APMA 1650: Homework 6 Solutions

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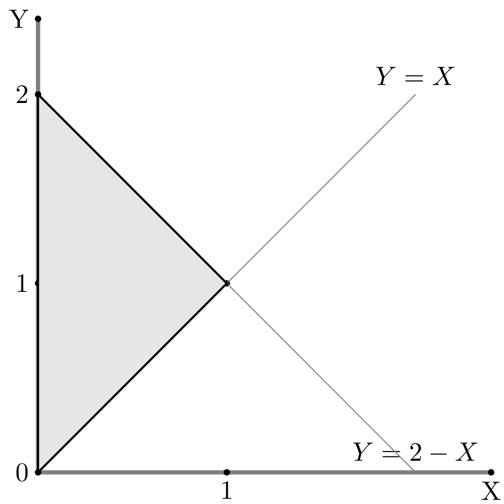
1. Suppose X and Y have joint probability density function $f(x, y)$ given by

$$f(x,y) = \begin{cases} cx^2y, & \text{if } 0 \leq x \leq y,\ x+y \leq 2 \\ 0, & \text{elsewhere} \end{cases}$$

We begin by drawing the region.



(a) Find the value of c that makes $f(x, y)$ a probability density function.

$$\begin{aligned} 1 &= \int_0^1 \int_x^{2-x} cx^2 y dy dx \\ &= \int_0^1 xc^2 \left(\frac{1}{2}y^2\right)\Big|_{y=x}^{y=2-x} \\ &= \int_0^1 c(2x^2 - 2x^3)dx \\ &= c\left(\frac{2}{3}x^3 - \frac{1}{2}x^4\right)\Big|_0^1 \\ &= \frac{c}{6} \end{aligned}$$

So $c = 6$.

(b) Find the marginal density functions for X and Y. Using $f(x, y) = 6x^2y$ from the previous part, we have

$$
\begin{aligned}
f_X(x) &= \int_x^{2-x} 6x^2 y dy \\
&= \left(3x^2y^2\right)\Big|_{y=x}^{y=2-x} \\
&= 12x^2 - 12x^3
\end{aligned}
$$

Taking into account our bounds, we have $f_X(x) = (12x^2 - 12x^3)\mathbb{1}_{[0,1]}(x)$.

To find $f_Y(y)$, we have to split the region in two. First, when $0 \le y \le 1$:

$$
\begin{aligned}
f_Y(y) &= \int_0^y 6x^2 y dx \\
&= 2x^3y\Big|_{x=0}^{x=y} \\
&= 2y^4
\end{aligned}
$$

Second, when $1 < y \le 2$:

$$
\begin{aligned}
f_Y(y) &= \int_0^{2-y} 6x^2 y dx \\
&= 2x^3y\Big|_{x=0}^{x=2-y} \\
&= 2y(2-y)^3
\end{aligned}
$$

Putting these together, we have $f_Y(y) = 2y^4\mathbb{1}_{[0,1]}(y) + 2y(2-y)^3\mathbb{1}_{(1,2]}(y)$.

(c) Are X and Y independent?

X and Y are not independent since their joint density is not the product of their marginals.

(d) Find the conditional density of Y given $X = x$.

$$
\begin{aligned}
f_{Y|X=x}(y|X=x) &= \frac{f(x,y)}{f_X(x)} \\
&= \frac{6x^2y}{12x^2 - 12x^3} \\
&= \frac{y}{2(1-x)}
\end{aligned}
$$

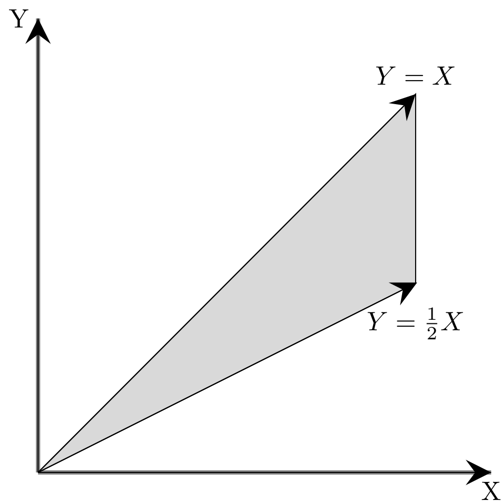
With the appropriate bounds, $f_{Y|X=x}(y|X=x) = \frac{y}{2(1-x)}\mathbb{1}_{[x,2-x]}(y)$.

(e) Find $P(Y < 1.1|X = 0.6)$.

$$
\begin{aligned}
P(Y < 1.1|X = 0.6) &= \int_{0.6}^{1.1} \frac{y}{2(0.4)} dy \\
&= \frac{y^2}{4(0.4)}\Big|_{0.6}^{1.1} \\
&= \frac{17}{32} \\
&= 0.53125
\end{aligned}
$$

2. Let X and Y be independent exponentially distributed random variables with mean $\mu = 1$. Find $P(X > Y|X < 2Y)$.

As always, first we draw the region!



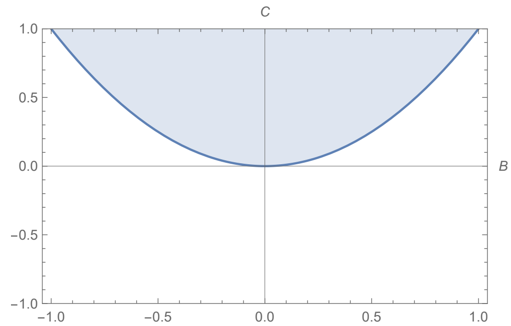
Using these bounds, we find that
$$\begin{aligned}
P(Y < X < 2Y) &= \int_0^\infty \int_y^{2y} e^{-(x+y)} dxdy \\
&= \int_0^\infty e^{-y}\left(-e^{-x}\right)\Big|_{x=y}^{x=2y} dy \\
&= \int_0^\infty -e^{-3y} + e^{-2y} dy \\
&= \left(\frac{1}{3}e^{-3y} - \frac{1}{2}e^{-2y}\right)\Big|_{y=0}^{y=\infty} \\
&= \frac{1}{6}
\end{aligned}$$

and similarly
$$\begin{aligned}
P(X < 2Y) &= \int_0^\infty \int_0^{2y} e^{-(x+y)} dxdy \\
&= \int_0^\infty e^{-y}(-e^{-x})\Big|_{x=0}^{x=2y} dy \\
&= \int_0^\infty -e^{-3y} + e^{-y} dy \\
&= \left(\frac{1}{3}e^{-3y} - e^{-y}\right)\Big|_{y=0}^{y=\infty} \\
&= \frac{2}{3}
\end{aligned}$$

Thus, $P(X > Y | X < 2Y) = \dfrac{P(Y < X < 2Y)}{P(X < 2Y)} = \dfrac{\frac{1}{6}}{\frac{2}{3}} = \dfrac{1}{4}$.

3. Consider the following quadratic equation: $x^2 + 2Bx + C = 0$. Suppose the coefficients B, C are independent random variables such that $B \sim U(-1, 1)$ and $C \sim U(-1, 1)$. Find the probability that the equation does not have real solutions.

   By the quadratic formula, $x = -B \pm \sqrt{B^2 - C}$, so this equation has no real roots if $B^2 < C$. We can solve this geometrically by drawing the region:



We can find the area of the shaded region using integration:

$$\begin{aligned}\text{Area of the shaded region} &= 2 - \int_{-1}^{1} x^2 dx \\ &= 2 - \frac{2}{3} \\ &= \frac{4}{3}\end{aligned}$$

Since B and C are uniform, the probability that $B^2 < C$ is the ratio of the shaded area to the total area:

$$P(B^2 < C) = \frac{\text{Area of the shaded region}}{\text{Total area}} = \frac{\frac{4}{3}}{4} = \frac{1}{3}.$$

Alternatively, we could solve this by finding the joint distribution of B and C. Since B and C are independent, the joint density of B and C is given by $f(b,c) = f_B(b)f_C(c) = \frac{1}{2}\mathbb{1}_{[-1,1]}(b)\frac{1}{2}\mathbb{1}_{[-1,1]}(c)$. Thus,

$$\begin{aligned}P(B^2 < C) &= \int_0^1 \int_{-\sqrt{c}}^{\sqrt{c}} \frac{1}{4} db dc \\ &= \int_0^1 \frac{1}{2}\sqrt{c} dc \\ &= \frac{1}{3}\end{aligned}$$

4. A building has n floors numbers $1, 2, \ldots, n$. At floor 0, m people get on the elevator together. Each gets off at one of the n floors, uniformly at random (and independently of everybody else). What is the expected number of floors the elevator stops at?

   Let X be the number of floors the elevator stops at, and let X_i $i = 1, \ldots, n$ be the indicator random variable such that $X_i = 1$ if the elevator stops at floor i and $X_i = 0$ otherwise. Then $X = \sum_{i=1}^{n} X_i$, so we can use linearity of expectations to find $E[X]$.

   Floor i will not be stopped at if every passenger stops at a different floor. Because the people are independent, $P(X_i = 0) = \left(\frac{n-1}{n}\right)^m$. Since $X_i \in \{0, 1\}$, it follows that $P(X_i = 1) = 1 - \left(\frac{n-1}{n}\right)^m$. Therefore,

$$E[X_i] = 0\left(\frac{n-1}{n}\right)^m + 1\left(1 - \left(\frac{n-1}{n}\right)^m\right) = 1 - \left(\frac{n-1}{n}\right)^m.$$

   Finally, this allows us to compute $E[X]$ using linearity:

$$E[X] = E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E[X_i] = n\left(1 - \left(\frac{n-1}{n}\right)^m\right).$$

5. Suppose that a sequence of n 1's and m 0's is randomly permuted so that each of the possible arrangements is equally likely. Any consecutive string of 1's is said to constitute a run a 1's. For example, if $n = 6, m = 4$, and the ordering is

$$1, 1, 1, 0, 1, 1, 0, 0, 1, 0$$

then, there are 3 runs of 1's. Find the expected number of runs of 1's.

Note that excluding the case of a run at the end of the sequence, each run of 1's must end in $1, 0$. Thus, we can count the number of runs not at the end by counting the number of sequences of $1, 0$. Let X be the number of runs of 1's, and define X_k by

$$X_k = \begin{cases} 1, & \text{if digit } k-1 = 1 \text{ and digit } k = 0 \\ 0, & \text{otherwise} \end{cases}$$

for $k = 2, \ldots, n+m$. We have a run at the end of the sequence if the sequence ends in 1. Define Y by $Y = \begin{cases} 1, & \text{if the last digit is 1} \\ 0, & \text{else} \end{cases}$

Then, $X = \sum_{k=2}^{n+m} X_k + Y$, so we can use linearity of expectation to find $E[X]$.

The probability of the sequence $1, 0$ is $\frac{n}{n+m}\frac{m}{n+m-1}$, so $E[X_k] = \frac{nm}{(n+m)(n+m-1)}$. Similarly, the probability that the last digit is a 1 is $\frac{n}{n+m}$, so $E[Y] = \frac{n}{n+m}$. Then,

$$E[X] = \sum_{k=2}^{n+m} E[X_k] + E[Y] = (n+m-1)\frac{nm}{(n+m)(n+m-1)} + \frac{n}{n+m} = \frac{nm+n}{(n+m)}.$$

6. Supposed that the number of people who enter the CVS on Thayer street on a given day follows a Poisson distribution with λ. Also suppose each person who enters the CVS is a male with probability p and a female with probability $q = 1 - p$. Show that the number of males and females entering CVS are independent Poisson random variables with parameters λp and λq, respectively.

Let X be the number of people entering CVS. Then it is given that $X \sim \text{Poiss}(\lambda)$. Let Y and Z be the number of men and women that enter CVS, respectively. Note that $X = Y + Z$.

First, we reason that if $Y = y$, and $Z = z$ given that $X = y + z$, then we know that y men and z women entered the CVS. The probability of a man entering is p, and the probability of a woman entering is q, so the probability that y men and z women entered in any order is

$$P(Y = y, Z = z | X = y + z) = \binom{y+z}{y} p^y q^z. \tag{1}$$

Then, the joint pmf of Y and Z is given by

$$\begin{aligned}
P(Y = y, Z = z) &= P(Y = y, Z = z | Y + Z = y + z) P(Y + Z = y + z) \\
&\quad + P(Y = y, Z = z | Y + Z \neq y + z) P(Y + Z \neq y + z) \\
&= P(Y = y, Z = z | Y + Z = y + z) P(Y + Z = y + z) \\
&= P(Y = y, Z = z | X = y + z) P(X = y + z) \text{ since } X = Y + Z \\
&= \left(\binom{y+z}{y} p^y q^z\right) \left(\frac{e^{-\lambda}\lambda^{y+z}}{(y+z)!}\right) \text{ by (1) and } X \sim \text{Pois}(\lambda) \\
&= \left(\frac{(\lambda p)^y}{y!}\right) \left(\frac{e^{-\lambda}(\lambda q)^z}{z!}\right) \\
&= \left(\frac{(e^{-\lambda p}\lambda p)^y}{y!}\right) \left(\frac{e^{-\lambda q}(\lambda q)^z}{z!}\right) \text{ by multiplying by } 1 = e^{-\lambda p + \lambda p}
\end{aligned}$$

Note that we have factored the joint pmf of Y and Z into the product of two Poisson variables with rates λp and λq. Thus, Y and Z are independent with $Y \sim \text{Pois}(\lambda p)$ and $Z \sim \text{Pois}(\lambda q)$.