DSP Homework

Xu, Minhuan

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Abstract

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2 IDFT

To derive the IDFT formula, we only should derive the expression of A^{-1}. And since A can be expressed as below

$$\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & w^1 & w^2 & \cdots & w^{N-1} \\ 1 & w^2 & w^4 & \cdots & w^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & w^{2(N-1)} & w^{4(N-1)} & \cdots & w^{(N-1)^2} \end{bmatrix}$$

we note rows in A as α_i, and rewrite it as below

$$A = \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{N-1} \end{bmatrix}$$

See $\alpha_i \cdot \alpha_j^H$, if $i \neq j$, we let $\lambda = i - j \in (0, N)$, and

$$\begin{aligned} \alpha_i \cdot \alpha_j^H &= \begin{bmatrix} 1 & w^i & w^{2i} & w^{3i} & \cdots & w^{(N-1)i} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ w^j \\ w^{2j} \\ w^{3j} \\ \vdots \\ w^{(N-1)j} \end{bmatrix} \\ &= 1 + w^\lambda + w^{2\lambda} + \cdots + w^{(N-1)\lambda} \\ &= \frac{1 - w^{N\lambda}}{1 - w^\lambda} \\ &= \frac{1 - e^{j2\pi\lambda} \quad (= 0)}{1 - e^{-j\frac{2\pi}{N}\lambda} \quad (\neq 0)} \\ &= 0 \end{aligned} \tag{1}$$

If $i = j$, we have

$$\alpha_i \cdot \alpha_j^H = 1 + w^0 + w^{2\times 0} + \cdots + w^{(N-1)\times 0} = N \tag{2}$$

So, conclude (1) and (2), we have

$$\alpha_i \cdot \alpha_j^H = \begin{cases} 0 & \text{if } i \neq j \\ N & \text{if } i = j \end{cases} \tag{3}$$

Therefore, see $A \cdot A^H$

$$\begin{aligned} A \cdot A^H &= \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{N-1} \end{bmatrix} \cdot \begin{bmatrix} \alpha_0^H & \alpha_1^H & \alpha_2^H & \cdots & \alpha_{N-1}^H \end{bmatrix} \\ &= N \cdot \begin{bmatrix} \alpha_0 \cdot \alpha_0^H & \alpha_0 \cdot \alpha_1^H & \cdots & \alpha_0 \cdot \alpha_{N-1}^H \\ \alpha_1 \cdot \alpha_0^H & \alpha_1 \cdot \alpha_1^H & \cdots & \alpha_1 \cdot \alpha_{N-1}^H \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{N-1} \cdot \alpha_0^H & \alpha_{N-1} \cdot \alpha_1^H & \cdots & \alpha_{N-1} \cdot \alpha_{N-1}^H \end{bmatrix} \\ &= N \cdot \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \\ &= N \cdot I \end{aligned} \tag{4}$$

If we multiply A^{-1} in both side of (4), we have

$$A^{-1} = \frac{1}{N} A^H \tag{5}$$

Therefore, IDFT formula can be expressed as below

$$\begin{aligned} x &= A^{-1}\tilde{x} \\ &= \frac{1}{N} A^H \tilde{x} \\ &= \frac{1}{N} (A\tilde{x}^H)^H \\ &= \frac{1}{N} \sum_{k=0}^{N-1} \tilde{x}(k) e^{j2\pi kn/N} \end{aligned} \tag{6}$$

Also, a more symmetric definition is as below

$$\tilde{x}(k) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}$$

$$x(n) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \tilde{x}(k) e^{j2\pi kn/N}$$

3 Find Analog Spectrum Value in DFT

The DTFT formula can be rewritten as below

$$\begin{aligned} \tilde{x}_s(f) &= \mathcal{F}\left[x_s(t)\right] \\ &= \mathcal{F}\left[x_a(t) \times s(t)\right] \\ &= \tilde{x}_a(f) * \tilde{s}(f) \\ &= \tilde{x}_a(f) * \left[f_s \sum_n \delta(f - nf_s) \right] \\ &= f_s \cdot \sum_n \tilde{x}_a(f - nf_s) \end{aligned} \tag{7}$$

To make the frequency discrete, we should do sampling in frequency domain. The period of (7) is f_s, so we make the analog frequency be $f = f_s \cdot \frac{k}{N}$.

Therefore, we have

$$\begin{aligned}\tilde{x}(\frac{k \cdot f_s}{N}) &= f_s \cdot \sum_n \tilde{x}_a[f_s \cdot \frac{k}{N} - f_s \cdot n] \\ &= f_s \cdot \sum_n \tilde{x}_a[f_s(\frac{k}{N} - n)]\end{aligned} \tag{8}$$

In order to compute the exact value of $\tilde{x}_a(f)$ for $f = 3010.3$ Hz, we should make $f_s(\frac{k}{N} - n) = 3010.3$ Hz in (8). Usually, we know that $f_s > B$ where B is the bandwidth of $x_a(t)$, so n must be 0, therefore, in order to compute the exact frequency spectrum value, we only should to compute:

$$f_s \cdot \frac{k}{n} = 3010.3 \text{ Hz}$$

4 Comparison of Codes Running Speed

Assuming N is the total number of the sampling points, I drew a picture of how much time it takes for 3 methods to do the calculations. The x-axis is Log$N$, and the y-axis is Time t. See Fig. 1

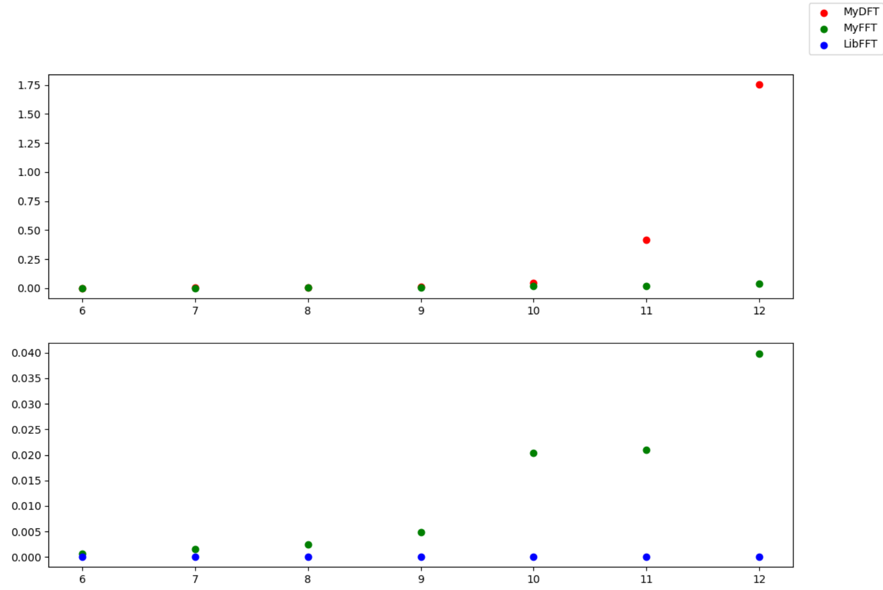

Figure 1: Comparison of the Codes Running Speed

Appendix A Code Listing

```
import numpy as np
import time
from matplotlib import pyplot as plt

def dft(x):
    N = len(x)
    xt = np.matmul(
        np.exp(-1j * 2 * np.pi * np.matmul(np.arange(N).reshape(N, 1), np.arange(N).reshape(1, N)/N)),
        x
        )
    return xt

def idft(xt):
    return np.conj(dft(np.conj(xt))) / N

def fft(x):
```

```python
    N = len(x)
    if N == 1:
        return x
    else:
        x0, x1 = x[::2] , x[1::2]
        xt0 = fft(x0)
        xt1 = fft(x1)
        tmp = np.exp(-1j*2*np.pi*np.arange(N/2)/N) * xt1
        xt = np.concatenate([xt0 + tmp, xt0 - tmp])
    return xt

def ifft(xt):
    return np.conj(fft(np.conj(xt))) / N

def test(m):
    x = np.random.rand(2**m)

    # MyDFT
    st1 = time.time()
    y1 = dft(x)
    et1 = time.time()
    # xx1 = idft(y1)

    # MyFFT
    st2 = time.time()
    y2 = fft(x)
    et2 = time.time()
    # xx2 = idft(y2)

    # LibFFT
    st3 = time.time()
    y3 = np.fft.fft(x)
    et3 = time.time()
    # xx3 = idft(y3)

    # err1 = xx1 - x
    # err2 = xx2 - x
    # err3 = xx3 - x

    t1 = et1 - st1
    t2 = et2 - st2
    t3 = et3 - st3

    return t1, t2, t3


M = [i for i in range(6, 13)]
T1 = []
T2 = []
T3 = []
for m in M:
    tmp1, tmp2, tmp3 = test(m)
    T1.append(tmp1)
    T2.append(tmp2)
    T3.append(tmp3)

fig = plt.figure()
plt.xlabel('LogN')
plt.ylabel('Time')
plt.subplot(211)
plt.scatter(M, T1, label='MyDFT', c='r')
plt.scatter(M, T2, label='MyFFT', c='g')
plt.subplot(212)
plt.scatter(M, T2, c='g')
plt.scatter(M, T3, label='LibFFT', c='b')
fig.legend()
plt.show()
```