1. Solution:

a. We integrate to find the CDF of $f(x)$. In particular, we have that

$$F(k) = P(X < k) = \int_0^k \frac{2(\theta - x)}{\theta^2} dx = \frac{1}{\theta^2} \int_0^k 2(\theta - x) dx = \frac{2k}{\theta} - \frac{k^2}{\theta^2}$$

Therefore, the CDF of this function is :

$$F(k) = \begin{cases} 0, & k < 0, \\ \frac{2k}{\theta} - \frac{k^2}{\theta^2}, & 0 < k < \theta \\ 1, & k > \theta \end{cases}$$

b. This is very similar to part a, except instead of calculating the probability $P(X < k)$, we calculate the probability $P(\frac{X}{\theta} < k)$. The calculations are as follows:

$$F_2(k) = P(\frac{X}{\theta} < k) = P(X < k\theta) = F(k\theta) = \frac{2k\theta}{\theta} - \frac{k^2\theta^2}{\theta^2} = 2k - k^2$$

$$F_2(k) = \begin{cases} 0, & k < 0, \\ 2k - k^2, & 0 < k < 1 \\ 1, & k > 1 \end{cases}$$

c. We wish to calculate $P(\frac{X}{\theta} \leq k) = 0.9$ for some k. The LHS is equivalent to the CDF from part b. So we can solve the equation $2k - k^2 = 0.9$ for k between 0 and 1. Solving this equation (through Wolfram Alpha) gives us $k = 0.684$ (we reject the value that is not between 0 and 1). This gives us the inequality $\frac{X}{\theta} \leq 0.684$ and rearranging for θ gives us $\theta \geq \frac{X}{0.684}$.

d. We wish to calculate $P(\frac{X}{\theta} \leq k) = 0.1$ for some k. The LHS is equivalent to the CDF from part b. So we can solve the equation $2k - k^2 = 0.1$ for k between 0 and 1. Solving this equation (through Wolfram Alpha) gives us $k = 0.051$ (we reject the negative value because we know k is bounded between 0 and 1). This gives us the inequality $\frac{X}{\theta} \leq 0.051$ and rearranging for θ gives us $\theta \geq \frac{X}{0.051}$.

2. Solution:

1. We can use large sample approximation for confidence interval using the values $\alpha = 0.95, n = 500, \mu = 5.4$,and $\sigma = 3.1$. This gives us the following interval

$$[\bar{x} - z_{0.025} * \frac{\sigma}{\sqrt{n}} \leq \mu \leq \bar{x} + z_{0.025} * \frac{\sigma}{\sqrt{n}}]$$
$$[5.4 - 1.96 * \frac{3.1}{\sqrt{500}} \leq \mu \leq 5.4 + 1.96 * \frac{3.1}{\sqrt{500}}]$$
$$[5.4 - 0.2717 \leq \mu \leq 5.4 + 0.2717]$$
$$[5.1283 \leq \mu \leq 5.6717]$$

2. Because n is sufficiently small, we use a t-test to solve for the confidence interval. We can use the values $2.086, n = 30, \mu = 6.58$, and $\sigma = 7.4$. This gives us the following interval

$$[\bar{x} - t_{0.025} * \frac{\sigma}{\sqrt{n}} \leq \mu \leq \bar{x} + t_{0.025} * \frac{\sigma}{\sqrt{n}}]$$
$$[6.58 - 2.086 * \frac{7.4}{\sqrt{21}} \leq \mu \leq 6.58 + 2.086 * \frac{7.4}{\sqrt{21}}]$$
$$[6.58 - 3.37 \leq \mu \leq 6.58 + 3.37]$$
$$[3.21 \leq \mu \leq 9.95]$$

3. Solution:

a. We first calculate the sample mean and the sample variance from the given samples.

$$\bar{x} = \frac{29+44+61+72+59}{5} = \frac{265}{5} = 53$$
$$S^2 = \frac{\sum (X_i - \bar{x})^2}{n-1} = \frac{24^2+9^2+8^2+19^2+6^2}{4} = 279.5$$

Now in order to calculate the confidence interval, we utilize the t-test because n is small. This gives us the following interval.

$$[\bar{x} - t_{0.025} * \frac{\sigma}{\sqrt{n}} \leq \mu \leq \bar{x} + t_{0.025} * \frac{\sigma}{\sqrt{n}}]$$
$$[53 - 4.596 * \frac{\sqrt{279.5}}{\sqrt{5}} \leq \mu \leq 53 + 4.596 * \frac{\sqrt{279.5}}{\sqrt{5}}]$$
$$[53 - 34.363 \leq \mu \leq 53 + 34.363]$$
$$[18.637 \leq \mu \leq 87.363]$$

b. We use the chi-square test in order to calculate the variance confidence interval (the values were provided in the homework problem set). Therefore, we calculate the interval as follows:

$$\sqrt{\frac{(n-1)s^2}{\chi^2_{\frac{\alpha}{2}}}} \leq \sigma \leq \sqrt{\frac{(n-1)s^2}{\chi^2_{1-\frac{\alpha}{2}}}}$$

These are all known values, so we calculate the interval by plugging in and simplifying.

$$\sqrt{\frac{4 * 279.5}{14.8602}} \leq \sigma \leq \sqrt{\frac{4 * 279.5}{0.20699}}$$
$$\sqrt{75.235} \leq \sigma \leq \sqrt{5401.227}$$
$$8.674 \leq \sigma \leq 73.49$$