DSP Homework

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Abstract

1 Videos

2 FIR And IIR

2.1 (a)

2.1.1 When $Q = 0$

When $Q = 0$, the system can be represented as

$$y(n) = a_0 x(n) + \cdots + a_P x(n - P)$$

Then, the $h(n)$ can be written as

$$h(n) = \sum_{i}^{P} a_i \times \delta(n - i)$$

So, $h(n)$ has finite duration, the system is FIR.

2.1.2 When $Q > 0$

When $Q > 0$, the system can be represented as

$$y(n) + b_1 y(n-1) + \cdots + b_Q y(n - Q) = a_0 x(n) + a_1 x(n-1) + \cdots + a_P x(n - P) \quad (1)$$

We can know that the $y(n), y(n-1), y(n-2), \cdots$ is

$$y(n) = a_0x(n) + a_1x(n-1) + \cdots + a_Px(n-P) - b_1y(n-1) - \cdots - b_Qy(n-Q) \tag{2}$$
$$y(n-1) = a_0x(n-1) + a_1x(n-2) + \cdots + a_Px(n-P-1) - b_1y(n-2) - \cdots - b_Qy(n-Q-1) \tag{3}$$
$$y(n-2) = a_0x(n-2) + a_1x(n-3) + \cdots + a_Px(n-P-2) - b_1y(n-3) - \cdots - b_Qy(n-Q-2) \tag{4}$$
$$y(n-3) = a_0x(n-3) + a_1x(n-4) + \cdots + a_Px(n-P-3) - b_1y(n-4) - \cdots - b_Qy(n-Q-3)$$
$$\cdots$$

Equation (2) contains all the terms in (3), and (3) contains all the terms in (4), all the same below, so (2) seems to have more terms than (3). However, actually there are always the same number of terms in each equation, so (2) does contain all the terms in (3), but also it has the same number of terms compared with the equations below. Only possibility is that $y(n)$ and $y(n-1)$ and so on have infinite number of terms. In one word, the system which (1) reveals is IIR.

2.2 (b)

2.2.1 Digital Filter Stability

When a digital filter has the feature that when input is bounded, the output must be bounded, this digital filter is stable.

2.2.2 FIR Stability

FIR can be expressed as below

$$y(n) = a_0x(n) + \cdots + a_Px(n-P)$$

If $x(n)$ is bounded, we can always find a maximum of $|x(n)|$ which we call it $max\{|x|\}$. And we can find the maximum of $|a_i|$ which we call it A as well. So

$$\begin{aligned} y(n) &= a_0x(n) + \cdots + a_Px(n-P) \\ &< |a_0||x(n)| + \cdots + |a_P||x(n-P)| \\ &< A \times max\{x\} \times P \\ &< \infty \end{aligned}$$

2.2.3 IIR Stability

We have

$$\begin{aligned} |y(n)| &= |x(n) * h(n)| \\ &\leq \sum_m |x(m) \times h(m-n)| \\ &\leq \sum_m |x(m)| \times |h(n-m)| \\ &\leq \sum_m max\{x\} \times h(n-m) \\ &= max\{x\} \sum_m h(m) \end{aligned}$$

and if we need $y(n) < \infty$, we should have

$$\sum_{-\infty}^{\infty} h(n) < \infty \tag{5}$$

Look at the $\tilde{h}(z)$ we can get from (2):

$$\tilde{h}(z) = \frac{\sum_{p=0}^{P} a_p z^{-p}}{\sum_{q=0}^{Q} b_q z^{-q}} \tag{6}$$

According to mathematics, we can rewrite $h(z)$ as below

$$\tilde{h}(z) = z^{\lambda-1} \left[\frac{z}{z-\alpha_0} + \frac{z}{z-\alpha_1} + \cdots + \frac{z}{z-\alpha_N} \right] \tag{7}$$

First, we know in frequency domain, the $\frac{z}{z-\alpha} = \frac{1}{1-\alpha/z} = \sum_{-\infty}^{\infty} az^{-1}$ means α^n in time domain.

Second, see (7) again, and in order to satisfy the requirement in (5), we need to make all poles in (7) (which are α here) less than 1, so that all α^n will converge, at last $h(z)$ will also converge.

2.2.4 An Example in Class

When $Q = 2, P = 0$, we have

$$y(n) + b_1 y(n-1) + b_2 y(n-2) = a_0 x(n) \tag{8}$$

In this condition, $\tilde{h}(z)$ is like

$$\tilde{h}(z) = \frac{a_0}{1 + b_1 z^{-1} + b_2 z^{-2}} = \frac{a_0 z^2}{z^2 + b_1 z + b_2} \tag{9}$$

Here, the expression of $H(z)$ is as below

$$H(z) = z^2 + b_1 z + b_2 \tag{10}$$

Roots of this equation is $z = -b_1 \pm \sqrt{b_1^2 - 4b_2}$.

First, if (10) has real root(s), to make the root(s) be in (-1, 1), the sufficient and necessary conditions is as below

$$\begin{cases} H(-1) = 1 - b_1 + b_2 > 0 \\ H(1) = 1 + b_1 + b_2 > 0 \\ -\frac{b_1}{2} \in (-1, 1) \end{cases}$$

It is easy using linear programming to prove that all the conditions above can be derived from $|b_1| + |b_2| < 1$.

Second, if (10) has complex root(s), to make the root(s) be in (-1, 1), the sufficient and necessary conditions is as below

$$\begin{aligned} |\alpha_1| = |\alpha_2| &= |\frac{-b_1 \pm j\sqrt{4b_2 - b_1^2}}{2}| \\ &= \sqrt{b_2} \in (-1, 1) \end{aligned}$$

It is easy to find that $\sqrt{b_2} \in (-1, 1)$ according to $|b_1| + |b_2| < 1$.

In conclusion, we can find that if $|b_1| + |b_2| < 1$ is true, the roots of $H(z)$ is all within the unit circle which means the system in (6) is stable. So, $|b_1| + |b_2| < 1$ is a sufficient condition for the IIR system to be stable when $Q = 2, P = 0$.

3 Why Some Voices Sound nicer

Appendix A Code Listing