APMA 1650 Homework 8 Solutions

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1. Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from a normal distribution with mean μ and variance σ^2.

   (a) Find the method-of-moments estimators of μ and σ^2.

   For the method of moments, we set $m_i' = \mu_i'$ for $i = 1, \ldots, k$ where k is the number of parameters we are estimating. Recall that μ_i' is the ith population moment and m_i' is the ith sample moment. Since we have two parameters, we set $\mu_1' = m_1'$ and $\mu_2' = m_2'$. Thus, we have the system

$$\bar{X} = E[X] = \mu,$$
$$\frac{1}{n}\sum_{i=1}^{n} X_i^2 = E[X^2] = \sigma^2 + \mu^2.$$

   Solving this system, we find that $\hat{\mu} = \bar{X}$ and $\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2 - \bar{X}^2$.

   (b) Determine whether the method-of-moments estimators for μ and σ^2 are consistent or not.

   By the Weak Law of Large Numbers, we know that the sample moments $\frac{1}{n}\sum_{i=1}^{n} X_i^k$ converge in probability to the population moments $E[X^k]$ (which we will denote by $\frac{1}{n}\sum_{i=1}^{n} X_i^k \xrightarrow{p} E[X^k]$). Thus,

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \xrightarrow{p} E[X] = \mu$$
$$\text{and } \frac{1}{n}\sum_{i=1}^{n} X_i^2 \xrightarrow{p} E[X^2] = \mathrm{Var}(X) + E[X]^2 = \sigma^2 + \mu^2$$

   From this, we can see that

$$\hat{\mu} = \bar{X} \xrightarrow{p} \mu$$
$$\hat{\sigma}^2 = \sum_{i=1}^{n} X_i^2 - \bar{X}^2 \xrightarrow{p} \sigma^2 + \mu^2 - \mu^2 = \sigma^2$$

   Since $\hat{\mu} \xrightarrow{p} \mu$ and $\hat{\sigma}^2 \xrightarrow{p} \sigma^2$, the estimators are consistent.

2. Compute the method of moments estimator and the MLE for λ, the parameter of an exponential distribution:

$$f(x|\lambda) = \lambda \exp(-\lambda x)$$

from a random sample of size n.

(a) For the method of moments, we set $m_1' = \mu_1'$, where μ_1' is the first population moment and m_1' is the first sample moment. Recall that for an exponential distribution, $\mu_1' = \frac{1}{\lambda}$, and the first sample moment is the sample mean $m_1' = \bar{X}$. Setting these equal and solving for λ, we find that the method of moments estimator for λ is:

$$\hat{\lambda} = \frac{1}{\bar{X}}.$$

(b) To get the likelihood function, we plug the data X_i in to the exponential density and multiply them together:

$$\begin{aligned} L(\theta|x) &= \prod_{i=1}^{n} f(X_i|\lambda) \\ &= \prod_{i=1}^{n} \lambda e^{-\lambda X_i} \\ &= \lambda^n e^{-\lambda \sum_{i=1}^{n} X_i} \\ &= \lambda^n e^{-\lambda n \bar{X}} \end{aligned}$$

We want to maximize $L(\theta|x)$. The easiest way to do this is to maximize the log likelihood function instead:

$$\begin{aligned} \log L(\theta|x) &= \log\left(\lambda^n e^{-\lambda n \bar{X}}\right) \\ &= n \log \lambda - \lambda n \bar{X} \end{aligned}$$

To maximize this, we take the derivative and set it equal to 0:

$$\frac{d}{d\lambda} \log L(\theta|x) = \frac{n}{\lambda} - n\bar{X} = 0$$

If we solve for λ, then we get

$$\lambda = \frac{1}{\bar{X}}.$$

So in this case the MLE is the same as the method of moments estimator.

3. The geometric probability mass function is given by

$$p(y|\theta) = \theta(1-\theta)^{y-1}, y = 1, 2, 3...$$

A random sample of size n is taken from a population with a geometric distribution.

(a) Find the method-of-moment estimator θ for when n = 1.
Recall that the first moment of the geometric distribution is given by $E[Y|\theta] = \frac{1}{\theta}$. For the method-of-moments estimator, we set $m_1' = \mu_1'$ to get $\bar{Y} = \frac{1}{\theta}$. Therefore,

$$\hat{\theta} = \frac{1}{\bar{Y}}.$$

When $n = 1$, we only have one sample Y. Thus, $m_1' = Y$ and the method-of-moment estimator for θ is $\hat{\theta} = \frac{1}{Y}$.

(b) Find the MLE for θ.
Like the previous question, we first find the likelihood function:

$$
\begin{aligned}
L(\theta|y) &= \prod_{i=1}^{n} p(Y_i|\theta) \\
&= \prod_{i=1}^{n} \theta(1-\theta)^{Y_i - 1} \\
&= \theta^n (1-\theta)^{\sum_{i=1}^{n} Y_i - 1}
\end{aligned}
$$

The log-likelihood function is then

$$
\begin{aligned}
\log L(\theta|y) &= \log\left(\theta^n (1-\theta)^{\sum_{i=1}^{n} Y_i - n}\right) \\
&= n \log\theta + \left(\sum_{i=1}^{n} Y_i - n\right) \log(1-\theta)
\end{aligned}
$$

To maximize this, we take the derivative with respect to 0 and set it equal to o:

$$
\frac{d}{d\theta} \log L(\theta|y) = \frac{n}{\theta} - \frac{\sum_{i=1}^{n} Y_i - n}{1-\theta} = 0.
$$

Rearranging this, we have

$$
n - n\theta = \left(\sum_{i=1}^{n} Y_i - n\right)\theta,
$$

and solving for θ gives the MLE

$$
\hat{\theta} = \frac{n}{\sum_{i=1}^{n} Y_i} = \frac{1}{\bar{Y}}.
$$

When $n = 1$, $\hat{\theta} = \frac{1}{Y}$, which matches the method-of-moments estimator in (a).