DSP Homework 06

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Abstract

1 Summary and Thoughts

2 My Sampling Method

2.1 Restatement

Use the Fourier series to develop a sampling method and compare it with the Shannon/Nyquist sampling method through examples.

2.2 Improvement

In class, we are considering the energy of the Fourier transform of the $s(t)$, we have

$$s(t) = \sum_{n=-\infty}^{\infty} c_n \, e^{j2\pi n fT} \tag{1}$$

and we have

$$E = \int_{-\frac{T}{2}}^{\frac{T}{2}} s(t) \cdot s^*(t) \, \mathrm{d}t$$

$$= \int_{-\frac{T}{2}}^{\frac{T}{2}} \sum_{n=-\infty}^{\infty} c_n \, e^{-j2\pi n fT} \cdot \sum_{m=-\infty}^{\infty} c_m e^{j2\pi m fT} \, \mathrm{d}t$$

$$= \int_{-\frac{T}{2}}^{\frac{T}{2}} \sum_{n=-\infty}^{\infty} c_m \cdot c_n \, e^{j2\pi(n-m)fT} \, \mathrm{d}t$$

$$= \sum_{n=-\infty}^{\infty} c_m \cdot c_n \int_{-\frac{T}{2}}^{\frac{T}{2}} e^{j2\pi(n-m)fT} \, \mathrm{d}t$$

and the integral of $e^{j2\pi(n-m)fT}$ usually be 0 except that $n-m$, so

$$E = \sum_{n=-\infty}^{\infty} c_n^2 \cdot T$$

I will try to prove that if $E < \infty$, we must ensure that $n \to \infty, c_n \to 0$, which means

$$E < \infty \Leftarrow n \to \infty, c_n \to 0 \tag{2}$$

if that

$$n \to \infty, c_n \to C \, (C \neq 0)$$

we can always find a big number N which makes $c_n^2 > 0 \, (n > N)$, and we can easily find that

$$\sum_{n=N}^{\infty} c_n^2 > (\infty - N) \cdot c_{n(min)}^2 \to \infty$$

So, $E \to \infty$ when $n \to \infty, c_n \to C \, (C \neq 0)$ Therefore, Equa.2 is proved. And we know that

$$c_n = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} s(t) \, e^{-j2\pi n f_s t} \, \mathrm{d}t \tag{3}$$

So, as I find out last week in my weekly report, ff we let the square wave last the length of 2τ, we now have

$$
\begin{aligned}
s(t) &= \sum_{n=-\infty}^{\infty} \mathrm{rect}(\frac{t - nT}{\tau}) \\
&= \sum_{n=-\infty}^{\infty} F_n e^{j2\pi n f_s t} \\
&= \sum_{n=-\infty}^{\infty} \left[\frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \mathrm{rect}(\frac{t}{\tau}) e^{-j2\pi n f_s t} \mathrm{d}t \right] e^{j2\pi n f_s t} \\
&= \sum_{n=-\infty}^{\infty} \left[f_s \int_{-\tau}^{\tau} e^{-j2\pi n f_s t} \mathrm{d}t \right] e^{j2\pi n f_s t} \\
&= \sum_{n=-\infty}^{\infty} 2 f_s \, \mathrm{sinc}\,(2 f_s n \tau) \, e^{j2\pi n f_s t}
\end{aligned}
\tag{4}
$$

Here,

$$
\begin{aligned}
s(t) &= \sum_{n=-\infty}^{\infty} \mathrm{rect}(\frac{t - nT}{\tau}) \\
c_n &= 2 f_s \, \mathrm{sinc}\,(2 f_s n \tau) < \infty
\end{aligned}
$$

Therefore

$$E = T \cdot \sum_{n=-\infty}^{\infty} c_n^2$$

$$= T \cdot \sum_{n=-\infty}^{\infty} sinc^2 (2 f_s n \tau)$$

In help of code, I know that

$$\int_{-\infty}^{\infty} sinc^2 (2 f_s n \tau) = \frac{\pi}{2 f_s \tau} < \infty$$

Therefore

$$\sum_{n=-\infty}^{\infty} sinc^2 (2 f_s n \tau) < \infty$$

we can now say that the $s(t)$ mentioned in Equa.4 is energy-limited.

If we look back on the $\delta(t)$ sampling, we have the energy E of $s(t)$:

$$E = \sum_{n=-\infty}^{\infty} T \to \infty \tag{5}$$

which is not realizable.

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3 Conclusion

Appendix A   Code Listing

```python
import numpy as np
from sympy import symbols, integrate, sinc

x, a = symbols('x, a')

print(integrate((sinc(a*x))**2,(x,-np.inf,np.inf)))
```