1. Solution:

a. Consider the probability distribution $P(X = 2^k) = 2^{-k}$. for all $k \geq 1$ For example, $P(X = 1) = 0.5, P(X = 2) = 0.25$ and so on. This satisfies the axioms of probability and I claim that this has infinite expectation.

Indeed, $E[X] = \sum_{k=1}^{\infty}(2^k)(2^{-k}) = \sum_{k=1}^{\infty} 1 = \infty$

This sum diverges so this probability distribution has infinite expectation.

b. We will manipulate the left side until it eventually equals our desired quantity.

$$
\begin{aligned}
Var[\alpha X + \beta] &= E[(\alpha X + \beta)^2] - E[(\alpha X + \beta)]^2 && (1)\\
&= E[\alpha^2 X^2 + 2X\alpha\beta + \beta^2] - (\alpha E[x] + \beta)^2 \text{ by expansion} && (2)\\
&= \alpha^2 E[X^2] + 2\alpha\beta E[x] + \beta^2 - (\alpha^2 E[x]^2 + 2\alpha\beta E[x] + \beta^2) \text{ linearity of expectation} && (3)\\
&= \alpha^2 E[X^2] + 2\alpha\beta E[x] + \beta^2 - \alpha^2 E[x]^2 - 2\alpha\beta E[x] - \beta^2 && (4)\\
&= \alpha^2 E[X^2] - \alpha^2 E[x]^2 && (5)\\
&= \alpha^2 Var[X] \text{ by formula for variance} && (6)
\end{aligned}
$$

and we are done.

c. The important thing to recognize her is that $E[X]$ and $Var[X]$ are numbers that can be pulled out of the expectation. Indeed,

$$E[Y] = E[\frac{X - E[X]}{\sqrt{Var[X]}}] = \frac{E[X] - E[X]}{\sqrt{Var[X]}} = 0$$

Therefore, $E[Y] = 0$.

$$Var[Y] = E[Y^2] - E[Y]^2 = E[Y^2]$$

since $E[Y] = 0$ from previous.

$$E[Y^2] = E[\frac{(X - E[X])^2}{Var[X]}] = \frac{E[X^2 - 2XE[X] + E[X]^2]}{Var[X]} = \frac{E[X^2] - E[X]^2}{Var[X]} = 1.$$

2. Solution:

We note that the expectation for each of the values 1 through 6 is the same because the probability one of these numbers is the same for fair dice.

We calculate the probability that we roll j of a number, for $j = 0, 1, 2, 3$

The probability of rolling 3 of a chosen number is $(\frac{1}{6})^3 = \frac{1}{216}$.

To calculate the probability of rolling 2 of a chosen number can be calculated in the following fashion. The number we roll "one" of can be chosen in 5 ways and it can appear on one of three dice. Therefore, the probability of rolling 2 of a chosen number if $\frac{3*5}{216} = \frac{15}{216}$.

Similarly, when calculating the probability of 1 of our chosen number appearing, it can appear in one of three dice. The other two dice can display one of the other 5 numbers. Therefore, the probability is $\frac{3*5*5}{216} = \frac{75}{216}$.

Finally, the probability of 0 of our chosen number is $\frac{5*5*5}{216} = \frac{125}{216}$.

Thus, the expectation of this game is

$$\frac{1}{216} * 3 + \frac{15}{216} * 2 + \frac{75}{216} * 1 + \frac{125}{216} * (-1) = \frac{3}{216} + \frac{30}{216} + \frac{75}{216} - \frac{125}{216} = \frac{-17}{216}$$

This game has a negative expectation so I would choose not to play it.

3. Solution:

a. The probability distribution of X_3 is:

$$X = \begin{cases} 3, & p^3 \\ 1, & 3p^2q \\ -1, & 3pq^2 \\ -3, & q^3 \end{cases} \tag{7}$$

Note that the probabilities are the same as Y_3. This means that all we have to do is translate the random variable such that $X_3 = Y_3$. Indeed, if we have $2 * Y_3 - 3$, this gives us the same probability distribution as X_3.

b. The two random variables X_n and Y_n are identical in distribution. Therefore, it satisfies to translate Y_n, similar to the way we did in part(a). Thus, I get the equality $X_n = 2 * Y_n - n$ We can calculate the expectation and variance by using the binomial distribution. Thus,

$$E[X] = E[2Y_n - n] = 2np - n$$
$$Var[X] = Var[2 * Y_n - n] = 4Var[Y_n] = 4np(1-p).$$

4. Solution:

For each of the 32 first-round games, we have a 0.5 chance of guessing correctly if we are using a fair coin. For each of the 16 second-round games, we would guess correctly if we had two correct coin flips, for a probability of 0.25. Generalizing, for the n-th round game, we have $\frac{1}{2^n}$ probability of guessing it correctly. Hence, the expectation of the number of points is:

$$32 * (\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64}) = \frac{63}{2}$$

5. Solution:

a. We can use the hypergeometric distribution here with the parameters $N = 50, n = 15, r = 30$. For a given k, where we are calculating the probability that exactly k of the graded projects are from the second group, we can use the hypergeometric distribution:

$$P(X = k) = \frac{\binom{30}{k}\binom{20}{15-k}}{\binom{50}{15}}$$

Since we are calculating the probability that there are at least 10, we can take the sum from $k = 10$ to $k = 15$ to achieve our desired result.

$$P(X \geq 10) = \sum_{k=10}^{15} \frac{\binom{30}{k}\binom{20}{15-k}}{\binom{50}{15}} = 0.3798$$

b. The complement to this event is the probability that at least ten do not come from the same section. This can only occur if 6,7,8,or 9 come from the same section (If there were less than 5, then at least 10 would come from the first section). Thus, the probability is equal to:

$$1 - \sum_{k=6}^{9} \frac{\binom{30}{k}\binom{20}{15-k}}{\binom{50}{15}} = 0.3938$$

c. We can use the formulas for mean and variance that were presented in lecture. Using the parameters $N = 50, n = 15, r = 30$, we have:

$$E[X] = \frac{nr}{N} = \frac{15 * 30}{50} = 9$$
$$Stdev[X] = \sqrt{Var[X]} = \sqrt{n\frac{r}{N}\frac{N-r}{N}\frac{N-n}{N-1}} = \sqrt{15\frac{30}{50}\frac{20}{50}\frac{35}{49}} = 1.604$$

d. We can use the formulas for mean and variance that were presented in lecture except using 35 for n instead of 15. Using the parameters $N = 50, n = 35, r = 30$, we have:

$$E[X] = \frac{nr}{N} = \frac{35 * 30}{50} = 21$$
$$Stdev[X] = \sqrt{Var[X]} = \sqrt{n\frac{r}{N}\frac{N-r}{N}\frac{N-n}{N-1}} = \sqrt{35\frac{30}{50}\frac{20}{50}\frac{15}{49}} = 1.604$$

6. Solution:

The probability for $X = k$ in a geometric distribution is $(1-p)^{k-1}p$. Therefore the probability of X being equal to an even number is

$$\begin{aligned}
P(even) &= P(X = 2) + P(X = 4) + ... && (8)\\
&= (1-p)p + (1-p)^3 p + ... && (9)\\
&= \sum_{k=0}^{\infty}(1-p)^{2k+1}p && (10)\\
&= \frac{(1-p)p}{1-(1-p)^2} && (11)\\
&= \frac{(1-p)p}{1-(1-2p+p^2)} && (12)\\
&= \frac{(1-p)p}{2p-p^2} && (13)\\
&= \frac{1-p}{2-p} && (14)
\end{aligned}$$

where we were able to simplify the sum by using the sum of an infinite geometric series.