APMA 1650: Homework 5 Solutions

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1. Let Z be a standard normal random variable, i.e., $Z \sim \mathcal{N}(0,1)$. Find the value z_0 such that

   (a) $P(Z > z_0) = 0.5$.

   $P(Z \leq z_0) = 1 - P(Z > z_0) = 0.5$. Equivalently, we want $P(Z > z_o) = 0.5$ Then, using the standard normal distribution table to find the corresponding z_0 value, we find that $z_0 = 0$.

   (b) $P(Z < z_0) = 0.8643$.

   Equivalently, we want $P(Z > z_0) = .1357$. Using the standard normal distribution table to find the corresponding z_0 value, we find that $z_0 = 1.1$.

   (c) $P(-z_0 < Z < z_0) = 0.90$.

   We rewrite this probability in the form $P(Z > z_0)$ so we can use the z-table:

   $$\begin{aligned} &P(-z_0 < Z < z_0) = 0.90 \\ &\iff P(Z \leq -z_0) + P(Z \geq z_0) = 0.1 \text{ since this is the complement} \\ &\iff P(Z \geq z_0) = 0.05 \text{ by symmetry} \end{aligned}$$

   Using standard normal distribution table, we find that $z_0 = 1.645$.

   (d) $P(-z_0 < Z < z_0) = 0.99$.

   We rewrite this probability in the form $P(Z > z_0)$ so we can use the z-table:

   $$\begin{aligned} &P(-z_0 < Z < z_0) = 0.99 \\ &\iff P(Z \leq -z_0) + P(Z \geq z_0) = 0.01 \\ &\iff P(Z \geq z_0) = 0.005 \text{ by symmetry} \end{aligned}$$

   Using standard normal distribution table, we find that $z_0 = 2.576$.

2. An electrical firm manufactures light bulbs that have a life, before burn-out, that is normally distributed with mean equal to 800 hours and a standard deviation of 40 hours. Find the probability that a bulb burns between 778 and 834 hours.

   Let X=life(in years) of a bulb before burn-out. We know $X \sim N(800, 40^2)$ and we want to find $P(778 < X < 834)$. For easier computation, we first transform this into a standard normal distribution. Recall that the z-score is given by $Z = \frac{X-\mu}{\sigma}$. Then,

   $$\begin{aligned} P(778 < X < 834) &= P\left(\frac{778-800}{40} < \frac{X-800}{40} < \frac{834-800}{40}\right) \\ &= P(-0.55 < Z < 0.85) \\ &= P(Z > -0.55) - P(Z > 0.85) \\ &= (1 - P(Z > 0.55)) - P(Z > 0.85) \text{ by symmetry} \\ &= (1 - 0.2912) - 0.1977 \text{ from the z-table} \\ &= 0.511 \end{aligned}$$

3. (Chernoff bounds) Let X be a random variable and $m_X(t)$ be the mgf of X. Show that

$$P(X \geq a) \leq e^{-ta} m_X(t) \quad \text{for all} \quad t > 0,$$
$$P(X \leq a) \leq e^{-ta} m_X(t) \quad \text{for all} \quad t < 0.$$

Hint: Use Markov's inequality.

We know that:

- Moment-generating function of X: $m_X(t) = E[e^{tX}]$
- Markov's inequality: if $E[g(X)] < \infty$, then for $a > 0$, $P(|g(X)| \geq a) \leq \frac{E[|g(X)|]}{a}$

We also know that e^{tX} is a nonnegative random variable and $e^{ta} > 0$; therefore by Markov's inequality:

$$\begin{aligned}
\forall t > 0, &\quad X \geq a \Leftrightarrow e^{tX} \geq e^{ta} \\
\Rightarrow &\quad P(X \geq a) = P(e^{tX} \geq e^{ta}) \leq \frac{E[e^{tX}]}{e^{ta}} = e^{-ta} m_X(t) \\
\forall t < 0, &\quad X \leq a \Leftrightarrow e^{tX} \geq e^{ta} \\
\Rightarrow &\quad P(X \leq a) = P(e^{tX} \geq e^{ta}) \leq \frac{E[e^{tX}]}{e^{ta}} = e^{-ta} m_X(t)
\end{aligned}$$

4. A machine used to fill cereal boxes dispenses, on the average, μ ounces per box. The manufacturer wants the actual ounces dispensed X to be within 1 ounce of μ at least 75% of the time. What is the largest value of σ, the standard deviation of X, that can be tolerated if the manufacturer's objectives are to be met?

Let X = cereal in ounces. Since we don't know the distribution of X, we need to use Chebyshev's inequality. We want to find σ such that we can guarantee $P(\mu - 1 < X < \mu + 1) \geq 0.75$. Equivalently, we want $P(|X - \mu| \geq 1) \leq 0.25$. Plugging this into Chebyshev's inequality, which states that for $k > 0$, $P(| X - \mu |\geq k) \leq \frac{\sigma^2}{k^2}$, we get

$$\begin{aligned}
P(| X - \mu |\geq 1) \leq \frac{\sigma^2}{1^2} &\leq 0.25 \\
\Rightarrow \quad \sigma^2 &\leq 0.25 \\
\Rightarrow \quad \sigma &\leq 0.5
\end{aligned}$$

$\therefore$ The largest value of σ that can be tolerated is 0.5.

5. The gamma function is defined to be

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx.$$

(a) Using the integration by parts, show that

$$\Gamma(z+1) = z\Gamma(z).$$

$$\begin{aligned}
\Gamma(z+1) &= \int_0^\infty x^z e^{-x} dx \\
&= [-x^z e^{-x}]_0^\infty - \int_0^\infty -zx^{z-1} e^{-x} dx \text{ by integration by parts} \\
&= \lim_{n\to\infty} \frac{-n^z}{e^n} + z\int_0^\infty x^{z-1} e^{-x} dx \\
&= 0 + z\Gamma(z) \\
&= z\Gamma(z)
\end{aligned}$$

(b) By using the above relation, show that for any positive integer n,

$$\Gamma(n+1) = n!.$$

We proceed via induction.

Base case (n=0):

$$\begin{aligned}
\Gamma(1) &= \int_0^\infty x^0 e^{-x} dx \\
&= \int_0^\infty e^{-x} dx \\
&= [-e^{-x}]_0^\infty \\
&= 1 \\
&= 0!
\end{aligned}$$

Assume this holds for $n = k$, i.e. $\Gamma(k) = (k-1)!$ Then we want to show this is true for $n = k+1$.

$$\begin{aligned}
\Gamma(k+1) &= k\Gamma(k) \text{ by part (a)} \\
&= k \cdot (k-1)! \text{ by the inductive hypthesis} \\
&= k!
\end{aligned}$$

$\therefore$ By mathematical induction, $\Gamma(n+1) = n!$ for any positive integer n.

6. Let consider two pdfs

$$f_1(x) = \mathbb{1}_{[0,1]}(x), \qquad f_2(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}.$$

For some $0 < \alpha < 1$, let define

$$f(x) = \alpha f_1(x) + (1-\alpha) f_2(x).$$

(a) Show that $f(x)$ is a probability density function.

Since $f_1(x)$ and $f_2(x)$ are probability density functions, we know $f_1(x) \geq 0$, $f_2(x) \geq 0$, $\int_{-\infty}^{\infty} f_1(x)dx = 1$, and $\int_{-\infty}^{\infty} f_2(x)dx = 1$. Since $0 < \alpha < 1$ (and so $0 < 1 - \alpha < 1$), we know $f(x) \geq 0$. Also,

$$\begin{aligned}
\int_{-\infty}^{\infty} f(x)dx &= \int_{-\infty}^{\infty} \alpha f_1(x) + (1-\alpha) f_2(x)dx \\
&= \alpha \int_{-\infty}^{\infty} f_1(x)dx + (1-\alpha) \int_{-\infty}^{\infty} f_2(x)dx \text{ by linearity of integrals} \\
&= \alpha + (1-\alpha) \\
&= 1
\end{aligned}$$

$\therefore f(x)$ is a probability density function.

(b) Let X_1 be a random variable whose pdf is $f_1(x)$ and X_2 be a random variable whose pdf is $f_2(x)$ where

$$E[X_1] = \mu_1, \quad \mathrm{Var}[X_1] = \sigma_1^2, \qquad E[X_2] = \mu_2, \quad \mathrm{Var}[X_2] = \sigma_2^2.$$

Let X be a random variable whose pdf is $f(x)$. Find $E[X]$ and $\mathrm{Var}[X]$.

Since $X_1 \sim \text{Uniform}([0,1])$, $\mu_1 = \frac{1}{2}$ and $\sigma_1^2 = \frac{1}{12}$. Similarly, X_2 follows a standard normal distribution, so we know $\mu_2 = 0, \sigma_2^2 = 1$.

Thus,

$$\begin{aligned}
E[X] &= \int_{-\infty}^{\infty} x f(x)dx \\
&= \alpha \int_{-\infty}^{\infty} x f_1(x)dx + (1-\alpha) \int_{-\infty}^{\infty} x f_2(x)dx \\
&= \alpha E[X_1] + (1-\alpha) E[X_2] \\
&= \alpha \left(\frac{1}{2}\right) + (1-\alpha)(0) \\
&= \frac{1}{2}\alpha
\end{aligned}$$

To find the variance, we will use $\text{Var}(X) = E[X^2] - E[X]^2$. First, we compute $E[X^2]$.

$$\begin{aligned}
E[X^2] &= \int_{-\infty}^{\infty} x^2 f(x)dx \\
&= \int_{-\infty}^{\infty} \alpha x^2 f_1(x) + (1-\alpha)x^2 f_2(x)dx \\
&= \alpha E[X_1^2] + (1-\alpha)E[X_2^2] \\
&= \alpha(\text{Var}(X_1) + E[X_1]^2) + (1-\alpha)(\text{Var}(X_2) + E[X_2]^2) \\
&= \alpha(\frac{1}{12} + \frac{1}{4}) + (1-\alpha)(1) \\
&= 1 - \frac{2}{3}\alpha
\end{aligned}$$

Therefore,

$$\begin{aligned}
\text{Var}(X) &= E[X^2] - E[X]^2 \\
&= 1 - \frac{2}{3}\alpha - \left(\frac{1}{2}\alpha\right)^2 \\
&= 1 - \frac{2}{3}\alpha - \frac{1}{4}\alpha^2
\end{aligned}$$