APMA 1650: Homework 7 Solutions

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1. (Hat Check problem). n people enter the restaurant and put their hats at the reception. Each person gets a random hat back when going back after having dinner. Find the expected value and variance of the number of people who get their right hat back.

   First, we will deal with the expected value. In order to solve this problem, we will want to use an indicator variable. Our indicator variable X_i will be 1 when a person gets the correct hat back and 0 when they don't.

   The probability that a person gets the correct hat back is: $P(X_i{=}1){=}\frac{1}{n}$. The probability that a person gets the wrong hat, which is the complement event, is therefore $1 - \frac{1}{n}$.

   If we say X represents the number of people who receive the correct hats, then we can also say that:

$$X = \sum_{i=1}^{n}(X_i)$$

   Therefore:

$$E[X] = E[\sum_{i=1}^{n}(X_i)]$$

   We can use our principle of linearity to rewrite this as:

$$E[X] = \sum_{i=1}^{n}(E[X_i])$$

   We know $E[X_i] = 1 * \frac{1}{n} + 0 * (1 - \frac{1}{n}) = \frac{1}{n}$

   Therefore:

$$E[X] = \sum_{i=1}^{n}(\frac{1}{n})$$

$$E[X] = n * \frac{1}{n} = 1$$

   As for the variance of the number of correct hats, we will set up a similar problem:

   In order to solve this problem, we will want to use an indicator variable. Our indicator variable X_i will be 1 when a person gets the correct hat back and 0 when they don't. In order to calculate $V(X)$, we will use our formula $V(X) = E(X^2) - (E(X)))^2$. In this case, that formula becomes:

$$V(X) = \sum_{i=1}^{n}(E[X_i^2]) + \sum_{i=1}^{n}\sum_{i\neq j}^{n}(E[X_i * X_j]) - \sum_{i=1}^{n}(E[X_i]^2)$$

Note: Our $E(X^2)$ is expanded to the two separate sums because X^2 is really $(X_1 + X_2 + X_3 + X_4 + ... + X_n)^2$ which can be rewritten as $\sum_{i=1}^n X_i^2 + \sum_{i=1}^n \sum_{i\neq j}^n X_i * X_j$.

We can simplify this to:

$$V(X) = n * \frac{1}{n} + (n-1) * n * \frac{1}{n(n-1)} + (n * \frac{1}{n})^2 = 1 + 1 - 1 = 1$$

So we get expected value of 1 and variance of 1.

2. (a) Let $Z \sim N(0,1)$. Find the mgf of Z

   Our moment generating function is

$$m_Z(t) = E[e^{Zt}] = \int_{-\infty}^{\infty} e^{zt} e^{-\frac{z^2}{2}} \frac{1}{\sqrt{2\pi}} dz$$

   To simplify this, we need to rewrite our exponents. We proceed by completing the square:

$$zt - \frac{z^2}{2} = -\frac{1}{2}(z^2 - 2zt + t^2 - t^2) = -\frac{1}{2}(z-t)^2 + \frac{t^2}{2}$$

   Once we substitute this square back in we have

$$m_Z(t) = e^{\frac{t^2}{2}} \int_{-\infty}^{\infty} e^{\frac{-(z-t)^2}{2}} \frac{1}{\sqrt{2\pi}} dz.$$

   This integral evaluates to one because it is the integral of a normal pdf with mean t and variance 1, leaving us with the simplified form:

$$m_Z(t) = e^{\frac{t^2}{2}}$$

   (b) Let $X \sim N(\mu, \sigma^2)$. Find the mgf of X. Hint: $X = \mu + \sigma Z$.

   As before, we begin with the definition of mgf:

$$\begin{aligned} m_X(t) &= E[e^{tX}] \\ &= E[e^{t(\mu+\sigma Z)}] \text{ because } X = \mu + \sigma Z \\ &= e^{t\mu} E[e^{(\sigma t)Z}] \text{ since } e^{t\mu} \text{ is a constant} \\ &= e^{t\mu} m_Z(\sigma t) \text{ by the definition of mgf} \\ &= e^{t\mu} e^{\frac{t^2\sigma^2}{2}} \text{ by (a)} \end{aligned}$$

   (c) Let $X_1, X_2, ..., X_n$ be i.i.d. random variables of $N \sim (\mu, \sigma^2)$. Let

$$\bar{X} = \frac{X_1 + X_2 + ... + X_n}{n}$$

   Find the mgf of $\bar{X}$.

We begin with the definition of mgf:

$$\begin{aligned} m_{\bar{X}}(t) &= E[e^{t\bar{X}}] \\ &= E[e^{\frac{t}{n}\sum_{i=1}^{n} X_i}] \text{ by the definition of } \bar{X} \\ &= E[\Pi_{i=1}^{n} e^{\frac{tX_i}{n}}] \\ &= \Pi_{i=1}^{n} E[e^{\frac{tX_i}{n}}] \text{ by independence} \\ &= E[e^{\frac{tX}{n}}]^n \text{ since the } X_i s \text{ are identically distributed} \\ &= m_X\left(\frac{t}{n}\right)^n \\ &= \left[e^{\frac{\mu t}{n}} e^{\frac{\sigma^2 t^2}{2n^2}}\right]^n \text{ by (b)} \\ &= e^{\mu t} e^{\frac{\sigma^2 t^2}{2n}} \end{aligned}$$

(d) Use (c), find $E[\bar{X}]$ and $Var[\bar{X}]$.

Note that $m_{\bar{X}}(t) = e^{\mu t} e^{\frac{\sigma^2 t^2}{2n}}$ is the MGF of a normal random variable with mean μ and variance $\frac{\sigma^2}{n}$, so $\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$. Thus, $E[\bar{X}] = \mu$ and $\text{Var}(\bar{X}) = \frac{\sigma^2}{n}$.

3. Let $Z \sim N(0,1)$. Let $X_1 = Z$ and $X_2 = Z^2$

(a) Find $E[X_1]$ and $E[X_2]$.

$E[X_1] = E[Z] = 0$ by the definition of the distribution.

$E[X_2] = E[Z^2] = V[Z] + E[Z]^2 = 1 + 0 = 1$

(b) Find $E[X_1 X_2]$

$E[X_1 X_2] = E[Z * Z^2] = E[Z^3] = M_Z'''(0)$ where $M_Z(t)$ is the mgf for $Z \sim N(0,1)$ that we found in 2a.

$$M_Z'''(t) = (t^3 + 3t)(e^{\frac{t^2}{2}})$$

$$M_Z'''(0) = (0^3 + 3 * 0)(e^{\frac{0^2}{2}}) = 0$$

(c) Find $Cov(X_1, X_2)$.

$$Cov(X_1, X_2) = E[X_1 X_2] - E[X_1] * E[X_2]$$

$$Cov(X_1, X_2) = 0 - 0 * 1 = 0$$

(d) Are X_1 and X_2 independent?

Because our co-variance is zero, we know that our two variables are uncorrelated. However, they are not necessarily independent. In this case, we know they are not independent because X_2 is X_1^2, so if we know the value of one, we can easily determine the value of the other.

4. A die is thrown 12 times (independently).

(a) Find the probability that every face appears twice.

This question is very similar to the monomial problem that we dealt with earlier in the year.

The number of combinations of successful trials can be written as $\frac{12!}{2!2!2!2!2!2!}$

$P(2 \text{ of each}) = (\frac{1}{6})^{12} * \frac{12!}{2^6}$

(b) Let X be the number of appearances of 6 and Y the number of appearances of 1. Find the joint probability distribution of X and Y.

P(X=x,Y=y) $= (\frac{1}{6})^x * (\frac{1}{6})^y * (\frac{4}{6})^{12-x-y} * \frac{12!}{x!y!(12-x-y)!}$

This function only exists for the domain $X \geq 0, Y \geq 0, X+Y \leq 12$

(c) Find Cov(X,Y).

Based on our pmf from part b., we know we are dealing with a bi-variate binomial distribution. This distribution has specific values for E[XY], E[X], and E[Y].

$$Cov(X,Y) = E[XY] - E[X] * E[Y]$$

$$E[XY] = \sum_{i=0}^{12} \sum_{j=0}^{12-i} i * j * \frac{12!}{i!j!(12-i-j)!} * (\frac{1}{6})^i * (\frac{1}{6})^j * (\frac{4}{6})^{12-i-j}$$

Now let k=j-1:

$$= \sum_{i=1}^{11} \frac{12!}{(i-1)!(12-i-1)!} * (\frac{1}{6})^{i+1} \sum_{k=0}^{12-i-1} \frac{(12-i-1)!}{k!(12-i-k-1)!} * (\frac{1}{6})^k (\frac{4}{6})^{12-i-k-1}$$

Using binomial theorem:

$$= \sum_{i=1}^{11} \frac{12!}{(i-1)!(12-i-1)!} * (\frac{1}{6})^{i+1} * (\frac{5}{6})^{12-i-1}$$

Let k=i-1:

$$= \sum_{k=0}^{10} \frac{12!}{(k)!(10-k)!} * (\frac{1}{6})^{k+2} * (\frac{5}{6})^{10-k}$$

Using binomial theorem:

$$E[XY] = (\frac{1}{6})^2 * 11 * 12 * 1^{10} = \frac{11}{3}$$

$$E[X] = n * P(x) = 12 * \frac{1}{6} = 2$$

$$E[Y] = n * P(y) = 12 * \frac{1}{6} = 2$$

$$Cov(X,Y) = E[XY] - E[X] * E[Y] = \frac{11}{3} - 2 * 2 = -\frac{1}{3}$$

Another way to approach this problem is to break it down into smaller problems. Let (X_i, Y_i) for $i = 1, \ldots, 12$ be the indicator random variables such that

$$X_i = \begin{cases} 1, & \text{if roll } i \text{ is a 6} \\ 0, & \text{otherwise} \end{cases}$$

and

$$Y_i = \begin{cases} 1, & \text{if roll } i \text{ is a 1} \\ 0, & \text{otherwise} \end{cases}$$

Then, $(X, Y) = \left(\sum_{i=1}^{12} X_i, \sum_{j=1}^{12} Y_j\right)$. For one pair (X_i, Y_i), we have that

$$
\begin{aligned}
E[X_iY_i] &= (0)(0)P(X=0, Y=0) + (0)(1)P(X=0, Y=1) + (1)(0)P(X=1, Y=0) \\
&= 0
\end{aligned}
$$

Also, $P(X_i = 1) = P(Y_i = 1) = \frac{1}{6}$, so $\text{Cov}(X_i, Y_i) = 0 - \frac{1}{6}\frac{1}{6} = -\frac{1}{36}$.

Next, we explore a property of covariances. We claim that $\text{Cov}(\sum_{i=1}^{n} X_i, \sum_{j=1}^{m} Y_j) = \sum_{i=1}^{n}\sum_{j=1}^{m} \text{Cov}(X_i, Y_j)$. Indeed,

$$
\begin{aligned}
\text{Cov}\left(\sum_{i=1}^{n} X_i, \sum_{j=1}^{m} Y_j\right) &= E\left[\left(\sum_{i=1}^{n} X_i - E\left[\sum_{i=1}^{n} X_i\right]\right)\left(\sum_{j=1}^{m} Y_j - E\left[\sum_{j=1}^{m} Y_j\right]\right)\right] \\
&= E\left[\left(\sum_{i=1}^{n} (X_i - \mu_{X_i})\right)\left(\sum_{j=1}^{m} (Y_j - \mu_{Y_j})\right)\right] \\
&= \sum_{i=1}^{n}\sum_{j=1}^{m} E\left[(X_i - \mu_{X_i})(Y_j - \mu_{Y_j})\right] \\
&= \sum_{i=1}^{n}\sum_{j=1}^{m} \text{Cov}(X_i, Y_j)
\end{aligned}
$$

Applying this to our problem, we have

$$
\begin{aligned}
Cov(X, Y) &= Cov\left(\sum_{i=1}^{12} X_i, \sum_{j=1}^{12} Y_i\right) \\
&= \sum_{i=1}^{12}\sum_{j=1}^{12} \text{Cov}(X_i, Y_j) \\
&= \sum_{i=1}^{12} \text{Cov}(X_i, Y_i) \text{ since dice rolls are independent for } i \neq j \\
&= 12\left(\frac{-1}{36}\right) \\
&= -\frac{1}{3}
\end{aligned}
$$

5. Twenty people consisting of 10 married couples are to be seated at 5 different tables, with 4 people at each table.

   (a) If all seats are randomly assigned, what is the expected number of married couples that are seated at the same table?

   Similar to problem 1, we will set up an indicator variable X_i for each of the ten couples that has a value of 1 when a couple is together at a table and zero otherwise. First, we will say that our husband is seated at the table we are looking at. The probability that the wife sits down at the same table is given by the hypergeometric probability below:

$$
P(X_i = 1) = \frac{\binom{18}{2} * \binom{1}{1}}{\binom{19}{3}} = \frac{3}{19}
$$

   Because we have declared X_i an indicator variable with $X = X_1 + X_2 + ... + X_10$, calculating our expected value is really easy.

$$E[X] = E[\sum_{n=1}^{10} 1 * \frac{3}{19}] = \sum_{n=1}^{10} E[\frac{3}{19}] = 10 * \frac{3}{19} = \frac{30}{19}$$

We expect $\frac{30}{19}$ couples to be seated at the same table.

(b) If 2 men and 2 women are randomly chosen to be seated at each table, what is the expected number of married couples that are seated at the same table?

We will use the same methodology for this question. We have the same indicator variable X_i that takes the values of 1 and 0 for the same events.

In this case we have a new $P(X_i = 1)$

$$P(X_i = 1) = \frac{\binom{1}{1} * \binom{9}{1}}{\binom{10}{2}} = \frac{1}{5}$$

Just as before, we calculate our expected number of couples as:

$$E[X] = E[\sum_{n=1}^{10} 1 * \frac{1}{5}] = \sum_{n=1}^{10} E[\frac{1}{5}] = 10 * \frac{1}{5} = 2$$

6. . Let $X_1, ..., X_n$ be i.i.d. random variables having expected value μ and variance σ^2. Let consider the following functions of random variables:

$$\hat{\mu}(X) = \frac{1}{n}\sum_{i=1}^{n} X_i S^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2 \hat{\sigma^2}(X) = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$$

(a) Find $E[\hat{\mu}(X)]$ $E[S^2]$ and $E[\hat{\sigma^2}(X)]$

$$E[\hat{\mu}(X)] = E[\frac{1}{n}\sum_{i=1}^{n} X_i] = \frac{1}{n}\sum_{i=1}^{n} E[X_i] = \frac{n}{n} * \mu = \mu$$

For S^2, we first make the following calculation that we will need later:

$$\begin{aligned}
E[X_i\bar{X}] &= \frac{1}{n}E\left[X_i \sum_{j=1}^{n} X_j\right] \\
&= \frac{1}{n}\left(E[X_i^2] + \sum_{j\neq i} E[X_i X_j]\right) \\
&= \frac{1}{n}\left(E[X_i]^2 + \mathrm{Var}(X_i) + (n-1)E[X_i]^2\right) \text{ by } X_i s \text{ are iid} \\
&= \frac{1}{n}\left(\mu^2 + \sigma^2 + (n-1)\mu^2\right) \\
&= \frac{\sigma^2}{n} + \mu^2
\end{aligned}$$

Now, for $E[S^2]$, we add and subtract μ to make calculations easier:

$$
\begin{aligned}
E[S^2] &= E\left[\frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2\right] \\
&= E\left[\frac{1}{n}\sum_{i=1}^{n}((X_i - \mu) + (\mu - \bar{X}))^2\right] \\
&= E\left[\frac{1}{n}\sum_{i=1}^{n}((X_i - \mu)^2 + 2(X_i - \mu)(\mu - \bar{X}) + (\mu - \bar{X})^2)\right] \text{ by expansion} \\
&= E[(X_i - \mu)^2 + 2(X_i - \mu)(\mu - \bar{X}) + (\mu - \bar{X})^2)] \text{ since the } X_i s \text{ are identically distributed} \\
&= \mathrm{Var}(X_i) + 2E[(X_i - \mu)(\mu - \bar{X})] + \mathrm{Var}(\bar{X})
\end{aligned}
$$

which follows because the first square is the expression fro $\mathrm{Var}(X_i)$, and the second square is $\mathrm{Var}(\bar{X})$ since $E[\bar{X}] = \mu$. If we expand the middle term, by our previous calculations, we get

$$E[(X_i - \mu)(\mu - \bar{X})] = \mu E[X_i] - \mu^2 - E[X_i\bar{X}] + \mu E[\bar{X}] = E[X_i\bar{X}] - \mu^2 = -\frac{\sigma^2}{n}$$

Plugging this back into our original expression for $E[S^2]$, we get

$$
\begin{aligned}
E[S^2] &= \mathrm{Var}(X_i) + 2E[(X_i - \mu)(\mu - \bar{X})] + \mathrm{Var}(\bar{X}) \\
&= \sigma^2 - \frac{2\sigma^2}{n} + \frac{\sigma^2}{n} \\
&= \frac{n-1}{n}\sigma^2
\end{aligned}
$$

Because $\hat{\sigma^2}(X) = \frac{n}{n-1}S^2$, we can just multiply our answer for $E[S^2]$ by that same constant. That leaves us with following solution.

$$E[\hat{\sigma^2}(X)] = \frac{n}{n-1}E[S^2] = \frac{n}{n-1}\left(\frac{n-1}{n}\sigma^2\right) = \sigma^2$$

(b) By using the weak law of large numbers, show that for any $\epsilon > 0$,

$$\lim_{n\to\infty} P(|\hat{\mu}(X) - \mu| > \epsilon) = 0$$

Our definition of the weak law of large numbers actually guarantees that this inequality holds, so if we let our n approach infinity in our weak law of large numbers, then we have satisfied our inequality.