Elements of Set Theory

NOT REQUIRED UNTIL LATER IN THE COURSE

The later parts of this course assume you (the student) are familiar with basic set theory. This brief document summarizes what is required. Although you won't need this material until you are well into the course (when you will have been introduced to the various notions and terms used in this summary), it's probably a good idea to look through the supplement at the start of the course to get a general overview, and then review it as and when you need it. (In particular completing many of the exercises in this summary assumes material to be covered in the first part of the course, so you may not be able to do them. That should not be a problem. By the time you need knowledge of set theory, in the final week of lectures, you should be able to complete all of the exercises.)

The concept of a *set* is extremely basic and pervades the whole of present-day mathematical thought. Any well-defined collection of objects is a set. For instance we have:

- the set of all students in your class
- the set of all prime numbers
- the set whose only member is you.

All it takes to determine a set is some way of specifying the collection. (Actually, that is not correct. In the mathematical discipline called abstract set theory, arbitrary collections are allowed, where there is no defining property.)

If A is a set, then the objects in the collection A are called either the *members* of A or the *elements* of A. We write

$$x \in A$$

to denote that x is an element of A.

Some sets occur frequently in mathematics, and it is convenient to adopt a standard notation for them:

$\mathcal{N}$: the set of all natural numbers (i.e., the numbers 1, 2, 3, etc.)[1]

$\mathcal{Z}$: the set of all integers (0 and all positive and negative whole numbers)

$\mathcal{Q}$: the set of all rational numbers (fractions)

$\mathcal{R}$: the set of all real numbers

Thus, for example,

$$x \in \mathcal{R}$$

Seluruh x adalah bagian dari bilangan real

means that x is a real number. And

$$(x \in \mathcal{Q}) \wedge (x > 0)$$

And

means that x is a positive rational number.

There are several ways of specifying a set. If it has a small number of elements we can list them. In this case we denote the set by enclosing the list of the elements in curly brackets; thus, for example,

$$\{1, 2, 3, 4, 5\}$$

[1]Note that in this course, 0 is not regarded as a natural number. This issue can cause confusion, as some authors include 0 among the natural numbers.

denotes the set consisting of the natural numbers 1, 2, 3, 4 and 5.

By use of 'dots' we can extend this notation to any finite set; e.g.

$$\{1, 2, 3, \ldots, n\}$$

Only Use 3 dots,remember that

denotes the set of the first n natural numbers. Again

$$\{2, 3, 5, 7, 11, 13, 17, \ldots, 53\}$$

could (given the right context) be used to denote the set of all primes up to 53.

Certain infinite sets can also be described by the use of dots (only now the dots have no end), e.g.

$$\{2, 4, 6, 8, \ldots, 2n, \ldots\}$$

denotes the set of all even natural numbers. Again,

$$\{\ldots, -8, -6, -4, -2, 0, 2, 4, 6, 8, \ldots\}$$

denotes the set of all even integers.

In general, however, except for finite sets with only a small number of elements, sets are best described by giving the property which defines the set. If $A(x)$ is some property, the set of all those x which satisfy $A(x)$ is denoted by

$$\{x \mid A(x)\}$$

Or, if we wish to restrict the x to those which are members of a certain set X, we would write

$$\{x \in X \mid A(x)\}$$

adalah himpunan bagian dari bilangan Integer, sedemikian sehingga himpunannya memenuhi x > 0

This is read "the set of all x in X such that $A(x)$". For example:

$$\begin{aligned}
\mathcal{N} &= \{x \in \mathcal{Z} \mid x > 0\} \\
\mathcal{Q} &= \{x \in \mathcal{R} \mid (\exists m, n \in \mathcal{Z})[(m > 0) \wedge (mx = n)]\} \\
\{\sqrt{2}, -\sqrt{2}\} &= \{x \in \mathcal{R} \mid x^2 = 2\} \\
\{1, 2, 3\} &= \{x \in \mathcal{N} \mid x < 4\}
\end{aligned}$$

Two sets, A, B are *equal*, written $A = B$, if they have exactly the same elements. As the above example shows, equality of sets does not mean they have identical definitions; there are often many different ways of describing the same set. The definition of equality reflects rather the fact that a set is just a collection of objects.

If we have to prove that the sets A and B are equal, we usually split the proof into two parts:

(a) Show that every member of A is a member of B.

(b) Show that every member of B is a member of A.

A = {3,2,1} , B = {1,2,3} So iyap A = B ,
Berlaku juga di piton kok

Taken together, (a) and (b) clearly imply $A = B$. (The proof of both (a) and (b) is usually of the 'take an arbitrary element' variety. To prove (a), for instance, we must prove $(\forall x \in A)(x \in B)$; so we take an arbitrary element x of A and show that x must be an element of B.)

Untuk seluruh x himpunan bagian dari A

The set notations introduced have obvious extensions. For instance we can write

$$\mathcal{Q} = \{m/n \mid m, n \in \mathcal{Z}, n \neq 0\}$$

and so on.

It is convenient in mathematics to introduce a set which has no elements: the *empty set* (or *null set*). There will only be one such set, of course, since any two such will have exactly the same elements and thus be (by definition) equal. The empty set is denoted by the Scandinavian letter

$$\emptyset$$

Untuk seluruh m/n sedemikian sehingga m, n elemen dari bilangan integer,dimana n tidak boleh 0

Natural != 0

[Note that this is not the Greek letter ϕ.] The empty set can be specified in many ways; e.g.

$$\begin{aligned}
\emptyset &= \{x \in \mathcal{R} \mid x^2 < 0\} \\
\emptyset &= \{x \in \mathcal{N} \mid 1 < x < 2\} \\
\emptyset &= \{x \mid x \neq x\}
\end{aligned}$$

Empty Set / Null

LOL

Notice that $\emptyset$ and $\{\emptyset\}$ are quite different sets. $\emptyset$ is the empty set: it has NO members. $\{\emptyset\}$ is a set which has ONE member. Hence

$$\emptyset \neq \{\emptyset\}$$

What is the case here is that

$$\emptyset \in \{\emptyset\}$$

(The fact that the single element of $\{\emptyset\}$ is the empty set is irrelevant in this connection: $\{\emptyset\}$ *does* have an element, $\emptyset$ does *not*.)

A set A is called a *subset* of a set B if every element of A is a member of B. For example, $\{1, 2\}$ is a subset of $\{1, 2, 3\}$. We write

$$A \subseteq B$$

$\subseteq$

to mean that A is a subset of B. If we wish to emphasize that A and B are unequal here, we write

$$A \subset B$$

and say that A is a *proper subset* of B (This usage compares with the ordering relations $\leq$ and $<$ on $\mathcal{R}$.)

Clearly, for any sets A, B, we have

$$A = B \text{ iff } (A \subseteq B) \wedge (B \subseteq A)$$

$A = \{1,2,4\}$
$B = \{4,2,1\}$
$= A \subseteq B \wedge B \subseteq A$

Exercises 1

1. What well-known set is this:

$$\{n \in \mathcal{N} \mid (n > 1) \wedge (\forall x, y \in \mathcal{N})[(xy = n) \Rightarrow (x = 1 \vee y = 1)]\}$$

2. Let

$$P = \{x \in \mathcal{R} \mid \sin(x) = 0\} \quad , \quad Q = \{n\pi \mid n \in \mathcal{Z}\}$$

What is the relationship between P and Q ?

3. Let

$$A = \{x \in \mathcal{R} \mid (x > 0) \wedge (x^2 = 3)\}$$

Give a simpler definition of the set A. $A = \{0.0 \ldots 1.73\}$

4. Prove that for any set A:

$$\emptyset \subseteq A \quad \text{and} \quad A \subseteq A$$

if $A = \{1,2,3\}$ then: A

5. Prove that if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$

6. List all subsets of the set $\{1, 2, 3, 4\}$.

7. List all subsets of the set $\{1, 2, 3, \{1, 2\}\}$.

8. Let $A = \{x \mid P(x)\}, B = \{x \mid Q(x)\}$, where P, Q are formulas such that $\forall x[P(x) \Rightarrow Q(x)]$. Prove that $A \subseteq B$.

9. Prove (by induction) that a set with exactly n elements has 2^n subsets.

$A = \{1,2,3\} = n = 3 \;//\; 2^n = 8$

10. Let

$$A = \{o, t, f, s, e, n\}$$

Give an alternative definition of the set A. (Hint: this is connected with $\mathcal{N}$ but is not entirely mathematical.)

There are various natural operations we can perform on sets. (They correspond *roughly* to addition, multiplication, and negation for integers.)

Given two sets A, B we can form the set of all objects which are members of either one of A and B. This set is called the *union* of A and B and is denoted by

$$A \cup B$$

Formally, this set has the definition

$$A \cup B = \{x \mid (x \in A) \vee (x \in B)\}$$

(Note how this is consistent with our decision to use the word 'or' to mean inclusive-or.)

The *intersection* of the sets A, B is the set of all members which A and B have in common. It is denoted by

$$A \cap B$$

and has the formal definition

$$A \cap B = \{x \mid (x \in A) \wedge (x \in B)\}$$

Two sets A, B are said to be *disjoint* if they have no elements in common: that is, if $A \cap B = \emptyset$.

The set-theoretic analog of negation requires the concept of a *universal set*. Often, when we are dealing with sets, they all consist of objects of the same kind. For example, in number theory we may focus on sets of natural numbers or sets of rationals; in real analysis we usually focus on sets of reals. A *universal set* for a particular discussion is simply the set of all objects of the kind being considered. It is frequently the domain over which the quantifiers range.

Once we have fixed a universal set we can introduce the notion of the *complement* of the set A. Relative to the universal set U, the complement of a set A is the set of all elements of U that are not in A. This set is denoted by A', and has the formal definition

$$A' = \{x \in U \mid x \notin A\}$$

[Notice that we write $x \notin A$ instead of $\neg(x \in A)$, for brevity.]

For instance, if the universal set is the set $\mathcal{N}$ of natural numbers, and E is the set of even (natural) numbers, then E' is the set of odd (natural) numbers.

The following theorem sums up the basic facts about the three set operations just discussed.

Theorem Let A, B, C be subsets of a universal set U.

(1) $A \cup (B \cup C) = (A \cup B) \cup C$

(2) $A \cap (B \cap C) = (A \cap B) \cap C$
((1) and (2) are the associative laws)

(3) $A \cup B = B \cup A$

(4) $A \cap B = B \cap A$
((3) and (4) are the commutative laws)

(5) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(6) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
((5) and (6) are the distributive laws)

(7) $(A \cup B)' = A' \cap B'$

(8) $(A \cap B)' = A' \cup B'$
((7) and (8) are called the De Morgan laws)

(9) $A \cup A' = U$

(10) $A \cap A' = \emptyset$
((9) and (10) are the complementation laws)

(11) $(A')' = A$
(self-inverse law)

Proof: Left as an exercise.

Exercises 2

1. Prove all parts of the above theorem.
2. Find a resource that explains *Venn diagrams* and use them to illustrate and help you understand the above theorem.

NOTE: This supplement is abridged from the course textbook, *Introduction to Mathematical Thinking*, by me (Keith Devlin), available from Amazon as a low-cost, print-on-demand book. You don't need to purchase the book to complete the course, but I know many students like to have a complete textbook. In developing this course, I first wrote the textbook, and then used it to construct all the course materials.