APMA 1650 Homework 05

Due: October 25, 2018

Due before class on Thursday, Oct 25, 2018. It can be dropped off in the APMA 1650 homework box on the first floor of the APMA department, 182 George St OR at class (before it starts).

Please attach the HW cover sheet to *the front of your HW* assignment. It can be found on Canvas. **Show all work and you MUST write up your own solutions.**

1. Let Z be a standard normal random variable, i.e., $Z \sim \mathcal{N}(0, 1)$. Find the value z_0 such that
   (a) $P(Z > z_0) = 0.5$.
   (b) $P(Z < z_0) = 0.8643$.
   (c) $P(-z_0 < Z < z_0) = 0.90$.
   (d) $P(-z_0 < Z < z_0) = 0.99$.

   Use the table 4 in the Appendix 3 of the textbook (page. 848) or see Canvas.

2. An electrical firm manufactures light bulbs that have a life, before burn-out, that is normally distributed with mean equal to 800 hours and a standard deviation of 40 hours. Find the probability that a bulb burns between 778 and 834 hours.

3. (Chernoff bounds) Let X be a random variable and $m_X(t)$ be the mgf of X. Show that

$$P(X \geq a) \leq e^{-ta} m_X(t) \quad \text{for all} \quad t > 0,$$
$$P(X \leq a) \leq e^{-ta} m_X(t) \quad \text{for all} \quad t < 0.$$

   Hint: Use Markov's inequality.

4. A machine used to fill cereal boxes dispenses, on the average, μ ounces per box. The manufacturer wants the actual ounces dispensed X to be within 1 ounce of μ at least 75% of the time. What is the largest value of σ, the standard deviation of X, that can be tolerated if the manufacturer's objectives are to be met?

5. The gamma function is defined to be

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx.$$

   (a) Using the integration by parts, show that

$$\Gamma(z + 1) = z\Gamma(z).$$

(b) By using the above relation, show that for any positive integer n,

$$\Gamma(n+1) = n!.$$

6. Let consider two pdfs

$$f_1(x) = \mathbb{1}_{[0,1]}(x), \qquad f_2(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}.$$

For some $0 < \alpha < 1$, let define

$$f(x) = \alpha f_1(x) + (1-\alpha) f_2(x).$$

(a) Show that $f(x)$ is a probability density function.

(b) Let X_1 be a random variable whose pdf is $f_1(x)$ and X_2 be a random variable whose pdf is $f_2(x)$ where

$$E[X_1] = \mu_1, \quad \text{Var}[X_1] = \sigma_1^2, \qquad E[X_2] = \mu_2, \quad \text{Var}[X_2] = \sigma_2^2.$$

Let X be a random variable whose pdf is $f(x)$. Find $E[X]$ and $\text{Var}[X]$.