第一章 行 列 式

主要内容

n阶行列式的定义、性质及其计算.

重点内容 行列式的计算

第一节　二阶与三阶行列式

主要内容

- 二阶行列式
- 三阶行列式
- 小结

二阶行列式的引入

用消元法解二元线性方程组 $\begin{cases} a_{11}x_1+a_{12}x_2=b_1\,, \\ a_{21}x_1+a_{22}x_2=b_2\,, \end{cases}$ 得

$$x_1=\frac{b_1a_{22}-a_{12}b_2}{a_{11}a_{22}-a_{12}a_{21}}\,,$$

提 示：

$$[a_{11}x_1+a_{12}x_2=b_1]\times a_{22}\Rightarrow a_{11}a_{22}x_1+a_{12}a_{22}x_2=b_1a_{22}\,,$$

$$[a_{21}x_1+a_{22}x_2=b_2]\times a_{12}\Rightarrow a_{12}a_{21}x_1+a_{12}a_{22}x_2=a_{12}b_2\,,$$

$$\Rightarrow(a_{11}a_{22}-a_{12}a_{21})x_1=b_1a_{22}-a_{12}b_2\,.$$

用消元法解二元线性方程组 $\begin{cases} a_{11}x_1+a_{12}x_2=b_1, \\ a_{21}x_1+a_{22}x_2=b_2, \end{cases}$ 得

$$x_1=\frac{b_1a_{22}-a_{12}b_2}{a_{11}a_{22}-a_{12}a_{21}}, \quad x_2=\frac{a_{11}b_2-b_1a_{21}}{a_{11}a_{22}-a_{12}a_{21}}.$$

提 示：

$$[a_{11}x_1+a_{12}x_2=b_1]\times a_{21}\Rightarrow a_{11}a_{21}x_1+a_{12}a_{21}x_2=b_1a_{21},$$

$$[a_{21}x_1+a_{22}x_2=b_2]\times a_{11}\Rightarrow a_{11}a_{21}x_1+a_{11}a_{22}x_2=a_{11}b_2,$$

$$\Rightarrow(a_{11}a_{22}-a_{12}a_{21})\,x_2=a_{11}b_2-b_1a_{21}.$$

用消元法解二元线性方程组 $\begin{cases} a_{11}x_1+a_{12}x_2=b_1, \\ a_{21}x_1+a_{22}x_2=b_2, \end{cases}$ 得

$$x_1=\frac{b_1a_{22}-a_{12}b_2}{a_{11}a_{22}-a_{12}a_{21}}, \quad x_2=\frac{a_{11}b_2-b_1a_{21}}{a_{11}a_{22}-a_{12}a_{21}}.$$

我们用符号 $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ 表示代数和 $a_{11}a_{22}-a_{12}a_{21}$ 这样就有

$$x_1=\frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}, x_2=\frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}.$$

定义　由四个数排成二行二列（横排称行、竖排称列）的数表

$$\begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{matrix} \qquad (4)$$

表达式 $a_{11}a_{22}-a_{12}a_{21}$ 称为数表（4）所确定的二阶行列式，并记作

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \qquad (5)$$

即 $D=\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}=a_{11}a_{22}-a_{12}a_{21}.$

术语：元素

行、列

二阶行列式的计算——对角线法则

主对角线

副对角线

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}.$$

（了解）对于二元线性方程组
$$\begin{cases} a_{11}x_1 + a_{12}x_2 = b_1, \\ a_{21}x_1 + a_{22}x_2 = b_2. \end{cases}$$

若记

$$D = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix},$$

系数行列式

$$\begin{cases} a_{11}x_1 + a_{12}x_2 = \boxed{b_1}, \\ a_{21}x_1 + a_{22}x_2 = \boxed{b_2}. \end{cases}$$

$$D = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix},$$

$$\begin{cases} a_{11}x_1 + a_{12}x_2 = \boxed{b_1}, \\ a_{21}x_1 + a_{22}x_2 = \boxed{b_2}. \end{cases}$$

$$D_1 = \begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix} = b_1 a_{22} - a_{12} b_2,$$

$$\begin{cases} a_{11}x_1 + a_{12}x_2 = \boxed{b_1}, \\ a_{21}x_1 + a_{22}x_2 = \boxed{b_2}. \end{cases}$$

$$\Downarrow$$

$$D = \begin{vmatrix} a_{11} & \boxed{a_{12}} \\ a_{21} & \boxed{a_{22}} \end{vmatrix},$$

$$\begin{cases} a_{11}x_1 + a_{12}x_2 = \boxed{b_1}, \\ a_{21}x_1 + a_{22}x_2 = \boxed{b_2}. \end{cases}$$

$$D_1 = \begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix} = b_1a_{22} - a_{12}b_2,$$

$$\begin{cases} a_{11}x_1 + a_{12}x_2 = \boxed{b_1}, \\ a_{21}x_1 + a_{22}x_2 = \boxed{b_2}. \end{cases}$$

$$D_2 = \begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix} = a_{11}b_2 - b_1a_{21}.$$

则二元线性方程组的解为

$$x_1 = \frac{D_1}{D} = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}, \qquad x_2 = \frac{D_2}{D} = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}.$$

注意 分母都为原方程组的系数行列式.

例1 求解二元线性方程组

$$\begin{cases} 3x_1 - 2x_2 = 12, \\ 2x_1 + x_2 = 1. \end{cases}$$

解

$$D = \begin{vmatrix} 3 & -2 \\ 2 & 1 \end{vmatrix} = 3 - (-4) = 7 \neq 0,$$

$$D_1 = \begin{vmatrix} 12 & -2 \\ 1 & 1 \end{vmatrix} = 14, \quad D_2 = \begin{vmatrix} 3 & 12 \\ 2 & 1 \end{vmatrix} = -21,$$

$$\therefore \quad x_1 = \frac{D_1}{D} = \frac{14}{7} = 2, \quad x_2 = \frac{D_2}{D} = \frac{-21}{7} = -3.$$

二、三阶行列式

定义 设有9个数排成3行3列的数表

$$\begin{matrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{matrix} \qquad (5)$$

记

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}, \qquad (6)$$

（6）式称为数表（5）所确定的**三阶行列式**.

三阶行列式的计算 --对角线法则

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$$
$$- a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}.$$

注意 红线上三元素的乘积冠以正号，蓝线上三元素的乘积冠以负号.

说明 对角线法则只适用于二阶与三阶行列式.

利用三阶行列式求解三元线性方程组（了解）

如果三元线性方程组 $\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1, \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2, \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3; \end{cases}$

的系数行列式 $D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \neq 0,$

$$D=\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \qquad D_1=\begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix},$$

$$D_3=\begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}. \qquad D_2=\begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix},$$

则三元线性方程组的解为:

$$x_1=\frac{D_1}{D}, \qquad x_2=\frac{D_2}{D}, \qquad x_3=\frac{D_3}{D}.$$

三、小结

二阶与三阶行列式的计算 ——对角线法则

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}.$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}.$$

了解二阶与三阶行列式计算公式的特点.

思考题

求一个二次多项式$f(x)$,使

$$f(1)=0,\quad f(2)=3,\quad f(-3)=28.$$

思考题解答

解 设所求的二次多项式为

$$f(x)=ax^2+bx+c,$$

由题意得 $f(1)=a+b+c=0,$

$f(2)=4a+2b+c=3,\quad f(-3)=9a-3b+c=28,$

得一个关于未知数 a,b,c 的线性方程组,

又 $D=-20\neq 0,\ \ D_1=-40, D_2=60, D_3=-20.$

得 $a=D_1/D=2, b=D_2/D=-3,\quad c=D_3/D=1$

故所求多项式为

$$f(x)=2x^2+-3x+1.$$