The cut property of Minimum Spanning Trees

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In this handout we use "graph" to mean "undirected, connected graph", and "tree" to mean "free tree" (i.e., an undirected, connected, acyclic graph).

You already know that a tree with n nodes has $n-1$ edges. Using this, you can easily prove the following fact: Adding an edge to a tree creates a unique cycle; removing any edge from that cycle yields a tree again. More precisely:

Fact 1 *Let $T=(V,E)$ be a tree and e be an edge not in E. Then the graph $T^+=(V,E\cup\{e\})$ has a unique cycle. Furthermore, if e' is any edge on that cycle, the graph $T'=\big(V,(E\cup\{e\})-e'\big)$ is a tree.*

A ***cut*** of a graph $G=(V,E)$ is a partition of the set of nodes V into two sets of nodes, $S\subseteq V$ and $V-S$; we denote this cut as the pair $(S,V-S)$. An edge e of G ***crosses*** the cut $(S,V-S)$ if one of the endpoints of e is in S and the other is in $V-S$.

Theorem 2 (Cut property of MSTs) *Let $G=(V,E)$ be a graph, $(S,V-S)$ be a cut of G, F be a set of edges of G, and e be an edge of G such that*
(a) F is contained in a MST of G,
(b) no edge in F crosses the cut $(S,V-S)$, and
(c) e is a minimum weight edge that crosses the cut $(S,V-S)$.
Then $F\cup\{e\}$ is also contained in a MST of G.

Intuitively, this theorem says that we can extend a partial MST (i.e., a subset of the edges of an MST of a graph) to a better partial MST (i.e., an even bigger subset of the edges of an MST of the graph) as follows: Find ***any*** cut that no edge in F crosses, and pick a minimum weight edge e that crosses the cut; then $F\cup\{e\}$ is the expanded partial MST. It is clear how such a property leads to greedy MST algorithms: We start with the trivial partial solution (an empty set of edges), and then we use the cut property to greedily expand this partial solution, until our "partial" solution contains $n-1$ edges — i.e., is a full MST. The correctness of Prim's and Kruskal's MST algorithms is based on the cut property. (Seeing how is left as an exercise.) We now prove Theorem 2.

Proof. Let T be a MST that contains the edges in F. (Such a MST exists by hypothesis (a) of the theorem.) If T contains e, we are done. So, suppose T does not contain e.

Let T^+ be the graph that results when we add e to T. By Fact 1, T^+ has a unique cycle; let u and v be the endpoints of e. By hypothesis (c), e crosses the cut $(S,V-S)$, so without loss of generality, suppose that $u\in S$ and $v\in V-S$. The unique cycle of T^+ consists of e, and a path of edges in T that connects u to v. This path contains an edge, say e', that has one endpoint in S and the other in $V-S$; this is because $u\in S$ and $v\in V-S$. So, e' also crosses the cut $(S,V-S)$.

Since both e and e' cross the cut $(S,V-S)$, and e is a minimum weight edge that crosses this cut (see hypothesis (c)), $\textbf{weight}(e)\le\textbf{weight}(e')$. By Fact 1, removing e' from T^+ results in a spanning tree T' of G. Recall that T' is constructed by adding e to T and then removing e' from the resulting graph. So,

$$\textbf{weight}(T')=\textbf{weight}(T)+\textbf{weight}(e)-\textbf{weight}(e')\le\textbf{weight}(T)$$

(where the last inequality follows from the fact that $\mathbf{weight}(e) \leq \mathbf{weight}(e')$). Since (i) T' is a spanning tree of G, (ii) T is a MST of G, and (iii) $\mathbf{weight}(T') \leq \mathbf{weight}(T)$, it follows that

$$T' \text{ is also a MST of } G. \tag{1}$$

Next we argue that T' contains all the edges in F. To see this recall that T contains all the edges in F, and e', the only edge of T that T' does not contain, is not in F. (This is because e' crosses $(S, V-S)$ and no edge in F crosses $(S, V-S)$, by hypothesis (b).) So, the edges of T excluding e' contain the edges in F. T' contains all the edges of T except e', and so

$$T' \text{ contains the edges in } F. \tag{2}$$

By its defintion,

$$T' \text{ contains the edge } e. \tag{3}$$

By (1), (2), and (3), T' is a MST of G that contains the edges in $F \cup \{e\}$, as wanted. $\square$