APMA 1650: Homework 1 Solutions

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1. (a) Show the distributive laws without drawing Venn diagrams.

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C), \qquad A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

We proceed by showing set inclusion both ways. (Recall that $A = B$ if and only if $A \subset B$ and $A \supset B$)

i. <u>Need to show $A \cap (B \cup C) \subset (A \cap B) \cup (A \cap C)$:</u> Let $x \in A \cap (B \cup C)$. Then $x \in A$, and $x \in B \cup C$, so either $x \in B$ or $x \in C$. If $x \in B$, then $x \in A \cap B$. Otherwise, if $x \in C$, then $x \in A \cap C$. Thus, $x \in (A \cap B)$ or $x \in (A \cap C)$, so we have shown that $x \in (A \cap B) \cup (A \cap C)$.

<u>Need to show $A \cap (B \cup C) \supset (A \cap B) \cup (A \cap C)$:</u> Let $x \in (A \cap B) \cup (A \cap C)$. Then either $x \in A \cap B$, in which case $x \in A \cap (B \cup C)$ since $B \subset B \cup C$ implies $A \cap B \subset A \cap (B \cup C)$, or $x \in A \cap C$, which implies $x \in A \cap (B \cup C)$ by the same reasoning. Thus, $x \in A \cap (B \cup C)$.

ii. <u>Need to show $A \cup (B \cap C) \subset (A \cup B) \cap (A \cup C)$:</u> Let $x \in A \cup (B \cap C)$. Then $x \in A$ or $x \in B \cap C$. If $x \in A$, then $x \in A \cup B$ and $x \in A \cup C$ since $A \subset A \cup B$ and $A \subset A \cup C$, so $x \in (A \cup B) \cap (A \cup C)$. Otherwise, if $x \in B \cap C$, then $x \in B$ and $x \in C$, so $x \in A \cup B$ and $x \in A \cup C$ since $B \subset A \cup B$ and $C \subset A \cup C$. Hence $x \in (A \cup B) \cap (A \cup C)$.

<u>Need to show $A \cup (B \cap C) \supset (A \cup B) \cap (A \cup C)$:</u> Let $x \in (A \cup B) \cap (A \cup C)$. Then $x \in A \cup B$ and $x \in A \cup C$. If $x \in A$, then $x \in A \cup (B \cap C)$ since $A \subset A \cup (B \cap C)$. Otherwise, we must have $x \in B$ and $x \in C$, so $x \in B \cap C$. Since $B \cap C \subset A \cup (B \cap C)$, $x \in A \cup (B \cap C)$.

Since we have shown set inclusion both ways, these sets must be equal.

(b) Assuming the above distributive laws, prove that

$$A \cap \bigcup_{i=1}^{k} B_i = \bigcup_{i=1}^{k} A \cap B_i.$$

We proceed by induction, taking the first law in (a) as our base case. Assume that this holds for $k = n$, i.e. that $A \cap \bigcup_{i=1}^{n} B_i = \bigcup_{i=1}^{n} A \cap B_i$. Then

$$\begin{aligned}
A \cap \bigcup_{i=1}^{n+1} B_i &= A \cap (\bigcup_{i=1}^{n} B_i \cup B_{n+1}) \\
&= (A \cap \bigcup_{i=1}^{n} B_i) \cup (A \cap B_{n+1}) \text{ by (a) part (i)} \\
&= (\bigcup_{i=1}^{n} A \cap B_i) \cup (A \cap B_{n+1}) \text{ by the inductive hypothesis} \\
&= \bigcup_{i=1}^{n+1} A \cap B_i.
\end{aligned}$$

2. We have a die such that the probability of the die showing a number is proportional to the number itself. For example, showing a six is two times as likely as showing a three.

First, we find the associated probabilities. Let p be the probability of rolling one, i.e. $P(\{1\}) = p$. Then we have that $P(\{2\}) = 2p$, $P(\{3\}) = 3p$, etc. Since $P(\Omega) = 1$ where Ω is the sample space, we must have that $1 = P(\{1\}) + \cdots + P(\{6\}) = p + 2p + 3p + 4p + 5p + 6p$. Solving gives $p = \frac{1}{21}$.

(a) When the die is tossed once, what is the probability we pick a odd number?

Let A be the event that we pick an odd number. Then $A = \{1, 3, 5\}$, so

$P(A) = P(\{1\}) + P(\{3\}) + P(\{5\}) = \frac{1}{21} + \frac{3}{21} + \frac{5}{21} = \frac{3}{7} \approx 0.4286$.

(b) When the die is tossed two times, what is the probability that the sum of the two is more than 9?

We first make a table of the possible outcomes where the face the first die is on the top and the face of the second die is on the left with their sum in the middle:

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Now we can see that if B is the event that the sum of the dice is larger than 9, then

$$B = \{(4,6), (6,4), (5,5), (5,6), (6,5), (6,6)\}.$$

Noting that the probability of each outcome is the same regardless of order, i.e.

$$P(\{(x, y)\}) = P(\{(y, x)\}),$$

we find that

$$\begin{aligned}
P(B) &= 2P(\{(4,6)\}) + P(\{(5,5)\}) + 2P(\{(5,6)\}) + P(\{(6,6)\}) \\
&= 2\left(\frac{4}{21}\right)\left(\frac{6}{21}\right) + \left(\frac{5}{21}\right)\left(\frac{5}{21}\right) + 2\left(\frac{5}{21}\right)\left(\frac{6}{21}\right) + \left(\frac{6}{21}\right)\left(\frac{6}{21}\right) \\
&= \frac{169}{441} \\
&\approx .3832
\end{aligned}$$

(c) When the die is tossed two times, what is the probability that the sum of the two is a prime number? (Note that 2, 3, 5, 7, 11 are prime numbers).

Using the table of outcomes again, we see that if C is the probability of getting a prime sum, then

$$C = \{(1,1), (1,2), (2,1), (2,3), (3,2), (1,4), (4,1), (1,6), (6,1), (2,5), (5,2), (3,4), (4,3), (5,6), (6,5)\}.$$

Once again, noting that order does not matter, we have

$$\begin{aligned}
P(C) &= P(\{(1,1)\}) + 2P(\{(1,2)\}) + 2P(\{(2,3)\}) + 2P(\{(1,4)\}) + 2P(\{(1,6)\}) + 2P(\{(2,5)\}) \\
&\quad + 2P(\{(3,4)\}) + 2P(\{(5,6)\}) \\
&= \left(\frac{1}{21}\right)\left(\frac{1}{21}\right) + 2\left(\frac{1}{21}\right)\left(\frac{2}{21}\right) + 2\left(\frac{2}{21}\right)\left(\frac{3}{21}\right) + 2\left(\frac{1}{21}\right)\left(\frac{4}{21}\right) + 2\left(\frac{1}{21}\right)\left(\frac{6}{21}\right) \\
&\quad + 2\left(\frac{2}{21}\right)\left(\frac{5}{21}\right) + 2\left(\frac{3}{21}\right)\left(\frac{4}{21}\right) + 2\left(\frac{5}{21}\right)\left(\frac{6}{21}\right) \\
&= \frac{141}{441} \\
&\approx 0.3197
\end{aligned}$$

3. (a) Let toss a fair coin 20 times. What is the probability of flipping the first Head after 9th toss? That is, starting at the 10th toss, the first Head is observed.

   Our sequence must start TTTTTTTTT (9 tails) and then must have a heads somewhere in the remaining 11 tosses; the remaining tosses can be either heads or tails. Flipping 9 tails in a sequence has probability $\left(\frac{1}{2}\right)^9$ since there is only one way it can happen. The only arrangement we know we cannot have for the remaining 11 flips is all tails, which has probability $\left(\frac{1}{2}\right)^{11}$. Thus, the probability of getting at least one heads in the last 11 flips is $1-\left(\frac{1}{2}\right)^{11}$. Putting these pieces together we get get

$$P(\text{first head after 9 tails}) = \left(\frac{1}{2}\right)^9\left(1-\left(\frac{1}{2}\right)^{11}\right) \approx 0.00195.$$

   (b) Toss a fair coin 20 times. What is the probability that you do not flip two Heads nor two Tails in a row?

   The key to this question is to notice that having no heads or tails in a row is the same as alternating heads and tails. This can only happen two ways: either the first toss is heads or it is tails. Thus,

$$P(\text{no two heads nor tails in a row}) = P(\text{alternate heads and tails}) = 2\left(\frac{1}{2}\right)^{20}.$$

   (c) Toss a fair coin 5 times. What is the probability of flipping two or more Heads in a row?

   We solve this problem in two ways, the first by brute force with some insight, and the second by recurrence. First, we split the problem into two cases: that we have heads first or last, or we don't have heads first or last. The first case is either HH— or —HH, where — can be any combination of heads and tails. In both cases, there are 3 open spaces, so the probability of either arrangement is $\left(\frac{1}{2}\right)^5 2^3$. If there are no heads first or last, we have three possible arrangements, THHHT, TTHHT, or THHTT, each with probability $\left(\frac{1}{2}\right)^5$. Thus

$$P(\text{at least two heads in a row}) = 2\left(\frac{1}{2}\right)^5 2^3 + 3\left(\frac{1}{2}\right)^5 = 0.59375.$$

   Note that there are many other ways to solve this problem with brute force and logic.

   To solve this problem by finding a recurrence relation, we set p_n to be the probability of having at least two heads in a row by flip n. Consider the nth flip. We will have two heads in a row if there are already two heads in a row at the $n-1st$ flip, or if there are not yet two in a row but the sequence ends THH. The probability of THH is $\frac{1}{8}$, so we have the relation $p_n = p_{n-1} + \frac{1}{8}(1-p_{n-3})$. For $n = 0, 1$, it is obvious that $p_0 = p_1 = 0$ since two heads in a row is not possible. We can also find that $p_2 = P(HH) = \frac{1}{4}$. Using these initial conditions in the rucurrence relation, we find that $p_5 = 0.59375$.

4. Assume there are 365 distinct possible birthdays. Suppose we ask the birthday of n students. Assume the probability of having any given birthday is equally likely.

   (a) What is the probability that there are no two students who have the same birth date?

   If we look at the n students in any order, then the first student can have any birthday, the second can have any birthday besides the first's, the third can have any birthday besides the first's or the second's, and so on. Thus, we get

$$P(\text{no same birthdays}) = \frac{365}{365}\frac{364}{365}\frac{363}{365}\cdots\frac{365-n+1}{365} = \frac{365!}{365^n\cdot(365-n)!}.$$

(b) What is the probability that at least two students share a birthday?

This is the complement of the event in part (a), so we have

$$P(\text{at least two have the same birthday}) = 1 - P(\text{no same birthdays}) = 1 - \frac{365!}{365^n \cdot (365-n)!}$$

5. A bag contains 100 balls of 75 white balls and 25 black balls. At each time, we randomly pick a ball from the bag and check the color and throw the ball away. Let's say we repeat this n times.

   (a) What is the probability that the number of observed black balls among $n = 10$ trials is $k = 4$?

   We want to pick 4 black balls out of 25 and 6 white balls out of 75. Thus, there are $\binom{25}{4} \cdot \binom{75}{4}$ ways to do this, but there are $\binom{100}{10}$ ways to choose any 10 balls from 100, so this means the probability is

$$P(4 \text{ black balls}) = \frac{\binom{25}{4}\binom{75}{6}}{\binom{100}{10}} = \frac{25! \cdot 75! \cdot 10! \cdot 90!}{4! \cdot 21! \cdot 69! \cdot 6! \cdot 100!} \approx 0.1471$$

   (b) What is the probability that the number of observed black balls among n trials is k?

   We generalize the approach from the previous problem:

$$P(k \text{ black balls out of } n) = \frac{\binom{25}{k}\binom{75}{n-k}}{\binom{100}{n}} = \frac{25! \cdot 75! \cdot (100-n)! \cdot n!}{k! \cdot (25-k)! \cdot (75-n+k)! \cdot (n-k)! \cdot 100!}$$

6. The probabilities are 0.4, 0.2, 0.3, and 0.1, respectively, that a delegate to a certain convention arrived by air, bus, automobile, or train. What is the probability that among 9 delegates randomly selected at this convention, 3 arrived by air, 3 arrived by bus, 1 arrived by automobile, and 2 arrived by train?

   Recall that the probability of forming groups of size $k_1, k_2, \ldots, k_j$ from n people is $\frac{n!}{k_1!k_2!\ldots k_j!}$. Thus,

$$P(3 \text{ air, } 3 \text{ bus, } 1 \text{ automobile, } 2 \text{ train out of } 9) = \frac{9!}{3!3!1!2!}(.4)^3(.2)^3(.3)^1(.1)^2 = 0.0077.$$