APMA 1650 Homework 07

Due: November 08, 2018

Due before class on Thursday, Nov 08, 2018. It can be dropped off in the APMA 1650 homework box on the first floor of the APMA department, 182 George St OR at class (before it starts).

Please attach the HW cover sheet to *the front of your HW* assignment. It can be found on Canvas. **Show all work and you MUST write up your own solutions.**

1. (Hat Check problem). n people enter the restaurant and put their hats at the reception. Each person gets a random hat back when going back after having dinner. Find the expected value and variance of the number of people who get their right hat back.

2. (a) Let $Z \sim \mathcal{N}(0, 1)$. Find the mgf of Z.
   (b) Let $X \sim \mathcal{N}(\mu, \sigma^2)$. Find the mgf of X. Hint: $X = \mu + \sigma Z$.
   (c) Let $X_1, X_2, \cdots, X_n$ be i.i.d. random variables of $\mathcal{N}(\mu, \sigma^2)$. Let
   $$\bar{X} = \frac{X_1 + X_2 + \cdots + X_n}{n}.$$
   Find the mgf of X.
   (d) Use (c), find $E[\bar{X}]$ and Var$[\bar{X}]$.

3. Let $Z \sim \mathcal{N}(0, 1)$. Let $X_1 = Z$ and $X_2 = Z^2$.
   (a) Find $E[X_1]$ and $E[X_2]$.
   (b) Find $E[X_1 X_2]$.
   (c) Find Cov(X_1, X_2).
   (d) Are X_1 and X_2 independent?

4. (Updated) A die is thrown 12 times (independently).
   (a) Find the probability that every face appears twice.
   (b) Let X be the number of appearances of 6 and Y the number of appearances of 1. Find the joint probability distribution of X and Y.
   (c) Find Cov(X, Y).

5. Twenty people consisting of 10 married couples are to be seated at 5 different tables, with 4 people at each table.

   (a) If all seats are randomly assigned, what is the expected number of married couples that are seated at the same table?

   (b) If 2 men and 2 women are randomly chosen to be seated at each table, what is the expected number of married couples that are seated at the same table?

6. Let $X_1, \cdots, X_n$ be i.i.d. random variables having expected value μ and variance σ^2. Let consider the following functions of random variables:

$$\hat{\mu}_n(\mathbf{X}) := \frac{1}{n}\sum_{i=1}^{n} X_i, \qquad S^2 := \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2, \qquad \hat{\sigma}_n^2(\mathbf{X}) := \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2.$$

   **Note that $\hat{\mu}_n(\mathbf{X})$ is called the sample mean and $\hat{\sigma}_n^2(\mathbf{X})$ is called the sample variance.

   (a) Find $E[\hat{\mu}_n(\mathbf{X})]$, $E[S^2]$, and $E[\hat{\sigma}_n^2(\mathbf{X})]$.

   (b) By using the weak law of large number, show that for any $\epsilon > 0$,

$$\lim_{n\to\infty} P(|\hat{\mu}_n(\mathbf{X}) - \mu| > \epsilon) = 0.$$