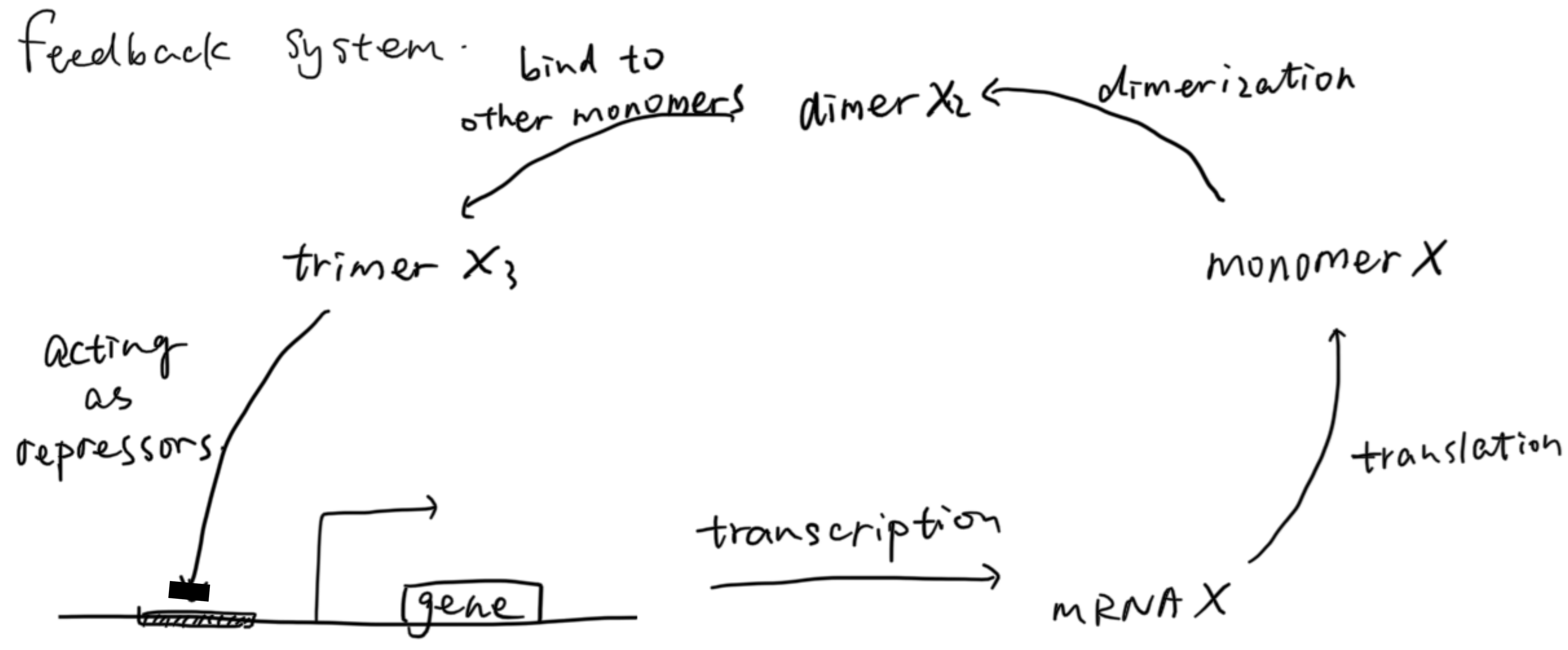
Q1

(a). feedback system.



(b). reaction list.

D: DNA.

m : mRNA.

X : monomer

X_2 : dimer

X_3 : trimer

y : DNA - trimer complex.

Binding reactions: $X_3 + D \underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} y$

Transcription: $D \xrightarrow{\beta} D + m$

$y \xrightarrow{\alpha\beta} y + m$.

translation: $m \xrightarrow{T_m} m + X$

formation of trimers: $X + X \underset{k_{-2}}{\overset{k_2}{\rightleftharpoons}} X_2$

$X_2 + X \underset{k_{-3}}{\overset{k_3}{\rightleftharpoons}} X_3$

degradation: $m \xrightarrow{\gamma_m} \phi$; $X \xrightarrow{\gamma_x} \phi$

(C). ODE System.

$$\frac{dD}{dt} = -k_1 D \cdot X_3 + k_{-1} y$$

$$\frac{dX_3}{dt} = k_3 \cdot X_2 \cdot X - k_{-3} \cdot X_3 + k_{-1} y - k_1 X_3 \cdot D$$

$$\frac{dm}{dt} = \beta D + \alpha \beta y - \gamma_m m$$

$$\frac{dy}{dt} = k_1 D X_3 - k_{-1} y$$

$$\frac{dX}{dt} = 2k_{-2} X_2 - 2k_2 X^2 + T_m m - \gamma_x X$$

$$\frac{dX_2}{dt} = k_2 X^2 - k_{-2} X_2 + k_{-3} X_3 - k_3 X_2 X$$

(d). Simplified 1-equation system.

① binding reactions are fast. $\Rightarrow -k_1 D X_3 + k_{-1} y = 0$

$$y = \frac{k_1}{k_{-1}} D \cdot X_3$$

let $\frac{k_1}{k_{-1}} = k$

$$y = k D X_3$$

② Conservation Law.

$$D_T = D + y$$
$$= D + KDX_3$$
$$= D(1 + KX_3)$$

$$D = \frac{D_T}{1 + kX_3}$$

③ Assume $\frac{dm}{dt} = 0$.

$$\frac{dm}{dt} = \beta D + \alpha\beta y - \gamma_m m = 0$$

$$\gamma_m m = \beta D + \alpha \beta y$$
$$= \beta(D + \alpha y)$$
$$= \beta\left(\frac{D_T}{1+kX_3} + \alpha k D X_3\right)$$
$$= \beta \cdot \frac{D_T + \alpha k X_3 D_T}{1 + kX_3}$$

$$m = \frac{\beta D_T}{\gamma_m} \cdot \frac{1 + \alpha k X_3}{1 + kX_3}$$

Simplify by making assumptions on time scale:

$$X_3 = \frac{k_3}{k_{-3}}(X_2 x)$$

$$X_2 = \frac{k_2}{k_{-2}}(x^2)$$

$$X_3 = \frac{k_3}{k_{-3}} \cdot \frac{k_2}{k_{-2}} x^3 \qquad \therefore m = \frac{1 + \alpha \frac{k_1}{k_{-1}} \cdot \frac{k_2}{k_{-2}} \cdot \frac{k_3}{k_{-3}} x^3}{1 + \frac{k_1}{k_{-1}} \cdot \frac{k_2}{k_{-2}} \cdot \frac{k_3}{k_{-3}} x^3} \cdot \frac{\beta D_T}{\gamma_m}$$

Substitutions: $\frac{dx}{dt} = T_m m - \gamma_x x$

$$\frac{dx}{dt} = \frac{T_m \beta D_T}{\gamma_m} \cdot \frac{1 + \alpha k' x^3}{1 + k' x^3} - \gamma_x x \quad \left(k' = \frac{k_1}{k_{-1}} \cdot \frac{k_2}{k_{-2}} \cdot \frac{k_3}{k_{-3}}\right)$$

Because transcription is negligible when repressors are bound, $\alpha \to 0$.

$$\frac{dx}{dt} = \frac{T_m \beta D_T}{\gamma_m} \cdot \frac{1}{1 + k' x^3} - \gamma_x x \quad \left(k' = \frac{k_1}{k_{-1}} \frac{k_2}{k_{-2}} \frac{k_3}{k_{-3}}\right)$$

(e). In (d), we assume the binding reaction is very fast ($\frac{dD}{dt} = 0$), this is the Quasi-equilibrium assumption.

We assume that total DNA = free DNA + bound DNA.

We assume that mRNA dynamics fast compared to protein dynamics ($\frac{dm}{dt} = 0$)