APMA 1650: Homework 2 Solutions

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1. Let A, B, and C be events such that $P(A) > P(B)$ and $P(C) > 0$.

   Note: There are many possible correct answers for this question.

   (a) Construct an example to demonstrate that it is possible that $P(A \mid C){<}P(A)$.

   We will be creating an experiment where we randomly pick from 100 students who have filled out a survey. The survey had three questions. Each question asked whether the student disliked class A, B and C. Each of these answers is independent of the others, as the students are allowed say they enjoy as many classes as they want. Our sample space will each possible combination of classes a student can enjoy (A, B, C, A and B, B and C, A and C, and A, B,and C). Our probability function is just dividing the the number of students who selected a specific event by 100. This means our function represents the likelihood that we pick a student who enjoyed a certain class or combination of classes from the 100 surveyed. For example, $P(A) = .2$ when 20 out of 100 students enjoyed class A.

   Our function and sample space satisfy the three axioms of probability. First, we know that regardless of the number of students who select a combination of the classes, the number can not be negative, meaning our probability function will always be greater than zero. Secondly, we know that each unique student is represented by one of our seven possible combinations. While our enjoying classes A, B, and C are not mutually exclusive, the probability function we defined does not double count these events when a student enjoys more than one. If we add up all of the sample space outcomes and divide by 100, we will get $P(S) = 1$, which satisfies our second axiom. Lastly, if no students responds favorably to two specific classes A and B, the probability of picking a student who enjoyed A or B will be equal to the probability of selecting a student who enjoyed A plus the probability of selecting a student who enjoyed B. This property will hold for any combination of A, B, and C, meaning we have satisfied our third axiom of probability.

   We will be using this same sample space throughout, but we will change the values of A, B, and C throughout. These changes will be described as specific probabilities in each excercise.

   Let $P(A) = .6$, $P(C) = .4$, and $P(A \cap C) = .2$ We know that $P(A \mid C) = \frac{P(A \cap C)}{P(C)}$ Thus, when we plug our numbers in the above expression and get $P(A \mid C) = .5$, which is less than $P(A) = 0.6$.

   (b) Construct an example to demonstrate that it is possible that $P(A \mid C){>}P(A)$.

   Let $P(A) = 0.5$, $P(C) = 0.2$, and $P(A \cap C) = 0.2$ We know that $P(A \mid C) = \frac{P(A \cap C)}{P(C)}$ Thus, when we plug our numbers in the above expression and get $P(A \mid C) = 1$, which is greater than $P(A) = 0.5$.

   (c) Construct an example to demonstrate that it is possible that $P(A \mid C){<}P(B \mid C)$

   Let Let $P(A) = 0.5$, $P(B) = 0.45$ $P(C) = 0.4$, $P(A \cap C) = 0.15$, $P(A \cap B) = 0$, $P(B \cap C) = 0.2$, and $P(A \cap B \cap C) = 0$ We know that $P(A \mid C) = \frac{P(A \cap C)}{P(C)}$ and that $P(B \mid C) = \frac{P(B \cap C)}{P(B)}$. Once we plug in our values of for A, B, and C, we get $P(A \mid C) = \frac{.15}{.4} = \frac{3}{8}$, and $P(B \mid C) = \frac{.2}{.4} = \frac{1}{2}$ Because $\frac{1}{2}{>}\frac{3}{8}$, we have satisfied all of the conditions of the example.

2. Suppose 1 in 1000 people carry a disease. There is a diagnostic test for the disease and it has the following accuracy. If an individual carries the disease, the test correctly detects this 99% of the time.

If an individual does not carry the disease, the test incorrectly reads positive with probability 5%. Suppose a person is given the test and the outcome is positive. What is the probability that he/she carries the disease? (Does the answer surprise you?)

In order to solve this problem, we are going to want to use Bayes' Rule:

$$P(A \mid B) = \frac{P(B \mid A)P(A)}{P(B)}$$

We can begin by identifying the four possible combinations of events. If a patient has the disease, the event will be denoted at D, with $P(D) = .001$, whereas if they are healthy, it will be denoted as H, with $P(H) = .999$. If they test positively for the disease, the event is labeled as P with $P(P) = .99 or .05$ depending on if the person has the disease, whereas a negative test is the event N with $P(N) = .01 or .95$ depending on if the person has the disease.

The probability we seek is $P(D \mid P)$. In order to get this solution using Bayes' rule, we need $P(P \mid D)$, $P(P)$ and $P(D)$. $P(P \mid D) and P(D)$ are both provided in the problem, so we just need to solve for $P(P)$. We know that $P(P)$ can be written as the sum of all probabilities where event P occurs. There are two possibilities where the person tests positive. This includes being diseased and testing positive or being healthy a testing positive. This equals $P(P \mid D)P(D) + P(P \mid H)P(H)$.

$$P(D \mid P) = \frac{P(P \mid D)P(D)}{P(P \mid D)P(D) + P(P \mid H)P(H)}$$

$$P(D \mid P) = \frac{.99 * .001}{.99 * .001 + .05 * .999} = .01943$$

3. Suppose we have identical biased two coins with the probability of Heads p and the probability of Tails 1 p. Let flip both. If the coins match, we record nothing and flip both coins again. If the coins do not match, we stop and record the outcome of the second. What is the probability of recording Heads?

   The probability of recording a single head on the second flip on any one set of flips is $p(1-p)$. The probability of not recording anything in one set is $(p^2 + (1-p)^2)$. Knowing these two facts, we can write the following expression to express all possibilities where heads is recorded:

   $(p^2 + (1-p)^2)^k * p(1-p)$

   where k is the number of times the two coins are flipped.

   In order to get the probability of heads being recorded, we must sum this expression for all values of k. This expression is:

$$\sum_{k=0}^{\infty} (p^2 + (1-p)^2)^k * p(1-p)$$

   This sum (which is of a geometric series) equals:

$$\frac{p(1-p)}{1 - (p^2 + (1-p)^2)}$$

   This can be expanded to:

$$\frac{p(1-p)}{1 - p^2 - 1 + 2p - p^2}$$

   Which simplifies to:

$$\frac{p - p^2}{2p - 2p^2}$$

This can be simplified to

$$\frac{1}{2}$$

4. Two people play a game. They take turns flipping a 'fair' coin, the first to flip H wins. A round constitutes just one flip (only one person gets to "go".)

   (a) What is the probability that the second player wins?

   In order for the second player to win, the coin must be flipped an even number of times, and every flip before the last flip must be tails.

   A way to express this would be that player two's probability of winning is for the 2kth round is $(\frac{1}{2})^{2k}$. This statement holds because it implies a tail has been flipped 2k-1 times and a head is flipped once at the end with each event occurring with a probability of $\frac{1}{2}$.

   In order to determine the probability of winning for Player 2, we must sum up this probability for all k. This sum can be written as a geometric sum following the form:

$$\sum_{k=1}^{\infty}(\frac{1}{2})^{2k}$$

   Which simplifies to upon squaring our inside term to drop down the exponent:

$$\sum_{k=1}^{\infty}(\frac{1}{4})^{k}$$

   Because this is a geometric series, we can evaluate the sum as long as we are careful about subtracting the k=0 term since our sum starts at k=1. This leads us to:

$$P(P2Victory) = \frac{1}{1 - \frac{1}{4}} - (\frac{1}{4})^0 = \frac{4}{3} - 1 = \frac{1}{3}$$

   (b) What is the probability that the game lasts at least 4 rounds?

   The probability that the game lasts at least 4 games is the same as saying that we have flipped 3 tails in a row with a fair coin, because that means nobody can win the game until round four. The probability of 3 tails in a row being flipped, given that they are independent events, is the probability of a single tail being flipped to the third power.

   This gives us a probability of:

$$P(Rounds \geq 4) = (\frac{1}{2})^3 = \frac{1}{8}$$

   (c) Given they have played 4 rounds, what is the probability that the game lasts at least 6 rounds?

   Using conditional probability, the problem can be written as:

$$P(Rounds \geq 6 \mid Rounds \geq 4) = \frac{P(Rounds \geq 6 \cap Rounds \geq 4)}{P(Rounds \geq 4)}$$

If they played at least 6 rounds, they will have played 4 rounds for sure. Therefore,

$$P(Rounds \geq 6 \mid Rounds \geq 4) = \frac{P(Rounds \geq 6)}{P(Rounds \geq 4)}$$

$P(Rounds \geq 4)$ was calculated in part b) to be $\frac{1}{8}$. , and using the complement rule,

$$P(Rounds \geq 6) = 1 - P(Rounds \leq 6)$$

$$P(Rounds \geq 6) = 1 - (P(Rounds = 1) + P(Rounds = 2) ... + P(Rounds = 5))$$

$$P(Rounds \geq 6) = 1 - (\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32})$$

$$P(Rounds \geq 6) = \frac{1}{32}$$

Substitute into the original equation,

$$P(Rounds \geq 6 \mid Rounds \geq 4) = \frac{\frac{1}{32}}{\frac{1}{8}} = \frac{1}{4}$$

(d) Suppose that the coin is biased with the probability of Heads equal to p. Suppose the game lasts exactly 4 rounds. What value of p maximizes the chances of the game lasting exactly 4 rounds? For k rounds?

For the game to last 4 rounds, we need four tails, and then a head to be flipped. This probability function will look like:

$$P(Rounds = 4) = p * (1 - p)^3$$

In order to maximize the chance of this happening, we want to find the maximum of our function $p*(1-p)^3$. To do this, we first take the derivative of our function and set it equal to zero as seen below:

$$\frac{d}{dp}(p * (1 - p)^3) = 0$$

$$-(p - 1)^2(4p - 1) = 0$$

Using basic algebra skills, we know p must either be $\frac{1}{4}$ or 1. We can rule out $p = 1$ because that would mean the first player must win, so we determine that we maximize the chance of playing 4 games by setting:

$$p = \frac{1}{4}$$

As for the case of k rounds, we set up the same steps:

$$P(Rounds = k) = p * (1 - p)^{k-1}$$

$$\frac{d}{dp}(p * (1 - p)^{k-1}) = 0$$

$$\frac{-(1 - p)^k(kp - 1)}{(p - 1)^2} = 0$$

Once again, we use algebra to determine that our solutions are either $p = 1$ or $p = \frac{1}{k}$. We can rule out our first solution again, leaving us with only our second solution. In order to make sure that this value yields a relative maximum, we want to complete the second derivative test.

The second derivative of the function is:

$$\frac{-(k-1)(1-p)^k * (kp-2)}{(p-1)^3}$$

When we plug in $\frac{1}{k}$ for p, we will always get a negative value for our second derivative, meaning that $\frac{1}{k}$ will always be a relative max. So our solution to the problem is:

$$p = \frac{1}{k}$$

5. Suppose you only have one dollar. One day, on the Thayer street, a guy suggests you play a game. The game is very simple. First, you pay x dollars and toss a coin. If the coin shows a head, you got 2x dollars. If the coin shows a tail, you got 0.5x dollars. The probability of the coin showing ahead is p and p is unknown to you. At each game, you have to pay all money you got.

   (a) Suppose you play this game 3 times. Find the probability distribution for the total amount of money you have.

   If we play the game three times, there are 4 possible outcomes that we could have. We can win 3 times, win 2 times (lose once), win 1 time (lose twice) or win 0 times (lose 3 times). This variable takes the form of a binomial distribution when you fix the number of games you will play.

   As for how much money each result yields for an initial 1 dollar wager, winning three times correlates with winning 8 dollars (2*2*2*1), winning twice correlates with winning 2 dollars (2*2*.5*1), winning once correlates with winning 50 cents (2*.5*.5*1), and winning zero times correlates with winning 12.5 cents (.5*.5*.5*1). Our distribution of the variable X (The amount of money won) is as follows:

$$P(X = .125) = P(Tails)^3 = (1-p)^3$$
$$P(X = .5) = \binom{3}{1} * P(Tails)^2 * P(Heads) = 3 * (1-p)^2 * p$$
$$P(X = 2) = \binom{3}{2} * P(Tails) * P(Heads)^2 = 3 * (1-p) * p^2$$
$$P(X = 8) = P(Heads)^3 = p^3$$

   (b) What is the expected amount of money after the k-th game play?

   Knowing that we are working with a binomial distribution is extremely helpful here. Also, because our two events are multiplicative inverses, solving for the expected value becomes much easier.

   By definition $E(X)$ for all outcomes is

$$E(X) = \sum_{allx} (x * P(x))$$

   Because our probability for each event is a binomial, we can replace $P(x)$ with the expression $\binom{k}{x} * P(Heads)^x * P(Tails)^{k-x}$. We can also replace x (our winnings) with an expression written as a result of the the number of games won and lost. We know that for every win doubles our money and each loss halves it. This means our winnings would amount to our initial money times 2^i (i is the number of wins) times $.5^{k-i}$ (k-i is the number of times we lose). In our case where we have one dollar, we get the expression $x = 2^i * .5^{k-i}$ where i indicates the number of victories. For ease of notation, we are going to call $1-p$ the variable q.

Now, we can start to put together a concrete solution.

Substituting our expressions for P(x) and x, we get:

$$E(X) = \sum_{x=0}^{k}(\binom{k}{x} * 2^x * 0.5^{k-x} * p^x * q^{k-x})$$

Which simplifies to:

$$E(X) = \sum_{x=0}^{k}(\binom{k}{x} * (2p)^x * (.5q)^{k-x})$$

Because this is a binomial distribution that we are summing, we can simplify the format to:

$$E(X) = (2p + .5q)^k$$

for k games

Which we can substitute $1 - p$ for q one more time to get:

$$E(X) = (\frac{3p}{2} + .5)^k$$

for k games

6. Let X be a discrete random variable that assigns positive probabilities to only the positive integers, 1, 2, 3, $\cdots$. Show that

$$E(X) = \sum_{k=1}^{\infty}(P(X \geq k))$$

We once again want to examine the definition of $E(X)$ for this distribution.

$$E(X) = \sum_{x=1}^{\infty}(x * P(X = x))$$

Now, we will substitute k for x

$$E(X) = \sum_{k=1}^{\infty} k * P(X = k)$$

Rewrite k*P(X=k) in summation notation

$$E(X) = \sum_{k=1}^{\infty} \sum_{j=1}^{k} P(X = k)$$

Switch the order of summations

$$E(X) = \sum_{j=1}^{\infty} \sum_{k=j}^{\infty} P(X = k)$$

Simplifying the sum

$$E(X) = \sum_{j=1}^{\infty} P(X \geq k)$$