APMA 1650 Homework 4 Solutions

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Due October 18, 2018

1. We showed in homework 3 that if $Z_n \sim$Bi$(n, \frac{1}{2})$ then $X_n = 2Z_n - n$. From this, we get $Y_n = \frac{1}{\sqrt{n}}(2Z_n - n) = \frac{2}{\sqrt{n}}Z_n - \sqrt{n}$. Then if E_X denotes that we are taking the expectation with respect to X,

$$\begin{aligned}
m_{Y_n}(t) &= E_{Y_n}[e^{Y_n t}] \\
&= E_{Z_n}[e^{(\frac{2}{\sqrt{n}}Z_n - \sqrt{n})t}] \\
&= \sum_{k=0}^{n} \binom{n}{k} \left(\frac{1}{2}\right)^k \left(\frac{1}{2}\right)^{n-k} e^{(\frac{2}{\sqrt{n}}Z_n - \sqrt{n})t} \text{ since } Z_n \sim \text{Bi}(n, \frac{1}{2}) \\
&= e^{-t\sqrt{n}} \sum_{k=0}^{n} \binom{n}{k} (\frac{1}{2} e^{\frac{2t}{\sqrt{n}}})^k \left(\frac{1}{2}\right)^{n-k} \\
&= e^{-t\sqrt{n}} \left(\frac{1}{2} + \frac{1}{2} e^{\frac{2t}{\sqrt{n}}}\right)^n \text{ by the binomial theorem} \\
&= (e^{-t/\sqrt{n}})^n \left(\frac{1}{2} + \frac{e^{\frac{2t}{\sqrt{n}}}}{2}\right)^n \\
&= \left(\frac{e^{-t/\sqrt{n}} + e^{t/\sqrt{n}}}{2}\right)^n
\end{aligned}$$

Taking n to infinity and Taylor expanding, we get:

$$\begin{aligned}
\lim_{n\to\infty} m_{Y_n}(t) &= \lim_{n\to\infty} \left(\frac{e^{-t/\sqrt{n}} + e^{t/\sqrt{n}}}{2}\right)^n \\
&= \lim_{n\to\infty} \left(\frac{1 - \frac{t}{\sqrt{n}} + \frac{t^2}{2n} - \frac{t^3}{3!n^{3/2}} + \frac{t^4}{4!n^2} - \cdots + 1 + \frac{t}{\sqrt{n}} + \frac{t^2}{2n} + \frac{t^3}{3!n^{3/2}} + \frac{t^4}{4!n^2} + \ldots}{2}\right)^n \\
&= \left(1 + \frac{t^2}{n}\left(\frac{1}{2!} + \frac{t^2}{4!n} + \frac{t^4}{6!n^2} + \ldots\right)\right)^n \text{ from simplifying} \\
&= e^{\lim_{n\to\infty} t^2\left(\frac{1}{2!} + \frac{t^2}{4!n} + \frac{t^4}{6!n^2} + \ldots\right)} \text{ by the hint} \\
&= e^{t^2/2}
\end{aligned}$$

2. (a) Note that this distribution has the form of a Beta distribution with $\alpha = 3, \beta = 5$. Thus, the constant is $c = \frac{\Gamma(8)}{\Gamma(3)\Gamma(5)} = \frac{7!}{2!4!} = 105$.

   If you didn't see that this was a Beta distribution, we could find c by requiring that the area under f is equal to 1. Then,

$$\int_0^1 cx^2(1-x)^4 dx = c \cdot (\frac{1}{105}) \quad = 1 \text{ (via your choice of integration method)}$$
$$c = 105$$

   (b) Since $X \sim \beta(3,5)$, $E[X] = \frac{3}{3+5} = \frac{3}{8}$. If you didn't notice this connection, you can find $E[X]$ by integration: $E[X] = \int_0^1 105x^3(1-x)^4 dx$

$$= \frac{3}{8}$$

3. (a) By definition,

$$\begin{aligned} m_X(t) &= \int_0^1 e^{tx} \cdot f(x)dx \\ &= \int_0^1 e^{tx} dx \\ &= \frac{e^{tx}}{t}\Big|_0^1 \\ &= \frac{e^t - 1}{t} \end{aligned}$$

   (b) To find the median, the area under f must be divided into 2 parts of area $\frac{1}{2}$. Thus, we must find m such that $\int_0^m f(x)dx = \frac{1}{2}$.

$$\begin{aligned} \int_0^m dx &= \frac{1}{2} \\ x\Big|_0^m &= \frac{1}{2} \\ m &= \frac{1}{2} \end{aligned}$$

   This median is unique, as the cdf of $f(x)$, which is $\int f(x)dx = x, x \in [0,1]$ is continuous and strictly increasing on $[0,1]$.

   (c) Notice that for $y \in [a,b]$ and $x \in [0,1]$, $\frac{y-a}{b-a} = x$. Thus, $f(\frac{y-a}{b-a}) = f(x)$ and $(b-a) \cdot g(y) = f(x) = 1$ for $y \in [a,b]$. Therefore, $g(y) = \frac{1}{b-a}$ for $y \in [a,b]$

$$\begin{aligned} E[Y] &= \int_a^b y \cdot g(y)dy \\ &= \int_a^b y \cdot \frac{1}{b-a} \cdot 1dx \\ &= \frac{y^2}{2(b-a)}\Big|_a^b = \frac{b^2 - a^2}{2(b-a)} \\ &= \frac{b+a}{2} \end{aligned}$$

$$
\begin{aligned}
E[Y^2] &= \int_a^b y^2 \cdot \frac{1}{b-a} \cdot 1dx \\
&= \frac{y^3}{3(b-a)} a^b = \frac{b^3 - a^3}{3(b-a)} \\
&= \frac{b^2 + ab + a^2}{3} \\
Var[Y] &= E[Y^2] - E[Y]^2 \\
&= \left(\frac{b^2 + ab + a^2}{3}\right) - \left(\frac{b+a}{2}\right)^2 \\
&= \left(\frac{b^2 + ab + a^2}{3}\right) - \left(\frac{b^2 + 2ab + a^2}{4}\right) \\
&= \frac{b^2 - 2ab + a^2}{12} \\
&= \frac{(b-a)^2}{12}
\end{aligned}
$$

4. Let V represent the volume of the ball. The pdf for R is $f(r) = \frac{1}{9-4} = \frac{1}{5}, r \in [4, 9]$.

$$
\begin{aligned}
E[V] &= \int_4^9 V \cdot f(r)dr = \int_4^9 \left(\frac{4}{3}\pi r^3\right)\left(\frac{1}{5}\right) dr \\
&= \frac{4\pi}{15} \cdot \int_4^9 r^3 dr \\
&= \frac{4\pi}{15} \cdot \left(\frac{r^4}{4}\right)\Big|_4^9 \\
&= \frac{1261\pi}{3} \approx 1320.52 \text{ in.}^3
\end{aligned}
$$

5. Without loss of generality, assume $A < B$
(a) To find the probability of landing closer to A than to B, note that there are only 2 possibilities: closer to A or closer to B. Both occurrences are equally likely, so the probability is equal to $\frac{1}{2}$.
Call the point where the parachutist lands point X. If X is twice as far from A than from B, X must be at least $\frac{2}{3}$ of the way from A to B.

Clearly, X must be in the last third of the segment from A to B. Thus, the probability is equal to $\frac{1}{3}$.

We can also approach this problem in a purely probabilistic way. The probability that X

lands closer to A than to B is $P(X - A < B - X) = P\left(X < \frac{A+B}{2}\right)$

$$= \int_A^{\frac{A+B}{2}} \frac{1}{B-A} dx$$

$$= \frac{1}{2}$$

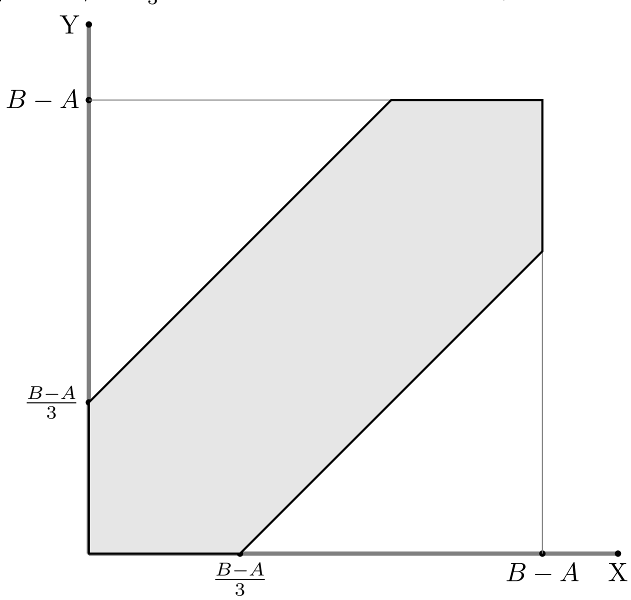
Similarly, the probability that X is more than twice as far from A as from B is

$$P(X - A > 2(B - X)) = P\left(X > \frac{2B+A}{3}\right)$$

$$= \int_{\frac{2B+A}{3}}^{B} \frac{1}{B-A} dx$$

$$= \frac{1}{3}.$$

(b) Let's call the location of the second parachutist Y. For Y to be within $\frac{B-A}{3}$ of X, $|X - Y| \leq \frac{B-A}{3}$. We can plot this inequality on a two-dimensional plane:



The total set of outcomes is represented by the square of size $(B - A) \times (B - A)$, and the desired set of outcomes is represented by the filled-in area of the inequality. Via simple geometry, the area of the inequality is equal to $\frac{5(B-A)^2}{9}$. Thus, the desired probability is equal to $\frac{\frac{5(B-A)^2}{9}}{(B-A)^2} = \frac{5}{9}$.

6. This problem is similar to HW2 Question 6, but with continuous variables instead of discrete ones. Note that the variable x is equal to $\int_0^x dt$ for some dummy variable t. Thus,

$$
\begin{aligned}
E[X] &= \int_0^{\infty} x f(x) dx \\
&= \int_0^{\infty} \int_0^{x} dt \cdot f(x) dx \\
&= \int_0^{\infty} \int_t^{\infty} f(x) dx dt \text{ switching the order of integration by Fubini/Tonelli's Theorem for positive } f(x) \\
&= \int_0^{\infty} F(x)\Big|_t^{\infty} dt \text{ by definition of } F(x) \\
&= \int_0^{\infty} 1 - F(t) dt, \text{ as } F(\infty) = 1 \text{ by definition}
\end{aligned}
$$

And we are done.

An alternate proof without double integration can be done in the following manner using integration by parts:

$$
\begin{aligned}
\int_0^{\infty} (1 - F(x)) dx &= x(1 - F(x))\Big|_0^{\infty} + \int_0^{\infty} x f(x) dx \text{ since } \frac{d}{dx} F(x) = f(x) \\
&= 0 + E[X]
\end{aligned}
$$

where the first term is 0 because
$$
\begin{aligned}
x(1 - F(x))\Big|_0^{a} &= a(1 - F(a)) \\
&= a \int_a^{\infty} f(x) dx \\
&\leq \int_a^{\infty} x f(x) dx \\
&\to 0 \text{ as } a \to \infty
\end{aligned}
$$