APMA 1650 Homework 04

Due: October 18, 2018

Due before class on Thursday, Oct 18, 2018. It can be dropped off in the APMA 1650 homework box on the first floor of the APMA department, 182 George St OR at class (before it starts).

Please attach the HW cover sheet to *the front of your HW* assignment. It can be found on Canvas. **Show all work and you MUST write up your own solutions.**

1. Recall 1D Random Walk (HW 3, Problem 3). Let X_n be a 1D random walk with $p = q = 0.5$. Let $Y_n = \frac{1}{\sqrt{n}} X_n$. Prove that

$$\lim_{n\to\infty} m_{Y_n}(t) = e^{t^2/2}.$$

   Hint: Let $g(n)$ and $h(n)$ be functions satisfying

$$\lim_{n\to\infty} g(n) = 0, \qquad \lim_{n\to\infty} h(n) = \infty.$$

   Then the following is true:

$$\lim_{n\to\infty} (1 + g(n))^{h(n)} = e^{\lim_{n\to\infty} g(n)h(n)}.$$

2. The proportion of time per day that all checkout counters in a supermarket are busy is a random variable X with pdf

$$f(x) = cx^2(1-x)^4 \cdot \mathbb{1}_{[0,1]}(x).$$

   (a) Find c which makes f a probability density function.

   (b) Find $E[X]$.

3. Let $f(x)$ be a piecewise constant function defined as follow:

$$f(x) = \mathbb{1}_{[0,1]}(x)$$

   where $\mathbb{1}_A(x)$ is the indicator function of A. Let X be a continuous random variable whose probability density function (pdf) is f.

   (a) Find the moment generating function of X.

   (b) Find a median. Is it unique?

   (c) Let Y be a continuous random variable whose pdf is

$$g(y) = \frac{1}{b-a} f\left(\frac{y-a}{b-a}\right), \qquad \text{for some } a < b.$$

   Find $E[Y]$ and Var$[Y]$. Note: Y is known as a uniform random variable in $[a, b]$. It is often referred to as $Y \sim \mathcal{U}(a, b)$.

4. Suppose you run a gym ball business having a special policy. If a customer make an order, you just randomly uniformly choose the size of a ball from 8 - 18 inches, (4 - 9 inches in terms of radius). That is, $R \sim \mathcal{U}(4, 9)$ representing the radius of a ball. What is the expected volume of a gym ball? Note that the volume of a ball of radius r is $\frac{4}{3}\pi r^3$.

5. (a) If a parachutist lands at a uniformly random point on a line between markers A and B, find the probability that she is closer to A than to B. Find the probability that her distance to A is more than twice her distance to B.

   (b) A second parachutist lands uniformly on the line segment between A and B. What is the probability that he lands within $\frac{|A-B|}{3}$ of the first parachutist?

6. Let X be a continuous random variable with density f. The density $f(x)$ is positive if $x \geq 0$ and 0 otherwise. If $F(x)$ is the cumulative distribution function of X, show that

$$E[X] = \int_0^\infty x f(x) dx = \int_0^\infty (1 - F(x))\, dx.$$

   Remark: It is worth to compare HW 2, Problem 6.