DSP Homework 06

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Abstract

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2 My Sampling Method

2.1 Restatement

Use the Fourier series to develop a sampling method and compare it with the Shannon/Nyquist sampling method through examples.

2.2 Improvement

In class, we are considering the energy of the Fourier transform of the $s(t)$, we have

$$s(t) = \sum_{n=-\infty}^{\infty} c_n\, e^{j2\pi nfT} \tag{1}$$

and we have

$$\begin{aligned}
E &= \int_{-\frac{T}{2}}^{\frac{T}{2}} s(t) \cdot s^*(t)\,\mathrm{d}t \\
&= \int_{-\frac{T}{2}}^{\frac{T}{2}} \sum_{n=-\infty}^{\infty} c_n\, e^{-j2\pi nfT} \cdot \sum_{m=-\infty}^{\infty} c_m e^{j2\pi mfT}\,\mathrm{d}t \\
&= \int_{-\frac{T}{2}}^{\frac{T}{2}} \sum_{n=-\infty}^{\infty} c_m \cdot c_n\; e^{j2\pi(n-m)fT}\,\mathrm{d}t \\
&= \sum_{n=-\infty}^{\infty} c_m \cdot c_n \int_{-\frac{T}{2}}^{\frac{T}{2}} e^{j2\pi(n-m)fT}\,\mathrm{d}t
\end{aligned}$$

and the integral of $e^{j2\pi(n-m)fT}$ usually be 0 except that $n-m$, so

$$E = \sum_{n=-\infty}^{\infty} c_n^2 \cdot T$$

I will try to prove that if $E < \infty$, we must ensure that $n \to \infty, c_n \to 0$, which means

$$E < \infty \Leftarrow n \to \infty, c_n \to 0 \tag{2}$$

if that

$$n \to \infty, c_n \to C\,(C \neq 0)$$

we can always find a big number N which makes $c_n^2 > 0\,(n > N)$, and we can easily find that

$$\sum_{n=N}^{\infty} c_n^2 > (\infty - N) \cdot c_{n(min)}^2 \to \infty$$

So, $E \to \infty$ when $n \to \infty, c_n \to C\,(C \neq 0)$. Therefore, Equa.2 is proved. And we know that

$$c_n = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} s(t)\, e^{-j2\pi n f_s t}\, \mathrm{d}t \tag{3}$$

So, as I find out last week in my weekly report, ff we let the square wave last the length of 2τ, we now have

$$\begin{aligned} s(t) &= \sum_{n=-\infty}^{\infty} \mathrm{rect}(\frac{t-nT}{\tau}) \\ &= \sum_{n=-\infty}^{\infty} F_n e^{j2\pi n f_s t} \\ &= \sum_{n=-\infty}^{\infty} \left[\frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \mathrm{rect}(\frac{t}{\tau}) e^{-j2\pi n f_s t} \mathrm{d}t \right] e^{j2\pi n f_s t} \\ &= \sum_{n=-\infty}^{\infty} \left[f_s \int_{-\tau}^{\tau} e^{-j2\pi n f_s t} \mathrm{d}t \right] e^{j2\pi n f_s t} \\ &= \sum_{n=-\infty}^{\infty} 2 f_s\, \mathrm{sinc}\,(2 f_s n \tau)\, e^{j2\pi n f_s t} \end{aligned} \tag{4}$$

Here,

$$\begin{array}{lcl} s(t) & = & \sum_{n=-\infty}^{\infty} \mathrm{rect}(\frac{t-nT}{\tau}) \\ c_n & = & 2 f_s\, \mathrm{sinc}\,(2 f_s n \tau) \end{array}$$

Therefore

$$\begin{aligned} E &= T \cdot \sum_{n=-\infty}^{\infty} c_n^2 \\ &= T \cdot \sum_{n=-\infty}^{\infty} sinc^2(2 f_s n \tau) \end{aligned}$$

In help of Wolfram Alpha [1], I know that when a is real, shown in Fig. 1.

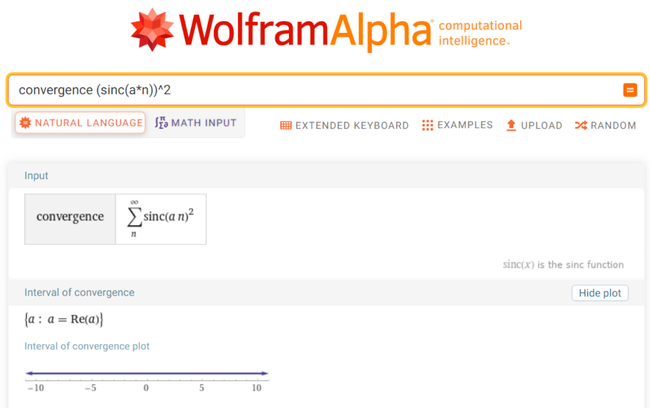

Figure 1: Convergence of $sinc^2(a \times n)$

Therefore

$$\sum_{n=-\infty}^{\infty} sinc^2(2f_s n\tau) < \infty \tag{5}$$

we can now say that the $s(t)$ mentioned in Equa.4 is energy-limited.
If we look back on the $\delta(t)$ sampling, we have the energy E of $s(t)$:

$$E = \sum_{n=-\infty}^{\infty} T \to \infty \tag{6}$$

which is not realizable.

3 Conclusion

References

[1] https://www.wolframalpha.com/.

Appendix A Code Listing

```
import numpy as np
from sympy import symbols, integrate, sinc

x, a = symbols('x, a')

print(integrate((sinc(a*x))**2,(x,-np.inf,np.inf)))
```