DSP Homework

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Abstract

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2 IDFT

To derive the IDFT formula, we only should derive the expression of A^{-1}. And since A can be expressed as below

$$\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & w^1 & w^2 & \cdots & w^{N-1} \\ 1 & w^2 & w^4 & \cdots & w^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & w^{2(N-1)} & w^{4(N-1)} & \cdots & w^{(N-1)^2} \end{bmatrix}$$

we note rows in A as α_i, and rewrite it as below

$$A = \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{N-1} \end{bmatrix}$$

See $\alpha_i \cdot \alpha_j^H$, if $i \neq j$, we let $\lambda = i - j \in (0, N)$, and

$$\begin{aligned} \alpha_i \cdot \alpha_j^H &= \begin{bmatrix} 1 & w^i & w^{2i} & w^{3i} & \cdots & w^{(N-1)i} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ w^j \\ w^{2j} \\ w^{3j} \\ \vdots \\ w^{(N-1)j} \end{bmatrix} \\ &= 1 + w^\lambda + w^{2\lambda} + \cdots + w^{(N-1)\lambda} \\ &= \frac{1 - w^{N\lambda}}{1 - w^\lambda} \\ &= \frac{1 - e^{j2\pi\lambda} \quad (= 0)}{1 - e^{-j\frac{2\pi}{N}\lambda} \quad (\neq 0)} \\ &= 0 \end{aligned} \tag{1}$$

If $i = j$, we have

$$\alpha_i \cdot \alpha_j^H = 1 + w^0 + w^{2\times 0} + \cdots + w^{(N-1)\times 0} = N \tag{2}$$

So, conclude (1) and (2), we have

$$\alpha_i \cdot \alpha_j^H = \begin{cases} 0 & \text{if } i \neq j \\ N & \text{if } i = j \end{cases} \tag{3}$$

Therefore, see $A \cdot A^H$

$$\begin{aligned} A \cdot A^H &= \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{N-1} \end{bmatrix} \cdot \begin{bmatrix} \alpha_0^H & \alpha_1^H & \alpha_2^H & \cdots & \alpha_{N-1}^H \end{bmatrix} \\ &= N \cdot \begin{bmatrix} \alpha_0 \cdot \alpha_0^H & \alpha_0 \cdot \alpha_1^H & \cdots & \alpha_0 \cdot \alpha_{N-1}^H \\ \alpha_1 \cdot \alpha_0^H & \alpha_1 \cdot \alpha_1^H & \cdots & \alpha_1 \cdot \alpha_{N-1}^H \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{N-1} \cdot \alpha_0^H & \alpha_{N-1} \cdot \alpha_1^H & \cdots & \alpha_{N-1} \cdot \alpha_{N-1}^H \end{bmatrix} \\ &= N \cdot \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \\ &= N \cdot I \end{aligned} \tag{4}$$

If we multiply A^{-1} in both side of (4), we have

$$A^{-1} = \frac{1}{N} A^H \tag{5}$$

Therefore, IDFT formula can be expressed as below

$$\begin{aligned} x &= A^{-1} \tilde{x} \\ &= \frac{1}{N} A^H \tilde{x} \\ &= \frac{1}{N} (A \tilde{x}^H)^H \\ &= \frac{1}{N} \sum_{k=0}^{N-1} \tilde{x}(k) e^{j2\pi kn/N} \end{aligned} \tag{6}$$

Also, a more symmetric definition is as below

$$\tilde{x}(k) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}$$

$$x(n) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \tilde{x}(k) e^{j2\pi kn/N}$$

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