APMA 1650 Midterm 1 Solutions

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1. (15 points) Let toss two fair dice.
   (a) (3 points) What is the probability space?
   (b) (3 points) What is the probability that at least one die is greater or equal to 3?
   (c) (3 points) What is the probability that the ratio of the first die to the second die is greater than 1/2?
   (d) (3 points) Let X be the number on the first die. Find the probability mass function of X.
   (e) (3 points) Compute $E[X]$.

Solutions.

(a) The probability space is $(\Omega, \mathcal{F}, P)$ where Ω is the sample space defined by

$$\Omega = \{(i,j) | 1 \le i, j \le 6\},$$

$\mathcal{F}$ is the set of all events of Ω, (more precisely, $\mathcal{F} = \mathcal{P}(\Omega)$), and P is the probability function such that $P((i,j)) = 1/36$.

(b) Let A be the event where at least one die is greater or equal to 3. Let consider A^c. Then

$$A^c = \{(i,j) \in \Omega | 1 \le i, j \le 2\}.$$

Since

$$P(A^c) = \sum_{(i,j)\in A^c} P((i,j)) = \frac{|A^c|}{36} = \frac{4}{36},$$

we have

$$P(A) = 1 - P(A^c) = \frac{32}{36} = \frac{8}{9}.$$

(c) Let consider

$$B_i = \{(i,j) \in \Omega | 1 \le j < 2i\}, \qquad B = \bigcup_{i=1}^{6} B_i.$$

Then B is the event of our interest. Since B_i's are disjoint, and

$$P(B_i) = \frac{|B_i|}{36} = \begin{cases} \frac{2i-1}{36} & \text{if } i \le 3 \\ \frac{1}{6} & \text{if } i > 3 \end{cases}$$

we have

$$P(B) = \sum_{i=1}^{6} P(B_i) = \sum_{i=1}^{3} \left(\frac{2i-1}{36}\right) + \frac{1}{2} = \frac{1}{3} - \frac{1}{12} + \frac{1}{2} = \frac{9}{12} = \frac{3}{4}.$$

(d) Since

$$P(X = k) = \sum_{j=1}^{6} P((k, j)) = \frac{1}{6},$$

the pmf of X is

$$P(X = k) = \frac{1}{6} \qquad \text{for} \quad k = 1, 2, 3, 4, 5, 6$$

and $P(X = k) = 0$ otherwise.

(e) By definition,

$$E[X] = \sum_{k=1}^{6} k \cdot \frac{1}{6} = \frac{7}{2}.$$

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2. (15 points) Suppose we have a bag which contains 25 green balls and 175 yellow balls. We draw a ball randomly. We check the color of the ball. After checking the color, we put the ball back to the bag.

   (a) (2.5 points) What is the probability that the number of times observing yellow balls among 11 draws is k? (Note that $k = 0, 1, 2, \cdots, 11$).

   (b) (5 points) What is the probability that the number of the draw on which the first yellow ball is obtained is at least 5?

   (c) (2.5 points) What is the expected number of the draw on which the first yellow ball is obtained?

   (d) (5 points) What is the probability that the number of the draw on which the k-th yellow ball is obtained is 11? (Note that $k = 1, 2, \cdots, 11$).

Solutions.

   (a) Let X be a random variable representing the number of times observing yellow balls among 11 draws. Let $p = \frac{175}{200}$ and $q = 1 - p$. Since $X \sim B(11, p)$,

$$P(X = k) = \binom{11}{p} p^k q^{11-k}$$

   for $k = 0, 1, 2, \cdots, 11$.

   (b) Let X be a random variable representing the number of the draw on which the first yellow ball is obtained. Then $X \sim Geo(p)$. Thus

$$P(X \geq 5) = \sum_{k=5}^{\infty} P(X = k) = \sum_{k=5}^{\infty} pq^{k-1} = pq^4 \sum_{k=1}^{\infty} q^{k-1} = q^4.$$

   (c) Since $X \sim Geo(p)$,

$$E[X] = \frac{1}{p} = \frac{200}{175} = \frac{8}{7}.$$

   (d) Note that $k-1$ yellow balls have to be chosen among the first 10 draws and k-th yellow ball has to be drawn exactly at 11th draw. Therefore, the probability that the number of the draw on which the k-th yellow ball is obtained is 11 is

$$\binom{10}{k-1} p^{k-1} q^{10-(k-1)} \cdot p = \binom{10}{k-1} p^k q^{10-(k-1)}.$$

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3. (10 points) Five balls are randomly chosen, without replacement, from an urn that contains 5 red, 6 white, and 7 blue balls. Find the probability that at least one ball of each color is chosen.

Solution. Recall that

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) \\ - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C).$$

Let consider the following events:

$$A = \text{no red balls are chosen},$$
$$B = \text{no whilte balls are chosen},$$
$$C = \text{no blue balls are chosen}.$$

Then we know that

$$P(A) = \frac{\binom{5}{0}\binom{13}{5}}{\binom{18}{5}}, \qquad P(B) = \frac{\binom{6}{0}\binom{12}{5}}{\binom{18}{5}}, \qquad P(C) = \frac{\binom{7}{0}\binom{11}{5}}{\binom{18}{5}}.$$

Furthermore,

$$P(A \cap B) = \frac{\binom{11}{0}\binom{7}{5}}{\binom{18}{5}}, \qquad P(B \cap C) = \frac{\binom{13}{0}\binom{5}{5}}{\binom{18}{5}}, \qquad P(C \cap A) = \frac{\binom{12}{0}\binom{6}{5}}{\binom{18}{5}}.$$

And $P(A \cap B \cap C) = 0$. Therefore,

$$P(A \cup B \cup C) = \frac{\binom{5}{0}\binom{13}{5}}{\binom{18}{5}} + \frac{\binom{6}{0}\binom{12}{5}}{\binom{18}{5}} + \frac{\binom{7}{0}\binom{11}{5}}{\binom{18}{5}} - \frac{\binom{11}{0}\binom{7}{5}}{\binom{18}{5}} - \frac{\binom{13}{0}\binom{5}{5}}{\binom{18}{5}} - \frac{\binom{12}{0}\binom{6}{5}}{\binom{18}{5}}.$$

Since we are looking for $P(A^c \cap B^c \cap C^c)$, the probability of our interest is

$$P(A^c \cap B^c \cap C^c) = 1 - P(A \cup B \cup C) \\ = 1 - \left(\frac{\binom{5}{0}\binom{13}{5}}{\binom{18}{5}} + \frac{\binom{6}{0}\binom{12}{5}}{\binom{18}{5}} + \frac{\binom{7}{0}\binom{11}{5}}{\binom{18}{5}} - \frac{\binom{11}{0}\binom{7}{5}}{\binom{18}{5}} - \frac{\binom{13}{0}\binom{5}{5}}{\binom{18}{5}} - \frac{\binom{12}{0}\binom{6}{5}}{\binom{18}{5}} \right)$$

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4. (15 points) Let $X \sim B(n, p)$.

   (a) (10 points) Find the moment generating function (mgf) of X.

   (b) (5 points) Using the mgf of X, check that $E[X] = np$ and Var$[X] = np(1-p)$.

Solutions.

(a) By definition and the binomial identity, we have

$$m_X(t) = E[e^{tX}] = \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} e^{tk} = \sum_{k=0}^{n} \binom{n}{k} (pe^t)^k q^{n-k} = \left(pe^t + q\right)^n .$$

(b) Since

$$\frac{d^k}{d^k t} m_X(0) = \mu'_k = E[X^k],$$

we have

$$\begin{aligned} m'_X(0) &= n(pe^t+q)^{n-1}pe^t \mid_{t=0} = np = E[X], \\ m''_X(0) &= \left(n(n-1)(pe^t+q)^{n-2}p^2e^{2t} + n(pe^t+q)^{n-1}pe^t\right) \mid_{t=0} \\ &= n(n-1)p^2 + np = E[X^2]. \end{aligned}$$

Therefore, Var$[X] = E[X^2] - (E[X])^2 = n(n-1)p^2 + np - (np)^2 = np(1-p)$.

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5. (10 points) There are 15 pencils in a box, of which 9 have not previously been used. Three of the pencils are randomly chosen, used, and then returned to the box. Later, another 3 pencils are randomly chosen from the box. Find the probability that none of these pencils has ever been used.

Solution. Let E be the event that none of the later 3 pencils has ever been used. Let consider the following events:

$$\begin{aligned} A_0 &= \text{Among the first 3 chosen pencils, no pencils are previously used,} \\ A_1 &= \text{Among the first 3 chosen pencils, one pencil is previously used,} \\ A_2 &= \text{Among the first 3 chosen pencils, two pencils are previously used,} \\ A_3 &= \text{Among the first 3 chosen pencils, three pencils are previously used.} \end{aligned}$$

Then

$$P(E) = \sum_{i=0}^{3} P(E \cap A_i).$$

For each i, since

$$P(E \cap A_i) = \frac{\binom{6}{i}\binom{9}{3-i}}{\binom{15}{3}} \frac{\binom{6+(3-i)}{0}\binom{9-(3-i)}{3}}{\binom{15}{3}},$$

we have

$$P(E) = \sum_{i=0}^{3} \frac{\binom{6}{i}\binom{9}{3-i}}{\binom{15}{3}} \frac{\binom{6+(3-i)}{0}\binom{9-(3-i)}{3}}{\binom{15}{3}}.$$

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6. (10 points) A certain typing agency employs 2 typists. The average number of errors per article is 3 when typed by the first typist and 4.2 when typed by the second under Poisson distributions. If your article is equally likely to be typed by either typist, find the probability that it will have no errors.

Solution. Let E be the event that the number of errors is 0. Let T_1 be the event that the article is typed by the first typist and T_2 be the event that the article is typed by the second typist. Note that

$$P(E|T_1) = \frac{3^0}{0!}e^{-3}, \qquad P(E|T_2) = \frac{4.2^0}{0!}e^{-4.2}, \qquad P(T_1) = P(T_2) = \frac{1}{2}.$$

Since

$$P(E) = P(E|T_1)P(T_1) + P(E|T_2)P(T_2)$$

we conclude that

$$P(E) = \frac{1}{2}\left(e^{-3} + e^{-4.2}\right).$$

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7. (10 points) Urn A has 5 white and 7 black balls. Urn B has 3 white and 12 black balls. Urn C has 7 white and 14 black balls. We flip a fair die. If the outcome is 1 or 2, then a ball from Urn A selected. If the outcome is 3 or 4, then a ball from Urn B selected. If the outcome is 5 or 6, then a ball from Urn C selected. Suppose a ball is randomly drawn. Given a white ball is selected, what is the probability that the ball is **not** from Urn A?

   Solution. Let A, B, C be the events that a ball is from Urn A, B, C, respectively. Let W be the event that a white ball is selected. We want to compute $P(A^c|W)$. By using the total law of probability, we have

$$P(W) = P(W|A)P(A) + P(W|B)P(B) + P(W|C)P(C).$$

   Since $P(A) = P(B) = P(C) = 1/3$, and $P(W|A) = 5/12, P(W|B) = 3/15, P(W|C) = 7/21$, we conclude that

$$P(W) = \frac{5}{12} \cdot \frac{1}{3} + \frac{3}{15} \cdot \frac{1}{3} + \frac{7}{21} \cdot \frac{1}{3} = \frac{19}{60}.$$

   Since

$$\begin{aligned} P(A^c|W) &= \frac{P(A^c \cap W)}{P(W)} = \frac{P(B \cap W) + P(C \cap W)}{P(W)} \\ &= \frac{P(W|B)P(B) + P(W|C)P(C)}{P(W)} \end{aligned}$$

   we conclude that

$$P(A^c|W) = \frac{3/15 \cdot 1/3 + 7/21 \cdot 1/3}{19/60} = \frac{96}{171}.$$

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8. (15 points) Let f be the probability density function of a random variable X;

$$f(x) = (a + bx^2) \cdot \mathbb{1}_{[0,1]}(x)$$

where $\mathbb{1}_{[0,1]}(x)$ is the indicator function.

(a) (7.5 points) If $E[X] = 3/5$, find a and b.

(b) (7.5 points) Compute $P[X \geq 1/2]$ and $\text{Var}[X]$.

Solutions.

(a) Since $E[X] = 3/5$, we have

$$\frac{3}{5} = E[X] = \int_0^1 x(a + bx^2)dx = \frac{1}{2}a + \frac{1}{4}b.$$

Since $\int f(x)dx = 1$, we have

$$\int_0^1 f(x)dx = a + \frac{1}{3}b = 1.$$

By solving the linear system of equations, we have

$$a = \frac{3}{5}, \qquad b = \frac{6}{5}.$$

(b) By definition,

$$P[X \geq 1/2] = \int_{1/2}^1 \left(\frac{3}{5} + \frac{6}{5}x^2\right) dx = \frac{9}{4}.$$

For variance, since we know $E[X]$, it suffices to compute $E[X^2]$. Since

$$E[X^2] = \int_0^1 x^2(a + bx^2)dx = \frac{1}{3}a + \frac{1}{5}b = \frac{11}{25},$$

we have

$$\text{Var}[X] = E[X^2] - (E[X])^2 = \frac{11}{25} - \frac{9}{25} = \frac{2}{25}.$$

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