(\beta_\rho b)Y_m(\beta_\rho a) = 0$$

$$E_\rho = -j\frac{1}{\omega\mu\varepsilon}\frac{\partial^2 A_z}{\partial\rho\,\partial z} = -\frac{\beta_\rho\beta_z}{\omega\mu\varepsilon}\left[A_1 J_m'(\beta_\rho\rho)+B_1 Y_m'(\beta_\rho\rho)\right]\left[C_2\cos(m\phi)+D_2\sin(m\phi)\right]e^{-j\beta_z z}$$

$$E_\phi = -j\frac{1}{\omega\mu\varepsilon}\frac{1}{\rho}\frac{\partial^2 A_z}{\partial\phi\,\partial z} = -\frac{m\beta_z}{\omega\mu\varepsilon\rho}\left[A_1 J_m(\beta_\rho\rho)+B_1 Y_m(\beta_\rho\rho)\right]\left[-C_2\sin(m\phi)+D_2\cos(m\phi)\right]e^{-j\beta_z z}$$

$$E_z = -j\frac{1}{\omega\mu\varepsilon}\left(\frac{\partial^2}{\partial z^2}+\beta^2\right)A_z = -j\frac{\beta_\rho^2}{\omega\mu\varepsilon}\left[A_1 J_m(\beta_\rho\rho)+B_1 Y_m(\beta_\rho\rho)\right]\left[C_2\cos(m\phi)+D_2\sin(m\phi)\right]e^{-j\beta_z z}$$

$$H_\rho = \frac{1}{\mu\rho}\frac{\partial A_z}{\partial\phi} = \frac{m}{\mu\rho}\left[A_1 J_m(\beta_\rho\rho)+B_1 Y_m(\beta_\rho\rho)\right]\left[-C_2\sin(m\phi)+D_2\cos(m\phi)\right]e^{-j\beta_z z}$$

$$H_\phi = -\frac{1}{\mu}\frac{\partial A_z}{\partial\rho} = -\frac{\beta_\rho}{\mu}\left[A_1 J_m'(\beta_\rho\rho)+B_1 Y_m'(\beta_\rho\rho)\right]\left[C_2\cos(m\phi)+D_2\sin(m\phi)\right]e^{-j\beta_z z}$$

$$H_z = 0$$

$\boxed{9.12}$ The solution to this problem is identical to that of Problem 9.10 with the following exceptions:

$$E_\rho = -\frac{1}{\varepsilon}\frac{1}{\rho}\frac{\partial F_z}{\partial\phi} = -\frac{m}{\varepsilon}\frac{1}{\rho}\left[A_1 J_m(\beta_\rho\rho)+B_1 Y_m(\beta_\rho\rho)\right]\left[-C_2\sin(m\phi)+D_2\cos(m\phi)\right]e^{-j\beta_z z}$$

$$E_\rho(\phi=0) = -\frac{m}{\varepsilon}\frac{1}{\rho}\left[A_1 J_m(\beta_\rho\rho)+B_1 Y_m(\beta_\rho\rho)\right]\left[-C_2(0)+D_2(1)\right]e^{-j\beta_z z} = 0 \Rightarrow D_2 = 0$$

$$E_\phi(\phi=\phi_0) = -\frac{m}{\varepsilon}\frac{1}{\rho}\left[A_1 J_m(\beta_\rho\rho)+B_1 Y_m(\beta_\rho\rho)\right]\left[-C_2\sin(m\phi_0)\right]e^{-j\beta_z z} = 0$$

$$\sin(m\phi_0) = 0 \Rightarrow m\phi_0 = \sin^{-1}(0) = p\pi \Rightarrow m = \frac{p\pi}{\phi_0},\quad p = 0,1,2,\ldots$$

$$F_z = A_{mn}\left[A_1 J_m(\beta_\rho\rho)+B_1 Y_m(\beta_\rho\rho)\right]\cos(m\phi)\,e^{-j\beta_z z}$$

$\boxed{9.13}$ The solution to this problem is identical to that of Problem 9.11 with the following exceptions:

$$E_z = -j\frac{1}{\omega\mu\varepsilon}\left(\frac{\partial^2}{\partial z^2}+\beta^2\right)A_z = -j\frac{\beta_\rho^2}{\omega\mu\varepsilon}\left[A_1 J_m(\beta_\rho\rho)+B_1 Y_m(\beta_\rho\rho)\right]\left[C_2\cos(m\phi)+D_2\sin(m\phi)\right]e^{-j\beta_z z}$$

$$E_z(\phi=0) = -j\frac{\beta_\rho^2}{\omega\mu\varepsilon}\left[A_1 J_m(\beta_\rho\rho)+B_1 Y_m(\beta_\rho\rho)\right]\left[C_2(1)+D_2(0)\right]e^{-j\beta_z z} = 0 \Rightarrow C_2 = 0$$

$$E_z(\phi=\phi_0) = -j\frac{\beta_\rho^2}{\omega\mu\varepsilon}\left[A_1 J_m(\beta_\rho\rho)+B_1 Y_m(\beta_\rho\rho)\right]\left[D_2\sin(m\phi_0)\right]e^{-j\beta_z z} = 0 \Rightarrow \sin(m\phi_0) = 0$$

$$m\phi_0 = \sin^{-1}(0) = p\pi \Rightarrow m = \frac{p\pi}{\phi_0},\quad p = 1,2,3,\ldots$$

$$A_z = B_{mn}\left[A_1 J_m(\beta_\rho\rho)+B_1 Y_m(\beta_\rho\rho)\right]\sin(m\phi)\,e^{-j\beta_z z}$$

9.14 $f = 7\times10^9$ Hz, $\sigma = 5.76\times10^7$ S/m, $\epsilon=\epsilon_0$, $a = 3\times10^{-2}$ m

<u>TE_{11} Mode</u>

$$f_c = \frac{\chi'_{11}}{2\pi a\sqrt{\mu_0\epsilon_0}} = \frac{1.8412(3\times10^8)}{2\pi(3\times10^{-2})} = 2.93\times10^9$$

$$(\alpha_c)_{11}^{TE} = \frac{R_s}{a\eta_0\sqrt{1-\left(\frac{f_c}{f}\right)^2}}\left[\left(\frac{f_c}{f}\right)^2 + \frac{1}{(1.8412)^2-1}\right]$$

$$\left(\frac{f_c}{f}\right)^2 = \left(\frac{2.93}{7}\right)^2 = (0.4186)^2 = 0.1752$$

$$R_s = \sqrt{\frac{\omega\mu}{2\sigma}} = \sqrt{\frac{2\pi\times10^9\times7\times4\pi\times10^{-7}}{2(5.76\times10^7)}} = 2\pi\sqrt{\frac{70}{5.76}}\times10^{-3} = 0.0219$$

$$(\alpha_c)_{11}^{TE} = \frac{0.0219}{3\times10^{-2}(377)\sqrt{1-0.1752}}\left[0.1752 + \frac{1}{(1.8412)^2-1}\right]$$

$$(\alpha_c)_{11}^{TE} = 2.132\times10^{-3}\left[0.5936\right] = 1.266\times10^{-3}\ \text{Np/m} = 10.99\times10^{-3}\ \text{dB/m}$$

<u>TE_{01} Mode</u>

$$f_c = \frac{\chi'_{01}}{2\pi a\sqrt{\mu_0\epsilon_0}} = \frac{3.8318(3\times10^8)}{2\pi(3\times10^{-2})} = 6.0985\times10^9$$

$$(\alpha_c)_{01}^{TE} = \frac{R_s}{a\eta}\frac{(f_c/f)^2}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}}$$

$$\left(\frac{f_c}{f}\right)^2 = \left(\frac{6.0985}{7}\right)^2 = (0.8712)^2 = 0.759$$

$$(\alpha_c)_{01}^{TE} = \frac{0.0219}{3\times10^{-2}(377)}\frac{0.759}{\sqrt{1-0.759}} = 2.9938\times10^{-3}\ \text{Np/m} = 26\times10^{-3}\ \text{dB/m}$$

9.15 $$\underline{J}_s = \hat{n}\times\underline{H} = -\hat{a}_\rho\times(\hat{a}_\rho H_\rho + \hat{a}_\phi H_\phi) = -\hat{a}_z H_\phi = -\hat{a}_z\left(-\frac{\partial A_z}{\partial\rho}\right) = \hat{a}_z\frac{\partial A_z}{\partial\rho}$$

$$\underline{J}_s = \hat{a}_z B_{01}\frac{\beta_\rho}{\mu}J_0'(\beta_\rho\rho) = \hat{a}_z B_{01}\frac{\chi_{01}}{\mu a}J_0'\left(\frac{\chi_{01}}{a}\rho\right)$$

$$P_c = \frac{R_s}{2}\iint_S \underline{J}_s\cdot\underline{J}_s^*\Big|_{\rho=a} ds = \frac{R_s}{2}\left(\frac{\beta_\rho}{\mu}\right)^2|B_{01}|^2\int_0^{\ell}\int_0^{2\pi}\left[J_0'(\chi_{01})\right]^2 a\,d\phi\,dz$$

$$P_c = a\pi R_s\ell\left(\frac{\beta_\rho}{\mu}\right)^2|B_{01}|^2\left[J_0'(\chi_{01})\right]^2$$

Cont'd.

9.15 Cont'd.

$$\frac{P_c}{\ell} = \pi a R_s\left(\frac{\beta_\rho}{\mu}\right)^2|B_{01}|^2\left[J_0'(\chi_{01})\right]^2$$

However

$$\left[J_0'(\chi_{01})\right]^2 = \left\{\frac{d}{d(\beta_\rho\rho)}J_0(\beta_\rho\rho)\right\}^2_{\rho=a} = \left\{-J_1(\beta_\rho\rho)\right\}^2_{\rho=a} = \left[J_1(\chi_{01})\right]^2$$

Thus

$$\frac{P_c}{\ell} = \pi R_s\left(\frac{\beta_\rho}{\mu}\right)^2|B_{01}|^2\left[J_1(\chi_{01})\right]^2 a$$

$$P_{01} = \frac{1}{2}\iint_{S_0} \mathrm{Re}\left[\underline{E}\times\underline{H}^*\right]\cdot d\underline{s} = \frac{1}{2}\int_0^{2\pi}\int_0^a \left\{\left[\hat{a}_\rho E_\rho + \hat{a}_\phi E_\phi + \hat{a}_z E_z\right]\times\left[\hat{a}_\rho H_\rho^* + \hat{a}_\phi H_\phi^*\right]\right\}\cdot d\underline{s}$$

$$P_{01} = \frac{1}{2}\int_0^{2\pi}\int_0^a \mathrm{Re}\,\hat{a}_z\left(E_\rho H_\phi^* - E_\phi H_\rho^*\right)\cdot\hat{a}_z\,\rho\,d\rho\,d\phi = \frac{1}{2}\int_0^{2\pi}\int_0^a \mathrm{Re}\left\{\left[E_\rho H_\phi^* - E_\phi H_\rho^*\right]\rho\,d\rho\,d\phi\right\}$$

$$\left.\begin{aligned} E_\rho &= -B_{01}\frac{\beta_\rho\beta_z}{\omega\mu\varepsilon}J_m'(\beta_\rho\rho)e^{-j\beta_z z} \\ H_\phi &= -B_{01}\frac{\beta_\rho}{\mu}J_m'(\beta_\rho\rho)e^{-j\beta_z z}\end{aligned}\right\}\quad \begin{aligned} E_\rho H_\phi^* - E_\phi H_\rho^* &= E_\rho H_\phi^* \\ &= |B_{01}|^2\left(\frac{\beta_\rho}{\mu}\right)^2\frac{\beta_z}{\omega\varepsilon}\left[J_m'(\beta_\rho\rho)\right]^2\end{aligned}$$

$$P_{01} = \frac{1}{2}|B_{01}|^2\left(\frac{\beta_\rho}{\mu}\right)^2\frac{\beta_z}{\omega\varepsilon}\int_0^{2\pi}\int_0^a\left[J_m'(\beta_\rho\rho)\right]^2\rho\,d\rho\,d\phi = |B_{01}|^2\left(\frac{\beta_\rho}{\mu}\right)^2\frac{\pi\beta_z}{\omega\varepsilon}\int_0^a\left[J_0'(\beta_\rho\rho)\right]^2\rho\,d\rho$$

$$J_0'(\beta_\rho\rho) = -J_1(\beta_\rho\rho)$$

$$P_{01} = |B_{01}|^2\left(\frac{\beta_\rho}{\mu}\right)^2\frac{\pi\beta_z}{\omega\varepsilon}\int_0^a J_1^2(\beta_\rho\rho)\,\rho\,d\rho = |B_{01}|^2\left(\frac{\beta_\rho}{\mu}\right)^2\frac{\pi\beta_z}{\omega\varepsilon}\left\{\frac{\rho^2}{2}\left[J_1^2(\beta_\rho\rho) - J_0(\beta_\rho\rho)J_2(\beta_\rho\rho)\right]\right\}_0^a$$

$$P_{01} = |B_{01}|^2\left(\frac{\beta_\rho}{\mu}\right)^2\frac{\pi\beta_z a^2}{2\omega\varepsilon}\left\{J_1^2(\chi_{01}) - J_0(\chi_{01})J_2(\chi_{01})\right\}$$

However $J_0(\chi_{01}) = 0$ by definition. Thus

$$P_{01} = |B_{01}|^2\left(\frac{\beta_\rho}{\mu}\right)^2\frac{\pi a^2\beta_z}{2\omega\varepsilon}J_1^2(\chi_{01})$$

Therefore

$$(\alpha_c)_{01}^{TM} = \frac{P_c/\ell}{2P_{01}} = \frac{a\pi R_s\left(\frac{\beta_\rho}{\mu}\right)^2|B_{01}|^2J_1^2(\chi_{01})}{|B_{01}|^2\left(\frac{\beta_\rho}{\mu}\right)^2\frac{\pi\beta_z a^2}{\omega\varepsilon}J_1^2(\chi_{01})} = \frac{aR_s\,\omega\varepsilon}{a^2\beta_z}$$

$$(\alpha_c)_{01}^{TM} = \frac{R_s}{a\beta_z}\frac{\beta}{\eta} = \frac{R_s}{a\eta}\frac{\beta}{\beta_z} = \frac{R_s}{a\eta}\frac{1}{\sqrt{1-(f_c/f)^2}}$$

9.16 $a = 6$ cm, $h = 10$ cm, $\varepsilon_r = 4$

a. $$(f_r)^{TE}_{mnp} = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{\chi'_{mn}}{a}\right)^2 + \left(\frac{p\pi}{h}\right)^2} = \frac{1}{2\pi a\sqrt{\mu\varepsilon}}\sqrt{(\chi'_{mn})^2 + \left(\frac{p\pi a}{h}\right)^2}$$

$$(f_r)^{TM}_{mnp} = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{\chi_{mn}}{a}\right)^2 + \left(\frac{p\pi}{h}\right)^2} = \frac{1}{2\pi a\sqrt{\mu\varepsilon}}\sqrt{(\chi_{mn})^2 + \left(\frac{p\pi a}{h}\right)^2}$$

$$(f_r)^{TM}_{010} = \frac{1}{2\pi a\sqrt{\mu\varepsilon}}(\chi_{01}) = \frac{2.4049\times 3\times 10^9}{2\pi(6\times 10^{-2})\,2} = 0.9569\times 10^9$$

$$(f_r)^{TE}_{111} = \frac{1}{2\pi a\sqrt{\mu\varepsilon}}\sqrt{(\chi'_{11})^2 + \left(\frac{\pi a}{h}\right)^2} = \frac{3\times 10^9}{2\pi(6\times 10^{-2})(2)}\sqrt{(1.8412)^2 + (0.6\pi)^2} = 1.0482\times 10^9$$

$$(f_r)^{TM}_{011} = \frac{1}{2\pi a\sqrt{\mu\varepsilon}}\sqrt{(\chi_{01})^2 + \left(\frac{\pi a}{h}\right)^2} = \frac{3\times 10^9}{2\pi(6\times 10^{-2})(2)}\sqrt{(2.4049)^2 + (0.6\pi)^2} = 1.2158\times 10^9$$

$$(f_r)^{TE}_{211} = \frac{1}{2\pi a\sqrt{\mu\varepsilon}}\sqrt{(\chi'_{21})^2 + \left(\frac{\pi a}{h}\right)^2} = \frac{3\times 10^9}{2\pi(6\times 10^{-2})(2)}\sqrt{(3.0542)^2 + (0.6\pi)^2} = 1.428\times 10^9$$

b. $$Q = \frac{1.2025\,\eta}{R_s\left(1 + \frac{a}{h}\right)}$$

$$R_s = \sqrt{\frac{\omega\mu}{2\sigma}} = \sqrt{\frac{2\pi(0.9569\times 10^9)(4\pi\times 10^{-7})}{2(5.76\times 10^7)}} = 2\pi\sqrt{\frac{9.569}{5.76}}\times 10^{-3} = 8.098\times 10^{-3}$$

$$Q = \frac{1.2025(377/2)\times 10^3}{8.098(1+0.6)} = 17.4944\times 10^3 = 17{,}494.4 \simeq 17{,}495$$

9.17 From Table 9-4 it is seen that for $h/a = 1$ the dominant mode is the TM_{010} and that the next higher order mode is the TE_{111} such that $(f_r)^{TE}_{111} = 1.514\,(f_r)^{TM}_{010} \simeq 1.5\,(f_r)^{TM}_{010}$

a. There $h/a = 1 \Rightarrow h = a = 4$ cm

b. $$(f_r)^{TM}_{010} = \frac{2.4049}{2\pi a\sqrt{\mu\varepsilon}} = \frac{2.4049\times 3\times 10^9}{2\pi(4\times 10^{-2})} = 2.8706\times 10^9$$

c. $$(f'_r)^{TM}_{010} = \frac{1}{1.5}(f_r)^{TM}_{010} = \frac{1}{1.5}\,\frac{2.4049}{2\pi a\sqrt{\mu_0\varepsilon_0}} = \frac{2.4049}{2\pi a\sqrt{\mu_0\varepsilon_0}\sqrt{\varepsilon_r}} \Rightarrow \sqrt{\varepsilon_r} = 1.5$$

$$\sqrt{\varepsilon_r} = 1.5 \Rightarrow \varepsilon_r = (1.5)^2 = 2.25$$

$\boxed{9.18}$

$$(f_r)^{TM^z}_{010} = \frac{1}{2\pi\sqrt{\mu_d\varepsilon_d}}\frac{2.4049}{a} = \frac{1}{2\pi a\sqrt{\mu_d\varepsilon_d}}(2.4049)$$

$$(f_r)^{TE^z}_{111} = \frac{1}{2\pi\sqrt{\mu_d\varepsilon_d}}\sqrt{\left(\frac{1.8412}{a}\right)^2+\left(\frac{\pi}{h}\right)^2} = \frac{1}{2\pi a\sqrt{\mu_d\varepsilon_d}}\sqrt{(1.8412)^2+\left(\frac{a}{h}\pi\right)^2}$$

(a) When $h/a < 2.03$ the dominant mode is the TM_{010} mode. Therefore for $h/a = 1.9$ the dominant mode is the TM_{010} mode.

(b) The next higher order mode is the TE^z_{111}. The difference in the resonant frequencies is

$$\frac{1}{2\pi a\sqrt{\mu_d\varepsilon_d}}\left\{\sqrt{(1.8412)^2+\left(\frac{a}{h}\pi\right)^2} - 2.4049\right\} = 50\times10^6$$

$$\frac{1}{2\pi a\sqrt{\mu_0\varepsilon_0}\sqrt{\varepsilon_r}}\left\{\sqrt{(1.8412)^2+\left(\frac{\pi}{1.9}\right)^2} - 2.4049\right\} = 50\times10^6$$

$$\sqrt{\varepsilon_r} = \frac{1}{2\pi a(50\times10^6)\sqrt{\mu_0\varepsilon_0}}\left\{\sqrt{(1.8412)^2+\left(\frac{\pi}{1.9}\right)^2} - 2.4049\right\}$$

$$= \frac{30\times10^9}{2\pi(2)(50\times10^6)}\left\{\sqrt{(1.8412)^2+\left(\frac{\pi}{1.9}\right)^2} - 2.4049\right\}$$

$$\sqrt{\varepsilon_r} = \frac{3\times10^3}{20\pi}\left\{2.4747-2.4049\right\} = \frac{300}{2\pi}(0.06977) = 3.331$$

$$\varepsilon_r = 11.097 \simeq 11$$

9.19

$$\underline{TE^z_n}$$

$$(f_c)^{TE}_n = \frac{n}{2h\sqrt{\mu_0\varepsilon_0}}\Bigg|_{n=2} = \frac{2}{2h\sqrt{\mu_0\varepsilon_0}} = \frac{2\times 3\times 10^8}{2h} = 300\times 10^6 = 3\times 10^8 \Rightarrow h = 1 \text{ m}$$

$$h = 1 \text{ m} = 100\text{cm} \Rightarrow h \quad 1 \text{ m}$$

$$\underline{TM^z_n}$$

$$(f_c)^{TM}_n = \frac{n}{2h\sqrt{\mu_0\varepsilon_0}}\Bigg|_{n=1} = \frac{3\times 10^8}{2h} = 3\times 10^8 \Rightarrow h = 0.5 \text{ m}$$

9.20

$$F_z^+(\rho,\phi,z) = D_1 H_m^{(2)}(\beta_\rho\rho)\left[C_2\cos(m\phi)+D_2\sin(m\phi)\right]\left[C_3\cos(\beta_z z)+D_3\sin(\beta_z z)\right]$$

$$\beta_\rho^2+\beta_z^2=\beta^2$$

$$E_\rho^+ = -\frac{1}{\varepsilon}\frac{1}{\rho}\frac{\partial F_z^+}{\partial\phi} = -D_1\frac{m}{\varepsilon\rho}H_m^{(2)}(\beta_\rho\rho)\left[-C_2\sin(m\phi)+D_2\cos(m\phi)\right]\left[C_3\cos(\beta_z z)+D_3\sin(\beta_z z)\right]$$

$$E_\rho^+(\phi=0) = -D_1\frac{m}{\varepsilon\rho}H_m^{(2)}(\beta_\rho\rho)\left[-C_2(0)+D_2(1)\right]\left[\quad\right]=0 \Rightarrow D_2=0$$

$$E_\rho^+(\phi=\phi_0) = -D_1\frac{m}{\varepsilon\rho}H_m^{(2)}(\beta_\rho\rho)\left[-C_2\sin(m\phi_0)\right]\left[\quad\right]=0 \Rightarrow \sin(m\phi_0)=0 \Rightarrow m=\frac{p\pi}{\phi_0},\ p=0,1,2,$$

$$E_\rho^+(z=0) = -D_1\frac{m}{\varepsilon\rho}H_m^{(2)}(\beta_\rho\rho)\left[-C_2\sin(m\phi)\right]\left[C_3(1)+D_3(0)\right]=0 \Rightarrow C_3=0$$

$$E_\rho^+(z=h) = -D_1\frac{m}{\varepsilon\rho}H_m^{(2)}(\beta_\rho\rho)\left[-C_2\sin(m\phi)\right]\left[D_3\sin(\beta_z h)\right]=0 \Rightarrow \sin(\beta_z h)=0 \Rightarrow \beta_z=\frac{n\pi}{h},\ n=1,2,\ldots$$

Therefore

$$F_z^+(\rho,\phi,z) = A_{pn}H_m^{(2)}(\beta_\rho\rho)\cos(m\phi)\sin(\beta_z z)$$

9.21

$$A_z^+(\rho,\phi,z) = D_1 H_m^{(2)}(\beta_\rho\rho)\left[C_2\cos(m\phi)+D_2\sin(m\phi)\right]\left[C_3\cos(\beta_z z)+D_3\sin(\beta_z z)\right]$$

$$E_\rho^+ = -j\frac{1}{\omega\mu\varepsilon}\frac{\partial^2 A_z}{\partial\rho\partial z} = -j\frac{D_1\beta_\rho\beta_z}{\omega\mu\varepsilon}H_m^{(2)}(\beta_\rho\rho)\left[C_2\cos(m\phi)+D_2\sin(m\phi)\right]\left[-C_3\sin\beta_z z+D_3\cos\beta_z z\right]$$

$$E_\rho^+(\phi=0) = -jD_1\frac{\beta_\rho\beta_z}{\omega\mu\varepsilon}H_m^{(2)}(\beta_\rho\rho)\left[C_2(1)+D_2(0)\right]\left[\quad\right]=0 \Rightarrow C_2=0$$

$$E_\rho^+(\phi=\phi_0) = -jD_1\frac{\beta_\rho\beta_z}{\omega\mu\varepsilon}H_m^{(2)}(\beta_\rho\rho)\left[D_2\sin(m\phi_0)\right]\left[\quad\right]=0 \Rightarrow \sin(m\phi_0)=0 \Rightarrow m=\frac{p\pi}{\phi_0},\ p=1,2,\ldots$$

$$E_\rho^+(z=0) = -jD_1\frac{\beta_\rho\beta_z}{\omega\mu\varepsilon}H_m^{(2)}(\beta_\rho\rho)\left[D_2\sin(m\phi)\right]\left[-C_3(0)+D_3(1)\right]=0 \Rightarrow D_3=0$$

$$E_\rho^+(z=h) = -jD_1\frac{\beta_\rho\beta_z}{\omega\mu\varepsilon}H_m^{(2)}(\beta_\rho\rho)\left[D_2\sin(m\phi)\right]\left[-C_3\sin(\beta_z h)\right]=0 \Rightarrow \sin(\beta_z h)=0 \Rightarrow \beta_z=\frac{n\pi}{h},\ n=0,1,2\ldots$$

Therefore $\quad A_z^+(\rho,\phi,z) = B_{pn}H_m^{(2)}(\beta_\rho\rho)\sin(m\phi)\cos(\beta_z z)$

9.22

Given: $\varepsilon_r \cong 130, \quad \mu_r = 1,$ surface of the dielectric can be approximated by a PMC
radius of the rod, a = 3 cm

For a Circular Dielectric Waveguide, the TE^z modes can be constructed using the vector potential F_z, where,

$$F_z(\rho,\phi,z) = \left[A_1 J_m(\beta_\rho^d \rho) + B_1 Y_m(\beta_\rho^d \rho)\right] \cdot \left[C_2 \cos(m\phi) + D_2 \sin(m\phi)\right] \cdot \left[A_3 e^{-j\beta_z z} + B_3 e^{+j\beta_z z}\right]$$

$$(\beta_\rho^d)^2 + \beta_z^2 = \beta^2$$

$f_1(\rho) = A_1 J_m(\beta_\rho^d \rho) + B_1 Y_m(\beta_\rho^d \rho)$ represents standing waves in the $\pm\rho$ directions.
$g_2(\phi) = C_2 \cos(m\phi) + D_2 \sin(m\phi)$ represents periodic waves in the $\pm\phi$ directions.
$h_3(z) = A_3 e^{-j\beta_z z} + B_3 e^{+j\beta_z a}$ represents traveling waves in the $\pm z$ directions.

Since the surface can be modeled as a PMC, the following boundary conditions exist:

$$BC\ \#1)\ H_\phi(\rho = a, 0 \le \phi \le 2\pi, z) = 0$$

or

$$BC\#2)\ H_z(\rho = a, 0 \le \phi \le 2\pi, z) = 0$$

using the following conditions...

1) The fields must be finite everywhere.
2) The fields must repeat every 2π radians in ϕ.

According to 1), $\boxed{B_1 = 0}$ since $Y_m[\rho = 0] = -\infty$.
According to 2), $\boxed{m = 1, 2, 3, \ldots}$
If we only consider only waves that propagate in the + z direction. Then $\boxed{B_3 = 0.}$

$$\boxed{F_z^+(\rho,\phi,z) = A_{mn} J_m(\beta_\rho^d \rho) \cdot \left[C_2 \cos(m\phi) + D_2 \sin(m\phi)\right] \cdot e^{-j\beta_z z}}$$

$$\text{with } \left(\beta_\rho^d\right)^2+\beta_z^2=\beta^2$$

Problem 9-22 cont.

$$E_\rho^+=-\frac{1}{\varepsilon_d\rho}\frac{\partial F_z^+}{\partial\phi}=-A_{mn}\frac{m}{\varepsilon_d\rho}J_m\left(\beta_\rho^d\rho\right)\cdot\left[-C_2\sin(m\phi)+D_2\cos(m\phi)\right]\cdot e^{-j\beta_z z}$$

$$E_\phi^+=\frac{1}{\varepsilon_d}\frac{\partial F_z^+}{\partial\rho}=A_{mn}\frac{\beta_\rho^d}{\varepsilon_d}J_m'\left(\beta_\rho^d\rho\right)\cdot\left[C_2\cos(m\phi)+D_2\sin(m\phi)\right]\cdot e^{-j\beta_z z}$$

$$E_z^+=0$$

$$H_\rho^+=-j\frac{1}{\omega\mu_d\varepsilon_d}\frac{\partial^2F_z^+}{\partial\rho\partial z}=-A_{mn}\frac{\beta_\rho^d\beta_z}{\omega\mu_d\varepsilon_d}J_m'\left(\beta_\rho^d\rho\right)\cdot\left[C_2\cos(m\phi)+D_2\sin(m\phi)\right]\cdot e^{-j\beta_z z}$$

$$H_\phi^+=-j\frac{1}{\omega\mu_d\varepsilon_d}\frac{1}{\rho}\frac{\partial^2F_z^+}{\partial\phi\partial z}=-A_{mn}\frac{m\beta_z}{\omega\mu_d\varepsilon_d}\frac{1}{\rho}J_m\left(\beta_\rho^d\rho\right)\cdot\left[-C_2\sin(m\phi)+D_2\cos(m\phi)\right]\cdot e^{-j\beta_z z}$$

$$H_z^+=-j\frac{1}{\omega\mu_d\varepsilon_d}\left(\frac{\partial^2}{\partial z^2}+\beta^2\right)F_z^+=-jA_{mn}\frac{\left(\beta_\rho^d\right)^2}{\omega\mu_d\varepsilon_d}J_m\left(\beta_\rho^d\rho\right)\cdot\left[C_2\cos(m\phi)+D_2\sin(m\phi)\right]\cdot e^{-j\beta_z z}$$

$$\text{with, } '=\frac{\partial}{\partial\left(\beta_\rho^d\rho\right)}$$

<u>Apply BC # 1)</u> : $H_\phi\left(\rho=a,0\le\phi\le2\pi,z\right)=0$

$$H_\phi\left(\rho=a,0\le\phi\le2\pi,z\right)=-A_{mn}\frac{m\beta_z}{\omega\mu_d\varepsilon_d}\frac{1}{a}J_m\left(\beta_\rho^d a\right)\cdot\left[-C_2\sin(m\phi)+D_2\cos(m\phi)\right]\cdot e^{-j\beta_z z}=0$$

$$\Rightarrow J_m\left(\beta_\rho^d a\right)=0\quad\Rightarrow\quad\beta_\rho^d a=\chi_{mn}\quad\Rightarrow\quad\boxed{\beta_\rho^d=\frac{\chi_{mn}}{a}}$$

where χ_{mn} represents the nth zero $(n=1,2,3,\ldots)$ of the Bessel function J_m of the first kind of order m $(m=0,1,2,3,\ldots)$.

Cutoff is defined when $\left(\beta_z\right)_{mn}=0$. Therefore,

$$\left(\beta_\rho^d\right)^2+\beta_z^2=\beta^2\quad\Rightarrow\quad\beta_z=\sqrt{\beta^2-\left(\beta_\rho^d\right)^2}\quad when\quad\beta>\beta_\rho^d=\frac{\chi_{mn}}{a}$$

$$\beta_c=\omega_c\sqrt{\mu_d\varepsilon_d}=2\pi f_c\sqrt{\mu_d\varepsilon_d}=\beta_\rho^d=\frac{\chi_{mn}}{a}$$

<u>Cutoff Frequency</u>: $\boxed{\left(f_c\right)_{mn}=\frac{\chi_{mn}}{2\pi a\sqrt{\mu_d\varepsilon_d}}}$ $\quad m=0,1,2,3,\ldots \quad n=1,2,3,\ldots$

Problem 9.22 cont.

Determine the cutoff frequencies of the lowest two modes when the radius of the rod is 3 cm.

For the TE^z modes, the lowest two modes are the TE^z_{01} and TE^z_{11}.

$$TE^z_{01} \Rightarrow \chi_{01} = 2.409$$

$$(f_c)_{01} = \frac{\chi_{01}}{2\pi a\sqrt{\mu\varepsilon}} = \frac{\chi_{mn} \cdot c}{2\pi a\sqrt{\mu_d \varepsilon_d}} = \frac{(2.409)\cdot(30\times10^9 \text{ cm/s})}{2\pi(3\text{ cm})\sqrt{(1)\cdot(130)}} = 0.336\text{ GHz}$$

$$TE^z_{11} \Rightarrow \chi_{11} = 3.8318$$

$$(f_c)_{11} = \frac{\chi_{11} \cdot c}{2\pi a\sqrt{\mu_d \varepsilon_d}} = \frac{(3.8318)\cdot(30\times10^9 \text{ cm/s})}{2\pi(3\text{ cm})\sqrt{(1)\cdot(130)}} = 0.535\text{ GHz}$$

$$\boxed{\begin{array}{l} TE^z_{01} \Rightarrow (f_c)_{01} = 0.336\text{ GHz} \\ TE^z_{11} \Rightarrow (f_c)_{11} = 0.535\text{ GHz} \end{array}}$$

$\boxed{9.23}$

Given: $\varepsilon_r \cong 130$, $\mu_r = 1$, surface of the dielectric can be approximated by a PMC
radius of the rod, a = 3 cm

For a Circular Dielectric Waveguide, the TM^z modes can be constructed using the vector potential A_z, where,

$$A_z(\rho,\phi,z) = [A_1 J_m(\beta_\rho^d \rho) + B_1 Y_m(\beta_\rho^d \rho)] \cdot [C_2 \cos(m\phi) + D_2 \sin(m\phi)] \cdot [A_3 e^{-j\beta_z z} + B_3 e^{+j\beta_z z}]$$

$$(\beta_\rho^d)^2 + \beta_z^2 = \beta^2$$

$f_1(\rho) = A_1 J_m(\beta_\rho^d \rho) + B_1 Y_m(\beta_\rho^d \rho)$ represents standing waves in the $\pm\rho$ directions.
$g_2(\phi) = C_2 \cos(m\phi) + D_2 \sin(m\phi)$ represents periodic waves in the $\pm\phi$ directions.
$h_3(z) = A_3 e^{-j\beta_z z} + B_3 e^{+j\beta_z a}$ represents traveling waves in the $\pm z$ directions.

Since the surface can be modeled as a PMC, the following boundary conditions exist:

$$BC\ \#1)\ H_\phi(\rho = a, 0 \le \phi \le 2\pi, z) = 0$$

or

$$BC\#2)\ H_z(\rho = a, 0 \le \phi \le 2\pi, z) = 0$$

using the following conditions...

1) The fields must be finite everywhere.
2) The fields must repeat every 2π radians in ϕ.

According to 1), $\boxed{B_1 = 0}$ since $Y_m[\rho = 0] = -\infty$.
According to 2), $\boxed{m = 1, 2, 3, \ldots}$
If we only consider only waves that propagate in the + z direction. Then $\boxed{B_3 = 0.}$

$$\boxed{A_z^+(\rho,\phi,z) = A_{mn} J_m(\beta_\rho^d \rho) \cdot [C_2 \cos(m\phi) + D_2 \sin(m\phi)] \cdot e^{-j\beta_z z}}$$

with $(\beta_\rho^d)^2 + \beta_z^2 = \beta^2$

Problem 9-23 cont.

$$E_\rho^+ = -j\frac{1}{\omega_r\mu_d\varepsilon_d}\frac{\partial^2 A_z^+}{\partial\rho\partial z} = -B_{mn}\frac{\beta_\rho^d\beta_z}{\omega\mu_d\varepsilon_d}J_m'(\beta_\rho^d\rho)\cdot[C_2\cos(m\phi)+D_2\sin(m\phi)]\cdot e^{-j\beta_z z}$$

$$E_\phi^+ = -j\frac{1}{\omega_r\mu_d\varepsilon_d}\frac{1}{\rho}\frac{\partial^2 A_z^+}{\partial\phi\partial z} = -B_{mn}\frac{m\beta_z}{\omega\mu_d\varepsilon_d}\frac{1}{\rho}J_m(\beta_\rho^d\rho)\cdot[-C_2\sin(m\phi)+D_2\cos(m\phi)]\cdot e^{-j\beta_z z}$$

$$E_z^+ = -j\frac{1}{\omega\mu_d\varepsilon_d}\left(\frac{\partial^2}{\partial z^2}+\beta^2\right)A_z^+ = -jB_{mn}\frac{(\beta_\rho^d)^2}{\omega\mu_d\varepsilon_d}J_m(\beta_\rho^d\rho)\cdot[C_2\cos(m\phi)+D_2\sin(m\phi)]\cdot e^{-j\beta_z z}$$

$$H_\rho^+ = \frac{1}{\mu_d}\frac{1}{\rho}\frac{\partial A_z^+}{\partial\phi} = B_{mn}\frac{m}{\mu_d}\frac{1}{\rho}J_m(\beta_\rho^d\rho)\cdot[-C_2\sin(m\phi)+D_2\cos(m\phi)]\cdot e^{-j\beta_z z}$$

$$H_\phi^+ = -\frac{1}{\mu_d}\frac{\partial A_z^+}{\partial\rho} = -B_{mn}\frac{\beta_\rho^d}{\mu_d}J_m'(\beta_\rho^d\rho)\cdot[C_2\cos(m\phi)+D_2\sin(m\phi)]\cdot e^{-j\beta_z z}$$

$$H_z^+ = 0$$

with, $' = \dfrac{\partial}{\partial(\beta_\rho^d\rho)}$

Apply BC # 1) : $H_\phi(\rho = a, 0\le\phi\le 2\pi, z) = 0$

$$H_\phi(\rho = a, 0\le\phi\le 2\pi, z) = -B_{mn}\frac{\beta_\rho^d}{\mu_d}J_m'(\beta_\rho^d\rho)\cdot[C_2\cos(m\phi)+D_2\sin(m\phi)]\cdot e^{-j\beta_z z}$$

$$\Rightarrow J_m'(\beta_\rho^d a) = 0 \quad\Rightarrow\quad \beta_\rho^d a = \chi_{mn}' \quad\Rightarrow\quad \boxed{\beta_\rho^d = \frac{\chi_{mn}'}{a}}$$

where χ_{mn}' represents the nth zero $(n = 1,2,3,\ldots)$ of the derivative of the Bessel function J_m of the first kind of order m $(m = 0,1,2,3,\ldots)$.

Cutoff is defined when $(\beta_z)_{mn} = 0$. Therefore,

$$(\beta_\rho^d)^2 + \beta_z^2 = \beta^2 \quad\Rightarrow\quad \beta_z = \sqrt{\beta^2 - (\beta_\rho^d)^2} \quad when \quad \beta > \beta_\rho^d = \frac{\chi_{mn}'}{a}$$

$$\beta_c = \omega_c\sqrt{\mu_d\varepsilon_d} = 2\pi f_c\sqrt{\mu_d\varepsilon_d} = \beta_\rho^d = \frac{\chi_{mn}'}{a}$$

Cutoff Frequency: $\boxed{(f_c)_{mn} = \dfrac{\chi_{mn}'}{2\pi a\sqrt{\mu_d\varepsilon_d}}}$ $\quad m = 0,1,2,3,\ldots$ $\quad n = 1,2,3,\ldots$

Problem 9-23 cont.

Determine the cutoff frequencies of the lowest two modes when the radius of the rod is 3 cm.

For the TM^z modes, the lowest two modes are the TM_{11}^z and TM_{21}^z.

$$\text{TM}_{11}^z \Rightarrow \chi_{11}' = 1.8412$$

$$(f_c)_{11} = \frac{\chi_{11}}{2\pi a\sqrt{\mu\varepsilon}} = \frac{\chi_{11}' \cdot c}{2\pi a\sqrt{\mu_d \varepsilon_d}} = \frac{(1.8412)\cdot(30\times10^9 \text{ cm/s})}{2\pi(3\text{ cm})\sqrt{(1)\cdot(130)}} = 0.257\text{ GHz}$$

$$\text{TM}_{21}^z \Rightarrow \chi_{21}' = 3.0542$$

$$(f_c)_{21} = \frac{\chi_{21}'}{2\pi a\sqrt{\mu\varepsilon}} = \frac{\chi_{21}' \cdot c}{2\pi a\sqrt{\mu_d \varepsilon_d}} = \frac{(3.0542)\cdot(30\times10^9 \text{ cm/s})}{2\pi(3\text{ cm})\sqrt{(1)\cdot(130)}} = 0.425\text{ GHz}$$

$$\boxed{\begin{array}{l}\text{TM}_{11}^z \Rightarrow (f_c)_{11} = 0.257\text{ GHz}\\ \text{TM}_{21}^z \Rightarrow (f_c)_{21} = 0.425\text{ GHz}\end{array}}$$

$\boxed{9.24}$

Given: $a = 3\,\text{cm}$, $\varepsilon_r = 20, 38, \text{and } 130$

Dominant Mode: $\text{HEM}_{11}\ (\text{HE}_{11})$ with zero cutoff frequency. $(f_c)_{11}^{\text{HE}} = 0\,\text{Hz}$

$$\zeta = \beta_z a = \sqrt{(\beta_0 a)^2 \varepsilon_r - \chi^2}$$
$$\xi = \sqrt{(\beta_0 a)^2 (\varepsilon_r \mu_r - 1) - \chi^2}$$

At Cutoff: $\beta_z = \beta_0$

$$\zeta = \beta_0 a = \sqrt{(\beta_0 a)^2 \varepsilon_r - \chi^2}$$
$$(\beta_0 a)^2 = (\beta_0 a)^2 \varepsilon_r - \chi^2$$
$$\chi^2 = (\beta_0 a)^2 \varepsilon_r - (\beta_0 a)^2$$
$$\chi = \beta_0 a \sqrt{\varepsilon_r - 1}$$

$$slope = \frac{\chi_{mn}}{\beta_0 a} = \sqrt{\varepsilon_r - 1}$$

At Cutoff: $\xi = 0 \Rightarrow \beta_0 = \beta_{oc}$

$$\xi\big|_{\beta_0 = \beta_{oc}} = 0 = \sqrt{(\beta_{0c} a)^2 (\varepsilon_r \mu_r - 1) - \chi_{mn}^2}$$
$$0 = (\beta_{0c} a)^2 (\varepsilon_r \mu_r - 1) - \chi_{mn}^2$$
$$\chi_{mn}^2 = (\beta_{0c} a)^2 (\varepsilon_r \mu_r - 1)$$
$$\chi_{mn} = \beta_{0c} a \sqrt{\varepsilon_r \mu_r - 1}$$
$$\chi_{mn} = \frac{\omega_c}{v} a \sqrt{\varepsilon_r \mu_r - 1} = \omega_c \sqrt{\mu_0 \varepsilon_0} \cdot a \sqrt{\varepsilon_r \mu_r - 1}$$
$$\chi_{mn} = 2\pi \cdot f_c \sqrt{\mu_0 \varepsilon_0} \cdot a \sqrt{\varepsilon_r \mu_r - 1}$$

$$(f_c)_{mn} = \frac{\chi_{mn}}{2\pi \cdot a \sqrt{\mu_0 \varepsilon_0} \sqrt{\varepsilon_r \mu_r - 1}} = \frac{\chi_{mn} \cdot c}{2\pi \cdot a \sqrt{\varepsilon_r \mu_r - 1}}$$

---

1. $\text{HE}_{11}\ (\text{HEM}_{11})$ $\quad J_1(\chi_{10}) = 0 \quad \chi_{10} = 0$

$(f_c)_{11}^{\text{HE}} = 0\,\text{Hz}$

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2. $\text{TE}_{01}, \text{TM}_{01}, \text{HE}_{21}\ (\text{HEM}_{21})$ $\quad J_0(\chi_{01}) = 0 \quad \chi_{01} = 2.4049$

$$\varepsilon_r = 20 \Rightarrow \quad (f_c)_{01} = \frac{(2.4049) \cdot (30 \times 10^9\ \text{cm/s})}{2\pi \cdot (3\,\text{cm}) \sqrt{(20) \cdot (1) - 1}} = 0.878\,\text{GHz}$$

$$\varepsilon_r = 38 \Rightarrow \quad (f_c)_{01} = \frac{(2.4049) \cdot (30 \times 10^9\ \text{cm/s})}{2\pi \cdot (3\,\text{cm}) \sqrt{38 \cdot (1) - 1}} = 0.629\,\text{GHz}$$

$$\varepsilon_r = 130 \Rightarrow \quad (f_c)_{01} = \frac{(2.4049) \cdot (30 \times 10^9\ \text{cm/s})}{2\pi \cdot (3\,\text{cm}) \sqrt{(130) \cdot (1) - 1}} = 0.337\,\text{GHz}$$

Problem 9-24 Cont.

3. $HE_{12}(HEM_{13}), EH_{11}(HEM_{12}), HE_{13}(HEM_{31})$ $\qquad J_1(\chi_{11})=0$ $\qquad \chi_{11}=3.8318$

$$\varepsilon_r = 20 \quad \Rightarrow \quad (f_c)_{01} = \frac{(3.8318)\cdot(30\times10^9 \text{ cm/s})}{2\pi\cdot(3\text{ cm})\sqrt{(20)\cdot(1)-1}} = 1.40\text{ GHz}$$

$$\varepsilon_r = 38 \quad \Rightarrow \quad (f_c)_{01} = \frac{(3.8318)\cdot(30\times10^9 \text{ cm/s})}{2\pi\cdot(3\text{ cm})\sqrt{38\cdot(1)-1}} = 1.00\text{ GHz}$$

$$\varepsilon_r = 130 \quad \Rightarrow \quad (f_c)_{01} = \frac{(3.8318)\cdot(30\times10^9 \text{ cm/s})}{2\pi\cdot(3\text{ cm})\sqrt{(130)\cdot(1)-1}} = 0.537\text{ GHz}$$

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4. $EH_{21}(HEM_{22}), EH_{11}, HE_{41}(HEM_{41})$ $\qquad J_2(\chi_{21})=0$ $\qquad \chi_{21}=5.1357$

$$\varepsilon_r = 20 \quad \Rightarrow \quad (f_c)_{01} = \frac{(5.1357)\cdot(30\times10^9 \text{ cm/s})}{2\pi\cdot(3\text{ cm})\sqrt{(20)\cdot(1)-1}} = 1.88\text{ GHz}$$

$$\varepsilon_r = 38 \quad \Rightarrow \quad (f_c)_{01} = \frac{(5.1357)\cdot(30\times10^9 \text{ cm/s})}{2\pi\cdot(3\text{ cm})\sqrt{38\cdot(1)-1}} = 1.34\text{ GHz}$$

$$\varepsilon_r = 130 \quad \Rightarrow \quad (f_c)_{01} = \frac{(5.1357)\cdot(30\times10^9 \text{ cm/s})}{2\pi\cdot(3\text{ cm})\sqrt{(130)\cdot(1)-1}} = 0.720\text{ GHz}$$

$\boxed{9.25}$

$$f_c = \frac{\chi_{mn}}{2\pi \cdot a\sqrt{\mu_0\varepsilon_0}\sqrt{\varepsilon_r - 1}} = \frac{\chi_{mn}\cdot c}{2\pi\cdot a\sqrt{\varepsilon_r - 1}} \quad\Rightarrow\quad \boxed{a = \frac{\chi_{mn}\cdot c}{2\pi\cdot f_c\sqrt{\varepsilon_r - 1}}}$$

$$\varepsilon_r = 2.56 \quad\Rightarrow\quad a = \frac{(2.4049)\cdot(30\times10^9\text{ cm/s})}{2\pi\cdot(3\times10^9\text{ Hz})\sqrt{2.56-1}} = 3.064\text{ cm}$$

$$\varepsilon_r = 4 \quad\Rightarrow\quad a = \frac{(2.4049)\cdot(30\times10^9\text{ cm/s})}{2\pi\cdot(3\times10^9\text{ Hz})\sqrt{4-1}} = 2.21\text{ cm}$$

$$\varepsilon_r = 9 \quad\Rightarrow\quad a = \frac{(2.4049)\cdot(30\times10^9\text{ cm/s})}{2\pi\cdot(3\times10^9\text{ Hz})\sqrt{9-1}} = 1.35\text{ cm}$$

$$\varepsilon_r = 16 \quad\Rightarrow\quad a = \frac{(2.4049)\cdot(30\times10^9\text{ cm/s})}{2\pi\cdot(3\times10^9\text{ Hz})\sqrt{16-1}} = 0.988\text{ cm}$$

$\boxed{9.26}$ $\gamma = \beta_z a\big|_{\beta_z=\beta_0} = \beta_0 a = \sqrt{(\beta_0 a)^2\varepsilon_r - \chi^2} \Rightarrow \varepsilon_r = \frac{(\beta_0 a)^2 + \chi^2}{(\beta_0 a)^2}$, $\chi = 2.4049$

At $f = 5$ GHz $\Rightarrow \lambda_0 = \frac{v_0}{f} = \frac{30\times10^9}{5\times10^9} = 6$ cm $\Rightarrow \beta_0 = 2\pi/\lambda_0$

a. $a = 1.315$ cm: $\beta_0 a = \frac{2\pi}{6}(1.315) = 1.377 \Rightarrow \varepsilon_r = \frac{(1.377)^2 + (2.4049)^2}{(1.377)^2} = 4.05$

b. $a = 1.838$ cm: $\beta_0 a = \frac{2\pi}{6}(1.838) = 1.9247 \Rightarrow \varepsilon_r = \frac{(1.9247)^2 + (2.4049)^2}{(1.9247)^2} = 2.56$

$\boxed{9.27}$ $\varepsilon_{reff} = (\beta_z/\beta_0)^2 = 2.78$, $\varepsilon_r = 4$, $f = 3$ GHz, $a/\lambda_0 = 0.3$

(a) $\beta_z/\beta_0 = \sqrt{2.78} = 1.667 \Rightarrow \beta_z = 1.667\beta_0 = 1.667\left(\frac{2\pi}{\lambda_0}\right) = 1.667\left(\frac{2\pi}{0.1}\right) = 104.74$ rad/m

$\beta_z = 104.74$ rad/m $= 1.0474$ rad/cm

$$(\beta_\rho^d)^2 + (\beta_z)^2 = \beta_d^2 = \varepsilon_r\beta_0^2 \Rightarrow (\beta_\rho^d)^2 = \varepsilon_r\beta_0^2 - \beta_z^2 = \varepsilon_r\beta_0^2 - \varepsilon_{reff}\beta_0^2 = \beta_0^2(\varepsilon_r - \varepsilon_{reff})$$

$$\beta_\rho^d = \beta_0\sqrt{\varepsilon_r - \varepsilon_{reff}} = 20\pi\sqrt{4-2.78} = 20\pi(1.1045) = 69.4 \text{ rad/m}$$

$$\beta_\rho^d = 69.4 \text{ rad/m} = 0.694 \text{ rad/cm}$$

(b)

$$-(\alpha_\rho^0)^2 + \beta_z^2 = \beta_0^2 \Rightarrow (\alpha_\rho^0)^2 = \beta_z^2 - \beta_0^2 = (1.667\beta_0)^2 - \beta_0^2 = \beta_0^2[(1.667)^2 - 1] = \beta_0^2(2.78-1)$$

$$\alpha_\rho^0 = \beta_0\sqrt{2.78-1} = 1.334\beta_0 = 1.334(20\pi) = 26.683\pi = 83.82 \text{ Nepers/m}$$

$$\alpha_\rho^0 = 83.82 \text{ Nepers/m} = 0.8382 \text{ Nepers/cm}$$

9.28

$$\eta = \beta_z a = \sqrt{(\beta_0 a)^2 \varepsilon_r - \chi^2}$$

At cutoff $\quad \eta = \beta_z a = \beta_0 a$

$$\beta_0 a = \sqrt{(\beta_0 a)^2 \varepsilon_r - \chi^2}$$

$$(\beta_0 a)^2 = (\beta_0 a)^2 \varepsilon_r - \chi^2$$

$$(\beta_0 a)^2 (\varepsilon_r - 1) = \chi^2$$

$$(\beta_0 a)^2 = \frac{\chi^2}{\varepsilon_r - 1}$$

$$\beta_0 a = \frac{\chi}{\sqrt{\varepsilon_r - 1}}$$

$$a = \frac{\chi}{\beta_0 \sqrt{\varepsilon_r - 1}} = \frac{\lambda_0}{2\pi} \frac{\chi}{\sqrt{\varepsilon_r - 1}}$$

$$\lambda_0 = \frac{v_0}{f} = \frac{30 \times 10^9}{4 \times 10^9} = 7.5 \text{ cm}$$

The next modes after the HE_{11} mode are TE_{01}, TM_{01}, HE_{21}

$\chi = 2.4049$.

$$a = \frac{7.5}{2\pi} \frac{2.4049}{\sqrt{4-1}} = \frac{7.5}{2\pi} \frac{2.4049}{\sqrt{3}} = 1.6574 \text{ cm}$$

$$\boxed{a = 1.6574 \text{ cm}}$$

$\boxed{9.29}$

(a) $$\frac{2a}{\lambda_0} \geq \frac{1}{\pi}\frac{2.4049}{\sqrt{\varepsilon_r - 1}}$$

$$\sqrt{\varepsilon_r - 1} \geq \frac{\lambda_0(2.4049)}{2\pi a} = \frac{2.4049\,\lambda_0}{2\pi a}$$

$$\varepsilon_r - 1 \geq \left(\frac{2.4049\lambda_0}{2\pi a}\right)^2$$

$$\varepsilon_r \geq 1 + \left(\frac{2.4049\lambda_0}{2\pi a}\right)^2$$

$$\lambda_0 = \frac{c}{f} = \frac{3\times10^8}{100\times10^9} = \frac{3\times10^8}{10^{11}} = 3\times10^{-3}$$

$$a = 1\times10^{-3}$$

$$\varepsilon_r \geq 1 + \left(\frac{2.4049\times3\times10^{-3}}{2\pi\times10^{-3}}\right)^2 = 1 + (1.148255)^2 \; 1 + 1.3185$$

$$\varepsilon_r \geq 2.3185$$

(b).. $\chi_{01} = 2.4049$ $\qquad \xi = \sqrt{(\beta_0 a)^2(\varepsilon_r - 1) - \chi^2}$

At cutoff $\xi = 0$, $\beta_{0c} = \omega_{c0}\sqrt{\mu_0\varepsilon_0} = 2\pi f_c\sqrt{\mu_0\varepsilon_0}$

$$(\beta_{0c}a)^2(\varepsilon_r - 1) = \chi^2 \Rightarrow \beta_{0c}a = \frac{\chi}{\sqrt{\varepsilon_r - 1}} \Rightarrow 2\pi f_{c0}\sqrt{\mu_0\varepsilon_0}\,a = \frac{\chi}{\sqrt{\varepsilon_r - 1}}$$

$$f_{c0} = \frac{1}{2\pi a}\frac{1}{\sqrt{\mu_0\varepsilon_0}}\frac{\chi}{\sqrt{\varepsilon_r - 1}} = \frac{2.4049(3\times10^8)}{2\pi(10^{-3})\sqrt{2.3185-1}} = \frac{2.4049(3)\times10^{11}}{2\pi\sqrt{1.3185}}$$

$$f_{c0} = 10^{11} = 100\times10^9 = 100 \text{ GHz}$$

$\boxed{9.30}$

For TE^z modes

$$F_z = A_{mn} J_m(\beta_\rho \rho)\left[C_2 \cos(m\phi) + D_2 \sin(m\phi)\right] e^{-j\beta_z z} \quad (9\text{-}7)$$

(a). The B.C.s for PMC walls are that the tangential magnetic fields must vanish. Choosing the H_z component from (6-80), we can write that

$$H_z = -j\frac{1}{\omega\mu\varepsilon}\left(\frac{\partial^2}{\partial z^2}+\beta^2\right)F_z = -j\frac{1}{\omega\mu\varepsilon}\left[-\beta_z^2+\beta^2\right]A_{mn}J_m(\beta_\rho\rho)\left[\quad\right]e^{-j\beta_z z}$$

$$H_z(\rho=a) = 0 = J_m(\beta_\rho a)\left[-j\frac{1}{\omega\mu\varepsilon}\left(-\beta_z^2+\beta^2\right)\right]A_{mn}\left[\quad\right]e^{-j\beta_z z}$$

$$J_m(\beta_\rho a) = 0 \Rightarrow \beta_\rho a = \chi_{mn} \Rightarrow \beta_\rho = \left(\frac{\chi_{mn}}{a}\right),\quad n=1,2,3,\ldots$$

$$H_z(\phi=0) = 0 = J_m(\beta_\rho\rho)\left[-j\frac{1}{\omega\mu\varepsilon}\left(-\beta_z^2+\beta^2\right)\right]A_{mn}\left[C_2(1)+D_2(0)\right]e^{-j\beta_z z} \Rightarrow \boxed{C_2=0}$$

$$H_z(\phi=\pi) = 0 = J_m(\beta_\rho\rho)\left[-j\frac{1}{\omega\mu\varepsilon}\left(-\beta_z^2+\beta^2\right)\right]A_{mn}\left[D_2\sin(m\pi)\right]e^{-j\beta_z z} \Rightarrow \sin(m\pi)=0$$

$$\sin(m\pi)=0 \Rightarrow m\pi = \sin^{-1}(0) = p\pi \Rightarrow m = p = 1,2,3,\ldots$$

$$\therefore \quad F_z = B_{mn} J_m(\beta_\rho\rho)\sin(m\phi)\,e^{-j\beta_z z}$$

$$m=1,2,3,\ldots;\quad n=1,2,3,\ldots$$

$$\beta_c = \omega_c\sqrt{\mu\varepsilon} = 2\pi f_c\sqrt{\mu\varepsilon} = \beta_\rho = \chi_{mn}/a$$

$$\boxed{(f_c) = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\left(\frac{\chi_{mn}}{a}\right) \quad \begin{cases} m=1,2,3,\ldots \\ n=1,2,3,\ldots\end{cases}}$$

(b). From Table 9-2, for the TE_{11} dominant mode ($\chi_{11}=3.8318$)

$$(f_c)_{11}^{TE} = \frac{30\times10^9(3.8318)}{2\pi(1.5)\sqrt{81}} = \boxed{1.3552\times10^9 = 1.3552\text{ GHz}}$$

9.31

For TM^z Modes

$$A_z = B_{mn} J_m(\beta_\rho \rho)\left[C_2\cos(m\phi)+D_2\sin(m\phi)\right]e^{-j\beta_z z}$$

(a). The B.C.s for PMC walls are that the tangential magnetic fields must vanish. Choosing the H_ρ and H_ϕ components from (6-70), we can write that

$$H_\rho = \frac{1}{\mu}\frac{1}{\rho}\frac{\partial A_z}{\partial \phi} = \frac{m B_{mn}}{\mu\rho} J_m(\beta_\rho \rho)\left[-C_2\sin(m\phi)+D_2\cos(m\phi)\right]e^{-j\beta_z z}$$

$$H_\phi = -\frac{1}{\mu}\frac{\partial A_z}{\partial \rho} = -\frac{\beta_\rho B_{mn}}{\mu} J_m'(\beta_\rho \rho)\left[C_2\cos(m\phi)+D_2\sin(m\phi)\right]e^{-j\beta_z z}$$

$$H_\rho(\phi=0) = 0 = \frac{m B_{mn}}{\mu\rho} J_m(\beta_\rho \rho)\left[-C_2(0)+D_2(1)\right]e^{-j\beta_z z} \Rightarrow D_2 = 0$$

$$H_\rho(\phi=\pi) = 0 = \frac{m B_{mn}}{\mu\rho} J_m(\beta_\rho \rho)\left[C_2\sin(m\pi)\right]e^{-j\beta_z z} \Rightarrow \sin(m\pi)=0 \Rightarrow m\pi = \sin^{-1}(0)$$

$$m\pi = \sin^{-1}(0) = p\pi \Rightarrow \boxed{m = p = 0, 1, 2, \ldots}$$

$$H_\phi(\rho=a) = 0 = -\frac{\beta_\rho B_{mn}}{\mu} J_m'(\beta_\rho a)\left[C_2\cos(m\phi)\right]e^{-j\beta_z z} \Rightarrow J_m'(\beta_\rho a) = 0$$

$$\beta_\rho a = \chi_{mn}' \Rightarrow \boxed{\beta_\rho = \left(\frac{\chi_{mn}'}{a}\right), \quad m = 0,1,2,\ldots \quad n = 1,2,3,\ldots}$$

$$\beta_c = \omega_c\sqrt{\mu\varepsilon} = 2\pi f_c\sqrt{\mu\varepsilon} = \beta_\rho = \left(\frac{\chi_{mn}'}{a}\right) \Rightarrow (f_c)_{mn} = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\left(\frac{\chi_{mn}'}{a}\right)$$

$$\boxed{(f_c)_{mn} = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\left(\frac{\chi_{mn}'}{a}\right) \Bigg\} \begin{array}{l} m = 0,1,2,\ldots \\ n = 1,2,3,\ldots \end{array}}$$

(b). From Table 9-1, the smallest χ_{mn}' is $\chi_{11}' = 1.8412$. Therefore the lowest order mode is the TM_{11} or

$$(f_c)_{11}^{TM} = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\left(\frac{\chi_{11}'}{a}\right) = \frac{30\times10^9}{2\pi\sqrt{81}}\left(\frac{1.8412}{1.5}\right) = 0.6511\times10^9$$

$$\boxed{(f_c)_{11}^{TM} = 0.6512\times10^9 = 0.6512\ \text{GHz}}$$

9.32

TE^z

$$F_z = [A_1 J_m(\beta_\rho \rho) + B_1 Y_m(\beta_\rho \rho)][C_2\cos(m\phi) + D_2\sin(m\phi)][C_3\cos(\beta_z z) + D_3\sin(\beta_z z)]$$

with $\beta_\rho^2 + \beta_z^2 = \beta^2$

$B_1 = 0$ because field must be finite at $\rho = 0$. Because of periodicity $m = 0, 1, 2, \ldots$

Thus $F_z(\rho,\phi,z) = A_1 J_m(\beta_\rho \rho)[C_2\cos(m\phi) + D_2\sin(m\phi)][C_3\cos(\beta_z z) + D_3\sin(\beta_z z)]$

$$H_\phi = -j\frac{1}{\omega\mu\varepsilon\rho}\frac{\partial^2 F_z}{\partial\phi\,\partial z} = -jA_1\frac{m\beta_z}{\omega\mu\varepsilon\rho}J_m(\beta_\rho\rho)[-C_2\sin(m\phi) + D_2\cos(m\phi)][-C_3\sin(\beta_z z) + D_3\cos(\beta_z z)]$$

$$H_\phi(\rho = a) = -j\frac{1}{\omega\mu\varepsilon}J_m(\beta_\rho a)[-C_2\sin m\phi + D_2\cos m\phi][-C_3\sin\beta_z z + D_3\cos\beta_z z] = 0$$

$$J_m(\beta_\rho a) = 0 \Rightarrow \beta_\rho a = \chi_{mn} \Rightarrow \beta_\rho = \left(\frac{\chi_{mn}}{a}\right)$$

$$H_\phi(z=0) = -j\frac{1}{\omega\mu\varepsilon\rho}J_m(\beta_\rho\rho)[-C_2\sin(m\phi) + D_2\cos(m\phi)][-C_3(0) + D_3(1)] = 0 \Rightarrow D_3 = 0$$

$$H_\phi(z=h) = -j\frac{1}{\omega\mu\varepsilon\rho}J_m(\beta_\rho\rho)[-C_2\sin(m\phi) + D_2\cos(m\phi)][-C_3\sin(\beta_z h)] = 0$$

$$\sin(\beta_z h) = 0 \Rightarrow \beta_z h = \sin^{-1}(0) = p\pi \Rightarrow \beta_z = \left(\frac{p\pi}{h}\right),\quad p = 0, 1, 2, \ldots$$

$$\beta_\rho^2 + \beta_z^2 = \left(\frac{\chi_{mn}}{a}\right)^2 + \left(\frac{p\pi}{h}\right)^2 = \beta_r^2 = \omega_r^2\mu\varepsilon \Rightarrow (f_r)_{mnp} = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{\chi_{mn}}{a}\right)^2 + \left(\frac{p\pi}{h}\right)^2},\quad \begin{matrix} m = 0,1,2,3,\ldots \\ n = 1,2,3,\ldots \\ p = 0,1,2,\ldots \end{matrix}$$

$$F_z = A_{mnp}J_m(\beta_\rho\rho)[C_2\cos(m\phi) + D_2\sin(m\phi)]\cos(\beta_z z)$$

The electric and magnetic fields can be obtained using (6-80) and F_z from above.

TM^z

$$A_z = A_1 J_m(\beta_\rho\rho)[C_2\cos(m\phi) + D_2\sin(m\phi)][C_3\cos(\beta_z z) + D_3\sin(\beta_z z)],\quad m = 0,1,2,\ldots$$

$$H_\phi = -\frac{1}{\mu}\frac{\partial A_z}{\partial\rho} = -A_1\frac{\beta_\rho}{\mu}J_m'(\beta_\rho\rho)[C_2\cos(m\phi) + D_2\sin(m\phi)][C_3\cos(\beta_z z) + D_3\sin(\beta_z z)]$$

$$H_\phi(\rho = a) = -A_1\frac{\beta_\rho}{\mu}J_m'(\beta_\rho a)[C_2\cos(m\phi) + D_2\sin(m\phi)][C_3\cos(\beta_z z) + D_3\sin(\beta_z z)] = 0$$

$$J_m'(\beta_\rho a) = 0 \Rightarrow \beta_\rho a = \chi_{mn}' \Rightarrow \beta_\rho = \left(\frac{\chi_{mn}'}{a}\right)$$

$$H_\phi(z=0) = -A_1\frac{\beta_\rho}{\mu}J_m'(\beta_\rho\rho)[C_2\cos(m\phi) + D_2\sin(m\phi)][C_3(1) + D_3(0)] = 0 \Rightarrow C_3 = 0$$

$$H_\phi(z=h) = -A_1\frac{\beta_\rho}{\mu}J_m'(\beta_\rho\rho)[C_2\cos(m\phi) + D_2\sin(m\phi)][D_3\sin(\beta_z h)] = 0$$

$$\sin(\beta_z h) = 0 \Rightarrow \beta_z h = \sin^{-1}(0) = p\pi \Rightarrow \beta_z = \left(\frac{p\pi}{h}\right),\quad p = 1,2,3,\ldots$$

$$\beta_\rho^2 + \beta_z^2 = \left(\frac{\chi_{mn}'}{a}\right)^2 + \left(\frac{p\pi}{h}\right)^2 = \beta_r^2 = \omega_r^2\mu\varepsilon \Rightarrow (f_r)_{mnp} = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{\chi_{mn}'}{a}\right)^2 + \left(\frac{p\pi}{h}\right)^2},\quad \begin{matrix} m = 0,1,2,\ldots \\ n = 1,2,3,\ldots \\ p = 1,2,3,\ldots \end{matrix}$$

$$A_z = B_{mnp}J_m(\beta_\rho\rho)[C_2\cos(m\phi) + D_2\sin(m\phi)]\sin(\beta_z z)$$

The electric and magnetic fields can be obtained using (6-70) and A_z from above.

$\boxed{9.33}$

(a) PMC modeling predicts the lowest TE^z mode to be the $\boxed{TE^z_{010} \text{ mode}}$ provided that $h/a < 2.03$.

(b)

$$Q = \omega \frac{2W_e}{P_d} = \omega \frac{2\left[\frac{\varepsilon'}{4}\iiint_V |\bar{E}|^2 dv\right]}{\frac{1}{2}\iiint_V \sigma |\bar{E}|^2 dv} = \frac{\omega\varepsilon'}{\sigma}$$

In the above simplified expression, ε is actually ε(ω) which can be further expanded into a real and imaginary part,

$$\varepsilon(\omega) = \varepsilon'(\omega) - j\varepsilon''(\omega)$$

$$\boxed{Q = \frac{\omega\varepsilon'}{\sigma} = \frac{1}{\tan\delta_e}}$$

(c) The resonant frequency is calculated using equation (9-101) on p. 529 for the TE^z_{010} mode as follows,

$$(f_r)^{TE^z}_{mnp} = \frac{1}{2\pi\sqrt{\mu_d\varepsilon_d}}\sqrt{\left(\frac{\chi_{mn}}{a}\right)^2 + \left(\frac{p\pi}{h}\right)^2} \qquad m = 0,1,2,\ldots$$

$$n = 1,2,3\ldots$$
$$p = 0,1,2,\ldots$$

where $\chi_{01} = 2.4049$ and $m = 0,\ n = 1,\ p = 0$

$$(f_r)^{TE^z}_{010} = \frac{\chi_{01}c}{2\pi a\sqrt{\varepsilon_d}} = \frac{(2.4049)(3\times10^8 m/s)}{2\pi(0.1148\times10^{-2} m)\sqrt{130}} = \boxed{8.773\ GHz}$$

(d) Using the expression derived in part (b), the value of Q is,

$$Q = \frac{\omega\varepsilon'}{\sigma} = \frac{1}{\tan\delta_e} = \frac{1}{(4\times10^{-4})} = \boxed{2{,}500}$$

$\boxed{9.34}$

For TE^z: $\underline{A} = 0$, $\underline{F} = \hat{a}_z F_z(\rho, \phi, z)$

(a). $F_z = [A_1 J_m(\beta_\rho \rho) + B_1 Y_m(\beta_\rho \rho)][C_2 \cos(m\phi) + D_2 \sin(m\phi)][C_3 \cos(\beta_z z) + D_3 \sin(\beta_z z)]$

B.C.'s: 1. $E_\rho(0 \le \rho \le a, \phi = 0, z) = E_\rho(0 \le \rho \le a, \phi = \pi, z) = 0$

2. $H_\phi(\rho = a, 0 \le \phi \le \pi, z) = 0$ or $H_z(\rho = a, 0 \le \phi \le \pi, z) = 0$

3. $H_\phi(0 \le \rho \le a, 0 \le \phi \le \pi, z = 0) = 0$ or $H_\rho(0 \le \rho \le a, 0 \le \phi \le \pi, z = 0) = 0$

$H_\phi(0 \le \rho \le a, 0 \le \phi \le \pi, z = \ell) = 0$ or $H_\rho(0 \le \rho \le a, 0 \le \phi \le \pi, z = \ell) = 0$

4. $\underline{E}$ & $\underline{H}$ field components finite everywhere $\Rightarrow \boxed{B_1 = 0}$

$$F_z = A_{mn} J_m(\beta_\rho \rho)[C_2 \cos(m\phi) + D_2 \sin(m\phi)][C_3 \cos(\beta_z z) + D_3 \sin(\beta_z z)]$$

(b). $E_\rho = -\frac{1}{\varepsilon\rho}\frac{\partial F_z}{\partial \phi} = -\frac{m}{\varepsilon\rho} A_{mn} J_m(\beta_\rho \rho)[-C_2 \sin(m\phi) + D_2 \cos(m\phi)][\quad]$

Using (1): $E_\rho(\phi = 0) = -\frac{m}{\varepsilon\rho} A_{mn} J_m(\beta_\rho \rho)[-C_2(0) + D_2(1)][\quad] = 0 \Rightarrow \boxed{D_2 = 0}$

Using (1): $E_\rho(\phi = \pi) = -\frac{m}{\varepsilon\rho} A_{mn} J_m(\beta_\rho \rho)[-C_2 \sin(m\pi)][\quad] = 0$

$\Rightarrow \sin(m\pi) = 0 \Rightarrow m\pi = \sin^{-1}(0) = q\pi \Rightarrow m = q,\ q = m = 0, 1, 2, 3, \ldots$

$$\boxed{m = q = 0, 1, 2, 3, \ldots}$$

$H_\phi = -j\frac{1}{\omega\mu\varepsilon\rho}\frac{\partial^2 F_z}{\partial\phi\,\partial z} = +j\frac{m\beta_z A_{mn}}{\omega\mu\varepsilon\rho} J_m(\beta_\rho \rho) C_2 \sin(m\phi)[-C_3 \sin(\beta_z z) + D_3 \cos(\beta_z z)]$

Using (2): $H_\phi(\rho = a) = +j\frac{m\beta_z A_{mn} C_2}{\omega\mu\varepsilon\rho} J_m(\beta_\rho a)\sin(m\phi)[\quad] = 0$

$$\Rightarrow J_m(\beta_\rho a) = 0 \Rightarrow \beta_\rho a = \chi_{mn} \Rightarrow \beta_\rho = \left(\frac{\chi_{mn}}{a}\right)$$

$$\boxed{\beta_\rho = \left(\frac{\chi_{mn}}{a}\right),\quad m = 0, 1, 2, \ldots,\ n = 1, 2, 3, \ldots}$$

Using (3): $H_\phi(z = 0) = +j\frac{m\beta_z A_{mn} C_2}{\omega\mu\varepsilon\rho} J_m(\beta_\rho a)\sin(m\phi)[-C_3(0) + D_3(1)] = 0 \Rightarrow \boxed{D_3 = 0}$

Using (3): $H_\phi(z = \ell) = +j\frac{m\beta_z A_{mn} C_2}{\omega\mu\varepsilon\rho} J_m(\beta_\rho a)\sin(m\phi)[-C_3 \sin(\beta_z \ell)] = 0$

$\sin(\beta_z \ell) = 0 \Rightarrow \beta_z \ell = \sin^{-1}(0) = p\pi$

$$\boxed{\beta_z = \left(\frac{p\pi}{\ell}\right),\quad p = 0, 1, 2, \ldots}$$

$\boxed{9.34 \text{ cont'd.}}$

9.34 cont'd.

$$\boxed{F_z(\rho,\phi,z) = A_{mnp} J_m(\beta_\rho \rho)\cos(m\phi)\cos(\beta_z z)}$$

$$\beta_\rho = \left(\frac{\chi_{mn}}{a}\right) \qquad m = 0,1,2,\ldots$$

$$n = 1,2,3,\ldots$$

$$\beta_z = \left(\frac{p\pi}{\ell}\right) \qquad p = 0,1,2,\ldots$$

(c). $$(\beta_\rho)^2 + (\beta_z)^2 = \beta_r^2 = \omega_r^2 \mu\varepsilon = (2\pi f_r)^2 \mu\varepsilon$$

$$(f_r)_{mnp} = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\sqrt{(\beta_\rho)^2 + (\beta_z)^2} = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{\chi_{mn}}{a}\right)^2 + \left(\frac{p\pi}{\ell}\right)^2}$$

$$(f_r)_{mnp} = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{\chi_{mn}}{a}\right)^2 + \left(\frac{p\pi}{\ell}\right)^2}$$

Dominant: From Table 9-2 $\Rightarrow \chi_{mn}\Big|_{\substack{m=0\\p=1}} = \chi_{01} = 2.4049$

$$p = 0 \Rightarrow \beta_z = 0$$

$$(f_r)^{TE}_{010} = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{\chi_{mn}}{a}\right)^2 + 0} = \frac{\chi_{01}}{2\pi a\sqrt{\mu_0\varepsilon_0\varepsilon_r}}$$

$$= \frac{\chi_{01}}{2\pi a\sqrt{\mu_0\varepsilon_0}\sqrt{\varepsilon_r}} = \frac{(2.4049)(30\times10^9)}{2\pi(2)\sqrt{81}}$$

$$\boxed{(f_r)^{TE}_{010} = 0.63792\times10^9}$$

$\boxed{9.35}$ For TM^z: $\underline{A} = \hat{a}_z A_z(\rho,\phi,z)$, $\underline{F} = 0$

(a) $A_z = \left[A_1 J_m(\beta_\rho \rho) + B_1 Y_m(\beta_\rho \rho)\right]\left[C_2\cos(m\phi) + D_2\sin(m\phi)\right]\left[C_3\cos(\beta_z z) + D_3\sin(\beta_z z)\right]$

B.C.s: 1. $E_\rho(0\le\rho\le a, \phi=0, z) = E_\rho(0\le\rho\le a, \phi=\pi, z) = 0$ or $E_z(\phi=0) = E_z(\phi=\pi) = 0$

2. $H_\phi(\rho=a, 0\le\phi\le\pi, z) = 0$ or $H_z(\rho=a, 0\le\phi\le\pi, z) = 0$

3. $H_\phi(0\le\rho\le a, 0\le\phi\le\pi, z=0) = 0$ or $H_\rho(0\le\rho\le a, 0\le\phi\le\pi, z=0) = 0$

$H_\phi(0\le\rho\le a, 0\le\phi\le\pi, z=\ell) = 0$ or $H_\rho(0\le\rho\le a, 0\le\phi\le\pi, z=\ell) = 0$

4. $\underline{E}$ & $\underline{H}$ field components are finite everywhere $\Rightarrow B_1 = 0$

Thus $A_z = B_{mn} J_m(\beta_\rho \rho)\left[C_2\cos(m\phi) + D_2\sin(m\phi)\right]\left[C_3\cos(\beta_z z) + D_3\sin(\beta_z z)\right]$

(b) $E_z = -j\frac{1}{\omega\mu\varepsilon}\left(\frac{\partial^2}{\partial z^2} + \beta^2\right)A_z = -j\frac{\beta_z B_{mn}}{\omega\mu\varepsilon}\underbrace{\left[-(\beta_z)^2 + \beta^2\right]}_{\beta_\rho^2} J_m(\beta_\rho\rho)\left[C_2\cos(m\phi) + D_2\sin(m\phi)\right]\left[C_3\cos(\beta_z z) + D_3\sin(\beta_z z)\right]$

(1) $E_z(\phi=0) = jB_{mn}\frac{\beta_z\beta_\rho^2}{\omega\mu\varepsilon}J_m(\beta_\rho\rho)\left[C_2(1) + D_2(0)\right]\left[C_3\cos(\beta_z z) + D_3\sin(\beta_z z)\right] = 0 \Rightarrow \boxed{C_2 = 0}$

(1) $E_z(\phi=\pi) = -jB_{mn}\frac{\beta_z\beta_\rho^2}{\omega\mu\varepsilon}J_m(\beta_\rho\rho)D_2\sin(m\pi)\left[C_3\cos(\beta_z z) + D_3\sin(\beta_z z)\right] = 0$

$\sin(m\pi) = 0 \Rightarrow m\pi = \sin(0) = q\pi$, $q = m = \cancel{0}, 1, 2, \ldots$

$\boxed{m = q = 1, 2, 3, \ldots}$

$H_\phi = -\frac{1}{\mu}\frac{\partial A_z}{\partial\rho} = -B_{mn}D_2\frac{1}{\mu}J_m'(\beta_\rho\rho)\sin(m\phi)\left[C_3\cos(\beta_z z) + D_3\sin(\beta_z z)\right]$

(2) $H_\phi(\rho=a) = -B_{mn}D_2\frac{1}{\mu}J_m'(\beta_\rho a)\sin(m\phi)\left[C_3\cos(\beta_z z) + D_3\sin(\beta_z z)\right] = 0$

$J_m'(\beta_\rho a) = 0 \Rightarrow \beta_\rho a = \chi_{mn}' \Rightarrow \beta_\rho = \left(\frac{\chi_{mn}'}{a}\right)$

$\boxed{\beta_\rho = \left(\frac{\chi_{mn}'}{a}\right),\ m = 1, 2, 3\ldots,\ n = 1, 2, 3, \ldots}$

(3)

$H_\phi(z=0) = -B_{mn}D_2\left(\frac{1}{\mu}\right)J_m'(\beta_\rho\rho)\sin(m\phi)\left[C_3(1) + D_3(0)\right] = 0 \Rightarrow \boxed{C_3 = 0}$

$H_\phi(z=\ell) = -B_{mn}D_2\left(\frac{1}{\mu}\right)J_m'(\beta_\rho\rho)\sin(m\phi)D_3\sin(\beta_z\ell) = 0$

$\sin(\beta_z\ell) = 0 \Rightarrow \beta_z\ell = \sin^{-1}(0) = p\pi$, $p = \cancel{0}, 1, 2, \ldots$

$\boxed{\beta_z = \left(\frac{p\pi}{\ell}\right),\ p = 1, 2, 3, \ldots}$

$$\boxed{A_z(\rho,\phi,z) = B_{mnp}J_m(\beta_\rho\rho)\sin(m\phi)\sin(\beta_z z)}$$

$\beta_\rho = \left(\frac{\chi_{mn}'}{a}\right)$, $m = 1, 2, 3, \ldots$, $n = 1, 2, 3, \ldots$

$\beta_z = \left(\frac{p\pi}{\ell}\right)$, $p = 1, 2, 3, \ldots$

Cont'd

9.35 cont'd

(c) $(\beta_\rho)^2 + (\beta_z)^2 = \beta_r^2 = \omega_r \mu\varepsilon = (2\pi f_r)^2 \mu\varepsilon$

$$(f_r)_{mnp} = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\sqrt{(\beta_\rho)^2 + \beta_z^2} = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{\chi'_{mn}}{a}\right)^2 + \left(\frac{p\pi}{\ell}\right)^2}$$

$$(f_r)_{mnp} = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{\chi'_{mn}}{a}\right)^2 + \left(\frac{p\pi}{\ell}\right)^2}$$

The dominant mode, based on the values of χ'_{mn} from Table 9-1, are: $m=1, n=1 \Rightarrow \chi'_{11} = 1.841$, $p=1$

$$(f_r)_{111} = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{1.841}{a}\right)^2 + \left(\frac{\pi}{\ell}\right)^2}$$

$a = 2$ cm, $\ell = 4$ cm, $\varepsilon_r = 81$

$$(f_r)_{111} = \frac{1}{2\pi\sqrt{\varepsilon_r}\sqrt{\mu_0\varepsilon_0}}\sqrt{\left(\frac{1.841}{2}\right)^2 + \left(\frac{\pi}{4}\right)^2}$$

$$= \frac{30\times10^9}{2\pi\sqrt{81}}\sqrt{\left(\frac{1.841}{2}\right)^2 + \left(\frac{\pi}{4}\right)^2} = \frac{30\times10^9}{2\pi(9)}\sqrt{(0.9205)^2 + (0.7854)^2}$$

$$= 0.5305\sqrt{0.8473 + 0.6169}\times10^9 = \left(0.5305\sqrt{1.4642}\right)\times10^9$$

$$= (0.5305)1.21\times10^9 = 0.6419\times10^9$$

$$\boxed{(f_r)^{TM}_{111} = 0.6419\times10^9 = 0.6419 \text{ GHz}}$$

9.36

(a),(b) For TE^z $(E_z=0)$

$$F_z(\rho,\phi,z)=\left[A_1 J_m(\beta_\rho^d \rho)+B_1 Y_m(\beta_\rho^d\rho)\right]\left[C_2\cos(m\phi)+D_2\sin(m\phi)\right]\left[C_3\cos(\beta_z z)+D_3\sin(\beta_z z)\right]$$

B.C.'s: Fields must be finite everywhere including $\rho=0$.

$$H_\rho(0\le\rho\le a,\ 0\le\phi\le\phi_0,\ z=0)=H_\rho(0\le\rho\le a,\ 0\le\phi\le\phi_0,\ z=h)=0$$

$$H_\rho(0\le\rho\le a,\ \phi=0,\ 0\le z\le h)=H_\rho(0\le\rho\le a,\ \phi=\phi_0,\ 0\le z\le h)=0$$

$$H_\phi(\rho=a,\ 0\le\phi\le\phi_0,\ 0\le z\le h)=0$$

For the fields to be finite everywhere, including $\rho=0$, $B_1=0$.

$$F_z(\rho,\phi,z)=A_1 J_m(\beta_\rho^d\rho)\left[\cancel{C_2}\cos(m\phi)+D_2\sin(m\phi)\right]\left[C_3\cos(\beta_z z)+\cancel{D_3}\sin(\beta_z z)\right]$$

$$H_\rho=-j\frac{1}{\omega\mu\varepsilon}\frac{\partial^2F_z}{\partial\rho\,\partial z},\quad H_\phi=-j\frac{1}{\omega\mu\varepsilon\rho}\frac{\partial^2F_z}{\partial\phi\,\partial z}$$

$$H_\rho=-jA_1\frac{\beta_\rho\beta_z}{\omega\mu\varepsilon}J_m'(\beta_\rho\rho)\left[C_2\cos(m\phi)+D_2\sin(m\phi)\right]\left[-C_3\sin(\beta_z z)+D_3\cos(\beta_z z)\right]$$

$$H_\rho(z=0)=-jA_1\frac{\beta_\rho\beta_z}{\omega\mu\varepsilon}J_m'(\beta_\rho\rho)\left[\quad\right]\left[C_3(0)+D_3(1)\right]=0\Rightarrow\boxed{D_3=0}$$

$$H_\rho(z=h)=-jA_1\frac{\beta_\rho\beta_z}{\omega\mu\varepsilon}J_m'(\beta_\rho\rho)\left[\quad\right]\left[C_3\sin(\beta_z h)\right]=0$$

$$\sin(\beta_z h)=0\Rightarrow\beta_z h=\sin^{-1}(0)=p\pi\Rightarrow\boxed{\beta_z=\left(\frac{p\pi}{h}\right),\ p=0,1,2,\ldots}$$

$$H_\rho(\phi=0)=-jA_1\frac{\beta_\rho\beta_z}{\omega\mu\varepsilon}J_m'(\beta_\rho\rho)\left[C_2(1)+D_2(0)\right]\left[\quad\right]=0\Rightarrow\boxed{C_2=0}$$

$$H_\rho(\phi=\phi_0)=-jA_1\frac{\beta_\rho\beta_z}{\omega\mu\varepsilon}J_m'(\beta_\rho\rho)\left[D_2\sin(m\phi_0)\right]\left[\quad\right]=0$$

$$\sin(m\phi_0)=0\Rightarrow m\phi_0=\sin^{-1}(0)=q\pi\Rightarrow\boxed{m=\frac{q\pi}{\phi_0},\ q=1,2,3,\ldots}$$

$$F_z=A_{mn}J_m(\beta_\rho\rho)\sin(m\phi)\cos(\beta_z z)$$

$$H_\phi=-j\frac{1}{\omega\mu\varepsilon\rho}\frac{\partial^2F_z}{\partial\phi\,\partial z}=+jA_{mn}\frac{m\beta_z}{\omega\mu\varepsilon\rho}J_m(\beta_\rho\rho)\cos(m\phi)\sin(\beta_z z)$$

$$H_\phi(\rho=a)=jA_{mn}\frac{m\beta_z}{\omega\mu\varepsilon\rho}J_m(\beta_\rho a)\cos(m\phi)\sin(\beta_z z)=0\Rightarrow J_m(\beta_\rho a)=0\Rightarrow\beta_\rho a=\chi_{mn}$$

$$\boxed{\beta_\rho=\left(\frac{\chi_{mn}}{a}\right),\ \begin{matrix}m=q\pi/\phi_0,\ q=1,2,3,\ldots\\ n=1,2,\ldots\end{matrix}}$$

$$\boxed{F_z=A_{mn}J_m(\beta_\rho\rho)\sin(m\phi)\cos(\beta_z z)}$$

9.37

TE^z

Since the circumferential surface is identical to that of Problem 9.32 the boundary conditions over this surface are identical to those of Problem 9.32. From the solution of 9.32 we have that

$$F_z = A_1 J_m(\beta_\rho \rho)[C_2 \cos(m\phi) + D_2 \sin(m\phi)][C_3 \cos(\beta_z z) + D_3 \sin(\beta_z z)]$$

with

$$\beta_\rho = \left(\frac{\chi_{mn}}{a}\right), \quad m = 0, 1, 2, 3, \ldots$$

$$E_\rho = -\frac{1}{\varepsilon}\frac{1}{\rho}\frac{\partial F_z}{\partial \phi} = -A_1 \frac{m}{\varepsilon}\frac{1}{\rho} J_m(\beta_\rho \rho)[-C_2 \sin(m\phi) + D_2 \cos(m\phi)][C_3 \cos(\beta_z z) + D_3 \sin(\beta_z z)]$$

Applying the boundary conditions at $z=0$ and $z=h$ leads to

$$E_\rho(z=0) = -A_1 \frac{m}{\varepsilon}\frac{1}{\rho} J_m(\beta_\rho \rho)[-C_2 \sin(m\phi) + D_2 \cos(m\phi)][C_3(1) + D_3(0)] = 0 \Rightarrow C_3 = 0$$

$$E_\rho(z=h) = -A_1 \frac{m}{\varepsilon}\frac{1}{\rho} J_m(\beta_\rho \rho)[-C_2 \sin(m\phi) + D_2 \cos(m\phi)][D_3 \sin(\beta_z h)] = 0$$

$$\sin(\beta_z h) = 0 \Rightarrow \beta_z h = \sin^{-1}(0) = p\pi \Rightarrow \beta_z = \frac{p\pi}{h}, \quad p = 1, 2, 3, \ldots$$

Therefore

$$F_z = A_{mn} J_m(\beta_\rho \rho)[C_2 \cos(m\phi) + D_2 \sin(m\phi)] \sin(\beta_z z), \quad \begin{matrix} m = 0, 1, 2, 3, \ldots \\ n = 1, 2, 3, \ldots \\ p = 1, 2, 3, \ldots \end{matrix}$$

$$(f_r)^{TE}_{mnp} = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{\chi_{mn}}{a}\right)^2 + \left(\frac{p\pi}{h}\right)^2}$$

TM^z

Again from Problem 9.32 solution

$$A_z = A_1 J_m(\beta_\rho \rho)[C_2 \cos(m\phi) + D_2 \sin(m\phi)][C_3 \cos(\beta_z z) + D_3 \sin(\beta_z z)]$$

$$\beta_\rho = \left(\frac{\chi'_{mn}}{a}\right), \quad \begin{matrix} m = 0, 1, 2, \ldots \\ n = 1, 2, 3, \ldots \end{matrix}$$

Applying the boundary conditions at $z=0$ and $z=h$ leads to

$$E_\rho = -j\frac{1}{\omega\mu\varepsilon}\frac{\partial^2 A_z}{\partial\rho\,\partial z} = -jA_1 \frac{\beta_\rho \beta_z}{\omega\mu\varepsilon} J'_m(\beta_\rho \rho)[C_2 \cos(m\phi) + D_2 \sin(m\phi)][-C_3 \sin(\beta_z z) + D_3 \cos(\beta_z z)]$$

$$E_\rho(z=0) = -jA_1 \frac{\beta_\rho \beta_z}{\omega\mu\varepsilon} J'_m(\beta_\rho \rho)[C_2 \cos(m\phi) + D_2 \sin(m\phi)][-C_3(0) + D_3(1)] = 0 \Rightarrow D_3 = 0$$

$$E_\rho(z=h) = -jA_1 \frac{\beta_\rho \beta_z}{\omega\mu\varepsilon} J'_m(\beta_\rho \rho)[C_2 \cos(m\phi) + D_2 \sin(m\phi)][-C_3 \sin(\beta_z h)] = 0$$

$$\sin(\beta_z h) = 0 \Rightarrow \beta_z h = \sin(0) = p\pi \Rightarrow \beta_z = \frac{p\pi}{h}, \quad p = 0, 1, 2, \ldots$$

Therefore

$$A_z = B_{mn} J_m(\beta_\rho \rho)[C_2 \cos(m\phi) + D_2 \sin(m\phi)] \cos(\beta_z z), \quad \begin{matrix} m = 0, 1, 2, 3, \ldots \\ n = 1, 2, 3, \ldots \\ p = 0, 1, 2, \ldots \end{matrix}$$

$$(f_r)^{TM}_{mnp} = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{\chi'_{mn}}{a}\right) + \left(\frac{p\pi}{h}\right)^2}$$

$\boxed{9.38}$ Since the only difference between Problems 9.37 and 9.38 is the angular extend of the cross section, then from Problems 9.32 and 9.37 sol'ns

$\underline{TE^z}$

$$H_\rho = -j\frac{1}{\omega\mu\varepsilon}\frac{\partial^2 F_z}{\partial\rho\partial z} = -jA_1\frac{\beta_\rho\beta_z}{\omega\mu\varepsilon}J_m'(\beta_\rho\rho)\left[C_2\cos(m\phi)+D_2\sin(m\phi)\right]\left[-C_3\sin(\beta_z z)+D_3\cos(\beta_z z)\right]$$

$$H_\rho(\phi=0) = -jA_1\frac{\beta_\rho\beta_z}{\omega\mu\varepsilon}J_m'(\beta_\rho\rho)\left[-C_3\sin(\beta_z z)+D_3\cos(\beta_z z)\right]\left[C_2(1)+D_2(0)\right]=0 \Rightarrow C_2=0$$

$$H_\rho(\phi=\phi_0) = -jA_1\frac{\beta_\rho\beta_z}{\omega\mu\varepsilon}J_m'(\beta_\rho\rho)\left[-C_3\sin(\beta_z z)+D_3\cos(\beta_z z)\right]D_2\sin(m\phi_0)=0$$

$$\sin(m\phi_0)=0 \Rightarrow m\phi_0=\sin^{-1}(0)=p\pi \Rightarrow m=p\left(\frac{\pi}{\phi_0}\right),\quad p=1,2,3,\ldots$$

From Problem 9.32 solution

$$\beta_\rho=\left(\frac{\chi_{mn}}{a}\right),\quad n=1,2,3,\ldots$$

$\underline{TM^z}$

$$H_\rho = \frac{1}{\mu\rho}\frac{\partial A_z}{\partial\phi} = A_1 J_m(\beta_\rho\rho)\frac{m}{\mu\rho}\left[-C_2\sin(m\phi)+D_2\cos(m\phi)\right]\left[C_3\cos(\beta_z z)+D_3\sin(\beta_z z)\right]$$

$$H_\rho(\phi=0) = A_1\frac{m}{\mu\rho}J_m(\beta_\rho\rho)\left[-C_2(0)+D_2(1)\right]\left[C_3\cos(\beta_z z)+D_3\sin(\beta_z z)\right]=0 \Rightarrow D_2=0$$

$$H_\rho(\phi=\phi_0) = A_1\frac{m}{\mu\rho}J_m(\beta_\rho\rho)\left[-C_2\sin(m\phi_0)\right]\left[C_3\cos(\beta_z z)+D_3\sin(\beta_z z)\right]=0$$

$$\sin(m\phi_0)=0 \Rightarrow m\phi_0=\sin^{-1}(0)=p\pi \Rightarrow m=\frac{p\pi}{\phi_0},\quad p=0,1,2,3,\ldots$$

From the solution of Problem 9.32

$$\beta_\rho=\left(\frac{\chi'_{mn}}{a}\right),\quad n=1,2,3,\ldots$$

$\boxed{9.39}$ (a) **$\underline{TE^z}$**

For the region: $(a \le \rho \le b, 0 \le \phi \le 2\pi, z)$

$$F_z(\rho,\phi,z) = \left[A_1 J_m(\beta_\rho^d\rho)+B_1 Y_m(\beta_\rho^d\rho)\right]\cdot\left[C_2\cos(m\phi)+D_2\sin(m\phi)\right]\cdot\left[C_3\cos(\beta_z z)+D_3\sin(\beta_z z)\right]$$

$$E_\phi = \frac{1}{\varepsilon_d}\frac{\partial F_z}{\partial\rho} = \frac{\beta_\rho^d}{\varepsilon_d}\left[A_1 J_m'(\beta_\rho^d\rho)+B_1 Y_m'(\beta_\rho^d\rho)\right]\cdot\left[C_2\cos(m\phi)+D_2\sin(m\phi)\right]$$
$$\times\left[C_3\cos(\beta_z z)+D_3\sin(\beta_z z)\right]$$

$$H_\phi = -j\frac{1}{\omega\mu_d\varepsilon_d}\frac{1}{\rho}\frac{\partial^2 F_z}{\partial\phi\,\partial z} = -j\frac{m\beta_z}{\omega\mu_d\varepsilon_d}\frac{1}{\rho}\left[A_1 J_m(\beta_\rho^d\rho)+B_1 Y_m(\beta_\rho^d\rho)\right]$$
$$\times\left[-C_2\sin(m\phi)+D_2\cos(m\phi)\right]\cdot\left[-C_3\sin(\beta_z z)+D_3\cos(\beta_z z)\right]$$

(Cont'd)

Prob. 9.39 cont.

Apply BC #1: $E_\phi(a \le \rho \le b, 0 \le \phi \le 2\pi, z=0) = 0$

$$E_\phi(a \le \rho \le b, 0 \le \phi \le 2\pi, z=0) = \frac{\beta_\rho^d}{\varepsilon_d}\left[A_1 J_m'(\beta_\rho^d \rho) + B_1 Y_m'(\beta_\rho^d \rho)\right]\cdot\left[C_2 \cos(m\phi) + D_2 \sin(m\phi)\right]\cdot\left[C_3 \cos(0) + D_3 \sin(0)\right] = 0$$

$$\frac{\beta_\rho^d}{\varepsilon_d}\left[A_1 J_m'(\beta_\rho^d \rho) + B_1 Y_m'(\beta_\rho^d \rho)\right]\cdot\left[C_2 \cos(m\phi) + D_2 \sin(m\phi)\right]\cdot\left[C_3(1) + D_3(0)\right] = 0 \quad \Rightarrow \quad \boxed{C_3 = 0}$$

Apply BC #2: $E_\phi(a \le \rho \le b, 0 \le \phi \le 2\pi, z=h) = 0$

$$E_\phi(a \le \rho \le b, 0 \le \phi \le 2\pi, z=0) = \frac{\beta_\rho^d}{\varepsilon_d}\left[A_1 J_m'(\beta_\rho^d \rho) + B_1 Y_m'(\beta_\rho^d \rho)\right]\cdot\left[C_2 \cos(m\phi) + D_2 \sin(m\phi)\right]\cdot\left[D_3 \sin(\beta_z h)\right] = 0$$

$$\sin(\beta_z h) = 0 \quad \Rightarrow \quad \beta_z h = p\pi \quad \Rightarrow \quad \boxed{\beta_z = \frac{p\pi}{h} \qquad p = 0, 1, 2, \ldots}$$

Apply BC #3: $H_\phi(\rho = b, 0 \le \phi \le 2\pi, 0 \le z \le h) = 0$

$$H_\phi(\rho = b, 0 \le \phi \le 2\pi, 0 \le z \le h) = -j\frac{m\beta_z}{\omega\mu_d\varepsilon_d}\frac{1}{b}\left[A_1 J_m(\beta_\rho^d \cdot b) + B_1 Y_m(\beta_\rho^d \cdot b)\right]$$
$$\times\left[-C_2 \sin(m\phi) + D_2 \cos(m\phi)\right]\cdot\left[D_3 \cos(\beta_z z)\right] = 0$$

$$A_1 J_m(\beta_\rho^d \cdot b) + B_1 Y_m(\beta_\rho^d \cdot b) = 0$$

Apply BC #4: $H_\phi(\rho = a, 0 \le \phi \le 2\pi, 0 \le z \le h) = 0$

$$H_\phi(\rho = a, 0 \le \phi \le 2\pi, 0 \le z \le h) = -j\frac{m\beta_z}{\omega\mu_d\varepsilon_d}\frac{1}{a}\left[A_1 J_m(\beta_\rho^d \cdot a) + B_1 Y_m(\beta_\rho^d \cdot a)\right]$$
$$\times\left[-C_2 \sin(m\phi) + D_2 \cos(m\phi)\right]\cdot\left[D_3 \cos(\beta_z z)\right] = 0$$

$$A_1 J_m(\beta_\rho^d \cdot a) + B_1 Y_m(\beta_\rho^d \cdot a) = 0$$

$$\left[A_1 J_m(\beta_\rho^d \cdot a) + B_1 Y_m(\beta_\rho^d \cdot a)\right] = 0$$
$$\left[A_1 J_m(\beta_\rho^d \cdot b) + B_1 Y_m(\beta_\rho^d \cdot b)\right] = 0$$

(Cont'd)

Prob. 9-23 cont.

$$Fg = \begin{bmatrix} J_m(\beta_\rho^d \cdot a) & Y_m(\beta_\rho^d \cdot a) \\ J_m(\beta_\rho^d \cdot b) & Y_m(\beta_\rho^d \cdot b) \end{bmatrix} \cdot \begin{bmatrix} A_1 \\ B_1 \end{bmatrix} = 0$$

The equation $Fg = 0$ has a nontrivial solution provided that the determinant of F is equal to zero. $\det(F) = 0$

$$\det(F) = J_m(\beta_\rho^d \cdot a) \cdot Y_m(\beta_\rho^d \cdot b) - J_m(\beta_\rho^d \cdot b) \cdot Y_m(\beta_\rho^d \cdot a) = 0$$

Therefore, the eigenvalues are obtained as solutions to:

$$J_m(\beta_\rho^d \cdot a) \cdot Y_m(\beta_\rho^d \cdot b) - J_m(\beta_\rho^d \cdot b) \cdot Y_m(\beta_\rho^d \cdot a) = 0, \qquad m = 0, 1, 2, \ldots$$

$$(\beta_\rho^d)^2 + \beta_z^2 = \beta_d^2$$

$$\beta_r = \omega_r \sqrt{\mu_d \varepsilon_d} = 2\pi f_r \sqrt{\mu_d \varepsilon_d} = \sqrt{(\beta_\rho^d)^2 + \left(\frac{p\pi}{h}\right)^2}$$

Expression for the resonant frequency for TE^z modes:

$$\boxed{f_r = \frac{1}{2\pi\sqrt{\mu_d \varepsilon_d}} \sqrt{(\beta_\rho^d)^2 + \left(\frac{p\pi}{h}\right)^2}}$$

(b) **TM^z MODES**

For the region: $(a \le \rho \le b, 0 \le \phi \le 2\pi, z)$

$$A_z(\rho, \phi, z) = [A_1 J_m(\beta_\rho^d \rho) + B_1 Y_m(\beta_\rho^d \rho)] \cdot [C_2 \cos(m\phi) + D_2 \sin(m\phi)] \cdot [C_3 \cos(\beta_z z) + D_3 \sin(\beta_z z)]$$

$$E_\phi = -j \frac{1}{\omega_r \mu_d \varepsilon_d} \frac{1}{\rho} \frac{\partial^2 A_z}{\partial \phi \, \partial z} = -j \frac{m\beta_z}{\omega_r \mu_d \varepsilon_d} \frac{1}{\rho} [A_1 J_m(\beta_\rho^d \rho) + B_1 Y_m(\beta_\rho^d \rho)]$$
$$\times [-C_2 \sin(m\phi) + D_2 \cos(m\phi)] \cdot [-C_3 \sin(\beta_z z) + D_3 \cos(\beta_z z)]$$

$$H_\varphi = -\frac{1}{\mu_d} \frac{\partial A_z}{\partial \rho} = -\frac{\beta_\rho^d}{\mu_d} [A_1 J'_m(\beta_\rho^d \rho) + B_1 Y'_m(\beta_\rho^d \rho)]$$
$$\times [C_2 \cos(m\phi) + D_2 \sin(m\phi)] \cdot [C_2 \cos(\beta_z z) + D_2 \sin(\beta_z z)]$$

(Cont'd)

Prob. 9.39 Cont.

Apply BC #1: $E_\phi(a \le \rho \le b, 0 \le \phi \le 2\pi, z = 0) = 0$

$$E_\phi(a \le \rho \le b, 0 \le \phi \le 2\pi, z = 0) = -j\frac{m\beta_z}{\omega_r\mu_d\varepsilon_d}\frac{1}{\rho}\left[A_1J_m(\beta_\rho^d\rho) + B_1Y_m(\beta_\rho^d\rho)\right]$$
$$\times\left[-C_2\sin(m\phi) + D_2\cos(m\phi)\right]\cdot\left[-C_3\sin(\beta_z 0) + D_3\cos(\beta_z 0)\right] = 0$$

$$-j\frac{m\beta_z}{\omega_r\mu_d\varepsilon_d}\frac{1}{\rho}\left[A_1J_m(\beta_\rho^d\rho) + B_1Y_m(\beta_\rho^d\rho)\right]\cdot\left[-C_2\sin(m\phi) + D_2\cos(m\phi)\right]\cdot\left[-C_3(0) + D_3(1)\right] = 0 \quad \Rightarrow \quad \boxed{D_3 = 0}$$

Apply BC #2: $E_\phi(a \le \rho \le b, 0 \le \phi \le 2\pi, z = h) = 0$

$$E_\phi(a \le \rho \le b, 0 \le \phi \le 2\pi, z = 0) = -j\frac{m\beta_z}{\omega_r\mu_d\varepsilon_d}\frac{1}{\rho}\left[A_1J_m(\beta_\rho^d\rho) + B_1Y_m(\beta_\rho^d\rho)\right]$$
$$\times\left[-C_2\sin(m\phi) + D_2\cos(m\phi)\right]\cdot\left[-C_3\sin(\beta_z h)\right] = 0$$

$$\sin(\beta_z h) = 0 \quad \Rightarrow \quad \beta_z h = p\pi \quad \Rightarrow \quad \boxed{\beta_z = \frac{p\pi}{h} \qquad p = 0, 1, 2, \ldots}$$

Apply BC #3: $H_\phi(\rho = b, 0 \le \phi \le 2\pi, 0 \le z \le h) = 0$

$$H_\phi(\rho = a, 0 \le \phi \le 2\pi, 0 \le z \le h) = -\frac{\beta_\rho^d}{\mu_d}\left[A_1J_m'(\beta_\rho^d \cdot b) + B_1Y_m'(\beta_\rho^d \cdot b)\right]$$
$$\times\left[C_2\cos(m\phi) + D_2\sin(m\phi)\right]\cdot\left[C_2\cos(\beta_z z) + D_2\sin(\beta_z z)\right] = 0$$

$$A_1J_m'(\beta_\rho^d \cdot b) + B_1Y_m'(\beta_\rho^d \cdot b) = 0$$

Apply BC #4: $H_\phi(\rho = a, 0 \le \phi \le 2\pi, 0 \le z \le h) = 0$

$$H_\phi(\rho = a, 0 \le \phi \le 2\pi, 0 \le z \le h) = -\frac{\beta_\rho^d}{\mu_d}\left[A_1J_m'(\beta_\rho^d \cdot a) + B_1Y_m'(\beta_\rho^d \cdot a)\right]$$
$$\times\left[C_2\cos(m\phi) + D_2\sin(m\phi)\right]\cdot\left[C_2\cos(\beta_z z) + D_2\sin(\beta_z z)\right] = 0$$

$$A_1J_m'(\beta_\rho^d \cdot a) + B_1Y_m'(\beta_\rho^d \cdot a) = 0$$

(Cont'd)

$$\left[A_1 J_m'\left(\beta_\rho^d \cdot a\right) + B_1 Y_m'\left(\beta_\rho^d \cdot a\right)\right] = 0$$
$$\left[A_1 J_m'\left(\beta_\rho^d \cdot b\right) + B_1 Y_m'\left(\beta_\rho^d \cdot b\right)\right] = 0$$

$$Fg = \begin{bmatrix} J_m'\left(\beta_\rho^d \cdot a\right) & Y_m'\left(\beta_\rho^d \cdot a\right) \\ J_m'\left(\beta_\rho^d \cdot b\right) & Y_m'\left(\beta_\rho^d \cdot b\right) \end{bmatrix} \cdot \begin{bmatrix} A_1 \\ B_1 \end{bmatrix} = 0$$

The equation $Fg = 0$ has a nontrivial solution provided that the determinant of F is equal to zero. $\det(F) = 0$

$$\det(F) = J_m'\left(\beta_\rho^d \cdot a\right) \cdot Y_m'\left(\beta_\rho^d \cdot b\right) - J_m'\left(\beta_\rho^d \cdot b\right) \cdot Y_m'\left(\beta_\rho^d \cdot a\right) = 0$$

Therefore, the eigenvalues are obtained as solutions to:

$$J_m'\left(\beta_\rho a\right) \cdot Y_m'\left(\beta_\rho b\right) - Y_m'\left(\beta_\rho a\right) \cdot J_m'\left(\beta_\rho b\right) = 0, \qquad m = 0, 1, 2, \ldots$$

$$\left(\beta_\rho^d\right)^2 + \beta_z^2 = \beta^2$$

$$\beta_r = \omega_r \sqrt{\mu_d \varepsilon_d} = 2\pi f_r \sqrt{\mu_d \varepsilon_d} = \sqrt{\left(\beta_\rho^d\right)^2 + \left(\frac{p\pi}{h}\right)^2}$$

Expression for the resonant frequency for TM^z modes:

$$\boxed{f_r = \frac{1}{2\pi\sqrt{\mu_d \varepsilon_d}} \sqrt{\left(\beta_\rho^d\right)^2 + \left(\frac{p\pi}{h}\right)^2}}$$

9.40

TE^z

Since this structure is a combination of the structure for Problems 9.38 and 9.39, it must satisfy the boundary conditions of Problems 9.38 and 9.39. Therefore from the solutions of Problems 9.38 and 9.39, we can write that

$$J_m(\beta_\rho a) Y_m(\beta_\rho b) - Y_m(\beta_\rho a) J_m(\beta_\rho b) = 0$$
$$m = p\left(\frac{\pi}{\phi_0}\right), \quad p = 1, 2, 3, \ldots\ldots$$

TM^z

For the same reasons as for the TE^z modes, we can write from the solutions of Problems 9.38 and 9.39 that

$$J_m'(\beta_\rho a) Y_m'(\beta_\rho b) - Y_m'(\beta_\rho a) J_m'(\beta_\rho b) = 0$$
$$m = p\left(\frac{\pi}{\phi_0}\right), \quad p = 0, 1, 2, 3, \ldots\ldots$$

9.41 TE^z: $F_z = [A_1 J_m(\beta_\rho \rho) + B_1 Y_m(\beta_\rho \rho)][\cancel{C_2}^{0}\cos(m\phi) + D_2 \sin(m\phi)][C_3\cos(\beta_z z) + \cancel{D_3}^{0}\sin(\beta_z z)]$

$$\left.\begin{aligned} H_\rho &= -j\frac{1}{\omega\mu\varepsilon}\frac{\partial^2 F_z}{\partial\rho\,\partial z} \\ H_\phi &= -j\frac{1}{\omega\mu\varepsilon}\frac{1}{\rho}\frac{\partial^2 F_z}{\partial\phi\,\partial z} \\ H_z &= -j\frac{1}{\omega\mu\varepsilon}\left(\frac{\partial^2}{\partial z^2}+\beta^2\right)F_z \end{aligned}\right\}$$

B.C.'s

$H_\phi(\phi=0) = H_\phi(\phi=\phi_0) = 0 = H_\rho(z=0) = H_\rho(z=0)$

$H_\phi(\rho=a) = H_\phi(\rho=b) = 0 = H_\phi(z=0) = H_\phi(z=h)$

or

$H_z(\phi=0) = H_z(\phi=\phi_0) = 0$

$H_z(\rho=a) = H_z(\rho=b) = 0$

$H_\rho(z=0) = H_\rho(z=h) = H_\phi(z=0) = H_\phi(z=h)$

$$H_z = -j\frac{1}{\omega\mu\varepsilon}(-\beta_z^2+\beta^2)[A_1 J_m(\beta_\rho\rho) + B_1 Y_m(\beta_\rho\rho)][\cancel{C_2}\cos(m\phi)+D_2\sin(m\phi)][C_3\cos(\beta_z z)+D_3\sin(\beta_z z)]$$

$$= -j\frac{\beta_\rho^2}{\omega\mu\varepsilon}[A_1 J_m(\beta_\rho\rho) + B_1 Y_m(\beta_\rho\rho)][\cancel{C_2}^{0}\cos(m\phi)+D_2\sin(m\phi)][C_3\cos(\beta_z z)+D_3\sin(\beta_z z)]$$

① $H_z(\phi=0) = 0 = -j\frac{\beta_\rho^2}{\omega\mu\varepsilon}[\quad][C_2(1)+D_2(0)][\quad] = 0 \Rightarrow \boxed{C_2 = 0}$

② $H_z(\phi=\phi_0) = 0 = -j\frac{\beta_\rho^2}{\omega\mu\varepsilon}[\quad][D_2\sin(m\phi_0)][\quad] = 0 \Rightarrow \sin(m\phi_0) = 0$

$$m\phi_0 = \sin^{-1}(0) = q\pi \Rightarrow m = \frac{q\pi}{\phi_0},\quad q = 1,2,3,\ldots$$

$H_\rho = -j\frac{\beta_z}{\omega\mu\varepsilon}[A_1 \quad][D_2\sin(m\phi)][-C_3\sin(\beta_z z)+D_3\cos(\beta_z z)]$

③ $H_\rho(z=0) = -j\frac{\beta_z}{\omega\mu\varepsilon}[\quad][\quad][-C_3(0)+D_3(1)] = 0 \Rightarrow D_3 = 0$

④ $H_\rho(z=h) = -j\frac{\beta_z}{\omega\mu\varepsilon}[\quad][\quad][-C_3\sin(\beta_z h)] = 0 \Rightarrow \beta_z h = \sin^{-1}(0) = p\pi$

$$\beta_z = \left(\frac{p\pi}{h}\right),\ p = 0,1,2,\ldots$$

⑤ $H_z(\rho=a) = -j\frac{\beta_\rho^2}{\omega\mu\varepsilon}D_2 C_3[A_1 J_m(\beta_\rho a)+B_1 Y_m(\beta_\rho a)][\sin(m\phi)\cos(\beta_z z)] = 0$

$$\left.\begin{aligned} &A_1 J_m(\beta_\rho a) + B_1 Y_m(\beta_\rho a) = 0 \\ ⑥\ H_z(\rho=b) = 0 \Rightarrow\ &A_1 J_m(\beta_\rho b) + B_1 Y_m(\beta_\rho b) = 0 \end{aligned}\right\} \begin{bmatrix} J_m(\beta_\rho a) & Y_m(\beta_\rho a) \\ J_m(\beta_\rho b) & Y_m(\beta_\rho b)\end{bmatrix}\begin{bmatrix}A_1\\B_1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$$\boxed{J_m(\beta_\rho a)Y_m(\beta_\rho b) - J_m(\beta_\rho b)Y_m(\beta_\rho a) = 0} \qquad \boxed{\beta_\rho^2+\beta_z^2 = \beta_r^2}$$

$m = \frac{q\pi}{\phi_0},\ q = 1,2,3,\ldots$; $\beta_z = \frac{p\pi}{h},\ p = 0,1,2,\ldots$

$\boxed{9.42}$

TM^z: $A_z = [A_1 J_m(\beta_\rho \rho) + B_1 Y_m(\beta_\rho \rho)][C_2 \cos(m\phi) + \cancel{D_2 \sin(m\phi)}][\cancel{C_3 \cos(\beta_z z)} + D_3 \sin(\beta_z z)]$

$$H_\rho = \frac{1}{\mu}\frac{1}{\rho}\frac{\partial A_z}{\partial \phi} = \frac{m}{\mu\rho}[A_1 J_m(\beta_\rho \rho) + B_1 Y_m(\beta_\rho \rho)][-C_2 \sin(m\phi) + D_2 \cos(m\phi)][C_3 \cos(\beta_z z) + D_3 \sin(\beta_z z)]$$

$$H_\phi = -\frac{1}{\mu}\frac{\partial A_z}{\partial \rho} = -\frac{\beta_\rho}{\mu}[A_1 J_m'(\beta_\rho \rho) + B_1 Y_m'(\beta_\rho \rho)][C_2 \cos(m\phi) + D_2 \sin(m\phi)][C_3 \cos(\beta_z z) + D_3 \sin(\beta_z z)]$$

$$H_\rho(\phi = 0) = \frac{m}{\mu\rho}[\quad][-C_2(0) + D_2(1)][\quad] = 0 \Rightarrow \boxed{D_2 = 0}$$

$$H_\rho(\phi = \phi_0) = \frac{m}{\mu\rho}[\quad][-C_2 \sin(m\phi_0)][\quad] = 0 \Rightarrow \sin(m\phi_0) = 0$$

$$m\phi_0 = \sin^{-1}(0) = q\pi \Rightarrow \boxed{m = \frac{q\pi}{\phi_0}, \quad q = 0, 1, 2, \ldots}$$

$$H_\rho(z = 0) = \frac{m}{\mu\rho}[\quad][\quad][C_3(1) + D_3(0)] = 0 \Rightarrow \boxed{C_3 = 0}$$

$$H_\rho(z = h) = \frac{m}{\mu\rho}[\quad][\quad][D_3 \sin(\beta_z h)] = 0 \Rightarrow \sin(\beta_z h) = 0$$

$$\beta_z h = \sin^{-1}(0) = p\pi \Rightarrow \boxed{\beta_z = \left(\frac{p\pi}{h}\right), \quad p = 1, 2, 3, \ldots}$$

$$H_\phi(\rho = a) = -\frac{\beta_\rho}{\mu}[A_1 J_m'(\beta_\rho a) + B_1 Y_m'(\beta_\rho a)][\quad][\quad] = 0$$

$$\boxed{A_1 J_m'(\beta_\rho a) + B_1 Y_m'(\beta_\rho a) = 0}$$

$$H_\phi(\rho = b) = 0 \Rightarrow \boxed{A_1 J_m'(\beta_\rho b) + B_1 Y_m'(\beta_\rho b) = 0}$$

$$\begin{bmatrix} J_m'(\beta_\rho a) & Y_m'(\beta_\rho a) \\ J_m'(\beta_\rho b) & Y_m'(\beta_\rho b) \end{bmatrix}\begin{bmatrix} A_1 \\ B_1 \end{bmatrix} = 0 \Rightarrow \boxed{J_m'(\beta_\rho a) Y_m'(\beta_\rho b) - J_m'(\beta_\rho b) Y_m'(\beta_\rho a) = 0}$$

$$\boxed{\beta_\rho^2 + \beta_z^2 = \beta_r^2}$$

9.43

$$a \le \rho \le b$$

$$F_z^d = \left[A_o^d J_o(\beta_\rho^d \rho) + B_o^d Y_o(\beta_\rho^d \rho)\right], \quad (\beta_\rho^d)^2 + (\beta_z)^2 = (\beta_d)^2$$

$$b \le \rho \le \infty$$

$$F_z^o = \left[A_o^o K_o(\alpha_\rho^o \rho)\right] e^{-j\beta_z z}, \quad -(\alpha_\rho^o)^2 + (\beta_z)^2 = (\beta_o)^2$$

$$\left.\begin{aligned}
E_\rho^d &= -\frac{1}{\varepsilon_d}\frac{1}{\rho}\frac{\partial F_z^d}{\partial \phi} = 0 \\
E_\phi^d &= \frac{1}{\varepsilon_d}\frac{\partial F_z^d}{\partial \rho} = \frac{\beta_\rho^d}{\varepsilon_d}\left[A_o^d J_o'(\beta_\rho^d \rho) + B_o^d Y_o'(\beta_\rho^d \rho)\right] e^{-j\beta_z z} \\
E_z^d &= 0 \\
H_\rho^d &= -j\frac{1}{\omega\mu_d\varepsilon_d}\frac{\partial^2 F_z^d}{\partial\rho\partial z} = -\frac{\beta_\rho^d \beta_z}{\omega\mu_d\varepsilon_d}\left[A_o^d J_o'(\beta_\rho^d \rho) + B_o^d Y_o'(\beta_\rho^d \rho)\right] e^{-j\beta_z z} \\
H_\phi^d &= -j\frac{1}{\omega\mu_d\varepsilon_d}\frac{1}{\rho}\frac{\partial^2 F_z^d}{\partial\phi\partial z} = 0 \\
H_z^d &= -j\frac{1}{\omega\mu_d\varepsilon_d}\left(\frac{\partial^2}{\partial z^2} + \beta_d^2\right) F_z^d = -j\frac{(\beta_\rho^d)^2}{\omega\mu_d\varepsilon_d}\left[A_o^d J_o(\beta_\rho^d \rho) + B_o^d Y_o(\beta_\rho^d \rho)\right] e^{-j\beta_z z}
\end{aligned}\right\} a \le \rho \le b$$

$$\left.\begin{aligned}
E_\rho^o &= -\frac{1}{\varepsilon_o}\frac{1}{\rho}\frac{\partial F_z^o}{\partial \phi} = 0 \\
E_\phi^o &= \frac{1}{\varepsilon_o}\frac{\partial F_z^o}{\partial \rho} = \frac{\alpha_\rho^o}{\varepsilon_o} A_o^o K_o'(\alpha_\rho^o \rho) e^{-j\beta_z z} \\
E_z^o &= 0 \\
H_\rho^o &= -j\frac{1}{\omega\mu_o\varepsilon_o}\frac{\partial^2 F_z^o}{\partial\rho\partial z} = -\frac{\alpha_\rho^o \beta_z}{\omega\mu_o\varepsilon_o} A_o^o K_o'(\alpha_\rho^o \rho) e^{-j\beta_z z} \\
H_\phi^o &= -j\frac{1}{\omega\mu_o\varepsilon_o}\frac{1}{\rho}\frac{\partial^2 F_z^o}{\partial\phi\partial z} = 0 \\
H_z^o &= -j\frac{1}{\omega\mu_o\varepsilon_o}\left(\frac{\partial^2}{\partial z^2} + \beta_o^2\right) F_z^o = j\frac{(\alpha_\rho^o)^2}{\omega\mu_o\varepsilon_o} A_o^o K_o(\alpha_\rho^o \rho) e^{-j\beta_z z}
\end{aligned}\right\} b \le \rho \le \infty$$

The boundary conditions are the following:

$$E_\phi^d(\rho = a, 0 \le \phi \le 2\pi, z) = 0 \qquad (1)$$

$$E_\phi^d(\rho = b, 0 \le \phi \le 2\pi, z) = E_\phi^o(\rho = b, 0 \le \phi \le 2\pi, z) \qquad (2)$$

$$H_z^d(\rho = b, 0 \le \phi \le 2\pi, z) = H_z^o(\rho = b, 0 \le \phi \le 2\pi, z) \qquad (3)$$

Applying boundary condition (1) leads to

$$\frac{\beta_\rho^d}{\varepsilon_d}\left[A_o^d J_o'(\beta_\rho^d a) + B_o^d Y_o'(\beta_\rho^d a)\right] e^{-j\beta_z z} = 0 \Rightarrow B_o^d = -A_o^d \frac{J_o'(\beta_\rho^d a)}{Y_o'(\beta_\rho^d a)} \qquad (4)$$

Applying boundary condition (2) leads to

$$\frac{\beta_\rho^d}{\varepsilon_d}\left[A_o^d J_o'(\beta_\rho^d b) + B_o^d Y_o'(\beta_\rho^d b)\right] = \frac{\alpha_\rho^o}{\varepsilon_o} A_o^o K_o'(\alpha_\rho^o b)$$

cont'd.

$\boxed{9.43 \text{ cont'd.}}$

$$A_o^o K_o'(\alpha_\rho^o b) = \frac{\varepsilon_o}{\varepsilon_d}\frac{\beta_\rho^d}{\alpha_\rho^o}\left[A_o^d J_o'(\beta_\rho^d b) + B_o^d Y_o'(\beta_\rho^d b)\right] \tag{5}$$

which can be written using (4) as

$$A_o^o K_o'(\alpha_\rho^o b) = \frac{\varepsilon_o}{\varepsilon_d}\frac{\beta_\rho^d}{\alpha_\rho^o} A_o^d\left[J_o'(\beta_\rho^d b) - Y_o'(\beta_\rho^d b)\frac{J_o'(\beta_\rho^d a)}{Y_o'(\beta_\rho^d a)}\right] \tag{6}$$

Applying boundary condition (3) leads to

$$-j\frac{(\beta_\rho^d)^2}{\omega\mu_d\varepsilon_d}\left[A_o^d J_o(\beta_\rho^d b) + B_o^d Y_o(\beta_\rho^d b)\right]e^{-j\beta_z z} = j\frac{(\alpha_\rho^o)^2}{\omega\mu_o\varepsilon_o}A_o^o K_o(\alpha_\rho^o b)\, e^{-j\beta_z z}$$

$$A_o^o K_o(\alpha_\rho^o b) = -\frac{\mu_o\varepsilon_o}{\mu_d\varepsilon_d}\left(\frac{\beta_\rho^d}{\alpha_\rho^o}\right)^2\left[A_o^d J_o(\beta_\rho^d b) + B_o^d Y_o(\beta_\rho^d b)\right]$$

which can be written as

$$A_o^o K_o(\alpha_\rho^o b) = -A_o^d\frac{\mu_o\varepsilon_o}{\mu_d\varepsilon_d}\left(\frac{\beta_\rho^d}{\alpha_\rho^o}\right)^2\left[J_o(\beta_\rho^d b) - Y_o(\beta_\rho^d b)\frac{J_o'(\beta_\rho^d a)}{Y_o'(\beta_\rho^d a)}\right] \tag{7}$$

Dividing (6) by (7) leads to

$$\frac{K_o'(\alpha_\rho^o b)}{K_o(\alpha_\rho^o b)} = -\frac{\mu_d}{\mu_o}\left(\frac{\alpha_\rho^o}{\beta_\rho^d}\right)\frac{\left[J_o'(\beta_\rho^d b)Y_o'(\beta_\rho^d a) - J_o'(\beta_\rho^d a)Y_o'(\beta_\rho^d b)\right]}{\left[J_o(\beta_\rho^d b)Y_o'(\beta_\rho^d a) - J_o'(\beta_\rho^d a)Y_o(\beta_\rho^d b)\right]} \tag{8}$$

In addition

$$(\beta_\rho^d)^2 + \beta_z^2 = \beta_d^2$$

$$-(\alpha_\rho^o)^2 + \beta_z^2 = \beta_o^2$$

Subtracting these two leads to

$$(\beta_\rho^d)^2 + (\alpha_\rho^o)^2 = \beta_d^2 - \beta_o^2 = \beta_o^2\mu_r\varepsilon_r - \beta_o^2 = \beta_o^2(\mu_r\varepsilon_r - 1) \tag{9}$$

Let us now examine the bracketed terms in the numerator and denominator of (8), using $J_o'(\beta_\rho^d a) = -J_1(\beta_\rho^d a)$, $Y_o'(\beta_\rho^d a) = -Y_1(\beta_\rho^d a)$, etc.

$$\text{NUM} = J_o'(\beta_\rho^d b)Y_o'(\beta_\rho^d a) - J_o'(\beta_\rho^d a)Y_o'(\beta_\rho^d b) \simeq Y_1(\beta_\rho^d a)\left\{J_1(\beta_\rho^d a)\left[1 - \frac{b-a}{a}\right] + \beta_\rho^d(b-a)J_o(\beta_\rho^d a)\right\}$$

$$- J_1(\beta_\rho^d a)\left\{Y_1(\beta_\rho^d a)\left[1 - \frac{b-a}{a}\right] + \beta_\rho^d(b-a)Y_o(\beta_\rho^d a)\right\}$$

$$\text{NUM} \simeq -(\beta_\rho^d)(b-a)\left[J_1(\beta_\rho^d a)Y_o(\beta_\rho^d a) - Y_1(\beta_\rho^d a)J_o(\beta_\rho^d a)\right] \tag{10}$$

cont'd.

$\boxed{\text{9.43 cont'd.}}$

$$\text{DEN} = J_0(\beta_\rho^d b)Y_0'(\beta_\rho^d a) - J_0'(\beta_\rho^d a)Y_0(\beta_\rho^d b) \simeq -Y_1(\beta_\rho^d a)\left[J_0(\beta_\rho^d a) - \beta_\rho^d(b-a)J_1(\beta_\rho^d a)\right]$$
$$+ J_1(\beta_\rho^d a)\left[Y_0(\beta_\rho^d a) - \beta_\rho^d(b-a)Y_1(\beta_\rho^d a)\right]$$

$$\text{DEN} \simeq \left[J_1(\beta_\rho^d a)Y_0(\beta_\rho^d a) - Y_1(\beta_\rho^d a)J_0(\beta_\rho^d a)\right. \qquad (11)$$

Using (10) and (11) as well as (9-119a) and (9-119b), we can write (8) as

$$\frac{K_0'(\alpha_\rho^o b)}{K_0(\alpha_\rho^o b)} = \frac{-1/b\alpha_\rho^o}{-\ln(0.89\alpha_\rho^o b)} = -\frac{\mu_d}{\mu_o}\frac{\alpha_\rho^o}{\beta_\rho^d}\left[-\beta_\rho^d(b-a)\right]$$

$$\frac{1}{b\alpha_\rho^o \ln(0.89\alpha_\rho^o b)} = \mu_r \alpha_\rho^o(b-a)$$

$$1 = \mu_r b(\alpha_\rho^o)^2(b-a)\ln(0.89\alpha_\rho^o b)$$

CHAPTER 10

10.1 $A_z^t = A_z^{(1)}\left(x-\frac{s}{2},y,z\right) - A_z^{(2)}\left(x+\frac{s}{2},y,z\right) \overset{s\to 0}{\simeq} -s\frac{\partial A_z^{(1)}}{\partial x}$

$$A_z^t \simeq j\,\frac{s\mu\beta I_e\Delta\ell}{4\pi}\frac{\partial}{\partial x}h_0^{(2)}(\beta r) = j\,\frac{s\mu\beta I_e\Delta\ell}{4\pi}\frac{\partial r}{\partial x}\frac{\partial}{\partial r}h_0^{(2)}(\beta r) = j\,\frac{s\mu\beta^2 I_e\Delta\ell}{4\pi}\frac{\partial r}{\partial x}h_0^{(2)\prime}(\beta r)$$

$$r=\sqrt{x^2+y^2+z^2}\,,\qquad x = r\sin\theta\cos\phi$$

$$\frac{\partial r}{\partial x} = \frac{1}{2}(x^2+y^2+z^2)^{-1/2}(2x) = \frac{x}{\sqrt{x^2+y^2+z^2}} = \frac{r\sin\theta\cos\phi}{r} = \sin\theta\cos\phi$$

$$A_z^t \simeq j\,\frac{s\mu\beta^2 I_e\Delta\ell}{4\pi}h_0^{(2)\prime}(\beta r)\sin\theta\cos\phi$$

Since according to (V-19a)

$$P_1^1(x) = -(1-x^2)^{1/2}$$

$$P_1^1(\cos\theta) = -(1-\cos^2\theta)^{1/2} = -\sin\theta$$

then

$$A_z^t \simeq -j\,\frac{s\mu\beta^2 I_e\Delta\ell}{4\pi}h_0^{(2)\prime}(\beta r)P_1^1(\cos\theta)\cos\phi = +j\,\frac{s\mu\beta^2 I_e\Delta\ell}{4\pi}h_1^{(2)}(\beta r)P_1^1(\cos\theta)\cos\phi$$

10.2 $A_z^t = A_z^{(1)}\left(x,y-\frac{s}{2},z\right) - A_z^{(2)}\left(x,y+\frac{s}{2},z\right) \overset{s\to 0}{\simeq} -s\frac{\partial A_z^{(1)}}{\partial y}$

$$A_z^t \simeq +j\,\frac{s\mu\beta I_e\Delta\ell}{4\pi}\frac{\partial}{\partial y}h_0^{(2)}(\beta r) = j\,\frac{s\mu\beta I_e\Delta\ell}{4\pi}\frac{\partial r}{\partial y}\frac{\partial}{\partial r}h_0^{(2)}(\beta r) = j\,\frac{s\mu\beta^2 I_e\Delta\ell}{4\pi}\frac{\partial r}{\partial y}h_0^{(2)\prime}(\beta r)$$

$$r=\sqrt{x^2+y^2+z^2}\,,\qquad y = r\sin\theta\sin\phi$$

$$\frac{\partial r}{\partial y} = \frac{1}{2}(x^2+y^2+z^2)^{-1/2}(2y) = \frac{y}{\sqrt{x^2+y^2+z^2}} = \frac{r\sin\theta\sin\phi}{r} = \sin\theta\sin\phi$$

$$A_z^t \simeq j\,\frac{s\mu\beta^2 I_e\Delta\ell}{4\pi}h_0^{(2)\prime}(\beta r)\sin\theta\sin\phi$$

Since from solution of Problem 10.1 $P_1^1(\cos\theta) = -\sin\theta$

then

$$A_z^t \simeq -j\,\frac{s\mu\beta^2 I_e\Delta\ell}{4\pi}h_0^{(2)\prime}(\beta r)P_1^1(\cos\theta)\sin\phi = +j\,\frac{s\mu\beta^2 I_e\Delta\ell}{4\pi}h_1^{(2)}(\beta r)P_1^1(\cos\theta)\sin\phi$$

10.3 $A_z^t = A_z^{(1)}\left(x,y,z-\frac{s}{2}\right) - A_z^{(2)}\left(x,y,z+\frac{s}{2}\right) \overset{s\to 0}{\simeq} -s\frac{\partial A_z^{(1)}}{\partial z}$

$$A_z^t \simeq j\,\frac{s\mu\beta I_e\Delta\ell}{4\pi}\frac{\partial}{\partial z}h_0^{(2)}(\beta r) = j\,\frac{s\mu\beta I_e\Delta\ell}{4\pi}\frac{\partial r}{\partial z}\frac{\partial}{\partial r}h_0^{(2)}(\beta r) = j\,\frac{s\mu\beta^2 I_e\Delta\ell}{4\pi}\frac{\partial r}{\partial z}h_0^{(2)\prime}(\beta r)$$

$$r=\sqrt{x^2+y^2+z^2}\,,\qquad z = r\cos\theta$$

$$\frac{\partial r}{\partial z} = \frac{1}{2}(x^2+y^2+z^2)^{-1/2}(2z) = \frac{z}{\sqrt{x^2+y^2+z^2}} = \frac{r\cos\theta}{r} = \cos\theta$$

Since according to (V-19a)

$$P_1^0(x) = P_1(x) = x \Rightarrow P_1(\cos\theta) = \cos\theta$$

then

$$A_z^t \simeq j\,\frac{s\mu\beta^2 I_e\Delta\ell}{4\pi}h_0^{(2)\prime}(\beta r)\cos\theta = -j\,\frac{s\mu\beta^2 I_e\Delta\ell}{4\pi}h_1^{(2)}(\beta r)P_1(\cos\theta)$$

$\boxed{10.4}$ $\underline{E} = -\frac{1}{\epsilon}\nabla\times\underline{F}$, $\underline{H} = -j\omega\underline{F} - j\frac{1}{\omega\mu\epsilon}\nabla(\nabla\cdot\underline{F})$

For TE^z: $\underline{F} = \hat{a}_z F_z = (\hat{a}_r\cos\theta - \hat{a}_\theta\sin\theta)F_z(r,\theta,\phi)$ using (II-13b)

$$F_r = \cos\theta\, F_z(r,\theta,\phi)$$
$$F_\theta = -\sin\theta\, F_z(r,\theta,\phi)$$

$$\nabla\cdot\underline{F} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2F_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}[\sin\theta F_\theta] + \frac{1}{r\sin\theta}\cancelto{0}{\frac{\partial F_\phi}{\partial\phi}}$$
$$= \frac{1}{r^2}\frac{\partial}{\partial r}[r^2\cos\theta F_z] + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}[-\sin^2\theta F_z]$$
$$\nabla\cdot\underline{F} = \frac{\cos\theta}{r^2}\frac{\partial}{\partial r}(r^2F_z) - \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin^2\theta F_z) = DF$$

$$\nabla(\nabla\cdot\underline{F}) = \hat{a}_r\frac{\partial}{\partial r}(DF) + \hat{a}_\theta\frac{1}{r}\frac{\partial}{\partial\theta}(DF) + \hat{a}_\phi\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(DF)$$

$$\nabla\times\underline{F} = \hat{a}_r\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\cancelto{0}{(F_\phi\sin\theta)} - \frac{\partial F_\theta}{\partial\phi}\right] + \frac{\hat{a}_\theta}{r}\left[\frac{1}{\sin\theta}\frac{\partial F_r}{\partial\phi} - \frac{\partial}{\partial r}\cancelto{0}{(rF_\phi)}\right] + \frac{\hat{a}_\phi}{r}\left[\frac{\partial}{\partial r}(rF_\theta) - \frac{\partial F_r}{\partial\theta}\right]$$

$$\nabla\times\underline{F} = \hat{a}_r\frac{1}{r\sin\theta}\left[+\frac{\partial}{\partial\phi}(\sin\theta F_z)\right] + \hat{a}_\theta\frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}(\cos\theta F_z)\right]$$
$$+\hat{a}_\phi\frac{1}{r}\left\{\frac{\partial}{\partial r}(-r\sin\theta F_z) - \frac{\partial}{\partial\theta}(\cos\theta F_z)\right\}$$

Expanding all these operations, it can be shown after many lengthy manipulations that

$$E_r = -\frac{1}{\epsilon r}\frac{\partial F_z}{\partial\phi}$$
$$E_\theta = -\frac{\cot\theta}{\epsilon}\frac{\partial F_z}{\partial\phi}$$
$$E_\phi = +\frac{1}{\epsilon r}\left[\sin\theta\frac{\partial}{\partial r}(rF_z) + \frac{\partial}{\partial\theta}(F_z\cos\theta)\right]$$
$$H_r = -j\omega F_z\cos\theta - j\frac{1}{\omega\mu\epsilon}\frac{\partial}{\partial r}\left[\frac{\cos\theta}{r^2}\frac{\partial}{\partial r}(r^2F_z) - \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(F_z\sin^2\theta)\right]$$
$$H_\theta = j\omega F_z\sin\theta + j\frac{1}{\omega\mu\epsilon r}\frac{\partial}{\partial\theta}\left[\frac{\cos\theta}{r^2}\frac{\partial}{\partial r}(r^2F_z) - \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(F_z\sin^2\theta)\right]$$
$$H_\phi = -j\frac{1}{\omega\mu\epsilon r\sin\theta}\frac{\partial}{\partial\phi}\left[\frac{\cos\theta}{r^2}\frac{\partial}{\partial r}(r^2F_z) - \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(F_z\sin^2\theta)\right]$$

with $\nabla^2F_z + \beta^2F_z = 0$

$\boxed{10.5}$ $\underline{H} = \frac{1}{\mu}\nabla\times\underline{A}$, $\underline{E} = -j\omega\underline{A} - j\frac{1}{\omega\mu\varepsilon}\nabla(\nabla\cdot\underline{A})$

For TM^z: $\underline{A} = \hat{a}_z A_z = (\hat{a}_r\cos\theta - \hat{a}_\theta\sin\theta)A_z(r,\theta,\phi)$ using (II-13b)

$$A_r = \cos\theta\, A_z(r,\theta,\phi)$$
$$A_\theta = -\sin\theta\, A_z(r,\theta,\phi)$$

Using the duality theorem of Section 7.2, Chapter 7, we can write utilizing the solution of Problem 10.4 that

$$E_r = -j\omega A_z\cos\theta - j\frac{1}{\omega\mu\varepsilon}\frac{\partial}{\partial r}\left[\frac{\cos\theta}{r^2}\frac{\partial}{\partial r}(r^2A_z) - \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A_z\sin^2\theta)\right]$$

$$E_\theta = +j\omega A_z\sin\theta - j\frac{1}{\omega\mu\varepsilon r}\frac{\partial}{\partial\theta}\left[\frac{\cos\theta}{r^2}\frac{\partial}{\partial r}(r^2A_z) - \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A_z\sin^2\theta)\right]$$

$$E_\phi = -j\frac{1}{\omega\mu\varepsilon r\sin\theta}\frac{\partial}{\partial\phi}\left[\frac{\cos\theta}{r^2}\frac{\partial}{\partial r}(r^2A_z) - \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A_z\sin^2\theta)\right]$$

$$H_r = \frac{1}{\mu r}\frac{\partial A_z}{\partial\phi}$$

$$H_\theta = \frac{\cot\theta}{\mu r}\frac{\partial A_z}{\partial\phi}$$

$$H_\phi = -\frac{1}{\mu r}\left[\sin\theta\frac{\partial}{\partial r}(rA_z) + \frac{\partial}{\partial\theta}(A_z\cos\theta)\right]$$

with $\nabla^2 A_z + \beta^2 A_z = 0$

$\boxed{10.6}$ According to (6-4a)

$$\boxed{\underline{H}_A = \frac{1}{\mu}\nabla\times\underline{A}} \qquad (1)$$

Away from the source $\underline{J}$

$$\boxed{\nabla\times\underline{H}_A = j\omega\varepsilon\underline{E}_A \Rightarrow \underline{E}_A = \frac{1}{j\omega\varepsilon}\nabla\times\underline{H}_A = \frac{1}{j\omega\mu\varepsilon}\nabla\times\nabla\times\underline{A}} \qquad (2)$$

Taking the curl of both sides of (1) leads to

$$\nabla\times\underline{H}_A = \frac{1}{\mu}\nabla\times\nabla\times\underline{A} \qquad (3)$$

Using Maxwell's equation of

$$\nabla\times\underline{H}_A = \underline{J} + j\omega\varepsilon E_A \qquad (4)$$

and equating it to (3) leads to

$$\nabla\times\nabla\times\underline{A} = \mu J + j\omega\varepsilon\mu\underline{E}_A \qquad (5)$$

the electric and magnetic fields are also related by Maxwell's equation

$$\nabla\times\underline{E}_A = -j\omega\mu\underline{H}_A \qquad (6)$$

cont'd.

$\boxed{\text{10.6 cont'd.}}$ Substituting (1) into (6) and regrouping leads to

$$\nabla\times\underline{E}_A = -j\omega\mu\underline{H}_A = -j\omega\mu\left(\frac{1}{\mu}\nabla\times\underline{A}\right) = -j\omega\nabla\times\underline{A} \qquad (7)$$

or

$$\nabla\times\left(\underline{E}_A + j\omega\underline{A}\right) = 0 \qquad (7a)$$

Using the vector identity of

$$\nabla\times(-\nabla\psi_e) = 0 \qquad (8)$$

and equating (7a) and (8) leads to

$$\underline{E}_A + j\omega\underline{A} = -\nabla\psi_e \Rightarrow \underline{E}_A = -j\omega\underline{A} - \nabla\psi_e \qquad (9)$$

Substituting (9) into (5) leads to

$$\nabla\times\nabla\times\underline{A} = \mu\underline{J} + j\omega\epsilon\mu\left(-j\omega\underline{A} - \nabla\psi_e\right) = \mu\underline{J} + \omega^2\mu\epsilon\underline{A} - j\omega\mu\epsilon\nabla\psi_e \qquad (10)$$

or

$$\boxed{\nabla\times\nabla\times\underline{A} - \omega^2\mu\epsilon\,\underline{A} = \mu\underline{J} - j\omega\mu\epsilon\nabla\psi_e} \qquad (10a)$$

For source-free region ($\underline{J}=0$) (10a) reduces to

$$\nabla\times\nabla\times\underline{A} - \omega^2\mu\epsilon\,\underline{A} = -j\omega\mu\epsilon\nabla\psi_e \qquad (11)$$

$\boxed{\text{10.7}}$ The required solution can be obtained by following the procedure of (10-17a)-(10-22a) which is used to derive (10-22b). Another alternative procedure is to use the duality theorem of Section 7.2, and Tables 7-1 and 7-2. The results follow easily.

$\boxed{\text{10.8}}$ Using (10-43) we can write (10-27) as

$$E_\phi = \frac{1}{j\omega\mu\epsilon}\frac{1}{r\sin\theta}\frac{\partial^2 A_r}{\partial r\partial\phi} = \frac{\beta m}{j\omega\mu\epsilon}\frac{1}{r\sin\theta}D_1\hat{H}_n^{(2)\prime}(\beta r)\left[A_2P_n^m(\cos\theta)+B_2P_n^m(-\cos\theta)\right]$$
$$\times\left[C_3\sin(m\phi)+D_3\cos(m\phi)\right]$$

Applying (10-44b) we have that

$$E_\phi(\theta=\alpha_1) = \frac{\beta m}{j\omega\mu\epsilon}\frac{D_1}{r\sin(\alpha_1)}\hat{H}_n^{(2)\prime}(\beta r)\left[A_2P_n^m(\cos\alpha_1)+B_2P_n^m(-\cos\alpha_1)\right]\left[\quad\right] = 0$$

$$A_2P_n^m(\cos\alpha_1)+B_2P_n^m(-\cos\alpha_1)=0$$

$$E_\phi(\theta=\alpha_2) = \frac{\beta m}{j\omega\mu\epsilon}\frac{D_1}{r\sin\alpha_1}\hat{H}_n^{(2)\prime}(\beta r)\left[A_2P_n^m(\cos\alpha_2)+B_2P_n^m(-\cos\alpha_2)\right]\left[\quad\right] = 0$$

$$A_2P_n^m(\cos\alpha_2)+B_2P_n^m(-\cos\alpha_2)=0$$

or
$$\begin{bmatrix} P_n^m(\cos\alpha_1) & P_n^m(-\cos\alpha_1) \\ P_n^m(\cos\alpha_2) & P_n^m(-\cos\alpha_2)\end{bmatrix}\begin{bmatrix}A_2\\B_2\end{bmatrix} = 0 \Rightarrow P_n^m(\cos\alpha_1)P_n^m(-\cos\alpha_2) - P_n^m(-\cos\alpha_1)P_n^m(\cos\alpha_2) = 0$$

10.9
$$\nabla \times \underline{E} = -j\omega\mu\underline{H} \qquad (1)$$
$$\nabla \times \underline{H} = j\omega\varepsilon\underline{E} \qquad (2)$$

Expanding (1) in spherical coordinates and assuming that the E-field has only an E_θ component independent of ϕ, reduces to

$$\nabla \times \underline{E} = \hat{a}_\phi \frac{1}{r}\frac{\partial}{\partial r}(rE_\theta) = -j\omega\mu(\hat{a}_r H_r + \hat{a}_\theta H_\theta + \hat{a}_\phi H_\phi) \qquad (3)$$

Since $\underline{H}$ has only an H_ϕ component, necessary to form the TEM mode with E_θ, (3) can be written as

$$\frac{1}{r}\frac{\partial}{\partial r}(rE_\theta) = -j\omega\mu H_\phi \qquad (4)$$

From Ampere's law of (2), we have when expanded in spherical coordinates, and assuming only E_θ and H_ϕ components independent of ϕ, that

$$\hat{a}_r \frac{1}{r^2 \sin\theta}\left[\frac{\partial}{\partial\theta}(r\sin\theta H_\phi)\right] - \hat{a}_\theta \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial r}(r\sin\theta H_\phi)\right] = j\omega\varepsilon(\hat{a}_\theta E_\theta) \qquad (5)$$

which can also be written as

$$\frac{\partial}{\partial\theta}(r\sin\theta H_\phi) = 0 \qquad (6a)$$

$$\frac{1}{r\sin\theta}\frac{\partial}{\partial r}(r\sin\theta H_\phi) = -j\omega\varepsilon E_\theta \qquad (6b)$$

Rewriting (6b) as

$$\frac{1}{r}\frac{\partial}{\partial r}(rH_\phi) = -j\omega\varepsilon E_\theta \qquad (7)$$

and substituting it into (4) we form a differential equation for H_ϕ as

$$-\frac{1}{j\omega\varepsilon r}\frac{\partial}{\partial r}\left[\frac{\partial}{\partial r}(rH_\phi)\right] = -j\omega\mu H_\phi \qquad (8)$$

or

$$\frac{\partial^2}{\partial r^2}(rH_\phi) = -\omega^2\mu\varepsilon(rH_\phi) = -\beta^2(rH_\phi) \qquad (8a)$$

A solution for (8a) must be obtained to satisfy (6a). To meet the condition of (6a), the θ variations of H_ϕ must be of the form

$$H_\phi = \frac{f(r)}{\sin\theta} \qquad (9)$$

A solution of (8a), which also meets the requirements of (9) and represents an outward traveling wave, is

$$H_\phi = \frac{H_0}{\sin\theta}\frac{e^{-j\beta r}}{r} \quad \text{where} \quad f(r) = H_0\frac{e^{-j\beta r}}{r} \qquad (10)$$

An inward traveling wave is also a solution but does not apply to the infinitely long structure.

Since the field is of TEM mode, the electric field is related to the magnetic field by the intrinsic impedance, and we can write it as

$$E_\theta = \eta H_\phi = \eta\frac{H_0}{\sin\theta}\frac{e^{-j\beta r}}{r} \qquad (11)$$

$\boxed{10.10}$ $E_\theta = \eta \frac{H_0}{\sin\theta}\frac{e^{-j\beta r}}{r}$, $H_\phi = \frac{H_0}{\sin\theta}\frac{e^{-j\beta r}}{r}$ From solution of Problem 10.9.

$$\underline{S}_{ave} = \frac{1}{2}\text{Re}[\underline{E}\times\underline{H}^*] = \frac{1}{2}\text{Re}[\hat{a}_\theta E_\theta \times \hat{a}_\phi H_\phi^*] = \frac{\hat{a}_r}{2}\text{Re}[E_\theta H_\phi^*] = \hat{a}_r \frac{|H_0|^2}{2r^2}\frac{\eta}{\sin^2\theta}$$

$$P_{rad} = \oiint \underline{S}_{ave}\cdot d\underline{s} = \int_0^{2\pi}\int_{\alpha/2}^{\pi-\alpha/2} \hat{a}_r S_r \cdot \hat{a}_r r^2 \sin\theta\, d\theta\, d\phi = |H_0|^2 \pi\eta \int_{\alpha/2}^{\pi-\alpha/2}\frac{d\theta}{\sin\theta} = 2\pi\eta|H_0|^2 \ln\left[\cot\left(\frac{\alpha}{4}\right)\right]$$

$\boxed{10.11}$ From solution of Problem 10.9

$$H_\phi = \frac{H_0}{\sin\theta}\frac{e^{-j\beta r}}{r}$$

The current on the surface of the cones, a distance r from the origin, is

$$I(r) = \int_0^{2\pi} H_\phi\, r\sin\theta\, d\phi = H_0 e^{-j\beta r}\int_0^{2\pi} d\phi = 2\pi H_0 e^{-j\beta r}$$

At $r=0$

$$I(r=0) = 2\pi H_0$$

Using $I(r=0)$ from above and the P_{rad} from the solution of Problem 10.10, we can write the radiation resistance as

$$R_r = \frac{2P_{rad}}{|I(r=0)|^2} = \frac{4\pi\eta|H_0|^2 \ln[\cot(\frac{\alpha}{4})]}{(2\pi)^2|H_0|^2} = \frac{\eta}{\pi}\ln\left[\cot\left(\frac{\alpha}{4}\right)\right]$$

$\boxed{10.12}$

(a) $R_r = 300 = \frac{120\pi}{\pi}\ln\left[\cot\left(\frac{\alpha}{2}\right)\right] = 120\ln\left[\cot\left(\frac{\alpha}{2}\right)\right] \Rightarrow \ln\left[\cot\left(\frac{\alpha}{2}\right)\right] = \frac{300}{120} = 2.5$

$$\cot\left(\frac{\alpha}{2}\right) = \ln^{-1}(2.5) = e^{2.5} = 12.18 \Rightarrow \frac{\alpha}{2} = \cot^{-1}(12.18) = 4.69^\circ$$

$$\alpha = 9.38^\circ \Rightarrow 2\alpha = 18.76^\circ \text{ total cone angle}$$

(b) $R_r = 50 = 120\ln\left[\cot\left(\frac{\alpha}{2}\right)\right] \Rightarrow \ln\left[\cot\left(\frac{\alpha}{2}\right)\right] = \frac{50}{120} = 0.4166 \Rightarrow \cot\left(\frac{\alpha}{2}\right) = \ln^{-1}(0.4166)$

$$\cot\left(\frac{\alpha}{2}\right) = 1.5169 \Rightarrow \frac{\alpha}{2} = \cot^{-1}(1.5169) = 33.39^\circ \Rightarrow \alpha = 66.789^\circ$$

$$2\alpha = 133.58^\circ \text{ total cone angle}$$

$\boxed{10.13}$

a. For $\frac{TE^r}{TE^r} \Rightarrow (F_r)_{mnp} = [A_2 P_n^m(\cos\theta) + B_2 Q_n^m(\cos\theta)][C_1\hat{H}_n^{(1)}(\beta r) + D_1\hat{H}_n^{(2)}(\beta r)][C_3\cos(m\phi) + D_3\sin(m\phi)]$

Since the fields must be finite at $\theta = 0^\circ$ where $Q_n^m(\cos\theta)$ has a singularity, then $B_2 = 0$ and $(F_r)_{mnp}$ reduces to

$$(F_r)_{mnp} = A_2 P_n^m(\cos\theta)[C_1\hat{H}_n^{(1)}(\beta r) + D_1\hat{H}_n^{(2)}(\beta r)][C_3\cos(m\phi) + D_3\sin(m\phi)]$$

b. The boundary condition is

$$E_\phi(0\le r\le\infty, \theta=\alpha, 0\le\phi\le 2\pi)$$

Using (10-23c) $\Rightarrow E_\phi = \frac{1}{\varepsilon}\frac{1}{r}\frac{\partial F_r}{\partial\theta} = \frac{A_2}{\varepsilon r}\frac{\partial P_n^m(\cos\theta)}{\partial\theta}[C_1\hat{H}_n^{(1)}(\beta r) + D_1\hat{H}_n^{(2)}(\beta r)][C_3\cos m\phi + D_3\sin m\phi]$

Thus $E_\phi(\theta=\alpha) = \frac{A_2}{\varepsilon r}\frac{\partial P_n^m(\cos\theta)}{\partial\theta}[\quad][\quad]\Big|_{\theta=\alpha} = 0 \Rightarrow \frac{\partial P_n(\cos\theta)}{\partial\theta}\Big|_{\theta=\alpha} = 0$

$\boxed{10.14}$

TM^r

a. For TM$^r \Rightarrow (A_r)_{mnp} = \left[A_2 P_n^m(\cos\theta) + B_2 Q_n^m(\cos\theta)\right]\left[C_1 \hat{H}_n^{(1)}(\beta r) + D_1 \hat{H}_n^{(2)}(\beta r)\right]\left[C_3\cos(m\phi) + D_3\sin(m\phi)\right]$

Since the fields must be finite at $\theta = 0^\circ$ where $Q_n^m(\cos\theta)$ has a singularity, then $B_2 = 0$ and $(A_r)_{mnp}$ reduces to

$$(A_r)_{mnp} = A_2 P_n^m(\cos\theta)\left[C_1 \hat{H}_n^{(1)}(\beta r) + D_1 \hat{H}_n^{(2)}(\beta r)\right]\left[C_3\cos(m\phi) + D_3\sin(m\phi)\right]$$

b. The boundary condition is

$$E_\phi(0 \le r \le \infty, \theta = \alpha, 0 \le \phi \le 2\pi) = 0 \quad \underline{\text{or}} \quad E_r(0 \le r \le \infty, \theta = \alpha, 0 \le \phi \le 2\pi) = 0$$

Using (10-27c) we can write that

$$E_\phi = \frac{1}{j\omega\mu\varepsilon r}\frac{1}{\sin\theta}\frac{\partial^2 A_r}{\partial r \partial\phi} = \frac{A_2 m\phi}{j\omega\mu\varepsilon r\sin\theta} P_n^m(\cos\theta)\left[C_1 \hat{H}_n^{(1)\prime}(\beta r) + D_1 \hat{H}_n^{(2)\prime}(\beta r)\right]\left[C_3\cos(m\phi) + D_3\sin(m\phi)\right]$$

thus

$$E_\phi(\theta = \alpha) = \frac{A_2 m\phi}{j\omega\mu\varepsilon r\sin\theta} P_n^m(\cos\theta)\left[\quad\right]\left[\quad\right]\Big|_{\theta=\alpha} = 0 \Rightarrow P_n^m(\cos\theta)\Big|_{\theta=\alpha} = 0$$

$\boxed{10.15}$

TE^r

$$(F_r)_{mnp} = \left[A_2 P_n^m(\cos\theta) + B_2 Q_n^m(\cos\theta)\right]\left[C_1 \hat{H}_n^{(1)}(\beta r) + D_1 \hat{H}_n^{(2)}(\beta r)\right]\left[C_3\cos(m\phi) + D_3\sin(m\phi)\right]$$

$$n = 1, 2, 3, \ldots$$

Since the fields must be finite at $\theta = 0$ and $\theta = \pi$ where $Q_n^m(\cos\theta)$ has a singularity, then $B_2 = 0$ and $(F_r)_{mnp}$ reduces to

$$(F_r)_{mnp} = A_2 P_n^m(\cos\theta)\left[C_1 \hat{H}_n^{(1)} + D_1 \hat{H}_n^{(2)}(\beta r)\right]\left[C_3\cos(m\phi) + D_3\sin(m\phi)\right]$$

The boundary condition is

$$E_\theta(0 \le r \le \infty, 0 \le \theta \le \pi, \phi = 0) = E_\theta(0 \le r \le \infty, 0 \le \theta \le \pi, \phi = \alpha) = 0$$

Using (10-23b) we can write

$$E_\theta = -\frac{1}{\varepsilon r}\frac{1}{\sin\theta}\frac{\partial F_r}{\partial\phi} = -\frac{m A_2}{\varepsilon r\sin\theta} P_n^m(\cos\theta)\left[C_1 \hat{H}_n^{(1)}(\beta r) + D_1 \hat{H}_n^{(2)}(\beta r)\right]\left[-C_3\sin(m\phi) + D_3\cos(m\phi)\right]$$

$$E_\theta(\phi = 0) = -\frac{m A_2}{\varepsilon r\sin\theta} P_n^m(\cos\theta)\left[\quad\right]\left[-C_3(0) + D_3(1)\right] = 0 \Rightarrow D_3 = 0$$

$$E_\theta(\phi = \alpha) = -\frac{m A}{\varepsilon r\sin\theta} P_n^m(\cos\theta)\left[\quad\right]\left[-C_3\sin(m\alpha)\right] = 0 \Rightarrow \sin(m\alpha) = 0$$

$$m\alpha = \sin^{-1}(0) = p\pi \Rightarrow m = \frac{p\pi}{\alpha}, \quad p = 0, 1, 2, \ldots$$

Thus $\quad (F_r)_{mnp} = A_{mnp} P_n^m(\cos\theta)\left[C_1 \hat{H}_n^{(1)}(\beta r) + D_1 \hat{H}_n^{(2)}(\beta r)\right]\cos(m\phi)$

$$m = \frac{p\pi}{\alpha}, \quad p = 0, 1, 2, \ldots$$
$$n = 1, 2, 3, \ldots$$

10.16

TM^r

$$(A_r)_{mnp} = [A_2 P_n^m(\cos\theta) + B_2 Q_n^m(\cos\theta)][C_1 \hat{H}_n^{(1)}(\beta r) + D_1 \hat{H}_n^{(2)}(\beta r)][C_3 \cos(m\phi) + D_3 \sin(m\phi)]$$
$$n = 1, 2, 3, \ldots$$

Since the fields must be finite at $\theta = 0$ and $\theta = \pi$ where $Q_n^m(\cos\theta)$ has singularities, then $B_2 = 0$ and $(A_r)_{mnp}$ reduces to

$$(A_r)_{mnp} = A_2 P_n^m(\cos\theta)[C_1 \hat{H}_n^{(1)}(\beta r) + D_1 \hat{H}_n^{(2)}(\beta r)][C_3 \cos(m\phi) + D_3 \sin(m\phi)]$$

The boundary condition is

$$E_\theta(0 \le r \le \infty, 0 \le \theta \le \pi, \phi = 0) = E_\theta(0 \le r \le \infty, 0 \le \theta \le \pi, \phi = \alpha) = 0$$

Using (10-27b) we can write

$$E_\theta = \frac{1}{j\omega\mu\varepsilon r}\frac{\partial^2 A_r}{\partial r \partial\theta} = \frac{\beta A_2}{j\omega\mu\varepsilon r}\frac{\partial P_n^m(\cos\theta)}{\partial\theta}[C_1 \hat{H}_n^{(1)\prime}(\beta r) + D_1 \hat{H}_n^{(2)\prime}(\beta r)][C_3 \cos(m\phi) + D_3 \sin(m\phi)]$$

$$E_\theta(\phi = 0) = \frac{\beta A_2}{j\omega\mu\varepsilon r}\frac{\partial P_n^m(\cos\theta)}{\partial\theta}[\quad][C_3(1) + D_3(0)] = 0 \Rightarrow C_3 = 0$$

$$E_\theta(\phi = \alpha) = \frac{\beta A_2}{j\omega\mu\varepsilon r}\frac{\partial P_n^m(\cos\theta)}{\partial\theta}[\quad][D_3 \sin(m\alpha)] = 0 \Rightarrow \sin(m\alpha) = 0 \Rightarrow m\alpha = \sin^{-1}(0) = p\pi$$

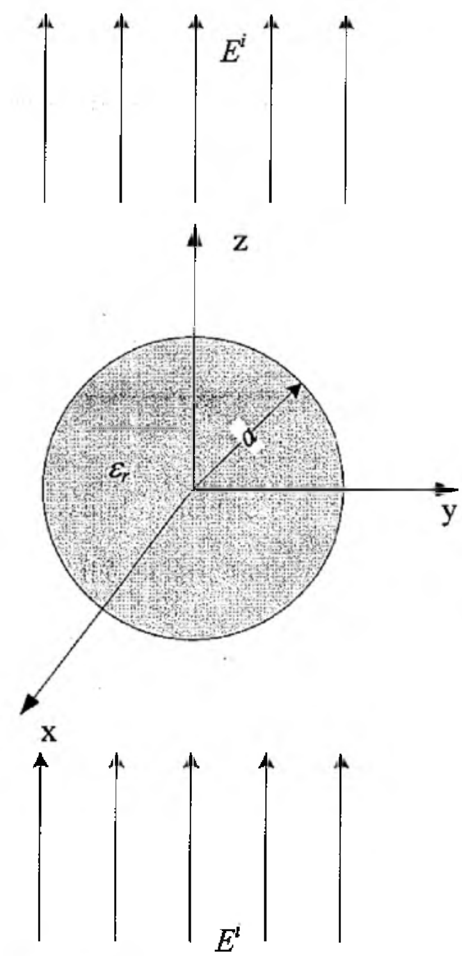
$$m = \frac{p\pi}{\alpha}, \quad p = 1, 2, 3, \ldots$$

Thus

$$(A_r)_{mnp} = A_{mnp} P_n^m(\cos\theta)[C_1 \hat{H}_n^{(1)}(\beta r) + D_1 \hat{H}_n^{(2)}(\beta r)]\sin(m\phi)$$
$$m = \frac{p\pi}{\alpha}, \quad p = 1, 2, 3, \ldots$$
$$n = 1, 2, 3, \ldots$$

10.17

A dielectric sphere of radius a and dielectric constant ε_r is placed in a uniform static electric field $\vec{E}^i = \hat{a}_z E_0$. Find the electric field $\vec{E}$ inside and outside the sphere.



The electric field $\bar{E}$ can be found by using (6-9a) with $\omega = 0$.

$$\bar{E} = -\nabla \psi \qquad (1)$$

where ϕ_e in (6-9a) is replaced by ψ.

To find ψ, we need to solve (3-75) with $\beta = 0$ which reduces to the Laplace equation

$$\nabla^2 \psi = 0 \qquad (2)$$

For our problem, (2) needs to be solved twice: first for the potential inside the sphere $\psi_{in}(r,\theta)$, and then for the potential outside the sphere $\psi_{out}(r,\theta)$. These two solutions are connected by the boundary conditions (1-26a) and (1-30a) on the surface of the sphere.

Eq. (3-75) in spherical coordinates is given by (3.77). Assuming the solution of the form (3-76) and applying the separation of variables method, partial differential equation (3-77) can be separated into three ordinary differential equations (3-86).

Since our problem is fully symmetric (invariant) with respect to ϕ,

$$\frac{\partial \psi}{\partial \phi} = fg \frac{\partial h}{\partial \phi} = \frac{\partial h}{\partial \phi} = 0 \qquad (3)$$

Therefore, (3-86c) can only be satisfied if $m = 0$. From now on we can ignore (3-86c) and function $h(\phi)$ in (3-76).

Substituting $m = 0$ into (3-86b) and $\beta = 0$ into (3-86a), and expanding the derivative in (3-86a), we get two ordinary differential equation:

$$r^2 \frac{d^2 f}{dr^2} + 2r \frac{df}{dr} - n(n+1) f = 0 \qquad (4)$$

$$\frac{1}{\sin\theta} \frac{d}{d\theta}\left(\sin\theta \frac{dg}{d\theta} \right) + n(n+1) g = 0 \qquad (5)$$

Cont'd

Cont'd

Equation (4) is known as the Euler's differential equation [1]. It has a formal solution

$$f(r) = Ar^n + Br^{-(n+1)} \quad (6)$$

Using substitution $x = \cos\theta$, (5) can be transformed into (V-1) which is known as the Legendre's differential equation. It's formal solution is given by (V-4) or, equivalently, by

$$g(\theta) = CP_n(\cos\theta) + DQ_n(\cos\theta) \qquad (7)$$

Thus, the general solution to (2) is given by

$$\psi(r,\theta) = \left(Ar^n + Br^{-(n+1)}\right) \times \left[CP_n(\cos\theta) + DQ_n(\cos\theta)\right] \quad (8)$$

As was already mentioned, (2) needs to be solved for the potentials ψ_{in} and ψ_{out}. Both of these solutions have the same general form (8), but they differ in the choice of the coefficients *A*, *B*, *C*, and *D*. The choice of the coefficients depends on the boundary condition for the electric field $\bar{E}$ on the surface of the sphere and the asymptotic behavior of $\bar{E}$ at infinity as well as at the point $r = 0$ inside the sphere.

Coefficient *D* is easily fixed for both potentials, ψ_{in} and ψ_{out}, by realizing that Legendre function $Q_n(\cos\theta)$ exhibits singularities at $\theta = 0, \pi$ or $x = \pm 1$ (see Figure V-2), therefore it should be removed from (8) by setting $D = 0$.

As far as potential $\psi_{in}(r,\theta)$ is concerned, it must be finite for all r, including $r = 0$. This means that $B_{in} = 0$, where subscript '*in*' indicates that this coefficient is only for the potential $\psi_{in}(r,\theta)$. Thus, our two solutions to (2) are

$$\psi_{in}(r,\theta) = A_{in} r^n P_n(\cos\theta) \qquad (9a)$$

$$\psi_{out}(r,\theta) = \left(A_{out} r^n + B_{out} r^{-(n+1)}\right) P_n(\cos\theta) \qquad (9b)$$

Next, we use the knowledge of the asymptotic behavior of $\bar{E}$ at infinity. Namely, for $(r = \infty)$ the electric field $\bar{E}$ should be equal to the original uniform field $\bar{E}^i$.

$$\bar{E}(r \to \infty, \theta) = -\nabla \psi_{out} = \bar{E}^i = \hat{a}_z E_0 \quad (10)$$

Cont'd

Cont'd

This condition requires that

$$\psi_{out}(r \to \infty, \theta) = -E_0 z = -E_0 r\cos\theta \quad (11)$$

Using (V-9), we make a substitution $\cos\theta = P_1(\cos\theta)$ and then equate (11) and (9b)

$$\left(A_{out} r^n + B_{out} r^{-(n+1)}\right) P_n(\cos\theta) = -E_0 r P_1(\cos\theta)\Big|_{r\to\infty} \quad (12)$$

Clearly, this equality is only possible if $n = 1$ on the left hand side. Thus,

$$\left(A_{out} r + B_{out} r^{-2}\right) P_1(\cos\theta) = -E_0 r P_1(\cos\theta)\Big|_{r\to\infty} \quad (13)$$

$$A_{out} + \frac{B_{out}}{r^3} = -E_0\Big|_{r\to\infty} \quad (14)$$

Therefore, letting $(r = \infty)$ in (14), we get $A_{out} = -E_0$. Thus, potentials (9) become

$$\psi_{in}(r,\theta) = A_{in} r\cos\theta \quad (15a)$$

$$\psi_{out}(r,\theta) = \left(-E_0 r + B_{out} r^{-2}\right)\cos\theta \quad (15b)$$

The remaining two coefficient are fixed by using the boundary conditions (1-26a) and (1-30a). Using (1) and (II-24), the electric field $\bar{E}$ can be written as

$$\bar{E}_{in} = A_{in}\left(-\hat{a}_r \cos\theta + \hat{a}_\theta \sin\theta\right) \quad (16a)$$

$$\bar{E}_{out}(r,\theta) = \hat{a}_r\left(E_0 + B_{out}\frac{2}{r^3}\right)\cos\theta - \hat{a}_\theta\left(E_0 - B_{out}\frac{1}{r^3}\right)\sin\theta \quad (16b)$$

Thus, conditions (1-26a) and (1-30a) on the surface of the sphere, $(r = a)$, give

$$-\hat{a}_\theta\left(E_0 - B_{out}\frac{1}{a^3}\right)\sin\theta = \hat{a}_\theta A_{in}\sin\theta \quad (17a)$$

$$\hat{a}_r \varepsilon_1\left(E_0 + B_{out}\frac{2}{a^3}\right)\cos\theta = -\hat{a}_r \varepsilon_2 A_{in}\cos\theta \quad (17b)$$

where $\varepsilon_1 = \varepsilon_0$, and $\varepsilon_2 = \varepsilon_0\varepsilon_r$. From here, it is easy to show that

Cont'd

Cont'd

$$B_{out} = a^3\left(\frac{\varepsilon_r - 1}{\varepsilon_r + 2}\right)E_0$$

$$A_{in} = -\left(\frac{3}{\varepsilon_r + 2}\right)E_0 \qquad (19)$$

Thus, potentials (15) becomes

$$\psi_{in}(r,\theta) = -\left(\frac{3}{\varepsilon_r + 2}\right)E_0 r\cos\theta \qquad (20a)$$

$$\psi_{out}(r,\theta) = -E_0 r\cos\theta + \left(\frac{\varepsilon_r - 1}{\varepsilon_r + 2}\right)E_0\frac{a^3}{r^2}\cos\theta \qquad (20b)$$

Finally, substituting (19) into (16), the electric field $\bar{E}$ inside and outside the sphere is

$$\bar{E}_{in} = \left(\frac{3}{\varepsilon_r + 2}\right)E_0\left(\hat{a}_r\cos\theta - \hat{a}_\theta\sin\theta\right) \qquad (21a)$$

$$\bar{E}_{out}(r,\theta) = \hat{a}_r E_0\left(1 + 2\varepsilon\frac{a^3}{r^3}\right)\cos\theta - \hat{a}_\theta E_0\left(1 - \varepsilon\frac{a^3}{r^3}\right)\sin\theta \qquad (21b)$$

where $\varepsilon = \left(\dfrac{\varepsilon_r - 1}{\varepsilon_r + 2}\right)$.

Using (II-13a), the fields in (21) can be transformed into rectangular coordinate system

$$\boxed{\begin{array}{l} E_{(x)in}(x,y,z) = E_{(y)in}(x,y,z) = 0 \\ E_{(z)in}(x,y,z) = E_0\left(\dfrac{3}{\varepsilon_r + 2}\right) \end{array}} \qquad (22a)$$

Cont'd

cont'd

$$E_{(x)out}(x,y,z) = E_0 3\left(\frac{\varepsilon_r - 1}{\varepsilon_r + 2}\right)\frac{a^3}{r^5} xz$$

$$E_{(y)out}(x,y,z) = E_0 3\left(\frac{\varepsilon_r - 1}{\varepsilon_r + 2}\right)\frac{a^3}{r^5} yz \qquad (22b)$$

$$E_{(z)out}(x,y,z) = E_0 + E_0 \frac{a^3}{r^5}\left(2z^2 - x^2 - y^2\right)$$

$$r = \sqrt{x^2 + y^2 + z^2}$$

[1] N.H. Asmar, "Applied Complex Analysis with Partial Differential Equations", Prentice-Hall, Inc., New Jersey, 2002

10.18 Using duality as outlined in Table 7-2, we can write the magnetic fields both internal and external to the magnetic sphere, as

$$H_{(x)in}(x,y,z) = H_{(y)in}(x,y,z) = 0$$

$$H_{(z)in}(x,y,z) = H_0\left(\frac{3}{\mu_r + 2}\right)$$

$$H_{(x)out}(x,y,z) = H_0 3\left(\frac{\mu_r - 1}{\mu_r + 2}\right)\frac{a^3}{r^5} xz$$

$$H_{(y)out}(x,y,z) = H_0 3\left(\frac{\mu_r - 1}{\mu_r + 2}\right)\frac{a^3}{r^5} yz$$

$$H_{(z)out}(x,y,z) = H_0 + H_0 \frac{a^3}{r^5}\left(2z^2 - x^2 - y^2\right)$$

$$r = \sqrt{x^2 + y^2 + z^2}$$

10.19

$$(f_r)^{TM}_{011} = \frac{2.744}{2\pi a\sqrt{\mu\varepsilon}} \qquad (f_r)^{TM}_{021} = \frac{3.870}{2\pi a\sqrt{\mu\varepsilon}}$$

$$\frac{(f_r)^{TM}_{021}}{(f_r)^{TM}_{011}} = \frac{3.870/(2\pi a\sqrt{\mu\varepsilon})}{2.744/(2\pi a\sqrt{\mu\varepsilon})} = \frac{3.870}{2.744} = 1.41$$

$$(f_r)^{TM}_{011} = \frac{2.744(3\times10^{10})}{2\pi a\sqrt{2.56}} = 1\times10^9 \Rightarrow a = \frac{2.744(30)}{2\pi(1.6)} = 8.188 \text{ cm}$$

10.20

a) Dominant degenerate modes:

TM^r_{001} (even), TM^r_{111} (even), TM^r_{111} (odd)

$$(f_r)^{TM^r}_{mnp} = \frac{\zeta'_{np}}{2\pi a\sqrt{\mu\varepsilon}} \qquad \zeta'_{11} = 2.744 \qquad \text{m} = 0,1 \text{ (even, odd)} - 3 \text{ modes}$$

$$(f_r)^{TM^r}_{011}(even) = (f_r)^{TM^r}_{111}(even) = (f_r)^{TM^r}_{111}(odd) = \frac{\zeta'_{11}}{2\pi a\sqrt{\mu\varepsilon}}$$

$$\frac{\zeta'_{11}}{2\pi a\sqrt{\mu\varepsilon}} = \frac{\zeta'_{11}\cdot c}{2\pi a\sqrt{\mu_r\varepsilon_r}} = \frac{2.744\cdot(30\times10^9 \text{ cm/s})}{2\pi(2\text{ cm})\sqrt{(1)(1)}} = 6.551\times10^9\text{ Hz}$$

$$\boxed{(f_r)^{TM^r}_{011}(even) = (f_r)^{TM^r}_{111}(even) = (f_r)^{TM^r}_{111}(odd) = 6.551\text{ GHz}}$$

b) Next higher order modes:

TM^r_{021} (even), TM^r_{121} (even), TM^r_{121} (odd), TM^r_{221} (even), TM^r_{221} (odd)

$$\zeta'_{21} = 3.870 \qquad \text{m} = 0, 1, 2 \text{ (even, odd)} - 8 \text{ modes}$$

$$\frac{\zeta'_{11}}{2\pi a\sqrt{\mu\varepsilon}} = \frac{3.870\cdot(30\times10^9 \text{ cm/s})}{2\pi(2\text{ cm})\sqrt{(1)(1)}} = 9.239\times10^9\text{ Hz}$$

BW = 9.239 GHz − 6.551 GHz = 2.688 GHz

$$\boxed{\text{BW} = 2.668 \text{ GHz}}$$

c) The dominant degenerate modes can operate over a bandwidth of 2.688 GHz before the next higher order degenerate modes begin to operate.

$$\frac{(f_r)^{TM}_{001}}{(f_r)^{TM}_{011}} = \frac{1}{\sqrt{\varepsilon_r}} = \frac{1}{2} \Rightarrow \boxed{\varepsilon_r = 4}$$

$\boxed{10.21}$ $$Q = 1.004\frac{\eta}{R_s} = 1.004\frac{\sqrt{\frac{\mu_0}{\varepsilon_r\varepsilon_0}}}{R_s} = \frac{1.004}{\sqrt{\varepsilon_r}}\frac{\eta_0}{R_s}$$

$$R_s = \sqrt{\frac{\omega\mu_0}{2\sigma}} = \sqrt{\frac{2\pi f(4\pi\times10^{-7})}{2\sigma}} = 2\pi\sqrt{\frac{f(10^{-7})}{\sigma}} = \frac{2\pi}{1}\sqrt{\frac{10\times10^9(10^{-7})}{5.7\times10^7}}$$

$$R_s = \frac{2\pi\times10^{-2}}{\sqrt{5.7}} = 2.6317\times10^{-2}$$

$$Q = \frac{1.004}{\sqrt{\varepsilon_r}}\frac{377}{2.631\times10^{-2}} = 10{,}000 \Rightarrow \sqrt{\varepsilon_r} = \frac{1.004(377)}{2.631(10{,}000)(10^{-2})}$$

$$\sqrt{\varepsilon_r} = 1.4383 \Rightarrow \boxed{\varepsilon_r = 2.0686}$$

$\boxed{10.22}$ $\sigma = 5.76\times10^7$ S/m, $Q = 10{,}000$

(a) The resonant frequency of the dominant TM_{011} mode is

$$(f_r)_{011}^{TM} = \frac{J'_{11}}{2\pi a\sqrt{\mu_0\varepsilon_0}} = \frac{2.744}{2\pi a\sqrt{\mu_0\varepsilon_0}}$$

$$Q = 1.004\frac{\eta_0}{R_s} = 1.004\frac{377}{\sqrt{\frac{2\pi f(4\pi\times10^{-7})}{2(5.76\times10^7)}}} = \frac{1.004(377)}{2\pi\times10^{-7}\sqrt{\frac{f}{5.76}}} = 10^4$$

$$\sqrt{f} = \frac{1.004(377)\times10^7\sqrt{5.76}}{2\pi(10^4)} = 1.4458\times10^5$$

$$f_r = 20.903\text{ GHz} = 20.903\times10^9$$

Thus $$\frac{2.744}{2\pi a\sqrt{\mu_0\varepsilon_0}} = \frac{2.744(30\times10^9)}{2\pi a} = 20.903\times10^9$$

$$a = \frac{2.744(30)}{2\pi(20.903)} = 0.6268\text{ cm}$$

(b) From the solution of Problem 10.21

$$R_s = \frac{2\pi\times10^{-2}}{\sqrt{5.76}} = 2.618\times10^{-2},\quad Q = \frac{1.004(377)}{\sqrt{\varepsilon_r}(2.618\times10^{-2})} = \frac{10{,}000}{3}$$

$$\sqrt{\varepsilon_r} = \frac{1.004(377)(3)}{2.618\times10^{2}} = 4.3374 \Rightarrow \boxed{\varepsilon_r = 18.813}$$

$$(f_r)_{011}^{TM} = \frac{2.744}{2\pi a\sqrt{\mu_0\varepsilon_0\varepsilon_r}} = \frac{2.744\times10^9(30)}{2\pi(0.6268)\sqrt{18.813}} = 4.8191\times10^9$$

$$(f_r)_{011}^{TM} = 4.8191\times10^9 = \boxed{4.8191\text{ GHz}}$$

$\boxed{10.23}$ $(A_r)_{111}(\text{even}) = -B_{111}C_3\hat{J}_1(2.744\frac{r}{a})\sin\theta\cos\phi = -A_{111}\hat{J}_1(2.744\frac{r}{a})\sin\theta\cos\phi$

$$H_r = 0$$

$$H_\theta = \frac{1}{\mu r}\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi} = +\frac{A_{111}}{\mu r\sin\theta}\hat{J}_1(2.744\frac{r}{a})\sin\phi\sin\theta = \frac{A_{111}}{\mu r}\hat{J}_1(2.744\frac{r}{a})\sin\phi$$

$$H_\phi = -\frac{1}{\mu r}\frac{\partial A_r}{\partial\theta} = \frac{A_{111}}{\mu r}\hat{J}_1(2.744\frac{r}{a})\cos\theta\cos\phi$$

$$|\underline{H}| = \sqrt{H_\theta^2 + H_\phi^2} = \frac{A_{111}}{\mu r}\hat{J}_1(2.744\frac{r}{a})\sqrt{\sin^2\phi + \cos^2\theta\cos^2\phi} = \frac{A_{111}}{\mu r}\hat{J}_1(2.744\frac{r}{a})H_o(\theta,\phi)$$

$$H_o(\theta,\phi) = \sqrt{\sin^2\phi + \cos^2\theta\cos^2\phi}$$

$$W = 2W_e = 2W_m = 2\left\{\frac{\mu}{4}\iiint_V |\underline{H}|^2 dv\right\} = \frac{\mu}{2}\frac{|A_{111}|^2}{\mu^2}\int_0^a \frac{1}{r^2}\hat{J}_1^2(2.744\frac{r}{a})r^2dr\left\{\int_0^{2\pi}\int_0^\pi |H_o(\theta,\phi)|^2\sin\theta d\theta d\phi\right\}$$

$$W = \frac{|A_{111}|^2}{2\mu}\int_0^a \hat{J}_1^2(2.744\frac{r}{a})dr\left[\int_0^{2\pi}\int_0^\pi |H_o(\theta,\phi)|^2\sin\theta d\theta d\phi\right] \text{ where } |H_o(\theta,\phi)|^2 = \sin^2\phi + \cos^2\theta\cos^2\phi$$

$$P_d = \frac{R_s}{2}\int_0^{2\pi}\int_0^\pi \underline{J}_s\cdot\underline{J}_s^* r^2\sin\theta d\theta d\phi\Big|_{r=a} = \frac{R_s}{2}\int_0^{2\pi}\int_0^\pi \underline{J}_s\cdot\underline{J}_s^* a^2\sin\theta d\theta d\phi$$

$$\underline{J}_s = \hat{n}\times\underline{H}\Big|_{r=a} = -\hat{a}_r\times(\hat{a}_\theta H_\theta + \hat{a}_\phi H_\phi)\Big|_{r=a} = (\hat{a}_\theta H_\phi + \hat{a}_\phi H_\theta)\Big|_{r=a}$$

$$= \left[\hat{a}_\theta\frac{A_{111}}{\mu r}\hat{J}_1(2.744\frac{r}{a})\cos\theta\cos\phi + \hat{a}_\phi\frac{A_{111}}{\mu r}\hat{J}_1(2.744\frac{r}{a})\sin\phi\right]_{r=a}$$

$$\underline{J}_s = \hat{a}_\theta\frac{1}{\mu a}\hat{J}_1(2.744)\cos\theta\cos\phi + \hat{a}_\phi\frac{A_{111}}{\mu a}\hat{J}_1(2.744)\sin\phi$$

$$\underline{J}_s\cdot\underline{J}_s^* = \frac{|A_{111}|^2}{\mu^2a^2}\hat{J}_1^2(2.744)\left[\cos^2\theta\cos^2\phi + \sin^2\phi\right] = \frac{|A_{111}|^2}{\mu^2a^2}\hat{J}_1^2(2.744)|H_o(\theta\phi)|^2$$

$$P_d = \frac{R_s}{2}\frac{|A_{111}|^2}{\mu^2}\hat{J}_1^2(2.744)\int_0^{2\pi}\int_0^\pi |H_o(\theta,\phi)|^2 d\theta d\phi$$

Therefore

$$Q = \omega_r\frac{W}{P_d} = \omega_r\frac{\frac{|A_{111}|^2}{2\mu}\int_0^a \hat{J}_1^2(2.744\frac{r}{a})dr\left[\int_0^{2\pi}\int_0^\pi |\underline{H}_o(\theta,\phi)|^2\sin\theta d\theta d\phi\right]}{\frac{R_s}{2}\frac{|A_{111}|^2}{\mu^2}\hat{J}_1^2(2.744)\left[\int_0^{2\pi}\int_0^\pi |H_o(\theta,\phi)|^2\sin\theta d\theta d\phi\right]}$$

$$Q = \frac{\mu\omega_r}{R_s}\frac{\int_0^a \hat{J}_1^2(2.744\frac{r}{a})dr}{\hat{J}_1^2(2.744)}$$

From Pg. 567 $\Rightarrow \int_0^a \hat{J}_1^2(2.744\frac{r}{a})dr = \frac{1.137}{\beta_r}$, $\hat{J}_1^2(2.744) = (1.0640)^2 = 1.132$

$$Q = \frac{\mu\omega_r}{R_s}\frac{1.137}{\beta_r\, 1.132} = 1.004\frac{\mu\omega_r}{\beta_r R_s} = 1.004\frac{\sqrt{\mu/\varepsilon}}{R_s} = 1.004\frac{\eta}{R_s}\quad\left[\begin{matrix}\text{Same as}\\ (10\text{-}73)\end{matrix}\right]$$

10.24 $(A_r)_{111}(odd) = B_{111} D_3 \hat{J}_1\left(2.744\frac{r}{a}\right)\sin\theta\sin\phi = -C_{111}\hat{J}_r\left(2.744\frac{r}{a}\right)\sin\theta\sin\phi$

By comparing the solution of this to that of Problem 10.23, we see that the two have the same solution which is also the same as that of (10-73). Therefore

$$Q = 1.004\frac{\eta}{R_s}$$

10.25

$$Q = 1.004\frac{\eta}{R_s} \qquad \Rightarrow \quad \eta = \sqrt{\frac{\mu}{\varepsilon}} = \eta_0\sqrt{\frac{\mu_r}{\varepsilon_r}} = 120\pi\sqrt{\frac{\mu_r}{\varepsilon_r}} \qquad \text{and} \qquad R_s = \sqrt{\frac{\omega\mu}{2\sigma}}$$

$$Q = 1.004\frac{120\pi\sqrt{\dfrac{\mu_r}{\varepsilon_r}}}{\sqrt{\dfrac{\omega\mu}{2\sigma}}} = 1.004\frac{120\pi}{\sqrt{\dfrac{\omega\mu_0\varepsilon_r}{2\sigma}}}$$

(a) The medium within the cavity is air: $\mu = \mu_0, \ \varepsilon = \varepsilon_0$

The resonant frequency for the dominant mode of a spherical cavity is:

$$(f_r)_{011}^{TM^r}(even) = (f_r)_{111}^{TM^r}(even) =$$

$$= (f_r)_{111}^{TM^r}(odd) = \frac{\zeta'_{11}}{2\pi a\sqrt{\mu\varepsilon}}\frac{\zeta'_{11}\cdot c}{2\pi a\sqrt{\mu_r\varepsilon_r}} = \frac{2.744\cdot(30\times10^9 \text{ cm/s})}{2\pi(2\text{ cm})\sqrt{(1)(1)}} = 6.551\times10^9\text{ Hz}$$

$$Q = 1.004\frac{120\pi}{\sqrt{\dfrac{\omega\mu_0\varepsilon_r}{2\sigma}}} = 1.004\frac{120\pi}{\sqrt{\dfrac{2\pi(6.551\times10^9\text{Hz})(4\pi\times10^{-7})(1)}{2\ (5.76\times10^7\text{ S/m})}}} = 17{,}862$$

(b) The medium within the cavity is polystyrene: $\mu = \mu_0, \ \varepsilon_r = 2.56$

The resonant frequency for the dominant mode of a spherical cavity is:

$$(f_r)_{011}^{TM^r}(even) = (f_r)_{111}^{TM^r}(even) =$$

$$= (f_r)_{111}^{TM^r}(odd) = \frac{\zeta'_{11}}{2\pi a\sqrt{\mu\varepsilon}}\frac{\zeta'_{11}\cdot c}{2\pi a\sqrt{\mu_r\varepsilon_r}} = \frac{2.744\cdot(30\times10^9 \text{ cm/s})}{2\pi(2\text{ cm})\sqrt{(1)(2.56)}} = 4.094\times10^9\text{ Hz}$$

$$Q = 1.004\frac{120\pi}{\sqrt{\dfrac{\omega\mu_0\varepsilon_r}{2\sigma}}} = 1.004\frac{120\pi}{\sqrt{\dfrac{2\pi(4.094\times10^9\text{Hz})(4\pi\times10^{-7})(2.56)}{2\ (5.76\times10^7\text{ S/m})}}} = 14{,}122$$

$$\boxed{\begin{array}{ll} air & \Rightarrow Q = 17{,}862 \\ polystyrene & \Rightarrow Q = 14{,}122 \end{array}}$$

10.26 $Q = 1.004\,\frac{\eta}{R_s} = 1.004\,\frac{377}{\sqrt{\frac{\pi f_r \mu_0}{\sigma}}} = 10{,}000 = 10^4 = \frac{1.004(377)\sqrt{\sigma}}{\sqrt{\pi f_r \mu_0}}$

$$\sqrt{f_r} = \frac{1.004(377)\sqrt{\sigma}}{\sqrt{\pi\mu_0}}\times 10^{-4} \Rightarrow f_r = \left[1.004(377)\times10^{-4}\right]^2 \frac{\sigma}{\pi\mu_0} = 20.89\text{ GHz}$$

$$f_r = 20.89\times10^9 = 20.89\text{ GHz}$$

$$f_r = \frac{2.744}{2\pi a\sqrt{\mu_0\varepsilon_0}} = 20.89\times10^9 \Rightarrow a = \frac{2.744}{2\pi\sqrt{\mu_0\varepsilon_0}(20.89\times10^9)} = \frac{2.744(30\times10^9)}{2\pi(20.89\times10^9)}$$

$$a = 0.6272\text{ cm}$$

$$Q = 1.004\,\frac{\sqrt{\mu/\varepsilon}}{\sqrt{\frac{\pi f_r\mu_0}{\sigma}}} = \frac{1.004\sqrt{\mu_0/\varepsilon_0}}{\sqrt{\varepsilon_r}}\sqrt{\frac{\sigma}{\pi\mu_0}}\frac{1}{\sqrt{f_r}} = 1.004\eta_0\sqrt{\frac{\sigma}{\pi\mu_0}}\frac{1}{\sqrt{\varepsilon_r}}\left\{\frac{2\pi a\sqrt{\mu_0\varepsilon_0\varepsilon_r}}{2.744}\right\}^{1/2}$$

$$Q \sim (\varepsilon_r)^{1/4}/(\varepsilon_r)^{1/2} = \frac{1}{(\varepsilon_r)^{1/4}} \Rightarrow (\varepsilon_r)^{1/4} = 3 \Rightarrow \varepsilon_r = (3)^4 = 81$$

10.27 $\underline{\text{TE}^r\text{ Modes}}$: $(F_r)_{mnp} = A_{mnp}\hat{J}_n(\beta r)P_n^m(\cos\theta)\left[C_3\cos(m\phi) + D_3\sin(m\phi)\right]$

(a) Boundary Conditions: $H_\theta(r=a) = H_\phi(r=a) = 0$

Because the B.C.s are on the tangential magnetic fields, then through duality, the TE^r modes for the PMC are the same as the TM^r modes for the PEC. Thus

$$(f_r)_{mnp}^{TE^r} = \frac{\zeta'_{np}}{2\pi a\sqrt{\mu\varepsilon}},\quad \begin{aligned} m &= 0,1,2,\dots \le n \\ n &= 1,1,2,\dots \\ p &= 1,2,3,\dots \end{aligned}$$

(b) The resonant frequency of the dominant TE_{011}^r mode is

$$(f_r)_{011}^{TE^r} = \frac{2.744}{2\pi a\sqrt{\mu\varepsilon}}\quad \text{based on Table 10-2.}$$

10.28 $\underline{\text{TM}^r\text{ Modes}}$: $(A_r)_{mnp} = B_{mnp}\hat{J}_n(\beta r)P_n^m(\cos\theta)\left[C_3\cos(m\phi) + D_3\sin(m\phi)\right]$

(a) Boundary Conditions: $H_\theta(r=a) = H_\phi(r=a) = 0$

Because the B.C.s are on the tangential magnetic fields, then thru duality the TM^r modes for the PMC are the same as the TE^r modes for the PEC. Thus

$$(f_r)_{mnp}^{TM^r} = \frac{\zeta_{np}}{2\pi a\sqrt{\mu\varepsilon}},\quad \begin{aligned} m &= 0,1,2,\dots \le n \\ n &= 1,2,3,\dots \\ p &= 1,2,3,\dots \end{aligned}$$

(b) The resonant frequency of the dominant TM_{011}^r mode is

$$(f_r)_{011}^{TM^r} = \frac{4.493}{2\pi a\sqrt{\mu\varepsilon}}\quad \text{based on Table 10-1.}$$

$\boxed{10.29}$ A spherical cavity with $\varepsilon_r \gg 1$ the boundary conditions

(a) are similar to those of a PEC spherical cavity. Therefore, through duality, the TE^r_{mnp} modes for the PMC ($\varepsilon_r \gg 1$) are the same as the TM^r_{mnp} of the PEC. Therefore the resonant frequency is given by (10-64) or

$$(f_r)^{TE^r}_{mnp}(PMC) = \frac{\zeta'_{np}}{2\pi a\sqrt{\mu\varepsilon}} \qquad \begin{array}{l} m=0,1,2,\ldots \le n \\ n=1,2,3,\ldots \\ p=1,2,3,\ldots \end{array}$$

The dominant mode is the TE^r_{011} or

$$(f_r)^{TE^r}_{011}(PMC) = \frac{\zeta'_{11}}{2\pi a\sqrt{\mu\varepsilon}} = \frac{2.744}{2\pi a\sqrt{\mu\varepsilon}} \quad \text{based on Table 10-2}$$

(b) For $a = 3$ cm, $\varepsilon_r = 81$, $\mu_r = 1$

$$(f_r)^{TE^r}_{011}(PMC) = \frac{2.744(30\times10^9)}{2\pi(3)\sqrt{81}} = 0.4852\times10^9 = 0.4852 \text{ GHz}$$

$\boxed{10.30}$ A spherical cavity with $\varepsilon_r \gg 1$ the boundary conditions are

(a) similar to those of a PEC spherical cavity. Therefore, through duality, the TM^r_{mnp} modes for the PMC ($\varepsilon_r \gg 1$) are the same as the TE^r_{mnp} of the PEC. Therefore the resonant frequency is given by (10-59a), or

$$(f_r)^{TM^r}_{mnp}(PMC) = \frac{\zeta_{np}}{2\pi a\sqrt{\mu\varepsilon}} \qquad \begin{array}{l} m=0,1,2,\ldots \le n \\ n=1,2,3,\ldots \\ p=1,2,3,\ldots \end{array}$$

The dominant mode is the TM^r_{011} or

$$(f_r)^{TM^r}_{011}(PMC) = \frac{\zeta_{11}}{2\pi a\sqrt{\mu\varepsilon}} = \frac{4.493}{2\pi a\sqrt{\mu\varepsilon}} \quad \text{based on Table 10-1}$$

(b) For $a = 3$ cm, $\varepsilon_r = 81$, $\mu_r = 1$

$$(f_r)^{TM^r}_{011}(PMC) = \frac{4.493(30\times10^9)}{2\pi(3)\sqrt{81}} = 0.7945\times10^9 = 0.7945 \text{ GHz}$$

10.31

a. From Sections 10.4.1 and 10.4.2 we saw that the dominant mode for the complete spherical cavity is the TM^r_{011} with the reduced vector potential given by (10-65a), or

$$(A_r)^{TM}_{011} = B_{011}C_3\hat{J}_1(2.744\tfrac{r}{a})\cos\theta = A_{011}\hat{J}_1(2.744\tfrac{r}{a})\cos\theta$$

The additional boundary conditions that the hemispherical cavity introduces are

$$E_r(0\le r\le a,\ \theta=\pi/2,\ 0\le\phi\le 2\pi)=0$$

$$E_\phi(0\le r\le a,\ \theta=\pi/2,\ 0\le\phi\le 2\pi)=0$$

Since the E_r and E_ϕ components for the TM^r modes are given by (10-27a) and (10-27c), or

$$E_r=\frac{1}{j\omega\mu\varepsilon}\left(\frac{\partial^2}{\partial r^2}+\beta^2\right)A_r \sim \cos\theta$$

$$E_\phi=\frac{1}{j\omega\mu\varepsilon}\frac{1}{r\sin\theta}\frac{\partial^2 A_r}{\partial r\partial\phi} \sim \frac{\cos\theta}{\sin\theta}=\cot\theta$$

we see that both satisfy the additional boundary conditions introduced by the bottom of the hemispherical cavity. There the TM^r_{011} mode is also the dominant mode of the hemispherical cavity. Its resonant frequency is given by (10-64), or

$$(f_r)^{TM}_{011}=\frac{2.744}{2\pi a\sqrt{\mu\varepsilon}}$$

b. The stored energy of the hemispherical cap is one half of the entire spherical cap, or one half of (10-70). In addition the power dissipated on the upper curved surface of the hemispherical cap is one half of that for the entire spherical cavity, or one half of (10-72). The power dissipated on the bottom flat surface is equal to

$$P_{df}=\frac{R_s}{2}\int_0^{2\pi}\int_0^a \underline{J}_s\cdot\underline{J}_s^*\Big|_{\theta=\pi/2} r\,dr\,d\phi$$

$$\underline{J}_s=\hat{n}\times\underline{H}\Big|_{\theta=\pi/2}=-\hat{a}_\theta\times\hat{a}_\phi H_\phi\Big|_{\theta=\pi/2}=-\hat{a}_r H_\phi\Big|_{\theta=\pi/2}=-\hat{a}_r\frac{B'_{011}}{\mu r}\hat{J}_1(2.744\tfrac{r}{a})$$

$$P_{df}=\frac{R_s}{2}\frac{|B'_{011}|^2}{\mu^2}\int_0^{2\pi}\int_0^a \frac{1}{r}\hat{J}_1^2(2.744\tfrac{r}{a})\,dr\,d\phi=\frac{R_s}{2}\frac{|B'_{011}|^2}{\mu^2}(2\pi)\int_0^a\frac{1}{r}\hat{J}_1^2(2.744\tfrac{r}{a})\,dr$$

$$\int_0^a\frac{1}{r}\hat{J}_1^2\left(2.744\frac{r}{a}\right)dr=\int_0^a\frac{1}{r}\left(\frac{\pi\beta r}{2}\right)J_{3/2}^2\left(2.744\frac{r}{a}\right)dr,\quad \beta=\frac{2.744}{a}$$

$$=\frac{\pi\beta}{2}\int_0^a J_{3/2}^2\left(2.744\frac{r}{a}\right)dr$$

cont'd

$\boxed{\text{10.31 cont'd.}}$ Let $r' = \frac{r}{a} \Rightarrow a\,dr' = dr \Rightarrow dr = a\,dr'$

$$\int_0^a \frac{1}{r}\hat{J}_1^2\left(2.744\frac{r}{a}\right)dr = \frac{\pi\beta a}{2}\int_0^1 J_{3/2}^2(2.744r')\,dr'$$

Using numerical integration to approximate the integral (see page that follows)

$$\int_0^a \frac{1}{r}\hat{J}_1^2\left(2.744\frac{r}{a}\right)dr = \frac{\pi\beta a}{2}\int_0^1 J_{3/2}^2(2.744r')\,dr' = \frac{\pi\beta a}{4}J_{3/2}^2(2.744)$$

$$\int_0^a \frac{1}{r}\hat{J}_1^2\left(2.744\frac{r}{a}\right)dr \overset{a=\frac{2.744}{\beta}}{=\!=\!=} \frac{\pi\beta}{4}\left(\frac{2.744}{\beta}\right)J_{3/2}^2 = \frac{2.744\pi}{4}J_{3/2}^2(2.744)$$

$$\therefore\ P_{df} = \frac{R_s}{2}\frac{|B'_{011}|^2}{\mu^2}(2\pi)\left(\frac{2.744\pi}{4}\right)J_{3/2}^2(2.744)$$

The power dissipated on the curved surface:

$$P_{dc} = \frac{|B'_{011}|^2}{\mu^2}\,\frac{1.132(2\pi)R_s}{3}$$

Total Power Dissipated (P_d):

$$P_d = P_{dc} + P_{df} = \frac{|B'_{011}|^2}{\mu^2}R_s\left[\frac{1.132(2\pi)}{3} + \frac{2.744\pi^2}{4}J_{3/2}^2(2.744)\right]$$

$$J_{3/2}^2(2.744) = 0.2627$$

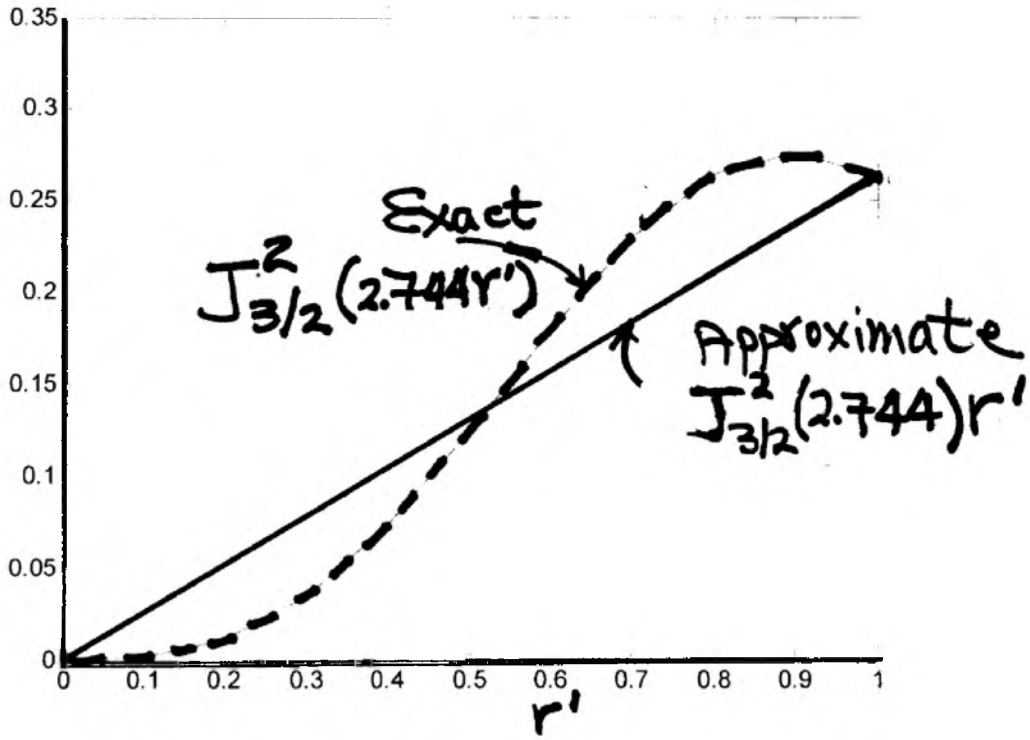
$$P_d = \frac{|B'_{011}|^2}{\mu^2}R_s\left[2.3709 + 1.7786\right] = \frac{|B'_{011}|^2}{\mu^2}R_s(4.1495)$$

$$W = \frac{|B'_{011}|^2}{\mu}\left(\frac{2\pi}{3}\right)\left(\frac{1.137}{\beta_r}\right) = \frac{|B'_{011}|^2}{\mu}\left(\frac{2.3813}{\beta_r}\right)$$

$$\therefore\ Q_h = \omega_r\frac{W}{P_d} = \omega_r\frac{\frac{|B'_{011}|^2}{\mu}\left(\frac{2.3813}{\omega_r\sqrt{\mu\epsilon}}\right)}{\frac{|B'_{011}|^2}{\mu^2}R_s(4.1495)} = 0.5739\left(\frac{\eta}{R_s}\right)$$

cont'd.

10.31 Cont'd.



$J^2_{3/2}(2.744r')$ was approximated by $J^2_{3/2}(2.744)r'$ so that the area of each from $0 \le r' \le 1$ is equal to that of the other; i.e.,

$$\int_0^1 J^2_{3/2}(2.744r')\,dr' = \int_0^1 J^2_{3/2}(2.744)\,r'\,dr'$$

10.31 cont'd

$$Q_h = 0.5739 \frac{\sqrt{\mu/\varepsilon}}{R_s} = 0.574 \frac{\eta}{R_s}$$ (Hemispherical Cavity)

c. For a spherical cavity

$$Q_s = 1.004 \frac{\eta}{R_s}$$

For a cylindrical cavity with $h=d$ by (9-57) we have

$$Q_c = 0.8017 \frac{\eta}{R_s}$$

For a rectangular cavity with $a=b=c$ (cube) by (8-88a) we have

$$Q_r = 0.7405 \frac{\eta}{R_s}$$

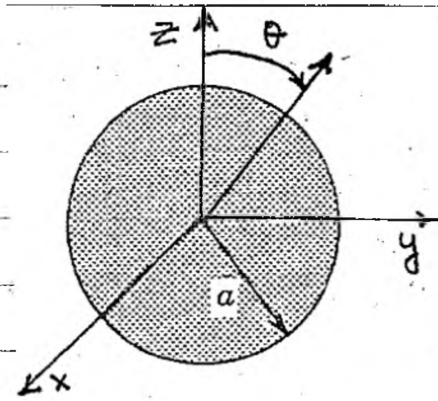
Thus

$$\frac{Q_h}{Q_s} = \frac{0.574}{1.004} \times 100 = 57.17\ \%$$

$$\frac{Q_h}{Q_c} = \frac{0.574}{0.8017} \times 100 = 71.6\ \%$$

$$\frac{Q_h}{Q_r} = \frac{0.574}{0.7405} \times 100 = 77.52\ \%$$

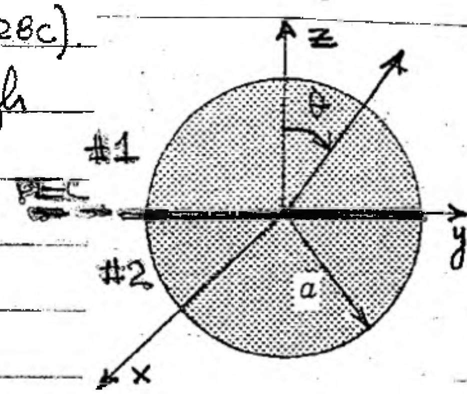
10.32 For a complete spherical cavity the TM^r_{mnp} mode, with the lowest resonant mode, is given in Section 10.4.2 by (10-65a), or



$$(A_r)_{011}(\text{even}) = B_{011} C_3 \hat{J}_1(\beta r) P_1^0(\cos\theta)$$
$$= B_{011} C_3 \hat{J}_1(\beta r)\cos\theta$$

The TM^r_{mnp} electric and magnetic field components are given, respectively, by (10-27a)-(10-27c) and (10-28a)-(10-28c).

Now by placing a PEC plate right through the center in a horizontal plane reduces the complete spherical cap to 2 independent hemi-spherical PEC cavities, as shown in the Figure P10-3 for one of them.



In order to find the dominant TM^r_{mnp} mode, we check to see if the TM^r_{mnp} dominant mode $[TM^r_{001}(\text{even})]$ holds. The additional boundary condition is $[A_r]_{011}(\text{even}) = B_{011} C_3 \hat{J}_1(\beta_r r)\cos\theta$

$$E_r(0 \le r \le a,\ \theta=\pi/2,\ 0\le\phi\le 2\pi) = 0 = \frac{1}{j\omega\mu\varepsilon}\left(\frac{\partial^2}{\partial r^2}+\beta^2\right)A_r \sim \cos\theta\Big|_{\theta=\pi/2} = 0$$

$$E_\phi(0 \le r \le a,\ \theta=\pi/2,\ 0\le\phi\le 2\pi) = 0 = \frac{1}{j\omega\mu\varepsilon}\frac{1}{r\sin\theta}\frac{\partial^2 A_r}{\partial r\partial\phi} \sim \cos\theta\Big|_{\theta=\pi/2} = 0$$

Looking at E_r, we have, from (10-27a)

$$E_r = \frac{1}{j\omega\mu\varepsilon}\left(\frac{\partial^2}{\partial r^2}+\beta^2\right)A_r = \frac{1}{j\omega\mu\varepsilon}\left(\frac{\partial^2}{\partial r^2}+\beta^2\right)\left[B_{011}C_3\hat{J}_1(\beta r)\right]\cos\theta$$

which is satisfied for $\theta=\pi/2$ $[E_r(\theta=\pi/2)=0]$. The same applies if we use E_ϕ from (10-27c) or

$$E_\phi = \frac{1}{j\omega\mu\varepsilon}\frac{1}{r\sin\theta}\frac{\partial^2 A_r}{\partial r\partial\phi} = \frac{1}{j\omega\mu\varepsilon}\frac{1}{r\sin\theta}\frac{\partial^2}{\partial r\partial\phi}\left[B_{011}C_3\hat{J}_1(\beta r)\cos\theta\right]$$

which is also satisfied for $\theta=\pi/2$ $[E_\phi(\theta=\pi/2)=0]$.

The other 2 modes, TM^r_{111}(even) and TM^r_{111}(odd) do not satisfy the additional BCs on E_r and E_ϕ because their potential is proportional to $\sin\theta$, which is not equal to zero when $\theta=\pi/2$.

(a) Therefore the dominant TM^r_{mnp} mode for the hemi-spherical cavity is the same as for the complete spherical cavity or TM^r_{011}(even)

TM^r_{011}(even)

(b) The reduced vector potential is that given by (10-65a) or

$$TM^r_{011}(\text{even}) = B_{011}\, C_3\, \hat{J}_1(\beta r)\cos\theta$$

(c) The lowest resonant frequency is given by (10-64) and Table 10-2, or

$$(f_r)^{TM^r}_{mnp} = \left.\frac{\zeta'_{np}}{2\pi a\sqrt{\mu\varepsilon}}\right|_{\substack{m=0\\n=1\\p=1}} = \frac{\zeta'_{11}}{2\pi a\sqrt{\mu_0\varepsilon_0}} = \left.\frac{2.744}{2\pi a\sqrt{\mu_0\varepsilon_0}}\right|_{a=3\text{ cm}}$$

$$(f_r)^{TM^r}_{011} = \frac{2.744(3\times10^{10})}{2\pi(3)} = \frac{2.744}{2\pi}\times10^{10} = 0.43672\times10^{10}$$

$$(f_r)^{TM^r}_{011} = 4.3672\times10^{9} = \boxed{4.3672\text{ GHz}}$$

$\boxed{10.33}$ For a complete spherical cavity the TE^r_{mnp} mode, with the lowest resonant mode, is given in Section 10.4.1 by (10-60a) or $(F_r)^{(\text{even})}_{011} = A_{011}\, C_3\, \hat{J}_1(4.493\frac{r}{a})\cos\theta$. The corresponding, to the TE^r_{011} mode, electric fields are given by (10-56a)-(10-56c), or

$E_r = 0$

$E_\theta = -\frac{1}{\varepsilon}\frac{1}{r\sin\theta}\frac{\partial F_r}{\partial\phi} = 0$

$E_\phi = \frac{1}{\varepsilon}\frac{1}{r}\frac{\partial F_r}{\partial\theta} = -\frac{A_{011}C_3}{\varepsilon r}\hat{J}_1(4.493\frac{r}{a})\sin\theta$

By placing a conductor along the xy-plane, as shown in

Cont'd

10.33 cont'd

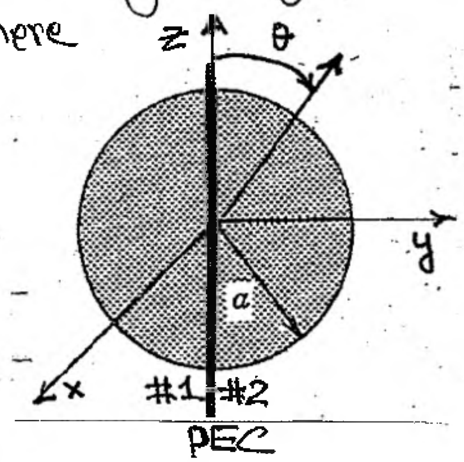
the second figure in the solution of Problem 10.32, to reduce the sphere to a hemi-sphere as shown in Figure P10-32, does not satisfy the additional boundary condition on the newly created boundary; i.e., tangential electric fields to vanish at $\theta = \pi/2$.

$$E_r = 0$$

$$E_\phi(\theta = \pi/2) = -\frac{A_{011}C_3}{\varepsilon r}\hat{J}_1\left(4.493\frac{r}{a}\right)\sin\theta\Big|_{\theta=\pi/2} \neq 0$$

* However, the other 2 modes, TE^r_{111}(even) and TE^r_{111}(odd) do satisfy the BCs.

Therefore the dominant TE^r_{011} mode cannot exist, because it does not satisfy the boundary conditions, for the geometry of Figure P10-32. However, when the sphere is split by placing the PEC plate on the xz-plane (vertically instead of horizontally) as shown in this figure, then the dominant TE^r_{011} mode can exist because the additional boundary condition on the newly created boundary is satisfied; i.e.,



$$E_\phi(\theta = q\pi) = -\frac{A_{011}C_3}{\varepsilon r}\hat{J}_1\left(4.493\frac{r}{a}\right)\sin\theta\Big|_{\theta=q\pi} = 0$$

Therefore, for this modified geometry

(a) The dominant TE^r_{011} mode does exist, but those of the TE^r_{111}(even) and TE^r_{111}(odd) do not exist.

(b) The reduced vector potential for the modified geometry is

$$(F_r)^{(even)}_{011} = A_{011}C_3\hat{J}_1\left(4.493\frac{r}{a}\right)\sin\theta$$

(c) The corresponding resonant frequency is given by (10-59a), or

$$(f_r)^{TE^r}_{mnp}\Big|_{\substack{m=0\\n=p=1}} = (f_r)^{TE}_{011} = \frac{J_{np}}{2\pi a\sqrt{\mu_0\varepsilon_0}}\Bigg|_{\substack{J_{11}=4.493\\a=3\text{cm}}} = \frac{4.493(30\times10^9)}{2\pi(3)}$$

$$(f_r)^{TE}_{011} = 7.151\times10^9 = \boxed{7.151\text{ GHz}}$$

$\boxed{10.34}$

For a PMC spherical cavity with $\varepsilon_r >> 1$ the modes with the lowest resonant frequency are; with vector potentials of

$TE^r(PMC) \sim TM^r(PEC)$ [When $\varepsilon_r >> 1$, BCs are those of PMC (See Sol'n to Problem 10.27)]

$$(F_r)_{011}(\text{even}) \simeq B_{011} C_3 \hat{J}_1\left(2.744 \frac{r}{a}\right)\cos\theta \qquad (10\text{-}65a)$$

$$(F_r)_{111}(\text{even}) \simeq -B_{111} C_3 \hat{J}_1\left(2.744 \frac{r}{a}\right)\sin\theta\cos\phi \qquad (10\text{-}65b)$$

$$(F_r)_{111}(\text{odd}) \simeq -B_{111} D_3 \hat{J}_1\left(2.744 \frac{r}{a}\right)\sin\theta\sin\phi \qquad (10\text{-}65c)$$

By placing a PEC in the middle, we introduce an additional boundary condition of $E_r(\theta = \pi/2,\ 0 \le \phi \le 2\pi) = E_\phi(\theta = \pi/2,\ 0 \le \phi \le 2\pi) = 0$ where

$$E_r = 0 \qquad (10\text{-}23a)$$

$$E_\phi = \frac{1}{\varepsilon}\frac{1}{r}\frac{\partial F_r}{\partial \theta} \qquad (10\text{-}23c)$$

Since $E_r = 0$, then:

$TE^r_{011}(\text{even})$: $E_\phi = -\dfrac{B_{011} C_3}{\varepsilon r}\hat{J}_1\left(2.744\dfrac{r}{a}\right)\sin\theta\Big|_{\substack{\theta=\pi/2 \\ 0\le\phi\le 2\pi}} \neq 0$ which is not satisfied.

Thus it does not exist.

$TE^r_{111}(\text{even})$: $E_\phi = -\dfrac{B_{111} C_3}{\varepsilon r}\hat{J}_1\left(2.744\dfrac{r}{a}\right)\cos\theta\cos\phi\Big|_{\substack{\theta=\pi/2 \\ 0\le\phi\le 2\pi}} = 0$ which is satisfied.

$TE^r_{111}(\text{odd})$: $E_\phi = -\dfrac{B_{111} D_3}{\varepsilon r}\hat{J}_1\left(2.744\dfrac{r}{a}\right)\cos\theta\sin\phi\Big|_{\substack{\theta=\pi/2 \\ 0\le\phi\le 2\pi}} = 0$ which is satisfied.

Thus the TE^r_{mnp} modes with the lowest resonant frequency are:

$$\boxed{\begin{array}{l} TE^r_{111}(\text{even}) \\ TE^r_{111}(\text{odd}) \end{array}} \Bigg\} \text{ with resonant frequency of } (f_r)^{TE^r}_{111} = \frac{J_{np}'}{2\pi a\sqrt{\mu\varepsilon}} = \frac{J_{11}'}{2\pi a\sqrt{\mu_0\varepsilon_0\varepsilon_r}}$$

$$(f_r)^{TE^r}_{111} = \frac{2.744(30\times10^9)}{2\pi(3)(9)} = 0.48525\times10^9$$

$$\boxed{(f_r)^{TE^r}_{111} = 0.48525 \text{ GHz}}$$

10.35

$TM^r(PMC) \sim TE^r(PEC)$ [When $\varepsilon_r >> 1$, BCs are those of PMC; see solution to Problem 10.28]

For complete sphere with $\varepsilon_r >> 1$

$$(A_r)_{011}\text{ (even)} \simeq A_{011} C_3 \hat{J}_1\left(4.493\frac{r}{a}\right)\cos\theta \qquad (10\text{-}60a)$$

$$(A_r)_{111}\text{ (even)} \simeq -A_{111} C_3 \hat{J}_1\left(4.493\frac{r}{a}\right)\sin\theta\cos\phi \qquad (10\text{-}60b)$$

$$(A_r)_{111}\text{ (odd)} \simeq -A_{111} D_3 \hat{J}_1\left(4.493\frac{r}{a}\right)\sin\theta\sin\phi \qquad (10\text{-}60c)$$

By placing a PEC in the middle of the sphere, we introduce an additional boundary condition of $E_r(\theta=\pi/2, 0\le\phi\le 2\pi) = E_\phi(\theta=\pi/2, 0\le\phi\le 2\pi) = 0$ where

$$E_r = \frac{1}{j\omega\mu\varepsilon}\left(\frac{\partial^2}{\partial r^2}+\beta^2\right)A_r \qquad (10\text{-}27a)$$

$$E_\phi = \frac{1}{j\omega\mu\varepsilon}\frac{1}{r\sin\theta}\frac{\partial^2 A_r}{\partial r\,\partial\phi} \qquad (10\text{-}27c)$$

TM_{011} (even):

$$E_r \sim \cos\theta\Big|_{\substack{\theta=\pi/2\\0\le\phi\le 2\pi}} = 0 \qquad \text{which is satisfied}$$

$$E_\phi = 0 \qquad \text{which is satisfied}$$

TM_{111} (even)

$$E_r \sim \sin\theta\cos\phi\Big|_{\substack{\theta=\pi/2\\0\le\phi\le 2\pi}} \ne 0 \qquad \text{which is not satisfied}$$

$$E_\phi \sim \sin\theta\sin\phi\Big|_{\substack{\theta=\pi/2\\0\le\phi\le 2\pi}} \ne 0 \qquad \text{which is not satisfied}$$

TM_{111} (odd):

$$E_r \sim \sin\theta\sin\phi\Big|_{\substack{\theta=\pi/2\\0\le\phi\le 2\pi}} \ne 0 \qquad \text{which is not satisfied}$$

$$E_\phi \sim \sin\theta\cos\phi\Big|_{\substack{\theta=\pi/2\\0\le\phi\le 2\pi}} \ne 0 \qquad \text{which is not satisfied}$$

Thus the TM^r_{mnp} mode with the lowest resonant frequency is

$\boxed{TM_{011}\text{(even)}}$ with $(f_r)^{TM^r}_{011} = \dfrac{\zeta_{11}}{2\pi a\sqrt{\mu_0\varepsilon_0\varepsilon_r}} = \dfrac{4.493(30\times10^9)}{2\pi(3)(9)} = 0.79454\times10^9$

$$\boxed{(f_r)^{TM^r}_{011} = 0.79454\text{ GHz}}$$

$\boxed{10.36}$ From the soution of Problem 10.27, the dominant TE^r_{mnp} modes for an entire sphere with PMC surface are the TM^r_{mnp} of a PEC, or $(TE^r)_{011}(\text{even}) = (TE^r)_{111}(\text{even}) = (TE^r)_{111}(\text{odd})$

with a resonant frequency of

$$(f_r)^{TE^r}_{011}(\text{even}) = (f_r)^{TE^r}_{111}(\text{even}) = (f_r)^{TE^r}_{111}(\text{odd}) = \frac{2.744}{2\pi a\sqrt{\mu\varepsilon}}$$

(c) The corresponding vector potential is analogous to (10-65a)-(10-65c), while the electric and magnetic fields are given by (10-23)-(10-24c). Now by placing a PMC plate right through the center in a horizontal plane reduces the entire spherical cavity to two independent hemi-spherical PMC cavities.

In order to find the dominant TE^r_{mnp} mode, we check to see if the TE^r_{mnp} dominant mode $[TE^r_{011}(\text{even})]$ holds. The additional BC is

$$H_r(0 \le r \le a,\ \theta = \pi/2,\ 0 \le \phi \le 2\pi) = 0$$

$$H_\phi(0 \le r \le a,\ \theta = \pi/2,\ 0 \le \phi \le 2\pi) = 0$$

$$(F_r)_{mnp} = (F_r)_{011}(\text{even}) = \left[A_{011} C_3 \hat{J}_1\left(4.493\frac{r}{a}\right)\right]\cos\theta \qquad (10\text{-}60a)$$

$$H_r = \frac{1}{j\omega\mu\varepsilon}\left(\frac{\partial^2}{\partial r^2} + \beta^2\right)F_r \sim [\qquad]\cos\theta \qquad (10\text{-}24a)$$

$$H_\phi = \frac{1}{j\omega\mu\varepsilon}\frac{1}{r\sin\theta}\frac{\partial^2 F_r}{\partial r\,\partial\phi} \sim [\qquad]\frac{\cos\theta}{\sin\theta} \qquad (10\text{-}24e)$$

$$H_r(0 \le r \le a,\ \theta = \pi/2,\ 0 \le \phi \le 2\pi) = [\qquad]\cos\theta\Big|_{\theta=\pi/2} = 0$$

$$H_\phi(0 \le r \le a,\ \theta = \pi/2,\ 0 \le \phi \le 2\pi) = [\qquad]\frac{\cos\theta}{\sin\theta}\Big|_{\theta=\pi/2} = 0$$

However for the $TE_{111}(\text{even})$ mode, where F_r is

$$(F_r)_{111}(\text{even}) = -A_{111} C_3 \hat{J}_1\left(4.493\frac{r}{a}\right)\sin\theta\cos\phi \qquad (10\text{-}60b)$$

and for the $TE_{111}(\text{odd})$ mode, where F_r is

$$(F_r)_{111}(\text{odd}) = -A_{111} D_3 \hat{J}_1\left(4.493\frac{r}{a}\right)\sin\theta\sin\phi \qquad (10\text{-}60c)$$

$\boxed{\text{Cont'd}}$

10.36 cont'd

The two boundary conditions on H_r and H_ϕ for the TE_{111}(even) and TE_{111}(odd) are not satisfied. Thus these two modes cannot be supported by the geometry of Fig. P10-36; only the TE_{011} (even) can be supported, and it is the dominant mode.

(b) The reduced vector potential for the dominant TE^r_{011}(even) mode is

$$(F_r)_{011}(\text{even}) = A_{011} C_3 \hat{J}_1\left(4.493\frac{r}{a}\right)\cos\theta$$

(c) The resonant frequency of the dominant TE^r_{011}(even) mode is

$$(f_r)^{TE^r}_{011} = \frac{\zeta'_{11}}{2\pi a\sqrt{\mu\varepsilon}} = \frac{2.744(30\times10^9)}{2\pi(3)} = 4.3672\times10^9 = 4.3672 \text{ GHz}.$$

10.37 From the solution of Problem 10.28, the dominant TM^r_{mnp} modes for an entire sphere with PMC surface are the TE^r_{mnp} of a PEC surface, or

$$(TM^r)_{011}(\text{even}) = (TM^r)_{111}(\text{even}) = (TM^r_{111})(\text{odd})$$

with a resonant frequency of

$$(f_r)^{TM^r}_{011}(\text{even}) = (f_r)^{TM^r}_{111}(\text{even}) = (f_r)^{TM^r}_{111}(\text{odd}) = \frac{4.493}{2\pi a\sqrt{\mu\varepsilon}}$$

(a) The corresponding vector potential is analogous to (10-60a)-(10-60c), while the electric and magnetic fields are given by (10-27a)-(10-28c). Now placing a PMC plate right through the center in the horizontal plane reduces the entire spherical cavity to two independent hemi-spherical PMC cavities.

In order to find the dominant TM^r_{mnp} mode, we check to see if the TM^r_{mnp} dominant mode [TM^r_{011} (even)] holds. The additional BCs are

$$H_r(0\le r\le a,\ \theta=\pi/2,\ 0\le\phi\le 2\pi) = 0$$

$$H_\phi(0\le r\le a,\ \theta=\pi/2,\ 0\le\phi\le 2\pi) = 0$$

$$(A_r)_{mnp} = (A_r)_{011}(\text{even}) = B_{011} C_3 \hat{J}_1\left(4.493\frac{r}{a}\right)\cos\theta \qquad (10\text{-}65a)$$

(cont'd)

10.37 Cont'd

Checking the additional boundary condition created by the PMC plate placed horizontally on the xy-plane, using (10-28a)-(10-28c)

$$H_r = 0$$

$$H_\theta = \frac{1}{\mu}\frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\phi} = 0$$

$$H_\phi\Big|_{\theta=\pi/2} = -\frac{1}{\mu}\frac{1}{r}\frac{\partial A_r}{\partial\theta} \sim [\quad]\sin\theta\Big|_{\theta=\pi/2} \neq 0$$

which is not satisfied. However for the TM_{111}(even) mode, where A_r is $(A_r)_{111}(\text{even}) = -B_{111}C_3\hat{J}_1(2.744\frac{r}{a})\sin\theta\cos\phi$ (10-65b)

and for the TM_{111}(odd), where A_r is

$$(A_r)_{111}(\text{odd}) = -B_{111}D_3\hat{J}_1(2.744\tfrac{r}{a})\sin\theta\sin\phi \qquad (10\text{-}65c)$$

the additional boundary condition on $H_\phi(\theta=\pi/2)$ is satisfied by both the TM^r_{111}(even) and TM^r_{111}(odd) modes.

(b) Therefore the reduced vector potential for the dominant TM^r_{111}(even) and TM^r_{111}(odd) is, respectively

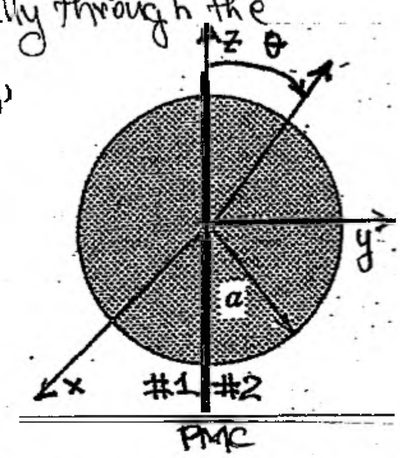
$$(A_r)_{111}(\text{even}) = -B_{111}C_3\hat{J}_1(2.744\tfrac{r}{a})\sin\theta\cos\phi$$

$$(A_r)_{111}(\text{odd}) = -B_{111}D_3\hat{J}_1(2.744\tfrac{r}{a})\sin\theta\sin\phi$$

(c) The resonant frequency of the dominant TM^r_{111}(even) and TM^r_{111}(odd) modes is

$$(f_r)^{TM^r}_{111}(\text{even, odd}) = \frac{\zeta_{11}}{2\pi a\sqrt{\mu_0\varepsilon_0}} = \frac{4.493(30\times10^9)}{2\pi(3)} = 7.151\times10^9 = 7.151\text{ GHz}$$

P.S. It should be pointed out that if the PMC hemi-spherical cavity was formed by placing the PMC ground plane vertically through the center of the sphere, as shown graphically, then the TM^r_{011}(even) mode would exist and the TM^r_{111}(even) and TM_{111}(odd) would not satisfy the newly created boundary conditions.



CHAPTER 11

11.1 For a displaced line source, the solution for an electric source is displayed in detail in Example 11-1. The solution for a magnetic line source follows the same procedure, and it does not to be detailed as it can be obtained easier by using duality. Using Table 7-2 as a guide, the electric fields of the electric line source are replaced by the magnetic, the magnetic by the negative electric, ε by μ, μ by ε, η by $1/\eta$, I_e by I_m, ect. Therefore, from Example 11-1, we can write for the magnetic line source displaced from the origin that

$$E_\rho = -j\frac{\beta I_m}{4}\frac{\rho' \sin(\phi-\phi')}{\sqrt{|\underline{\rho}-\underline{\rho}'|^2}} H_1^{(2)}(\beta|\underline{\rho}-\underline{\rho}'|)$$

$$E_\phi = j\frac{\beta I_m}{4}\frac{[\rho-\rho'\cos(\phi-\phi')]}{\sqrt{|\underline{\rho}-\underline{\rho}'|^2}} H_1^{(2)}(\beta|\underline{\rho}-\underline{\rho}'|)$$

$$E_z = H_\rho = H_\phi = 0, \quad |\underline{\rho}-\underline{\rho}'| = \sqrt{\rho^2+(\rho')^2-2\rho\rho'\cos(\phi-\phi')}$$

$$H_z = -\frac{\omega\varepsilon I_m}{4} H_0^{(2)}(\beta|\underline{\rho}-\underline{\rho}'|)$$

11.2 Using the geometry of Figure 11-2, but with the electric

(a) line source replaced with a magnetic line source, we can write

$$H_z^t = H_z^i + H_z^r = -\frac{\omega\varepsilon I_m}{4}\left[H_0^{(2)}(\beta\rho_i) + H_0^{(2)}(\beta\rho_r)\right]$$
$$= -I_m\frac{\beta}{4\eta}\left[H_0^{(2)}(\beta\rho_i) + H_0^{(2)}(\beta\rho_r)\right]$$

because the image, to represent the reflected fields, has the same magnitude and phase as the actual source.

(b) For far-field observations the Hankel functions can be replaced by their asymptotic expansions of (IV-17). Therefore we can write the total magnetic field as

$$H_z^t = -I_m\frac{\beta}{4\eta}\sqrt{\frac{j2}{\pi\beta\rho}}\left[\frac{e^{-j\beta\rho_i}}{\sqrt{\rho_i}} + \frac{e^{-j\beta\rho_r}}{\sqrt{\rho_r}}\right] = -\frac{I_m}{\eta}\sqrt{\frac{j\beta}{8\pi}}\left[\frac{e^{-j\beta\rho_i}}{\sqrt{\rho_i}} + \frac{e^{-j\beta\rho_r}}{\sqrt{\rho_r}}\right]$$

For far-field observations

$$\rho_i \simeq \rho - h\cos\left(\frac{\pi}{2}-\phi\right) = \rho - h\sin\phi$$
$$\rho_r \simeq \rho + h\cos\left(\frac{\pi}{2}-\phi\right) = \rho + h\sin\phi$$
for phase terms

$$\rho_i \simeq \rho_r \simeq \rho \qquad \text{for amplitude terms}$$

Therefore the total field can be written in a more compact form, with the radial distances referenced to the origin, as

$$H_z^t = -\frac{I_m}{\eta}\sqrt{\frac{j\beta}{8\pi}}\left\{\frac{e^{-j\beta\rho}}{\sqrt{\rho}}2\left[\frac{e^{j\beta h\sin\phi}+e^{-j\beta h\sin\phi}}{2}\right]\right\}$$

$$H_z^t = -\frac{I_m}{\eta}\sqrt{\frac{j\beta}{8\pi}}\frac{e^{-j\beta\rho}}{\sqrt{\rho}}\left\{2\cos(\beta h\sin\phi)\right\}$$

(c)

To find the smallest height where the field will attain nulls at certain observation angles, we set the cosine function (usually referred to as the array factor) to zero. That is

$$\cos(\beta h\sin\phi_n) = 0 \Rightarrow \beta h\sin\phi_n = \cos^{-1}(0) = \frac{n\pi}{2}, \quad n=1,3,5,\ldots$$

For the smallest height h, we choose $n=1$, or

$$\beta h\sin\phi_n = \frac{\pi}{2} \Rightarrow h = \frac{\pi/2}{\beta\sin\phi_n} = \frac{\pi}{2}\frac{\lambda}{2\pi}\frac{1}{\sin\phi_n} = \frac{\lambda}{4\sin\phi_n}$$

$$h = \frac{\lambda}{4\sin\phi_n}$$

- For $\phi = 30°$: $h = \frac{\lambda}{4}\frac{1}{0.5} = \frac{\lambda}{2}$
- For $\phi = 90°$: $h = \frac{\lambda}{4}$

$\boxed{11.3}$ For an electric line source placed above a PMC, the magnitude and phase of the image are the same as the actual source. Therefore the solution of this problem is the same as that of 11.2 except with the magnetic field replaced by the electric field using duality.

(a) $E_z^t = -I_e\eta\frac{\beta}{4}\left[H_0^{(2)}(\beta\rho_i) + H_0^{(2)}(\beta\rho_r)\right]$

(b) $H_z^t = -I_e\eta\sqrt{\frac{j\beta}{8\pi}}\left[\frac{e^{-j\beta\rho}}{\sqrt{\rho}}\left\{2\cos(\beta h\sin\phi)\right\}\right]$

(c) • $\phi = 30°$: $h = \lambda/2$ • $\phi = 90°$: $h = \lambda/4$

11.4 For a magnetic line source placed a height h above a PMC, the magnitude of the image is the same as that of the actual source while its phase is 180° (out of phase). Therefore, using the solution of Problem 11.2, we can write the solution of this problem as:

(a) $H_z^t = -I_m \frac{\beta}{4\eta}\left[H_0^{(2)}(\beta\rho_i) - H_0^{(2)}(\beta\rho_r)\right]$

(b) $$H_z^t = -\frac{I_m}{\eta}\sqrt{\frac{j\beta}{8\pi}}\,\frac{e^{-j\beta\rho}}{\sqrt{\rho}}\left\{2j\left[\frac{e^{+j\beta h\sin\phi} - e^{-j\beta h\sin\phi}}{2j}\right]\right\}$$

$$= -\frac{I_m}{\eta}\sqrt{\frac{j\beta}{8\pi}}\,\frac{e^{-j\beta\rho}}{\sqrt{\rho}}\left[2j\sin(\beta h\sin\phi)\right]$$

(c) For the smallest height h for which the field will vanish

$$\sin(\beta h\sin\phi_n) = 0 \Rightarrow \beta h\sin\phi_n = \sin^{-1}(0) = n\pi, \quad n = 0,1,2,\ldots$$

The smallest nontrivial ($h \neq 0$) height will be for $n = 1$. Therefore

$$h = \frac{\pi}{\beta\sin\phi_n} = \frac{\pi\lambda}{2\pi\sin\phi_n} = \frac{\lambda}{2\sin\phi_n}$$

- $\phi = 30°$: $h = \frac{\lambda}{2(0.5)} = \lambda$
- $\phi = 90°$: $h = \frac{\lambda}{2}$

11.5 For an electric current element with current I_e located at

(a) $x = 0$, $y = 0$, the potential is given by

$$A_z^{(0)} = -j\frac{\mu I_e}{4} H_0^{(2)}(\beta\rho)$$

For a current element located at $x = x'$, $y = y'$, the potential is given by

$$A_z = -j\frac{\mu I_e}{4} H_0^{(2)}(\beta|\underline{\rho} - \underline{\rho}'|) = -j\frac{\mu I_e}{4} H_0^{(2)}\left(\beta\sqrt{(x-x')^2 + (y-y')^2}\right)$$

Thus for $x' = 0$, $y' = h$ and current I_e

$$A_z^{(1)} = -j\frac{\mu I_e}{4} H_0^{(2)}\left(\beta\sqrt{x^2 + (y-h)^2}\right) = A_z^{(1)}(x, y-h)$$

(b) For the image at $x' = 0$, $y = -h$ and current $-I_e$

$$A_z^{(2)} = +j\frac{\mu I_e}{4} H_0^{(2)}\left(\beta\sqrt{x^2 + (y+h)^2}\right) = A_z^{(2)}(x, y+h)$$

Therefore the total field is equal to

$$A_z^t = A_z^{(1)}(x, y-h) - A_z^{(2)}(x, y+h) = -\left[A_z^{(0)}(x, y+h) - A_z^{(0)}(x, y-h)\right]$$

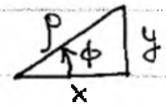
(cont'd)

11.5 cont'd

$$\lim_{h\to 0} A_z^t = -\lim_{h\to 0}\left[A_z^{(1)}(x,y+h) - A_z^{(1)}(x,y-h)\right] = -2h \lim_{h\to 0}\left[\frac{A_z^{(1)}(x,y+h) - A_z^{(1)}(x,y-h)}{2h}\right]$$

$$= -2h\frac{\partial A_z^{(1)}}{\partial y} = -2h\frac{\partial}{\partial y}\left[-j\frac{\mu I_e}{4}H_0^{(2)}(\beta\rho)\right] = j\frac{2\mu h I_e}{4}\frac{\partial}{\partial y}H_0^{(2)}(\beta\rho)$$

where $\rho = \sqrt{x^2+y^2}$



$$\frac{\partial}{\partial y} = \frac{\partial}{\partial\rho}\frac{\partial\rho}{\partial y} = \frac{\partial\rho}{\partial y}\frac{\partial}{\partial\rho}$$

$$\frac{\partial\rho}{\partial y} = \frac{\partial}{\partial y}(x^2+y^2)^{1/2} = \frac{1}{2}(x^2+y^2)^{-1/2}(2y) = \frac{y}{\sqrt{x^2+y^2}} = \frac{y}{\rho} = \sin\phi$$

$$\frac{\partial}{\partial\rho}H_0^{(2)}(\beta\rho) = -\beta H_1^{(2)}(\beta\rho)$$

Therefore

$$\lim_{h\to 0} A_z^t = j\frac{\mu h I_e}{2}\sin\phi\left[-\beta H_1^{(2)}(\beta\rho)\right] = -j\frac{\beta\mu h I_e}{2}H_1^{(2)}(\beta\rho)\sin\phi$$

(c) For TM^z

$$E_\rho = -j\frac{1}{\omega\mu\varepsilon}\frac{\partial^2 A_z^t}{\partial\rho\,\partial z} = 0 \qquad H_\rho = \frac{1}{\mu\rho}\frac{\partial A_z^t}{\partial\phi} = -j\frac{\beta h I_e}{2\rho}H_1^{(2)}(\beta\rho)\cos\phi$$

$$E_\phi = -j\frac{1}{\omega\mu\varepsilon}\frac{1}{\rho}\frac{\partial^2 A_z^t}{\partial\phi\,\partial z} = 0 \qquad H_\phi = -\frac{1}{\mu}\frac{\partial A_z^t}{\partial\rho} = -\frac{1}{\mu}\frac{\partial}{\partial\rho}\left[-j\frac{\beta\mu h I_e}{2}H_1^{(2)}(\beta\rho)\sin\phi\right]$$

$$E_z = -j\frac{1}{\omega\mu\varepsilon}\left(\cancel{\frac{\partial^2}{\partial z^2}}^{0}+\beta^2\right)A_z^t = -j\frac{\beta^2}{\omega\mu\varepsilon}A_z^t \qquad = +j\frac{\beta h I_e\sin\phi}{2}\frac{\partial}{\partial\rho}\left[H_1^{(2)}(\beta\rho)\right]$$

$$= -j\frac{\beta^2}{\omega\mu\varepsilon}\left(-j\frac{\beta\mu h I_e}{2}\right)H_1^{(2)}(\beta\rho)\sin\phi \qquad H_\phi = j\frac{\beta h I_e}{2}\sin\phi\left[\beta H_0^{(2)}(\beta\rho) - \frac{1}{\rho}H_1^{(2)}(\beta\rho)\right]$$

$$E_z = -\frac{\beta^2\eta h I_e}{2}H_1^{(2)}(\beta\rho)\sin\phi \qquad H_z = 0$$

$$E_z = 0 = -\frac{\beta^2\eta h I_e}{2}H_1^{(2)}(\beta\rho)\sin\phi = 0 \Rightarrow \phi = 0°, 180°.$$

11.6 Since a magnetic line source above a PMC is the dual of the solution of Problem 11.5 (electric line source above PEC), using duality

(a) $F_z^{(1)} = -j\frac{\varepsilon I_m}{4}H_0^{(2)}\left(\beta\sqrt{x^2+(y-h)^2}\right) = F_z^{(1)}(x, y-h)$

(b) $\lim_{h\to 0} F_z^t = -j\frac{\varepsilon\beta^2 h I_m}{2}H_1^{(2)}(\beta\rho)\sin\phi$

(c) $H_\rho = H_\phi = 0$

$$H_z = -\frac{\beta^2 h I_m}{2\eta}H_1^{(2)}(\beta\rho)\sin\phi \qquad H_z = 0 = \sin\phi = 0 \Rightarrow \phi = 0°, 180°$$

$\boxed{11.7}$

a. For a current element located at $x=0, y=0$, the potential is given by

$$A_z^{(0)} = -j\frac{\mu I}{4} H_0^{(2)}(\beta\rho)$$

For a current element located at $x=x', y=y'$, the potential is given by

$$A_z = -j\frac{\mu I}{4} H_0^{(2)}(\beta|\underline{\rho}-\underline{\rho}'|) = -j\frac{\mu I}{4} H_0^{(2)}\left(\beta\sqrt{(x-x')^2+(y-y')^2}\right)$$

Thus for $x'=+s/2, y'=0$ and with $+I$ current

$$A_z^{(1)} = -j\frac{\mu I}{4} H_0^{(2)}\left(\beta\sqrt{(x-x')^2+y^2}\right) = A_z^{(1)}(x-\tfrac{s}{2},y)$$

while for $x'=-s/2, y'=0$ and with $-I$ current

$$A_z^{(2)} = +j\frac{\mu I}{4} H_0^{(2)}\left(\beta\sqrt{(x+x')^2+y^2}\right) = -A_z^{(2)}(x+\tfrac{s}{2},y)$$

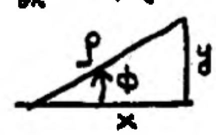
Therefore the total is equal to

$$A_z^t = A_z^{(1)}(x-\tfrac{s}{2},y) - A_z^{(2)}(x+\tfrac{s}{2},y) = -\left[A_z^{(0)}(x+\tfrac{s}{2},y) - A_z^{(0)}(x-\tfrac{s}{2},y)\right]$$

b.
$$\lim_{s\to 0} A_z^t = -\lim_{s\to 0}\left[A_z^{(0)}(x+\tfrac{s}{2},y) - A_z^{(0)}(x-\tfrac{s}{2},y)\right] = -s\lim_{s\to 0}\left[\frac{A_z^{(0)}(x+\frac{s}{2},y)-A_z^{(0)}(x-\frac{s}{2},y)}{s}\right]$$

$$= -s\frac{\partial A_z^{(0)}}{\partial x} = -s\frac{\partial}{\partial x}\left\{-j\frac{\mu I}{4}H_0^{(2)}(\beta\rho)\right\} = +j\frac{\mu s I}{4}\frac{\partial}{\partial x}H_0^{(2)}(\beta\rho)$$

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \rho}\frac{\partial \rho}{\partial x} = \frac{\partial\rho}{\partial x}\frac{\partial}{\partial\rho} \qquad \text{where } \rho = \sqrt{x^2+y^2}$$



$$\frac{\partial\rho}{\partial x} = \frac{1}{2}(x^2+y^2)^{-1/2}(2x) = \frac{x}{\sqrt{x^2+y^2}} = \frac{x}{\rho} = \cos\phi$$

$$\frac{\partial}{\partial\rho}H_0^{(2)}(\beta\rho) = -\beta H_1^{(2)}(\beta\rho)$$

Therefore

$$\lim_{s\to 0} A_z^t = j\frac{\mu s I}{4}\cos\phi\left[-\beta H_1^{(2)}(\beta\rho)\right] = -j\frac{\mu\beta s I}{4}H_1^{(2)}(\beta\rho)\cos\phi$$

c. For TM^z

$$E_\rho = -j\frac{1}{\omega\mu\varepsilon}\frac{\partial^2 A_z^t}{\partial\rho\partial z} = 0$$

$$E_\phi = -j\frac{1}{\omega\mu\varepsilon}\frac{1}{\rho}\frac{\partial^2 A_z^t}{\partial\phi\partial z} = 0$$

$$E_z = -j\frac{1}{\omega\mu\varepsilon}\left(\frac{\partial^2}{\partial z^2}+\beta^2\right)A_z^t = -j\frac{\beta^2}{\omega\mu\varepsilon}A_z^t$$

$$= -\frac{\beta^2 s I}{4\omega\varepsilon}H_1^{(2)}(\beta\rho)\cos\phi = -\frac{\beta^2\eta s I}{4}H_1^{(2)}(\beta\rho)\cos\phi$$

$$H_\rho = \frac{1}{\mu\rho}\frac{\partial A_z^t}{\partial\phi} = +j\frac{\beta s I}{4\rho}H_1^{(2)}(\beta\rho)\sin\phi$$

$$H_\phi = -\frac{1}{\mu}\frac{\partial A_z^t}{\partial\rho} = +j\frac{\beta s I}{4}\cos\phi\frac{\partial}{\partial\rho}H_1^{(2)}(\beta\rho)$$

$$= j\frac{\beta s I}{4}\cos\phi\left[\beta H_0^{(2)}(\beta\rho) - \frac{1}{\rho}H_1^{(2)}(\beta\rho)\right]$$

$$= j\frac{\beta s I}{4\rho}\cos\phi\left[(\beta\rho)H_0^{(2)}(\beta\rho) - H_1^{(2)}(\beta\rho)\right]$$

$$H_z = 0$$

11.8 Two magnetic line sources displaced symmetrically along the x-axis a distance s apart, as shown in Figure P11-7, is the dual of the solution of Problem 11.7 (2 electric line sources displaced symmetrically along the x-axis a distance s apart). Therefore using the solution of Problem 11.7 and duality, we can write:

(a) $F_z^t = F_z^{(1)}(x-\frac{s}{2},y) - F_z^{(2)}(x+\frac{s}{2},y) = -\left[F_z^{o}(x+\frac{s}{2},y) - F_z^{o}(x-\frac{s}{2},y)\right]$

(b) $\lim\limits_{s\to 0} F_z^t = -j\frac{\beta\epsilon s I_m}{4} H_1^{(2)}(\beta\rho)\cos\phi$

(c)
$$E_\rho = -j\frac{\beta s I_m}{4\rho}H_1^{(2)}(\beta\rho)\sin\phi \qquad H_\rho = 0$$
$$E_\phi = -j\frac{\beta s I_m}{4\rho}\cos\phi\left[(\beta\rho)H_0^{(2)}(\beta\rho) - H_1^{(2)}(\beta\rho)\right] \qquad H_\phi = 0$$
$$E_z = 0 \qquad H_z = -\frac{\beta^2 s I_m}{4\eta}H_1^{(2)}(\beta\rho)\cos\phi$$

11.9

The solution of this problem is identical to the solution of the equivalent problem of Problem 11.5 (an electric line source placed a height h above a PEC ground plane). Therefore using the solution of Problem 11.5 with $h \to s/2$, we can write:

(a) $A_z^{(1)} = -j\frac{\mu I_e}{4}H_0^{(2)}\left(\beta\sqrt{x^2+(y-s/2)^2}\right) = A_z^{(1)}(x, y-s/2)$

$$A_z^t = A_z^{(1)}(x, y-s/2) - A_z^{(2)}(x, y+s/2) = -\left[A_z^{o}(x, y+s/2) - A_z^{o}(x, y-s/2)\right]$$

(b) $A_z^t = -j\frac{\beta\mu s I_e}{4}H_1^{(2)}(\beta\rho)\sin\phi$

(c)
$$E_\rho = 0 \qquad H_\rho = -j\frac{\beta s I_e}{4\rho}H_1^{(2)}(\beta\rho)\cos\phi$$
$$E_\phi = 0 \qquad H_\phi = +j\frac{\beta s I_e}{4}\sin\phi\frac{\partial}{\partial\rho}\left[H_1^{(2)}(\beta\rho)\right]$$
$$E_z = -\frac{\beta^2\eta s I_e}{4}H_1^{(2)}(\beta\rho)\sin\phi \qquad = j\frac{\beta s I_e}{4}\sin\phi\left[\beta H_0^{(2)}(\beta\rho) - \frac{1}{\rho}H_1^{(2)}(\beta\rho)\right]$$
$$H_z = 0$$

$\boxed{11.10}$ The solution of this problem can be obtained from the solution of Problem 11.9 by using duality (since the two problems are the duals of each other). Therefore, using duality as outlined in Table 7-2 and the solution of Problem 11.9, we can write that:

(a) $F_z^{(1)} = -j\dfrac{\varepsilon I_m}{4} H_0^{(2)}\left(\beta\sqrt{x^2+(y-s/2)^2}\right) = {}_z^{(1)}(x, y-s/2)$

$$F_z^t = F_z^{(1)}(x, y-s/2) - F^{2}(x, y+s/2) = -\left[F_z^{(0)}(x, y+s/2) - F_z^{(0)}(x, y-s/2)\right]$$

(b) $F_z^t = -j\dfrac{\beta\varepsilon s I_m}{4} H_1^{(2)}(\beta\rho)\sin\phi$

(c) $E_\rho = j\dfrac{\beta s I_m}{4\rho} H_1^{(2)}(\beta\rho)\cos\phi$ $\qquad H_\rho = 0$

$H_\phi = 0$

$E_\phi = -j\dfrac{\beta s I_m}{4}\sin\phi\left[\beta H_0^{(2)}(\beta\rho) - \dfrac{1}{\rho}H_1^{(2)}(\beta\rho)\right]$

$E_z = 0$ $\qquad H_z = -\dfrac{\beta^2 s I_m}{4\eta} H_1^{(2)}(\beta\rho)\sin\phi$

$\boxed{11.11}$ a. Using the results from the solution of Problem 11.7, we can consider the two line sources along the negative x axis as one set similar to the one of Problem 11.7 and the two line sources along the positive x axis as another set similar to the one of Problem 11.7. Thus for each set

$$A_z^t(\text{left set}) = A_z^{t\ell} = -\frac{\mu\beta I s_1}{4j} H_1^{(2)}(\beta\rho)\cos\phi = -A_z^{o}(s_1)$$

$$A_z^t(\text{right set}) = A_z^{tr} = \frac{\mu\beta I s_1}{4j} H_1^{(2)}(\beta\rho)\cos\phi = A_z^{(o)}(s_1)$$

Since the two sets are displaced relative to each other along the x axis by a separation s_2, we can write that as $s_2 \to 0$ the total potential for all four sources (2 sets) can be written by treating each set as a source and using the results of Problem 11.7 as

$$A_z^t = A_z^{tr} + A_z^{tr} = A_z^{(o)}(x-\tfrac{s_2}{2},y) - A_z^{(o)}(x+\tfrac{s_2}{2},y) = -\left[A_z^{(o)}(x+\tfrac{s_2}{2},y) - A_z^{(o)}(x-\tfrac{s_2}{2},y)\right]$$

$$\lim_{s_2\to 0} A_z^t = s_2 \lim_{s_2\to 0}\left\{\frac{A_z^t}{s_2}\right\} = -s_2 \lim_{s_2\to 0}\left\{\frac{A_z^{(o)}(x+\frac{s_2}{2},y) - A_z^{(o)}(x-\frac{s_2}{2},y)}{s_2}\right\} = -s_2\frac{\partial A_z^{(o)}(s_1)}{\partial x}$$

$$= -s_2\frac{\partial}{\partial x}\left\{\frac{\mu\beta I s_1}{4j} H_1^{(2)}(\beta\rho)\cos(\phi)\right\} = -\frac{\mu\beta s_1 s_2 I}{4j}\frac{\partial}{\partial x}\left\{H_1^{(2)}(\beta\rho)\cos\phi\right\}$$

$$\lim_{s_2\to 0} A_z^t = -\frac{\mu\beta s_1 s_2 I}{4j}\left\{H_1^{(2)}(\beta\rho)\frac{\partial}{\partial x}\cos\phi + \cos\phi\frac{\partial}{\partial x}H_1^{(2)}(\beta\rho)\right\}$$

$$\frac{\partial}{\partial x}\cos\phi = \frac{\partial}{\partial x}\left\{\frac{x}{\sqrt{x^2+y^2}}\right\} = \frac{\partial}{\partial x}\left\{x(x^2+y^2)^{-1/2}\right\} = (x^2+y^2)^{-1/2} + x\left(-\tfrac{1}{2}\right)(x^2+y^2)^{-3/2}(2x)$$

$$= \frac{1}{\sqrt{x^2+y^2}} - \frac{x^2}{(x^2+y^2)^{3/2}} = \frac{1}{\rho} - \frac{(\rho\cos\phi)^2}{\rho^3} = \frac{1}{\rho} - \frac{\cos^2\phi}{\rho} = \frac{1-\cos^2\phi}{\rho} = \frac{\sin^2\phi}{\rho}$$

$$\frac{\partial}{\partial x}H_1^{(2)}(\beta\rho) = \frac{\partial\rho}{\partial x}\frac{\partial}{\partial\rho}H_1^{(2)}(\beta\rho) = \cos\phi\frac{\partial}{\partial\rho}H_1^{(2)}(\beta\rho) = \cos\phi\left[-\beta H_2^{(2)}(\beta\rho) + \frac{1}{\rho}H_1^{(2)}(\beta\rho)\right]$$

$$\lim_{s_2\to 0} A_z^t = -\frac{\mu\beta s_1 s_2 I}{4j}\left\{H_1^{(2)}(\beta\rho)\frac{\sin^2\phi}{\rho} + \cos^2\phi\left[-\beta H_2^{(2)}(\beta\rho) + \frac{1}{\rho}H_1^{(2)}(\beta\rho)\right]\right\}$$

$$= -\frac{\mu\beta s_1 s_2 I}{4j}\left\{\frac{H_1^{(2)}(\beta\rho)}{\rho} - \beta\cos^2\phi\, H_2^{(2)}(\beta\rho)\right\}$$

b. For TM^z

$$E_\rho = -j\frac{1}{\omega\mu\varepsilon}\frac{\partial^2 A_z^t}{\partial\rho\,\partial z} \qquad H_\rho = \frac{1}{\mu\rho}\frac{\partial A_z^t}{\partial\phi}$$

$$E_\phi = -j\frac{1}{\omega\mu\varepsilon}\frac{1}{\rho}\frac{\partial^2 A_z^t}{\partial\phi\,\partial z} \qquad H_\phi = -\frac{1}{\mu}\frac{\partial A_z^t}{\partial\rho}$$

$$E_z = -j\frac{1}{\omega\mu\varepsilon}\left(\frac{\partial^2}{\partial z^2}+\beta^2\right)A_z^t \qquad H_z = 0$$

11.12 a. Using the results from the solution of Problem 11.7, we can consider the two line sources along the negative y axis as one set similar to the one of Problem 11.7 and the two line sources along the positive y axis as another set similar to the one of Problem 11.7. Thus for each set

$$A_z^t(\text{upper}) = A_z^{tu} = \frac{\mu\beta I s}{4j} H_1^{(2)}(\beta\rho)\cos\phi = A_z^{(0)}(s)$$

$$A_z^t(\text{lower}) = A_z^{tl} = -\frac{\mu\beta I s}{4j} H_1^{(2)}(\beta\rho)\cos\phi = -A_z^{(0)}(s)$$

Since the two sets are displaced relative to each other along the y axis by a separation h, we can write that as $h\to 0$ the total potential for all four sources (2 sets) can be written by treating each set as a source and use a procedure similar to Problem 11.7. Therefore

$$A_z^t = A_z^{tl} + A_z^{tu} = -A_z^{(0)}\left(x, y+\frac{h}{2}\right) + A_z^{(0)}\left(x, y-\frac{h}{2}\right) = -\left[A_z^{(0)}\left(x,y+\frac{h}{2}\right) - A_z^{(0)}\left(x,y-\frac{h}{2}\right)\right]$$

$$\lim_{h\to 0} A_z^t = h\lim_{h\to 0}\left(\frac{A_z^t}{h}\right) = -h\lim_{h\to 0}\left\{\frac{A_z^{(0)}\left(x,y+\frac{h}{2}\right) - A_z^{(0)}\left(x,y-\frac{h}{2}\right)}{h}\right\} = -h\frac{\partial A_z^{(0)}(s)}{\partial y}$$

$$= -h\frac{\partial}{\partial y}\left\{\frac{\mu\beta I s}{4j} H_1^{(2)}(\beta\rho)\cos\phi\right\} = -\frac{\mu\beta h s I}{4j}\frac{\partial}{\partial y}\left\{H_1^{(2)}(\beta\rho)\cos\phi\right\}$$

$$\lim_{h\to 0} A_z^t = -\frac{\mu\beta h s I}{4j}\left\{H_1^{(2)}(\beta\rho)\frac{\partial}{\partial y}\cos\phi + \cos\phi\frac{\partial}{\partial y}H_1^{(2)}(\beta\rho)\right\}$$

$$\frac{\partial}{\partial y}\cos\phi = \frac{\partial}{\partial y}\left[\frac{x}{(x^2+y^2)^{1/2}}\right] = \frac{\partial}{\partial y}\left[x(x^2+y^2)^{-1/2}\right] = x\left(-\frac{1}{2}\right)(x^2+y^2)^{-3/2}(2y) = -\frac{xy}{(x^2+y^2)^{3/2}}$$

$$= -\frac{1}{\sqrt{x^2+y^2}}\frac{x}{\sqrt{x^2+y^2}}\frac{y}{\sqrt{x^2+y^2}} = -\frac{1}{\rho}\cos\phi\sin\phi = -\frac{\cos\phi\sin\phi}{\rho}$$

$$\frac{\partial}{\partial y}H_1^{(2)}(\beta\rho) = \frac{\partial\rho}{\partial y}\frac{\partial}{\partial\rho}H_1^{(2)}(\beta\rho) = \sin\phi\left[-\beta H_2^{(2)}(\beta\rho) + \frac{1}{\rho}H_1^{(2)}(\beta\rho)\right]$$

$$\lim_{h\to 0} A_z^t = -\frac{\mu\beta h s I}{4j}\left\{-\frac{H_1^{(2)}(\beta\rho)}{\rho}\cos\phi\sin\phi + \cos\phi\sin\phi\left[-\beta H_2^{(2)}(\beta\rho) + \frac{1}{\rho}H_1^{(2)}(\beta\rho)\right]\right\}$$

$$= \frac{\mu\beta^2 h s I}{4j}H_2^{(2)}(\beta\rho)\cos\phi\sin\phi = \frac{\mu\beta^2 h s I}{8j}H_2^{(2)}(\beta\rho)\sin(2\phi)$$

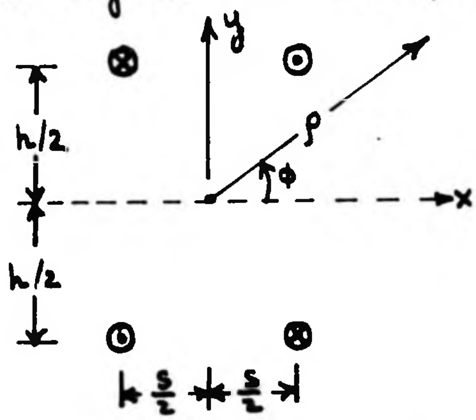
b. For TM^z

$$E_\rho = -j\frac{1}{\omega\mu\varepsilon}\frac{\partial^2 A_z^t}{\partial\rho\partial z} \qquad H_\rho = \frac{1}{\mu\rho}\frac{\partial A_z^t}{\partial\phi}$$

$$E_\phi = -j\frac{1}{\omega\mu\varepsilon}\frac{1}{\rho}\frac{\partial^2 A_z^t}{\partial\phi\partial z} \qquad H_\phi = -\frac{1}{\mu}\frac{\partial A_z^t}{\partial\rho}$$

$$E_z = -j\frac{1}{\omega\mu\varepsilon}\left(\frac{\partial^2}{\partial z^2}+\beta^2\right)A_z^t \qquad H_z = 0$$

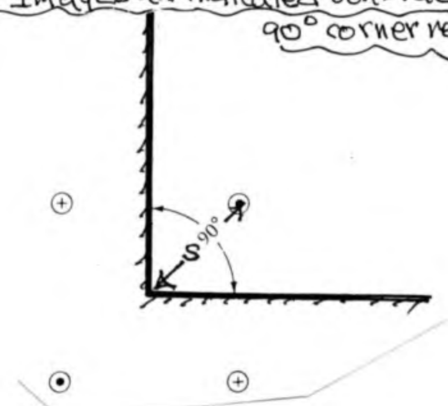
$\boxed{11.13}$ The solution to this problem can be obtained using image theory. Since below the interface the fields are zero, the solution is only valid above the interface. The equivalent model for the geometry of Problem 11.13 is shown here where we replace the conductor with two image sources. Then the solution to this equivalent is identical to that of Problem 11.12, and it is valid on and above the interface; below the interface the fields are zero.



$\boxed{11.14}$ The solution to this problem can be obtained using image theory. Since the

a. fields within the conductor are zero, the solution using image theory is only valid within the 90° sector. The equivalent model for the geometry of Problem 11.14 is shown here where we replace the conductor with three image sources. Then the solution to this equivalent is identical to that of Problem 11.12, and it is valid within the 90° sector; within the conductor the fields are zero. The only substitution that needs to be made is to replace s and h in the solution of Problem 11.12 by $1.414s = \sqrt{2}s$. Thus we can write the total potential as

[Three Images are indicated outside the 90° corner reflector]



$$A_z^t = \frac{(\sqrt{2})^2\beta^2 s^2 I}{j8} H_2^{(2)}(\beta\rho)\sin(2\phi) = \frac{(\beta s)^2 I}{j4} H_2^{(2)}(\beta\rho)\sin(2\phi)$$

b. For TM^z

$$E_\rho = \frac{1}{j\omega\mu\varepsilon}\frac{\partial^2 A_z^t}{\partial\rho\,\partial z} = 0 \qquad H_\rho = \frac{1}{\mu\rho}\frac{\partial A_z^t}{\partial\phi} = \frac{(\beta s)^2 I}{j2\mu\rho} H_2^{(2)}(\beta\rho)\cos(2\phi)$$

$$E_\phi = \frac{1}{j\omega\varepsilon\mu}\frac{1}{\rho}\frac{\partial^2 A_z^t}{\partial\phi\,\partial z} = 0 \qquad H_z = 0$$

$$E_z = \frac{1}{j\omega\mu\varepsilon}\left(\frac{\partial^2}{\partial z^2}+\beta^2\right)A_z^t = \frac{\beta^2}{j\omega\mu\varepsilon}A_z^t = -j\frac{\beta^2(\beta s)^2 I}{4\omega\mu\varepsilon}H_2^{(2)}(\beta\rho)\sin(2\phi)$$

$$H_\phi = -\frac{1}{\mu}\frac{\partial A_z^t}{\partial\rho} = \frac{(\beta s)^2 I}{j4\mu}\sin(2\phi)\left[\beta H_1^{(2)}(\beta\rho) - \frac{2}{\rho}H_2^{(2)}(\beta\rho)\right]$$

$0 \le \phi \le 90°$

c. $E_z = 0 = -j\dfrac{\beta^2(\beta s)^2 I}{4\omega\mu\varepsilon}H_2^{(2)}(\beta\rho)\sin(2\phi) \Rightarrow \sin(2\phi) = 0 \Rightarrow 2\phi = \sin^{-1}(0) = 0, \pi$

$\phi = 0, \frac{\pi}{2}$

11.15 The parallel slots can be replaced by equivalent magnetic current densities. Using the geometry of Figure P11-15, we represent each of the slots with magnetic current densities of

$x = s/2$: $\underline{M}_1 = -2\hat{n} \times \underline{E}_{a1} = -2\hat{a}_y \times (-\hat{a}_x E_0) = -\hat{a}_z 2E_0$

$x = -s/2$: $\underline{M}_2 = -2\hat{n} \times \underline{E}_{a2} = -2\hat{a}_y \times (\hat{a}_x E_0) = +\hat{a}_z 2E_0$

Each slot can then be modeled with magnetic line sources (with TE^z modes) with equivalent magnetic currents of

$x = s/2$: $I_{m1} = -2E_0$

$x = -s/2$: $I_{m2} = 2E_0$

(a) The equivalent problem is then identical to Problem 11.8. Therefore, based on the solution of Problem 11.8, we can write

$$F_z^t = F_z^{(1)}(x - s/2, y) - F_z^{(2)}(x + s/2, y) = -\left[F_z^{(0)}(x + s/2, y) - F_z^{0}(x - s/2, y)\right]$$

(b) $$\lim_{s \to 0} F_z^t = -j\frac{\varepsilon\beta s(-2E_0)}{4} H_1^{(2)}(\beta\rho)\cos\phi = j\frac{\varepsilon\beta s E_0}{2} H_1^{(2)}(\beta\rho)\cos\phi$$

(c) $$E_\rho = +j\frac{\beta^2 s E_0}{2\rho} H_1^{(2)}(\beta\rho)\cos\phi \qquad H_\rho = 0$$

$$E_\phi = j\frac{\beta s E_0}{2\rho}\cos\phi\left[(\beta\rho)H_0^{(2)}(\beta\rho) - H_1^{(2)}(\beta\rho)\right] \qquad H_\phi = 0$$

$$E_z = 0 \qquad H_z = \frac{\beta^2 s E_0}{2\eta} H_1^{(2)}(\beta\rho)\cos\phi$$

11.16

Based on the geometry of Figure 11-2 and (11-20a), we can write the total field above the ground plane as (for $\rho = \rho_0$)

$$E_z^t = j\eta I_e\sqrt{\frac{j\beta}{2\pi}}\sin(\beta h\sin\phi)\frac{e^{-j\beta\rho_0}}{\sqrt{\rho_0}}$$

which vanishes when (for $\rho = \rho_0$)

$$\sin(\beta h\sin\phi) = 0 \Rightarrow \beta h\sin\phi = \sin^{-1}(0) = n\pi, \quad n = 0, 1, 2, \ldots$$

$$h = \frac{n\pi}{\beta\sin\phi} = \frac{n\pi}{(2\pi/\lambda)\sin\phi}$$

The smallest height for $h \neq 0$, $n = 1$. Thus the smallest height

$$h = \left.\frac{\pi\lambda}{2\pi\sin\phi}\right|_{\phi=60^\circ} = \left.\frac{\lambda}{2\sin\phi}\right|_{\phi=60^\circ} = \frac{\lambda}{2(\sqrt{3}/2)} = \frac{\lambda}{\sqrt{3}}$$

$$h = 0.5774\lambda$$

11.17 The normalized field for Problem 11.16 is represented by

$$E^{t}_{zn} = \sin(\beta h \sin\phi)$$

whose maximum value is 1 (or 0 dB). For the field to be −3dB from the maximum, its dimensionless value at that point is $1/\sqrt{2} = 0.70711$.

To determine the 2 smallest height that will result for the field to be 0.70711 (or −3 dB) at $\phi = 60°$, we set the normalized field to

$$\sin(\beta h \sin\phi) = 0.70711 \Rightarrow \beta h \sin\phi = \sin^{-1}(0.70711) = \frac{\pi}{4}, \frac{3\pi}{4}, \cdots$$

therefore $h = \left.\frac{n\pi/4}{\beta\sin\phi}\right|_{\phi=60°} = \left.\frac{n\pi/4}{(2\pi/\lambda)\sin\phi}\right|_{\phi=60°} = \left.\frac{n\lambda}{4\sin\phi}\right|_{\phi=60°} = \frac{n\lambda}{4\left(\frac{\sqrt{3}}{2}\right)} = \frac{n\lambda}{\sqrt{3}}$ $\quad n = 1, 3, 5, \ldots$

$n = 1$: $h_1 = \frac{\lambda}{\sqrt{3}} = 0.5774\lambda$

$n = 3$: $h_3 = \frac{3\lambda}{\sqrt{3}} = 1.7321\lambda$

11.18

(a) From Example 11-2

$$\underline{H}^{i} = \hat{a}_z H_0 e^{j\beta(x\cos\phi_i + y\sin\phi_i)}$$

(b) The reflected magnetic field, in terms of the incident magnetic field H_0, can be written as [using Figure 11-4(c)]

$$\underline{H}^{r} = -\hat{a}_z \Gamma H_0 e^{-j\beta(x\cos\phi_r + y\sin\phi_r)}$$

(c) Using Snell's law of reflection ($\phi_i = \pi - \phi_r$), and that the tangential components of the magnetic field vanish on the PMC surface ($y=0$), we have that

$$H_0 e^{j\beta x\cos(\pi-\phi_r)} - \Gamma H_0 e^{-j\beta\cos\phi_r} = 0$$

$$e^{-j\beta x\cos\phi_r} - \Gamma e^{-j\beta\cos\phi_r} = 0$$

$$1 - \Gamma = 0 \Rightarrow \boxed{\Gamma = +1} \text{ for a PMC surface.}$$

11.19 For normal incidence, the 3-D RCS of a flat plate is equal to

$$\sigma_{3\text{-}D} = 4\pi\left(\frac{A}{\lambda}\right)^2 \quad \text{where } A = \text{area of the flat plate}$$

(a) TE^x: The bistatic 3-D RCS at any observation angle is given by (11-44a), or

$$\sigma_{3\text{-}D} = 4\pi\left(\frac{A}{\lambda}\right)^2 \left\{ \cos^2\theta_s \frac{\sin\left[\frac{\beta b}{2}(\sin\theta_s - \sin\theta_i)\right]}{\frac{\beta b}{2}(\sin\theta_s - \sin\theta_i)} \right\}$$

To find its maximum, we will assume the sinc function varies much faster than the $\cos^2\theta_s$ function, whis is usually the case, so that the $\cos^2\theta_s$ is basically a constant at the maximum of the sinc.

The maximum value of the sinc function occurs when its argument is zero; in this case $\sin\theta_s = \sin\theta_i \Rightarrow \theta_s = \theta_i$ (specular direction). Therefore for the 3-D bistatic RCS to be reduced by 10 dB along the specular direction ($\theta_s = \theta_i$), then the cosine squared function must be equal to

$$\cos^2\theta_s\Big|_{\theta_s = \theta_i} = \cos^2\theta_i = 0.1 \quad (-10\,dB)$$

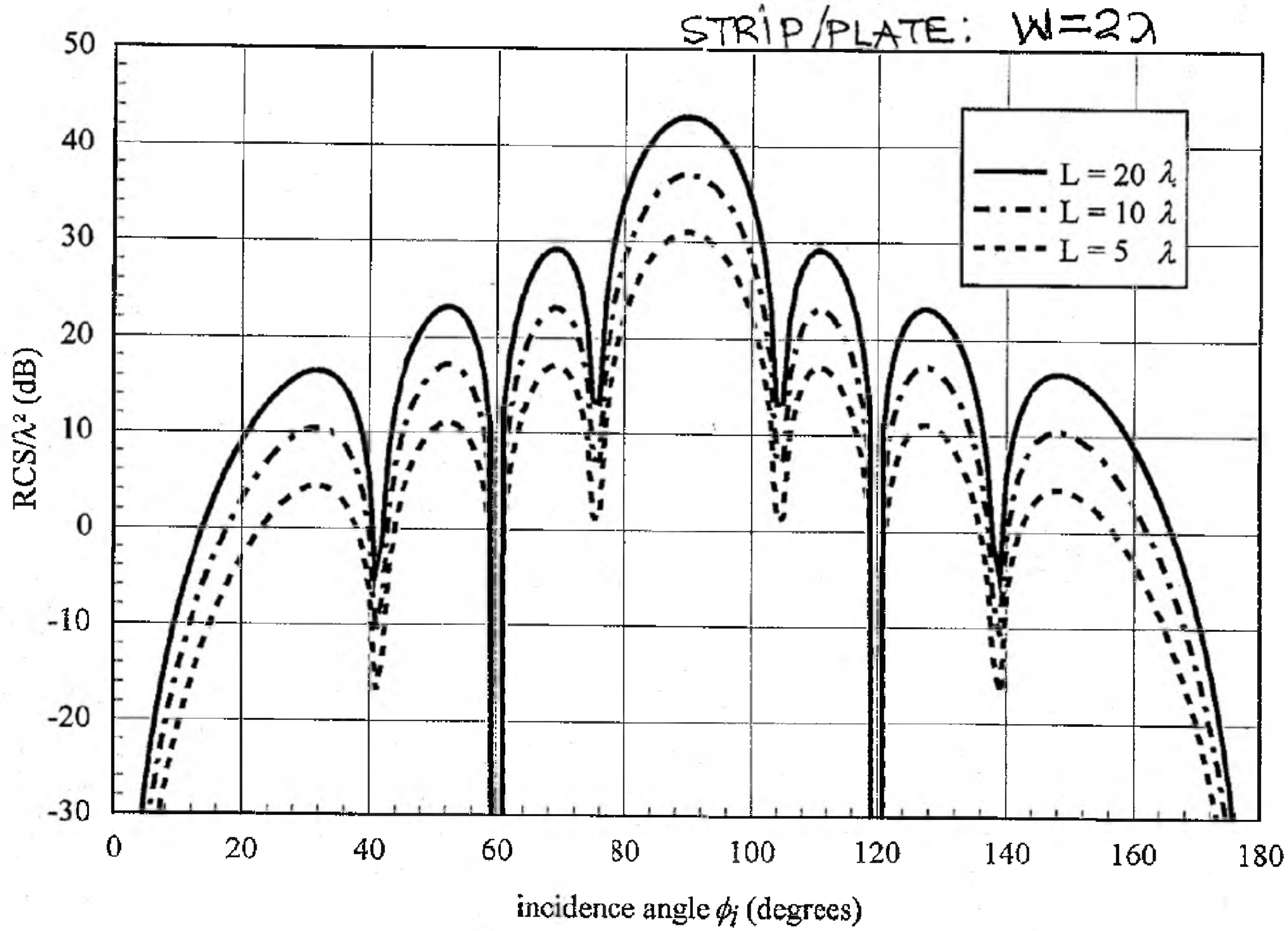
$$\theta_i = \cos^{-1}(0.1)^{1/2} = \cos^{-1}(0.3162) = 71.565^\circ$$

(b) TM^x: Similarly for the TM^x polarization, the 3-D RCS is given (from Example 11-3) by (selecting $\phi_s = 90^\circ$)

$$\sigma_{3\text{-}D} = 4\pi\left(\frac{A}{\lambda}\right)^2 \left[\cos^2\theta_i \frac{\sin\left[\frac{\beta b}{2}(\sin\theta_s - \sin\theta_i)\right]}{\frac{\beta b}{2}(\sin\theta_s - \sin\theta_i)} \right]$$

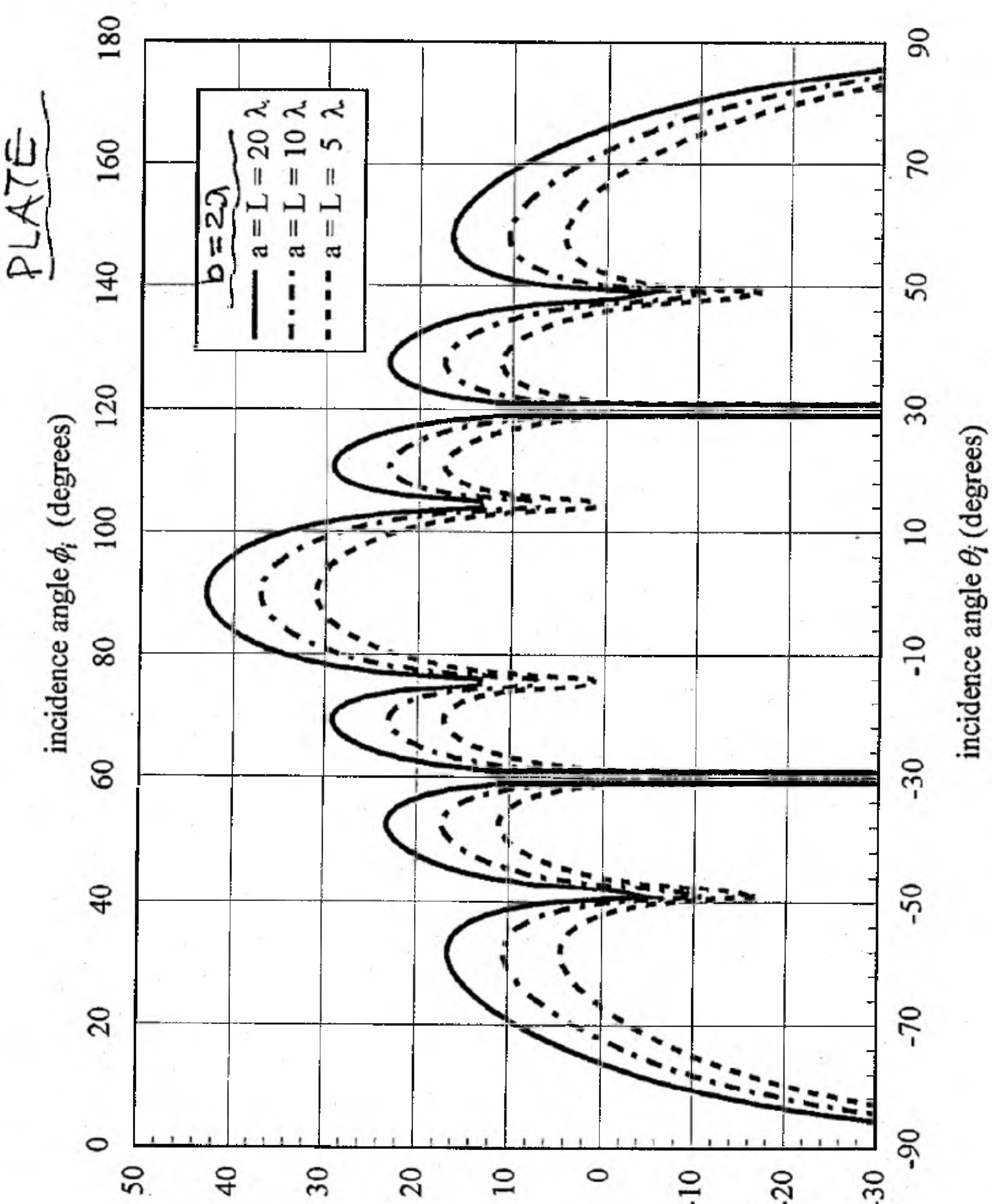
which is identical to the one for the TE^x polarization, except that the cosine squared function is $\cos^2\theta_i$ (instead of $\cos^2\theta_s$ for TE^x). However, again, to maximize the sinc function, $\theta_s = \theta_i$. Therefore to reduce the maximum 3-D RCS toward the specular direction ($\theta_s = \theta_i$) by $-10\,dB$, the incidence angle $\theta_i = 71.565^\circ$, which is the same as for TE^x.

11.20



Normalized, monostatic radar cross section RCS/λ^2 versus incidence angle ϕ_i

11.21



Normalized, monostatic radar cross section RCS/λ^2 versus incidence angle for strip ϕ_i and plate θ_i

11.22

The maximum monostatic 3-D RCS of a flat plate is given by

$$\sigma_{3-D} = 4\pi\left(\frac{A}{\lambda}\right)^2 = 4\pi\frac{A^2}{\lambda^2}, \text{ where } A = \text{area of plate}$$

If the area of the plate is represented as

$$A = n\lambda^2$$

then the σ_{3-D} can be written as

$$\sigma_{3-D} = 4\pi\left(\frac{n^2\lambda^4}{\lambda^2}\right) = 4\pi n^2\lambda^2$$

so that its normalized expression be represented by

$$\frac{\sigma_{3-D}}{\lambda^2} = 4\pi n^2$$

• If it is desired to have the normalized 3-D RCS to be +20 dB (a factor of 100), then

$$\frac{\sigma_{3-D}}{\lambda^2} = 4\pi n^2 = 100 \Rightarrow n^2 = \frac{100}{4\pi} \Rightarrow n = \frac{10}{2\sqrt{\pi}} = 2.8209$$

Thus the flat plate must be $2.8209\lambda^2$ in order for its normalized maximum monostatic RCS to be +20 dB.

• At $f = 10$ GHz $\Rightarrow \lambda = 3\times10^8/10\times10^9 = 0.03$ meters

Thus σ_{3-D} is equal to

$$\sigma_{3-D} = 4\pi\left(\frac{A}{\lambda}\right)^2\Bigg|_{\substack{A = 2.8209\lambda^2 \\ \lambda = 0.03\text{ m}}} = 4\pi(7.9575\lambda^2) = 4\pi(7.9575)(0.03)^2$$

$$\sigma_{3-D} = 0.09 \text{ m}^2$$

which normalized to 1 m² is equal to

$$\sigma_{3-D} = 0.09 = 10\log_{10}(0.09) = \boxed{-10.458 \text{ dBsm}}$$

$\boxed{11.23}$ The field scattered by a two-dimensional scatterer when it is illuminated by a plane wave at normal incidence can be written as

$$\underline{E}^s(\rho,\phi,\phi') \sqrt{\frac{2}{\pi\beta\rho}}\, e^{-j(\beta\rho-\frac{\pi}{4})}\, \underline{E}_2^s(\phi,\phi') \qquad (1)$$

with the corresponding 2-D RCS expressed as

$$\sigma_{2\text{-}D} = \lim_{\rho\to\infty}\left\{2\pi\rho\frac{|\underline{E}^s|^2}{|\underline{E}^i|^2}\right\} = \frac{4}{\beta}\left|\underline{E}_2^s(\phi,\phi')\right|^2 = \frac{2\lambda}{\pi}\left|\underline{E}_2^s(\phi,\phi')\right|^2 \qquad (2)$$

In a similar manner for a three-dimensional scatterer, the scattered field is in the same plane as the 2-D fields

$$\underline{E}^s(r,\phi,\phi') \sim \frac{e^{-j\beta r}}{\beta r}\underline{E}_3^s(\phi,\phi') \qquad (3)$$

with its corresponding RCS written as

$$\sigma_{3\text{-}D} = \lim_{r\to\infty}\left\{4\pi r^2\frac{|\underline{E}^s|^2}{|\underline{E}^i|^2}\right\} = \frac{4\pi}{\beta^2}\left|\underline{E}_3^s(\phi,\phi')\right|^2 = \frac{\lambda^2}{\pi}\left|\underline{E}_3^s(\phi,\phi')\right|^2 \qquad (4)$$

Now let us assume that the 3-D scatterer is segmented into 2-D strips and the currents in the 3-D scatterer are identical to those of the 2-D when observations are made in a plane perpendicular to the scatterer. Then the scattered field can be written as a summation or integral; thus

$$\underline{E}^s = C\iint \underline{J}(r')\frac{e^{-j\beta R}}{R}ds' \simeq C\frac{e^{-j\beta r}}{r}\iint \underline{J}(r')e^{j\beta\hat{a}_r\cdot\underline{r}'}ds' \Rightarrow \underline{E}_3^s = \beta C\iint \underline{J}(r')e^{j\beta\hat{a}_r\cdot\underline{r}'}ds' \qquad (5)$$

If $\underline{J}$ is independent of z (the axial coordinate), the integration with respect to z can be carried out and we can write for a scatterer of length L as

$$\underline{E}_3^s = \beta C\iint \underline{J}(r')e^{j\beta\hat{a}_r\cdot\underline{r}'}ds' = \beta L C\int \underline{J}(\rho')e^{j\beta\hat{a}_\rho\cdot\underline{\rho}'}d\rho' \qquad (6)$$

If the z integration is carried out in

$$\underline{E}^s = C\iint \underline{J}(r')\frac{e^{-j\beta R}}{R}ds' = -j\pi C\int \underline{J}(\rho')H_0^{(2)}(\beta|\underline{\rho}-\underline{\rho}'|)d\rho'$$

$$\simeq -j\pi C\, e^{-j(\beta r-\frac{\pi}{4})}\sqrt{\frac{2}{\pi\beta r}}\int \underline{J}(\rho')\, e^{+j\beta\hat{a}_\rho\cdot\underline{\rho}'}d\rho' \qquad (7)$$

which when compared with (1) we can write

$$\underline{E}_2^s(\phi,\phi') = -j\pi C\int \underline{J}(\rho')e^{+j\beta\hat{a}_\rho\cdot\underline{\rho}'}d\rho' \qquad (8)$$

By comparing (6) and (8)

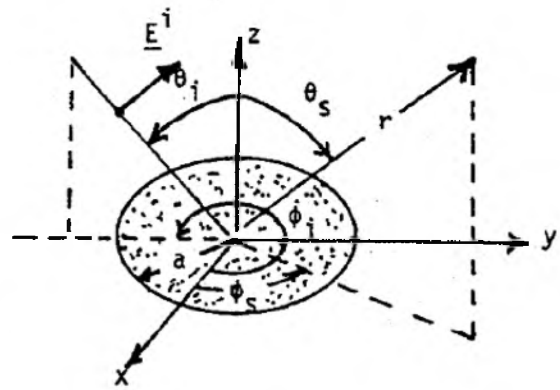
$$\underline{E}_3^s = j\frac{\beta L}{\pi}\underline{E}_2 \qquad (9)$$

Now using (9) we can write (4) as

$$\sigma_{3\text{-}D} = \frac{\lambda^2}{\pi}\frac{(\beta L)^2}{\pi^2}|\underline{E}_2^s|^2 = \frac{4L^2}{\pi}|\underline{E}_2^s|^2 = \frac{4L^2}{\pi}\left(\frac{\pi}{2\lambda}\right)\left[\frac{2\lambda}{\pi}|\underline{E}_2^s|^2\right] = \frac{2L^2}{\lambda}\sigma_{2\text{-}D}$$

$\boxed{11.24}$

a. $\underline{E}^i = E_0(\hat{a}_y \cos\theta_i + \hat{a}_z \sin\theta_i) e^{-j\underline{\beta}\cdot\underline{r}_i}$



$$\underline{E}^i = E_0(\hat{a}_y \cos\theta_i + \hat{a}_z \sin\theta_i) \cdot$$
$$\cdot e^{-j\beta(\hat{a}_y \sin\theta_i - \hat{a}_z\cos\theta_i)\cdot(\hat{a}_x x + \hat{a}_y y + \hat{a}_z z)}$$

$$\underline{E}^i = E_0(\hat{a}_y \cos\theta_i + \hat{a}_z \sin\theta_i) e^{-j\beta(y\sin\theta_i - z\cos\theta_i)}$$

$$\underline{H}^i = \hat{a}_x \frac{E_0}{\eta} e^{-j\beta(y\sin\theta_i - z\cos\theta_i)}$$

$$\underline{J}_s = 2\hat{n}\times\underline{H}^i\Big|_{\substack{z=0\\y=y'}} = 2\hat{a}_z\times\hat{a}_x \frac{E_0}{\eta} e^{-j\beta y'\sin\theta_i} = \hat{a}_y \frac{2E_0}{\eta} e^{-j\beta y'\sin\theta_i}$$

$$J_x = J_z = 0, \quad J_y = \frac{2E_0}{\eta} e^{-j\beta y'\sin\theta_i}$$

b. $$N_\theta = \iint_S [J_x \cos\theta_s\cos\phi_s + J_y\cos\theta_s\sin\phi_s - J_z\sin\theta_s] e^{j\beta r'\cos\psi} ds'$$

$$N_\phi = \iint_S [-J_x\sin\phi_s + J_y\cos\phi_s] e^{j\beta r'\cos\psi} ds'$$

$r'\cos\psi = \rho'\sin\theta_s\cos(\phi_s - \phi') = x'\sin\theta_s\cos\phi_s + y'\sin\theta_s\sin\phi_s$ [see (6-132a)]

$ds' = \rho' d\rho' d\phi'$ [see (6-132b)]

Therefore

$$N_\theta = \iint_S J_y \cos\theta_s\sin\phi_s e^{j\beta[x'\sin\theta_s\cos\phi_s + y'\sin\theta_s\sin\phi_s]} \rho' d\rho' d\phi'$$

$$N_\theta = \frac{2E_0}{\eta}\cos\theta_s\sin\phi_s \int_0^{2\pi}\int_0^a e^{j\beta[x'\sin\theta_s\cos\phi_s + y'(\sin\theta_s\sin\phi_s - \sin\theta_i)]} \rho' d\rho' d\phi'$$

$$\begin{cases} x' = \rho'\cos\phi' \\ y' = \rho'\sin\phi' \end{cases}$$

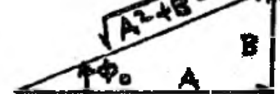
Thus $x'\sin\theta_s\cos\phi_s + y'(\sin\theta_s\sin\phi_s - \sin\theta_i) = \rho'\{\underbrace{\sin\theta_s\cos\phi_s}_{A}\cos\phi' + \underbrace{(\sin\theta_s\sin\phi_s - \sin\theta_i)}_{B}\sin\phi'\}$

$$= \rho'[A\cos\phi' + B\sin\phi']$$

$$x'\sin\theta_s\cos\phi_s + y'(\sin\theta_s\sin\phi_s - \sin\theta_i) = \rho'\sqrt{A^2+B^2}\left[\frac{A}{\sqrt{A^2+B^2}}\cos\phi' + \frac{B}{\sqrt{A^2+B^2}}\sin\phi'\right]$$

$A = \sin\theta_s\cos\phi_s$

$B = \sin\theta_s\sin\phi_s - \sin\theta_i$



$\cos\phi_0 = \dfrac{A}{\sqrt{A^2+B^2}}$

$\sin\phi_0 = \dfrac{B}{\sqrt{A^2+B^2}}$

$$x'\sin\theta_s\cos\phi_s + y'(\sin\theta_s\sin\phi_s - \sin\theta_i) = \rho'\sqrt{A^2+B^2}[\cos\phi'\cos\phi_0 + \sin\phi'\sin\phi_0]$$
$$= \rho'\sqrt{A^2+B^2}\cos(\phi' - \phi_0)$$

cont'd.

$\boxed{11.24\text{ cont'd.}}$ $N_\theta = \frac{2E_0}{\eta}\cos\theta_s\sin\phi_s\int_0^a\left\{\int_0^{2\pi} e^{j\beta\rho'\sqrt{A^2+B^2}\cos(\phi'-\phi_0)}d\phi'\right\}\rho' d\rho'$

$$N_\theta = \frac{2E_0}{\eta}\cos\theta_s\sin\phi_s\int_0^a [I]\rho' d\rho'$$

where $I = \int_0^{2\pi} e^{j\beta\sqrt{A^2+B^2}\rho'\cos(\phi'-\phi_0)}d\phi'$

The function $e^{j\beta\sqrt{A^2+B^2}\rho'\cos(\phi'-\phi_0)}$ is a periodic function of ϕ' with a period of 2π. Therefore the integration over one period is the same no matter where the integration begins. That is

$$\int_{\phi_0}^{2\pi+\phi_0} e^{j\beta\sqrt{A^2+B^2}\rho'\cos(\phi-\phi')}d\phi' = \int_0^{2\pi} e^{j\beta\sqrt{A^2+B^2}\rho'\cos(\phi'-\phi_0)}d\phi' = 2\pi J_0\left(\beta\sqrt{A^2+B^2}\rho'\right)$$

Thus

$$N_\theta = \frac{2E_0}{\eta}\cos\theta_s\sin\phi_s\int_0^a 2\pi J_0\left(\beta\sqrt{A^2+B^2}\rho'\right)\rho' d\rho' = \frac{2\pi E_0}{\eta}\cos\theta_s\sin\phi_s\int_0^a J_0\left(\beta\sqrt{A^2+B^2}\rho'\right)\rho' d\rho'$$

Let $z = \beta\sqrt{A^2+B^2}\rho' \Rightarrow dz = \beta\sqrt{A^2+B^2}d\rho'$

$\rho' = z/\beta\sqrt{A^2+B^2} \Rightarrow d\rho' = dz/\beta\sqrt{A^2+B^2}$

$$N_\theta = \frac{4\pi E_0}{\eta}\cos\theta_s\sin\phi_s\int_0^{\beta\sqrt{A^2+B^2}a}\frac{J_0(z)\,z\,dz}{\left(\beta\sqrt{A^2+B^2}\right)^2} = \frac{4\pi E_0}{\eta\left(\beta\sqrt{A^2+B^2}\right)^2}\cos\theta_s\sin\phi_s\int_0^{\beta\sqrt{A^2+B^2}a} z J_0(z)\,dz$$

Since

$$\int_0^\gamma z J_0(z)\,dz = \gamma J_1(\gamma)$$

then

$$N_\theta = \frac{4\pi E_0}{\eta}\cos\theta_s\sin\phi_s\frac{\beta\sqrt{A^2+B^2}a}{\left(\beta\sqrt{A^2+B^2}\right)^2}J_1\left(\beta a\sqrt{A^2+B^2}\right) = \frac{4\pi a^2}{\eta}\cos\theta_s\sin\phi_s\left[\frac{J_1\left(\beta a\sqrt{A^2+B^2}\right)}{\beta a\sqrt{A^2+B^2}}\right]$$

Similarly

$$N_\phi = \iint_S J_y\cos\phi_s e^{j\beta\rho'\sin\theta_s\cos(\phi_s-\phi')}\rho' d\rho' d\phi' = \frac{4\pi a^2 E_0}{\eta}\cos\phi_s\left[\frac{J_1\left(\beta a\sqrt{A^2+B^2}\right)}{\beta a\sqrt{A^2+B^2}}\right]$$

Thus

$$E_\theta^s = -j\frac{\beta e^{-j\beta r}}{4\pi r}\left[\cancel{L_\phi}^{\,0}+\eta N_\theta\right] = -j\eta\frac{\beta e^{-j\beta r}}{4\pi r}N_\theta = -j\frac{\beta a^2 E_0 e^{-j\beta r}}{r}\cos\theta_s\sin\phi_s\frac{J_1\left(\beta a\sqrt{A^2+B^2}\right)}{\beta a\sqrt{A^2+B^2}}$$

$$E_\phi^s = j\frac{\beta e^{-j\beta r}}{4\pi r}\left[\cancel{L_\theta}^{\,0}-\eta N_\phi\right] = -j\eta\frac{\beta e^{-j\beta r}}{4\pi r}N_\phi = -j\frac{\beta a^2 E_0 e^{-j\beta r}}{r}\cos\phi_s\frac{J_1\left(\beta a\sqrt{A^2+B^2}\right)}{\beta a\sqrt{A^2+B^2}}$$

c.

$$|E^s| = \sqrt{|E_\theta^s|^2+|E_\phi^s|^2} = E_0\frac{\beta a^2}{r}\left\{\sqrt{\cos^2\theta_s\sin^2\phi_s+\cos^2\phi_s}\left|\frac{J_1\left(\beta a\sqrt{A^2+B^2}\right)}{\beta a\sqrt{A^2+B^2}}\right|\right\} = \frac{\beta a^2}{r}|F|E_0$$

$$\sigma_{3\text{-}D} = \lim_{r\to\infty}\left\{4\pi r^2\frac{|E^s|^2}{|E^i|^2}\right\} = 4\pi r^2\left(\frac{\beta a^2}{r}\right)^2|F|^2 = 4\pi\left(\frac{\pi a^2}{\lambda}\right)^2|2F|^2$$

where $|F| = \sqrt{\cos^2\theta_s\sin^2\phi_s+\cos^2\phi_s}\left|\frac{J_1\left(\beta a\sqrt{A^2+B^2}\right)}{\beta a\sqrt{A^2+B^2}}\right|$

For monostatic observations $\phi_s = 3\pi/2$, $\theta_s = \theta_i$.

$A = \sin\theta_s\cos\phi_s = 0$

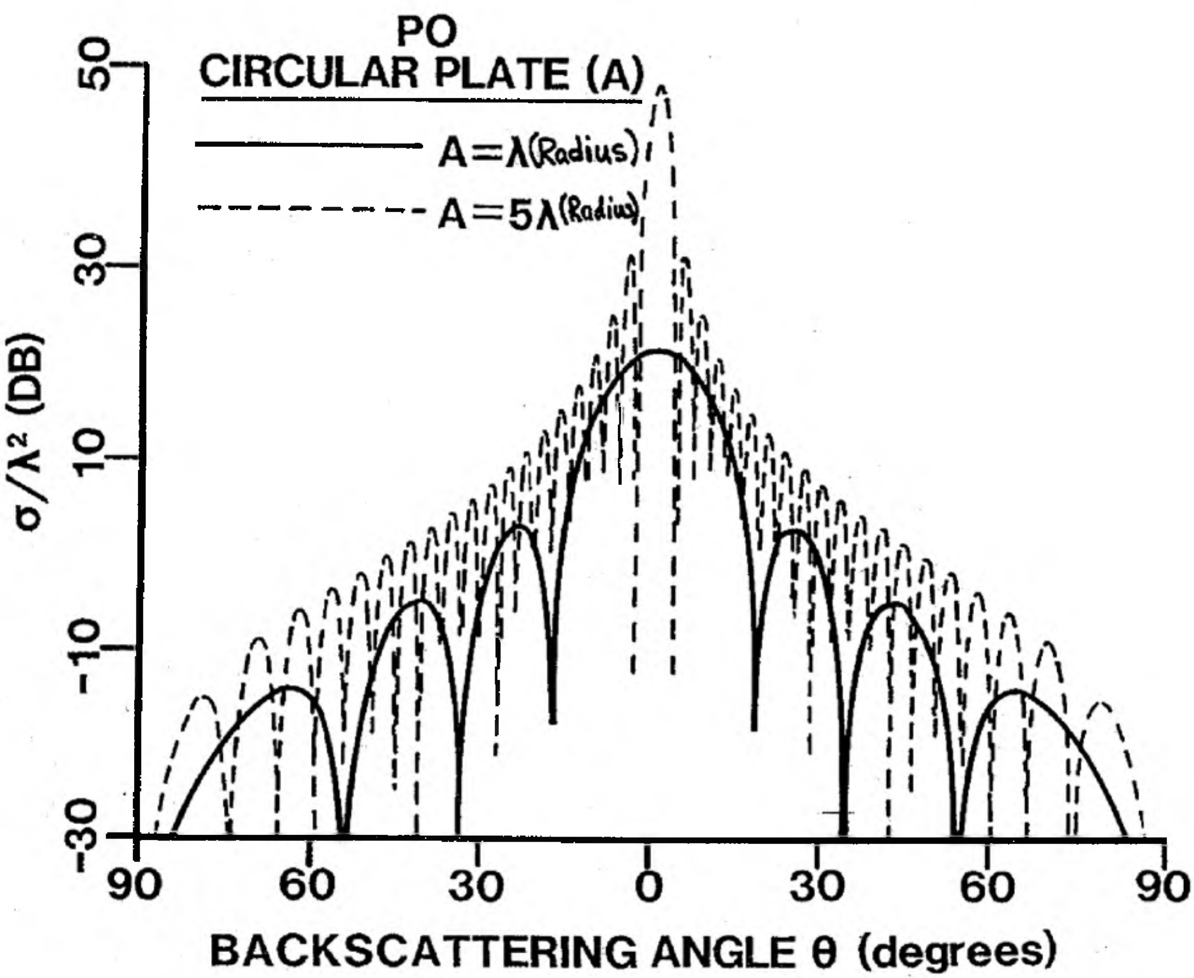
$B = \sin\theta_s\sin\phi_s - \sin\theta_i = \sin\theta_i(-1)-\sin\theta_i = -2\sin\theta_i$ $\Big\}\sqrt{A^2+B^2} = 2\sin\theta_i$

cont'd.

$\boxed{11.24\text{ cont'd.}}$ Thus

$$E_\theta^s = j\frac{\beta a^2 E_0 e^{-j\beta r}}{r}\cos\theta_i \frac{J_1(2\beta a\sin\theta_i)}{2\beta a\sin\theta_i}, \quad E_\phi^s = 0$$

$$\sigma_{3\text{-}D} = \lim_{r\to\infty}\left\{4\pi r^2 \frac{|E_\theta^s|^2}{|E^i|^2}\right\} = 4\pi(\beta a^2)^2\left[\cos\theta_i \frac{J_1(2\beta a\sin\theta_i)}{2\beta a\sin\theta_i}\right]^2 = 4\pi\left(\frac{\pi a^2}{\lambda}\right)^2\left[2\cos\theta_i \frac{J_1(2\beta a\sin\theta_i)}{2\beta a\sin\theta_i}\right]^2$$



11.25 Using the geometry for the solution of Problem 11.24 but for TM^x plane wave incidence, we can write

a.
$$\underline{E}^i = \hat{a}_x E_0 e^{-j\beta(y\sin\theta_i - z\cos\theta_i)}, \quad \underline{H}^i = \frac{E_0}{\eta}(-\hat{a}_y\cos\theta_i - \hat{a}_z\sin\theta_i)e^{-j\beta(y\sin\theta_i - z\cos\theta_i)}$$

$$\underline{J}_s = 2\hat{n}\times\underline{H}^i\Big|_{z=0,\,y=y'} = 2\hat{a}_z\times(-\hat{a}_y H_y - \hat{a}_z H_z) = \hat{a}_x\frac{2E_0}{\eta}\cos\theta_i e^{-j\beta y'\sin\theta_i}$$

$$J_y = J_z = 0, \quad J_x = \frac{2E_0}{\eta}\cos\theta_i e^{-j\beta y'\sin\theta_i}$$

b. From the solution of Problem 11.24

$$N_\theta = \iint_S J_x\cos\theta_s\cos\phi_s e^{j\beta r'\cos\psi}ds' = \frac{2E_0}{\eta}\cos\theta_i\cos\theta_s\cos\phi_s\int_0^{2\pi}\int_0^a e^{-j\beta y'}e^{j\beta(x'\sin\theta_s\cos\phi_s + y'\sin\theta_s\sin\phi_s)}\rho'd\rho'd\phi'$$

$$N_\phi = \iint_S -J_x\sin\phi_s e^{j\beta r'\cos\psi}ds' = -\frac{2E_0}{\eta}\cos\theta_i\sin\phi_s\int_0^{2\pi}\int_0^a e^{-j\beta y'}e^{j\beta(x'\sin\theta_s\cos\phi_s + y'\sin\theta_s\sin\phi_s)}\rho'd\rho'd\phi'$$

Using the integration technique for the solution of Problem 11.24 we can write

$$N_\theta = \frac{4\pi a^2 E_0}{\eta}\cos\theta_i\cos\theta_s\cos\phi_s\frac{J_1(\beta a\sqrt{A^2+B^2})}{\beta a\sqrt{A^2+B^2}}, \quad N_\phi = -\frac{4\pi a^2 E_0}{\eta}\cos\theta_i\sin\phi_s\frac{J_1(\beta a\sqrt{A^2+B^2})}{\beta a\sqrt{A^2+B^2}}$$

$$E_\theta^s = -j\frac{\beta a^2 E_0 e^{-j\beta r}}{r}\cos\theta_i\cos\theta_s\cos\phi_s\frac{J_1(\beta a\sqrt{A^2+B^2})}{\beta a\sqrt{A^2+B^2}}, \quad E_\phi^s = j\frac{\beta a^2 E_0 e^{-j\beta r}}{r}\cos\theta_i\sin\phi_s\frac{J_1(\beta a\sqrt{A^2+B^2})}{\beta a\sqrt{A^2+B^2}}$$

$$A = \sin\theta_s\cos\phi_s$$
$$B = \sin\theta_s\sin\phi_s - \sin\theta_i$$

c.
$$|\underline{E}^s| = \sqrt{|E_\theta^s|^2 + |E_\phi^s|^2} = E_0\frac{\beta a^2}{r}\left\{\sqrt{\cos^2\theta_i\cos^2\theta_s\cos^2\phi_s + \cos^2\theta_i\sin^2\phi_s}\left|\frac{J_1(\beta a\sqrt{A^2+B^2})}{\beta a\sqrt{A^2+B^2}}\right|\right.$$

$$|\underline{E}|^2 = \sqrt{|E_\theta^s|^2 + |E_\phi^s|^2} = E_0\frac{\beta a^2}{r}\left\{\cos\theta_i\sqrt{\cos^2\theta_s\cos^2\phi_s + \sin^2\phi_s}\left|\frac{J_1(\beta a\sqrt{A^2+B^2})}{\beta a\sqrt{A^2+B^2}}\right|\right\} = E_0\frac{\beta a^2}{r}|G|$$

$$G = \cos\theta_i\sqrt{\cos^2\theta_s\cos^2\phi_s + \sin^2\phi_s}\left|\frac{J_1(\beta a\sqrt{A^2+B^2})}{\beta a\sqrt{A^2+B^2}}\right|$$

$$\sigma_{3\text{-}D} = \lim_{r\to\infty}\left\{4\pi r^2\frac{|\underline{E}^s|^2}{|\underline{E}^i|^2}\right\} = 4\pi r^2\left(\frac{\beta a^2}{r}\right)^2|G|^2 = 4\pi\left(\frac{\pi a^2}{\lambda}\right)^2|2G|^2$$

For monostatic observations $\phi_s = 3\pi/2$, $\theta_s = \theta_i$.

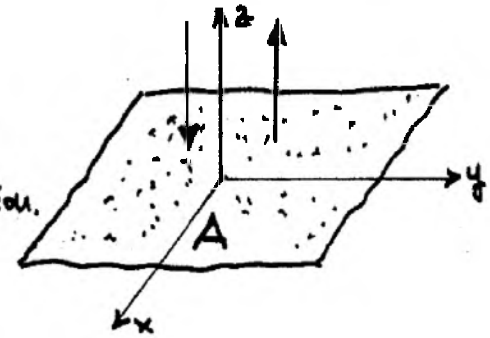
$$\left.\begin{aligned} A &= \sin\theta_s\cos\phi_s = 0 \\ B &= \sin\theta_s\sin\phi_s - \sin\theta_i = \sin\theta_i(-1) - \sin\theta_i = -2\sin\theta_i \end{aligned}\right\}\sqrt{A^2+B^2} = 2\sin\theta_i$$

$$E_\theta^s = 0, \quad E_\phi^s = -j\frac{\beta a^2 E_0 e^{-j\beta r}}{r}\cos\theta_i\frac{J_1(2\beta a\sin\theta_i)}{2\beta a\sin\theta_i}$$

$$\sigma_{3\text{-}D} = \lim_{r\to\infty}\left\{4\pi r^2\frac{|E_\phi^s|^2}{|\underline{E}^i|^2}\right\} = 4\pi\left(\frac{\pi a^2}{\lambda}\right)^2\left[2\cos\theta_i\frac{J_1(2\beta a\sin\theta_i)}{2\beta a\sin\theta_i}\right]^2$$

The monostatic for this polarization (TM^x) is the same as that of the TE^x polarization of Problem 11.24. Therefore the monostatic RCS plots for this polarization for $a = \lambda, 5\lambda$ are those shown for the solution of Problem 11.24.

$\boxed{11.26}$ To solve this problem let us assume that the incident electric field is polarized in the x direction. The same answer is obtained using any other polarization.



$$\underline{E}^i = \hat{a}_x E_0 e^{j\beta z}, \quad \underline{H}^i = -\hat{a}_y \frac{E_0}{\eta} e^{+j\beta z}$$

$$\underline{J}_s \simeq 2\hat{n}\times\underline{H}^i\Big|_{z=0} = 2\hat{a}_z \times\left(-\hat{a}_y \frac{E_0}{\eta}\right) = \hat{a}_x \frac{2E_0}{\eta}$$

$$J_x = \frac{2E_0}{\eta}, \; J_y = J_z = 0, \; \theta_s = 0$$

$$N_\theta = \iint_S \left[J_x \cancelto{1}{\cos\theta_s}\cos\phi_s + \cancelto{0}{J_y}\cos\theta_s \sin\phi_s - \cancelto{0}{J_z}\sin\theta_s\right] e^{j\beta(x'\cancelto{0}{\sin\theta_s}\cos\phi_s + y'\cancelto{0}{\sin\theta_s}\sin\phi_s)} dx'dy'$$

$$= \frac{2E_0}{\eta}\cos\phi_s \iint_S dx'dy' = \frac{2E_0 A}{\eta}\cos\phi_s$$

$$N_\phi = \iint_S \left[-J_x \sin\phi_s + \cancelto{0}{J_y}\cos\phi_s\right] e^{j\beta(x'\cancelto{0}{\sin\theta_s}\cos\phi_s + y'\cancelto{0}{\sin\theta_s}\sin\phi_s)} dx'dy' = -\frac{2E_0}{\eta}\sin\phi_s \iint_S dx'dy' = -\frac{2E_0 A}{\eta}\sin\phi_s$$

$$E_\theta^s \simeq -j\frac{\beta e^{-j\beta r}}{4\pi r}\left[\cancelto{0}{L_\phi} + \eta N_\theta\right] = -j\frac{\beta e^{-j\beta r}}{4\pi r}\left(2E_0 A\cos\phi_s\right) = -j\frac{E_0 A\beta e^{-j\beta r}}{2\pi r}\cos\phi_s$$

$$E_\phi^s \simeq j\frac{\beta e^{-j\beta r}}{4\pi r}\left[\cancelto{0}{L_\theta} - \eta N_\phi\right] = j\frac{\beta e^{-j\beta r}}{4\pi r}\left[2E_0 A\sin\phi_s\right] = j\frac{E_0 A\beta e^{-j\beta r}}{2\pi r}\sin\phi_s$$

$$|\underline{E}^s| = \sqrt{|E_\theta^s|^2 + |E_\phi^s|^2} = \frac{|E_0|A\beta}{2\pi r}\sqrt{\cos^2\phi_s + \sin^2\phi_s} = \frac{|E_0|A\beta}{2\pi r}$$

$$\sigma = \lim_{r\to\infty}\left[4\pi r^2 \frac{|\underline{E}^s|^2}{|\underline{E}^i|^2}\right] = \lim_{r\to\infty}\left[4\pi r^2 \frac{|E_0|^2 A^2\beta^2}{4\pi^2 r^2 |E_0|^2}\right] = \frac{(A\beta)^2}{\pi} = 4\pi\left(\frac{A}{\lambda}\right)^2$$

$$\sigma = 4\pi\left(\frac{A}{\lambda}\right)^2$$

$\boxed{11.27}$

a. With the PO approximation, the current on the surface of the perfectly conducting object can be expressed as

$$\underline{J}_s = 2H_0 \hat{n}\times\hat{a}_h e^{-j\beta R_1} \tag{1}$$

where $\hat{a}_h$ represents the polarization of the H field and R_1 is the distance from the source to the scattering point. Then

$$\underline{A} = \frac{\mu}{4\pi}\iint_S \underline{J}_s \frac{e^{-j\beta R_2}}{R_2} ds' = \frac{\mu_0 H_0}{2\pi}\iint_S (\hat{n}\times\hat{a}_h)\frac{e^{-j\beta(R_1+R_2)}}{R_2} ds' \tag{2}$$

$$\underline{H}^s = \frac{1}{\mu_0}\nabla\times\underline{A} = \frac{H_0}{2\pi}\iint_S \nabla\times\left[(\hat{n}\times\hat{a}_h)\frac{e^{-j\beta(R_1+R_2)}}{R_2}\right] ds' \tag{3}$$

where the integration is over the surface of the object. For far field and monostatic observations

$$R_1 = R_2 = R - z \;,\quad \nabla \Leftrightarrow -j\beta\hat{a}_z \tag{4}$$

where R is the distance from the origin to the observation point, and z is the z axis projection of the position of the scattering point. With the above approximation, the scattered field of (3) can be reduced to

$$\underline{H}^S = -j\frac{\beta H_0}{2\pi R}\iint \hat{a}_z\times(\hat{n}\times\hat{a}_h)\, e^{-j2\beta(R-z)} ds' \tag{5}$$

Since

$$\hat{a}_z\times(\hat{n}\times\hat{a}_h) = (\hat{a}_z\cdot\hat{a}_h)\hat{n} - (\hat{a}_z\cdot\hat{n})\hat{a}_h\,,\quad \hat{a}_z\cdot\hat{a}_h = 0 \tag{6}$$

then

$$\underline{H}^S = j\frac{\beta H_0 e^{-j\beta R}}{2\pi R}\iint_A (\hat{a}_z\cdot\hat{n})\hat{a}_h e^{+j2\beta z} ds' \tag{7}$$

Because

$$(\hat{a}_z\cdot\hat{n})\, ds' = dA \tag{8}$$

then

$$\underline{H}^s = j\frac{\beta H_0 e^{-j\beta R}}{2\pi R}\iint_A \hat{a}_h e^{j2\beta z} dA \tag{9}$$

where the integration is performed in the xy-plane.

b. The monostatic RCS can now be written as

$$\sigma_{3\text{-}D} = \lim_{R\to\infty}\left[4\pi R^2 \frac{|\underline{H}^s|^2}{|\underline{H}^i|^2}\right] = \left(\frac{\beta|H_0|}{\pi}\right)^2 \left|\iint_A \hat{a}_h e^{j2\beta z} dA\right|^2$$

$$\sigma_{3\text{-}D} = \frac{4\pi}{\lambda^2}\left|\iint_A e^{j2\beta z} dA\right|^2$$

Cont'd.

$\boxed{11.27 \text{ cont'd.}}$

c. It is sometimes more convenient to write the integral in Part b in parameter form since many regular geometries have pretty simple expressions in terms of some special parameters. Thus by a change of variables we have

$$dA = dx\,dy = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} du\,dv = J(x,y;u,v)\,du\,dv \quad \text{where } J \text{ is the Jacobian.}$$

In many cases it is also possible to write the integral as

$$dA = J[u(z),v]\frac{du}{dz}dz\,dv$$

which allows an independent integral

$$\sigma_{3\text{-}D} = \frac{4\pi}{\lambda^2}\left|\iint e^{j2\beta z} J[u(z),v]\frac{du}{dz}dz\,dv\right|^2$$

Taking the cylindrical coordinates as a special case,

$$\left.\begin{array}{l} u=\rho \\ v=\phi \end{array}\right\} J(x,y;u,v)=\rho$$

$$dA = \rho\,d\rho\,d\phi = \frac{1}{2}\frac{d\rho^2}{dz}dz\,d\phi$$

Thus

$$\sigma_{3\text{-}D} = \frac{4\pi}{\lambda^2}\frac{1}{4}\left|\iint e^{j2\beta z}\frac{d\rho^2}{dz}dz\,d\phi\right|^2 = \frac{\pi}{\lambda^2}\left|\int_0^{2\pi}\int_0^{z(\phi)} e^{j2\beta z}\frac{d\rho^2}{dz}dz\,d\phi\right|^2$$

$\boxed{11.28}$ $\quad \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = -\frac{z}{c}, \quad z < h$

a. $\quad z = -c\left[\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2\right]$

For $y=0$ (i.e. xz-plane)

$$z = -c\left(\frac{x}{a}\right)^2$$

$$z' = \frac{dz}{dx} = -2c\left(\frac{x}{a}\right)\left(\frac{1}{a}\right) = -2\frac{c}{a^2}x$$

$$z'' = \frac{d^2z}{dx^2} = -\frac{2c}{a^2}$$

$$R_1\Big|_{x=0} = \text{radius of curvature}\Big|_{x=0} = \frac{[1+(z')^2]^{3/2}}{|z''|}\Bigg|_{x=0} = \frac{\left[1+\left(\frac{2c}{a^2}\right)^2x^2\right]^{3/2}}{\frac{2c}{a^2}}\Bigg|_{x=0} = \frac{a^2}{2c} = a_1$$

$$R_1(x=0) = \frac{a^2}{2c} = a_1$$

For $x=0$ (i.e. yz-plane)

$$z = -c\left(\frac{y}{b}\right)^2$$

$$z' = \frac{dz}{dy} = -2c\left(\frac{y}{b}\right)\left(\frac{1}{b}\right) = -2\frac{c}{b^2}y$$

$$z'' = \frac{d^2z}{dy^2} = -\frac{2c}{b^2}$$

$$R_2\Big|_{y=0} = \frac{[1+(z')^2]^{3/2}}{|z''|}\Bigg|_{y=0} = \frac{\left[1+\left(\frac{2c}{b^2}\right)^2\right]^{3/2}}{2c/b^2}\Bigg|_{y=0} = \frac{b^2}{2c} = a_2$$

11.28 cont'd. $x=\rho\cos\phi,\ y=\rho\sin\phi$

b.

$$z=-c\left[\frac{x^2}{a^2}+\frac{y^2}{b^2}\right]=-c\rho^2\left[\left(\frac{\cos\phi}{a}\right)^2+\left(\frac{\sin\phi}{b}\right)^2\right]=-c\rho^2/a^2\left[\cos^2\phi+\left(\frac{a}{b}\right)^2\sin^2\phi\right]$$

$$=-\frac{c}{a^2}\rho^2\left[1-\sin^2\phi+\left(\frac{a}{b}\right)^2\sin^2\phi\right]=-\frac{2c}{a^2}\left(\frac{\rho^2}{2}\right)\left\{1-\left[\sin^2\phi-\left(\frac{a}{b}\right)^2\sin^2\phi\right]\right\}$$

$$z=-\frac{\rho^2}{2a_1}\left\{1-\left[1-\frac{a_1}{a_2}\right]\sin^2\phi\right\}\quad \text{where } a_1=\frac{a^2}{2c},\ a_2=\frac{b^2}{2c}$$

c. From Part b

$$dz=-\frac{d\rho^2}{2a_1}\left\{1-\left[1-\frac{a_1}{a_2}\right]\sin^2\phi\right\}\Rightarrow\frac{d\rho^2}{dz}=-\frac{2a_1}{1-\left[1-\frac{a_1}{a_2}\right]\sin^2\phi}$$

Then

$$\sigma_{3-D}=\frac{\pi}{\lambda^2}\left|-\int_0^{2\pi}\int_0^{h}e^{j2\beta z}\frac{2a_1}{1-\left[1-\frac{a_1}{a_2}\right]\sin^2\phi}\,dz\,d\phi\right|^2$$

Using the integral

$$\int_0^{2\pi}\frac{d\phi}{1-\alpha\sin^2\phi}=\frac{2\pi}{\sqrt{1-\alpha}}$$

we can write the RCS as

$$\sigma_{3-D}=\frac{\pi}{\lambda^2}\left|\left(\frac{e^{j2\beta z}}{j2\beta}\right)\Bigg|_0^h\frac{2a_1(2\pi)}{\sqrt{1-(1-a_1/a_2)}}\right|^2=\frac{\pi}{(\beta\lambda)^2}\left|\left(\frac{e^{j2\beta h}-1}{j2}\right)4\pi\sqrt{a_1a_2}\right|^2$$

$$\sigma_{3-D}=4\pi a_1a_2\sin^2(\beta h)$$

d. From Part c of the solution it is apparent that elimination of the exponential reduces the RCS to

$$\sigma_{3-D}=\pi a_1a_2\sin^2(\beta h)$$

11.29 $\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2+\left(\frac{z}{c}\right)^2=1$, $x=\rho\cos\phi$, $y=\rho\sin\phi$

a. $\left(\frac{z}{c}\right)^2=1-\left\{\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2\right\}=1-\frac{1}{a^2}\left[x^2+\left(\frac{a}{b}\right)^2y^2\right]=1-\frac{\rho^2}{a^2}\left\{1-\left[1-\frac{a_1}{a_2}\right]\sin^2\phi\right\}$

$z^2=c^2-\frac{c^2\rho^2}{a^2}\left\{1-\left[1-\frac{a_1}{a_2}\right]\sin^2\phi\right\}=c^2-\frac{c\rho^2}{a^2/c}\left\{1-\left[1-\frac{a_1}{a}\right]\sin^2\phi\right\}=c^2-\rho^2\frac{c}{a_1}\left\{1-\left(1-\frac{a_1}{a_2}\right)\sin^2\phi\right\}$

b. To find the RCS we perform the integration using parameter integral by assuming that

$x=at\cos\phi$
$y=bt\sin\phi$

then the ellipsoid equation can be expressed as

$$\left(\frac{z}{c}\right)^2=1-\left[\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2\right]=1-\left[t^2\cos^2\phi+t^2\sin^2\phi\right]=1-t^2$$

$$dA=J(x,y;t,\phi)\,dt\,d\phi=-\frac{abz\,dz\,d\phi}{c^2}$$

Now from Part c of Problem 11.27 we evaluate the integral from $z=h$ to $z=c$. Thus

$$\sigma_{3\text{-}D}=\frac{4\pi}{\lambda^2}\left|\int_0^{2\pi}\int_h^c e^{j2\beta z}J[u(z),v]\frac{du}{dz}dz\,dv\right|^2=\frac{4\pi}{\lambda^2}\left|-\int_0^{2\pi}\int_h^c e^{j2\beta z}\frac{abz\,dz}{c^2dz}dz\,d\phi\right|^2$$

$$\sigma_{3\text{-}D}=\frac{4\pi}{\lambda^2}\left|\frac{ab(2\pi)}{c^2}\int_h^c e^{j2\beta z}z\,dz\right|^2$$

Neglecting the term with $(2\beta c)^{-1}$, the above equation for the RCS reduces to

$$\sigma_{3\text{-}D}=\frac{\pi(ab)^2}{c^4}\left|c\,e^{+j2\beta c}-h\,e^{+j2\beta h}\right|^2=\frac{\pi a^2b^2}{c^4}\left\{c^2+h^2-2hc\cos[2\beta(c-h)]\right\}$$

$$\sigma_{3\text{-}D}=\pi\frac{a^2}{c}\frac{b^2}{c}\left\{1+\left(\frac{h}{c}\right)^2-2\frac{h}{c}\cos[2\beta(c-h)]\right\}=\pi a_1a_2\left\{1+\left(\frac{h}{c}\right)^2-2\frac{h}{c}\cos[2\beta(c-h)]\right\}$$

If $h=0$ or if h is very irregular around the periphery of the ellipsoid, then the above expression for $\sigma_{3\text{-}D}$ reduces to

$$\sigma_{3\text{-}D}=\pi a_1a_2\overset{a_1=a_2=a}{=}\pi a^2$$

11.30 $H_n^{(2)}=J_n(x)-jY_n(x)\Rightarrow H_{-n}^{(2)}=J_{-n}(x)-jY_{-n}(x)$

For integer n: $J_{-n}(x)=(-1)^nJ_n(x)$, $Y_{-n}(x)=(-1)^nY_n(x)$

Therefore

$$H_{-n}^{(2)}=J_{-n}(x)-jY_{-n}(x)=(-1)^nJ_n(x)-j(-1)^nY_n(x)$$

$$H_{-n}^{(2)}=(-1)^n\left[J_n(x)-jY_n(x)\right]=(-1)^nH_n^{(2)}(x)$$

* In the first printing, there was an typographical error in the statement of the problem; an extra +1 appeared on the exponent of the (−1).

11.31 $E_z^+ = e^{-j\beta x} = e^{-j\beta\rho\cos\phi} = \sum_{n=-\infty}^{+\infty} a_n J_n(\beta\rho)e^{jn\phi}$

Using (11-54a) or $a_n = j^{-n}$, we can write the above expression as

$$E_z^+ = e^{-j\beta x} = e^{-j\beta\rho\cos\phi} = \sum_{n=-\infty}^{+\infty} j^{-n} J_n(\beta\rho)e^{jn\phi}$$

11.32 Since the only difference between $H_0^{(2)}$ and $H_0^{(1)}$ is the superscript, the expansion for $H_0^{(1)}$ in terms of cylindrical wave function originating at the origin proceeds along the same lines as the expansion $H_0^{(2)}$ with the replacement of the $H_0^{(2)}$ type of wave functions by $H_0^{(1)}$. Therefore (11-71a) is easily obtained by referring to (11-69a) and (11-71b) is obtained by referring to (11-69b).

11.33 $\underline{E}^i = \hat{a}_z E_z^i = \hat{a}_z E_0 \sum_{n=-\infty}^{+\infty} j^{-n} J_n(\beta\rho)e^{jn\phi}$

$$\underline{E}^i = \hat{a}_z E_0 \Big\{ \underbrace{J_0(\beta\rho)}_{n=0} + \underbrace{j^{-1} J_1(\beta\rho)e^{j\phi}}_{n=1} + \underbrace{j^{-2} J_2(\beta\rho)e^{j2\phi}}_{n=2} + \cdots$$

$$+ j^{1} \underbrace{J_{-1}(\beta\rho)e^{-j\phi}}_{n=-1} + j^2 J_{-2}(\beta\rho)e^{-j2\phi} + \cdots \Big\}$$

Since $J_{-n}(x) = (-1)^n J_n(x) \Rightarrow J_{-1}(\beta\rho) = -J_1(\beta\rho),\ J_{-3}(\beta\rho) = -J_3(\beta\rho)$
$J_{-2}(\beta\rho) = J_2(\beta\rho),\ J_{-4}(\beta\rho) = J_4(\beta\rho)$

Thus

$$\underline{E}^i = \hat{a}_z E_0 \left\{ J_0(\beta\rho) + j^{-1} 2 J_1(\beta\rho)\left[\frac{e^{j\phi} - j^2 e^{-j\phi}}{2}\right] + j^{-2} 2 J_2(\beta\rho)\left[\frac{e^{j2\phi} + j^4 e^{-j2\phi}}{2}\right] + \cdots \right\}$$

$$= \hat{a}_z E_0 \{ J_0(\beta\rho) - j2 J_1(\beta\rho)\cos(\phi) - 2 J_2(\beta\rho)\cos(2\phi) \cdots \}$$

$$\underline{E}^i = \hat{a}_z E_0 \sum_{n=0}^{\infty} (-j)^n \epsilon_n J_n(\beta\rho)\cos(n\phi), \quad \epsilon_n = \begin{cases} 1 & n=0 \\ 2 & n\neq 0 \end{cases}$$

11.34 $\underline{J}_s = \hat{a}_z \frac{2E_0}{\pi a\omega\mu} \sum_{n=-\infty}^{+\infty} j^{-n} \frac{e^{jn\phi}}{H_n^{(2)}(\beta a)}$

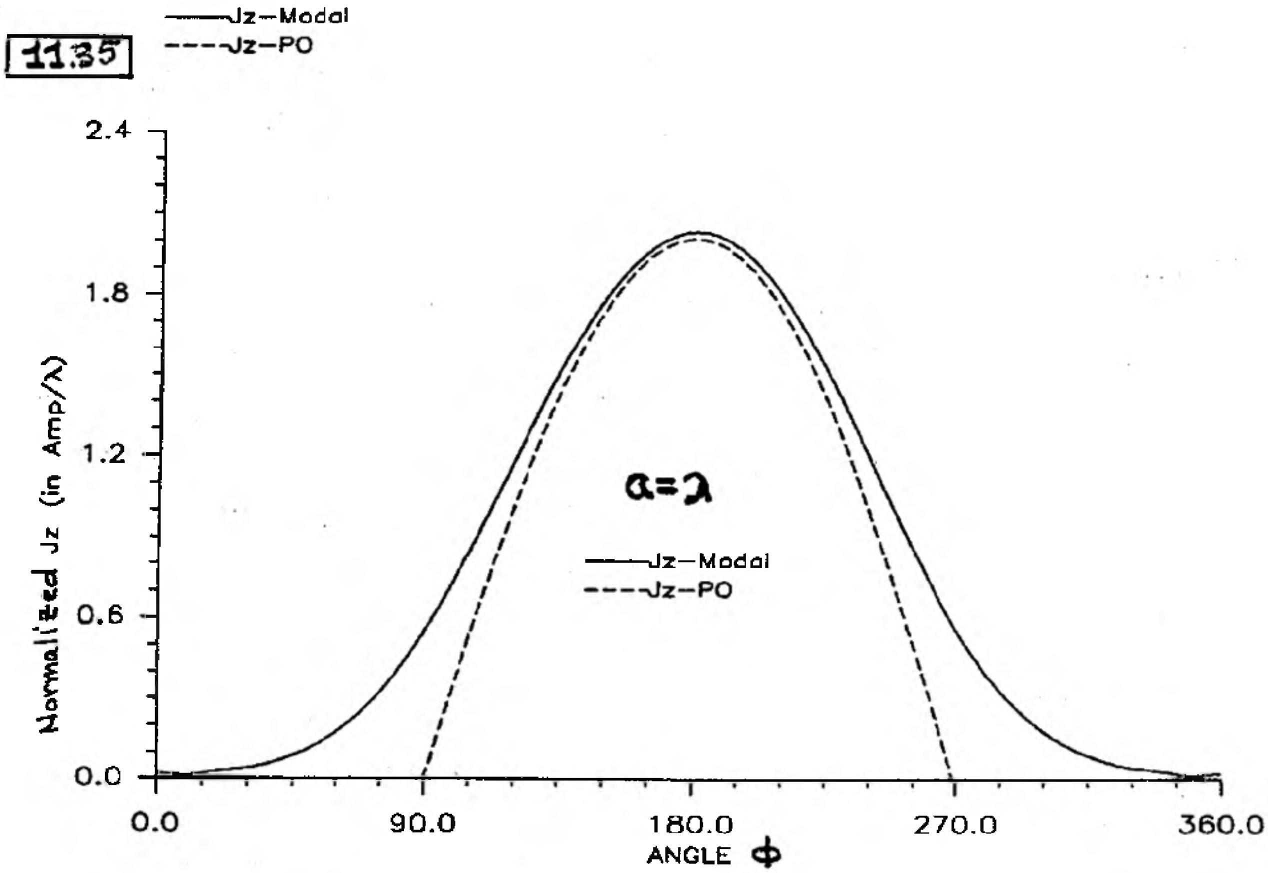
Using a procedure similar to that for the solution of Problem 11.33 we can write the above expression as

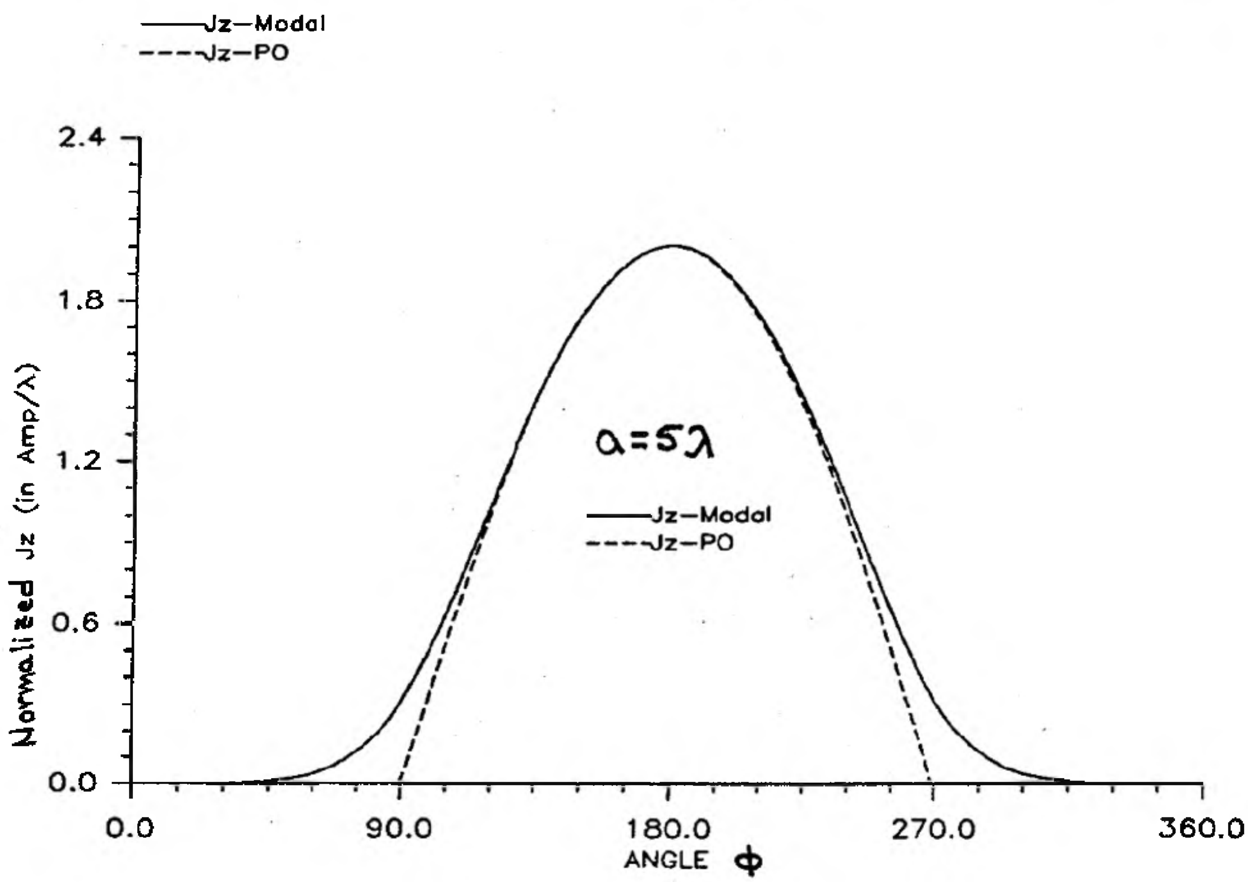
$$\underline{J}_s = \hat{a}_z \frac{2E_0}{\pi a\omega\mu} \sum_{n=0}^{\infty} (-j)^n \epsilon_n \frac{\cos(n\phi)}{H_n^{(2)}(\beta a)}$$

where

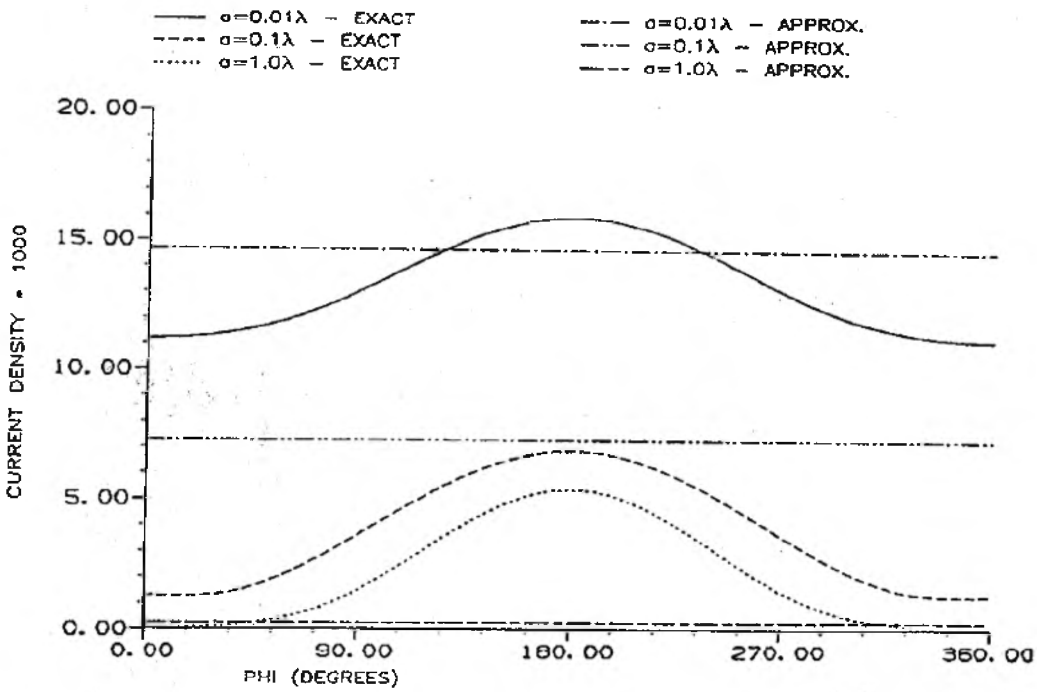
$$\epsilon_n = \begin{cases} 1 & n=0 \\ 2 & n\neq 0 \end{cases}$$

11.35


Jz—Modal
Jz—PO
2.4
1.8
1.2
0.6
0.0
Normalized Jz (in Amp/λ)
a=λ
Jz—Modal
Jz—PO
0.0
90.0
180.0
270.0
360.0
ANGLE Φ


Jz—Modal
Jz—PO
2.4
1.8
1.2
0.6
0.0
Normalized Jz (in Amp/λ)
a=5λ
Jz—Modal
Jz—PO
0.0
90.0
180.0
270.0
360.0
ANGLE Φ

$\boxed{11.36}$



INDUCED CURRENT DENSITY
CIRCULAR CYLINDER

TM^z POLARIZATION

$\boxed{11.37}$

$$\sigma_{2\text{-}D} = \frac{4}{\beta}\left|\sum_{n=-\infty}^{+\infty}\frac{J_n(\beta a)}{H_n^{(2)}(\beta a)}e^{jn\phi}\right|^2 = \frac{4}{\beta}\Bigg|\underbrace{\frac{J_0(\beta a)}{H_0^{(2)}(\beta a)}}_{n=0}+\underbrace{\frac{J_1(\beta a)}{H_1^{(2)}(\beta a)}e^{j\phi}}_{n=1}+\underbrace{\frac{J_2(\beta a)}{H_2^{(2)}(\beta a)}e^{j2\phi}}_{n=2}+\cdots$$

$$+\underbrace{\frac{J_{-1}(\beta a)}{H_{-1}^{(2)}(\beta a)}e^{-j\phi}}_{n=-1}+\underbrace{\frac{J_{-2}(\beta a)}{H_{-2}^{(2)}(\beta a)}e^{-j2\phi}}_{n=-2}+\cdots\Bigg|^2$$

Since $\left.\begin{array}{l} J_{-n}(x) = (-1)^n J_n(x) \\ H_{-n}^{(2)}(x) = (-1)^n H_n^{(2)}(x) \end{array}\right\} \begin{array}{l} J_{-1}(x) = -J_1(x),\ J_{-2}(x) = J_2(x),\ J_{-3}(x) = -J_3(x),\ \ldots \\ H_{-1}^{(2)}(x) = -H_1^{(2)}(x),\ H_{-2}^{(2)}(x) = H_2^{(2)}(x),\ H_{-3}^{(2)}(x) = -H_3^{(2)}(x),\ \ldots \end{array}$

then

$$\sigma_{2\text{-}D} = \frac{4}{\beta}\left|\frac{J_0(\beta a)}{H_0^{(2)}(\beta a)}+2\frac{J_1(\beta a)}{H_1^{(2)}(\beta a)}\underbrace{\left(\frac{e^{j\phi}+e^{-j\phi}}{2}\right)}_{\cos\phi}+2\frac{J_2(\beta a)}{H_2^{(2)}(\beta a)}\underbrace{\left(\frac{e^{j2\phi}+e^{-j2\phi}}{2}\right)}_{\cos 2\phi}+\cdots\right|^2$$

$$\sigma_{2\text{-}D} = \frac{2\lambda}{\pi}\left|\sum_{n=0}^{\infty}\epsilon_n\frac{J_n(\beta a)}{H_n^{(2)}(\beta a)}\cos(n\phi)\right|^2$$

$\boxed{11.38}$ According to (11-92a) and (11-92b) we see that the ρ component of the magnetic field has a $1/\rho$ variation outside the summation while the ϕ component has no ρ variations outside the summation. Within the summation both have similar ρ variations when $\rho \gg \lambda$ (far field). Therefore we can approximate the far zone scattered magnetic field by

$$\underline{H}^s = \hat{a}_\rho H_\rho^s + \hat{a}_\phi H_\phi^s \overset{\rho\gg\lambda}{\simeq} \hat{a}_\phi H_\phi^s = -\hat{a}_\phi \frac{\beta E_0}{j\omega\mu} \sum_{n=-\infty}^{+\infty} j^{-n} \frac{J_n(\beta a)}{H_n^{(2)}(\beta a)} H_n^{(2)\prime}(\beta\rho) e^{jn\phi}$$

$$H_n^{(2)\prime}(\beta\rho) = \frac{d}{d(\beta\rho)} H_n^{(2)}(\beta\rho) \overset{\rho\gg\lambda}{\simeq} \frac{d}{d(\beta\rho)}\left\{\sqrt{\frac{2j}{\pi}} (\beta\rho)^{-1/2} e^{-j\beta\rho} j^n\right\}$$

$$\overset{\rho\gg\lambda}{\simeq} j^n \sqrt{\frac{2j}{\pi}} \left\{ \beta\rho^{-1/2}(-j) e^{-j\beta\rho} - \frac{1}{2}(\beta\rho)^{-3/2} e^{-j\beta\rho}\right\} = j^n \sqrt{\frac{2j}{\pi}} \left\{ -j \frac{e^{-j\beta\rho}}{(\beta\rho)^{1/2}} - \frac{e^{-j\beta\rho}}{2(\beta\rho)^{3/2}} \right\}$$

$$H_n^{(2)}(\beta\rho) \simeq -j^n \sqrt{\frac{2j}{\pi}}\, j \frac{e^{-j\beta\rho}}{\sqrt{\beta\rho}} = -j^{n+1} \sqrt{\frac{2j}{\pi}} \frac{e^{-j\beta\rho}}{\sqrt{\beta\rho}} = -j^{n+1} \sqrt{\frac{2j}{\pi\beta\rho}} e^{-j\beta\rho}$$

Therefore the scattered magnetic field can be approximated by

$$\underline{H}^s \simeq \hat{a}_\phi \frac{\beta E_0}{\omega\mu} \sqrt{\frac{2j}{\pi\beta\rho}} e^{-j\beta\rho} \sum_{n=-\infty}^{+\infty} \frac{J_n(\beta a)}{H_n^{(2)}(\beta a)} e^{jn\phi} = \hat{a}_\phi \frac{E_0}{\eta} \sqrt{\frac{2j}{\pi\beta\rho}} e^{-j\beta\rho} \sum_{n=-\infty}^{+\infty} \frac{J_n(\beta a)}{H_n^{(2)}(\beta a)} e^{jn\phi}$$

The RCS can now be written as

$$\sigma_{2\text{-}D} = \lim_{\rho\to\infty}\left[2\pi\rho \frac{|\underline{H}^s|^2}{|\underline{H}^i|^2}\right] = \frac{4}{\beta} \left| \sum_{n=-\infty}^{+\infty} \frac{J_n(\beta a)}{H_n^{(2)}(\beta a)} e^{jn\phi} \right|^2 \text{ where } \underline{H}^i = -\hat{a}_y \frac{E_0}{\eta}$$

$\boxed{11.39}$ $$\underline{J}_s = \hat{a}_\phi j \frac{2H_0}{\pi\beta a} \sum_{n=-\infty}^{+\infty} j^{-n} \frac{e^{jn\phi}}{H_n^{(2)\prime}(\beta a)} = \hat{a}_\phi j \frac{2H_0}{\pi\beta a} \Bigg\{ \underbrace{\frac{1}{H_0^{(2)\prime}(\beta a)}}_{n=0} +$$

$$+ \underbrace{j^{-1}\frac{e^{j\phi}}{H_1^{(2)\prime}(\beta a)}}_{n=1} + \underbrace{j^{-2}\frac{e^{j2\phi}}{H_2^{(2)\prime}(\beta a)}}_{n=2} + \underbrace{j^{-3}\frac{e^{j3\phi}}{H_3^{(2)\prime}(\beta a)}}_{n=3} + \underbrace{j^{-4}\frac{e^{j4\phi}}{H_4^{(2)\prime}(\beta a)}}_{n=4} + \cdots$$

$$+ \underbrace{j^{+1}\frac{e^{-j\phi}}{H_{-1}^{(2)\prime}(\beta a)}}_{n=-1} + \underbrace{j^{+2}\frac{e^{-j2\phi}}{H_{-2}^{(2)\prime}(\beta a)}}_{n=-2} + \underbrace{j^{+3}\frac{e^{-j3\phi}}{H_{-3}^{(2)\prime}(\beta a)}}_{n=-3} + \underbrace{j^{+4}\frac{e^{-j4\phi}}{H_{-4}^{(2)\prime}(\beta a)}}_{n=-4} + \cdots$$

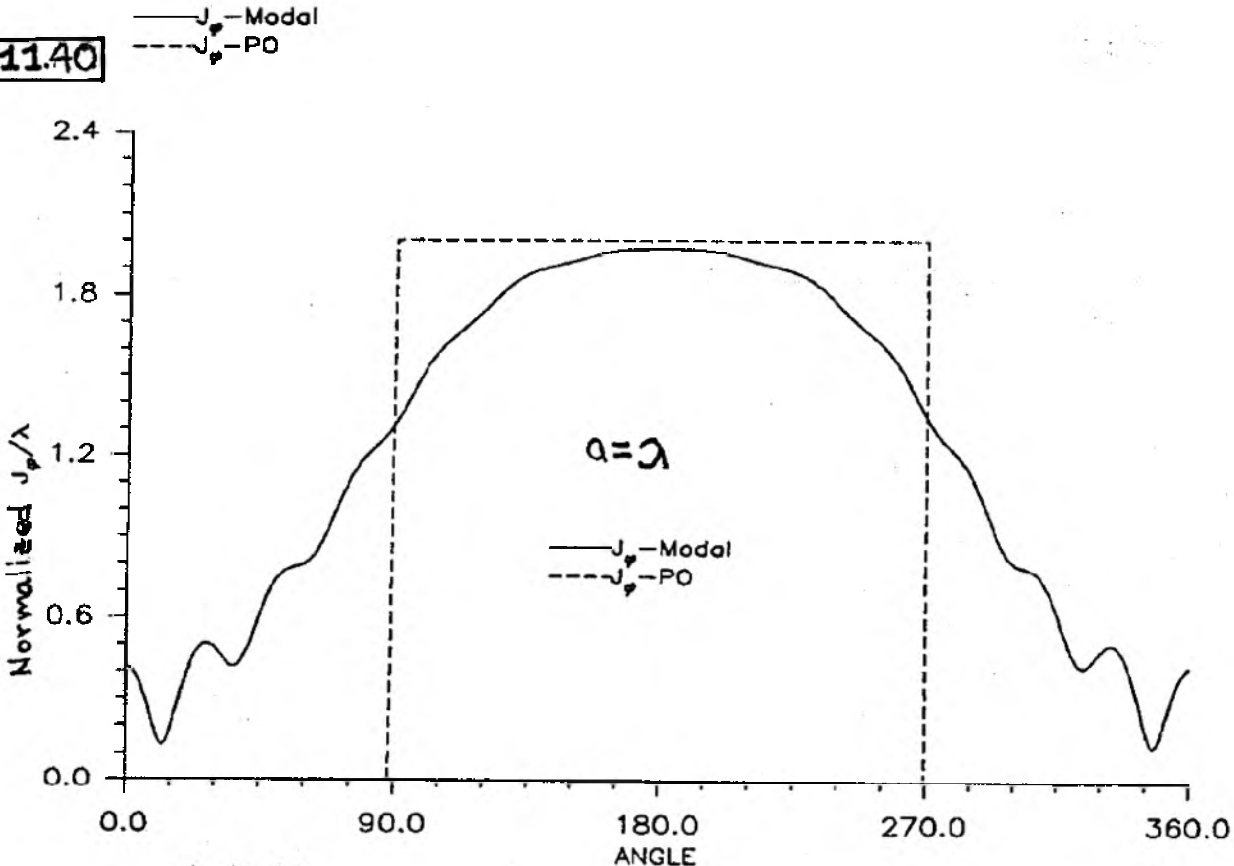
$$\underline{J}_s = \hat{a}_\phi j \frac{2H_0}{\pi\beta a} \left\{ \frac{1}{H_0^{(2)\prime}(\beta a)} + j^{-1} \frac{2}{H_1^{(2)\prime}(\beta a)} \left[\frac{e^{j\phi} - j^{+2} e^{-j\phi}}{2}\right] + j^{-2} \frac{2}{H_2^{(2)\prime}(\beta a)} \left[\frac{e^{j2\phi} + j^{+4} e^{-j2\phi}}{2}\right] + \cdots\cdots \right\}$$

$$= \hat{a}_\phi j \frac{2H_0}{\pi\beta a} \left\{ \frac{1}{H_0^{(2)\prime}(\beta a)} + j^{-1} \frac{2\cos\phi}{H_1^{(2)\prime}(\beta a)} - \frac{2\cos(2\phi)}{H_2^{(2)\prime}(\beta a)} + j^{-3} \frac{2\cos(3\phi)}{H_3^{(2)\prime}(\beta a)} \cdots \right\}$$

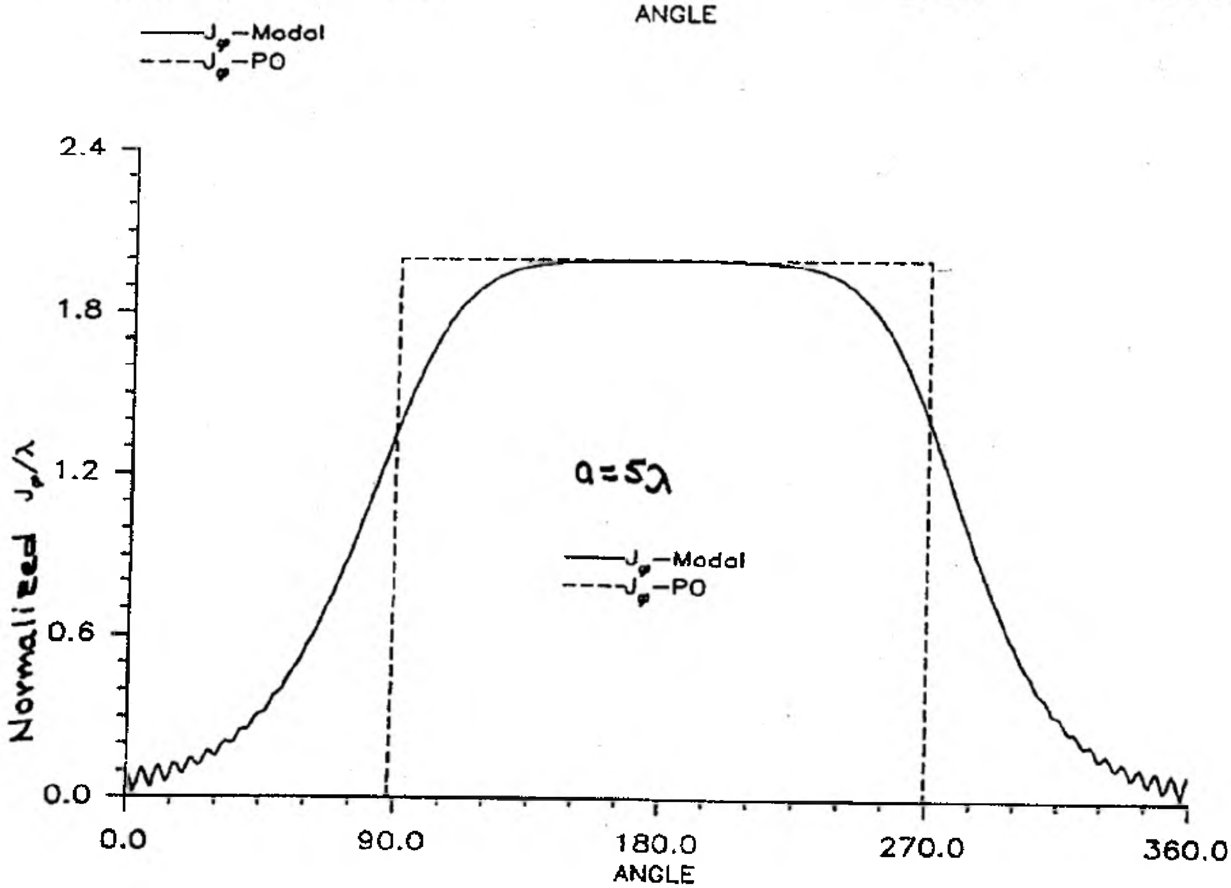
$$\underline{J}_s = \hat{a}_\phi j \frac{2H_0}{\pi\beta a} \left\{ \sum_{n=0}^{\infty} \epsilon_n (-j)^n \frac{\cos(n\phi)}{H_n^{(2)\prime}(\beta a)} \right.$$

where $$\epsilon_n = \begin{cases} 1, & n=0 \\ 2, & n\neq 0 \end{cases}$$

11.40

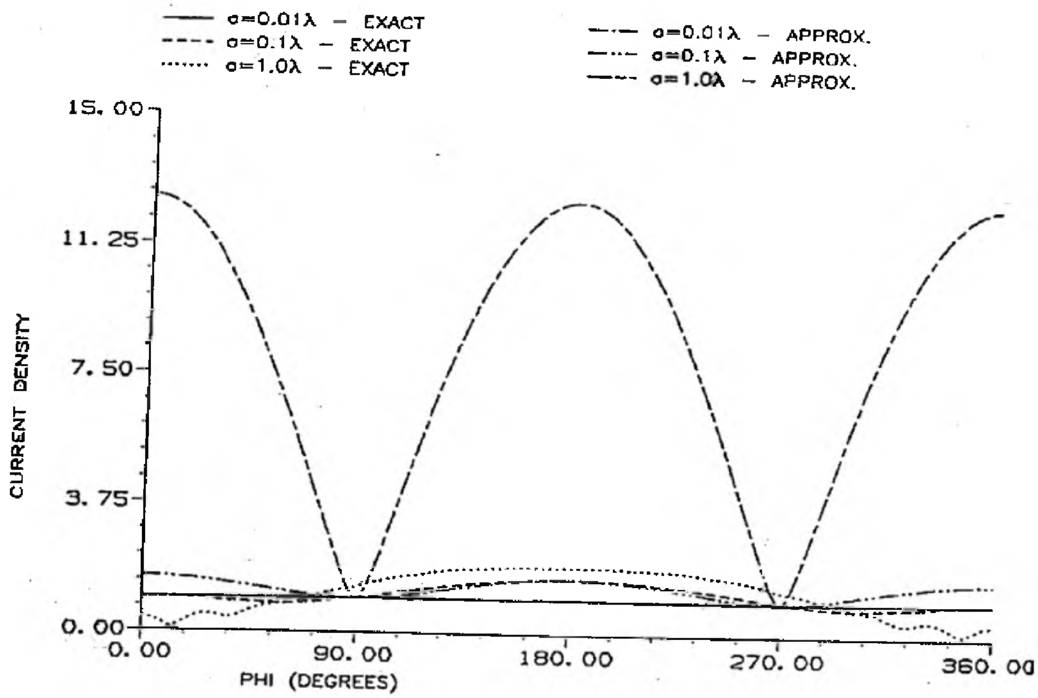


——J_φ—Modal
- - - -J_φ—PO
2.4
1.8
1.2
0.6
0.0
Normalized J_φ/λ
$a=\lambda$
——J_φ—Modal
- - - -J_φ—PO
0.0
90.0
180.0
270.0
360.0
ANGLE



——J_φ—Modal
- - - -J_φ—PO
2.4
1.8
1.2
0.6
0.0
Normalized J_φ/λ
$a=5\lambda$
——J_φ—Modal
- - - -J_φ—PO
0.0
90.0
180.0
270.0
360.0
ANGLE

[11.41]



INDUCED CURRENT DENSITY (C)
CIRCULAR CYLINDER

TE^z POLARIZATION

$\boxed{11.42}$

$$\sigma_{2\text{-}D} = \frac{4}{\beta}\left|\sum_{n=-\infty}^{+\infty} \frac{J_n'(\beta a)}{H_n^{(2)\prime}(\beta a)} e^{jn\phi}\right|^2 = \frac{4}{\beta}\left|\underbrace{\frac{J_0'(\beta a)}{H_0^{(2)\prime}(\beta a)}}_{n=0} + \underbrace{\frac{J_1'(\beta a)}{H_1^{(2)\prime}(\beta a)}}_{n=1} e^{j\phi} + \underbrace{\frac{J_2'(\beta a)}{H_2^{(2)\prime}(\beta a)}}_{n=2} e^{j2\phi} + \cdots \right.$$

$$\left. + \underbrace{\frac{J_{-1}'(\beta a)}{H_{-1}^{(2)\prime}(\beta a)}}_{n=-1} e^{-j\phi} + \underbrace{\frac{J_{-2}'(\beta a)}{H_{-2}^{(2)\prime}(\beta a)}}_{n=-2} e^{-j2\phi} + \cdots \right|^2$$

$$\sigma_{2\text{-}D} = \frac{4}{\beta}\left|\frac{J_0'(\beta a)}{H_0^{(2)\prime}(\beta a)} + 2\frac{J_1'(\beta a)}{H_1^{(2)\prime}(\beta a)}\left(\frac{e^{j\phi}+e^{-j\phi}}{2}\right) + 2\frac{J_2'(\beta a)}{H_2^{(2)\prime}(\beta a)}\left(\frac{e^{j2\phi}+e^{-j2\phi}}{2}\right) + \cdots \right|^2$$

$$\sigma_{2\text{-}D} = \frac{4}{\beta}\left|\sum_{n=0}^{\infty} \epsilon_n \frac{J_n'(\beta a)}{H_n^{(2)\prime}(\beta a)} \cos(n\phi)\right|^2 = \frac{2\lambda}{\pi}\left|\sum_{n=0}^{\infty} \epsilon_n \frac{J_n'(\beta a)}{H_n^{(2)\prime}(\beta a)} \cos(n\phi)\right|^2$$

where

$$\epsilon_n = \begin{cases} 1 & n=0 \\ 2 & n\neq 0 \end{cases}$$

$\boxed{11.43}$ According to (11-110b) and (11-110c) we see that the ρ component of the scattered electric field has a $1/\rho$ variation outside the summation while the ϕ component has no ρ variations outside the summation. Within the summation both have the same ρ variations. Therefore we can approximate the far zone scattered electric field by

$$\underline{E}^s = \hat{a}_\rho E_\rho^s + \hat{a}_\phi E_\phi^s \overset{\rho\gg\lambda}{\simeq} \hat{a}_\phi E_\phi^s = \hat{a}_\phi \frac{\beta H_0}{j\omega\varepsilon} \sum_{n=-\infty}^{+\infty} j^{-n} \frac{J_n'(\beta a)}{H_n^{(2)\prime}(\beta a)} H_n^{(2)}(\beta\rho) e^{jn\phi}$$

Since

$$H_n^{(2)}(\beta\rho) \overset{\rho\gg\lambda}{\simeq} \sqrt{\frac{2j}{\pi\beta\rho}}\, j^n e^{-j\beta\rho}$$

$$\underline{E}^s \overset{\rho\gg\lambda}{\simeq} \hat{a}_\phi \frac{\beta H_0}{j\omega\varepsilon}\sqrt{\frac{2j}{\pi\beta\rho}}\, e^{-j\beta\rho} \sum_{n=-\infty}^{+\infty} \frac{J_n'(\beta a)}{H_n^{(2)\prime}(\beta a)} e^{jn\phi} = -\hat{a}_\phi j H_0 \eta \sqrt{\frac{2j}{\pi\beta\rho}}\, e^{-j\beta\rho} \sum_{n=-\infty}^{+\infty} \frac{J_n'(\beta a)}{H_n^{(2)\prime}(\beta a)} e^{jn\phi}$$

$$\underline{E}^s \overset{\rho\gg\lambda}{\simeq} -\hat{a}_\phi j H_0 \eta \sqrt{\frac{2j}{\pi\beta\rho}}\, e^{-j\beta\rho} \sum_{n=-\infty}^{+\infty} \frac{J_n'(\beta a)}{H_n^{(2)\prime}(\beta a)} e^{jn\phi}, \quad \underline{E}^i = \hat{a}_y \eta H_0$$

$$\sigma_{2\text{-}D} = \lim_{\rho\to\infty}\left\{2\pi\rho \frac{|\underline{E}^s|^2}{|\underline{E}^i|^2}\right\} = \lim_{\rho\to\infty}\left\{2\pi\rho \frac{2}{\pi\beta\rho}\left|\sum_{n=-\infty}^{+\infty} \frac{J_n'(\beta a)}{H_n^{(2)\prime}(\beta a)} e^{jn\phi}\right|^2\right\}$$

$$\sigma_{2\text{-}D} = \frac{4}{\beta}\left|\sum_{n=-\infty}^{+\infty} \frac{J_n'(\beta a)}{H_n^{(2)\prime}(\beta a)} e^{jn\phi}\right|^2 = \frac{2\lambda}{\pi}\left|\sum_{n=-\infty}^{+\infty} \frac{J_n'(\beta a)}{H_n^{(2)\prime}(\beta a)} e^{jn\phi}\right|^2$$

$\boxed{11.44}$ $\underline{E}^i = (\hat{a}_y + j\hat{a}_z)e^{-j\beta x}$: Circular Polarization, CCW

(a) TE^z: $E_y^i = e^{-j\beta x}$ Because plane wave $\Rightarrow H_z^i = \frac{1}{\eta}e^{-j\beta x}$

According to (11-115), the scattered far-zone H-field is

$$H_z^s = -\frac{1}{\eta}\sqrt{\frac{j\beta}{\pi\beta}}\frac{e^{-j\beta\rho}}{\sqrt{\rho}}\sum_{n=-\infty}^{+\infty}\frac{J_n'(\beta a)}{H_n^{(2)\prime}(\beta a)}e^{jn\phi} = -\frac{1}{\eta}\sqrt{\frac{j\beta}{\pi\beta}}\frac{e^{-j\beta\rho}}{\sqrt{\rho}}\sum_{n=0}^{+\infty}\varepsilon_n\frac{J_n'(\beta a)}{H_n^{(2)\prime}(\beta a)}\cos(n\phi)$$

and the scattered far-zone electric field is $\varepsilon_n = \begin{cases} 1 & n=0 \\ 2 & n\neq 0 \end{cases}$

$$E_\phi^s \simeq \eta H_z^s = -\sqrt{\frac{j\beta}{\pi\beta}}\frac{e^{-j\beta\rho}}{\sqrt{\rho}}\sum_{n=-\infty}^{+\infty}\frac{J_n'(\beta a)}{H_n^{(2)\prime}(\beta a)}e^{jn\phi} = -\sqrt{\frac{j\beta}{\pi\beta}}\frac{e^{-j\beta\rho}}{\sqrt{\rho}}\sum_{n=-\infty}^{+\infty}\frac{J_n'(\beta a)}{H_n^{(2)\prime}(\beta a)}\cos(n\phi)$$

TM^z: $E_z^i = j\,e^{-j\beta x}$ According to (11-100), the far-zone E-field is

$$E_z^s = -j\sqrt{\frac{j\beta}{\pi\beta}}\frac{e^{-j\beta\rho}}{\sqrt{\rho}}\sum_{n=-\infty}^{+\infty}\frac{J_n(\beta a)}{H_n(\beta a)}e^{jn\phi} = -j\sqrt{\frac{j\beta}{\pi\beta}}\frac{e^{-j\beta\rho}}{\sqrt{\rho}}\sum_{n=0}^{+\infty}\varepsilon_n\frac{J_n(\beta a)}{H_n^{(2)}(\beta a)}\cos(n\phi)$$

TOTAL (TE^z+TM^z):

$$\underline{E}^s(\text{total}) \overset{\beta\rho\to\text{large}}{\simeq} -\sqrt{\frac{j\beta}{\pi\beta}}\frac{e^{-j\beta\rho}}{\sqrt{\rho}}\sum_{n=0}^{+\infty}\varepsilon_n\left[\hat{a}_\phi\frac{J_n'(\beta a)}{H_n^{(2)\prime}(\beta a)} + j\hat{a}_z\frac{J_n(\beta a)}{H_n^{(2)}(\beta a)}\right]\cos(n\phi)$$

$\varepsilon_n = \begin{cases} 1 & n=0 \\ 2 & n\neq 0 \end{cases}$

(b) • $\phi = 0°$: According to (11-6): $\hat{a}_\phi = \left[-\hat{a}_x\sin\phi + \hat{a}_y\cos\phi\right]_{\phi=0} = \hat{a}_y$

$$\underline{E}^s(\text{total})\Big|_{\phi=0} \simeq \underline{E}^s(\text{total}) = \hat{a}_y E_y^s + j\hat{a}_z E_z^s$$

$$\begin{pmatrix}\text{forward} \\ \text{direction}\end{pmatrix} \simeq -j\sqrt{\frac{j\beta}{\pi\beta}}\frac{e^{-j\beta\rho}}{\sqrt{\rho}}\sum_{n=0}^{\infty}\varepsilon_n\left[\hat{a}_y\frac{J_n'(\beta a)}{H_n^{(2)\prime}(\beta a)} + j\hat{a}_z\frac{J_n(\beta a)}{H_n^{(2)}(\beta a)}\right]$$

$\therefore$ Elliptical Polarization, CCW which agrees with what is expected (see Figure 5.16, Section 5.6)

• $\phi = 180°$: According to (11-6): $\hat{a}_\phi = \left[-\hat{a}_x\sin\phi + \hat{a}_y\cos\phi\right]_{\phi=180°} = -\hat{a}_y$

$$\underline{E}^s(\text{total})\Big|_{\phi=180°} \simeq \underline{E}^s(\text{total}) = \hat{a}_y E_y^s + j\hat{a}_z E_z^s$$

Backscattered direction

$$\simeq -j\sqrt{\frac{j\beta}{\pi\beta}}\frac{e^{-j\beta\rho}}{\sqrt{\rho}}\sum_{n=0}^{\infty}(-1)^n\varepsilon_n\left[-\hat{a}_y\frac{J_n'(\beta a)}{H_n^{(2)\prime}(\beta a)} + j\hat{a}_z\frac{J_n(\beta a)}{H_n^{(2)}(\beta a)}\right]$$

$\therefore$ Elliptical Polarization, CW which agrees with what is expected (See Figure 5.16, Section 5.6 discussion).

$\underline{E}^{i} = (\hat{a}_y + j\hat{a}_z)e^{-j\beta x}$: Circular Polarization, CCW

$\boxed{11.45}$ Based on duality, the scattering characteristics

(a) of a PMC cylinder can be obtained from those of a PEC cylinder. Accordingly, the:

TM^z scattering from a PMC cylinder is the same as the TE^z scattering from a PEC cylinder.

See solution of Problem 11.56.

TE^z scattering from a PMC cylinder is the same as the TM^z scattering from a PEC cylinder.

See solution of Problem 11.57.

Based upon that, and without rederiving the expressions of the scattered field but rather using duality and referring to the solution of Problem 11.44, we can write the far-zone scattered field from a PMC cylinder of circular cross section and radius a, as:

TOTAL (TE^z + TM^z):

$$\underline{E}^s(\text{total}) \underset{\beta\rho \to \text{large}}{\simeq} +\sqrt{\frac{j\beta}{\pi\beta}}\frac{e^{-j\beta\rho}}{\sqrt{\rho}}\sum_{n=0}^{\infty}\varepsilon_n\left[\hat{a}_\phi\frac{J_n(\beta a)}{H_n^{(2)}(\beta a)} + j\hat{a}_z\frac{J_n'(\beta a)}{H_n^{(2)'}(\beta a)}\right]\cos(n\phi)$$

(b) • $\phi = 0°$: According to (11-6): $\hat{a}_\phi = \left[-\hat{a}_x\sin\phi + \hat{a}_y\cos\phi\right]_{\phi=0} = \hat{a}_y$

$$\underline{E}^s(\text{total})\Big|_{\phi=0°} \simeq j\sqrt{\frac{j\beta}{\pi\beta}}\frac{e^{-j\beta\rho}}{\sqrt{\rho}}\sum_{n=0}^{\infty}\varepsilon_n\left[\hat{a}_y\frac{J_n(\beta a)}{H_n^{(2)}(\beta a)} + j\hat{a}_z\frac{J_n'(\beta a)}{H_n^{(2)'}(\beta a)}\right]$$

(forward direction)

$\therefore$ Elliptical Polarization, CCW which with what is expected (see Figure 5.16, Section 5.6 discussion).

• $\phi = 180°$: According to (11-6): $\hat{a}_\phi = \left[-\hat{a}_x\sin\phi + \hat{a}_y\cos(\phi)\right]_{\phi=180°} = -\hat{a}_y$

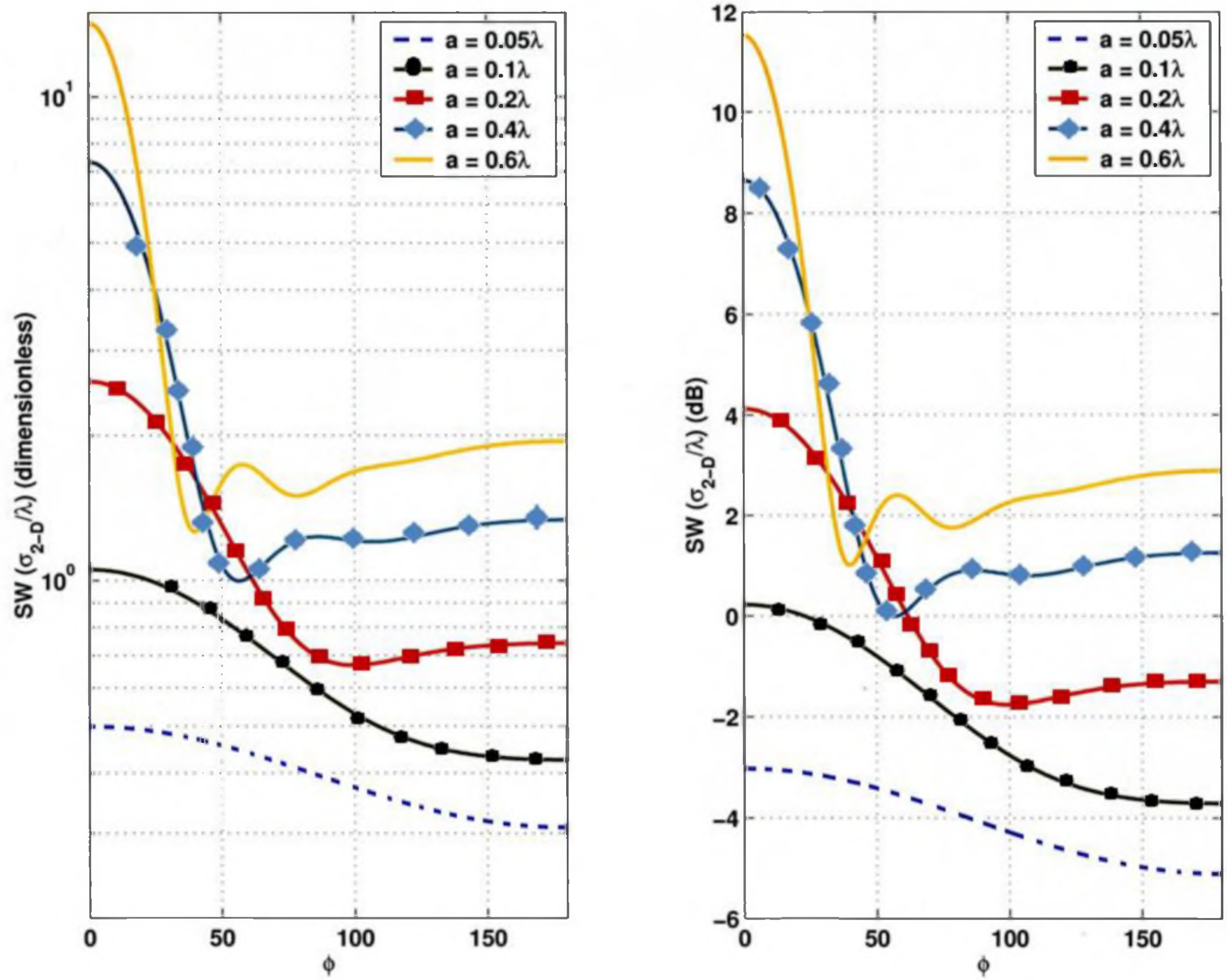
$$\underline{E}^s(\text{total})\Big|_{\phi=180°} \simeq +j\sqrt{\frac{j\beta}{\pi\beta}}\frac{e^{-j\beta\rho}}{\sqrt{\rho}}\sum_{n=0}^{\infty}(-1)^n\varepsilon_n\left[-\hat{a}_y\frac{J_n(\beta a)}{H_n^{(2)}(\beta a)} + j\hat{a}_z\frac{J_n'(\beta a)}{H_n^{(2)'}(\beta a)}\right]$$

Backscattered direction

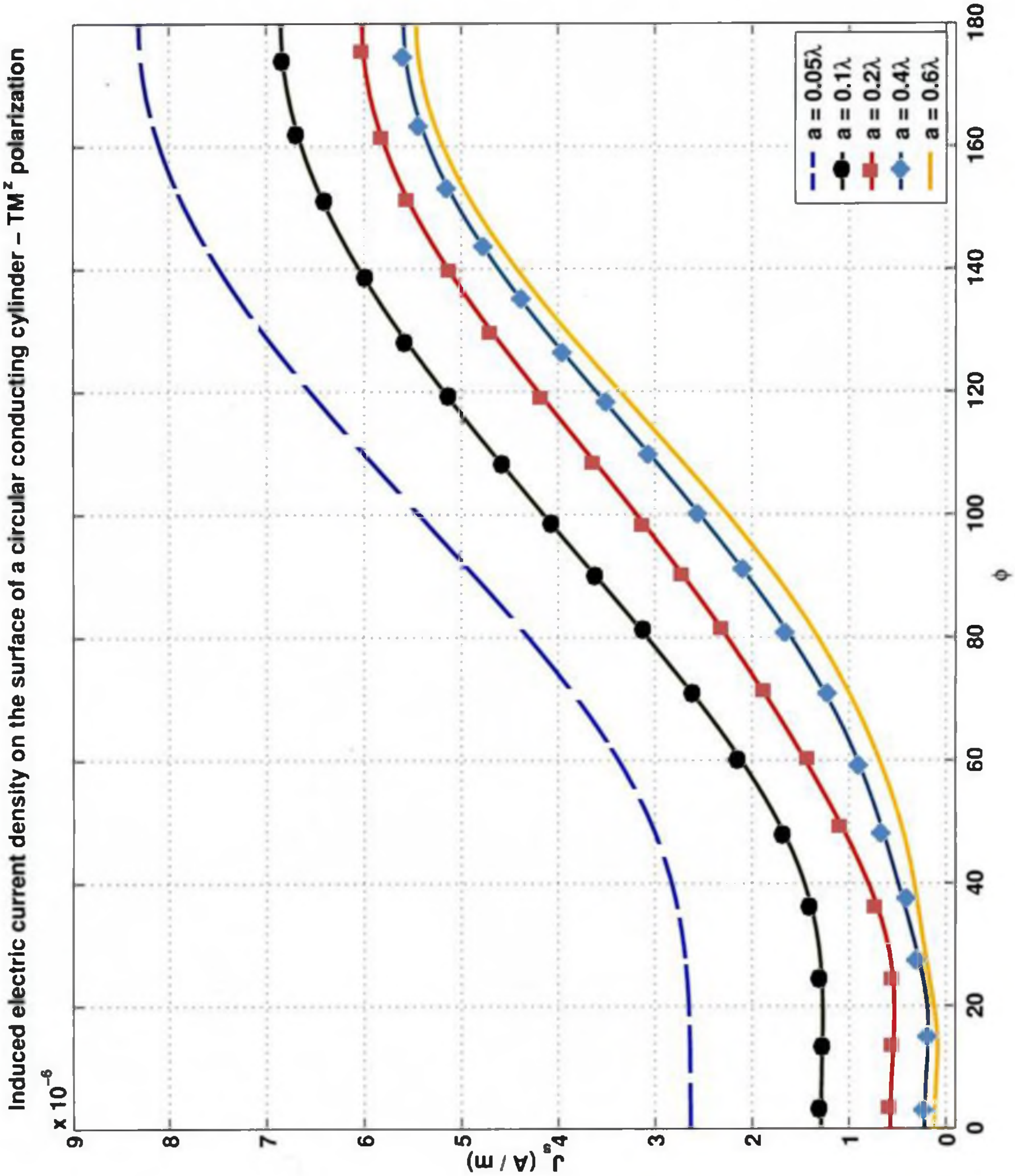
$\therefore$ Elliptical Polarization, CW which agrees with what is expected (see Figure 5.16, Section 5.6 discussion).

11.46

Two-dimensional TM^z bistatic scattering width (SW) of a circular conducting cylinder

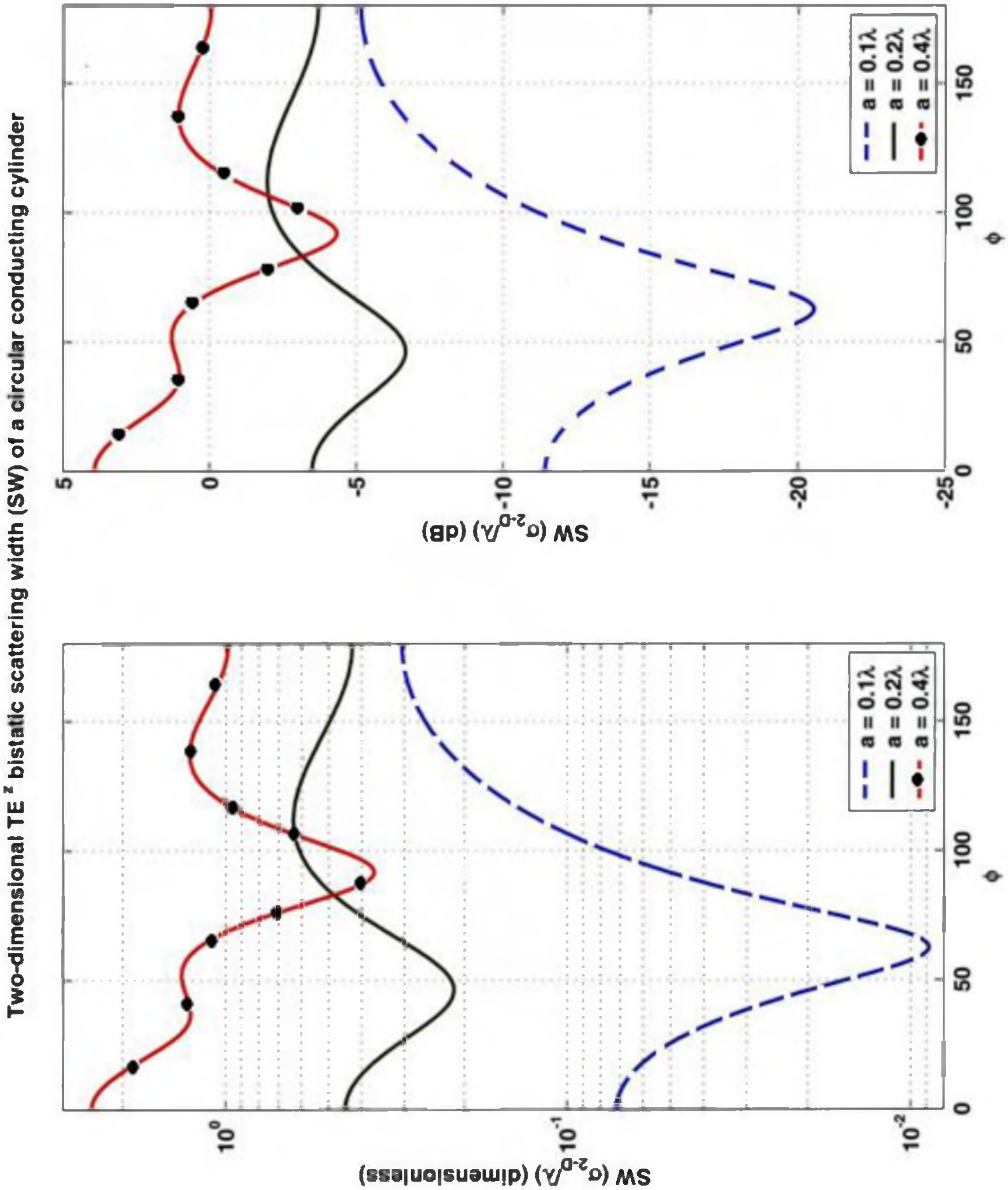


1147



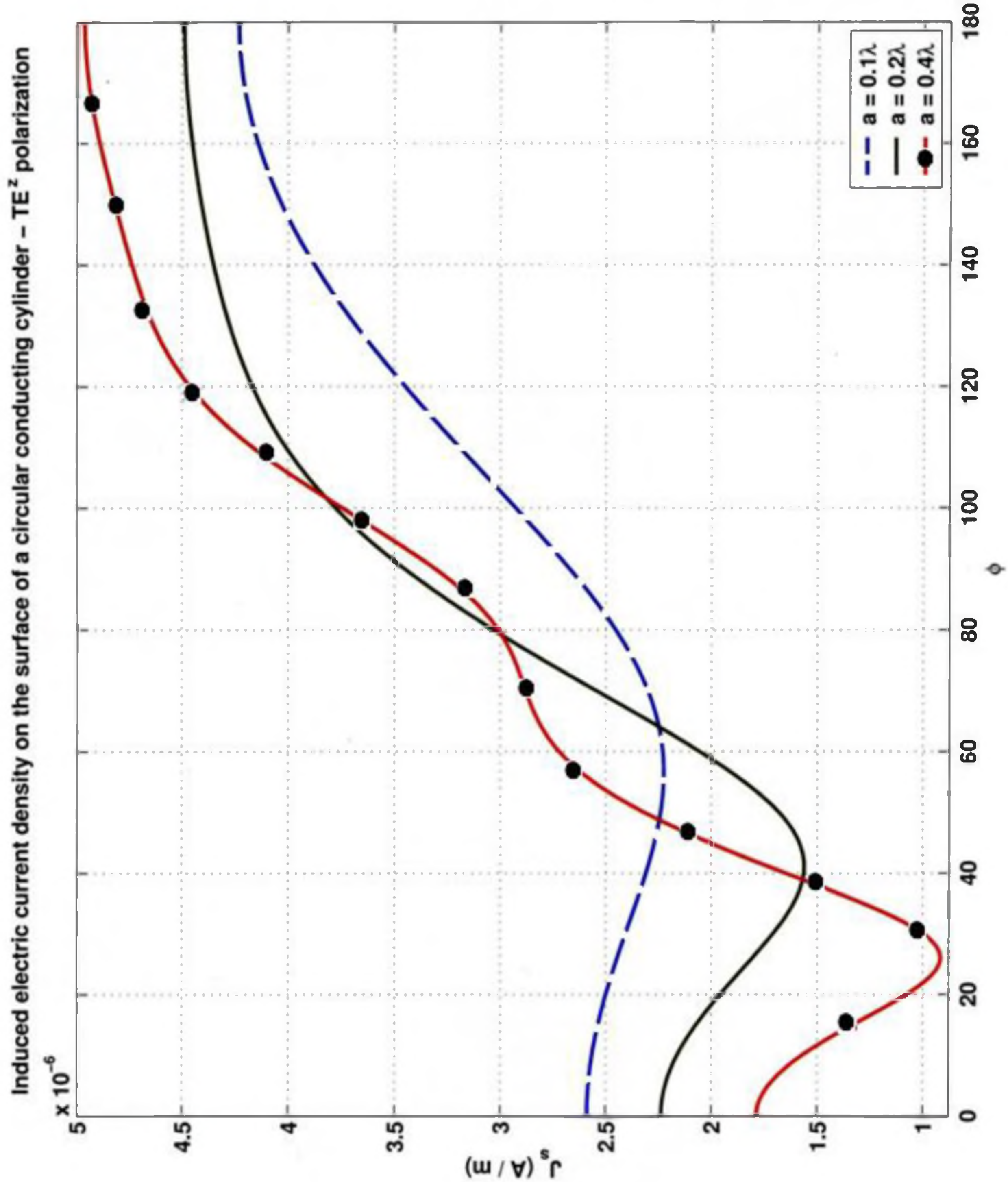
Induced electric current density on the surface of a circular conducting cylinder – TM^z polarization
x 10^{-6}
J$_s$ (A / m)
φ
0
1
2
3
4
5
6
7
8
9
0
20
40
60
80
100
120
140
160
180
a = 0.05λ
a = 0.1λ
a = 0.2λ
a = 0.4λ
a = 0.6λ

11.48



Two-dimensional TE z bistatic scattering width (SW) of a circular conducting cylinder
SW (σ2-D/λ) (dimensionless)
SW (σ2-D/λ) (dB)
φ
a = 0.1λ
a = 0.2λ
a = 0.4λ

11.49



Induced electric current density on the surface of a circular conducting cylinder – TE^z polarization
x 10^{-6}
J_s (A / m)
ϕ
a = 0.1λ
a = 0.2λ
a = 0.4λ

$\boxed{11.50}$ For a small ($a \ll \lambda$) PEC cylinder the bistatic scattering TM^z width (SW) is given by (11-103a), or

$$\sigma_{2-D} \overset{a\ll\lambda}{\simeq} \frac{\pi\lambda}{2}\left|\frac{1}{\ln(0.89\beta a)}\right|^2$$

which is independent of the observation angle ϕ. Therefore there is no maximum or minimum on the pattern based on this approximate expression.

However based on the results of Figure 11.46, there seems to be a

(a) Maximum at $\phi = 0°$.

(b) Minimum at $\phi = 180°$.

$\boxed{11.51}$ For a small ($a \ll \lambda$) PEC cylinder the bistatic TE^z scattering width (SW) is given by (11-118d), or

$$\sigma_{2-D} \overset{a\ll\lambda}{\simeq} \frac{\pi\lambda}{8}(\beta a)^2\left[1-2\cos(\phi)\right]^2$$

(a) Its maximum value occur when

$$|1-2\cos(\phi)| = 3 \Rightarrow \cos\phi = -1 \Rightarrow \phi = 180°$$

This seems to be confirmed by the curves of Figure 11.48.

(b) Its minimum value occurs when

$$|1-2\cos(\phi)| = 0 \Rightarrow 2\cos\phi = 1 \Rightarrow \cos\phi = 1/2$$

$$\phi = \cos^{-1}(1/2) = 60°$$

This seems to be confirmed by the curves of Fig. 11.48.

11.52 TM^z:

For a small ($a \ll \lambda$) PEC cylinder the TM^z bistatic SW is given by

$$\sigma_{2-D} \simeq \frac{\pi\lambda}{2}\left|\frac{1}{\ln(0.89\beta a)}\right|^2$$

which is independent of the observation angle ϕ. To maintain the normalized SW (σ/λ) to less than -20dB (0.01 dimensionless), the radius must be such that

$$\frac{\sigma}{\lambda} = \frac{\pi}{2}\left|\frac{1}{\ln(0.89\beta a)}\right|^2 = \frac{1}{100} \Rightarrow |\ln(0.89\beta a)| = \pm 12.5331$$

Choosing

$$\ln(0.89\beta a) = -12.5331 \Rightarrow 0.89\beta a = 3.6053\times10^{-6} \Rightarrow \boxed{a = 6.4472\times10^{-7}\lambda}$$

11.53 TE^z:

For a small ($a \ll \lambda$) PEC cylinder the TE^z bistatic SW is given by

$$\sigma_{2-D}\Big|_{max} = \frac{\pi\lambda}{8}(\beta a)^2\left[1-2\cos(\phi)\right]^2\Big|_{max} = \frac{\pi\lambda}{8}(\beta a)^2(3) \text{ @ } \phi = 180^\circ$$

To maintain the normalized SW (σ/λ) to less than -20dB (0.01 dimensionless), the radius must be such that

$$\frac{\sigma}{\lambda} = \frac{\pi}{8}(\beta a)^2(3) = 0.01 \Rightarrow \beta a = \sqrt{\frac{8}{300\pi}} \Rightarrow a = \frac{\lambda}{2\pi}\sqrt{\frac{8}{300\pi}}$$

$$\boxed{a = 0.01471\lambda}$$

11.54 TM^z:

Based on the solution of Problem 11.52, to main the normalized SW (σ/λ) to less than -10dB (0.1 dimensionless), the radius must be

$$\frac{\sigma}{\lambda} = \frac{\pi}{2}\left|\frac{1}{\ln(0.89\beta a)}\right|^2 = \frac{1}{10} \Rightarrow |\ln(0.89\beta a)| = \pm 3.9633$$

Choosing $\ln(0.89\beta a) = -3.9633 \Rightarrow 0.89\beta a = 0.019 \Rightarrow \beta a = \frac{0.019}{0.89}$

$$a = \frac{\lambda}{2\pi}\left(\frac{0.019}{0.89}\right) = \boxed{3.398\times10^{-3}\lambda}$$

11.55 TE^z:

Based on the solution of Problem 11.53, to maintain the normalized SW (σ/λ) to less than -10 dB (0.1 dimensionless), the radius must be

$$\frac{\sigma}{\lambda}\Big|_{max} = \frac{\pi}{8}(\beta a)^2\left[1-2\cos\phi\right]^2\Big|_{max} = \frac{\pi}{8}(\beta a)^2(3) = 0.1 \text{ @ } \phi = 180^\circ$$

$$\beta a = \sqrt{\frac{8}{30\pi}} \Rightarrow a = \frac{\lambda}{2\pi}\sqrt{\frac{8}{30\pi}} = \boxed{4.637\times10^{-2}\lambda}$$

11.56 $\underline{TM^z}$: $\underline{E}^i = \hat{a}_z E_0 e^{-j\beta x} = \hat{a}_z E_0 e^{-j\beta\rho\cos\phi}$

Using duality, the magnetic fields are obtained from (11-111a)–(11-111d). We can write that (from the TE^z PEC and using duality)

$$H_z^t = 0$$

$$H_\rho = \frac{H_0}{j\omega\mu}\frac{1}{\rho}\sum_{n=-\infty}^{+\infty} n j^{-n+1}\left[J_n(\beta\rho) - \frac{J_n'(\beta a)}{H_n^{(2)'}(\beta a)} H_n^{(2)}(\beta\rho)\right] e^{jn\phi}$$

$$H_\phi^t = -\frac{\beta H_0}{j\omega\mu}\sum_{n=-\infty}^{+\infty} j^{-n}\left[J_n'(\beta\rho) - \frac{J_n'(\beta a)}{H_n^{(2)'}(\beta a)} H_n^{(2)'}(\beta\rho)\right] e^{jn\phi}$$

(a) The 2-D RCS(SW) for the TM^z PMC surface is the same, due to duality, as that of the TE^z PEC surface, or from (11-117)

$$\sigma_{2-D} = \frac{4}{\beta}\left|\sum_{n=-\infty}^{+\infty}\frac{J_n'(\beta a)}{H_n^{(2)'}(\beta a)} e^{jn\phi}\right|^2 = \frac{2\lambda}{\pi}\left|\sum_{n=0}^{\infty}\varepsilon_n \frac{J_n'(\beta a)}{H_n^{(2)'}(\beta a)}\cos(n\phi)\right|^2$$

$$\varepsilon_n = \begin{cases} 1 & n=0 \\ 2 & n\neq 0\end{cases}$$

(b) The current density on the surface of the cylinder ($\rho = a$), it is given by

$$\underline{J}_s = \hat{n}\times\underline{H}^t\Big|_{\rho=a} = \hat{a}_\rho\times(\hat{a}_\rho H_\rho^t + \hat{a}_\phi H_\phi)\Big|_{\rho=a} = \hat{a}_z H_\phi^t\Big|_{\rho=a}$$

$$\underline{J}_s = \hat{a}_z H_\phi^t(\rho=a) = \hat{a}_z\left[-\frac{\beta H_0}{j\omega\mu}\sum_{n=-\infty}^{+\infty} j^{-n}\left[J_n'(\beta a) - \frac{J_n'(\beta a)}{H_n^{(2)'}(\beta a)} H_n^{(2)'}(\beta a)\right] e^{jn\phi}\right.$$

$$= \hat{a}_z\left\{-\frac{\beta H_0}{j\omega\mu}\sum_{n=-\infty}^{+\infty} j^{-n}\left[\frac{J_n'(\beta a) H_n^{(2)'}(\beta a) - J_n'(\beta a) H_n^{(2)'}(\beta a)}{H_n^{(2)'}(\beta a)}\right] e^{jn\phi}\right\} = 0$$

as it should be because the tangential $\underline{H}$ field vanishes next to a PMC surface; i.e., PMC surface cannot support an electric current density.

$\boxed{11.57}$ TE^z: $\underline{H}^i = \hat{a}_z H_0 \sum_{n=-\infty}^{+\infty} j^{-n} J_n(\beta\rho) e^{jn\phi}$

Using duality, the magnetic fields are obtained from (11-93a)-(11-93d). We can write that (from the TM^z PEC and using duality)

$$H_\rho^t = 0 = H_\phi^t$$

$$H_z^t = H_0 \sum_{n=-\infty}^{+\infty} j^{-n} \left[J_n(\beta\rho) - \frac{J_n(\beta a)}{H_n^{(2)}(\beta a)} H_n^{(2)}(\beta\rho) \right] e^{jn\phi}$$

(a) The 2-D RCS (SW) for the TE^z PMC surface is the same, due to duality, as that of the TM^z PEC surface, or from (11-102)

$$\sigma_{2-D} = \frac{4}{\beta} \left| \sum_{n=-\infty}^{+\infty} \frac{J_n(\beta a)}{H_n^{(2)}(\beta a)} e^{jn\phi} \right|^2 = \frac{2\lambda}{\pi} \left| \sum_{n=0}^{\infty} \varepsilon_n \frac{J_n(\beta a)}{H_n^{(2)}(\beta a)} \cos(n\phi) \right|^2$$

$$\varepsilon_n = \begin{cases} 1 & n=0 \\ 2 & n \neq 0 \end{cases}$$

(b) The current density on the surface of the cylinder ($\rho=a$), it is given by

$$\underline{J}_s = \hat{n} \times \underline{H}^t \Big|_{\rho=a} = \hat{a}_\rho \times (\hat{a}_z H_z^t) \Big|_{\rho=a} = -\hat{a}_\phi H_z^t \Big|_{\rho=a}$$

$$= -\hat{a}_\phi H_z^t(\rho=a) = -\hat{a}_\phi \left[H_0 \sum_{n=-\infty}^{+\infty} j^{-n} \left[J_n(\beta a) - \frac{J_n(\beta a)}{H_n^{(2)}(\beta a)} H_n^{(2)}(\beta a) \right] e^{jn\phi} \right]$$

$$\underline{J}_s = -\hat{a}_\phi \left\{ H_0 \sum_{n=-\infty}^{+\infty} \left[\frac{J_n(\beta a) H_n^{(2)}(\beta a) - J_n(\beta a) H_n^{(2)}(\beta a)}{H_n^{(2)}(\beta a)} \right] e^{jn\phi} \right\} = 0$$

as it should be because the tangential $\underline{H}$ field vanishes next to a PMC surface; i.e., PMC surface cannot support an electric current density.

$\boxed{11.58}$ $\underline{E}^i = \hat{a}_z E_0 \sum_{n=-\infty}^{+\infty} j^{-n} J_n(\beta_0\rho) e^{jn\phi}$ $\rho \geq a$

$\underline{E}^s = \hat{a}_z E_0 \sum_{n=-\infty}^{+\infty} a_n H_n^{(2)}(\beta_0\rho) e^{jn\phi}$ $\rho \geq a$

$\underline{E}^d = \hat{a}_z E_0 \sum_{n=-\infty}^{+\infty} \left[b_n J_n(\beta_1\rho) + c_n Y_n(\beta_1\rho) \right] e^{jn\phi}$ $a \geq \rho \geq 0$

a. $\nabla \times \underline{E} = -j\omega\mu \underline{H} \Rightarrow \underline{H} = -\frac{1}{j\omega\mu} \nabla \times \underline{E} = -\frac{1}{j\omega\mu} \left\{ \hat{a}_\rho \frac{1}{\rho} \frac{\partial E_z}{\partial \phi} - \hat{a}_\phi \frac{\partial E_z}{\partial \rho} \right\}$

$H_\rho = -\frac{1}{j\omega\mu} \frac{1}{\rho} \frac{\partial E_z}{\partial \phi}$, $H_\phi = \frac{1}{j\omega\mu} \frac{\partial E_z}{\partial \rho}$

Therefore

$$H_\rho^i = -\frac{E_0}{j\omega\mu_0} \frac{1}{\rho} \sum_{n=-\infty}^{+\infty} n j^{-n+1} J_n(\beta_0\rho) e^{jn\phi}$$

$$H_\phi^i = \frac{E_0 \beta_0}{j\omega\mu_0} \sum_{n=-\infty}^{+\infty} j^{-n} J_n'(\beta_0\rho) e^{jn\phi}$$

$$H_\rho^s = -\frac{E_0}{j\omega\mu_0} \frac{1}{\rho} \sum_{n=-\infty}^{+\infty} jn a_n H_n^{(2)}(\beta_0\rho) e^{jn\phi}$$

$$H_\phi^s = \frac{E_0 \beta_0}{j\omega\mu_0} \sum_{n=-\infty}^{+\infty} a_n H_n^{(2)\prime}(\beta_0\rho) e^{jn\phi}$$

$$H_\rho^d = -\frac{E_0}{j\omega\mu_1} \frac{1}{\rho} \sum_{n=-\infty}^{+\infty} jn \left[b_n J_n(\beta_1\rho) + c_n Y_n(\beta_1\rho) \right] e^{jn\phi}$$

$$H_\phi^d = \frac{E_0 \beta_1}{j\omega\mu_1} \sum_{n=-\infty}^{+\infty} \left[b_n J_n'(\beta_1\rho) + c_n Y_n'(\beta_1\rho) \right] e^{jn\phi}$$

b. Since the fields within the dielectric cylinder must be finite everywhere (including $\rho=0$), then $c_n = 0$. Matching tangential components of electric and magnetic fields at $\rho = a$, leads to

$(E_z^i + E_z^s)_{\rho=a} = E_z^d)_{\rho=a}$; $(H_\phi^i + H_\phi^s)_{\rho=a} = H_\phi^d)_{\rho=a}$

$$\sum_{n=-\infty}^{+\infty} \left[j^{-n} J_n(\beta_0 a) + a_n H_n^{(2)}(\beta_0 a) \right] e^{jn\phi} = \sum_{n=-\infty}^{+\infty} b_n J_n(\beta_1 a) e^{jn\phi}$$

$$\frac{\beta_0}{\mu_0} \sum_{n=-\infty}^{+\infty} \left[j^{-n} J_n'(\beta_0 a) + a_n H_n^{(2)\prime}(\beta_0 a) \right] = \frac{\beta_1}{\mu_1} \sum_{n=-\infty}^{+\infty} b_n J_n'(\beta_1 a) e^{jn\phi}$$

$$a_n H_n^{(2)}(\beta_0 a) - b_n J_n(\beta_1 a) = -j^{-n} J_n(\beta_0 a)$$

$$a_n H_n^{(2)\prime}(\beta_0 a) - b_n \frac{\beta_1}{\beta_0} \frac{\mu_0}{\mu_1} J_n'(\beta_1 a) = -j^{-n} J_n'(\beta_0 a)$$

or

$$a_n H_n^{(2)}(\beta_0 a) - b_n J_n(\beta_1 a) = -j^{-n} J_n(\beta_0 a) \quad (1)$$

$$a_n H_n^{(2)\prime}(\beta_0 a) - b_n \sqrt{\frac{\epsilon_r}{\mu_r}} J_n'(\beta_1 a) = -j^{-n} J_n'(\beta_0 a) \quad (2)$$

Cont'd.

$\boxed{11.58 \text{ cont'd.}}$ Multiplying (1) by $\sqrt{\frac{\epsilon_r}{\mu_r}}J_n'(\beta_1 a)$ and (2) by $J_n(\beta_1 a)$, we get

$$a_n\sqrt{\epsilon_r/\mu_r}\,H_n^{(2)}(\beta_0 a)J_n'(\beta_1 a) - b_n\sqrt{\epsilon_r/\mu_r}\,J_n(\beta_1 a)J_n'(\beta_1 a) = -j^{-n}J_n(\beta_0 a)J_n'(\beta_1 a)\sqrt{\epsilon_r/\mu_r} \quad (3)$$

$$a_n\, H_n^{(2)\prime}(\beta_0 a)J_n(\beta_1 a) - b_n\sqrt{\epsilon_r/\mu_r}\,J_n(\beta_1 a)J_n'(\beta_1 a) = -j^{-n}J_n'(\beta_0 a)J_n(\beta_1 a) \quad (4)$$

Subtracting (4) from (3) leads to

$$a_n\left[\sqrt{\epsilon_r/\mu_r}\,H_n^{(2)}(\beta_0 a)J_n'(\beta_1 a) - H_n^{(2)\prime}(\beta_0 a)J_n(\beta_1 a)\right] = -j^{-n}\left[\sqrt{\epsilon_r/\mu_r}\,J_n(\beta_0 a)J_n'(\beta_1 a) - J_n'(\beta_0 a)J_n(\beta_1 a)\right]$$

$$a_n = j^{-n}\,\frac{J_n'(\beta_0 a)J_n(\beta_1 a) - \sqrt{\epsilon_r/\mu_r}\,J_n(\beta_0 a)J_n'(\beta_1 a)}{\sqrt{\epsilon_r/\mu_r}\,J_n'(\beta_1 a)H_n^{(2)}(\beta_0 a) - J_n(\beta_1 a)H_n^{(2)\prime}(\beta_0 a)} \quad (5)$$

Multiplying (1) by $H_n^{(2)\prime}(\beta_0 a)$ and (2) by $H_n^{(2)}(\beta_0 a)$, we get

$$a_n H_n^{(2)}(\beta_0 a)H_n^{(2)\prime}(\beta_0 a) - b_n J_n(\beta_1 a)H_n^{(2)\prime}(\beta_0 a) = -j^{-n}J_n(\beta_0 a)H_n^{(2)\prime}(\beta_0 a) \quad (6)$$

$$a_n H_n^{(2)}(\beta_0 a)H_n^{(2)\prime}(\beta_0 a) - b_n\sqrt{\frac{\epsilon_r}{\mu_r}}J_n'(\beta_1 a)H_n^{(2)}(\beta_0 a) = -j^{-n}J_n'(\beta_0 a)H_n^{(2)}(\beta_0 a) \quad (7)$$

Subtracting (7) from (6) leads to

$$b_n\left[\sqrt{\epsilon_r/\mu_r}\,J_n'(\beta_1 a)H_n^{(2)}(\beta_0 a) - J_n(\beta_1 a)H_n^{(2)\prime}(\beta_0 a)\right] = j^{-n}\left[J_n'(\beta_0 a)H_n^{(2)}(\beta_0 a) - J_n(\beta_0 a)H_n^{(2)\prime}(\beta_0 a)\right]$$

$$b_n = j^{-n}\,\frac{J_n(\beta_0 a)H_n^{(2)\prime}(\beta_0 a) - J_n'(\beta_0 a)H_n^{(2)}(\beta_0 a)}{J_n(\beta_1 a)H_n^{(2)\prime}(\beta_0 a) - \sqrt{\epsilon_r/\mu_r}\,J_n'(\beta_1 a)H_n^{(2)}(\beta_0 a)} \quad (8)$$

$\boxed{11.59}$ c.

$$\sigma_{2D} = \lim_{\rho\to\infty}\left[2\pi\rho\,\frac{|E_z^s|^2}{|E_z^i|^2}\right]$$

$$E_z^s = E_0\sum_{n=-\infty}^{+\infty} a_n H_n^{(2)}(\beta_0\rho)e^{jn\phi} \overset{\beta_0\rho\to\infty}{\simeq} E_0\sum_{n=-\infty}^{+\infty} a_n\sqrt{\frac{2j}{\pi\beta_0\rho}}\,j^n e^{-j\beta_0\rho}e^{jn\phi}$$

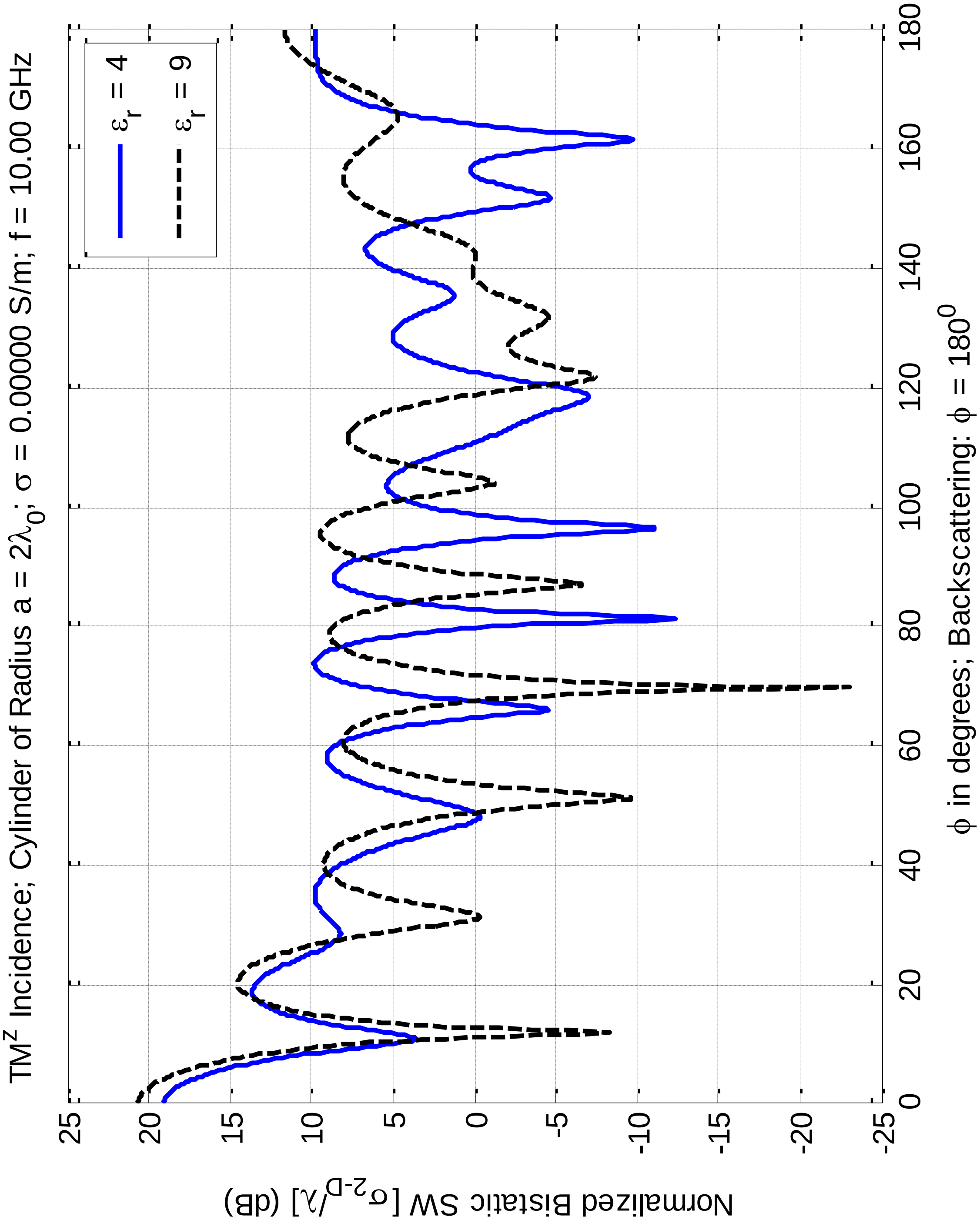
$$= E_0\sqrt{\frac{2j}{\pi\beta_0\rho}}\,e^{-j\beta_0\rho}\sum_{n=-\infty}^{+\infty} a_n j^n e^{jn\phi}$$

$$E_z^s = E_0\sqrt{\frac{2j}{\pi\beta_0\rho}}\,e^{-j\beta_0\rho}\sum_{n=0}^{\infty}\varepsilon_n a_n j^n\cos(n\phi), \quad \varepsilon_n = \begin{cases}1 & n=0\\ 2 & n\neq 0\end{cases}$$

$$\sigma_{2D} = \frac{4}{\beta_0}\left|\sum_{n=0}^{\infty}\varepsilon_n a_n j^n\cos(n\phi)\right|^2 = \frac{2\lambda_0}{\pi}\left|\sum_{n=0}^{\infty}\varepsilon_n a_n j^n\cos(n\phi)\right|^2$$

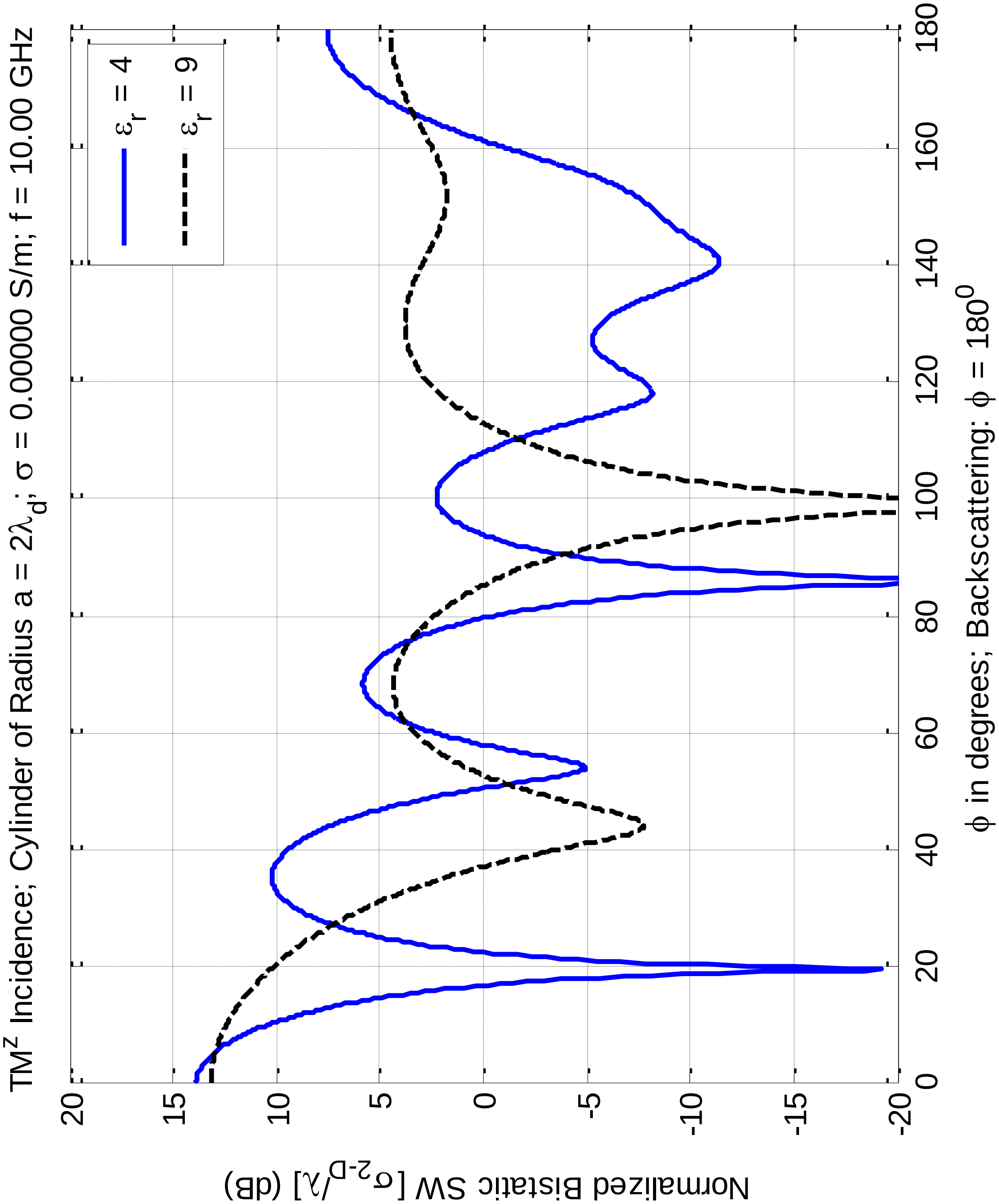
d. See attached figures.

11.59



TM^z Incidence; Cylinder of Radius a = 2λ_0; σ = 0.00000 S/m; f = 10.00 GHz
Normalized Bistatic SW [$\sigma_{2\text{-}D}/\lambda$] (dB)
ϕ in degrees; Backscattering: ϕ = 180^0
ε_r = 4
ε_r = 9
25
20
15
10
5
0
-5
-10
-15
-20
-25
0
20
40
60
80
100
120
140
160
180

11.59


TM^z Incidence; Cylinder of Radius a = 2λ_d; σ = 0.00000 S/m; f = 10.00 GHz
ε_r = 4
ε_r = 9
ϕ in degrees; Backscattering: ϕ = 180^0
Normalized Bistatic SW [$\sigma_{2\text{-D}}/\lambda$] (dB)
0
20
40
60
80
100
120
140
160
180
20
15
10
5
0
-5
-10
-15
-20

$\boxed{11.60}$ The reflection coefficient of a plane wave from a planar interface formed by two media is given by (5-4a) (for normal incidence)

$$\Gamma = \frac{\eta_2-\eta_1}{\eta_2+\eta_1} = \frac{\sqrt{\frac{\mu_2}{\varepsilon_2}}-\sqrt{\frac{\mu_1}{\varepsilon_1}}}{\sqrt{\frac{\mu_2}{\varepsilon_2}}+\sqrt{\frac{\mu_1}{\varepsilon_1}}}$$

where η_2 is the intrinsic impedance of the medium the wave is impinging upon while η_1 is the intrinsic impedance of the wave is traveling in. Assuming medium 2 has a dielectric constant $\varepsilon_{r2}/\varepsilon_{r1} = \varepsilon_r \gg 1$, while their permeabilities are the same, the reflection coefficient can be written as

$$\Gamma = \frac{1-\sqrt{\varepsilon_2/\varepsilon_1}}{1+\sqrt{\varepsilon_2/\varepsilon_1}} = \frac{1-\sqrt{\varepsilon_2/\varepsilon_0}}{1+\sqrt{\varepsilon_2/\varepsilon_0}} = \frac{1-\sqrt{\varepsilon_r}}{1+\sqrt{\varepsilon_r}} \overset{\varepsilon_r\gg 1}{\simeq} -1$$

Therefore the reflecting/scattering surface has reflection characteristics similar to those of a PEC. Therefore a cylinder with $\varepsilon_r \gg 1, \mu_r = 1$ has scattering characteristics similar to those of a PEC. Thus we can approximate the SW and $\underline{J}_s$ by those of a PEC.

(a) $\underline{TM^z}$:

From (11-102), (11-102a)

$$\sigma_{2-D} \simeq \frac{4}{\beta}\left|\sum_{n=-\infty}^{+\infty} \frac{J_n(\beta a)}{H_n^{(2)}(\beta a)} e^{jn\phi}\right|^2 = \frac{2\lambda}{\pi}\left|\sum_{n=0}^{\infty} \varepsilon_n \frac{J_n(\beta a)}{H_n^{(2)}(\beta a)} \cos(n\phi)\right|^2$$

$$\varepsilon_n = \begin{cases} 1 & n=0 \\ 2 & n\neq 0 \end{cases}$$

(b) $\underline{TM^z}$:

From (11-97)

$$\underline{J}_s \simeq \hat{a}_z \frac{2E_0}{\pi a \omega \mu} \sum_{n=-\infty}^{+\infty} j^{-n} \frac{e^{jn\phi}}{H_n^{(2)}(\beta a)}$$

This can also be confirmed by examining the solution of Problem 11.58, especially the coefficient a_n when $\varepsilon_r \gg 1, \mu_r = 1$.

11.61 When $\mu_r >> 1$, $\varepsilon_r = 1$, the reflection coefficient of the solution of Problem 11.60 reduces to ($\varepsilon_1 = \varepsilon_2$)

$$\Gamma = \frac{\sqrt{\mu_2/\varepsilon_2} - \sqrt{\mu_1/\varepsilon_1}}{\sqrt{\mu_2/\varepsilon_2} + \sqrt{\mu_1/\varepsilon_1}} = \frac{\sqrt{\mu_2/\mu_1} - 1}{\sqrt{\mu_2/\mu_1} + 1} = \frac{\sqrt{\mu_2/\mu_0} - 1}{\sqrt{\mu_2/\mu_0} + 1} = \frac{\sqrt{\mu_r} - 1}{\sqrt{\mu_r} + 1} \overset{\mu_r >> 1}{\simeq} +1$$

Therefore the reflecting/scattering surface has reflection characteristics similar to those of a PMC. Therefore a cylinder with $\mu_r >> 1$, $\varepsilon_r = 1$ has scattering characteristics similar to those of a PMC. Thus we can approximate the SW and $\underline{J}_s$ by those of a PMC.

(a) TM^z:

From the solution of Problem 11.56 (scattering from a PMC cylinder) we can write the normalized scattering width (SW) as

$$\sigma_{2-D} \simeq \frac{4}{\beta} \left| \sum_{n=-\infty}^{+\infty} \frac{J_n'(\beta a)}{H_n^{(2)\prime}(\beta a)} e^{jn\phi} \right|^2 = \frac{2\lambda}{\pi} \left| \sum_{n=0}^{\infty} \varepsilon_n \frac{J_n'(\beta a)}{H_n^{(2)\prime}(\beta a)} \cos(n\phi) \right|^2$$

$$\varepsilon_n = \begin{cases} 1 & n = 0 \\ 2 & n \neq 0 \end{cases}$$

(b) ~~TM^z~~ Also from the solution of Problem 11.56 (scattering from a PMC cylinder) we can write the current density $\underline{J}_s$ on the surface ($\rho = a$) of the PMC cylinder as

$$\underline{J}_s = \hat{a}_z \left\{ -\frac{\beta H_0}{j\omega\mu} \sum_{n=-\infty}^{+\infty} j^{-n} \left[\frac{J_n'(\beta a) H_n^{(2)\prime}(\beta a) - J_n'(\beta a) H_n^{(2)\prime}(\beta a)}{H_n^{(2)\prime}(\beta a)} \right] e^{jn\phi} \right\} = 0$$

as it should be because the tangential $\underline{H}$ field vanishes on the surface of a PMC surface; PMC surface cannot support an electric current density.

This can also be confirmed by examining the solution of Problem 11.58, especially the coefficient a_n when $\mu_r >> 1$, $\varepsilon_r = 1$.

$\boxed{11.62}$ $\underline{H}^i = \hat{a}_z H_0 \sum_{n=-\infty}^{+\infty} j^{-n} J_n(\beta_0 \rho) e^{jn\phi}$ $\qquad \rho \geq a$

$$\underline{H}^s = \hat{a}_z H_0 \sum_{n=-\infty}^{+\infty} a_n H_n^{(2)}(\beta_0 \rho) e^{jn\phi} \qquad \rho \geq a$$

$$\underline{H}^d = \hat{a}_z H_0 \sum_{n=-\infty}^{+\infty} \left[b_n J_n(\beta_1 \rho) + c_n Y_n(\beta_1 \rho) \right] e^{jn\phi} \qquad a \geq \rho \geq 0$$

a. $\nabla \times \underline{H} = j\omega\varepsilon \underline{E} \Rightarrow \underline{E} = \frac{1}{j\omega\varepsilon} \nabla \times \underline{H} = \frac{1}{j\omega\varepsilon} \left\{ \hat{a}_\rho \frac{1}{\rho} \frac{\partial H_z}{\partial \phi} - \hat{a}_\phi \frac{\partial H_z}{\partial \rho} \right\}$

$$E_\rho = \frac{1}{j\omega\varepsilon} \frac{1}{\rho} \frac{\partial H_z}{\partial \phi}, \quad E_\phi = -\frac{1}{j\omega\varepsilon} \frac{\partial H_z}{\partial \rho}$$

Therefore

$$E_\rho^i = \frac{H_0}{j\omega\varepsilon_0} \frac{1}{\rho} \sum_{n=-\infty}^{+\infty} n j^{-n+1} J_n(\beta_0 \rho) e^{jn\phi}$$

$$E_\phi^i = -\frac{H_0 \beta_0}{j\omega\varepsilon_0} \sum_{n=-\infty}^{+\infty} j^{-n} J_n'(\beta_0 \rho) e^{jn\phi}$$

$$E_\rho^s = \frac{H_0}{j\omega\varepsilon_0} \frac{1}{\rho} \sum_{n=-\infty}^{+\infty} jn a_n H_n^{(2)}(\beta_0 \rho) e^{jn\phi}$$

$$E_\phi^s = -\frac{H_0 \beta_0}{j\omega\varepsilon_0} \sum_{n=-\infty}^{+\infty} a_n H_n^{(2)\prime}(\beta_0 \rho) e^{jn\phi}$$

cont'd.

11.62 cont'd.

$$E_\rho^d = \frac{H_0}{j\omega\varepsilon_1}\frac{1}{\rho}\sum_{n=-\infty}^{+\infty} jn\left[b_n J_n(\beta_1\rho)+c_n Y_n(\beta_1\rho)\right]e^{jn\phi}$$

$$E_\phi^d = -\frac{H_0\beta_1}{j\omega\varepsilon_1}\sum_{n=-\infty}^{+\infty}\left[b_n J_n'(\beta_1\rho)+c_n Y_n'(\beta_1\rho)\right]e^{jn\phi}$$

b. Since the fields within the dielectric cylinder must be finite everywhere (including $\rho=0$), then $c_n=0$. Matching tangential components of electric and magnetic fields at $\rho=a$, leads to

$$\left(E_\phi^i+E_\phi^s\right)_{\rho=a} = E_\phi^d\Big)_{\rho=a} \; ; \; \left(H_z^i+H_z^s\right)_{\rho=a} = H_z^d\Big)_{\rho=a}$$

$$\frac{\beta_0}{\varepsilon_0}\sum_{n=-\infty}^{+\infty}\left[j^{-n}J_n'(\beta_0 a)+a_n H_n^{(2)\prime}(\beta_0 a)\right]e^{jn\phi} = \frac{\beta_1}{\varepsilon_1}\sum_{n=-\infty}^{+\infty} b_n J_n'(\beta_1 a)e^{jn\phi}$$

$$\sum_{n=-\infty}^{+\infty}\left[j^{-n}J_n(\beta_0 a)+a_n H_n^{(2)}(\beta_0 a)\right]e^{jn\phi} = \sum_{n=-\infty}^{+\infty} b_n J_n(\beta_1 a)e^{jn\phi}$$

$$a_n H_n^{(2)\prime}(\beta_0 a) - \frac{\beta_1}{\beta_0}\frac{\varepsilon_0}{\varepsilon_1} b_n J_n'(\beta_1 a) = -j^{-n}J_n'(\beta_0 a)$$

$$a_n H_n^{(2)}(\beta_0 a) - b_n J_n(\beta_1 a) = -j^{-n}J_n(\beta_0 a)$$

or

$$a_n H_n^{(2)\prime}(\beta_0 a) - \sqrt{\frac{\mu_r}{\varepsilon_r}}\, b_n J_n'(\beta_1 a) = -j^{-n}J_n'(\beta_0 a) \qquad (1)$$

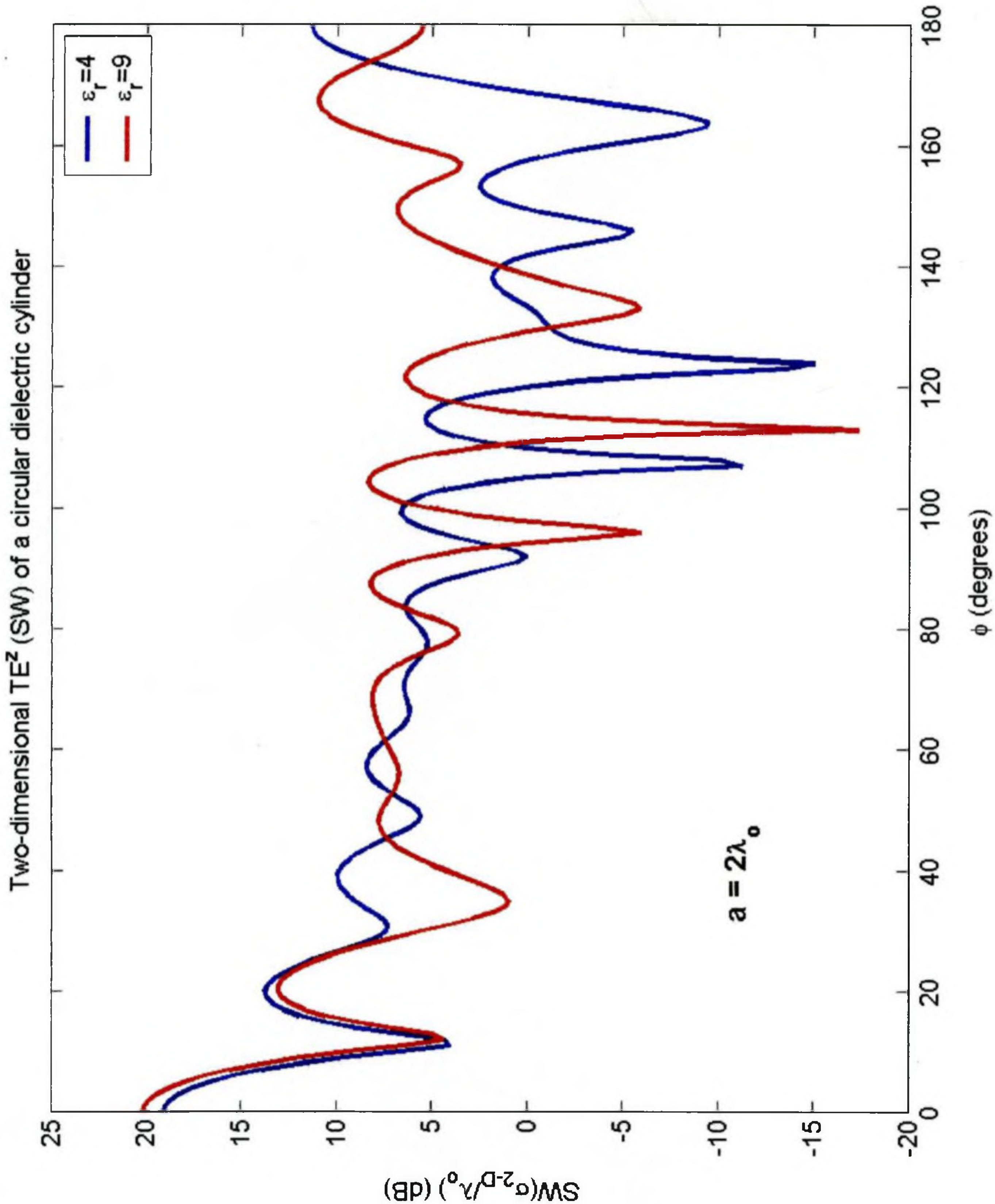
$$a_n H_n^{(2)}(\beta_0 a) - b_n J_n(\beta_1 a) = -j^{-n}J_n(\beta_0 a) \qquad (2)$$

Comparing (1) and (2) of this problem solution with (1) and (2) of the solution of Problem 11.58 we see that they are identical except at μ_r should be replaced by ε_r and ε_r should be replaced with μ_r. Therefore by referring to (5) and (6) of the solution of Problem 11.58, we can write that

$$a_n = j^{-n}\frac{J_n'(\beta_0 a)J_n(\beta_1 a)-\sqrt{\mu_r/\varepsilon_r}\,J_n(\beta_0 a)J_n'(\beta_1 a)}{\sqrt{\mu_r/\varepsilon_r}\,J_n'(\beta_1 a)H_n^{(2)}(\beta_0 a)-J_n(\beta_1 a)H_n^{(2)\prime}(\beta_0 a)} \qquad (3)$$

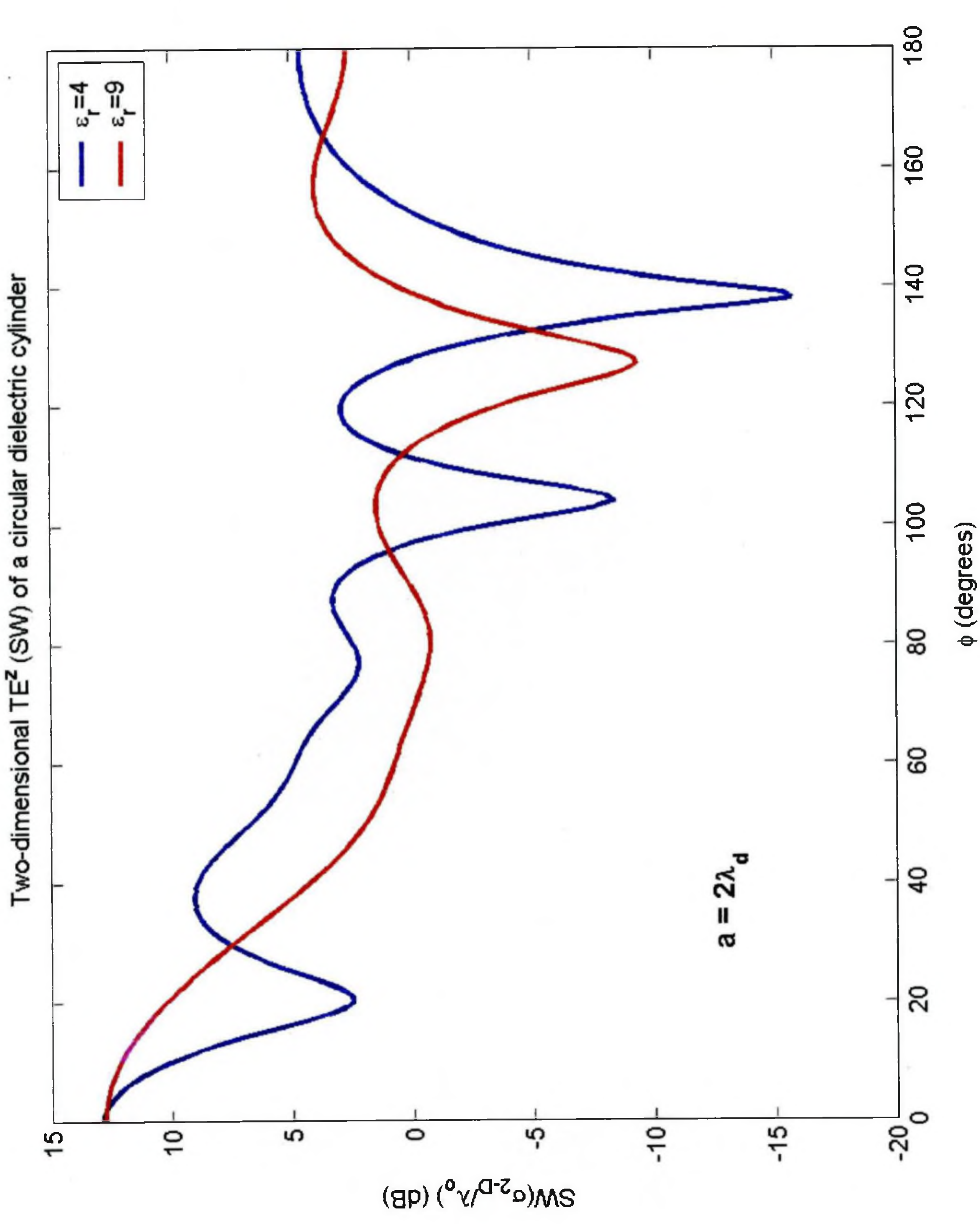
$$b_n = j^{-n}\frac{J_n(\beta_0 a)H_n^{(2)\prime}(\beta_0 a)-J_n'(\beta_0 a)H_n^{(2)}(\beta_0 a)}{J_n(\beta_1 a)H_n^{(2)\prime}(\beta_0 a)-\sqrt{\mu_r/\varepsilon_r}\,J_n'(\beta_1 a)H_n^{(2)}(\beta_0 a)} \qquad (4)$$

11.62



Two-dimensional TE^z (SW) of a circular dielectric cylinder
$\varepsilon_r=4$
$\varepsilon_r=9$
$a = 2\lambda_o$
ϕ (degrees)
SW($\sigma_{2\text{-}D}/\lambda_o$) (dB)

11.62



Two-dimensional TE^z (SW) of a circular dielectric cylinder
$\varepsilon_r=4$
$\varepsilon_r=9$
$a = 2\lambda_d$
ϕ (degrees)
SW($\sigma_{2\text{-}D}/\lambda_o$) (dB)

$\boxed{11.63}$ The reflection coefficient of a plane wave from a planar interface from by two media, under normal wave incidence, is given by

$$\Gamma = \frac{\eta_2-\eta_1}{\eta_2+\eta_1} = \frac{\sqrt{\mu_2/\varepsilon_2}-\sqrt{\mu_1/\varepsilon_1}}{\sqrt{\mu_2/\varepsilon_2}+\sqrt{\mu_1/\varepsilon_1}}$$

where η_2 is the intrinsic impedance of the medium the wave is impinging upon while η_1 is the intrinsic impedance of the wave is traveling in. Assuming medium 2 has a dielectric constant $\varepsilon_{r2}/\varepsilon_{r1}=\varepsilon_r \gg 1$, while the permeabities are the same, the reflection coefficient can be written as

$$\Gamma = \frac{1-\sqrt{\varepsilon_2/\varepsilon_0}}{1+\sqrt{\varepsilon_2/\varepsilon_0}} = \frac{1-\sqrt{\varepsilon_2/\varepsilon_0}}{1+\sqrt{\varepsilon_2/\varepsilon_0}} = \frac{1-\sqrt{\varepsilon_r}}{1+\sqrt{\varepsilon_r}} \overset{\varepsilon_r \gg 1}{\simeq} -1$$

Therefore the reflecting/scattering surface has reflection characteristics similar to those of a PEC. Therefore the cylinder with $\varepsilon_r \gg 1, \mu_r=1$ has scattering characteristics similar to those of a PEC. Thus we can approximate the SW and $\underline{J}_s$ by those of a PEC.

(a) TE^z:

From (11-117), (11-117a)

$$\sigma_{2-D} \simeq \frac{4}{\beta}\left|\sum_{n=-\infty}^{+\infty} \frac{J_n'(\beta a)}{H_n^{(2)\prime}(\beta a)} e^{jn\phi}\right|^2 = \frac{2\lambda}{\pi}\left|\sum_{n=0}^{\infty} \varepsilon_n \frac{J_n'(\beta a)}{H_n^{(2)\prime}(\beta a)} \cos(n\phi)\right|^2$$

(b) TE^z:

From (11-113)

$$\underline{J}_s \simeq \hat{a}_\phi j\frac{2H_0}{\pi\beta a}\sum_{n=-\infty}^{+\infty} j^{-n}\frac{e^{jn\phi}}{H_n^{(2)\prime}(\beta a)}$$

This can be confirmed by examining the solution of Problem 11.62, especially the coefficient a_n when $\varepsilon_r \gg 1, \mu_r=1$.

11.64 When $\mu_r >> 1$, $\varepsilon_r = 1$, the reflection coefficient of the solution of Problem 11.63 reduces to ($\varepsilon_1 = \varepsilon_2$)

$$\Gamma = \frac{\sqrt{\mu_2/\varepsilon_2} - \sqrt{\mu_1/\varepsilon_1}}{\sqrt{\mu_2/\varepsilon_2} + \sqrt{\mu_1/\varepsilon_1}} = \frac{\sqrt{\mu_2/\mu_1} - 1}{\sqrt{\mu_2/\mu_1} + 1} = \frac{\sqrt{\mu_2/\mu_0} - 1}{\sqrt{\mu_2/\mu_0} + 1} = \frac{\sqrt{\mu_r} - 1}{\sqrt{\mu_r} + 1} \overset{\mu_r >> 1}{\simeq} +1$$

Therefore the reflecting/scattering surface has reflection characteristics similar to those of a PMC. Therefor a cylinder with $\mu_r >> 1$, $\varepsilon_r = 1$ has characteristics similar to those of a PMC. Thus we can approximate the SW and $\underline{J}_s$ by those of a PMC.

(a) TE^z:

From the solution of Problem 11.57 (scattering from a PMC cylinder) we can write the normalized scattering width (SW) as

$$\sigma_{2-D} = \frac{4}{\beta}\left|\sum_{n=-\infty}^{+\infty} \frac{J_n(\beta a)}{H_n^{(2)}(\beta a)} e^{jn\phi}\right|^2 = \frac{2\lambda}{\pi}\left|\sum_{n=0}^{\infty} \varepsilon_n \frac{J_n(\beta a)}{H_n^{(2)}(\beta a)} \cos(n\phi)\right|^2$$

$$\varepsilon_n = \begin{cases} 1 & n=0 \\ 2 & n\neq 0 \end{cases}$$

(b) TE^z: The current density on the surface of the cylinder ($\rho = a$), it is given by (also from the solution of Problem 11.57; scattering of a TE^z plane wave from a PMC cylinder)

$$\underline{J}_s = -\hat{a}_\phi \left\{ H_0 \sum_{n=-\infty}^{+\infty} \left[\frac{J_n(\beta a) H_n^{(2)}(\beta a) - J_n(\beta a) H_n^{(2)}(\beta a)}{H_n^{(2)}(\beta a)}\right] e^{jn\phi} \right\} = 0$$

as it should be because the tangential $\underline{H}$ field vanishes on a PMC surface; i.e., a PMC surface cannot support an electric current density.

$\boxed{11.65}$ $\underline{E}^i = \hat{a}_z E_0 \sum_{n=-\infty}^{+\infty} j^{-n} J_n(\beta_0\rho) e^{jn\phi}$ $\rho \geq$

$\underline{E}^s = \hat{a}_z E_0 \sum_{n=-\infty}^{+\infty} a_n H_n^{(2)}(\beta_0\rho) e^{jn\phi}$ $\rho \geq b$

$\underline{E}^d = \hat{a}_z E_0 \sum_{n=-\infty}^{+\infty} [b_n J_n(\beta_1\rho) + c_n Y_n(\beta_1\rho)]$ $b \geq \rho \geq a$

a. From the solution of Problem 11.58, we can write (by using Maxwell's equation) the corresponding magnetic field components as

$$H_\rho^i = -\frac{E_0}{j\omega\mu_0}\frac{1}{\rho}\sum_{n=-\infty}^{+\infty} n j^{-n+1} J_n(\beta_0\rho) e^{jn\phi}$$

$$H_\phi^i = \frac{E_0\beta_0}{j\omega\mu_0}\sum_{n=-\infty}^{+\infty} j^{-n} J_n'(\beta_0\rho) e^{jn\phi}$$

$$H_\rho^s = -\frac{E_0}{j\omega\mu_0}\frac{1}{\rho}\sum_{n=-\infty}^{+\infty} n j a_n H_n^{(2)}(\beta\rho) e^{jn\phi}$$

$$H_\phi^s = \frac{E_0\beta_0}{j\omega\mu_0}\sum_{n=-\infty}^{+\infty} a_n H_n^{(2)\prime}(\beta_0\rho) e^{jn\phi}$$

$$H_\rho^d = -\frac{E_0}{j\omega\mu_1}\frac{1}{\rho}\sum_{n=-\infty}^{+\infty} n j [b_n J_n(\beta_1\rho) + c_n Y_n(\beta_1\rho)] e^{jn\phi}$$

$$H_\phi^d = \frac{E_0\beta_1}{j\omega\mu_1}\sum_{n=-\infty}^{+\infty} [b_n J_n'(\beta_1\rho) + c_n Y_n'(\beta_1\rho)] e^{jn\phi}$$

b. To find the wave amplitude coefficients a_n, b_n, c_n, the following boundary conditions must be used.

$$E_z^d(\rho=a) = 0 \qquad (1)$$

$$\left(E_z^i + E_z^s\right)_{\rho=b} = E_z^d\Big)_{\rho=b} \qquad (2)$$

$$\left(H_\phi^i + H_\phi^s\right)_{\rho=b} = H_\phi^d\Big)_{\rho=b} \qquad (3)$$

Using (1), we can write that

$$E_0\sum_{n=-\infty}^{+\infty}[b_n J_n(\beta_1 a) + c_n Y_n(\beta_1 a)] = 0 \Rightarrow c_n = -b_n\frac{J_n(\beta_1 a)}{Y_n(\beta_1 a)} \qquad (4)$$

Using boundary condition (2), we can write that

$$E_0\sum_{n=-\infty}^{+\infty}[j^{-n}J_n(\beta_0 b) + a_n H_n^{(2)}(\beta_0 b)]e^{jn\phi} = E_0\sum_{n=-\infty}^{+\infty}[b_n J_n(\beta_1 b) + c_n Y_n(\beta_1 b)]$$

$$j^{-n}J_n(\beta_0 b) + a_n H_n^{(2)}(\beta_0 b) = b_n J_n(\beta_1 b) + c_n Y_n(\beta_1 b) \qquad (5)$$

Using (4) ~~reduces~~ (5) to

$$j^{-n}J_n(\beta_0 b) + a_n H_n^{(2)}(\beta_0 b) = b_n\left[J_n(\beta_1 b) - \frac{J_n(\beta_1 a)}{Y_n(\beta_1 a)}Y_n(\beta_1 b)\right]$$

$$j^{-n}J_n(\beta_0 b) + a_n H_n^{(2)}(\beta_0 b) = b_n\left[\frac{J_n(\beta_1 b)Y_n(\beta_1 a) - J_n(\beta_1 a)Y_n(\beta_1 b)}{Y_n(\beta_1 a)}\right]$$

cont'd.

$\boxed{\text{11.65 cont'd.}}$ which may be written as

$$a_n H_n^{(2)}(\beta_0 b) - b_n F_n(\beta_1 a, \beta_1 b) = -j^{-n} J_n(\beta_0 b) \tag{6}$$

where $$F_n(\beta_1 a, \beta_1 b) = \frac{J_n(\beta_1 b) Y_n(\beta_1 a) - J_n(\beta_1 a) Y_n(\beta_1 b)}{Y_n(\beta_1 a)} \tag{6a}$$

Using boundary condition (3), we can write that

$$\frac{\beta_0}{\mu_0} \sum_{n=-\infty}^{+\infty} \left[j^{-n} J_n'(\beta_0 b) + a_n H_n^{(2)'}(\beta_0 b)\right] e^{jn\phi} = \frac{\beta_1}{\mu_1} \sum_{n=-\infty}^{+\infty} \left[b_n J_n'(\beta_1 b) + c_n Y_n'(\beta_1 b)\right]$$

$$j^{-n} J_n'(\beta_0 b) + a_n H_n^{(2)'}(\beta_0 b) = \frac{\beta_1}{\beta_0}\frac{\mu_0}{\mu_1}\left[b_n J_n'(\beta_1 b) + c_n Y_n'(\beta_1 b)\right] \tag{7}$$

which by using (4) reduces to

$$j^{-n} J_n'(\beta_0 b) + a_n H_n^{(2)'}(\beta_0 b) = \sqrt{\frac{\varepsilon_r}{\mu_r}}\, b_n \left[J_n'(\beta_1 b) - \frac{J_n(\beta_1 a)}{Y_n(\beta_1 a)} Y_n'(\beta_1 b)\right]$$

$$= \sqrt{\frac{\varepsilon_r}{\mu_r}}\, b_n \left[\frac{J_n'(\beta_1 b) Y_n(\beta_1 a) - J_n(\beta_1 a) Y_n'(\beta_1 b)}{Y_n(\beta_1 b)}\right]$$

which may be written as

$$a_n H_n^{(2)'}(\beta_0 b) - \sqrt{\frac{\varepsilon_r}{\mu_r}}\, G_n(\beta_1 a, \beta_1 b) = -j^{-n} J_n'(\beta_0 b) \tag{8}$$

where $$G_n(\beta_1 a, \beta_1 b) = \frac{J_n'(\beta_1 b) Y_n(\beta_1 a) - J_n(\beta_1 a) Y_n'(\beta_1 b)}{Y_n(\beta_1 b)} \tag{8a}$$

Now we need to solve for a_n and b_n using (6) and (8), or

$$a_n H_n^{(2)}(\beta_0 b) - b_n F_n(\beta_1 a, \beta_1 b) = -j^{-n} J_n(\beta_0 b) \tag{6}$$

$$a_n H_n^{(2)'}(\beta_0 b) - \sqrt{\frac{\varepsilon_r}{\mu_r}}\, b_n G_n(\beta_1 a, \beta_1 b) = -j^{-n} J_n'(\beta_0 b) \tag{8}$$

Equations (6) and (8) of this problem are identical in form to (1) and (2) of the solution of Problem 11.58 Therefore by proper interface of functions we write from the solution of Problem 11.58 that

$$a_n = j^{-n} \frac{J_n'(\beta_0 b) F_n(\beta_1 a, \beta_1 b) - \sqrt{\varepsilon_r/\mu_r}\, J_n(\beta_0 b) G_n(\beta_1 a, \beta_1 b)}{\sqrt{\varepsilon_r/\mu_r}\, G_n(\beta_1 a, \beta_1 b) H_n^{(2)}(\beta_0 b) - F_n(\beta_1 a, \beta_1 b) H_n^{(2)'}(\beta_0 b)} \tag{9}$$

$$b_n = j^{-n} \frac{J_n(\beta_0 b) H_n^{(2)'}(\beta_0 b) - J_n'(\beta_0 b) H_n^{(2)}(\beta_0 b)}{F_n(\beta_1 a, \beta_1 b) H_n^{(2)'}(\beta_0 b) - \sqrt{\varepsilon_r/\mu_r}\, G_n(\beta_1 a, \beta_1 b) H_n^{(2)}(\beta_0 b)} \tag{10}$$

$$c_n = -b_n \frac{J_n(\beta_1 a)}{Y_n(\beta_1 a)} \tag{4}$$

$$F_n = \frac{J_n(\beta_1 b) Y_n(\beta_1 a) - J_n(\beta_1 a) Y_n(\beta_1 b)}{Y_n(\beta_1 a)} \quad (6a), \quad G_n = \frac{J_n'(\beta_1 b) Y_n(\beta_1 a) - J_n(\beta_1 a) Y_n'(\beta_1 b)}{Y_n(\beta_1 b)} \quad (7a)$$

11.66

$$\underline{H}^i = \hat{a}_z H_0 \sum_{n=-\infty}^{+\infty} j^{-n} J_n(\beta_0\rho) e^{jn\phi} \qquad \rho \geq a$$

$$\underline{H}^s = \hat{a}_z H_0 \sum_{n=-\infty}^{+\infty} a_n H_n^{(2)}(\beta_0\rho) e^{jn\phi} \qquad \rho \geq a$$

$$\underline{H}^d = \hat{a}_z H_0 \sum_{n=-\infty}^{+\infty} \left[b_n J_n(\beta_1\rho) + c_n Y_n(\beta_1\rho)\right] e^{jn\phi} \qquad b \geq \rho \geq a$$

a. From the solution of Problem 11.62, we can write (by using Maxwell's equation) the corresponding electric field components

$$E_\rho^i = \frac{H_0}{j\omega\varepsilon_0}\frac{1}{\rho}\sum_{n=-\infty}^{+\infty} n j^{-n+1} J_n(\beta_0\rho) e^{jn\phi}$$

$$E_\phi^i = -\frac{H_0\beta_0}{j\omega\varepsilon_0}\sum_{n=-\infty}^{+\infty} j^{-n} J_n'(\beta_0\rho) e^{jn\phi}$$

$$E_\rho^s = \frac{H_0}{j\omega\varepsilon_0\rho}\sum_{n=-\infty}^{+\infty} n j a_n H_n^{(2)}(\beta_0\rho) e^{jn\phi}$$

$$E_\phi^s = -\frac{H_0\beta_0}{j\omega\varepsilon_0}\sum_{n=-\infty}^{+\infty} a_n H_n^{(2)\prime}(\beta_0\rho) e^{jn\phi}$$

$$E_\rho^d = \frac{H_0}{j\omega\varepsilon_1}\frac{1}{\rho}\sum_{n=-\infty}^{+\infty} jn\left[b_n J_n(\beta_1\rho) + c_n Y_n(\beta_1\rho)\right] e^{jn\phi}$$

$$E_\phi^d = -\frac{H_0\beta_1}{j\omega\varepsilon_1}\sum_{n=-\infty}^{+\infty}\left[b_n J_n'(\beta_1\rho) + c_n Y_n'(\beta_1\rho)\right] e^{jn\phi}$$

b. To find the wave amplitude coefficients a_n, b_n, c_n the following boundary conditions must be used.

$$E_\phi^d(\rho = a) = 0 \qquad (1)$$

$$\left(E_\phi^i + E_\phi^s\right)_{\rho=b} = E_\phi^d\Big)_{\rho=b} \qquad (2)$$

$$\left(H_z^i + H_z^s\right)_{\rho=b} = H_z^d\Big)_{\rho=b} \qquad (3)$$

Using (1), we can write

$$-\frac{H_0\beta_1}{j\omega\varepsilon_1}\sum_{n=-\infty}^{+\infty}\left[b_n J_n'(\beta_1 a) + c_n Y_n'(\beta_1 a)\right] e^{jn\phi} = 0 \Rightarrow c_n = -b_n\frac{Y_n'(\beta_1 a)}{J_n'(\beta_1 a)} \qquad (4)$$

Using boundary condition (2), we can write

$$\frac{\beta_0}{\varepsilon_0}\sum_{n=-\infty}^{+\infty}\left[j^{-n}J_n'(\beta_0 b) + a_n H_n^{(2)\prime}(\beta_0 b)\right] e^{jn\phi} = \frac{\beta_1}{\varepsilon_1}\sum_{n=-\infty}^{+\infty}\left[b_n J_n'(\beta_1 b) + c_n Y_n'(\beta_1 b)\right]$$

$$\left[j^{-n}J_n'(\beta_0 b) + a_n H_n^{(2)\prime}(\beta_0 b)\right] = \frac{\beta_1}{\beta_0}\frac{\varepsilon_0}{\varepsilon_1}\left[b_n J_n'(\beta_1 b) + c_n Y_n'(\beta_1 b)\right] \qquad (5)$$

which by using (4) reduces to

$$j^{-n}J_n'(\beta_0 b) + a_n H_n^{(2)\prime}(\beta_0 b) = \frac{\mu_r}{\varepsilon_r} b_n\left[J_n'(\beta_1 b) - \frac{Y_n'(\beta_1 a)}{J_n'(\beta_1 a)} Y_n'(\beta_1 b)\right]$$

$$= \sqrt{\frac{\mu_r}{\varepsilon_r}}\, b_n\left[\frac{J_n'(\beta_1 b)J_n'(\beta_1 a) - Y_n'(\beta_1 a)Y_n'(\beta_1 b)}{J_n'(\beta_1 a)}\right]$$

cont'd.

11.66 cont'd. which may be written as

$$a_n H_n^{(2)'}(\beta_0 b) - \sqrt{\frac{\mu_r}{\varepsilon_r}}\, b_n G_n'(\beta_1 a, \beta_1 b) = -j^{-n} J_n'(\beta_0 b) \qquad (6)$$

where
$$G_n' = \frac{J_n'(\beta_1 b) J_n'(\beta_1 a) - Y_n'(\beta_1 a) Y_n'(\beta_1 b)}{J_n'(\beta_1 a)} \qquad (6a)$$

Using boundary condition (3), we can write

$$\sum_{n=-\infty}^{+\infty}\left[j^{-n} J_n(\beta_0 b) + a_n H_n^{(2)}(\beta_0 b)\right] e^{jn\phi} = \sum_{n=-\infty}^{+\infty}\left[b_n J_n(\beta_1 b) + c_n Y_n(\beta_1 b)\right] e^{jn\phi}$$

$$j^{-n} J_n(\beta_0 b) + a_n H_n^{(2)}(\beta_0 b) = b_n J_n(\beta_1 b) + c_n Y_n(\beta_1 b) \qquad (7)$$

Using (4) reduces (7) to

$$j^{-n} J_n(\beta_0 b) + a_n H_n^{(2)}(\beta_0 b) = b_n\left[J_n(\beta_1 b) - \frac{Y_n'(\beta_1 a)}{J_n'(\beta_1 a)} Y_n(\beta_1 b)\right]$$
$$= b_n\left[\frac{J_n(\beta_1 b) J_n'(\beta_1 a) - Y_n'(\beta_1 a) Y_n(\beta_1 b)}{J_n'(\beta_1 a)}\right]$$

which may be written as

$$a_n H_n^{(2)}(\beta_0 b) - b_n F_n'(\beta_1 a, \beta_1 b) = -j^{-n} J_n(\beta_0 b) \qquad (8)$$

where
$$F_n' = \frac{J_n(\beta_1 b) J_n'(\beta_1 a) - Y_n'(\beta_1 a) Y_n(\beta_1 b)}{J_n'(\beta_1 a)} \qquad (8a)$$

Now we need to solve for a_n and b_n using (6) and (8), or

$$a_n H_n^{(2)}(\beta_0 b) - b_n F_n'(\beta_1 a, \beta_1 b) = -j^{-n} J_n(\beta_0 b) \qquad (8)$$

$$a_n H_n^{(2)'}(\beta_0 b) - b_n\sqrt{\frac{\mu_r}{\varepsilon_r}}\, G_n'(\beta_1 a, \beta_1 b) = -j^{-n} J_n'(\beta_0 b) \qquad (6)$$

These two equations are identical to (6) and 8, respectively, of the solution of Problem 11.65. Therefore using (9) and (10) of Problem 11.65 solution, we can write by replacing ε_r/μ_r by μ_r/ε_r that

$$a_n = j^{-n}\frac{J_n'(\beta_0 b) F_n'(\beta_1 a, \beta_1 b) - \sqrt{\mu_r/\varepsilon_r}\, J_n(\beta_0 b) G_n'(\beta_1 a, \beta_1 b)}{\sqrt{\mu_r/\varepsilon_r}\, G_n'(\beta_1 a, \beta_1 b) H_n^{(2)}(\beta_0 b) - F_n'(\beta_1 a, \beta_1 b) H_n^{(2)'}(\beta_0 b)} \qquad (9)$$

$$b_n = j^{-n}\frac{J_n(\beta_0 b) H_n^{(2)'}(\beta_0 b) - J_n'(\beta_0 b) H_n^{(2)}(\beta_0 b)}{F_n'(\beta_1 a, \beta_1 b) H_n^{(2)'}(\beta_0 b) - \sqrt{\mu_r/\varepsilon_r}\, G_n'(\beta_1 a, \beta_1 b) H_n^{(2)}(\beta_0 b)} \qquad (10)$$

$$c_n = -b_n J_n'(\beta_1 a)/Y_n'(\beta_1 a) \qquad (4)$$

$$F_n' = \frac{J_n(\beta_1 b) J_n'(\beta_1 a) - Y_n'(\beta_1 a) Y_n(\beta_1 b)}{J_n'(\beta_1 a)} \quad (8a), \quad G_n' = \frac{J_n'(\beta_1 b) J_n'(\beta_1 a) - Y_n'(\beta_1 a) Y_n'(\beta_1 b)}{J_n'(\beta_1 a)} \quad (6a)$$

$\boxed{11.67}$ Examining (11-134a)-(11-134c) we see that the ϕ component of the electric field has a $1/\rho$ variation outside the summation while the ρ and z components have no ρ variations outside the summation. Thus for large values of ρ ($\rho \gg \lambda$) the ϕ component is small compared to the other two; so small that we assume that it is negligible. Thus

$$\underline{E}_t^s = \underline{E}_\rho^s + \underline{E}_\phi^s + \underline{E}_z^s \overset{\rho\gg\lambda}{\simeq} \underline{E}_\rho^s + \underline{E}_z^s$$

Using (11-135a) and (11-135b) we can write the ρ and z components of the electric field, as given by (11-134a) and (11-134c), as

$$E_\rho^s \overset{\rho\gg\lambda}{\simeq} E_0 \cos\theta_i\, e^{j\beta(z\cos\theta_i - \rho\sin\theta_i)} \sqrt{\frac{2j}{\pi\beta\rho\sin\theta_i}} \sum_{n=-\infty}^{+\infty} a_n e^{jn\phi}$$

$$E_z^s \overset{\rho\gg\lambda}{\simeq} E_0 \sin\theta_i\, e^{j\beta(z\cos\theta_i - \rho\sin\theta_i)} \sqrt{\frac{2j}{\pi\beta\rho\sin\theta_i}} \sum_{n=-\infty}^{+\infty} a_n e^{jn\phi}$$

$$|\underline{E}_t^s| \simeq \sqrt{|E_\rho^s|^2 + |E_z^s|^2} = |E_0| \sqrt{\frac{2}{\pi\beta\rho\sin\theta_i}} \left| \sum_{n=-\infty}^{+\infty} a_n e^{jn\phi} \right|$$

Therefore using (11-21b)

$$\sigma_{2-D} = \lim_{\rho\to\infty}\left[2\pi\rho \frac{|\underline{E}_t^s|^2}{|\underline{E}^i|^2}\right] = \lim_{\rho\to\infty}\left[2\pi\rho \frac{2}{\pi\beta\rho\sin\theta_i} \left|\sum_{n=-\infty}^{+\infty} a_n e^{jn\phi}\right|^2\right] = \frac{4}{\beta\sin\theta_i} \left|\sum_{n=-\infty}^{+\infty} a_n e^{jn\phi}\right|^2$$

$\boxed{11.68}$ Examining (11-153d)-(11-153f) we see that the ϕ component of the magnetic of the magnetic field has a $1/\rho$ variation outside the summation while the ρ and z components have no ρ variations outside the summation. Thus for large values of ρ ($\rho \gg \lambda$) the ϕ component is small compared to the other two; so small that we assume that it is negligible. Thus

$$\underline{H}_t^s = \underline{H}_\rho^s + \underline{H}_\phi^s + \underline{H}_z^s \overset{\rho\gg\lambda}{\simeq} \underline{H}_\rho^s + \underline{H}_z^s$$

Using (11-135a) and (11-135b) we can write the ρ and z components of the magnetic field, as given by (11-153d) and (11-153f), as

$$H_\rho^s \overset{\rho\gg\lambda}{\simeq} H_0 \cos\theta_i\, e^{j\beta(z\cos\theta_i - \rho\sin\theta_i)} \sqrt{\frac{2j}{\pi\beta\rho\sin\theta_i}} \sum_{n=-\infty}^{+\infty} b_n e^{jn\phi}$$

$$H_z^s \overset{\rho\gg\lambda}{\simeq} H_0 \sin\theta_i\, e^{j\beta(z\cos\theta_i - \rho\sin\theta_i)} \sqrt{\frac{2j}{\pi\beta\rho\sin\theta_i}} \sum_{n=-\infty}^{+\infty} b_n e^{jn\phi}$$

$$|\underline{H}_t^s| \simeq \sqrt{|H_\rho^s|^2 + |H_z^s|^2} = |H_0| \sqrt{\frac{2}{\pi\beta\rho\sin\theta_i}} \left| \sum_{n=-\infty}^{+\infty} b_n e^{jn\phi} \right|$$

Therefore using (11-21c)

$$\sigma_{2-D} = \lim_{\rho\to\infty}\left[2\pi\rho \frac{|\underline{H}_t^s|^2}{|\underline{H}^i|^2}\right] = \lim_{\rho\to\infty}\left[2\pi\rho \frac{2}{\pi\beta\rho\sin\theta_i} \left|\sum_{n=-\infty}^{+\infty} b_n e^{jn\phi}\right|^2\right] = \frac{4}{\beta\sin\theta_i} \left|\sum_{n=-\infty}^{+\infty} b_n e^{jn\phi}\right|^2$$

11.69 The total field radiated by a line source next to a cylinder with large radius $a \gg \lambda$ (in this case $a = 50\lambda$) is approximately the same as an electric line source a height h above a PEC flat ground-plane, as shown in Figure 11-2.

Therefore the normalized radiated field is represented by the array factor $[\sin(\beta h \sin\phi)]$ of (11-20a) where the line source is placed directly above the PEC ground plane ($\phi' = 90°$).

(a) To determine the smallest height of the line source above the cylinder, we set the sine function (array factor) of (11-20a) equal to unity; that is (for a maximum)

$$\sin(\beta h \sin\phi)\Big|_{\phi=90°} \simeq \sin(\beta h) = 1 \Rightarrow \beta h \simeq \sin^{-1}(1) \simeq \frac{n\pi}{2}, \quad n = 1, 3, \cdots$$

For n = 1 (smallest height):

$$h \simeq \frac{\pi/2}{\beta} = \frac{\pi/2}{2\pi/\lambda} = \frac{\lambda}{4}$$

Thus the line source must be placed at a height $\lambda/4$ above the cylinder surface.

(b) To determine the smallest height for a null along $\phi = 90°$, we set the sine function (array factor) of (11-20a) equal to zero; that is (for a null):

$$\sin(\beta h \sin\phi)\Big|_{\phi=90} = \sin(\beta h) = 0 \Rightarrow \beta h = \sin^{-1}(0) = n\pi, \quad n = 0, 1, 2, \cdots$$

For other than h = 0 (trivial solution), n = 1. Therefore

$$h \simeq \frac{\pi}{\beta} = \frac{\pi}{2\pi/\lambda} = \frac{\lambda}{2}$$

(c) To find the smallest when the pattern at $\phi = 90°$ is -3 dB (0.707 field amplitude), we set the sine function [array factor of (11-20a)] equal to 0.707; that is

$$\sin(\beta h \sin\phi)\Big|_{\phi=90°} = \sin(\beta h) = 0.707 \Rightarrow \beta h = \sin^{-1}(0.707) = \frac{n\pi}{4}, \quad n = 1, 3, \cdots$$

For the smallest height, n = 1. Therefore

$$h = \frac{\pi/4}{\beta} = \frac{\pi/4}{2\pi/\lambda} = \frac{\lambda}{8}$$

$\boxed{11.70}$ According to (11-164b) when $\rho > \rho' = s$

a. $$E_z^t = -\frac{\beta^2 I_e}{4\omega\varepsilon}\sum_{n=-\infty}^{+\infty} H_n^{(2)}(\beta\rho)\left[J_n(\beta s) - \frac{J_n(\beta a)}{H_n^{(2)}(\beta a)} H_n^{(2)}(\beta s)\right] e^{jn(\phi-\phi')}$$

For the two sources (one at $\phi'=0$ and the other at $\phi'=\pi$), the total electric field can be written as

$$E_z^t(\text{total}) = -\frac{\beta^2 I_e}{4\omega\varepsilon}\sum_{n=-\infty}^{+\infty} H_n^{(2)}(\beta\rho)\left[J_n(\beta s) - \frac{J_n(\beta a)}{H_n^{(2)}(\beta a)} H_n^{(2)}(\beta s)\right]\left[e^{jn\phi} + e^{jn\phi}e^{-jn\pi}\right]$$

$$E_z^t(\text{total}) = -\frac{\beta^2 I_e}{4\omega\varepsilon}\sum_{n=-\infty}^{+\infty} H_n^{(2)}(\beta\rho)\left[J_n(\beta s) - \frac{J_n(\beta a)}{H_n^{(2)}(\beta a)} H_n^{(2)}(\beta s)\right] e^{jn\phi}\left(1 + e^{-jn\pi}\right)$$

b. For large observation distances ($\rho \gg \lambda$)

$$H_n^{(2)}(\beta\rho) \overset{\rho\gg\lambda}{\simeq} \sqrt{\frac{2j}{\pi\beta\rho}}\, j^n e^{-j\beta\rho}$$

Thus

$$E_z^t(\text{total}) \overset{\rho\gg\lambda}{\simeq} -\frac{\beta^2 I_e}{4\omega\varepsilon}\sqrt{\frac{2j}{\pi\beta\rho}}\, e^{-j\beta\rho}\sum_{n=-\infty}^{+\infty} j^n\left[J_n(\beta s) - \frac{J_n(\beta a)}{H_n^{(2)}(\beta a)} H_n^{(2)}(\beta s)\right] e^{jn\phi}\left(1 + e^{-jn\pi}\right)$$

When $s \ll \lambda$ and $a \ll \lambda$, then only the first and second terms of the infinite summation are sufficient. Therefore

$$E_z^t(\text{total}) \underset{\substack{s\ll\lambda\\ a\ll\lambda}}{\overset{\rho\gg\lambda}{\simeq}} -\frac{\beta^2 I_e}{4\omega\varepsilon}\sqrt{\frac{2j}{\pi\beta\rho}}\, e^{-j\beta\rho}\left\{\left[J_0(\beta s) - \frac{J_0(\beta a)}{H_0^{(2)}(\beta a)} H_0^{(2)}(\beta s)\right]2 \right.$$
$$+ j\left[J_1(\beta s) - \frac{J_1(\beta a)}{H_1^{(2)}(\beta a)} H_1^{(2)}(\beta s)\right] e^{j\phi}\,\overset{0}{\cancel{\left(1+e^{-j\pi}\right)}}$$
$$\left. + j^{-1}\left[J_{-1}(\beta s) - \frac{J_{-1}(\beta a)}{H_{-1}^{(2)}(\beta a)} H_{-1}^{(2)}(\beta s)\right] e^{-j\phi}\,\overset{0}{\cancel{\left(1+e^{j\pi}\right)}}\right\}$$

$$E_z^t(\text{total}) \underset{\substack{s\ll\lambda\\ a\ll\lambda}}{\overset{\rho\gg\lambda}{\simeq}} -\frac{\beta^2 I_e}{4\omega\varepsilon}\sqrt{\frac{2j}{\pi\beta\rho}}\, e^{-j\beta\rho}\left\{2\left[J_0(\beta s) - \frac{J_0(\beta a)}{H_0^{(2)}(\beta a)} H_0^{(2)}(\beta s)\right]\right\} = 0$$

For this to vanish

$$\left[J_0(\beta s) - \frac{J_0(\beta a)}{H_0^{(2)}(\beta a)} H_0^{(2)}(\beta s)\right] = 0 \Rightarrow s = a$$

Therefore for the field to vanish the source must be placed next to the cylinder. Of course this is obvious since the electric source does not radiate when it is placed tangentially next to a perfect electric conductor. Thus because the cylinder is small there is no other position of the source which will lead to a vanishing far zone field.

$\boxed{11.71}$ According to (11-176g) when $\rho > \rho' = b$

a. $$H_z^t = -\frac{\beta^2 I_m}{4\omega\mu}\sum_{n=-\infty}^{+\infty} H_n^{(2)}(\beta\rho)\left[J_n(\beta b) - \frac{J_n'(\beta a)}{H_n^{(2)\prime}(\beta a)} H_n^{(2)}(\beta b)\right] e^{jn(\phi-\phi')}$$

For the three sources of Figure P11-33

$$H_{z0}^t = -2\,\frac{I_m\beta^2}{4\omega\mu}\sum_{n=-\infty}^{+\infty} H_n^{(2)}(\beta\rho)\left[J_n(\beta b) - \frac{J_n'(\beta a)}{H_n^{(2)\prime}(\beta a)} H_n^{(2)}(\beta b)\right] e^{jn\phi}\,, \quad \phi' = 0$$

$$H_{z1}^t = -\frac{I_m\beta^2}{4\omega\mu}\sum_{n=-\infty}^{+\infty} H_n^{(2)}(\beta\rho)\left[J_n(\beta b) - \frac{J_n'(\beta a)}{H_n^{(2)\prime}(\beta a)} H_n^{(2)}(\beta b)\right] e^{jn\phi} e^{-j\phi_0 n}\,, \quad \phi' = \phi_0$$

$$H_{z-1}^t = -\frac{I_m\beta^2}{4\omega\mu}\sum_{n=-\infty}^{+\infty} H_n^{(2)}(\beta\rho)\left[J_n(\beta b) - \frac{J_n'(\beta a)}{H_n^{(2)\prime}(\beta a)} H_n^{(2)}(\beta b)\right] e^{jn\phi} e^{j\phi_0 n}\,, \quad \phi = 2\pi - \phi_0$$

Thus the total field for all the three sources is equal to

$$H_z^t(\text{total}) = -\frac{I_m\beta^2}{4\omega\mu}\sum_{n=-\infty}^{+\infty} H_n^{(2)}(\beta\rho)\left[J_n(\beta b) - \frac{J_n'(\beta a)}{H_n^{(2)\prime}(\beta a)} H_n^{(2)}(\beta b)\right] e^{jn\phi}\left[2 + 2\left(\frac{e^{j\phi_0 n} + e^{-j\phi_0 n}}{2}\right)\right]$$

$$= -\frac{I_m\beta^2}{4\omega\mu}\sum_{n=-\infty}^{+\infty} H_n^{(2)}(\beta\rho)\left[J_n(\beta b) - \frac{J_n'(\beta a)}{H_n^{(2)\prime}(\beta a)} H_n^{(2)}(\beta b)\right] 2\, e^{jn\phi}\left[1 + \cos(n\phi_0)\right]$$

$$H_z^t(\text{total}) = -\frac{I_m\beta^2}{\omega\mu}\sum_{n=-\infty}^{+\infty} H_n^{(2)}(\beta\rho)\left[J_n(\beta b) - \frac{J_n'(\beta a)}{H_n^{(2)\prime}(\beta a)} H_n^{(2)}(\beta b)\right] e^{jn\phi}\cos^2\left(\frac{n\phi_0}{2}\right)$$

b. $$\frac{|H_z^s|}{|H_z^i|} = \frac{\left|\sum\limits_{n=-\infty}^{+\infty} H_n^{(2)}(\beta\rho)\,\frac{J_n'(\beta a)}{H_n^{(2)\prime}(\beta a)} H_n^{(2)}(\beta b)\, e^{jn\phi}\cos^2\left(\frac{n\phi_0}{2}\right)\right|}{\left|\sum\limits_{n=-\infty}^{+\infty} H_n^{(2)}(\beta\rho)\, J_n(\beta b)\, e^{jn\phi}\cos^2\left(\frac{n\phi_0}{2}\right)\right|}$$

c. For $\rho \gg \lambda \Rightarrow H_n^{(2)}(\beta\rho) \overset{\rho\gg\lambda}{\simeq} \sqrt{\frac{2j}{\pi\beta\rho}}\, j^n e^{-j\beta\rho}$

Thus $$H_z^t(\text{total}) \simeq -\frac{I_m\beta^2}{\omega\mu}\sqrt{\frac{2j}{\pi\beta\rho}}\, e^{-j\beta\rho}\sum_{n=-\infty}^{+\infty} j^n\left[J_n(\beta b) - \frac{J_n'(\beta a)}{H_n^{(2)\prime}(\beta a)} H_n^{(2)}(\beta b)\right] e^{jn\phi}\cos^2\left(\frac{n\phi_0}{2}\right)$$

$$H_z^t(\text{total}) \simeq -\frac{I_m\beta^2}{\omega\mu}\sqrt{\frac{2j}{\pi\beta\rho}}\, e^{-j\beta\rho}\sum_{n=-\infty}^{+\infty} j^n\left[\frac{J_n(\beta b) H_n^{(2)\prime}(\beta a) - J_n'(\beta a) H_n^{(2)}(\beta b)}{H_n^{(2)\prime}(\beta a)}\right] e^{jn\phi}\cos^2\left(\frac{n\phi_0}{2}\right)$$

$$H_z^t(\text{total})\Big|_{\substack{\text{norm}\\ \text{constant }\rho\\ \rho = C}} \sim \sum_{n=-\infty}^{+\infty} j^n\left[\frac{J_n(\beta b) H_n^{(2)\prime}(\beta a) - J_n'(\beta a) H_n^{(2)}(\beta b)}{H_n^{(2)\prime}(\beta a)}\right] e^{jn\phi}\cos^2\left(\frac{n\phi_0}{2}\right)$$

$\boxed{11.72}$ According to (11-191(a))

$$E_z^t \underset{\beta\rho' \to \infty}{\overset{\alpha=0}{\simeq}} E_0 \sum_{m=1}^{\infty} j^{m/2} J_{m/2}(\beta\rho) \sin\left(\frac{m}{2}\phi'\right) \sin\left(\frac{m}{2}\phi\right)$$

The corresponding magnetic field is obtained using Maxwell's equation of

$$\nabla \times \underline{E} = -j\omega\mu \underline{H} \Rightarrow \underline{H} = -\frac{1}{j\omega\mu} \nabla \times \underline{E} \Rightarrow \begin{cases} H_\rho = -\dfrac{1}{j\omega\mu}\dfrac{1}{\rho}\dfrac{\partial E_z}{\partial \phi} \\ H_\phi = \dfrac{1}{j\omega\mu}\dfrac{\partial E_z}{\partial \rho} \\ H_z = 0 \end{cases}$$

with the corresponding current density of

$$\underline{J} = \hat{n} \times \underline{H}\Big|_{\phi=0} = \hat{a}_y \times (\hat{a}_\rho H_\rho + \hat{a}_\phi H_\phi)\Big|_{\phi=0} = \hat{a}_y \times (\hat{a}_x H_\rho + \hat{a}_y H_\phi) = -\hat{a}_z H_\rho$$

$$J_z = -H_\rho\Big|_{\phi=0} = \frac{1}{j\omega\mu}\frac{1}{\rho}\frac{\partial E_z}{\partial \phi}\Big|_{\phi=0} = \frac{E_0}{j2\omega\mu}\frac{1}{\rho}\sum_{m=1}^{\infty} j^{m/2}\, m\, J_{m/2} \sin\left(\frac{m}{2}\phi'\right)\cos\left(\frac{m}{2}\phi\right)\Big|_{\phi=0}$$

$$J_z = \frac{E_0}{j2\omega\mu\rho}\sum_{m=1}^{\infty} m\, j^{m/2} J_{m/2}(\beta\rho) \sin\left(\frac{m}{2}\phi'\right)$$

When $\beta\rho \to 0$

$$J_{m/2}(\beta\rho) \underset{m \neq 0}{\overset{\beta\rho \to 0}{\simeq}} \frac{1}{\frac{m}{2}!}\left(\frac{\beta\rho}{2}\right)^{m/2}$$

Considering only the $m=1$ term (since the $m=0$ term does not contribute)

$$J_z \underset{m=1}{\overset{\beta\rho \to 0}{\simeq}} \frac{E_0}{j2\omega\mu\rho}\, j^{1/2} \frac{1}{\frac{1}{2}!}\left(\frac{\beta\rho}{2}\right)^{1/2} \sin\left(\frac{\phi'}{2}\right)$$

Since

$$\frac{1}{2}! = \Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}$$

then

$$J_z \underset{m=1}{\overset{\beta\rho \to 0}{\simeq}} \frac{E_0}{j2\omega\mu\rho}\, j^{1/2} \frac{2}{\sqrt{\pi}}\sqrt{\frac{\beta\rho}{2}} \sin\left(\frac{\phi'}{2}\right) = \frac{E_0}{\eta\beta\rho}\sqrt{\frac{\beta\rho}{2j\pi}} \sin\left(\frac{\phi'}{2}\right)$$

$$J_z \simeq \frac{E_0}{\eta}\sqrt{\frac{1}{j2\pi\beta\rho}} \sin\left(\frac{\phi'}{2}\right)$$

$\boxed{11.73}$ $H_z^t = \sum_s b_s J_s(\beta\rho) H_s^{(2)}(\beta\rho') \cos[s(\phi'-\alpha)] \cos[s(\phi-\alpha)]$

When $\beta\rho' \to \infty \Rightarrow H_s^{(2)}(\beta\rho') = \sqrt{\frac{2j}{\pi\beta\rho'}}\, j^s e^{-j\beta\rho'}$. Therefore using also (11-196)

$$H_z^t \simeq \frac{\pi\omega\varepsilon I_m}{4(\pi-\alpha)}\sqrt{\frac{2j}{\pi\beta\rho'}}\, e^{-j\beta\rho'} \sum_s j^s \varepsilon_s J_s(\beta\rho) \cos[s(\phi'-\alpha)] \cos[s(\phi-\alpha)]$$

$$\simeq H_0 \sum_s \varepsilon_s j^s J_s(\beta\rho) \cos[s(\phi'-\alpha)] \cos[s(\phi-\alpha)] \text{ where } H_0 = I_m \sqrt{\frac{\pi j}{8\beta}}\, \frac{\omega\varepsilon}{\pi-\alpha}\, \frac{e^{-j\beta\rho'}}{\sqrt{\rho'}}$$

$\boxed{11.74}$ According to (11-199a)

$$H_z^t \underset{\alpha=0}{\overset{\beta\rho\to\infty}{\simeq}} H_0 \sum_{m=0}^{\infty} \epsilon_{m/2}\, j^{m/2} J_{m/2}(\beta\rho)\cos\left(\frac{m}{2}\phi'\right)\cos\left(\frac{m}{2}\phi\right)$$

Therefore the current density is given by

$$\underline{J} = \hat{n}\times\underline{H}\Big|_{\phi=0} = \hat{a}_\phi \times \hat{a}_z H_z\Big|_{\phi=0} = \hat{a}_\rho H_z\Big|_{\phi=0} = \hat{a}_\rho H_0 \sum_{m=0}^{\infty} \epsilon_{m/2}\, j^{m/2} J_{m/2}(\beta\rho)\cos\left(\frac{m}{2}\phi'\right)$$

$$J_\rho = H_0 \sum_{m=0}^{\infty} \epsilon_{m/2}\, j^{m/2} J_{m/2}(\beta\rho)\cos\left(\frac{m}{2}\phi'\right)$$

When $\beta\rho\to 0$ only the first term of the summation is necessary. Therefore

$$J_\rho \overset{\beta\rho\to 0}{\simeq} H_0\, \epsilon_0\, j^0 J_0(\beta\rho)(1)$$

When $\beta\rho\to 0$ $\quad J_0(\beta\rho) \overset{\beta\rho\to 0}{\simeq} 1$

Thus $\quad J_\rho \overset{\beta\rho\to 0}{\simeq} H_0(1)(1)(1)(1) = H_0$

$\boxed{11.75}$ The procedure follows that for the derivation of (11-221a) from (11-217a). The only difference for the derivation of (11-222a) from (11-222) is that instead of (11-219a) we need to use the integral of

$$\int_0^\pi e^{j\beta r\cos\theta} P_m(\cos\theta)\sin\theta\, d\theta = 2j^{+m} j_m(\beta r)$$

Then the expected solution of (11-222) and (11-222a) follows.

$\boxed{11.76}$ When $f = 8.5$ GHz $\Rightarrow \lambda = \dfrac{3\times10^8}{8.5\times10^9} = 0.353\times10^{-1} = 0.0353$ m. Therefore the

a. diameter of 1.128 m is much greater than the wavelength. Thus the RCS of the sphere is equal to its geometric cross sectional area of

$$\sigma = \pi a^2 = \pi(0.564)^2 = 0.999 \simeq 1 \text{ m}^2 \text{ or } 0 \text{ dB/sm (dBsm)}$$

b. For normal incidence the RCS of any flat-plate is equal to

$$\sigma = 4\pi\left(\frac{A}{\lambda}\right)^2 \Rightarrow A = \lambda\sqrt{\frac{\sigma}{4\pi}} = 0.0353\sqrt{\frac{1}{4\pi}} = 0.0353(0.2821)$$

$$A \simeq 0.01 \text{ m}^2$$

11.77

(a) The RCS of a sphere of radius a_s in the GO region is given by

$$RCS_{sphere} = \pi a_s^2$$

The maximum RCS of a flat circular plate of radius a_p, based on physical optics, is equal to

$$RCS_{plate} = 4\pi\left(\frac{A}{\lambda}\right)^2 = 4\pi\left(\frac{\pi a_p^2}{\lambda}\right)^2$$

Equating the 2 RCSs

$$\pi a_s^2 = 4\pi\left(\frac{\pi a_p^2}{\lambda}\right)^2 \Rightarrow a_s^2 = 4\left(\frac{\pi a_p^2}{\lambda}\right)^2 \Rightarrow a_s = 2\left(\frac{\pi a_p^2}{\lambda}\right)$$

$$\boxed{a_s(\text{sphere}) = 2\left[\frac{\pi a_p^2(\text{plate})}{\lambda}\right]}$$

(b) For $a_p = 3\lambda$: $a_s(\text{sphere}) = 2\left[\frac{\pi(3\lambda)^2}{\lambda}\right] = 2\left(\frac{9\pi\lambda^2}{\lambda}\right) = 18\pi\lambda$

$$\boxed{a_s(\text{sphere}) = 18\pi\lambda = 56.5487\lambda}$$

11.78

(a) For a flat plate, the maximum RCS at normal incidence is

$$RCS = 4\pi\left(\frac{A}{\lambda}\right)^2 \underset{A=25\lambda^2}{=} 4\pi(25\lambda)^2 = 4(625)\pi\lambda^2 = 7{,}853.9816\lambda^2$$

At $f = 10$ GHz, $\lambda = 3\times10^8/10\times10^9 = 0.03$ m

$$RCS = 7{,}853.9816\,\lambda^2 = 7{,}853.9816(0.03)^2 = 7.0686\ m^2$$

which, based on $1\ m^2$, is equal to

$$RCS(\text{in dBsm}) = 10\log_{10}(7.0686) = 8.4933\ \text{dBsm}$$

(b) The RCS of a sphere in the GO region is πa_s^2, a_s = radius

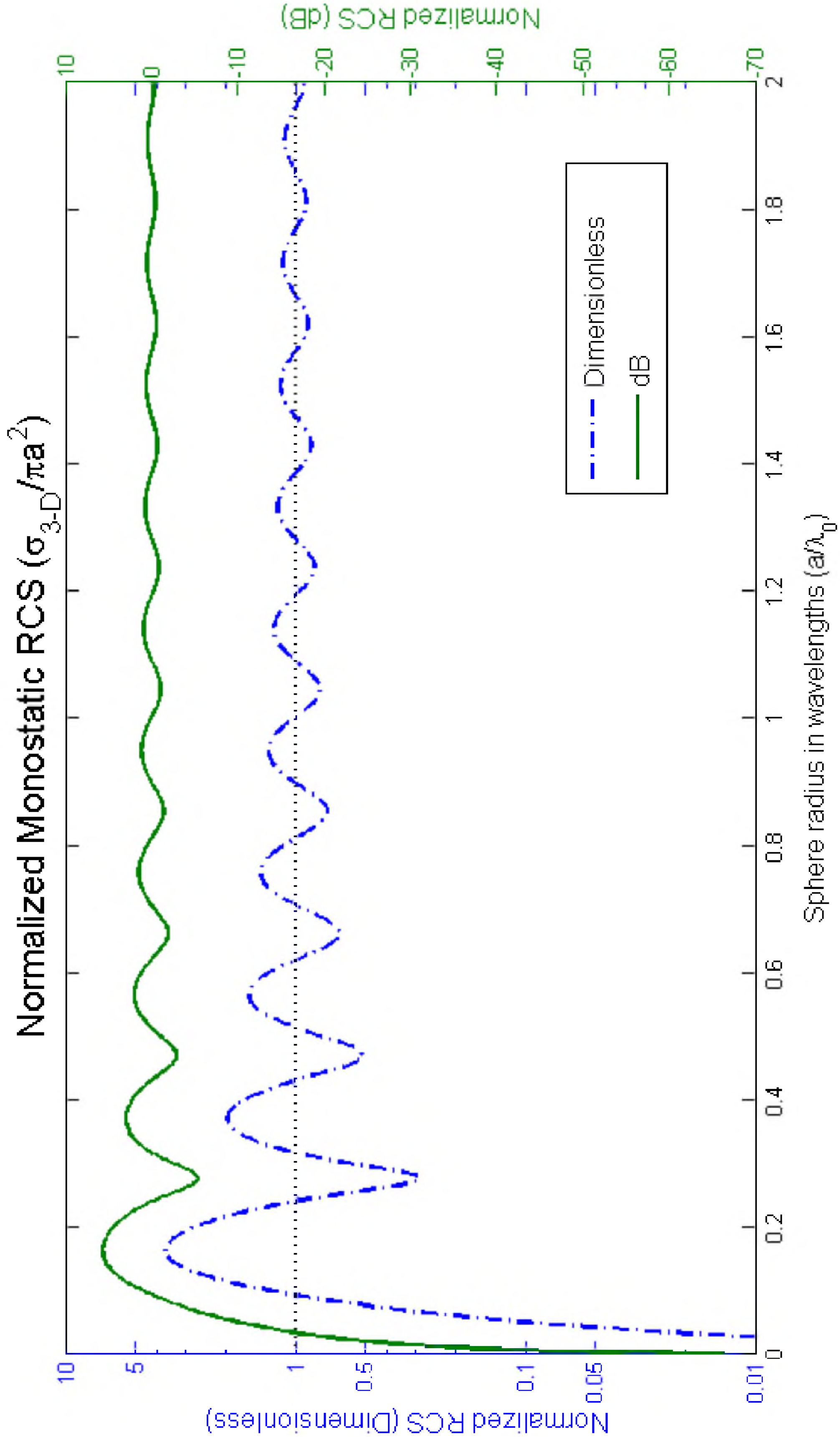
Therefore

$$\pi a_s^2 = 7{,}853.9816\,\lambda^2$$

$$a_s = [7{,}853.9816\,\lambda^2/\pi]^{1/2} = 50\lambda$$

Therefore the radius of the sphere must be 50λ in order for the sphere to have the same RCS at normal incidence as a flat plate of area $25\lambda^2$.

11.79



Normalized Monostatic RCS ($\sigma_{3\text{-}D}/\pi a^2$)
Normalized RCS (dB)
10
0
-10
-20
-30
-40
-50
-60
-70
Normalized RCS (Dimensionless)
10
5
1
0.5
0.1
0.05
0.01
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
Sphere radius in wavelengths (a/λ_0)
Dimensionless
dB

$\boxed{11.80}$

For a plane wave, with only an x-component of the electric field incident along $\theta_i = \pi$ (180°) on a PEC sphere of radius a, the far-zone scattered spherical components of the electric field are given by (11-242b) and (11-242c), or

$$E_\theta^s \simeq jE_0 \frac{e^{-j\beta r}}{\beta r} \cos\phi \sum_{n=1}^{\infty} j^n \left[b_n \sin\theta P_n^{(1)'}(\cos\theta) - c_n \frac{P_n^{(1)}(\cos\theta)}{\sin\theta} \right] \tag{1a}$$

$$E_\phi^s \simeq jE_0 \frac{e^{-j\beta r}}{\beta r} \sin\phi \sum_{n=1}^{\infty} j^n \left[b_n \frac{P_n^{(1)}(\cos\theta)}{\sin\theta} - c_n \sin\theta P_n^{(1)'}(\cos\theta) \right] \tag{1b}$$

where b_n and c_n are given, respectively, by (11-238a) and (11-238b).

The corresponding rectangular components of the scattered electric field are given by

$$E_x^s = \cos\theta\cos\phi\, E_\theta^s - \sin\phi\, E_\phi^s \tag{2a}$$

$$E_y^s = \cos\theta\sin\phi\, E_\theta^s + \cos\phi\, E_\phi^s \tag{2b}$$

Along the monostatic scattering direction $(\theta=\pi)$, (2a) and (2b) reduce to

$$E_x^s\Big|_{\theta=\pi} = -\cos\phi\, E_\theta^s(\theta=\pi) - \sin\phi\, E_\phi^s(\theta=\pi) \tag{3a}$$

$$E_y^s\Big|_{\theta=\pi} = -\sin\phi\, E_\theta^s(\theta=\pi) + \cos\phi\, E_\phi^s(\theta=\pi) \tag{3b}$$

When $\phi=\pi$, (3a) and (3b) reduce, using (1a) and (1b), to

$$E_x^s\Big|_{\substack{\theta=\pi\\ \phi=\pi}} = E_\theta^s(\theta=\pi, \phi=\pi)$$

$$E_y^s\Big|_{\substack{\theta=\pi\\ \phi=\pi}} = -E_\phi^s(\theta=\pi, \phi=\pi) = 0$$

When $\phi = 3\pi/2$, (3a) and (3b) reduce, using (1a) and (1b), to

$$E_x^s\Big|_{\substack{\theta=\pi\\ \phi=3\pi/2}} = E_\phi^s(\theta=\pi, \phi=3\pi/2)$$

$$E_y^s\Big|_{\substack{\theta=\pi\\ \phi=3\pi/2}} = E_\theta^s(\theta=\pi, \phi=3\pi/2) = 0$$

In both cases, $E_y^s = 0$. Thus there is NO cross-polarized field along the monostatic scattering direction!!

/4

11.81

From Equations (11-231) and (11-232), the vector potentials of the incident fields outside the dielectric sphere take the form of

$$A_r^i = E_0 \frac{\cos\phi}{\omega} \sum_{n=1}^{\infty} a_n \hat{J}_n(\beta_0 r) P_n^{(1)}(\cos\theta)$$

$$F_r^i = E_0 \frac{\sin\phi}{\omega\eta_0} \sum_{n=1}^{\infty} a_n \hat{J}_n(\beta_0 r) P_n^{(1)}(\cos\theta)$$

where, from Equation (11-231a),

$$a_n = j^{-n} \frac{(2n+1)}{n(n+1)}$$

From Equation (11-233), the vector potentials of the scattered fields outside the dielectric sphere take the form of

$$A_r^s = E_0 \frac{\cos\phi}{\omega} \sum_{n=1}^{\infty} b_n \hat{H}_n^{(2)}(\beta_0 r) P_n^{(1)}(\cos\theta)$$

$$F_r^s = E_0 \frac{\sin\phi}{\omega\eta_0} \sum_{n=1}^{\infty} c_n \hat{H}_n^{(2)}(\beta_0 r) P_n^{(1)}(\cos\theta)$$

The combined vector potentials of the fields outside the dielectric sphere equal the sum of the incident and scattered potentials, and are reproduced from Equation (11-235) as

$$A_r^{t+} = A_r^i + A_r^s = E_0 \frac{\cos\phi}{\omega} \sum_{n=1}^{\infty} \left[a_n \hat{J}_n(\beta_0 r) + b_n \hat{H}_n^{(2)}(\beta_0 r) \right] P_n^{(1)}(\cos\theta)$$

$$F_r^{t+} = F_r^i + F_r^s = E_0 \frac{\sin\phi}{\omega\eta_0} \sum_{n=1}^{\infty} \left[a_n \hat{J}_n(\beta_0 r) + c_n \hat{H}_n^{(2)}(\beta_0 r) \right] P_n^{(1)}(\cos\theta)$$

In the above equations,

$$\beta_0 = \omega\sqrt{\mu_0\varepsilon_0}$$

and

$$\eta_0 = \sqrt{\mu_0/\varepsilon_0}$$

Cont'd

11.81 cont'd

Inside the dielectric sphere, the vector potentials take the form of

$$A_r^{t-} = E_0 \frac{\cos\phi}{\omega} \sum_{n=1}^{\infty} d_n \hat{J}_n(\beta_d r) P_n^{(1)}(\cos\theta)$$

$$F_r^{t-} = E_0 \frac{\sin\phi}{\omega \eta_d} \sum_{n=1}^{\infty} e_n \hat{J}_n(\beta_d r) P_n^{(1)}(\cos\theta)$$

with

$$\beta_d = \omega\sqrt{\mu_d \varepsilon_d}$$

and

$$\eta_d = \sqrt{\mu_d/\varepsilon_d}$$

where

$$\mu_d = \dot{\mu}_r \mu_0$$

and

$$\varepsilon_d = \dot{\varepsilon}_r \varepsilon_0$$

Thus

$$\beta_d = \omega\sqrt{\mu_d \varepsilon_d} = \omega\sqrt{\dot{\mu}_r \mu_0 \dot{\varepsilon}_r \varepsilon_0} = \omega\sqrt{\mu_0 \varepsilon_0}\sqrt{\dot{\mu}_r \dot{\varepsilon}_r} = \beta_0\sqrt{\dot{\mu}_r \dot{\varepsilon}_r}$$

and

$$\eta_d = \sqrt{\mu_d/\varepsilon_d} = \sqrt{\dot{\mu}_r \mu_0/\dot{\varepsilon}_r \varepsilon_0} = \sqrt{\mu_0/\varepsilon_0}\sqrt{\dot{\mu}_r/\dot{\varepsilon}_r} = \eta_0\sqrt{\dot{\mu}_r/\dot{\varepsilon}_r}$$

Cont'd

11.81 cont'd

Drawing from Equation (11-234), the electric and magnetic fields inside or outside the sphere, using the proper vector potentials, are given by

$$E_r = \frac{1}{j\omega\mu\varepsilon}\left(\frac{\partial^2}{\partial r^2} + \beta^2\right)A_r$$

$$E_\theta = \frac{1}{j\omega\mu\varepsilon}\frac{1}{r}\frac{\partial^2 A_r}{\partial r \partial\theta} - \frac{1}{\varepsilon}\frac{1}{r\sin\theta}\frac{\partial F_r}{\partial\phi}$$

$$E_\phi = \frac{1}{j\omega\mu\varepsilon}\frac{1}{r\sin\theta}\frac{\partial^2 A_r}{\partial r\partial\phi} + \frac{1}{\varepsilon}\frac{1}{r}\frac{\partial F_r}{\partial\theta}$$

$$H_r = \frac{1}{j\omega\mu\varepsilon}\left(\frac{\partial^2}{\partial r^2} + \beta^2\right)F_r$$

$$H_\theta = \frac{1}{\mu}\frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\phi} + \frac{1}{j\omega\mu\varepsilon}\frac{1}{r}\frac{\partial^2 F_r}{\partial r\partial\theta}$$

$$H_\phi = -\frac{1}{\mu}\frac{1}{r}\frac{\partial A_r}{\partial\theta} + \frac{1}{j\omega\mu\varepsilon}\frac{1}{r\sin\theta}\frac{\partial^2 F_r}{\partial r\partial\phi}$$

Now, for any Bessel function, Equation (IV-19) can be used to show

$$\frac{\partial Z_n(\beta r)}{\partial r} = -\beta Z_{n+1}(\beta r) + \frac{n}{r}Z_n(\beta r)$$

and

$$\frac{\partial Z_n(\beta r)}{\partial(\beta r)} = -Z_{n+1}(\beta r) + \frac{n}{\beta r}Z_n(\beta r)$$

then following relation will be used in the expansion of the electric and magnetic fields below:

$$\frac{\partial Z_n(\beta r)}{\partial r} = -\beta Z_{n+1}(\beta r) + \frac{n}{r}Z_n(\beta r) = \beta\left[-Z_{n+1}(\beta r) + \frac{n}{\beta r}Z_n(\beta r)\right] = \beta\frac{\partial Z_n(\beta r)}{\partial(\beta r)}$$

Cont'd

11.81 Cont'd

Using Equation (11-234) with Equation (11-235), the fields transverse to the radial direction outside the sphere are

$$E_\theta^{t+} = \frac{1}{j\omega\mu_0\varepsilon_0}\frac{1}{r}\frac{\partial^2 A_r^{t+}}{\partial r\partial\theta} - \frac{1}{\varepsilon_0}\frac{1}{r\sin\theta}\frac{\partial F_r^{t+}}{\partial\phi}$$

$$= -j\frac{E_0}{\omega\mu_0\varepsilon_0 r}\left\{\frac{\beta_0}{\omega}\cos\phi\sum_{n=1}^{\infty}\left[a_n\hat{J}'_n(\beta_0 r) + b_n\hat{H}'^{(2)}_n(\beta_0 r)\right]P'^{(1)}_n(\cos\theta)\right\}$$

$$-\frac{E_0}{\varepsilon_0 r\sin\theta}\left\{\frac{1}{\omega\eta_0}\cos\phi\sum_{n=1}^{\infty}\left[a_n\hat{J}_n(\beta_0 r) + c_n\hat{H}^{(2)}_n(\beta_0 r)\right]P^{(1)}_n(\cos\theta)\right\}$$

$$E_\phi^{t+} = \frac{1}{j\omega\mu_0\varepsilon_0}\frac{1}{r\sin\theta}\frac{\partial^2 A_r^{t+}}{\partial r\partial\phi} + \frac{1}{\varepsilon_0}\frac{1}{r}\frac{\partial F_r^{t+}}{\partial\theta}$$

$$= j\frac{E_0}{\omega\mu_0\varepsilon_0 r\sin\theta}\left\{\frac{\beta_0}{\omega}\sin\phi\sum_{n=1}^{\infty}\left[a_n\hat{J}'_n(\beta_0 r) + b_n\hat{H}'^{(2)}_n(\beta_0 r)\right]P^{(1)}_n(\cos\theta)\right\}$$

$$+\frac{E_0}{\varepsilon_0 r}\left\{\frac{1}{\omega\eta_0}\sin\phi\sum_{n=1}^{\infty}\left[a_n\hat{J}_n(\beta_0 r) + c_n\hat{H}^{(2)}_n(\beta_0 r)\right]P'^{(1)}_n(\cos\theta)\right\}$$

$$H_\theta^{t+} = \frac{1}{\mu_0}\frac{1}{r\sin\theta}\frac{\partial A_r^{t+}}{\partial\phi} + \frac{1}{j\omega\mu_0\varepsilon_0}\frac{1}{r}\frac{\partial^2 F_r^{t+}}{\partial r\partial\theta}$$

$$= -\frac{E_0}{\mu_0 r\sin\theta}\left\{\frac{1}{\omega}\sin\phi\sum_{n=1}^{\infty}\left[a_n\hat{J}_n(\beta_0 r) + b_n\hat{H}^{(2)}_n(\beta_0 r)\right]P^{(1)}_n(\cos\theta)\right\}$$

$$-j\frac{E_0}{\omega\mu_0\varepsilon_0 r}\left\{\frac{\beta_0}{\omega\eta_0}\sin\phi\sum_{n=1}^{\infty}\left[a_n\hat{J}'_n(\beta_0 r) + c_n\hat{H}'^{(2)}_n(\beta_0 r)\right]P'^{(1)}_n(\cos\theta)\right\}$$

$$H_\phi^{t+} = -\frac{1}{\mu_0}\frac{1}{r}\frac{\partial A_r^{t+}}{\partial\theta} + \frac{1}{j\omega\mu_0\varepsilon_0}\frac{1}{r\sin\theta}\frac{\partial^2 F_r^{t+}}{\partial r\partial\phi}$$

$$= -\frac{E_0}{\mu_0 r}\left\{\frac{1}{\omega}\cos\phi\sum_{n=1}^{\infty}\left[a_n\hat{J}_n(\beta_0 r) + b_n\hat{H}^{(2)}_n(\beta_0 r)\right]P'^{(1)}_n(\cos\theta)\right\}$$

$$-j\frac{E_0}{\omega\mu_0\varepsilon_0 r\sin\theta}\left\{\frac{\beta_0}{\omega\eta_0}\cos\phi\sum_{n=1}^{\infty}\left[a_n\hat{J}'_n(\beta_0 r) + c_n\hat{H}'^{(2)}_n(\beta_0 r)\right]P^{(1)}_n(\cos\theta)\right\}$$

Cont'd

11.81 contd

Similarly, Equation (11-234) is used to find the fields transverse to the radial direction inside the sphere, which are

$$E_\theta^{t-} = \frac{1}{j\omega\mu_d\varepsilon_d}\frac{1}{r}\frac{\partial^2 A_r^{t-}}{\partial r\partial\theta} - \frac{1}{\varepsilon_d}\frac{1}{r\sin\theta}\frac{\partial F_r^{t-}}{\partial\phi}$$

$$= -j\frac{E_0}{\omega\mu_d\varepsilon_d r}\left\{\frac{\beta_d}{\omega}\cos\phi\sum_{n=1}^{\infty} d_n\hat{J}'_n(\beta_d r)P'^{(1)}_n(\cos\theta)\right\}$$

$$- \frac{E_0}{\varepsilon_d r\sin\theta}\left\{\frac{1}{\omega\eta_d}\cos\phi\sum_{n=1}^{\infty} e_n\hat{J}_n(\beta_d r)P_n^{(1)}(\cos\theta)\right\}$$

$$E_\phi^{t-} = \frac{1}{j\omega\mu_d\varepsilon_d}\frac{1}{r\sin\theta}\frac{\partial^2 A_r^{t-}}{\partial r\partial\phi} + \frac{1}{\varepsilon_d}\frac{1}{r}\frac{\partial F_r^{t-}}{\partial\theta}$$

$$= j\frac{E_0}{\omega\mu_d\varepsilon_d r\sin\theta}\left\{\frac{\beta_d}{\omega}\sin\phi\sum_{n=1}^{\infty} d_n\hat{J}'_n(\beta_d r)P_n^{(1)}(\cos\theta)\right\}$$

$$+ \frac{E_0}{\varepsilon_d r}\left\{\frac{1}{\omega\eta_d}\sin\phi\sum_{n=1}^{\infty} e_n\hat{J}_n(\beta_d r)P'^{(1)}_n(\cos\theta)\right\}$$

$$H_\theta^{t-} = \frac{1}{\mu_d}\frac{1}{r\sin\theta}\frac{\partial A_r^{t-}}{\partial\phi} + \frac{1}{j\omega\mu_d\varepsilon_d}\frac{1}{r}\frac{\partial^2 F_r^{t-}}{\partial r\partial\theta}$$

$$= -\frac{E_0}{\mu_d r\sin\theta}\left\{\frac{1}{\omega}\sin\phi\sum_{n=1}^{\infty} d_n\hat{J}_n(\beta_d r)P_n^{(1)}(\cos\theta)\right\}$$

$$- j\frac{E_0}{\omega\mu_d\varepsilon_d r}\left\{\frac{\beta_d}{\omega\eta_d}\sin\phi\sum_{n=1}^{\infty} e_n\hat{J}'_n(\beta_d r)P'^{(1)}_n(\cos\theta)\right\}$$

$$H_\phi^{t-} = -\frac{1}{\mu_d}\frac{1}{r}\frac{\partial A_r^{t-}}{\partial\theta} + \frac{1}{j\omega\mu_d\varepsilon_d}\frac{1}{r\sin\theta}\frac{\partial^2 F_r^{t-}}{\partial r\partial\phi}$$

$$= -\frac{E_0}{\mu_d r}\left\{\frac{1}{\omega}\cos\phi\sum_{n=1}^{\infty} d_n\hat{J}_n(\beta_d r)P'^{(1)}_n(\cos\theta)\right\}$$

$$- j\frac{E_0}{\omega\mu_d\varepsilon_d r\sin\theta}\left\{\frac{\beta_d}{\omega\eta_d}\cos\phi\sum_{n=1}^{\infty} e_n\hat{J}'_n(\beta_d r)P_n^{(1)}(\cos\theta)\right\}$$

Contd

11.81 cont'd

where

$$' = \frac{\partial}{\partial(\beta r)}$$

for the spherical Bessel or Hankel function, and

$$' = \frac{\partial}{\partial\theta}$$

for the associated Legendre functions.

The appropriate boundary equations require the continuity of the tangential electric and magnetic fields, and are

$$E_\theta^{t-}(r = a, 0 \leq \theta \leq \pi, 0 \leq \emptyset \leq 2\pi) = E_\theta^{t+}(r = a, 0 \leq \theta \leq \pi, 0 \leq \emptyset \leq 2\pi)$$

$$E_\phi^{t-}(r = a, 0 \leq \theta \leq \pi, 0 \leq \emptyset \leq 2\pi) = E_\phi^{t+}(r = a, 0 \leq \theta \leq \pi, 0 \leq \emptyset \leq 2\pi)$$

$$H_\theta^{t-}(r = a, 0 \leq \theta \leq \pi, 0 \leq \emptyset \leq 2\pi) = H_\theta^{t+}(r = a, 0 \leq \theta \leq \pi, 0 \leq \emptyset \leq 2\pi)$$

$$H_\phi^{t-}(r = a, 0 \leq \theta \leq \pi, 0 \leq \emptyset \leq 2\pi) = H_\phi^{t+}(r = a, 0 \leq \theta \leq \pi, 0 \leq \emptyset \leq 2\pi)$$

Cont'd

11.81 cont'd

Applying the first boundary condition yields

$$-j\frac{E_0}{\omega\mu_d\varepsilon_d a}\left\{\frac{\beta_d}{\omega}\cos\phi\sum_{n=1}^{\infty} d_n\hat{J}'_n(\beta_d a)P'^{(1)}_n(\cos\theta)\right\}$$
$$-\frac{E_0}{\varepsilon_d a\sin\theta}\left\{\frac{1}{\omega\eta_d}\cos\phi\sum_{n=1}^{\infty} e_n\hat{J}_n(\beta_d a)P^{(1)}_n(\cos\theta)\right\}$$
$$=-j\frac{E_0}{\omega\mu_0\varepsilon_0 a}\left\{\frac{\beta_0}{\omega}\cos\phi\sum_{n=1}^{\infty}\left[a_n\hat{J}'_n(\beta_0 a)+b_n\hat{H}'^{(2)}_n(\beta_0 a)\right]P'^{(1)}_n(\cos\theta)\right\}$$
$$-\frac{E_0}{\varepsilon_0 a\sin\theta}\left\{\frac{1}{\omega\eta_0}\cos\phi\sum_{n=1}^{\infty}\left[a_n\hat{J}_n(\beta_0 a)+c_n\hat{H}^{(2)}_n(\beta_0 a)\right]P^{(1)}_n(\cos\theta)\right\}$$

which leads to the following two relations

$$-j\frac{E_0}{\omega\mu_d\varepsilon_d a}\left\{\frac{\beta_d}{\omega}\cos\phi\sum_{n=1}^{\infty} d_n\hat{J}'_n(\beta_d a)P'^{(1)}_n(\cos\theta)\right\}$$
$$=-j\frac{E_0}{\omega\mu_0\varepsilon_0 a}\left\{\frac{\beta_0}{\omega}\cos\phi\sum_{n=1}^{\infty}\left[a_n\hat{J}'_n(\beta_0 a)+b_n\hat{H}'^{(2)}_n(\beta_0 a)\right]P'^{(1)}_n(\cos\theta)\right\}$$

$$-\frac{E_0}{\varepsilon_d a\sin\theta}\left\{\frac{1}{\omega\eta_d}\cos\phi\sum_{n=1}^{\infty} e_n\hat{J}_n(\beta_d a)P^{(1)}_n(\cos\theta)\right\}$$
$$=-\frac{E_0}{\varepsilon_0 a\sin\theta}\left\{\frac{1}{\omega\eta_0}\cos\phi\sum_{n=1}^{\infty}\left[a_n\hat{J}_n(\beta_0 a)+c_n\hat{H}^{(2)}_n(\beta_0 a)\right]P^{(1)}_n(\cos\theta)\right\}$$

Using

$$\frac{\beta}{\omega^2\mu\varepsilon}=\frac{1}{\omega\eta\varepsilon}=\frac{1}{\beta}$$

The above equations reduce to

$$d_n\beta_0\hat{J}'_n(\beta_d a)=a_n\beta_d\hat{J}'_n(\beta_0 a)+b_n\beta_d\hat{H}'^{(2)}_n(\beta_0 a)$$

$$e_n\beta_0\hat{J}_n(\beta_d a)=a_n\beta_d\hat{J}_n(\beta_0 a)+c_n\beta_d\hat{H}^{(2)}_n(\beta_0 a)$$

Cont'd

11.81 cont'd

Solving for each of the unknown coefficients yields

$$d_n = \frac{a_n\beta_d\hat{J}'_n(\beta_0 a) + b_n\beta_d\hat{H}'^{(2)}_n(\beta_0 a)}{\beta_0\hat{J}'_n(\beta_d a)}$$

$$e_n = \frac{a_n\beta_d\hat{J}_n(\beta_0 a) + c_n\beta_d\hat{H}^{(2)}_n(\beta_0 a)}{\beta_0\hat{J}_n(\beta_d a)}$$

$$b_n = \frac{d_n\beta_0\hat{J}'_n(\beta_d a) - a_n\beta_d\hat{J}'_n(\beta_0 a)}{\beta_d\hat{H}'^{(2)}_n(\beta_0 a)}$$

$$c_n = \frac{e_n\beta_0\hat{J}_n(\beta_d a) - a_n\beta_d\hat{J}_n(\beta_0 a)}{\beta_d\hat{H}^{(2)}_n(\beta_0 a)}$$

Applying the second boundary condition from above will lead to the same equations. Similarly, it is only necessary to apply one of the two boundary conditions equating the tangential magnetic fields. Applying the third boundary condition yields

$$\begin{aligned}
-\frac{E_0}{\mu_d a\sin\theta}&\left\{\frac{1}{\omega}\sin\phi\sum_{n=1}^{\infty} d_n\hat{J}_n(\beta_d a)P_n^{(1)}(\cos\theta)\right\}\\
&-j\frac{E_0}{\omega\mu_d\varepsilon_d a}\left\{\frac{\beta_d}{\omega\eta_d}\sin\phi\sum_{n=1}^{\infty} e_n\hat{J}'_n(\beta_d a)P'^{(1)}_n(\cos\theta)\right\}\\
&=-\frac{E_0}{\mu_0 a\sin\theta}\left\{\frac{1}{\omega}\sin\phi\sum_{n=1}^{\infty}\left[a_n\hat{J}_n(\beta_0 a)+b_n\hat{H}^{(2)}_n(\beta_0 a)\right]P_n^{(1)}(\cos\theta)\right\}\\
&-j\frac{E_0}{\omega\mu_0\varepsilon_0 a}\left\{\frac{\beta_0}{\omega\eta_0}\sin\phi\sum_{n=1}^{\infty}\left[a_n\hat{J}'_n(\beta_0 a)+c_n\hat{H}'^{(2)}_n(\beta_0 a)\right]P'^{(1)}_n(\cos\theta)\right\}
\end{aligned}$$

11.81 cont'd

which leads to the following two relations

$$-\frac{E_0}{\mu_d a \sin\theta}\left\{\frac{1}{\omega}\sin\phi \sum_{n=1}^{\infty} d_n \hat{J}_n(\beta_d a) P_n^{(1)}(\cos\theta)\right\}$$
$$= -\frac{E_0}{\mu_0 a \sin\theta}\left\{\frac{1}{\omega}\sin\phi \sum_{n=1}^{\infty}\left[a_n \hat{J}_n(\beta_0 a) + b_n \hat{H}_n^{(2)}(\beta_0 a)\right] P_n^{(1)}(\cos\theta)\right\}$$

$$-j\frac{E_0}{\omega\mu_d\varepsilon_d a}\left\{\frac{\beta_d}{\omega\eta_d}\sin\phi \sum_{n=1}^{\infty} e_n \hat{J}'_n(\beta_d a) P'^{(1)}_n(\cos\theta)\right\}$$
$$= -j\frac{E_0}{\omega\mu_0\varepsilon_0 a}\left\{\frac{\beta_0}{\omega\eta_0}\sin\phi \sum_{n=1}^{\infty}\left[a_n \hat{J}'_n(\beta_0 a) + c_n \hat{H}'^{(2)}_n(\beta_0 a)\right] P'^{(1)}_n(\cos\theta)\right\}$$

Using

$$\frac{1}{\omega\mu} = \frac{\beta}{\omega^2\mu\varepsilon\eta} = \frac{1}{\beta\eta}$$

These reduce to

$$d_n\beta_0\eta_0\hat{J}_n(\beta_d a) = a_n\beta_d\eta_d\hat{J}_n(\beta_0 a) + b_n\beta_d\eta_d\hat{H}_n^{(2)}(\beta_0 a)$$

$$e_n\beta_0\eta_0\hat{J}'_n(\beta_d a) = a_n\beta_d\eta_d\hat{J}'_n(\beta_0 a) + c_n\beta_d\eta_d\hat{H}'^{(2)}_n(\beta_0 a)$$

Solving for each of the unknown coefficients yields

$$d_n = \frac{a_n\beta_d\eta_d\hat{J}_n(\beta_0 a) + b_n\beta_d\eta_d\hat{H}_n^{(2)}(\beta_0 a)}{\beta_0\eta_0\hat{J}_n(\beta_d a)}$$

$$e_n = \frac{a_n\beta_d\eta_d\hat{J}'_n(\beta_0 a) + c_n\beta_d\eta_d\hat{H}'^{(2)}_n(\beta_0 a)}{\beta_0\eta_0\hat{J}'_n(\beta_d a)}$$

$$b_n = \frac{d_n\beta_0\eta_0\hat{J}_n(\beta_d a) - a_n\beta_d\eta_d\hat{J}_n(\beta_0 a)}{\beta_d\eta_d\hat{H}_n^{(2)}(\beta_0 a)}$$

$$c_n = \frac{e_n\beta_0\eta_0\hat{J}'_n(\beta_d a) - a_n\beta_d\eta_d\hat{J}'_n(\beta_0 a)}{\beta_d\eta_d\hat{H}'^{(2)}_n(\beta_0 a)}$$

Cont'd

11.81 cont'd

Equating the expressions of the coefficients found from applying the boundary conditions,

$$d_n = \frac{a_n\beta_d\hat{J}'_n(\beta_0 a) + b_n\beta_d\hat{H}'^{(2)}_n(\beta_0 a)}{\beta_0\hat{J}'_n(\beta_d a)} = \frac{a_n\beta_d\eta_d\hat{J}_n(\beta_0 a) + b_n\beta_d\eta_d\hat{H}^{(2)}_n(\beta_0 a)}{\beta_0\eta_0\hat{J}_n(\beta_d a)}$$

$$e_n = \frac{a_n\beta_d\hat{J}_n(\beta_0 a) + c_n\beta_d\hat{H}^{(2)}_n(\beta_0 a)}{\beta_0\hat{J}_n(\beta_d a)} = \frac{a_n\beta_d\eta_d\hat{J}'_n(\beta_0 a) + c_n\beta_d\eta_d\hat{H}'^{(2)}_n(\beta_0 a)}{\beta_0\eta_0\hat{J}'_n(\beta_d a)}$$

$$b_n = \frac{d_n\beta_0\hat{J}'_n(\beta_d a) - a_n\beta_d\hat{J}'_n(\beta_0 a)}{\beta_d\hat{H}'^{(2)}_n(\beta_0 a)} = \frac{d_n\beta_0\eta_0\hat{J}_n(\beta_d a) - a_n\beta_d\eta_d\hat{J}_n(\beta_0 a)}{\beta_d\eta_d\hat{H}^{(2)}_n(\beta_0 a)}$$

$$c_n = \frac{e_n\beta_0\hat{J}_n(\beta_d a) - a_n\beta_d\hat{J}_n(\beta_0 a)}{\beta_d\hat{H}^{(2)}_n(\beta_0 a)} = \frac{e_n\beta_0\eta_0\hat{J}'_n(\beta_d a) - a_n\beta_d\eta_d\hat{J}'_n(\beta_0 a)}{\beta_d\eta_d\hat{H}'^{(2)}_n(\beta_0 a)}$$

Cross-multiplying and factoring out the propagation constants when possible,

$$\begin{aligned} &a_n\eta_0\hat{J}'_n(\beta_0 a)\hat{J}_n(\beta_d a) + b_n\eta_0\hat{H}'^{(2)}_n(\beta_0 a)\hat{J}_n(\beta_d a) \\ &\qquad = a_n\eta_d\hat{J}_n(\beta_0 a)\hat{J}'_n(\beta_d a) + b_n\eta_d\hat{H}^{(2)}_n(\beta_0 a)\hat{J}'_n(\beta_d a) \end{aligned}$$

$$\begin{aligned} &a_n\eta_0\hat{J}_n(\beta_0 a)\hat{J}'_n(\beta_d a) + c_n\eta_0\hat{H}^{(2)}_n(\beta_0 a)\hat{J}'_n(\beta_d a) \\ &\qquad = a_n\eta_d\hat{J}'_n(\beta_0 a)\hat{J}_n(\beta_d a) + c_n\eta_d\hat{H}'^{(2)}_n(\beta_0 a)\hat{J}_n(\beta_d a) \end{aligned}$$

$$\begin{aligned} &d_n\beta_0\eta_d\hat{H}^{(2)}_n(\beta_0 a)\hat{J}'_n(\beta_d a) - a_n\beta_d\eta_d\hat{J}'_n(\beta_0 a)\hat{H}^{(2)}_n(\beta_0 a) \\ &\qquad = d_n\beta_0\eta_0\hat{H}'^{(2)}_n(\beta_0 a)\hat{J}_n(\beta_d a) - a_n\beta_d\eta_d\hat{J}_n(\beta_0 a)\hat{H}'^{(2)}_n(\beta_0 a) \end{aligned}$$

$$\begin{aligned} &e_n\beta_0\eta_d\hat{H}'^{(2)}_n(\beta_0 a)\hat{J}_n(\beta_d a) - a_n\beta_d\eta_d\hat{J}_n(\beta_0 a)\hat{H}'^{(2)}_n(\beta_0 a) \\ &\qquad = e_n\beta_0\eta_0\hat{H}^{(2)}_n(\beta_0 a)\hat{J}'_n(\beta_d a) - a_n\beta_d\eta_d\hat{J}'_n(\beta_0 a)\hat{H}^{(2)}_n(\beta_0 a) \end{aligned}$$

Rearranging the terms,

$$\begin{aligned} &b_n\eta_0\hat{H}'^{(2)}_n(\beta_0 a)\hat{J}_n(\beta_d a) - b_n\eta_d\hat{H}^{(2)}_n(\beta_0 a)\hat{J}'_n(\beta_d a) \\ &\qquad = -a_n\eta_0\hat{J}'_n(\beta_0 a)\hat{J}_n(\beta_d a) + a_n\eta_d\hat{J}_n(\beta_0 a)\hat{J}'_n(\beta_d a) \end{aligned}$$

$$\begin{aligned} &c_n\eta_0\hat{H}^{(2)}_n(\beta_0 a)\hat{J}'_n(\beta_d a) - c_n\eta_d\hat{H}'^{(2)}_n(\beta_0 a)\hat{J}_n(\beta_d a) \\ &\qquad = -a_n\eta_0\hat{J}_n(\beta_0 a)\hat{J}'_n(\beta_d a) + a_n\eta_d\hat{J}'_n(\beta_0 a)\hat{J}_n(\beta_d a) \end{aligned}$$

Cont'd

11.81 Cont'd

$$d_n\beta_0\eta_0\hat{H}'^{(2)}_n(\beta_0 a)\hat{J}_n(\beta_d a) - d_n\beta_0\eta_d\hat{H}^{(2)}_n(\beta_0 a)\hat{J}'_n(\beta_d a)$$
$$= a_n\beta_d\eta_d\hat{J}_n(\beta_0 a)\hat{H}'^{(2)}_n(\beta_0 a) - a_n\beta_d\eta_d\hat{J}'_n(\beta_0 a)\hat{H}^{(2)}_n(\beta_0 a)$$

$$e_n\beta_0\eta_0\hat{H}^{(2)}_n(\beta_0 a)\hat{J}'_n(\beta_d a) - e_n\beta_0\eta_d\hat{H}'^{(2)}_n(\beta_0 a)\hat{J}_n(\beta_d a)$$
$$= a_n\beta_d\eta_d\hat{J}'_n(\beta_0 a)\hat{H}^{(2)}_n(\beta_0 a) - a_n\beta_d\eta_d\hat{J}_n(\beta_0 a)\hat{H}'^{(2)}_n(\beta_0 a)$$

Solving for the coefficients,

$$b_n = \frac{-\eta_0\hat{J}'_n(\beta_0 a)\hat{J}_n(\beta_d a) + \eta_d\hat{J}_n(\beta_0 a)\hat{J}'_n(\beta_d a)}{\eta_0\hat{H}'^{(2)}_n(\beta_0 a)\hat{J}_n(\beta_d a) - \eta_d\hat{H}^{(2)}_n(\beta_0 a)\hat{J}'_n(\beta_d a)}a_n$$

$$c_n = \frac{-\eta_0\hat{J}_n(\beta_0 a)\hat{J}'_n(\beta_d a) + \eta_d\hat{J}'_n(\beta_0 a)\hat{J}_n(\beta_d a)}{\eta_0\hat{H}^{(2)}_n(\beta_0 a)\hat{J}'_n(\beta_d a) - \eta_d\hat{H}'^{(2)}_n(\beta_0 a)\hat{J}_n(\beta_d a)}a_n$$

$$d_n = \frac{\beta_d\eta_d\hat{J}_n(\beta_0 a)\hat{H}'^{(2)}_n(\beta_0 a) - \beta_d\eta_d\hat{J}'_n(\beta_0 a)\hat{H}^{(2)}_n(\beta_0 a)}{\beta_0\eta_0\hat{H}'^{(2)}_n(\beta_0 a)\hat{J}_n(\beta_d a) - \beta_0\eta_d\hat{H}^{(2)}_n(\beta_0 a)\hat{J}'_n(\beta_d a)}a_n$$

$$e_n = \frac{\beta_d\eta_d\hat{J}'_n(\beta_0 a)\hat{H}^{(2)}_n(\beta_0 a) - \beta_d\eta_d\hat{J}_n(\beta_0 a)\hat{H}'^{(2)}_n(\beta_0 a)}{\beta_0\eta_0\hat{H}^{(2)}_n(\beta_0 a)\hat{J}'_n(\beta_d a) - \beta_0\eta_d\hat{H}'^{(2)}_n(\beta_0 a)\hat{J}_n(\beta_d a)}a_n$$

Factoring out the propagation constant and impedance of free space leaves

$$b_n = \frac{-\hat{J}'_n(\beta_0 a)\hat{J}_n(\beta_d a) + \sqrt{\frac{\dot{\mu}_r}{\dot{\varepsilon}_r}}\hat{J}_n(\beta_0 a)\hat{J}'_n(\beta_d a)}{\hat{H}'^{(2)}_n(\beta_0 a)\hat{J}_n(\beta_d a) - \sqrt{\frac{\dot{\mu}_r}{\dot{\varepsilon}_r}}\hat{H}^{(2)}_n(\beta_0 a)\hat{J}'_n(\beta_d a)}a_n$$

$$c_n = \frac{-\hat{J}_n(\beta_0 a)\hat{J}'_n(\beta_d a) + \sqrt{\frac{\dot{\mu}_r}{\dot{\varepsilon}_r}}\hat{J}'_n(\beta_0 a)\hat{J}_n(\beta_d a)}{\hat{H}^{(2)}_n(\beta_0 a)\hat{J}'_n(\beta_d a) - \sqrt{\frac{\dot{\mu}_r}{\dot{\varepsilon}_r}}\hat{H}'^{(2)}_n(\beta_0 a)\hat{J}_n(\beta_d a)}a_n$$

$$d_n = \frac{\dot{\mu}_r\left[\hat{J}_n(\beta_0 a)\hat{H}'^{(2)}_n(\beta_0 a) - \hat{J}'_n(\beta_0 a)\hat{H}^{(2)}_n(\beta_0 a)\right]}{\hat{H}'^{(2)}_n(\beta_0 a)\hat{J}_n(\beta_d a) - \sqrt{\frac{\dot{\mu}_r}{\dot{\varepsilon}_r}}\hat{H}^{(2)}_n(\beta_0 a)\hat{J}'_n(\beta_d a)}a_n$$

$$e_n = \frac{\dot{\mu}_r\left[\hat{J}'_n(\beta_0 a)\hat{H}^{(2)}_n(\beta_0 a) - \hat{J}_n(\beta_0 a)\hat{H}'^{(2)}_n(\beta_0 a)\right]}{\hat{H}^{(2)}_n(\beta_0 a)\hat{J}'_n(\beta_d a) - \sqrt{\frac{\dot{\mu}_r}{\dot{\varepsilon}_r}}\hat{H}'^{(2)}_n(\beta_0 a)\hat{J}_n(\beta_d a)}a_n$$

Cont'd

11.81 Cont'd

Now, from Equation (IV-49), a relation for spherical Bessel and Hankel functions is

$$\hat{B}_n(\beta r) = \sqrt{\frac{\pi\beta r}{2}} B_{n+1/2}(\beta r)$$

which leads to

$$\begin{aligned}
&\hat{J}_n(\beta_0 a)\hat{H}'^{(2)}_n(\beta_0 a) - \hat{J}'_n(\beta_0 a)\hat{H}^{(2)}_n(\beta_0 a) \\
&= \sqrt{\frac{\pi\beta_0 a}{2}} J_{n+1/2}(\beta_0 a) \sqrt{\frac{\pi\beta_0 a}{2}} H'^{(2)}_{n+1/2}(\beta_0 a) \\
&- \sqrt{\frac{\pi\beta_0 a}{2}} J'_{n+1/2}(\beta_0 a) \sqrt{\frac{\pi\beta_0 a}{2}} H^{(2)}_{n+1/2}(\beta_0 a) \\
&= \frac{\pi\beta_0 a}{2}\left[J_{n+1/2}(\beta_0 a)H'^{(2)}_{n+1/2}(\beta_0 a) - J'_{n+1/2}(\beta_0 a)H^{(2)}_{n+1/2}(\beta_0 a)\right]
\end{aligned}$$

From Equation (IV-15), the Hankel function of the second kind is defined as

$$H^{(2)}_n(\beta r) = J_n(\beta r) - jY_n(\beta r)$$

This leads to

$$\begin{aligned}
&J_{n+1/2}(\beta_0 a)H'^{(2)}_{n+1/2}(\beta_0 a) - J'_{n+1/2}(\beta_0 a)H^{(2)}_{n+1/2}(\beta_0 a) \\
&= J_{n+1/2}(\beta_0 a)\left[J'_{n+1/2}(\beta_0 a) - jY'_{n+1/2}(\beta_0 a)\right] \\
&- J'_{n+1/2}(\beta_0 a)\left[J_{n+1/2}(\beta_0 a) - jY_{n+1/2}(\beta_0 a)\right] \\
&= J_{n+1/2}(\beta_0 a)J'_{n+1/2}(\beta_0 a) - jJ_{n+1/2}(\beta_0 a)Y'_{n+1/2}(\beta_0 a) \\
&- J'_{n+1/2}(\beta_0 a)J_{n+1/2}(\beta_0 a) + jJ'_{n+1/2}(\beta_0 a)Y_{n+1/2}(\beta_0 a) \\
&= -j\left[J_{n+1/2}(\beta_0 a)Y'_{n+1/2}(\beta_0 a) - J'_{n+1/2}(\beta_0 a)Y_{n+1/2}(\beta_0 a)\right]
\end{aligned}$$

Finally, using the Wronskian from Equation (IV-21), which is,

$$J_n(\beta r)Y'_n(\beta r) - J'_n(\beta r)Y_n(\beta r) = \frac{2}{\pi\beta r}$$

leads to

$$-j\left[J_{n+1/2}(\beta_0 a)Y'_{n+1/2}(\beta_0 a) - J'_{n+1/2}(\beta_0 a)Y_{n+1/2}(\beta_0 a)\right] = -j\frac{2}{\pi\beta_0 a}$$

Cont'd

11.81 cont'd

It follows that

$$\hat{J}_n(\beta_0 a)\hat{H}'^{(2)}_n(\beta_0 a) - \hat{J}'_n(\beta_0 a)\hat{H}^{(2)}_n(\beta_0 a)$$
$$= \frac{\pi\beta_0 a}{2}\left[J_{n+1/2}(\beta_0 a)H'^{(2)}_{n+1/2}(\beta_0 a) - J'_{n+1/2}(\beta_0 a)H^{(2)}_{n+1/2}(\beta_0 a)\right]$$
$$= \frac{\pi\beta_0 a}{2}\left\{-j\left[J_{n+1/2}(\beta_0 a)Y'_{n+1/2}(\beta_0 a) - J'_{n+1/2}(\beta_0 a)Y_{n+1/2}(\beta_0 a)\right]\right\}$$
$$= \frac{\pi\beta_0 a}{2}\left\{-j\frac{2}{\pi\beta_0 a}\right\} = -j$$

Inserting this back into the equations for the coefficients, and multiplying by the complex relative permittivity leaves

$$b_n = \frac{-\sqrt{\dot{\varepsilon}_r}\hat{J}'_n(\beta_0 a)\hat{J}_n(\beta_d a) + \sqrt{\dot{\mu}_r}\hat{J}_n(\beta_0 a)\hat{J}'_n(\beta_d a)}{\sqrt{\dot{\varepsilon}_r}\hat{H}'^{(2)}_n(\beta_0 a)\hat{J}_n(\beta_d a) - \sqrt{\dot{\mu}_r}\hat{H}^{(2)}_n(\beta_0 a)\hat{J}'_n(\beta_d a)}a_n$$

$$c_n = \frac{-\sqrt{\dot{\varepsilon}_r}\hat{J}_n(\beta_0 a)\hat{J}'_n(\beta_d a) + \sqrt{\dot{\mu}_r}\hat{J}'_n(\beta_0 a)\hat{J}_n(\beta_d a)}{\sqrt{\dot{\varepsilon}_r}\hat{H}^{(2)}_n(\beta_0 a)\hat{J}'_n(\beta_d a) - \sqrt{\dot{\mu}_r}\hat{H}'^{(2)}_n(\beta_0 a)\hat{J}_n(\beta_d a)}a_n$$

$$d_n = -j\frac{\dot{\mu}_r\sqrt{\dot{\varepsilon}_r}}{\sqrt{\dot{\varepsilon}_r}\hat{H}'^{(2)}_n(\beta_0 a)\hat{J}_n(\beta_d a) - \sqrt{\dot{\mu}_r}\hat{H}^{(2)}_n(\beta_0 a)\hat{J}'_n(\beta_d a)}a_n$$

$$e_n = +j\frac{\dot{\mu}_r\sqrt{\dot{\varepsilon}_r}}{\sqrt{\dot{\varepsilon}_r}\hat{H}^{(2)}_n(\beta_0 a)\hat{J}'_n(\beta_d a) - \sqrt{\dot{\mu}_r}\hat{H}'^{(2)}_n(\beta_0 a)\hat{J}_n(\beta_d a)}a_n$$

11.82

For a dielectric sphere of small radius, the first term may be sufficient to represent the fields. The small-argument approximations of the spherical Bessel and Hankel functions and their derivatives can be found using multiple methods. One is to use Equation (IV-49), which gives us

$$\hat{J}_n(\beta r) = (\beta r) j_n(\beta r)$$

$$\hat{H}_n^{(2)}(\beta r) = (\beta r) h_n^{(2)}(\beta r) = (\beta r) j_n(\beta r) - j(\beta r) y_n(\beta r)$$

where, from Equation (IV-48), for small arguments,

$$j_n(\beta r) \simeq \frac{(\beta r)^n}{1 \cdot 3 \cdot 5 \cdots (2n+1)}$$

$$y_n(\beta r) \simeq -1 \cdot 3 \cdot 5 \cdots (2n-1)(\beta r)^{-(n+1)}$$

Then

$$j_1(\beta r) \simeq \frac{\beta r}{3}$$

$$y_1(\beta r) \simeq -\frac{1}{(\beta r)^2}$$

and

$$\hat{J}_1(\beta r) = (\beta r) j_1(\beta r) \simeq \frac{(\beta r)^2}{3}$$

$$\hat{H}_1^{(2)}(\beta r) = (\beta r) h_1^{(2)}(\beta r) = (\beta r) j_1(\beta r) - j(\beta r) y_1(\beta r) \simeq \frac{(\beta r)^2}{3} + j\frac{1}{\beta r} \approx j\frac{1}{\beta r}$$

It follows that

$$\hat{J}'_1(\beta r) = \frac{\partial \hat{J}_1(\beta r)}{\partial(\beta r)} \simeq \frac{\partial}{\partial(\beta r)}\left[\frac{(\beta r)^2}{3}\right] = \frac{2(\beta r)}{3}$$

and

$$\hat{H}'^{(2)}_1(\beta r) = \frac{\partial \hat{H}_1^{(2)}(\beta r)}{\partial(\beta r)} \simeq \frac{\partial}{\partial(\beta r)}\left[j\frac{1}{\beta r}\right] = -j\frac{1}{(\beta r)^2}$$

Cont'd

11.82 cont'd

Therefore, the small-argument approximations for the spherical Bessel and Hankel functions and their derivatives to be used are

$$\hat{J}_1(\beta r) \simeq \frac{(\beta r)^2}{3}$$

$$\hat{H}_1^{(2)}(\beta r) \simeq j\frac{1}{\beta r}$$

$$\hat{J}'_1(\beta r) \simeq \frac{2(\beta r)}{3}$$

$$\hat{H}'^{(2)}_1(\beta r) \simeq -j\frac{1}{(\beta r)^2}$$

Also,

$$a_1 = -j1.5$$

Inserting the above into the coefficient equations,

$$b_1 = \frac{-\sqrt{\dot{\varepsilon}_r}\hat{J}'_1(\beta_0 a)\hat{J}_1(\beta_d a) + \sqrt{\dot{\mu}_r}\hat{J}_1(\beta_0 a)\hat{J}'_1(\beta_d a)}{\sqrt{\dot{\varepsilon}_r}\hat{H}'^{(2)}_1(\beta_0 a)\hat{J}_1(\beta_d a) - \sqrt{\dot{\mu}_r}\hat{H}_1^{(2)}(\beta_0 a)\hat{J}'_1(\beta_d a)}a_1$$

$$\simeq \frac{-\sqrt{\dot{\varepsilon}_r}\left[\frac{2}{3}(\beta_0 a)\right]\left[\frac{1}{3}(\beta_d a)^2\right] + \sqrt{\dot{\mu}_r}\left[\frac{1}{3}(\beta_0 a)^2\right]\left[\frac{2}{3}(\beta_d a)\right]}{\sqrt{\dot{\varepsilon}_r}\left[-j\frac{1}{(\beta_0 a)^2}\right]\left[\frac{1}{3}(\beta_d a)^2\right] - \sqrt{\dot{\mu}_r}\left[j\frac{1}{\beta_0 a}\right]\left[\frac{2}{3}(\beta_d a)\right]}(-j1.5)$$

$$= (-j1.5)\frac{\frac{2\dot{\mu}_r\dot{\varepsilon}_r}{9}(\beta_0 a)^3\left[-\sqrt{\dot{\varepsilon}_r} + \frac{1}{\sqrt{\dot{\varepsilon}_r}}\right]}{\left(-j\frac{\dot{\mu}_r\dot{\varepsilon}_r}{3}\right)\left[\sqrt{\dot{\varepsilon}_r} + \frac{2}{\sqrt{\dot{\varepsilon}_r}}\right]} = -(\beta_0 a)^3\frac{\dot{\varepsilon}_r - 1}{\dot{\varepsilon}_r + 2}$$

$$c_1 = \frac{-\sqrt{\dot{\varepsilon}_r}\hat{J}_1(\beta_0 a)\hat{J}'_1(\beta_d a) + \sqrt{\dot{\mu}_r}\hat{J}'_1(\beta_0 a)\hat{J}_1(\beta_d a)}{\sqrt{\dot{\varepsilon}_r}\hat{H}_1^{(2)}(\beta_0 a)\hat{J}'_1(\beta_d a) - \sqrt{\dot{\mu}_r}\hat{H}'^{(2)}_1(\beta_0 a)\hat{J}_1(\beta_d a)}a_1$$

$$\simeq \frac{-\sqrt{\dot{\varepsilon}_r}\left[\frac{2}{3}(\beta_0 a)^2\right]\left[\frac{1}{3}(\beta_d a)\right] + \sqrt{\dot{\mu}_r}\left[\frac{1}{3}(\beta_0 a)\right]\left[\frac{2}{3}(\beta_d a)^2\right]}{\sqrt{\dot{\varepsilon}_r}\left[j\frac{1}{\beta_0 a}\right]\left[\frac{2}{3}(\beta_d a)\right] - \sqrt{\dot{\mu}_r}\left[-j\frac{1}{(\beta_0 a)^2}\right]\left[\frac{1}{3}(\beta_d a)^2\right]}(-j1.5)$$

$$= (-j1.5)\frac{\frac{2\dot{\mu}_r\dot{\varepsilon}_r}{9}(\beta_0 a)^3\left[-\frac{1}{\sqrt{\dot{\mu}_r}} + \sqrt{\dot{\mu}_r}\right]}{\left(j\frac{\dot{\mu}_r\dot{\varepsilon}_r}{3}\right)\left[\frac{2}{\sqrt{\dot{\mu}_r}} + \sqrt{\dot{\mu}_r}\right]} = -(\beta_0 a)^3\frac{\dot{\mu}_r - 1}{\dot{\mu}_r + 2}$$

Cont'd

11.82 cont'd

$$d_1 = -j \frac{\dot{\mu}_r\sqrt{\dot{\varepsilon}_r}}{\sqrt{\dot{\varepsilon}_r}\hat{H}'^{(2)}_1(\beta_0 a)\hat{J}_1(\beta_d a) - \sqrt{\dot{\mu}_r}\hat{H}^{(2)}_1(\beta_0 a)\hat{J}'_1(\beta_d a)} a_1$$

$$\simeq -j \frac{\dot{\mu}_r\sqrt{\dot{\varepsilon}_r}}{\sqrt{\dot{\varepsilon}_r}\left[-j\frac{1}{(\beta_0 a)^2}\right]\left[\frac{1}{3}(\beta_d a)^2\right] - \sqrt{\dot{\mu}_r}\left[j\frac{1}{\beta_0 a}\right]\left[\frac{2}{3}(\beta_d a)\right]}(-j1.5)$$

$$= (-1.5)\frac{\dot{\mu}_r\sqrt{\dot{\varepsilon}_r}}{\left(-j\frac{\dot{\mu}_r\dot{\varepsilon}_r}{3}\right)\left[\sqrt{\dot{\varepsilon}_r} + \frac{2}{\sqrt{\dot{\varepsilon}_r}}\right]} = \frac{9}{2}\frac{\dot{\mu}_r\dot{\varepsilon}_r}{j\dot{\mu}_r\dot{\varepsilon}_r(\dot{\varepsilon}_r + 2)} = \frac{9}{j2(\dot{\varepsilon}_r + 2)}$$

$$e_1 = +j \frac{\dot{\mu}_r\sqrt{\dot{\varepsilon}_r}}{\sqrt{\dot{\varepsilon}_r}\hat{H}^{(2)}_1(\beta_0 a)\hat{J}'_1(\beta_d a) - \sqrt{\dot{\mu}_r}\hat{H}'^{(2)}_1(\beta_0 a)\hat{J}_1(\beta_d a)} a_1$$

$$\simeq +j \frac{\dot{\mu}_r\sqrt{\dot{\varepsilon}_r}}{\sqrt{\dot{\varepsilon}_r}\left[j\frac{1}{\beta_0 a}\right]\left[\frac{2}{3}(\beta_d a)\right] - \sqrt{\dot{\mu}_r}\left[-j\frac{1}{(\beta_0 a)^2}\right]\left[\frac{1}{3}(\beta_d a)^2\right]}(-j1.5)$$

$$= (1.5)\frac{\dot{\mu}_r\sqrt{\dot{\varepsilon}_r}}{\left(j\frac{\dot{\mu}_r\dot{\varepsilon}_r}{3}\right)\left[\frac{2}{\sqrt{\dot{\mu}_r}} + \sqrt{\dot{\mu}_r}\right]} = \frac{9\sqrt{\dot{\mu}_r}}{j2\left(\sqrt{\dot{\varepsilon}_r}\right)(\dot{\mu}_r + 2)}$$

Thus for a dielectric sphere of small radius,

$$b_1 \approx -(\beta_0 a)^3\frac{\dot{\varepsilon}_r - 1}{\dot{\varepsilon}_r + 2}$$

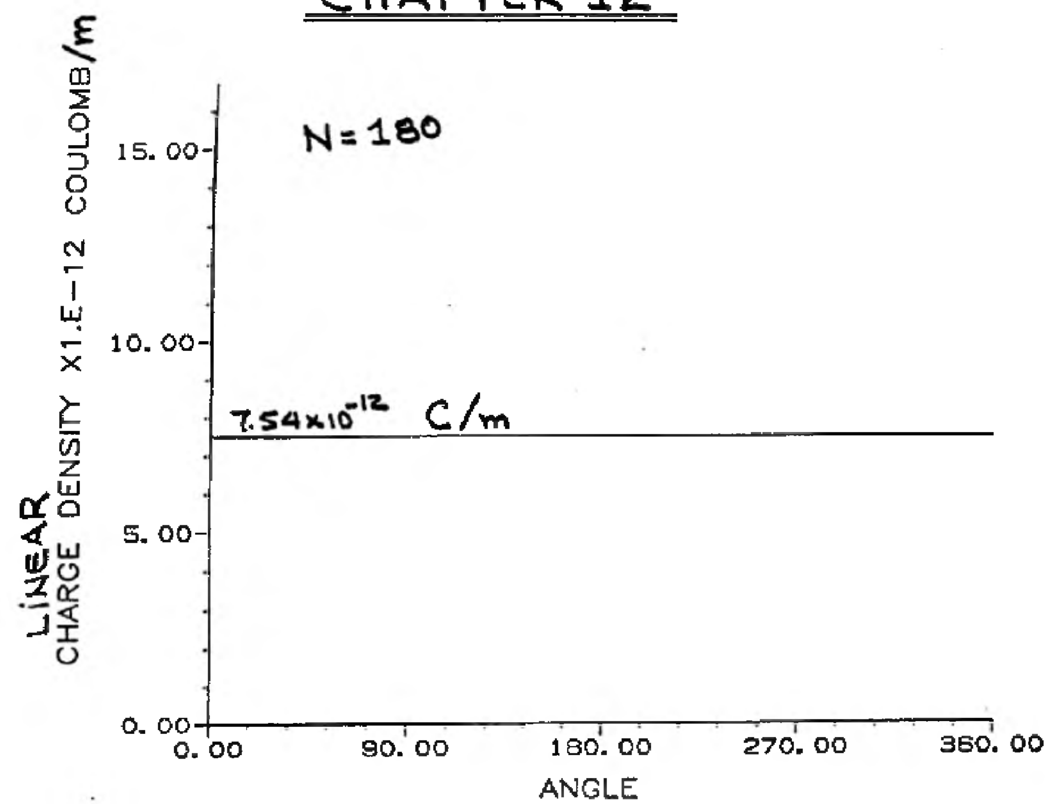
$$c_1 \approx -(\beta_0 a)^3\frac{\dot{\mu}_r - 1}{\dot{\mu}_r + 2}$$

$$d_1 \approx \frac{9}{j2(\dot{\varepsilon}_r + 2)}$$

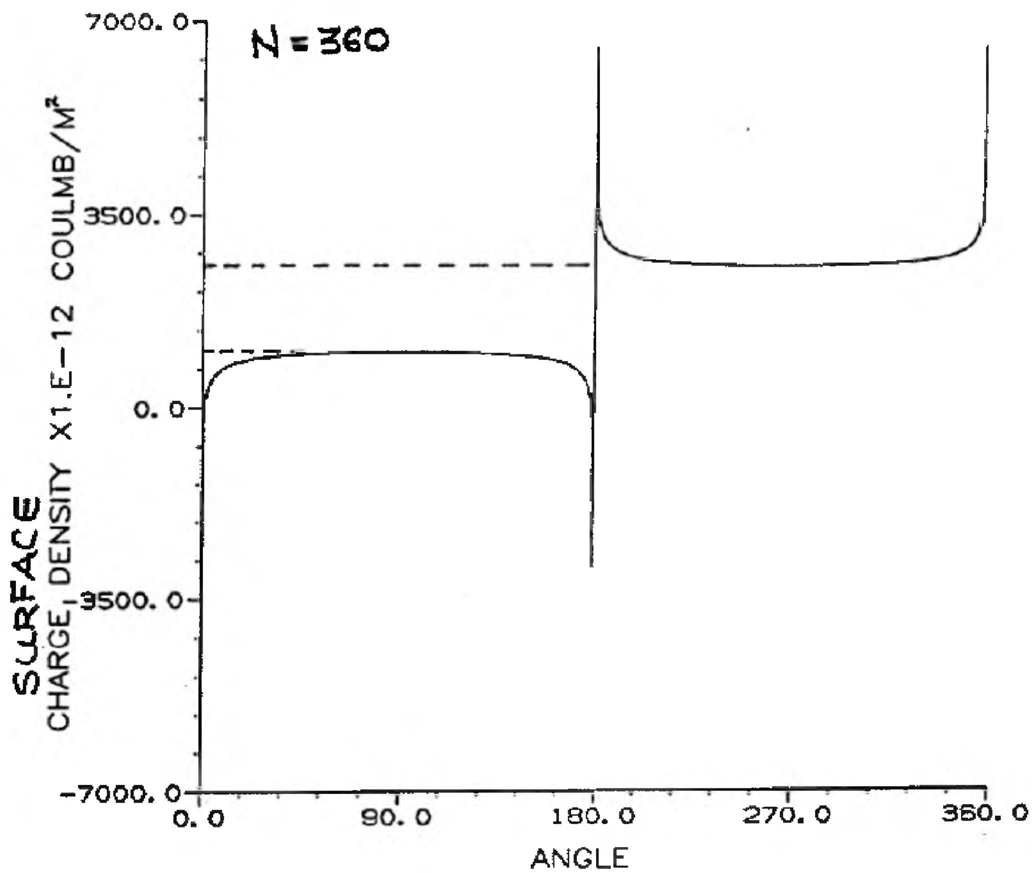
$$e_1 \approx \frac{9\sqrt{\dot{\mu}_r}}{j2\left(\sqrt{\dot{\varepsilon}_r}\right)(\dot{\mu}_r + 2)}$$

CHAPTER 12

12.1



12.2



12.3

Potential

$$V_z = \frac{10a}{\sqrt{a^2 + z^2}} \quad 0 \le z \le 7 \qquad (1)$$

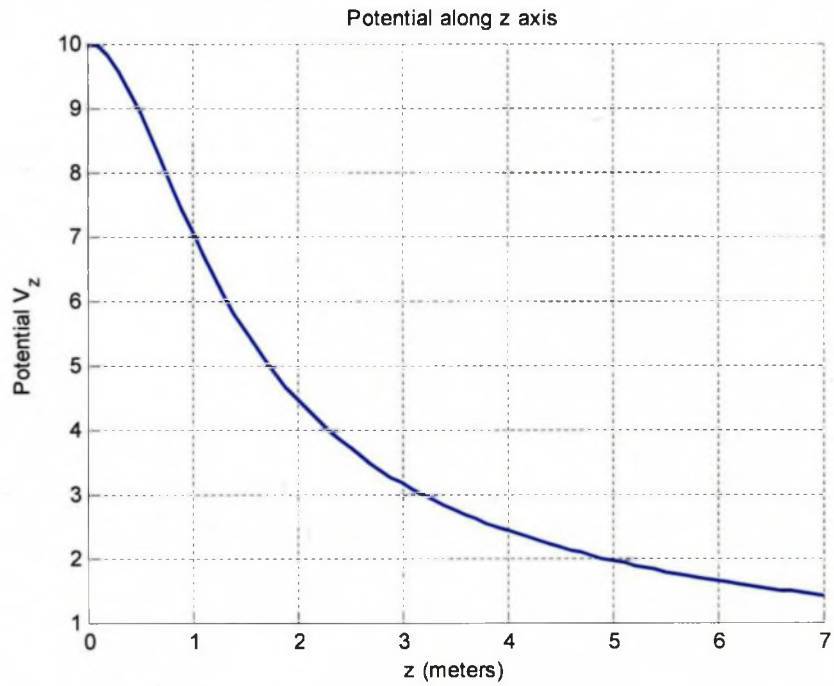


Figure 1

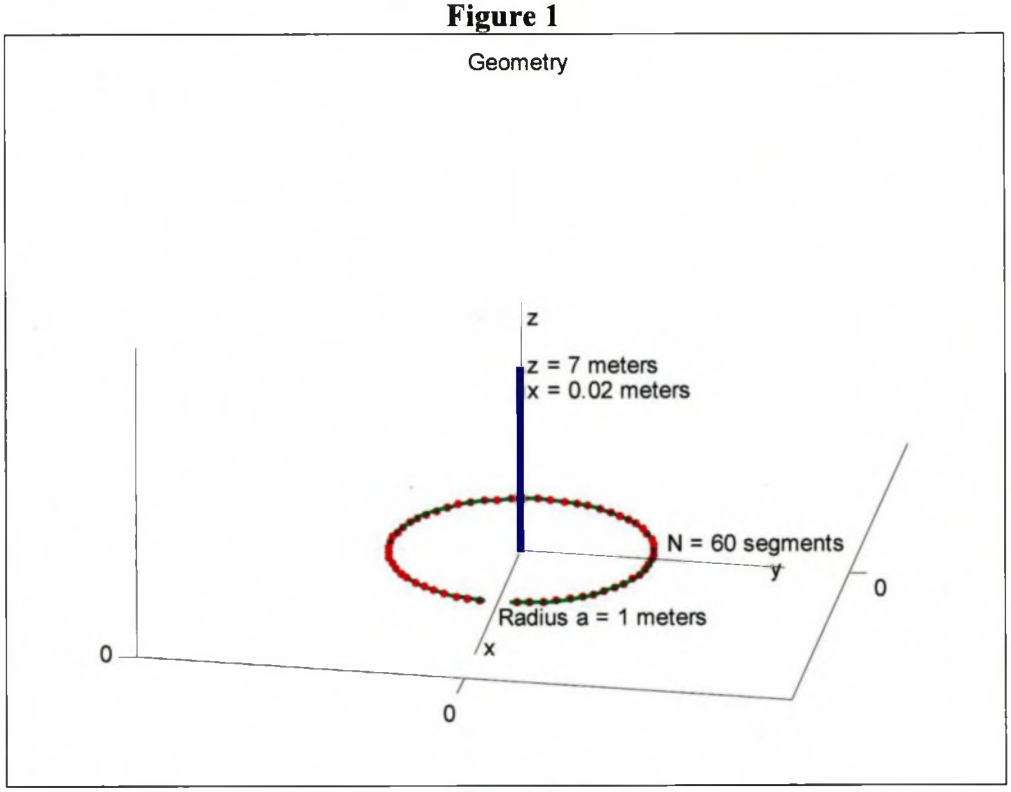


Figure 2

Cont'd

12.3 cont'd

Formulation

Integral equation based on Poisson's differential equation.

$$V = \frac{a}{4\pi\varepsilon_o} \int_{\phi_i'}^{\phi_f'} \frac{\rho(\phi')}{R} d\phi' \qquad (2)$$

$$\text{Assume } \rho(\phi') = \sum_{n=1}^{N} a_n g_n \qquad (3)$$

$$\text{where: } g_n = \begin{cases} 1 & \phi'_{n-1} \leq \phi' \leq \phi'_n \\ 0 & elsewhere \end{cases} \qquad (4)$$

Then,

$$\frac{10a}{\sqrt{a^2+z^2}} = \frac{a}{4\pi\varepsilon_o} \sum_{n=1}^{N} a_n \left(\int_{\phi'_{ni}}^{\phi'_{nf}} \frac{1}{\sqrt{a^2+x^2+z_m^2-2ax\cos(\phi')}} d\phi' \right) \qquad (5)$$

Therefore, a system of equations can be set up using

$$V_m = 4\pi\varepsilon_o \frac{10}{\sqrt{a^2+z_m^2}} \qquad (6a)$$

$$I_n = a_n \qquad (6b)$$

$$Z_{mn} = \sum_{n=1}^{N} \left(\int_{\phi'_{ni}}^{\phi'_{nf}} \frac{1}{\sqrt{a^2+x^2+z_m^2-2ax\cos(\phi')}} d\phi' \right) \qquad (6c)$$

The system of equations is

$$\{V_m\} = [Z_{mn}]\{I_{nm}\} \qquad (7)$$

This system can be solved in MatLab for the unknown coefficients $a_n = I_n$.

Cont'd

12.3 Cont'd

Program Output

```
Radius of the disconnected ring (meters)
=> 1
Ring starts at angle (degrees)
=>5
Ring ends at angle (degrees)
=>355
Displacement of the potential from the center along x axis (meters)
=> 0.02
Number of segments
=>60
Warning: Matrix is close to singular or badly scaled.
         Results may be inaccurate. RCOND = 6.444138e-020.
> In TakeHome1b_Kononov at 46
>>
```

Charge Density on the Ring

This problem is the inverse of a problem where it is required to find potential due to a given charge distribution. In fact, V_z in (1) is the potential along z axis due to a uniform charge distribution along a closed ring of radius *a*. Since our geometry is perturbed only slightly from the geometry of a closed ring, i.e. the ring is discontinuous (see Figure 2), it is reasonable to expect that the charge distribution found using the Integral Equation method will also be nearly uniform. Based on this assumption, we can predict the charge distribution as follows. Using (2)

$$V = \frac{a}{4\pi\varepsilon_o}\int_{5^\circ}^{355^\circ}\frac{\rho(\phi')}{R}d\phi' = \frac{a1}{4\pi\varepsilon_o R}\int_{5^\circ}^{355^\circ}\rho(\phi')d\phi' = \frac{\rho(\phi')}{4\pi\varepsilon_o R}(355-5)\frac{\pi}{180} \quad (8)$$

Equating to (8) to (1),

$$V_z = \frac{10a}{\sqrt{a^2+z^2}} = \frac{a\rho(\phi')}{4\varepsilon_o R}\frac{35}{18} \quad (9)$$

where

$$a = 1$$
$$R = \sqrt{a^2+z^2} \quad (10)$$

Then,

$$\rho(\phi') = 10\frac{18}{35}4\varepsilon_o = 182.14\times 10^{-12} \quad (11)$$

Cont'd

12.3 cont'd

By solving this problem using Integral equation method, we get the following charge density:

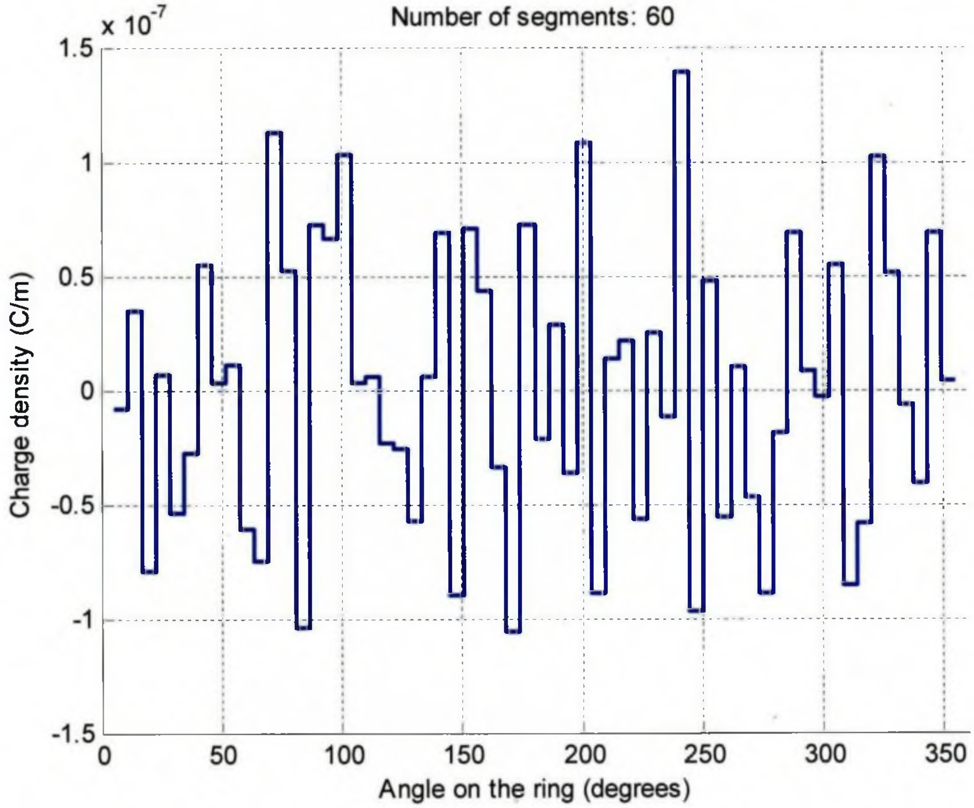


Figure 3

which is obviously non-uniform. This charge density, however, does reproduce the potential specified by (1) as can be seen in Figure 4. This observation suggests that the problem of finding the charge distribution given a potential does not always have a unique solution. In other words, it is possible to find different charge distributions that produce the same potential. In such a case it is said that the problem is 'ill posed'. In order to rectify this problem, more (or different) boundary conditions are required.

Another indication of the problem not having a unique solution is the fact that the matrix $\left[Z_{mn}\right]$ is close to singular as is indicated by the following warning message produced by MatLab:

```
Warning: Matrix is close to singular or badly scaled.
Results may be inaccurate. RCOND = 6.444138e-020.
```

Cont'd

12.3 cont'd

Potential
Integral Equation and Equation (1)

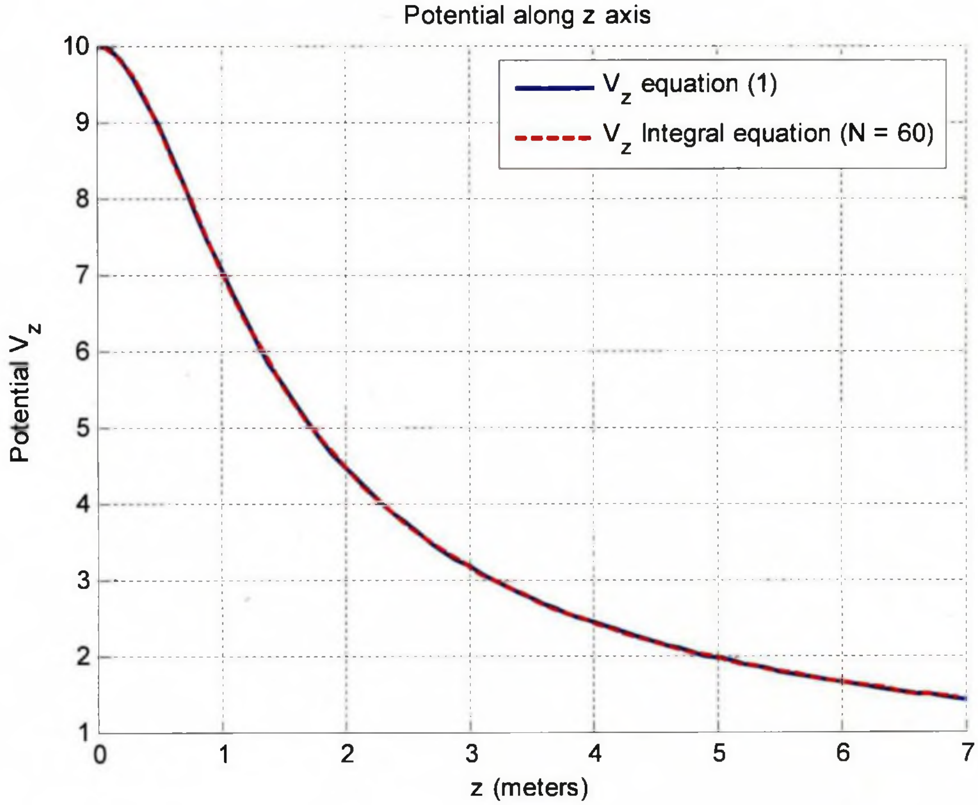
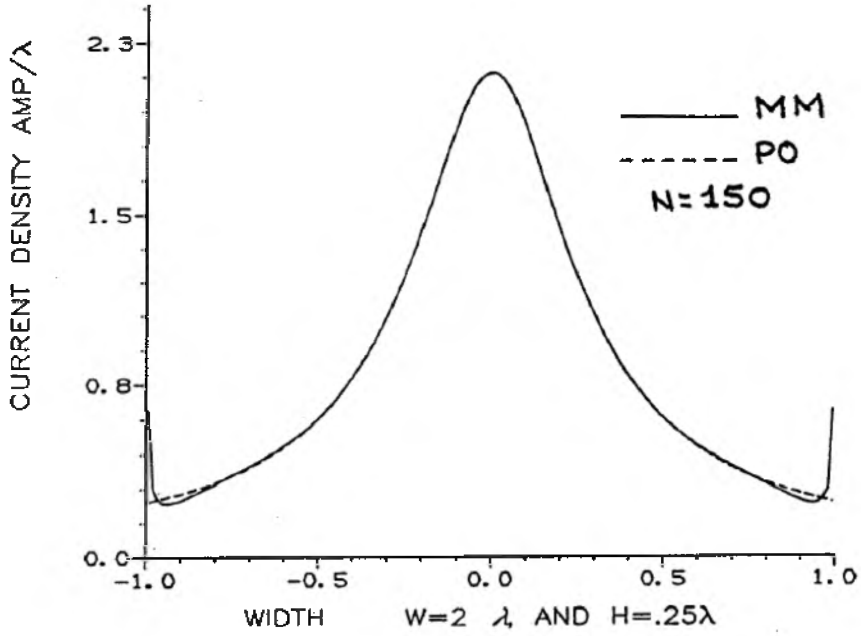
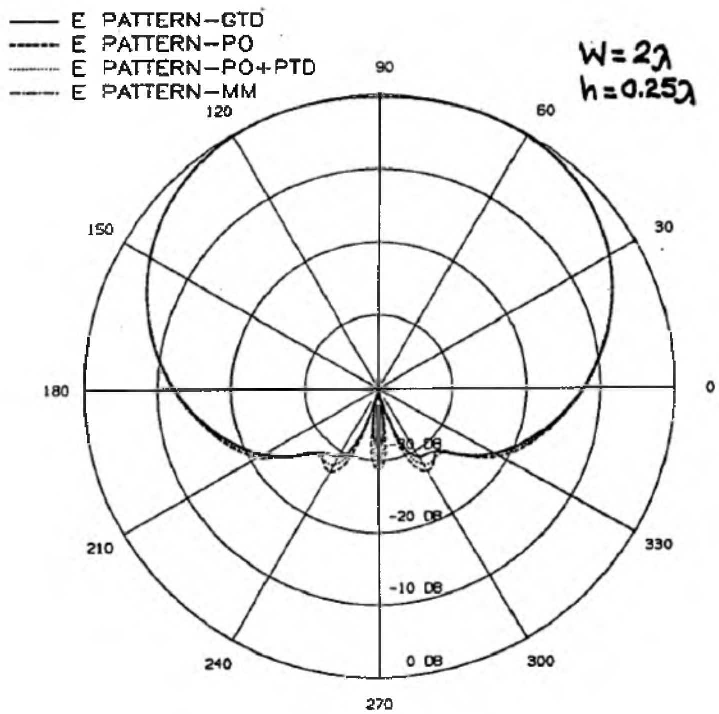


Figure 4

Cont'd

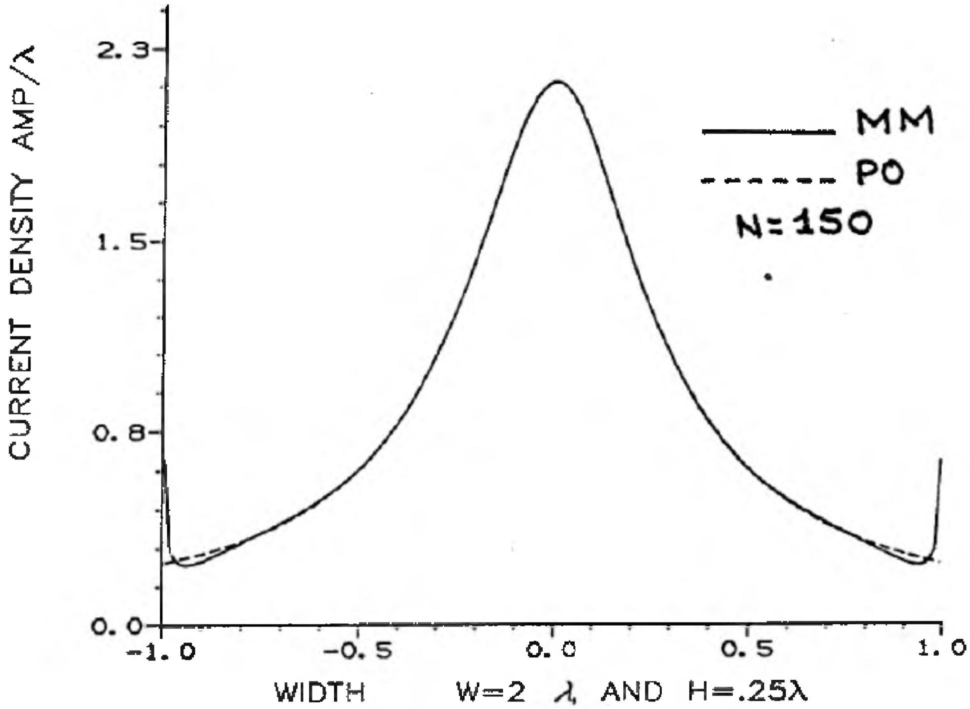


CURRENT DENSITY AMP/λ
2.3
1.5
0.8
0.0
-1.0
-0.5
0.0
0.5
1.0
WIDTH W=2 λ AND H=.25λ
MM
PO
N=150

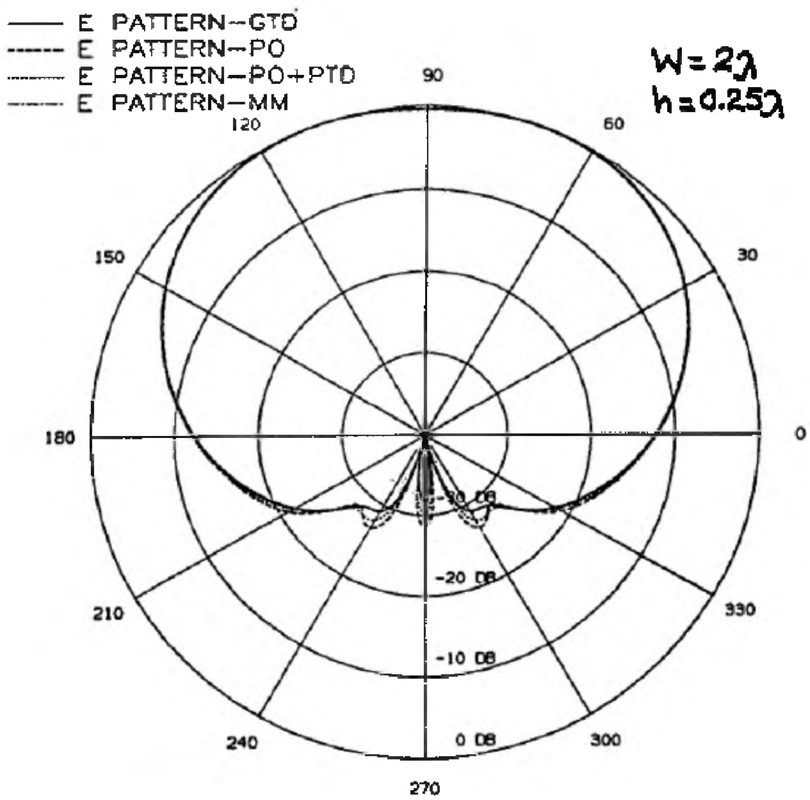


E PATTERN–GTD
E PATTERN–PO
E PATTERN–PO+PTD
E PATTERN–MM
W=2λ
h=0.25λ
90
60
30
0
330
300
270
240
210
180
150
120
-30 DB
-20 DB
-10 DB
0 DB

[124]



2.3
1.5
0.8
0.0
CURRENT DENSITY AMP/λ
MM
PO
N=150
-1.0
-0.5
0.0
0.5
1.0
WIDTH W=2 λ AND H=.25λ



E PATTERN—GTD
E PATTERN—PO
E PATTERN—PO+PTD
E PATTERN—MM
W=2λ
h=0.25λ
90
120
60
150
30
180
0
210
330
240
300
270
-30 DB
-20 DB
-10 DB
0 DB

12.5 The H_z component of a magnetic line source of constant current I_m is given by (11-15b)

$$H_z = -I_m \frac{\beta^2}{4\omega\mu} H_0^{(2)}(\beta\rho) \qquad (11\text{-}5b)$$

which is identical in form with the E_z component of an electric line source of constant current I_e as given by (12-12), or

$$E_z = -I_e \frac{\beta^2}{4\omega\varepsilon} H_0^{(2)}(\beta\rho) \qquad (12\text{-}12)$$

The boundary conditions for a magnetic line source on a PMC, vanishing $\underline{H}^{tan}$ on the strip, are identical to those of an electric line source on a PEC, vanishing $\underline{E}^{tan}$ on the strip as given by (12-16).

Therefore the Integral Equation for the two problems, for a normalized current $I_m = I_e = 1$ is identical, and it is given by (12-17a) which in this case can be written, to solve for the linear magnetic current density M_z, as

$$\boxed{H_0^{(2)}(\beta\rho_m) = -\int_{-w/2}^{w/2} M_z(x') H_0^{(2)}(\beta|\underline{\rho}_m - \underline{\rho}'|)\,dx'}$$

The above equation can also be obtained directly from (12-17a), using duality by referring to Table 7-2; replacing $J_z(x')$ by $M_z(x')$ in (12-17a), or

$$H_0^{(2)}(\beta\rho_m) = -\int_{-w/2}^{w/2} J_z(x') H_0^{(2)}(\beta|\underline{\rho}_m - \underline{\rho}'|)\,dx' \qquad (12\text{-}17a)$$

12-6 The H_z component of a TE^z polarization plane wave is identical in form to the electric field E_z for the TM^z polarization, which is given by (12-57), or

$$\underline{E}^i = \hat{a}_z E_0 e^{j\beta(x\cos\phi_i + y\sin\phi_i)} \qquad (12\text{-}57)$$

Therefore H_z^i can be written for the TE^z polarization as

$$\underline{H}^i = \hat{a}_z H_0 e^{j\beta(x\cos\phi_i + y\sin\phi_i)}$$

The vector potential F_z for the TE^z polarization of a 2-D PMC strip can be written, using (12-59) and (12-60) as a guide

$$A_z = -j\frac{\mu}{4}\int_0^W J_z(x')\, H_0^{(2)}(\beta|\underline{\rho}-\underline{\rho}'|)\,dx' \qquad (12\text{-}59)$$

$$\underline{E}^s = -\hat{a}_z \frac{\beta\eta}{4}\int_0^W J_z(x')\, H_0^{(2)}(\beta|\underline{\rho}-\underline{\rho}'|)\,dx' \qquad (12\text{-}60)$$

and (6-96a) with (6-96b)

$$\underline{A} = \frac{\mu}{4\pi}\iint_S \underline{J}_s(x',y',z')\,\frac{e^{-j\beta R}}{R}\,ds' \qquad (6\text{-}96a)$$

$$\underline{F} = \frac{\varepsilon}{4\pi}\iint_S \underline{M}_s(x',y',z')\,\frac{e^{-j\beta R}}{R}\,ds' \qquad (6\text{-}96b)$$

$$F_z = \frac{\varepsilon}{4\pi}\iint_S M_z(x)\,\frac{e^{-j\beta R}}{R}\,ds' = -j\frac{\varepsilon}{4}\int_0^W M_z(x')\, H_0^{(2)}(\beta|\underline{\rho}-\underline{\rho}'|)\,dx'$$

while by referring to (12-60), and comparing (6-32b) with (6-33b)

$$\underline{H}^s = -\hat{a}_z\, j\omega F_z = -\hat{a}_z \frac{\omega\varepsilon}{4}\int_0^W M_z(x')\, H_0^{(2)}(\beta|\underline{\rho}-\underline{x}'|)\,dx'$$

$$= -\hat{a}_z j\,\frac{\omega\sqrt{\mu\varepsilon}}{4\sqrt{\mu/\varepsilon}}\int_0^W M_z(x')\, H_0^{(2)}(\beta|\underline{\rho}-x'|)\,dx'$$

$$\underline{H}^s = -\hat{a}_z \frac{\beta}{4\eta}\int_0^W M_z(x')\, H_0^{(2)}(\beta|\underline{\rho}-\underline{x}'|)\,dx'$$

cont'd

12.6 cont'd

The boundary conditions for the PMC strip TE^z polarization (vanishing tangential magnetic field) are identical in form for the PEC strip TM^z polarization (vanishing tangential electric field) as given by (12-63), or

$$\frac{\beta\eta_c}{4}\int_0^W J_z(x')\,H_0^{(2)}(\beta|x_m-x'|)\,dx' = e^{j\beta x_m\cos\phi_i} \qquad (12\text{-}63)$$

Using duality

$$\beta \rightarrow \beta,\quad \eta \rightarrow \frac{1}{\eta},\quad J_z \rightarrow M_z$$

and (12-63), we can write the integral equation as

$$\boxed{\frac{\beta}{4\eta}\int_0^W M_z(x')\,H_0^{(2)}(\beta|x_m-x'|)\,dx' = e^{j\beta x_m\cos\phi_i}}$$

Easy Way:

The above integral equation can be obtained directly by using (12-63) and duality (Table 7-2), without having to go through all the steps. However having gone through all the steps in deriving it, it gives the reader a better appreciation and understanding.

12.7 **Show that :**

$$\left(\frac{\partial^2}{\partial x^2}+\beta^2\right)H_0^{(2)}(\beta R)$$

$$=\frac{\beta^2}{2}\left[H_0^{(2)}(\beta R)+H_2^{(2)}(\beta R)\cos(2\phi'')\right]$$

$$\frac{\partial^2}{\partial x \partial y}H_0^{(2)}(\beta R)=\frac{\beta^2}{2}H_2^{(2)}(\beta R)\sin(2\phi'')$$

....

$$\frac{d}{dx}H_n^{(2)}(dx)=\alpha H_{n-1}^{(2)}(\alpha x)-\frac{n}{x}H_n^{(2)}(\alpha x) \quad (1)$$

$$\frac{d}{dx}H_n^{(2)}(\alpha x)=-\alpha H_{n+1}^{(2)}(\alpha x)+\frac{n}{x}H_n^{(2)}(\alpha x) \quad (2)$$

Using (2) with n=0

$$\frac{d}{dx}H_0^{(2)}(\alpha x)=-\alpha H_1^{(2)}(\alpha x) \quad (3)$$

Using (2) with n=1

$$\frac{d}{dx}H_1^{(2)}(\alpha x)=-\alpha H_2^{(2)}+\frac{1}{x}H_1^{(2)}(\alpha x)$$

Dividing by x

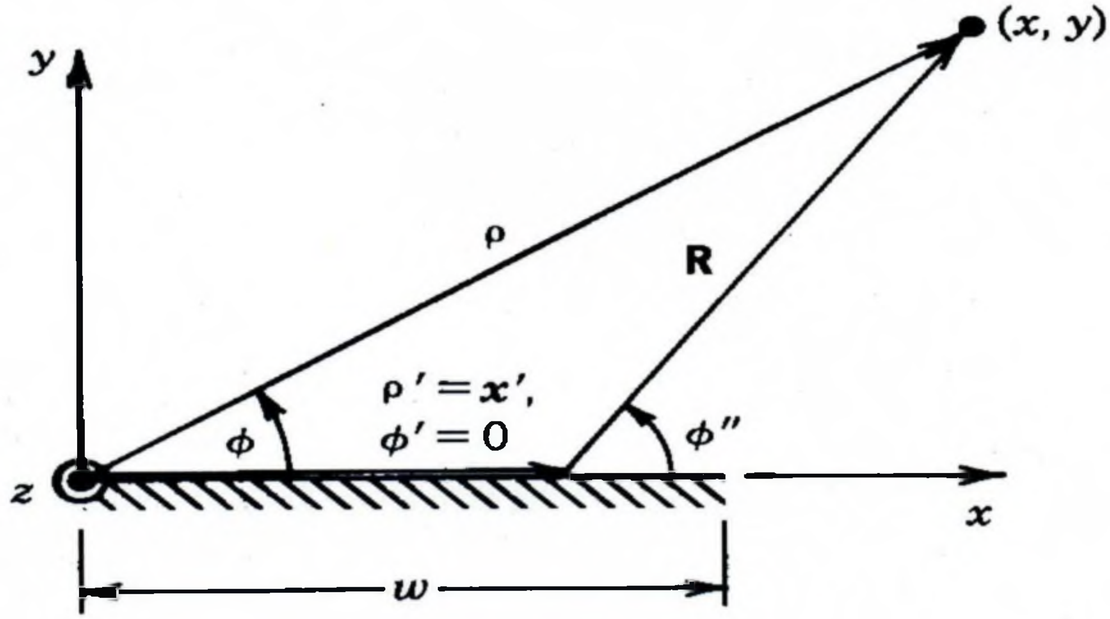
$$\frac{1}{x}\frac{dH_1^{(2)}(\alpha x)}{dx}=-\frac{\alpha}{x}H_2^{(2)}(\alpha x)+\frac{1}{x^2}H_1^{(2)}(\alpha x)$$

or

$$\frac{1}{x}\frac{dH_1^{(2)}(\alpha x)}{dx}=-\frac{1}{x^2}H_1^{(2)}(\alpha x)=-\frac{\alpha}{x}H_2^{(2)}(\alpha x)$$

Contd

12.7 cont'd



$$R = \left|\underline{\rho} - \underline{\rho}'\right| = \sqrt{\rho^2 + (\rho')^2 - 2\rho\rho'\cos(\phi_s - \phi')}\,\Big|_{\phi'=0}$$

$$\simeq \rho - \rho'\cos(\phi_s) = \rho - x'\cos\phi_s$$

Figure 12.4 Geometry for scattering from a strip.

cont'd

12.7 cont'd

or

$$\underbrace{\frac{1}{\alpha x}\frac{d}{dx}H_1^{(2)}(\alpha x)-\frac{1}{\alpha x^2}H_1^{(2)}(\alpha x)}_{\frac{d}{dx}\left[\frac{1}{\alpha x}H_1^{(2)}(\alpha x)\right]}=-\frac{1}{x}H_2^{(2)}(\alpha x)$$

$$\frac{d}{dx}\left[\frac{1}{\alpha x}H_1^{(2)}(\alpha x)\right]=-\frac{1}{x}H_2^{(2)}(\alpha x) \qquad (4)$$

$$\frac{\partial^2}{\partial x^2}H_0^{(2)}\left(\beta\left|\underline{\rho}-\underline{x}'\right|\right)=\frac{\partial^2}{\partial x^2}H_0^{(2)}(\beta R)=\frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}H_0^{(2)}(\beta R)\right]$$

$$\frac{\partial}{\partial x}H_0^{(2)}(\beta R)=\frac{\partial R}{\partial x}\left[\frac{\partial}{\partial R}H_0^{(2)}(\beta R)\right]=\frac{\partial R}{\partial x}\left[\beta H_1^{(2)}(\beta R)\right]$$

$$\frac{\partial H_0^{(2)}}{\partial x}(\beta R)=\frac{\partial R}{\partial x}\left[-\beta H_1^{(2)}(\beta R)\right]$$

$$R=\left[\left(x-x'\right)^2+\left(y-y'\right)^2\right]^{1/2}$$

$$=\left[\left(x-x'\right)^2+y^2\right]^{1/2}$$

$$\frac{\partial R}{\partial x}=\frac{1}{2}\left[\left(x-x'\right)^2+y^2\right]^{-1/2}2\left(x-x'\right)$$

$$=\frac{x-x'}{\sqrt{\left(x-x'\right)^2+y^2}}=\frac{x-x'}{R}$$

<u>Therefore:</u> $$\frac{\partial H_0^{(2)}(\beta R)}{\partial x}=\left(\frac{x-x'}{R}\right)\left[-\beta H_1^{(2)}(\beta R)\right]$$

$$=-\beta\frac{x-x'}{R}H_1^{(2)}(\beta R)$$

Cont'd

12.7 cont'd

$$\frac{\partial^2}{\partial x^2} H_0^{(2)}(\beta R) = \frac{\partial}{\partial x}\left[\frac{\partial H_0^{(2)}(\beta R)}{\partial x}\right]$$

$$= -\beta\left[\frac{\partial}{\partial x}\left\{\frac{x-x'}{R} H_1^{(2)}(\beta R)\right\}\right]$$

$$= -\beta \frac{\partial}{\partial x}\left\{\frac{x-x'}{R} H_1^{(2)}(\beta R)\right\}$$

$$\frac{\partial^2}{\partial x^2} H_0^{(2)}(\beta R) = -\beta \frac{\partial}{\partial x}\left\{(x-x')\frac{H_1^{(2)}(\beta R)}{R}\right\}$$

$$\frac{\partial^2}{\partial x^2} H_0^{(2)}(\beta R) = -\beta \frac{\partial}{\partial x}\left\{(x-x')\left[\frac{H_1^{(2)}(\beta R)}{R}\right]\right\}$$

$$\frac{\partial^2}{\partial x^2} H_0^{(2)}(\beta R) = -\beta \left\{(x-x')\frac{\partial}{\partial x}\left[\frac{H_1^{(2)}(\beta R)}{R}\right] + \frac{H_1^{(2)}(\beta R)}{R}\right\}$$

$$\frac{\partial^2}{\partial x^2} H_0^{(2)}(\beta R) = -\beta \left\{(x-x')\frac{\partial R}{\partial x}\frac{\partial}{\partial R}\left[\frac{H_1^{(2)}(\beta R)}{R}\right] + \frac{H_1^{(2)}(\beta R)}{R}\right\}$$

$$\frac{\partial^2}{\partial x^2} H_0^{(2)}(\beta R) = -\beta \left\{(x-x')\underbrace{\frac{\partial R}{\partial x}}_{\frac{(x-x')}{R}}\underbrace{\frac{\partial}{\partial R}\left[\frac{H_1^{(2)}(\beta R)}{R}\right]}_{-\frac{\beta}{R}H_2^{(2)}(\beta R)} + \frac{H_1^{(2)}(\beta R)}{R}\right\}$$

$$\frac{\partial^2}{\partial x^2} H_0^{(2)}(\beta R) = -\beta \left\{\frac{(x-x')^2}{R}\left[-\frac{\beta}{R}H_2^{(2)}(\beta R)\right] + \frac{H_1^{(2)}(\beta R)}{R}\right\}$$

$$= +\beta^2 \frac{H_2^{(2)}(\beta R)}{R^2}(x-x')^2 - \beta\frac{H_1^{(2)}(\beta R)}{R}$$

cont'd

12.7 cont'd

Subtracting (1) from (2)

$$0 = -\alpha\left[H_{n+1}^{(2)}\alpha x + H_{n-1}^{(2)}(\alpha x)\right] + \frac{2n}{x}H_n^{(2)}\alpha x$$

or

$$\frac{1}{x}H_n^{(2)}(\alpha x) = \frac{\alpha}{2}\left[H_{n+1}^{(2)}\alpha x + H_{n-1}^{(2)}(\alpha x)\right]$$

For n=1: $$\frac{1}{R}H_1^{(2)}(\beta R) = \frac{\beta}{2}\left[H_2^{(2)}(\beta R) + H_0^{(2)}(\beta R)\right]$$

$$\frac{\partial^2}{\partial x^2}H_0^{(2)}(\beta R) = \beta^2\frac{H_2^{(2)}(\beta R)}{R^2}(x-x')^2$$
$$-\beta\frac{\beta}{2}\left[H_2^{(2)}(\beta R) + H_0^{(2)}(\beta R)\right]$$

$$\frac{\partial^2}{\partial x^2}H_0^{(2)}(\beta R) = \frac{\beta^2}{2}\left[\frac{2(x-x')^2}{R^2} - 1\right]H_2^{(2)}(\beta R) - \frac{\beta^2}{2}H_0^{(2)}(\beta R)$$

$$\frac{\partial^2}{\partial x^2}H_0^{(2)}(\beta R) = \frac{\beta^2}{2}\left[2\cos^2\phi'' - 1\right]H_2^{(2)}(\beta R) - \frac{\beta^2}{2}H_0^{(2)}(\beta R)$$

$$\left(\frac{\partial^2}{\partial x^2} + \beta^2\right)H_0^{(2)}(\beta R) = \left\{\begin{array}{l}\frac{\beta^2}{2}\underbrace{\left[2\cos^2\phi'' - 1\right]}_{\cos(2\phi'')}H_2^{(2)}(\beta R) \\ -\frac{\beta^2}{2}H_0^{(2)}(\beta R) + \beta^2 H_0^{(2)}(\beta R)\end{array}\right\}$$

$$\boxed{\left(\frac{\partial^2}{\partial x^2} + \beta^2\right)H_0^{(2)}(\beta R) = \left\{\frac{\beta^2}{2}\left[H_0^{(2)}(\beta R) + H_2^{(2)}(\beta R)\cos(2\phi'')\right]\right\}} \quad (5)$$

cont'd

12.7 cont'd

Using (5):

$$E_x^s = -\frac{\eta}{4\beta}\int_0^w J_x(x')\left\{\left[\frac{\partial^2}{\partial x^2}+\beta^2\right]H_0^{(2)}(\beta R)\right\}dx'$$

$$\boxed{E_x^s = -\frac{\eta\beta}{8}\int_0^w J_x(x')\left\{H_2^{(2)}(\beta R)\cos(2\phi'') + H_0^{(2)}(\beta R)\right\}dx'}$$

....

$$\frac{\partial^2 H_0^{(2)}(\beta R)}{\partial y\partial x} = \frac{\partial}{\partial y}\left[\frac{\partial H_0^{(2)}}{\partial x}(\beta R)\right] = \frac{\partial}{\partial y}\left[-\beta\frac{x-x'}{R}H_1^{(2)}(\beta R)\right]$$

$$= -\beta\frac{\partial}{\partial y}\left[\frac{x-x'}{R}H_1^{(2)}(\beta R)\right] = -\beta\frac{\partial}{\partial y}\left[(x-x')\frac{H_1^{(2)}(\beta R)}{R}\right]$$

$$\frac{\partial^2 H_0^{(2)}(\beta R)}{\partial y\partial x} = -\beta\left[(x-x')\frac{\partial}{\partial y}\left\{\frac{H_1^{(2)}(\beta R)}{R}\right\} + \underbrace{\frac{\partial(x-x')}{\partial y}}_{0}\frac{H_1^{(2)}(\beta R)}{R}\right]$$

$$\frac{\partial^2 H_0^{(2)}(\beta R)}{\partial y\partial x} = -\beta(x-x')\frac{\partial R}{\partial y}\frac{\partial}{\partial R}\left[\frac{H_1^{(2)}(\beta R)}{R}\right]$$

$$= -\beta(x-x')\frac{\partial R}{\partial y}\frac{\partial}{\partial R}\left[\frac{H_1^{(2)}(\beta R)}{R}\right]$$

$$\frac{\partial R}{\partial y} = \frac{\partial}{\partial R}\left[\left(x-x'\right)^2+y^2\right]^{1/2} = \frac{1}{2}\left[\left(x-x'\right)^2+y^2\right]^{-1/2}(2y)$$

$$\frac{\partial R}{\partial y} = \frac{y}{\sqrt{(x-x')^2+y^2}} = \frac{y}{R}$$

Cont'd

12.7 cont'd

$$\frac{\partial^2 H_0^{(2)}(\beta R)}{\partial y \partial x} = -\beta(x-x')\frac{y}{R}\underbrace{\frac{\partial}{\partial R}\left[\frac{H_1^{(2)}(\beta R)}{R}\right]}_{-\frac{\beta}{R}H_2^{(2)}(\beta R)}$$

$$= +\frac{\beta^2(x-x')y}{R^2}H_2^{(2)}(\beta R)$$

$$\frac{\partial^2 H_0^{(2)}(\beta R)}{\partial y \partial x} = \beta^2\left(\frac{x-x'}{R}\right)\left(\frac{y}{R}\right)H_2^{(2)}(\beta R)$$

$$\frac{\partial^2 H_0^{(2)}(\beta R)}{\partial y \partial x} = \beta^2 \underbrace{\cos\phi''\sin\phi''}_{\frac{1}{2}\sin(2\phi'')}H_2^{(2)}(\beta R)$$

$$\boxed{\frac{\partial^2 H_0^{(2)}(\beta R)}{\partial y \partial x} = \frac{1}{2}\beta^2 H_2^{(2)}(\beta R)\sin(2\phi'')} \quad (6)$$

Using (6):

$$E_y^s = -\frac{\eta}{4\beta}\int_0^w J_x(x')\left\{\frac{\partial^2}{\partial y \partial x}H_0^{(2)}(\beta R)\right\}dx'$$

$$= -\frac{\eta}{4\beta}\int_0^w J_x(x')\left\{\frac{\beta^2}{2}H_2^{(2)}(\beta R)\sin(2\phi'')\right\}dx'$$

$$\boxed{E_y^s = -\frac{\eta\beta}{8}\int_0^w J_x(x')\left\{H_2^{(2)}(\beta R)\sin(2\phi'')\right\}dx'}$$

$\boxed{12.8}$ Using Figure 12.13(b), we can write the incident magnetic field as

$$\underline{H}^i = \hat{a}_z H_0 e^{j\beta(x\cos\phi_i + y\sin\phi_i)}$$

$$\underline{J}_s^{total} = \hat{n}\times(\underline{H}^i + \underline{H}^r)\Big|_{\substack{strip\\ y=0}} = \hat{a}_y \times (\hat{a}_z H_z^i + \hat{a}_z H_z^r)\Big|_{\substack{strip\\ y=0}}$$

Assuming the current density on the strip is the same as that of a strip with infinite width, then the total current density on the strip is equal to twice that due to the incident field. Thus

$$\underline{J}_s^t = \hat{a}_y \times (2\hat{a}_z H_z^i) = 2\hat{a}_x H_0 e^{j\beta x\cos\phi_i} = \hat{a}_x J_x^t$$

$$J_x^t = 2H_0 e^{j\beta x\cos\phi_i}$$

Using (12-59) but for J_x, we can write the corresponding potential

$$A_x = \frac{\mu}{4\pi}\iint_S J_x(x')\frac{e^{-j\beta R}}{R}\,ds' = \frac{\mu}{4\pi}\int_0^W J_x(x')\left[\int_{-\infty}^{+\infty}\frac{e^{-j\beta\sqrt{|\underline{\rho}-\underline{\rho}'|^2+(z-z')^2}}}{\sqrt{|\underline{\rho}-\underline{\rho}'|^2+(z-z')^2}}\,dz'\right]dx'$$

$$= -j\frac{\mu}{4}\int_0^W J_x(x')\,H_0^{(2)}(\beta|\underline{\rho}-\underline{\rho}'|)\,dx'$$

$$\underline{H}^s = \frac{1}{\mu}\nabla\times A = \hat{a}_z\left(-\frac{1}{\mu}\frac{\partial A_x}{\partial y}\right) = -\hat{a}_z\frac{1}{\mu}\left(-j\frac{\mu}{4}\right)\int_0^W J_x(x')\frac{\partial}{\partial y}\left[H_0^{(2)}(|\underline{\rho}-\underline{\rho}'|)\right]dx'$$

According to Figure 12-14

$$\frac{\partial}{\partial y}H_0^{(2)}(\beta|\underline{\rho}-\underline{\rho}'|) = \frac{\partial}{\partial y}H_0^{(2)}(\beta R) = -\beta\frac{y-y'}{R}H_1^{(2)}(\beta R) = -\beta\sin\phi'' H_1^{(2)}(\beta R)$$

Therefore

$$\underline{H}^s = +\hat{a}_z j\frac{1}{4}\int_0^W J_x(x')\left[-\beta\sin\phi'' H_1^{(2)}(\beta R)\right]dx' = -j\frac{\beta}{4}\int_0^W J_x(x')H_1^{(2)}(\beta R)\sin\phi''\,dx'$$

$$\underline{H}^s = -j\frac{\beta}{4}\int_0^W J_x(x')\,H_1^{(2)}(\beta R)\sin\phi''\,dx'$$

Since $J_x = 2H_0 e^{j\beta x'\cos\phi_i}$ $\Rightarrow$ $\boxed{\underline{H}^s = -jH_0\frac{\beta}{2}\int_0^W H_1^{(2)}(\beta R)\,e^{j\beta x'\cos\phi_i}\sin\phi''\,dx'}$

$\boxed{\text{cont'd}}$

12.8 cont'd

For far-field observations, according to Figure 12-14,

$$\phi' = 0, \ \rho' = x', \ \phi'' = \phi$$

and the Hankel function is approximated by its asymptotic expansion of

$$R = \sqrt{\rho^2 + (\rho')^2 - 2\rho\rho'\cos(\phi - \phi')} \ \underset{\substack{\phi'=0 \\ \rho'=x'}}{\overset{\rho >> \rho'}{=}} \ \rho - x'\cos\phi$$

$$H_1^{(2)}(\beta R) = \sqrt{\frac{2}{\pi\beta\rho}} \, e^{-j(\beta\rho - 3\pi/4)} e^{+j\beta x'\cos\phi}$$

$$= \sqrt{\frac{2}{\pi\beta\rho}} \, e^{-j\beta\rho} e^{+j\pi/4} e^{+j\beta x'\cos\phi}$$

$$H_1^{(2)}(\beta R) = \sqrt{\frac{2j}{\pi\beta\rho}} \, e^{-j\beta\rho} e^{+j\beta x'\cos\phi}$$

Therefore the far zone scattered field is

$$\underline{H}^s = -jH_0 \frac{\beta}{2} \int_0^W H_1^{(2)}(\beta R) \, e^{j\beta x'\cos\phi^i} \sin\phi'' \, dx'$$

$$\simeq -jH_0 \sin\phi \left(\frac{\beta}{2}\right) \sqrt{\frac{2j}{\pi\beta\rho}} \, e^{-j\beta\rho} \int_0^W e^{j\beta x'(\cos\phi + \cos\phi^i)} dx'$$

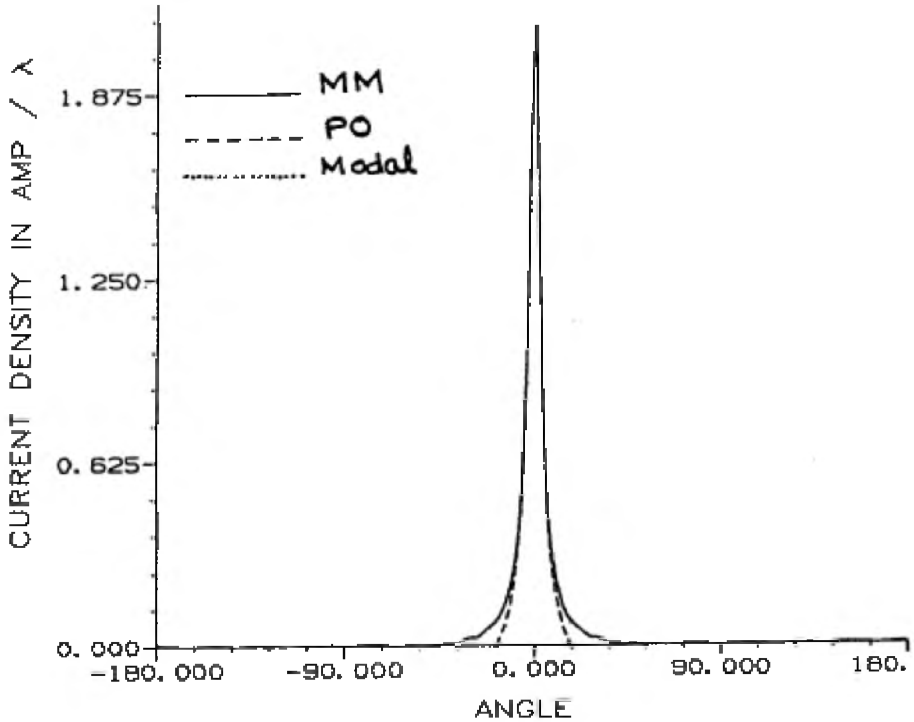
$$\boxed{\underline{H}^s \simeq -H_0 \sqrt{\frac{\beta \sin\phi}{j2\pi\rho}} \, e^{-j\beta\rho} \int_0^W e^{j\beta x'(\cos\phi + \cos\phi^i)} dx'}$$

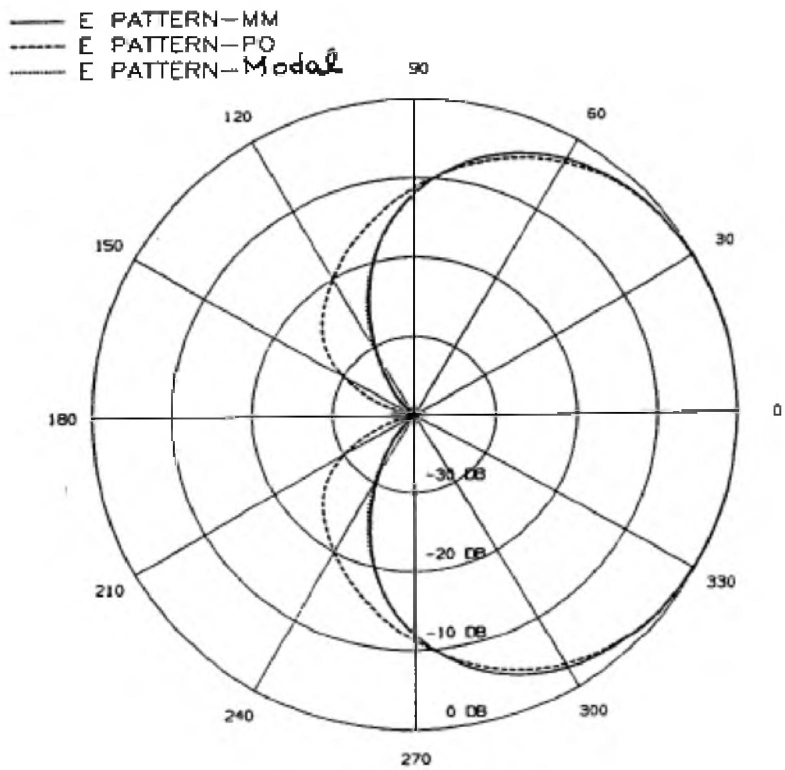
According to (11-21c)

$$\sigma_{2-D} = \lim_{\rho \to \infty} \left[2\pi\rho \frac{|H^s|^2}{|H^i|^2} \right] = \left[\beta \sin^2\phi \left| \int_0^W e^{j\beta x'(\cos\phi + \cos\phi^i)} dx' \right|^2 \right]$$

$$\boxed{\sigma_{2D} = \frac{2\pi}{\lambda} \sin^2\phi \left| \int_0^W e^{j\beta x'(\cos\phi + \cos\phi^i)} dx' \right|^2}$$

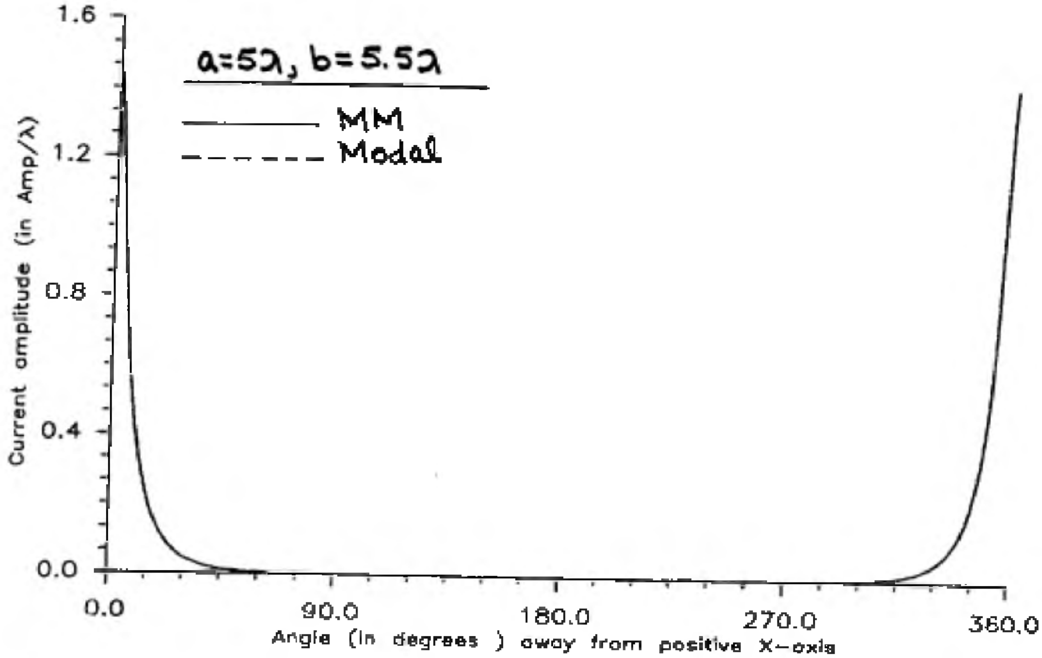
12.9

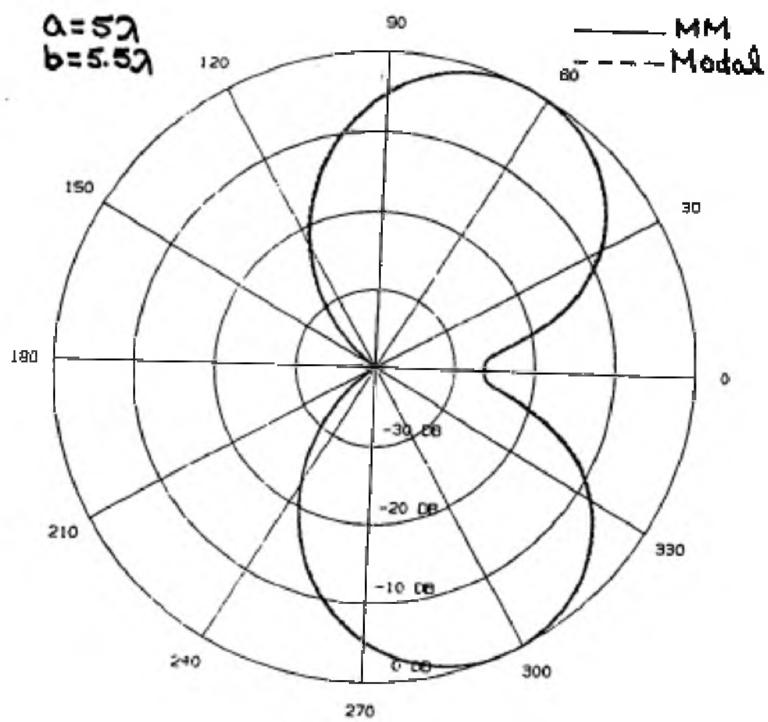




This problem can be solved using the TDRS computer program found at the end of this chapter.

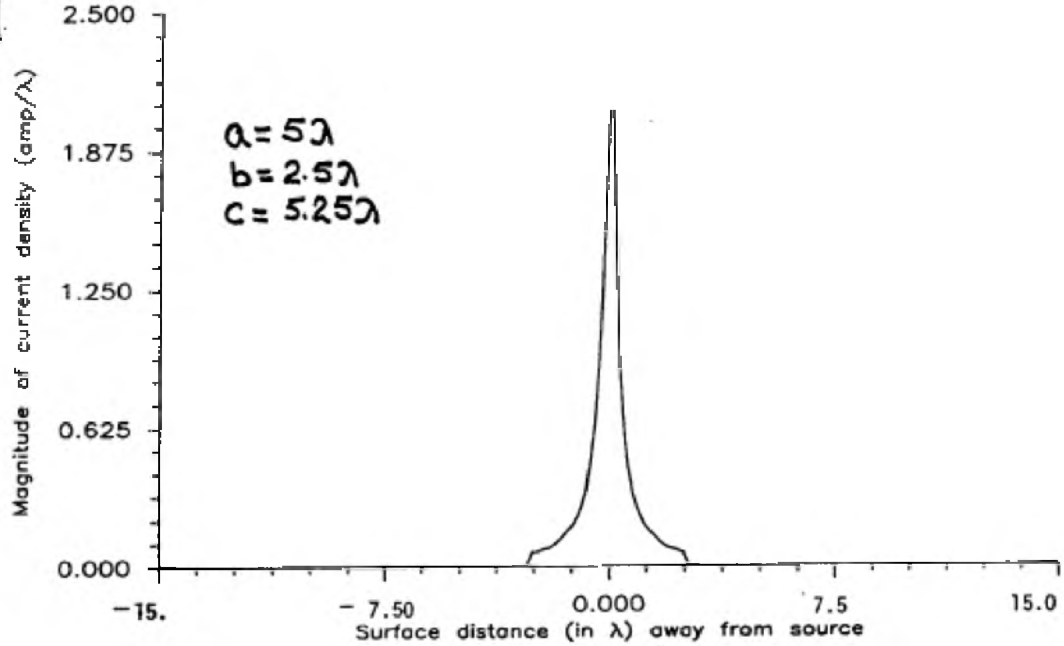
12.10

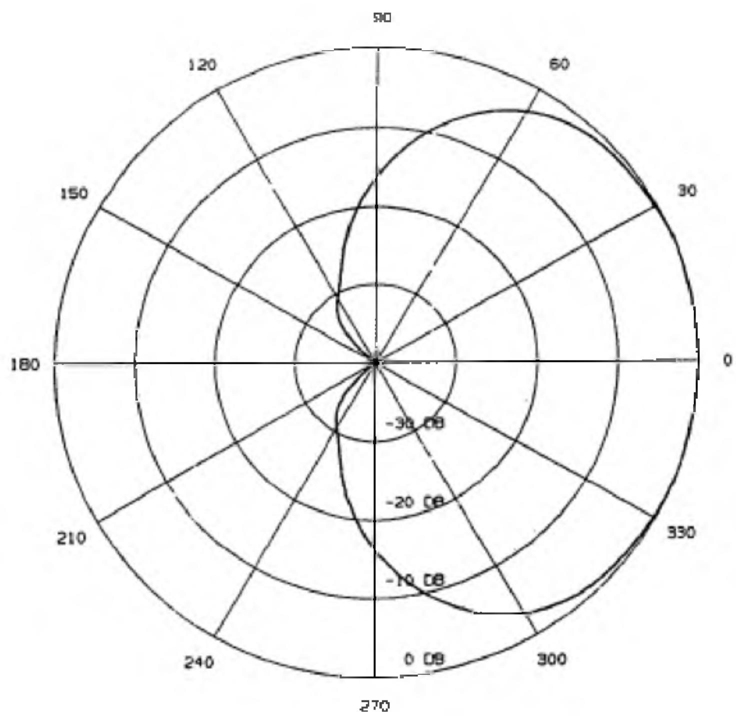




This problem can be solved using the TDRS computer program at the end of this chapter.

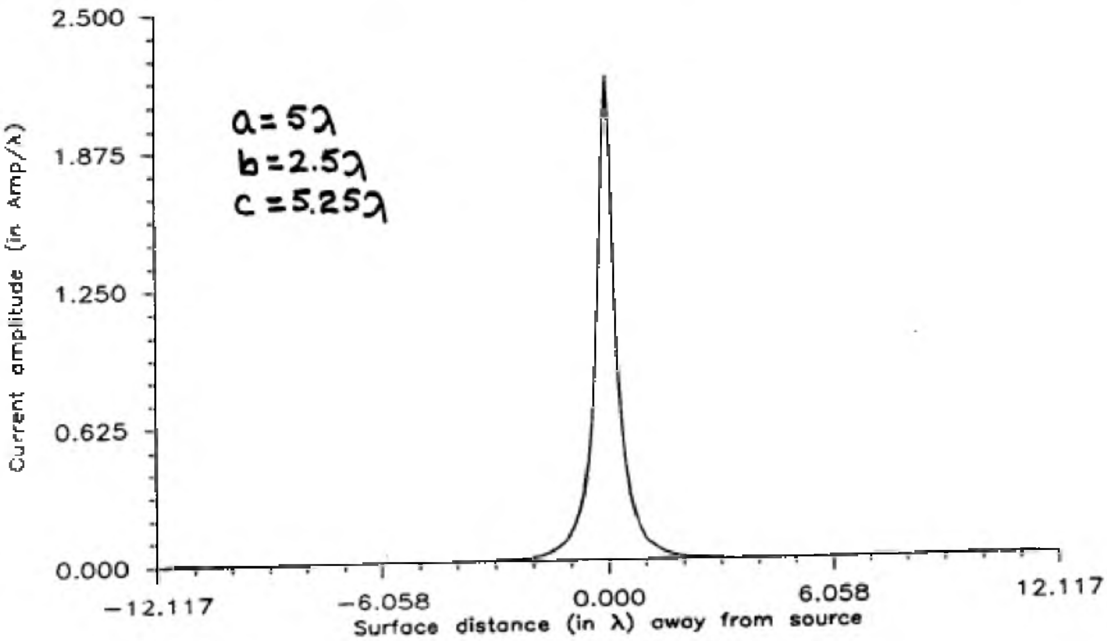
12.11

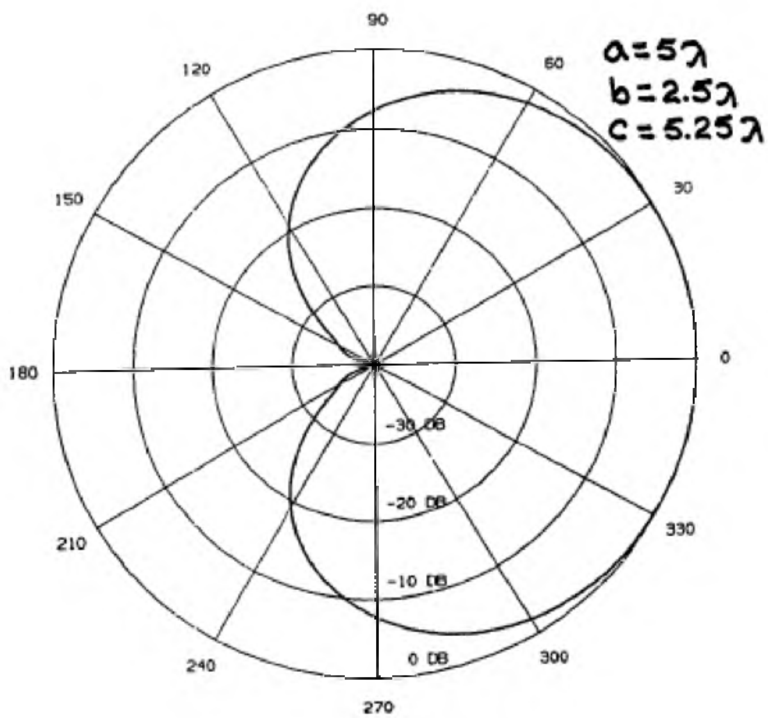




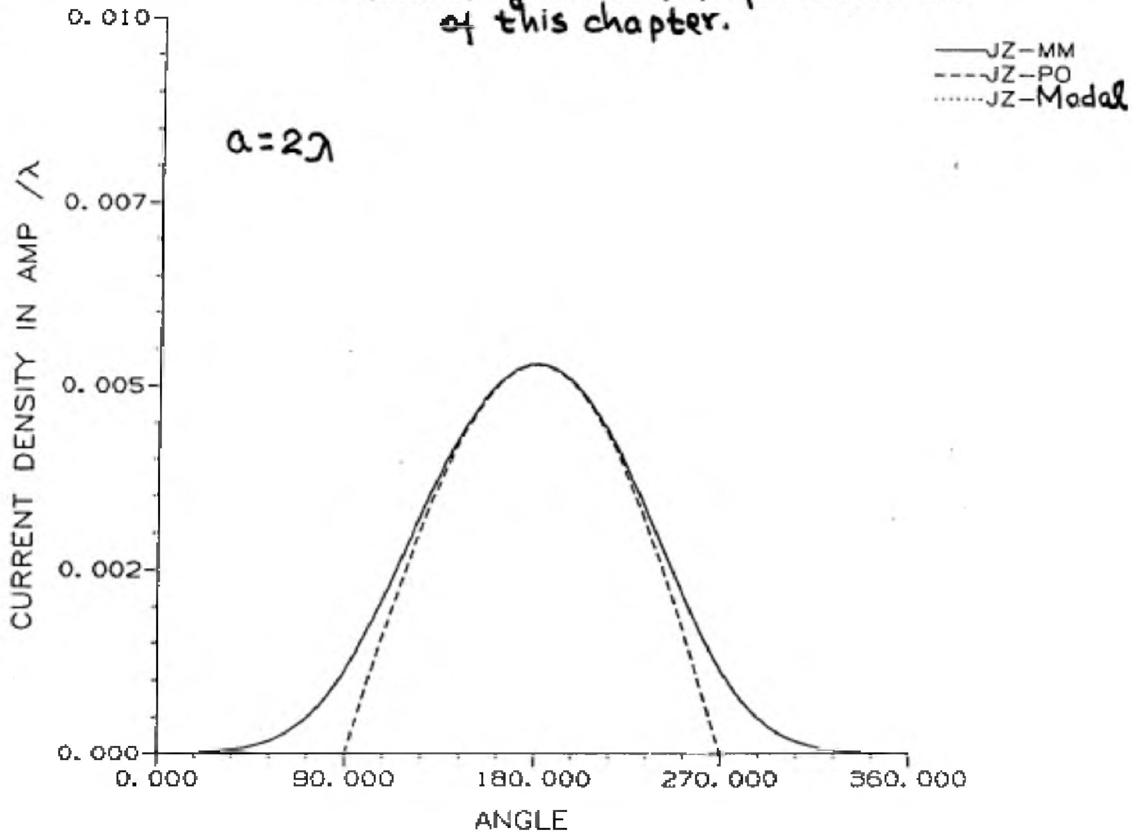
This problem can be solved using the TDRS program at the end of this chapter.

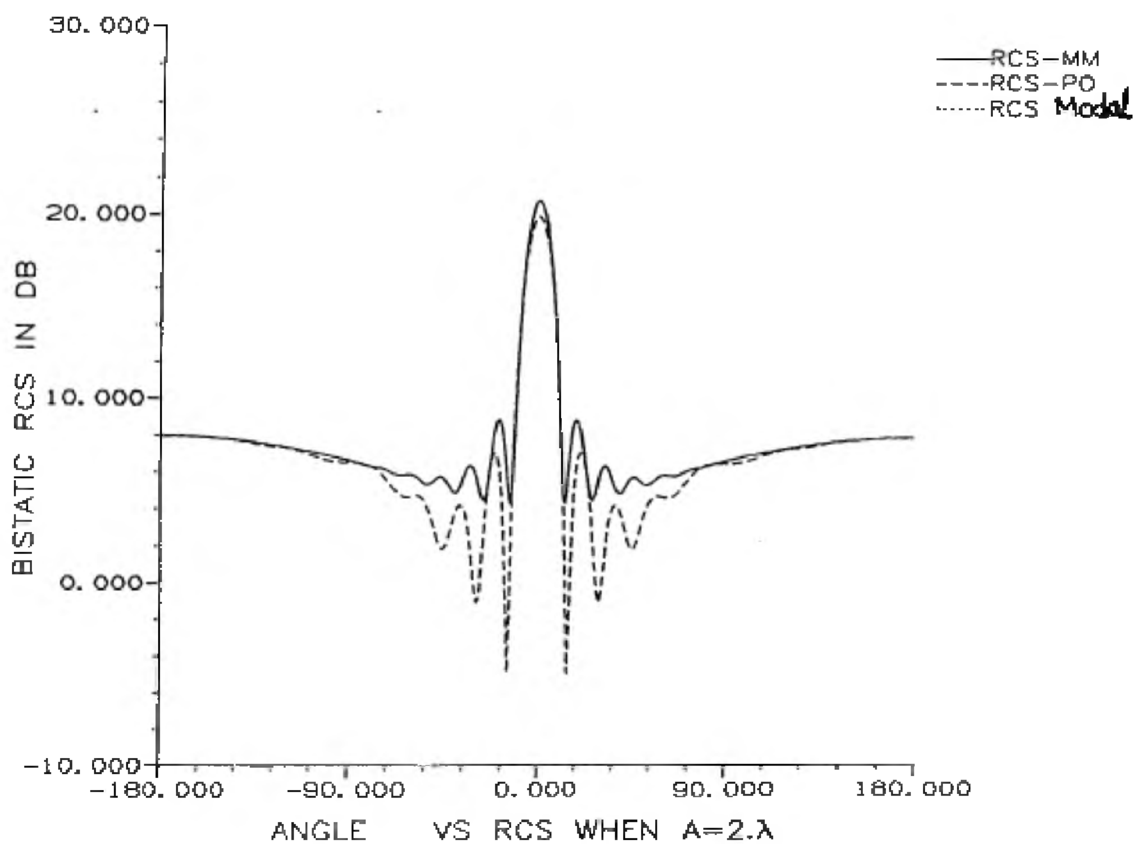
12.12 This problem can be solved using the TDRS program at the end of this chapter.



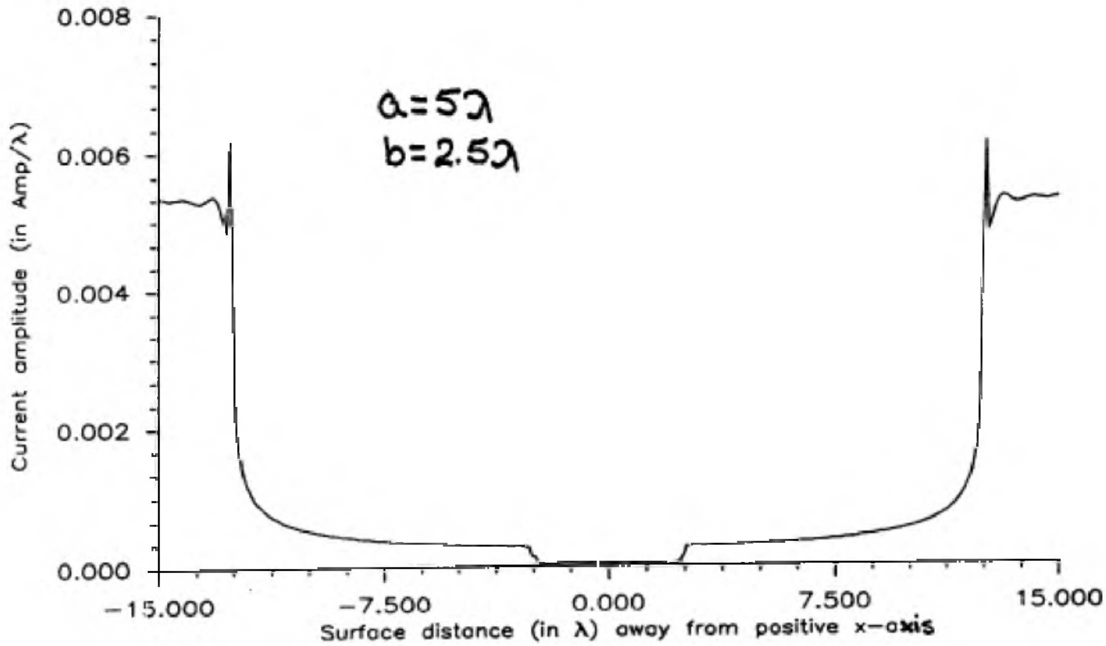


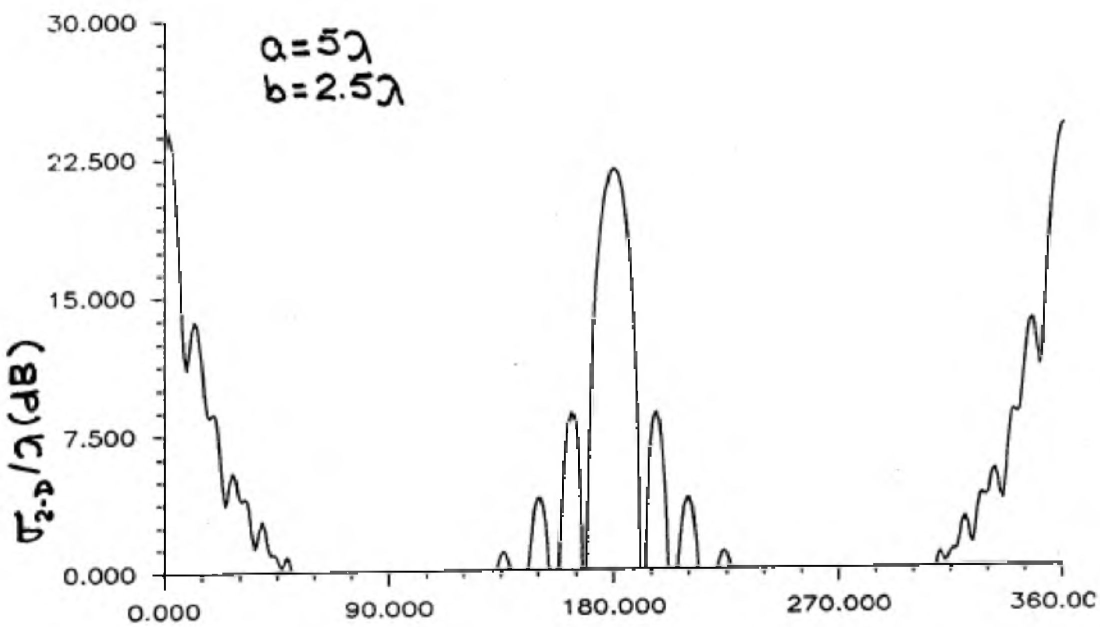
12.13 This problem can be solved using the TDRS program at the end of this chapter.





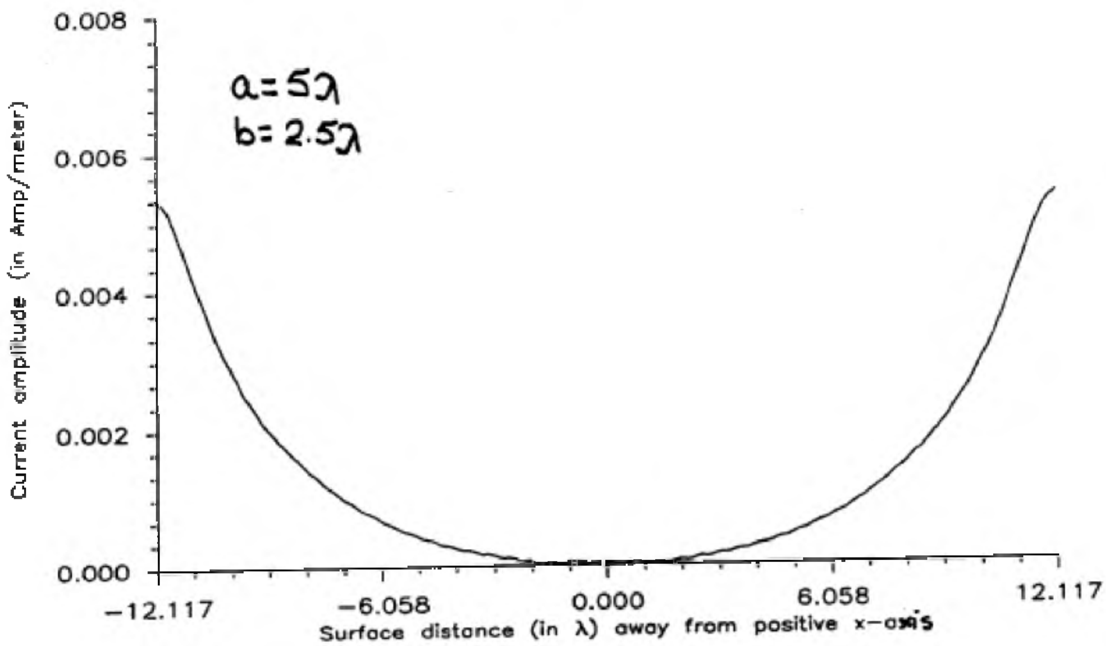
12.14

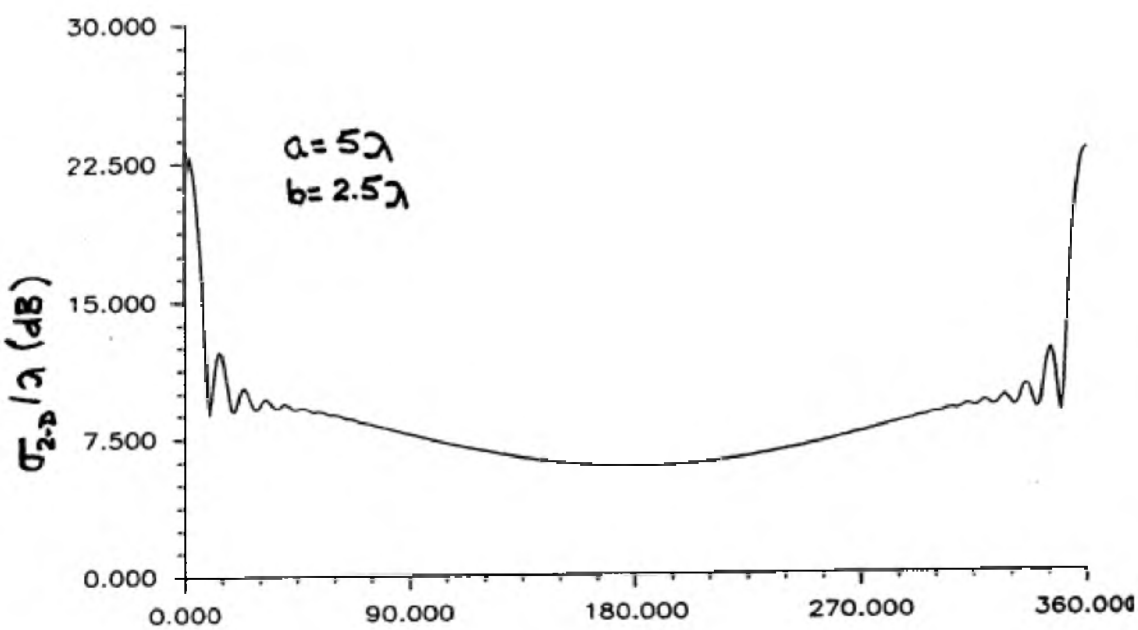




This problem can be solved using the TDRS computer program at the end of this chapter.

$\boxed{12.15}$ This problem can be solved using the TDRS program at the end of the chapter.





12.16

$$\frac{d}{dx} H_0^{(2)}(\alpha x) = -\alpha H_1^{(2)}(\alpha x); \qquad (1)$$

$$\frac{d}{dx} H_1^{(2)}(\alpha x) = -\alpha H_2^{(2)}(\alpha x) + \frac{1}{x} H_1^{(2)}(\alpha x)$$

Dividing both sides by x

$$\frac{1}{x}\frac{d}{dx} H_1^{(2)}(\alpha x) = -\frac{\alpha}{x} H_2^{(2)}(\alpha x) + \frac{1}{x^2} H_1^{(2)}(\alpha x)$$

or

$$\frac{1}{x}\frac{dH_1^{(2)}(\alpha x)}{dx} - \frac{1}{x^2} H_1^{(2)}(\alpha x) = -\frac{\alpha}{x} H_2^{(2)}(\alpha x)$$

or

$$\underbrace{\frac{1}{\alpha x}\frac{d}{dx} H_1^{(2)}(\alpha x) - \frac{1}{\alpha x^2} H_1^{(2)}(\alpha x)}_{\frac{d}{dx}\left[\frac{1}{\alpha x} H_1^{(2)}(\alpha x)\right]} = -\frac{1}{x} H_2^{(2)}(\alpha x)$$

Thus

$$\frac{d}{dx}\left[\frac{1}{\alpha x} H_1^{(2)}(\alpha x)\right] = -\frac{1}{x} H_2^{(2)}(\alpha x) \qquad (2)$$

$$\frac{\partial^2}{\partial x^2} H_0^{(2)}(\beta|\underline{\rho}-\underline{x}'|) = \frac{\partial^2}{\partial x^2} H_0^{(2)}(\beta R) = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x} H_0^{(2)}(\beta R)\right] \qquad (3)$$

$$\frac{\partial}{\partial x} H_0^{(2)}(\beta R) = \frac{\partial R}{\partial x}\left[\frac{\partial}{\partial R} H_0^{(2)}(\beta R)\right] = \frac{\partial R}{\partial x}\left[-\beta H_1^{(2)}(\beta R)\right] \qquad (4)$$

Since $R = \left[(x-x')^2 + (y-y')^2\right]^{1/2} = \left[(x-x')^2 + y^2\right]^{1/2}$

then

$$\frac{\partial R}{\partial x} = \frac{1}{2}\left[(x-x')^2 + y^2\right]^{-1/2} 2(x-x') = \frac{x-x'}{\sqrt{(x-x')^2 + y^2}} = \frac{x-x'}{R} \qquad (5)$$

Thus we can write (4) as

$$\frac{\partial}{\partial x} H_0^{(2)}(\beta R) = \frac{\partial R}{\partial x}\left[-\beta H_1^{(2)}(\beta R)\right] = -\beta \frac{x-x'}{R} H_1^{(2)}(\beta R) \qquad (6)$$

Using (6) we can write (3) as

$$\frac{\partial^2}{\partial x^2} H_0^{(2)}(\beta R) = \frac{\partial}{\partial x}\left[-\beta \frac{x-x'}{R} H_1^{(2)}(\beta R)\right] = -\beta \frac{\partial}{\partial x}\left[\frac{x-x'}{R} H_1^{(2)}(\beta R)\right] = -\beta \frac{\partial}{\partial x}\left[(x-x')\frac{H_1^{(2)}(\beta R)}{R}\right]$$

$$= -\beta\left\{(x-x')\frac{\partial}{\partial x}\left[\frac{H_1^{(2)}(\beta R)}{R}\right] + \frac{H_1^{(2)}(\beta R)}{R}\right\} = -\beta\left\{(x-x')\frac{\partial R}{\partial x}\frac{\partial}{\partial R}\left[\frac{H_1^{(2)}(\beta R)}{R}\right] + \frac{H_1^{(2)}(\beta R)}{R}\right\}$$

Using (2) and (4) we can write it as

$$\frac{\partial^2}{\partial x^2} H_0^{(2)}(\beta R) = -\beta\left\{(x-x')\frac{(x-x')}{R}\left[-\frac{\beta}{R} H_2^{(2)}(\beta R)\right] + \frac{H_1^{(2)}(\beta R)}{R}\right\} = \beta^2 \frac{H_2^{(2)}(\beta R)}{R^2}(x-x')^2 - \beta \frac{H_1^{(2)}(\beta R)}{R} \qquad (7)$$

Cont'd.

$\boxed{12.16 \text{ Cont'd.}}$ For the Hankel functions we can write (IV-18) and (IV-19) as

$$\frac{d}{dx}H_n^{(2)}(\alpha x) = \alpha\, H_{n-1}^{(2)}(\alpha x) - \frac{n}{x}H_n^{(2)}(\alpha x) \tag{8a}$$

$$\frac{d}{dx}H_n^{(2)}(\alpha x) = -\alpha\, H_{n+1}^{(2)}(\alpha x) + \frac{n}{x}H_n^{(2)}(\alpha x) \tag{8b}$$

Subtracting (8a) from (8b) reduces to

$$0 = -\alpha\left[H_{n+1}^{(2)}(\alpha x) + H_{n-1}^{(2)}(\alpha x)\right] + \frac{2n}{x}H_n^{(2)}(\alpha x)$$

or

$$\frac{1}{x}H_n^{(2)}(\alpha x) = \frac{\alpha}{2}\left[H_{n+1}^{(2)}(\alpha x) + H_{n-1}^{(2)}(\alpha x)\right] \tag{9}$$

which for $n=1$ reduces to

$$\frac{1}{x}H_1^{(2)}(\beta R) = \frac{\beta}{2}\left[H_2^{(2)}(\beta R) + H_0^{(2)}(\beta R)\right] \tag{9a}$$

Using (9a) we can write (7) as

$$\frac{\partial^2}{\partial x^2}H_0^{(2)}(\beta R) = \beta^2\frac{H_2^{(2)}(\beta R)}{R^2}(x-x')^2 - \beta\left\{\frac{\beta}{2}\left[H_2^{(2)}(\beta R) + H_0^{(2)}(\beta R)\right]\right\}$$

$$\frac{\partial^2}{\partial x^2}H_0^{(2)}(\beta R) = \frac{\beta^2}{2}\left\{2\frac{(x-x')^2}{R^2} - 1\right\}H_2^{(2)}(\beta R) - \frac{\beta^2}{2}H_0^{(2)}(\beta R) = \frac{\beta^2}{2}\left\{\underbrace{\left[2\cos^2\phi'' - 1\right]}_{\cos(2\phi'')}H_2^{(2)}(\beta R) - H_0^{(2)}(\beta R)\right\}$$

$$\frac{\partial^2}{\partial x^2}H_0^{(2)}(\beta R) = \frac{\beta^2}{2}\left\{H_2^{(2)}(\beta R)\cos(2\phi'') - H_0^{(2)}(\beta R)\right\} \tag{10}$$

Using (10) we can write (12-70a) as

$$E_x^s = -\frac{\eta}{4\beta}\int_0^W J_x(x')\left\{\frac{\beta^2}{2}\left[H_2^{(2)}(\beta R)\cos(2\phi'') - H_0^{(2)}(\beta R)\right] + \beta^2 H_0^{(2)}(\beta R)\right\}dx'$$

$$E_x^s = -\frac{\eta\beta}{8}\int_0^W J_x(x')\left\{H_2^{(2)}(\beta R)\cos(2\phi'') + H_0^{(2)}(\beta R)\right\}dx' \tag{11}$$

Now let us look at (12-71b).

$$\frac{\partial^2 H_0^{(2)}(\beta R)}{\partial y\,\partial x} = \frac{\partial}{\partial y}\left[\frac{\partial}{\partial x}H_0^{(2)}(\beta R)\right] = \frac{\partial}{\partial y}\left[-\beta\frac{(x-x')}{R}H_1^{(2)}(\beta R)\right] = -\beta\frac{\partial}{\partial y}\left[(x-x')\frac{H_1^{(2)}(\beta R)}{R}\right]$$

$$= -\beta\left\{(x-x')\frac{\partial}{\partial y}\left[\frac{H_1^{(2)}(\beta R)}{R}\right] + \cancelto{0}{\frac{\partial}{\partial y}(x-x')}\frac{H_1^{(2)}(\beta R)}{R}\right\} = -\beta(x-x')\frac{\partial R}{\partial y}\frac{\partial}{\partial R}\left[\frac{H_1^{(2)}(\beta R)}{R}\right]$$

$$= -\beta(x-x')\frac{y}{R}\left[-\frac{\beta}{R}H_2^{(2)}(\beta R)\right] = \beta^2\frac{(x-x')}{R}\left(\frac{y}{R}\right)H_2^{(2)}(\beta R) = \beta^2\underbrace{\cos(\phi'')\sin(\phi'')}_{\frac{1}{2}\sin(2\phi'')}H_2^{(2)}(\beta R)$$

$$\frac{\partial^2 H_0^{(2)}(\beta R)}{\partial y\,\partial x} = \frac{\beta^2}{2}H_2^{(2)}(\beta R)\sin(2\phi'') \tag{12}$$

Therefore we can write (12-70b) as

$$E_y^s = -\frac{\eta}{4\beta}\int_0^W J_x(x')\left\{\frac{\beta^2}{2}H_2^{(2)}(\beta R)\sin(2\phi'')\right\}dx' = -\frac{\eta\beta}{8}\int_0^W J_x(x')H_2^{(2)}(\beta R)\sin(2\phi'')\,dx'$$

$\boxed{12.17}$ $Z_{mn} = \frac{\beta\eta}{8}\int_{x_n}^{x_{n+1}} \left[H_0^{(2)}(\beta R_{mn}) + H_2^{(2)}(\beta R_{mn})\cos(2\phi''_{mn})\right] dx'$

a. For $m=n$

$$\cos(2\phi'_{mn}) = \frac{(x')^2+t^2}{(x')^2+t^2} \overset{t\to 0}{\simeq} 1$$

$$Z_{mn} \simeq \frac{\beta\eta}{8}\int_{-\Delta x_n/2}^{\Delta x_n/2}\left[H_0^{(2)}(\beta R) + H_2^{(2)}(\beta R)\right]dx'$$

$$H_0^{(2)}(\beta R) \simeq 1 - j\frac{2}{\pi}\ln\left(\frac{\beta\gamma|x'|}{2}\right) + O(x)^2 \Rightarrow \int_{-\Delta x_n/2}^{\Delta x_n/2} H_0^{(2)}(\beta R)\,dx' \simeq \int_{-\Delta x_n/2}^{\Delta x_n/2}\left[1 - j\frac{2}{\pi}\ln\left(\frac{\beta\gamma|x'|}{2}\right)\right]dx' = \Delta\ell\left[1 - j\frac{2}{\pi}\ln\left(\frac{\beta\gamma\Delta x_n}{4e}\right)\right]$$

$\gamma = 1.781$ $\qquad e = 2.718$

$$H_2^{(2)}(\beta R) \simeq j\frac{1}{\pi}\left[\frac{4}{\beta^2(x'^2+t^2)} + 1\right] + O(x)^2 \Rightarrow \int_{-\Delta x_n/2}^{\Delta x_n/2} H_2^{(2)}(\beta R)\cos(2\phi')\,dx' \simeq \lim_{t\to 0}\int_{-\Delta x_n/2}^{\Delta x_n/2} j\frac{1}{\pi}\left[\frac{4(x'^2-t^2)}{\beta^2(x'^2+t^2)} + 1\right]dx'$$

$$= j\frac{\Delta x_n}{\pi}\left[-\frac{16}{(\beta\Delta x_n)^2} + 1\right]$$

Therefore

$$Z_{mn} = \frac{\beta\eta\,\Delta\ell}{8}\left\{1 - j\frac{1}{\pi}\left[-1 + 2\ln\left(\frac{\beta\gamma\Delta x_n}{4e}\right) + \frac{16}{(\beta\Delta x_n)^2}\right]\right\}, \quad m=n$$

b. $|m-n| \le 2$, $m \ne n$ $\Rightarrow \cos(2\phi''_{mn}) = 1$

$$Z_{mn} = \frac{\beta\eta}{8}\int_{x_n}^{x_{n+1}}\left[H_0^{(2)}(\beta R_{mn}) + H_2^{(2)}(\beta R_{mn})\right]dx' = \frac{\beta\eta}{8}\int_{x_n-\frac{\Delta x_n}{2}}^{x_n+\frac{\Delta x_n}{2}}\left[H_0^{(2)}(\beta|x'-x_m|) + H_2^{(2)}(\beta|x'-x_m|)\right]dx'$$

Using (IV-18) and (IV-19)

$$\frac{dH_1^{(2)}(\beta R)}{dR} = \beta H_0^{(2)}(\beta R) - \frac{1}{R}H_1^{(2)}(\beta R) \qquad (1)$$

$$\frac{dH_1^{(2)}(\beta R)}{dR} = -\beta H_2(\beta R) + \frac{1}{R}H_1^{(2)}(\beta R) \qquad (2)$$

Subtrating (1) and (2), we get

$$\beta\left[H_0^{(2)}(\beta R) + H_2^{(2)}(\beta R)\right] - \frac{2}{R}H_1^{(2)}(\beta R) \Rightarrow H_0^{(2)}(\beta R) + H_2^{(2)}(\beta R) = \frac{2}{\beta R}H_1^{(2)}(\beta R) \qquad (3)$$

Thus

$$Z_{mn} = \frac{\beta\eta}{4}\int_{x_n-\frac{\Delta x_n}{2}}^{x_n+\Delta x_n/2}\frac{H_1^{(2)}(\beta|x'-x_m|)}{(\beta|x'-x_m|)}dx' \simeq \frac{\beta\eta}{4}\int_{x_n-\frac{\Delta x_n}{2}}^{x_n+\Delta x_n/2}\frac{\left(\frac{\beta|x'-x_m|}{2} + j\frac{2}{\pi}\frac{1}{\beta|x'-x_m|}\right)}{\beta|x'-x_m|}dx'$$

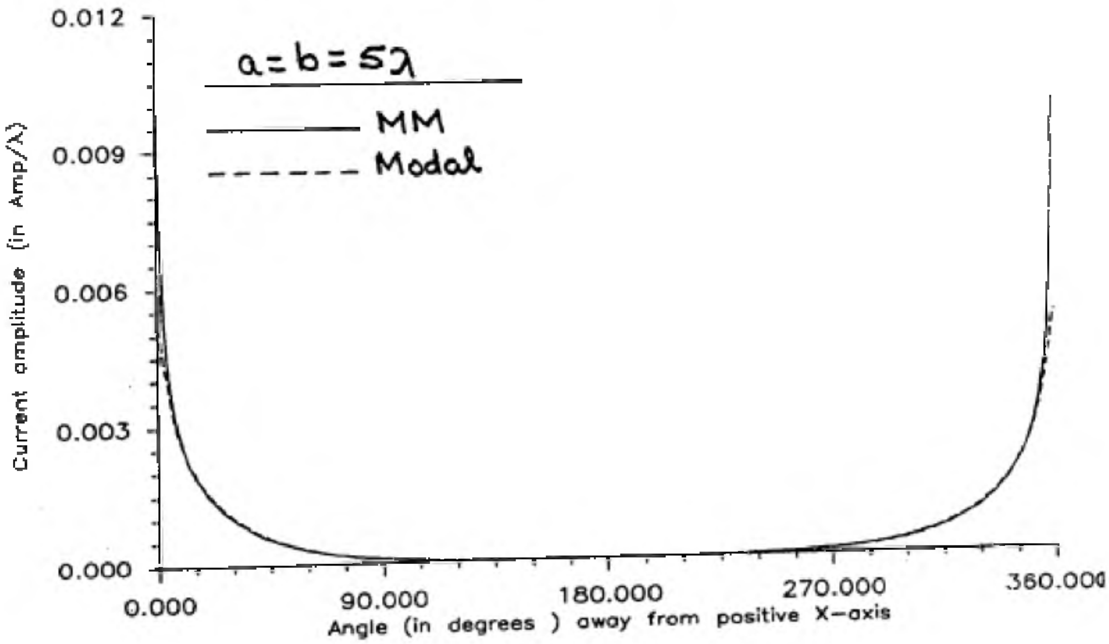
$$Z_{mn} = \frac{\beta\eta}{4}\int_{x_n-\frac{\Delta x_n}{2}}^{x_n+\Delta x_n/2}\frac{1}{2}\left[1 + j\frac{4}{\pi}\frac{1}{\beta^2|x'-x_m|^2}\right]dx' \qquad \text{Let } x' = x'' + x_n. \text{ Then } x'' = x' - x_n$$

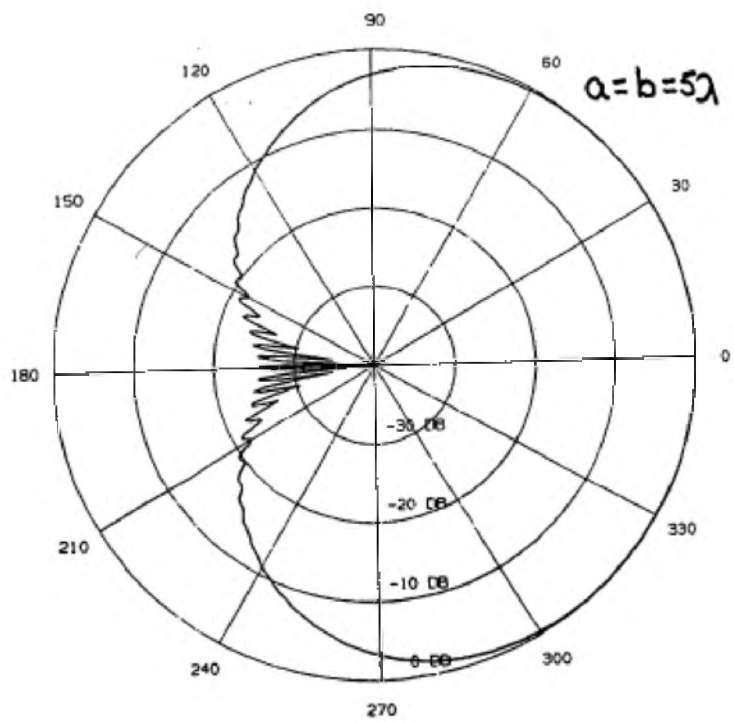
$$Z_{mn} = \frac{\beta\eta}{8}\int_{-\Delta x_n/2}^{\Delta x_n/2}\left[1 + j\frac{4}{\pi}\frac{1}{\beta^2\left[|x_m-x_n| + x'\right]^2}\right]dx' = \frac{\beta\eta\Delta x_n}{8}\left\{1 - j\frac{4}{\pi\beta^2\Delta x_n}\left[\frac{1}{|x_m-x_n| + \frac{\Delta x_n}{2}} - \frac{1}{|x_m-x_n| - \frac{\Delta x_n}{2}}\right]\right\}$$

$$Z_{mn} = \frac{\beta\eta\,\Delta x_n}{8}\left\{1 + j\frac{4}{\pi\beta^2}\frac{1}{|x_m-x_n|^2 + (\Delta x_n)^2/4}\right\}, \quad |m-n| \le 2,\ m \ne n$$

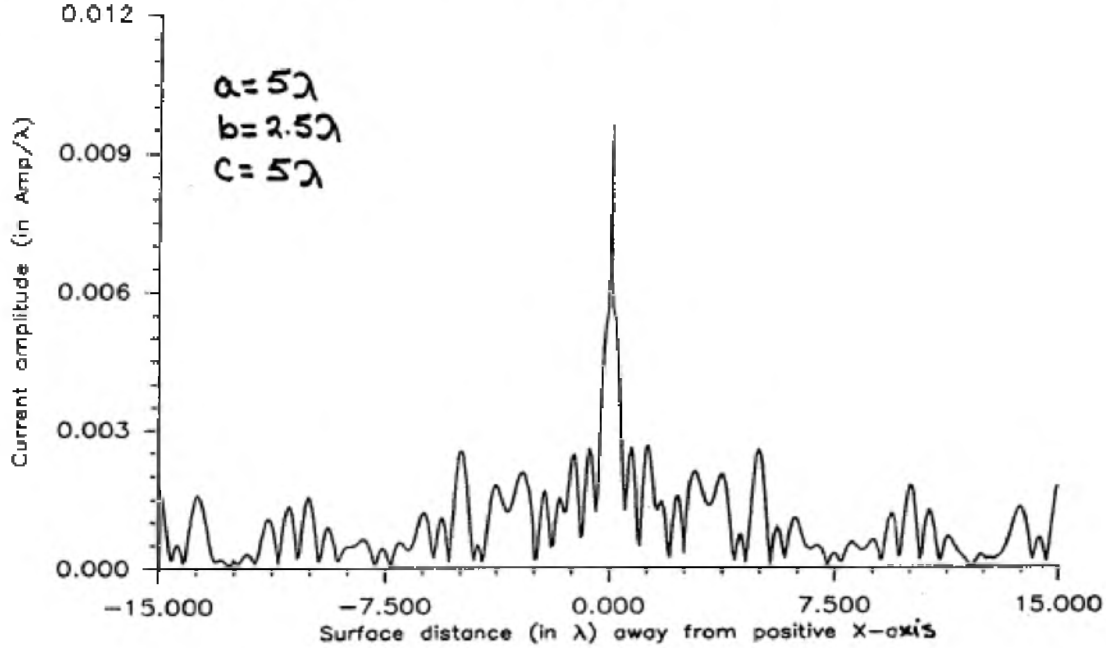
c. $|m-n| > 2$: $\quad Z_{mn} = \frac{\beta\eta}{4}\int_{x_n-\Delta x_n/2}^{x_n+\Delta x_n/2}\frac{H_1^{(2)}(\beta|x'-x_m|)}{(\beta|x'-x_m|)}dx' = \frac{\beta\eta}{4}\int_{-\Delta x_n/2}^{\Delta x_n/2}\frac{H_1^{(2)}\left[\beta(|x_m-x_n| + x')\right]}{\beta(|x_m-x_n| + x')}dx'$

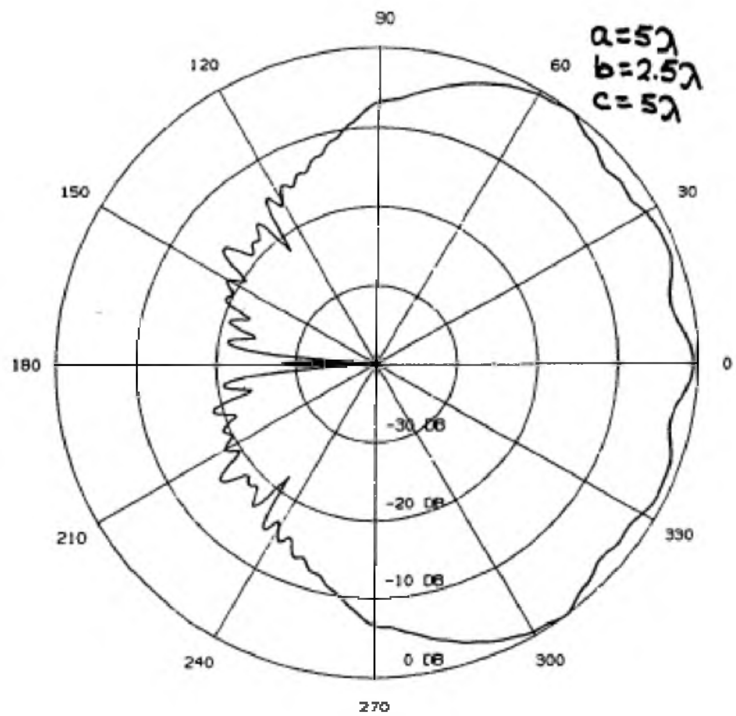
12.18 This problem can be solved using the TDRS computer program at the end of this chapter.



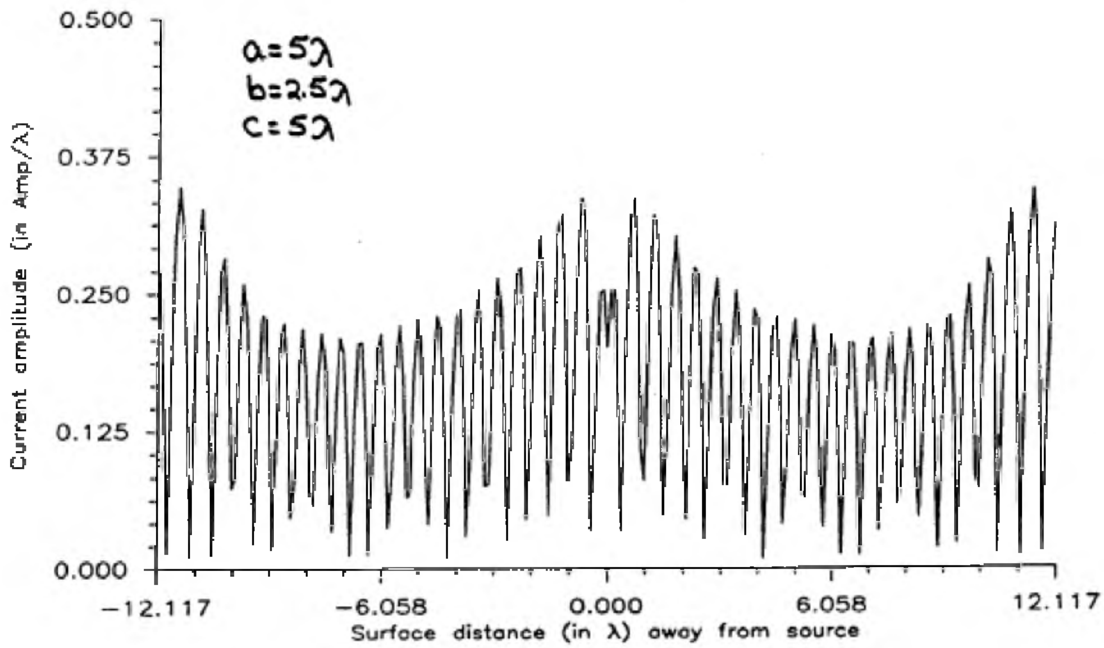


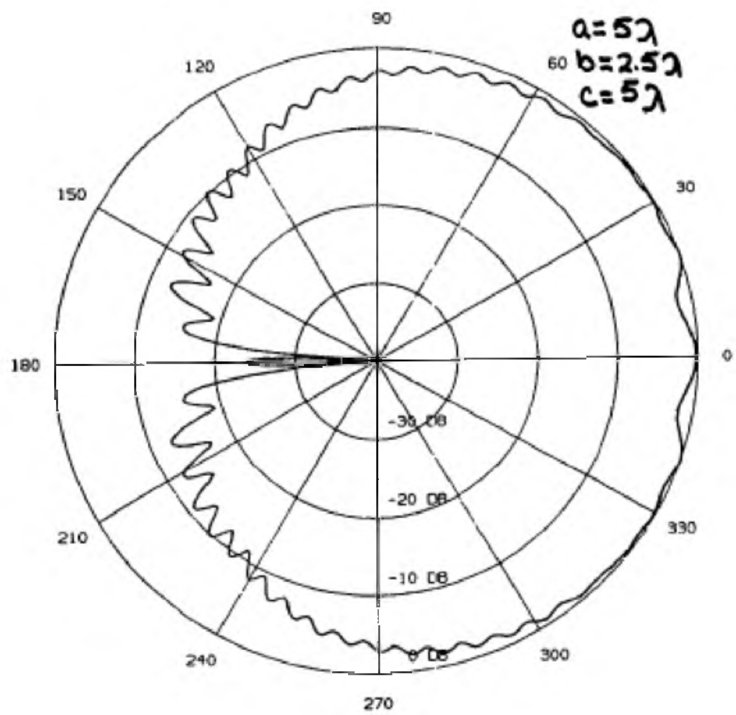
12.19 This problem can be solved using the TDRS computer program at the end of this chapter.



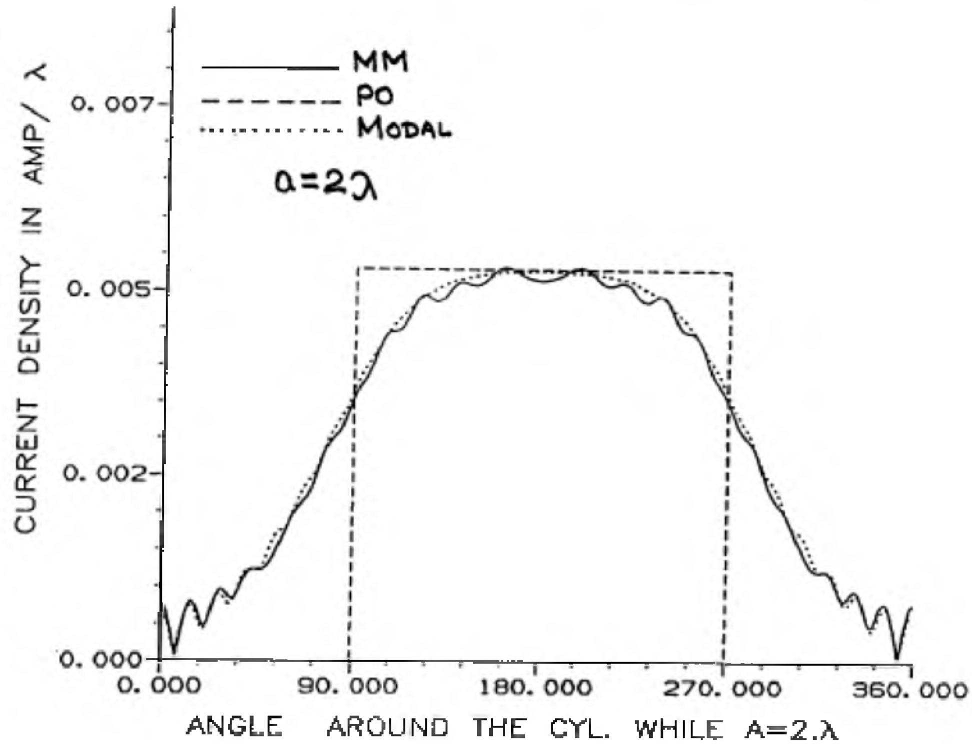


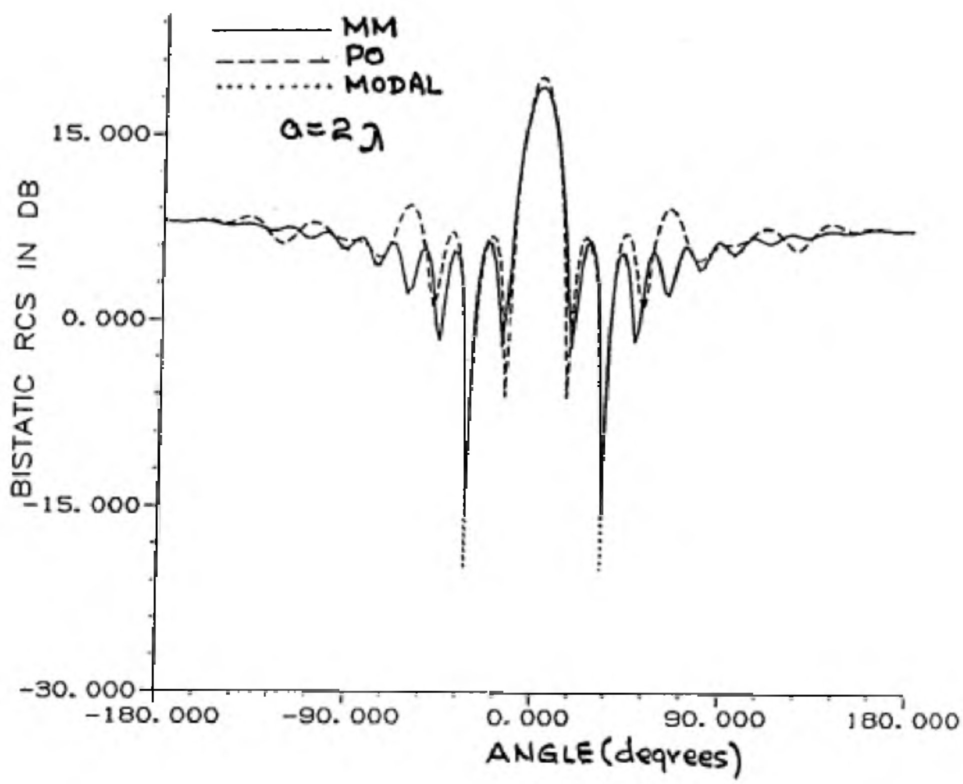
$\boxed{12.20}$ This problem can be solved using the TDRS computer program at the end of this chapter.



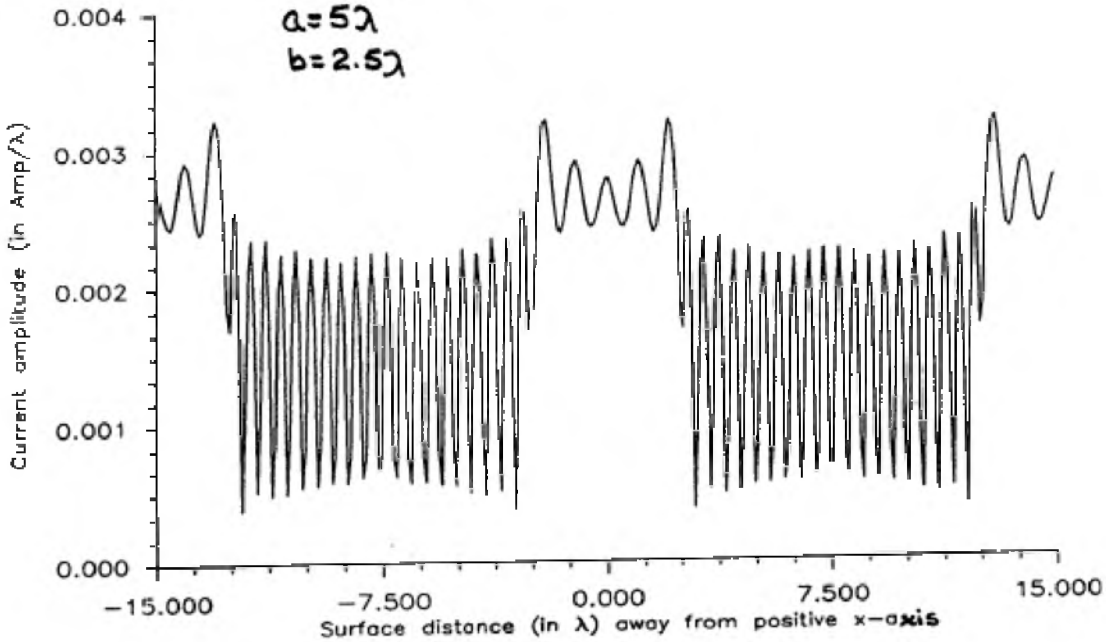


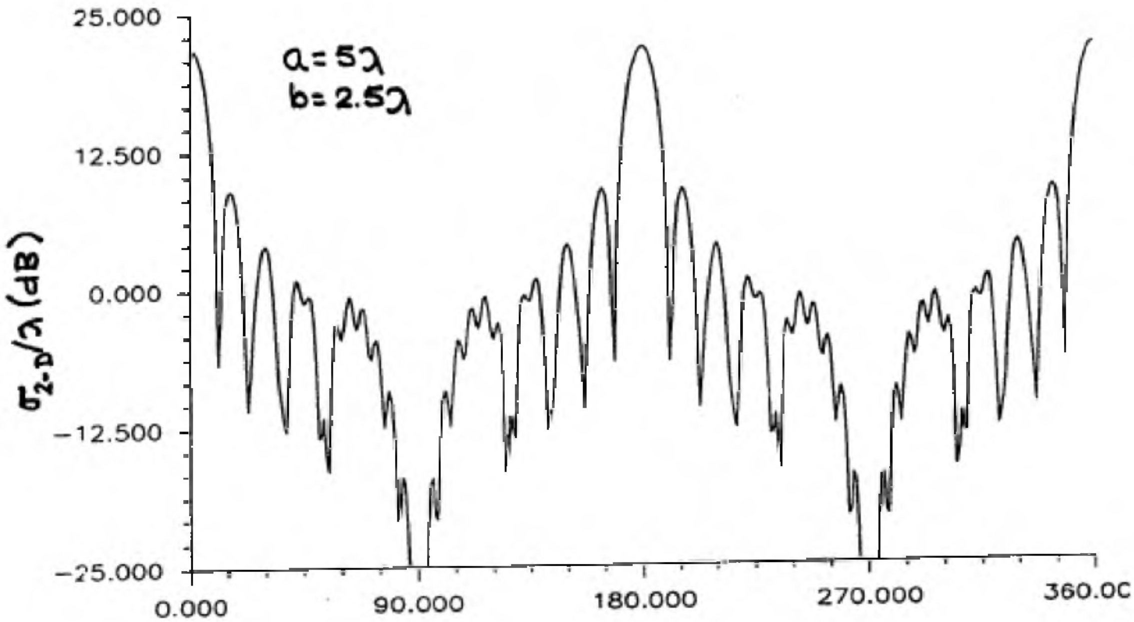
12.21 This problem can be solved using the TDRS computer program at the end of this chapter.



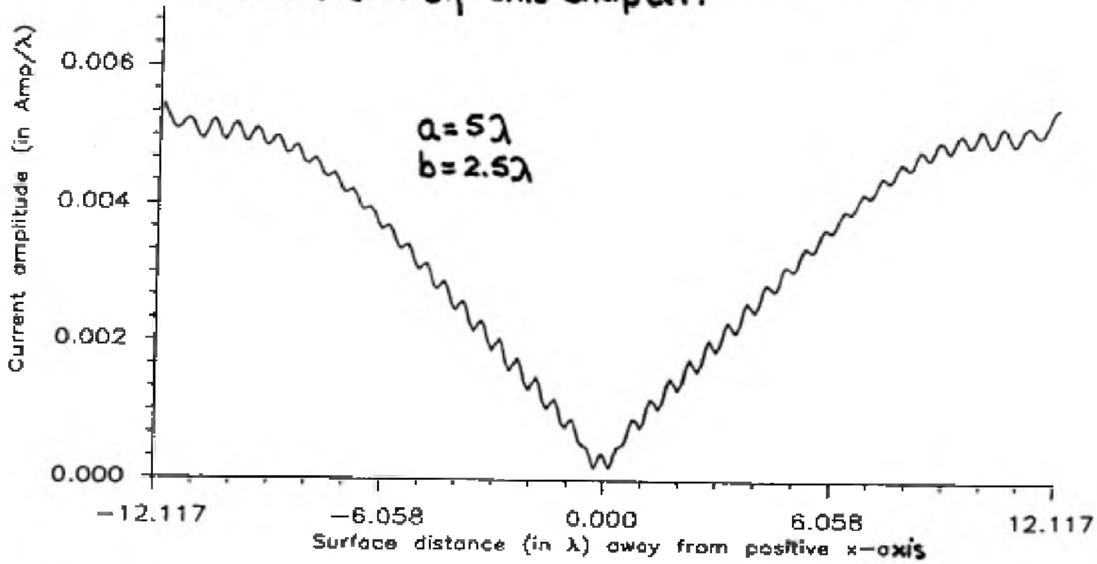


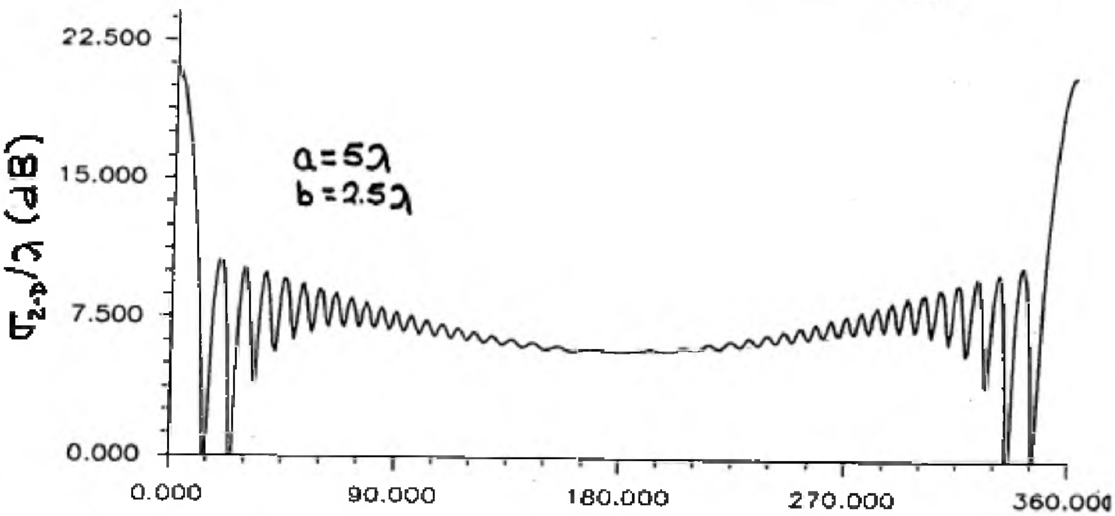
12.22 This problem can be solved using the TDRS computer program at the end of this chapter.





12.23 This problem can be solved using the TDRS computer program at the end of this chapter.





12.24

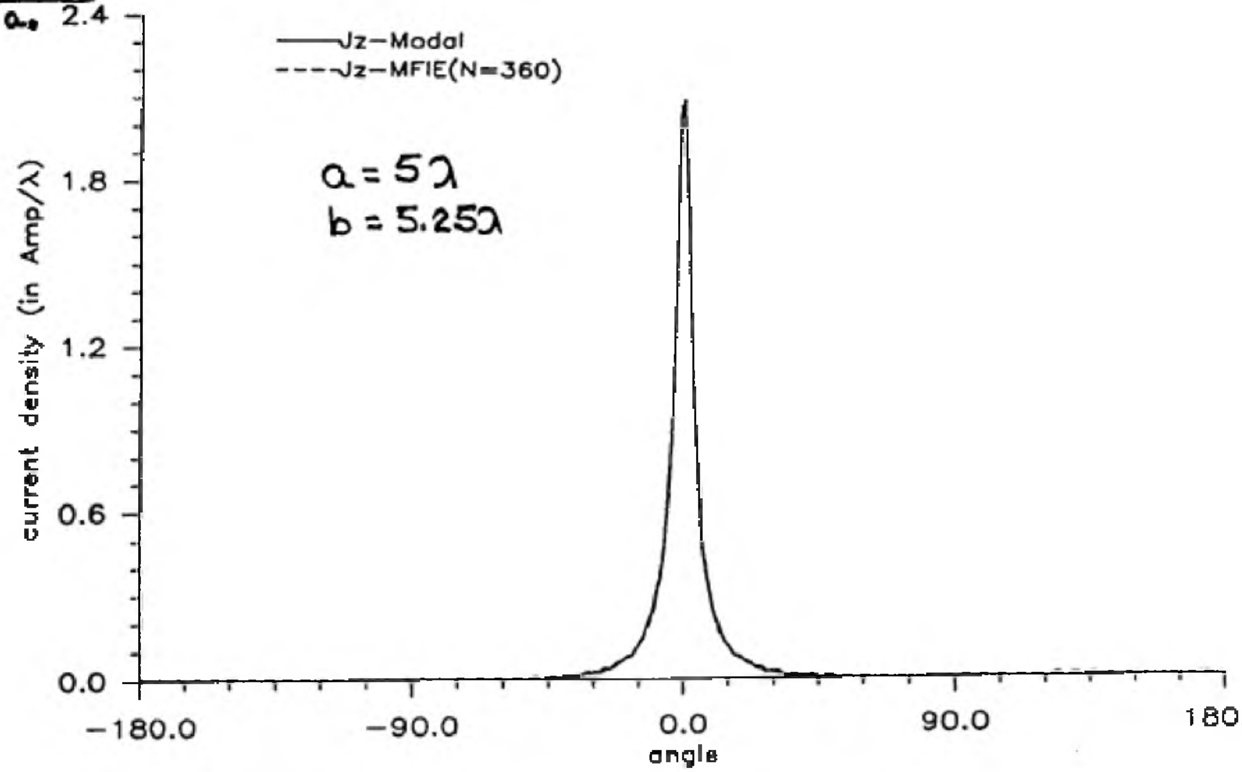
$$\hat{n}\cdot\nabla H_0^{(2)}(\beta R)=\hat{n}\cdot\left\{\hat{a}_R\frac{\partial}{\partial R}+\hat{a}_\phi\frac{1}{R}\cancel{\frac{\partial}{\partial\phi}}^{0}+\hat{a}_z\cancel{\frac{\partial}{\partial z}}^{0}\right\}H_0^{(2)}(\beta R)=(\hat{n}\cdot\hat{a}_R)\frac{\partial}{\partial R}H_0^{(2)}(\beta R)$$

$$=(\hat{n}\cdot\hat{a}_R)\left\{-\beta H_1^{(2)}(\beta R)\right\}=-\beta(\hat{n}\cdot\hat{a}_R)H_1^{(2)}(\beta R)$$

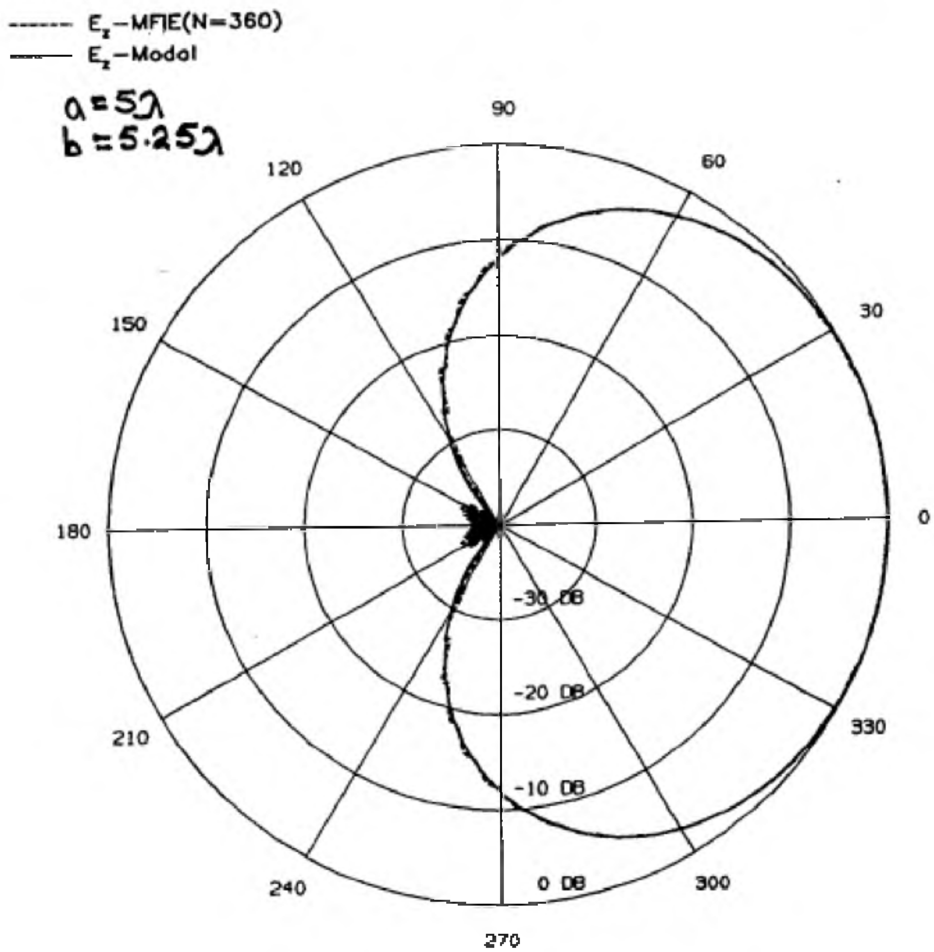
$$\hat{n}\cdot\nabla H_0^{(2)}(\beta R)=-\beta\cos\psi\, H_1^{(2)}(\beta R)$$

12.25

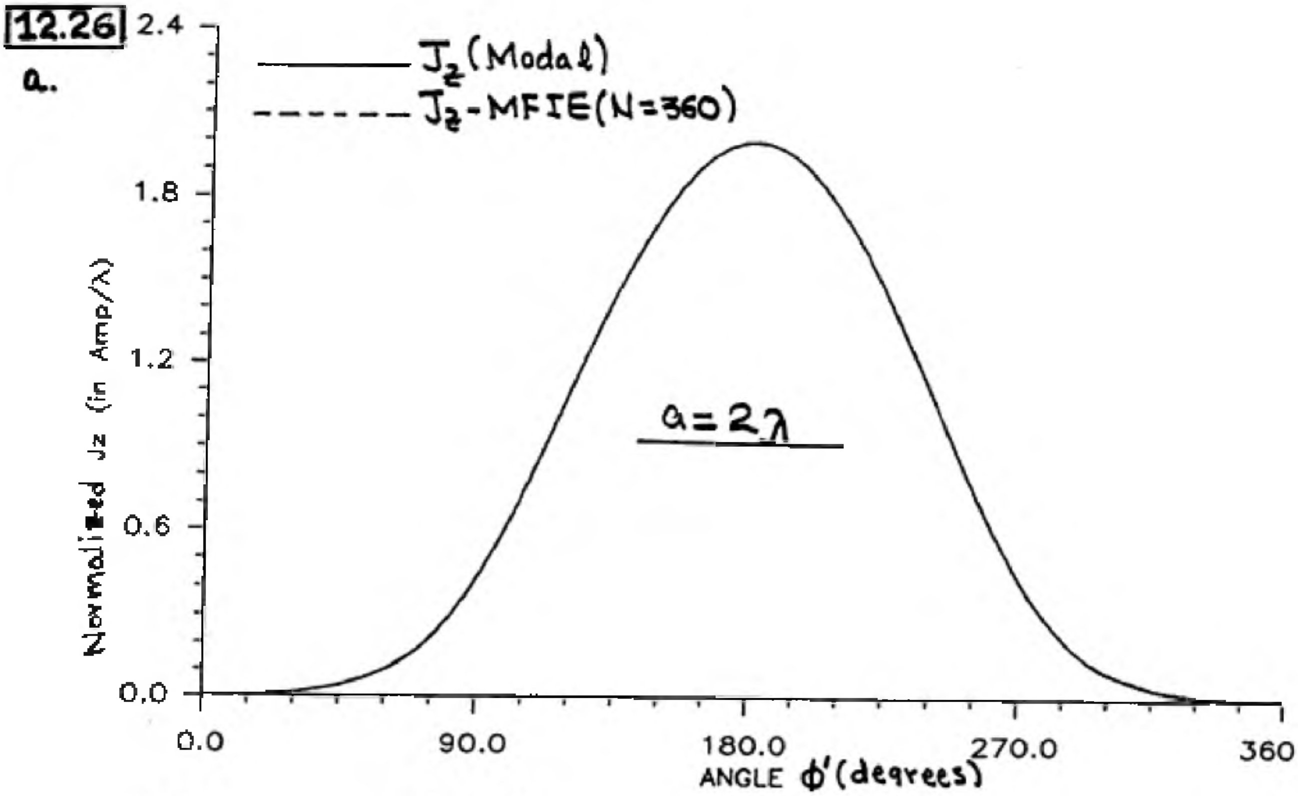
a.

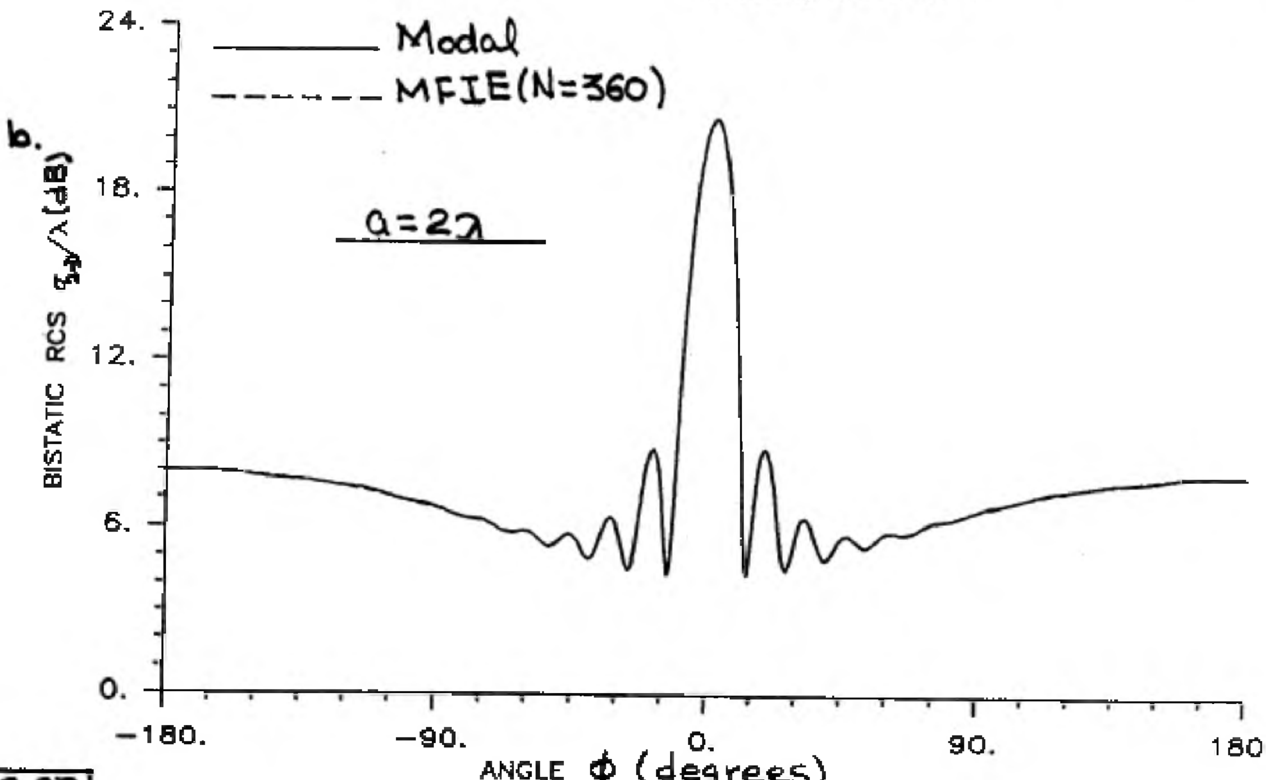

Jz-Modal
Jz-MFIE(N=360)
a = 5λ
b = 5.25λ
current density (in Amp/λ)
2.4
1.8
1.2
0.6
0.0
-180.0
-90.0
0.0
90.0
180.
angle

b.


Ez-MFIE(N=360)
Ez-Modal
a=5λ
b=5.25λ
90
60
30
0
330
300
270
240
210
180
150
120
-30 DB
-20 DB
-10 DB
0 DB

12.26





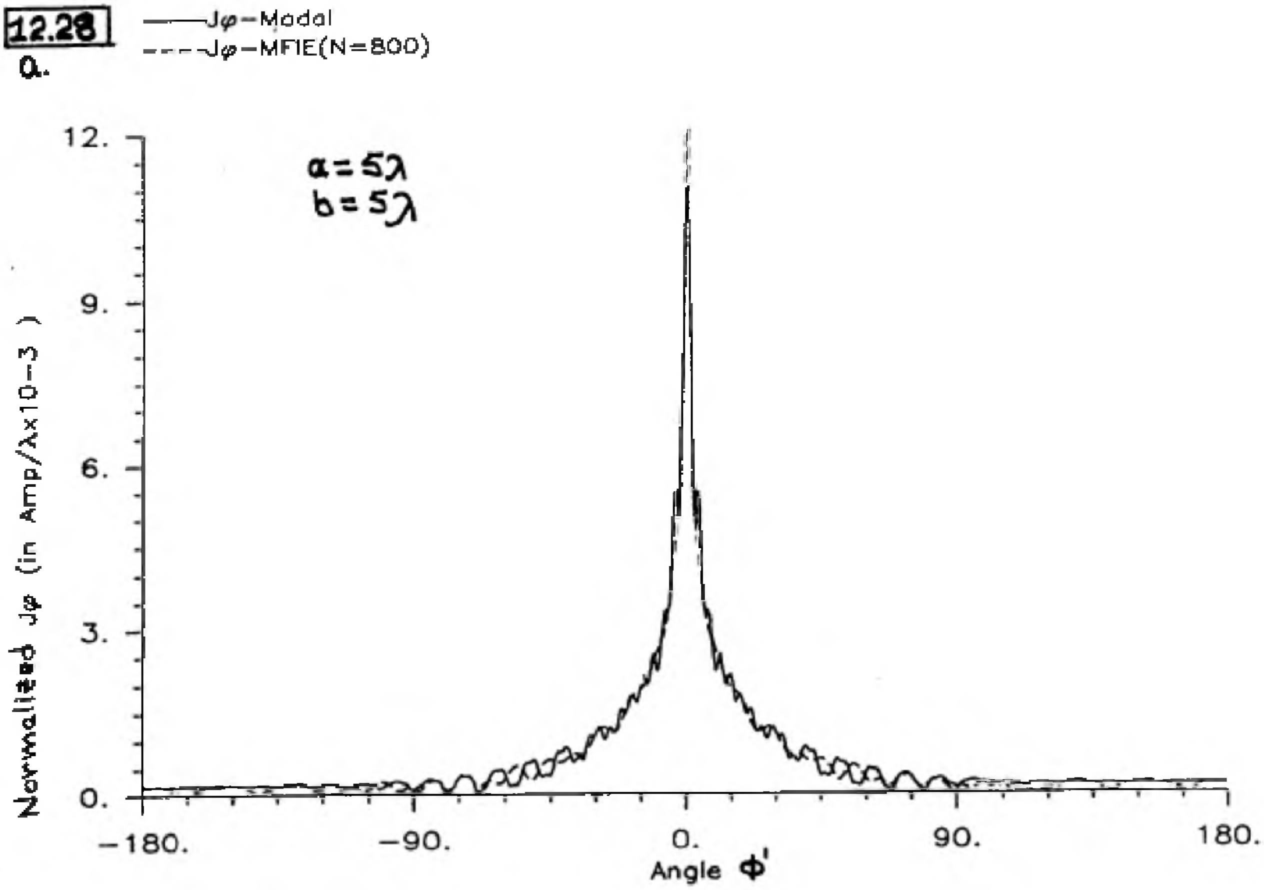
12.27

$$\hat{n}'\cdot\nabla H_0^{(2)}(\beta R) = \hat{n}'\cdot\left[\hat{a}_R \frac{\partial}{\partial R} + \hat{a}_\phi \cancel{\frac{\partial}{\partial \phi}} + \hat{a}_z \cancel{\frac{\partial}{\partial z}}\right] H_0^{(2)}(\beta R) = (\hat{n}'\cdot\hat{a}_R)\frac{\partial}{\partial R} H_0^{(2)}(\beta R)$$

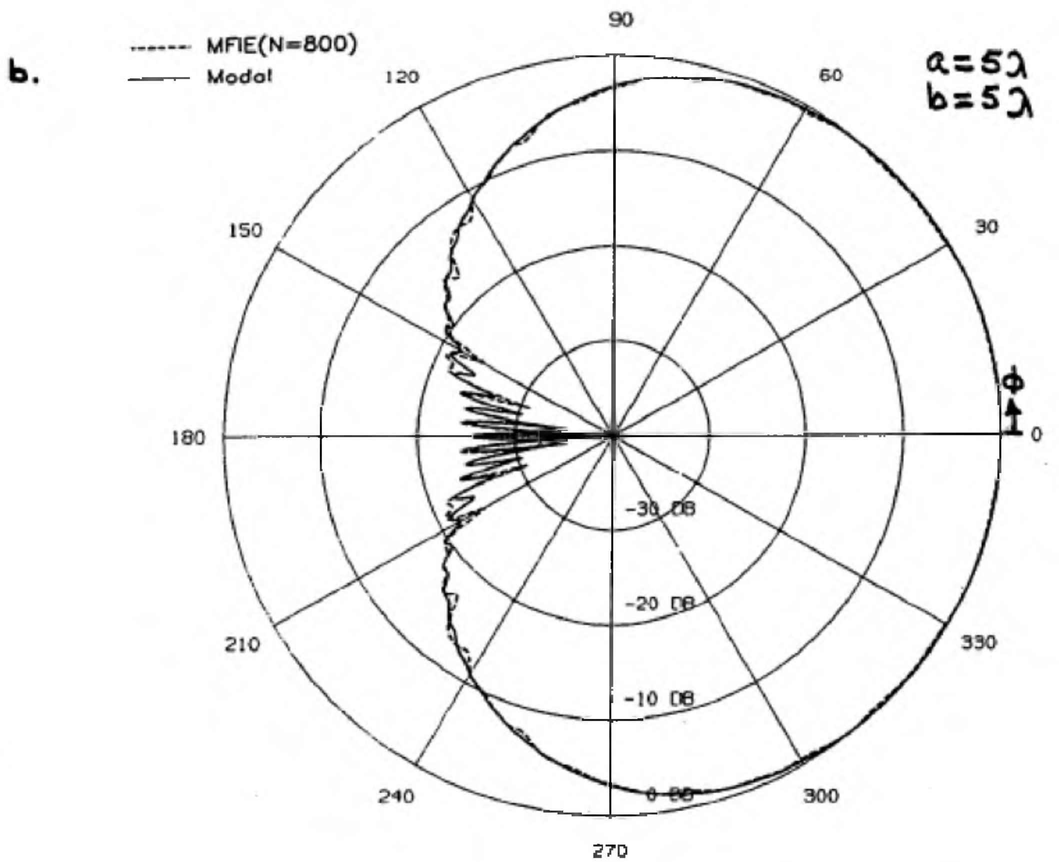
$$\hat{n}'\cdot\nabla H_0^{(2)}(\beta R) = (\hat{n}'\cdot\hat{R})\left[-\beta H_1^{(2)}(\beta R)\right] = -\beta\cos\psi' H_1^{(2)}(\beta R)$$

12.28

a.


——Jφ—Modal
---Jφ—MFIE(N=800)
a=5λ
b=5λ
Normalized Jφ (in Amp/λx10-3)
12.
9.
6.
3.
0.
-180.
-90.
0.
90.
180.
Angle Φ'

b.


MFIE(N=800)
Modal
a=5λ
b=5λ
90
60
30
0
330
300
270
240
210
180
150
120
-30 DB
-20 DB
-10 DB
0 DB
Φ

$\boxed{12.29}$ $\frac{1}{4\pi}\int_{-\ell/2}^{\ell/2} I(z')\left[\left(\frac{\partial^2}{\partial z^2}+\beta^2\right)\frac{e^{-j\beta R}}{R}\right]dz' = -j\omega\varepsilon E_z^i(\rho=a)$, $R^2=(z-z')^2+a^2$

$$\frac{\partial}{\partial z}\left(\frac{e^{-j\beta R}}{R}\right) = -(1+j\beta R)\left(\frac{z-z'}{R^3}\right)e^{-j\beta R}$$

$$\frac{\partial^2}{\partial z^2}\left(\frac{e^{-j\beta R}}{R}\right) = -(1+j\beta R)\frac{e^{-j\beta R}}{R^3} - \left[j\beta\frac{(z-z')^2}{R^4} - (1+j\beta R)(3+j\beta R)\frac{(z-z')^2}{R^5}\right]e^{-j\beta R}$$

$$= -(1+j\beta R)\frac{e^{-j\beta R}}{R^3} - \left[j\beta(R^2-a^2)R - (3+j4\beta R-(\beta R)^2)(R^2-a^2)\right]\frac{e^{-j\beta R}}{R^5}$$

$$= -(1+j\beta R)\frac{e^{-j\beta R}}{R^3} - (R^2-a^2)\left[(\beta R)^2 - 3(1+j\beta R)\right]\frac{e^{-j\beta R}}{R^5}$$

$$= \frac{e^{-j\beta R}}{R^5}\left[-R^2(1+j\beta R) - (R^2-a^2)(\beta R)^2 + 3(R^2-a^2)(1+j\beta R)\right]$$

$$\frac{\partial^2}{\partial z^2}\left(\frac{e^{-j\beta R}}{R}\right) = \frac{e^{-j\beta R}}{R^5}\left\{(1+j\beta R)(2R^2-3a^2) - (R^2-a^2)(\beta R)^2\right\}$$

Therefore

$$\frac{1}{4\pi}\int_{-\ell/2}^{\ell/2} I(z')\left[\left(\frac{\partial^2}{\partial z^2}+\beta^2\right)\frac{e^{-j\beta R}}{R}\right]dz' = \frac{1}{4\pi}\int_{-\ell/2} I(z')\frac{e^{-j\beta R}}{R^5}\left\{(1+j\beta R)(2R^2-3a^2) - k^2(kR)^2 + a^2(\beta R)^2 + \beta^2 R^4\right\}dz'$$

$$= \frac{1}{4\pi}\int_{-\ell/2}^{+\ell/2} I(z')\frac{e^{-j\beta R}}{R^5}\left[(1+j\beta R)(2R^2-3a^2) + (\beta a R)^2\right]dz'$$

$\boxed{12.30}$ According to (12-121) we have that

$$\left(\frac{\partial^2}{\partial z^2}+\beta^2\right)\int_{-\ell/2}^{+\ell/2} I_z(z')\frac{e^{-j\beta R}}{4\pi R}dz' = -j\omega\varepsilon E_z^i(\rho=a) = -j\frac{\beta}{\eta}E_z^i \qquad (1)$$

The above equation is an inhomogeneous ordinary harmonic differential equation whose solution is obtained as the sum of complementary and particular solutions. Thus we can write the solution as

$$\int_{-\ell/2}^{\ell/2} I_z(z')\frac{e^{-j\beta R}}{4\pi R}dz' = -j\frac{\beta}{\eta}\underbrace{\left[B\cos(\beta z) + C\sin\beta z\right]}_{\text{Complementary}} \underbrace{- j\frac{1}{\eta}\int_0^z E_z^i\sin[\beta(z-z')]dz'}_{\text{Particular Solution}} \qquad (2)$$

At $z=0$ the particular solution vanishes. Therefore (2) then reduces to

$$\int_{-\ell/2}^{\ell/2} I_z(z')\frac{e^{-j\beta R}}{4\pi R}dz' = -j\frac{\beta}{\eta}\left[B\cos(\beta z) + C\sin(\beta z)\right] = -j\frac{1}{\eta}\left[\beta B\cos(\beta z) + \beta C\sin(\beta z)\right]$$

$$= -j\sqrt{\frac{\varepsilon}{\mu}}\left[B_1\cos(\beta z) + C_1\sin(\beta z)\right]$$

1231 $\underline{E}_f = \hat{a}_\rho \dfrac{V_i}{2\rho' \ln(b/a)}$, $\underline{M}_f = -2\hat{n}\times\underline{E}_f = -2\hat{a}_z\times\hat{a}_\rho E_\rho = -\hat{a}_\phi \dfrac{V_i}{\rho' \ln(b/a)}$

$$M_\phi = -\frac{V_i}{\rho' \ln(b/a)} \tag{1}$$

The solution presented here is that reported in the following journal paper: L. L. Tsai, "A numerical solution for the near and far fields of an annular ring of magnetic current", IEEE Transactions Antennas and Propagation, Vol. AP-20, No. 5, Sept. 1972, pp. 569-576.

The radiated electric and magnetic fields can be found using

$$\underline{E} = -j\omega\underline{A} - j\frac{1}{\omega\mu\varepsilon}\nabla(\nabla\cdot\underline{A}) - \frac{1}{\varepsilon}\nabla\times\underline{F} \tag{2a}$$

$$\underline{H} = -j\omega\underline{F} - j\frac{1}{\omega\mu\varepsilon}\nabla(\nabla\cdot\underline{F}) + \frac{1}{\mu}\nabla\times\underline{A} \tag{2b}$$

where

$$\underline{F} = \frac{\varepsilon}{4\pi}\iint_S \underline{M}_f \frac{e^{-j\beta R}}{R}\, ds' \tag{2c}$$

$$\underline{A} = \frac{\mu}{4\pi}\iint_S \underline{J} \frac{e^{-j\beta R}}{R}\, ds' \tag{2d}$$

Since $\underline{J} = 0$, then $\underline{A} = 0$ and (2a) and (2b) reduce to

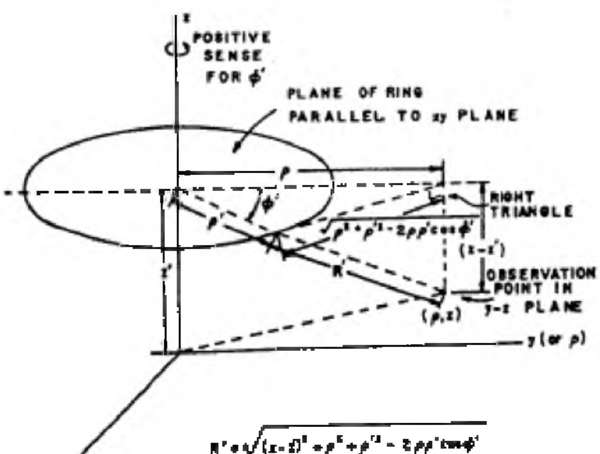
$$\underline{E} = -\frac{1}{\varepsilon}\nabla\times\underline{F} \tag{3a}$$

$$\underline{H} = -j\omega\underline{F} - j\frac{1}{\omega\mu\varepsilon}\nabla(\nabla\cdot\underline{F}) \tag{3b}$$

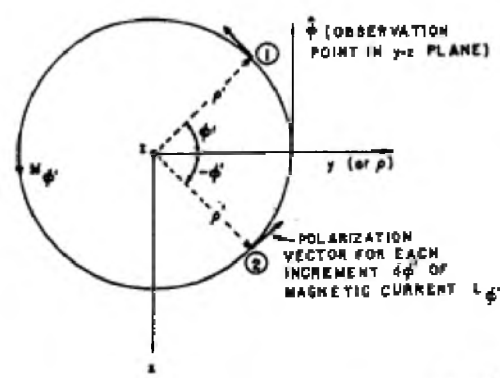
To evaluate (2c) the differential area ds' on the frill is

$$ds' = \rho' d\rho' d\phi' \tag{4}$$

Dividing the frill into $d\rho'$ wide magnetic ring current, the contribution of each ring will be evaluated first. This is accomplished by integrating with respect to ϕ' before summing over all the rings.



Geometry for ring.



Polarization for source and observation.

cont'd.

cont'd

1231 cont'd. The problem possesses symmetry since there are no ϕ variations. Therefore the observation can be located at any plane. Without loss of generality we choose the yz plane for the observations. For this plane $\rho = y$ and the $\underline{F}$ potential is ϕ-polarized (F_ϕ). Using (1) and (2c) we can then write that

$$F_\phi = -\frac{\varepsilon}{2\pi}\frac{1}{\ln(b/a)}\int_0^b\int_0^\pi \cos\phi' \,\frac{e^{-j\beta R'}}{R'}\,d\phi' d\rho' \qquad (5)$$

where
$$R' = \left[(z-z')^2 + \rho^2 - (\rho')^2 - 2\rho\rho'\cos\phi'\right]^{1/2} \qquad (5a)$$

Because the ring is on the xy-plane, each pair of incremental current element $d\phi'$ (symmetrically located at $\pm\phi'$) has only a net $\hat{a}_\phi$ contribution. This leads to the $2\cos\phi'$ factor with an integration in ϕ' from 0 to π.
Now using (2a) and (2b) in cylindrical coordinates, we can write that

$$E_z = -\frac{1}{\varepsilon}\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho F_\phi) \qquad (6a)$$

$$E_\rho = \frac{1}{\varepsilon}\frac{\partial}{\partial z}(F_\phi) \qquad (6b)$$

$$H_\phi = -j\omega F_\phi \qquad (6c)$$

$$E_\phi = H_\rho = H_z = 0 \qquad (6d)$$

Closed form expressions for the electric and magnetic fields can be derived when $\rho \gg b$ without making far field approximations. From (5a)

$$R_0 = \left[(z-z')^2 + \rho^2\right]^{1/2}$$

After suitable approximations on (5) we can write that

$$F_\phi(\rho, z) \simeq -\frac{\varepsilon}{4\ln(b/a)}\beta\rho\,\frac{e^{-j\beta R_0}}{R_0^2}(b^2-a^2)\left\{-j\frac{(b^2+a^2)}{4R_0^2} + \frac{1}{2\beta R_0} + j\frac{1}{2}\right\} \qquad (7)$$

Using (7), we can write (6a) as

$$E_z \simeq +V_i\,\frac{\beta(b^2-a^2)}{8\ln(b/a)}\,\frac{e^{-j\beta R_0}}{R_0^2}\left\{2\left[\frac{1}{\beta R_0} + j - j\frac{(b^2+a^2)}{2R_0^2}\right] + \frac{\rho^2}{R_0}\left[\left(\frac{1}{\beta R_0} + j - j\frac{(b^2+a^2)}{2R_0^2}\right)\left(-j\beta - \frac{2}{R_0}\right) + \left(-\frac{1}{\beta R_0^2} + j\frac{(b^2+a^2)}{R_0^3}\right)\right]\right\} \qquad (8)$$

and (6b) as

$$E_\rho \simeq -V_i\frac{(b^2-a^2)\rho}{8\ln(b/a)}\frac{(z-z')}{R_0}\frac{e^{-j\beta R_0}}{R_0^2}\left\{\beta - \left[\frac{3}{\beta} + \frac{\beta(b^2+a^2)}{2}\right]\frac{1}{R_0^2} + j\left[\frac{2(b^2+a^2)}{R_0^3} - \frac{3}{R_0}\right]\right\} \qquad (9)$$

$\boxed{12.32}$ On the axis of the ring ($\rho=0$), the symmetry of the problem along the ϕ direction requires that $E_\rho(\rho=0,z)=0$. However equations (8) and (9) of the solution of Problem 12.31 are no longer accurate when $\rho \to 0$. Therefore we will derive a simpler form for the field which will be accurate only for $\rho=0$. When $\rho=0$, we have that

$$R' = \left[(z-z')^2+(\rho')^2\right]^{1/2} \qquad (1)$$

Therefore

$$F_\phi(0,z) = -\frac{\varepsilon}{4\pi}\frac{1}{\ln(b/a)}\int_a^b \frac{e^{-j\beta R'}}{R'}\left[\int_0^{2\pi}\cos\phi' d\phi'\right]d\rho' \equiv 0 \qquad (2)$$

then

$$H_\phi(0,z) = -j\omega F_\phi = 0 \qquad (3)$$

From (6a) of the solution of Problem 12.32

$$E_z(0,z) = -\frac{1}{\varepsilon}\frac{F_\phi}{\rho} - \frac{1}{\varepsilon}\frac{\partial F_\phi}{\partial \rho} = -\frac{2}{\varepsilon}\frac{\partial F_\phi}{\partial \rho}\bigg|_{\rho=0} \qquad (4)$$

In deriving (4), l' Hospital's rule was used to take care of the 0/0 term.

Using (2) and (4) and interchanging the order of integration and differentiation, it can be shown that (4) can be written as

$$E_z(0,z) = \frac{1}{\pi}\frac{1}{\ln(b/a)}\int_0^b\int_0^\pi \cos\phi'\left[\frac{\partial}{\partial \rho}\frac{e^{-j\beta R'}}{R'}\right]_{\rho=0} d\phi' d\rho'$$

$$E_z(0,z) = \frac{1}{2\ln(b/a)}\left\{\frac{e^{-j\beta\left[(z-z')^2+a^2\right]^{1/2}}}{\left[(z-z')^2+a^2\right]^{1/2}} - \frac{e^{-j\beta\left[(z-z')^2+b^2\right]^{1/2}}}{\left[(z-z')^2+b^2\right]^{1/2}}\right\} \qquad (5)$$

The above solution is that found in:

1. L. L. Tsai, "A numerical solution for the near and far fields of an annular ring of magnetic current", IEEE Transactions Antennas and Propagation, Vol. AP-20, No. 5, Sept. 1972, pp. 569-576.

1233

$\ell = 0.47\lambda,\ a = 0.005\lambda,\ N = 21$

```
Z( 1, 1) = -0.96043E+03+J-0.29443E+00
Z( 2, 1) =  0.41849E+03+J-0.29383E+00
Z( 3, 1) =  0.44756E+02+J-0.29209E+00
Z( 4, 1) =  0.13232E+02+J-0.28921E+00
Z( 5, 1) =  0.57993E+01+J-0.28520E+00
Z( 6, 1) =  0.31310E+01+J-0.28012E+00
Z( 7, 1) =  0.19197E+01+J-0.27398E+00
Z( 8, 1) =  0.12790E+01+J-0.26686E+00
Z( 9, 1) =  0.90103E+00+J-0.25880E+00
Z(10, 1) =  0.65908E+00+J-0.24988E+00
Z(11, 1) =  0.49402E+00+J-0.24016E+00
Z(12, 1) =  0.37563E+00+J-0.22974E+00
Z(13, 1) =  0.28728E+00+J-0.21868E+00
Z(14, 1) =  0.21923E+00+J-0.20708E+00
Z(15, 1) =  0.16554E+00+J-0.19502E+00
Z(16, 1) =  0.12239E+00+J-0.18262E+00
Z(17, 1) =  0.87235E-01+J-0.16995E+00
Z(18, 1) =  0.58352E-01+J-0.15711E+00
Z(19, 1) =  0.34510E-01+J-0.14419E+00
Z(20, 1) =  0.14817E-01+J-0.13130E+00
Z(21, 1) = -0.13950E-02+J-0.11852E+00
Z( 1, 2) =  0.41849E+03+J-0.29383E+00
Z( 2, 2) = -0.96043E+03+J-0.29443E+00
Z( 3, 2) =  0.41849E+03+J-0.29383E+00
Z( 4, 2) =  0.44756E+02+J-0.29209E+00
Z( 5, 2) =  0.13232E+02+J-0.28921E+00
Z( 6, 2) =  0.57993E+01+J-0.28520E+00
Z( 7, 2) =  0.31310E+01+J-0.28012E+00
Z( 8, 2) =  0.19197E+01+J-0.27398E+00
Z( 9, 2) =  0.12790E+01+J-0.26686E+00
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Z(11, 2) =  0.65908E+00+J-0.24988E+00
Z(12, 2) =  0.49402E+00+J-0.24016E+00
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Z(15, 2) =  0.21923E+00+J-0.20708E+00
Z(16, 2) =  0.16554E+00+J-0.19502E+00
Z(17, 2) =  0.12239E+00+J-0.18262E+00
Z(18, 2) =  0.87235E-01+J-0.16995E+00
Z(19, 2) =  0.58352E-01+J-0.15711E+00
Z(20, 2) =  0.34510E-01+J-0.14419E+00
Z(21, 2) =  0.14817E-01+J-0.13130E+00
Z( 1, 3) =  0.44756E+02+J-0.29209E+00
Z( 2, 3) =  0.41849E+03+J-0.29383E+00
Z( 3, 3) = -0.96043E+03+J-0.29443E+00
Z( 4, 3) =  0.41849E+03+J-0.29383E+00
Z( 5, 3) =  0.44756E+02+J-0.29209E+00
Z( 6, 3) =  0.13232E+02+J-0.28921E+00
Z( 7, 3) =  0.57993E+01+J-0.28520E+00
Z( 8, 3) =  0.31310E+01+J-0.28012E+00
Z( 9, 3) =  0.19197E+01+J-0.27398E+00
Z(10, 3) =  0.12790E+01+J-0.26686E+00
Z(11, 3) =  0.90103E+00+J-0.25880E+00
Z(12, 3) =  0.65908E+00+J-0.24988E+00
Z(13, 3) =  0.49402E+00+J-0.24016E+00
Z(14, 3) =  0.37563E+00+J-0.22974E+00
Z(15, 3) =  0.28728E+00+J-0.21868E+00
Z(16, 3) =  0.21923E+00+J-0.20708E+00
Z(17, 3) =  0.16554E+00+J-0.19502E+00
Z(18, 3) =  0.12239E+00+J-0.18262E+00
Z(19, 3) =  0.87235E-01+J-0.16995E+00
Z(20, 3) =  0.58352E-01+J-0.15711E+00
Z(21, 3) =  0.34510E-01+J-0.14419E+00
Z( 1, 4) =  0.13232E+02+J-0.28921E+00
Z( 2, 4) =  0.44756E+02+J-0.29209E+00
Z( 3, 4) =  0.41849E+03+J-0.29383E+00
Z( 4, 4) = -0.96043E+03+J-0.29443E+00
Z( 5, 4) =  0.41849E+03+J-0.29383E+00
Z( 6, 4) =  0.44756E+02+J-0.29209E+00
Z( 7, 4) =  0.13232E+02+J-0.28921E+00
Z( 8, 4) =  0.57993E+01+J-0.28520E+00
Z( 9, 4) =  0.31310E+01+J-0.28012E+00
Z(10, 4) =  0.19197E+01+J-0.27398E+00
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Z(16, 4) =  0.28728E+00+J-0.21868E+00
Z(17, 4) =  0.21923E+00+J-0.20708E+00
Z(18, 4) =  0.16554E+00+J-0.19502E+00
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Z( 4, 5) =  0.41849E+03+J-0.29383E+00
Z( 5, 5) = -0.96043E+03+J-0.29443E+00
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Z( 1, 6) =  0.31310E+01+J-0.28012E+00
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Z(14, 6) =  0.90103E+00+J-0.25880E+00
Z(15, 6) =  0.65908E+00+J-0.24988E+00
Z(16, 6) =  0.49402E+00+J-0.24016E+00
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Z(15, 9) =  0.19197E+01+J-0.27398E+00
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Z(17, 9) =  0.90103E+00+J-0.25880E+00
Z(18, 9) =  0.65908E+00+J-0.24988E+00
Z(19, 9) =  0.49402E+00+J-0.24016E+00
Z(20, 9) =  0.37563E+00+J-0.22974E+00
Z(21, 9) =  0.28728E+00+J-0.21868E+00
Z( 1,10) =  0.65908E+00+J-0.24988E+00
Z( 2,10) =  0.90103E+00+J-0.25880E+00
Z( 3,10) =  0.12790E+01+J-0.26686E+00
Z( 4,10) =  0.19197E+01+J-0.27398E+00
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Z( 7,10) =  0.13232E+02+J-0.28921E+00
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Z(10,10) = -0.96043E+03+J-0.29443E+00
Z(11,10) =  0.41849E+03+J-0.29383E+00
Z(12,10) =  0.44756E+02+J-0.29209E+00
Z(13,10) =  0.13232E+02+J-0.28921E+00
Z(14,10) =  0.57993E+01+J-0.28520E+00
Z(15,10) =  0.31310E+01+J-0.28012E+00
Z(16,10) =  0.19197E+01+J-0.27398E+00
Z(17,10) =  0.12790E+01+J-0.26686E+00
Z(18,10) =  0.90103E+00+J-0.25880E+00
Z(19,10) =  0.65908E+00+J-0.24988E+00
Z(20,10) =  0.49402E+00+J-0.24016E+00
Z(21,10) =  0.37563E+00+J-0.22974E+00
Z( 1,11) =  0.49402E+00+J-0.24016E+00
Z( 2,11) =  0.65908E+00+J-0.24988E+00
Z( 3,11) =  0.90103E+00+J-0.25880E+00
Z( 4,11) =  0.12790E+01+J-0.26686E+00
Z( 5,11) =  0.19197E+01+J-0.27398E+00
Z( 6,11) =  0.31310E+01+J-0.28012E+00
Z( 7,11) =  0.57993E+01+J-0.28520E+00
Z( 8,11) =  0.13232E+02+J-0.28921E+00
Z( 9,11) =  0.44756E+02+J-0.29209E+00
Z(10,11) =  0.41849E+03+J-0.29383E+00
Z(11,11) = -0.96043E+03+J-0.29443E+00
Z(12,11) =  0.41849E+03+J-0.29383E+00
Z(13,11) =  0.44756E+02+J-0.29209E+00
Z(14,11) =  0.13232E+02+J-0.28921E+00
Z(15,11) =  0.57993E+01+J-0.28520E+00
Z(16,11) =  0.31310E+01+J-0.28012E+00
Z(17,11) =  0.19197E+01+J-0.27398E+00
Z(18,11) =  0.12790E+01+J-0.26686E+00
Z(19,11) =  0.90103E+00+J-0.25880E+00
Z(20,11) =  0.65908E+00+J-0.24988E+00
Z(21,11) =  0.49402E+00+J-0.24016E+00
Z( 1,12) =  0.37563E+00+J-0.22974E+00
Z( 2,12) =  0.49402E+00+J-0.24016E+00
Z( 3,12) =  0.65908E+00+J-0.24988E+00
Z( 4,12) =  0.90103E+00+J-0.25880E+00
Z( 5,12) =  0.12790E+01+J-0.26686E+00
Z( 6,12) =  0.19197E+01+J-0.27398E+00
Z( 7,12) =  0.31310E+01+J-0.28012E+00
Z( 8,12) =  0.57993E+01+J-0.28520E+00
Z( 9,12) =  0.13232E+02+J-0.28921E+00
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cont'd.

12.33 cont'd.

```
Z(10,12) =  0.44756E+02+J-0.29209E+00
Z(11,12) =  0.41849E+03+J-0.29383E+00
Z(12,12) = -0.96043E+03+J-0.29443E+00
Z(13,12) =  0.41849E+03+J-0.29383E+00
Z(14,12) =  0.44756E+02+J-0.29209E+00
Z(15,12) =  0.13232E+02+J-0.28921E+00
Z(16,12) =  0.57993E+01+J-0.28520E+00
Z(17,12) =  0.31310E+01+J-0.28012E+00
Z(18,12) =  0.19197E+01+J-0.27398E+00
Z(19,12) =  0.12790E+01+J-0.26686E+00
Z(20,12) =  0.90103E+00+J-0.25880E+00
Z(21,12) =  0.65908E+00+J-0.24988E+00
Z( 1,13) =  0.28728E+00+J-0.21868E+00
Z( 2,13) =  0.37563E+00+J-0.22974E+00
Z( 3,13) =  0.49402E+00+J-0.24016E+00
Z( 4,13) =  0.65908E+00+J-0.24988E+00
Z( 5,13) =  0.90103E+00+J-0.25880E+00
Z( 6,13) =  0.12790E+01+J-0.26686E+00
Z( 7,13) =  0.19197E+01+J-0.27398E+00
Z( 8,13) =  0.31310E+01+J-0.28012E+00
Z( 9,13) =  0.57993E+01+J-0.28520E+00
Z(10,13) =  0.13232E+02+J-0.28921E+00
Z(11,13) =  0.44756E+02+J-0.29209E+00
Z(12,13) =  0.41849E+03+J-0.29383E+00
Z(13,13) = -0.96043E+03+J-0.29443E+00
Z(14,13) =  0.41849E+03+J-0.29383E+00
Z(15,13) =  0.44756E+02+J-0.29209E+00
Z(16,13) =  0.13232E+02+J-0.28921E+00
Z(17,13) =  0.57993E+01+J-0.28520E+00
Z(18,13) =  0.31310E+01+J-0.28012E+00
Z(19,13) =  0.19197E+01+J-0.27398E+00
Z(20,13) =  0.12790E+01+J-0.26686E+00
Z(21,13) =  0.90103E+00+J-0.25880E+00
Z( 1,14) =  0.21923E+00+J-0.20708E+00
Z( 2,14) =  0.28728E+00+J-0.21868E+00
Z( 3,14) =  0.37563E+00+J-0.22974E+00
Z( 4,14) =  0.49402E+00+J-0.24016E+00
Z( 5,14) =  0.65908E+00+J-0.24988E+00
Z( 6,14) =  0.90103E+00+J-0.25880E+00
Z( 7,14) =  0.12790E+01+J-0.26686E+00
Z( 8,14) =  0.19197E+01+J-0.27398E+00
Z( 9,14) =  0.31310E+01+J-0.28012E+00
Z(10,14) =  0.57993E+01+J-0.28520E+00
Z(11,14) =  0.13232E+02+J-0.28921E+00
Z(12,14) =  0.44756E+02+J-0.29209E+00
Z(13,14) =  0.41849E+03+J-0.29383E+00
Z(14,14) = -0.96043E+03+J-0.29443E+00
Z(15,14) =  0.41849E+03+J-0.29383E+00
Z(16,14) =  0.44756E+02+J-0.29209E+00
Z(17,14) =  0.13232E+02+J-0.28921E+00
Z(18,14) =  0.57993E+01+J-0.28520E+00
Z(19,14) =  0.31310E+01+J-0.28012E+00
Z(20,14) =  0.19197E+01+J-0.27398E+00
Z(21,14) =  0.12790E+01+J-0.26686E+00
Z( 1,15) =  0.16554E+00+J-0.19502E+00
Z( 2,15) =  0.21923E+00+J-0.20708E+00
Z( 3,15) =  0.28728E+00+J-0.21868E+00
Z( 4,15) =  0.37563E+00+J-0.22974E+00
Z( 5,15) =  0.49402E+00+J-0.24016E+00
Z( 6,15) =  0.65908E+00+J-0.24988E+00
Z( 7,15) =  0.90103E+00+J-0.25880E+00
Z( 8,15) =  0.12790E+01+J-0.26686E+00
Z( 9,15) =  0.19197E+01+J-0.27398E+00
Z(10,15) =  0.31310E+01+J-0.28012E+00
Z(11,15) =  0.57993E+01+J-0.28520E+00
Z(12,15) =  0.13232E+02+J-0.28921E+00
Z(13,15) =  0.44756E+02+J-0.29209E+00
Z(14,15) =  0.41849E+03+J-0.29383E+00
Z(15,15) = -0.96043E+03+J-0.29443E+00
Z(16,15) =  0.41849E+03+J-0.29383E+00
Z(17,15) =  0.44756E+02+J-0.29209E+00
Z(18,15) =  0.13232E+02+J-0.28921E+00
Z(19,15) =  0.57993E+01+J-0.28520E+00
Z(20,15) =  0.31310E+01+J-0.28012E+00
Z(21,15) =  0.19197E+01+J-0.27398E+00
Z( 1,16) =  0.12239E+00+J-0.18262E+00
Z( 2,16) =  0.16554E+00+J-0.19502E+00
Z( 3,16) =  0.21923E+00+J-0.20708E+00
Z( 4,16) =  0.28728E+00+J-0.21868E+00
Z( 5,16) =  0.37563E+00+J-0.22974E+00
Z( 6,16) =  0.49402E+00+J-0.24016E+00
Z( 7,16) =  0.65908E+00+J-0.24988E+00
Z( 8,16) =  0.90103E+00+J-0.25880E+00
Z( 9,16) =  0.12790E+01+J-0.26686E+00
Z(10,16) =  0.19197E+01+J-0.27398E+00
Z(11,16) =  0.31310E+01+J-0.28012E+00
Z(12,16) =  0.57993E+01+J-0.28520E+00
Z(13,16) =  0.13232E+02+J-0.28921E+00
Z(14,16) =  0.44756E+02+J-0.29209E+00
Z(15,16) =  0.41849E+03+J-0.29383E+00
Z(16,16) = -0.96043E+03+J-0.29443E+00
Z(17,16) =  0.41849E+03+J-0.29383E+00
Z(18,16) =  0.44756E+02+J-0.29209E+00
Z(19,16) =  0.13232E+02+J-0.28921E+00
Z(20,16) =  0.57993E+01+J-0.28520E+00
Z(21,16) =  0.31310E+01+J-0.28012E+00
Z( 1,17) =  0.87235E-01+J-0.16995E+00
Z( 2,17) =  0.12239E+00+J-0.18262E+00
Z( 3,17) =  0.16554E+00+J-0.19502E+00
Z( 4,17) =  0.21923E+00+J-0.20708E+00
Z( 5,17) =  0.28728E+00+J-0.21868E+00
Z( 6,17) =  0.37563E+00+J-0.22974E+00
Z( 7,17) =  0.49402E+00+J-0.24016E+00
Z( 8,17) =  0.65908E+00+J-0.24988E+00
Z( 9,17) =  0.90103E+00+J-0.25880E+00
Z(10,17) =  0.12790E+01+J-0.26686E+00
Z(11,17) =  0.19197E+01+J-0.27398E+00
Z(12,17) =  0.31310E+01+J-0.28012E+00
Z(13,17) =  0.57993E+01+J-0.28520E+00
Z(14,17) =  0.13232E+02+J-0.28921E+00
Z(15,17) =  0.44756E+02+J-0.29209E+00
Z(16,17) =  0.41849E+03+J-0.29383E+00
Z(17,17) = -0.96043E+03+J-0.29443E+00
Z(18,17) =  0.41849E+03+J-0.29383E+00
Z(19,17) =  0.44756E+02+J-0.29209E+00
Z(20,17) =  0.13232E+02+J-0.28921E+00
Z(21,17) =  0.57993E+01+J-0.28520E+00
Z( 1,18) =  0.58352E-01+J-0.15711E+00
Z( 2,18) =  0.87235E-01+J-0.16995E+00
Z( 3,18) =  0.12239E+00+J-0.18262E+00
Z( 4,18) =  0.16554E+00+J-0.19502E+00
Z( 5,18) =  0.21923E+00+J-0.20708E+00
```

Cont'd.

12.33 cont'd.

```
Z( 6,18) =  0.28728E+00+J-0.21868E+00
Z( 7,18) =  0.37563E+00+J-0.22974E+00
Z( 8,18) =  0.49402E+00+J-0.24016E+00
Z( 9,18) =  0.65908E+00+J-0.24988E+00
Z(10,18) =  0.90103E+00+J-0.25880E+00
Z(11,18) =  0.12790E+01+J-0.26686E+00
Z(12,18) =  0.19197E+01+J-0.27398E+00
Z(13,18) =  0.31310E+01+J-0.28012E+00
Z(14,18) =  0.57993E+01+J-0.28520E+00
Z(15,18) =  0.13232E+02+J-0.28921E+00
Z(16,18) =  0.44756E+02+J-0.29209E+00
Z(17,18) =  0.41849E+03+J-0.29383E+00
Z(18,18) = -0.96043E+03+J-0.29443E+00
Z(19,18) =  0.41849E+03+J-0.29383E+00
Z(20,18) =  0.44756E+02+J-0.29209E+00
Z(21,18) =  0.13232E+02+J-0.28921E+00
Z( 1,19) =  0.34510E-01+J-0.14419E+00
Z( 2,19) =  0.58352E-01+J-0.15711E+00
Z( 3,19) =  0.87235E-01+J-0.16995E+00
Z( 4,19) =  0.12239E+00+J-0.18262E+00
Z( 5,19) =  0.16554E+00+J-0.19502E+00
Z( 6,19) =  0.21923E+00+J-0.20708E+00
Z( 7,19) =  0.28728E+00+J-0.21868E+00
Z( 8,19) =  0.37563E+00+J-0.22974E+00
Z( 9,19) =  0.49402E+00+J-0.24016E+00
Z(10,19) =  0.65908E+00+J-0.24988E+00
Z(11,19) =  0.90103E+00+J-0.25880E+00
Z(12,19) =  0.12790E+01+J-0.26686E+00
Z(13,19) =  0.19197E+01+J-0.27398E+00
Z(14,19) =  0.31310E+01+J-0.28012E+00
Z(15,19) =  0.57993E+01+J-0.28520E+00
Z(16,19) =  0.13232E+02+J-0.28921E+00
Z(17,19) =  0.44756E+02+J-0.29209E+00
Z(18,19) =  0.41849E+03+J-0.29383E+00
Z(19,19) = -0.96043E+03+J-0.29443E+00
Z(20,19) =  0.41849E+03+J-0.29383E+00
Z(21,19) =  0.44756E+02+J-0.29209E+00
Z( 1,20) =  0.14817E-01+J-0.13130E+00
Z( 2,20) =  0.34510E-01+J-0.14419E+00
Z( 3,20) =  0.58352E-01+J-0.15711E+00
Z( 4,20) =  0.87235E-01+J-0.16995E+00
Z( 5,20) =  0.12239E+00+J-0.18262E+00
Z( 6,20) =  0.16554E+00+J-0.19502E+00
Z( 7,20) =  0.21923E+00+J-0.20708E+00
Z( 8,20) =  0.28728E+00+J-0.21868E+00
Z( 9,20) =  0.37563E+00+J-0.22974E+00
Z(10,20) =  0.49402E+00+J-0.24016E+00
Z(11,20) =  0.65908E+00+J-0.24988E+00
Z(12,20) =  0.90103E+00+J-0.25880E+00
Z(13,20) =  0.12790E+01+J-0.26686E+00
Z(14,20) =  0.19197E+01+J-0.27398E+00
Z(15,20) =  0.31310E+01+J-0.28012E+00
Z(16,20) =  0.57993E+01+J-0.28520E+00
Z(17,20) =  0.13232E+02+J-0.28921E+00
Z(18,20) =  0.44756E+02+J-0.29209E+00
Z(19,20) =  0.41849E+03+J-0.29383E+00
Z(20,20) = -0.96043E+03+J-0.29443E+00
Z(21,20) =  0.41849E+03+J-0.29383E+00
Z( 1,21) = -0.13950E-02+J-0.11852E+00
Z( 2,21) =  0.14817E-01+J-0.13130E+00
Z( 3,21) =  0.34510E-01+J-0.14419E+00
Z( 4,21) =  0.58352E-01+J-0.15711E+00
Z( 5,21) =  0.87235E-01+J-0.16995E+00
Z( 6,21) =  0.12239E+00+J-0.18262E+00
Z( 7,21) =  0.16554E+00+J-0.19502E+00
Z( 8,21) =  0.21923E+00+J-0.20708E+00
Z( 9,21) =  0.28728E+00+J-0.21868E+00
Z(10,21) =  0.37563E+00+J-0.22974E+00
Z(11,21) =  0.49402E+00+J-0.24016E+00
Z(12,21) =  0.65908E+00+J-0.24988E+00
Z(13,21) =  0.90103E+00+J-0.25880E+00
Z(14,21) =  0.12790E+01+J-0.26686E+00
Z(15,21) =  0.19197E+01+J-0.27398E+00
Z(16,21) =  0.31310E+01+J-0.28012E+00
Z(17,21) =  0.57993E+01+J-0.28520E+00
Z(18,21) =  0.13232E+02+J-0.28921E+00
Z(19,21) =  0.44756E+02+J-0.29209E+00
Z(20,21) =  0.41849E+03+J-0.29383E+00
Z(21,21) = -0.96043E+03+J-0.29443E+00
```

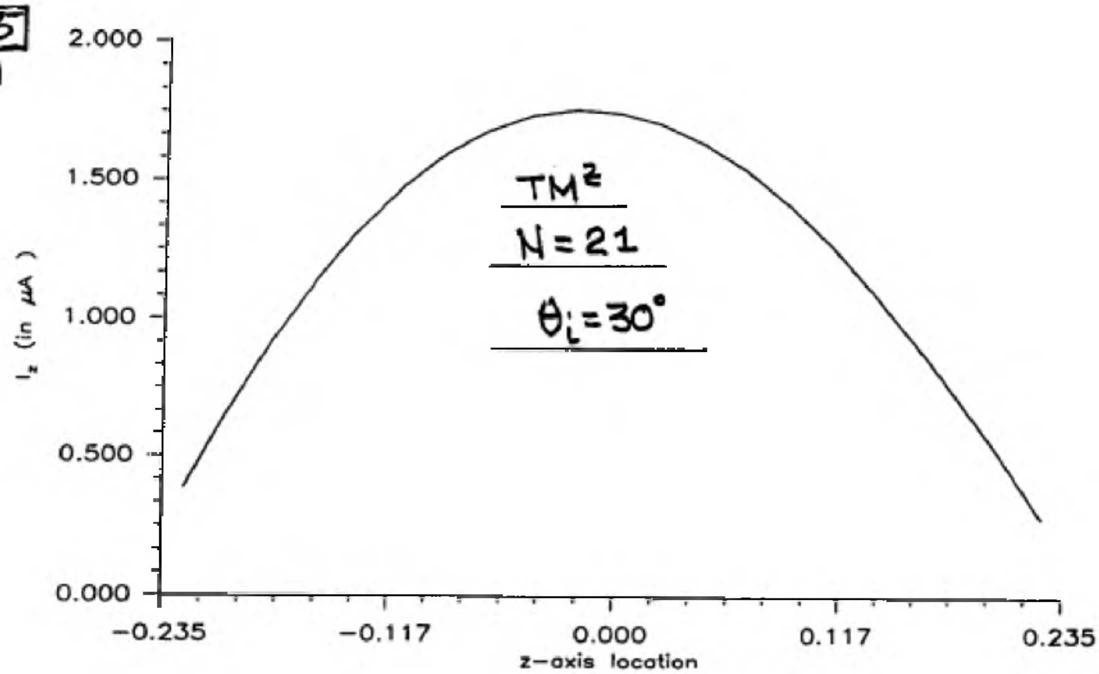
12.34 $\ell = \lambda/50,\ a = 0.005\lambda,\ N = 21$

Delta Gap — **Magnetic Frill**

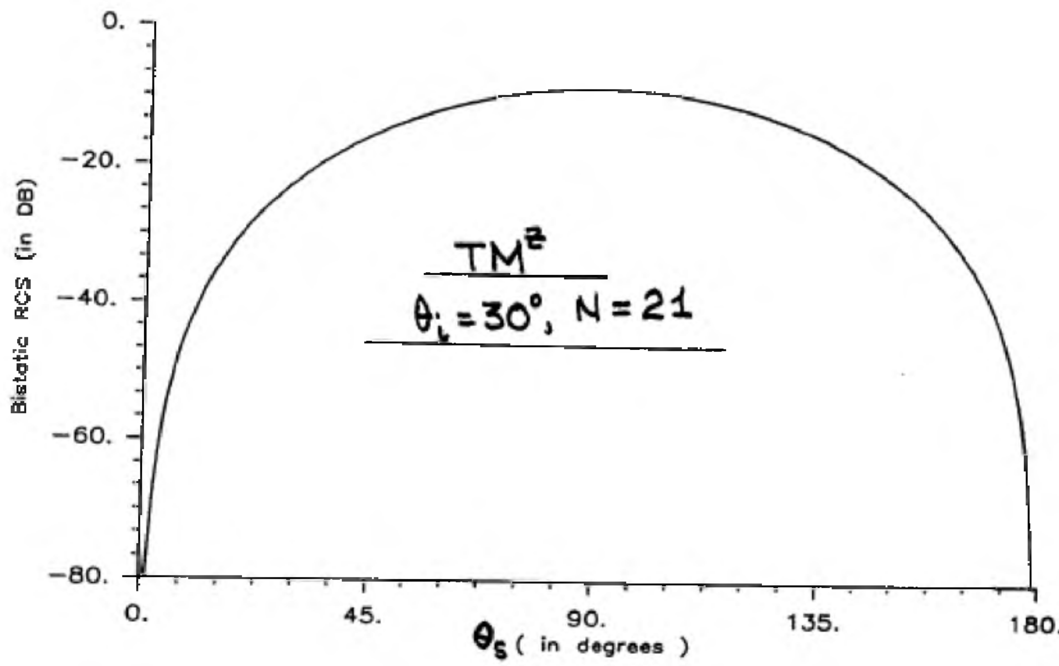
$Z_{in} = 8.17\times10^{-6} - j2.75\times10^{-2}$ Ohms; $Z_{in} = 0.2199 - j421.4$ Ohms

12.35

(a)



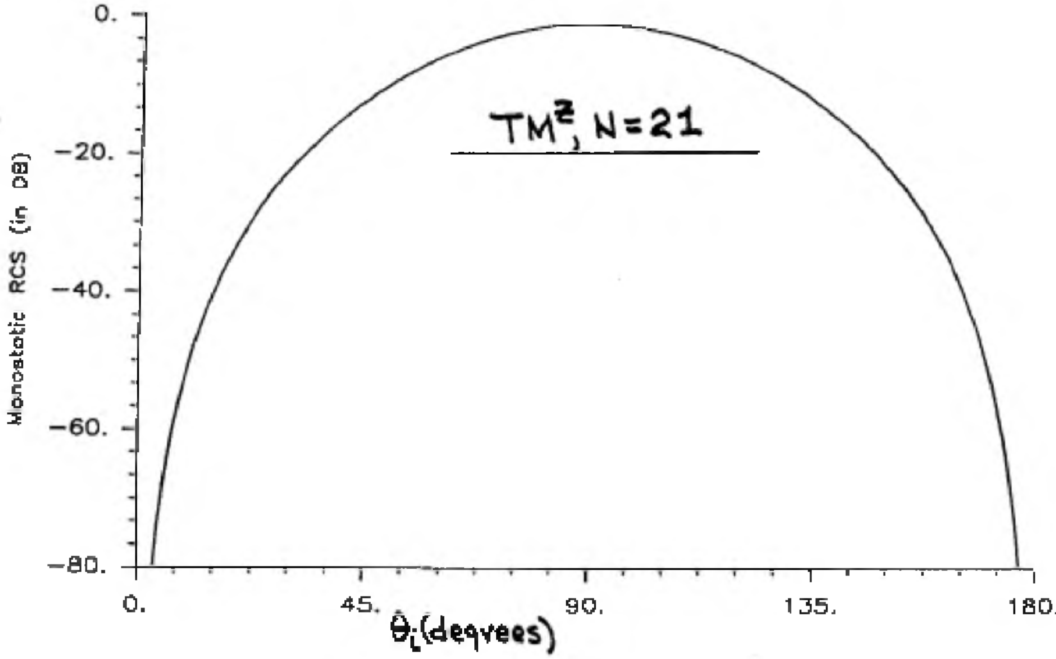
(b)



cont'd.

12.35 cont'd.

(c)



Stability is achieved and maintained when the number of segments does not exceed about 100 ($N \leq 100$). However $N=21$ gives very good results.

12.36

For TE^z uniform plane wave incidence, the solutions of parts a, b, c of Problem 12.35 are all zero because the incident electric field is perpendicular to the wire.

CHAPTER 13

13.1 Let α = angular space from Q-Q' along z axis

β = angular space from P-P' along z axis

Therefore each area, dA and dA_0, is defined as the product of the arc lengths in the xz and yz planes. Thus

$$dA_0 = (\rho_1 \beta)(\rho_2 \alpha) = \alpha\beta\,\rho_1\rho_2$$

$$dA = [(\rho_1+s)\beta][(\rho_2+s)\alpha] = \alpha\beta(\rho_1+s)(\rho_2+s)$$

Therefore we can write that

$$\frac{|\underline{E}|}{|\underline{E}_0|} = \sqrt{\frac{dA_0}{dA}} = \sqrt{\frac{\alpha\beta\rho_1\rho_2}{\alpha\beta(\rho_1+s)(\rho_2+s)}} = \sqrt{\frac{\rho_1\rho_2}{(\rho_1+s)(\rho_2+s)}}$$

13.2 $z = f - \dfrac{x^2+y^2}{4f}$. Since $z = g(u)$ where $u = \dfrac{x^2+y^2}{2}$

then $z = f - \dfrac{1}{2f}u$

$$\frac{dg(u)}{du} = \frac{d}{du}\left[f - \frac{1}{2f}u\right] = -\frac{1}{2f}$$

$$\frac{d^2g(u)}{du^2} = \frac{d}{du}\left[-\frac{1}{2f}\right] = 0$$

Thus

$$K^2 = 1 + 2u\left[\frac{dg(u)}{du}\right]^2 = 1 + 2u\left[-\frac{1}{2f}\right]^2 = 1 + \frac{2u}{4f^2} = 1 + \frac{u}{2f^2} = 1 + \frac{x^2+y^2}{4f^2}$$

$$K = \sqrt{1 + \frac{x^2+y^2}{4f^2}}$$

$$\frac{1}{R_1} = \frac{1}{K}\frac{dg(u)}{du} = -\frac{1}{2f}\frac{1}{\sqrt{1+\frac{x^2+y^2}{4f^2}}}$$

$$\frac{1}{R_2} = \frac{1}{K^3}\left[\frac{dg(u)}{du} + 2u\cancelto{0}{\frac{d^2g(u)}{du^2}}\right] = \frac{1}{K^3}\frac{dg(u)}{du}$$

$$= -\frac{1}{2f}\frac{1}{\left[1 + \frac{x^2+y^2}{4f^2}\right]^{3/2}}$$

$\boxed{13.3}$ $S_o(0) = 10$ mwatts/cm^2

$r_o = 5 \text{ m} = 500 \text{ cm}$

$r_1 = 50 \text{ m} = 5{,}000 \text{ cm}$

$$\frac{|E|^2}{|E_o|^2} = \frac{dA_o}{dA} = \frac{S(s)}{S_o(0)}$$

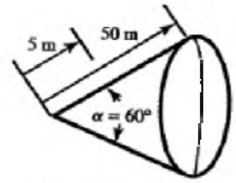
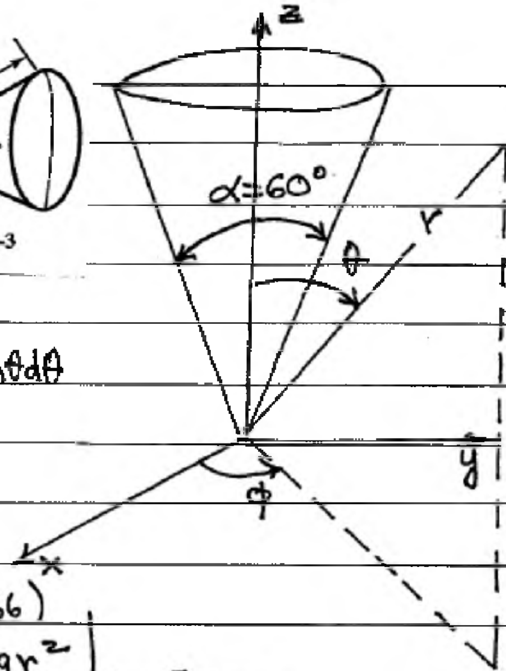


Figure P13-3



(a) $$dA_o = \int_0^{2\pi}\int_0^{\theta_o} r_o^2 \sin\theta \, d\theta \, d\phi = 2\pi \int_0^{\alpha/2} r_o^2 \sin\theta \, d\theta$$

$$= 2\pi r_o^2 \left(-\cos\theta\right)\Big|_0^{\pi/6}$$

$$= 2\pi r_o^2 \left(-\cos 30^\circ + \cos 0\right)$$

$$= 2\pi r_o^2 \left(-\frac{\sqrt{3}}{2} + 1\right) = 2\pi r_o^2 (1 - 0.866)$$

$$dA_o = 2\pi r_o^2 (0.13397) = 0.84179 r_o^2 \Big|_{r_o = 500 \text{ cm}}$$

$$dA_o = 0.841797 (500)^2 = 21.04468 \times 10^4 = 210.4468 \times 10^3 \text{ cm}^2$$

$$W = S_o(r_o) \, dA_o = 10 \times 10^{-3} (210.4468 \times 10^3) = 2{,}104.468 \text{ Watts}$$

$$\boxed{W = 2{,}104.468 \text{ Watts}}$$

(b) $$\frac{S(r_1)}{S(r_0)} = \frac{dA_0}{dA_1} = \frac{0.841797 r_0^2}{0.841797 r_1^2} = \left(\frac{r_0}{r_1}\right)^2 = \left(\frac{5}{50}\right)^2 = \left(\frac{1}{10}\right)^2 = \frac{1}{100}$$

$$S(r_1) = \frac{1}{100} S(r_0) = \frac{10 \times 10^{-3}}{100} = 0.1 \times 10^{-3}$$

$$\boxed{S(r_1) = 0.1 \times 10^{-3} = 0.1 \text{ mwatts/cm}^2}$$

alternate:

$$\frac{S(r_1)}{S(r_0)} = \frac{\rho_1 \rho_2}{(\rho_1 + s)(\rho_2 + s)} = \frac{5(5)}{(5+45)(5+45)} = \frac{5(5)}{(50)(50)} = \frac{1}{100}$$

$$\boxed{S(r_1) = \frac{1}{100} S(r_0) = 0.1 \times 10^{-3} = 0.1 \text{ mwatts/cm}^2}$$

$\boxed{13.4}$ To derive (13-30) we look at the geometry of Figure 13-8.

The length of the arc between points $Q_R - Q_R$ is equal to

$$\rho_a \Delta\phi = \frac{\Delta\psi_1 \rho_0}{\cos\theta_i} = \frac{\rho_0 \Delta\psi_1}{\cos\theta_i}$$

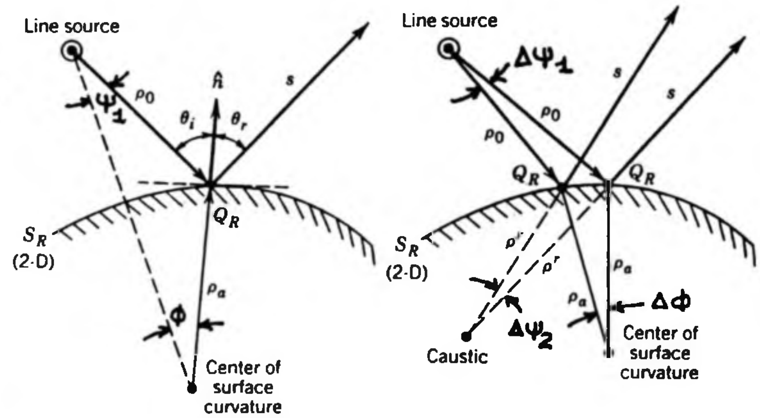
Since

$$\psi_1 = \pi - \left[\phi + \tfrac{1}{2}(2\pi - 2\theta_i)\right] = \pi - \left[\phi + (\pi - \theta_i)\right] = \theta_i - \phi$$

then

$$\Delta\psi_1 = \Delta\theta_i - \Delta\phi$$

and we can write that



(a) *(b)*

$$(1) \quad \rho_a \Delta\phi = \frac{\rho_0(\Delta\theta_i - \Delta\phi)}{\cos\theta_i}$$

Also

$$\rho_a \Delta\phi = \frac{\rho^r \Delta\psi_2}{\cos\theta_i}$$

Since

$$\Delta\psi_2 = \Delta\theta_i + \Delta\phi$$

then

$$(2) \quad \rho_2 \Delta\phi = \frac{\rho^r \Delta\psi_2}{\cos\theta_i} = \frac{\rho^r(\Delta\theta_i + \Delta\phi)}{\cos\theta_i}$$

Equating (1) and (2) we have that

$$\rho_a \Delta\phi \cos\theta_i = \rho_0(\Delta\theta_i - \Delta\phi) = \rho^r(\Delta\theta_i + \Delta\phi)$$

or

$$(3) \quad \Delta\theta_i - \Delta\phi = \frac{\rho_a}{\rho_0}\Delta\phi\cos\theta_i \Rightarrow \Delta\phi\left(1 + \frac{\rho_a}{\rho_0}\cos\theta_i\right) = \Delta\theta_i \Rightarrow \Delta\phi = \frac{\rho_0 \Delta\theta_i}{\rho_0 + \rho_a\cos\theta_i}$$

$$(4) \quad \Delta\theta_i + \Delta\phi = \frac{\rho_a}{\rho^r}\Delta\phi\cos\theta_i \Rightarrow \Delta\phi = \left(-1 + \frac{\rho_a}{\rho^r}\cos\theta_i\right) = \Delta\theta_i \Rightarrow \Delta\phi = \frac{\rho^r \Delta\theta_i}{-\rho^r + \rho_a\cos\theta_i}$$

Equating (3) and (4) leads to

$$(5) \quad \frac{\rho_0 \Delta\theta_i}{\rho_0 + \rho_a\cos\theta_i} = \frac{\rho^r \Delta\theta_i}{-\rho^r + \rho_a\cos\theta_i} \Rightarrow \frac{\rho_0}{\rho_0 + \rho_a\cos\theta_i} = \frac{\rho^r}{-\rho^r + \rho_a\cos\theta_i}$$

Solving (5) leads to

$$\frac{1}{\rho^r} = \frac{1}{\rho_0} + \frac{2}{\rho_a\cos\theta_i}$$

$\boxed{13.5}$ (a) $\frac{1}{\rho^r}=\frac{1}{\rho_o}+\frac{2}{\rho_a\cos\theta_i}$, $\rho_o=a$, $\rho_a=+a$, $\theta_i=0$

Therefore

$$\frac{1}{\rho^r}=\frac{1}{a}+\frac{2}{a}=+\frac{3}{a}\Rightarrow \rho^r=+a/3$$

Thus the caustic is at the origin of the circle.

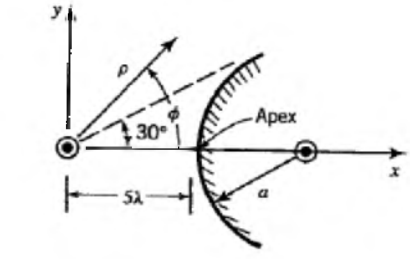


Figure P13-5

(b)
$$\underline{E}^r=\underline{E}^i(Q_R)\cdot\underset{\sim}{R}\sqrt{\frac{\rho^r}{\rho^r+s}}\,e^{-j\beta s}$$

$$\underline{E}^i(Q_R)=\hat{a}_z 10^{-3}\ \text{V/m}$$

$$\underset{\sim}{R}=-\hat{a}_z\hat{a}_z$$

$$\rho^r=+a/3=5\lambda/3$$

$$s=\rho+a=50\lambda+5\lambda=55\lambda$$

$$\underline{E}^r(\rho=50\lambda,\phi=180^\circ)=(\hat{a}_z10^{-3})\cdot(-\hat{a}_z\hat{a}_z)\sqrt{\frac{+5\lambda/3}{5\lambda/3+55\lambda}}\,e^{-j\frac{2\pi}{\lambda}(55\lambda)}$$

$$=-\hat{a}_z10^{-3}\sqrt{\frac{5}{170}}\,e^{-j110\pi}=-\hat{a}_z0.1715\times10^{-3}$$

$\boxed{13.6}$ (a) $\frac{1}{\rho^r}=\frac{1}{\rho_o}+\frac{2}{\rho_a\cos\theta_i}$, $\rho_o=a$, $\rho_a=-a$, $\theta_i=0$

Therefore

$$\frac{1}{\rho^r}=\frac{1}{a}-\frac{2}{a}=-\frac{1}{a}\Rightarrow \rho^r=-a$$

Thus the caustic is at the origin of the circle.

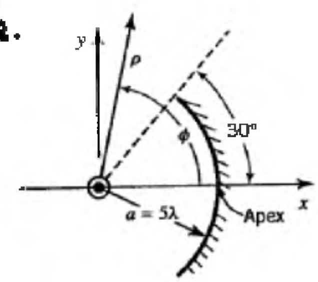


Figure P13-6

(b)
$$\underline{E}^r=\underline{E}^i(Q_R)\cdot\underset{\sim}{R}\sqrt{\frac{\rho^r}{\rho^r+s}}\,e^{-j\beta s}$$

$$\underline{E}^i(Q_R)=\hat{a}_z10^{-3}\ \text{V/m}$$

$$\underset{\sim}{R}=-\hat{a}_z\hat{a}_z$$

$$\rho^r=-a=-5\lambda$$

$$s=\rho+a=50\lambda+5\lambda=55\lambda$$

$$\underline{E}^r(\rho=50\lambda,\phi=180^\circ)=(\hat{a}_z10^{-3})\cdot(-\hat{a}_z\hat{a}_z)\sqrt{\frac{-5\lambda}{-5\lambda+55\lambda}}\,e^{-j\frac{2\pi}{\lambda}(55\lambda)}$$

$$=-\hat{a}_z10^{-3}\sqrt{-\frac{5}{50}}\,e^{-j110\pi}=-\hat{a}_z0.3162\times10^{-3}e^{\pm j\pi/2}$$

$\boxed{13.7}$

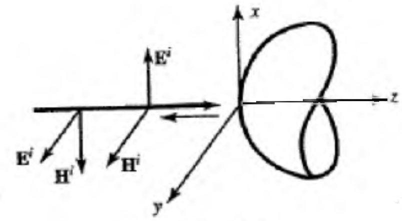


Figure P13-7

According to (13-20)

$$\underline{E}^r(s) = \underline{E}^i(z=0)\cdot \overline{R}\sqrt{\frac{\rho_1^r\rho_2^r}{(\rho_1^r+s)(\rho_2^r+s)}}\, e^{-j\beta s}$$

$$\overset{s\gg\rho_1^r\rho_2^r}{\simeq} \underline{E}^i(z=0)\cdot \overline{R}\,\frac{\sqrt{\rho_1^r\rho_2^r}}{s}e^{-j\beta s} = \underline{E}^i(z=0)\cdot\overline{R}\sqrt{\rho_1^r\rho_2^r}\,\frac{e^{-j\beta s}}{s}$$

According to (13-23a)

$$\rho_1^r\rho_2^r = \frac{R_1R_2}{4}$$

(a) $\underline{E}_{\parallel}^i = \hat{a}_x e^{-j\beta z}\Big|_{z=0} = \hat{a}_x,\ \overline{R}_{\parallel} = -\hat{a}_x\hat{a}_x$

$$\underline{E}^s = \hat{a}_x\cdot(-\hat{a}_x\hat{a}_x)\sqrt{\rho_1^r\rho_2^r}\,\frac{e^{-j\beta s}}{s} = -\hat{a}_x\sqrt{\frac{R_1R_2}{4}}\,\frac{e^{-j\beta s}}{s} = -\hat{a}_x\sqrt{\frac{10\lambda(19.6\lambda)}{4}}\,\frac{e^{-j\beta s}}{s}$$

$$\simeq -\hat{a}_x\sqrt{\frac{196\lambda^2}{4}}\,\frac{e^{-j\beta s}}{s} = -\hat{a}_x 7\lambda\frac{e^{-j\beta s}}{s}$$

$$\sigma = \lim_{s\to\infty}\left[4\pi s^2\frac{|\underline{E}^s|^2}{|\underline{E}^i|}\right] = 4\pi s^2\left[\frac{(7\lambda)^2}{s^2}\right] = 4\pi(49)\lambda^2 = 196\pi\lambda^2$$

$$\frac{\sigma}{\lambda^2} = 196\pi = 615.752 = 10\log_{10}(615.752) = 27.894\ \text{dB}$$

$$\frac{\sigma}{\lambda^2} = 615.752 = 27.894\ \text{dB}$$

(b) $\underline{E}_{\perp}^i = \hat{a}_y e^{-j\beta z}\Big|_{z=0} = \hat{a}_y,\ \overline{R}_{\perp} = -\hat{a}_y\hat{a}_y$

$$\underline{E}^s \simeq \hat{a}_y\cdot(-\hat{a}_y\hat{a}_y)\sqrt{\rho_1^r\rho_2^r}\,\frac{e^{-j\beta s}}{s} = -\hat{a}_y\sqrt{\rho_1^r\rho_2^r}\,\frac{e^{-j\beta s}}{s} = -\hat{a}_y\sqrt{\frac{R_1R_2}{4}}\,\frac{e^{-j\beta s}}{s}$$

$$\underline{E}^s \simeq -\hat{a}_y\sqrt{\frac{196\lambda^2}{4}}\,\frac{e^{-j\beta s}}{s} = -\hat{a}_y 7\lambda\frac{e^{-j\beta s}}{s}$$

$$\sigma = \lim_{s\to\infty}\left[4\pi s^2\frac{|\underline{E}|^2}{|\underline{E}^i|^2}\right] = 4\pi s^2\left[\frac{(7\lambda)^2}{s^2}\right] = 196\pi\lambda^2$$

$$\frac{\sigma}{\lambda^2} = 196\pi = 615.752 = 27.894\ \text{dB}$$

$\boxed{13.8}$

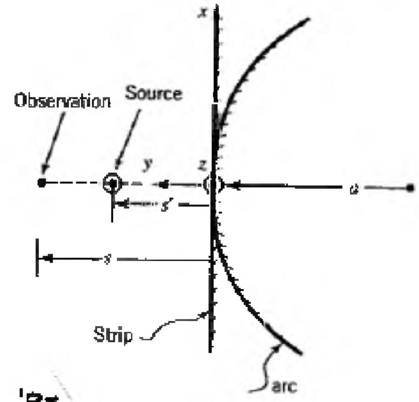


Figure P13-8

$$E^r_{GO}(\text{strip}) = E^i(z=0) \cdot \bar{R} A e^{-j\beta s}$$
$$= E^i(z=0)(-1)\sqrt{\frac{\rho^r}{\rho^r+s}}\, e^{-j\beta s}$$

$$E^r_{GO}(\text{strip}) = -E^i(z=0)\sqrt{\frac{s'}{s'+s}}\, e^{-j\beta s}$$
$$= -E^i(z=0)\sqrt{\frac{s'/s}{s'/s+1}}\, e^{-j\beta s} \overset{s \gg s'}{\simeq} -E^i(z=0)\sqrt{\frac{s'}{s}}\, e^{-j\beta s}$$

$$E^r_{GO}(\text{cylinder}) = E^i(z=0) \cdot \bar{R} A e^{-j\beta s}$$
$$= E^i(z=0)(-1)\sqrt{\frac{\rho^r}{\rho^r+s}}\, e^{-j\beta s} = -E^i(z=0)\sqrt{\frac{\rho^r}{\rho^r+s}}\, e^{-j\beta s}$$

$$\frac{1}{\rho^r} = \frac{1}{\rho_0} + \frac{2}{\rho_a \cos\theta_i} = \frac{1}{s'} + \frac{2}{a} = \frac{a+s'(2)}{as'} \overset{s' \gg a}{\simeq} \frac{2}{a} \Rightarrow \rho^r = \frac{a}{2}$$

$$\sqrt{\frac{\rho^r}{\rho^r+s}} = \sqrt{\frac{a/2}{a/2+s}} \overset{s \gg a}{\simeq} \sqrt{\frac{a}{2s}}$$

$$E^r_{GO}(\text{cylinder}) = -E^i(z=0)\sqrt{\frac{\rho^r}{\rho^r+s}}\, e^{-j\beta s} = -E^i(z=0)\sqrt{\frac{a}{2s}}\, e^{-j\beta s} = -E^i(z=0)\sqrt{\frac{a/2}{s}}\, e^{-j\beta s}$$

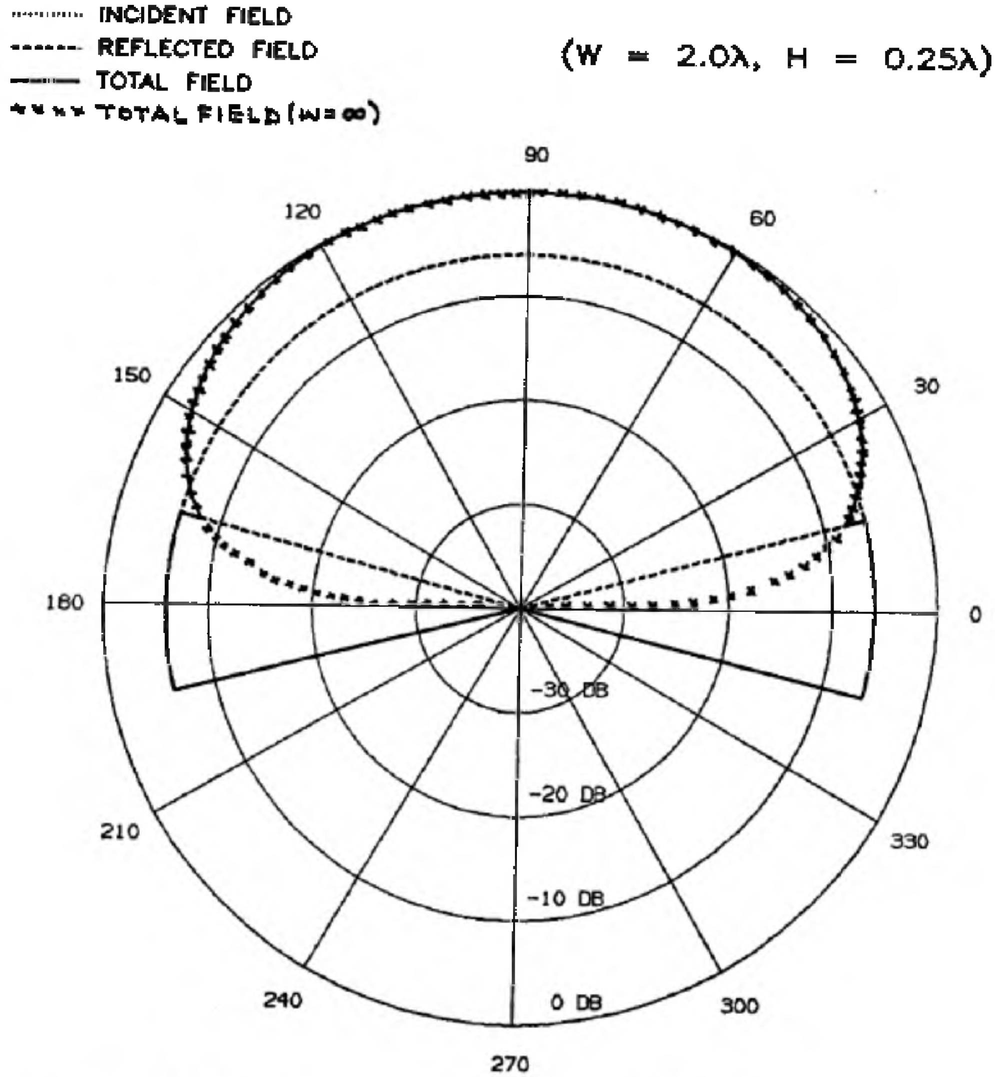
$$\frac{|E^r_{GO}(\text{cylinder})|}{|E^r_{GO}(\text{strip})|} = \left| \frac{-E^i_z(z=0)\sqrt{\frac{a/2}{s}}\, e^{-j\beta s}}{-E^i_z(z=0)\sqrt{s'/s}\, e^{-j\beta s}} \right| = \sqrt{\frac{a/2}{s}\left(\frac{s}{s'}\right)} = \sqrt{\frac{a/2}{s'}}$$

$$\left|\frac{E^r_{GO}(\text{cylinder})}{E^r_{GO}(\text{strip})}\right| = -20\text{ dB} = +20\log_{10}\left|\frac{\sqrt{a/2}}{s'}\right| \Rightarrow \frac{\sqrt{a/2}}{s'} = \frac{1}{10} \Rightarrow \frac{a/2}{s'} = \frac{1}{100}$$

$$\frac{a}{2} = \frac{s'}{100} \Rightarrow a = \left.\frac{2s'}{100}\right|_{s'=10\lambda} = \frac{2(10\lambda)}{100} = \frac{20\lambda}{100} = \frac{\lambda}{5}$$

$$\therefore \quad a = \frac{\lambda}{5}, \quad s' = 10\lambda$$

$\boxed{13.9}$



$\boxed{13.10}$ $$F_1(\beta\rho)\Big|_{SDP_{\pm\pi}} = \frac{e^{-j\pi/4}}{2n\sqrt{2\pi\beta}}\left\{\cot\left[\frac{\pi+(\phi-\phi')}{2n}\right]+\cot\left[\frac{\pi-(\phi-\phi')}{2n}\right]\right\}\frac{e^{-j\beta\rho}}{\sqrt{\rho}}$$

Since
$$\cot(x+y)=\frac{\cos(x+y)}{\sin(x+y)},\quad \cot(x-y)=\frac{\cos(x-y)}{\sin(x-y)}$$

then
$$\cot(x+y)+\cot(x-y)=\frac{\cos(x+y)}{\sin(x+y)}+\frac{\cos(x-y)}{\sin(x-y)}=\frac{\cos(x+y)\sin(x-y)+\sin(x+y)\cos(x-y)}{\sin(x+y)\sin(x-y)}$$

Also
$$\cos(x+y)\sin(x-y)=\frac{1}{2}\left[\sin(2x)-\sin(2y)\right]$$
$$\sin(x+y)\cos(x-y)=\frac{1}{2}\left[\sin(2x)+\sin(2y)\right]$$
$$\sin(x+y)\sin(x-y)=-\frac{1}{2}\left[\cos(2x)-\cos(2y)\right]$$

Thus
$$\cot(x+y)+\cot(x-y)=\frac{\frac{1}{2}\left[\sin(2x)-\sin(2y)\right]+\frac{1}{2}\left[\sin(2x)+\sin(2y)\right]}{-\frac{1}{2}\left[\cos(2x)-\cos(2y)\right]}$$
$$=-\frac{2\sin(2x)}{\cos(2x)-\cos(2y)}$$

cont'd.

13.10 Cont'd. Therefore

$$\cot\Big(\underbrace{\tfrac{\pi}{2n}}_{x}+\underbrace{\tfrac{\phi-\phi'}{2n}}_{y}\Big)+\cot\Big(\underbrace{\tfrac{\pi}{2n}}_{x}-\underbrace{\tfrac{\phi-\phi'}{2n}}_{y}\Big) = -\frac{2\sin(\frac{\pi}{n})}{\cos(\frac{\pi}{n})-\cos(\frac{\phi-\phi'}{n})}$$

Thus $$F_1(\beta\rho) = -\frac{e^{-j\pi/4}}{n\sqrt{2\pi\beta}}\frac{\sin(\frac{\pi}{n})}{\cos(\frac{\pi}{n})-\cos(\frac{\phi-\phi'}{n})}\frac{e^{-j\beta\rho}}{\sqrt{\rho}}$$

13.11 Following the procedure used to evaluate (13-48a) along $SDP_{\pm\pi}$ and along C_T in order to derive, respectively, (13-61) and (13-58) we can show that by simply replacing $\phi-\phi'$ by $\phi+\phi'$ everywhere in that derivation we arrive at (13-62a) when we evaluate (13-48b) along C_T and at (13-62b) when we evaluate (13-48b) along $SDP_{\pm\pi}$.

13.12 (a) For $\rho\geq\rho'$ the Green's function corresponding to (13-38) is given by (11-208b) which for the geometry of Figure 13-13(b) can be written as

$$G(\rho,\rho',\phi,\phi') = \frac{1}{n}\sum_{m=0}^{\infty}\epsilon_m J_{\frac{m}{n}}(\beta\rho')H^{(2)}_{\frac{m}{n}}(\beta\rho)\left\{\cos\left[\tfrac{m}{n}(\phi-\phi')\right]\pm\cos\left[\tfrac{m}{n}(\phi+\phi')\right]\right\}$$

where $$\epsilon_m=\begin{cases}1 & m=0\\ 2 & m\neq 0\end{cases}$$

It is obvious that the above Green's function is identical to that of (13-38) except that ρ and ρ' have been interchanged. Therefore for $\rho\gg\rho'$ the Green's function is identical to (13-40) and (13-40a) but with ρ and ρ' interchanged.

(b) The same reasoning as in Part a is used to justify that the geometrical optics for $\rho\gg\rho'$ are those given by (13-65) but with ρ and ρ' interchanged.

(c) Similarly the diffracted fields for $\rho\gg\rho'$ are identical to of (13-67) but with ρ and ρ' interchanged.

The answers to all three parts above also justified by the application of the reciprocity principle as demonstrated graphically in Figure 13-17.

13.13 Because the ISB and RSB are far away from the observation angle, we can use Keller's forms to compute the diffracted fields.

(a) $E^i_{Go} = e^{j\beta\rho\cos(\phi-\phi')} = e^{j\frac{2\pi}{\lambda}(5\lambda)\cos(180°-45°)} = e^{j10\pi\cos(135°)} = -0.9752+j0.2214$

(b) $E^r_{Go} = 0$ because the reflected GO field does not exist at $\phi = 180°$.

(c) $E^t_{Go} = E^i_{Go} - E^r_{Go} = -0.9752+j0.2214$

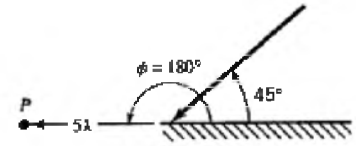


Figure P13-13

(d)
$$E^i_D = \frac{e^{-j(\beta\rho+\pi/4)}}{\sqrt{2\pi\beta\rho}}\,\frac{1}{n}\sin\left(\frac{\pi}{n}\right)\left\{\frac{1}{\cos\left(\frac{\pi}{n}\right)-\cos\left(\frac{\phi-\phi'}{n}\right)}\right\}_{\substack{n=2\\ \phi=180°\\ \phi'=45°}}$$
$$= \frac{e^{-j(\frac{2\pi}{\lambda}5\lambda+\pi/4)}}{2\sqrt{2\pi(\frac{2\pi}{\lambda})5\lambda}}\left\{-\frac{1}{\cos\left(\frac{135}{2}\right)}\right\} = -\frac{e^{-j(10\pi+\pi/4)}}{4\pi\sqrt{5}\,(0.3827)} = -\frac{e^{-j10.25\pi}}{10.753}$$
$$E^i_D = -0.093(0.707-j0.707) = -0.0657+j0.0657$$

(e)
$$E^r_D = \frac{e^{-j(\beta\rho+\pi/4)}}{\sqrt{2\pi\beta\rho}}\,\frac{1}{n}\sin\left(\frac{\pi}{n}\right)\left\{\frac{1}{\cos\left(\frac{\pi}{n}\right)-\cos\left(\frac{\phi+\phi'}{n}\right)}\right\}_{\substack{n=2\\ \phi=180°\\ \phi'=45°}}$$
$$E^r_D = \frac{e^{-j(10\pi+\pi/4)}}{4\pi\sqrt{5}}\left\{-\frac{1}{\cos\left(\frac{225°}{2}\right)}\right\} = \frac{e^{-j(10\pi+\pi/4)}}{4\pi\sqrt{5}(0.3827)} = 0.0657-j0.0657$$

(f) $E^t_D = E^i_D - E^r_D = 2E^i_D = -0.1314+j0.1314$

(g) $E^t = E^t_{Go}+E^t_D = (-0.9752+j0.2214)+(-0.1314+j0.1314) = -1.1066+j0.3568$

13.14 For the hard polarization only the total diffracted field and the total field are different from those of the soft polarization. From Problem 13.13

(a) $E^i_{Go} = -0.9752+j0.2214$

(b) $E^r_{Go} = 0$

(c) $E^t_{Go} = E^i_{Go}+E^r_{Go} = -0.9752+j0.2214$

(d) $E^i_D = -0.0657+j0.0657$

(e) $E^r_D = 0.0657-j0.0657$

(f) $E^t_D = E^i_D+E^r_D = 0$

(g) $E^t = E^t_{Go}+E^t_D = -0.9752+j0.2214$

$\boxed{13.15}$ The solution of this problem can be obtained by referring to the reciprocity principle of Figure 13-17 and to the solutions of Problems 13.12 and 13.13. Doing this, we can write the following:

(a) $E^{i}_{G0} = -0.9752 + j0.2214$

(b) $E^{r}_{G0} = 0$

(c) $E^{t}_{G0} = E^{i}_{G0} - E^{r}_{G0} = -0.9752 + j0.2214$

(d) $E^{i}_{D} = -0.0657 + j0.0657$

(e) $E^{r}_{D} = 0.0657 - j0.0657$

(f) $E^{t}_{D} = E^{i}_{D} - E^{r}_{D} = 2E^{i}_{D} = -0.1314 + j0.1314$

(g) $E^{t} = E^{t}_{G0} + E^{t}_{D} = -1.1066 + j0.3568$

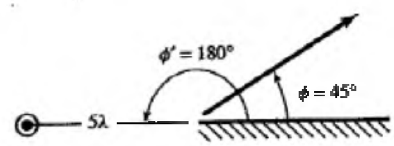


Figure P13-15

$\boxed{13.16}$ The solution of this problem can be obtained by referring to the reciprocity principle of Figure 13-17 and to the solutions of Problems 13.12 and 13.14. Doing this, we can write the following:

(a) $E^{i}_{G0} = -0.9752 + j0.2214$

(b) $E^{r}_{G0} = 0$

(c) $E^{t}_{G0} = E^{i}_{G0} + E^{r}_{G0} = -0.9752 + j0.2214$

(d) $E^{i}_{D} = -0.0657 + j0.0657$

(e) $E^{r}_{D} = 0.0657 - j0.0657$

(f) $E^{t}_{D} = E^{i}_{D} + E^{r}_{D} = 0$

(g) $E^{t} = E^{t}_{G0} + E^{t}_{D} = -0.9752 + j0.2214$

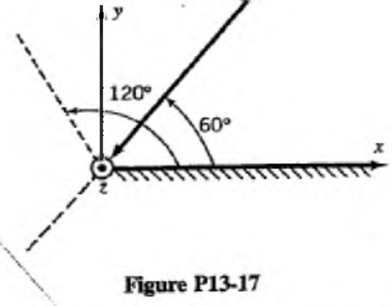


Figure P13-17

$\boxed{13.17}$ $\underline{E} = \hat{a}_z E^i$ (soft polarization)

Reflection Shadow Boundary (RSB):

RSB: $\pi - \phi' = 180° - 60° = 120°$

(a) $\underline{E}^i_{GO}(\phi = 120^{°-}) = \hat{a}_z E^i e^{j\beta\rho\cos(\phi-\phi')}\Big|_{\substack{\phi=120° \\ \rho=5.5\lambda,\ \phi'=60°}}$

$= \hat{a}_z E^i e^{j\frac{2\pi}{\lambda}(5.5\lambda)\cos(120-60)} = \hat{a}_z e^{j11\pi\cos(60°)} E^i = \hat{a}_z E^i e^{j11\pi(0.5)}$

$\underline{E}^i_{GO}(\phi=120^{°-}) = \hat{a}_z E^i e^{j5.5\pi} = \hat{a}_z E^i e^{j3\pi/2} = -j\hat{a}_z E^i$

$\underline{E}^i_{GO}(\phi=120^{°+}) = \underline{E}^i_{GO}(\phi=120^{°-}) = -j\hat{a}_z E^i$

(b) $\underline{E}^r_{GO}(\phi=120^{°-}) = \hat{a}_z E^i e^{j\beta\rho\cos(\phi+\phi')} = \hat{a}_z E^i e^{j11\pi\cos(120°+60°)} = \hat{a}_z E^i e^{j11\pi\cos(\pi)}$

$= \hat{a}_z E^i e^{-j11\pi} = \hat{a}_z E^i e^{-j\pi} = -\hat{a}_z E^i$

$\underline{E}^r_{GO}(\phi=120^{°+}) = 0$

(c) $\underline{E}^i_D(\phi=120^{°-}) = \text{small} \simeq 0$ because far away for ISB

$\underline{E}^i_D(\phi=120^{°+}) = \text{small} \simeq$ because far away from ISB

} Continuous across RSB

(d) $E^r_D(\phi=120^{°-}) = \hat{a}_z E^i\left[-0.5 e^{-j\beta\rho}\,\mathrm{sgn}(\underset{\varepsilon>0}{\varepsilon})\right] = \hat{a}_z E^i\left[-0.5 e^{-j(\frac{2\pi}{\lambda})(5.5\lambda)}(+1)\right]$

$= \hat{a}_z E^i\left(-0.5 e^{-j11\pi}\right) = \hat{a}_z E^i\left(-0.5 e^{-j\pi}\right) = +\hat{a}_z E^i(0.5)$

$E^r_D(\phi=120^{°-}) = 0.5 E^i \hat{a}_z$

$E^r_D(\phi=120^{°+}) = \hat{a}_z E^i\left[-0.5 e^{-j\beta\rho}\,\mathrm{sgn}\,\underset{\varepsilon<0}{\varepsilon}\right] = -0.5 E^i \hat{a}_z$

(e) $\underline{E}^{total}(\phi=120^{°-}) = \underline{E}^i_{GO}(\phi=120^{°-}) + \underline{E}^r_{GO}(\phi=120^{°-}) + \underline{E}^i_D(\phi=120^{°-}) + \underline{E}^r_D(\phi=120^{°-})$

$= -j\hat{a}_z E^i - \hat{a}_z E^i + 0.5 E^i \hat{a}_z = -\hat{a}_z(0.5+j)E^i$

$\underline{E}^{total}(\phi=120^{°+}) = \underline{E}^i_{GO}(\phi=120^{°+}) + \underline{E}^r_{GO}(\phi=120^{°+}) + \underline{E}^i_D(\phi=120^{°+}) + \underline{E}^r_D(\phi=120^{°+})$

$= -j\hat{a}_z E^i + 0 + 0 + (-0.5 E^i \hat{a}_z) = -\hat{a}_z(0.5+j)E^i$

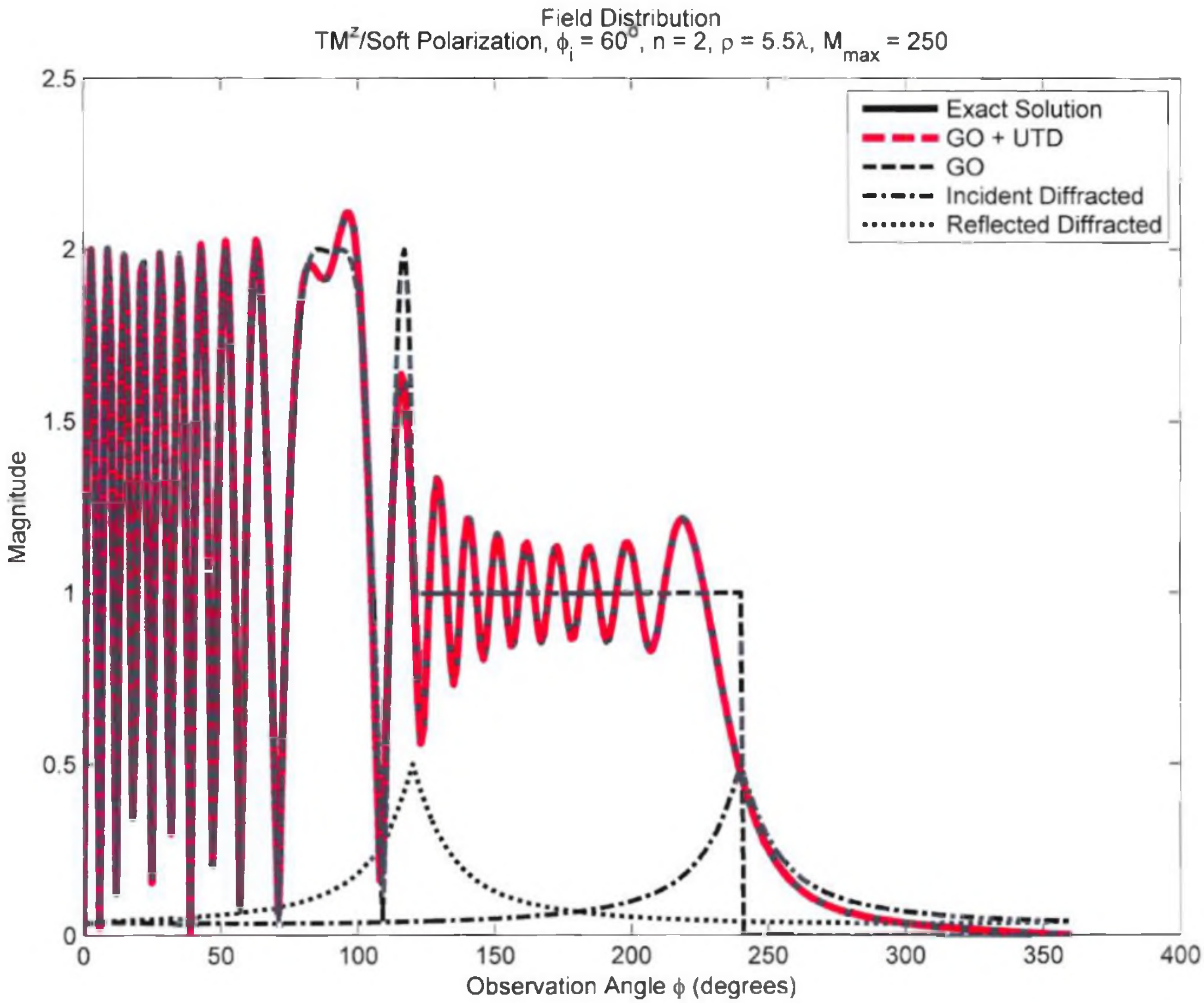
which shows continuity of the field across the RSB ($\phi = 120°$).

A similar procedure can be used to compute the fields and illustrate the continuity of them at the ISB ($\phi = 240°$).

* A complete set of linear and polar amplitude patterns follow, assuming $E^i = 1$.

13.17 cont'd

TM^z – Soft Polarization



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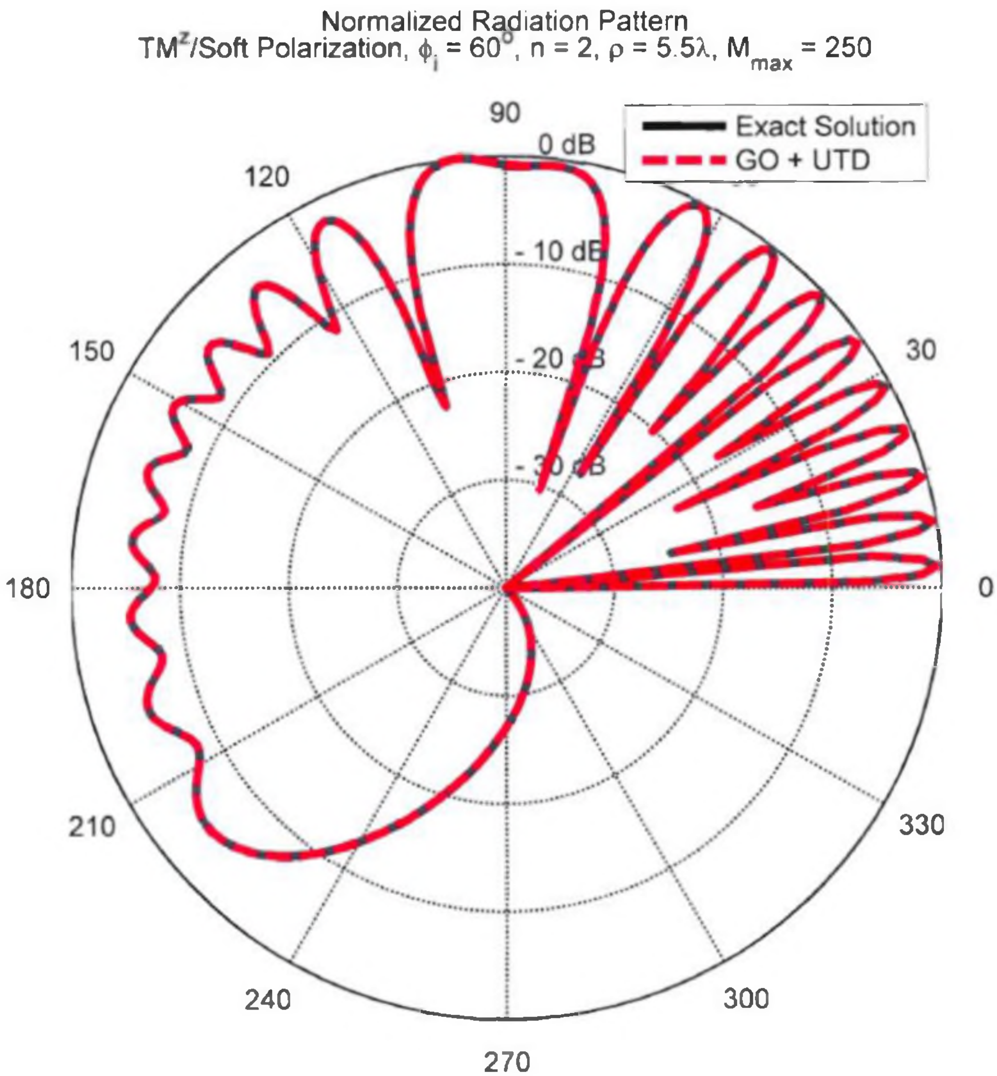
13.17 cont'd

TM^z – Soft Polarization



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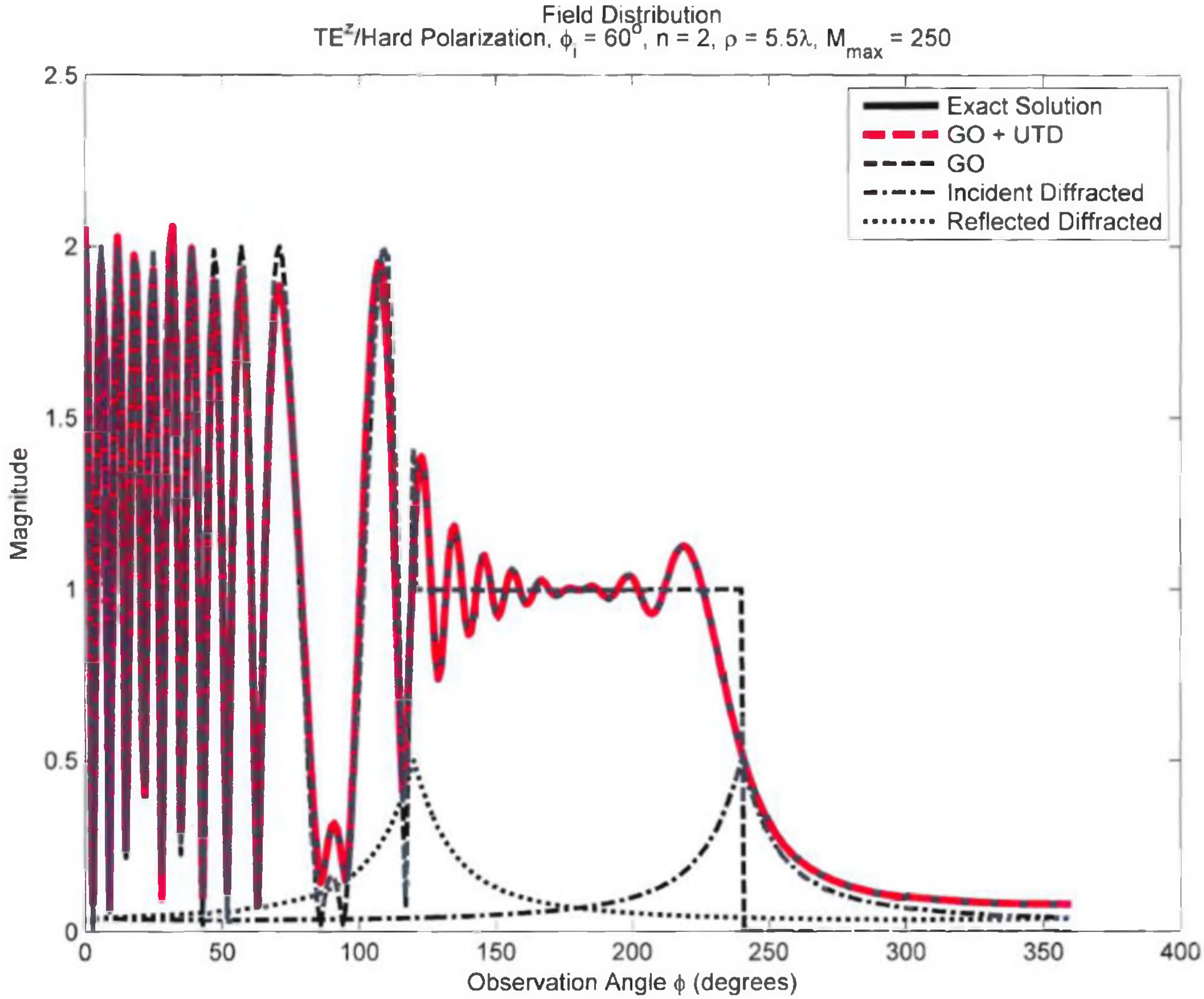
13.17 cont'd

TE^z- Hard Polarization



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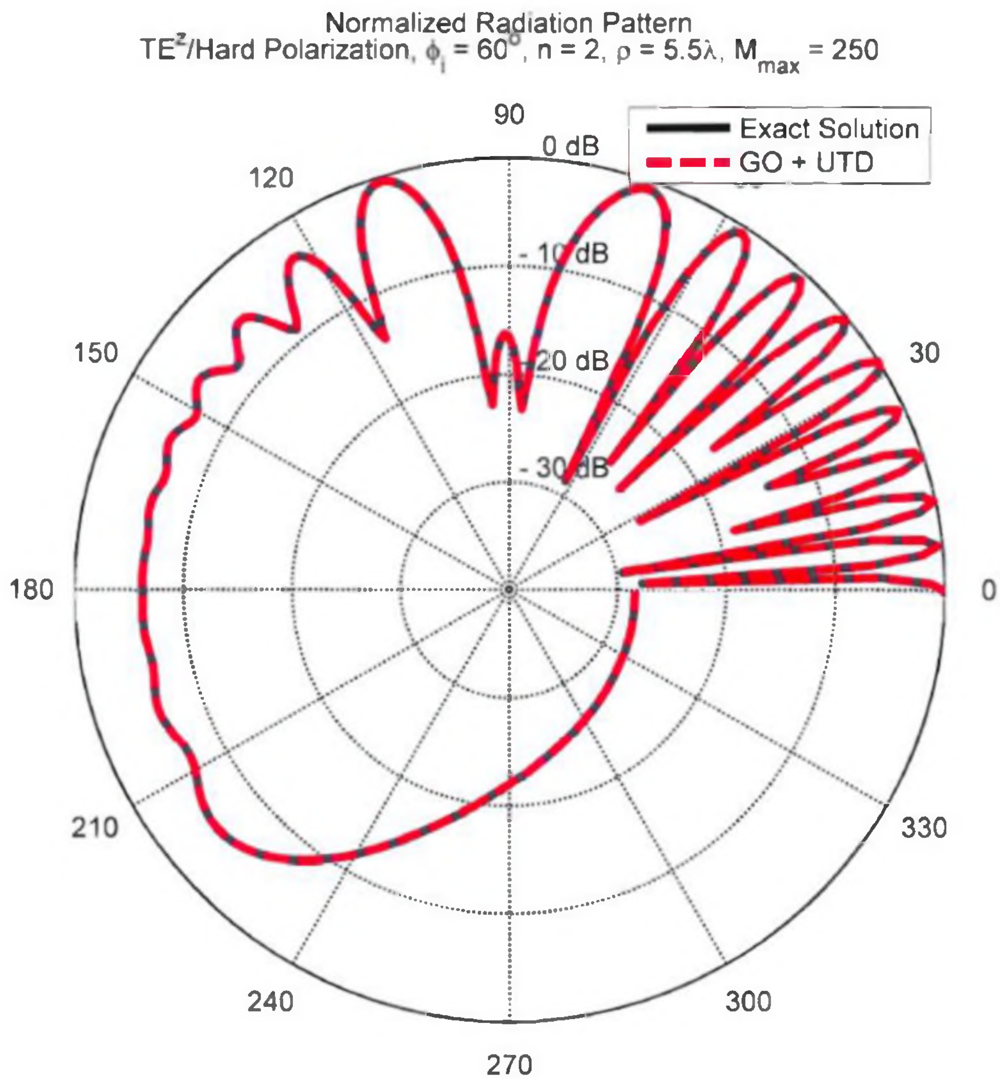
13.17 cont'd

TE^z-Hard Polarization



$$\boxed{13.18} \quad V_B^i = \frac{e^{-j(\pi/4+\beta\rho)}}{\sqrt{2\pi\beta\rho}} \; \frac{\frac{1}{n}\sin\left(\frac{\pi}{n}\right)}{\cos\left(\frac{\pi}{n}\right)-\cos\left(\frac{\phi-\phi'}{n}\right)}$$

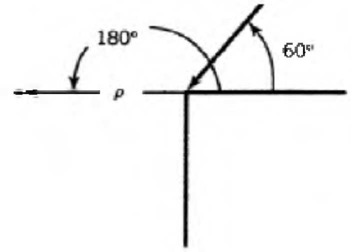


Figure P13-18

$$n = 3/2, \; \phi' = 60^\circ, \; \phi = 180^\circ, \; \rho = 81\lambda$$

(a) $$V_B^i = \frac{e^{-j(\pi/4+\frac{2\pi}{\lambda}81\lambda)}}{\sqrt{2\pi\left(\frac{2\pi}{\lambda}\right)81\lambda}} \; \frac{\frac{2}{3}\sin\left(\frac{2}{3}\pi\right)}{\cos\left(\frac{2}{3}\pi\right)-\cos\left[\frac{2}{3}(180-60)\right]}$$

$$= \frac{2e^{-j\pi/4}}{3(2\pi)(9)} \; \frac{\sin(120^\circ)}{\cos(120^\circ)-\cos(80^\circ)} = \frac{e^{-j\pi/4}}{27\pi} \; \frac{0.866}{-0.5-0.17365} = \frac{e^{-j\pi/4}}{27\pi}(-1.28554)$$

$$\boxed{V_B^i = -0.01516(0.707-j0.707) = -0.01071(1-j)} \quad \underline{\text{GTD}}$$

(b) $$V_B^i = \frac{e^{-j\beta\rho}}{\sqrt{\rho}}\left\{-\frac{e^{-j\pi/4}}{2n\sqrt{2\pi\beta}}\left[\cot\left[\frac{\pi+(\phi-\phi')}{2n}\right]\overbrace{F\left[\beta\rho g^+(\phi-\phi')\right]}^{\simeq 1} + \cot\left[\frac{\pi-(\phi-\phi')}{2n}\right]\overbrace{F\left[\beta\rho g^-(\phi-\phi')\right]}^{\simeq 1}\right]\right\}$$

Away from ISB & RSB

$$\simeq -\frac{e^{-j(\beta\rho+\pi/4)}}{2n\sqrt{2\pi\beta\rho}}\left\{\cot\left[\frac{\pi+(\phi-\phi')}{2n}\right]+\cot\left[\frac{\pi-(\phi-\phi')}{2n}\right]\right\}$$

$$\cot\left[\frac{\pi+(\phi-\phi')}{2n}\right] = \cot\left[\frac{180^\circ+(180^\circ-60^\circ)}{2(3/2)}\right] = \cot\left(\frac{300}{3}\right) = \cot(100^\circ) = -0.17633$$

$$\cot\left[\frac{\pi-(\phi-\phi')}{2n}\right] = \cot\left[\frac{180^\circ-(180^\circ-60^\circ)}{2(3/2)}\right] = \cot\left(\frac{60^\circ}{3}\right) = \cot(20^\circ) = 2.74748$$

$$V_B^i = -\frac{e^{-j\left[\frac{2\pi}{\lambda}(81\lambda)+\frac{\pi}{4}\right]}}{2(3/2)\sqrt{2\pi\left(\frac{2\pi}{\lambda}\right)81\lambda}}\left[-0.17633+2.74748\right] = -\frac{e^{-j\pi/4}}{3(2\pi)(9)}\left[2.57116\right] = -0.01516e^{-j\pi/4}$$

$$\boxed{V_B^i = -0.01516(0.707-j0.707) = -0.01072(1-j)} \quad \underline{\text{UTD}}$$

$\therefore$ $V_B^i(\text{GTD}) = V_B^i(\text{UTD})$ because observations are made away from the ISB and RSB.

$$\xi^- = \phi-\phi' = 180^\circ-60^\circ = 120^\circ = \frac{2\pi}{3}, \; n = 1.5 \Rightarrow \begin{cases} \text{Using Fig. 13-19(a)} \Rightarrow N^- = 0 \\ g^- = 1+\cos(\xi^- - 2\pi n N^-) = 1+\cos(120^\circ) = 0.5 \\ \text{Using Fig. 13-19(b)} \Rightarrow N^+ = 1 \\ g^+ = 1+\cos(\xi^- - 2\pi n N^+) = 1+\cos\left(\frac{2\pi}{3}-2\pi\frac{3}{2}1\right) \\ \quad = 1+\cos\left(\frac{2\pi}{3}-3\pi\right) = 1+\cos\left(-\frac{7\pi}{3}\right) = 1+0.5 \\ g^+ = 1.5 \end{cases}$$

$$\left.\beta\rho g^-\right|_{g^-=0.5} = \frac{2\pi}{\lambda}(81\lambda)(0.5) = 81\pi \gg 1 \quad \text{Thus } F(\beta\rho g^-) = F(81\pi) \simeq 1$$

$$\left.\beta\rho g^+\right|_{g^+=1.5} = \frac{2\pi}{\lambda}(81\lambda)(1.5) = 243\pi \gg 1 \quad \text{Thus } F(\beta\rho g^+) = F(243\pi) \simeq 1$$

Because observations are made far away from ISB

13.19

$\vec{H}^i = \hat{a}_z e^{-j\beta x}$ Half plane diffraction $(n=2)$

(a) $\underline{H}^i = \hat{a}_z e^{-j\beta_0 x}$

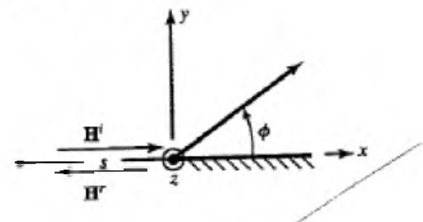


Figure P13-19

(b) $\underline{E}^i = \hat{a}_y \eta_0 e^{-j\beta_0 x}$

(c) $\underline{H}^d = \hat{a}_z (1) D^h(s, \phi, \phi', n=2) \frac{1}{\sqrt{s}} e^{-j\beta_0 s}$

$$D_1^h = \frac{e^{-j\pi/4}}{\sqrt{2\pi\beta}} \frac{1}{n}\sin\left(\frac{\pi}{n}\right)\left[\frac{1}{\cos\left(\frac{\pi}{n}\right)-\cos\left(\frac{\phi-\phi'}{n}\right)} + \frac{1}{\cos\left(\frac{\pi}{n}\right)-\cos\left(\frac{\phi+\phi'}{n}\right)}\right]$$

$$= \frac{e^{-j\pi/4}\sin\left(\frac{\pi}{2}\right)}{2\sqrt{2\pi\beta}}\left[\frac{1}{\cos\left(\frac{\pi}{2}\right)-\cos\left(\frac{0}{2}\right)} + \frac{1}{\cos\left(\frac{\pi}{2}\right)-\cos\left(\frac{2\pi}{2}\right)}\right]_{\substack{n=2\\ \phi=\phi'=180^\circ}}$$

$$D_1^h = \frac{e^{-j\pi/4}}{2\sqrt{2\pi\beta}}\left[-1+1\right] = 0$$

$\underline{H}^d = 0$

(d) $\underline{E}^d = 0$

(e) $\frac{|\underline{H}^d|}{|\underline{H}^i|} = \frac{0}{1} = 0 = 10\log_{10}(0) = -\infty \text{ dB}$

13.20

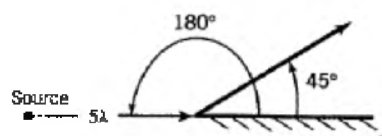


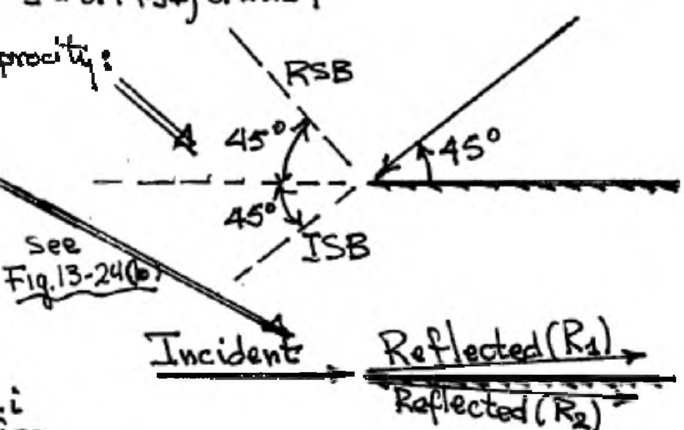
Figure P13-20

(a) $E^i_{G0} = e^{j\beta\rho'\cos(\phi-\phi')} = e^{j\beta\rho\cos(45^\circ-180^\circ)}$

$= e^{j\frac{2\pi}{\lambda}(5\lambda)\cos(-135^\circ)} = e^{j10\pi\cos(-135^\circ)}$

$E^i_{G0} = e^{j10\pi(-0.707)} = e^{-j7.07\pi} = -0.975 + j0.2214$

(b) $E^r_{G0} = 0$ Because of reciprocity:



Also because of RSBs:

$\phi' = \pi - \left.\frac{(2-n)\pi}{2}\right|_{n=2} = \pi - 0 = \pi$

$\phi = \pi - \phi' = \pi - \pi = 0$

(c)

$E^t_{G0} = E^i_{G0} + E^r_{G0} = E^i_{G0} + 0 = E^i_{G0}$

$= -0.975 + j0.2214$

$\boxed{13.20 \text{ cont'd}}$

(d) $$E_D^i = \frac{e^{-j(\beta\rho'+\pi/4)}}{\sqrt{2\pi\beta\rho'}} \frac{\frac{1}{n}\sin\left(\frac{\pi}{n}\right)}{\cos\left(\frac{\pi}{n}\right)-\cos\left(\frac{\phi-\phi'}{n}\right)}\Bigg|_{\substack{n=2,\ \phi=45^\circ \\ \phi'=180^\circ,\ \rho'=5\lambda}} = \frac{e^{-j\left(\frac{2\pi}{\lambda}5\lambda+\frac{\pi}{4}\right)}}{\sqrt{2\pi\left(\frac{2\pi}{\lambda}\right)(5\lambda)}} \frac{\frac{1}{2}\sin\left(\frac{\pi}{2}\right)}{\cos\left(\frac{\pi}{2}\right)-\cos\left(\frac{45^\circ-180^\circ}{2}\right)}$$

$$= \frac{e^{-j(10\pi+\pi/4)}}{2\pi\sqrt{5}\,(2)} \frac{1}{-\cos(135/2)} = -\frac{e^{-j\pi/4}}{4\pi\sqrt{5}} \frac{1}{\cos(-67.5^\circ)} = -\frac{e^{-j\pi/4}}{4\pi\sqrt{5}\,(0.38268)}$$

$$E_D^i = 0.06575(-1+j)$$

(e) $$E_D^r = \frac{e^{-j(\beta\rho'+\pi/4)}}{\sqrt{2\pi\beta\rho'}} \frac{\frac{1}{n}\sin\left(\frac{\pi}{n}\right)}{\cos\left(\frac{\pi}{n}\right)-\cos\left(\frac{\phi+\phi'}{n}\right)} = \frac{e^{-j\pi/4}}{28.0993\cos(112.5^\circ)} = 0.06575(1-j)$$

(f) $$E_D^t = E_D^i - E_D^r = 0.06575(-1+j) - 0.06575(1-j) = 2\left[0.06575(-1+j)\right]$$

$$E_D^t = 0.1315(-1+j) = -0.1315(1-j)$$

(g) $$E^t = E_{GO}^t + E_D^t = (-0.975+j0.2214)+(-0.1315+j0.1315)$$

$$= -1.1065+j0.3529$$

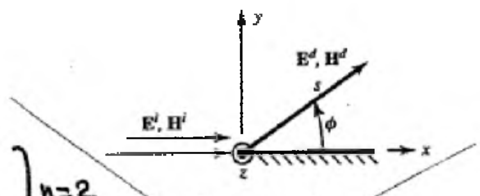


Figure P13-21

$\boxed{13.21}$

(a) $$\underline{E}_D = \hat{a}_z \frac{e^{-j(\beta\rho+\pi/4)}}{\sqrt{2\pi\beta\rho}} \left\{ \frac{\frac{1}{n}\sin\left(\frac{\pi}{n}\right)}{\cos\left(\frac{\pi}{n}\right)-\cos\left(\frac{\phi-\phi'}{n}\right)} - \frac{\frac{1}{n}\sin\left(\frac{\pi}{n}\right)}{\cos\left(\frac{\pi}{n}\right)-\cos\left(\frac{\phi+\phi'}{n}\right)} \right\}_{\substack{n=2 \\ \rho=s=9\lambda \\ \phi'=180^\circ,\ \phi=90^\circ}}$$

$$= \hat{a}_z \frac{e^{-j(\beta s+\pi/4)}\frac{1}{2}\sin\left(\frac{\pi}{2}\right)}{\sqrt{2\pi\beta s}} \left\{ -\frac{1}{\cos\left(\frac{90-180}{2}\right)} + \frac{1}{\cos\left(\frac{90+180}{2}\right)} \right\}$$

$$= \hat{a}_z \frac{e^{-j\left(\frac{2\pi}{\lambda}9\lambda+\frac{\pi}{4}\right)}}{2\sqrt{2\pi\left(\frac{2\pi}{\lambda}\right)9\lambda}} \left\{ -\frac{1}{\cos(-45^\circ)} + \frac{1}{\cos(135^\circ)} \right\} = \hat{a}_z \frac{e^{-j\pi/4}}{2(2\pi)(3)} \left\{ -\frac{1}{0.707} - \frac{1}{0.707} \right\}$$

$$\underline{E}_D = -\hat{a}_z \frac{e^{-j\pi/4}(2)}{2(2\pi)(3)(0.707)} = -\hat{a}_z \frac{0.707(1-j)}{2\pi(3)(0.707)} = -\hat{a}_z \frac{1}{6\pi}(1-j) = -\hat{a}_z 0.053(1-j)$$

(b) $$\underline{H}_D = \hat{a}_z \frac{e^{-j(\beta\rho+\pi/4)}}{\sqrt{2\pi\beta\rho}} \left\{ \frac{\frac{1}{n}\sin\left(\frac{\pi}{n}\right)}{\cos\left(\frac{\pi}{n}\right)-\cos\left(\frac{\phi-\phi'}{n}\right)} + \frac{\frac{1}{n}\sin\left(\frac{\pi}{n}\right)}{\cos\left(\frac{\pi}{n}\right)-\cos\left(\frac{\phi+\phi'}{n}\right)} \right\}_{\substack{n=2 \\ \rho=s=9\lambda \\ \phi'=180^\circ,\ \phi=90^\circ}}$$

Based on the calculation of Part (a), by changing the minus (−) sign within the brackets to a plus (+), we can write the diffracted magnetic field as

$$\underline{H}_D = \hat{a}_z \frac{e^{-j\pi/4}}{2(2\pi)(3)} \left\{ -\frac{1}{0.707} + \frac{1}{0.707} \right\} = 0$$

$\boxed{13.22}$ (a) $\underline{E}^i_{G0}\Big|_{\phi=225^-} = \hat{a}_z e^{j\beta\rho\cos(\phi-\phi')} = \hat{a}_z e^{j\frac{2\pi}{\lambda}5.5\lambda\cos(225^\circ-45^\circ)} = \hat{a}_z e^{j11\pi} = -\hat{a}_z$

(b) $\underline{E}^i_{G0}\Big|_{\phi=225^+} = 0$

(c) $\underline{E}^i_D\Big|_{\phi=225^-} \simeq \hat{a}_z\left[-0.5e^{-j\beta\rho}\operatorname{sgn}(\epsilon)\right] = \hat{a}_z\left(-0.5e^{-j11\pi}\operatorname{sgn}\epsilon\right) = \hat{a}_z(0.5),\quad \epsilon>0$

(d) $\underline{E}^i_D\Big|_{\phi=225^+} \simeq \hat{a}_z\left[-0.5e^{-j\beta\rho}\operatorname{sgn}(\epsilon)\right] = \hat{a}_z\left(-0.5e^{-j11\pi}\operatorname{sgn}\epsilon\right) = -\hat{a}_z(0.5),\quad \epsilon<0$

(e) $\underline{E}^t\Big|_{\phi=225^-} = \underline{E}^i_{G0}+\underline{E}^t_D = \underline{E}^i_{G0}+\underline{E}^i_D-\overset{\simeq 0}{\cancel{\underline{E}^r_D}} \simeq \underline{E}^i_{G0}+\underline{E}^i_D = \hat{a}_z(-1+0.5) = -\hat{a}_z(0.5)$

(f) $\underline{E}^t\Big|_{\phi=225^+} = \underline{E}^i_{G0}+\underline{E}^t_D = \underline{E}^i_{G0}+\underline{E}^i_D-\overset{\simeq 0}{\cancel{\underline{E}^r_D}} = \underline{E}^i_{G0}+\underline{E}^i_D = \hat{a}_z(0-0.5) = -\hat{a}_z(0.5)$

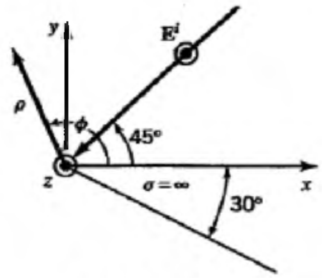


Figure P13-22

$\boxed{13.23}$

(a) $\underline{H}^i_{G0}\Big|_{\phi=225^-} = \hat{a}_z e^{j\beta\rho\cos(\phi-\phi')} = \hat{a}_z e^{j\frac{2\pi}{\lambda}5.5\lambda\cos(225^\circ-45^\circ)} = \hat{a}_z e^{-j11\pi} = -\hat{a}_z$

(b) $\underline{H}^i_{G0}\Big|_{\phi=225^+} = 0$

(c) $\underline{H}^i_D\Big|_{\phi=225^-} \simeq \hat{a}_z\left[-0.5e^{-j\beta\rho}\operatorname{sgn}\epsilon\right] = \hat{a}_z\left(-0.5e^{-j11\pi}\operatorname{sgn}\epsilon\right) = \hat{a}_z 0.5,\quad \epsilon>0$

(d) $\underline{H}^i_D\Big|_{\phi=225^+} \simeq \hat{a}_z\left[-0.5e^{-j\beta\rho}\operatorname{sgn}\epsilon\right] = \hat{a}_z\left(-0.5e^{-j11\pi}\operatorname{sgn}\epsilon\right) = -\hat{a}_z 0.5,\quad \epsilon<0$

(e) $\underline{H}^t\Big|_{\phi=225^-} = \underline{H}^i_{G0}+\underline{H}^t_D = \underline{H}^i_{G0}+\underline{H}^i_D+\overset{\simeq 0}{\cancel{\underline{H}^r_D}} \simeq \underline{H}^i_{G0}+\underline{H}^i_D = \hat{a}_z(0.5-1) = -\hat{a}_z 0.5$

(f) $\underline{H}^t\Big|_{\phi=225^+} = \underline{H}^i_{G0}+\underline{H}^t_D = \underline{H}^i_{G0}+\underline{H}^i_D+\overset{\simeq 0}{\cancel{\underline{H}^r_D}} \simeq \underline{H}^i_{G0}+\underline{H}^i_D = \hat{a}_z(0-0.5) = -\hat{a}_z 0.5$

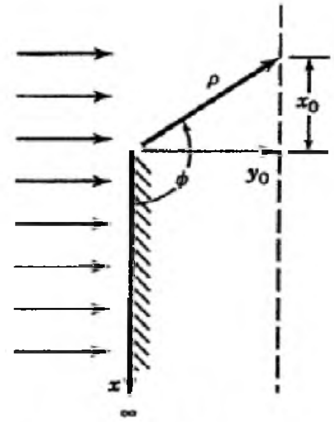


Figure P13-24

$\boxed{13.24}$ (a)

$$E^i_{G0} = \begin{cases} e^{j\beta\rho\cos(\phi-90^\circ)} & x_0 \geq 0 \\ 0 & x_0 < 0 \end{cases}$$

$$E^r_{G0} = 0 \qquad -\infty < x_0 < +\infty$$

$$E^i_D = V^i_B = D^i \frac{e^{-j\beta\rho}}{\sqrt{\rho}} = -\frac{e^{-j(\beta\rho+\pi/4)}}{2n\sqrt{2\pi\beta\rho}} \Big\{ C^+(\xi^-, n=2) F[\beta\rho g^+(\xi^-)] + C^-(\xi^-, n=2) F[\beta\rho g^-(\xi^-)] \Big\}_{n=2}$$

$$E^r_D = V^r_B = D^r \frac{e^{-j\beta\rho}}{\sqrt{\rho}} = -\frac{e^{-j(\beta\rho+\pi/4)}}{2n\sqrt{2\pi\beta\rho}} \Big\{ C^+(\xi^+, n=2) F[\beta\rho g^+(\xi^+)] + C^-(\xi^+, n=2) F[\beta\rho g^-(\xi^+)] \Big\}_{n=2}$$

$$\underline{E}^t = E^i_{G0} + E^i_D - E^r_D = e^{j\beta\rho\cos(\phi-90^\circ)} + E^i_D - E^r_D, \qquad x_0 \geq 0$$

$$\underline{E}^t = E^i_{G0} + E^i_D - E^r_D = 0 + E^i_D - E^r_D, \qquad x_0 < 0$$

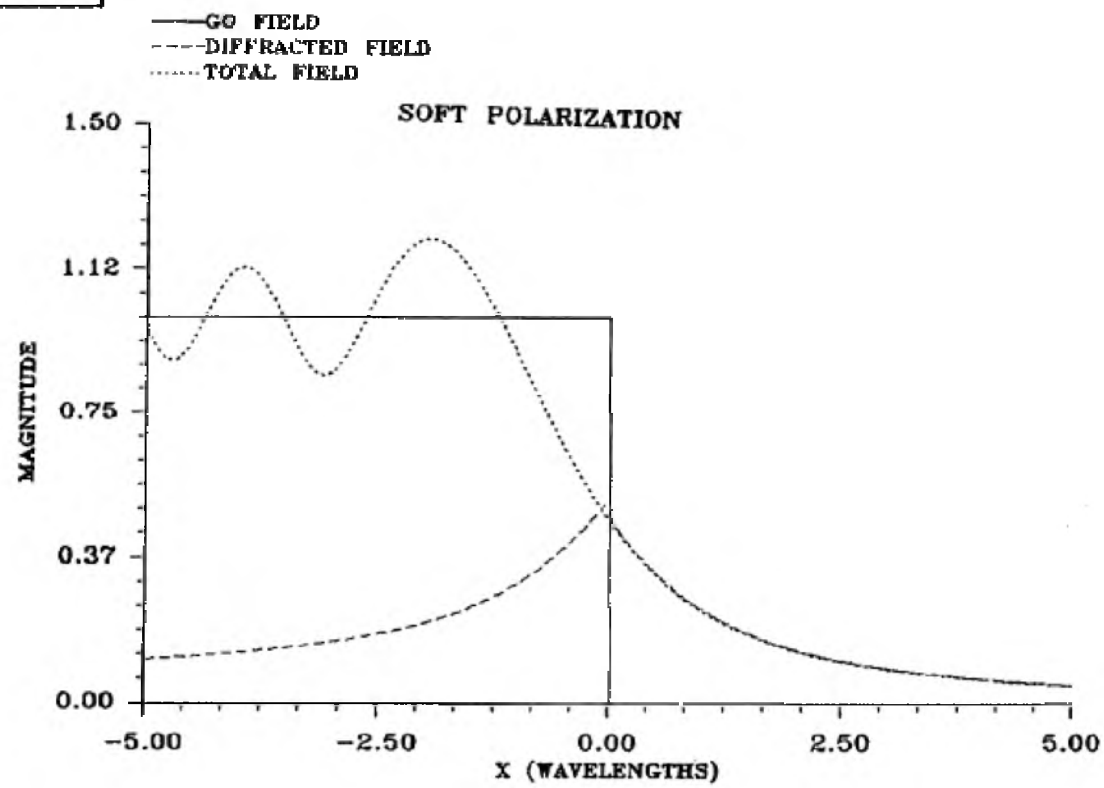
where $\rho = \sqrt{x_0^2 + y_0^2}$, $\phi = 90^\circ + \tan^{-1}\left(\frac{x_0}{y_0}\right)$

(b) See Graphs on the next page.

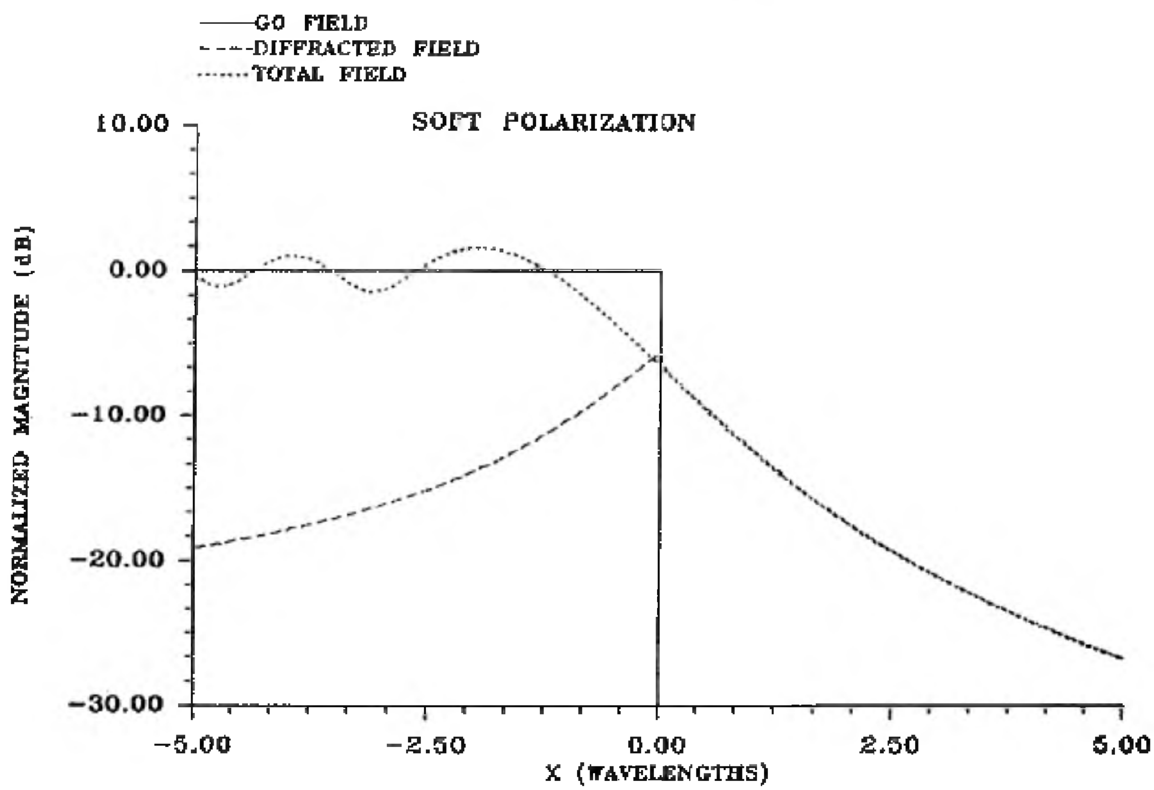
(c) At $y_0 = 5\lambda$ and $x_0 = 0$ the total field $|E^t| = 0.475$ V/m while $|E^t_D| = 0.526$ V/m. When $y_0 = 500\lambda$ and $x_0 = 0$, the field is $|E^t| = 0.497$ V/m while $|E^t_D| = 0.502$ V/m. We see that as y_0 becomes very large $|E^t|$ approaches the expected value of 0.5 or −6 dB.

cont'd.

13.24 cont'd.



GO FIELD
DIFFRACTED FIELD
TOTAL FIELD
SOFT POLARIZATION
1.50
1.12
0.75
0.37
0.00
MAGNITUDE
-5.00
-2.50
0.00
2.50
5.00
X (WAVELENGTHS)



GO FIELD
DIFFRACTED FIELD
TOTAL FIELD
SOFT POLARIZATION
10.00
0.00
-10.00
-20.00
-30.00
NORMALIZED MAGNITUDE (dB)
-5.00
-2.50
0.00
2.50
5.00
X (WAVELENGTHS)

$\boxed{13.25}$

(a) $$E^i_{Go} = \begin{cases} e^{j\beta\rho\cos(\phi-90^\circ)} & x_o \geq 0 \\ 0 & x_o < 0 \end{cases}$$

$$E^r_{Go} = 0 \qquad -\infty \leq x_o \leq +\infty$$

$$E^i_D = V^i_B = D^i \frac{e^{-j\beta\rho}}{\sqrt{\rho}} = -\frac{e^{-j(\beta\rho+\pi/4)}}{2n\sqrt{2\pi\beta\rho}} \left\{ C^+(\xi^-, n=2) F[\beta\rho g^+(\xi^-)] + C^-(\xi^-, n=2) F[\beta\rho g^-(\xi^-)] \right\}_{n=2}$$

$$E^i_D = V^r_B = D^r \frac{e^{-j\beta\rho}}{\sqrt{\rho}} = -\frac{e^{-j(\beta\rho+\pi/4)}}{2n\sqrt{2\pi\beta\rho}} \left\{ C^+(\xi^+, n=2) F[\beta\rho g^+(\xi^+)] + C^-(\xi^+, n=2) F[\beta\rho g^-(\xi^+)] \right\}_{n=2}$$

$$E^t = E^i_{Go} + E^i_D + E^r_D = e^{j\beta\rho\cos(\phi-90^\circ)} + E^i_D + E^r_D \qquad , x_o \geq 0$$

$$E^t = E^i_{Go} + E^i_D + E^r_D = 0 + E^i_D + E^r_D \qquad , x_o < 0$$

where $\rho = \sqrt{x_o^2 + y_o^2}$, $\phi = 90^\circ + \tan^{-1}\left(\frac{x_o}{y_o}\right)$

(b) See Graphs on the next page

(c) At $y_o = 5\lambda$ and $x_o = 0$ the total field $|E^t| = 0.526$ while $|E^t_D| = 0.475$. When $y_o = 500\lambda$ and $x_o = 0$, the total field is $|E^t| = 0.502$ while $|E^t_D| = 0.497$ V/m. We see that as y_o becomes very large $|E^t|$ approaches the expected value of 0.5 or -6 dB.

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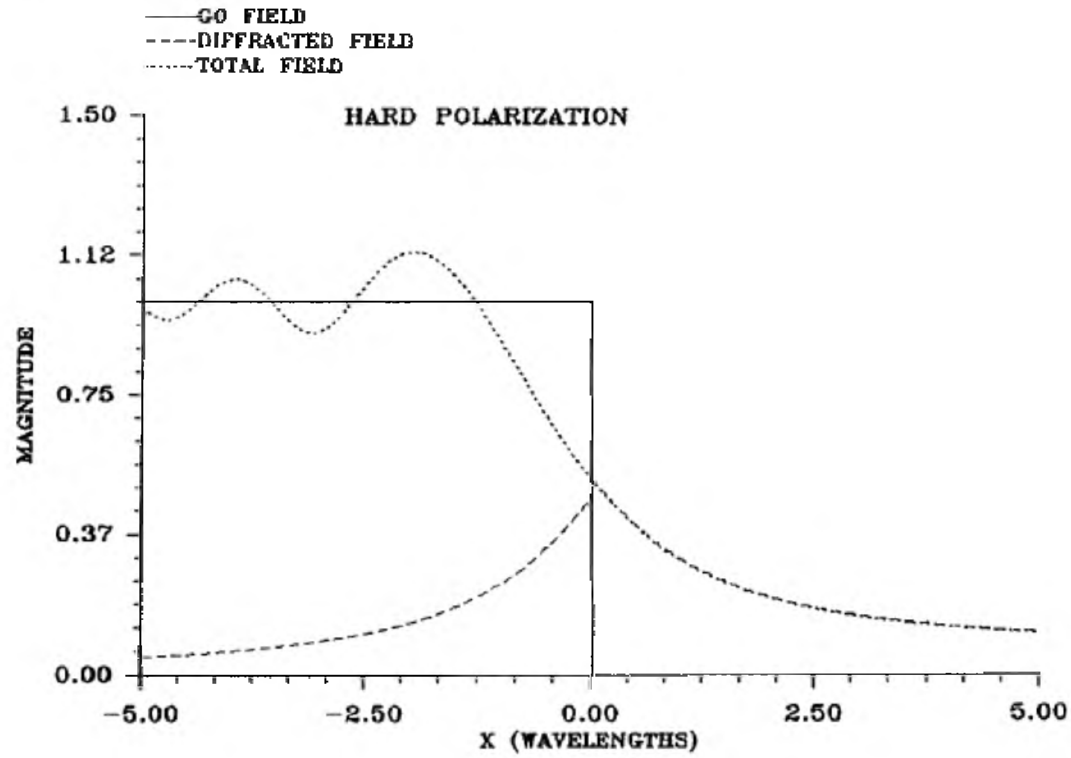
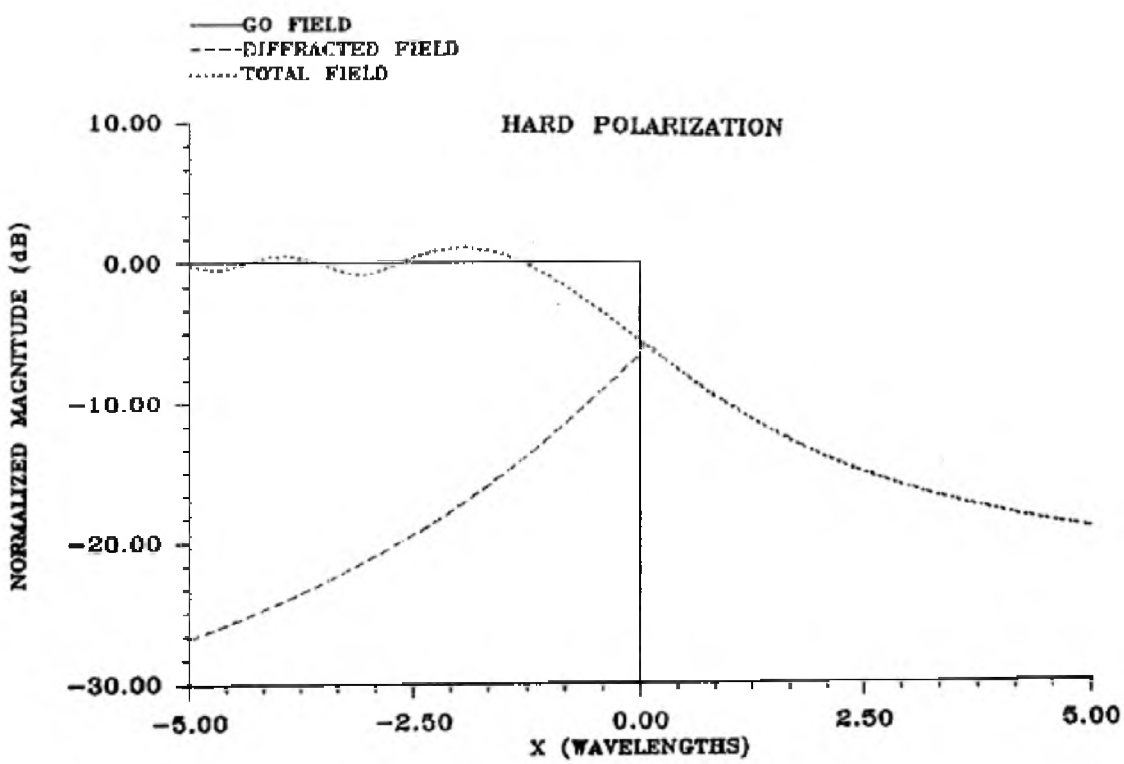
13.25 Cont'd.



GO FIELD
DIFFRACTED FIELD
TOTAL FIELD
HARD POLARIZATION
1.50
1.12
0.75
0.37
0.00
MAGNITUDE
-5.00
-2.50
0.00
2.50
5.00
X (WAVELENGTHS)



GO FIELD
DIFFRACTED FIELD
TOTAL FIELD
HARD POLARIZATION
10.00
0.00
-10.00
-20.00
-30.00
NORMALIZED MAGNITUDE (dB)
-5.00
-2.50
0.00
2.50
5.00
X (WAVELENGTHS)

$\boxed{13.26}$

$\phi' = \frac{n\pi}{2}$

$0 \le \phi \le n\pi$

(a) Away from the incident and reflected shadow boundaries we can use Keller's diffraction coefficients and functions.

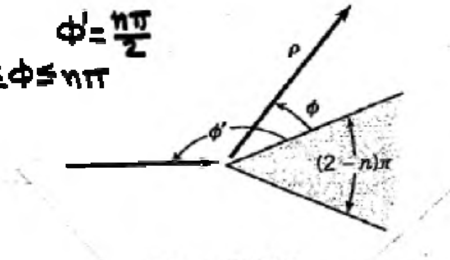


Figure P13-26

$\phi' = \frac{n\pi}{2}$

$0 \le \phi \le n\pi$

$$V_B^i = \frac{e^{-j(\beta\rho+\pi/4)}}{\sqrt{2\pi\beta\rho}} \frac{\frac{1}{n}\sin\left(\frac{\pi}{n}\right)}{\cos\left(\frac{\pi}{n}\right)-\cos\left(\frac{\phi-\phi'}{n}\right)} = D^i \frac{e^{-j\beta\rho}}{\sqrt{\rho}}$$

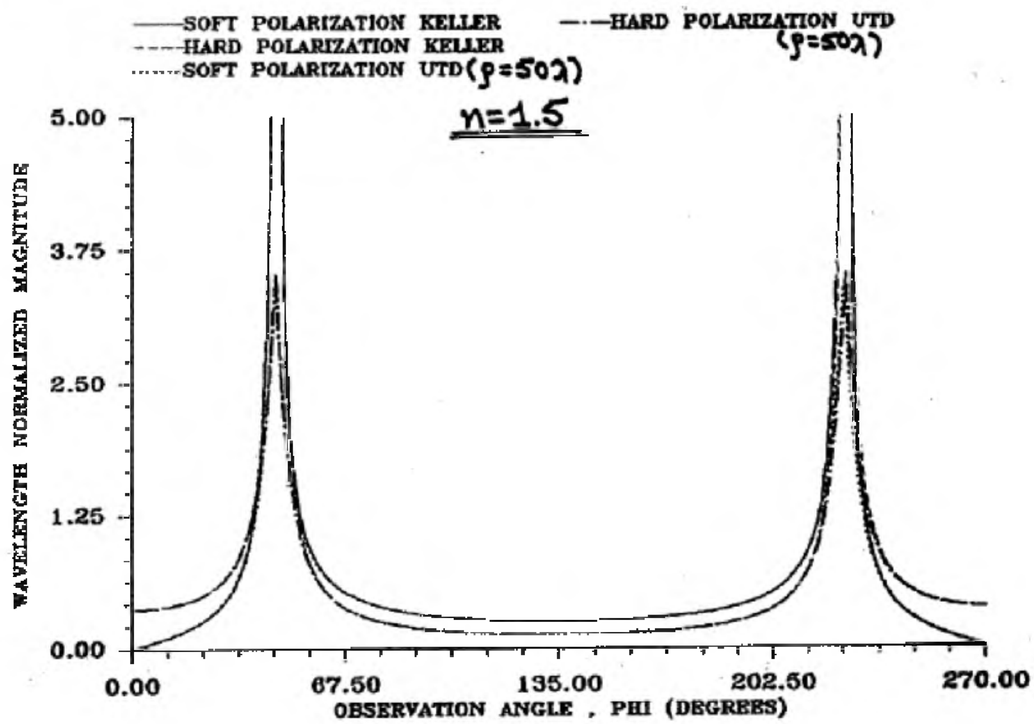
$$V_B^r = \frac{e^{-j(\beta\rho+\pi/4)}}{\sqrt{2\pi\beta\rho}} \frac{\frac{1}{n}\sin\left(\frac{\pi}{n}\right)}{\cos\left(\frac{\pi}{n}\right)-\cos\left(\frac{\phi+\phi'}{n}\right)} = D^r \frac{e^{-j\beta\rho}}{\sqrt{\rho}}$$

$$V_B^{s,h} = \frac{e^{-j(\beta\rho+\pi/4)}}{n\sqrt{2\pi\beta\rho}} \sin\left(\frac{\pi}{n}\right)\left\{\frac{1}{\cos\left(\frac{\pi}{n}\right)-\cos\left(\frac{\phi-\phi'}{n}\right)} \mp \frac{1}{\cos\left(\frac{\pi}{n}\right)-\cos\left(\frac{\phi+\phi'}{n}\right)}\right\} = D^{s,h}\frac{e^{-j\beta\rho}}{\sqrt{\rho}}$$

$$D^{s,h} = \frac{e^{-j\pi/4}}{n\sqrt{2\pi\beta}} \sin\left(\frac{\pi}{n}\right)\left\{\frac{1}{\cos\left(\frac{\pi}{n}\right)-\cos\left(\frac{\phi-\phi'}{n}\right)} \mp \frac{1}{\cos\left(\frac{\pi}{n}\right)-\cos\left(\frac{\phi+\phi'}{n}\right)}\right\}$$

$$D^{s,h}\sqrt{\lambda} = \frac{e^{-j\pi/4}}{n(2\pi)^{1/2}} \sin\left(\frac{\pi}{n}\right)\left\{\frac{1}{\cos\left(\frac{\pi}{n}\right)-\cos\left(\frac{\phi-\phi'}{n}\right)} \mp \frac{1}{\cos\left(\frac{\pi}{n}\right)-\cos\left(\frac{\phi+\phi'}{n}\right)}\right\}$$

(b)



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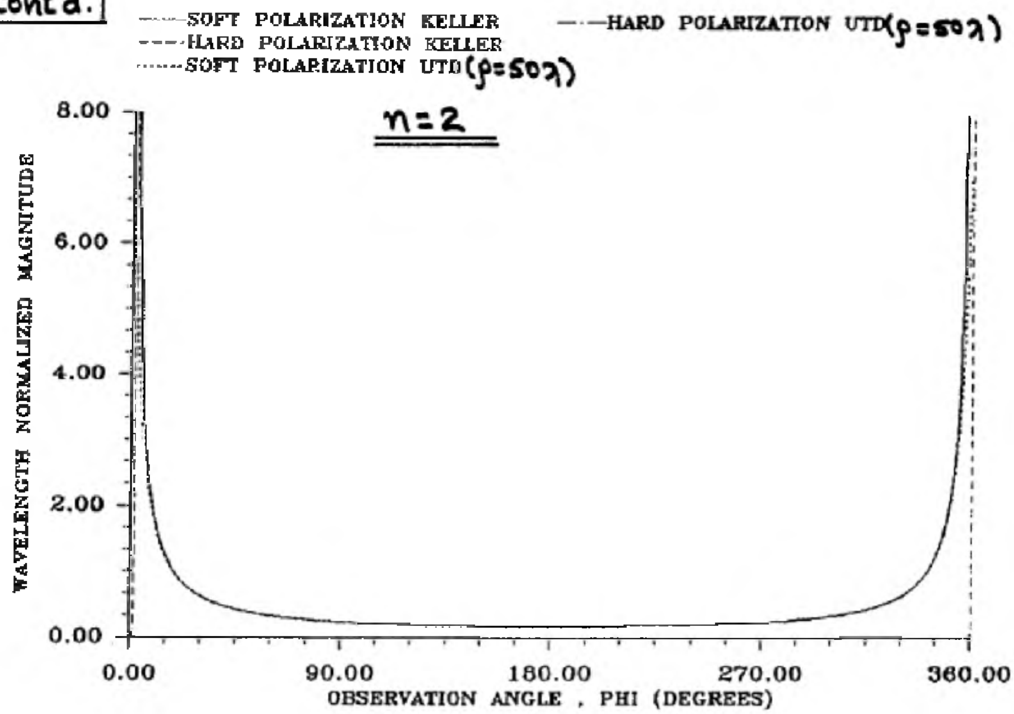
13.26 cont'd.



(c) For n=2

$$V_B^{s,h} = -\frac{e^{-j(\beta\rho+\pi/4)}}{2\sqrt{2\pi\beta\rho}}\left\{\sec\left(\frac{\phi-\phi'}{2}\right) \mp \sec\left(\frac{\phi+\phi'}{2}\right)\right\}$$

(d)

$$\sigma_w(2\text{-D}) = \lim_{\rho\to\infty}\left\{2\pi\rho\frac{|V_B^{s,h}|^2}{|1|^2}\right\} = \frac{1}{4\beta}\left\{\sec\left(\frac{\phi-\phi'}{2}\right) \mp \sec\left(\frac{\phi+\phi'}{2}\right)\right\}^2$$

$$\sigma_w(2\text{-D}) = \frac{\lambda}{8\pi}\left|\sec\left(\frac{\phi-\phi'}{2}\right) \mp \sec\left(\frac{\phi+\phi'}{2}\right)\right|^2$$

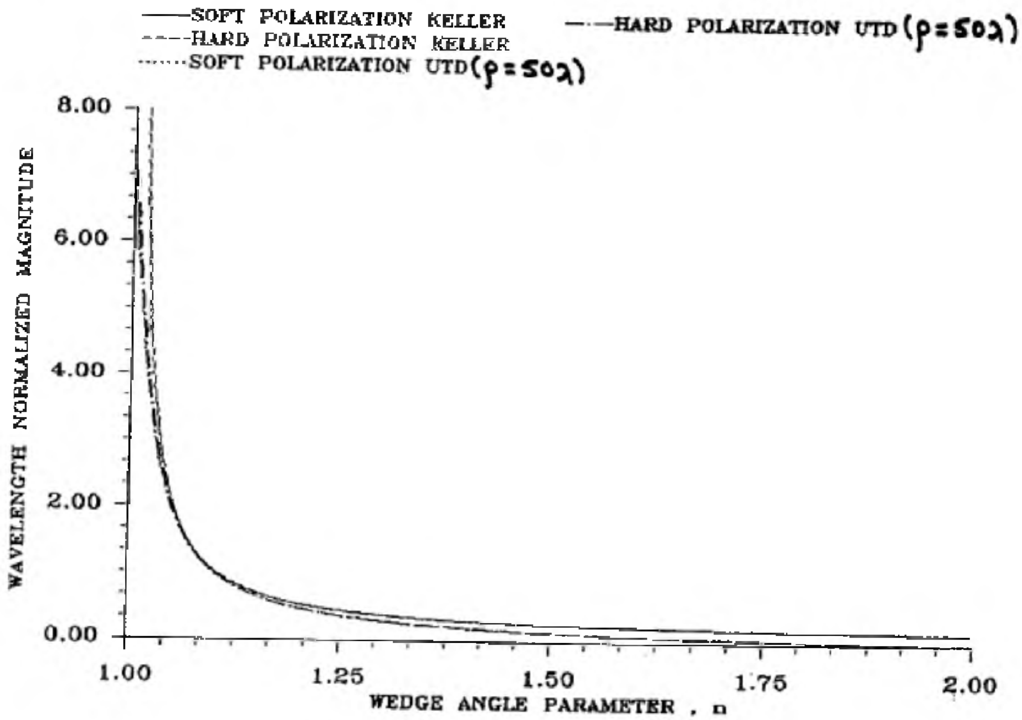
(e) For $\phi=\phi'$

$$\sigma_w(2\text{-D}) = \frac{\lambda}{8\pi}\left|1 \mp \sec(\phi')\right|^2$$

It is seen that when $\phi'=180°$ the hard polarized scattering width vanishes. This implies that the leading edge of a half-plane at grazing angle ($\phi=180°$) is invisible to the radar for hard polarized incident waves. This can be used to design low observable targets.

13.27 From the formulations of Problem 13.26, we can write for $\phi=\phi'$

$$D^{s,h}\sqrt{\lambda} = \frac{e^{-j\pi/4}}{n(2\pi)^{1/2}}\sin\left(\frac{\pi}{n}\right)\left\{\frac{1}{\cos\left(\frac{\pi}{n}\right)-1} \mp \frac{1}{\cos\left(\frac{\pi}{n}\right)-\cos\left(\frac{2\phi'}{n}\right)}\right\}$$



13.28 From the formulations of Problem 13.26, we can write for $\phi=0$ and $\phi'=n\pi/2$

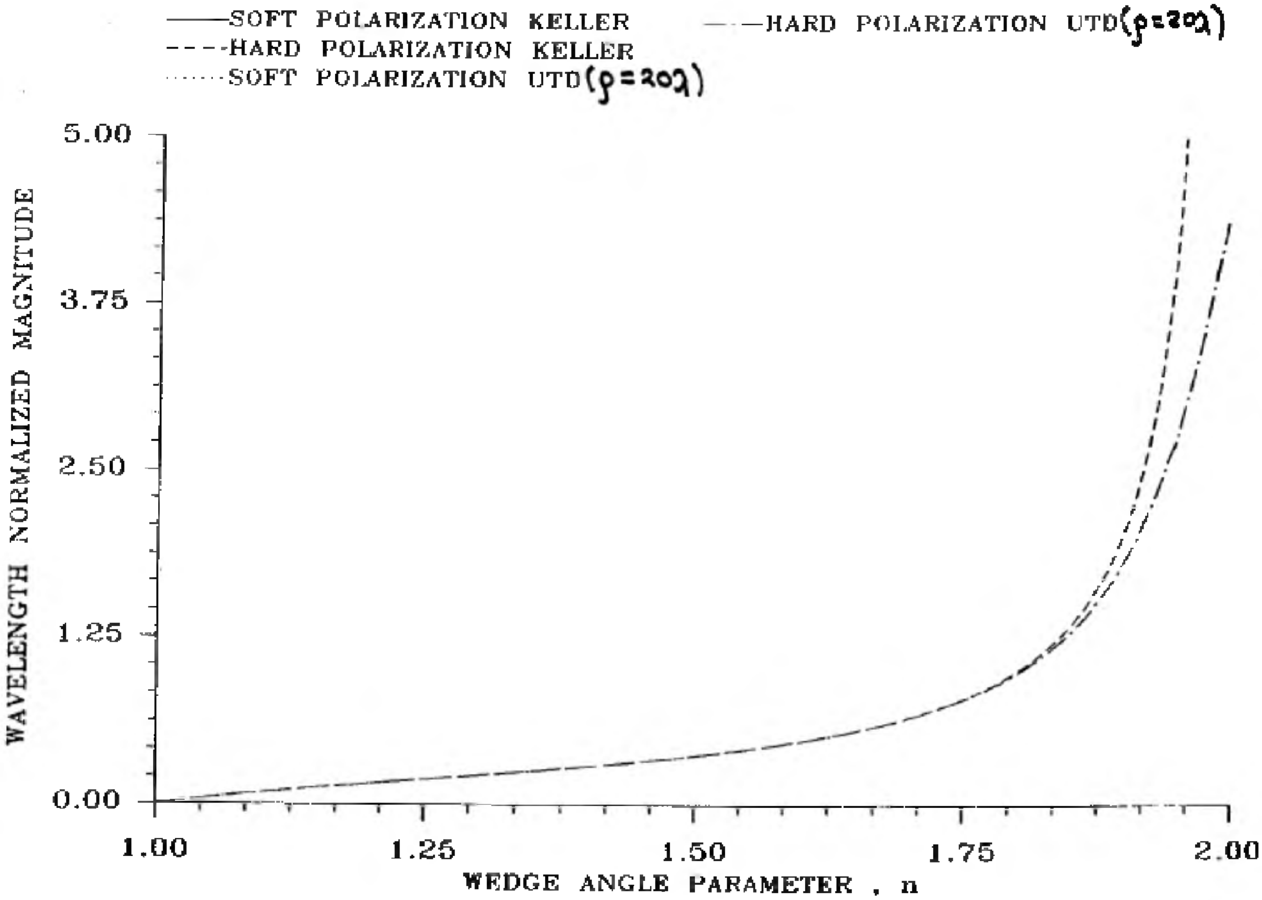
$$D^{s,h}\sqrt{\lambda} = \frac{e^{-j\pi/4}}{n(2\pi)^{1/2}}\sin\left(\frac{\pi}{n}\right)\left\{\frac{1}{\cos\left(\frac{\pi}{n}\right)-\cos\left(\frac{\phi'}{n}\right)} \mp \frac{1}{\cos\left(\frac{\pi}{n}\right)-\cos\left(\frac{\phi'}{n}\right)}\right\}_{\phi=\frac{n\pi}{2}}$$

$$= \frac{e^{-j\pi/4}}{n(2\pi)^{1/2}}\sin\left(\frac{\pi}{n}\right)\left\{\frac{1}{\cos\left(\frac{\pi}{n}\right)-\cos\left(\frac{\pi}{2}\right)} \mp \frac{1}{\cos\left(\frac{\pi}{n}\right)-\cos\left(\frac{\pi}{n}\right)}\right\}$$

The plots of these are shown on the next page.

Cont'd.

13.28 Cont'd.


SOFT POLARIZATION KELLER
HARD POLARIZATION KELLER
SOFT POLARIZATION UTD(ρ=20λ)
HARD POLARIZATION UTD(ρ=20λ)
5.00
3.75
2.50
1.25
0.00
WAVELENGTH NORMALIZED MAGNITUDE
1.00
1.25
1.50
1.75
2.00
WEDGE ANGLE PARAMETER , n

$\boxed{13.29}$ $F(x) = 2j\sqrt{x}\, e^{jx} \int_{\sqrt{x}}^{\infty} e^{-j\tau^2} d\tau$

Let us write that

$$2j\sqrt{x}\, e^{jx} \int_0^{\infty} e^{-j\tau^2} d\tau = 2j\sqrt{x}\, e^{jx} \int_0^{\sqrt{x}} e^{-j\tau^2} d\tau + 2j\sqrt{x}\, e^{jx} \int_{\sqrt{x}}^{\infty} e^{-j\tau^2} d\tau$$

or

$$\underbrace{2j\sqrt{x}\, e^{jx} \int_{\sqrt{x}}^{\infty} e^{-j\tau^2} d\tau}_{F(x)} = \underbrace{2j\sqrt{x}\, e^{jx} \int_0^{\infty} e^{-j\tau^2} d\tau}_{F_\infty(x)} - \underbrace{2j\sqrt{x}\, e^{jx} \int_0^{\sqrt{x}} e^{-j\tau^2} d\tau}_{F_0(x)}$$

$$F_\infty(x) = 2j\sqrt{x}\, e^{jx} \int_0^{\infty} e^{-j\tau^2} d\tau = 2j\sqrt{x}\, e^{jx} \left[\frac{\sqrt{\pi}}{2} e^{-j\pi/4}\right] = j\sqrt{\pi x}\, e^{-j\pi/4} e^{jx} = \sqrt{\pi x}\, e^{j\pi/4} e^{jx}$$

$$F_0(x) = 2j\sqrt{x}\, e^{jx} \int_0^{\sqrt{x}} e^{-j\tau^2} d\tau \simeq 2j\sqrt{x}\, e^{jx} \int_0^{\sqrt{x}} \sum_{n=0}^{M} \frac{(-j\tau^2)^n}{n!} d\tau \simeq 2j\sqrt{x}\, e^{jx} \sum_{n=0}^{M} \int_0 \frac{(-j\tau^2)^n}{n!} d\tau$$

$$F_0(x) \simeq 2j\sqrt{x}\, e^{jx} \sum_{n=0}^{M} \frac{(-j\tau^2)^n \tau}{n!(2n+1)} \Bigg|_0^{\sqrt{x}} \simeq 2jx\, e^{jx} \sum_{n=0}^{M} \frac{(-jx)^n}{n!(2n+1)}$$

Therefore

$$F(x) \simeq \sqrt{\pi x}\, e^{j\pi/4} e^{jx} - 2jx\, e^{jx} \sum_{n=0}^{\infty} \frac{(-jx)^n}{n!(2n+1)} = e^{jx}\left\{\sqrt{\pi x}\, e^{j\pi/4} - 2jx \sum_{n=0}^{M} \frac{(-jx)^n}{n!(2n+1)}\right\}$$

$$\simeq e^{j(x+\pi/4)}\left\{\sqrt{\pi x} - 2x\, e^{j\pi/4} \sum_{n=0}^{M} \frac{(-jx)^n}{n!(2n+1)}\right\}$$

$\boxed{13.30}$ $F(x) = 2j\sqrt{x}\, e^{jx} \int_{\sqrt{x}}^{\infty} e^{-j\tau^2} d\tau$

Let us first look at the integral which can be written by letting $w = \sqrt{x}$, and repeatedly integrating by parts as

$$\int_w^{\infty} e^{-j\tau^2} d\tau = -\frac{e^{-j\tau^2}}{j2\tau}\Bigg|_w^{\infty} - \frac{1}{2j} \int_w^{\infty} \frac{e^{-j\tau^2}}{\tau^2} d\tau$$

$$= -\frac{e^{-j\tau^2}}{j2\tau}\Bigg|_w^{\infty} - \frac{1}{2j}\left\{-\frac{e^{-j\tau^2}}{j2\tau^3}\Bigg|_w^{\infty} - \frac{3}{j2} \int_w^{\infty} \frac{e^{-j\tau^2}}{\tau^4} d\tau\right\}$$

$$\int_w^{\infty} e^{-j\tau^2} d\tau = -\frac{e^{-j\tau^2}}{j2\tau}\Bigg|_w^{\infty} - \frac{1}{2j}\left\{-\frac{e^{j\tau^2}}{j2\tau^3}\Bigg|_w^{\infty} - \frac{3}{j2}\left[-\frac{e^{-j\tau^2}}{j2\tau^5}\Bigg|_w^{\infty} - \frac{5}{j2} \int_w^{\infty} \frac{e^{-j\tau^2}}{\tau^6} d\tau\right]\right\}$$

Continuing this process we can write that

$$\int_w^{\infty} e^{-j\tau^2} d\tau = \frac{e^{-jw^2}}{j2w}\left[1 + \sum_{m=1}^{N} \frac{1.2.3.\cdots(2m-1)}{(-j2w^2)^m}\right] + R_N \quad \text{where } R_N = \frac{1.3.5\ldots(2N+1)}{(-j2)^{N+1}} \int_w^{\infty} \frac{e^{-j\tau^2}}{\tau^{2N+2}} d\tau$$

Since R_N decreases as w increases, then

$$F(x) = 2j\sqrt{x}\, e^{jx} \int_{\sqrt{x}}^{\infty} e^{-j\tau^2} d\tau \simeq 2j\sqrt{x}\, e^{jx}\left\{\frac{e^{-jx}}{j2\sqrt{x}}\left[1 + \sum_{m=1}^{N} \frac{1\cdot3\cdot5\cdots(2m-1)}{(-j2x)^m}\right]\right\} \simeq 1 + \sum_{m=1}^{N} \frac{1\cdot3\cdot5\cdots(2m-1)}{(-j2x)^m}$$

13.31

a. The procedure to show that there is a finite discontinuity of the same amplitude and phase along the RSB of Figure 13-24a, you follow the same procedure as was done on pages 789 and 790 for the ISB of Figure 13-24a. Along the RSB of Figure 13-24a

$$\xi^+ = \phi + \phi' = \pi \quad (1)$$

and in the neighborhood of it

$$\xi^+ = \phi + \phi' = \pi - \varepsilon \quad (2)$$

where ε is positive in the illuminated side of the reflection shadow boundary. Using (2) we can write (13-66f)

$$2n\pi N^- - (\phi+\phi') = 2n\pi N^- - (\pi - \varepsilon) = (2nN^- - 1)\pi + \varepsilon = -\pi \quad (3)$$

For this situation the cotangent function that becomes singular is that shown in second row of Table 13-1 whose $N^- = 0$. Thus (3) reduces for $N^- = 0$ to

$$\phi + \phi' = \pi - \varepsilon \quad (4)$$

which we can use to write the cotangent of the second row of Table 13-1 as

$$C^-(\phi+\phi', n) = \cot\left[\frac{\pi - (\phi+\phi')}{2n}\right] = \cot\left[\frac{\pi - (\pi - \varepsilon)}{2n}\right] = \cot\left(\frac{\varepsilon}{2n}\right) \simeq \frac{2n}{\varepsilon} = \frac{2n}{|\varepsilon|\,\mathrm{sgn}(\varepsilon)} \quad (5)$$

According to (13-66d)

$$g^-(\xi^+) = g^-(\phi+\phi') = 1 + \cos[(\phi+\phi') - 2n\pi N^-] = 1 + \cos(\phi+\phi') = 1 + \cos(\pi - \varepsilon) = 1 - \cos(\varepsilon)$$
$$\simeq 1 - \left[1 - \frac{\varepsilon^2}{2}\right] = \frac{\varepsilon^2}{2} \quad (6)$$

The transition function of (13-69f) can also be written using (6) as

$$F[\beta\rho g^-(\xi^+)] = F[\beta\rho g^-(\phi+\phi')] = F\left[\beta\rho\left(\frac{\varepsilon^2}{2}\right)\right] \quad (7)$$

which for small values of its argument can be approximated by the first term of (13-74a). Thus

$$F\left[\beta\rho\left(\frac{\varepsilon^2}{2}\right)\right] \simeq \sqrt{\frac{\pi\beta\rho\varepsilon^2}{2}}\, e^{j\pi/4} = |\varepsilon|\sqrt{\frac{\pi\beta\rho}{2}}\, e^{j\pi/4} \quad (8)$$

Thus the product of $C^-(\phi+\phi', n)\, F[\beta\rho g^-(\phi+\phi')]$ can be approximated by

$$C^-(\phi+\phi', n)\, F[\beta\rho g^-(\phi+\phi')] = \frac{2n}{|\varepsilon|\,\mathrm{sgn}(\varepsilon)} |\varepsilon| \sqrt{\frac{\pi\beta\rho}{2}}\, e^{j\pi/4} = n\sqrt{2\pi\beta\rho}\,\mathrm{sgn}(\varepsilon)\, e^{j\pi/4} \quad (9)$$

Therefore the corresponding reflected diffracted field of (13-69) can be approximated along the reflection shadow boundary by the second term within the brackets of (13-69b). Thus we can write that

$$V_B^r(\rho, \phi+\phi' = \pi - \varepsilon, n) \simeq \frac{e^{-j\beta\rho}}{\sqrt{\rho}}\left\{-\frac{e^{-j\pi/4}}{2n\sqrt{2\pi\beta}}\, n\sqrt{2\pi\beta\rho}\,\mathrm{sgn}(\varepsilon)\, e^{j\pi/4}\right\} = -\frac{e^{-j\beta\rho}}{2}\,\mathrm{sgn}(\varepsilon)$$

Cont'd.

13.31 Cont'd.

b. The same procedure is used along the ISB of Figure 13-24 b. The only difference is that you use (13-66e) with $\phi-\phi'$, the third row of Table 13-1, (13-66c) with $(\phi-\phi')$, (13-68e), and (13-68) can be approximated by the first term within the brackets of (13-68b). The result is the same as (13-83).

c. The same procedure is used along the RSB of Figure 13-24b. The only difference is that you use (13-66e) with $\phi+\phi'$, the fourth row of Table 13-1, (13-66c) with $\phi+\phi'$, (13-69e), and (13-69) can be approximated by the first term within the brackets of (13-69b). The result is the same as (13-83)

13.32

(a) ISB$(\phi-\phi')=\pi$ See Figure 13-24(a)

$$\cot\left[\frac{\pi-(\phi-\phi')}{2n}\right]_{\phi-\phi'=\pi} = \cot\left(\frac{\pi-\pi}{2n}\right) = \cot\left(\frac{0}{2n}\right) = \infty, \quad F\left[\beta\rho g^-(\phi-\phi')\right]_{\phi-\phi'=\pi} = 0$$

Product: $\cot\left[\frac{\pi-(\phi-\phi')}{2n}\right]F\left[\beta\rho g^-(\phi-\phi')\right]_{\phi-\phi'=\pi} =$ Discontinuous but finite bounded; removes GO discontinuity

(b)

RSB$(\phi+\phi')=\pi$ See Figure 13-24(a)

$$\cot\left[\frac{\pi-(\phi+\phi')}{2n}\right]_{\phi+\phi'=\pi} = \cot\left(\frac{\pi-\pi}{2n}\right) = \cot\left(\frac{0}{2n}\right) = \infty, \quad F\left[\beta\rho g^-(\phi+\phi')\right]_{\phi+\phi'=\pi} = 0$$

Product: $\cot\left[\frac{\pi-(\phi+\phi')}{2n}\right]F\left[\beta\rho g^-(\phi+\phi')\right]_{\phi+\phi'=\pi} =$ Discontinuous but finite bounded; removes GO discontinuity

13.33

(a) ISB$(\phi-\phi'=-\pi)$ See Figure 13-24(b)

$$\cot\left[\frac{\pi+(\phi-\phi')}{2n}\right]_{\phi-\phi'=-\pi} = \cot\left(\frac{\pi-\pi}{2n}\right) = \cot\left(\frac{0}{2n}\right) = \infty; \quad F\left[\beta\rho g^+(\phi-\phi')\right]_{\phi-\phi'=-\pi} = 0$$

Product: $\cot\left[\frac{\pi+(\phi-\phi')}{2n}\right]F\left[\beta\rho g^+(\phi-\phi')\right]_{\phi-\phi'=-\pi} =$ Discontinuous but finite bounded; removes GO discontinuity

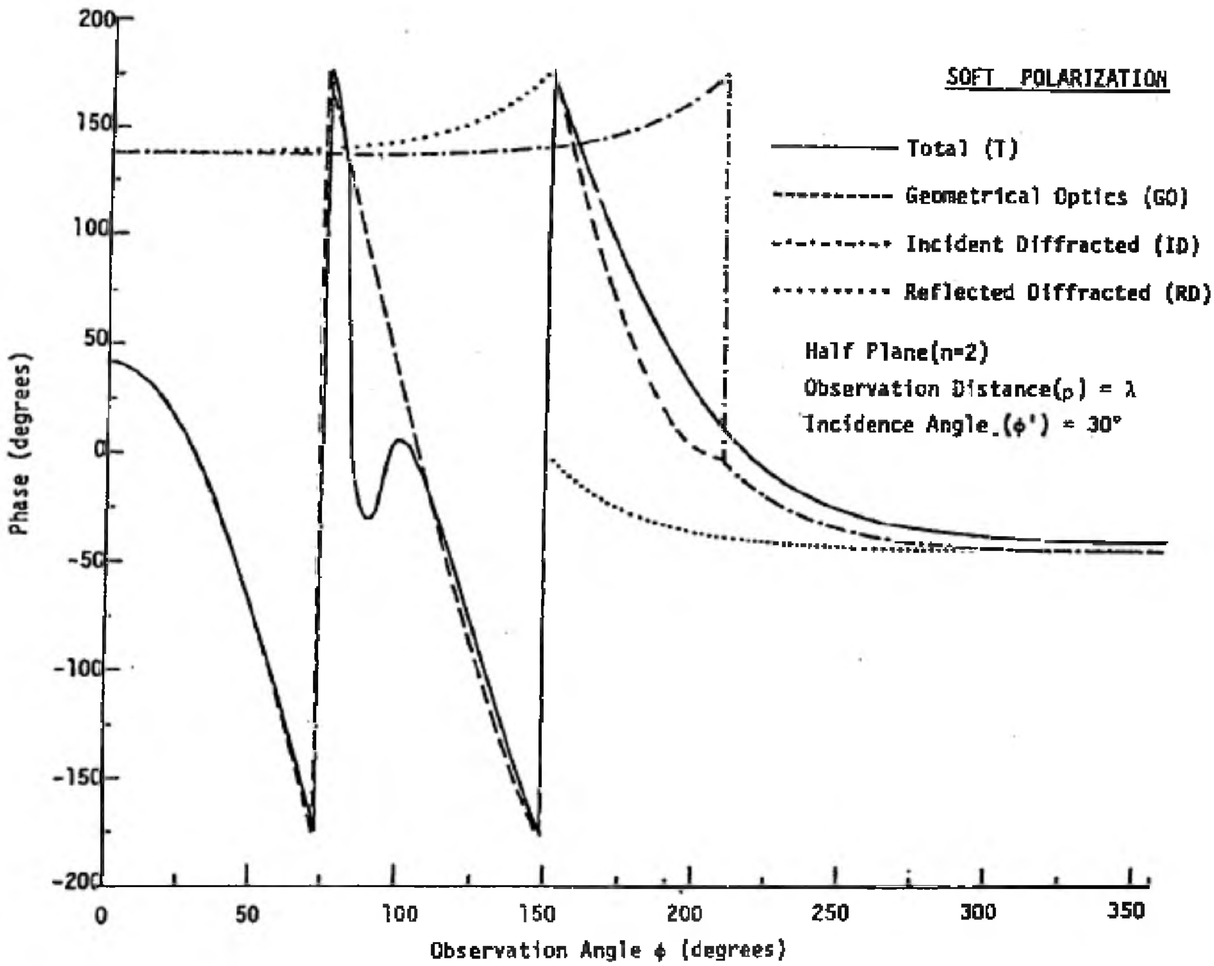
(b) RSB$\left[\phi+\phi'=(2n-1)\pi\right]$ See Figure 13-24(b)

$$\cot\left[\frac{\pi+(\phi+\phi')}{2n}\right]_{\phi+\phi'=(2n-1)\pi} = \cot\left[\frac{\pi+(2n-1)\pi}{2n}\right] = \cot(\pi) = \infty; \quad F\left[\beta\rho g^+(\phi+\phi')\right]_{\phi+\phi'=(2n-1)\pi} = 0$$

Product: $\cot\left[\frac{\pi+(\phi+\phi')}{2n}\right]F\left[\beta\rho g^+(\phi+\phi')\right]_{\phi+\phi'=(2n-1)\pi} =$ Discontinuous but finite bounded; removes GO discontinuity.

13.3.4

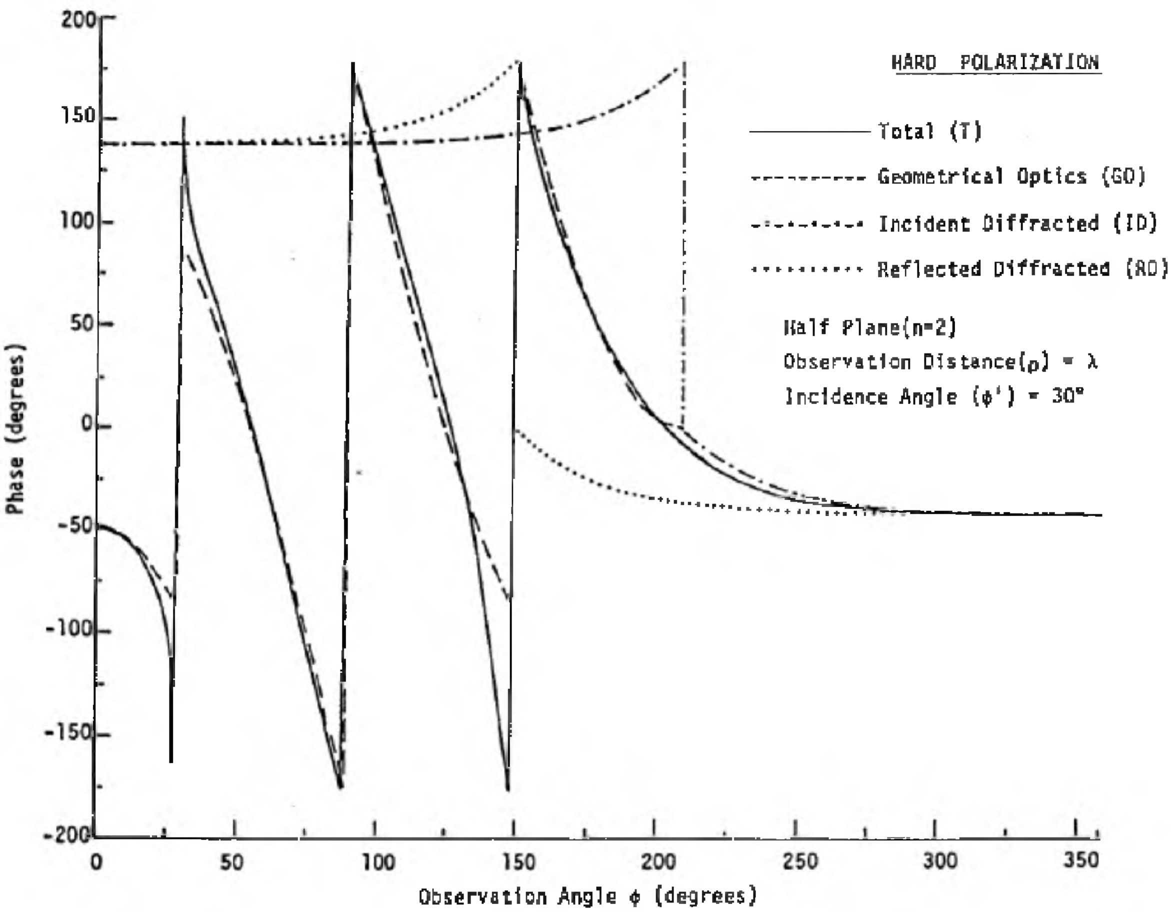
(a) Soft Polarization Phase



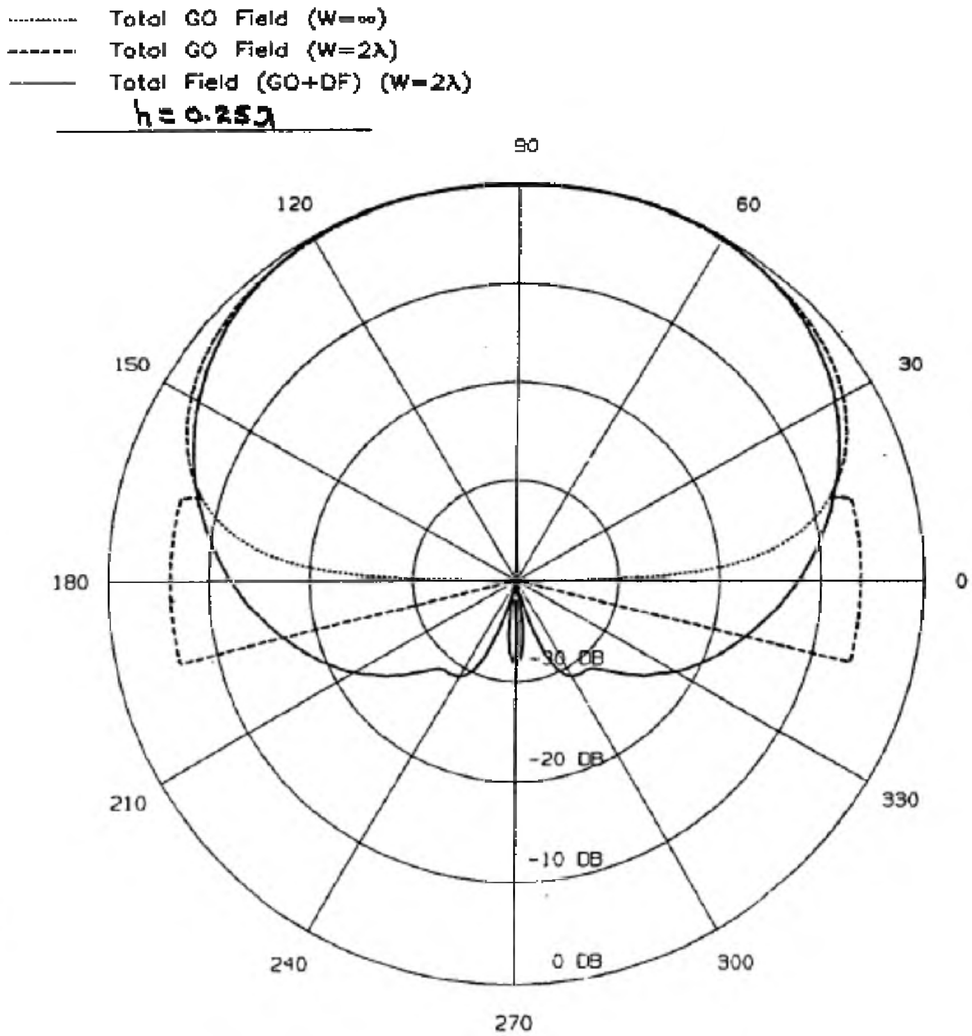
Cont'd.

13.34 Cont'd.

(b) Hard Polarization Phase



13.35



13.36 The only difference between the soft and hard polarized fields is sign in front of the second term within the brackets in the diffraction coefficient on pages 796-798. Changing that sign, we can eventually write the total magnetic field based on the electric field of page 798 as

$$\underline{H}^d = -\hat{a}_z H_0 \frac{e^{-j\pi/4}}{\sqrt{2\pi\beta}}\left[\cos(\beta w\cos\phi) - j\beta w\,\frac{\sin(\beta w\cos\phi)}{\beta w\cos\phi}\right]\frac{e^{-j\beta\rho}}{\sqrt{\rho}}$$

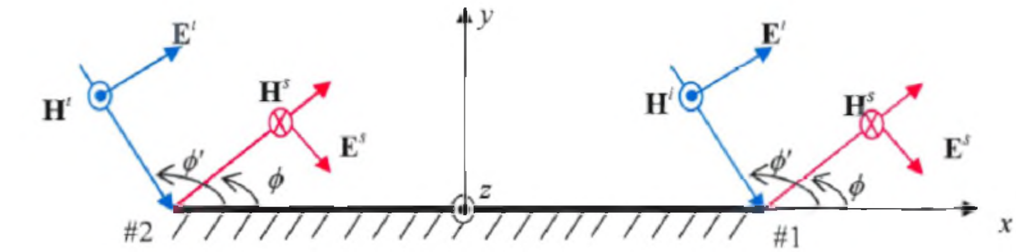
They are identical in form except for the sign (minus) in front of the second term within the brackets. Thefore the scattering width for the hard polarization is identical polarization since it involves the magnitude of the diffracted field.

13.37

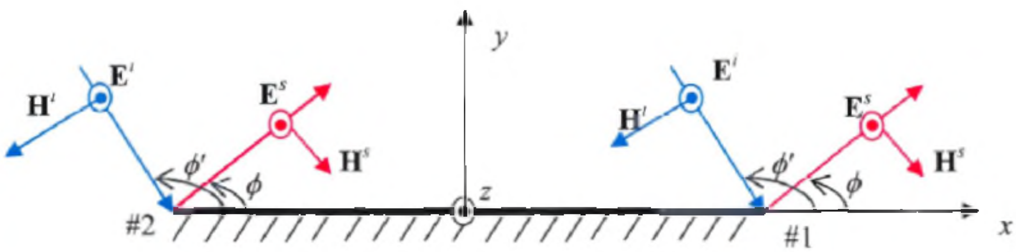
Geometrical/Uniform Theory of Diffraction (GTD/UTD):

Geometry

Hard Polarization:



Soft Polarization:



(a) (Hard Polarization)

- **Backscattered /Monostatic SW for a hard polarized uniform plane wave incident upon a PEC strip of width *w*.**

Using the geometry of the hard polarization, we can state that the incident field is:

$$\mathbf{H}^i = \hat{a}_z H_0 e^{j\beta(x\cos\phi' + y\sin\phi')}$$

Wedge #1:

In the wedge #1, the incident field is given by,

$$\mathbf{H}^i\Big|_{\substack{x=\frac{w}{2} \\ y=0}} = \hat{a}_z H_0 e^{j\beta(w/2\cos\phi')}$$

The backscattered field is:

$$\mathbf{H}_1^d = \mathbf{H}^i \cdot \overline{\mathbf{D}}_1^h \cdot A_1(\rho_1) e^{-j\beta\rho_1}$$

cont'd

13.37 cont'd

where

$$D_1^h = \frac{e^{-\frac{j\pi}{4}}\left(\frac{1}{n}\right)\sin\left(\frac{\pi}{n}\right)}{\sqrt{2\pi\beta}}\left[\frac{1}{cos\left(\frac{\pi}{n}\right)-\cos\left(\frac{\psi_1-\psi_1'}{n}\right)}+\frac{1}{cos\left(\frac{\pi}{n}\right)-\cos\left(\frac{\psi_1+\psi_1'}{n}\right)}\right]$$

For this geometry and wedge #1, $n = 2$ and $\psi_1 = \pi - \phi$ and $\psi_1' = \pi - \phi'$

$$D_1^h = \frac{e^{-\frac{j\pi}{4}}\left(\frac{1}{2}\right)\sin\left(\frac{\pi}{2}\right)}{\sqrt{2\pi\beta}}\left[\frac{1}{cos\left(\frac{\pi}{2}\right)-\cos\left(\frac{\pi-\phi-\pi+\phi'}{2}\right)}\right.$$
$$\left.+\frac{1}{cos\left(\frac{\pi}{2}\right)-\cos\left(\frac{\pi-\phi+\pi-\phi'}{2}\right)}\right]$$
$$= \frac{e^{-\frac{j\pi}{4}}}{2\sqrt{2\pi\beta}}\left[\frac{1}{-\cos\left(\frac{-\phi+\phi'}{2}\right)}+\frac{1}{-\cos\left(\frac{2\pi-\phi-\phi'}{2}\right)}\right]$$
$$= \frac{e^{-\frac{j\pi}{4}}}{2\sqrt{2\pi\beta}}\left[-\frac{1}{\cos\left(\frac{-\phi+\phi'}{2}\right)}+\frac{1}{-\cos\left(\frac{2\pi-\phi-\phi'}{2}\right)}\right]$$
$$D_1^h = \frac{-e^{-\frac{j\pi}{4}}}{2\sqrt{2\pi\beta}}\left[\frac{1}{\cos\left(\frac{-\phi+\phi'}{2}\right)}+\frac{1}{\cos\left(\frac{2\pi-\phi-\phi'}{2}\right)}\right]$$

Then, the diffracted field from wedge #1 is given by,

$$\mathbf{H}_1^d = -\hat{a}_z H_0 e^{j\beta(w/2cos\phi')}\frac{e^{-\frac{j\pi}{4}}}{2\sqrt{2\pi\beta}}\left[\frac{1}{\cos\left(\frac{-\phi+\phi'}{2}\right)}+\frac{1}{\cos\left(\frac{2\pi-\phi-\phi'}{2}\right)}\right]\frac{e^{-j\beta(\rho-w/2cos\phi)}}{\sqrt{\rho}}$$

or,

$$\boxed{\mathbf{H}_1^d = -\hat{a}_z H_0 e^{j\beta w/2(cos\phi'+cos\phi)}\frac{e^{-\frac{j\pi}{4}}}{2\sqrt{2\pi\beta}}\left[\frac{1}{\cos\left(\frac{-\phi+\phi'}{2}\right)}+\frac{1}{\cos\left(\frac{2\pi-\phi-\phi'}{2}\right)}\right]\frac{e^{-j\beta\rho}}{\sqrt{\rho}}}$$

cont'd

13.37 cont'd

Wedge #2:

In the wedge #2, the incident field is given by,

$$\mathbf{H}^i\Big|_{\substack{x=-\frac{w}{2}\\ y=0}} = \hat{a}_z H_0 e^{-j\beta(w/2\cos\phi\prime)}$$

The backscattered field is:

$$\mathbf{H}_2^d = \mathbf{H}^i \cdot \overline{\mathbf{D}}_2^h \cdot A_2(\rho_2) e^{-j\beta\rho_2}$$

where

$$D_2^h = \frac{e^{-\frac{j\pi}{4}}\left(\frac{1}{n}\right)\sin\left(\frac{\pi}{n}\right)}{\sqrt{2\pi\beta}}\left[\frac{1}{cos\left(\frac{\pi}{n}\right) - \cos\left(\frac{\psi_2 - \psi_2{}'}{n}\right)} + \frac{1}{cos\left(\frac{\pi}{n}\right) - \cos\left(\frac{\psi_2 + \psi_2{}'}{n}\right)}\right]$$

For this geometry and wedge #2, $n = 2$ and $\psi_2 = \phi, \psi_2' = \phi'$

$$D_2^h = \frac{e^{-\frac{j\pi}{4}}\left(\frac{1}{2}\right)\sin\left(\frac{\pi}{2}\right)}{\sqrt{2\pi\beta}}\left[\frac{1}{cos\left(\frac{\pi}{2}\right) - \cos\left(\frac{\phi - \phi'}{2}\right)} + \frac{1}{cos\left(\frac{\pi}{2}\right) - \cos\left(\frac{\phi + \phi'}{2}\right)}\right]$$

$$D_2^h = \frac{e^{-\frac{j\pi}{4}}}{2\sqrt{2\pi\beta}}\left[\frac{1}{-\cos\left(\frac{\phi - \phi'}{2}\right)} + \frac{1}{-\cos\left(\frac{\phi + \phi'}{2}\right)}\right]$$

$$D_2^h = -\frac{e^{\frac{j\pi}{4}}}{2\sqrt{2\pi\beta}}\left[\frac{1}{\cos\left(\frac{\phi - \phi'}{2}\right)} + \frac{1}{\cos\left(\frac{\phi + \phi'}{2}\right)}\right]$$

Then, the diffracted field from wedge #2 is given by,

$$\mathbf{H}_2^d = -\hat{a}_z H_0 e^{-j\beta(w/2\cos\phi\prime)} \frac{e^{-\frac{j\pi}{4}}}{2\sqrt{2\pi\beta}}\left[\frac{1}{\cos\left(\frac{\phi - \phi'}{2}\right)} + \frac{1}{\cos\left(\frac{\phi + \phi'}{2}\right)}\right]\frac{1}{\sqrt{\rho}} e^{-j\beta(\rho + w/2\cos\phi)}$$

or,

$$\boxed{\mathbf{H}_2^d = -\hat{a}_z H_0 e^{-j\beta w/2(\cos\phi' + \cos\phi)} \frac{e^{-\frac{j\pi}{4}}}{2\sqrt{2\pi\beta}}\left[\frac{1}{\cos\left(\frac{\phi - \phi'}{2}\right)} + \frac{1}{\cos\left(\frac{\phi + \phi'}{2}\right)}\right]\frac{1}{\sqrt{\rho}} e^{-j\beta\rho}}$$

cont'd

13.37 cont'd

Now, the total diffracted field is given by,

$$\mathbf{H}^d = \mathbf{H}_1^d + \mathbf{H}_2^d$$

$$\mathbf{H}^d = -\hat{a}_z H_0 e^{j\beta w/2(\cos\phi'+\cos\phi)} \frac{e^{-\frac{j\pi}{4}}}{2\sqrt{2\pi\beta}} \left[\frac{1}{\cos\left(\frac{-\phi+\phi'}{2}\right)} + \frac{1}{\cos\left(\frac{2\pi-\phi-\phi'}{2}\right)}\right] \frac{1}{\sqrt{\rho}} e^{-j\beta\rho}$$

$$- \hat{a}_z H_0 e^{-j\beta w/2(\cos\phi'+\cos\phi)} \frac{e^{-\frac{j\pi}{4}}}{2\sqrt{2\pi\beta}} \left[\frac{1}{\cos\left(\frac{\phi-\phi'}{2}\right)} + \frac{1}{\cos\left(\frac{\phi+\phi'}{2}\right)}\right] \frac{1}{\sqrt{\rho}} e^{-j\beta\rho}$$

$$\mathbf{H}^d = -\hat{a}_z H_0 \frac{e^{-\frac{j\pi}{4}}}{2\sqrt{2\pi\beta}} \left\{ e^{j\beta w/2(\cos\phi'+\cos\phi)} \left[\frac{1}{\cos\left(\frac{-\phi+\phi'}{2}\right)} + \frac{1}{\cos\left(\frac{2\pi-\phi-\phi'}{2}\right)}\right] \right.$$

$$\left. + e^{-j\beta w/2(\cos\phi'+\cos\phi)} \left[\frac{1}{\cos\left(\frac{\phi-\phi'}{2}\right)} + \frac{1}{\cos\left(\frac{\phi+\phi'}{2}\right)}\right] \right\} \frac{1}{\sqrt{\rho}} e^{-j\beta\rho}$$

$$\mathbf{H}^d = -\hat{a}_z H_0 \frac{e^{-\frac{j\pi}{4}}}{2\sqrt{2\pi\beta}} \left\{ e^{j\beta w/2(\cos\phi'+\cos\phi)} \left[\frac{1}{\cos\left(\frac{\phi-\phi'}{2}\right)} - \frac{1}{\cos\left(\frac{\phi+\phi'}{2}\right)}\right] \right.$$

$$\left. + e^{-j\beta w/2(\cos\phi'+\cos\phi)} \left[\frac{1}{\cos\left(\frac{\phi-\phi'}{2}\right)} + \frac{1}{\cos\left(\frac{\phi+\phi'}{2}\right)}\right] \right\} \frac{1}{\sqrt{\rho}} e^{-j\beta\rho}$$

Regrouping similar terms,

$$\mathbf{H}^d = -\hat{a}_z H_0 \frac{e^{-\frac{j\pi}{4}}}{\sqrt{2\pi\beta}} \left\{ \frac{1}{\cos\left(\frac{\phi-\phi'}{2}\right)} \left[\frac{e^{j\beta w/2(\cos\phi'+\cos\phi)} + e^{-j\beta w/2(\cos\phi'+\cos\phi)}}{2}\right] \right.$$

$$\left. - \frac{1}{\cos\left(\frac{\phi+\phi'}{2}\right)} \left[\frac{e^{j\beta w/2(\cos\phi'+\cos\phi)} - e^{-j\beta w/2(\cos\phi'+\cos\phi)}}{2}\right] \right\} \frac{1}{\sqrt{\rho}} e^{-j\beta\rho}$$

$$\mathbf{H}^d = -\hat{a}_z H_0 \frac{e^{-\frac{j\pi}{4}}}{\sqrt{2\pi\beta}} \left\{ \frac{1}{\cos\left(\frac{\phi-\phi'}{2}\right)} [\cos(\beta w/2(\cos\phi' + \cos\phi))] \right.$$

$$\left. - \frac{j}{\cos\left(\frac{\phi+\phi'}{2}\right)} [\sin(\beta w/2(\cos\phi' + \cos\phi))] \right\} \frac{1}{\sqrt{\rho}} e^{-j\beta\rho}$$

cont'd

13.37 cont'd

For backscattered, when $\phi = \phi'$, the diffracted field, that is equal to the backscattered field, reduces to:

$$\boxed{\mathbf{H}^s = \mathbf{H}^d = -\hat{a}_z H_0 \frac{e^{-\frac{j\pi}{4}}}{\sqrt{2\pi\beta}}\left\{cos(\beta w cos\phi) - j\beta w \frac{sin(\beta w cos\phi)}{(\beta w cos\phi)}\right\}\frac{e^{-j\beta\rho}}{\sqrt{\rho}}}$$

Therefore, the **monostatic scattering width** can be obtained as

$$\sigma_{2-D} = \lim_{\rho\to\infty}\left[2\pi\rho\left|\frac{\mathbf{H}^s}{\mathbf{H}^i}\right|^2\right]$$

$$\sigma_{2-D} = \lim_{\rho\to\infty}\left[2\pi\rho\left|\frac{-\hat{a}_z H_0 \frac{e^{-\frac{j\pi}{4}}}{\sqrt{2\pi\beta}}\left\{cos(\beta w cos\phi) - j\beta w \frac{sin(\beta w cos\phi)}{(\beta w cos\phi)}\right\}\frac{e^{-j\beta\rho}}{\sqrt{\rho}}}{H_0 e^{j\beta(x cos\phi' + y sin\phi')}}\right|^2\right]$$

$$\sigma_{2-D} = \lim_{\rho\to\infty}\left[\frac{1}{\beta}\left|cos(\beta w cos\phi') - j\beta w \frac{sin(\beta w cos\phi')}{(\beta w cos\phi')}\right|^2\right]$$

$$\boxed{\boldsymbol{\sigma_{2-D} = \frac{\lambda}{2\pi}\left|cos(\beta w cos\phi') - j\beta w \frac{sin(\beta w cos\phi')}{(\beta w cos\phi')}\right|^2}}$$

- **Bistatic scattering field and its bistatic scattered width SW for a hard polarized uniform plane wave incident upon a PEC strip of width *w*.**

From the previous formulation, we have the total diffracted field is given by,

$$\boxed{\mathbf{H}^s = -\hat{a}_z H_0 \frac{e^{-\frac{j\pi}{4}}}{\sqrt{2\pi\beta}}\left\{\frac{\cos\left(\beta w/2(cos\phi' + cos\phi)\right)}{\cos\left(\frac{\phi - \phi'}{2}\right)} - \frac{j\sin\left(\beta w/2(cos\phi' + cos\phi)\right)}{\cos\left(\frac{\phi + \phi'}{2}\right)}\right\}\frac{1}{\sqrt{\rho}}e^{-j\beta\rho}}$$

cont'd

13.37 cont'd

Therefore, the **bistatic scattering width** is given by,

$$\sigma_{2-D} = \lim_{\rho\to\infty}\left[2\pi\rho\left|\frac{\mathbf{H}^s}{\mathbf{H}^i}\right|^2\right]$$

$$= \lim_{\rho\to\infty}\left[2\pi\rho\left|\frac{-\hat{a}_z H_0\frac{e^{-\frac{j\pi}{4}}}{\sqrt{2\pi\beta}}\left\{\frac{\cos\left(\beta w/2(\cos\phi'+\cos\phi)\right)}{\cos\left(\frac{\phi-\phi'}{2}\right)}-\frac{j\sin\left(\beta w/2(\cos\phi'+\cos\phi)\right)}{\cos\left(\frac{\phi+\phi'}{2}\right)}\right\}\frac{e^{-j\beta\rho}}{\sqrt{\rho}}}{H_0 e^{j\beta(x\cos\phi'+y\sin\phi')}}\right|^2\right]$$

$$\sigma_{2-D} = \lim_{\rho\to\infty}\left[\frac{1}{\beta}\left|\frac{\cos\left(\beta w/2(\cos\phi'+\cos\phi)\right)}{\cos\left(\frac{\phi-\phi'}{2}\right)}-\frac{j\sin\left(\beta w/2(\cos\phi'+\cos\phi)\right)}{\cos\left(\frac{\phi+\phi'}{2}\right)}\right|^2\right]$$

$$\boxed{\sigma_{2-D} = \frac{\lambda}{2\pi}\left|\frac{\cos\left(\beta w/2(\cos\phi'+\cos\phi)\right)}{\cos\left(\frac{\phi-\phi'}{2}\right)}-\frac{j\sin\left(\beta w/2(\cos\phi'+\cos\phi)\right)}{\cos\left(\frac{\phi+\phi'}{2}\right)}\right|^2}$$

(b) Bistatic scattering field and its bistatic scattered width for a soft polarized uniform plane wave incident upon a PEC strip of width *w*.

Using the geometry of the soft polarization, we can state that the incident field is:

$$\mathbf{E}^i = \hat{a}_z\eta H_0 e^{j\beta(x\cos\phi'+y\sin\phi')}$$

Wedge #1:

In the wedge #1, the incident field is given by,

$$\mathbf{E}^i\Big|_{\substack{x=\frac{w}{2}\\ y=0}} = \hat{a}_z\eta H_0 e^{j\beta(w/2\cos\phi_i)}$$

The backscattered field is:

$$\mathbf{E}_1^d = \mathbf{E}^i\cdot\overline{\mathbf{D}}_1^s\cdot A_1(\rho_1)e^{-j\beta\rho_1}$$

cont'd

13.37 cont'd

where

$$D_1^s = \frac{e^{-\frac{j\pi}{4}}\left(\frac{1}{n}\right)\sin\left(\frac{\pi}{n}\right)}{\sqrt{2\pi\beta}}\left[\frac{1}{\cos\left(\frac{\pi}{n}\right)-\cos\left(\frac{\psi_1-\psi_1'}{n}\right)}-\frac{1}{\cos\left(\frac{\pi}{n}\right)-\cos\left(\frac{\psi_1+\psi_1'}{n}\right)}\right]$$

For this geometry and wedge #1, n=2 and $\psi_1 = \pi - \phi$ and $\psi_1' = \pi - \phi'$

$$D_1^s = \frac{e^{-\frac{j\pi}{4}}\left(\frac{1}{2}\right)\sin\left(\frac{\pi}{2}\right)}{\sqrt{2\pi\beta}}\left[\frac{1}{\cos\left(\frac{\pi}{2}\right)-\cos\left(\frac{\pi-\phi-\pi+\phi'}{2}\right)}-\frac{1}{\cos\left(\frac{\pi}{2}\right)-\cos\left(\frac{\pi-\phi+\pi-\phi'}{2}\right)}\right]$$

$$D_1^s = \frac{e^{-\frac{j\pi}{4}}}{2\sqrt{2\pi\beta}}\left[\frac{1}{-\cos\left(\frac{-\phi+\phi'}{2}\right)}-\frac{1}{-\cos\left(\frac{2\pi-\phi-\phi'}{2}\right)}\right]$$

$$= \frac{e^{-\frac{j\pi}{4}}}{2\sqrt{2\pi\beta}}\left[-\frac{1}{\cos\left(\frac{-\phi+\phi'}{2}\right)}-\frac{1}{-\cos\left(\frac{2\pi-\phi-\phi'}{2}\right)}\right]$$

$$D_1^s = \frac{-e^{-\frac{j\pi}{4}}}{2\sqrt{2\pi\beta}}\left[\frac{1}{\cos\left(\frac{-\phi+\phi'}{2}\right)}-\frac{1}{\cos\left(\frac{2\pi-\phi-\phi'}{2}\right)}\right]$$

Then, the diffracted field from wedge #1 is given by,

$$\mathbf{E}_1^d = -\hat{a}_z \eta H_0 e^{j\beta(w/2\cos\phi')}\frac{e^{-\frac{j\pi}{4}}}{2\sqrt{2\pi\beta}}\left[\frac{1}{\cos\left(\frac{-\phi+\phi'}{2}\right)}-\frac{1}{\cos\left(\frac{2\pi-\phi-\phi'}{2}\right)}\right]\frac{e^{-j\beta(\rho-w/2\cos\phi)}}{\sqrt{\rho}}$$

or,

$$\boxed{\mathbf{E}_1^d = -\hat{a}_z \eta H_0 e^{j\beta w/2(\cos\phi'+\cos\phi)}\frac{e^{-\frac{j\pi}{4}}}{2\sqrt{2\pi\beta}}\left[\frac{1}{\cos\left(\frac{-\phi+\phi'}{2}\right)}+\frac{1}{\cos\left(\frac{\phi+\phi'}{2}\right)}\right]\frac{e^{-j\beta\rho}}{\sqrt{\rho}}}$$

cont'd

13.37 cont'd

Wedge #2:

In the wedge #2, the incident field is given by,

$$\mathbf{E}^i\Big|_{\substack{x=-\frac{w}{2}\\ y=0}} = \hat{a}_z \eta H_0 e^{-j\beta(w/2\cos\phi\prime)}$$

The backscattered field is:

$$\mathbf{E}_2^d = \mathbf{E}^i \cdot \overline{\mathbf{D}}_2^s \cdot A_2(\rho_2) e^{-j\beta\rho_2}$$

where

$$D_2^s = \frac{e^{-\frac{j\pi}{4}}\left(\frac{1}{n}\right)\sin\left(\frac{\pi}{n}\right)}{\sqrt{2\pi\beta}}\left[\frac{1}{cos\left(\frac{\pi}{n}\right)-\cos\left(\frac{\psi_2-\psi_2{}'}{n}\right)}-\frac{1}{cos\left(\frac{\pi}{n}\right)-\cos\left(\frac{\psi_2+\psi_2{}'}{n}\right)}\right]$$

For this geometry and wedge #2, $n = 2$ and $\psi_2 = \phi, \psi_2' = \phi'$

$$D_2^s = \frac{e^{-\frac{j\pi}{4}}\left(\frac{1}{2}\right)\sin\left(\frac{\pi}{2}\right)}{\sqrt{2\pi\beta}}\left[\frac{1}{cos\left(\frac{\pi}{2}\right)-\cos\left(\frac{\phi-\phi'}{2}\right)}-\frac{1}{cos\left(\frac{\pi}{2}\right)-\cos\left(\frac{\phi+\phi'}{2}\right)}\right]$$

$$= \frac{e^{-\frac{j\pi}{4}}}{2\sqrt{2\pi\beta}}\left[\frac{1}{-\cos\left(\frac{\phi-\phi'}{2}\right)}-\frac{1}{-\cos\left(\frac{\phi+\phi'}{2}\right)}\right]$$

$$D_2^s = -\frac{e^{-\frac{j\pi}{4}}}{2\sqrt{2\pi\beta}}\left[\frac{1}{\cos\left(\frac{\phi-\phi'}{2}\right)}-\frac{1}{\cos\left(\frac{\phi+\phi'}{2}\right)}\right]$$

Then, the diffracted field from wedge #2 is given by,

$$\mathbf{E}_2^d = -\hat{a}_z \eta H_0 e^{-j\beta(w/2\cos\phi\prime)}\frac{e^{-\frac{j\pi}{4}}}{2\sqrt{2\pi\beta}}\left[\frac{1}{\cos\left(\frac{\phi-\phi'}{2}\right)}-\frac{1}{\cos\left(\frac{\phi+\phi'}{2}\right)}\right]\frac{1}{\sqrt{\rho}}e^{-j\beta(\rho+w/2\cos\phi)}$$

Cont'd

13.37 cont'd

or,

$$\boxed{\mathbf{E}_2^d = -\hat{a}_z\eta H_0 e^{-j\beta w/2(\cos\phi'+\cos\phi)}\frac{e^{-\frac{j\pi}{4}}}{2\sqrt{2\pi\beta}}\left[\frac{1}{\cos\left(\frac{\phi-\phi'}{2}\right)}-\frac{1}{\cos\left(\frac{\phi+\phi'}{2}\right)}\right]\frac{1}{\sqrt{\rho}}e^{-j\beta\rho}}$$

Now, the total diffracted field is given by,

$$\mathbf{E}^d = \mathbf{E}_1^d + \mathbf{E}_2^d$$

$$\mathbf{E}^d = -\hat{a}_z\eta H_0 e^{j\beta w/2(\cos\phi'+\cos\phi)}\frac{e^{-\frac{j\pi}{4}}}{2\sqrt{2\pi\beta}}\left[\frac{1}{\cos\left(\frac{\phi-\phi'}{2}\right)}+\frac{1}{\cos\left(\frac{\phi+\phi'}{2}\right)}\right]\frac{e^{-j\beta\rho}}{\sqrt{\rho}}$$

$$-\hat{a}_z\eta H_0 e^{-j\beta w/2(\cos\phi'+\cos\phi)}\frac{e^{-\frac{j\pi}{4}}}{2\sqrt{2\pi\beta}}\left[\frac{1}{\cos\left(\frac{\phi-\phi'}{2}\right)}-\frac{1}{\cos\left(\frac{\phi+\phi'}{2}\right)}\right]\frac{1}{\sqrt{\rho}}e^{-j\beta\rho}$$

$$\mathbf{E}^d = -\hat{a}_z\eta H_0\frac{e^{-\frac{j\pi}{4}}}{2\sqrt{2\pi\beta}}\left\{e^{j\beta w/2(\cos\phi'+\cos\phi)}\left[\frac{1}{\cos\left(\frac{\phi-\phi'}{2}\right)}+\frac{1}{\cos\left(\frac{\phi+\phi'}{2}\right)}\right]\right.$$

$$\left.+e^{-j\beta w/2(\cos\phi'+\cos\phi)}\left[\frac{1}{\cos\left(\frac{\phi-\phi'}{2}\right)}-\frac{1}{\cos\left(\frac{\phi+\phi'}{2}\right)}\right]\right\}\frac{1}{\sqrt{\rho}}e^{-j\beta\rho}$$

Regrouping similar terms,

$$\mathbf{E}^d = -\hat{a}_z\eta H_0\frac{e^{-\frac{j\pi}{4}}}{\sqrt{2\pi\beta}}\left\{\frac{1}{\cos\left(\frac{\phi-\phi'}{2}\right)}\left[\frac{e^{j\beta w/2(\cos\phi'+\cos\phi)}+e^{-j\beta w/2(\cos\phi'+\cos\phi)}}{2}\right]\right.$$

$$\left.+\frac{1}{\cos\left(\frac{\phi+\phi'}{2}\right)}\left[\frac{e^{j\beta w/2(\cos\phi'+\cos\phi)}-e^{-j\beta w/2(\cos\phi'+\cos\phi)}}{2}\right]\right\}\frac{1}{\sqrt{\rho}}e^{-j\beta\rho}$$

$$\boxed{\mathbf{E}^d = -\hat{a}_z\eta H_0\frac{e^{-\frac{j\pi}{4}}}{\sqrt{2\pi\beta}}\left\{\frac{\cos\left(\beta w/2(\cos\phi'+\cos\phi)\right)}{\cos\left(\frac{\phi-\phi'}{2}\right)}+\frac{j\sin\left(\beta w/2(\cos\phi'+\cos\phi)\right)}{\cos\left(\frac{\phi+\phi'}{2}\right)}\right\}\frac{1}{\sqrt{\rho}}e^{-j\beta\rho}}$$

cont'd

13.37 cont'd

Then, the bistatic scattering width is given by,

$$\sigma_{2-D} = \lim_{\rho\to\infty}\left[2\pi\rho\left|\frac{E^s}{E^i}\right|^2\right]$$

$$\sigma_{2-D} = \lim_{\rho\to\infty}\left[2\pi\rho\left|\frac{-\hat{a}_z\eta H_0\dfrac{e^{-\frac{j\pi}{4}}}{\sqrt{2\pi\beta}}\left\{\dfrac{\cos\left(\beta w/2(cos\phi'+cos\phi)\right)}{\cos\left(\dfrac{\phi-\phi'}{2}\right)}+\dfrac{j\sin\left(\beta w/2(cos\phi'+cos\phi)\right)}{\cos\left(\dfrac{\phi+\phi'}{2}\right)}\right\}\dfrac{e^{-j\beta\rho}}{\sqrt{\rho}}}{H_0e^{j\beta(xcos\phi\prime+ysin\phi\prime)}}\right|^2\right]$$

$$\sigma_{2-D} = \lim_{\rho\to\infty}\left[\frac{1}{\beta}\left|\frac{\cos\left(\beta w/2(cos\phi'+cos\phi)\right)}{\cos\left(\dfrac{\phi-\phi'}{2}\right)}+\frac{j\sin\left(\beta w/2(cos\phi'+cos\phi)\right)}{\cos\left(\dfrac{\phi+\phi'}{2}\right)}\right|^2\right]$$

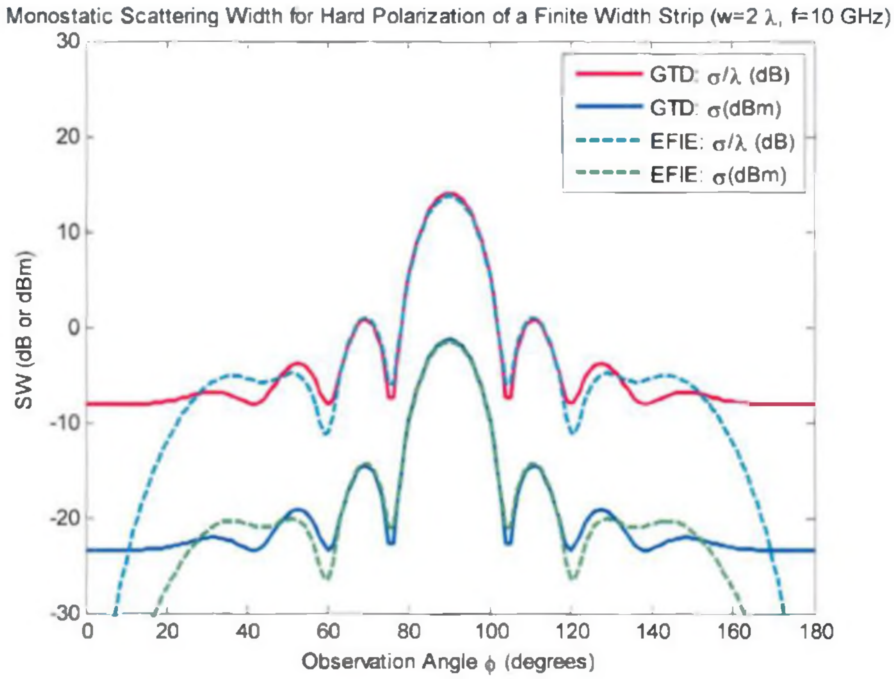
$$\boxed{\sigma_{2-D} = \frac{\lambda}{2\pi}\left|\frac{\cos\left(\beta w/2(cos\phi'+cos\phi)\right)}{\cos\left(\dfrac{\phi-\phi'}{2}\right)}+\frac{j\sin\left(\beta w/2(cos\phi'+cos\phi)\right)}{\cos\left(\dfrac{\phi+\phi'}{2}\right)}\right|^2}$$

cont'd

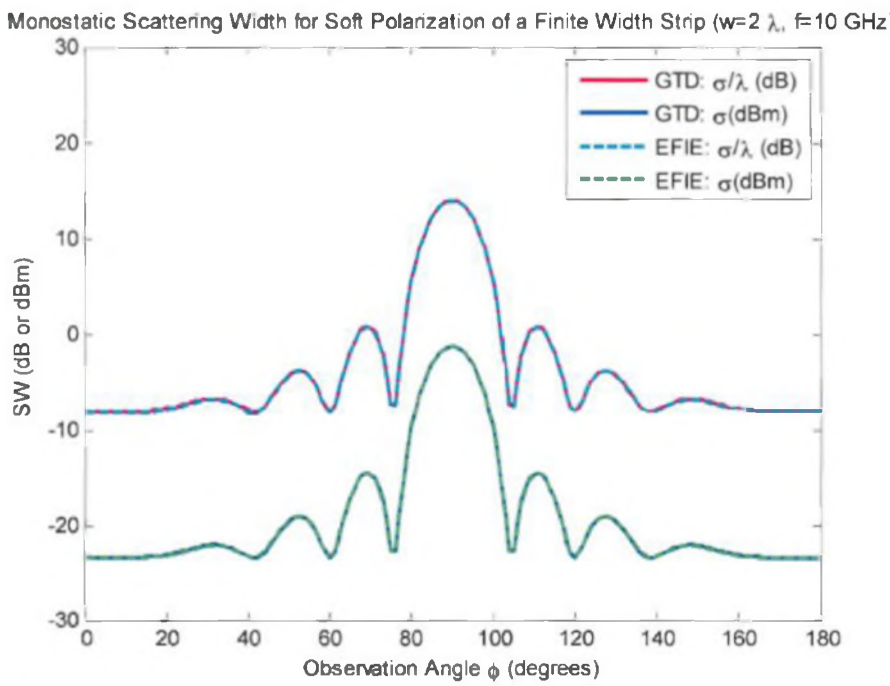
13.37 cont'd

(c) Scattering Patterns:

Case 1: $w = 2\lambda$, $f = 10$ GHz, $\phi' = 120°$, Monostatic, Hard



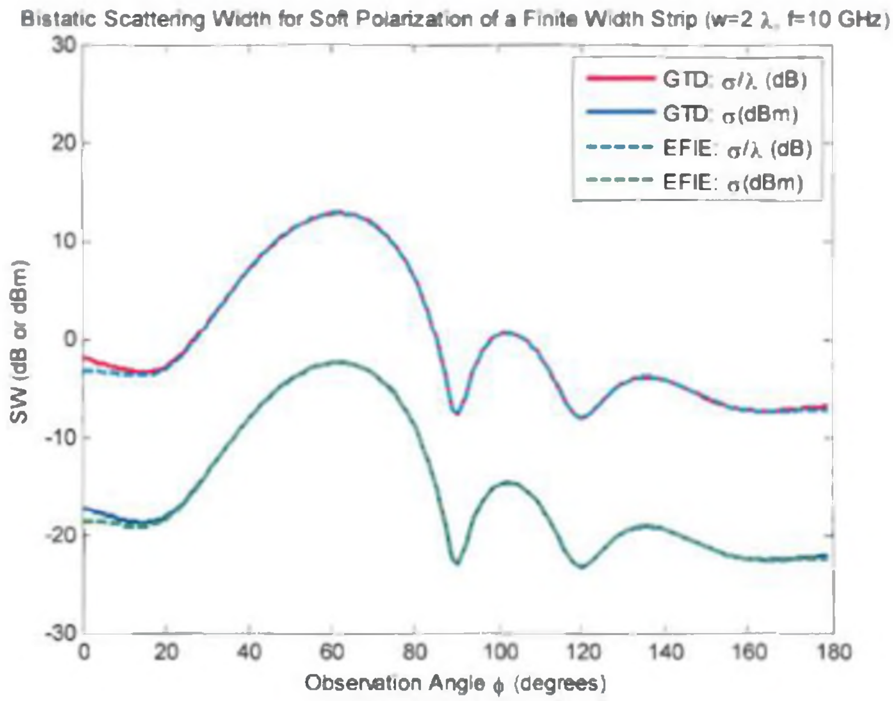
Case 2: $w = 2\lambda$, $f = 10$ GHz, $\phi' = 120°$, Monostatic, Soft



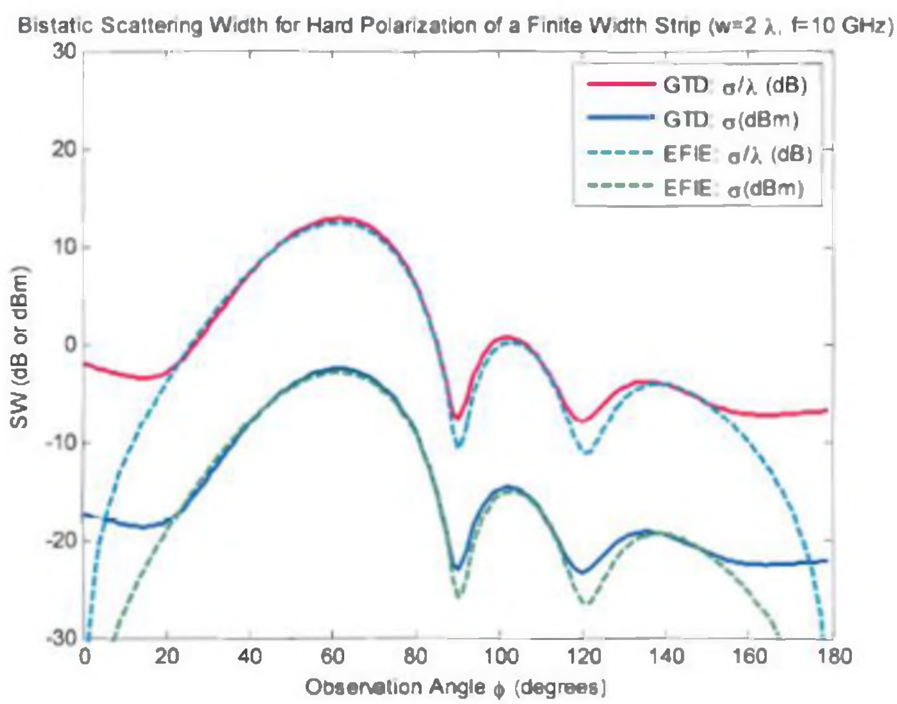
cont'd

13.37 Cont'd

Case 3: $w = 2\lambda$, $f = 10$ GHz, $\phi' = 120°$, Bistatic, Soft

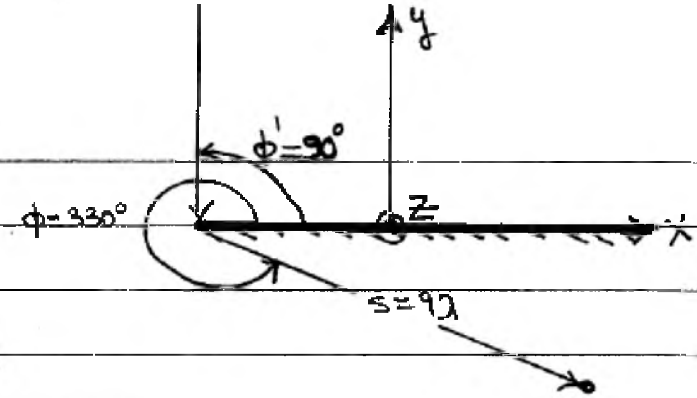


Case 4: $w = 2\lambda$, $f = 10$ GHz, $\phi' = 120°$, Bistatic, Hard



13.38

$$\underline{E}^i = \hat{a}_z E_0 e^{+j\beta y} = \hat{a}_z(1)e^{+j\beta y}$$



(a)

$$\underline{E}^d = \underline{E}^i(y=0)\cdot \tilde{\underline{D}}\,\frac{e^{-j\beta s}}{\sqrt{s}}$$

$$= \hat{a}_z E^i(y=0)\cdot \hat{a}_z\hat{a}_z D^s \frac{e^{-j\beta s}}{\sqrt{s}}$$

$$= \hat{a}_z(1)\cdot\hat{a}_z\hat{a}_z\,\frac{e^{-j\pi/4}}{\sqrt{2\pi\beta}}\,\frac{1}{n}\sin\left(\frac{\pi}{n}\right)\left[\frac{1}{\cos\left(\frac{\pi}{n}\right)-\cos\left(\frac{\phi-\phi'}{n}\right)}-\frac{1}{\cos\left(\frac{\pi}{n}\right)-\cos\left(\frac{\phi+\phi'}{n}\right)}\right]\frac{e^{-j\beta s}}{\sqrt{s}}$$

$$\underline{E}^d = \hat{a}_z\,\frac{e^{-j(\beta s+\pi/4)}}{\sqrt{2\pi\beta s}}\,\frac{1}{n}\sin\left(\frac{\pi}{n}\right)\left\{\frac{1}{\cos\left(\frac{\pi}{n}\right)-\cos\left(\frac{\phi-\phi'}{n}\right)}-\frac{1}{\cos\left(\frac{\pi}{n}\right)-\cos\left(\frac{\phi+\phi'}{n}\right)}\right\}_{\substack{n=2\\ s=9\lambda\\ \phi=330^\circ\\ \phi'=90^\circ}}$$

$$= \hat{a}_z\,\frac{e^{-j\left[\frac{2\pi}{\lambda}(9\lambda)+\frac{\pi}{4}\right]}}{\sqrt{2\pi\left(\frac{2\pi}{\lambda}\right)9\lambda}}\left(\frac{1}{2}\right)(1)\left\{\frac{1}{0-\cos\left(\frac{240}{2}\right)}-\frac{1}{0-\cos\left(\frac{420^\circ}{2}\right)}\right\}$$

$$= \hat{a}_z\,\frac{e^{-j\pi/4}}{2\pi(3)(2)}\left\{\frac{1}{(-1/2)}-\frac{1}{(-\sqrt{3}/2)}\right\} = +\hat{a}_z\,\frac{e^{-j\pi/4}}{6\pi(2)}\left\{\frac{2}{1}-\frac{2}{\sqrt{3}}\right\}$$

$$\underline{E}^d = \hat{a}_z\,\frac{e^{-j\pi/4}}{6\pi}\left\{1-\frac{1}{\sqrt{3}}\right\} = \hat{a}_z\,\frac{e^{-j\pi/4}}{6\pi}(1-0.57735) = \hat{a}_z\,\frac{e^{-j\pi/4}}{6\pi}(0.42265)$$

$$\underline{E}^d = \hat{a}_z\,0.02242\,e^{-j\pi/4} = \hat{a}_z\,0.01585(1-j)$$

$$\underline{E}^d(\phi=330^\circ, s=5\lambda) = \hat{a}_z\,0.02242 e^{-j\pi/4} = \hat{a}_z\,0.01585(1-j)$$

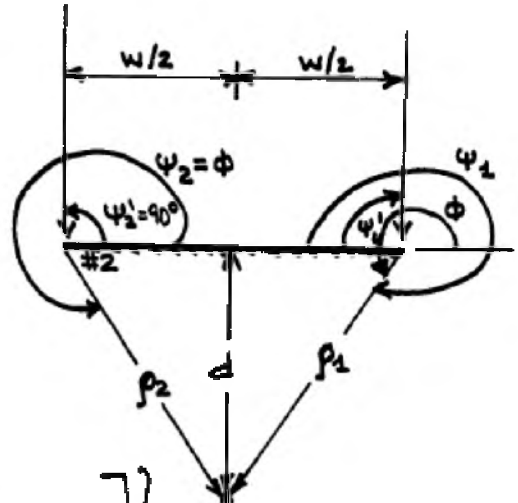
(b)

$$\frac{|\underline{E}^d|}{|\underline{E}^i|} = \frac{|0.02242e^{-j\pi/4}|}{|e^{+j\beta y}|} = 0.02242$$

$$\frac{|\underline{E}^d|}{|\underline{E}^i|}(\text{dB}) = 20\log_{10}(0.02242) = 20(-1.64936) = -32.987$$

$\boxed{13.39}$ $\quad U = U_1^d + U_2^d$

where U is used to represent the electric field for soft polarization and the magnetic field for the hard polarization.



(a)

$$U_1^d = U_0 \frac{e^{-j\beta\rho_1}}{\sqrt{\rho_1}} \left\{ \frac{e^{-j\pi/4}}{\sqrt{2\pi\beta}} \frac{1}{n} \sin\left(\frac{\pi}{n}\right) \cdot \left[\frac{1}{\cos\left(\frac{\pi}{n}\right) - \cos\left(\frac{\psi_1 - \psi_1'}{n}\right)} \pm \frac{1}{\cos\left(\frac{\pi}{n}\right) - \cos\left(\frac{\psi_1 + \psi_1'}{n}\right)} \right] \right\}_{\substack{n=2 \\ \psi_1' = 90°}}$$

$$U_1^d = U_0 \frac{e^{-j(\beta\rho_1 + \pi/4)}}{2\sqrt{2\pi\beta\rho_1}} \left[\frac{1}{\cos\left(\frac{\psi_1 - 90°}{2}\right)} \pm \frac{1}{\cos\left(\frac{\psi_1 + 90°}{2}\right)} \right]_{\psi_1 = \frac{5\pi}{2} - \phi}$$

$$U_2^d = -U_0 \frac{e^{-j(\beta\rho_2 + \pi/4)}}{2\sqrt{2\pi\beta\rho_1}} \left[\frac{1}{\cos\left(\frac{\psi_2 - 90°}{2}\right)} \pm \frac{1}{\cos\left(\frac{\psi_2 + 90°}{2}\right)} \right]_{\psi_2 = \phi}$$

(b) $w = 3\lambda$, $d = 5\lambda$ $\Rightarrow \rho_1 = \rho_2 = \sqrt{d^2 + (w/2)^2} = \sqrt{(5\lambda)^2 + (3\lambda/2)^2} = 5.22\lambda$

$$\psi_1 = \psi_2 = 360° - \tan^{-1}\left(d/w/2\right) = 360° - 73.3° = 286.7°$$

$$U_1^d = -\frac{e^{-j\left(\frac{2\pi}{\lambda} \cdot 5.22\lambda + \pi/4\right)}}{2(2\pi)\sqrt{5.22}} \left[\frac{1}{\cos\left(\frac{286.7 - 90}{2}\right)} \pm \frac{1}{\cos\left(\frac{286.7 + 90}{2}\right)} \right] = U_2^d$$

$$U^t = 2U_1^d = 2U_2^d = -\frac{e^{-j(10.69\pi)}}{2\pi\sqrt{5.22}} \left[\frac{1}{\cos(98.35)} \pm \frac{1}{\cos(188.35)} \right]$$

$$= -\frac{e^{-j0.69\pi}}{14.355} \left[-\frac{1}{0.1452} \pm \frac{1}{-0.9894} \right]$$

$$= -\frac{e^{-j124.2°}}{14.355} \left[-6.886 \mp 1.0107 \right]$$

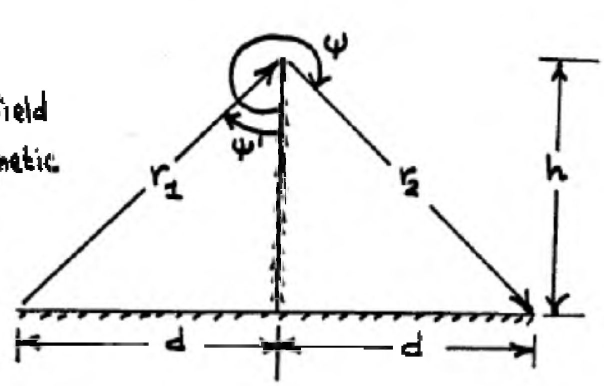
$$U^t = U_1^d + U_2^d = -0.0697\left(-6.886 \mp 1.0107\right) \underline{\angle -124.2°}$$

$$U^t(\text{soft}) = -0.0697(-6.886 + 1.0107) \underline{\angle -124.2°} = 0.4095 \underline{\angle -124.2°}$$

$$U^t(\text{hard}) = -0.0697(-6.886 - 1.0107) \underline{\angle -124.2°} = 0.5504 \underline{\angle -124.2°}$$

$\boxed{13.40}$ $U^d = U^i(Q_D)\, D^{s,h} A\, e^{-j\beta s}$

where U represents the electric field for soft polarization and the magnetic field for hard polarization.



(a) $U^i(Q_D) = \dfrac{e^{-j\beta r_1}}{r_1}$

$$A = \sqrt{\frac{s'}{s(s'+s)}} = \sqrt{\frac{r_1}{r_2(r_1+r_2)}}$$

$$A \overset{r_1=r_2}{=} \sqrt{\frac{1}{2r_1}} = \sqrt{\frac{1}{2r_2}}$$

$$D^{s,h} = \frac{e^{-j\pi/4}}{\sqrt{2\pi\beta}}\,\frac{1}{n}\sin\left(\frac{\pi}{n}\right)\left\{\frac{1}{\cos\left(\frac{\pi}{n}\right)-\cos\left(\frac{\psi-\psi'}{n}\right)} \mp \frac{1}{\cos\left(\frac{\pi}{n}\right)-\cos\left(\frac{\psi+\psi'}{n}\right)}\right\}_{n=2}$$

$$D^{s,h} = -\frac{e^{-j\pi/4}}{2\sqrt{2\pi\beta}}\left\{\frac{1}{\cos\left(\frac{\psi-\psi'}{2}\right)} \mp \frac{1}{\cos\left(\frac{\psi+\psi'}{2}\right)}\right\}$$

$$U^d_{s,h} = -\frac{e^{-j\beta r_1}}{r_1}\,\frac{e^{-j\pi/4}}{2\sqrt{2\pi\beta}}\left\{\frac{1}{\cos\left(\frac{\psi-\psi'}{2}\right)} \mp \frac{1}{\cos\left(\frac{\psi+\psi'}{2}\right)}\right\}\frac{1}{\sqrt{2r_2}}\,\frac{e^{-j\beta r_2}}{1}$$

$$U^d_{s,h} = -\frac{e^{-j[\beta(r_1+r_2)+\pi/4]}}{4r_1\sqrt{\pi\beta r_2}}\left[\frac{1}{\cos\left(\frac{\psi-\psi'}{2}\right)} \mp \frac{1}{\cos\left(\frac{\psi+\psi'}{2}\right)}\right]$$

(b) For $r_1 = r_2 = \sqrt{2}\,d$, $d = 5\lambda$, $h = 5\lambda$, $\psi' = 45°$, $\psi = 315° \Rightarrow \frac{\psi-\psi'}{2} = 135°$, $\frac{\psi+\psi'}{2} = 180°$

$$U^d_{s,h} = -\frac{e^{-j(2\beta r_2+\pi/4)}}{4r_2\sqrt{\pi\beta r_2}}\left[\frac{1}{\cos(135°)} \mp \frac{1}{\cos(180°)}\right] = -\frac{e^{-j(2\beta r_2+\pi/4)}}{4r_2\sqrt{\pi\beta r_2}}(-1.414 \pm 1)$$

$$\left|\frac{U^d_{s,h}}{U(2d)}\right| = \left|\frac{\frac{(-1.414\pm1)}{4r_2\sqrt{\pi\beta r_2}}}{\frac{1}{2d}}\right| = \left|\frac{(-1.414\pm1)(d)}{2r_2\sqrt{\pi\beta r_2}}\right|_{\substack{r_2=\sqrt{2}\,5\lambda \\ r_2=\sqrt{2}d}} = \left|\frac{(-1.414\pm1)}{2\sqrt{2}\sqrt{\pi\beta\sqrt{2}d}}\right| = \left|\frac{(-1.414\pm1)}{4\pi\sqrt{\sqrt{2}\,5}}\right|$$

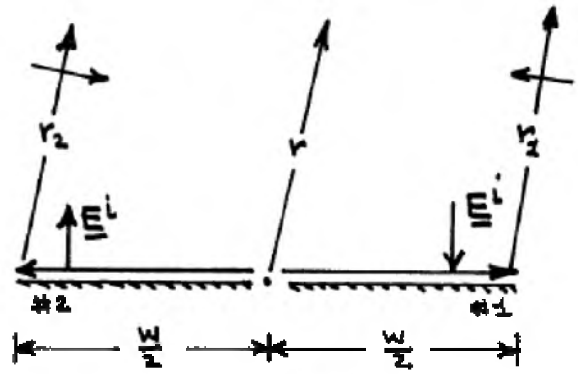
$$\left|\frac{U^d_{s,h}}{U(2d)}\right| = \left|\frac{(-1.414\pm1)}{33.416}\right|$$

$$\left|\frac{U^d_s}{U(2d)}\right| = \left|\frac{-1.414+1}{33.416}\right| = \frac{0.414}{33.416} = 0.01239 = -38.139\text{ dB}$$

$$\left|\frac{U^d_h}{U(2d)}\right| = \left|\frac{-1.414-1}{33.416}\right| = \frac{2.414}{33.416} = 0.07224 = -22.824\text{ dB}$$

13.41 The geometry for this problem

(a) is identical to that of Figure 13-32. Therefore the diffraction formulation is identical to that of Example 13-6 except for the following:



1. The incident field on wedge #2 is the opposite of that incident on wedge #1 (in Example 13-6 they were identical). Therefore the diffracted field for this problem will have an additional negative sign.

2. The incident GO field for this problem is given by

$$E^i(\theta_2) = \frac{1}{2} E_{\theta G}(\theta=\pi/2, r=w/2) = \frac{E_0}{2} \frac{\sin\left(\frac{\beta b}{2}\right)}{\frac{\beta b}{2}} \frac{e^{-j\beta w/2}}{w/2}$$

whereas for Example 13-6 was given by

$$E^i(\theta_1) = \frac{1}{2} E_{\theta G}(\theta=\pi/2, r=w/2) = \frac{E_0}{2} \frac{\sin\left(\frac{\beta b}{2}\right)}{\frac{\beta b}{2}} \frac{e^{-j\beta w/2}}{w/2}$$

Taking into account these differences and using the results of Example 13-6 on page 806, we can write for this problem that

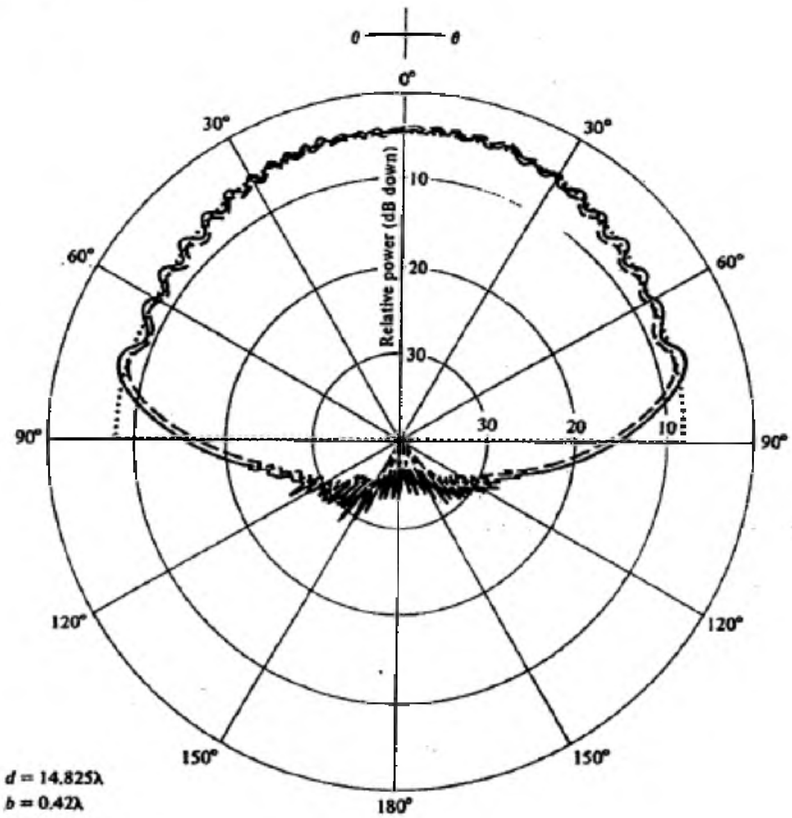
$$E^d_{\theta 1} = E_0 \frac{\sin\left(\frac{\beta b}{2}\right)}{\frac{\beta b}{2}} V_B^i\left(\frac{w}{2}, \psi_1, n_1=2\right) e^{+j(\beta w/2)\sin\theta} \frac{e^{-j\beta r}}{r}, \quad \psi_1 = \frac{\pi}{2}+\theta, \ 0\le\theta\le\pi$$

$$E^d_{\theta 2} = E_0 \frac{\sin\left(\frac{\beta b}{2}\right)}{\frac{\beta b}{2}} V_B^i\left(\frac{w}{2}, \psi_2, n_1=2\right) e^{-j(\beta w/2)\sin\theta} \frac{e^{-j\beta r}}{r}, \quad \psi_2 = \begin{cases} \frac{\pi}{2}-\theta, & 0\le\theta\le\pi/2 \\ \frac{5\pi}{2}-\theta, & \frac{\pi}{2}\le\theta\le\pi \end{cases}$$

(b) The normalized amplitude pattern (in decibels) for $0^\circ\le\theta\le180^\circ$ when $w/2 = 14.825\lambda$ and $b = 0.422\lambda$ is shown on the next page.

Cont'd.

13.41 cont'd.



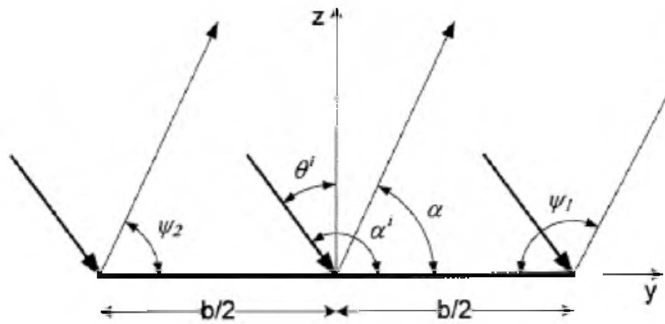
$d = 14.825\lambda$
$b = 0.42\lambda$
$f = 35$ GHz

———— Experiment

— — — Theory (GO and GTD)

········ Theory (GO)

$\boxed{13.42}$ Uniform plane wave incident upon a rectangular ground plane.



$\theta^i \in [0^\circ, 90^\circ]$; $\alpha \in [0^\circ, 180^\circ]$; Plate Dimensions $(a \times b)$

$\theta^i = \alpha^i - 90^\circ$; $\psi_1 = 180^\circ - \alpha$; $\psi_2 = \alpha$

$$\boxed{(\sigma_{3-D}) \cong \sigma_{2-D}(2l^2/\lambda) = \sigma_{2-D}(2a^2/\lambda)}$$

$$\boxed{\sigma_{2-D} = \lim_{r\to\infty}\left[2\pi\rho\frac{|E^s|^2}{|E^i|^2}\right] = \lim_{r\to\infty}\left[2\pi\rho\frac{|H^s|^2}{|H^i|^2}\right]}$$

$$E^i = E_0 e^{j\beta(y\cos\alpha' + z\sin\alpha')}; \qquad |H^i|^2 = \left(\frac{E_0}{\eta}\right)^2; \qquad |E^i|^2 = E_0^2$$

For UTD

Hard polarization	Soft polarization
$\lvert E^s\rvert^2 = \left\lvert E_{hard}^{1st(1,2)} + E_{hard}^{2nd(1,2)}\right\rvert^2$	$\lvert E^s\rvert^2 = \left\lvert E_{soft}^{1st(1,2)}\right\rvert^2$

For Physical Optics

Monostatic RCS (soft and hard polarizations)

$$\boxed{\sigma_{3-D} = 4\pi\left(\frac{ab}{\lambda}\right)^2 \cos^2\theta_i \left|\frac{\sin(\beta b\sin\theta_i)}{\beta b\sin\theta_i}\right|^2}$$

Bistatic RCS (hard/TE^x polarization)

$$\boxed{\sigma_{3-D} = 4\pi\left(\frac{ab}{\lambda}\right)^2 \cos^2\theta^s \left|\frac{\sin\left(\frac{\beta b}{2}(\sin\theta^s - \sin\theta^i)\right)}{\frac{\beta b}{2}(\sin\theta^s - \sin\theta^i)}\right|^2}$$

Cont'd

13.42 cont'd

Hard polarization (TE^x)

- **First-order diffractions from edges #1 and #2 for general bistatic scattering.**

Use H field.

$$H_{hard}^{1st(1,2)} = H_{hard}^{d1} = H_1^d + H_2^d$$

Hard polarization

H_1^d	H_2^d
$Q1 = (x=0, y=b/2, z=0)$	$Q2 = (x=0, y=-b/2, z=0)$
$H_1^d = \bar{H}^i(Q1) D_1^h A_1 e^{-j\beta r_1}$	$H_2^d = H^i(Q2) D_2^h A_2 e^{-j\beta r_2}$
$H^i(Q1) = \frac{E_0}{\eta} e^{j\beta(b/2)\cos\alpha'}$	$H^i(Q2) = \frac{E_0}{\eta} e^{-j\beta(b/2)\cos\alpha'}$
$r_1 = r - (b/2)\cos\alpha$ phase $r_1 = r$ amplitude	$r_2 = r + (b/2)\cos\alpha$ phase $r_2 = r$ amplitude
$A_1 = \frac{1}{\sqrt{r_1}} = \frac{1}{\sqrt{r}}$	$A_2 = \frac{1}{\sqrt{r_2}} = \frac{1}{\sqrt{r}}$
$A_1 e^{-j\beta r_1} = e^{j\beta(b/2)\cos\alpha} \frac{e^{-j\beta r}}{\sqrt{r}}$	$A_2 e^{j\beta r_2} = e^{-j\beta(b/2)\cos\alpha} \frac{e^{-j\beta r}}{\sqrt{r}}$
Keller $D_1^h(180^\circ - \alpha, 180^\circ - \alpha^i, 2)$	Keller $D_2^h(\alpha, \alpha^i, 2)$

$$\boxed{H_{hard}^{1st(1,2)} = \frac{E_0}{\eta}\left(D_1^h e^{j\beta(b/2)\gamma^+} + D_2^h e^{-j\beta(b/2)\gamma^+}\right)\frac{e^{-j\beta r}}{\sqrt{r}}}$$

$$\gamma^+ = \cos\alpha' + \cos\alpha$$

Cont'd

13.42 cont'd

- **Second-order bistatic diffractions from edges #1 and #2.**

Use H field

$$H_{hard}^{2nd(1,2)} = H_{hard}^{d2} = H_{21r}^{d} + H_{12r}^{d}$$

Hard polarization

H_{21r}^{d}	H_{12r}^{d}
$Q1 = (x = 0, y = b/2, z = 0)$	$Q2 = (x = 0, y = -b/2, z = 0)$
$H_{21r}^{d} = H_{i21}^{d} D_{21r}^{h} A_{21r} e^{-j\beta r_1}$	$H_{12r}^{d} = H_{i12}^{d} D_{12r}^{h} A_{12r} e^{-j\beta r_2}$
$r_1 = r - (b/2)\cos\alpha$ phase $r_1 = r$ amplitude	$r_2 = r + (b/2)\cos\alpha$ phase $r_2 = r$ amplitude
$A_{21r} = \frac{1}{\sqrt{r_1}} = \frac{1}{\sqrt{r}}$	$A_{12r} = \frac{1}{\sqrt{r_2}} = \frac{1}{\sqrt{r}}$
$A_{21r} e^{-j\beta r_1} = e^{j\beta(b/2)\cos\alpha} \frac{e^{-j\beta r}}{\sqrt{r}}$	$A_{12r} e^{-j\beta r_2} = e^{-j\beta(b/2)\cos\alpha} \frac{e^{-j\beta r}}{\sqrt{r}}$
UTD $D_{21r}^{h}(b, 180^\circ - \alpha, 0, 90, 2)$	UTD $D_{12r}^{h}(b, \alpha, 0, 90, 2)$

Cont'd

13.42 cont'd

where

H_{i21}^{d}	H_{i12}^{d}
$Q2=\left(x=0,y=-b/2,z=0\right)$	$Q1=\left(x=0,y=b/2,z=0\right)$
$H_{i21}^{d}=H^{i}(Q2)D_{i21}^{h}A_{i21}e^{-j\beta r_2}$	$H_{i12}^{d}=H^{i}(Q1)D_{i12}^{h}A_{i12}e^{-j\beta r_1}$
$H^{i}(Q2)=\dfrac{E_0}{\eta}e^{-j\beta(b/2)\cos\alpha^{i}}$	$H^{i}(Q1)=\dfrac{E_0}{\eta}e^{j\beta(b/2)\cos\alpha^{i}}$
$r_2=b$ phase, amplitude	$r_1=b$ phase, amplitude
$A_{i21}=\dfrac{1}{\sqrt{r_2}}=\dfrac{1}{\sqrt{b}}$	$A_{i12}=\dfrac{1}{\sqrt{r_1}}=\dfrac{1}{\sqrt{b}}$
$A_{i21}e^{-j\beta r_2}=\dfrac{e^{-j\beta b}}{\sqrt{b}}$	$A_{i12}e^{-j\beta r_1}=\dfrac{e^{-j\beta b}}{\sqrt{b}}$
UTD $D_{i21}^{h}\left(b,0,\alpha^{i},90,2\right)$	UTD $D_{i12}^{h}\left(b,0,180^\circ-\alpha^{i},90,2\right)$

$$\boxed{H_{21r}^{d}=\frac{E_0}{\eta}\frac{e^{-j\beta b}}{\sqrt{b}}e^{-j\beta(b/2)\gamma^{-}}D_{i21}^{h}D_{21r}^{h}\frac{e^{-j\beta r}}{\sqrt{r}}}$$

$$\boxed{H_{12r}^{d}=\frac{E_0}{\eta}\frac{e^{-j\beta b}}{\sqrt{b}}e^{j\beta(b/2)\gamma^{-}}D_{i12}^{h}D_{12r}^{h}\frac{e^{-j\beta r}}{\sqrt{r}}}$$

$$\boxed{H_{hard}^{2nd(1,2)}=\frac{E_0}{\eta}\frac{e^{-j\beta b}}{\sqrt{b}}\left(D_{i21}^{h}D_{21r}^{h}e^{-j\beta(b/2)\gamma^{-}}+D_{i12}^{h}D_{12r}^{h}e^{j\beta(b/2)\gamma^{-}}\right)\frac{e^{-j\beta r}}{\sqrt{r}}}$$

$$\gamma^{-}=\cos\alpha^{i}-\cos\alpha$$

- **Scattering bistatic patterns.**

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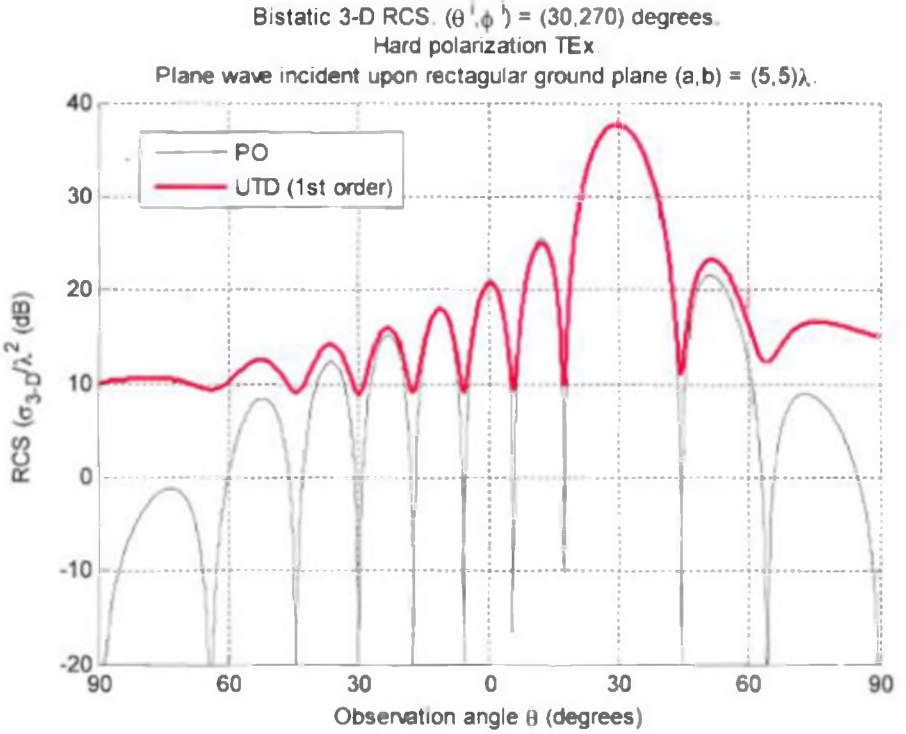
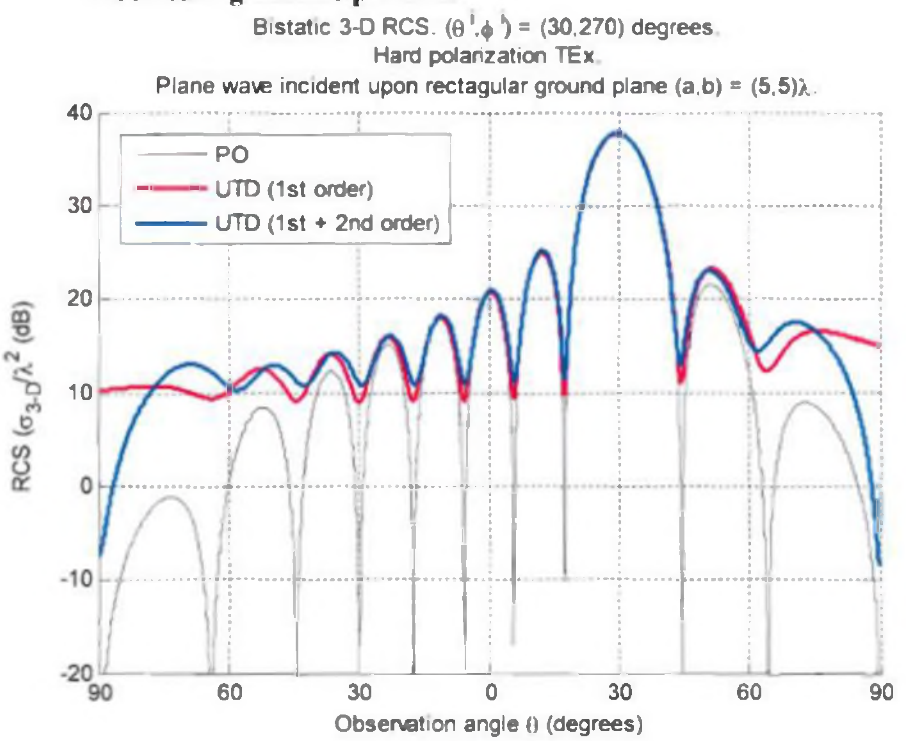
13.42 cont'd



• **Scattering bistatic patterns.**



• **Scattering monostatic patterns.**

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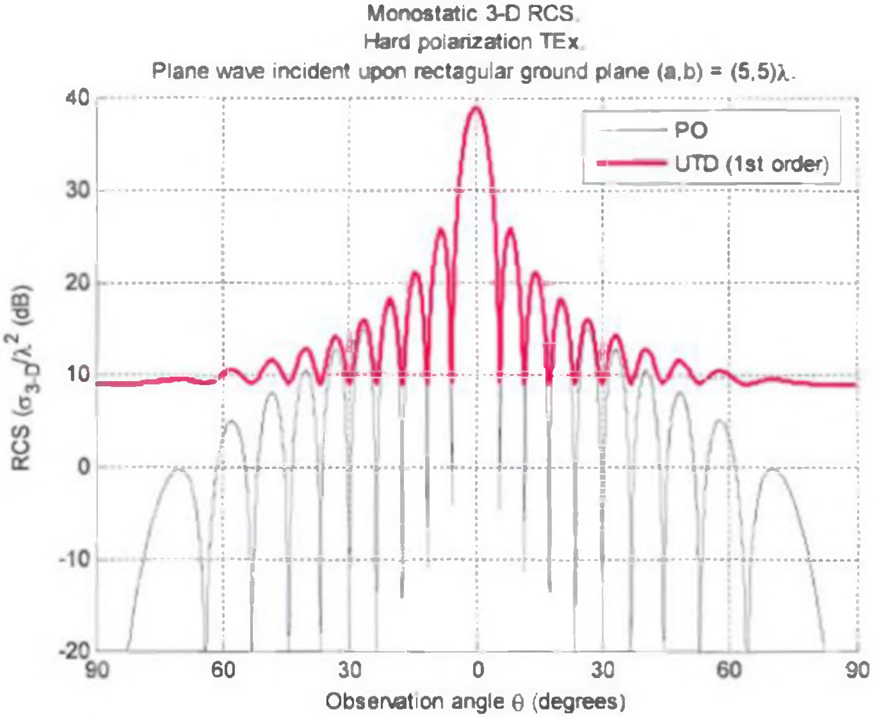
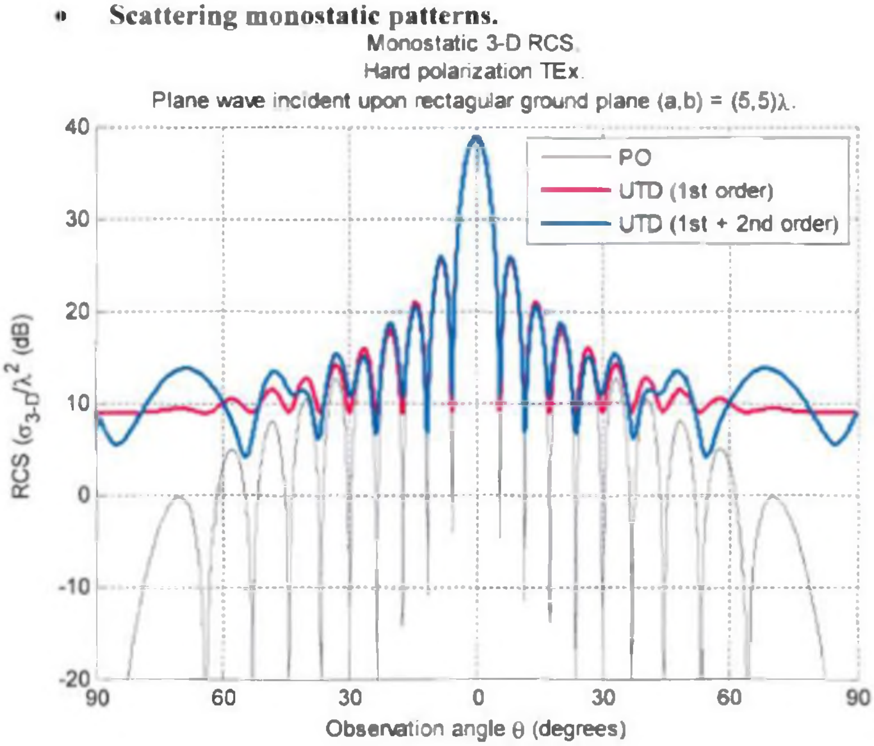
13.42 cont'd



- **Scattering monostatic patterns.**



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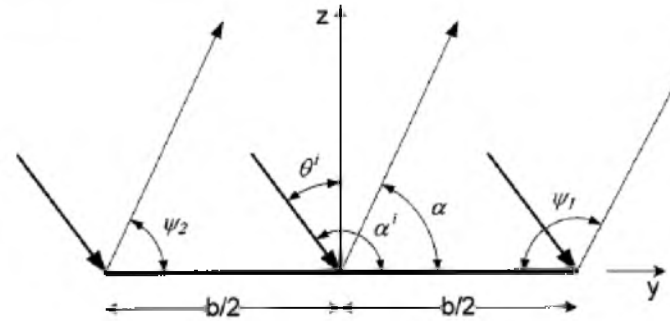
13.43 Uniform plane wave incident upon a rectangular ground plane.



$\theta^i \in [0^\circ, 90^\circ]$; $\alpha \in [0^\circ, 180^\circ]$; Plate Dimensions $(a \times b)$

$\theta^i = \alpha^i - 90^\circ$; $\quad \psi_1 = 180^\circ - \alpha$; $\quad \psi_2 = \alpha$

$$\boxed{(\sigma_{3-D} / \lambda^2) \cong \sigma_{2-D} 2l^2 = \sigma_{2-D} 2a^2}$$

$$\boxed{\sigma_{2-D} = \lim_{r \to \infty} \left[2\pi r \frac{|E^s|^2}{|E^i|^2} \right] = \lim_{r \to \infty} \left[2\pi r \frac{|H^s|^2}{|H^i|^2} \right]}$$

$$E^i = E_0 e^{j\beta(y\cos\alpha^i + z\sin\alpha^i)}; \quad |H^i|^2 = \left(\frac{E_0}{\eta} \right)^2; \quad |E^i|^2 = E_0^2$$

For UTD

Hard polarization	Soft polarization
$\lvert E^s \rvert^2 = \left\lvert E_{hard}^{1st(1,2)} + E_{hard}^{2nd(1,2)} \right\rvert^2$	$\lvert E^s \rvert^2 = \left\lvert E_{soft}^{1st(1,2)} \right\rvert^2$

For Physical optics

Monostatic RCS (soft/TM^x and hard/TE^x polarizations)

$$\boxed{\sigma_{3-D} = 4\pi \left(\frac{ab}{\lambda} \right)^2 \cos^2 \theta_i \left| \frac{\sin(\beta b \sin\theta_i)}{\beta b \sin\theta_i} \right|^2}$$

Bistatic RCS (soft/TM^x polarization)

$$\boxed{\sigma_{3-D} = 4\pi \left(\frac{ab}{\lambda} \right)^2 \cos^2 \theta^i \left| \frac{\sin\left(\frac{\beta b}{2} (\sin\theta^s - \sin\theta^i) \right)}{\frac{\beta b}{2} (\sin\theta^s - \sin\theta^i)} \right|^2}$$

Cont'd

13.43 cont'd

$\boxed{\text{Soft polarization (TM}^x\text{)}}$

- **First-order diffractions from edges #1 and #2 for general bistatic scattering.**

Use E field

$$E_{hard}^{1st(1,2)} = E_{hard}^{d1} = E_1^d + E_2^d$$

Soft polarization

E_1^d	E_2^d
$Q1 = (x = 0, y = b/2, z = 0)$	$Q2 = (x = 0, y = -b/2, z = 0)$
$E_1^d = E^i(Q1) D_1^s A_1 e^{-j\beta r_1}$	$E_2^d = E^i(Q2) D_2^s A_2 e^{-j\beta r_2}$
$E^i(Q1) = E_0 e^{j\beta(b/2)\cos\alpha'}$	$E^i(Q2) = E_0 e^{-j\beta(b/2)\cos\alpha'}$
$r_1 = r - (b/2)\cos\alpha$ phase $r_1 = r$ amplitude	$r_2 = r + (b/2)\cos\alpha$ phase $r_2 = r$ amplitude
$A_1 = \dfrac{1}{\sqrt{r_1}} = \dfrac{1}{\sqrt{r}}$	$A_2 = \dfrac{1}{\sqrt{r_2}} = \dfrac{1}{\sqrt{r}}$
$A_1 e^{-j\beta r_1} = e^{j\beta(b/2)\cos\alpha} \dfrac{e^{-j\beta r}}{\sqrt{r}}$	$A_2 e^{j\beta r_2} = e^{-j\beta(b/2)\cos\alpha} \dfrac{e^{-j\beta r}}{\sqrt{r}}$
Keller $D_1^s(180^\circ - \alpha, 180^\circ - \alpha^i, 2)$	Keller $D_2^s(\alpha, \alpha^i, 2)$

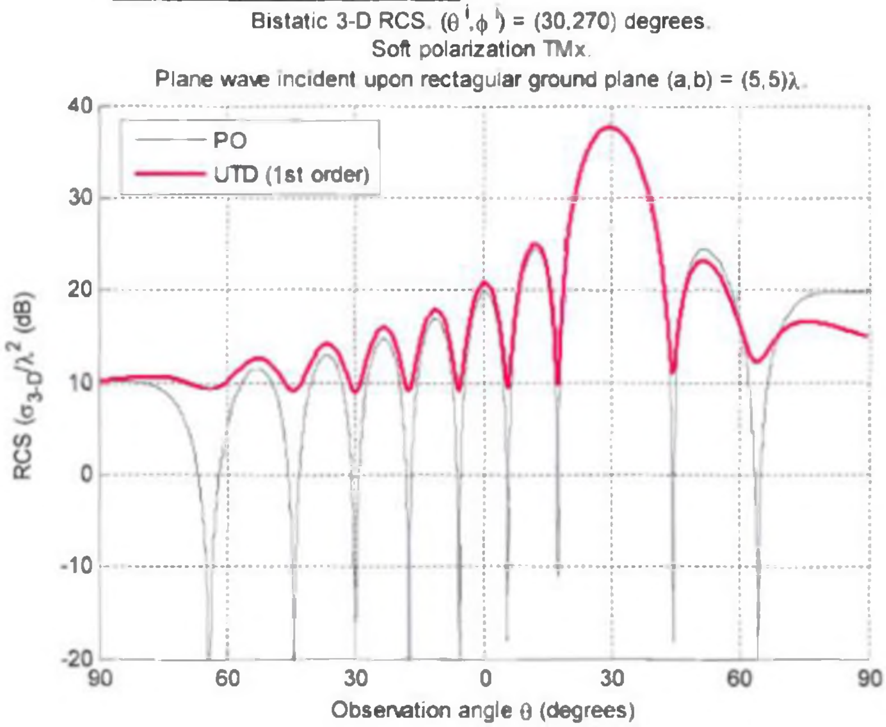
$$\boxed{E_{hard}^{1st(1,2)} = E_0 \left(D_1^s e^{j\beta(b/2)\gamma^+} + D_2^s e^{-j\beta(b/2)\gamma^+} \right) \frac{e^{-j\beta r}}{\sqrt{r}}}$$

$$\gamma^+ = \cos\alpha' + \cos\alpha$$

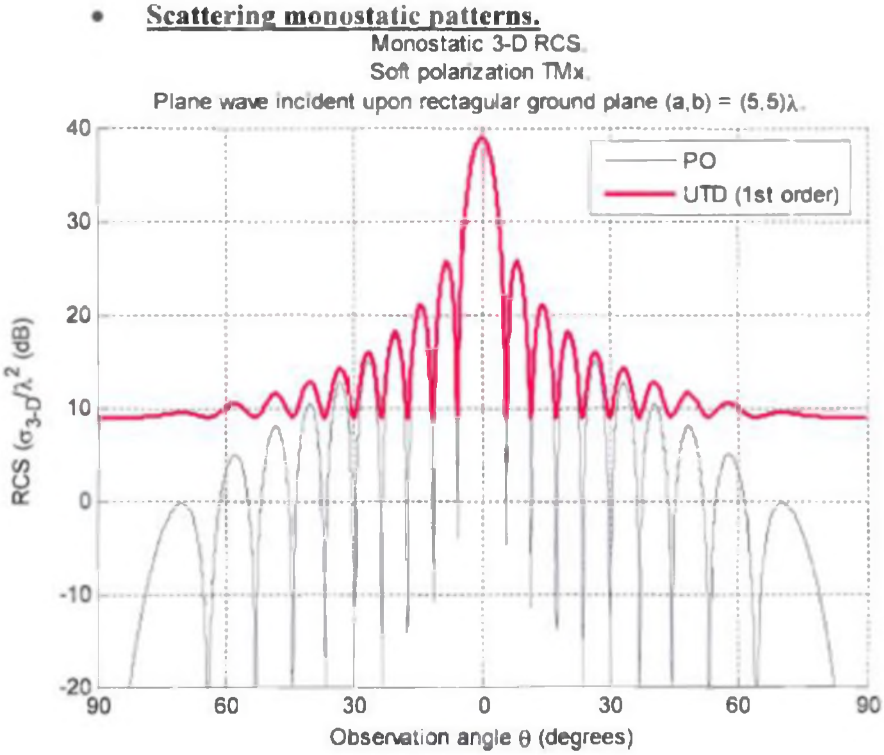
cont'd

13.43 cont'd

- **Scattering bistatic patterns.**



- **Scattering monostatic patterns.**



$\boxed{13.44}$ $\underline{E}^d(s) = \underline{E}^i(Q_D)\cdot\tilde{D}\,A(s',s)\,e^{-j\beta s}$

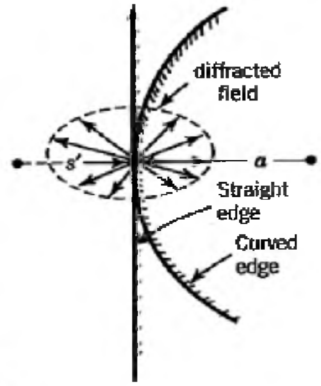


Figure P13-44

(a) For a straight edge: $A_s(s,s') = \frac{\sqrt{s'}}{s}$

For a curved edge: $A_c(s,s') = \frac{1}{s}\sqrt{\rho_c}$

$$\frac{1}{\rho_c} = \frac{1}{\rho_e} - \frac{\hat{n}_e\cdot(\hat{s}'-\hat{s})}{\rho_g} = \frac{1}{s'} - \frac{\hat{n}_e\cdot\hat{s}'-\hat{n}_e\cdot\hat{s}}{a}$$

$$\frac{1}{\rho_c} = \frac{1}{s'} - \frac{\cos(180^\circ)-\cos(0)}{a} = \frac{1}{s'} - \frac{-1-1}{a} = \frac{1}{s'}+\frac{2}{a} = \frac{a+2s'}{as'}$$

$$\frac{A_c(s,s')}{A_s(s,s')} = \frac{\sqrt{\rho_c}/s}{\sqrt{s'}/s} = \frac{\sqrt{\rho_c}}{\sqrt{s'}} = \frac{1}{\sqrt{s'}}\sqrt{\frac{as'}{a+2s'}} = \sqrt{\frac{a}{a+2s'}} \quad\Bigg|\quad \rho_c = \frac{as'}{a+2s'}$$

(b) $$\frac{A_c(s,s')}{A_s(s,s')} = \sqrt{\frac{a}{a+2s'}}\,\Bigg|_{\substack{a=25\lambda\\ s'=100\lambda}} = \sqrt{\frac{25}{25+200}} = \sqrt{\frac{25}{225}} = \sqrt{\frac{1}{9}} = \frac{1}{3} = 20\log_{10}\left(\frac{1}{3}\right) = 20(-0.47712)$$

$$\frac{A_c(s,s')}{A_s(s,s')} = \frac{1}{3} = 20(-0.47712) = -9.54 \text{ dB}$$

$\boxed{13.45}$ For uniform plane wave illumination

$$A_s(s,s')\Big|_{\substack{\text{straight}\\ \text{edge}}} = \frac{1}{\sqrt{s}}, \quad A_c(s,s')\Big|_{\substack{\text{curved}\\ \text{edge}}} = \frac{1}{s}\sqrt{\rho_c}$$

$$\frac{1}{\rho_c} = \frac{1}{\rho_e} - \frac{\hat{n}\cdot(\hat{s}'-\hat{s})}{\rho_g\sin^2\beta_0}\Bigg|_{\substack{\beta_0=90^\circ\\ \rho_e=\infty\text{ (plane wave)}\\ \rho_g=a}} = -\frac{\hat{n}\cdot\hat{s}'-\hat{n}\cdot\hat{s}}{a} = -\frac{-\hat{a}_x\cdot\hat{a}_x-[-\hat{a}_x\cdot(-\hat{a}_x)]}{a} = -\frac{-1-1}{a}$$

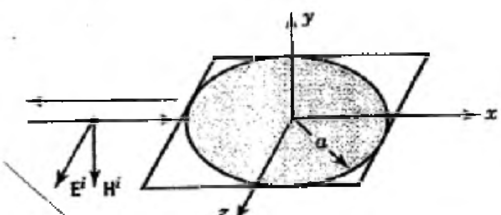


Figure P13-45

$$\frac{1}{\rho_c} = -\frac{-1-1}{a} = \frac{2}{a} \Rightarrow \rho_c = \frac{a}{2}$$

$$\frac{A_c(s,s')}{A_s(s,s')} = \frac{\sqrt{\rho_c}/s}{1/\sqrt{s}} = \sqrt{\frac{\rho_c}{s}} = \sqrt{\frac{a}{2s}}$$

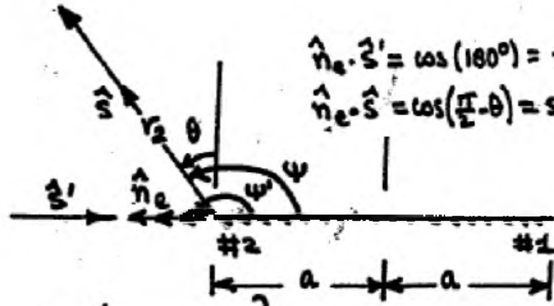
$$\frac{A_c}{A_s} = -20\text{ dB} = 20\log_{10}\left(\frac{A_c}{A_s}\right) \Rightarrow \frac{A_c}{A_s} = \frac{1}{10} = \sqrt{\frac{a}{2s}}$$

$$\frac{a}{2s} = \left(\frac{1}{10}\right)^2 = \frac{1}{100} \Rightarrow a = \frac{2s}{100} = \frac{2(50\lambda)}{100} = \lambda$$

$$\boxed{a=\lambda}$$

$\hat{n}_e \cdot \hat{s}' = \cos(180^\circ) = -1$

$\hat{n}_e \cdot \hat{s} = \cos\left(\frac{\pi}{2}-\theta\right) = \sin\theta$



$\boxed{13.46}$ $\underline{U}^d_{s,h} = \underline{U}^i(\theta_2)\cdot\underline{\underline{D}}^{s,h} A\, e^{-j\beta r_2}$

(a) $U^d_{s,h} = U^i(\theta_2) D^{s,h} A\, e^{-j\beta r_2}$

$$D^{s,h} = \frac{e^{-j\pi/4}}{\sqrt{2\pi\beta}}\,\frac{1}{n}\sin\left(\frac{\pi}{n}\right)\left\{\frac{1}{\cos\left(\frac{\pi}{n}\right)-\cos\left(\frac{\psi-\psi'}{n}\right)} \mp \frac{1}{\cos\left(\frac{\pi}{n}\right)-\cos\left(\frac{\psi+\psi'}{n}\right)}\right\}_{\substack{n=2\\ \psi'=\pi\\ \psi=\frac{\pi}{2}+\theta}} \left\}\begin{matrix}\psi-\psi'=\theta-\frac{\pi}{2}\\ \psi+\psi'=\theta+\frac{3\pi}{2}\end{matrix}\right.$$

$$D^{s,h} = -\frac{e^{-j\pi/4}}{2\sqrt{2\pi\beta}}\left\{\frac{1}{\cos\left(\frac{\theta-\pi/2}{2}\right)} \mp \frac{1}{\cos\left(\frac{\theta+3\pi/2}{2}\right)}\right\}$$

$$A = \sqrt{\frac{\rho_c}{r_2(\rho_c+r_2)}} \overset{r_2 \gg \rho_c}{\simeq} \frac{\sqrt{\rho_c}}{r_2}$$

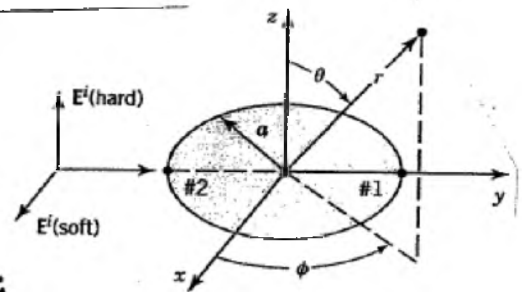


Figure P13-46

$$\frac{1}{\rho_c} = \frac{1}{\rho_e} - \left.\frac{\hat{n}_e\cdot(\hat{s}'-\hat{s})}{\rho_g \sin^2\beta_0'}\right|_{\beta_0'=90^\circ} = \frac{1}{\rho_e} - \frac{\hat{n}_e\cdot\hat{s}'-\hat{n}_e\cdot\hat{s}}{\rho_g},$$

$$\frac{1}{\rho_c} = \frac{1}{\infty} - \frac{-1-\sin\theta}{a} = \frac{1+\sin\theta}{a} \Rightarrow \rho_c = \frac{a}{1+\sin\theta}$$

$$A = \frac{1}{r_2}\sqrt{\frac{a}{1+\sin\theta}}$$

$$U^d_{s,h} = -\frac{e^{-j\pi/4}}{2\sqrt{2\pi\beta}}\sqrt{\frac{a}{1+\sin\theta}}\left\{\frac{1}{\cos\left(\frac{\theta-\pi/2}{2}\right)} \mp \frac{1}{\cos\left(\frac{\theta+3\pi/2}{2}\right)}\right\}\frac{e^{-j\beta r_2}}{r_2}$$

(b) The caustic is located at a point ρ_c to the right of point #2 (i.e., between the diffraction point and the center of the plate). From part a, the caustic is at

$$\rho_c = \frac{a}{1+\sin\theta}$$

from the edge of wedge #2.

(c) For $\theta = \pi/2$, $r_2 = 5\lambda$ and $a = 2\lambda$ when $U(\theta_1) = 10^{-3}$ V/m

$$U^d_{s,h} = \frac{10^{-3}e^{-j\pi/4}}{2\sqrt{2\pi\beta}}\sqrt{\frac{a}{2}}\left\{\frac{1}{\cos(0^\circ)} \mp \frac{1}{\cos(\pi)}\right\}\frac{e^{-j\beta r_2}}{r_2} = -\frac{10^{-3}e^{-j\pi/4}}{2\sqrt{2\pi\beta}}\sqrt{\frac{a}{2}}\left\{1 \pm 1\right\}\frac{e^{-j\beta r_2}}{r_2}$$

$$U^d_{s,h} = -\frac{10^{-3}e^{-j\pi/4}}{2(2\pi)}\sqrt{\frac{2}{2}}\left\{1 \pm 1\right\}\frac{e^{-j2\pi(5)}}{(5)}$$

$$U^d_s = -\frac{10^{-3}e^{-j\pi/4}}{2(2\pi)}(2)\frac{e^{-j10\pi}}{5} = -\frac{10^{-3}e^{-j\pi/4}}{10\pi} = -\frac{10^{-4}}{\pi}e^{-j\pi/4}$$

$$U^d_h = -\frac{10^{-3}e^{-j\pi/4}}{2(2\pi)}\left\{1-1\right\}\frac{e^{-j10\pi}}{5} = 0$$

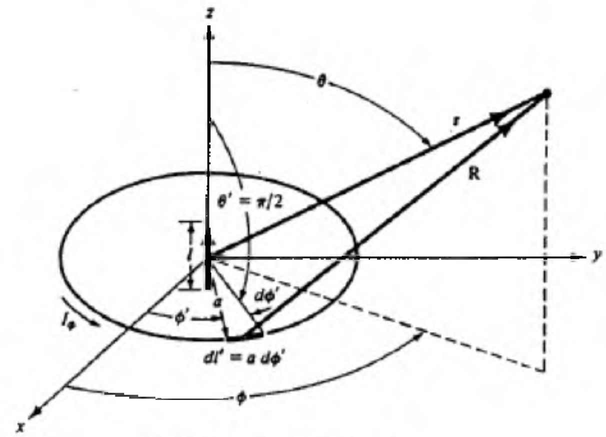
$\boxed{13.47}$ The vector potential is given by (6-97a) or

$$\underline{A} = \frac{\mu}{4\pi}\int_C \underline{I}_e \frac{e^{-j\beta R}}{R}\,dl' \qquad , \; dl' = a\,d\phi' \text{ for a circle}$$

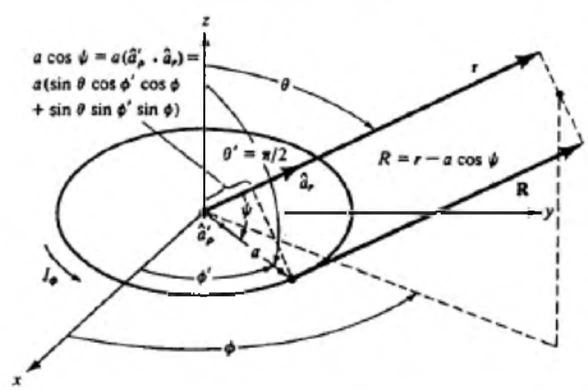
where is represented, in general, by

$$R = \left[r^2 + a^2 - 2ra\sin\theta\cos(\phi-\phi')\right]^{1/2} \underset{}{\overset{r \gg a}{\simeq}} r - a\sin\theta\cos(\phi-\phi')$$

R is the distance from any point on the loop to the observation point, r is the distance from the center to the observation point and a is the radius. See figure below for geometry.



(a) Geometry for circular loop



(b) Geometry for far-field observations

Geometrical arrangement for loop antenna analysis.

In general

$$\underline{I}_e(x',y',z') = \hat{a}_x I_x^e + \hat{a}_y I_y^e + \hat{a}_z I_z^e$$

Using the transformation of (II-7a) or (II-7b)

$$I_x^e = I_\rho^e \cos\phi' - I_\phi^e \sin\phi'$$
$$I_y^e = I_\rho^e \sin\phi' + I_\phi^e \cos\phi'$$
$$I_z^e = I_z^e$$

Cont'd.

$\boxed{\text{13.47 Cont'd.}}$ and that of (II-13b) for the unit vectors, or

$$\hat{a}_x = \hat{a}_r \sin\theta\cos\phi + \hat{a}_\theta \cos\theta\cos\phi - \hat{a}_\phi \sin\phi$$
$$\hat{a}_y = \hat{a}_r \sin\theta\sin\phi + \hat{a}_\theta \cos\theta\sin\phi + \hat{a}_\phi \cos\phi$$
$$\hat{a}_z = \hat{a}_r \cos\theta \qquad - \hat{a}_\theta \sin\theta$$

we can write that

$$\underline{I}_e = \hat{a}_r\left[I_\rho^e \sin\theta\cos(\phi-\phi') + I_\phi^e \sin\theta\sin(\phi-\phi') + I_z^e\cos\theta\right]$$
$$+\hat{a}_\theta\left[I_\rho^e\cos\theta\cos(\phi-\phi') + I_\phi^e\cos\theta\sin(\phi-\phi') - I_z^e\sin\theta\right]$$
$$+\hat{a}_\phi\left[-I_\rho^e\sin(\phi-\phi') + I_\phi^e\cos(\phi-\phi')\right]$$

For a circular loop the current is flowing in the ϕ direction (I_ϕ) so that

$$\underline{I}_e = \hat{a}_r I_\phi^e \sin\theta\sin(\phi-\phi') + \hat{a}_\theta I_\phi^e\cos\theta\sin(\phi-\phi') + \hat{a}_\phi I_\phi^e\cos(\phi-\phi')$$

The only component which contributes to the integral for the potential $\underline{A}$ is the ϕ component; the radial and θ components integrate to zero (see Chapter 5, pp. 232–237 of <u>Antenna Theory: Analysis and Design</u> by C.A. Balanis, John Wiley, 2005). Thus

$$\underline{A} \simeq \hat{a}_\phi \frac{\mu a e^{-j\beta r}}{4\pi r}\int_0^{2\pi} I_\phi^e\, e^{j\beta a\sin\theta\cos(\phi-\phi')}\cos(\phi-\phi')\,d\phi'$$

The far zone field is obtained by using (6-101a) or

$$E_r^e \simeq 0,\quad E_\theta^e \simeq -j\omega A_\theta = 0$$
$$E_\phi^e \simeq -j\omega A_\phi = -j\frac{\omega\mu a e^{-j\beta r}}{4\pi r}\int_0^{2\pi} I_\phi^e\cos(\phi-\phi')\, e^{j\beta a\sin\theta\cos(\phi-\phi')}\,d\phi'$$

Making use of the duality theorem as outlined in Table 7-2, we can write that for a magnetic current I_ϕ^m the magnetic field is given by

$$H_r^m \simeq H_\theta^m \simeq 0$$
$$H_\phi^m = -j\frac{\omega\varepsilon a e^{-j\beta r}}{4\pi r}\int_0^{2\pi} I_\phi^m\cos(\phi-\phi')\,e^{j\beta a\sin\theta\cos(\phi-\phi')}\,d\phi'$$

Cont'd.

$\boxed{13.47 \text{ cont'd.}}$ For a uniform current I_ϕ^e and I_ϕ^m are constants. Therefore we can choose any observation point because of symmetry (let $\phi = 0$). Therefore

$$E_\phi^e \simeq -j\frac{\omega \mu a e^{-j\beta r}}{4\pi r} I_\phi^e \times \int_0^{2\pi} e^{j\beta a \sin\theta\cos\phi'} \cos\phi' \, d\phi'$$

Since $\int_0^{2\pi} \cos\phi' e^{j\beta a \sin\theta \cos\phi'} d\phi' = j 2\pi J_1(\beta a \sin\theta)$ where $J_1(x)$ is the Bessel function of the first kind of order 1.

Thus

$$E_\phi^e = \frac{a\omega\mu e^{-j\beta r}}{2r} I_\phi^e \, J_1(\beta a \sin\theta)$$

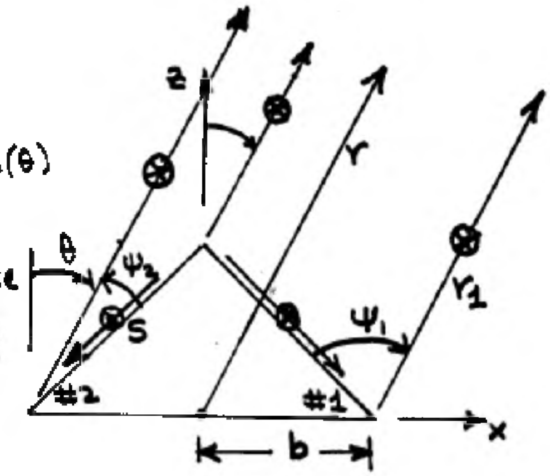
Similarly

$$H_\phi^m = \frac{a\omega\varepsilon e^{-j\beta r}}{2r} I_\phi^m \, J_1(\beta a \sin\theta)$$

$\boxed{13.48}$

(a)

$$H_{\phi G} = R(\theta) e^{j\beta S\cos(\alpha/2)\cos(\theta)}$$



The field radiated by the source in the direction of wedge #1 is given by

$$H_{\phi G}^i = R(\theta = \pi - \tfrac{\alpha}{2}) \frac{e^{-j\beta s}}{s}$$

The field diffracted by wedge #1 is given by

$$H_\phi^d = H_{\phi G}^i D_h(s, \psi, n) A(s, r_1) e^{-j\beta r_1}$$

$$= R(\theta = \pi - \tfrac{\alpha}{2}) \frac{e^{-j\beta s}}{s} D_h(s,\psi,n) \frac{\sqrt{s}}{r_1} e^{-j\beta r_1} = R(\theta = \pi - \tfrac{\alpha}{2}) \left[D_h(s,\psi,n) \frac{e^{-j\beta s}}{\sqrt{s}} \right] \frac{e^{-j\beta r_1}}{r_1}$$

$$= R(\theta = \pi - \tfrac{\alpha}{2}) \left[2 D^i(s,\psi,n) \frac{e^{-j\beta s}}{\sqrt{s}} \right] \frac{e^{-j\beta r_1}}{r_1} = 2R(\theta = \pi - \tfrac{\alpha}{2}) \left[D^i(s,\psi,n) \frac{e^{-j\beta s}}{\sqrt{s}} \right] \frac{e^{-j\beta r_1}}{r_1}$$

$$H_\phi^d = 2R(\theta = \pi - \tfrac{\alpha}{2}) V_B^i(s,\psi,n) \frac{e^{-j\beta r_1}}{r_1}$$

Since

$r_1 \simeq r - b\sin\theta\cos\phi'$ for phase terms

$r_1 \simeq r$ for amplitude terms

then

$$H_\phi^d = 2R(\theta = \pi - \tfrac{\alpha}{2}) V_B^i(s,\psi,n) \frac{e^{-j\beta r}}{r} e^{j\beta b \sin\theta\cos\phi'}$$

cont'd.

13.48 cont'd. $H_\phi^d = 2R(\theta=\pi-\frac{\alpha}{2})\frac{e^{-j\beta r}}{r} V_B^i(s,\psi,n)e^{j\beta b\sin\theta\cos\phi'}$

Since the above is only the contribution from only one point on the rim, the total field is the sum from all points or in the limit the integral around the periphery of the rim. Thus using the form of (13-105b)

$$H_\phi^d = 2R(\theta=\pi-\frac{\alpha}{2})\frac{e^{-j\beta r}}{r}\int_0^{2\pi} V_B^i(s,\psi,n)\cos\phi' e^{j\beta b\sin\theta\cos\phi'} b\,d\phi'$$

which by suppressing the $e^{-j\beta r}/r$ factor can be written as

$$H_{\phi n}^d = 2bR(\theta=\pi-\frac{\alpha}{2})\int_0^{2\pi} V_B^i(s,\psi,n)\cos\phi' e^{j\beta b\sin\theta\cos\phi'} d\phi'$$

(b) Since the integral

$$I(\beta)=\int_0^{2\pi} V_B^i(s,\psi,n)\cos\phi' e^{j\beta b\sin\theta\cos\phi'} d\phi' = \int_0^{2\pi} F(\phi')e^{\beta f(\phi')}d\phi'$$

$$F(\phi') = V_B^i(s,\psi,n)\cos\phi', \quad f(\phi') = jb\sin\theta\cos\phi'$$

is of the (VI-1) form, then using (VI-18), it can be written as

$$I(\beta) = \sqrt{\frac{2\pi}{\beta}}\sum_s \sqrt{-\frac{1}{f''(\phi_s')}}F(\phi_s')e^{\beta f(\phi_s')}$$

$$f'(\phi') = -jb\sin\theta\sin\phi' \Rightarrow f'(\phi_s') = -jb\sin\theta\sin\phi_s' = 0 \Rightarrow \phi_s' = 0, \pi$$

$\underline{\underline{\phi_s'=0}}$: $\psi=\psi_1=\frac{\alpha}{2}+\theta$, $f''(\phi_{s1}'=0) = -jb\sin(\theta)\cos(0) = -jb\sin\theta$

$F(\phi_{s1}'=0) = V_B^i(s,\frac{\alpha}{2}+\theta,n)\cos(0) = V_B^i(s,\frac{\alpha}{2}+\theta,n)$

$\underline{\underline{\phi_s'=\pi}}$: $\psi=\psi_2=\frac{\alpha}{2}-\theta$ $\quad f''(\phi_{s2}'=\pi) = -jb\sin\theta\cos(\pi) = +jb\sin\theta$

$F(\phi_{s2}'=\pi) = V_B^i(s,\frac{\alpha}{2}-\theta,n)\cos(\pi) = -V_B^i(s,\frac{\alpha}{2}-\theta,n)$

Thus

$$I(\beta)\simeq\sqrt{\frac{2\pi}{\beta}}\left\{\sqrt{\frac{1}{jb\sin\theta}}V_B^i(s,\frac{\alpha}{2}+\theta,n)e^{j\beta b\sin\theta} - \sqrt{\frac{-1}{jb\sin\theta}}V_B^i(s,\frac{\alpha}{2}-\theta,n)e^{-j\beta b\sin\theta}\right\}$$

$$\simeq\sqrt{\frac{2\pi}{\beta b\sin\theta}}\left\{e^{-j\pi/4}V_B^i(s,\frac{\alpha}{2}+\theta,n)e^{j\beta b\sin\theta} - e^{+j\pi/4}V_B^i(s,\frac{\alpha}{2}-\theta,n)e^{-j\beta b\sin\theta}\right\}$$

Therefore the diffracted field can be written as

$$H_{\phi n}^d = 2bR(\theta=\pi-\frac{\alpha}{2})e^{-j\pi/4}\sqrt{\frac{2\pi}{\beta b\sin\theta}}\left\{V_B^i(s,\frac{\alpha}{2}+\theta,n)e^{j\beta b\sin\theta} - jV_B^i(s,\frac{\alpha}{2}-\theta,n)e^{-j\beta b\sin\theta}\right\}$$

The first term represents the diffraction contribution from edge #1 while the second is the contribution from edge #2.

$\boxed{13.49}$ For the solution of this problem refer to Figure 13-37

$$\left.\begin{aligned} E_1^d &= E^i(Q_1)\mathcal{D}_h^1(L,\xi_1,n=2)\sqrt{\frac{\rho_{c1}}{r_1(\rho_{c1}+r_1)}}\,e^{-j\beta r_1} \\ E_2^d &= -E^i(Q_2)\mathcal{D}_h^2(L,\xi_2,n=2)\sqrt{\frac{\rho_{c2}}{r_2(\rho_{c2}+r_2)}}\,e^{-j\beta r_2} \end{aligned}\right\} \quad 0\le\theta\le\pi$$

where $E^i(Q_1)=E^i(Q_2)=\frac{1}{2}E_{\theta o}(r=a,\theta=\pi/2)=\frac{1}{2}\left.\frac{\cos(\frac{\pi}{2}\cos\theta)}{\sin\theta}\right|_{\theta=\pi/2}=\frac{1}{2}\frac{e^{-j\beta a}}{a}$

Q_1 and Q_2 are the diffraction points (#1 and #2).

According to (13-68)-(13-71d) for $n=2$

$$\mathcal{D}_h^1(L,\xi_1,2)=\mathcal{D}^i(L,\xi_1^-,2)+\mathcal{D}^r(L,\xi_1^+,2)$$

$$\mathcal{D}^i(L,\xi_1^-,2)=-\frac{e^{-j\pi/4}}{4\sqrt{2\pi\beta}}\left\{C_1^+(\xi_1^-,2)F_1[\beta L g_1^+(\xi_1^-)]+C_1^-(\xi_1^-,2)F_1[\beta L g_1^-(\xi_1^-)]\right\}$$

$$\mathcal{D}^r(L,\xi_1^-,2)=-\frac{e^{-j\pi/4}}{4\sqrt{2\pi\beta}}\left\{C_1^+(\beta_1^+,2)F_1[\beta L g_1^+(\xi_1^+)]+C_1^-(\xi_1^+,2)F_1[\beta L g_1^-(\xi_1^+)]\right\}$$

where $C_1^+(\xi,2)=\cot\left(\frac{\pi+\xi}{4}\right)$, $C_1^-(\xi,2)=\cot\left(\frac{\pi-\xi}{4}\right)$

$$F_1[\beta L g^+(\xi)]=2j\sqrt{\beta L g^+(\xi)}\,e^{j\beta L g^+(\xi)}\int_{\sqrt{\beta L g^+(\xi)}}^{\infty}e^{-j\tau^2}d\tau$$

$$\xi_1^+=\psi_1+\psi_1'=\psi_1=\pi/2+\theta \qquad g_1^+=1+[\cos(\psi_1)-2(2)\pi N^+]$$

$$\xi_1^-=\psi_1-\psi_1'=\psi_1=\pi/2+\theta \qquad g_1^-=1+[\cos(\psi_1)-2(2)\pi N^-]$$

with N^+ and N^- being positive or negative integers or zero which most closely satisfy the equations

$$2(2)\pi N^+-\psi_1=+\pi \qquad \text{and} \qquad 2(2)\pi N^--\psi_1=-\pi$$

For curved edge diffraction, the distance parameter L (see Reference [10]) is given for far-field observations by

$$L=\left.\sin\theta\sin^2\beta_o\right|_{\beta_o=\pi/2}=a\sin\theta$$

$$\mathcal{D}_h^2(L,\xi_2,2)=\mathcal{D}^i(L,\xi_2^-,2)+\mathcal{D}^r(L,\xi_2^+,2)$$

$$\mathcal{D}^i(L,\xi_2^-,2)=-\frac{e^{-j\pi/4}}{4\sqrt{2\pi\beta}}\left\{C_2^+(\xi_2^-,2)F_2[\beta L g_2^+(\xi_2^-)]+C_2^-(\xi_2^-,2)F_2[\beta L g_2^-(\xi_2^-)]\right\}$$

$$\mathcal{D}^r(L,\xi_2^+,2)=-\frac{e^{-j\pi/4}}{4\sqrt{2\pi\beta}}\left\{C_2^+(\xi_2^+,2)F_2[\beta L g_2^+(\xi_2^+)]+C_2^-(\xi_2^+,2)F_2[\beta L g_2^-(\xi_2^+)]\right\}$$

Cont'd.

13.49 Cont'd. where $C_2^+(\xi,2)=\cot\left(\frac{\pi+\xi}{4}\right)$, $C_2^-(\xi,2)=\cot\left(\frac{\pi-\xi}{4}\right)$

$$F_2[\beta L g^+(\xi)] = 2j\sqrt{\beta L g^+(\xi)}\, e^{j\beta L g^+(\xi)} \int_{\sqrt{\beta L g^+(\xi)}}^{\infty} e^{-j\tau^2} d\tau$$

$$\left.\begin{aligned} \xi_2^+ &= \psi_2+\psi_2' = \psi_2 \\ \xi_2^- &= \psi_2-\psi_2' = \psi_2 \\ g_2^+ &= 1+[\cos\psi_2 - 2(2)\pi N^+] \\ g_2^- &= 1+[\cos\psi_2 - 2(2)\pi N^-] \end{aligned}\right\} \begin{aligned} &\psi_2 = \frac{\pi}{2}-\theta, \quad 0\le\theta\le\pi/2 \\ &\psi_2 = \frac{5\pi}{2}-\theta, \quad \pi/2\le\theta\le\pi \end{aligned}$$

with N^+ or N^- being a positive or negative integer or zero which most closely satisfies the equation

$$2(2)\pi N^+ - \psi_2 = +\pi \quad \text{and} \quad 2(2)\pi N^- - \psi_2 = -\pi$$

For far field observations

$$\left.\begin{aligned} r_1 &\simeq r - a\cos\left(\frac{\pi}{2}-\theta\right) = r - a\sin\theta \\ r_2 &\simeq r + a\cos\left(\frac{\pi}{2}-\theta\right) = r + a\sin\theta \end{aligned}\right\} \text{For phase terms}$$

$$r_1 \simeq r_2 \simeq r \qquad \text{For amplitude terms}$$

ρ_{c1} and ρ_{c2} can be derived by using (13-100) and (13-100a). These are given on pages 821-822 as

$$\rho_{c1} = \frac{a}{\sin\theta} \Rightarrow A_1 = \frac{\sqrt{\rho_{c1}}}{r_1} = \frac{1}{r_1}\sqrt{\frac{a}{\sin\theta}}$$

$$\rho_{c2} = -\frac{a}{\sin\theta} \Rightarrow A_2 = \frac{\sqrt{\rho_{c2}}}{r_2} = \frac{1}{r_2}\sqrt{-\frac{a}{\sin\theta}}$$

To model the ground plane as a ring radiator, the equivalent current concept of Section 13.3.5 will be used. Toward $\theta=0,\pi$

$$\rho_{c1} = \frac{a}{0} = \infty \Rightarrow A_1 = \frac{1}{r_1}\sqrt{\frac{a}{0}} = \infty$$

Therefore

$$\sqrt{\frac{\rho_{c1}}{r_1(\rho_{c1}+r_1)}} \overset{r_1\simeq r}{\simeq} \sqrt{\frac{\rho_{c1}}{r(\rho_{c1}+r)}} = \frac{1}{\sqrt{r}}\sqrt{\frac{\rho_{c1}}{\rho_{c1}+r}} = \frac{1}{\sqrt{r}}\sqrt{\frac{1}{1+\frac{r}{\rho_{c1}}}} \overset{\rho_{c1}\to\infty}{\simeq} \frac{1}{\sqrt{r}}$$

and

$$E_1^d = E^i(Q_1)D_h^1(L,\xi,2)\sqrt{\frac{\rho_{c1}}{r_1(\rho_{c1}+r_1)}}\,e^{-j\beta r_1} \overset{r_1\simeq r}{\simeq} E^i(Q_1)D_h^1(L,\xi,2)\sqrt{\frac{\rho_{c1}}{r(\rho_{c1}+r)}}\,e^{-j\beta r}$$

which for $\theta\simeq 0,\pi$ reduces to

$$E_1^d\Big|_{\theta\simeq 0} \simeq E^i(Q_1)D_h^1(L,\xi,2)\Big|_{\theta=0}\frac{e^{-j\beta r}}{\sqrt{r}}\,;\quad E_2^d\Big|_{\theta\simeq\pi} \simeq E^i(Q_1)D_h^1(L,\xi,2)\Big|_{\theta=\pi}\frac{e^{-j\beta r}}{\sqrt{r}}$$

Cont'd.

13.49 Cont'd. and each has the same form as the left side of (13-102a)

Equating each of the above to the right side of (13-101a) or (13-102a), we can write that

$$E_{axial} = E_1^d(\theta = 0,\pi) \simeq E^i(\theta_1) D_h^1(L,\xi,2)\Big|_{\theta=0,\pi} \frac{e^{-j\beta r}}{\sqrt{r}} = I_e^m \frac{\beta}{2\eta}\sqrt{\frac{j}{2\pi\beta}}\,\frac{e^{-j\beta r}}{\sqrt{r}}$$

where I_e^m is the equivalent magnetic current, and it is equal to

$$I_e^m = 2\eta\sqrt{\frac{2\pi}{j\beta}}\,E^i(\theta_1)D_h^1(L,\xi,2)\Big|_{\theta=0,\pi} = 2\eta\sqrt{\frac{2\pi}{\beta}}\,e^{-j\pi/4}\,E^i(\theta_1)D_h^1(L,\xi,2)\Big|_{\theta=0,\pi}$$

which is identical in form to (13-103b).

Since the equivalent current is distributed along a circular ring at the edge of the ground plane, the field radiated by this equivalent current is given by (13-104b), or

$$E_\theta^m = j\,\frac{\omega\varepsilon a e^{-j\beta r}}{4\pi r}\int_0^{2\pi} I_e^m(\phi')\cos(\phi-\phi')\,e^{j\beta a\sin\theta\cos(\phi-\phi')}\,d\phi'$$

The ground plane is symmetrical, and the diffractions at any point along the rim are identical to those of any other point. Thus the current can be assumed to be uniform (with respect to ϕ'). Also the observations are not a function of ϕ. Choosing $\phi = 0$, reduces to

$$E_\theta^m = -\frac{a\omega\varepsilon e^{-j\beta r}}{2r}J_1(\beta a\sin\theta)$$

which is of the form of (13-106b). Using the equivalent current from above, we can write the diffracted field due to the ring source of the ground plane as

$$E_\theta^m = -\frac{a\omega\varepsilon e^{-j\beta r}}{2r}\,2\eta\sqrt{\frac{2\pi}{\beta}}\,e^{-j\pi/4}E^i(\theta_1)D_h^i(L,\xi,2)\Big|_{\theta=0,\pi} J_1(\beta a\sin\theta)$$

$$E_\theta^m = -a\sqrt{2\pi\beta}\,e^{-j\pi/4}E^i(\theta_1)D_h^i(L,\xi,2)\Big|_{\theta=0,\pi} J_1(\beta a\sin\theta)\frac{e^{-j\beta r}}{r}$$

(b) The pattern computed using the equivalent current near the axis ($\theta \simeq 0°$ and $180°$) of the ground plane when $d = 4.064\lambda$ is shown plotted in Figure 13-39 where it is combined with that of the 2-point diffraction (which is used for points removed from the axis).

13.50

Formulation

2-D monostatic scattering width of the wedge for any included angle WA for both soft and hard polarization

Soft polarization

Incident field

$$\vec{E}^i = \hat{a}_z e^{-j\beta x} \tag{1}$$

Diffracted field

$$\vec{E}^d = \vec{E}^i(Q) \cdot D^s A e^{-j\beta\rho} \tag{2}$$

where

$$\vec{E}^i(Q) = \vec{E}^i(x=0) = \hat{a}_z 1 \tag{3}$$

$$A = \frac{1}{\sqrt{\rho}} \tag{4}$$

$$D^s = \frac{e^{-j\pi/4}\sin\left(\frac{\pi}{4}\right)}{n\sqrt{2\pi\beta}}\left[\frac{1}{\cos\left(\frac{\pi}{n}\right)-\cos\left(\frac{\phi-\phi'}{n}\right)} - \frac{1}{\cos\left(\frac{\pi}{n}\right)-\cos\left(\frac{\phi+\phi'}{n}\right)}\right] \tag{5}$$

For monostatic scattering

$$\phi = \phi' \tag{6}$$

then,

$$D^s = \frac{e^{-j\pi/4}\sin\left(\frac{\pi}{4}\right)}{n\sqrt{2\pi\beta}}\left[\frac{1}{\cos\left(\frac{\pi}{n}\right)-1} - \frac{1}{\cos\left(\frac{\pi}{n}\right)-\cos\left(\frac{2\phi'}{n}\right)}\right] \tag{7}$$

Since

$$\sigma_{2-D} = \lim_{\rho\to\infty}\left[2\pi\rho\frac{|\vec{E}^s|^2}{|\vec{E}^i|^2}\right] \tag{8}$$

and

$$\vec{E}^s = \vec{E}^d \tag{9}$$

then,

$$\sigma_{2-D} = 2\pi\rho \left| D^s \frac{1}{\sqrt{\rho}} e^{-j\beta\rho} \right|^2 = 2\pi \, | \, D^s \, |^2$$

$$\sigma_{2-D} = \frac{\lambda}{n^2 2\pi} \left| \sin\left(\frac{\pi}{4}\right) \left(\frac{1}{\cos\left(\frac{\pi}{n}\right) - 1} - \frac{1}{\cos\left(\frac{\pi}{n}\right) - \cos\left(\frac{2\phi'}{n}\right)} \right) \right|^2 \tag{10}$$

Finally, for the **soft polarization**

$$\boxed{\frac{\sigma_{2-D}}{\lambda} = \frac{1}{n^2 2\pi} \left| \sin\left(\frac{\pi}{4}\right) \left(\frac{1}{\cos\left(\frac{\pi}{n}\right) - 1} - \frac{1}{\cos\left(\frac{\pi}{n}\right) - \cos\left(\frac{2\phi'}{n}\right)} \right) \right|^2} \tag{11}$$

Hard polarization

For the hard polarization we should choose $\vec{H}$ field instead of $\vec{E}$ and then follow the same procedure as for the soft polarization to obtain

$$\boxed{\frac{\sigma_{2-D}}{\lambda} = \frac{1}{n^2 2\pi} \left| \sin\left(\frac{\pi}{4}\right) \left(\frac{1}{\cos\left(\frac{\pi}{n}\right) - 1} + \frac{1}{\cos\left(\frac{\pi}{n}\right) - \cos\left(\frac{2\phi'}{n}\right)} \right) \right|^2} \tag{12}$$

Formulation

2-D monostatic scattering width of the cylinder for any radius a for both soft and hard polarization

Soft polarization

Incident field

$$\bar{E}^i = \hat{a}_z E_o \sum_{n=-\infty}^{\infty} j^{-n} J_n(\beta\rho) e^{jn\phi} \tag{1}$$

Scattered field

$$\bar{E}^s = \hat{a}_z E_o \sum_{n=-\infty}^{\infty} c_n H_n^{(2)}(\beta\rho) \tag{2}$$

Use the boundary condition

$$\bar{E}^t(\rho = a, 0 \le \phi \le 2\pi) = \bar{E}^i + \bar{E}^s = 0 \tag{3}$$

to solve for the unknown coefficient

$$E_o \sum_{n=-\infty}^{\infty} \left(j^{-n} J_n(\beta\rho) e^{jn\phi} + c_n H_n^{(2)}(\beta\rho) \right) = 0 \tag{4}$$

then,

$$c_n = -j^{-n} \frac{J_n(\beta a)}{H_n^{(2)}(\beta a)} e^{jn\phi} \tag{5}$$

The scattered field is

$$\bar{E}^s = -\hat{a}_z E_o \sum_{n=-\infty}^{\infty} j^{-n} \frac{J_n(\beta a)}{H_n^{(2)}(\beta a)} H_n^{(2)}(\beta\rho) e^{jn\phi} \tag{6}$$

Use the far-zone approximation

$$H_n^{(2)}(\beta\rho) \overset{\beta\rho\to\infty}{=} \sqrt{\frac{2j}{\pi\beta\rho}}\, j^n e^{-j\beta\rho} \tag{7}$$

then,

$$\vec{E}^s = -\hat{a}_z E_o \sqrt{\frac{2j}{\pi\beta}} \frac{e^{-j\beta\rho}}{\sqrt{\rho}} \sum_{n=-\infty}^{\infty} \frac{J_n(\beta a)}{H_n^{(2)}(\beta a)} e^{jn\phi} \tag{8}$$

Thus,

$$\sigma_{2-D} = \lim_{\rho\to\infty}\left[2\pi\rho \frac{|\vec{E}^s|^2}{|\vec{E}^i|^2}\right] = 2\pi \frac{2}{\pi\beta}\left|\sum_{n=-\infty}^{\infty} \frac{J_n(\beta a)}{H_n^{(2)}(\beta a)} e^{jn\phi}\right|^2 \tag{9}$$

For the monostatic scattering

$$\phi = \phi' \tag{10}$$

then,

$$\sigma_{2-D} = \frac{2\lambda}{\pi}\left|\sum_{n=-\infty}^{\infty} \frac{J_n(\beta a)}{H_n^{(2)}(\beta a)} e^{jn\phi'}\right|^2 \tag{11}$$

Finally, for the soft **polarization**

$$\boxed{\frac{\sigma_{2-D}}{\lambda} = \frac{2}{\pi}\left|\sum_{n=0}^{\infty} \varepsilon_n \frac{J_n(\beta a)}{H_n^{(2)}(\beta a)} \cos(n\phi')\right|^2} \tag{12}$$

where

$$\varepsilon_n = \begin{cases} 1 & n = 0 \\ 2 & n \neq 0 \end{cases} \tag{13}$$

Hard polarization

Use $\vec{H}$ field instead of $\vec{E}$.

Incident field

$$\vec{H}^i = \hat{a}_z H_o \sum_{n=-\infty}^{\infty} j^{-n} J_n(\beta\rho) e^{jn\phi} \tag{1}$$

Scattered field

$$\vec{H}^s = \hat{a}_z H_o \sum_{n=-\infty}^{\infty} d_n H_n^{(2)}(\beta\rho) \tag{2}$$

Use the boundary condition

$$\vec{E}_\phi^t(\rho = a, 0 \le \phi \le 2\pi) = \vec{E}_\phi^i + \vec{E}_\phi^s = 0 \tag{3}$$

where

$$\vec{E}_\phi^s = -\frac{1}{j\omega\varepsilon}\frac{\partial \vec{H}_z^s}{\partial\rho} = -\hat{a}_z \frac{\beta H_o}{j\omega\varepsilon} \sum_{n=-\infty}^{\infty} d_n H_n^{'(2)}(\beta\rho)$$

$$\vec{E}_\phi^i = -\frac{1}{j\omega\varepsilon}\frac{\partial \vec{H}_z^i}{\partial\rho} = -\hat{a}_z \frac{\beta H_o}{j\omega\varepsilon} \sum_{n=-\infty}^{\infty} j^{-n} J_n^{'}(\beta\rho) e^{jn\phi} \tag{4}$$

Solve for the unknown coefficients

$$-\frac{\beta H_o}{j\omega\varepsilon} \sum_{n=-\infty}^{\infty} \left(j^{-n} J_n^{'}(\beta\rho) e^{jn\phi} + d_n H_n^{'(2)}(\beta\rho) \right) = 0 \tag{5}$$

then,

$$d_n = -j^{-n} \frac{J_n^{'}(\beta a)}{H_n^{'(2)}(\beta a)} e^{jn\phi} \tag{6}$$

The scattered field is

$$\vec{H}^s = -\hat{a}_z H_o \sum_{n=-\infty}^{\infty} j^{-n} \frac{J_n^{'}(\beta a)}{H_n^{'(2)}(\beta a)} H_n^{(2)}(\beta\rho) e^{jn\phi} \tag{7}$$

Use the far-zone approximation

$$H_n^{(2)}(\beta\rho) \overset{\beta\rho\to\infty}{=} \sqrt{\frac{2j}{\pi\beta\rho}}\, j^n e^{-j\beta\rho} \tag{8}$$

then,

$$\vec{H}^s = -\hat{a}_z H_o \sqrt{\frac{2j}{\pi\beta}} \frac{e^{-j\beta\rho}}{\sqrt{\rho}} \sum_{n=-\infty}^{\infty} \frac{J_n'(\beta a)}{H_n^{'(2)}(\beta a)} e^{jn\phi} \tag{9}$$

For the monostatic scattering

$$\phi = \phi' \tag{10}$$

Thus,

$$\sigma_{2-D} = \lim_{\rho\to\infty}\left[2\pi\rho\frac{|\vec{H}^s|^2}{|\vec{H}^i|^2}\right] = 2\pi\frac{2}{\pi\beta}\left|\sum_{n=-\infty}^{\infty} \frac{J_n'(\beta a)}{H_n^{'(2)}(\beta a)} e^{jn\phi}\right|^2 \tag{11}$$

Finally, for the **hard polarization**

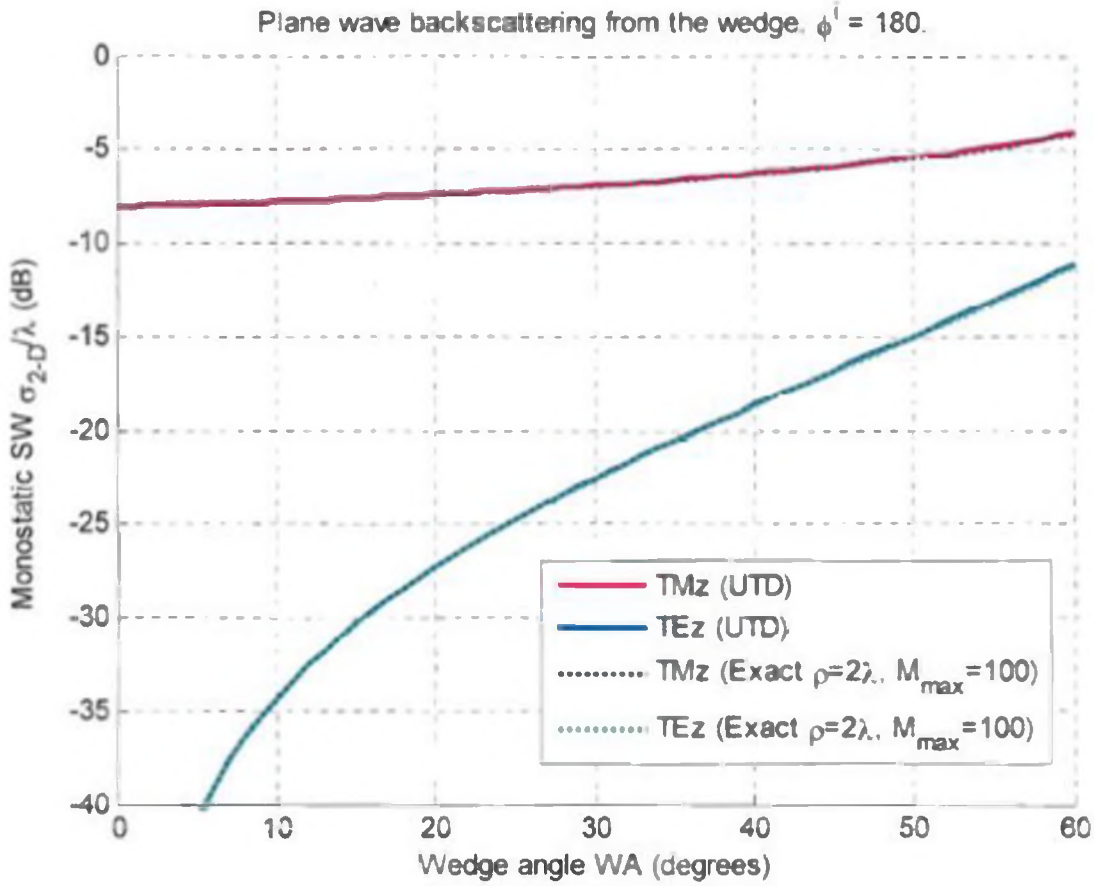
$$\boxed{\frac{\sigma_{2-D}}{\lambda} = \frac{2}{\pi}\left|\sum_{n=0}^{\infty} \varepsilon_n \frac{J_n'(\beta a)}{H_n^{'(2)}(\beta a)} \cos(n\phi')\right|^2} \tag{12}$$

where

$$\varepsilon_n = \begin{cases} 1 & n = 0 \\ 2 & n \neq 0 \end{cases} \tag{13}$$

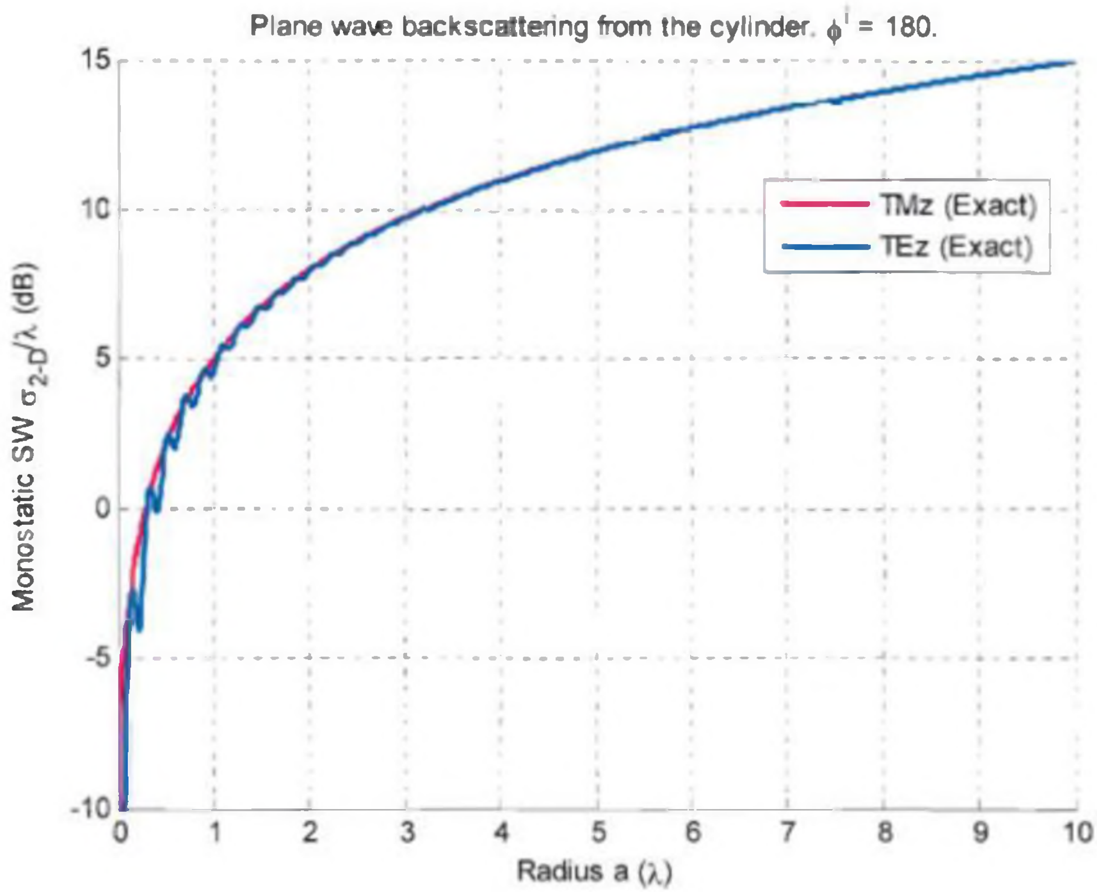
(13.50a)

Normalized 2-D monostatic scattering width SW of the wedge vs. wedge angle WA [$0 \leq WA \leq 60^{\circ}$] for both hard and soft polarization.



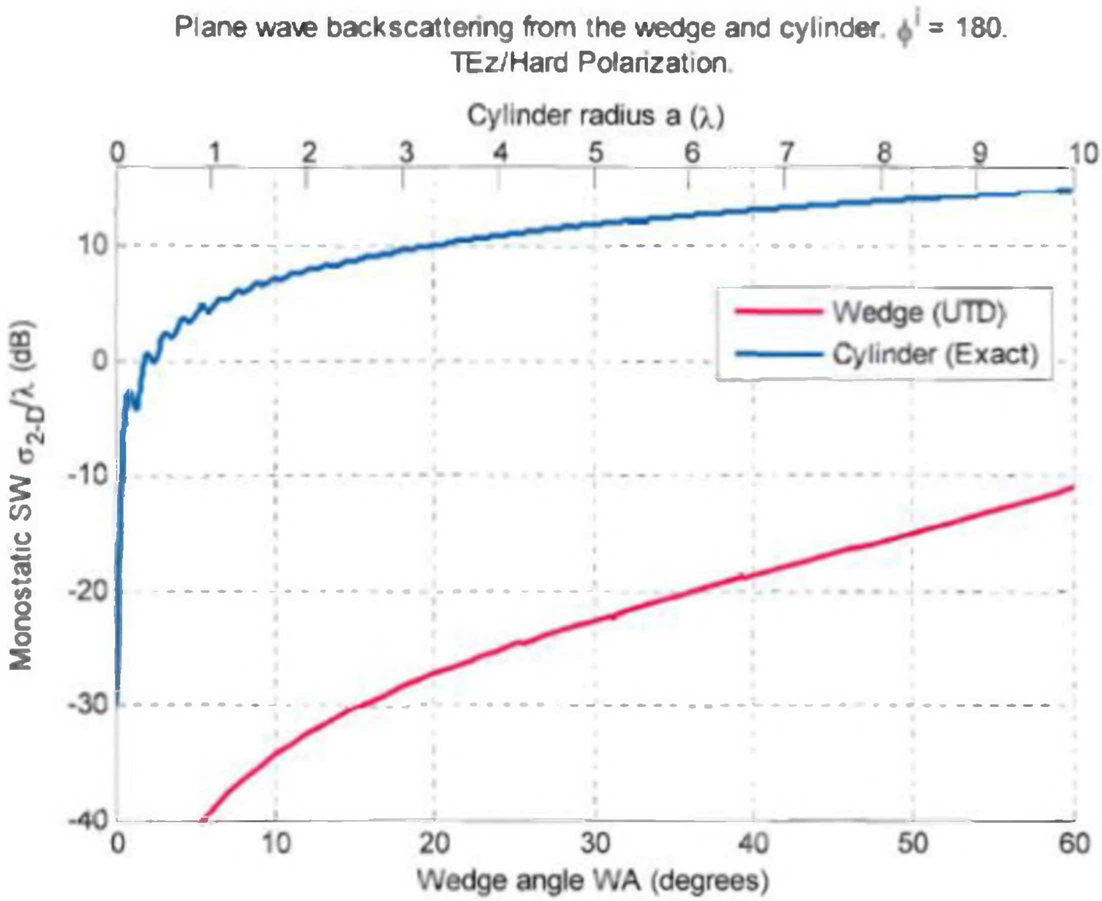
(13.50b)

Normalized 2-D monostatic scattering width SW of the cylinder vs. radius a $[0 \le a \le 10\lambda]$ for both hard and soft polarization



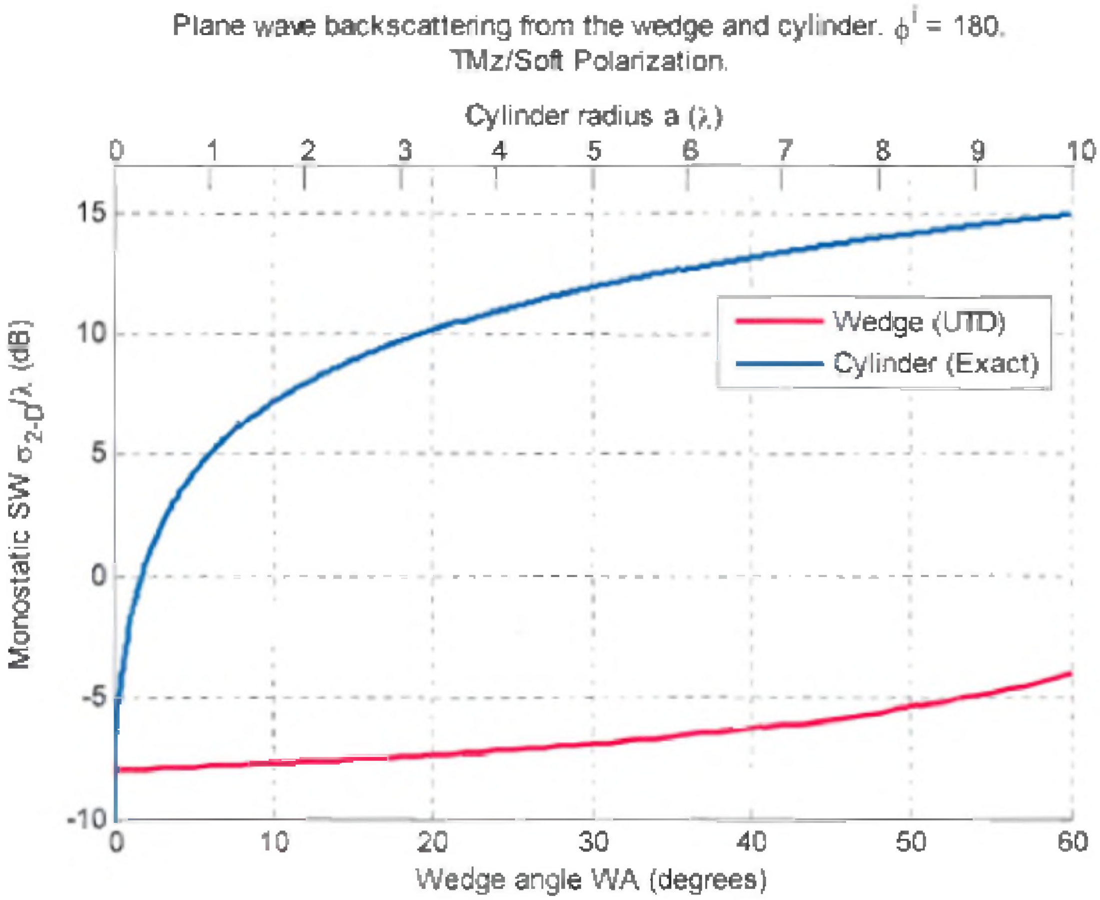
(13.50c)

Normalized 2-D monostatic scattering width SW of the wedge [$0 \le WA \le 60^\circ$] and the cylinder [$0 \le a \le 10\lambda$] for hard polarization



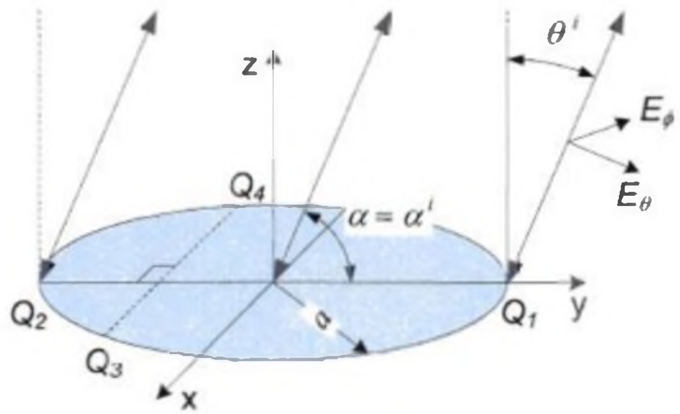
(13.50d)

Normalized 2-D monostatic scattering width SW of the wedge [$0 \le WA \le 60^\circ$] and the cylinder [$0 \le a \le 10\lambda$] for soft polarization



13.51 Uniform plane wave incident upon a circular ground plane.

The numbered equations below are the same equations as in [36]



$\theta' \in [0, 90^\circ]$; $\alpha \in [0, 180^\circ]$; $\alpha = 90^\circ - \theta$; $\theta = \theta'$

$$\boxed{\sigma_{3-D} = \lim_{r \to \infty}\left[4\pi r^2 \frac{|E^s|^2}{|E^i|^2}\right]}$$

$$\vec{E}^i_\theta = \hat{a}_\theta E_0 e^{j\beta(y\sin\theta' + z\cos\theta')}$$
$$\vec{E}^i_\phi = \hat{a}_\phi E_0 e^{j\beta(y\sin\theta' + z\cos\theta')} \qquad (1),\ (2)$$
$$|E^i|^2 = E_0^2$$

For Physical optics

Monostatic RCS (hard and soft polarizations)

$$\boxed{\sigma_{3-D} = 4\pi\left(\frac{\pi a^2}{\lambda}\right)^2 \cos^2\theta \left|2\frac{J_1(2\beta a\sin\theta)}{2\beta a\sin\theta}\right|^2}$$

For UTD

Hard polarization	Soft polarization
$\lvert E^s\rvert^2 = \left\lvert E^{1st(1,2)}_{hard} + E^{2nd(1,2)}_{hard} + E^{2nd(3,4)}_{hard}\right\rvert^2$	$\lvert E^s\rvert^2 = \left\lvert E^{1st(1,2)}_{soft} + E^{2nd(3,4)}_{hard}\right\rvert^2$

cont'd

13.51 cont'd

Hard polarization (TE^x)

- **First-order 3-D diffractions from edges #1 and #2 for monostatic scattering.**

$$E_{hard}^{1st(1,2)} = E_{hard}^{d1} = E_1^d + E_2^d$$

Hard polarization

E_1^d	E_2^d
$Q1 = (z=0, y=a)$	$Q2 = (z=0, y=-a)$
$E_1^d = E_\theta^i(Q1) D_1^h A_1 e^{-j\beta r_1}$	$E_2^d = E_\theta^i(Q2) D_2^h A_2 e^{-j\beta r_2}$
$E_\theta^i(Q1) = E_0 e^{j\beta a \sin\theta}$	$E_\theta^i(Q2) = E_0 e^{-j\beta a \sin\theta}$
$r_1 = r - a\sin\theta$ phase $r_1 = r$ amplitude	$r_2 = r + a\sin\theta$ phase $r_2 = r$ amplitude
$A_1 = \dfrac{\sqrt{\rho_{c1}}}{r_1} = \dfrac{\sqrt{\rho_{c1}}}{r}$	$A_2 = \dfrac{\sqrt{\rho_{c2}}}{r_2} = \dfrac{\sqrt{\rho_{c2}}}{r}$
$\rho_{c1} = \dfrac{a}{2\sin\theta}$ (7)	$\rho_{c2} = \dfrac{-a}{2\sin\theta}$ (8)
$A_1 e^{-j\beta r_1} = e^{j\beta a\sin\theta}\sqrt{\dfrac{a}{2\sin\theta}}\dfrac{e^{-j\beta r}}{r}$	$A_2 e^{-j\beta r_2} = e^{-j\beta a\sin\theta}\sqrt{\dfrac{-a}{2\sin\theta}}\dfrac{e^{-j\beta r}}{r}$
Keller $D_1^h(90^\circ+\theta, 90^\circ+\theta, 2)$ $D_1^h = \dfrac{e^{-j\pi/4}}{2\sqrt{2\pi\beta}}\left(-1+\dfrac{1}{\sin\theta}\right)$ (3)	Keller $D_2^h(90^\circ-\theta, 90^\circ-\theta, 2)$ $D_2^h = \dfrac{e^{-j\pi/4}}{2\sqrt{2\pi\beta}}\left(-1-\dfrac{1}{\sin\theta}\right)$ (4)

$$\boxed{E_{hard}^{1st(1,2)} = -E_0 \frac{1}{2}\sqrt{\frac{a}{\pi\beta\sin\theta}}\left(\cos\chi - j\frac{\sin\chi}{\sin\theta}\right)\frac{e^{-j\beta r}}{r}} \quad (10)$$

$$\chi = 2\beta a \sin\theta - \pi/4$$

cont'd

13.51 cont'd

- **Second-order 3-D diffractions from edges #1 and #2 for monostatic scattering.**

$$E_{hard}^{2nd(1,2)} = E_{hard}^{d2} = E_{21r}^{d} + E_{12r}^{d}$$

Hard polarization

E_{21r}^{d}	E_{12r}^{d}
$Q1 = (z = 0, y = a)$	$Q2 = (z = 0, y = -a)$
$E_{21r}^{d} = E_{i21}^{d} D_{21r}^{h} A_{21r} e^{-j\beta r_1}$	$E_{12r}^{d} = E_{i12}^{d} D_{12r}^{h} A_{12r} e^{-j\beta r_2}$
$r_1 = r - a\sin\theta$ phase $r_1 = r$ amplitude	$r_2 = r + a\sin\theta$ phase $r_2 = r$ amplitude
$A_{21r} = \dfrac{\sqrt{\rho_{21r}}}{r_1} = \dfrac{\sqrt{\rho_{21r}}}{r}$	$A_{12r} = \dfrac{\sqrt{\rho_{12r}}}{r_2} = \dfrac{\sqrt{\rho_{12r}}}{r}$
$\rho_{21r} = \dfrac{a(1+2\sin\theta)}{2\sin^2\theta}$ (27)	$\rho_{12r} = \dfrac{a(1-2\sin\theta)}{2\sin^2\theta}$ (26)
$A_{21r} e^{-j\beta r_1} = e^{j\beta a\sin\theta} \sqrt{\dfrac{a(1+2\sin\theta)}{2\sin^2\theta}} \dfrac{e^{-j\beta r}}{r}$	$A_{12r} e^{-j\beta r_2} = e^{-j\beta a\sin\theta} \sqrt{\dfrac{a(1-2\sin\theta)}{2\sin^2\theta}} \dfrac{e^{-j\beta r}}{r}$
UTD $D_{21r}^{h}(2a, 90^\circ + \theta, 0, 90, 2)$	UTD $D_{12r}^{h}(2a, 90^\circ - \theta, 0, 90, 2)$

cont'd

13.51 cont'd

where

E^d_{i21}	E^d_{i12}
$Q2=(z=0, y=-a)$	$Q1=(z=0, y=a)$
$E^d_{i21}=E^i_\theta(Q2)D^h_{i21}A_{i21}e^{-j\beta r_2}$	$E^d_{i12}=E^i_\theta(Q1)D^h_{i12}A_{i12}e^{-j\beta r_1}$
$E^i_\theta(Q2)=E_0e^{-j\beta a\sin\theta}$	$E^i_\theta(Q1)=E_0e^{j\beta a\sin\theta}$
$r_2=2a$ phase, amplitude	$r_1=2a$ phase, amplitude
$A_{i21}=\sqrt{\dfrac{\rho_c}{r_2(\rho_c+r_2)}}=\sqrt{\dfrac{\rho_{i21}}{2a(\rho_{i21}+2a)}}$	$A_{i12}=\sqrt{\dfrac{\rho_c}{r_1(\rho_c+r_1)}}=\sqrt{\dfrac{\rho_{i12}}{2a(\rho_{i12}+2a)}}$
$\rho_{i21}=\dfrac{-a}{1+\sin\theta}$ (25)	$\rho_{i12}=\dfrac{-a}{1-\sin\theta}$ (24)
$A_{i21}e^{-j\beta r_2}=\sqrt{\dfrac{-2a}{1+2\sin\theta}}\dfrac{e^{-j\beta 2a}}{2a}$	$A_{i12}e^{-j\beta r_1}=\sqrt{\dfrac{-2a}{1-2\sin\theta}}\dfrac{e^{-j\beta 2a}}{2a}$
UTD $D^h_{i21}(2a,0,90^\circ-\theta,90,2)$	UTD $D^h_{i12}(2a,0,90^\circ+\theta,90,2)$

$$E^d_{12r}=E_0e^{j\beta a\sin\theta}\sqrt{\frac{-2a}{1-2\sin\theta}}\frac{e^{-j\beta 2a}}{2a}e^{-j\beta a\sin\theta}\sqrt{\frac{a(1-2\sin\theta)}{2\sin^2\theta}}\frac{e^{-j\beta r}}{r}D^h_{i12}D^h_{12r}$$

$$E^d_{12r}=E_0\frac{ja}{\sin\theta}D^h_{i12}D^h_{12r}\frac{e^{-j\beta 2a}}{2a}\frac{e^{-j\beta r}}{r}$$

$$E^d_{12r}=E_0\frac{jae^{j\beta 2a}}{\sin\theta}D^h_{i12}D^h_{12r}\frac{e^{-j\beta 2a}}{\sqrt{2a}}\frac{e^{-j\beta 2a}}{\sqrt{2a}}\frac{e^{-j\beta r}}{r}$$

cont'd

13.51 cont'd

$$E^d_{21r} = E_0 e^{-j\beta a\sin\theta}\sqrt{\frac{-2a}{1+2\sin\theta}}\frac{e^{-j\beta 2a}}{2a}e^{j\beta a\sin\theta}\sqrt{\frac{a(1+2\sin\theta)}{2\sin^2\theta}}\frac{e^{-j\beta r}}{r}D^h_{i21}D^h_{21r}$$

$$E^d_{21r} = \frac{jE_0 a}{\sin\theta}D^h_{i21}D^h_{21r}\frac{e^{-j\beta 2a}}{2a}\frac{e^{-j\beta r}}{r}$$

$$E^d_{21r} = E_0\frac{jae^{j\beta 2a}}{\sin\theta}D^h_{i21}D^h_{21r}\frac{e^{-j\beta 2a}}{\sqrt{2a}}\frac{e^{-j\beta 2a}}{\sqrt{2a}}\frac{e^{-j\beta r}}{r}$$

Hard polarization

E^d_{21r}	E^d_{12r}
$E^d_{21r} = E_0\frac{jae^{j\beta 2a}}{\sin\theta}V^h_{i21}V^h_{21r}\frac{e^{-j\beta r}}{r}$	$E^d_{12r} = E_0\frac{jae^{j\beta 2a}}{\sin\theta}V^h_{i12}V^h_{12r}\frac{e^{-j\beta r}}{r}$

where

$$V^h_{i12} = \frac{e^{-j\beta 2a}}{\sqrt{2a}}D^h_{i12}\left(2a,0,90^\circ+\theta,90,2\right) \quad (20)$$

$$V^h_{i21} = \frac{e^{-j\beta 2a}}{\sqrt{2a}}D^h_{i21}\left(2a,0,90^\circ-\theta,90,2\right) \quad (21)$$

$$V^h_{12r} = \frac{e^{-j\beta 2a}}{\sqrt{2a}}D^h_{12r}\left(2a,90^\circ-\theta,0,90,2\right) \quad (22)$$

$$V^h_{21r} = \frac{e^{-j\beta 2a}}{\sqrt{2a}}D^h_{21r}\left(2a,90^\circ+\theta,0,90,2\right) \quad (23)$$

$$\boxed{E^{2nd(1,2)}_{hard} = E_0\frac{jae^{j\beta 2a}}{\sin\theta}\left(V^h_{i21}V^h_{21r}+V^h_{i12}V^h_{12r}\right)\frac{e^{-j\beta r}}{r}} \quad (28)$$

cont'd

13.51 cont'd

- **Second-order 3-D diffractions from the migrating points #3 and #4 for monostatic scattering.**

$$E_{hard}^{2nd(3,4)} = E_{hard}^{d2} = E_{34r}^{d} + E_{43r}^{d}$$

Hard polarization

E_{43r}^{d}	E_{34r}^{d}
$Q3 = (x = a\cos\phi_3, y = a\sin\phi_3, z = 0)$	$Q4 = (x = a\cos\phi_4, y = a\sin\phi_4, z = 0)$
$\phi_3 = \arccos\left(\dfrac{-1}{\sqrt{1+\sin^2\theta}}\right)$ (12)	$\phi_4 = \arccos\left(\dfrac{1}{\sqrt{1+\sin^2\theta}}\right)$ (13)
$E_{43r}^{d} = E_{i43}^{d} D_{43r}^{h} A_{43r} e^{-j\beta r_3}$	$E_{34r}^{d} = E_{i34}^{d} D_{34r}^{h} A_{34r} e^{-j\beta r_4}$
$r_3 = r - a\sin\theta\sin\phi_3$ phase $r_3 = r$ amplitude	$r_4 = r + a\sin\theta\sin\phi_4$ phase $r_4 = r$ amplitude
$A_{43r} = \dfrac{\sqrt{\rho_{43r}}}{r_3} = \dfrac{\sqrt{\rho_{43r}}}{r}$	$A_{34r} = \dfrac{\sqrt{\rho_{34r}}}{r_4} = \dfrac{\sqrt{\rho_{34r}}}{r}$
$\rho_{43r} = \dfrac{-a(1+2\sin^2\theta)}{2\sin^2(\theta)(1+\sin^2\theta)^{3/2}}$ (15)	$\rho_{34r} = \dfrac{-a(1+2\sin^2\theta)}{2\sin^2(\theta)(1+\sin^2\theta)^{3/2}}$ (15)
Keller D_{43r}^{h}	Keller D_{34r}^{h}

$$\boxed{E_{hard}^{2nd(3,4)} = -\frac{j\sin^2(\theta)e^{-j2\beta a\sqrt{1+\sin^2\theta}}}{\pi\beta\sin\theta\sqrt{1+\sin^2\theta}}\frac{e^{-j\beta r}}{r}} \quad (19)$$

cont'd

13.51 cont'd

- **Ring-radiator contributions.**

$$E_{hard}^{ring} = E_\theta^m + E_\theta^e$$

where

$$E_\theta^e = \frac{a}{4\pi}\frac{e^{-j\beta r}}{r}\cos^2\theta \int_0^{2\pi}\left(1+\frac{\sqrt{g}}{h}\right)\frac{\cos^2(\phi)e^{-j2\beta ah}}{g}\,d\phi = 0 \quad \text{(49a)}$$

$$E_\theta^m = -\frac{a}{4\pi}\frac{e^{-j\beta r}}{r}\int_0^{2\pi}\left(1-\frac{\sqrt{g}}{h}\right)\frac{\sin^2(\phi)e^{j2\beta ah}}{g}\,d\phi \quad \text{(49b)}$$

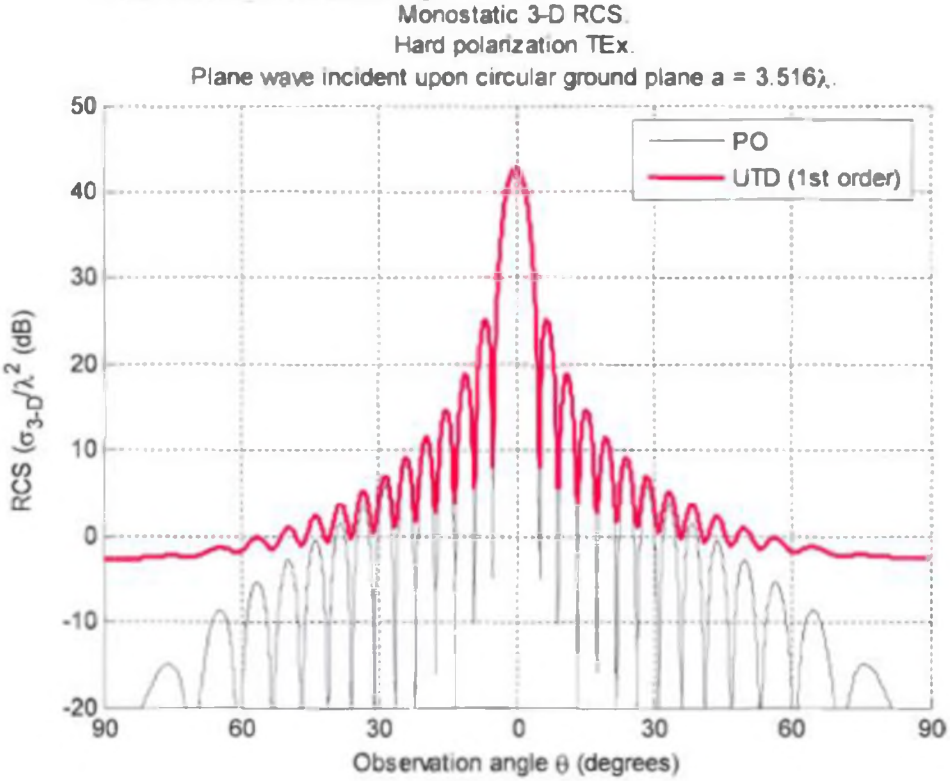
$$g = 1-\sin^2\theta\cos^2\phi$$
$$h = \sin\theta\sin\phi$$

$$\boxed{E_{hard}^{ring} = -\frac{a}{4\pi}\frac{e^{-j\beta r}}{r}\int_0^{2\pi}\left(1-\frac{\sqrt{1-\sin^2\theta\cos^2\phi}}{h}\right)\frac{\sin^2(\phi)e^{j2\beta a\sin\theta\sin\phi}}{1-\sin^2\theta\cos^2\phi}\,d\phi}$$

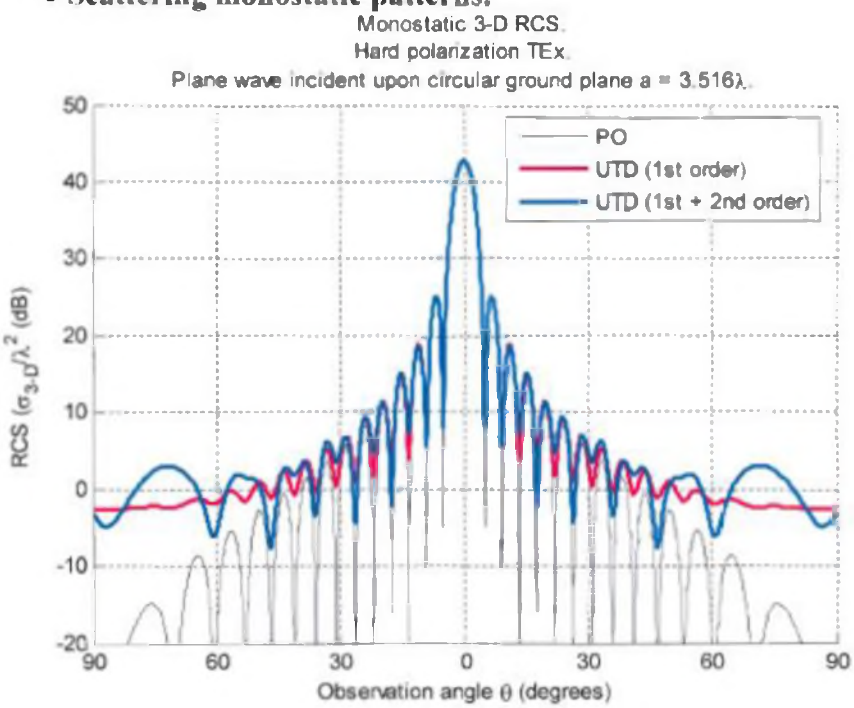
cont'd

13.51 cont'd

• **Scattering monostatic patterns.**

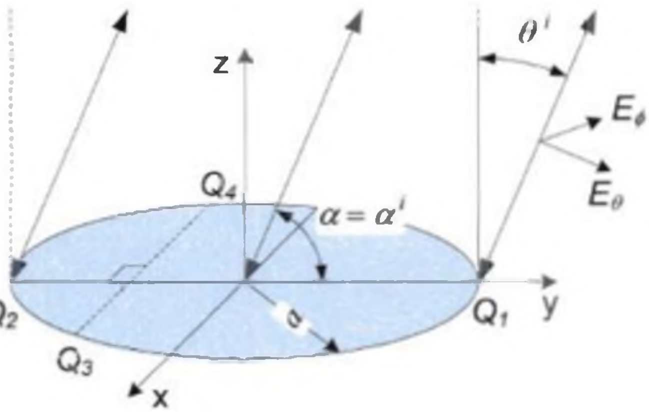


• **Scattering monostatic patterns.**



13.52 Uniform plane wave incident upon a circular ground plane.

The numbered equations below are the same equations as in [36].



$\theta^i \in [0, 90^\circ]$; $\alpha \in [0, 180^\circ]$; $\alpha = 90^\circ - \theta$; $\theta = \theta^i$

$$\boxed{\sigma_{3-D} = \lim_{r \to \infty}\left[4\pi r^2 \frac{|E^s|^2}{|E^i|^2}\right]}$$

$$\bar{E}^i_\theta = \hat{a}_\theta E_0 e^{j\beta(y\sin\theta' + z\cos\theta')}$$
$$\bar{E}^i_\phi = \hat{a}_\phi E_0 e^{j\beta(y\sin\theta' + z\cos\theta')} \qquad (1),\ (2)$$

$$|E^i|^2 = E_0^2$$

For Physical optics

Monostatic RCS (hard and soft polarizations)

$$\boxed{\sigma_{3-D} = 4\pi\left(\frac{\pi a^2}{\lambda}\right)^2 \cos^2\theta \left|2\frac{J_1(2\beta a\sin\theta)}{2\beta a\sin\theta}\right|^2}$$

For UTD

Hard polarization	Soft polarization
$\lvert E^s\rvert^2 = \left\lvert E^{1st(1,2)}_{hard} + E^{2nd(1,2)}_{hard} + E^{2nd(3,4)}_{hard}\right\rvert^2$	$\lvert E^s\rvert^2 = \left\lvert E^{1st(1,2)}_{soft} + E^{2nd(3,4)}_{hard}\right\rvert^2$

cont'd

13.52 Cont'd

Soft polarization (TM^x)

- **First-order 3-D diffractions from edges #1 and #2 for monostatic scattering.**

$$E_{soft}^{1st(1,2)} = E_{soft}^{d1} = E_1^d + E_2^d$$

Soft polarization

E_1^d	E_2^d
$Q1 = (z=0, y=a)$	$Q2 = (z=0, y=-a)$
$E_1^d = E_\phi^i(Q1) D_1^s A_1 e^{-j\beta r_1}$	$E_2^d = E_\phi^i(Q2) D_2^s A_2 e^{-j\beta r_2}$
$E_\phi^i(Q1) = E_0 e^{j\beta a \sin\theta^i}$	$E_\phi^i(Q2) = E_0 e^{-j\beta a \sin\theta^i}$
$r_1 = r - a\sin\theta$ phase $r_1 = r$ amplitude	$r_2 = r + a\sin\theta$ phase $r_2 = r$ amplitude
$A_1 = \dfrac{\sqrt{\rho_{c1}}}{r_1} = \dfrac{\sqrt{\rho_{c1}}}{r}$	$A_2 = \dfrac{\sqrt{\rho_{c2}}}{r_2} = \dfrac{\sqrt{\rho_{c2}}}{r}$
$\rho_{c1} = \dfrac{a}{2\sin\theta}$ (7)	$\rho_{c2} = \dfrac{-a}{2\sin\theta}$ (8)
$A_1 e^{-j\beta r_1} = e^{j\beta a\sin\theta}\sqrt{\dfrac{a}{2\sin\theta}}\dfrac{e^{-j\beta r}}{r}$	$A_2 e^{-j\beta r_2} = e^{-j\beta a\sin\theta}\sqrt{\dfrac{-a}{2\sin\theta}}\dfrac{e^{-j\beta r}}{r}$
Keller $D_1^s(90^\circ+\theta, 90^\circ+\theta, 2)$ $D_1^s = \dfrac{e^{-j\pi/4}}{2\sqrt{2\pi\beta}}\left(-1-\dfrac{1}{\sin\theta}\right)$ (3)	Keller $D_2^s(90^\circ-\theta, 90^\circ-\theta, 2)$ $D_2^s = \dfrac{e^{-j\pi/4}}{2\sqrt{2\pi\beta}}\left(-1+\dfrac{1}{\sin\theta}\right)$ (4)

$$\boxed{E_{soft}^{1st(1,2)} = E_0 \frac{1}{2}\sqrt{\frac{a}{\pi\beta\sin\theta}}\left(\cos\chi + j\frac{\sin\chi}{\sin\theta}\right)\frac{e^{-j\beta r}}{r}} \quad (11)$$

$$\chi = 2\beta a\sin\theta - \pi/4$$

Cont'd

13.52 cont'd

- **Second-order 3-D diffractions from the migrating points #3 and #4 for monostatic scattering.**

$$E_{soft}^{2nd(3,4)} = E_{soft}^{d2} = E_{34r}^{d} + E_{43r}^{d}$$

Soft polarization

E_{43r}^{d}	E_{34r}^{d}
$Q3 = \left(x = a\cos\phi_3, y = a\sin\phi_3, z = 0\right)$	$Q4 = \left(x = a\cos\phi_4, y = a\sin\phi_4, z = 0\right)$
$\phi_3 = \arccos\left(\dfrac{-1}{\sqrt{1+\sin^2\theta}}\right)$ (12)	$\phi_4 = \arccos\left(\dfrac{1}{\sqrt{1+\sin^2\theta}}\right)$ (13)
$E_{43r}^{d} = E_{i43}^{d} D_{43r}^{s} A_{43r} e^{-j\beta r_3}$	$E_{34r}^{d} = E_{i34}^{d} D_{34r}^{s} A_{34r} e^{-j\beta r_4}$
$r_3 = r - a\sin\theta\sin\phi_3$ phase $r_3 = r$ amplitude	$r_4 = r + a\sin\theta\sin\phi_4$ phase $r_4 = r$ amplitude
$A_{43r} = \dfrac{\sqrt{\rho_{43r}}}{r_3} = \dfrac{\sqrt{\rho_{43r}}}{r}$	$A_{34r} = \dfrac{\sqrt{\rho_{34r}}}{r_4} = \dfrac{\sqrt{\rho_{34r}}}{r}$
$\rho_{43r} = \dfrac{-a(1+2\sin^2\theta)}{2\sin^2(\theta)(1+\sin^2\theta)^{3/2}}$ (15)	$\rho_{34r} = \dfrac{-a(1+2\sin^2\theta)}{2\sin^2(\theta)(1+\sin^2\theta)^{3/2}}$ (15)
Keller D_{43r}^{s}	Keller D_{34r}^{s}

$$\boxed{E_{soft}^{2nd(3,4)} = -\frac{j\cos^2(\theta)e^{-j2\beta a\sqrt{1+\sin^2\theta}}}{\pi\beta\sin\theta\sqrt{1+\sin^2\theta}}\frac{e^{-j\beta r}}{r}} \quad (18)$$

cont'd

13.52 cont'd

- **Ring-radiator contributions.**

$$E_{soft}^{ring} = E_{\phi}^{m} + E_{\phi}^{e}$$

where

$$E_{\phi}^{e} = \frac{a}{4\pi}\frac{e^{-j\beta r}}{r}\int_{0}^{2\pi}\left(1+\frac{\sqrt{g}}{h}\right)\frac{\sin^{2}(\phi)e^{-j2\beta ah}}{g}d\phi \quad \text{(48a)}$$

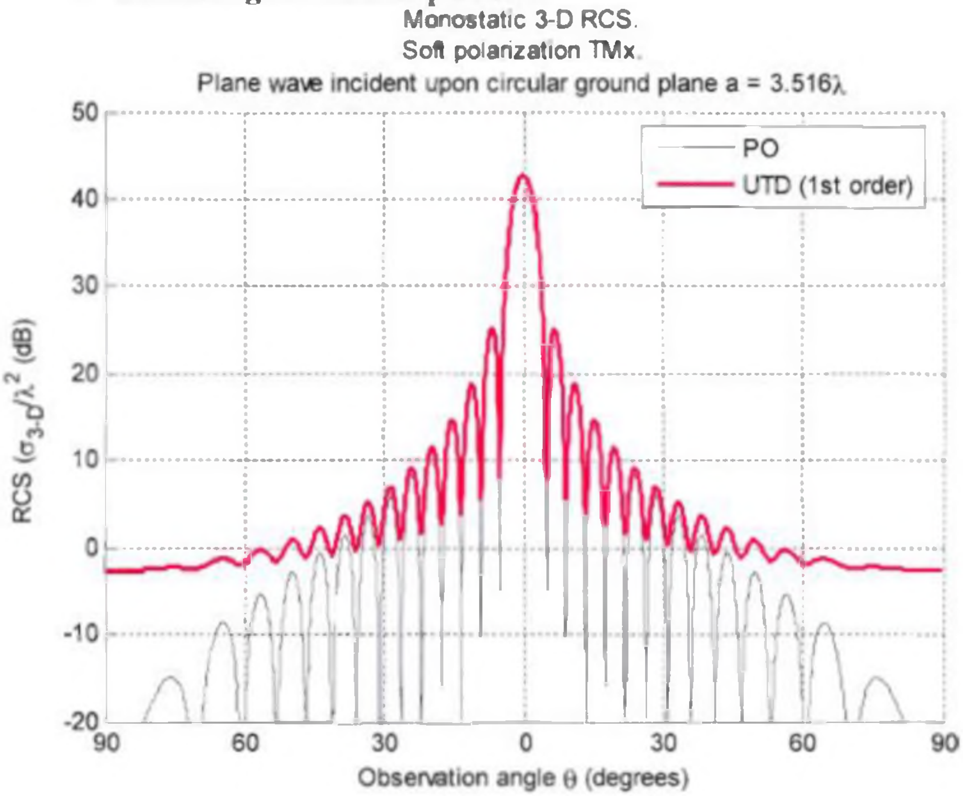
$$E_{\phi}^{m} = -\frac{a}{4\pi}\frac{e^{-j\beta r}}{r}\cos^{2}\theta\int_{0}^{2\pi}\left(1-\frac{\sqrt{g}}{h}\right)\frac{\cos^{2}(\phi)e^{j2\beta ah}}{g}d\phi = 0 \quad \text{(48b)}$$

$$g = 1-\sin^{2}\theta\cos^{2}\phi$$
$$h = \sin\theta\sin\phi$$

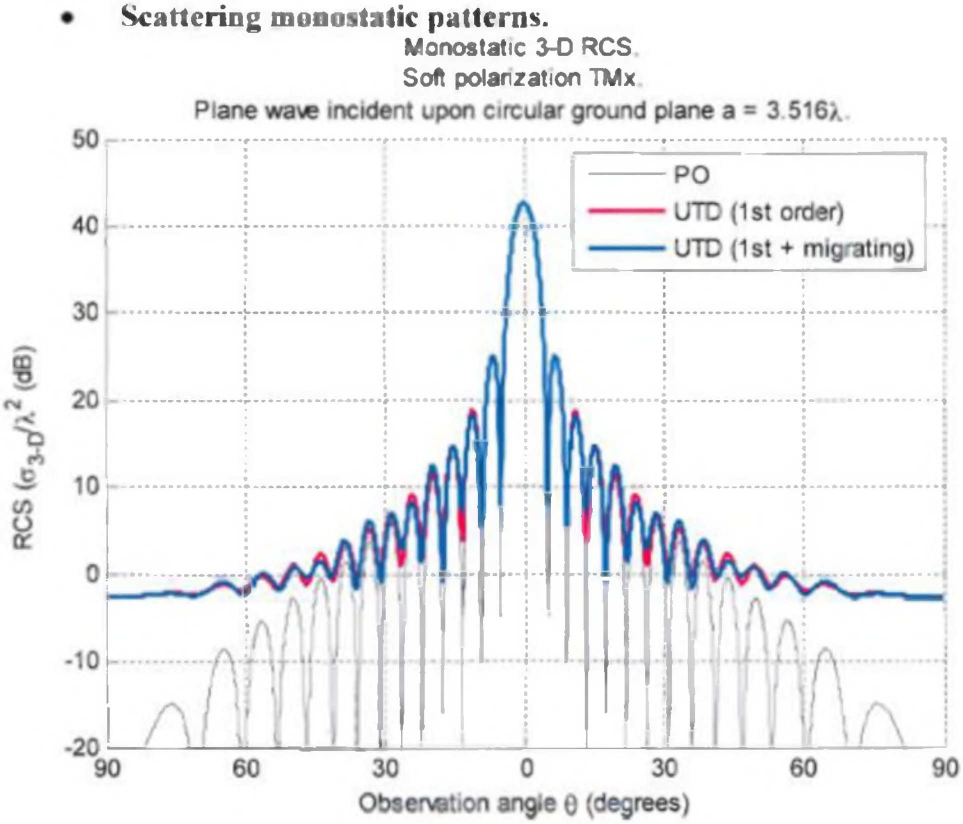
$$\boxed{E_{soft}^{ring} = \frac{a}{4\pi}\frac{e^{-j\beta r}}{r}\int_{0}^{2\pi}\left(1+\frac{\sqrt{1-\sin^{2}\theta\cos^{2}\phi}}{h}\right)\frac{\sin^{2}(\phi)e^{j2\beta a\sin\theta\sin\phi}}{1-\sin^{2}\theta\cos^{2}\phi}d\phi}$$

13.52 cont'd

- **Scattering monostatic patterns.**



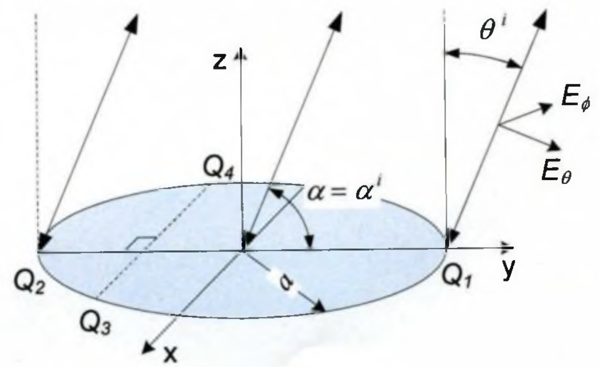
- **Scattering monostatic patterns.**



$\boxed{13.53}$ Uniform plane wave incident upon a circular and square ground plane.

The numbered equations below are the same equations as in [36].

Circular Ground Plane (radius = a)



$$\theta^i \in [0, 90^\circ];\ \alpha \in [0, 180^\circ];\ \alpha = 90^\circ - \theta;\ \theta = \theta^i$$

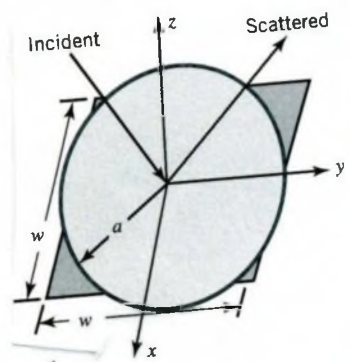
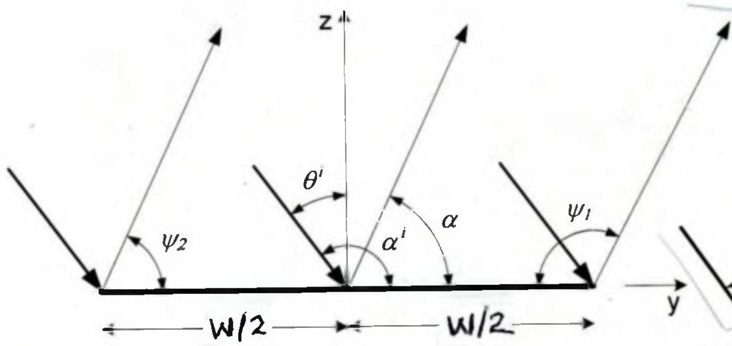
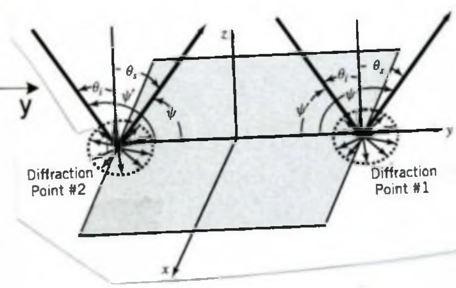


Figure P13-53

Square Ground Plane (w x w)





$$\theta^i \in [0^\circ, 90^\circ];\quad \alpha \in [0^\circ, 180^\circ]$$

$$\theta^i = \alpha^i - 90^\circ;\quad \psi_1 = 180^\circ - \alpha;\quad \psi_2 = \alpha$$

Area of a square ground plane with side length w

$$A_s = w^2$$

Area of a circular ground plane of radius a

$$A_c = \pi a^2$$

If $A_s = A_c$, then $\Rightarrow a = \dfrac{w}{\sqrt{\pi}} = \dfrac{5\lambda}{\sqrt{\pi}} = 2.8209\lambda$

cont'd

13.53 cont'd

$$\sigma_{3-D} = \lim_{r \to \infty} \left[4\pi r^2 \frac{|E^s|^2}{|E^i|^2} \right]$$

$$\vec{E}^i_\theta = \hat{a}_\theta E_0 e^{j\beta(y \sin\theta' + z\cos\theta')}$$
$$\vec{E}^i_\phi = \hat{a}_\phi E_0 e^{j\beta(y \sin\theta' + z\cos\theta')} \quad (1),\ (2)$$

$$|E^i|^2 = E_0^2$$

For Physical optics

Monostatic RCS (hard and soft polarizations)

$$\sigma_{3-D} = 4\pi \left(\frac{\pi a^2}{\lambda} \right)^2 \cos^2\theta \left| 2 \frac{J_1(2\beta a \sin\theta)}{2\beta a \sin\theta} \right|^2$$

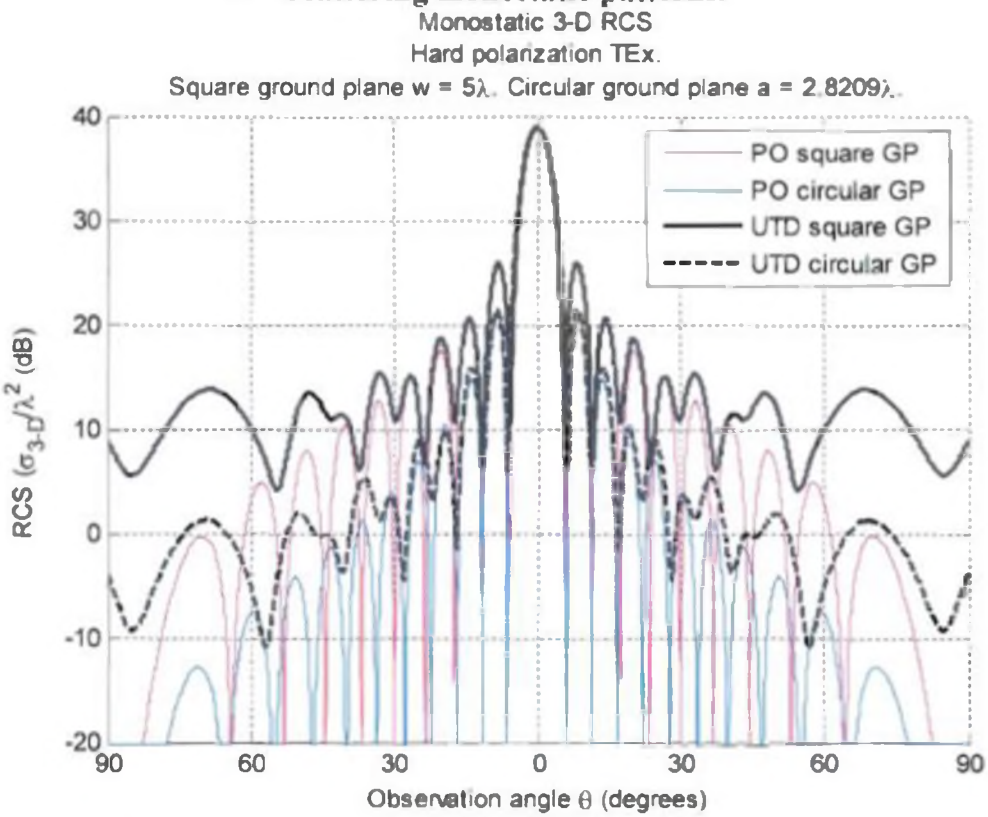
For UTD

Hard polarization	Soft polarization
$\|E^s\|^2 = \left\| E_{hard}^{1st(1,2)} + E_{hard}^{2nd(1,2)} + E_{hard}^{2nd(3,4)} \right\|^2$	$\|E^s\|^2 = \left\| E_{soft}^{1st(1,2)} + E_{hard}^{2nd(3,4)} \right\|^2$

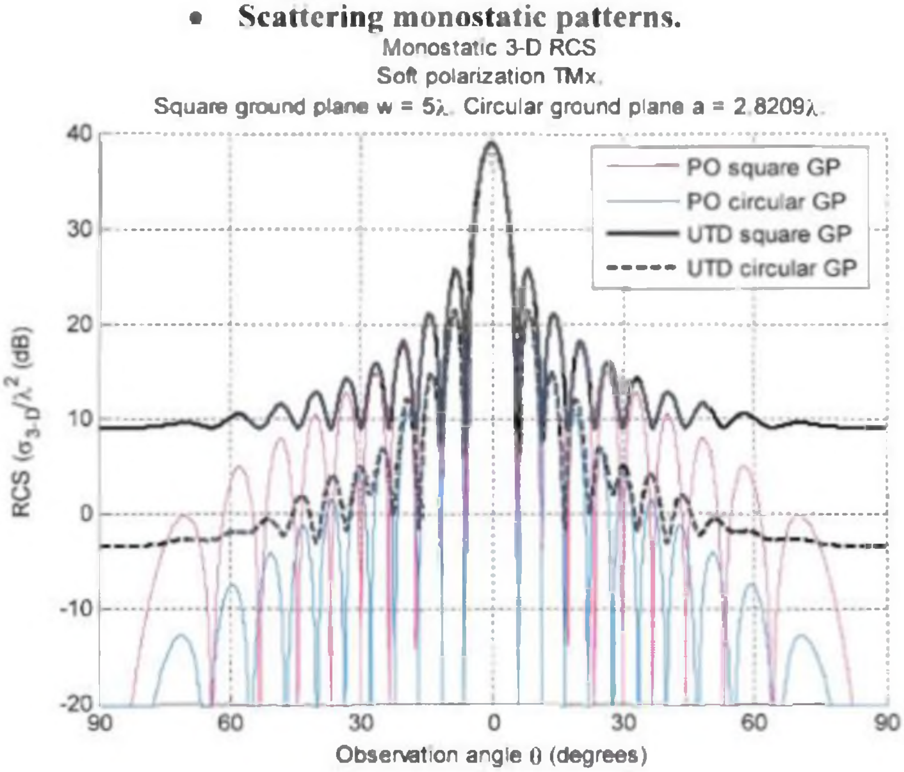
cont'd

13.53 cont'd

- **Scattering monostatic patterns.**



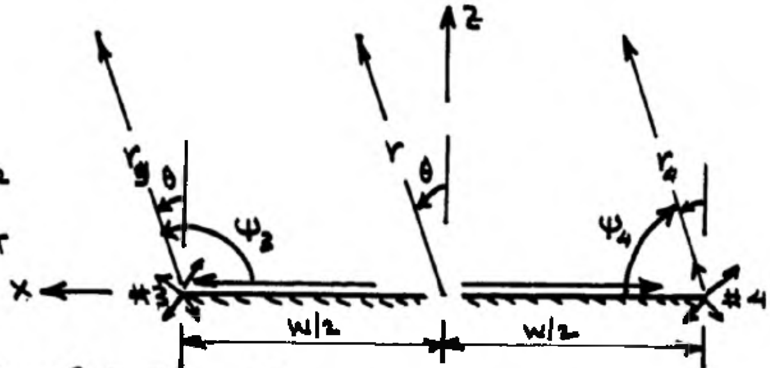
- **Scattering monostatic patterns.**



13.54

(a) $\psi_3 = \pi/2 + \theta$

$$\psi_4 = \begin{cases} \frac{\pi}{2} - \theta, & 0 \le \theta \le \pi/2 \\ \frac{5\pi}{2} - \theta, & \pi/2 \le \theta \le \pi \end{cases}$$



$$E^d = \frac{1}{j\beta} \frac{\partial E^i}{\partial n}(Q_D) \frac{\partial D^s}{\partial \phi'} \sqrt{\frac{\rho_c}{s(\rho_c+s)}}\, e^{-j\beta s}$$

$$\frac{\partial E^i}{\partial n}(Q_D) = \frac{1}{s'} \frac{\partial E^i}{\partial \phi'}\Big|_{Q_D}, \quad \frac{\partial D^s}{\partial \phi'} = \text{slope diffraction coefficient}$$

For Wedge #3

$$\frac{\partial E^i}{\partial n}(Q_3) = \frac{1}{s'} \frac{\partial E^i}{\partial \psi_3}\Big|_{\substack{s'=w/2 \\ \psi_3 = \theta + \pi/2}} = \frac{1}{s'} \frac{\partial E^i}{\partial \theta}\Big|_{\substack{\theta = 90^\circ \\ s' = w/2}}$$

For Wedge #4

$$\frac{\partial E^i}{\partial n}(Q_4) = \frac{1}{s'} \frac{\partial E^i}{\partial \psi_4}\Big|_{\substack{s'=w/2 \\ \psi_4 = \{\pi/2-\theta,\ 0 \le \theta \le 90^\circ;\ 5\pi/2-\theta,\ 90^\circ \le \theta \le 180^\circ\}}} = -\frac{1}{s'} \frac{\partial E^i}{\partial \theta}\Big|_{\substack{\theta = 90^\circ \\ s' = w/2}}$$

$$\frac{\partial E^i}{\partial \theta} = \frac{\partial}{\partial \theta}\left[\cos\theta \frac{\cos\left(\frac{\beta a}{2}\sin\theta\right)}{\left(\frac{\beta a}{2}\sin\theta\right)^2 - \left(\frac{\pi}{2}\right)^2}\right] \frac{e^{-j\beta r}}{r} = \frac{\partial}{\partial \theta}\left[\frac{N}{D}\right]\frac{e^{-j\beta r}}{r} = \left[\frac{DN' - ND'}{D^2}\right]\frac{e^{-j\beta r}}{r}$$

where $N = \cos\theta \cos\left(\frac{\beta a}{2}\sin\theta\right)$

$D = \left(\frac{\beta a}{2}\sin\theta\right)^2 - \left(\frac{\pi}{2}\right)^2$

$N' = \frac{\partial}{\partial \theta} N = -\sin\theta \cos\left(\frac{\beta a}{2}\sin\theta\right) - \cos\theta \sin\left(\frac{\beta a}{2}\sin\theta\right)\left(\frac{\beta a}{2}\cos\theta\right)$

$D' = \frac{\partial}{\partial \theta} D = 2\left(\frac{\beta a}{2}\sin\theta\right)\left(\frac{\beta a}{2}\cos\theta\right)$

Therefore

$$\frac{\partial E^i}{\partial \theta} = \frac{e^{-j\beta r}}{r} \frac{\left[\left(\frac{\beta a}{2}\sin\theta\right)^2 - \left(\frac{\pi}{2}\right)^2\right]\left\{-\sin\theta\cos\left(\frac{\beta a}{2}\sin\theta\right) - \cos\theta\sin\left(\frac{\beta a}{2}\sin\theta\right)\left(\frac{\beta a}{2}\cos\theta\right)\right\} - \cos\theta\cos\left(\frac{\beta a}{2}\sin\theta\right)\left[2\left(\frac{\beta a}{2}\sin\theta\right)\left(\frac{\beta a}{2}\cos\theta\right)\right]}{\left[\left(\frac{\beta a}{2}\sin\theta\right)^2 - \left(\frac{\pi}{2}\right)^2\right]^2}$$

$$\frac{\partial E^i}{\partial \theta}\Big|_{\theta = 90^\circ} = \frac{\left[\left(\frac{\beta a}{2}\right)^2 - \left(\frac{\pi}{2}\right)^2\right]\left[-\cos\left(\frac{\beta a}{2}\right)\right]}{\left[\left(\frac{\beta a}{2}\right)^2 - \left(\frac{\pi}{2}\right)^2\right]^2} \frac{e^{-j\beta r}}{r} = -\frac{\cos(\beta a/2)}{[(\beta a/2)^2 - (\pi/2)^2]} \frac{e^{-j\beta r}}{r}$$

cont'd.

13.54 Cont'd. Since $\left.\frac{\partial E^i}{\partial\theta}\right|_{\theta=90^\circ}$ becomes indeterminate when $\frac{\beta a}{2}=\frac{\pi}{2}$ or $a=\frac{\lambda}{2}$, then using L' Hospital's rule leads to

$$\left.\frac{\partial E^i}{\partial\theta}\right|_{\substack{\theta=90^\circ\\ a=\lambda/2}} = \left.\frac{\sin\left(\frac{\beta a}{2}\right)}{2\left(\frac{\beta a}{2}\right)}\right|_{a=\lambda/2} \frac{e^{-j\beta r}}{r} = \frac{1}{\pi}\frac{e^{-j\beta r}}{r}$$

Thus the field diffracted from edge #3 can be written as

$$E_3^d = \frac{1}{j\beta}\frac{1}{2}\frac{\partial E^i}{\partial n}(Q_3)\frac{\partial D^s}{\partial\phi'}\frac{\sqrt{s'}}{s}e^{-j\beta s} = \left.\frac{1}{2j\beta}\left[\frac{1}{s'}\frac{\partial E^i(Q_3)}{\partial\theta}\right]\frac{\partial D^s}{\partial\phi'}\frac{\sqrt{s'}}{s}e^{-j\beta s}\right|_{\substack{s'=w/2\\ s=r_3}}$$

Since

$r_3 \simeq r - \frac{w}{2}\sin\theta$ for phase terms

$r_3 \simeq r$ for amplitude terms

$$E_3^d = \frac{1}{2j\beta}\left[\frac{1}{w/2}\ \frac{-\cos(\beta a/2)}{\left(\frac{\beta a}{2}\right)^2-\left(\frac{\pi}{2}\right)^2}\frac{e^{-j\beta w/2}}{w/2}\right]\frac{\partial D^s(\phi,\phi',n;\beta_0')}{\partial\phi'}\frac{\sqrt{w/2}}{r}e^{-j\beta\left(r-\frac{w}{2}\sin\theta\right)}$$

$$E_3^d = -j\frac{(2/w)^{3/2}}{2\beta}\frac{\cos(\beta a/2)}{(\beta a/2)^2-(\pi/2)^2}\frac{\partial D^s(\phi,\phi',n;\beta_0')}{\partial\phi'}e^{-j\beta\frac{w}{2}(1-\sin\theta)}\frac{e^{-j\beta r}}{r}$$

where $\phi=\psi_3,\ \phi'=\psi_3'=0^\circ,\ n=2,\ \beta_0'=90^\circ$

Similarly

$$E_4^d = \left.\frac{1}{j\beta 2}\left[-\frac{1}{s'}\ \frac{\partial E^i(Q_4)}{\partial\theta}\right]\frac{\partial D^s(\phi,\phi',n;\beta_0')}{\partial\phi'}\frac{\sqrt{s'}}{s}e^{-j\beta s}\right|_{\substack{s'=w/2\\ s=r_3}}$$

Since

$r_4 \simeq r + \frac{w}{2}\sin\theta$ for phase terms

$r_4 \simeq r$ for amplitude terms

then

$$E_4^d = -j\frac{(2/w)^{3/2}}{2\beta}\frac{\cos\left(\frac{\beta a}{2}\right)}{\left(\frac{\beta a}{2}\right)^2-\left(\frac{\pi}{2}\right)^2}\frac{\partial D^s(\phi,\phi',n;\beta_0')}{\partial\phi'}e^{-j\beta\frac{w}{2}(1+\sin\theta)}\frac{e^{-j\beta r}}{r}$$

where $\phi=\psi_4,\ \phi'=\psi_4'=0^\circ,\ n=2,\ \beta_0'=90^\circ$

The total diffracted field is then equal to

$$E_t^d = E_3^d + E_4^d$$

To compute the total field, the diffracted field must be combined with the geometrical optics field.

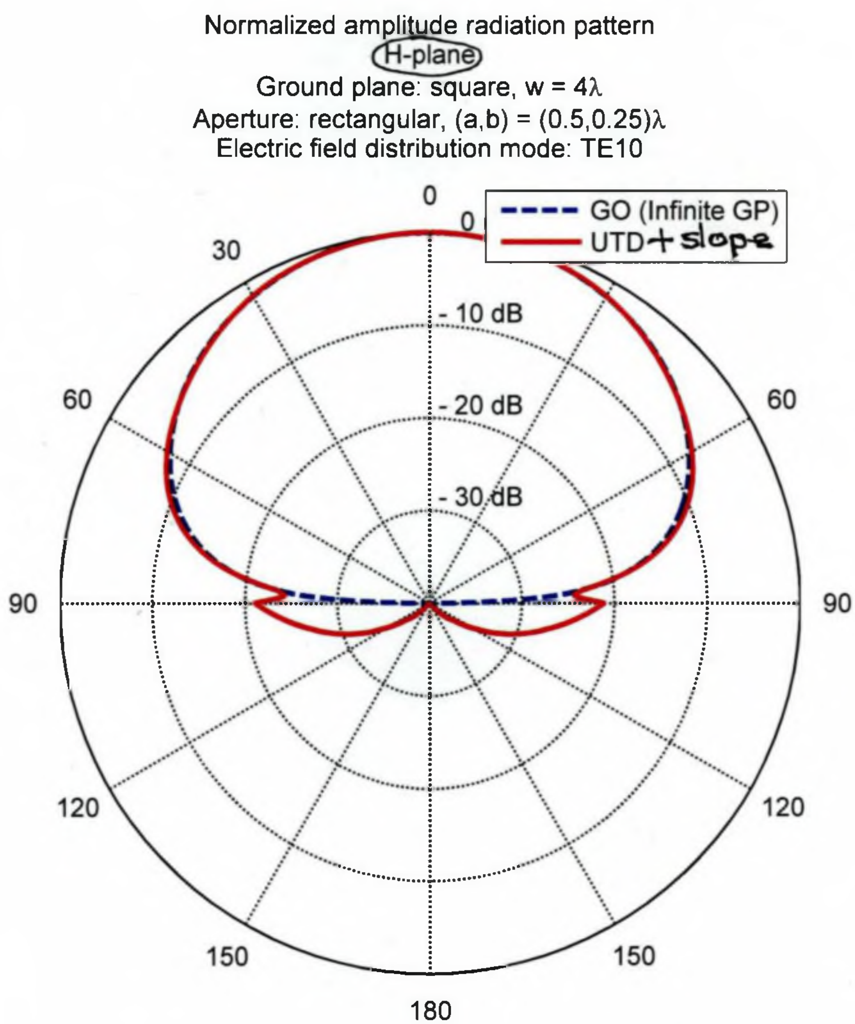
cont'd.

$\boxed{\text{13.54 Cont'd.}}$

(b) To plot the field for $a = \lambda/2$, the term

$\dfrac{\cos(\beta a/2)}{\left(\frac{\beta a}{2}\right)^2 - \left(\frac{\pi}{2}\right)^2}$ is indeterminate and must be evaluated by

L'Hopital's rule as $\displaystyle\lim_{a \to \lambda/2} \frac{\cos(\beta a/2)}{\left(\frac{\beta a}{2}\right)^2 - \left(\frac{\pi}{2}\right)^2} = -\left.\frac{\sin(\beta a/2)}{2(\beta a/2)}\right|_{\beta a/2 = \pi/2} = -\frac{1}{\pi}$



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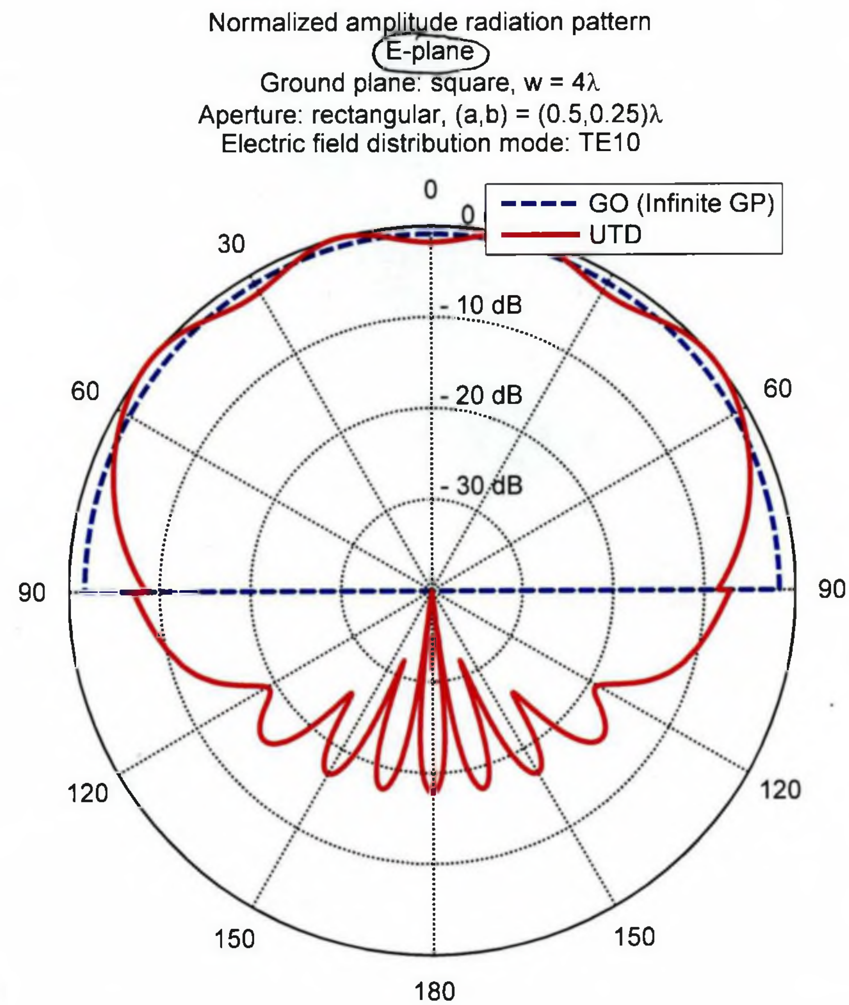
13.54 cont'd



Normalized amplitude radiation pattern
E-plane
Ground plane: square, w = 4λ
Aperture: rectangular, (a,b) = (0.5,0.25)λ
Electric field distribution mode: TE10
GO (Infinite GP)
UTD
0
30
60
90
120
150
180
- 10 dB
- 20 dB
- 30 dB

13.55

(a) $H_z^i = H_0 H_0^{(2)}(\beta\rho) \overset{\beta\rho\to\infty}{\simeq} H_0\sqrt{\frac{2j}{\pi\beta\rho}}\, e^{-j\beta\rho} = H_1 \frac{e^{-j\beta\rho}}{\sqrt{\beta\rho}}$ where $H_1 = H_0\sqrt{\frac{2j}{\pi}}$

$H_{z1}^d = H_z^i(\rho=\frac{w}{2}) D_h A_1 e^{-j\beta r_1}$, $H_z^i(\rho=\frac{w}{2}) = H_1 \frac{e^{-j\beta w/2}}{\sqrt{\beta w/2}}$

$$D_h^1 = \frac{e^{-j\pi/4}\frac{1}{n}\sin(\frac{\pi}{n})}{\sqrt{2\pi\beta}}\left\{\frac{1}{\cos(\frac{\pi}{n})-\cos(\frac{\psi_1-\psi_1'}{n})}+\frac{1}{\cos(\frac{\pi}{n})-\cos(\frac{\psi_1+\psi_1'}{n})}\right\}_{\substack{n=3/2\\ \psi_1'=0}}$$

$$= \frac{2e^{-j\pi/4}}{3}\frac{\sin(\frac{2\pi}{3})}{\sqrt{2\pi\beta}}\left\{\frac{1}{\cos(\frac{2\pi}{3})-\cos(\frac{2\psi_1}{3})}+\frac{1}{\cos(\frac{2\pi}{3})-\cos(\frac{2\psi_1}{3})}\right\}$$

For observation at edge #2, $\psi_1 = 270°$. Thus

$$D_h^1 = \frac{4e^{-j\pi/4}}{3}\frac{\sin(\frac{2\pi}{3})}{\sqrt{2\pi\beta}}\left\{\frac{1}{\cos(120°)-\cos(180°)}\right\} = \frac{4(0.867)e^{-j\pi/4}}{3\sqrt{2\pi\beta}}\left\{\frac{1}{-0.5-(-1)}\right\}$$

$$D_h^1 = \frac{2.312\, e^{-j\pi/4}}{\sqrt{2\pi\beta}}$$

$A_1 = \frac{1}{\sqrt{w}}$,

Thus at the edge #2, the field is given by

$$H_{z1}^d(\text{at }\#2) = H_1 \frac{e^{-j\beta w/2}}{\sqrt{\beta w/2}} \frac{2.312\, e^{-j\pi/4}}{\sqrt{2\pi\beta}} \frac{e^{-j\beta w}}{\sqrt{w}} = H_1 \frac{2.312}{\sqrt{\pi}}\frac{e^{-j3\beta w/2}}{\beta w} e^{-j\pi/4}$$

$$H_{z1}^d = 1.3044\, H_1 \frac{e^{-j3\beta w/2}}{\beta w} e^{-j\pi/4}$$

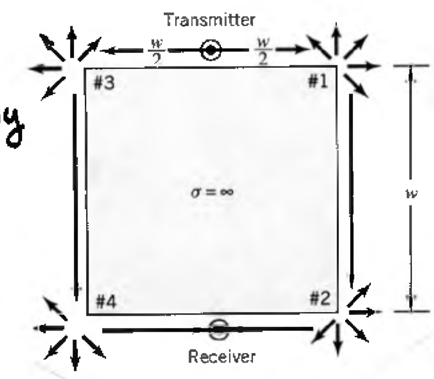


Figure P13-55

The field diffracted by wedge #2 is given by

$$H_{z2}^d = \frac{H_{z1}^d(\text{at }\#2)}{2} D_h^2 A_2 e^{-j\beta w/2}$$

Because of the geometry

$$D_h^2 = D_h^1 = \frac{2.312\, e^{-j\pi/4}}{\sqrt{2\pi\beta}}, \quad A_2 = \frac{1}{\sqrt{w/2}}$$

Thus

$$H_{z2}^d = \frac{1.3044\, H_1}{2}\frac{e^{-j3\beta w/2}}{\beta w} e^{-j\pi/4}\left[\frac{2.312\, e^{-j\pi/4}}{\sqrt{2\pi\beta}}\right]\frac{1}{\sqrt{w/2}} e^{-j\beta w/2}$$

$$H_{z2}^d = \frac{0.6522(2.312)}{\sqrt{\pi}} H_1 \frac{e^{-j2\beta w}}{(\beta w)^{3/2}} e^{-j\pi/2} = -j\,0.85073\, H_1 \frac{e^{-j2\beta w}}{(\beta w)^{3/2}}$$

Using the symmetry of the problem

$$H_{z4}^d = H_{z2}^d = -j\,0.85073\, H_1 \frac{e^{-j2\beta w}}{(\beta w)^{3/2}}$$

cont'd.

$\boxed{\text{13.55 cont'd.}}$ Therefore the total field at the observation is

$$H_{zt}^{d} = H_{z2}^{d} + H_{z4}^{d} = -j\,1.70147\,H_1 \frac{e^{-j2\beta W}}{(\beta W)^{3/2}}$$

(b) $H_z^{i} = H_1 \dfrac{e^{-j\beta\rho}}{\sqrt{\beta\rho}}\Bigg|_{\rho=W} = H_1 \dfrac{e^{-j\beta W}}{\sqrt{\beta W}}$

$$\left|\frac{H_{zt}^{d}}{H_z^{i}}\right| = \left|\frac{-j\,1.70147\,H_1 \dfrac{e^{-j2\beta W}}{(\beta W)^{3/2}}}{H_1 \dfrac{e^{-j\beta W}}{(\beta W)^{1/2}}}\right| = \frac{1.70147}{\beta W}\Bigg|_{W=5\lambda} = \frac{1.70147}{10\pi} = 0.05416$$

$$\left|\frac{H_{zt}^{d}}{H_z^{i}}\right| = 0.05416 \quad \text{or} \quad 20\log_{10}(0.05416) = -25.326 \text{ dB}$$

CHAPTER 14

14.1 The duality in electromagnetics, as summarized in Table 7-2, going from the fields due to electric current $\underline{I}$ to those due to a magnetic current density $\underline{M}$, is accomplished by basically replacing $\underline{E}$ by $\underline{H}$, $\underline{H}$ by $-\underline{E}$, η by $1/\eta$, μ by ε, ε by μ, β by β, $\underline{I}$ by $\underline{M}$. When this is done in (14-2), we can write its dual as

$$\underline{H} - (\hat{n}\cdot\underline{H})\hat{n} = -\frac{Z_o}{\eta}\hat{n}\times\mathbf{E}$$

where $\eta = Z/Z_o$, where Z_o is the free space value of 377 (constant)

The basic objective of this is that if we obtain the solution of one polarization with normalized impedance η, this solution is identical to the solution of a polarization with normalized impedance $1/\eta$.

14.2 For TM^z (soft polarization), the impedance boundary conditions based on the geometry of 14-1 (both interior and exterior wedges) are given by

$$E_z = -\eta_o Z_o H_\rho \quad \text{for } \phi = 0$$
$$E_z = \eta_n Z_o H_\rho \quad \text{for } \phi = n\pi$$

TM^z: Using Maxwell's equations

$$\frac{1}{\rho}\frac{\partial E_z}{\partial\phi} = -j\omega\mu H_\rho \Rightarrow H_\rho = -\frac{1}{j\omega\mu}\frac{1}{\rho}\frac{\partial E_z}{\partial\phi}$$

the above two conditions can be reduced to

$$E_z = -\eta_o Z_o\left(-\frac{1}{j\omega\mu}\frac{1}{\rho}\frac{\partial E_z}{\partial\phi}\right) \Rightarrow \frac{\partial E_z}{\partial\phi} - j\frac{\beta\rho}{\eta_o}E_z = 0 \quad @\ \phi = 0$$

$$E_z = +\eta_n Z_o\left(-\frac{1}{j\omega\mu}\frac{1}{\rho}\frac{\partial E_z}{\partial\phi}\right) \Rightarrow \frac{\partial E_z}{\partial\phi} + j\frac{\beta\rho}{\eta_n}E_z = 0 \quad @\ \phi = n\pi$$

$\boxed{14.3}$ For the TE^z polarization (hard), the impedance boundary conditions based on the geometry of (14-1) (both interior and exterior wedges) are given by

$$E_\rho = \eta_0 Z_0 H_z \quad \text{for } \phi = 0$$

$$E_\rho = -\eta_n Z_0 H_z \quad \text{for } \phi = n\pi$$

TE^z: Using Maxwell's equations

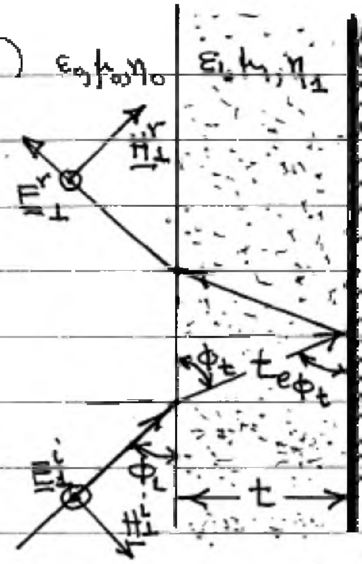
$$\frac{1}{\rho}\frac{\partial H_z}{\partial \phi} = j\omega\varepsilon E_\rho \Rightarrow E_\rho = \frac{1}{j\omega\varepsilon}\frac{1}{\rho}\frac{\partial H_z}{\partial \phi}$$

the above two conditions can be reduced to

$$\frac{1}{j\omega\varepsilon}\frac{1}{\rho}\frac{\partial H_z}{\partial \phi} = \eta_0 Z_0 H_z \Rightarrow \frac{\partial H_z}{\partial \phi} - j\beta\rho\eta_0 H_z = 0 \quad @\ \phi = 0$$

$$\frac{1}{j\omega\varepsilon}\frac{1}{\rho}\frac{\partial H_z}{\partial \phi} = -\eta_n Z_0 H_z \Rightarrow \frac{\partial H_z}{\partial \phi} + j\beta\rho\eta_0 H_z = 0 \quad @\ \phi = n\pi$$

$\boxed{14.4}$ Soft Polarization ($\perp$ polarization)



Based on (14-4a), the reflection coefficient of a flat surface with a normalized surface impedance η is

$$\Gamma_s = \frac{E^r}{E^i} = \frac{\eta \sin\phi_i - 1}{\eta \sin\phi_i + 1} \qquad (1)$$

If we allow for refraction, the above equation can be written based on (15-17a) as

$$\Gamma_s = \frac{E^r}{E^i} = \frac{\eta \sin\phi_i - \sin\phi_t}{\eta \sin\phi_i + \sin\phi_t} \qquad (2)$$

Based on Snell's law of refraction

$$\beta_0 \cos^2\phi_i = \beta_1 \cos^2\phi_t \Rightarrow \phi_t = \cos^{-1}\left[\frac{\beta_0}{\beta_1}\cos\phi_i\right]^{1/2} = \cos^{-1}\left[\sqrt{\frac{\mu_0\varepsilon_0}{\mu_1\varepsilon_1}}\cos\phi_i\right]^{1/2} \qquad (3)$$

If ε_1, μ_1 or $\varepsilon_1\mu_1$ are each large ($\varepsilon_1 \gg 1$, $\mu_1 \gg 1$, $\mu_1\varepsilon_1 \gg 1$), then (3) reduces to

$$\phi_t = \cos^{-1}\left[\sqrt{\frac{\mu_0\varepsilon_0}{\mu_1\varepsilon_1}}\cos\phi_i\right]^{1/2} \xrightarrow{\varepsilon_1, \mu_1, \varepsilon_1\mu_1 \gg 1} \cos^{-1}(0) = 90^\circ \Rightarrow \text{normal incidence.}$$

which indicates that (2) reduces to (1). This implies that **the impedance boundary conditions are most accurate near normal incidence** which will be most valid for **materials** with high permittivities and/or high permeabilities.

If we place a dielectric slab of thickness t to cover the PEC ground plane, as shown above, the reflection coefficient can be written as that of (2) but with η replaced with an equivalent normalized surface impedance η_e. For this case, we can rewrite (2) as

$$\Gamma_s = \frac{E^r}{E^i} = \frac{\eta_e \sin\phi_i - \sin\phi_t}{\eta_e \sin\phi_i + \sin\phi_t} \qquad (4)$$

One way to compute η_e is to use transmission line theory.

continued

14.4 cont'd

The input reflection coefficient of a short transmission line, with characteristic impedance Z_c and length ℓ, is

$$Z_{in} = jZ_c \tan(\beta\ell)$$

Treating the wave transmission in the dielectric covered ground plane as a shorted transmission line, we can write based on the figure of the previous page that the equivalent surface impedance η_e is

$$\boxed{\eta_e = j\eta \tan(\beta_1 t \sin\phi_t)} \qquad (5)$$

since the wave travels in the slab through an oblique distance t_e

$$\sin\phi_t = t/t_e \Rightarrow t = t_e \sin\phi_t \qquad (6)$$

and where $\beta_1 = \omega\sqrt{\mu_1\varepsilon_1}$.

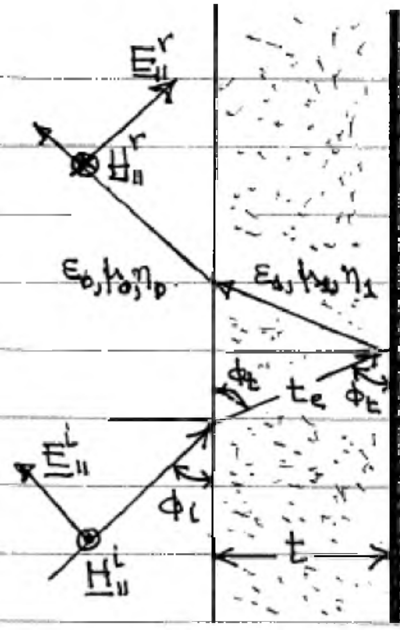
Therefore based on (5), the reflection coefficient of (4) can be written, using (5), as

$$\boxed{\Gamma_s = \frac{j\eta\tan(\beta_1 t\sin\phi_t)\sin\phi_i - \sin\phi_t}{j\eta\tan(\beta_1 t\sin\phi_t)\sin\phi_i + \sin\phi_t}} \qquad (7)$$

If the permeability μ_1 or permittivity ε_1 or both are very large, then (7) reduces to ($\phi_t \simeq 90^\circ$, normal incidence)

$$\boxed{\Gamma_s \simeq \frac{j\eta\tan(\beta_1 t)\sin\phi_i - \sin\phi_t}{j\eta\tan(\beta_1 t)\sin\phi_i + \sin\phi_t}} \qquad (8)$$

14.5 Hard Polarization (∥ Polarization)



Using (14-4b)

$$\Gamma_h = \frac{E^r}{E^i} = \frac{\eta - \sin\phi_i}{\eta + \sin\phi_i} \quad (14\text{-}4b)$$

(5-24c) but permit refraction, we can write it as (with $\phi_i = \frac{\pi}{2} - \theta_i$ and $\phi_t = \frac{\pi}{2} - \theta_t$)

$$\Gamma_h = \frac{E^r}{E^i} = \frac{\eta \sin\phi_t - \sin\phi_i}{\eta \sin\phi_t + \sin\phi_i}$$

Following the procedure of the solution of Problem 14.4, by using transmission line theory where we can write the above equation as

$$\Gamma_h = \frac{E^r}{E^i} = \frac{\eta_e \sin\phi_t - \sin\phi_i}{\eta_e \sin\phi_t + \sin\phi_i}$$

where η = equivalent surface impedance $= j\eta \tan(\beta_1 t \sin\phi_t)$
then it can be expressed as

$$\boxed{\Gamma_h = \frac{E^r}{E^i} = \frac{j\eta \tan(\beta_1 t \sin\phi_t) - \sin\phi_i}{j\eta \tan(\beta_1 t \sin\phi_t) + \sin\phi_i}}$$

which, if the permeability μ_1, or permittivity ε_1, or both are very large, then the above equation reduces (because $\phi_t \simeq 90°$, near normal incidence) to

$$\boxed{\Gamma_h = \frac{E^r}{E^i} \simeq \frac{j\eta \tan(\beta_1 t) - \sin\phi_i}{j\eta \tan(\beta_1 t) + \sin\phi_i}}$$

where $\beta_1 = \omega\sqrt{\mu_1 \varepsilon_1}$.

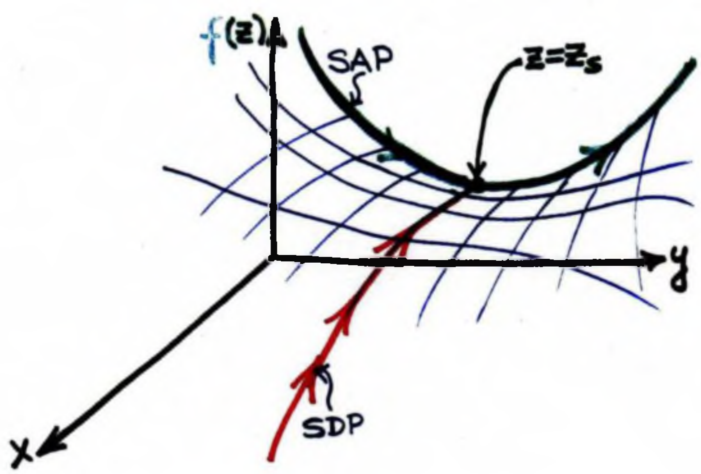
14.6

The saddle-point is so named because it lies at a point in the complex plane at which:

$$\left|e^{\beta\rho h(z)}\right|_{z=z_s} = \left|e^{\beta\rho h(z_s)}\right| = \begin{cases} \text{maximum} & \text{(along one direction)} \\ \text{minimum} & \text{(along a perpendicular direction)} \end{cases}$$

In three dimensions, a plot of $\left|e^{\beta\rho h(z)}\right|$ would take the shape of a saddle with $z = z_s$ at the center of the saddle.

If the function $H(z)$ is smoothly varying near the saddle point and if the contour C follows the steepest *descent* path through $z = z_s$, then the integrand is maximum at the saddle point and negligible elsewhere along the path. If the contour follows the steepest *ascent* path through $z = z_s$, then the integrand is minimum at the saddle point and large elsewhere along the path.



The steepest descent (ascent) path equation states that the *imaginary* part of $h(z)$ is *constant* (phase variations are nearly constant) because along these paths the *real part* of $h(z)$ (amplitude variation) is varying *most rapidly*. Hence $\left|e^{\beta\rho h(z)}\right|$ either increases (or decreases) as rapidly as possible along the path.

$$\boxed{h(z) = h_r(z) + h_i(z); \qquad z = x + jy}$$

The stepest descent path is obtained by equatin

$$\boxed{h_i(z) = h_i(z_s); \text{ Steepest Descent Path}}$$

The same equation is used for the ascent path.

Example: $e^{\beta\rho h(z)} = e^{j\beta\rho\cos(z)} = e^{\beta\rho[j\cos(z)]}$

$$h(z) = j\cos(z) = h_r(z) + h_i(z)$$

$$j\cos(z) = j\cos(x+jy) = h_r(z) + h_i(z)$$

$$j\cos(z) = j\cos(x+jy) = j\cos(x)\cos(jy) - j\sin(x)\sin(jy)$$

$$j\cos(z) = j\cos(x)\cosh(y) + \sin(x)\sinh(y)$$

$$h_i(z) = \cos(x)\cosh(y)$$

$$h_r(z) = \sin(x)\sinh(y)$$

$$I_{\max}\left[j\cos(z)\right] = I_{\max}\left[j\cos(\pm\pi)\right] = I_{\max}\left[j(-1)^m\right];\; m = \pm 1$$

$\boxed{h_i(z) = h_i(z_s)}$ $\Rightarrow$ Steepest Descent Paths

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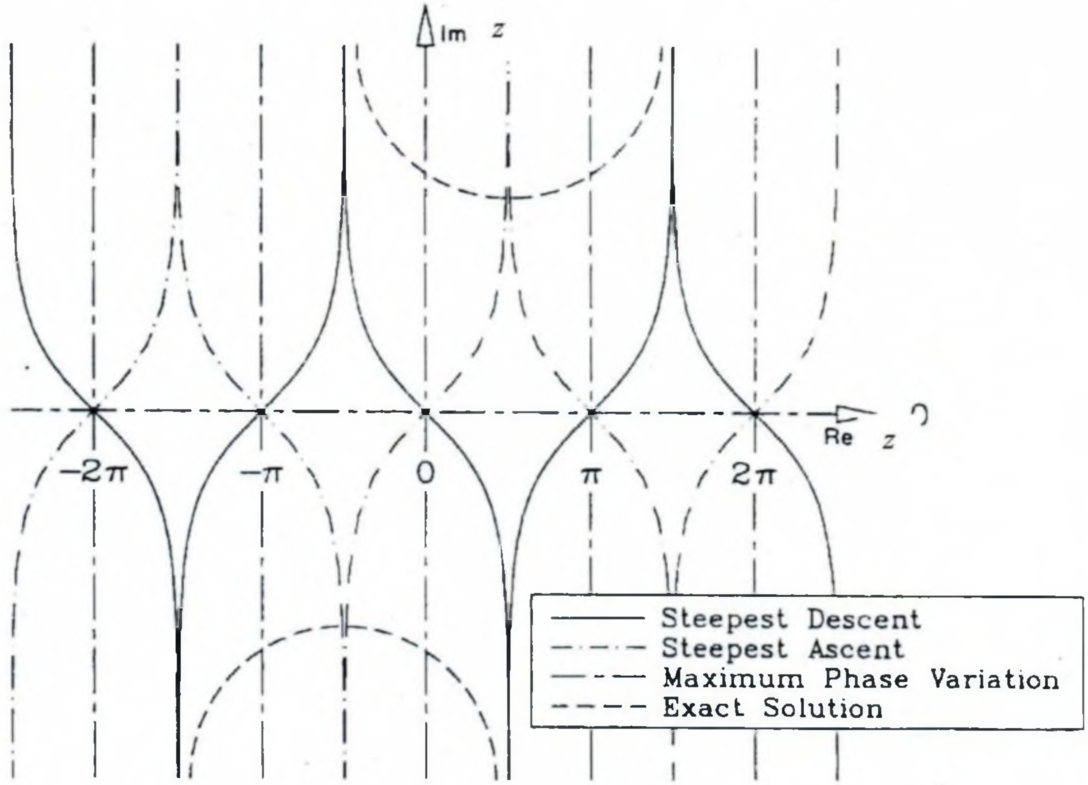
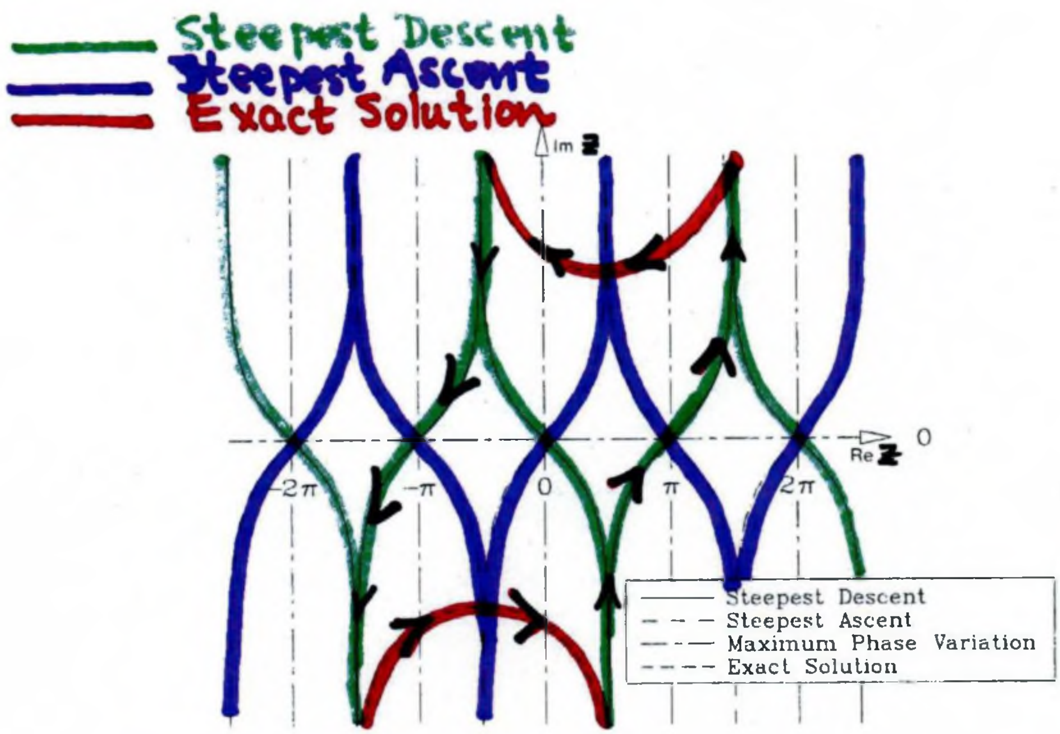
$\boxed{\cos(x)\cosh(y) = -1}$ Steepest Descent Paths

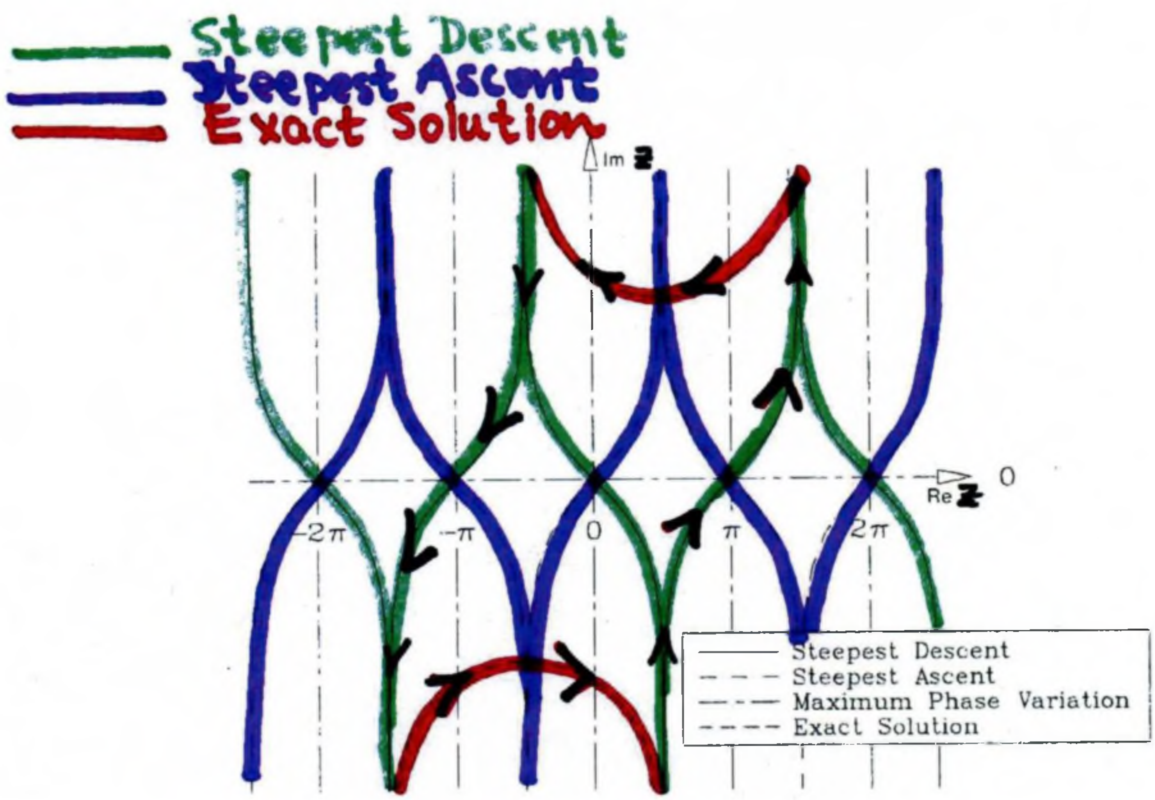

Im z
Re z
0
−2π
−π
0
π
2π
Steepest Descent
Steepest Ascent
Maximum Phase Variation
Exact Solution


Steepest Descent
Steepest Ascent
Exact Solution
Im z
Re z
0
−2π
−π
0
π
2π
Steepest Descent
Steepest Ascent
Maximum Phase Variation
Exact Solution

14.7

<u>Continuation of the Solution of Problem 14.6</u>

In contrast, the rapidly varying phase paths used for the Method of Stationary Phase are those for which the *real part* of $h(z)$ (amplitude variation) *is constant*, which means that the *imaginary* part of $h(z)$ (phase variations) vary as *rapidly as possible* away from the stationary saddle points.



Example: $e^{\beta\rho h(z)} = e^{j\beta\rho\cos(z)} = e^{\beta\rho[j\cos(z)]}$

$$h(z) = j\cos(z) = h_r(z) + h_i(z)$$

$$j\cos(z) = j\cos(x + jy) = h_r(z) + h_i(z)$$

$$j\cos(z) = j\cos(x + jy) = j\cos(x)\cos(jy) - j\sin(x)\sin(jy)$$

$$j\cos(z) = j\cos(x)\cosh(y) + \sin(x)\sinh(y)$$

$$h_i(z) = \cos(x)\cosh(y)$$

$$h_r(z) = \sin(x)\sinh(y)$$

$$I_{\max}\left[j\cos(z)\right] = I_{\max}\left[j\cos(\pm\pi)\right] = I_{\max}\left[j(-1)^m\right];\ m = \pm 1$$

$h_i(z) = h_i(z_s) \Rightarrow$ steepest descent paths

$\boxed{\cos(x)\cosh(y) = -1}$ Steepest Descent Paths

$h_r(z) = h_r(z_s) \Rightarrow$ stationary phase paths

$\boxed{\sin(x)\sinh(y) = 0}$ Stationary Phase Paths

$\boxed{14.8}$ Wedge $= 90^\circ \Rightarrow n = 3/2$, $\phi' = 60^\circ$, $\phi = 180^\circ$

$$\varepsilon_1 = 4\varepsilon_0,\ \mu_1 = \mu_0 \Rightarrow \eta_1 = \sqrt{\frac{\mu_1}{\varepsilon_1}} = \sqrt{\frac{\mu_0}{4\varepsilon_0}} = \frac{\eta_0}{2} \Rightarrow \eta = \frac{\eta_1}{\eta_0} = \frac{1}{2}$$

Hard Polarization

$$\theta_0 = \theta_n = \sin^{-1}(\eta_0) = \sin^{-1}(\eta_n) = \sin^{-1}(1/2) = 30^\circ \quad \begin{cases} \sin\theta_0 = 1/2 \\ \sin\theta_n = 1/2 \end{cases}$$

- The incident diffracted field, based on the asymptotic form, is given by the first term within the brackets of (14-59). Using a Matlab computer program developed by the author and his students, the computed value for the incident diffracted field based on the above parameters is:

$$U^i = (-9.4724 + j9.4339) \times 10^{-2}$$

It should be noted that the observation point $\phi = 180^\circ$ is far removed from the incident ($\phi = 240^\circ$) and the reflection ($\phi = 120^\circ$) shadow boundaries such that the Fresnel transition functions of (14-59) are basically nearly unity

- The reflected diffracted field, based on the asymptotic form, is given by the second term within the brackets of (14-59). Again using a Matlab program developed by the author and his students, the computed value for the reflected diffracted field based on the above parameters is:

$$U^r = (4.107 - j4.094) \times 10^{-2}$$

It should be noted again, that the observation point $\phi = 180^\circ$ is far removed from the incident ($\phi = 240^\circ$) and the reflection ($\phi = 120^\circ$) shadow boundaries such that the Fresnel transition functions of (14-59) are basically nearly unity.

$\boxed{14.9}$ Wedge $= 90^\circ \Rightarrow n = 3/2$, $\phi' = 60^\circ$, $\phi = 180^\circ$

$$\varepsilon_1 = 4\varepsilon_0,\ \mu_1 = \mu_0 \Rightarrow \eta_1 = \sqrt{\frac{\mu_1}{\varepsilon_1}} = \sqrt{\frac{\mu_0}{4\varepsilon_0}} = \frac{\eta_0}{2} \Rightarrow \eta = \frac{\eta_1}{\eta_0} = \frac{1}{2}$$

Soft Polarization:

$$\theta_0 = \theta_n = \sin^{-1}\left(\frac{1}{\eta_0}\right) = \sin^{-1}\left(\frac{1}{\eta_n}\right) = \sin^{-1}\left(\frac{1}{1/2}\right) = \sin^{-1}(2) \quad \left\{ \begin{array}{l} \sin\theta_0 = 2 \\ \sin\theta_n = 2 \end{array} \right.$$

- The incident diffracted field, based on the asymptotic form, is given by the first term within the brackets of (14-59). Using a Matlab computer program developed by the author and his students, the computed value for the incident diffracted field based on the above parameters is:

$$U^i = (-10.0249 + j\,8.4852) \times 10^{-2}$$

It should be noted that the observation point $\phi = 180^\circ$ is far removed from the incident ($\phi = 240^\circ$) and the reflection ($\phi = 120^\circ$) shadow boundaries such that the Fresnel transition functions of (14-59) are basically nearly unity

- The reflected diffracted field, based on the asymptotic form, is given by the second term within the brackets of (14-59). Again using a Matlab program developed by the author and his students, the computed value for the reflected diffracted field based on the above parameters is:

$$U^r = (-5.0221 + j\,6.5065) \times 10^{-2}$$

It should be noted again that the observation point $\phi = 180^\circ$ is far removed from the incident ($\phi = 240^\circ$) and the reflection ($\phi = 120^\circ$) shadow boundaries such that the Fresnel transition functions of (14-59) are basically nearly unity.

14.10

The surface wave residue for the z_n pole is given by

$$\mathcal{U}^n_{sw} = \frac{U_0}{n} \operatorname{Re}\left[\frac{\Psi\left(z+\frac{n\pi}{2}-\phi\right)}{\Psi\left(\frac{n\pi}{2}-\phi'\right)} \frac{\sin\left(\frac{\phi'}{n}\right)}{\cos\left(\frac{z-\phi}{n}\right)-\cos\left(\frac{\phi'}{n}\right)} e^{j\beta\rho\cos z}, z_n \right]$$

Assuming that no GO poles coincide with the surface wave pole, and since the exponential has no finite poles, the residue can be determined as

$$z_n = \phi - n\pi - \pi - \theta_n$$

$$U^n_{sw} = \frac{U_0}{n} \lim \left[(z-z_n) \frac{\Psi\left(z+\frac{n\pi}{2}-\phi\right)}{\Psi\left(\frac{n\pi}{2}-\phi'\right)} \frac{\sin(\phi'/n)}{\cos\left(\frac{z-\phi}{n}\right)-\cos(\phi'/n)} \cdot e^{j\beta\rho\cos z} \right]$$

$$= \frac{U_0}{n} \left[\frac{e^{j\beta\rho\cos(\phi-n\pi-\pi-\theta_n)}}{\Psi\left(\frac{n\pi}{2}-\phi'\right)} \frac{\sin(\phi'/2)}{\cos\left(\frac{-\pi-n\pi-\theta_n}{n}\right)-\cos(\phi'/n)} \right]$$

$$\cdot \lim_{z\to z_n} \left[(z-z_n)\Psi\left(z-\frac{n\pi}{2}-\phi\right) \right]$$

Since;

$$\cos(\phi-n\pi-\pi-\theta_n) = \cos\left[(\phi-n\pi-\theta_n)-\pi\right] = -\cos(\phi-n\pi-\theta_n)$$

$$\cos\left(\frac{-\pi-n\pi-\theta_n}{n}\right) = \cos\left(\frac{\pi+n\pi+\theta_n}{n}\right)$$

Thus

$$U^n_{sw} = \frac{U_0}{n} \left[\frac{e^{-j\beta\rho\cos(\phi-n\pi-\theta_n)}}{\Psi\left(\frac{n\pi}{2}-\phi'\right)} \frac{\sin(\phi'/2)}{\cos\left(\frac{\pi+n\pi+\theta_n}{n}\right)-\cos(\phi'/n)} \right]$$

$$\cdot \lim_{z\to z_n} \left[(z-z_n)\Psi\left(z+\frac{n\pi}{2}-\phi\right) \right]$$

$$\lim_{z\to z_n} \left[(z-z_n)\Psi\left(z+\frac{n\pi}{2}-\phi\right) \right] = \lim_{z\to z_n} \left[(z-z_n)\Psi_n\left(z+n\pi-\phi+\frac{\pi}{2}-\theta_0\right) \cdot \right.$$

$$\left. \Psi_n\left(z+n\pi-\phi-\frac{\pi}{2}+\theta_0\right) \cdot \Psi_n\left(z-\phi+\frac{\pi}{2}-\theta_n\right) \cdot \Psi_n\left(z-\phi-\frac{\pi}{2}+\theta_n\right) \right]$$

Because $\Psi_n(-z) = \Psi_n(z)$

Cont'd

$\boxed{14.10 \text{ cont'd}}$ Because $\Psi_n(-z) = \Psi_n(z)$

$$\Psi_n\left(z + n\pi - \phi + \frac{\pi}{2} - \theta_0\right)\Big|_{z \to z_n} = \Psi_n\left[-\frac{\pi}{2} - (\theta_n + \theta_0)\right] = \Psi_n\left(\frac{\pi}{2} + \theta_0 + \theta_n\right)$$

$$\Psi_n\left(z + n\pi - \phi - \frac{\pi}{2} + \theta_0\right)\Big|_{z \to z_n} = \Psi_n\left[-\frac{3\pi}{2} - (\theta_n - \theta_0)\right] = \Psi_n\left(\frac{3\pi}{2} + \theta_n - \theta_0\right)$$

$$\Psi_n\left(z - \phi + \frac{\pi}{2} - \theta_n\right)\Big|_{z \to z_n} = \Psi_n\left(-\frac{\pi}{2} - n\pi - 2\theta_n\right) = \Psi_n\left(\frac{\pi}{2} + n\pi + 2\theta_n\right)$$

$$\Psi_n\left(z - \phi - \frac{\pi}{2} + \theta_n\right)\Big|_{z \to z_n} = \Psi_n\left(-n\pi - \frac{3\pi}{2}\right) = \Psi_n\left(n\pi + \frac{3\pi}{2}\right)$$

As $z \to z_n$, the fourth Maliuzhinets function $\Psi_n\left(n\pi + \frac{3\pi}{2}\right)$ has a singularity. Therefore

$$\lim_{z \to z_n}\left[(z - z_n)\Psi\left(z + \frac{n\pi}{2} - \phi\right)\right] = \Psi_n\left(\frac{\pi}{2} + \theta_0 + \theta_n\right)\Psi_n\left(\frac{3\pi}{2} + \theta_n - \theta_0\right)\Psi_n\left(\frac{\pi}{2} + n\pi + 2\theta_n\right)$$

$$\lim_{z \to z_n}\left[(z - z_n)\Psi_n\left(z - \phi - \frac{\pi}{2} + \theta_n\right)\right]$$

$$\lim_{z \to z_n}\left[(z - z_n)\Psi_n\left(z - \phi - \frac{\pi}{2} + \theta_n\right)\right] = \lim_{z \to z_n}\left[(z - z_n)\Psi_n\left(z - z_0 + n\pi + \frac{3\pi}{2}\right)\right]$$

$$= \lim\left[z\,\Psi_n\left(z + n\pi + \frac{3\pi}{2}\right)\right] = \sin\left(\frac{\pi}{2n}\right)\Psi_n\left(n\pi - \frac{\pi}{2}\right)\lim_{z \to 0}\left[z \csc\left(\frac{z}{2n}\right)\right]$$

$$= 2n \sin\left(\frac{\pi}{2n}\right)\Psi\left(n\pi - \frac{\pi}{2}\right)$$

Substituting these $\lim_{z \to z_n}\left[(z - z_n)\Psi\left(z + \frac{n\pi}{2} - \phi\right)\right] = \lim_{z \to z_n}\left[(z - z_n)\Psi_n\left(z - \phi - \frac{\pi}{2} + \theta_n\right)\right]$

$$= \sin\left(\frac{\pi}{2n}\right)\Psi_n\left(n\pi - \frac{\pi}{2}\right)\lim_{z \to 0}\left[z \csc\left(\frac{z}{2n}\right)\right]$$

$$\lim_{z \to z_n}\left[(z - z_n)\Psi\left(z + \frac{n\pi}{2} - \phi\right)\right] = 2n \sin\left(\frac{\pi}{2n}\right)\Psi\left(n\pi - \frac{\pi}{2}\right)$$

we can write the surface wave contribution from face n as

$$U_{sw}^{n} = \frac{U_0}{n}\left[\frac{e^{-j\beta\rho\cos(\phi - n\pi - \theta_n)}}{\Psi\left(\frac{n\pi}{2} - \phi'\right)} \frac{\sin(\phi/2)}{\cos\left(\frac{\pi + n\pi - \theta_n}{n}\right) - \cos(\phi/n)} \cdot 2n \sin\left(\frac{\pi}{2n}\right)\right.$$

$$\left.\cdot \Psi_n\left(n\pi - \frac{\pi}{2}\right)\Psi_n\left(\frac{\pi}{2} + \theta_0 + \theta_n\right)\cdot\Psi_n\left(\frac{3\pi}{2} + \theta_n - \theta_0\right)\cdot\Psi_n\left(\frac{\pi}{2} + n\pi + 2\theta_n\right)\right)$$

cont'd

14.10 cont'd

The final form can be written, by rearranging the terms and their arguments, as:

$$U_{SW}^{n} = U_0 \left[\frac{2\sin\left(\frac{\pi}{2n}\right)}{\Psi\left(\frac{n\pi}{2}-\phi'\right)} \frac{\sin\left(\frac{\phi'}{2}\right) e^{j\beta\rho\cos(\phi-n\pi-\theta_n)}}{\cos\left(\frac{n\pi+\pi+\theta_n}{n}\right)-\cos\left(\frac{\phi'}{n}\right)} \right] \cdot$$

$$\cdot \Psi_n\left(n\pi-\frac{\pi}{2}\right)\cdot\Psi_n\left(\frac{\pi}{2}+n\pi+2\theta_n\right)\cdot\Psi_n\left(\frac{3\pi}{2}+\theta_n-\theta_0\right)\cdot\Psi_n\left(\frac{\pi}{2}+\theta_0+\theta_n\right)$$

14.11 Using (14-47) and the second term from (14-53)

$$H_2(z) = \frac{\Psi\left(z+\frac{n\pi}{2}-\phi\right)}{\Psi\left(\frac{n\pi}{2}-\phi'\right)}\cot\left(\frac{\xi^- - z}{2n}\right)$$

$$h_2(z) = j\cos(z),\quad z_s = +\pi,\quad z_p = \xi^- - 2\pi N^- n$$

$$h_2'(z) = -j\sin(z),\quad h_2''(z) = -j\cos(z),\quad h_2''(z_s=\pi) = +j$$

$$\left|\sqrt{\frac{-2\pi}{\beta\rho h_2''(z_s)}}\right| = \left|\sqrt{\frac{-2\pi}{\beta\rho(j)}}\right| = \sqrt{\frac{2\pi}{\beta\rho}}$$

$$U_{\substack{SDP\\+\pi}}^{(2)} = jU_0\frac{1}{4\pi n}\frac{e^{-j\beta\rho}}{\sqrt{\rho}}\sqrt{\frac{2\pi}{\beta}}\,e^{-j3\pi/4}\frac{\Psi\left(\pi+\frac{n\pi}{2}-\phi\right)}{\Psi\left(\frac{n\pi}{2}-\phi\right)}\cot\left(\frac{\xi^- - \pi}{2n}\right)F\{\beta\rho[g(\xi^-)]\}$$

$$= jU_0\frac{e^{-j\beta\rho}}{\sqrt{\rho}}\left[-j\frac{e^{-j\pi/4}}{2n\sqrt{2\pi\beta}}\right]\frac{\Psi\left(\pi+\frac{n\pi}{2}-\phi\right)}{\Psi\left(\frac{n\pi}{2}-\phi'\right)}(-1)\cot\left(\frac{\pi-\xi}{2n}\right)F\{\beta\rho[g(\xi^-)]\}$$

$$U_{SDP}^{(2)} = +U_0\frac{e^{-j\beta\rho}}{\sqrt{\rho}}\left[-\frac{e^{-j\pi/4}}{2n\sqrt{2\pi\beta}}\right]\frac{\Psi\left(\pi+\frac{n\pi}{2}-\phi\right)}{\Psi\left(\frac{n\pi}{2}-\phi'\right)}\cot\left(\frac{\pi-\xi^-}{2n}\right)$$

$$\cdot F\left\{\beta\rho\left[1+\cos(\xi^- - 2\pi N^- n)\right]\right\}$$

14.12 Using (14-47) and the third term from (14-53)

$$H_3(z) = \frac{\psi\left(z + \frac{n\pi}{2} - \phi\right)}{\psi\left(\frac{n\pi}{2} - \phi'\right)} \cot\left(\frac{\xi^+ - z}{2n}\right)$$

$h_3(z) = j\cos z$, $z_s = -\pi$, $z_p \Rightarrow \xi^+ - 2\pi N_+^+ n$, $h_3'(z) = -j\sin z$, $h_3''(z) = -j\cos(z)$

$$\left|\sqrt{\frac{-2\pi}{\pi\rho h_3''(z_s)}}\right| = \left|\sqrt{\frac{2\pi}{\pi\rho(j)}}\right| = \sqrt{\frac{2\pi}{\beta\rho}}$$

$$U^{(3)}_{\substack{SDP\\-\pi}} = -jU_0 \frac{1}{4\pi n} e^{-j\beta\rho} \frac{\psi\left(-\pi + \frac{n\pi}{2} - \phi\right)}{\psi\left(\frac{n\pi}{2} - \phi'\right)} \cot\left(\frac{\xi^+ + \pi}{2n}\right) e^{j\pi/4} \sqrt{\frac{2\pi}{\beta\rho}}\, F\left\{\beta\rho g(\xi^+)\right\}$$

$$= -jU_0 \frac{e^{-j\beta\rho}}{4\pi n} \frac{\psi\left(-\pi + \frac{n\pi}{2} - \phi\right)}{\psi\left(\frac{n\pi}{2} - \phi'\right)} \cot\left(\frac{\pi + \xi^+}{2n}\right) \frac{e^{-j\frac{\pi}{2}}}{-j} e^{j\pi/4} \sqrt{\frac{2\pi}{\beta\rho}}\, F\left\{\beta\rho g(\xi^+)\right\}$$

$$U^{(3)}_{\substack{SDP\\-\pi}} = -U_0 \frac{e^{-j\beta\rho}}{\sqrt{\rho}} \left[-\frac{e^{-j\pi/4}}{2n\sqrt{2\pi\beta}}\right] \frac{\psi\left(\pi + \frac{n\pi}{2} - \phi\right)}{\psi\left(\frac{n\pi}{2} - \phi'\right)} \cot\left(\frac{\pi + \xi^+}{2n}\right) F\left\{\beta\rho\left[1 + \cos\left(\xi^+ - 2\pi N_+^+ n\right)\right]\right\}$$

14.13 Using (14-47) and the fourth term from (14-53)

$$H_4(z) = \frac{\psi\left(z + \frac{n\pi}{2} - \phi\right)}{\psi\left(\frac{n\pi}{2} - \phi'\right)} \cot\left(\frac{\xi^+ - z}{2n}\right)$$

$h_4(z) = j\cos z$, $z_s = +\pi$, $z_p = \xi^+ - 2\pi N_+^- n$, $h_4'(z) = -j\sin(z)$, $h''(z) = -j\cos z$

$$\left|\sqrt{\frac{-2\pi}{\pi\rho h_4''(z_s)}}\right| = \left|\sqrt{\frac{-2\pi}{\pi\rho(j)}}\right| = \sqrt{\frac{2\pi}{\beta\rho}}$$

$$U^{4}_{\substack{SDP\\+\pi}} = -jU_0 \frac{e^{-j\beta\rho}}{4\pi n} \frac{\psi\left(\pi + \frac{n\pi}{2} - \phi\right)}{\psi\left(\frac{n\pi}{2} - \phi'\right)} \cot\left(\frac{\xi^+ - \pi}{2n}\right) e^{-j3\pi/4} \sqrt{\frac{2\pi}{\beta\rho}}\, F\left\{\beta\rho g(\xi^+)\right\}$$

$$= -jU_0 \frac{e^{-j\beta\rho}}{4\pi n} \frac{\psi\left(\pi + \frac{n\pi}{2} - \phi\right)}{\psi\left(\frac{n\pi}{2} - \phi'\right)} (-1) \cot\left(\frac{\pi - \xi^+}{2n}\right) e^{-j\frac{\pi}{2}} e^{-j\frac{\pi}{4}} F\left\{\beta\rho g(\xi^+)\right\}$$

$$U_{\substack{SDP\\+\pi}} = -U_0 \frac{e^{-j\beta\rho}}{\sqrt{\rho}} \left[-\frac{e^{-j\pi/4}}{2n\sqrt{2\pi\beta}}\right] \frac{\psi\left(\pi + \frac{n\pi}{2} - \phi\right)}{\psi\left(\frac{n\pi}{2} - \phi'\right)} \cot\left(\frac{\pi - \xi^+}{2n}\right) F\left\{\beta\rho\left[1 + \cos\left(\xi^+ - 2\pi N_+^- n\right)\right]\right\}$$

CHAPTER 15

$\boxed{15.1}$ $\quad \dfrac{d^2u}{dx^2} = \dfrac{1}{T}F(x) = f(x)$, $\dfrac{d^2G}{dx^2} = \dfrac{1}{T}\delta(x-x')$, $p(x') = T$

The homogeneous form is

$$\frac{d^2u}{dx^2} = 0 \Rightarrow \begin{array}{l} u_1(x) = A_1 x + B_1 \\ u_2(x) = A_2 x + B_2 \end{array}$$

Since $u_1(x=0) = 0 \Rightarrow u_1(x) = A_1 x \qquad , u_1'(x) = A_1$

$u_2(x=\ell) = 0 \Rightarrow u_2(x) = A_2(x-\ell) \qquad , u_2'(x) = A_2$

$$W(x') = u_1(x')u_2'(x') - u_2(x')u_1'(x') = A_1x'A_2 - A_2(x'-\ell)A_1 = A_1A_2\ell$$

$$G(x,x') = \begin{cases} \dfrac{1}{T}\dfrac{A_2(x'-\ell)}{A_1A_2\ell}A_1x = \dfrac{1}{T}\dfrac{x'-\ell}{\ell}x = \dfrac{1}{T}\left(\dfrac{x'-\ell}{\ell}\right)x \ , & 0 \le x \le x' \\[2ex] \dfrac{1}{T}\dfrac{A_1x'}{A_1A_2\ell}A_2(x-\ell) = \dfrac{1}{T}\dfrac{x'}{\ell}(x-\ell) = \dfrac{1}{T}\dfrac{(x-\ell)}{\ell}x' & x' \le x \le \ell \end{cases}$$

$\boxed{15.2}$ $\quad \dfrac{d^2u}{dx^2} + \beta^2 u(x) = f(x)$, $\beta^2 = \omega/c$, $u(x=0) = u(x=\ell) = 0$, $\lambda = \beta^2$

$$\frac{d^2u}{dx^2} + \beta^2 u(x) = 0 \Rightarrow \begin{array}{l} u_1(x) = A_1\cos(\beta x) + B_1\sin(\beta x) \\ u_2(x) = A_2\cos[\beta(x-\ell)] + B_2\sin[\beta(x-\ell)] \end{array}$$

Since $u_1(x=0) = 0 \Rightarrow u_1(x) = B_1\sin(\beta x) \qquad , u_1'(x) = B_1\beta\cos(\beta x)$

$u_2(x=\ell) = 0 \Rightarrow u_2(x) = B_2\sin[\beta(x-\ell)]$, $u_2'(x) = B_2\beta\cos[\beta(x-\ell)]$

$$W(x') = u_1(x')u_2'(x') - u_2(x')u_1'(x') = B_1B_2\beta\{\sin(\beta x')\cos[\beta(x'-\ell)] - \sin[\beta(x'-\ell)]\cos(\beta x')\}$$

$$= \beta B_1B_2\left\{\sin\beta x'\left[\cos\beta x'\cos\beta\ell + \sin\beta x'\sin\beta\ell\right] - \cos(\beta x')\left[\sin(\beta x')\cos\beta\ell - \cos\beta x'\sin\beta\ell\right]\right\}$$

$$W(x') = \beta B_1B_2\sin\beta\ell\left[\sin^2\beta x' + \cos^2\beta x'\right] = \beta B_1B_2\sin(\beta\ell)$$

$$G(x,x') = \begin{cases} \dfrac{B_2\sin[\beta(x'-\ell)]}{\beta B_1B_2\sin(\beta\ell)}B_1\sin(\beta x) = \dfrac{\sin[\beta(x'-\ell)]}{\beta\sin(\beta\ell)}\sin(\beta x) \ , & 0 \le x \le x' \\[2ex] \dfrac{B_1\sin(\beta x')}{\beta B_1B_2\sin(\beta\ell)}B_2\sin[\beta(x-\ell)] = \dfrac{\sin(\beta x')}{\beta\sin(\beta\ell)}\sin[\beta(x-\ell)] , & x' \le x \le \ell \end{cases}$$

$\boxed{15.3}$ Closed Form

$$\nabla^2 V(x,x') = -\frac{1}{\varepsilon_0} g(x') = \frac{d^2V(x)}{dx^2} \Rightarrow \frac{d^2V}{dx^2} = -\frac{1}{\varepsilon_0} g(x')$$

(a) $V(x=0) = V(x=w) = 0$

$p(x) = 1$

$$\nabla^2 G(x,x') = \delta(x-x') = \frac{d^2G(x,x')}{dx^2}$$

$$G(x=0,x') = G(x=w,x') = 0$$

(b) $$\frac{d^2G(x,x')}{dx^2} = 0 \Rightarrow G(x,x') = \begin{cases} A_1x+B_1 & 0\le x\le x' \\ A_2x+B_2 & x'\le x\le w \end{cases}$$

Applying the boundary conditions:

$$G(x=0,x') = A_1(0)+B_1 = 0 \Rightarrow B_1 = 0$$

$$G(x=w,x') = A_2(w)+B_2 = 0 \Rightarrow B_2 = -wA_2$$

$$G(x,x') = \begin{cases} A_1x & y_1 = A_1x,\ y_1' = A_1 \\ A_2(x-w) & y_2 = A_2(x-w),\ y_2' = A_2 \end{cases}$$

$$W = y_1(x')y_2'(x') - y_2(x')y_1'(x') = A_1x'(A_2) - A_2(x'-w)A_1$$

$$W = A_1A_2x' - A_1A_2(x'-w) = A_1A_2(x'-x'+w) = wA_1A_2$$

$$G(x,x') = \begin{cases} \dfrac{y_2(x')y_1(x)}{p(x')W(x')} = \dfrac{A_2(x'-w)A_1(x)}{(1)wA_1A_2} = \dfrac{1}{w}(x'-w)x, & 0\le x\le x' \\[2ex] \dfrac{y_1(x')y_2(x)}{p(x')W(x')} = \dfrac{A_1x'[A_2(x-w)]}{(1)A_1A_2w} = \dfrac{1}{w}x'(x-w), & x'\le x\le w \end{cases}$$

(c)

$$V(x) = \int_0^w f(x')G(x,x')dx' = \frac{-1}{\varepsilon_0}\int_0^w g(x')G(x,x')dx'$$

$$= -\frac{1}{\varepsilon_0}\left[\int_0^x g(x')\frac{1}{w}(x'-w)x\,dx' + \int_x^w g(x')\frac{1}{w}(x-w)x'dx'\right]$$

$$V(x) = -\frac{1}{\varepsilon_0 w}\left[\int_0^x g(x')(x'-w)x\,dx' + \int_x^w g(x')(x-w)x'dx'\right]$$

15.4 Series Form

$$\nabla^2 V(x,x') = -\frac{1}{\varepsilon_0} q(x') = \frac{d^2V}{dx^2} \Rightarrow \frac{d^2V}{dx^2} = -\frac{1}{\varepsilon_0} q(x')$$

(a) $V(x=0) = V(x=W) = 0, \quad p(x=1)$

$G(x=0,x') = G(x=W,x') = 0$

(b) $\frac{d^2\psi_n(x)}{dx^2} + \beta_n^2 \psi_n(x) = 0 \Rightarrow \psi_n(x) = A\cos(\beta_n x) + B\sin(\beta_n x)$

Boundary conditions: $\psi_n(x=0) = 0 = A_1(1) + B(0) \Rightarrow A = 0$

$\psi_n(x=W) = 0 = B\sin(\beta_n W) = 0 \Rightarrow \sin(\beta_n W) = 0$

$\beta_n W = \sin^{-1}(0) = n\pi \Rightarrow \beta_n = \frac{n\pi}{W}$

$\beta_n = \frac{n\pi}{W}, \quad n = 1, 2, 3, \ldots$

$\psi_n(x) = B\sin(\beta_n x), \quad \lambda_n = (\beta_n)^2 = \left(\frac{n\pi}{W}\right)^2$

$$G(x,x') = \sum_n \frac{\psi_n(x')\psi_n(x)}{\lambda - \lambda_n}$$

$$\int_0^W \psi_m(x)\psi_n(x)\,dx = \int_0^W B\sin(\sqrt{\lambda_m}x)\,B\sin(\sqrt{\lambda_n}\,x) = \begin{cases} 1 & m = n \\ 0 & m \neq n \end{cases}$$

$$\int_0^W \psi_m(x)\psi_n(x)\,dx = B^2\int_0^W \sin^2(\sqrt{\lambda_n}\,x)\,dx = B^2\left(\frac{W}{2}\right) = 1 \Rightarrow B^2 = 2/W$$

$$B = \sqrt{\frac{2}{W}}$$

$$G(x,x') = \frac{2}{W}\sum_{n=1}^{\infty} \frac{\sin\left(\frac{n\pi}{W}x'\right)}{\lambda - \lambda_n}\sin\left(\frac{n\pi}{W}x\right)\Bigg|_{\lambda=0} = -\frac{2}{W}\sum_{n=1}^{\infty}\frac{\sin\left(\frac{n\pi}{W}x'\right)}{\lambda_n}\sin\left(\frac{n\pi}{W}x\right)\Bigg|_{\lambda_n = \left(\frac{n\pi}{W}\right)^2}$$

$$G(x,x') = -\frac{2W}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\left(\frac{n\pi}{W}x'\right)\sin\left(\frac{n\pi}{W}x\right)$$

(c) $$V(x) = -\frac{1}{\varepsilon_0}\int_0^W G(x,x')q(x')\,dx' = -\frac{1}{\varepsilon_0}\int_0^W (-1)\frac{2W}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\left(\frac{n\pi}{W}x'\right)\sin\left(\frac{n\pi}{W}x\right)\cdot q\,\delta(x-x')\,dx'$$

$$V(x) = \frac{2Wq}{\varepsilon_0\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin^2\left(\frac{n\pi}{W}x\right)$$

$\boxed{15.5}$ $\frac{d^2V}{d\phi^2} = f(\phi) = -\frac{1}{\varepsilon_0} g(\phi)$, $V(\phi=0) = V(\phi=\alpha) = 0$, $p(\phi) = 1$

The homogeneous form can be written as

$$\frac{d^2V}{d\phi^2} = 0 \Rightarrow \begin{array}{l} V_1(\phi) = A_1\phi + B_1 \\ V_2(\phi) = A_2\phi + B_2 \end{array}$$

Since $V_1(\phi=0)=0 \Rightarrow V_1(\phi) = A_1\phi$, $V_1'(\phi) = A_1$

$V_2(\phi=\alpha)=0 \Rightarrow V_2(\phi) = A_2(\phi-\alpha)$, $V_2'(\phi) = A_2$

$$W(\phi') = V_1(\phi')V_2'(\phi') - V_2(\phi')V_1'(\phi')$$

$$W(\phi') = A_1\phi' A_2 - A_2(\phi'-\alpha)A_1 = A_1A_2\alpha$$

Now we can write the Green's function of (15-45a) and (15-45b) as

$$G(\phi,\phi') = \begin{cases} \dfrac{A_2(\phi'-\alpha)}{A_1A_2\alpha} A_1\phi = \left(\dfrac{\phi'-\alpha}{\alpha}\right)\phi & , \quad 0 \le \phi \le \phi' \\[2ex] \dfrac{A_1\phi'}{A_1A_2\alpha} A_2(\phi-\alpha) = \left(\dfrac{\phi'}{\alpha}\right)(\phi-\alpha) & , \quad \phi' \le \phi \le \alpha \end{cases}$$

$\boxed{15.6}$ $\frac{d^2V}{d\phi^2} = f(\phi) = -\frac{1}{\varepsilon} g(\phi)$, $V(\phi=0)=0$, $V(\phi=\alpha) = V_0$

$\frac{d^2G(\phi,\phi')}{d\phi^2} = \delta(\phi-\phi')$, $G(\phi=0)=0$, $G(\phi=\alpha) = V_0$

$V_1(\phi) = A_1\phi + B_1$, $V_1'(\phi) = A_1$, $V_1(\phi=0) = A_1(0) + B_1 = 0 \Rightarrow B_1 = 0$, $\phi<\phi'$

$V_2(\phi) = A_2\phi + B_2$, $V_2'(\phi) = A_2$, $V_2(\phi=\alpha) = A_2(\alpha) + B_2 = V_0 \Rightarrow B_2 = V_0 - \alpha A_2$, $\phi>\phi'$

$V_1(\phi) = A_1\phi$, $V_1'(\phi) = A_1$

$V_2(\phi) = V_0 + A_2(\phi-\alpha)$, $V_2'(\phi) = A_2$

<u>Continuity at $\phi=\phi'$</u>: (1) $A_1\phi' = V_0 + A_2(\phi'-\alpha)$: $G_1(\phi') = G_2(\phi')$

<u>Discontinuity at $\phi=\phi'$</u>: (2) $A_2 - A_1 = 1$: $\left.\frac{dG_2}{d\phi}\right|_{\phi=\phi'} - \left.\frac{dG_1}{d\phi}\right|_{\phi=\phi'} = 1$

Solving (1) and (2) for A_1 and A_2:

$$A_1 = \frac{1}{\alpha}\left[V_0 + (\phi'-\alpha)\right], \quad A_2 = \frac{1}{\alpha}\left[V_0 + \phi'\right]$$

Thus

$$V_1(\phi) = A_1\phi = \frac{\phi}{\alpha}\left[V_0 + (\phi'-\alpha)\right] \qquad \phi<\phi'$$

$$V_2(\phi) = V_0 + \frac{(\phi-\alpha)}{\alpha}\left[V_0 + \phi'\right] \qquad \phi>\phi'$$

$\boxed{15.7}$ $\dfrac{\partial^2 V}{\partial x^2}+\dfrac{\partial^2 V}{\partial y^2}=f(x,y)=-\dfrac{1}{\varepsilon_0}\varrho(x,y)$, $V(x=0,0\le y\le b)=V(x=a,0\le y\le b)=0$

$V(0\le x\le a, y=0)=0$

$V(0\le x\le a, y=b)=V_0$

The Green's function will be chosen to satisfy the partial differential equation of

$$\frac{\partial^2 G}{\partial x^2}+\frac{\partial^2 G}{\partial y^2}=\delta(x-x')\delta(y-y')$$

subject to the boundary conditions of

$$G(x=0,0\le y\le b)=G(x=a,0\le y\le b)=0$$
$$G(0\le x\le a, y=0)=0,\ G(0\le x\le a, y=b)=V_0$$

The potential can then be found using (15-127).

$$G(x,y;x',y')=\sum_{m=1,2,..}^{\infty} g_m(y;x',y')\sin\left(\frac{m\pi}{a}x\right)$$

Thus

$$\sum_{m=1,2,..}^{\infty}\left[-\left(\frac{m\pi}{a}\right)^2 g_m(y;x',y')\sin\left(\frac{m\pi}{a}x\right)+\sin\left(\frac{m\pi}{a}x\right)\frac{d^2 g_m(y;x',y')}{dy^2}\right]=\delta(x-x')\delta(y-y')$$

Multiplying both sides by $\sin(n\pi x/a)$ and integrating with respect to x from 0 to a, and using (15-48a) we can write that

$$\frac{d^2 g_m(y;x',y')}{dy^2}-\left(\frac{m\pi}{a}\right)^2 g_m(y;x',y')=\frac{2}{a}\sin\left(\frac{m\pi}{a}x'\right)\delta(y-y')$$

$$g_m^{(1)}=A_m(x',y')\cosh\left(\frac{m\pi}{a}y\right)+B_m(x',y')\sinh\left(\frac{m\pi}{a}y\right)\qquad y<y'$$
$$g_m^{(2)}=C_m(x',y')\cosh\left(\frac{m\pi}{a}y\right)+D_m(x',y')\sinh\left(\frac{m\pi}{a}y\right)\qquad y>y'$$
$$g_m^{(1)}(y=0)=0=A_m(1)+B_m(0)=0\Rightarrow A_m=0$$
$$g_m^{(2)}(y=b)=V_0=C_m\cosh\left(\frac{m\pi b}{a}\right)+D_m\sinh\left(\frac{m\pi b}{a}\right)\Rightarrow D_m=\frac{1}{\sinh\left(\frac{m\pi b}{a}\right)}\left\{V_0-C_m\cosh\left(\frac{m\pi b}{a}\right)\right\}$$

Thus

$$g_m^{(1)}=B_m(x',y')\sinh\left(\frac{m\pi}{a}y\right)$$
$$g_m^{(2)}=C_m\cosh\left(\frac{m\pi}{a}y\right)+\left[V_0-C_m\cosh\left(\frac{m\pi b}{a}\right)\right]\frac{\sinh\left(\frac{m\pi}{a}y\right)}{\sinh\left(\frac{m\pi b}{a}\right)}$$

<u>Continuity of $g_m^{(1)}$ and $g_m^{(2)}$ at $y=y'$:</u>

$$B_m\sinh\left(\frac{m\pi}{a}y'\right)=C_m\cosh\left(\frac{m\pi}{a}y'\right)+\left[V_0-C_m\cosh\left(\frac{m\pi b}{a}\right)\right]\frac{\sinh\left(\frac{m\pi}{a}y'\right)}{\sinh\left(\frac{m\pi b}{a}\right)}\quad(1)$$

<u>Discontinuity of $dg_m^{(1)}/d\phi|_{\phi=\phi'}$ and $dg_m^{(2)}/d\phi|_{\phi=\phi'}$:</u>

$$\frac{m\pi}{a}\left\{C_m\sinh\left(\frac{m\pi}{a}y'\right)+\left[V_0-C_m\cosh\left(\frac{m\pi b}{a}\right)\right]\frac{\cosh\left(\frac{m\pi}{a}y'\right)}{\sinh\left(\frac{m\pi b}{a}\right)}-B_m\cosh\left(\frac{m\pi}{a}y'\right)\right\}=1\quad(2)$$

cont'd.

$\boxed{15.7 \text{ cont'd.}}$ (1) and (2) can be written as:

$$B_m \sinh\left(\frac{m\pi}{a}y'\right) - C_m\left[\cosh\left(\frac{m\pi}{a}y'\right) - \coth\left(\frac{m\pi b}{a}\right)\sinh\left(\frac{m\pi}{a}y'\right)\right] = V_0\,\frac{\sinh\left(\frac{m\pi}{a}y'\right)}{\sinh\left(\frac{m\pi b}{a}\right)} \qquad (3)$$

$$-B_m \cosh\left(\frac{m\pi}{a}y'\right) + C_m\left[\sinh\left(\frac{m\pi}{a}y'\right) - \coth\left(\frac{m\pi b}{a}\right)\cosh\left(\frac{m\pi}{a}y'\right)\right] = \frac{a}{m\pi} - V_0\,\frac{\cosh\left(\frac{m\pi}{a}y'\right)}{\sinh\left(\frac{m\pi b}{a}\right)} \qquad (4)$$

These can be written as

$$\alpha_1 B_m - \beta_1 C_m = \gamma_1 \qquad (5)$$

$$-\alpha_2 B_m + \beta_2 C_m = \gamma_2 \qquad (6)$$

where $\alpha_1 = \sinh\left(\frac{m\pi}{a}y'\right)$, $\beta_1 = \left[\cosh\left(\frac{m\pi}{a}y'\right) - \coth\left(\frac{m\pi b}{a}\right)\sinh\left(\frac{m\pi}{a}y'\right)\right]$, $\gamma_1 = V_0\,\frac{\sinh\left(\frac{m\pi}{a}y'\right)}{\sinh\left(\frac{m\pi b}{a}\right)}$

$\alpha_2 = \cosh\left(\frac{m\pi}{a}y'\right)$, $\beta_2 = \left[\sinh\left(\frac{m\pi}{a}y'\right) - \coth\left(\frac{m\pi b}{a}\right)\cosh\left(\frac{m\pi}{a}y'\right)\right]$, $\gamma_2 = \frac{a}{m\pi} - V_0\,\frac{\cosh\left(\frac{m\pi}{a}y'\right)}{\sinh\left(\frac{m\pi b}{a}\right)}$

Solving (5) and (6) for B_m and C_m leads to:

$$C_m = \frac{\alpha_1\gamma_2 + \alpha_2\gamma_1}{\beta_2\alpha_1 - \beta_1\alpha_2} \qquad (7)$$

$$B_m = \frac{\beta_1\gamma_2 + \beta_2\gamma_1}{\beta_2\alpha_1 - \alpha_2\beta_1} \qquad (8)$$

Thus

$$g_m^{(1)} = B_m(x',y')\sinh\left(\frac{m\pi}{a}y\right), \qquad (9)$$

$$g_m^{(2)} = C_m(x',y')\cosh\left(\frac{m\pi}{a}y\right) + \left[V_0 - C_m\cosh\left(\frac{m\pi b}{a}\right)\right]\frac{\sinh\left(\frac{m\pi}{a}y\right)}{\sinh\left(\frac{m\pi b}{a}\right)}, \qquad y>y' \qquad (10)$$

Thus the Green's function

$$G(x,y;x',y') = \sum_{m=1,2,..}^{\infty} g_m(y;x',y')\sin\left(\frac{m\pi}{a}x\right)$$

can be written as

$$G(x,y;x',y') = \begin{cases} \displaystyle\sum_{m=1,2,..}^{\infty} B_m(x',y')\sinh\left(\frac{m\pi}{a}y\right)\sin\left(\frac{m\pi}{a}x\right) & ; \ y<y' \\[2ex] \displaystyle\sum_{m=1,2,..}^{\infty}\left\{C_m(x',y')\cosh\left(\frac{m\pi}{a}y\right) + \left[V_0 - C_m\cosh\left(\frac{m\pi b}{a}\right)\right]\frac{\sinh\left(\frac{m\pi}{a}y\right)}{\sinh\left(\frac{m\pi b}{a}\right)}\right\}\sin\left(\frac{m\pi}{a}x\right) & , \ y>y' \end{cases}$$

$\boxed{15.8}$ The Green's function will be chosen to satisfy the partial differential equation of

$$\frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2} = \delta(x-x')(y-y') \qquad (1)$$

subject to the boundary conditions of

$$G(x=0, 0 \le y \le \infty) = G(x=a, 0 \le y \le \infty) = 0 \qquad (2a)$$

$$G(0 \le x \le a, y=0) = 0\ ; \quad G(0 \le x \le a, y=\infty) = \text{Finite} \qquad (2b)$$

Choose the Green's function to satisfy the boundary conditions of (2a) at x=0 and x=a. Thus

$$G(x,y;x',y') = \sum_{m=1,2,\ldots}^{\infty} g_m(y;x',y') \sin\left(\frac{m\pi}{a}x\right) \qquad (3)$$

Substituting (3) into (1) leads to

$$\sum_{m=1,2,\ldots}^{\infty} \left[-\left(\frac{m\pi}{a}\right)^2 g_m(y;x',y') + \frac{d^2 g_m(y;x',y')}{dy^2}\right] \sin\left(\frac{m\pi}{a}x\right) = \delta(x-x')\delta(y-y') \qquad (4)$$

Multiplying both sides of (4) by $\sin(m\pi x/a)$, integrating with respect to x from 0 to a, and using (15-48a) and (15-48b), we can write that

$$\frac{d^2 g_m}{dy^2} - \left(\frac{m\pi}{a}\right)^2 g_m = \frac{2}{a}\sin\left(\frac{m\pi}{a}x'\right)\delta(y-y') \qquad (5)$$

For the homogeneous form of (5), the two solutions can be written as

$$g_m^{(1)}(y;x',y') = A_m(x',y')e^{-(m\pi/a)y} + B_m(x',y')e^{+(m\pi/a)y} \quad , y<y' \qquad (6)$$

$$g_m^{(2)}(y;x',y') = C_m(x',y')e^{-(m\pi/a)y} + D_m(x',y')e^{+(m\pi/a)y} \quad , y>y' \qquad (7)$$

Now apply on (6) and (7) the boundary conditions of (2b).

$$g_m^{(1)}(y=0) = A_m + B_m = 0 \Rightarrow B_m = -A_m \qquad (8)$$

$$g_m^{(2)}(y=\infty) = \text{finite} \Rightarrow D_m = 0 \qquad (9)$$

Thus

$$g_m^{(1)}(y,x,y') = 2A_m(x',y')\left[\frac{e^{-(m\pi/a)y} - e^{+(m\pi/a)y}}{2}\right] = 2A_m(x',y')\sinh\left(\frac{m\pi}{a}y\right) \quad y<y' \qquad (10)$$

$$g_m^{(2)}(y;x',y') = C_m(x',y')\, e^{-(m\pi/a)y} \quad y>y' \qquad (11)$$

<u>Continuity of $g_m^{(1)}$ and $g_m^{(2)}$ at $y=y'$:</u>

$$2A_m(x',y')\sinh\left(\frac{m\pi}{a}y'\right) = C_m(x',y')\, e^{-\left(\frac{m\pi}{a}y'\right)} \qquad (12)$$

<u>Discontinuity of $dg_m^{(1)}/dy|_{y=y'}$ and $dg_m^{(2)}/dy|_{y=y'}$:</u>

$$\left(\frac{m\pi}{a}\right)\left[-C_m e^{-(m\pi/a)y'} - 2A_m\cosh\left(\frac{m\pi}{a}y'\right)\right] = 1 \qquad (13)$$

cont'd.

$\boxed{\text{15.8 cont'd.}}$ Equations (12) and (13) can be written as

$$C_m = 2A_m \sinh\left(\frac{m\pi}{a}y'\right) e^{+(m\pi/a)y'} \tag{14}$$

$$C_m e^{-(m\pi/a)y'} + 2A_m \cosh\left(\frac{m\pi}{a}y'\right) = -\frac{a}{m\pi} \tag{15}$$

Solving these two equations for A_m and C_m leads to

$$A_m = -\frac{2}{m\pi}\,\frac{1}{2\left[\cosh\left(\frac{m\pi}{a}y'\right)+\sinh\left(\frac{m\pi}{a}y'\right)\right]} \tag{16}$$

$$C_m = -\frac{2}{m\pi}\,\frac{\sinh\left(\frac{m\pi}{a}y'\right) e^{(m\pi/a)y'}}{\cosh\left(\frac{m\pi}{a}y'\right)+\sinh\left(\frac{m\pi}{a}y'\right)} \tag{17}$$

Thus (10) and (11) can be written as

$$g_m^{(1)}(y;x',y') = -\frac{2}{m\pi}\,\frac{\sinh\left(\frac{m\pi}{a}y\right)}{\cosh\left(\frac{m\pi}{a}y'\right)+\sinh\left(\frac{m\pi}{a}y'\right)}\,, \quad y<y' \tag{18}$$

$$g_m^{(2)}(y;x',y') = -\frac{2}{m\pi}\,\frac{\sinh\left(\frac{m\pi}{a}y'\right) e^{(m\pi/a)y'}}{\cosh\left(\frac{m\pi}{a}y'\right)+\sinh\left(\frac{m\pi}{a}y'\right)}\, e^{-(m\pi/a)y}, \quad y>y' \tag{19}$$

Thus the Green's function of (3) can be expressed as

$$G(x,y;x',y') = \begin{cases} \displaystyle\sum_{m=1,2,\ldots}^{\infty} -\left(\frac{2}{m\pi}\right)\frac{1}{\cosh\left(\frac{m\pi}{a}y'\right)+\sinh\left(\frac{m\pi}{a}y'\right)}\sinh\left(\frac{m\pi}{a}y\right)\sin\left(\frac{m\pi}{a}x\right), & y<y' \\[2ex] \displaystyle\sum_{m=1,2,\ldots}^{\infty} -\left(\frac{2}{m\pi}\right)\frac{\sinh\left(\frac{m\pi}{a}y'\right)e^{(m\pi/a)y'}}{\cosh\left(\frac{m\pi}{a}y'\right)+\sinh\left(\frac{m\pi}{a}y'\right)}\, e^{\left(\frac{m\pi}{a}\right)y}\sin\left(\frac{m\pi}{a}x\right), & y>y' \end{cases}$$

$\boxed{15.9}$ $$G(x,y;x',y') = \sum_{n=1,2,\ldots}^{\infty} g_n(x;x',y')\sin\left(\frac{n\pi}{b}y\right) \qquad (1)$$

The coefficients $g_n(x;x',y')$ of the Fourier series will be determined by first substituting (1) into (15-72). This leads to

$$\sum_{n=1,2,\ldots}^{\infty}\left[-\left(\frac{n\pi}{b}\right)^2 g_n(x;x',y')\sin\left(\frac{n\pi}{b}y\right)+\sin\left(\frac{n\pi}{b}y\right)\frac{d^2 g_n(x;x',y')}{dx^2}\right]=\delta(x-x')\delta(y-y') \qquad (2)$$

Multiplying both sides of (2) by $\sin\left(\frac{m\pi}{b}y\right)$, integrating with respect to y from 0 to b, and using (15-48a) and (15-48b), we can write that

$$\frac{d^2 g_m(x;x',y')}{dx^2}-\left(\frac{n\pi}{b}\right)^2 g_n(x;x',y')=\frac{2}{a}\sin\left(\frac{m\pi}{b}y'\right)\delta(x-x') \qquad (3)$$

Equation (3) is recognized as a one-dimensional differential equation for $g_m(x;x',y')$ which can be solved using the recipe of Section 15.3.1 as provided by (15-44c) and (15-45a)-(15-45b).

For the homogeneous form of (3), or

$$\frac{d^2 g_m(x;x',y')}{dx^2}-\left(\frac{n\pi}{b}\right)^2 g_n(x;x',y') \qquad (4)$$

two solutions that satisfy, respectively, the boundary conditions at $x=0$ and $x=a$ are

$$g_n^{(1)} = A_n(x',y')\sinh\left(\frac{n\pi}{b}x\right) \qquad \text{for } x\le x' \qquad (5a)$$

$$g_n^{(2)} = B_n(x',y')\sinh\left[\frac{n\pi}{b}(a-x)\right] \qquad \text{for } x\ge x' \qquad (5b)$$

Using (14-44c) where $y_1=g_n^{(1)}$ and $y_2=g_n^{(2)}$, we can write the Wronskian as

$$W(x;x',y')=-\left(\frac{n\pi}{b}\right)A_nB_n\left\{\sinh\left(\frac{n\pi}{b}x'\right)\cosh\left[\frac{n\pi}{b}(a-x')\right]+\cosh\left(\frac{n\pi}{b}x'\right)\sinh\left[\frac{n\pi}{b}(a-x')\right]\right\}$$

$$=-\left(\frac{n\pi}{b}\right)A_nB_n\sinh\left(\frac{n\pi a}{b}\right) \qquad (6)$$

By comparing (3) with the form of (15-33), it is apparent that

$$p(x)=1,\quad q(x)=0,\quad r(x)=1,\qquad \lambda=-\left(\frac{n\pi}{b}\right)^2 \qquad (7)$$

Using (5a)-(7) the solution for $g_m(x;x',y')$ of (3) can be written by referring to (15-45a) and (15-45b) as

$$g_m(x;x',y')=\begin{cases}-\dfrac{2}{n\pi}\sin\left(\dfrac{n\pi}{b}y'\right)\dfrac{\sinh\left[\frac{n\pi}{b}(a-x')\right]}{\sinh\left(\frac{n\pi a}{b}\right)}\sinh\left(\dfrac{n\pi}{b}x\right), & 0\le x\le x' \quad (8a)\\[2ex] -\dfrac{2}{n\pi}\sin\left(\dfrac{n\pi}{b}y'\right)\dfrac{\sinh\left(\frac{n\pi}{b}x'\right)}{\sinh\left(\frac{n\pi a}{b}\right)}\sinh\left[\dfrac{n\pi}{b}(a-x)\right], & x'\le x\le a \quad (8b)\end{cases}$$

Using (1), (8a) and (8b) we can then write the Green's function of (15-83a) & (15-83b).

$\boxed{15.10}$ $\nabla^2 V = f(\rho,\phi) = -\frac{1}{\varepsilon_0} g(\rho,\phi) \Rightarrow \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} = f(\rho,\phi) = -\frac{1}{\varepsilon_0} g(\rho,\phi)$ (1)

$$\nabla^2 G = \delta(\underline{\rho}-\underline{\rho}') \Rightarrow \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial G}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 G}{\partial \phi^2} = \delta(\underline{\rho}-\underline{\rho}') = \frac{1}{\rho}\delta(\rho-\rho')\delta(\phi-\phi') \quad (2)$$

$$G(\rho=a,\ 0\le\phi\le 2\pi) = 0, \quad G = \text{finite everywhere within the cylinder} \quad (2a)$$

Choose a solution of the form

$$G(\rho,\phi;\rho',\phi') = \sum_{m=-\infty}^{\infty} g_m(\rho,\rho') e^{jm(\phi-\phi')} = \sum_{m=0,1,\dots}^{\infty} \epsilon_m g_m(\rho,\rho') \cos[m(\phi-\phi')],\ \epsilon_m = \begin{cases}1, & m=0\\ 2, & m\neq 0\end{cases} \quad (3)$$

Substituting (3) into (2) leads to

$$\sum_{m=-\infty}^{+\infty}\left[\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dg_m}{d\rho}\right) - \frac{1}{\rho^2}m^2 g_m\right] e^{jm(\phi-\phi')} = \frac{1}{\rho}\delta(\rho-\rho')\delta(\phi-\phi') \quad (4)$$

Multiplying both sides of (4) by $e^{-jn(\phi-\phi')}$, integrating from 0 to 2π, and using (15-155a) and (15-155b), we can write that

$$2\pi\left[\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dg_m}{d\rho}\right) - \frac{1}{\rho^2}m^2 g_m\right] = \frac{1}{\rho}\delta(\rho-\rho') \quad (5)$$

or

$$\rho\frac{d^2 g_m}{d\rho^2} + \frac{dg_m}{d\rho} - \frac{m^2}{\rho} g_m = \frac{1}{2\pi}\delta(\rho-\rho') \quad (5a)$$

The two solutions for the homogeneous form of (5a) are

$$g_m^{(1)} = A_m \rho^m + B_m \frac{1}{\rho^m} \quad \text{for } \rho<\rho' \quad (6a)$$

$$g_m^{(2)} = C_m \rho^m + D_m \frac{1}{\rho^m} \quad \text{for } \rho>\rho' \quad (6b)$$

Since the Green's function must be finite everywhere, including $\rho=0$, the $B_m=0$. Thus

$$g_m^{(1)} = A_m \rho^m \quad \text{for } \rho<\rho' \quad (7a)$$

$$g_m^{(2)} = C_m \rho^m + D_m \frac{1}{\rho^m} \quad \text{for } \rho>\rho' \quad (7b)$$

Also $g_m^{(2)}(\rho=a) = 0 = C_m a^m + D_m a^{-m} = 0 \Rightarrow D_m = -C_m a^{2m}$. Thus

$$g_m^{(1)} = A_m \rho^m \Rightarrow g_m^{(1)\prime} = A_m m \rho^{m-1} \quad (8a)$$

$$g_m^{(2)} = C_m\left[\rho^m - a^{2m}\rho^{-m}\right] \Rightarrow g_m^{(2)\prime} = C_m\left[m\rho^{m-1} + m a^{2m}\rho^{-(m+1)}\right] \quad (8b)$$

Using (14-44c), we can write the Wronskian to

$$W(\rho') = A_m C_m\left\{(\rho')^m\left[m(\rho')^{m-1} + m a^{2m}(\rho')^{-(m+1)}\right] - \left[(\rho')^m - a^{2m}(\rho')^{-m}\right] m(\rho')^{m-1}\right\} = A_m C_m 2m a^{2m}\frac{1}{\rho'} \quad (9)$$

cont'd.

$\boxed{15.10\ \text{cont'd.}}$ Therefore using (5a), (15-45a) and (15-45b), we can write $g_m(\rho,\rho')$ with $p(\rho')=\rho'$ as

$$g_m(\rho,\rho') = \begin{cases} \frac{1}{4\pi m a^{2m}}\left[(\rho')^m - a^{2m}(\rho')^{-m}\right]\rho^m & , \ \rho<\rho' \quad (10a) \\ \frac{1}{4\pi m a^{2m}}(\rho')^m\left[\rho^m - a^{2m}\rho^{-m}\right] & , \ \rho>\rho' \quad (10b) \end{cases}$$

with (10a) and (10b) we can express the Green's function of (3) to

$$G(\rho,\phi;\rho',\phi') = \begin{cases} \frac{1}{4\pi}\sum_{m=-\infty}^{+\infty}\frac{1}{m a^{2m}}\left[(\rho')^m - \frac{a^{2m}}{(\rho')^m}\right]\rho^m e^{jm(\phi-\phi')} & , \ \rho<\rho' \quad (11a) \\ \frac{1}{4\pi}\sum_{m=-\infty}^{+\infty}\frac{1}{m a^{2m}}(\rho')^m\left[\rho^m - \frac{a^{2m}}{\rho^m}\right] e^{jm(\phi-\phi')} & , \ \rho>\rho' \quad (11b) \end{cases}$$

$\boxed{15.11}$

$$\frac{\partial^2 G}{\partial x^2}+\frac{\partial^2 G}{\partial y^2}+\beta^2 G = \delta(x-x')\delta(y-y') \qquad (1)$$

$$G(x=0,\ 0\le y\le b) = G(x=a,\ 0\le y\le b)=0 \qquad (1a)$$

$$G(0\le x\le a,\ y=0) = G(0\le x\le a,\ y=b)=0 \qquad (1b)$$

$$G(x,y;x',y') = \sum_m g_m(y;x',y')\left[A_m\sin(\beta_x x)+B_m\cos(\beta_x x)\right] \qquad (2)$$

Applying (1a) on (2) leads to

$$G(x=0) = \sum_m g_m(y;x',y')\left[A_m(0)+B_m(1)\right]=0 \Rightarrow B_m=0$$

$$G(x=a) = \sum_m g_m(y;x',y')\left[A_m\sin(\beta_x a)\right]=0 \Rightarrow \beta_x a = \sin^{-1}(0) = m\pi \Rightarrow \beta_x = \left(\frac{m\pi}{a}\right), m=1,2,\dots$$

Thus

$$G(x,y;x',y') = \sum_{m=1,2,..}^{\infty} A_m g_m(y;x',y')\sin\left(\frac{m\pi}{a}x\right) \qquad (3)$$

Substituting (3) into (1), we can write that

$$\sum_{m=1,2,..}^{\infty} A_m\left\{-\left(\frac{m\pi}{a}\right)^2 g_m + \frac{d^2 g_m}{dy^2}+\beta^2 g_m\right\}\sin\left(\frac{m\pi}{a}x\right) = \delta(x-x')\delta(y-y')$$

$$\sum_{m=1,2,..}^{\infty} A_m\left\{\frac{d^2 g_m}{dy^2}+\left[\beta^2-\left(\frac{m\pi}{a}\right)^2\right]g_m\right\}\sin\left(\frac{m\pi}{a}x\right) = \delta(x-x')\delta(y-y') \qquad (4)$$

Multiplying both sides by $\sin(n\pi x/a)$, integrating between 0 to a in x, and using (15-48a) and (15-48b), leads to

cont'd.

$\boxed{\text{15.11 cont'd.}}$ $\frac{a}{2}\left\{\frac{d^2 g_m}{dy^2}+\left[\beta^2-\left(\frac{m\pi}{a}\right)^2\right]g_m\right\}=\sin\left(\frac{m\pi}{a}x'\right)\delta(y-y')$

or $$\frac{d^2 g_m}{dy^2}+\left[\beta^2-\left(\frac{m\pi}{a}\right)^2\right]g_m=\frac{d^2 g_m}{dy^2}+\beta_y^2 g_m=\frac{2}{a}\sin\left(\frac{m\pi}{a}x'\right)\delta(y-y'),\ \text{where } \beta_y^2=\beta^2-\left(\frac{m\pi}{a}\right)^2 \quad (5)$$

The two solutions for the homogeneous form of (5) are

$$g_m^{(1)}=A_1\cos(\beta_y y)+B_1\sin(\beta_y y) \qquad y<y' \qquad (6a)$$

$$g_m^{(2)}=C_1\cos[\beta_y(y-b)]+D_1\sin[\beta_y(y-b)] \qquad y>y' \qquad (6b)$$

Applying the boundary conditions of (1b) on (6a) and (6b) leads to

$$g_m^{(1)}(y=0)=A_1(1)+B_1(0)=0 \Rightarrow A_1=0$$

$$g_m^{(2)}(y=b)=C_1(1)+D_1(0)=0 \Rightarrow C_1=0$$

Thus (6a) and (6b) reduce to

$$g_m^{(1)}=B_1\sin(\beta_y y) \quad , g_m^{(1)\prime}=\beta_y B_1\cos(\beta_y y) \quad , y<y' \qquad (7a)$$

$$g_m^{(2)}=D_1\sin[\beta_y(y-b)] \quad , g_m^{(2)\prime}=\beta_y D_1\cos[\beta_y(y-b)] \quad , y>y' \qquad (7b)$$

The Wronskian of (15-44c) can now be written as

$$W(y')=\beta_y B_1 D_1\left[\sin(\beta_y y')\cos[\beta_y(y'-b)]-\sin[\beta_y(y'-b)]\cos(\beta_y y')\right]$$

$$W(y')=\beta_y B_1 D_1\left[\sin\left(\beta_y y'-\beta_y y'+\beta_y b\right)\right]=\beta_y B_1 D_1\sin(\beta_y b) \qquad (8)$$

Using (15-45a), (15-45b) and (5) with $p(y')=1$, we can write

$$g_m(y;x',y')=\begin{cases}\dfrac{2}{a\,\beta_y}\,\dfrac{\sin[\beta_y(y'-b)]}{\sin(\beta_y b)}\sin(\beta_y y)\sin\left(\frac{m\pi}{a}x'\right), & y<y' \quad (9a)\\[2ex] \dfrac{2}{a\beta_y}\,\dfrac{\sin(\beta_y y')}{\sin(\beta_y b)}\sin[\beta_y(y-b)]\sin\left(\frac{m\pi}{a}x'\right), & y>y' \quad (9b)\end{cases}$$

Thus the Green's function of (3) can now be expressed as

$$G(x,y;x',y')=\begin{cases}\displaystyle\sum_{m=1,2,\ldots}^{\infty}\frac{2}{a\beta_y}\,\frac{\sin[\beta_y(y'-b)]}{\sin(\beta_y b)}\sin(\beta_y y)\sin\left(\frac{m\pi}{a}x'\right)\sin\left(\frac{m\pi}{a}x\right), & y<y' \quad (10a)\\[3ex] \displaystyle\sum_{m=1,2,\ldots}^{\infty}\frac{2}{a\beta_y}\,\frac{\sin(\beta_y y')}{\sin(\beta_y b)}\sin[\beta_y(y-b)]\sin\left(\frac{m\pi}{a}x'\right)\sin\left(\frac{m\pi}{a}x\right), & y>y' \quad (10b)\end{cases}$$

15.12 Since the closed form recipe of (15-45a) and (15-45b) is only one-dimensional, and the subject problem is three-dimensional, the solution for Figure 15-7 can proceed in many different ways, including the one outlined in Section 15.6.1 which starts with a two-dimensional Fourier series of sine function in x and cosine function in y to represent the Green's function $G(x,y,z;x',y',z')$. The solution basically concludes by using the closed form recipe to represent the z-variations of $g_{mn}(z;x',y',z')$, which is represented by the differential equation of (15-143) with solutions of closed form as given by (15-146a) for $z<z'$ and (15-146b) for $z>z'$. The solution concludes with the final form of the Green's function as given by (15-147).

An alternate solution would be to represent the y-variations in (15-140) by a sine function $\left[\sin\left(\frac{n\pi}{b}y\right)\right]$ since we have no boundary conditions in the y-direction as represented by (15-137a).

15.13 The Green's function when the excitation in Example 15-5 is a magnetic line source is the same as that of the electric line source, as given by, in Example 15-5,

$$G(\rho,\phi;\rho',\phi') = -\frac{1}{4j} H_0^{(2)}(\beta_0|\underline{\rho}-\underline{\rho}'|)$$

However, for the electric line source the Green's function is related to the electric field E_z by

$$E_z = j\omega\mu I_e G = -\frac{\omega\mu}{4} I_e H_0^{(2)}(\beta|\underline{\rho}-\underline{\rho}'|)$$

while it is related to the magnetic field H_z for the magnetic line source by

$$H_z = j\omega\varepsilon I_m G = -\frac{\omega\varepsilon}{4} I_m H_0^{(2)}(\beta|\underline{\rho}-\underline{\rho}'|)$$

$\boxed{15.14}$ The Green's function for this problem must satisfy the partial differential equation that H_ϕ must satisfy; that is

$$\nabla^2 G + \beta_0^2 G = \delta(\underline{\rho} - \underline{\rho}') \qquad (1)$$

The boundary conditions for the Green's function are chosen to be the same as those for H_ϕ; that is

$$G(\rho=a, 0\le\phi\le 2\pi, 0\le z\le h) = G(\rho=b, 0\le\phi\le 2\pi, 0\le z\le h) = 0 \qquad (2)$$

Let

$$G(\rho,\phi;\rho',\phi') = \sum_{m=-\infty}^{+\infty} g_m(\rho;\rho',\phi')e^{jm\phi} \qquad (3)$$

Choosing (15-152d) to represent $\delta(\underline{\rho}-\underline{\rho}')$, we can write (1) in expanded form, assuming no z variations, as (15-153).

Following the procedure outlined on page 929-930 we can write the two solutions for $g_m(\rho;\rho',\phi')$ as given by (15-159a) and (15-159b), or

$$g_m^{(1)} = A_m J_m(\beta_0\rho) + B_m Y_m(\beta_0\rho) \qquad \text{for } \rho<\rho' \qquad (3a)$$

$$g_m^{(2)} = C_m J_m(\beta_0\rho) + D_m Y_m(\beta_0\rho) \qquad \text{for } \rho>\rho' \qquad (3b)$$

The Green's function of (3) must satisfy the boundary conditions of (2). Thus

$$g_m^{(1)}(\rho=a) = A_m J_m(\beta_0 a) + B_m Y_m(\beta_0 a) = 0 \Rightarrow B_m = -A_m\frac{J_m(\beta_0 a)}{Y_m(\beta_0 a)} \qquad (4a)$$

$$g_m^{(2)}(\rho=b) = C_m J_m(\beta_0 b) + D_m Y_m(\beta_0 b) = 0 \Rightarrow D_m = -C_m\frac{J_m(\beta_0 b)}{Y_m(\beta_0 b)} \qquad (4b)$$

Thus (3a) and (3b) reduce to

$$g_m^{(1)} = A_m\left[J_m(\beta_0\rho) - \frac{J_m(\beta_0 a)}{Y_m(\beta_0 a)}Y_m(\beta_0\rho)\right] = A_m\left[J_m(\beta_0\rho) - \alpha Y_m(\beta_0\rho)\right] \qquad (5a)$$

$$g_m^{(2)} = C_m\left[J_m(\beta_0\rho) - \frac{J_m(\beta_0 b)}{Y_m(\beta_0 b)}Y_m(\beta_0\rho)\right] = C_m\left[J_m(\beta_0\rho) - \gamma Y_m(\beta_0\rho)\right] \qquad (5b)$$

Using (15-44c) where $y_1 = g_m^{(1)}$ and $y_2 = g_m^{(2)}$, we can write the Wronskian as

$$W(\rho') = \beta_0 A_m\left[J_m(\beta_0\rho') - \alpha Y_m(\beta_0\rho')\right]C_m\left[J_m'(\beta_0\rho') - \gamma Y_m'(\beta_0\rho)\right]$$
$$-\beta_0 C_m\left[J_m(\beta_0\rho') - \gamma Y_m(\beta_0\rho')\right]A_m\left[J_m'(\beta_0\rho') - \alpha Y_m'(\beta_0\rho)\right]$$

$$W(\rho') = \beta_0 A_m C_m\left[J_m(\beta_0\rho')J_m'(\beta_0\rho') + \alpha\gamma Y_m(\beta_0\rho')Y_m'(\beta_0\rho) - \alpha Y_m(\beta_0\rho')J_m'(\beta_0\rho') - \gamma J_m(\beta_0\rho')Y_m'(\beta_0\rho')\right]$$
$$-\beta_0 C_m A_m\left[J_m(\beta_0\rho')J_m'(\beta_0\rho') + \alpha\gamma Y_m(\beta_0\rho')Y_m'(\beta_0\rho) - \gamma Y_m(\beta_0\rho')J_m'(\beta_0\rho') - \alpha J_m(\beta_0\rho')Y_m'(\beta_0\rho')\right]$$

cont'd.

15.14 Cont'd.

$$W(\rho') = \beta_0 A_m C_m \left\{ -\alpha\left[J_m'Y_m - J_m Y_m'\right] + \gamma\left[J_m'Y_m - J_m Y_m'\right]\right\}, \quad ' = \frac{\partial}{\partial(\beta_0\rho)} \qquad (6)$$

By using the Wronskian for Bessel functions of (11-95), (6) reduces to

$$W(\rho') = +\frac{2}{\pi} A_m C_m \frac{1}{\rho'}\left\{+\alpha - \gamma\right\} = \frac{2}{\pi\rho'} A_m C_m \left[\frac{J_m(\beta_0 a)}{Y_m(\beta_0 a)} - \frac{J_m(\beta_0 b)}{Y_m(\beta_0 b)}\right]$$

$$W(\rho') = \frac{2}{\pi}\frac{1}{\rho'} A_m C_m \frac{J_m(\beta_0 a)Y_m(\beta_0 b) - J_m(\beta_0 b)Y_m(\beta_0 a)}{Y_m(\beta_0 a)Y_m(\beta_0 b)} = \frac{2}{\pi}\frac{A_m C_m}{\rho'}(JY)_m \qquad (7)$$

$$\text{where} \quad (JY)_m = \frac{J_m(\beta_0 a)Y_m(\beta_0 b) - J_m(\beta_0 b)Y_m(\beta_0 a)}{Y_m(\beta_0 a)Y_m(\beta_0 b)} \qquad (7a)$$

Finally $g_m(\rho;\rho',\phi')$ of (15-156) can be written using (15-158), (5a)-(7a) by referring to (15-45a) and (15-45b), as

$$g_m(\rho;\rho',\phi') = \begin{cases} \dfrac{1}{4}\left[\dfrac{J_m(\beta_0\rho') - \gamma Y_m(\beta_0\rho')}{(JY)_m}\right]\left[J_m(\beta_0\rho) - \alpha Y_m(\beta_0\rho)\right] e^{-jm\phi'}, & a \le \rho \le \rho' \quad (8a) \\[2ex] \dfrac{1}{4}\left[\dfrac{J_m(\beta_0\rho') - \alpha Y_m(\beta_0\rho')}{(JY)_m}\right]\left[J_m(\beta_0\rho) - \gamma Y_m(\beta_0\rho)\right] e^{-jm\phi'}, & \rho' \le \rho \le b \quad (8b) \end{cases}$$

Thus the Green's function of (3) can be written as

$$G(\rho,\phi;\rho',\phi') = \begin{cases} \dfrac{1}{4}\displaystyle\sum_{m=-\infty}^{+\infty}\left[\dfrac{J_m(\beta_0\rho') - \gamma Y_m(\beta_0\rho')}{(JY)_m}\right]\left[J_m(\beta_0\rho) - \alpha Y_m(\beta_0\rho)\right] e^{jm(\phi-\phi')} & (9a) \\[3ex] \dfrac{1}{4}\displaystyle\sum_{m=-\infty}^{+\infty}\left[\dfrac{J_m(\beta_0\rho') - \alpha Y_m(\beta_0\rho')}{(JY)_m}\right]\left[J_n(\beta_0\rho) - \gamma Y_m(\beta_0\rho)\right] e^{jm(\phi-\phi')} & (9b) \end{cases}$$

$\boxed{15.15}$ The solution to this problem follows that of Problem 15.12. One of differences are the boundary conditions which for this problem are

$$G(\rho=a, 0\le\phi\le\phi_0, 0\le z\le h) = G(\rho=b, 0\le\phi\le\phi_0, 0\le z\le h) = 0 \qquad (1a)$$

$$G(a\le\rho\le b, \phi=0, 0\le z\le h) = G(a\le\rho\le b, \phi=\phi_0, 0\le z\le h) = 0 \qquad (1b)$$

To meet the boundary conditions on ϕ, we select a solution for the Green's function of the form

$$G(\rho,\phi;\rho',\phi') = \sum_{m=1,2,\ldots}^{+\infty} g_m(\rho;\rho',\phi')\sin\left(\frac{m\pi}{\phi_0}\phi\right) = \sum_{m=1,2,\ldots}^{\infty} g_m(\rho;\rho',\phi')\sin(r\phi) \qquad (2)$$

$$, \; r = \frac{m\pi}{\phi_0}, \quad m=1,2,3,\ldots$$

Substituting (2) into (15-153) leads to

$$\sum_{m=1,2,\ldots}^{\infty}\left[\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}-\frac{r^2}{\rho^2}+\beta_0^2\right]g_m(\rho;\rho',\phi')\sin(r\phi) = \frac{1}{\rho}\delta(\rho-\rho')\delta(\phi-\phi') \qquad (3)$$

Multiplying both sides of (3) by $\sin(s\phi)$, integrating from 0 to ϕ_0, and using the orthogonality conditions of (15-48a) and (15-48b) leads to

$$\rho\frac{d^2 g_m}{d\rho^2}+\frac{dg_m}{d\rho}+\left(\rho\beta_0^2-\frac{r^2}{\rho}\right)g_m = \frac{2}{\phi_0}\sin(r\phi')\delta(\rho-\rho') \qquad (4)$$

the homogeneous form of (4) is

$$\rho\frac{d^2 g_m}{d\rho^2}+\frac{dg_m}{d\rho}+\left(\rho\beta_0^2-\frac{r^2}{\rho}\right)g_m = 0 \qquad (5)$$

Thus its two solutions are

$$g_m^{(1)} = A_m J_m(\beta_0\rho)+B_m Y_m(\beta_0\rho) \qquad \text{for } \rho<\rho' \qquad (6a)$$

$$g_m^{(2)} = C_m J_m(\beta_0\rho)+D_m Y_m(\beta_0\rho) \qquad \text{for } \rho>\rho' \qquad (6b)$$

Since the boundary conditions in ρ for this problem are the same as those of Problem 15-12, thus we can write using (6)-(8b), (4) and (6a)-(6b) the solution for g_m as

$$g_m(\rho;\rho',\phi') = \begin{cases} \dfrac{\pi}{\phi_0}\left[\dfrac{J_r(\beta_0\rho')-\gamma Y_r(\beta_0\rho')}{(JY)_r}\right]\left[J_r(\beta_0\rho)-\alpha Y_r(\beta_0\rho)\right]\sin(r\phi') & a\le\rho\le\rho' \quad (7a) \\[2ex] \dfrac{\pi}{\phi_0}\left[\dfrac{J_r(\beta_0\rho')-\alpha Y_r(\beta_0\rho')}{(JY)_r}\right]\left[J_r(\beta_0\rho)-\gamma Y_r(\beta_0\rho)\right]\sin(r\phi') & \rho'\le\rho\le b \quad (7b) \end{cases}$$

cont'd.

15.15 cont'd. where $\alpha = \dfrac{J_r(\beta_o a)}{Y_r(\beta_o a)}$ (8a)

$$\gamma = \frac{J_r(\beta_o b)}{Y_r(\beta_o b)} \tag{8b}$$

$$(JY)_r = \frac{J_r(\beta_o a)Y_r(\beta_o b) - J_r(\beta_o b)Y_r(\beta_o a)}{Y_r(\beta_o a)\,Y_r(\beta_o b)} \tag{8c}$$

Thus the Green's function of (2) can be written as

$$G(\rho,\phi;\rho',\phi') = \begin{cases} \dfrac{\pi}{\phi_o}\displaystyle\sum_{m=1,2,\ldots}^{\infty}\left\{\dfrac{J_{m\pi/\phi_o}(\beta_o\rho') - \gamma Y_{m\pi/\phi_o}(\beta_o\rho')}{(JY)_r}\right\}\left[J_{\frac{m\pi}{\phi_o}}(\beta_o\rho) - \alpha Y_{\frac{m\pi}{\phi_o}}(\beta_o\rho)\right]\sin\left(\frac{m\pi}{\phi_o}\phi'\right)\sin\left(\frac{m\pi}{\phi_o}\phi\right) & a \le \rho \le \rho' \quad (9a) \\[2em] \dfrac{\pi}{\phi_o}\displaystyle\sum_{m=1,2,\ldots}^{\infty}\left\{\dfrac{J_{m\pi/\phi_o}(\beta_o\rho') - \alpha Y_{m\pi/\phi_o}(\beta_o\rho')}{(JY)_r}\right\}\left[J_{\frac{m\pi}{\phi_o}}(\beta_o\rho) - \gamma Y_{\frac{m\pi}{\phi_o}}(\beta_o\rho)\right]\sin\left(\frac{m\pi}{\phi_o}\phi'\right)\sin\left(\frac{m\pi}{\phi_o}\phi\right) & \rho' \le \rho \le b \quad (9b) \end{cases}$$

15.16 This problem is a special case of that of Problem 15.13 except that the boundary condition in ρ is

$$G(\rho = b,\ 0 \le \phi \le \phi_o,\ 0 \le z \le h) = 0 \tag{1}$$

Thus we can copy the solution from Problem 15.13 by letting $\alpha = \dfrac{J_r(\beta_o a)}{Y_r(\beta_o a)} = 0$.

Thus the Green's function can be written from (9a) and (9b) of Problem 9.16 as

$$G(\rho,\phi;\rho',\phi') = \begin{cases} \dfrac{\pi}{\phi_o}\displaystyle\sum_{m=1,2,\ldots}^{\infty}\left\{\dfrac{J_{m\pi/\phi_o}(\beta_o\rho') - \gamma Y_{m\pi/\phi_o}(\beta_o\rho')}{(JY)_r}\right\} J_{\frac{m\pi}{\phi_o}}(\beta_o\rho)\sin\left(\frac{m\pi}{\phi_o}\phi'\right)\sin\left(\frac{m\pi}{\phi_o}\phi\right) & a \le \rho \le \rho' \quad (2a) \\[2em] \dfrac{\pi}{\phi_o}\displaystyle\sum_{m=1,2,\ldots}^{\infty}\left\{\dfrac{J_{m\pi/\phi_o}(\beta_o\rho')}{(JY)_r}\right\}\left[J_{\frac{m\pi}{\phi_o}}(\beta_o\rho) - \gamma Y_{\frac{m\pi}{\phi_o}}(\beta_o\rho)\right]\sin\left(\frac{m\pi}{\phi_o}\phi'\right)\sin\left(\frac{m\pi}{\phi_o}\phi\right) & \rho' \le \rho \le b \quad (2b) \end{cases}$$

where $\gamma = \dfrac{J_{m\pi/\phi_o}(\beta_o b)}{Y_{m\pi/\phi_o}(\beta_o b)}$ (3a)

$$(JY)_r = -\frac{J_{m\pi/\phi_o}(\beta_o b)}{Y_{m\pi/\phi_o}(\beta_o b)} \tag{3b}$$

$\boxed{15.17}$ The solution to this problem proceeds in the same way until we reach (15-156), or

$$G(\rho,\phi;\rho',\phi') = \sum_{m=-\infty}^{+\infty} g_m(\rho;\rho',\phi')e^{jm\phi} \qquad (1)$$

$$\rho\frac{d^2g_m}{d\rho^2} + \frac{dg_m}{d\rho} + \left(\rho\beta_0^2 - \frac{m^2}{\rho}\right)g_m = \frac{e^{-jm\phi'}}{2\pi}\delta(\rho-\rho') \qquad (2)$$

Now instead of using a closed form for the solution of $g_m(\rho;\rho',\phi')$ of (2), we will select a series form, as outlined in (15-52)-(15-61). We write the desired series solution as

$$g_m(\rho;\rho',\phi') = \sum_{n=1,2,\ldots}^{\infty}\left[A_{mn}J_m(\beta_{mn}\rho) + B_{mn}Y_m(\beta_{mn}\rho)\right] \qquad (3)$$

Since the Green's function must be finite everywhere, including $\rho=0$, then $g_m(\rho;\rho',\phi')$ must also be finite everywhere (including $\rho=0$). Thus $B_{mn}=0$. Therefore

$$g_m(\rho;\rho',\phi') = \sum_{n=1,2,\ldots}^{\infty} A_{mn}J_m(\beta_{mn}\rho) \quad \text{of the form of (15-56)} \qquad (4)$$

Now we apply the boundary condition of (15-149a) which leads to

$$g_m(\rho=a;\rho',\phi') = \sum_{n=1,2,\ldots}^{\infty} A_{mn}(\rho',\phi')J_m(\beta_{mn}a) = 0 \Rightarrow \beta_{mn} = \chi_{mn} = \text{roots of Bessel function } J_m$$

Thus

$$\beta_{mn} = \left(\frac{\chi_{mn}}{a}\right), \quad \begin{array}{l} m=0,\pm1,\pm2,\ldots \\ n=1,2,3,\ldots \end{array} \qquad (5)$$

$$g_m(\rho;\rho',\phi') = \sum_{n=1,2,\ldots}^{\infty} A_{mn}(\rho',\phi')J_m(\beta_{mn}\rho), \quad \beta_{mn} = \frac{\chi_{mn}}{a} \qquad (6)$$

The solution for $g_m(\rho;\rho',\phi')$ as given by (3) or (4) was chosen to satisfy the homogeneous differential equation of

$$\rho\frac{d^2J_m}{d\rho^2} + \frac{dJ_m}{d\rho} + \left(\rho\beta_{mn}^2 - \frac{m^2}{\rho}\right)J_m = 0 \qquad (7)$$

where $\psi_m(\rho)$ in (15-56) is $\psi_m(\rho) = J_m(\beta_{mn}\rho)$ of the form of (14-54)

Substituting (6) in (2) we can write that

$$\sum_{n=1,2,\ldots}^{\infty}\left\{\rho\frac{d^2J_m}{d\rho^2} + \frac{dJ_m}{d\rho} + \left(\rho\beta_{mn}^2 - \frac{m^2}{\rho}\right)J_m + \left(\rho\beta_0^2 - \rho\beta_{mn}^2\right)J_m\right\}A_{mn} = \frac{e^{-jm\phi'}}{2\pi}\delta(\rho-\rho') \qquad (8)$$

Substituting (7) in (8) leads to

$$\sum_{n=1,2,\ldots}^{\infty} A_{mn}(\beta_0^2 - \beta_{mn}^2)\rho J_m(\beta_{mn}\rho) = \frac{e^{-jm\phi'}}{2\pi}\delta(\rho-\rho') \qquad (9)$$

Since the eigenfunction $J_{mn}(\beta_{mn}\rho)$ are orthogonal over the interval 0 to a for the same value of m but different value of n (with the

cont'd.

$\boxed{\text{15.17 cont'd.}}$ weighting function ρ) as [9], [13]

$$\int_0^a \rho\, J_m(\beta_{mn}\rho) J_m(\beta_{mp}\rho)\, d\rho = \delta_{np} N \quad \text{where } \delta_{np} = \begin{cases} 1 & p=n \\ 0 & p \neq n \end{cases} \qquad (10)$$

then we multiply both sides of (9) by $J_m(\beta_{mp}\rho)$ and integrate from 0 to a. Doing this and using that

$$\int_0^a \rho\, J_m(\beta_{mn}\rho) J_m(\beta_{mp}\rho)\, d\rho = \begin{cases} \frac{a^2}{2} J_{m+1}^2(\beta_{mn}a) & p=n \quad (11a) \\ 0 & p \neq n \quad (11b) \end{cases}$$

we can write that

$$A_{mn}(\beta_0^2 - \beta_{mn}^2) \frac{a^2}{2} J_{m+1}^2(\beta_{mn}a) = \frac{e^{-jm\phi'}}{2\pi} J_m(\beta_{mn}\rho') \qquad (12)$$

or

$$A_{mn} = \frac{1}{\pi a^2 (\beta_0^2 - \beta_{mn}^2) J_{m+1}^2(\beta_{mn}a)} e^{-jm\phi'} J_m(\beta_{mn}\rho') \qquad (12a)$$

Therefore $g_m(\rho; \rho', \phi')$ of (6) can be written as

$$g_m(\rho; \rho', \phi') = \sum_{n=1,2,\ldots}^{\infty} \frac{1}{\pi a^2 (\beta_0^2 - \beta_{mn}^2) J_{m+1}^2(\beta_{mn}a)} e^{-jm\phi'} J_m(\beta_{mn}\rho') J_m(\beta_{mn}\rho) \qquad (13)$$

Ultimately the Green's function $(\rho, \phi; \rho', \phi')$ of (15-151) can be written as

$$G(\rho, \phi; \rho', \phi') = \frac{1}{\pi a^2} \sum_{m=-\infty}^{+\infty} \left[\sum_{n=1}^{\infty} \frac{1}{(\beta_0^2 - \beta_{mn}^2) J_{m+1}^2(\beta_{mn}a)} J_m(\beta_{mn}\rho') J_m(\beta_{mn}\rho) \right] e^{jm(\phi - \phi')} \qquad (14)$$

[13] D.C. Stinson, Intermediate Mathematics of Electromagnetics, Prentice-Hall, 1976, pp. 13, 46–47, 262.

[9] R. Collin, Field Theory of Guided Waves, McGraw-Hill Book Co., p. 196.

15.18 The boundary conditions of this problem are

$$E_z(\rho=a, 0\le\phi\le 2\pi) = 0 \tag{1}$$

The Green's function will be selected to have the same boundary conditions as those of the electric field, or

$$G(\rho=a, 0\le\phi\le 2\pi) = 0 \tag{2}$$

The solution of this problem follows that on pages 928-930. Thus we can write using (15-151), and (15-159a)-(15-159b) that

$$G(\rho,\phi;\rho',\phi') = \sum_{m=-\infty}^{+\infty} g_m(\rho;\rho',\phi')\, e^{jm\phi} \tag{3}$$

$$g_m^{(1)} = A_m J_m(\beta_o\rho) + B_m H_m^{(2)}(\beta_o\rho) \tag{3a}$$

$$g_m^{(2)} = C_m H_m^{(2)}(\beta_o\rho) \tag{3b}$$

Applying the boundary condition of (2) leads to

$$g_m^{(1)}(\rho=a) = 0 = A_m J_m(\beta_o a) + B_m H_m^{(2)}(\beta_o a) \Rightarrow B_m = -A_m \frac{J_m(\beta_o a)}{H_m^{(2)}(\beta_o a)} \tag{4}$$

Thus

$$g_m^{(1)} = A_m\left\{J_m(\beta_o\rho) - \frac{J_m(\beta_o a)}{H_m^{(2)}(\beta_o a)} H_m^{(2)}(\beta_o\rho)\right\} = A_m\left\{J_m(\beta_o\rho) - \alpha H_m^{(2)}(\beta_o\rho)\right\} \tag{5a}$$

$$g_m^{(2)} = C_m H_m^{(2)}(\beta_o\rho) \qquad \alpha = \frac{J_m(\beta_o a)}{H_m^{(2)}(\beta_o a)} \tag{5b}$$

The Wronskian of (15-44c) can be written as

$$W(\rho') = \beta_o A_m C_m\left[J_m(\beta_o\rho') - \alpha H_m^{(2)}(\beta_o\rho')\right] H_m^{(2)\prime}(\beta_o\rho') - \beta_o A_m C_m H_m^{(2)}(\beta_o\rho')\left[J_m'(\beta_o\rho') - \alpha H_m^{(2)\prime}(\beta_o\rho')\right]$$

$$= \beta_o A_m C_m\left\{J_m(\beta_o\rho') H_m^{(2)\prime}(\beta_o\rho') - J_m'(\beta_o\rho') H_m^{(2)}(\beta_o\rho')\right\}$$

$$W(\rho') = -j\beta_o A_m C_m\left[J_m(\beta_o\rho') Y_m'(\beta_o\rho') - J_m'(\beta_o\rho') Y_m(\beta_o\rho')\right], \qquad ' = \frac{\partial}{\partial(\beta_o\rho')} \tag{6}$$

Using the Wronskian for Bessel functions of (11-95), we can write (6) as

$$W(\rho') = -j\frac{2}{\pi} A_m C_m \frac{1}{\rho'} \tag{7}$$

Thus we can write $g_m(\rho;\rho',\phi')$, using (7), (15-156), (15-45a) and (15-45b), as

$$g_m(\rho;\rho',\phi') = \begin{cases} j\dfrac{1}{4}\left[H_m^{(2)}(\beta_o\rho')\right]\left[J_m(\beta_o\rho) - \alpha H_m^{(2)}(\beta_o\rho)\right] e^{-jm\phi'}, & a\le\rho\le\rho' \quad (8a) \\[2ex] j\dfrac{1}{4}\left[J_m(\beta_o\rho') - \alpha H_m^{(2)}(\beta_o\rho')\right] H_m^{(2)}(\beta_o\rho)\, e^{-jm\phi'}, & \rho'\le\rho\le\infty \quad (8b) \end{cases}$$

cont'd.

15.18 cont'd. Thus the Green's function of (3) can now be written as

$$G(\rho,\phi;\rho',\phi')=\begin{cases} j\frac{1}{4}\sum_{m=-\infty}^{+\infty} H_m^{(2)}(\beta_0\rho')\left[J_m(\beta_0\rho)-\alpha H_m^{(2)}(\beta_0\rho)\right]e^{jm(\phi-\phi')}, & a\le\rho\le\rho' \quad (9a)\\ j\frac{1}{4}\sum_{m=-\infty}^{+\infty}\left[J_m(\beta_0\rho')-\alpha H_m^{(2)}(\beta_0\rho')\right]H_m^{(2)}(\beta_0\rho)e^{jm(\phi-\phi')}, & \rho'\le\rho\le\infty \quad (9b)\end{cases}$$

The forms of (9a) and (9b) which are representative of the electric field E_z are identical in form of (11-64a) and (11-64b).

15.19 The boundary conditions for this problem are

$$E_\phi(\rho=a,\ 0\le\phi\le 2\pi)=0 \tag{1}$$

For this polarization the ϕ component of the electric field is related to the z component of the magnetic field by (11-106b), or

$$E_\phi=-\frac{1}{j\omega\varepsilon}\frac{\partial H_z}{\partial\rho} \tag{2}$$

The Green's function will be selected for this problem to represent the z component of the magnetic field (H_z). Therefore according to (1) and (2) the Green's function will be selected to have the boundary condition of

$$\left.\frac{\partial}{\partial\rho}G\right|_{\rho=a}=G'(\rho=a)=0 \tag{3}$$

The solution of this problem follows that on pages 928-930. thus we can write using (15-151), and (15-159a)-(15-159b) that

$$G(\rho,\phi;\rho',\phi')=\sum_{m=-\infty}^{+\infty} g_m(\rho;\rho',\phi')e^{jm\phi} \tag{4}$$

$$g_m^{(1)}=A_mJ_m(\beta_0\rho)+B_mH_m^{(2)}(\beta_0\rho) \tag{4a}$$

$$g_m^{(2)}=C_mH_m^{(2)}(\beta_0\rho) \tag{4b}$$

Applying the boundary condition of (3) leads to

$$g_m'(\rho=a)=A_mJ_m'(\beta_0a)+B_mH_m^{(2)\prime}(\beta_0a)=0\Rightarrow B_m=-A_m\frac{J_m'(\beta_0a)}{H_m^{(2)\prime}(\beta_0a)} \tag{5}$$

cont'd.

$\boxed{\text{15.19 cont'd.}}$ Thus $g_m^{(1)}$ and $g_m^{(2)}$ can be expressed as

$$g_m^{(1)} = A_m\left\{J_m(\beta_0\rho) - \frac{J_m'(\beta_0 a)}{H_m^{(2)'}(\beta_0 a)}H_m^{(2)}(\beta_0\rho)\right\} = A_m\left\{J_m(\beta_0\rho) - \gamma H_m^{(2)}(\beta_0\rho)\right\} \quad (6a)$$

$$g_m^{(2)} = C_m H_m^{(2)}(\beta_0\rho) \quad (6b)$$

$$\gamma = \frac{J_m'(\beta_0 a)}{H_m^{(2)'}(\beta_0 a)} \quad (6c)$$

The Wronskian of (15-44c) can be written as

$$W(\rho') = \beta_0 A_m C_m\left\{\left[J_m(\beta_0\rho') - \gamma H_m^{(2)}(\beta_0\rho')\right]H_m^{(2)'}(\beta_0\rho') - H_m^{(2)}(\beta_0\rho')\left[J_m'(\beta_0\rho') - \gamma H_m^{(2)'}(\beta_0\rho')\right]\right\}$$

$$W(\rho') = \beta_0 A_m C_m\left\{J_m(\beta_0\rho')H_m^{(2)'}(\beta_0\rho') - J_m'(\beta_0\rho')H_m^{(2)}(\beta_0\rho')\right\} = -j\beta_0 A_m C_m \underbrace{\left[J_m(\beta_0\rho')Y_m'(\beta_0\rho') - J_m'(\beta_0\rho')Y_m(\beta_0\rho')\right]}_{=\frac{2}{\pi(\beta_0\rho')}} \quad (7)$$

Using the Wronskian for Bessel functions of (11-95), we can write (7) as

$$W(\rho') = -j\frac{2}{\pi}A_m C_m \frac{1}{\rho'} \quad (8)$$

Thus we can write $g_m(\rho;\rho',\phi')$, using (8), (15-156), (15-45a) and (15-45b) as

$$g_m(\rho;\rho',\phi') = \begin{cases} j\dfrac{1}{4}\left[H_m^{(2)}(\beta_0\rho')\right]\left[J_m(\beta_0\rho) - \gamma H_m^{(2)}(\beta_0\rho)\right]e^{-jm\phi'}, & a \le \rho \le \rho' \quad (9a) \\[2ex] j\dfrac{1}{4}\left[J_m(\beta_0\rho') - \gamma H_m^{(2)}(\beta_0\rho')\right]H_m^{(2)}(\beta_0\rho)\,e^{-jm\phi'}, & \rho' \le \rho \le \infty \quad (9b) \end{cases}$$

Thus the Green's function of (4) can now be written as

$$G(\rho,\phi;\rho',\phi') = \begin{cases} j\dfrac{1}{4}\displaystyle\sum_{m=-\infty}^{+\infty} H_m^{(2)}(\beta\rho')\left[J_m(\beta_0\rho) - \frac{J_m'(\beta_0 a)}{H_m^{(2)'}(\beta_0 a)}H_m^{(2)}(\beta_0\rho)\right]e^{jm(\phi-\phi')} \quad (10a) \\ \qquad a \le \rho \le \rho' \\[2ex] j\dfrac{1}{4}\displaystyle\sum_{m=-\infty}^{+\infty}\left[J_m(\beta_0\rho') - \frac{J_m'(\beta_0 a)}{H_m^{(2)'}(\beta_0 a)}H_m^{(2)}(\beta_0\rho')\right]H_m^{(2)}(\beta_0\rho)\,e^{jm(\phi-\phi')} \quad (10b) \\ \qquad \rho' \le \rho \le \infty \end{cases}$$

The forms of (10a) and (10b), which are representative of the z component of the magnetic field (H_z), are identical in form of (11-176f) and (11-176g).

$\boxed{15.20}$ The boundary conditions of this problem are

$$E_z(0\le\rho\le\infty,\ \phi=\alpha) = E_z(0\le\rho\le\infty,\ \phi=2\pi-\alpha)=0 \quad (1)$$

The Green's function will be selected to have the same boundary conditions as those of the electric field, or

$$G(0\le\rho\le\infty,\ \phi=\alpha) = G(0\le\rho\le\infty,\ \phi=2\pi-\alpha)=0 \quad (2)$$

To meet the boundary conditions on ϕ, we select a solution for the Green's function of the form

$$G(\rho,\phi;\rho',\phi') = \sum_m g_m(\rho;\rho',\phi')\left\{A_m\cos[m(\phi-\alpha)] + B_m\sin[m(\phi-\alpha)]\right\} \quad (3)$$

Applying (2) leads to

$$G(\phi=\alpha) = \sum_m g_m(\rho;\rho',\phi')\left\{A_m(1)+B_m(0)\right\}=0 \Rightarrow A_m=0 \quad (4a)$$

$$G(\phi=2\pi-\alpha) = \sum_m g_m(\rho;\rho',\phi')B_m\sin[2m(\pi-\alpha)]=0 \Rightarrow 2m(\pi-\alpha)=r\pi \quad (4b)$$

$$m=\frac{r\pi}{2(\pi-\alpha)}, \qquad r=1,2,3,\ldots.$$

Thus (3) reduces to

$$G(\rho,\phi;\rho',\phi') = \sum_m B_m\, g_m(\rho;\rho',\phi')\sin[m(\phi-\alpha)] \quad (5)$$

Substituting (5) into (15-153) leads to

$$\sum_m\left[\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}-\frac{m^2}{\rho^2}+\beta_0^2\right]g_m(\rho;\rho',\phi')\sin[m(\phi-\alpha)] = \frac{1}{\rho}\delta(\rho-\rho')\delta(\phi-\phi') \quad (6)$$

Multiplying both sides of (6) by $\sin[n(\phi-\alpha)]$, integrating both sides from α to $(2\pi-\alpha)$, and using the orthogonality of (15-48a) and (15-48b)

$$(\pi-\alpha)\left[\frac{\partial^2 g_m}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial g_m}{\partial\rho}+\left(\beta_0^2-\frac{m^2}{\rho^2}\right)g_m\right] = \frac{1}{\rho}\sin[m(\phi'-\alpha)]\delta(\rho-\rho') \quad (7)$$

or

$$\frac{d^2 g_m}{d\rho^2}+\frac{1}{\rho}\frac{dg_m}{d\rho}+\left(\beta_0^2-\frac{m^2}{\rho^2}\right)g_m = \frac{1}{\pi-\alpha}\frac{1}{\rho}\sin[m(\phi'-\alpha)]\delta(\rho-\rho') \quad (7a)$$

The homogeneous form of (7a) is

$$\frac{d^2 g_m}{d\rho^2}+\frac{1}{\rho}\frac{dg_m}{d\rho}+\left(\beta_0^2-\frac{m^2}{\rho^2}\right)g_m = 0 \quad (8)$$

whose two solutions can be written as

$$g_m^{(1)} = A_m J_m(\beta_0\rho) \qquad 0\le\rho\le\rho' \quad (9a)$$

$$g_m^{(2)} = C_m H_m^{(2)}(\beta_0\rho) \qquad \infty\ge\rho\ge\rho' \quad (9b)$$

cont'd.

15.20 cont'd. The Wronskian of (15-44c) can be written as

$$W(\rho') = A_m C_m \beta_0 J_m(\beta_0\rho') H_m^{(2)'}(\beta_0\rho') - A_m C_m \beta_0 H_m^{(2)}(\beta_0\rho') J_m'(\beta_0\rho')$$

$$W(\rho') = -j\beta_0 A_m C_m \left[J_m(\beta_0\rho') Y_m'(\beta_0\rho') - J_m'(\beta_0\rho') Y_m(\beta_0\rho') \right] \quad (10)$$

Using the Wronskian for Bessel functions of (11-95), we can write (10) as

$$W(\rho') = -j\frac{2}{\pi} A_m C_m \frac{1}{\rho'} \quad (11)$$

Thus we can write $g_m(\rho;\rho',\phi')$, using (11), (7a), (15-45a) and (15-45b) as

$$g_m(\rho;\rho',\phi') = \begin{cases} j\dfrac{\pi}{2(\pi-\alpha)} H_m^{(2)}(\beta_0\rho') J_m(\beta_0\rho) \sin[m(\phi'-\alpha)] & 0 \le \rho \le \rho' \quad (12a) \\ j\dfrac{\pi}{2(\pi-\alpha)} J_m(\beta_0\rho') H_m^{(2)}(\beta_0\rho) \sin[m(\phi'-\alpha)] & \rho' \le \rho \le \infty \quad (12b) \end{cases}$$

Thus the Green's function of (5) can then be written as

$$G(\rho,\phi;\rho',\phi') = \begin{cases} j\dfrac{\pi}{2(\pi-\alpha)} \sum\limits_m B_m H_m^{(2)}(\beta_0\rho') J_m(\beta_0\rho) \sin[m(\phi'-\alpha)] \sin[m(\phi-\alpha)], & 0 \le \rho \le \rho' \quad (13a) \\ j\dfrac{\pi}{2(\pi-\alpha)} \sum\limits_m B_m J_m(\beta_0\rho') H_m^{(2)}(\beta_0\rho) \sin[m(\phi'-\alpha)] \sin[m(\phi-\alpha)], & \rho' \le \rho \le \infty \quad (13b) \end{cases}$$

$$\text{where} \quad m = \frac{r\pi}{2(\pi-\alpha)}, \quad r = 1, 2, 3, \cdots \quad (13c)$$

The forms of (13a) and (13b), which are representative of the z component of the electric field (E_z), are identical in form to (11-182a) and (11-182b).

15.21 The boundary conditions for this problem are

$$E_\rho(0\le\rho\le\infty,\phi=\alpha)=E_\rho(0\le\rho\le\infty,\phi=2\pi-\alpha)=0 \quad (1)$$

For this polarization the ϕ component of the electric field is related to the z component of the magnetic field by (11-106a), or

$$E = +\frac{1}{j\omega\epsilon\rho}\frac{\partial H_z}{\partial\rho} \quad (2)$$

The Green's function will be selected for this problem to represent the z component of the magnetic field (H_z). Therefore according to (1) and (2) the Green's function will be selected to have the boundary conditions of

$$\left.\frac{\partial G}{\partial\phi}\right|_{\phi=\alpha} = G'(\phi=\alpha) = \left.\frac{\partial G}{\partial\phi}\right|_{\phi=2\pi-\alpha} = G'(\phi=2\pi-\alpha)=0 \quad (3)$$

To meet the boundary conditions on ϕ, we select a solution for the Green's function of the form

$$G(\rho,\phi;\rho',\phi')=\sum_m g_m(\rho;\rho',\phi')\left\{A_m\cos[m(\phi-\alpha)]+B_m\sin[m(\phi-\alpha)]\right\} \quad (4)$$

Applying (3) leads to

$$G'(\phi=\alpha)=\sum_m m g_m(\rho;\rho',\phi')\left\{-A_m\sin[m(\phi-\alpha)]+B_m\cos[m(\phi-\alpha)]\right\}_{\phi=\alpha}$$

$$=\sum_m m g_m(\rho;\rho',\phi')\left\{-A_m(0)+B_m(1)\right\}=0 \Rightarrow B_m=0$$

$$G'(\phi=2\pi-\alpha)=\sum_m m g_m(\rho;\rho',\phi')\left\{-A_m\sin[2m(\pi-\alpha)]\right\}=0\Rightarrow m=\frac{s\pi}{2(\pi-\alpha)},\quad s=0,1,2,\dots \quad (5)$$

Thus (4) is reduced to

$$G(\rho,\phi;\rho',\phi')=\sum_m A_m g_m(\rho;\rho',\phi')\cos[m(\phi-\alpha)] \quad (6)$$

$$m=\frac{s\pi}{2(\pi-\alpha)},\quad s=0,1,2,\dots \quad (6a)$$

Substituting (6) into (15-153) leads to

$$\sum_m\left[\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}-\frac{m^2}{\rho^2}+\beta_0^2\right]g_m(\rho;\rho',\phi')\cos[m(\phi-\alpha)]=\frac{1}{\rho}\delta(\rho-\rho')\delta(\phi-\phi') \quad (7)$$

Multiplying both sides of (7) by $\cos[n(\phi-\alpha)]$, integrating both sides in ϕ from α to $(2\pi-\alpha)$, and using the orthogonality conditions for cosine functions, we can write that

cont'd.

15.21 cont'd.

$$(\pi-\alpha)\left[\frac{\partial^2 g_m}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial g_m}{\partial\rho}+\left(\beta_0^2-\frac{m^2}{\rho^2}\right)g_m\right]=\frac{1}{\rho}\cos[m(\phi'-\alpha)]\,\delta(\rho-\rho') \quad (8)$$

or

$$\frac{d^2 g_m}{d\rho^2}+\frac{1}{\rho}\frac{dg_m}{d\rho}+\left(\beta_0^2-\frac{m^2}{\rho^2}\right)g_m=\frac{1}{\pi-\alpha}\sin[m(\phi'-\alpha)]\,\frac{1}{\rho}\delta(\rho-\rho') \quad (8a)$$

The homogeneous form of (8a) is

$$\frac{d^2 g_m}{d\rho^2}+\frac{1}{\rho}\frac{dg_m}{d\rho}+\left(\beta_0^2-\frac{m^2}{\rho^2}\right)g_m=0 \quad (9)$$

whose two solutions can be written as

$$g_m^{(1)}=A_m J_m(\beta_0\rho) \qquad 0\le\rho\le\rho' \quad (10a)$$

$$g_m^{(2)}=C_m H_m^{(2)}(\beta_0\rho) \qquad \rho'\le\rho\le\infty \quad (10b)$$

The Wronskian of (15-44c) can be written as

$$W(\rho')=\beta_0 A_m C_m\left[J_m(\beta_0\rho')H_m^{(2)\prime}(\beta_0\rho')-H_m^{(2)}(\beta_0\rho')J_m'(\beta_0\rho')\right]$$

$$W(\rho')=-j\beta_0 A_m C_m\left[J_m(\beta_0\rho')Y_m'(\beta_0\rho')-J_m'(\beta_0\rho')Y_m(\beta_0\rho')\right] \quad (11)$$

Using the Wronskian for Bessel functions of (11-95), we can write (11) as

$$W(\rho')=-j\frac{2}{\pi}A_m C_m\frac{1}{\rho'} \quad (12)$$

Thus we can write $g_m(\rho;\rho',\phi')$, using (12), (8a), (15-45a) and (15-45b), as

$$g_m(\rho;\rho',\phi')=\begin{cases} j\frac{\pi}{2(\pi-\alpha)}H_m^{(2)}(\beta_0\rho')J_m(\beta_0\rho)\cos[m(\phi'-\alpha)], & 0\le\rho\le\rho' \quad (13a)\\ j\frac{\pi}{2(\pi-\alpha)}J_m(\beta_0\rho')H_m^{(2)}(\beta_0\rho)\cos[m(\phi'-\alpha)], & \rho'\le\rho\le\infty \quad (13b)\end{cases}$$

Thus the Green's function of (6) can then be written as

$$G(\rho,\phi;\rho',\phi')=\begin{cases} j\frac{\pi}{2(\pi-\alpha)}\sum_m A_m H_m^{(2)}(\beta_0\rho')J_m(\beta_0\rho)\cos[m(\phi'-\alpha)]\cos[m(\phi-\alpha)], & 0\le\rho\le\rho' \quad (14a)\\ \qquad m=\frac{s\pi}{2(\pi-\alpha)},\ s=0,1,2,\dots & \\ j\frac{\pi}{2(\pi-\alpha)}\sum_m A_m J_m(\beta_0\rho')H_m^{(2)}(\beta_0\rho)\cos[m(\phi'-\alpha)]\cos[m(\phi-\alpha)], & \rho'\le\rho\le\infty \quad (14b)\end{cases}$$

The forms of (14a) and (14b), which are representative of the z component of the magnetic field (H_z), are identical in form to (11-192a) and (11-192b).

$\boxed{15.22}$ To solve this problem, it is easier to assume that the point source

(a). of Figure P15.20 is located at the origin. When that is done, the field radiated by the point source is only a function of r (not a function of θ or ϕ). Thus in expanded form (15-166) can be written as

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial G}{\partial r}\right)+\beta^2 G = \frac{\partial^2 G}{\partial r^2}+\frac{2}{r}\frac{\partial G}{\partial r}+\beta^2 G = \frac{1}{4\pi r^2}\delta(r-r') \qquad (1)$$

with $\quad G(r=\infty) = \text{finite} \qquad (1a)$

Since $G(r,r')$ is only a function of r, (1) can be written as

$$\frac{d^2G}{dr^2}+\frac{2}{r}\frac{dG}{dr}+\beta^2 G = \frac{1}{4\pi r^2}\delta(r-r') \Rightarrow r^2\frac{d^2G}{dr^2}+2r\frac{dG}{dr}+(\beta r)^2 G = \frac{\delta(r-r')}{4\pi} \qquad (2)$$

A solution of the homogeneous form of (2) takes the form of

$$G(r) = A\frac{e^{-j\beta r}}{r}+B\frac{e^{+j\beta r}}{r} \qquad (3)$$

For an $e^{j\omega t}$ time convention, the boundary condition of (1a) allows us to write that $B=0$. Thus (3) reduces to

$$G(r) = A\frac{e^{-j\beta r}}{r} \qquad (4)$$

To evaluate the coefficient A in (4), we can integrate (2) over an infinitesimal volume which surrounds the origin and to then allow the radial distance r to approach zero ($r\to 0$). Doing this we find that $A = -\frac{1}{4\pi}$ which allows us to write (4) as

$$G(r) = -\frac{e^{-j\beta r}}{4\pi r} \qquad (5)$$

When the source is removed from the origin, as shown in Figure P15.20, the Green's function of (5) can be written as

$$G = -\frac{1}{4\pi}\frac{e^{-j\beta R}}{R} = -\frac{1}{4\pi}\frac{e^{-j\beta|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \qquad (6)$$

(b). The reduction of the three-dimensional Green's function of (6) to that of the two-dimensional one of Example is obtained by the use of the integral of (11-28a). This allows us to reduce the three-dimensional problem to a two-dimensional one.