

# Least-Squares Rigid Motion Using SVD

Olga Sorkine

---

## Abstract

This note summarizes the steps to computing the rigid transformation that aligns two sets of points.

*Key words:* Shape matching, rigid alignment, rotation, SVD

---

## 1 Problem statement

Let  $P = \{\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_n\}$  and  $Q = \{\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_n\}$  be two sets of corresponding points in  $\mathbb{R}^d$ . We wish to find a rigid transformation that optimally aligns the two sets in the least squares sense, i.e., we seek a rotation  $R$  and a translation vector  $\mathbf{t}$  such that

$$(R, \mathbf{t}) = \operatorname{argmin}_{R, \mathbf{t}} \sum_{i=1}^n w_i \|(R\mathbf{p}_i + \mathbf{t}) - \mathbf{q}_i\|^2, \quad (1)$$

where  $w_i > 0$  are weights for each point pair.

In the following we will detail the derivation of  $R$  and  $\mathbf{t}$ ; readers that are interested in the final recipe may skip the proofs and go directly Section 4.

## 2 Computing the translation

Assume  $R$  is fixed and denote  $F(\mathbf{t}) = \sum_{i=1}^n w_i \|(R\mathbf{p}_i + \mathbf{t}) - \mathbf{q}_i\|^2$ . We can find the optimal translation by taking the derivative of  $F$  w.r.t.  $\mathbf{t}$  and searching for its roots:

$$\begin{aligned} 0 &= \frac{\partial F}{\partial \mathbf{t}} = \sum_{i=1}^n 2w_i (R\mathbf{p}_i + \mathbf{t} - \mathbf{q}_i) = \\ &= 2\mathbf{t} \left( \sum_{i=1}^n w_i \right) + 2R \left( \sum_{i=1}^n w_i \mathbf{p}_i \right) - 2 \sum_{i=1}^n w_i \mathbf{q}_i. \end{aligned} \quad (2)$$

Denote

$$\bar{\mathbf{p}} = \frac{\sum_{i=1}^n w_i \mathbf{p}_i}{\sum_{i=1}^n w_i}, \quad \bar{\mathbf{q}} = \frac{\sum_{i=1}^n w_i \mathbf{q}_i}{\sum_{i=1}^n w_i}. \quad (3)$$

By rearranging the terms of (2) we get

$$\mathbf{t} = \bar{\mathbf{q}} - R\bar{\mathbf{p}}. \quad (4)$$

In other words, the optimal translation  $\mathbf{t}$  maps the transformed weighted centroid of  $P$  to the weighted centroid of  $Q$ . Let us plug the optimal  $\mathbf{t}$  into our objective function:

$$\sum_{i=1}^n w_i \|(R\mathbf{p}_i + \mathbf{t}) - \mathbf{q}_i\|^2 = \sum_{i=1}^n w_i \|R\mathbf{p}_i + \bar{\mathbf{q}} - R\bar{\mathbf{p}} - \mathbf{q}_i\|^2 = \quad (5)$$

$$= \sum_{i=1}^n w_i \|R(\mathbf{p}_i - \bar{\mathbf{p}}) - (\mathbf{q}_i - \bar{\mathbf{q}})\|^2. \quad (6)$$

We can thus concentrate on computing the rotation  $R$  by restating the problem such that the translation would be zero:

$$\mathbf{x}_i := \mathbf{p}_i - \bar{\mathbf{p}}, \quad \mathbf{y}_i := \mathbf{q}_i - \bar{\mathbf{q}}. \quad (7)$$

So we look for the optimal rotation  $R$  such that

$$R = \operatorname{argmin}_R \sum_{i=1}^n w_i \|R\mathbf{x}_i - \mathbf{y}_i\|^2. \quad (8)$$

### 3 Computing the rotation

Let us simplify the expression we are trying to minimize in (8):

$$\begin{aligned} \|R\mathbf{x}_i - \mathbf{y}_i\|^2 &= (R\mathbf{x}_i - \mathbf{y}_i)^T (R\mathbf{x}_i - \mathbf{y}_i) = (\mathbf{x}_i^T R^T - \mathbf{y}_i^T) (R\mathbf{x}_i - \mathbf{y}_i) = \\ &= \mathbf{x}_i^T R^T R\mathbf{x}_i - \mathbf{y}_i^T R\mathbf{x}_i - \mathbf{x}_i^T R^T \mathbf{y}_i + \mathbf{y}_i^T \mathbf{y}_i = \\ &= \mathbf{x}_i^T \mathbf{x}_i - \mathbf{y}_i^T R\mathbf{x}_i - \mathbf{x}_i^T R^T \mathbf{y}_i + \mathbf{y}_i^T \mathbf{y}_i. \end{aligned} \quad (9)$$

We got the last step by remembering that rotation matrices imply  $R^T R = I$  ( $I$  is the identity matrix).

Note that  $\mathbf{x}_i^T R^T \mathbf{y}_i$  is a scalar:  $\mathbf{x}_i^T$  has dimension  $1 \times d$ ,  $R^T$  is  $d \times d$  and  $\mathbf{y}_i$  is  $d \times 1$ . For any scalar  $a$  we trivially have  $a = a^T$ , therefore

$$\mathbf{x}_i^T R^T \mathbf{y}_i = (\mathbf{x}_i^T R^T \mathbf{y}_i)^T = \mathbf{y}_i^T R\mathbf{x}_i. \quad (10)$$

Therefore we have

$$\|R\mathbf{x}_i - \mathbf{y}_i\|^2 = \mathbf{x}_i^T \mathbf{x}_i - 2\mathbf{y}_i^T R\mathbf{x}_i + \mathbf{y}_i^T \mathbf{y}_i. \quad (11)$$

Let us look at the minimization and substitute the above expression:

$$\begin{aligned} \operatorname{argmin}_R \sum_{i=1}^n w_i \|R\mathbf{x}_i - \mathbf{y}_i\|^2 &= \operatorname{argmin}_R \sum_{i=1}^n w_i (\mathbf{x}_i^T \mathbf{x}_i - 2\mathbf{y}_i^T R\mathbf{x}_i + \mathbf{y}_i^T \mathbf{y}_i) = \\ &= \operatorname{argmin}_R \left( \sum_{i=1}^n w_i \mathbf{x}_i^T \mathbf{x}_i - 2 \sum_{i=1}^n w_i \mathbf{y}_i^T R\mathbf{x}_i + \sum_{i=1}^n w_i \mathbf{y}_i^T \mathbf{y}_i \right) = \\ &= \operatorname{argmin}_R \left( -2 \sum_{i=1}^n w_i \mathbf{y}_i^T R\mathbf{x}_i \right). \end{aligned} \quad (12)$$

The last step (removing  $\sum_{i=1}^n w_i \mathbf{x}_i^T \mathbf{x}_i$  and  $\sum_{i=1}^n w_i \mathbf{y}_i^T \mathbf{y}_i$ ) holds because these expressions do not depend on  $R$  at all, so excluding them would not affect the minimizer. The same holds for multiplication of the minimization expression by a scalar, so we have

$$\operatorname{argmin}_R \left( -2 \sum_{i=1}^n w_i \mathbf{y}_i^T R\mathbf{x}_i \right) = \operatorname{argmax}_R \sum_{i=1}^n w_i \mathbf{y}_i^T R\mathbf{x}_i. \quad (13)$$

We note that

$$\sum_{i=1}^n w_i \mathbf{y}_i^T R\mathbf{x}_i = \operatorname{tr} (WY^T RX), \quad (14)$$

where  $W = \operatorname{diag}(w_1, \dots, w_n)$  is an  $n \times n$  diagonal matrix with the weight  $w_i$  on diagonal entry  $i$ ;  $Y$  is the  $d \times n$  matrix with  $\mathbf{y}_i$  as its columns and  $X$  is the  $d \times n$  matrix with  $\mathbf{x}_i$  as its columns. We remind the reader that the trace of a square matrix is the sum of the elements on the diagonal:  $\operatorname{tr}(A) = \sum_{i=1}^n a_{ii}$ . See Figure 1 for an illustration of the algebraic manipulation.

Therefore we are looking for a rotation  $R$  that maximizes  $\operatorname{tr}(WY^T RX)$ . Matrix trace has the property

$$\operatorname{tr}(AB) = \operatorname{tr}(BA) \quad (15)$$

for any matrices  $A, B$  of compatible dimensions. Therefore

$$\operatorname{tr}(WY^T RX) = \operatorname{tr}((WY^T)(RX)) = \operatorname{tr}(RXWY^T). \quad (16)$$

Let us denote the  $d \times d$  ‘‘covariance’’ matrix  $S = XWY^T$ . Take SVD of  $S$ :

$$S = U\Sigma V^T. \quad (17)$$

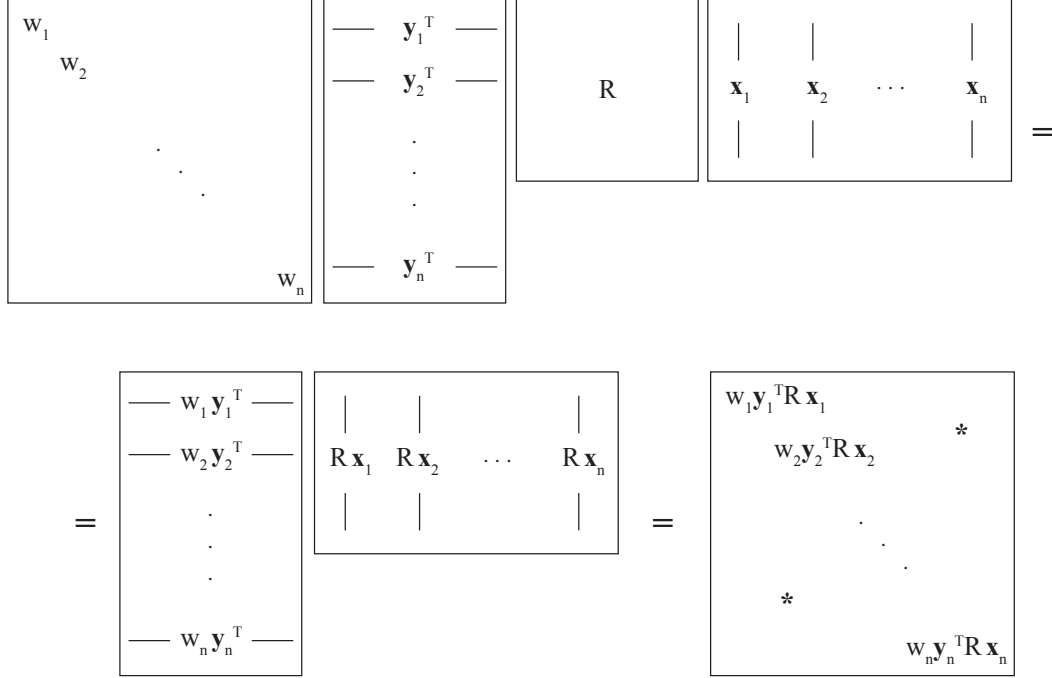


Fig. 1. Schematic explanation of  $\sum_{i=1}^n w_i \mathbf{y}_i^T R \mathbf{x}_i = \text{tr}(W Y^T R X)$ .

Now substitute the decomposition into the trace we are trying to maximize:

$$\text{tr}(R X W Y^T) = \text{tr}(R S) = \text{tr}(R U \Sigma V^T) = \text{tr}(\Sigma V^T R U). \quad (18)$$

The last step was achieved using the property of trace (15). Note that  $V$ ,  $R$  and  $U$  are all orthogonal matrices, so  $M = V^T R U$  is also an orthogonal matrix. This means that the columns of  $M$  are orthonormal vectors, and in particular,  $\mathbf{m}_j^T \mathbf{m}_j = 1$  for each  $M$ 's column  $\mathbf{m}_j$ . Therefore all entries  $m_{ij}$  of  $M$  are smaller than 1 in magnitude:

$$1 = \mathbf{m}_j^T \mathbf{m}_j = \sum_{i=1}^d m_{ij}^2 \Rightarrow m_{ij} \leq 1 \Rightarrow |m_{ij}| < 1. \quad (19)$$

So what is the maximum possible value for  $\text{tr}(\Sigma M)$ ? Remember that  $\Sigma$  is a diagonal matrix with non-negative values  $\sigma_1, \sigma_2, \dots, \sigma_d \geq 0$  on the diagonal. Therefore:

$$\text{tr}(\Sigma M) = \begin{pmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_d \end{pmatrix} \begin{pmatrix} m_{11} & m_{12} & \dots & m_{1d} \\ m_{21} & m_{22} & \dots & m_{2d} \\ \vdots & \vdots & \ddots & \vdots \\ m_{d1} & m_{d2} & \dots & m_{dd} \end{pmatrix} = \sum_{i=1}^d \sigma_i m_{ii} \leq \sum_{i=1}^d \sigma_i. \quad (20)$$

Therefore the trace is maximized if  $m_{ii} = 1$ . Since  $M$  is an orthogonal matrix, this means that  $M$  would have to be the identity matrix!

$$I = M = V^T R U \Rightarrow V = R U \Rightarrow R = V U^T. \quad (21)$$

**Orientation rectification.** The process we just described finds the optimal *orthogonal* matrix, which could potentially contain reflections in addition to rotations. Imagine that the point set  $P$  is a perfect reflection of  $Q$  – we will then find that reflection, which aligns the two point sets perfectly and yields zero energy (8) – the global minimum in this case. However, if we restrict ourselves to rotations only, there might not be a rotation that perfectly aligns the points.

Checking whether  $R = VU^T$  is a rotation is simple: if  $\det(VU^T) = -1$  it contains reflection, otherwise  $\det(VU^T) = +1$ . Assume  $\det(VU^T) = -1$ : this means that the global maximum of  $\text{tr}(\Sigma M)$  is generally not attainable by a rotation, and we need to look for the “next best thing”. Let us look for other (local) maxima of  $\text{tr}(\Sigma M)$  as a function of  $M$ ’s diagonal values  $m_{ii}$ :

$$\text{tr}(\Sigma M) = \sigma_1 m_{11} + \sigma_2 m_{22} + \dots + \sigma_d m_{dd} =: f(m_{11}, \dots, m_{dd}). \quad (22)$$

If we consider the  $m_{ii}$ ’s as variables, the domain of  $(m_{11}, \dots, m_{dd})$  is a subset of  $[-1, 1]^d$ . The function  $f$  is linear in the  $m_{ii}$ ’s, so it attains its extrema on the boundary of the domain (no extrema on the interior). Since our domain is rectilinear, the extrema will be attained at the vertices  $(\pm 1, \pm 1, \dots, \pm 1)$ . We had to rule out  $(1, 1, \dots, 1)$  since that gave a reflection, therefore the next best shot is  $(1, 1, \dots, 1, -1)$ :

$$\text{tr}(\Sigma M) = \sigma_1 + \sigma_2 + \dots + \sigma_{d-1} - \sigma_d. \quad (23)$$

This is larger than any other combination (except  $(1, 1, \dots, 1)$ ) because  $\sigma_d$  is the smallest singular value.

To summarize, we arrive at the fact that if  $\det(VU^T) = -1$ , we need

$$M = V^T R U = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 & \\ & & & & -1 \end{pmatrix} \Rightarrow R = V \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 & \\ & & & & -1 \end{pmatrix} U^T. \quad (24)$$

We can write a general formula that encompasses both cases,  $\det(VU^T) = 1$  and  $\det(VU^T) = -1$ :

$$R = V \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 & \\ & & & & \det(VU^T) \end{pmatrix} U^T. \quad (25)$$

## 4 Rigid motion computation – summary

Let us summarize the steps to computing the optimal translation  $\mathbf{t}$  and rotation  $R$  that minimize

$$\sum_{i=1}^n w_i \| (R\mathbf{p}_i + \mathbf{t}) - \mathbf{q}_i \|^2.$$

- (1) Compute the weighted centroids of both point sets:

$$\bar{\mathbf{p}} = \frac{\sum_{i=1}^n w_i \mathbf{p}_i}{\sum_{i=1}^n w_i}, \quad \bar{\mathbf{q}} = \frac{\sum_{i=1}^n w_i \mathbf{q}_i}{\sum_{i=1}^n w_i}.$$

- (2) Compute the centered vectors

$$\mathbf{x}_i := \mathbf{p}_i - \bar{\mathbf{p}}, \quad \mathbf{y}_i := \mathbf{q}_i - \bar{\mathbf{q}}, \quad i = 1, 2, \dots, n.$$

- (3) Compute the  $d \times d$  covariance matrix

$$S = XWY^T,$$

where  $X$  and  $Y$  are the  $d \times n$  matrices that have  $\mathbf{x}_i$  and  $\mathbf{y}_i$  as their columns, respectively, and  $W = \text{diag}(w_1, w_2, \dots, w_n)$ .

- (4) Compute the singular value decomposition  $S = U\Sigma V^T$ . The rotation we are looking for is then

$$R = V \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \\ & & & & \det(VU^T) \end{pmatrix} U^T.$$

- (5) Compute the optimal translation as

$$\mathbf{t} = \bar{\mathbf{q}} - R\bar{\mathbf{p}}.$$