Homework 2

菠萝包萝卜

2025 年 10 月 9 日

1 HW-2

Problem. 2.1.1 Suppose $(X_1, \cdots, X_n)$ has density $f(x_1, x_2 \cdots, x_n)$, that is

$$P((X_1, \cdots, X_n) \in A) = \int_A f(\boldsymbol{x}) d\boldsymbol{x} \ for \ A \in \mathcal{R}^n$$

If $f(\boldsymbol{x})$ can be written as $g_1(x_1) \cdots g_n(x_n)$ where the $g_m \geq 0$ and measurable, then $X_1, \cdots, X_n$ are Independent. Note that the g_m are not assumed to be probability densities.

Proof. 等价的，如果能从 $g_1(x_1) \cdots g_n(x_n), g_m(x_m) \geq 0$ ，得到对于任意 $x_1, \cdots, x_n \in \mathbb{R}$ 都有

$$\int_{t_1 \leq x_1, \cdots t_n \leq x_n} f(\boldsymbol{t}) d\boldsymbol{t} = \prod_{i=1}^{n} \int_{t_i \leq x_i} f_i(t_i) dt_i$$

就行了，由于 $f(\boldsymbol{x}) = \prod_{i=1}^{n} g_i(x_i)$，所以

$$\begin{aligned} LHS &= \int_{t_1 \leq x_1, \cdots t_n \leq x_n} f(\boldsymbol{t}) d\boldsymbol{t} = \int_{t_1 \leq x_1, \cdots t_n \leq x_n} \prod_{i=1}^{n} g(t_i) dt_1 \cdots dt_n \\ &= \prod_{i=1}^{n} \int_{t_i \leq x_i} g(t_i) dt_i \end{aligned}$$

现在由归一性，由于密度 $f(\boldsymbol{x})$ 可以变量分离

$$\left(\int_{\mathbb{R}} g(t_1) dt_1 \right) \left(\int_{\mathbb{R}} g(t_2) dt_2 \right) \cdots \left(\int_{\mathbb{R}} g(t_n) dt_n \right) = 1$$

分别把这 n 个积分值设为 $c_1, c_2, \cdots, c_n$，就有 $c_1 \cdots c_n = 1$，然后求某个 X_i 的边际分布，对其他 $n-1$ 个变量在 $\mathbb{R}$ 上积分

$$f_i(x_i) = g_i(x_i) \prod_{j \neq i} c_j = g_i(x_i) \tilde{c}_i$$

其中 $c_i \tilde{c}_i = 1$，这样回到前面就是

$$\begin{aligned} LHS &= \prod_{i=1}^{n} \int_{t_i \leq x_i} g(t_i) dt_i = \prod_{i=1}^{n} \int_{t_i \leq x_i} \frac{f_i(t_i)}{\tilde{c}_i} dt_i \\ &= \prod_{i=1}^{n} \int_{t_i \leq x_i} c_i f_i(t_i) dt_i \\ &= c_1 \cdot c_2 \cdots c_n \prod_{i=1}^{n} \int_{t_i \leq x_i} f_i(t_i) dt_i \ \textbf{这里注意到} c_1 \cdots c_n = 1 \\ &= \prod_{i=1}^{n} \int_{t_i \leq x_i} f_i(t_i) dt_i \end{aligned}$$

□

Problem. 2.1.6 Prove directly from the definition that if X and Y are independent and f and g are measurable functions then $f(X)$ and $g(Y)$ are independent.

Proof. 对于任意的 $T \in \mathcal{R}, S \in \mathcal{R}$，考虑

$$P(\{\omega : f(X(\omega)) \in T\} \cap \{\omega : g(Y(\omega)) \in S\})$$

由于 f 可测，所以

$$\{x : f(x) \in T\} = A \in \mathcal{R}$$

然后

$$\{\omega : f(X(\omega)) \in T\} = \{\omega : X(\omega) \in A\} \in \mathcal{R}$$

类似地

$$\{\omega : g(Y(\omega)) \in S\} = \{\omega : Y(\omega) \in B\} \in \mathcal{R}$$

由 X, Y 独立，所以

$$P(X \in A, Y \in B) = P(X \in A)P(Y \in B)$$

还原一下就是

$$P(f(X) \in T, g(Y) \in S) = P(f(X) \in T)P(g(Y) \in S)$$

进而独立. □

Problem. 2.1.14 Let $X, Y \geq 0$ be independent with distribution functions F and G. Find the distribution function of XY.

Solution. 设 XY 分布函数为 H ，首先对于任意的 $z < 0$ 始终成立

$$H(z) = P(XY \leq z) = 0$$

然后从 $z \geq 0$ 开始

$$\begin{aligned} P(XY \leq z) &= \int_{\{xy \leq z, x \geq 0, y \geq 0\}} 1dF(x)dG(y) \\ &= \int_0^{+\infty} 1dF(x) \int_0^{\frac{z}{x}} 1dG(y) = \int_0^{+\infty} G(\frac{z}{x})dF(x) \end{aligned}$$

当然也可以是

$$P(XY \leq z) = \int_0^{+\infty} F(\frac{z}{y})dG(y)$$

□

Problem. 1.4.1 Show that if $f \geq 0$ and $\int f d\mu = 0$ then $f = 0$ *a.e.*

Proof. 假设 f 不是几乎处处为 0，那么存在 $\epsilon > 0$ 使得 $\mu(\{f > \epsilon\}) > 0$

设在定义空间上的一个函数 $h(x) = \begin{cases} \frac{\epsilon}{2} & ,f(x) > \epsilon \\ 0 & ,f(x) \le \epsilon \end{cases}$，有 $0 < h < f, h \le \epsilon, \mu(\{h > 0\}) \le \mu(\{f > \epsilon\}) < \infty$ （当然这里如果 $\{f > \epsilon\}$ 是无穷测度集那么 f 积分也显然非 0 了）然后有

$$\int f d\mu > \int h d\mu = \int_{\{h>0\}} h d\mu = \frac{\epsilon}{2} \cdot \mu(\{f > \epsilon\}) > 0$$

矛盾！ □

Problem. 1.5.1 Let $||f||_\infty = \inf\{M : \mu(\{x : |f(x)| > M\}) = 0\}$. Prove that

$$\int |fg| d\mu \le ||f||_1 \cdot ||g||_\infty$$

分析一下，$||f||_\infty$ 就是使得 $|f(x)| \le M$ *a.e.* 的最小 M

Proof. 我们先证明 $A = \{x : |g| > ||g||_\infty\}$ 是一个零测集，注意到

$$\{x : |g| > ||g||_\infty\} = \bigcup_{i=1}^{\infty} \{x : |g| > ||g||_\infty + \frac{1}{i}\}$$

由 $||g||_\infty$ 定义右边是可列个零测集的并，必然是零测集，从而左边也是零测集，然后回到原问题

$$\begin{aligned} \int |fg| d\mu &= \int_{\{|g|>||g||_\infty\}} |fg| d\mu + \int_{\{|g|\le||g||_\infty\}} |fg| d\mu \\ &= \int_{|g|\le||g||_\infty} |fg| d\mu \\ &\le ||g||_\infty \int_{|g|\le||g||_\infty} |f| d\mu \le ||g||_\infty \int |f| d\mu = ||f||_1 \cdot ||g||_\infty \end{aligned}$$

□

Problem. 1.5.2 Show that if μ is a probability measure then

$$||f||_\infty = \lim_{p\to\infty} ||f||_p$$

Proof. 由 $||f||_\infty$ 定义以及 μ 是个概率测度，我们有 $\mu(\{|f| \le ||f||_\infty\}) = 1$，然后

$$|f|^p \le ||f||_\infty^p \ a.e.$$

从而积分

$$\int |f|^p d\mu \le \int ||f||_\infty^p d\mu = ||f||_\infty^p \cdot \mu(\Omega) = ||f||_\infty^p$$

即

$$\left(\int |f|^p d\mu\right)^{\frac{1}{p}} \le ||f||_\infty$$

对任意 $p > 1$ 成立，现在取个上极限就是

$$\limsup_{p\to\infty} \left(\int |f|^p d\mu\right)^{\frac{1}{p}} \le ||f||_\infty$$

另一方面，$\forall \epsilon > 0$，定义 $M = ||f||_\infty - \epsilon$, 由 $||f||_\infty$ 定义就有

$$A = \mu(\{x : |f(x)| > M\}) > 0$$

这样对于任意 $p > 1$ 就有

$$\int |f|^p d\mu \geq \int_A |f|^p d\mu > \int_A M^p d\mu = M^p \mu(A)$$

然后取 $1/p$ 次幂就是

$$\left(\int |f|^p d\mu \right)^{\frac{1}{p}} \geq M \mu^{\frac{1}{p}}(A)$$

两边同时取下极限

$$\liminf_{p\to\infty} ||f||_p \geq (||f||_\infty - \epsilon) \cdot 1 = ||f||_\infty - \epsilon$$

由 ϵ 任意性，取 $\epsilon \to 0$ 便有

$$\liminf_{p\to\infty} ||f||_p \geq ||f||_\infty$$

从而

$$||f||_\infty = \lim_{p\to\infty} ||f||_p$$

□