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Stochastic Calculus and Mean Field Game Theory

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1. INTRODUCTION

When driving on a crowded street, you want to change your speed and your position in the most efficient way to get rid of this traffic as soon as possible. You determine your speed and direction depending on where you are and the states of other surrounding cars. This is a strategic game with a large number of players. You cannot make decisions purely based on your situation because there is a "mean-field" around you. Then one natural question is that under such a traffic jam, how should we optimize our decisions? The Mean Field Game theory provides a solution.

As the name suggests, Mean Field Game is a large population dynamic game. If there are infinitely many players in the game, it is impossible to collect all the information from everyone and then make an optimal strategy. Fortunately, Mean Field Game theory tells us that we do not need this much detailed information. Instead, we just need to know the global distribution of all other players. The interactions among players become negligible and each player contributes to the global distribution. The Mean Field Game is a system of two PDEs: the first one is of Hamilton-Jacobi-Bellman type and takes care of the optimization of the cost function while the second one is the forward Fokker-Planck equation describing the evolution of mean-field density.

Before we get into details of Mean Field Game theory, we first introduce Itô process, Itô formula, and solutions to the stochastic differential equation, all of which are important tools that will be frequently used in the Mean Field Game theory.

2. Stochastic Calculus

2.1. **Itô Integral.** In this section, we explain how Itô integral is defined. First, we consider the one-dimensional case. Let $B_s(\omega)$ be the one-dimensional standard Brownian motion defined on the probability space $(\Omega, \mathcal{F}, P)$. Let $\{\mathcal{F}_t\}_{t\geq 0}$ be the filtration $\sigma(\{B_s : s \leq t\})$. Let T be a bounded time interval of $\mathbb{R}$. In the following, E is the same as $E^{\mathcal{P}^0}$, the expectation w.r.t the law $\mathcal{P}^0$ for Brownian motion starting at 0. Let U be a class of functions that satisfy the following conditions:

(1) $f(t,\omega) : T \times \Omega \to \mathbb{R}$ is $(\mathcal{B} \times \mathcal{F})$-measurable, where $\mathcal{B}$ is the Borel σ-algebra on T.
(2) $f(t,\cdot)$ is $\mathcal{F}_t$-adapted.
(3) $E\big[\int_T f(t,\omega)^2 dt\big] < \infty$.

Definition 2.1. For functions $f \in U$, the Itô integral of f is defined to be

$$\int_T f(t,\omega)dB_t(\omega) = \lim_{n\to\infty} \int_T \phi_n(t,\omega)dB_t(\omega), \tag{2.2}$$

where the limit is in L^2. The function ϕ_n has the form $\phi_n(t,\omega) = \sum_{j=0}^{n-1} c_j(\omega)\mathcal{X}_{[t_j,t_{j+1})}$ such that $\{t_0, t_1, \ldots, t_n\}$ forms a partition of T, $c_j(\omega)$ is a $\mathcal{F}_{t_j}$-measurable random variable and satisfies

$$\lim_{n\to\infty} E\left[\int_T (f(t,\omega) - \phi_n(t,\omega))^2 dt\right] = 0. \tag{2.3}$$

Since ϕ_n is an elementary process, its integral is defined in the natural way:

$$\int_T \phi_n(t,\omega)dB_t(\omega) = \sum_{j=0}^{n-1} c_j(\omega)(B_{t_{j+1}} - B_{t_j})(\omega).$$

For the above definition to make sense, we need to show that such a sequence of functions $\{\phi_n\}$ exists and the limit of the integral exists and is also almost surely unique.

Proposition 2.4. *If $f \in U$, then there exists a sequence of simple functions ϕ_n such that*

$$\lim_{n\to\infty} E\left[\int_T (f - \phi_n)^2 dt\right] = 0.$$

Proof. We start by approximating bounded and t-continuous functions by ϕ_n. Let $T = [a,b]$. Suppose that $f_1 \in U$ is bounded and $f_1(\cdot,\omega)$ is continuous for all ω. Define

$$\phi_n(t,\omega) = \sum_{j=0}^{n-1} f_1(t_j,\omega)\mathcal{X}_{[\frac{b-a}{n}j,\frac{b-a}{n}(j+1))}.$$

Since f_1 is continuous, ϕ_n converges to f_1 pointwise for each ω as n goes to infinity. Given that f_1 is bounded, applying dominated convergence theorem twice yields

$$\lim_{n\to\infty}\int_T (f_1-\phi_n)^2 dt = 0 \text{ for each } \omega \Rightarrow \lim_{n\to\infty} E\left[\int_T (f_1-\phi_n)^2 dt\right] = 0.$$

Next, we prove the same result for bounded functions. Suppose that $f_2 \in U$ is bounded. Construct a smooth function $\psi_n : \mathbb{R} \to \mathbb{R}$ such that

$$\psi_n \geq 0, \int_{-\frac{1}{n}}^{0} \psi_n(t)dt = 1, \text{ and } \psi_n(t) = 0 \text{ for } t \in \mathbb{R}\backslash[-\frac{1}{n}, 0].$$

Let $g_n(t,\omega) = \int_{\mathbb{R}} \psi_n(s-t) f_2(s,\omega) ds$. Notice that ψ_n is defined on $[-\frac{1}{n}, 0]$ because we want $g_n(t,\omega)$ to be $\mathcal{F}_t$-measurable. Since f_2 is bounded, g_n is bounded. We need to show that $g_n(\cdot,\omega)$ is t-continuous for each ω. Choose any convergent sequence $\{t_i\}$ such that $\lim_{i\to\infty} t_i = t$. For each t_i, $\psi_n(s-t_i) = 0$ when $|s-t_i| > \frac{1}{n}$. Since ψ_n is smooth, $\lim_{i\to\infty}\psi_n(s-t_i)f_2(s,\omega) = \psi_n(s-t)f_2(s,\omega)$. So there exists an upper bound integrable function for each t_i:

$$\psi_n(s-t_i)f_2(s,\omega) \leq \|\psi_n\|_\infty \|f_2\|_\infty \mathcal{X}_{[t_i-\frac{1}{n}, t_i]}.$$

By dominated convergence theorem, $\lim_{i\to\infty}\int_{\mathbb{R}}\psi_n(s-t_i)f_2(s,\omega)ds = \int_{\mathbb{R}}\psi_n(s-t)f_2(s,\omega)ds$. Therefore, g_n is t-continuous for each ω. Then we can show that

$$\lim_{n\to\infty}\int_T (f_2-g_n)^2 dt = 0 \text{ for all } \omega, \text{ and } \lim_{n\to\infty} E\left[\int_T (f_2-g_n)^2 dt\right] = 0.$$

Details are included in Lemma 4.1. Lastly, we show that for any function $f_3 \in U$, there is a sequence of bounded functions h_n defined to be

$$h_n(t,\omega) = f_3(t,\omega)\mathcal{X}_{\{-n\leq f_3\leq n\}} + n\mathcal{X}_{\{f_3>n\}} - n\mathcal{X}_{\{f_3<-n\}}$$

such that h_n converges to f_3 pointwise for each ω. Since f_3 is in L^2 almost surely with respect to P, we get the same limiting result by using dominated convergence theorem($(f_3-h_n)^2 \leq {f_3}^2$). Combining these three cases, we get that for any $f \in U$, there exists a sequence of simple functions ϕ_n such that $\lim_{n\to\infty} E\left[\int_T (f-\phi_n)^2 dt\right] = 0$. □

Lemma 2.5. *The Itô Isometry*
If $\phi(t)$ is defined as in Definition 2.1, then

$$E\left[\left(\int_T \phi_n(t,\omega)dB_t(\omega)\right)^2\right] = E\left[\int_T \phi_n(t,\omega)^2 dt\right]. \tag{2.6}$$

Proof. Let $\Delta B_j = B_{t_{j+1}} - B_{t_j} \overset{\mathrm{d}}{=} \mathcal{N}(0, t_{j+1}-t_j)$. By definition, $E[\Delta B_j] = 0$, $E[{\Delta B_j}^2] = t_{j+1}-t_j$, and ΔB_i, ΔB_j are independent if $i \neq j$. Therefore,

$$\begin{aligned} E\left[\left(\int_T \phi_n(t,\omega)dB_t(\omega)\right)^2\right] &= E\left[\sum_{i,j} c_i(\omega)c_j(\omega)\Delta B_i(\omega)\Delta B_j(\omega)\right] \\ &= \sum_j E[c_j(\omega)^2](t_{j+1}-t_j) \\ &= E\left[\int_T \phi_n(t,\omega)^2 dt\right]. \end{aligned} \tag{2.7}$$

□

Remark 2.8. The proof of Proposition 2.4 shows that for $f \in U$, $\lim_{n\to\infty} E[\int_T (f-\phi_n)^2 dt] = 0$. By applying Lemma 2.6, $\{\int_T \phi_n(t,\omega)dB_t(\omega)\}_{n\in\mathbb{N}}$ forms a Cauchy sequence in $L^2(P)$ because

$$\begin{aligned}
&\left\|\int_T \phi_m dB_t - \int_T \phi_n dB_t\right\|_2^2 = E\left[\left(\int_T \phi_m - \phi_n dB_t\right)^2\right] \\
&= E\left[\int_T (\phi_m - \phi_n)^2 dt\right] \le E\left[\int_T (\phi_m - f)^2 dt\right] + E\left[\int_T (f-\phi_n)^2 dt\right] \\
&\to 0 \text{ as } n \text{ and } m \to \infty.
\end{aligned} \tag{2.9}$$

Since $L^2(P)$ is complete, we know that the limit exists and is in $L^2(P)$.

Lemma 2.10. *Lemma 2.5 holds for all $f \in U$, i.e.,*

$$E\left[\left(\int_T f(t,\omega)dB_t(\omega)\right)^2\right] = E\left[\int_T f(t,\omega)^2 dt\right].$$

Proof. The theorem is a result of (2.2), (2.3) and (2.6) over a bounded interval T. ☐

To prove the next theorem, we need the following result for ϕ_n.

Lemma 2.11. *Let $T = [0,S]$. Suppose $\{\phi_n\}_{n\in\mathbb{N}}$ is a sequence of functions defined in Definition 2.1. Then for any ω, $\int_0^t \phi_n(s,\omega)dB_s(\omega)$ is a martingale with respect to $\mathcal{F}_t$ for all n.*

Proof. Choose any $0 \le v < t \le S$.

$$\begin{aligned}
E\left[\int_0^t \phi_n dB_s \middle| \mathcal{F}_v\right] &= E\left[\int_0^v \phi_n dB_s + \int_v^t \phi_n dB_s \middle| \mathcal{F}_v\right] \\
&= \int_0^v \phi_n dB_s + E\left[\sum_{v\le t_j\le t_{j+1}\le t} c_j(B_{t_{j+1}} - B_{t_j}) \middle| \mathcal{F}_v\right] \\
&= \int_0^v \phi_n dB_s + \sum_{v\le t_j\le t_{j+1}\le t} E[c_j E[\Delta B_j|\mathcal{F}_{t_j}]|\mathcal{F}_v] \\
&= \int_0^v \phi_n dB_s.
\end{aligned} \tag{2.12}$$

Since ϕ_n is a finite sum, we can exchange taking expected value and the summation. The last equality is because ΔB_j is independent of $\mathcal{F}_v$ for $v \le t_j$ and $\mathcal{F}_{t_j} \subset \mathcal{F}_v$. Therefore, it is a martingale. ☐

Definition 2.13. Let $\{X_t\}_{t\in T}$, $\{Y_t\}_{t\in T} : (\Omega, \mathcal{F}, P) \to (S, \mathcal{B})$ be stochastic processes. We say Y is a version of X if $P(X_t \neq Y_t) = 0$ for each $t \in T$.

Theorem 2.14. *Let $T = [0,S]$. Suppose $f \in U$. There exists a t-continuous version of $\int_0^t f(s,\omega)dB_s(\omega)$ for all $t \in T$.*

Proof. Choose any $t \in T$. By Proposition 2.4, there exists a sequence of simple functions ϕ_n such that $\lim_{n\to\infty} I_n(t,\omega) = I(t,\omega)$, where $I_n(t,\omega) = \int_0^t \phi_n(s,\omega)dB_s(\omega)$ and $I(t,\omega) = \int_0^t f(s,\omega)dB_s(\omega)$. We know that $I_n(t,\omega) = \sum_j c_j(\omega)(B_{t_{j+1}} - B_{t_j})(\omega)$ is t-continuous because B_t is continuous. I is the pointwise limit of I_n for each $s \in T$. Since there is an uncountable set of s, the continuity of I_s is not guaranteed.

By Lemma 2.11, we know that I_n is a t-continuous martingale. Hence, $I_n - I_m$ is also a t-continuous martingale with respect to $\mathcal{F}_t$. By the Doob's martingale inequality, we get for any $\epsilon > 0$,

$$
\begin{aligned}
&P\left(\sup_{t\in T}|I_n(t,\omega) - I_m(t,\omega)| > \epsilon\right) \leq \frac{1}{\epsilon^2}E[|I_n(S,\omega) - I_m(S,\omega)|^2] \\
&= \frac{1}{\epsilon^2}E\left[\int_0^S (\phi_n - \phi_m)^2 ds\right] \to 0 \text{ as } n, m \to \infty.
\end{aligned}
\tag{2.15}
$$

For each k, choose $n_{k+1} > n_k$ such that $P\left(\sup_{t\in T}|I_{n_{k+1}}(t,\omega) - I_{n_k}(t,\omega)| > 2^{-k}\right) \leq 2^{-k}$. Then by Borel-Cantelli Lemma, we get

$$
P\left(\sup_{t\in T}|I_{n_{k+1}}(t,\omega) - I_{n_k}(t,\omega)| > 2^{-k} \text{ for infinitely many } k\right) = 0. \tag{2.16}
$$

So on a full measure set(that is for almost all ω), there exists $k(\omega)$ for each ω such that $|I_{n_{k+1}}(t,\omega) - I_{n_k}(t,\omega)| \leq 2^{-k}$ for all large $k \geq k(\omega)$. Therefore, $I_{n_k}(t,\omega)$ converges uniformly over T to its limit $J(t,\omega)$ for almost all ω. Uniform convergence guarantees that $J(\cdot,\omega)$ is continuous a.s.. For each t, we already know that $\{I_n\}$ converges to I in L_2. So we get that for all t, $I(t,\omega) = J(t,\omega)$ for almost all ω. □

For future reference, we use $\int_T f(s,\omega)dB_s(\omega)$ to denote the t-continuous version. Recall the definition of U, we want to expand U by modifying the second condition.

Let V be a class of functions that satisfy, in addition to the first and the third conditions of U, that $f(t,\cdot)$ is $\mathcal{H}_t$-adapted and B_t is a martingale with respect to $\mathcal{H}_t$. Note that $\mathcal{H}_t$ contains $\mathcal{F}_t$ and ensures that $E[B_t - B_s|\mathcal{F}_v] = 0$ for any $v < s < t$. With this new class of functions, we can discuss the generalized n-dimensional Itô integral.

Definition 2.17. Let $B = (B_1, B_2, \ldots, B_n)$ be a n-dimensional Brownian motion. Let $\{\mathcal{H}_t\}$ be a filtration. Let $V^{m\times n}$ be a set of $m \times n$ matrices whose entries are in V. For any $f \in V^{m\times n}$, we define

$$
\int_T f dB = \int_T \begin{bmatrix} f_{11} & \cdots & f_{1n} \\ \vdots & & \vdots \\ f_{m1} & \cdots & f_{mn} \end{bmatrix} \begin{bmatrix} dB_1 \\ \vdots \\ dB_n \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^n \int_T f_{1j}(s,\omega)dB_j(s,\omega) \\ \vdots \\ \sum_{j=1}^n \int_T f_{mj}(s,\omega)dB_j(s,\omega) \end{bmatrix}.
$$

Next, we make one more extension. Let W be a class of functions that satisfy that

(1) $f(t,\omega) : T \times \Omega \to \mathbb{R}$ is $(\mathcal{B} \times \mathcal{F})$-measurable, where $\mathcal{B}$ is the Borel σ-algebra on T.
(2) $f(t,\cdot)$ is $\mathcal{H}_t$-adapted and B_t is a martingale with respect to $\mathcal{H}_t$.
(3) $P\left(\int_T f(s,\omega)^2 ds < \infty\right) = 1.$.

The Itô integral of $f \in W$ is defined in a similar way by a sequence of approximating simple functions. We use the similar idea as in Proposition 2.4 to prove the well-definedness of this integral. However, we no longer have convergence in L_2 since $E[\int_T (f - \phi_n)^2 dt]$ might be infinite. We can still prove convergence in probability, i.e., for all ϵ, $\lim_{n\to\infty} P\left(|\int_T \phi_n(t,\omega)dB_t(\omega) - \int_T f(t,\omega)dB_t(\omega)| > \epsilon\right) = 0$, and define the Itô integral for $f \in W$ as the limit in probability.

Next, we present the definition of one-dimensional Itô process.

2.2. **Itô Process and Itô Formula.**

Definition 2.18. Let B_t be the standard Brownian motion defined on the probability space $(\Omega, \mathcal{F}, P)$. The Itô process is a stochastic process $X_t : (\Omega, \mathcal{F}, P) \to (\mathbb{R}, \mathcal{B})$ of the form

$$X_t(\omega) = X_0(\omega) + \int_0^t u(s,\omega)ds + \int_0^t v(s,\omega)dB_s,$$
$$\text{or } dX_t = udt + vdB_t,$$

where $v \in W$ and u satisfies conditions $1, 2$ of W and $P\left(\int_T |u(s,\omega)|ds < \infty\right) = 1$.

To prove Ito formula, we need the following proposition.

Proposition 2.19. *Let $\Pi_m = \{0 = t_0^m < t_1^m < \cdots < t_{n_m}^m\}$ be a sequence of partitions of $T = [0, S]$ such that $\|\Pi_m\| := \max_{1\le j\le n_m} |t_j^m - t_{j-1}^m| \to 0$ as $m \to \infty$. Let $B = (B_t)_{t\in T}$ be a standard Brownian motion. Let $Q_m(B)(\omega) = \sum_{j=1}^{n_m}((B_{t_j^m} - B_{t_{j-1}^m})(\omega))^2$. Then $\lim_{m\to\infty} \|Q_m(B) - S\|_2 = 0$. Note that $Q_m(B)$ is defined to be the quadratic variation of the Brownian motion.*

Proof. Let $\Delta B_j^m = B_{t_j}^m - B_{t_{j-1}}^m$ and $\Delta t_j^m = t_j^m - t_{j-1}^m$. Notice that $\Delta B_j^m \overset{\text{d}}{=} \mathcal{N}(0, \Delta t_j^m)$ and $E[(\Delta B_j^m)^2] = \Delta t_j^m$. We have $Q_m(B) - S = \sum_{j=1}^{n_m}(\Delta B_j^m)^2 - \Delta t_j^m$. Then

$$\begin{aligned}
\|Q_m(B) - S\|_2^2 &= E\left[\left(\sum_{j=1}^{n_m}(\Delta B_j^m)^2 - \Delta t_j^m\right)^2\right] \\
&= \sum_{j,k=1}^{n_m} E[((\Delta B_j^m)^2 - \Delta t_j^m)((\Delta B_k^m)^2 - \Delta t_k^m)] \\
&= \sum_{j=1}^{n_m} E[((\Delta B_j^m)^2 - \Delta t_j^m)^2] \\
&= \sum_{j=1}^{n_m} E[(\Delta B_j^m)^4 - 2(\Delta B_j^m)^2\Delta t_j^m + (\Delta t_j^m)^2] \\
&= \sum_{j=1}^{n_m} E[(\Delta B_j^m)^4] - 2E[(\Delta B_j^m)^2]\Delta t_j^m + (\Delta t_j^m)^2 \\
&= \sum_{j=1}^{n_m} 3(\Delta t_j^m)^2 - 2(\Delta t_j^m)^2 + (\Delta t_j^m)^2 \\
&= \sum_{j=1}^{n_m} 2(\Delta t_j^m)^2 \le 2\|\Pi_m\| \sum_{j=1}^{n_m} |\Delta t_j^m| = 2S\|\Pi_m\| \to 0 \text{ as } m \to \infty.
\end{aligned}$$

We used the fact that the fourth moment of normal distribution is $E[(\Delta B_j^m)^4] = 3(\Delta t_j^m)^2$. □

The above theorem tells us the interesting fact that $(dB_t)^2 = dt$, which will be used in the following theorem.

Theorem 2.20. *Assume that X_t is an Itô process such that $dX_t = udt + vdB_t$. Suppose $g(t, x) \in C^2([0, \infty], \mathbb{R})$. Let $Y_t = g(t, X_t)$. Then Y_t is also an Itô process* *and satisfies*

$$dY_t = \frac{\partial g}{\partial t}(t, X_t)dt + \frac{\partial g}{\partial x}(t, X_t)dX_t + \frac{1}{2}\frac{\partial^2 g}{\partial^2 x}(t, X_t)v^2 dt,$$

where $(dX_t)^2 = v^2 dt$, $(dt)^2 = dt \cdot dB_t = dB_t \cdot dt = 0$.

Proof. We are going to prove the result for special functions g such that g, $\frac{\partial g}{\partial t}$, $\frac{\partial g}{\partial x}$, $\frac{\partial^2 g}{\partial t^2}$, $\frac{\partial^2 g}{\partial x^2}$, $\frac{\partial^2 g}{\partial t \partial x}$ are all bounded and for special X_t such that u, v are bounded elementary processes. Then we can use bounded C^2 functions to approximate unbounded functions and prove uniform convergence on a compact subset of $[0, \infty) \times \mathbb{R}$.

Let $\Delta t_j = t_{j+1} - t_j$ and $\Delta X_j = X_{t_{j+1}} - X_{t_j}$. Expanding $g(t, X_t)$ by using Taylor's theorem yields

$$\begin{aligned} g(t, X_t) =& g(0, X_0) + \sum_j \frac{\partial g}{\partial t}\Delta t_j + \sum_j \frac{\partial g}{\partial x}\Delta X_j + \frac{1}{2}\sum_j \frac{\partial^2 g}{\partial t^2}(\Delta t_j)^2 \\ &+ \sum_j \frac{\partial^2 g}{\partial t \partial x}(\Delta t_j)(\Delta X_j) + \frac{1}{2}\sum_j \frac{\partial^2 g}{\partial x^2}(\Delta X_j)^2 + \sum_j \mathcal{O}(|\Delta t_j|^2 + |\Delta X_j|^2), \end{aligned}$$

where all partial derivatives are evaluated at (t_j, X_{t_j}). Since u, v are elementary processes, we define $u = \sum_j u_j \mathcal{X}_{[t_j, t_{j+1})}$ and $v = \sum_j v_j \mathcal{X}_{[t_j, t_{j+1})}$, where $u_j = u(t_j, \omega)$, $v_j = v(t_j, \omega)$ are $\mathcal{F}_{t_j}$-measurable. By the definition of Riemann integral, as $\Delta t_j \to 0$, we know that

$$\sum_j \frac{\partial g}{\partial t}\Delta t_j \to \int_0^t \frac{\partial g}{\partial t}(s, X_s)ds \text{ and } \sum_j \frac{\partial g}{\partial x}u_j \Delta t_j \to \int_0^t \frac{\partial g}{\partial x}(s, X_s)u(s, \omega)ds.$$

Also, we claim that

$$\sum_j \frac{\partial g}{\partial x}v_j \Delta B_j \to \int_0^t \frac{\partial g}{\partial x}(s, X_s)v(s, \omega)dB_s.$$

Let $\hat{g}(t) = \frac{\partial g}{\partial x}(t_j, X_{t_j})v_j$ for $t \in [t_j, t_{j+1})$. Then as $\max_j \Delta t_j \to 0$,

$$E\left[\left(\int_0^t \hat{g}(s) - \frac{\partial g}{\partial x}(s, X_s)v(s, \omega)dB_s\right)^2\right] = E\left[\int_0^t (\hat{g}(s) - \frac{\partial g}{\partial x}(s, X_s)v(s, \omega))^2 ds\right] \to 0.$$

Therefore,

$$\sum_j \frac{\partial g}{\partial x}\Delta X_j \to \int_0^t \frac{\partial g}{\partial x}(s, X_s)dX_s.$$

Next, we have that as $\max_j \Delta t_j \to 0$,

$$\frac{1}{2}\sum_j \frac{\partial^2 g}{\partial t^2}(\Delta t_j)^2 \leq \frac{t}{2}\left\|\frac{\partial^2 g}{\partial t^2}\right\|_\infty \max_j \Delta t_j \to 0 \text{ and}$$

$$\begin{aligned} &E\left[\left(\sum_j \frac{\partial^2 g}{\partial t \partial x}(\Delta t_j)(\Delta X_j)\right)^2\right] \\ &= \sum_{i,j} E\left[\left(\frac{\partial^2 g}{\partial t \partial x}u_i u_j\right)^2\right](\Delta t_i)^2(\Delta t_j)^2 + \sum_j E\left[\left(\frac{\partial^2 g}{\partial t \partial x}v_j\right)^2\right](\Delta t_j)^3 \to 0 \end{aligned}$$

because all functions inside the expectation are bounded. Hence, $\sum_j \frac{\partial^2 g}{\partial t \partial x}(\Delta t_j)(\Delta X_j) \to 0$. There is one last term left to discuss.

$$\begin{aligned}\sum_j \frac{\partial^2 g}{\partial x^2}(\Delta X_j)^2 &= \sum_j \frac{\partial^2 g}{\partial x^2}(u_j \Delta t_j + v_j \Delta B_j)^2 \\ &= \sum_j \frac{\partial^2 g}{\partial x^2}\Big((u_j)^2(\Delta t_j)^2 + 2u_j v_j (\Delta t_j)(\Delta B_j) + (v_j)^2(\Delta B_j)^2\Big).\end{aligned}$$

As $\max_j \Delta t_j \to 0$,

$$\begin{aligned}&\sum_j \frac{\partial^2 g}{\partial x^2}(u_j)^2(\Delta t_j)^2 \le t\left\|\frac{\partial^2 g}{\partial x^2}\right\|_\infty (\max_j |u_j|)(\max_j \Delta t_j) \to 0 \text{ and} \\ &E\Big[\Big(\sum_j \frac{\partial^2 g}{\partial x^2} 2u_j v_j (\Delta t_j)(\Delta B_j)\Big)^2\Big] = \sum_j E\Big[\Big(2\frac{\partial^2 g}{\partial x^2}u_j v_j\Big)^2\Big](\Delta t_j)^3 \\ &\le 2t\left\|\frac{\partial^2 g}{\partial x^2}uv\right\|_\infty^2 (\max_j \Delta t_j)^2 \to 0.\end{aligned}$$

By Proposition 2.19, we know that as $\Delta t \to 0$,

$$E\Big[\Big(\sum_j \frac{\partial^2 g}{\partial x^2}(v_j)^2(\Delta B_j)^2 - \sum_j \frac{\partial^2 g}{\partial x^2}(v_j)^2 \Delta t_j\Big)^2\Big] \to 0,$$

which means that

$$\sum_j \frac{\partial^2 g}{\partial x^2}(v_j)^2(\Delta B_j)^2 \to \int_0^t \frac{\partial^2 g}{\partial x^2} v^2 ds.$$

So combining all terms together, we get that as $\Delta t \to 0$,

$$g(t, X_t) = g(0, X_0) + \int_0^t \frac{\partial g}{\partial t}(s, X_s) ds + \int_0^t \frac{\partial g}{\partial x}(s, X_s) dX_s + \int_0^t \frac{\partial^2 g}{\partial x^2} v^2 ds.$$

Hence, we finish the proof. □

Now, we generalize the Itô process and Itô formula to the m-dimensional case.

Definition 2.21. Let B be a m-dimensional Brownian motion. An n-dimensional Itô process X_t is given by

$$dX(t) = udt + vdB(t),$$

where $dB(t) = \begin{bmatrix} dB_1(t) \\ \vdots \\ dB_m(t) \end{bmatrix}$, v is a $n \times m$ matrix whose entries $\{v_{ij}\}$ are in W, $u = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$, and $dX(t) = \begin{bmatrix} dX_1(t) \\ \vdots \\ dX_n(t) \end{bmatrix}$. Given a C^2 function $g(t, x) : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^p$, the process $Y(t, \omega) = g(t, X(t, \omega))$ is also an Itô process and satisfies the Itô formula

$$dY_k = \frac{\partial g_k}{\partial t}(t, X) dt + \sum_{i=1}^n \frac{\partial g_k}{\partial x_i}(t, X) dX_i + \frac{1}{2}\sum_{i,j=1}^n \frac{\partial^2 g_k}{\partial x_i \partial x_j}(t, X) v_i {v_j}^T dt,$$

where Y_k is the k-th component of $Y(t)$, X_i is the i-th component of $X(t)$ and v_i is the i-th row of the matrix. The same computation rule applies, i.e., $(dB_i)(dB_j) = \delta_{ij}dt$.

2.3. **Existence & Uniqueness Theorem for Stochastic Differential Equations.**

Lemma 2.22. *Gronwall Inequality*

Let $[0, S]$ be a bounded time interval. Let $h(s)$ be a nonnegative continuous function such that

$$h(s) \leq A + B\int_0^s h(t)dt$$

for $0 \leq s \leq S$ and for some constants $A \in \mathbb{R}$, $B \geq 0$. Then $h(s) \leq Ae^{Bs}$.

Proof. It is obviously true when $B = 0$. So we consider the case when $B > 0$. Define $y(s) = \int_0^s h(t)dt$. Then $y'(s) = h(s)$ and $y'(s) \leq A + By(s)$. Define a new function $f(s) = y(s)e^{-Bs}, f(0) = y(0) = 0$. Then

$$f'(s) = y'(s)e^{-Bs} - By(s)e^{-Bs} = e^{-Bs}(y'(s) - By(s)) \leq Ae^{-Bs}.$$

We integrate $f'(s)$ to get

$$f(s) = \int_0^s f'(t)dt \leq \int_0^s Ae^{-Bt}dt = -\frac{A}{B}e^{-Bs} + \frac{A}{B}.$$

By the definition of f, we know that

$$y(s)e^{-Bs} \leq -\frac{A}{B}e^{-Bs} + \frac{A}{B} \Rightarrow y(s) \leq -\frac{A}{B} + \frac{A}{B}e^{Bs}.$$

Lastly, taking the derivative gives

$$h(s) \leq Ae^{Bs} \text{ for } 0 \leq s \leq S.$$

□

Next, we talk about the existence and uniqueness of the solution for stochastic differential equations.

As before, let $\{\mathcal{F}_t\}_{t\in T}$ be the filtration generated by the m-dimensional Brownian motion.

Theorem 2.23. *Define $u(s, x) : [0, S] \times \mathbb{R}^n \to \mathbb{R}^n$ and $v(s, x) : [0, S] \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$ be $\mathcal{F}_s$ measurable functions such that for some constant C, D, $x, y \in \mathbb{R}^n$ and $s \in [0, S]$,*

$$|u(s, x)| + |v(s, x)| \leq C(1 + |x|),$$

and

$$|u(s, x) - u(s, y)| + |v(s, x) - v(s, y)| \leq D|x - y|,$$

where $|v|^2 = \sum_{i,j} |v_{ij}|^2$. Let Z be a random variable that is independent of the σ-algebra $\mathcal{F}_\infty$ and $E[|Z|^2] < \infty$.

Then the stochastic differential equation

$$dX_s = u(s, X_s)ds + v(s, X_s)dB_s,\ 0 \leq s \leq S,\ X_0 = Z$$

has a unique s-continuous solution $X_s(w)$ which is adapted to the filtration generated by Z and $\{B_t\}_{0\leq t\leq s}$ and $E[\int_0^S |X_t|^2dt] < \infty$.

Proof. We first prove the uniqueness. Choose $0 \leq s \leq S$. Let $T = [0, s]$. Suppose there are two t-continuous solutions $X_s(\omega)$ and $\hat{X}_s(\omega)$ that satisfy the stochastic differential equation. Then $X_0(\omega) = \hat{X}_0(\omega) = Z(\omega)$ a.s.. Let $a(s,\omega) = u(s, X_s(\omega)) - u(s, \hat{X}_s(\omega))$ and $b(s,\omega) = v(s, X_s(\omega)) - v(s, \hat{X}_s(\omega))$. Then we get the following inequality:

$$
\begin{aligned}
&E[|X_s - \hat{X}_s|^2] = E\left[\left(X_0 - \hat{X}_0 + \int_T a dt + \int_T b dB_t\right)^2\right] \\
&\leq 3E[(X_0 - \hat{X}_0)^2] + 3E\left[\left(\int_T a dt\right)^2\right] + 3E\left[\left(\int_T b dB_t\right)^2\right] ((a+b+c)^2 \leq 3a^2 + 3b^2 + 3c^2) \\
&\leq 3E\left[\left(\int_T a dt\right)^2\right] + 3E\left[\left(\int_T b dB_t\right)^2\right] (X_0 = \hat{X}_0 \ a.s.) \\
&\leq 3sE\left[\int_T a^2 dt\right] + 3E\left[\int_T b^2 dt\right] (\text{by Holder's inequality and Itô isometry}) \\
&\leq 3sE\left[\int_T D^2 |X_t - \hat{X}_t|^2 dt\right] + 3E\left[\int_T D^2 |X_t - \hat{X}_t|^2 dt\right] (\text{by Lipschitz condition on u, v}) \\
&\leq 3D^2(s+1) \int_T E[|X_t - \hat{X}_t|^2] dt \ (\text{by Fubini's theorem}).
\end{aligned}
$$

Define $h(s) = E[|X_s - \hat{X}_s|^2]$ for $s \in [0, S]$. Following the inequality, we get $h(s) \leq 3D^2(s+1) \int_0^s h(t)dt$. By using Lemma 2.22, we get that

$$h(s) \leq 3D^2(s+1) \int_0^s h(t)dt \leq 0 + 3D^2(S+1) \int_0^s h(t)dt \Rightarrow h(s) = 0 \text{ for } 0 \leq s \leq S.$$

Hence, $E[|X_s - \hat{X}_s|^2] = 0$ implies that $P(X_s = \hat{X}_s) = 1$ for all $0 \leq s \leq S$. Then under countable intersection,

$$P(\cap_{s \in [0,S], s \in \mathbb{Q}} X_s = \hat{X}_s) = 1.$$

Since X_s and $\hat{X}_s$ are both time continuous processes, we get that $P(X_s = \hat{X}_s \text{ for all } s \in [0, S]) = 1$. This shows the uniqueness.

Next, we prove the existence of the solution. Define $Y_s^0 = X_0$ for $s \in [0, S]$ and inductively define Y_s^k as follows:

$$Y_s^k = X_0 + \int_0^s u(t, Y_t^{k-1}) dt + \int_0^s v(t, Y_t^{k-1}) dB_t, \ k \geq 1.$$

Then using the same computation as above, we can get the same inequality for Y_s^k :

$$E[|Y_s^{k+1} - Y_s^k|^2] \leq 3D^2(S+1) \int_0^s E[|Y_t^k - Y_t^{k-1}|^2] dt, \text{ for } k \geq 1, \ 0 \leq s \leq S.$$

Notice that we have the initial condition

$$
\begin{aligned}
E[|Y_s^1 - Y_s^0|^2] &\leq 2sE\left[\int_0^s |u(t, Y_t^0)|^2 dt\right] + 2E\left[\int_0^s |v(t, Y_t^0)|^2 dt\right] \\
&\leq 2s^2 C^2 E[(1 + |X_0|)^2] + 2sC^2 E[(1 + |X_0|)^2] \leq \alpha_1 s,
\end{aligned}
\tag{2.24}
$$

for some constant $\alpha_1 > 0$ that depends on C, S and $E[(1 + |X_0|)^2]$. By induction,

$$E[|Y_s^2 - Y_s^1|^2] \leq 3D^2(S+1)\alpha_1 \frac{s^2}{2},$$

$$E[|Y_s^{k+1} - Y_s^k|^2] \leq (3D^2(S+1))^k \alpha_1 \frac{s^{k+1}}{(k+1)!},$$

for $k \geq 0$ and $0 \leq s \leq S$. Let λ be the Lebesgue measure on $[0, S]$. We can obtain the following equation: for large $m < n$,
(2.25)

$$\begin{aligned}
&\|Y_s^n - Y_s^m\|_{L^2_{\lambda\times P}} = \left\| \sum_{j=m}^{n-1} Y_s^{j+1} - Y_s^j \right\|_{L^2(\lambda\times P)} \leq \sum_{j=m}^{n-1} \left\| Y_s^{j+1} - Y_s^j \right\|_{L^2(\lambda\times P)} \\
&= \sum_{j=m}^{n-1} \left(E\left[\int_0^S (Y_t^{j+1} - Y_t^j)^2 dt \right] \right)^{\frac{1}{2}} = \sum_{j=m}^{n-1} \left(\int_0^S E[(Y_t^{j+1} - Y_t^j)^2] dt \right)^{\frac{1}{2}} \\
&\leq \sum_{j=m}^{n-1} \left(\int_0^S (3D^2(S+1))^j \alpha_1 \frac{t^{j+1}}{(j+1)!} dt \right)^{\frac{1}{2}} = \sum_{j=m}^{n-1} \left((3D^2(S+1))^j \alpha_1 \frac{t^{j+2}}{(j+2)!} \right)^{\frac{1}{2}} \\
&\to 0 \text{ as } m, n \to \infty.
\end{aligned}$$

Therefore, we have a Cauchy sequence $\{Y_s^k\}_{k\in\mathbb{N}}$ in $L^2(\lambda \times P)$ which means that there exists a limit. Define $X_s = \lim_{k\to\infty} Y_s^k$ such that $X_0 = Z$ a.s., X_s is $\mathcal{F}_s$-measurable and $E[\int_0^S |X_t|^2 dt] < \infty$. We are left to check that X_s satisfies the stochastic differential equation. We know that

$$Y_s^k = X_0 + \int_0^s u(t, Y_t^{k-1}) dt + \int_0^s v(t, Y_t^{k-1}) dB_t, \ k \geq 1.$$

Then

$$\begin{aligned}
&E\left[\left(\int_0^s u(t, Y_t^{k-1}) - u(t, X_t) dt \right)^2 \right] \leq E\left[\left(\int_0^s D|Y_t^{k-1} - X_t| dt \right)^2 \right] \\
&\leq SD^2 E\left[\int_0^s |Y_t^{k-1} - X_t|^2 dt \right] \to 0 \text{ as } k \to \infty.
\end{aligned} \tag{2.26}$$

In addition,

$$\begin{aligned}
&E\left[\left(\int_0^s v(t, Y_t^{k-1}) - v(t, X_t) dB_t \right)^2 \right] \leq E\left[\int_0^s D^2 |Y_t^{k-1} - X_t|^2 dt \right] \\
&\leq D^2 E\left[\int_0^s |Y_t^{k-1} - X_t|^2 dt \right] \to 0 \text{ as } k \to \infty.
\end{aligned} \tag{2.27}$$

Hence, we know that $\int_0^s u(t, Y_t^{k-1}) dt \to \int_0^s u(t, X_t) dt$ in $L^2(P)$ and $\int_0^s v(t, Y_t^{k-1}) dt \to \int_0^s v(t, X_t) dt$ in $L^2(P)$. Then, we have the following

$$\begin{aligned}
X_s &= X_0 + \lim_{k\to\infty} \int_0^s u(t, Y_t^{k-1}) dt + \lim_{k\to\infty} \int_0^s v(t, Y_t^{k-1}) dB_t \\
&= X_0 + \int_0^s u(t, X_t) dt + \int_0^s v(t, X_t) dB_t.
\end{aligned} \tag{2.28}$$

This is the desired result. □

3. Introduction to Mean Field Game Theory

3.1. **The One Player Case.** Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{s\leq t\leq S}, P)$ be the probability space. The filtration $\{\mathcal{F}_t\}_{t\geq s}$ is generated by the n-dimensional Brownian motion B_t.

Consider the Itô process $Y_t(\omega) : [s, S] \times \Omega \to \mathbb{R}^d$ that satisfies

$$\begin{aligned} &dY_t = f(Y_t, \alpha_t, t)dt + \sigma(Y_t, \alpha_t, t)dB_t, \;\; s \le t \le S, \\ &Y_s = x \;\; a.s., \end{aligned} \tag{3.1}$$

where σ is a $d \times n$ matrix whose entries are measurable. We assume that f and σ are Lipschitz continuous, bounded and measurable functions such that by Theorem 2.23 there is a unique t-continuous solution Y_t. Note that f and σ depend on the control process α_t. Let $\alpha_t(\omega) : [s, S] \times \Omega \to \mathbb{R}^m$ be adapted to the filtration $\{\mathcal{F}_t\}_{s \le t \le S}$. Let $A_{s,S}$ denote the set of all possible control processes:

$$A_{s,S} = \{\alpha_t : [s, S] \times \Omega \to \mathbb{R}^m \mid \{\alpha_t\}_{s \le t \le S} \text{ is adapted to } \{\mathcal{F}_t\}_{s \le t \le S}\}.$$

Notice that Y_t is a.s. uniquely determined by its initial condition and α_t.

The problem we are interested in is the following: Given a bounded, t-continuous and measurable running cost function γ that depends on Y_t, α_t, t for some $s \le t \le S$ and a bounded, measurable terminal cost function g that only depends on Y_S, we want to find the optimal control process such that the expected total cost is minimized, that is

$$u(x, s) = \inf_{\alpha \in A_{s,S}} E\left[\int_s^S \gamma(Y_t, \alpha_t, t)dt + g(Y_S) \,\middle|\, Y_s = x\right]. \tag{3.2}$$

Here, $u(x, s) : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ is a function of the initial state x and the initial time s and is called the value function. Notice that s is the initial time of the Ito process but s is a variable in the definition of u.

Next, we introduce the Dynamic Programming Property which says that if we know the value function at time τ, we can determine $u(\cdot, s)$ for any $s \le \tau$ by choosing the optimal control process from s to τ and using $u(\cdot, \tau)$ as the terminal cost. Let $\alpha = \{\alpha_t\}_{s \le t \le S}$.

Theorem 3.3. *The Dynamic Programming Property*

Given a terminal time S, let $u(x, s)$ be defined as above. If $s \le z \le S$, then

$$u(x, s) = \inf_{\alpha \in A_{s,z}} E\left[\int_s^z \gamma(Y_t, \alpha_t, t)dt + u(Y_z, z) \,\middle|\, Y_s = x\right].$$

Proof. The control process can be split into two intervals $\alpha = \alpha_1 \oplus \alpha_2$ where $\alpha \in A_{s,S}$, $\alpha_1 \in A_{s,z}$ and $\alpha_2 \in A_{z,S}$. The addition is defined by concatenation. From this, we also have $A_{s,S} = A_{s,z} \oplus A_{z,S}$. Similarly, we can decompose the Itô process Y_t into $Y_t = Y_1(t) \oplus Y_2(t)$ where $Y_1(t) : [s, z] \times \Omega \to \mathbb{R}^d$ and $Y_2(t) : [z, S] \times \Omega \to \mathbb{R}^d$

satisfy (3.1) and $Y_1(z) = Y_2(z)$ a.s.. Then we get
(3.4)

$$u(x,s) = \inf_{\alpha\in A_{s,S}} E\left[\int_s^z \gamma(Y_t,\alpha_t,t)dt + \int_z^S \gamma(Y_t,\alpha_t,t)dt + g(Y_S)\,\middle|\, Y_s = x\right]$$

$$= \inf_{\alpha\in A_{s,S}} E\left[\int_s^z \gamma(Y_t,\alpha_t,t)dt + E\left[\int_z^S \gamma(Y_t,\alpha_t,t)dt + g(Y_S)\,\middle|\,\mathcal{F}_z\right]\,\middle|\, Y_s = x\right]$$

$$= \inf_{\alpha_1\in A_{s,z},\alpha_2\in A_{z,S},Y_1(z)=Y_2(z)} E\left[\int_s^z \gamma(Y_1(t),\alpha_1,t)dt + E\left[\int_z^S \gamma(Y_2(t),\alpha_2,t)dt + g(Y_S)\,\middle|\, Y_z\right]\,\middle|\, Y_s = x\right] \ (*)$$

$$= \inf_{\alpha_1\in A_{s,z}} E\left[\int_s^z \gamma(Y_1(t),\alpha_1,t)dt + \inf_{\alpha_2\in A_{z,S},Y_1(z)=Y_2(z)} E\left[\int_z^S \gamma(Y_2(t),\alpha_2,t)dt + g(Y_S)\,\middle|\, Y_z\right]\,\middle|\, Y_s = x\right] \ (**)$$

$$= \inf_{\alpha_1\in A_{s,z}} E\left[\int_s^z \gamma(Y_1(t),\alpha_1,t)dt + u(Y_z,z)\,\middle|\, Y_s = x\right] \text{ (by the definition of u).}$$

The third equality comes from the Markov property of the Itô process. For a proof, readers can refer to chapter 7 of [1]. To justify the fourth equality, we have $(*) \geq (**)$ because

$$\inf_{\alpha_2\in A_{z,S},Y_1(z)=Y_2(z)} E\left[\int_z^S \gamma(Y_2(t),\alpha_2,t)dt + g(Y_S)\,\middle|\, Y_z\right] \leq E\left[\int_z^S \gamma(Y_2(t),\alpha_2,t)dt + g(Y_S)\,\middle|\, Y_z\right]$$

for all values Y_z can take. For the opposite direction, we know that for any $\epsilon > 0$, there exists $\alpha_2(y)$ such that

$$E\left[\int_z^S \gamma(Y_2,\alpha_2(y),t)dt + g(Y_S)\,\middle|\, Y_z = y\right] < u(y,z) + \epsilon.$$

By the Markov property of Y_t, we can replace y by a random variable Y_z and get

$$E\left[\int_z^S \gamma(Y_2,\alpha_2(Y_z),t)dt + g(Y_S)\,\middle|\, Y_z\right] < u(Y_z,z) + \epsilon.$$

This $\alpha_2(Y_z)$ is contained in the set $\{\alpha_1 \in A_{s,z}, \alpha_2 \in A_{z,S}, Y_1(z) = Y_2(z)\}$ by appropriately choosing α_1. Therefore, $(*) \leq (**) + \epsilon$ for any $\epsilon > 0$. Intuitively speaking, the Markov property guarantees that conditioned on the present time Y_z, the past and future are independent. □

In general, the value function $u(x,s)$ may not be differentiable. However, to show that the value function satisfies a PDE, we assume $u(x,s)$ is twice differentiable and γ, g are both Lipschitz continuous and differentiable. In addition, we also need the assumption that α_t is continuous on a small interval $[s, s+h]$.

Theorem 3.5. *Suppose we have the Itô process Y_t as in (3.1) and its value function $u(x,s)$ as in (3.2). Assume that the value function is a C^2 function in both variables. Let $\nabla u(x,s)$ or Du denote the derivative with respect to x, $u_s(x,s)$ the derivative with respect to time s and $D^2u(x,s)$ the second derivative with respect to x. Then u satisfies the following PDE:*

$$u_s(x,s) + H(D^2u, Du, x, s) = 0,$$

where the function H is the Hamiltonian and is defined to be

$$H = \inf_{\alpha\in A'}\left(\gamma(x,\alpha,s) + \frac{1}{2}tr(D^2u(x,s)\sigma(x,\alpha,s)\sigma^T(x,\alpha,s)) + f(x,\alpha,s)\cdot Du(x,s)\right),$$

where A' is the set of all possible control values at time s.

Remark 3.6. For the Hamiltonian H, the infimum is taken over a deterministic subset of $\mathbb{R}^m$ but not over the set of admissible processes. To simplify the notation, we define the second order differential operator

$$\mathcal{L}^\alpha u = \frac{1}{2}\mathrm{tr}(D^2u(x,s)\sigma(x,\alpha,s)\sigma^T(x,\alpha,s)) + f(x,\alpha,s)\cdot Du(x,s).$$

Proof. Apply the Itô formula to u:

$$\begin{aligned}
&u(Y_\tau,\tau) - u(x,s)\\
&= \int_s^\tau u_s(Y_t,t)dt + \int_s^\tau \nabla u(Y_t,t)dY_t + \int_s^\tau \frac{1}{2}\sum_{i,j}\frac{\partial^2 u}{\partial x_i \partial x_j}\sigma_j(Y_t,\alpha_t,t)(\sigma_i(Y_t,\alpha_t,t))^T dt\\
&= \int_s^\tau u_s(Y_t,t) + \nabla u(Y_t,t)\cdot f(Y_t,\alpha_t,t) + \frac{1}{2}\mathrm{tr}(D^2u(x,t)\sigma(x,\alpha_t,t)(\sigma(x,\alpha_t,t))^T)dt\\
&+ \int_s^\tau (\nabla u(Y_t,t))^T\sigma(Y_t,\alpha_t,t)dB_t\\
&= \int_s^\tau u_s(Y_t,t) + \mathcal{L}^{\alpha_t}u(Y_t,t)\,dt + \int_s^\tau (\nabla u(Y_t,t))^T\sigma(Y_t,\alpha_t,t)dB_t.
\end{aligned}$$

From theorem 3.3, we know that for any $s \le \tau \le S$,

$$\begin{aligned}
0 &= \inf_{\{\alpha_t\}_{s\le t\le\tau}\in A_{s,\tau}} E\Big[\int_s^\tau \gamma(Y_t,\alpha_t,t)dt + u(Y_\tau,\tau) - u(x,s)\,\Big|\, Y_s = x\Big]\\
&= \inf_{\{\alpha_t\}_{s\le t\le\tau}\in A_{s,\tau}} E\Big[\int_s^\tau \gamma(Y_t,\alpha_t,t)dt + \int_s^\tau u_s(Y_t,t) + \mathcal{L}^{\alpha_t}u(Y_t,t)\,dt\,\Big|\, Y_s = x\Big]
\end{aligned}$$

because $E\big[\int_s^\tau (\nabla u(Y_t,t))^T\sigma(Y_t,\alpha_t,t)dB_t\big] = 0$. Let $\tau = s+h$, divide the whole equation by h and take the limit as $h\to 0$. By our assumptions on $u(x,s)$ and α_t, we can take this limit and get the PDE

$$u_s(x,s) + \inf_{\alpha\in A'}\big(\gamma(x,\alpha,s) + \mathcal{L}^\alpha u(x,s)\big) = 0.$$

This is exactly $u_s(x,s) + H(D^2u, Du, x, s) = 0$ provided how H is defined. $\square$

This PDE is called the Hamilton-Jacobi-Bellman equation. We will use this again later.

3.2. **Multiple Players Case.** We use $(Y_t^i)_{i=1^N}$ to denote N players. Assume that U is a compact subset of $\mathbb{R}^m$. For each player, there exist two functions f_i,$\sigma_i : \mathbb{R}^d\times U\times[s,S]\times\mathcal{P}(\mathbb{R}^d)$ such that $Y_t^i : [s,S]\times\Omega\to\mathbb{R}^d$ satisfies

$$dY_t^i = f_i(Y_t^i,\alpha_t^i,t,\nu_t^N)dt + \sigma_i(Y_t^i,\alpha_t^i,t,\nu_t^N)dB_t^i,$$

where $\nu_t^N = \frac{1}{N}\sum_{i=1}^N \delta_{Y_t^i}$ and δ_y is the point mass at y. We assume that initial conditions $\{Y_s^i\}_{i=1}^N$ are i.i.d. random variables and B_t^i is independent from B_t^j for $j\ne i$. Each player can choose a control process or a strategy α_t^i from a set of all admissible processes

$$\begin{aligned}
A^i = \big\{\alpha_t^i(\omega) : [s,S]\times\Omega\to U \,|\, E\big[\int_s^S |\alpha_t^i|^2 dt\big] < \infty,\ &\alpha_t^i \text{ is measurable w.r.t}\\
&\sigma(Y_s^1,\dots,Y_s^N, B^1_{s\le\tau\le t},\dots,B^N_{s\le\tau\le t})\big\}.
\end{aligned}$$

Note that A^i is the same for all $1\le i\le N$. Let A be the common set of admissible processes for all players. Each player evolves depending on his/her position and the empirical distribution of all other players' positions, which is the reason why it is called the Mean Field Game. There are a few notations to mention beforehand: $\alpha_t^{-i} = (\alpha_t^1, \dots, \alpha_t^{i-1}, \alpha_t^{i+1}, \dots, \alpha_t^N)$ and $\alpha^i = \{\alpha_t^i\}_{s \le t \le S}$.

Each player chooses the optimal strategy to minimize his/her cost function defined as

$$J_N^i(\alpha^i, \alpha^{-i}) = E\bigg[\int_s^S \gamma_i(Y_t^i, \alpha_t^i, t, \nu_t^N)dt + g_i(Y_S^i, \nu_S^N)\bigg].$$

Functions $\gamma_i, g_i : \mathbb{R}^d \times X \times [s, S] \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}$ apply to player i. Note that $\mathcal{P}(\mathbb{R}^d)$ is the space of probability measures defined on $\mathbb{R}^d$ equipped with the Borel σ-algebra generated by the topology of weak convergence of measures. From the definition, we see that the cost function not only depends on the player's own strategy but also indirectly on other players' overall strategies and positions through ν_t^N.

In real-life examples, we usually have a large dynamic system with an infinite number of players to deal with. So we will talk about the limiting behavior of the system. To make things easier, we replace $f_i, \sigma_i, \gamma_i, g_i$ by universal functions f, σ, γ, g for all players. Our assumption is that f, σ, γ, g are Lipschitz continuous and bounded.

Next, we introduce the criterion to determine the optimal strategy α_t^i and to minimize the cost function J_N^i.

We use $(\alpha^1, \dots, \alpha^N) \in A^N$ to denote $((\alpha_t^1)_{s \le t \le S}, \dots, (\alpha_t^N)_{s \le t \le S})$.

Definition 3.7. A N-tuple $(\alpha^{*,1}, \dots, \alpha^{*,N}) \in A^N$ is a Nash equilibrium in pure strategies if and only if

$$J_N^i(\alpha^{*,1}, \dots, \alpha^{*,N}) \le J_N^i(\alpha^i, \alpha^{*,-i}) \text{ for all } \alpha^i \in A.$$

Definition 3.8. Let Q be a set and N be a positive integer. A function $u : Q^N \to \mathbb{R}$ is symmetric if

$$u(x_1, \dots, x_N) = u(x_{\sigma(1)}, \dots, x_{\sigma(N)}) \text{ for any permutation } \sigma \text{ on } \{1, 2, \dots, N\}.$$

Theorem 3.9. *Let Q be a compact metric space. Let $u_N : Q^N \to \mathbb{R}$ be a symmetric function satisfying the following two conditions:*
1. there is a uniform bound on u_N, i.e., $\sup_{Q^N} |u_N| \le C$ for some $C > 0$.
2. u_N is uniformly continuous in the following sense. Let $X = (x_1, \dots, x_N), Y = (y_1, \dots, y_N)$. Let $\mu_X^N = \frac{1}{N}\sum_{i=1}^N \delta_{x_i}$ and $\mu_Y^N = \frac{1}{N}\sum_{i=1}^N \delta_{y_i}$. There is a modulus of continuity ω independent of N such that

$$|u_N(X) - u_N(Y)| \le \omega(d_1(\mu_X^N, \mu_Y^N)) \textit{ for all } X, Y \in Q^N, N \in \mathbb{N}.$$

The metric we are using is the Wasserstein distance on $\mathcal{P}(Q)$:

$$d_1(\mu, \nu) = \inf_\pi \int_{Q \times Q} d(x, y) d\pi(x, y),$$

where $d(x, y)$ is defined by the metric on Q and the infimum is taken over all probability measures π over $Q \times Q$ such that μ, ν are the marginals of π.

If the above two conditions are true, then there exists a subsequence u_{N_k} of u_N and a continuous map $U : \mathcal{P}(Q) \to \mathbb{R}$ such that

$$\lim_{k \to \infty} \sup_{X \in Q^{N_k}} |u_{N_k}(X) - U(\mu_X^{N_k})| = 0. \tag{3.10}$$

Remark 3.11. Alternatively, the dual representation of the Wasserstein distance is

$$d_1(\mu,\nu) = \sup\left\{ \int_Q f(x)d(\mu-\nu)(x) \,\middle|\, f : Q \to \mathbb{R} \text{ is continuous, } Lip(f) \le 1 \right\}.$$

Since Q is compact, we can assume that the set of such functions f is uniformly bounded.

Proof. By the usual properties of modulus of continuity for continuous functions on compact sets, we have that ω is concave, continuous, and nondecreasing. To begin with, we define a sequence of functions $U_N : \mathcal{P}(Q) \to \mathbb{R}$ as follows:

$$U_N(\mu) = \inf_{X \in Q^N} \{u_N(X) + \omega(d_1(\mu, \mu_X^N))\}.$$

By condition 2 of u_N, we know that

$$U_N(\mu_X^N) = \inf_{Y \in Q^N} \{u_N(Y) + \omega(d_1(\mu_X^N, \mu_Y^N))\} = u_N(X).$$

From this, we know that for any $N \in \mathbb{N}$, $|u_N(X) - U_N(\mu_X^N)| = 0$. We just need to show that there is a subsequence of $\{U_N\}$ that converges uniformly over all $X \in Q^N$ to some function U. Our goal is to use the Arzela-Ascoli theorem which requires that U_N is uniformly bounded and uniformly equicontinuous. Uniform boundedness of U_N is easy to show because u_N is uniformly bounded by condition 1. In addition, $\sup_{\mu,\nu \in \mathcal{P}(Q)} d_1(\mu,\nu)$ is bounded because Q is compact implies that $\mathcal{P}(Q)$ is a compact and complete metric space under d_1. Hence, U_N is uniformly bounded over $\mathcal{P}(Q)$. Next, we choose any $\mu, \nu \in \mathcal{P}(Q)$. For any $\epsilon > 0$, there exists $Z \in Q^N$ such that

$$u_N(Z) + \omega(d_1(\nu, \mu_Z^N)) \le U_N(\nu) + \epsilon.$$

Then

$$\begin{aligned}
U_N(\mu) &\le u_N(Z) + \omega(d_1(\mu, \mu_Z^N)) \\
&\le U_N(\nu) + \epsilon - \omega(d_1(\nu, \mu_Z^N)) + \omega(d_1(\mu, \mu_Z^N)) \\
&\le U_N(\nu) + \epsilon - \omega(d_1(\nu, \mu_Z^N)) + \omega(d_1(\mu,\nu) + d_1(\nu, \mu_Z^N)) \qquad (3.12)\\
&\le U_N(\nu) + \epsilon - \omega(d_1(\nu, \mu_Z^N)) + \omega(d_1(\mu,\nu)) + \omega(d_1(\nu, \mu_Z^N)) \\
&\le U_N(\nu) + \epsilon + \omega(d_1(\mu,\nu)),
\end{aligned}$$

where the third line comes from the triangle inequality of the metric d_1 and the fourth line is because ω is concave, hence subadditive. From the above inequality, we know that

$$U_N(\mu) - U_N(\nu) \le \omega(d_1(\mu,\nu)) + \epsilon$$

for all $\epsilon > 0$, $\mu, \nu \in \mathcal{P}(Q)$ and for some ω independent of N. Therefore, we claim that the sequence $\{U_N\}$ is equicontinuous on the compact set $\mathcal{P}(Q)$. Lastly, by using the Arzela-Ascoli theorem, we have proved that there exists a subsequence U_{N_k} that converges uniformly to a limit U which is also continuous. Since we have shown that for all $N \in \mathbb{N}$, $|u_N(X) - U_N(\mu_X^N)| = 0$, we get (3.10). □

Note that U is actually uniformly continuous because $\{U_N\}$ is equicontinuous and U is the limit under uniform convergence.

Assume that the cost functions $J_N^1, \dots, J_N^N$ are all symmetric. Recall how we initially defined the set of all possible strategies A. We assign the L_2 metric to the set A, i.e., $\forall \alpha, \alpha' \in A$,

$$d(\alpha, \alpha') = \left(E\Big[\int_s^S |\alpha_t - \alpha'_t|^2 dt \Big] \right)^{\frac{1}{2}}.$$

Now, we redefine A to be a compact subset of the original A. Let $\mathcal{P}(A)$ be the set of probability measures on A. Suppose $m \in \mathcal{P}(A)$. Then m is a probability measure on a set of various flows of random variables $\{\alpha_t\}_{s \le t \le S}$. Assume γ, g are nice functions such that the cost function satisfies the two conditions in Theorem 3.9.

Then by applying the previous theorem to the cost function, we get that $J_N^i(\alpha^i, \alpha^{-i})$ has a continuous limiting function $J(\alpha^i, m) : A \times \mathcal{P}(A) \to \mathbb{R}$ where α_t^i is to keep track of player i's strategy and m is the limit of a subsequence of the empirical distribution $\frac{1}{N}\sum_{i=1}^N \delta_{\alpha^i}$ as $N \to \infty$. Assume that J_N^i has the form

$$J_N^i(\alpha^i, \alpha^{-i}) = J\left(\alpha^i, \frac{1}{N-1}\sum_{j \ne i} \delta_{\alpha^j}\right) \text{ and up to a subsequence,}$$

$$\lim_{k \to \infty} J_{N_k}^i(\alpha^i, \alpha^{-i}) = J(\alpha^i, m). \tag{3.13}$$

Remark 3.14. Let $\nu_N = \frac{1}{N}\sum_{j=1}^n \delta_{\alpha^j}$ and $\nu_{N-1} = \frac{1}{N-1}\sum_{j \ne i} \delta_{\alpha^j}$. Then by the dual representation of the Wasserstein distance, we get

$$d_1(\nu_{N-1}, \nu_N) = d_1(\nu_{N-1}, \frac{N-1}{N}\nu_{N-1} + \frac{1}{N}\delta_{\alpha^i}) \le \frac{C}{N} \tag{3.15}$$

for some $C > 0$. Hence, $\lim_{N \to \infty} \nu_{N-1} = m$, which justifies (3.13).

For future reference, set $m_{\hat{\alpha}^N}^N = \frac{1}{N}\sum_{i=1}^N \delta_{\alpha^i}$ where $\hat{\alpha}^N = (\alpha^1, \dots, \alpha^N)$.

Theorem 3.16. *Assume that $\hat{\alpha}^{*,N} = (\alpha^{*,1}, \dots, \alpha^{*,N})$ is a Nash equilibrium in pure strategies for the game $J_N^1, \dots, J_N^N$. Then up to a subsequence, the sequence of empirical distributions $m_{\hat{\alpha}^{*,N}}^N$ converges to a measure $m \in \mathcal{P}(A)$ such that*

$$\int_A J(\alpha, m) dm(\alpha) = \inf_{\mu \in \mathcal{P}(A)} \int_A J(\alpha, m) d\mu(\alpha).$$

Proof. Without loss of generality, we assume that the whole sequence converges, i.e.,

$$\lim_{N \to \infty} m_{\hat{\alpha}^{*,N}}^N = m.$$

We first define

$$m_{\hat{\alpha}^{*,N}}^{N-1} = \frac{1}{N-1} \sum_{j \ne i, 1 \le j \le N} \delta_{\alpha^{*,j}}.$$

Since $\hat{\alpha}^{*,N} = (\alpha^{*,1}, \dots, \alpha^{*,N})$ is a Nash equilibrium, $J_N^i(\alpha^{*,i}, \alpha^{*,-i}) \le J_N^i(\alpha^i, \alpha^{*,-i})$ for all $\alpha^i \in A$. Then we get

$$\begin{aligned}\inf_{\mu \in \mathcal{P}(A)} \int_A J(\alpha, m_{\hat{\alpha}^{*,N}}^{N-1}) d\mu(\alpha) &= \int_A J(\alpha, m_{\hat{\alpha}^{*,N}}^{N-1}) d\delta_{\alpha^{*,i}}(\alpha) \\ &= J(\alpha^{*,i}, m_{\hat{\alpha}^{*,N}}^{N-1}) = J_N^i(\alpha^{*,i}, \alpha^{*,-i}).\end{aligned}$$

By (3.15) and uniform continuity of J, we get that for any $\epsilon > 0$, we can choose a sufficiently large N such that

$$\inf_{\mu\in\mathcal{P}(A)}\int_A J(\alpha, m^{N-1}_{\hat{\alpha}^*,N})d\mu(\alpha) \leq \inf_{\mu\in\mathcal{P}(A)}\int_A J(\alpha, m^{N}_{\hat{\alpha}^*,N})d\mu(\alpha) + \epsilon, \tag{3.17}$$

$$\int_A J(\alpha, m^{N}_{\hat{\alpha}^*,N})d\delta_{\alpha^{*,i}}(\alpha) = J(\alpha^{*,i}, m^{N}_{\hat{\alpha}^*,N}) \leq J(\alpha^{*,i}, m^{N-1}_{\hat{\alpha}^*,N}) + \epsilon.$$

Hence,

$$\int_A J(\alpha, m^{N}_{\hat{\alpha}^*,N})d\delta_{\alpha^{*,i}}(\alpha) \leq \inf_{\mu\in\mathcal{P}(A)}\int_A J(\alpha, m^{N}_{\hat{\alpha}^*,N})d\mu(\alpha) + 2\epsilon$$

which holds for all $1 \leq i \leq N$. By multiplying the previous inequality by $\frac{1}{N}$ and summing up from $i = 1$ to N, we get

$$\int_A J(\alpha, m^{N}_{\hat{\alpha}^*,N})dm^{N}_{\hat{\alpha}^*,N}(\alpha) \leq \inf_{\mu\in\mathcal{P}(A)}\int_A J(\alpha, m^{N}_{\hat{\alpha}^*,N})d\mu(\alpha) + 2\epsilon.$$

Since J is uniformly continuous, we can choose N large enough such that $|J(\alpha, m^{N}_{\hat{\alpha}^*,N}) - J(\alpha, m)| < \epsilon$ for any $\alpha \in A$. Then we get

$$\begin{aligned}&\int_A J(\alpha, m)dm^{N}_{\hat{\alpha}^*,N}(\alpha) - \epsilon \leq \int_A J(\alpha, m^{N}_{\hat{\alpha}^*,N})dm^{N}_{\hat{\alpha}^*,N}(\alpha)\\ &\leq \inf_{\mu\in\mathcal{P}(A)}\int_A J(\alpha, m^{N}_{\hat{\alpha}^*,N})d\mu(\alpha) + 2\epsilon \leq \inf_{\mu\in\mathcal{P}(A)}\int_A J(\alpha, m)d\mu(\alpha) + 3\epsilon.\end{aligned} \tag{3.18}$$

Hence,

$$\int_A J(\alpha, m)dm^{N}_{\hat{\alpha}^*,N}(\alpha) \leq \inf_{\mu\in\mathcal{P}(A)}\int_A J(\alpha, m)d\mu(\alpha) + 4\epsilon.$$

Taking the limit as N goes to infinity yields

$$\int_A J(\alpha, m)dm(\alpha) \leq \inf_{\mu\in\mathcal{P}(A)}\int_A J(\alpha, m)d\mu(\alpha) + 4\epsilon$$

for any $\epsilon > 0$, which proves the theorem. □

This theorem tells us that the Nash equilibrium for the finite game can be extended into the infinite game by taking the limit of the empirical distribution of strategies.

3.3. **Intuition Behind the MFG System.** We use the following specific example to illustrate the main idea of the Mean Field Game system which consists of a Hamilton-Jacobi-Bellman equation for the value function and a Fokker-Planck equation for the mean field density.

We start with a finite N-players game. Let $Y^i_t : [s, S] \times \Omega \to \mathbb{R}^d$ represent the i-th player who satisfies the stochastic differential equation of the form

$$dY^i_t = \alpha^i_t dt + dB^i_t,$$

where α_t is the strategy or, in this case, the player's velocity. As before, we define $\nu^N_t = \frac{1}{N}\sum_{i=1}^N \delta_{Y^i_t}$ and assume that $Y^1_s, \dots, Y^N_s$ are i.i.d. random variables and $\{B^i_t\}_{i=1}^N$ are also independent from each other. The player chooses his/her optimal velocity to minimize the following cost function

$$J^i_N(\alpha^i, \alpha^{-i}) = E\bigg[\int_s^S \big(f(Y^i_t, \nu^N_t) + \frac{1}{2}|\alpha^i_t|^2\big)dt + g(Y^i_T, \nu^N_T)\bigg].$$

The faster the player, the greater the cost.

Next, we are going to show the intuitive idea but not the rigorous proof that explains the MFG system.

By the definition of the cost function for player i, we see that functions f, g apply to all players and ν_t^N is symmetric in Y_t^j for $j \neq i$. Each player depends on all other players globally through ν_t^N and does not care about who is where. Each player follows the same cost function to determine his/her optimal strategy given that other players' strategies are fixed. There is plenty of symmetry in the system and we expect that this symmetry would appear at the equilibrium. So we guess that there is a function a_N such that $\alpha_t^{*,i} = a_N(Y_t^{*,i}, t, \nu_t^{*,N})$ at the equilibrium $(\alpha^{*,i}, \alpha^{*,-i})$. Here, $Y_t^{*,i}$ is the state of player i at time t when taking strategy $\alpha_t^{*,i}$ and $\nu_t^{*,N}$ is the empirical distribution of Y_t^{-i} at equilibrium. The fact that a_N depends on $Y_t^{*,i}, t, \nu_t^{*,N}$ is reasonable because each player determines his/her instant strategies depending on his/her state, the current time and the global distribution of other players' relative states. As $N \to \infty$, we hope that a_N converges to a function a that is independent of N. Without any proof, we assume that this function a is well-defined and is Lipschitz continuous and bounded such that the stochastic differential equation, $dY_t^i = a(Y_t^i, t, \nu_t^N)dt + dB_t^i$, is solvable.

Since the empirical distribution of Y_t^i is normalized by the factor $\frac{1}{N}$, we see that the influence of player j on player i becomes less and less as $N \to \infty$. So what we expect to happen is that players get less and less correlated as $N \to \infty$ and are exchangeable in the sense that as $N \to \infty$, they have the same distribution under the assumption that $\{Y_0^i\}$ are i.i.d. and players evolve under the same function a and the Brownian motion. In the end, we want to conclude that since players become independent and have the same distribution as $N \to \infty$, by the law of large numbers, ν_t^N converges to the law of any player $\mathcal{L}(Y_t^i)$ for $1 \leq i \leq N$.

Suppose $(Y_t^1, \ldots, Y_t^N)$ is the solution to $dY_t^i = a(Y_t^i, t, \nu_t^N)dt + dB_t^i$ for $1 \leq i \leq N$. Our conclusion is that for any integer k, as $N \to \infty$, $\mathcal{L}(Y_t^{n_1}, \ldots, Y_t^{n_k})$ converges weakly to $\mathcal{L}(\bar{Y}_t)^{\otimes k}$ for any given time t and for some stochastic process $(\bar{Y}_t)_{s \leq t \leq S}$ which solves

$$d\bar{Y}_t = a(\bar{Y}_t, t, \mathcal{L}(\bar{Y}_t))dt + dB_t.$$

This equation is called the Mckean-Vlasov stochastic differential equation. The diffusion coefficient inside depends on the distribution of the solution itself.

Then we introduce the following MFG equilibrium to fully characterize $\mathcal{L}(\bar{Y}_t)$.

Definition 3.19. We call $(m_t)_{s \leq t \leq S} \in \mathcal{P}(\mathbb{R}^d)$ a Mean Field Game equilibrium if the optimization problem

$$\begin{aligned} &\inf_{\{\alpha_t\}_{s \leq t \leq S} \in A} E\left[\int_s^S \big(f(Y_t, m_t) + \frac{1}{2}|\alpha_t|^2\big)dt + g(Y_T, m_t)\right] \\ &\text{subject to} \qquad dY_t = \alpha_t dt + dB_t \end{aligned} \tag{3.20}$$

has one solution $(\bar{Y}_t)_{s \leq t \leq S}$ and for all $t \in [s, S]$, $\mathcal{L}(\bar{Y}_t) = m_t$.

This definition tells us that the law of the best response under the global distribution $(m_t)_{s \leq t \leq S}$ is given by $(m_t)_{s \leq t \leq S}$.

3.4. **The PDE formulation of MFG.** Under the environment $(m_t)_{s \leq t \leq S}$, we have frozen the interactions among players. Then we can solve (3.20) by using the PDE for the value function, which we have developed in the previous chapter.

Consider the value function

$$u(x,s) = \inf_{\alpha_t \in A} E\Big[\int_s^S \big(f(Y_t, m_t) + \frac{1}{2}|\alpha_t|^2\big)dt + g(Y_T, m_t)\,\Big|\, Y_s = x\Big].$$

From Theorem 3.5, we know that the value function satisfies the Hamilton-Jacobi-Bellman equation, which, in this case, becomes:

$$\partial_s u(x,s) + \frac{1}{2}\Delta u(x,s) + \inf_{\alpha \in \mathbb{R}^d}\big(\frac{1}{2}\alpha^2 + \alpha \cdot \partial_x u(x,s)\big) + f(x, m_s) = 0,$$

$$\big(\partial_s u + \frac{1}{2}\Delta u\big)(x,s) - \frac{1}{2}|\partial_x u(x,s)|^2 + f(x, m_s) = 0,$$

where the optimal strategy is taken to be $\alpha = -\partial_x u(x,s)$. There is also a boundary condition $u(x,S) = g(x, m_S)$. So we get the corresponding optimal trajectory under the optimal strategy, that is

$$d\bar{Y}_t = -\partial_x u(\bar{Y}_t, t)dt + dB_t. \tag{3.21}$$

Notice that the value function u depends on m_t which is also unknown. So in order to solve the PDE for u, we need to develop an equation for the flow of probability measures m_t. From the definition of MFG equilibrium, we know that $\mathcal{L}(\bar{Y}_t) = m_t$. Since $\bar{Y}_t$ solves $d\bar{Y}_t = -\partial_x u(\bar{Y}_t, t)dt + dB_t$, we get the following Forward Kolmogorov equation or the Fokker-Planck equation:

$$\partial_t m_t = \operatorname{div}(m_t \partial_x u(x,t)) + \frac{1}{2}\Delta m_t.$$

For proof of the forward Kolmogorov equation, readers can refer to [5]. Now, we state the formal definition of the MFG system for this specific model:

$$\begin{cases} 1. \quad \big(\partial_t u + \frac{1}{2}\Delta u\big)(x,t) - \frac{1}{2}|\partial_x u(x,t)|^2 + f(x, m_t) = 0 \\ 2. \quad \partial_t m_t - \operatorname{div}(m_t \partial_x u(x,t)) - \frac{1}{2}\Delta m_t = 0 \\ 3. \quad m_s \text{ is fixed and } u(x,S) = g(x, m_S) \end{cases}. \tag{3.22}$$

Equation 1 is the HJB equation of the value function when the flow of distribution $(m_t)_{s \le t \le S}$ describing the equilibrium state of the population is fixed. This is a backward equation with a given terminal condition. Equation 2 is the forward Kolmogorov equation describing how the flow of measures evolves with time. This equation is forward in time and is equipped with an initial condition m_s which determines how players are distributed at time s. It is because these two directions are conflicting that the MFG system is very difficult to solve.

3.5. **Solving the MFG system.** Next, we are going to discuss solutions to the HJB equation under nice conditions and justify that $\mathcal{L}(\bar{Y}_t)$ solves the Fokker-Planck equation. Then we will briefly explain how to use Schauder's fixed point theorem to solve the MFG system.

We try to solve the Fokker-Planck equation when $u(x,t)$ is fixed. Let $h : \mathbb{R}^d \times [s,S] \to \mathbb{R}^d$ be a function that is continuous in time and Hölder continuous in space. We want to solve the following

$$\begin{cases} \partial_t m_t - \operatorname{div}(m_t h) - \frac{1}{2}\Delta m_t = 0 \\ m_s = m(x,s) \end{cases}. \tag{3.23}$$

Definition 3.24. We say that $m \in L^1(\mathbb{R}^d \times [s,S])$ is a weak solution to (3.23) if for any test function $\psi \in C_c^\infty(\mathbb{R}^d \times [s,S])$(compactly supported smooth function), we have

$$\begin{aligned}&\int_{\mathbb{R}^d} \psi(x,s)dm_s(x) - \int_{\mathbb{R}^d} \psi(x,S)dm_S(x)\\ &+ \int_s^S \int_{\mathbb{R}^d} \partial_t\psi(x,t) + \frac{1}{2}\Delta\psi(x,t) - \partial_x\psi(x,t)\cdot h(x,t)dm_t(x)dt = 0.\end{aligned} \tag{3.25}$$

Note that the above equation is obtained by multiplying ψ to (3.23) and then applying integration by parts. Consider the stochastic differential equation

$$\begin{cases} d\bar{Y}_t = -h(\bar{Y}_t,t)dt + dB_t \\ \bar{Y}_s = Z_s \end{cases}. \tag{3.26}$$

The initial condition $Z_s \in L^1(\Omega)$ is a random variable and is independent of the Brownian motion. Since h is continuous in time and Hölder continuous in space, we know that (3.26) has a unique solution $\{\bar{Y}_t\}_{s\le t\le S}$.

Theorem 3.27. *Under the above assumptions, if $\mathcal{L}(\bar{Y}_s) = m_s$, then $m_t := \mathcal{L}(\bar{Y}_t)$ is a weak solution of* (3.23).

Proof. Provided that ψ is smooth and compactly supported, by Itô formula, we have that

$$\begin{aligned}\psi(\bar{Y}_\tau,\tau) =& \psi(\bar{Y}_s,s) + \int_s^\tau \partial_t\psi(\bar{Y}_t,t) + \frac{1}{2}\Delta\psi(\bar{Y}_t,t) - \partial_x\psi(\bar{Y}_t,t)\cdot h(\bar{Y}_t,t)dt\\ &+ \int_s^\tau \partial_x\psi(\bar{Y}_t,t)dB_t.\end{aligned} \tag{3.28}$$

Then we take the expectation on both sides

$$E[\psi(\bar{Y}_\tau,\tau)] = E\left[\psi(\bar{Y}_s,s) + \int_s^\tau \partial_t\psi(\bar{Y}_t,t) + \frac{1}{2}\Delta\psi(\bar{Y}_t,t) - \partial_x\psi(\bar{Y}_t,t)\cdot h(\bar{Y}_t,t)dt\right].$$

$$\begin{aligned}&\int_{\mathbb{R}^d} \psi(y,\tau)dm_\tau(y) = \int_{\mathbb{R}^d} \psi(y,s)dm_s(y)\\ &+ \int_s^\tau \int_{\mathbb{R}^d} \partial_t\psi(y,t) + \frac{1}{2}\Delta\psi(y,t) - \partial_x\psi(y,t)\cdot h(y,t)dm_t(y)dt,\end{aligned}$$

which satisfies (3.25). Hence, m is a weak solution to (3.23). □

To show the uniqueness of the solution, we require the assumption that functions f, g satisfy the Lasry-Lions monotonicity condition, i.e.,

$$\forall m, m' \in \mathcal{P}(\mathbb{R}^d),\ \int_{\mathbb{R}^d} f(x,m) - f(x,m')d(m-m')(x) \ge 0$$
$$\text{and } \int_{\mathbb{R}^d} g(x,m) - g(x,m')d(m-m')(x) \ge 0.$$

We will skip the detailed proof.

To solve the HJB equation, we apply the Cole-Hopf transformation to linearize the PDE. Recall the HJB equation that we want to solve:

$$\begin{cases} \left(\partial_t u + \frac{1}{2}\Delta u\right)(x,t) - \frac{1}{2}|\partial_x u(x,t)|^2 + f(x,m_t) = 0 \\ u(x,S) = g(x,m_S) \end{cases} \tag{3.29}$$

for some determined flow of probability measures m_t. The Cole-Hopf transformation suggests the following idea. Let $\phi(x,t) = e^{-u(x,t)}$ or $u(x,t) = -\log \phi(x,t)$. Then we rewrite (3.29) as a function of ϕ:

$$\frac{\partial_t \phi}{\phi}(x,t) + \frac{1}{2}\frac{\Delta\phi}{\phi}(x,t) - f(x, m_t) = 0,$$

$$\phi_t(x,t) + \frac{1}{2}\Delta\phi(x,t) - f(x,m_t)\phi = 0,$$

which is a linear equation and is much easier to solve. We can solve for ϕ and then take the logarithm to get u. The Cole-Hopf transformation does not work for the general MFG system. To prove the uniqueness of the solution u, we need more conditions on u, f, g. For details of the uniqueness proof, readers can refer to chapter 2 of [2].

Lastly, the Schauder's fixed point theorem states that if $\mathcal{C}$ is a nonempty, convex, closed subset of a Hausdorff topological vector space and T is a continuous mapping from $\mathcal{C}$ into itself such that $T(\mathcal{C})$ is relatively compact in $\mathcal{C}$, then T has a fixed point. We construct a mapping from a convex closed subset of $C^0([s,S], \mathcal{P}(\mathbb{R}^d))$ to itself. $C^0([s,S], \mathcal{P}(\mathbb{R}^d))$ is a set of flows of probability measures $\{m_t\}_{s\leq t\leq S}$ that are continuous in time. For any fixed $\mu = \{\mu_t\}_{s\leq t\leq S} \in \mathcal{C}$, we solve the (3.29) for the unique solution u and then define $m = T(\mu)$ to be the solution of (3.23) with $h = \partial_x u(x,t)$. We can show that under nice continuity assumptions on f, g and u, T is a continuous mapping, hence a fixed point exists, which is the solution to the MFG system. For the detailed proof of this theorem, readers can refer to chapter 4 of [2].

4. Appendix

In this section, we present the proof of the property of the mollifier.

Lemma 4.1. *If $f \in U$ be a bounded function and $g_n(t,\omega) = \int_{\mathbb{R}} \psi_n(s-t) f(s,\omega) ds$, then $\lim_{n\to\infty} \int_T (f-g_n)^2 dt = 0$.*

Proof. We rewrite the function to be $g_n(t,\omega) = \int_{\mathbb{R}} \psi_n(a) f(a+t,\omega) da$. Then

$$\begin{aligned}|f - g_n| &\leq \int_{\mathbb{R}} |f(t,\omega) - f(a+t,\omega)|\psi_n(a) da,\\ |f-g_n|^2 &\leq \left(\int_{\mathbb{R}} |f(t,\omega) - f(a+t,\omega)|\psi_n(a) da\right)^2,\\ &\leq \int_{\mathbb{R}} |f(t,\omega) - f(a+t,\omega)|^2 \psi_n(a) da,\end{aligned}$$

where the third inequality comes from Jensen's inequality since $x \mapsto x^2$ is a convex function. Applying an integral over t and then using Fubini's theorem gives

$$\int_T (f-g_n)^2 dt \leq \int_{\mathbb{R}} \left(\int_T (f(t,\omega) - f(t+a,\omega))^2 dt\right) \psi_n(a) da.$$

Since f is defined on a bounded interval, by Theorem 2.24 of [6], we can find a function $h \in \mathcal{C}_c$ satisfying $\int_T (f-h)^2 dt < \frac{\epsilon}{3}$. Also, since h is continuous, when a is small enough $\int_T (h(t,\omega) - h(t+a,\omega))^2 dt < \frac{\epsilon}{3}$. Therefore, by triangle inequality, we get the following:

$$\int_T (f(t,\omega) - f(t+a,\omega))^2 dt \leq \int_T (f(t,\omega) - h(t,\omega))^2 dt$$
$$+ \int_T (h(t,\omega) - h(t+a,\omega))^2 dt + \int_T (f(t+a,\omega) - h(t+a,\omega))^2 dt < \epsilon,$$

when a is small enough. We also know that $a \to 0$ as $n \to \infty$. Hence,

$$\lim_{n\to\infty} \int_T (f - g_n)^2 dt \leq \int_{\mathbb{R}} \epsilon \psi_n(a) da = \epsilon \ \forall \epsilon.$$

This gives the result. □

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