ENGN2020 – HOMEWORK3

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Problem 2(K8-4-5)

Matrix A $=\begin{bmatrix}-5 & 0 & 15\\ 3 & 4 & -9\\ -5 & 0 & 15\end{bmatrix}$, Matrix P = $\begin{bmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}$

Then, P^{-1} = $\begin{bmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}$, therefore $\widehat{\mathrm{A}} = P^{-1}AP = \begin{bmatrix}4 & 3 & -9\\ 0 & -5 & 15\\ 0 & -5 & 15\end{bmatrix}$

For matrix A, the characteristic determinant gives the characteristic equation:

$$(4-\lambda)(-5-\lambda)(15-\lambda)+300=0$$

Then we get: $$\lambda^3 - 14\lambda^2 + 40\lambda = 0$$

The roots, which are also eigenvalues of A, are $\lambda_1 = 0,\ \lambda_2 = 10,\ \lambda_3 = 4$

For $\lambda = 0$:

$$A - \lambda\mathrm{I} = \mathrm{A} = \begin{bmatrix}-5 & 0 & 15\\ 3 & 4 & -9\\ -5 & 0 & 15\end{bmatrix} = \begin{bmatrix}-5 & 0 & 15\\ 3 & 4 & -9\\ 0 & 0 & 0\end{bmatrix} = \begin{bmatrix}-5 & 0 & 15\\ 3 & 4 & -9\\ 0 & 0 & 0\end{bmatrix} = \begin{bmatrix}-5 & 0 & 15\\ 0 & 4 & 0\\ 0 & 0 & 0\end{bmatrix}$$

Let $x_1 = 3$, then $x_2 = 0,\ x_3 = 1$

So, an eigenvector of A corresponding to $\lambda = 0$ is $[3 \quad 0 \quad -1]^T$

For $\lambda = 10$:

$$A - \lambda\mathrm{I} = \begin{bmatrix}-15 & 0 & 15\\ 3 & -6 & -9\\ -5 & 0 & 5\end{bmatrix} = \begin{bmatrix}-5 & 0 & 5\\ 1 & -2 & -3\\ 0 & 0 & 0\end{bmatrix} = \begin{bmatrix}1 & -2 & -3\\ -5 & 0 & 5\\ 0 & 0 & 0\end{bmatrix} = \begin{bmatrix}1 & -2 & -3\\ 0 & -10 & -10\\ 0 & 0 & 0\end{bmatrix}$$

Let $x_3 = 1$, then $x_1 = 1,\ x_2 = -1$

So, an eigenvector of A corresponding to $\lambda = 10$ is $[1 \quad -1 \quad 1]^T$

For $\lambda = 4$:

$$A - \lambda\mathrm{I} = \begin{bmatrix}-9 & 0 & 15\\ 3 & 0 & -9\\ -5 & 0 & 11\end{bmatrix} = \begin{bmatrix}-5 & 0 & 11\\ 3 & 0 & -9\\ 0 & 0 & -12\end{bmatrix}$$

$x_3 = 0$, and $x_1 = 0$, x_2 can be anything, let $x_2 = 1$

So, an eigenvector of A corresponding to $\lambda = 4$ is $[0 \quad 1 \quad 0]^T$

For matrix $\widehat{\mathrm{A}}$, the characteristic determinant gives the characteristic equation: $\lambda^3 - 14\lambda^2 + 40\lambda = 0$

The roots, which are the same eigenvalues of A, are $\lambda_1 = 0,\ \lambda_2 = 10,\ \lambda_3 = 4$

For $\lambda = 0$:

$$\widehat{\mathrm{A}} - \lambda\mathrm{I} = \mathrm{A} = \begin{bmatrix}4 & 3 & -9\\ 0 & -5 & 15\\ 0 & -5 & 15\end{bmatrix} = \begin{bmatrix}4 & 3 & -9\\ 0 & -5 & 15\\ 0 & 0 & 0\end{bmatrix}$$

Let $x_2 = 3$, then $x_3 = 1,\ x_1 = 0$

So, an eigenvector of $\widehat{\mathrm{A}}$ corresponding to $\lambda = 0$ is $[0 \quad 3 \quad 1]^T$

For $\lambda = 10$:

$$\widehat{\mathrm{A}} - \lambda\mathrm{I} = \begin{bmatrix}-6 & 3 & -9\\ 0 & -15 & 15\\ 0 & -5 & 5\end{bmatrix} = \begin{bmatrix}-2 & 1 & -3\\ 0 & -5 & 5\\ 0 & 0 & 0\end{bmatrix}$$

Let $x_2 = 1$, then $x_3 = 1,\ x_1 = -1$

So, an eigenvector of $\widehat{\mathrm{A}}$ corresponding to $\lambda = 10$ is $[-1 \quad 1 \quad 1]^T$

For $\lambda = 4$:

$$\widehat{\mathrm{A}} - \lambda \mathrm{I} = \begin{bmatrix} 4 & 3 & -9 \\ 0 & -5 & 15 \\ 0 & -5 & 15 \end{bmatrix} = \begin{bmatrix} 0 & 3 & 9 \\ 0 & -9 & 15 \\ 0 & -5 & 11 \end{bmatrix}$$

$x_2 = 0$, and $x_3 = 0$, x_2 can be anything, let $x_1 = 1$

So, an eigenvector of $\widehat{\mathrm{A}}$ corresponding to $\lambda = 4$ is $[1 \quad 0 \quad 0]^T$

It can be calculated that $[0 \quad 3 \quad 1]^T = P[3 \quad 0 \quad -1]^T$

$$[-1 \quad 1 \quad 1]^T = P[1 \quad -1 \quad 1]^T$$

$$[1 \quad 0 \quad 0]^T = P[0 \quad 1 \quad 0]^T$$

Problem 4(K20-7-1 to K20-7-6)

(a) K20-7-1:

$$\mathrm{A} = \begin{bmatrix} 5 & 2 & 4 \\ -2 & 0 & 2 \\ 2 & 4 & 7 \end{bmatrix}$$

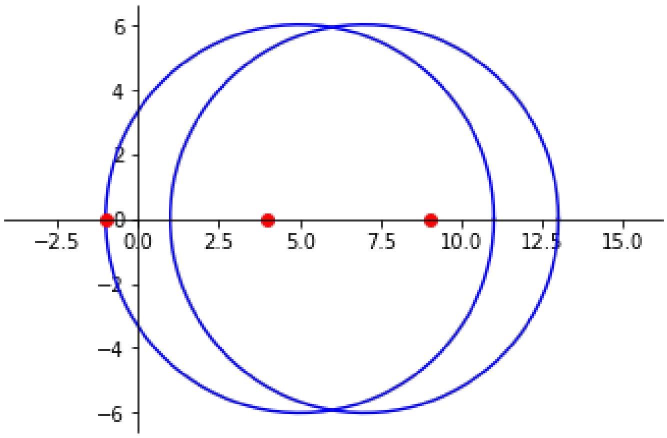

Figure 1: Gerschgorin disks of K20-7-1

(b) K20-7-2

$$\mathrm{A} = \begin{bmatrix} 5 & 0.01 & 0.01 \\ 0.01 & 8 & 0.01 \\ 0.01 & 0.01 & 9 \end{bmatrix}$$

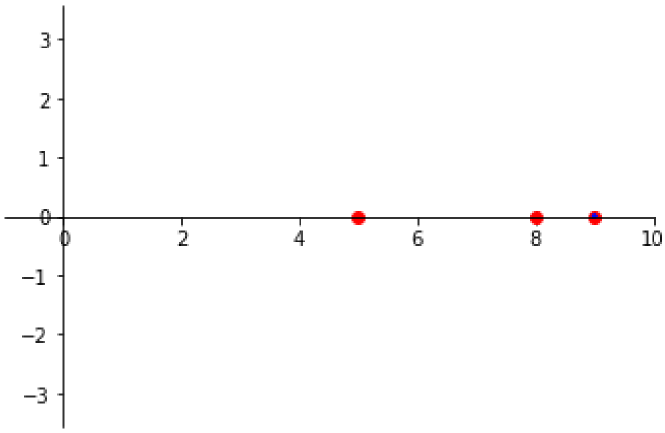

Figure 1: Gerschgorin disks of K20-7-2

(c) K20-7-3:

$$\mathrm{A} = \begin{bmatrix} 0 & 0.4 & -0.1 \\ -0.4 & 0 & 0.3 \\ 0.1 & -0.3 & 0 \end{bmatrix}$$

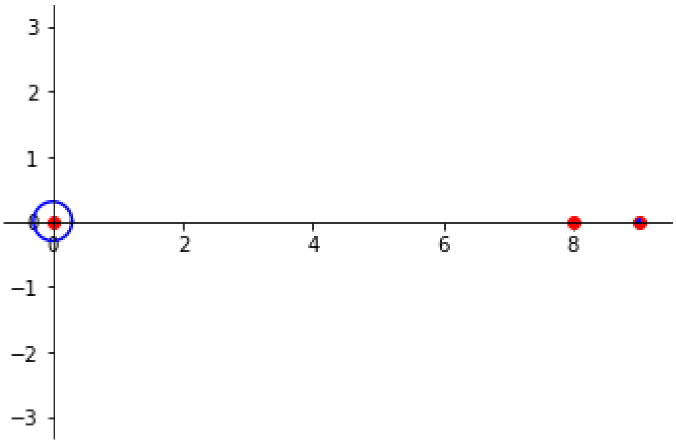

Figure 3: Gerschgorin disks of K20-7-3

(d) K20-7-4:

$$\mathrm{A} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 4 & 3 \\ 1 & 3 & 12 \end{bmatrix}$$

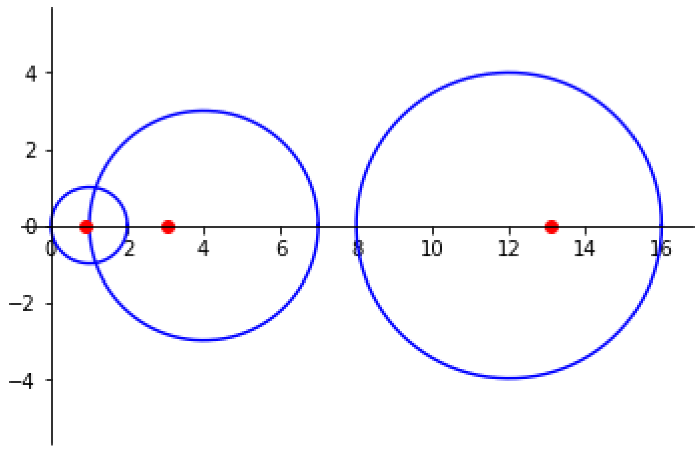

Figure 4: Gerschgorin disks of K20-7-4

(e) K20-7-5:

$$A = \begin{bmatrix} 2 & i & 1+i \\ -i & 3 & 0 \\ 1-i & 0 & 8 \end{bmatrix}$$

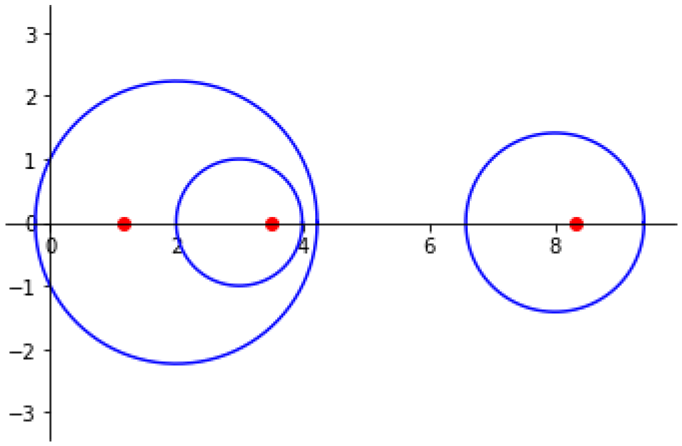

Figure 5: Gerschgorin disks of K20-7-5

(f) K20-7-6:

$$A = \begin{bmatrix} 10 & 0.1 & -0.2 \\ 0.1 & 6 & 0 \\ -0.2 & 0 & 3 \end{bmatrix}$$

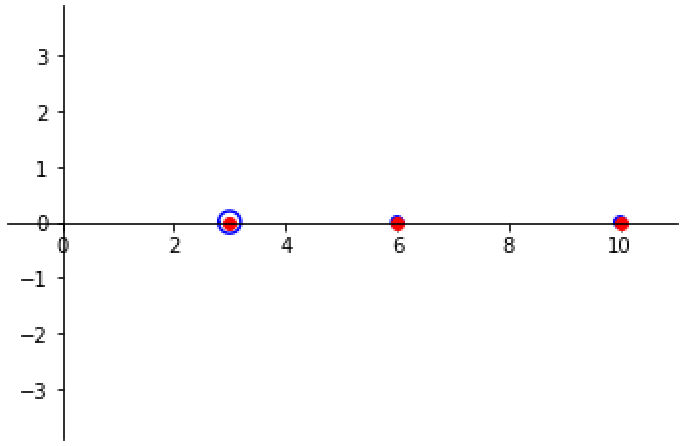

Figure 6: Gerschgorin disks of K20-7-6

Problem 5(K8-4-6)

(a)Trace:

Let A be a $n \times n$ matrix, $trace\ A = a_{11} + a_{22} + \cdots + a_{nn}$.

For $A = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$, trace A = 0, its eigen values are 5 and -5, the trace equals the sum of the eigenvalues.

For $A = \begin{bmatrix} 8 & -4 \\ 2 & 2 \end{bmatrix}$, trace A = 10, its eigen values are 6 and 4, the trace equals the sum of the eigenvalues.

For $A = \begin{bmatrix} -5 & 0 & 15 \\ 3 & 4 & -9 \\ -5 & 0 & 15 \end{bmatrix}$, trace A = 14, its eigen values are 0, 4 and 10, the trace equals the sum of the eigenvalues.

(b)Trace of product:

Proof:

Let A be a $n \times n$ matrix, $trace\ A = a_{11} + a_{22} + \cdots + a_{nn}$.

Let B be a $n \times n$ matrix, $trace\ B = b_{11} + b_{22} + \cdots + b_{nn}$.

$AB = \begin{bmatrix} c_{11} & \cdots & c_{n1} \\ \vdots & \ddots & \vdots \\ c_{1n} & \cdots & c_{nn} \end{bmatrix}$, where $c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$, then $trace\ AB = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} b_{ji}$

$BA = \begin{bmatrix} d_{11} & \cdots & d_{n1} \\ \vdots & \ddots & \vdots \\ d_{1n} & \cdots & d_{nn} \end{bmatrix}$, where $d_{ij} = \sum_{k=1}^{n} b_{ik} a_{kj}$,then $trace\ BA = \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij} a_{ji}$

Therefore, $trace\ AB = trace\ BA$

(c) Relationship between $\boldsymbol{PAP^{-1}}$ and $\boldsymbol{P^{-1}AP}$:

Let A and P be $n \times n$ matrices

Let $\check{A} = PAP^{-1}$ and $\hat{A} = P^{-1}AP$

$\check{A} = PPP^{-1}APP^{-1}P^{-1} = PP\hat{A}P^{-1}P^{-1} = P^2\hat{A}P^{-2}$

(d) Diagonalization:

To change the order of the eigenvalues in D, we should interchange the corresponding eigenvectors in matrix X in $D = XAX^{-1}$

Problem 6(K8-5-1 to K8-5-6)

(a) K8-5-1:

$A = \begin{bmatrix} -6 & i \\ -i & 6 \end{bmatrix}$, then $\bar{A} = \begin{bmatrix} -6 & -i \\ i & 6 \end{bmatrix}$, $\bar{A}^T = \begin{bmatrix} -6 & i \\ -i & 6 \end{bmatrix} = A$, so A is a Hermitian matrix.

According to Theorem 1, the eigenvalues are real numbers.

In fact, the eigenvalues are -6.08276253 and 6.08276253

(b) K8-5-2:

$A = \begin{bmatrix} i & 1+i \\ -1+i & 0 \end{bmatrix}$, then $\bar{A} = \begin{bmatrix} -i & 1-i \\ -1-i & 0 \end{bmatrix}$, $\bar{A}^T = \begin{bmatrix} -i & -1-i \\ 1-i & 0 \end{bmatrix} = -A$, so A is a skew-Hermitian matrix.

According to Theorem 1, the eigenvalues are pure imaginary numbers or zero.

In fact, the eigenvalues are $2i$ and $-i$

(c) K8-5-3:

$A = \begin{bmatrix} 1/2 & i\sqrt{3/4} \\ i\sqrt{3/4} & 1/2 \end{bmatrix}$, then $\bar{A} = \begin{bmatrix} 1/2 & -i\sqrt{3/4} \\ -i\sqrt{3/4} & 1/2 \end{bmatrix}$, $\bar{A}^T = \begin{bmatrix} 1/2 & -i\sqrt{3/4} \\ -i\sqrt{3/4} & 1/2 \end{bmatrix} = A^{-1}$, so A is a unitary matrix.

According to Theorem 1, the eigenvalues have absolute value 1

In fact, the eigenvalues are$1/2 + i\sqrt{3/4}$ and $1/2 - i\sqrt{3/4}$

(d) K8-5-4:

$A = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$, then $\bar{A} = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}$, $\bar{A}^T = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix} = -A$, so A is a skew-Hermitian matrix.

According to Theorem 1, the eigenvalues are pure imaginary numbers or zero.

In fact, the eigenvalues are i and $-i$

(e) K8-5-5:

$A = \begin{bmatrix} i & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{bmatrix}$, then $\bar{A} = \begin{bmatrix} -i & 0 & 0 \\ 0 & 0 & -i \\ 0 & -i & 0 \end{bmatrix}$, $\bar{A}^T = \begin{bmatrix} -i & 0 & 0 \\ 0 & 0 & -i \\ 0 & -i & 0 \end{bmatrix} = -A$, so A is a skew-Hermitian matrix.

According to Theorem 1, the eigenvalues are pure imaginary numbers or zero.

In fact, the eigenvalues are i, i and $-i$

(f) K8-5-6:

$A = \begin{bmatrix} 0 & 2+2i & 0 \\ 2-2i & 0 & 2+2i \\ 0 & 2-2i & 0 \end{bmatrix}$ then $\bar{A} = \begin{bmatrix} 0 & 2-2i & 0 \\ 2+2i & 0 & 2-2i \\ 0 & 2+2i & 0 \end{bmatrix}$, $\bar{A}^T = \begin{bmatrix} 0 & 2+2i & 0 \\ 2-2i & 0 & 2+2i \\ 0 & 2-2i & 0 \end{bmatrix} = A$, so A is a Hermitian matrix.

According to Theorem 1, the eigenvalues are real numbers.

In fact, the eigenvalues are 4, 0 and -4.

Problem 7

$f(x) = e^x - 3x - 5 = 0$

(a) Fixed-point iteration:

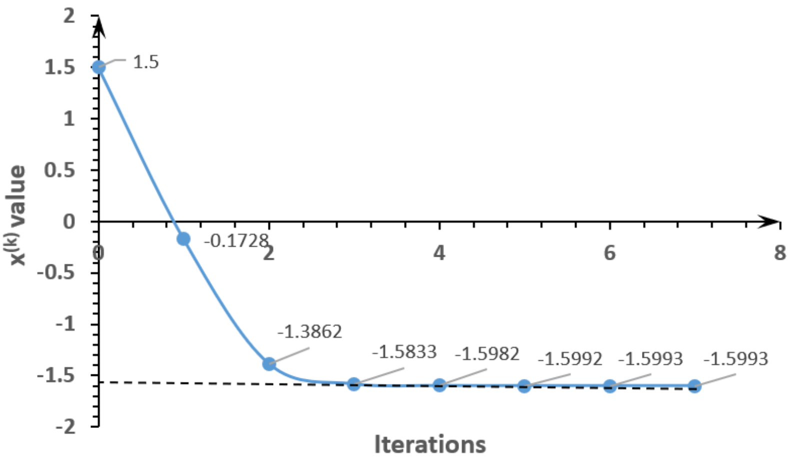

Figure 7: Fixed point iteration

(b)Newton-Raphson iteration:

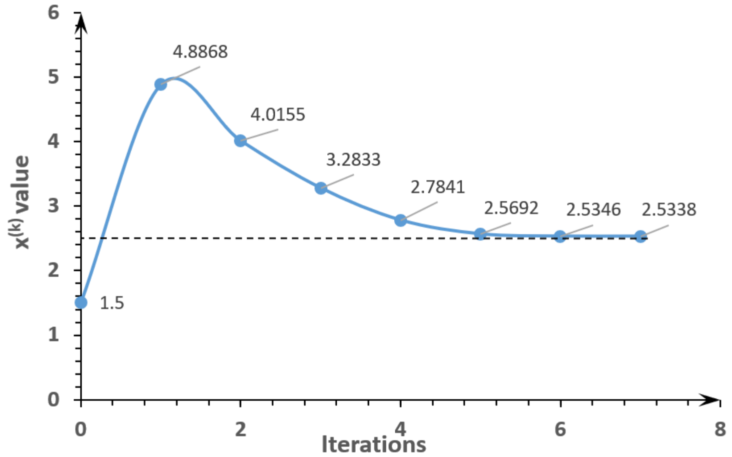

Figure 8: Newton-Raphson iteration

Problem 8

$f(x) = \sin x + \frac{x}{10} = 0$

(a) Plots and roots

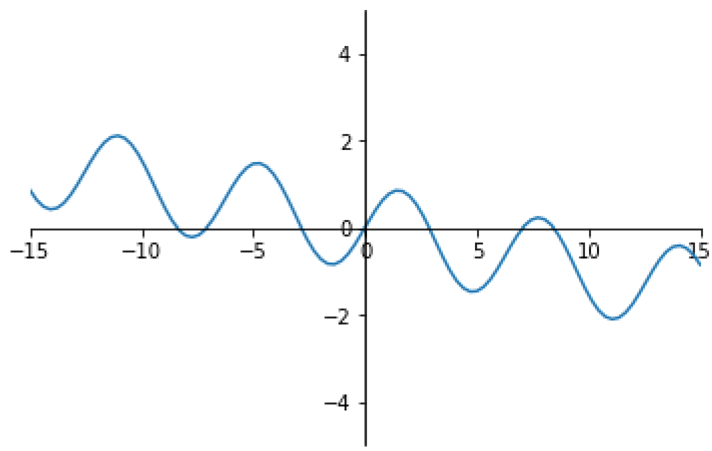

Figure 9: Plot of $f(x) = 0$

There are seven roots. The roots are -8.4232, -7.0681, - 2.8523, 0, 2.8523, 7.0681, 8.4232.

(b) Factor out one root

Choose 8.4232 as the root to be factored out.

Then $g(x) = f(x)/(x - 8.4232)$

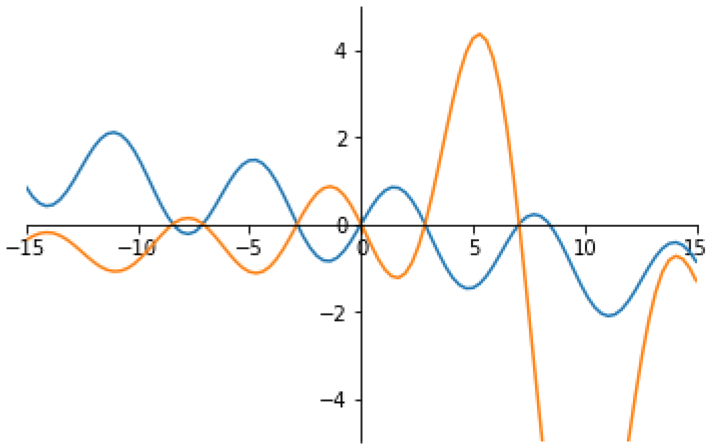

Figure 10: Plot of $f(x) = 0$ and $g(x) = 0$. Blue line for $f(x)$, and yellow line for $g(x)$.

For $g(x)$, there are six roots. The roots are -8.4232, -7.0681, - 2.8523, 0, 2.8523, and 7.0681. They are the same as the roots of $f(x)$, except for $x = 8.4232$

(c) Factor out three roots

Choose 8.4232, -7.0681 and 2.8523 as the roots to be factored out.

Then $g(x) = f(x)/(x - 8.4232)(x + 7.0681)(x - 2.8523)$

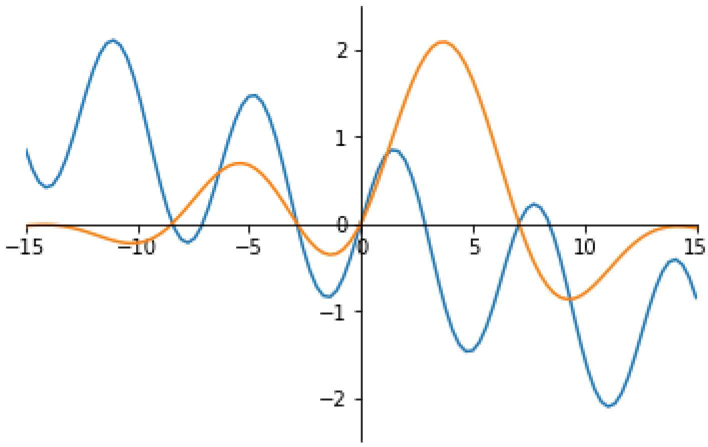

Figure 11: Plot of $f(x) = 0$ and $g(x) = 0$. Blue line for $f(x)$, and yellow line for $g(x)$.

For $g(x)$, there are four roots. The roots are -8.4232, - 2.8523, 0, and 7.0681. They are the same as the roots of $f(x)$, except for $x =$ 8.4232, -7.0681 and 2.8523.

(d) Problems

1.For some equations, x = 0 can be a possible root. However, it`s unable with that root.

2.By factoring out some roots, the function g(x) may become quite complicated, and it`s even harder to get the derivative.