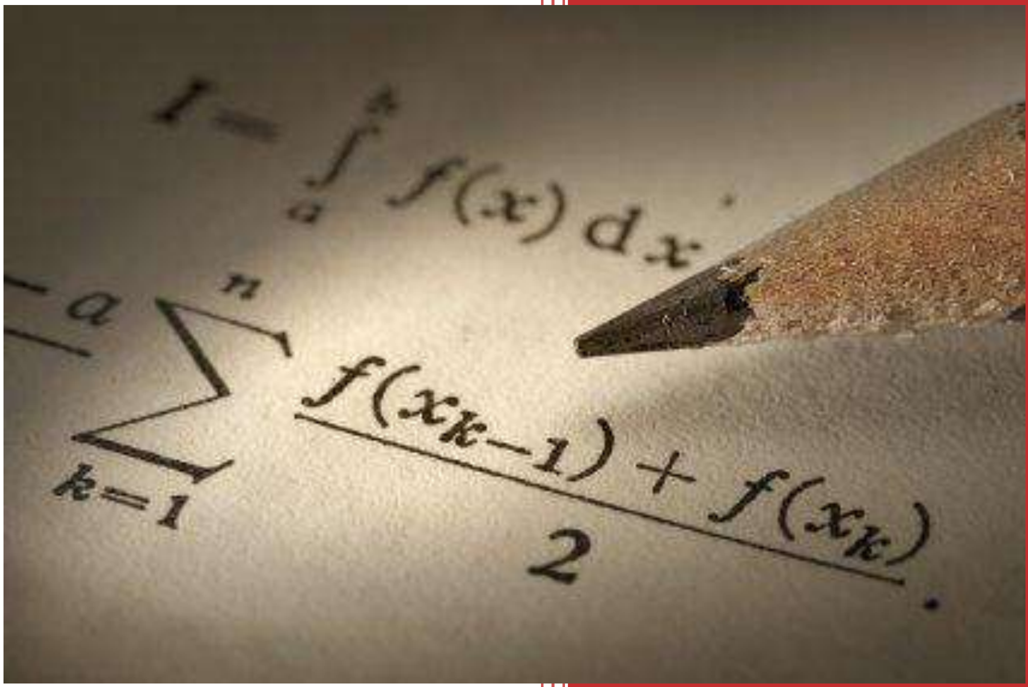
2010

CÁLCULO INTEGRAL SOLUCIÓN DE PROBLEMAS PROPUESTOS EN GUÍAS Y PROBLEMASESPECIALES



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PROBLEMAS RESUELTOS DE INTEGRALES INMEDIATAS .
Verificación por derivación

1. $\int 3x^4 dx$

Solución:

$$\int 3x^4 dx = \frac{3}{4+1}x^{4+1} + C \Leftrightarrow \int 3x^4 dx = \frac{3}{5}x^5 + C.$$

Verificación:

$$\frac{d}{dx}\left(\frac{3}{5}x^5 + C\right) = 5\times\frac{3}{5}x^{5-1} + 0 \Leftrightarrow \frac{d}{dx}\left(\frac{3}{5}x^5 + C\right) = 3x^4$$

2. $\int 2x^7 dx$

Solución:

$$\int 2x^7 dx = \frac{2}{7+1}x^{7+1} + C \Leftrightarrow \int 2x^7 dx = \frac{1}{4}x^8 + C.$$

Verificación:

$$\frac{d}{dx}\left(\frac{1}{4}x^8 + C\right) = 8\times\frac{1}{4}x^{8-1} + 0 \Leftrightarrow \frac{d}{dx}\left(\frac{1}{4}x^8 + C\right) = 2x^7$$

3. $\int \frac{1}{x^3} dx$

Solución:

$$\int \frac{1}{x^3} dx \Leftrightarrow \int x^{-3} dx \Leftrightarrow \frac{1}{-3+1}x^{-3+1} + C;$$

$$\therefore \quad \int \frac{1}{x^3} dx = -\frac{1}{2}x^{-2} + C = -\frac{1}{2x^2} + C.$$

Verificación:

$$\frac{d}{dx}\left(-\frac{1}{2}x^{-2} + C\right) = -2\left(-\frac{1}{2}\right)x^{-2-1} + 0 \Leftrightarrow \frac{d}{dx}\left(-\frac{1}{2}x^{-2} + C\right) = x^{-3}$$

4. $\int \frac{3}{t^5} dt$

Solución:

$$\int \frac{3}{t^5} dt \Leftrightarrow \int 3t^{-5} dt \Leftrightarrow \frac{3}{-5+1}t^{-5+1} + C;$$

$$\therefore \quad \int \frac{3}{t^5} dt = -\frac{3}{4}t^{-4} + C.$$

Verificación:

$$\frac{d}{dt}\left(-\frac{3}{4}t^{-4} + C\right) = -4\left(-\frac{3}{4}\right)t^{-4-1} + 0 \Leftrightarrow \frac{d}{dt}\left(-\frac{3}{4}t^{-4} + C\right) = 3t^{-5}$$

5. $\int 5u^{3/2}du$

Verificación:

$$D_u\left(2u^{5/2}+C\right)=\frac{5}{2}(2)u^{5/2-2/2}+0 \Leftrightarrow D_u\left(2u^{5/2}+C\right)=5u^{3/2}$$

Solución:

$$\int 5u^{3/2}du=\frac{5}{3/2+1}u^{3/2+1}+C,$$

$$\Rightarrow \quad \int 5u^{3/2}du=\frac{5}{5/2}u^{5/2}+C;$$

$$\therefore \quad \int 5u^{3/2}du=2u^{5/2}+C.$$

6. $\int 10\sqrt[3]{x^2}\,dx$

Verificación:

$$D_x\left(6x^{5/3}+C\right)=\frac{5}{3}(6)x^{5/3-3/3}+0 \Leftrightarrow D_x\left(3x^{5/3}+C\right)=10x^{2/3}$$

Solución:

$$\int 10\sqrt[3]{x^2}\,dx \Leftrightarrow \int 10x^{2/3}dx,$$

$$\Rightarrow \quad \int 10x^{2/3}dx=\frac{10}{2/3+1}x^{2/3+1}+C,$$

$$\Rightarrow \quad \int 10x^{2/3}dx=\frac{10}{5/3}x^{5/3}+C;$$

$$\therefore \quad \int 10x^{2/3}dx=6x^{5/3}+C.$$

7. $\int \frac{2}{\sqrt[3]{x}}dx$

Verificación:

$$D_x\left(3x^{2/3}+C\right)=\frac{2}{3}(3)x^{2/3-3/3}+0 \Leftrightarrow D_x\left(3x^{2/3}+C\right)=2x^{-1/3}$$

Solución:

$$\int \frac{2}{\sqrt[3]{x}}dx \Leftrightarrow \int \frac{2}{x^{1/3}}dx \Leftrightarrow \int 2x^{-1/3}dx,$$

$$\Rightarrow \quad \int 2x^{-1/3}dx=\frac{2}{-1/3+3/3}x^{-1/3+3/3}+C,$$

$$\Rightarrow \quad \int 2x^{-1/3}dx=\frac{2}{2/3}x^{2/3}+C;$$

$$\therefore \quad \int \frac{2}{\sqrt[3]{x}}dx=3x^{2/3}+C.$$

8. $\int \frac{3}{\sqrt{y}} dy$

Verificación:

$D_y\left(6y^{1/2}+C\right)=\frac{1}{2}(6)y^{1/2-2/2}+0 \Leftrightarrow D_y\left(6y^{1/2}+C\right)=3y^{-1/2}$

Solución:

$$\int \frac{3}{\sqrt{y}} dy \Leftrightarrow \int \frac{3}{y^{1/2}} dy \Leftrightarrow \int 3y^{-1/2} dy,$$

$$\Rightarrow \quad \int 3y^{-1/2} dy = \frac{3}{-1/2+2/2} y^{-1/2+2/2} + C,$$

$$\Rightarrow \quad \int 3y^{-1/2} dy = \frac{3}{1/2} y^{1/2} + C;$$

$$\therefore \quad \int \frac{3}{\sqrt{y}} dy = 6y^{1/2} + C.$$

9. $\int 6t^2 \sqrt[3]{t}\, dt$

Solución:

$$\int 6t^2 \sqrt[3]{t}\, dt \Leftrightarrow \int 6t^2 t^{1/3} dt \Leftrightarrow \int 6t^{6/3+1/3} dt \Leftrightarrow \int 6t^{7/3} dt,$$

$$\Rightarrow \quad \int 6t^{7/3} dt = \frac{6}{7/3+3/3} t^{7/3+3/3} + C \Leftrightarrow \int 6t^{7/3} dt = \frac{6}{10/3} t^{10/3} + C;$$

$$\therefore \quad \int 6t^2 \sqrt[3]{t}\, dt = \frac{9}{5} t^{10/3} + C.$$

10. $\int 7x^3 \sqrt{x}\, dx$

Solución:

$$\int 7x^3 \sqrt{x}\, dx \Leftrightarrow \int 7x^3 x^{1/2} dx \Leftrightarrow \int 7x^{6/2+1/2} dx \Leftrightarrow \int 7x^{7/2} dx,$$

$$\Rightarrow \quad \int 7x^{7/2} dx = \frac{7}{7/2+2/2} x^{7/2+2/2} + C \Leftrightarrow \int 7x^{7/2} dx = \frac{7}{9/2} x^{9/2} + C;$$

$$\therefore \quad \int 7x^3 \sqrt{x}\, dx = \frac{14}{9} x^{9/2} + C.$$

11. $\int \left(4x^3 + x^2\right) dx$

Solución:

$$\int \left(4x^3 + x^2\right) dx = \int 4x^3 dx + \int x^2 dx = \frac{4}{3+1} x^{3+1} + \frac{1}{3} x^{2+1} + C = \frac{4}{4} x^4 + \frac{1}{3} x^3 + C = x^4 + \frac{1}{3} x^3 + C.$$

12. $\int \left(3u^5 - 2u^3\right) du$

Solución:

$$\int \left(3u^5 - 2u^3\right) du = \int 3u^5 du - \int 2u^3 du = \frac{3}{5+1} u^{5+1} - \frac{2}{3+1} u^{3+1} + C = \frac{3}{6} u^6 - \frac{2}{4} u^4 + C = \frac{1}{2} u^6 - \frac{1}{2} u^4 + C.$$

13. $\int y^3 (2y^2 - 3) dy$

Solución:

$$\int y^3 (2y^2 - 3) dy \Leftrightarrow \int (2y^5 - 3y^3) dy,$$

$$\Rightarrow \quad \int (2y^5 - 3y^3) dy = \int 2y^5 dy - \int 3y^3 dy = \frac{2}{5+1} y^{5+1} - \frac{3}{3+1} y^{3+1} + C = \frac{2}{6} y^6 - \frac{3}{4} y^4 + C;$$

$$\therefore \quad \int y^3 (2y^2 - 3) dy = \frac{1}{3} y^6 - \frac{3}{4} y^4 + C.$$

14. $\int x^4 (5 - x^2) dx$

Solución:

$$\int x^4 (5 - x^2) dx \Leftrightarrow \int (5x^4 - x^6) dx,$$

$$\Rightarrow \quad \int (5x^4 - x^6) dx = \int 5x^4 dx - \int x^6 dx = \frac{5}{4+1} x^{4+1} - \frac{1}{6+1} x^{6+1} + C = \frac{5}{5} x^5 - \frac{1}{7} x^7 + C;$$

$$\therefore \quad \int x^4 (5 - x^2) dx = x^5 - \frac{1}{7} x^7 + C.$$

15. $\int (3 - 2t + t^2) dt$

Solución:

$$\int (3 - 2t + t^2) dt = \int 3dt - \int 2t dt + \int t^2 dt,$$

$$\Rightarrow \quad \int (3 - 2t + t^2) dt = 3t - \frac{2}{1+1} t^{1+1} + \frac{1}{2+1} t^{2+1} + C,$$

$$\Rightarrow \quad \int (3 - 2t + t^2) dt = 3t - \frac{2}{2} t^2 + \frac{1}{3} t^3 + C;$$

$$\therefore \quad \int (3 - 2t + t^2) dt = 3t - t^2 + \frac{1}{3} t^3 + C.$$

Verificación:

$$\frac{d}{dt}\left(3t - t^2 + \frac{1}{3} t^3\right) = 3 - 2t + \frac{3}{3} t^3 = 3 - 2t + t^3$$

16. $\int (4x^3 - 3x^2 + 6x - 1)dx$

Solución:

$$\int (4x^3 - 3x^2 + 6x - 1)dx = \int 4x^3 dx - \int 3x^2 dx + \int 6x dx - \int dx,$$

$$\Rightarrow \quad \int (4x^3 - 3x^2 + 6x - 1)dx = \frac{4}{3+1}x^{3+1} - \frac{3}{2+1}x^{2+1} + \frac{6}{1+1}x^{1+1} - x + C,$$

$$\Rightarrow \quad \int (4x^3 - 3x^2 + 6x - 1)dx = \frac{4}{4}x^4 - \frac{3}{3}x^3 + \frac{6}{2}x^2 - x + C;$$

$$\therefore \quad \int (4x^3 - 3x^2 + 6x - 1)dx = x^4 - x^3 + 3x^2 - x + C.$$

Verificación:

$$\frac{d}{dt}\left(x^4 - x^3 + 3x^2 - x + C\right) =$$
$$4x^3 - 3x^2 + 2(3)x - 1 + 0 =$$
$$4x^3 - 3x^2 + 6x - 1$$

17. $\int (8x^4 + 4x^3 - 6x^2 - 4x + 5)dx$

Solución:

$$\int (8x^4 + 4x^3 - 6x^2 - 4x + 5)dx = \int 8x^4 dx + \int 4x^3 dx - \int 6x^2 dx - \int 4x dx + \int 5 dx,$$

$$\Rightarrow \quad \int (8x^4 + 4x^3 - 6x^2 - 4x + 5)dx = \frac{8}{4+1}x^{4+1} + \frac{4}{3+1}x^{3+1} - \frac{6}{2+1}x^{2+1} - \frac{4}{1+1}x^{1+1} + 5x + C,$$

$$\Rightarrow \quad \int (8x^4 + 4x^3 - 6x^2 - 4x + 5)dx = \frac{8}{5}x^5 + \frac{4}{4}x^4 - \frac{6}{3}x^3 - \frac{4}{2}x^2 + 5x + C;$$

$$\therefore \quad \int (8x^4 + 4x^3 - 6x^2 - 4x + 5)dx = \frac{8}{5}x^5 + x^4 - 2x^3 - 2x^2 + 5x + C.$$

Verificación:

$$\frac{d}{dt}\left(\frac{8}{5}x^5 + x^4 - 2x^3 - 2x^2 + 5x + C\right) = 5\times\frac{8}{5}x^4 + 4x^3 - 3(2)x^2 - 2(2)x + 5 + 0 = 8x^4 + 4x^3 - 6x^2 - 4x + 5$$

18. $\int (2 + 3x^2 - 8x^3)dx$

Solución:

$$\int (2 + 3x^2 - 8x^3)dx = \int 2dx + \int 3x^2 dx - \int 8x^3 dx,$$

$$\Rightarrow \quad \int (8x^4 + 4x^3 - 6x^2 - 4x + 5)dx = 2x + \frac{3}{2+1}x^{2+1} - \frac{8}{3+1}x^{3+1} + C,$$

$$\Rightarrow \quad \int (8x^4 + 4x^3 - 6x^2 - 4x + 5)dx = 2x + \frac{3}{3}x^3 - \frac{8}{4}x^4 + C;$$

$$\therefore \quad \int (8x^4 + 4x^3 - 6x^2 - 4x + 5)dx = 2x + x^3 - 2x^4 + C.$$

Verificación:

$$\frac{d}{dt}\left(2x + x^3 - 2x^4 + C\right) = 2 + 3x^2 - 8x^3$$

19. $\int \sqrt{x}(x+1)dx$

Solución:

$$\int \sqrt{x}(x+1)dx \Leftrightarrow \int x^{1/2}(x+1)dx \Leftrightarrow \int \left(x^{3/2}+x^{1/2}\right)dx,$$

$$\int \left(x^{3/2}+x^{1/2}\right)dx = \int x^{3/2}dx + \int x^{1/2}dx = \frac{1}{3/2+2/2}x^{3/2+2/2} + \frac{1}{1/2+2/2}x^{1/2+2/2} + C,$$

$$\Rightarrow \quad \int \left(x^{3/2}+x^{1/2}\right)dx = \frac{1}{5/2}x^{5/2} + \frac{1}{3/2}x^{3/2} + C;$$

$$\therefore \quad \int \sqrt{x}(x+1)dx = \frac{2}{5}x^{5/2} + \frac{2}{3}x^{3/2} + C.$$

20. $\int (ax^2+bx+c)dx$

Solución:

$$\int (ax^2+bx+c)dx = \int ax^2dx + \int bxdx + \int cdx = \frac{a}{2+1}x^{2+1} + \frac{b}{1+1}x^{1+1} + cx + C = \frac{a}{3}x^3 + \frac{b}{2}x^2 + cx + C.$$

21. $\int (x^{3/2}-x)dx$

Solución:

$$\int (x^{3/2}-x)dx = \int x^{3/2}dx - \int xdx = \frac{1}{3/2+2/2}x^{3/2+2/2} - \frac{1}{1+1}x^{1+1} + C = \frac{1}{5/2}x^{5/2} - \frac{1}{2}x^2 + C;$$

$$\therefore \quad \int (x^{3/2}-x)dx = \frac{2}{5}x^{5/2} - \frac{1}{2}x^2 + C.$$

22. $\int \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)dx$

Solución:

$$\int \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)dx \Leftrightarrow \int \left(x^{1/2} - \frac{1}{x^{1/2}}\right)dx \Leftrightarrow \int \left(x^{1/2} - x^{-1/2}\right)dx,$$

$$\Rightarrow \quad \int \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)dx = \int x^{1/2}dx - \int x^{-1/2}dx = \frac{1}{1/2+2/2}x^{1/2+2/2} - \frac{1}{-1/2+2/2}x^{-1/2+2/2} + C;$$

$$\therefore \quad \int \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)dx = \frac{1}{3/2}x^{3/2} - \frac{1}{1/2}x^{1/2} + C = \frac{2}{3}x^{3/2} - 2x^{1/2} + C.$$

ACTIVIDAD I. PROBLEMAS PROPUESTOS EN LA GUÍA II

INTEGRALES QUE SE RESUELVEN EMPLEANDO IDENTIDADES TRIGONOMÉTRICAS FUNDAMENTALES PARA ***INTEGRAR POTENCIAS DE FUNCIONES TRIGONOMÉTRICAS Y PRODUCTOS DE POTENCIAS TRIGONOMÉTRICAS.***

La siguiente tabla de identidades trigonométricas es fundamental para realizar todas

1) $\int sen^4 dx =$	**6)** $\int \tan^3 xdx =$	**11)** $\int sen^2 x\cos^3 x\, dx =$	**16)** $\int \sqrt{tg^3 4x}\sec^4 4xdx$
2) $\int sen^5 dx =$	**7)** $\int \tan^4 3xdx =$	**12)** $\int sen^3 x\cos^4 x\, dx$	**17)** $\int sen^3 x\cos^2 xdx =$
3) $\int \cos^4 3xdx =$	**8)** $\int ctg^2 xdx =$	**13)** $\int sen^5 2x\cos^3 2xdx =$	**18)** $\int \tan^3 x\sec^4 xdx =$
4) $\int \cos^5 2xdx =$	**9)** $\int ctg^3 xdx =$	**14)** $\int \tan^3 x\sec^5 xdx =$	**19)** $\int \tan^5 x\sec^3 xdx =$
5) $\int \tan^2 xdx =$	**10)** $\int ctg^4 x\, dx =$	**15)** $\int \tan^3 x\sec^6 xdx =$	**20)** $\int sen^3 x\cos^3 xdx =$

las transformaciones necesarias para simplificar las expresiones trigonométricas contenidas en las integrales.

Identidades trigonométricas			
$\operatorname{sen}^2 u = 1-\cos^2 u$	$\cos^2 u = 1-\operatorname{sen}^2 u$	$\operatorname{sen}^2 u = \frac{1-\cos 2u}{2}$	$\cos^2 u = \frac{1+\cos 2u}{2}$
$\sec^2 u = 1+\tan^2 u$	$\csc^2 u = 1+\cot^2 u$	$\operatorname{sen} 2u = 2\operatorname{sen} u\cos u$	
$\operatorname{sen} mx\cos nx = \frac{1}{2}\operatorname{sen}(m-n)x+\frac{1}{2}\operatorname{sen}(m+n)x$		$\cos mx\cos nx = \frac{1}{2}\cos(m-n)x+\frac{1}{2}\cos(m+n)x$	

Problema 1

$$\int sen^4 xdx = \int \left(sen^2 x\right)^2 dx = \int \left[\frac{1}{2}(1-\cos 2x)\right]^2 dx$$

$$= \int \frac{1}{4}\left(1-2\cos 2x+\cos^2 2x\right)dx = \frac{1}{4}\int dx - \frac{1}{2}\underbrace{\int \cos 2xdx}_{\substack{u=2x\\ du=2dx\\ \frac{du}{2}=du}} + \frac{1}{4}\underbrace{\int \cos^2 2xdx}_{\substack{v=2x\\ dv=2dx\\ \frac{dv}{2}=dx}}$$

$$=\frac{1}{4}x-\frac{1}{2}\int\cos u\frac{du}{2}+\frac{1}{4}\int\cos^2 v\frac{dv}{2}$$

$$=\frac{1}{4}x-\frac{1}{4}\int\cos udu+\frac{1}{8}\int\cos^2 vdv=\frac{1}{4}x-\frac{1}{4}senu+\frac{1}{8}\int\frac{1}{2}(1+\cos 2v)dv$$

$$=\frac{1}{4}x-\frac{1}{4}sen2x+\frac{1}{16}\int(1+\cos 2v)dv=\frac{1}{4}x-\frac{1}{4}sen2x+\frac{1}{16}\int dv+\frac{1}{16}\underbrace{\int\cos 2vdv}_{\begin{array}{l}w=2v\\ dw=2dv\\ \frac{dw}{2}=dv\end{array}}$$

$$=\frac{1}{4}x-\frac{1}{4}sen2x+\frac{1}{16}v+\frac{1}{16}\int\cos w\frac{dw}{2}=\frac{1}{4}x-\frac{1}{4}sen2x+\frac{1}{16}\bullet 2x+\frac{1}{32}senw$$

$$=\frac{1}{4}x-\frac{1}{4}sen\,2x+\frac{1}{8}x+\frac{1}{32}sen\,4x$$

$$=\frac{3}{8}x-\frac{1}{4}sen\,2x+\frac{1}{32}sen\,4x+c$$

Problema 2

$$\int sen^5 xdx=\int senxsen^4 xdx=\int senx(sen^2 x)^2 dx$$

$$=\int(1-\cos^2 x)^2 senxdx=\int(1-2\cos^2 x+\cos^4 x)senxdx$$

$$=\int(senx-2\cos^2 xsenx+\cos^4 xsenx)dx$$

$$=\int senxdx-2\underbrace{\int\cos^2 xsenxdx}_{\begin{array}{l}u=\cos x\\ du=-senxdx\\ -du=senxdx\end{array}}+\underbrace{\int\cos^4 xsenxdx}_{\begin{array}{l}v=\cos x\\ dv=-senxdx\\ -dv=senxdx\end{array}}$$

$$=-\cos x-2\int u^2(-du)+\int v^4(-du)=-\cos x+2\int u^2du-\int v^4dv==-\cos x+\frac{2u^3}{3}-\frac{v^5}{5}$$

$$=-\cos x+\frac{2}{3}\cos^3 x-\frac{\cos^5 x}{5}+c$$

Problema 3

$$\int cos^4 3x\,dx = \int (cos^2 3x)^2 dx = \int \left(\frac{1+\cos 6x}{2}\right)^2 dx = \frac{1}{4}\int (1+\cos 6x)^2 dx =$$

$$\frac{1}{4}\int (1+2\cos 6x + cos^2 6x)\,dx = \frac{1}{4}\int dx + \frac{1}{2}\int \cos 6x\,dx + \frac{1}{4}\int cos^2 6x dx =$$

$$\frac{1}{4}x + \frac{1}{12}\int \cos 6x\,dx + \frac{1}{24}\int cos^2 6x\,dx = \frac{1}{4}x + \frac{1}{12}sen\ 6x + \frac{1}{24}\int \left(\frac{1+\cos 12x}{2}\right)dx =$$

$$\frac{1}{4}x + \frac{1}{12}sen\ 6x + \frac{1}{48}\int dx + \frac{1}{48}\int \cos 12x\,dx = \frac{1}{4}x + \frac{1}{12}sen\ 6x + \frac{1}{48}x + \frac{1}{576}\int \cos 12x\,dx =$$

$$\frac{13}{48}x + \frac{1}{12}sen\ 6x + \frac{1}{576}sen\ 12x + c$$

Problema 4

$$\int cos^5 2x\,dx = \int \cos 2x (cos^2 2x)^2\,dx = \int \cos 2x\,(1-sen^2 2x)^2 = \int \cos 2x\ \ (1-2sen^2 2x + sen^4 2x)dx = \int \cos 2x\,dx - 2\int sen^2 2x \cos 2x\,dx + \int sen^4 2x\ cos\ 2xdx =$$

$u = sen2x \qquad du = 2\ cos2xdx \qquad \frac{du}{2} = cos2xdx$

$$= \frac{1}{2}sen\ 2x - \frac{1}{3}sen^3 2x + \frac{1}{10}sen^5 2x + c$$

Problema 5

$$\int \tan^2 xdx = \int (\sec^2 x - 1)dx = \int \sec^2 xdx - \int dx$$

$$= \tan x - x + c$$

Problema 6

$$\int \tan^3 xdx = \int \tan^2 x \tan xdx = \int (\sec^2 x - 1)\tan xdx$$

$$= \underbrace{\int \sec^2 x \tan xdx}_{\substack{u=\tan x \\ du = \sec^2 xdx}} - \int \tan xdx$$

$$= \int udu - Ln|\sec x| + c = \frac{u^2}{2} - Ln|\sec x| + c = \frac{\tan^2 x}{2} - Ln|\sec x| + c$$

Problema 7

$$\underbrace{\int \tan^4 3x dx}_{\substack{u=3x \\ \frac{1}{3}du = dx}} = \frac{1}{3}\int \tan^2 u \tan^2 u du = \frac{1}{3}\int \tan^2 u\left(\sec^2 u - 1\right)du$$

$$= \frac{1}{3}\int \tan^2 u \sec^2 u \, du - \frac{1}{3}\int \tan^2 u \, du$$

v = tg u ; dv = sec^{2}u du

$$= \frac{1}{3}\int v^2 dv - \frac{1}{3}\int\left(\sec^2 u - 1\right)du = \frac{1}{3}\frac{v^3}{3} - \frac{1}{3}\int \sec^2 u du + \frac{1}{3}\int du = \frac{1}{9}v^3 - \frac{1}{3}tg\, u + \frac{1}{3}u$$

$$= \frac{1}{9}\tan^3 u - \frac{1}{3}v + x + c = \frac{1}{9}\tan^3 3x - \frac{1}{3}\tan 3x + \frac{1}{3}(3x)$$

$$= \frac{1}{9}\tan^3 3x - \frac{1}{3}\tan 3x + x + c$$

Problema 8

$$\int \cot^2 x dx = \int\left(\csc^2 x - 1\right)dx = \int \csc^2 x dx - \int dx = -ctgx - x + c$$

Problema 9

$$\int \cot^3 x dx = \int \cot x \cot^2 x dx = \int \cot x\left(\csc^2 x - 1\right)dx$$

$$= \underbrace{\int \cot x \csc^2 x dx}_{\substack{u = ctgx \\ du = -\csc^2 x dx \\ -du = \csc^2 x dx}} - \int \cot x dx = \int u(-du) - Ln|senx|$$

$$= -\int udu - Ln|senx| = -\frac{u^2}{2} - Ln|senx| \quad = -\frac{ctg^2 x}{2} - Ln|senx| + c$$

Problema 10

$$\int \cot^4 x dx = \int \cot^2 x \cot^2 x dx = \int \cot^2 x\left(\csc^2 x - 1\right)dx$$

$$= \underbrace{\int \cot^2 x \csc^2 x dx}_{\substack{u = \cot x \\ du = -\csc^2 x dx \\ -du = \csc^2 x dx}} - \int \cot^2 x dx$$

$$= \int u^2(-du) - \int (\csc^2 x - 1)\, dx = -\int u^2 du - \int \csc^2 x dx + \int dx = -\frac{u^3}{3} + ctg\ x + x + c$$

$$= -\frac{\cot^3 x}{3} + \cot x + x + c$$

Problema 11

$$\int sen^2 x\, cos^3 x\, dx = \int sen^2 x\, cos^2 x \cos x\, dx = \int sen^2 x\,(1 - sen^2 x) cosx\, dx =$$

$$\int sen^2 x \cos x\, dx - \int sen^4 x \cos x\, dx = \frac{1}{3}\, sen^3 x\ - \frac{1}{5} sen^5 x + c$$

Problema 12

$$\int sen^3 x cos^4 x\, dx = \int sen^2 x\ \ senx\, cos^4 x dx = \int (1 - cos^2 x) senx\, cos^4 x dx =$$

$$= \int senx\, cos^4 x\, dx - \int cos^6 x\ senx\, dx = -\int cos^4 x\,(-senx) dx + \int cos^6 x\,(-senx)\, dx$$

$$= \frac{1}{5} cos^5 x + \frac{1}{7} cos^7 x + c$$

Problema 13

$$\int sen^5 2x\, cos^3 2x\, dx = \int sen^5 2x\, cos^2 2x \cos 2x\, dx = \int sen^5 2x\,(1 - sen^2 2x) \cos 2x\, dx =$$

$$\int sen^5 2x\ \cos 2x\, dx - \int sen^7 2x \cos 2x\, dx = \int u^5 \cdot \frac{du}{2} - \int u^7 \cdot \frac{du}{2} = \frac{1}{2}\int u^5 du - \frac{1}{2}\int u^7 du =$$

$u = sen\ 2x$ $\qquad du = 2\cos 2x\, dx$ $\qquad \frac{du}{2} = \cos 2x\, dx$

$$= \frac{1}{2} \cdot \frac{u^6}{6} - \frac{1}{2} \cdot \frac{u^8}{8} + c = \frac{1}{12} sen^6\, 2x - \frac{1}{16} sen^8 2x + c$$

Problema 14

$$\int \tan^3 x \sec^5 x dx = \int \tan^2 x \sec^4 x dx \tan x \sec x dx = \int (\sec^2 x - 1) \sec^4 x (\sec x \tan x dx),$$

$$\Rightarrow \quad \int \tan^3 x \sec^5 x dx = \int \sec^6 x (\sec x \tan x dx) - \int \sec^4 x (\sec x \tan x dx)$$

Sea $\quad u = \sec x, \quad \Rightarrow \quad du = \sec x \tan x dx$

De tal manera, que al hacer las sustituciones respectivas, queda:

$$\int \tan^3 x \sec^5 x dx = \int u^6 du - \int u^4 du = \frac{1}{7} u^7 - \frac{1}{5} u^5 + c;$$

$$\therefore \int \tan^3 x \sec^5 x\,dx = \frac{1}{7}\sec^7 x - \frac{1}{5}\sec^5 x + c$$

Problema 15

$$\int \tan^3 x \sec^6 xdx = \int \tan^3 x \sec^4 x \underline{\sec^2 x\,dx} = \int \tan^3 x \left(\sec^2 x\right)^2 \sec^2 x\,dx$$

$$= \int \tan^3 x \left(1+\tan^2 x\right)^2 \underline{\sec^2 x\,dx} = \int \tan^3 x(1+2\tan^2 x + \tan^4 x)\sec^2 x\,dx$$

$$= \int \tan^3 x \sec^2 xdx + 2\int \tan^5 x \sec^2 xdx + \int \tan^7 x \sec^2 xdx$$

$u = \tan x$
$du = \sec^2 xdx$

$$= \int u^3 du + 2\int u^5 du + \int u^7 = \frac{u^4}{4} + \frac{2u^6}{6} + \frac{u^8}{8} + c = \frac{\tan^4 x}{4} + \frac{\tan^6 x}{3} + \frac{\tan^8 x}{8} + c$$

Problema 16

$$\int tg^4 x\, sec^4 x\, dx = \int tg^4 x\, sec^2 x\, sec^2 x\, dx = \int tg^4 x\, (1+tg^2 x)\, sec^2 x dx =$$

$$\int tg^4 x\, sec^2 x dx + \int tg^6 x\, sec^2 x dx = \int u^4 du + \int u^6 du = \frac{u^5}{5} + \frac{u^7}{7} + c = \frac{1}{5} tg^5 x + \frac{1}{7} tg^7 x + c$$

$u = tgx \qquad du = sec^2 x\, dx$

Problema 17

$$\int sen^3 x \cos^2 xdx = \int sen^2 x \cos^2 x \underline{senx\,dx}$$

$$= \int \left(1-\cos^2 x\right)\cos^2 x \underline{senx\,dx} = \int \cos^2 x \underline{senx\,dx} - \int \cos^4 x \underline{senx\,dx}$$

$u = \cos x$
$du = -senxdx$
$-du = senxdx$

$$= -\int u^2 du + \int u^4 du = -\frac{u^3}{3} + \frac{u^5}{5} + c = -\frac{\cos^3 x}{3} + \frac{\cos^5 x}{5} + c$$

Problema 18

$$\int \tan^3 x \sec^4 x\,dx = \int \tan^3 x \sec^2 x \underline{\sec^2 x\,dx}$$

$$= \int \tan^3 x(1+\tan^2 x)\underline{\sec^2 x\,dx} = \underbrace{\int \tan^3 x \sec^2 x\,dx + \int \tan^5 x \sec^2 x\,dx}$$

$$u = \tan x$$

$$du = \sec^2 x dx$$

$$= \int u^3 du + \int u^5 du = \frac{u^4}{4} + \frac{u^6}{6} + c = \frac{\tan^4 x}{4} + \frac{\tan^6 x}{6} + c$$

Problema 19

$$\int \tan^5 x \sec^3 x\,dx = \int \tan^4 x \sec^2 x \underline{\sec x \tan x\,dx} = \int (\tan^2 x)^2 \sec^2 x \underline{\sec x \tan x\,dx}$$

$$= \int (\sec^2 x - 1)^2 \underline{\sec^2 x \sec x \tan x\,dx} = \int (\sec^4 x - 2\sec^2 x + 1)\underline{\sec^2 x \sec x \tan x\,dx}$$

$$= \underbrace{\int \sec^6 x}\underline{\sec x \tan x\,dx} - \int 2\sec^4 \underline{\sec x \tan x\,dx} + \int \sec^2 x \underline{\sec x \tan x\,dx}$$

$$u = \sec x$$

$$du = \sec x \tan x$$

$$= \int u^6 du - 2\int u^4 du + \int u^2 du = \frac{u^7}{7} - \frac{2u^5}{5} + \frac{u^3}{3} + c$$

$$= \frac{1}{7}\sec^7 x - \frac{2}{5}\sec^5 x + \frac{1}{3}\sec^3 x + c$$

Problema 20

$$\int sen^3 x \cos^3 x\,dx = \int sen^3 x \cos^2 x \underline{\cos x\,dx} = \int sen^3 x(1 - sen^2 x)\underline{\cos x\,dx}$$

$$= \int sen^3 x \underline{\cos x\,dx} - \underbrace{\int sen^5 x}\underline{\cos x\,dx}$$

$$u = senx$$

$$du = \cos xdx$$

$$= \int u^3(du) - \int u^5(du) = \frac{u^4}{4} - \frac{u^6}{6} + c = \frac{sen^4 x}{4} - \frac{sen^6 x}{6} + c$$

ACTIVIDAD COMPLEMENTARIA I .

PROBLEMAS PROPUESTOS EN LA GUÍA II

INTEGRALES QUE SE RESUELVEN EMPLEANDO IDENTIDADES TRIGONOMÉTRICAS FUNDAMENTALES PARA INTEGRAR POTENCIAS DE FUNCIONES TRIGONOMÉTRICAS Y PRODUCTOS DE POTENCIAS TRIGONOMÉTRICAS.

S o l u c i o n e s

1. Solución:

$$\int \cos^3 4x \sin 4x dx = -\frac{1}{4}\int (\cos 4x)^3 (-4\sin 4x dx)$$

Sea

$$u = \cos 4x, \quad \Rightarrow \quad du = -4\sin x dx$$

De tal forma que

$$\int \cos^3 4x \sin 4x dx = -\frac{1}{4}\int u^3 du = -\frac{1}{4}\times\frac{1}{4}u^4 + c = -\frac{1}{16}u^4 + c;$$

$$\therefore \quad \int \cos^3 4x \sin 4x dx = -\frac{1}{16}\cos^4 4x + c.$$

2. Solución:

$$\int \sin^5 x\cos^2 x dx = \int \sin^4 x\cos^2 x \sin x dx = \int \left(\sin^2 x\right)^2 \cos^2 x \sin x dx,$$

$$\Rightarrow \quad \int \sin^5 x\cos^2 x dx = \int \left(1-\cos^2 x\right)^2 \cos^2 x\sin x dx = \int \left(1-2\cos^2 x+\cos^4 x\right)\cos^2 x\sin x dx,$$

$$\Rightarrow \quad \int \sin^5 x\cos^2 x dx = \int \left(\cos^2 x - 2\cos^4 x + \cos^6 x\right)\sin x dx,$$

$$\Rightarrow \quad \int \sin^5 x\cos^2 x dx = \int \cos^2 x\sin x dx - 2\int \cos^4 x\sin x dx + \int \cos^6 x\sin x dx,$$

$$\Rightarrow \quad \int \sin^5 x\cos^2 x dx = -\int \cos^2 x(-\sin x dx) + 2\int \cos^4 x(-\sin x dx) - \int \cos^6 x(-\sin x dx)$$

Sea

$$u = \cos x, \quad \Rightarrow \quad du = -\sin x dx$$

De tal forma que

$$\int \sin^5 x\cos^2 x dx = -\int u^2 du + 2\int u^4 du - \int u^6 du = -\frac{1}{3}u^3 + 2\times\frac{1}{5}u^5 - \frac{1}{7}u^7 + c;$$

$$\therefore \quad \int \sin^5 x\cos^2 x dx = -\frac{1}{3}\cos^3 x + \frac{2}{5}\cos^5 x - \frac{1}{7}\cos^7 x + c.$$

3. Solución:

$$\int \cos^4 x dx = \int \left(\cos^2 x\right)^2 dx = \int \left(\frac{1+\cos 2x}{2}\right)^2 dx = \frac{1}{4}\int 1 + 2\cos 2x + \cos^2 2x dx,$$

$$\Rightarrow \quad \int \cos^4 x dx = \frac{1}{4}\int 1 + 2\cos 2x + \frac{1+\cos 4x}{2} dx = \frac{1}{8}\int 2 + 4\cos 2x + 1 + \cos 4x dx,$$

$$\Rightarrow \quad \int \cos^4 x dx = \frac{1}{8}\int 3 + 4\cos 2x + \cos 4x dx = \frac{1}{8}\left[3x + 4\left(\frac{1}{2}\sin 2x\right) + \frac{1}{4}\sin 4x + c_1\right];$$

$$\therefore \quad \int \cos^4 x dx = \frac{3}{8}x + \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + c.$$

4. Solución:

$$\int \sin^2 3x \cos^2 3x dx = \int \left[\frac{1-\cos 6x}{2} \times \frac{1+\cos 6x}{2}\right] dx = \frac{1}{4}\int 1 - \cos^2 6x dx = \frac{1}{4}\int \sin^2 6x dx,$$

$$\Rightarrow \quad \int \sin^2 3x \cos^2 3x dx = \frac{1}{4}\int \frac{1-\cos 12x}{2} dx = \frac{1}{8}\int 1 - \cos 12x dx = \frac{1}{8}\left[x - \frac{1}{12}\sin 12x + c_1\right];$$

$$\therefore \quad \int \sin^2 3x \cos^2 3x dx = \frac{1}{8}x - \frac{1}{96}\sin 12x + c.$$

5. Solución:

$$\int \sin 3x \cos 5x dx$$

Aplicando la identidad trigonométrica $\sin mx \cos nx = \frac{1}{2}\sin(m-n)x + \frac{1}{2}\sin(m+n)x,$ se tiene:

$$\sin 3x \cos 5x dx = \frac{1}{2}\sin(3-5)x + \frac{1}{2}\sin(3+5)x = \frac{1}{2}\sin(-2x) + \frac{1}{2}\sin 8x,$$

$$\Rightarrow \quad \sin 3x \cos 5x dx = -\frac{1}{2}\sin 2x + \frac{1}{2}\sin 8x$$

De tal manera que

$$\int \sin 3x \cos 5x dx = \int -\frac{1}{2}\sin 2x + \frac{1}{2}\sin 8x dx = -\frac{1}{2}\left[-\frac{1}{2}\cos 2x\right] + \frac{1}{2}\left[-\frac{1}{8}\cos 8x\right] + c;$$

$$\therefore \quad \int \sin 3x \cos 5x dx = \frac{1}{4}\cos 2x - \frac{1}{16}\cos 8x + c.$$

6. Solución:

$$\int \cos 4x \cos 3x dx$$

Aplicando la identidad trigonométrica $\cos mx \cos nx = \frac{1}{2}\cos(m-n)x + \frac{1}{2}\cos(m+n)x,$ se tiene:

$$\cos 4x \cos 3x dx = \frac{1}{2}\cos(4-5)x + \frac{1}{2}\cos(4+3)x = \frac{1}{2}\cos(-x) + \frac{1}{2}\cos 7x,$$

$$\Rightarrow \quad \cos 4x \cos 3x = \frac{1}{2}\cos x + \frac{1}{2}\cos 7x$$

De tal manera que

$$\int \cos 4x \cos 3x dx = \int \frac{1}{2}\cos x + \frac{1}{2}\cos 7x dx = \frac{1}{2}\sin x + \frac{1}{2}\left[\frac{1}{7}\sin 7x\right] + c;$$

$$\therefore \quad \int \cos 4x \cos 3x dx = \frac{1}{2}\sin x + \frac{1}{14}\sin 7x + c.$$

7. Solución:

$$\int \cot^3 x dx = \int \cot^2 x \cot x dx = \int (\csc^2 x - 1)\cot x = \int (\csc^2 x \cot x - \cot x) dx,$$

$$\Rightarrow \quad \int \cot^3 x dx = \int \csc^2 x \cot x dx - \int \cot x dx = \int \csc^2 x \cot x dx - \ln|\sin x| + c_1$$

Hallemos ahora, por sustitución, $\int \csc^2 x \cot x dx$:

$$\int \csc^2 x \cot x dx = -\int \cot x(-\csc^2 x dx) \quad \text{y} \quad \int \csc^2 x \cot x dx = -\int \csc x(-\csc x \cot x dx);$$

$$\therefore \quad \int \csc^2 x \cot x dx = -\frac{1}{2}\cot^2 x + c_2 \quad \text{y} \quad \int \csc^2 x \cot x dx = -\frac{1}{2}\csc^2 x + c_3$$

De tal manera que se pueden dar dos respuestas, aparentemente *distintas*:

$$\int \cot^3 x dx = -\frac{1}{2}\cot^2 x - \ln|\sin x| + c \quad \text{y} \quad \int \cot^3 x dx = -\frac{1}{2}\csc^2 x - \ln|\sin x| + c.$$

<u>Nota</u>: la razón de la aparente ambigüedad de la respuesta radica en el hecho de que $\cot^2 x = \csc^2 x - 1$; esto es, $\cot^2 x = \csc^2 x + c$.

8. Solución:

$$\int \sec^4 x dx = \int \sec^2 x \sec^2 x dx = \int (\tan^2 x + 1)\sec^2 x dx = \int \tan^2 x \sec^2 x dx + \int \sec^2 x dx;$$

$$\therefore \quad \int \sec^4 x dx = \frac{1}{3}\tan^3 x + \tan x + c.$$

9. Solución:

$$\int \csc^3 x dx = \int \csc x \csc^2 x dx$$

Sea

$$u = \csc x, \quad \Rightarrow \quad du = -\csc x \cot x dx$$

$$dv = \csc^2 x dx, \quad \Rightarrow \quad v = -\cot x$$

Así

$$\int \csc^3 x dx = -\csc x \cot x - \int \csc x \cot^2 x dx = -\csc x \cot x - \int \csc x(\csc^2 x - 1)dx,$$

$$\Rightarrow \quad \int \csc^3 x dx = -\csc x \cot x - \int \csc^3 x dx + \int \csc x dx,$$

$$\Rightarrow \quad 2\int \csc^3 x dx = -\csc x \cot x + \ln|\csc x - \cot x| + c_1;$$

$$\therefore \quad \int \csc^3 x dx = -\frac{1}{2}\csc x \cot x dx + \frac{1}{2}\ln|\csc x - \cot x| + c.$$

10. Solución:

$$\int \tan^6 x \sec^4 x dx = \int \tan^6 x \sec^2 x \sec^2 x dx = \int \tan^6 x \sec^2 x(\tan^2 x + 1)dx,$$

$$\Rightarrow \quad \int \tan^6 x \sec^4 x dx = \int \tan^8 x \sec^2 x dx + \int \tan^6 x \sec^2 x dx;$$

$$\therefore \quad \int \tan^6 x \sec^4 x dx = \frac{1}{9}\tan^9 x + \frac{1}{7}\tan^7 x + c.$$

11. Solución:

$$\int \tan^3 x \sec^5 x dx = \int \tan^2 x \sec^4 x dx \tan x \sec x dx = \int (\sec^2 x - 1)\sec^4 x(\sec x \tan x dx),$$

$$\Rightarrow \quad \int \tan^3 x \sec^5 x dx = \int \sec^6 x(\sec x \tan x dx) - \int \sec^4 x(\sec x \tan x dx)$$

Sea $u = \sec x, \quad \Rightarrow \quad du = \sec x \tan x dx$

De tal manera, que al hacer las sustituciones respectivas, queda:

$$\int \tan^3 x \sec^5 x dx = \int u^6 du - \int u^4 du = \frac{1}{7}u^7 - \frac{1}{5}u^5 + c;$$

$$\therefore \quad \int \tan^3 x \sec^5 x dx = \frac{1}{7}\sec^7 x - \frac{1}{5}\sec^5 x + c.$$

En éste mismo espacio se resuelve la integral de la secante cúbica que se requiere para el siguiente ejercicio.

$$\int \sec^3 x dx = \int \sec x \sec^2 x dx$$

Sea $u = \sec x, \quad \Rightarrow \quad du = \sec x \tan x dx$

$dv = \sec^2 x dx, \quad \Rightarrow \quad v = \tan x$

De tal manera, que al hacer las sustituciones respectivas, queda:

$$\int \sec^3 x dx = \sec x \tan x - \int \sec x \tan^2 x dx = \sec x \tan x - \int \sec x(\sec^2 x - 1)dx,$$

$$\Rightarrow \quad \int \sec^3 x dx = \sec x \tan x - \int \sec^3 x dx + \int \sec x dx,$$

$$\Rightarrow \quad 2\int \sec^3 x dx = \sec x \tan x + \ln|\sec x + \tan x| + c_1;$$

$$\therefore \quad \int \sec^3 x dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x| + c.$$

12. Solución:

$$\int \tan^2 x \sec^3 x dx = \int (\sec^2 x - 1)\sec^3 x dx = \int \sec^5 x dx - \int \sec^3 x dx,$$

$$\Rightarrow \quad \int \tan^2 x \sec^3 x dx = \int \sec^3 x \sec^2 x dx - \left[\int \sec^3 x dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x| + c_1\right]$$

Nota:
El resultado $\int \sec^3 x dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x| + c_1$ se obtuvo en el ejercicio anterior.

Sea $u = \sec^3 x, \quad \Rightarrow \quad du = 3\sec^3 x \tan x dx$

$dv = \sec^2 x dx, \quad \Rightarrow \quad v = \tan x$

De tal manera, que al hacer las sustituciones respectivas, queda:

$$\int \tan^2 x \sec^3 x dx = \sec^3 x \tan x - 3\int \tan^2 x \sec^3 x dx - \frac{1}{2}\sec x \tan x - \frac{1}{2}\ln|\sec x + \tan x| - c_1,$$

$$\Rightarrow \quad 4\int \tan^2 x \sec^3 x dx = \sec^3 x \tan x - \frac{1}{2}\sec x \tan x - \frac{1}{2}\ln|\sec x + \tan x| - c_1;$$

$$\therefore \quad \int \tan^2 x \sec^3 x dx = \frac{1}{4}\sec^3 x \tan x - \frac{1}{8}\sec x \tan x - \frac{1}{8}\ln|\sec x + \tan x| - c.$$

SOLUCIÓN AL PROBLEMA PROPUESTO

$$\int sen^6 t cos^2 t\, dt$$

Solución:

$$\int \text{sen}^6 t\cos^2 t\,dt = \int(\text{sen}^2 t)^3\cos^2 t\,dt = \int\left(\frac{1-\cos 2t}{2}\right)^3\left(\frac{1+\cos 2t}{2}\right)dt,$$

$$\Rightarrow \quad = \frac{1}{16}\int(1-\cos 2t)^3(1+\cos 2t)\,tdt = \frac{1}{16}\int(1-\cos 2t)^2(1-\cos 2t)(1+\cos 2t)\,dt$$

$$\Rightarrow \quad = \frac{1}{16}\int(1-\cos 2t)^2(1-\cos^2 2t)\,tdt = \frac{1}{16}\int(1-2\cos 2t+\cos^2 2t)(1-\cos^2 2t)\,dt,$$

$$\Rightarrow \quad \int \text{sen}^6 t\cos^2 t\,dt = \frac{1}{16}\int\left(1-\cancel{\cos^2 2t}-2\cos 2t+2\cos^3 2t+\cancel{\cos^2 2t}-\cos^4 2t\right)dt,$$

$$\Rightarrow \quad \int \text{sen}^6 t\cos^2 t\,dt = \frac{1}{16}\int(1-2\cos 2t+2\cos^3 2t-\cos^4 2t)\,dt,$$

$$\Rightarrow \quad \int \text{sen}^6 t\cos^2 t\,dt = \frac{1}{16}\left[\int dt-\int 2\cos 2t\,dt+\int 2\cos^3 2t\,dt-\int\cos^4 2t\,dt\right],$$

$$\Rightarrow \quad \int \text{sen}^6 t\cos^2 t\,dt = \frac{1}{16}\left[t-\text{sen}\,2t+\int 2\cos^2 2t\cos 2t\,dt-\int(\cos^2 2t)^2\,dt\right],$$

$$\Rightarrow \quad \int \text{sen}^6 t\cos^2 t\,dt = \frac{1}{16}\left[t-\text{sen}\,2t+2\int(1-\text{sen}^2 2t)\cos 2t\,dt-\int\left(\frac{\cos 4t+1}{2}\right)^2 dt\right],$$

$$\Rightarrow \quad = \frac{1}{16}\left[t-\text{sen}\,2t+2\left[\int\cos 2t\,dt-\int\text{sen}^2 2t\cos 2t\,dt\right]-\frac{1}{4}\int(\cos^2 4t+2\cos 4t+1)\,dt\right],$$

$$\Rightarrow \quad = \frac{1}{16}\left[t-\text{sen}\,2t+2\left[\frac{1}{2}\text{sen}\,2t-\frac{1}{6}\text{sen}^3 2t\right]-\frac{1}{4}\left[\int\cos^2 4t\,dt+\int 2\cos 4t\,dt+\int dt\right]\right],$$

$$\Rightarrow \quad = \frac{1}{16}\left[t-\cancel{\text{sen}\,2t}+\cancel{\text{sen}\,2t}-\frac{1}{3}\text{sen}^3 2t-\frac{1}{4}\left[\int\frac{\cos 8t+1}{2}dt+\frac{1}{2}\text{sen}\,4t+t\right]\right],$$

$$\Rightarrow \quad \int \text{sen}^6 t\cos^2 t\,dt = \frac{1}{16}\left[t-\frac{1}{3}\text{sen}^3 2t-\frac{1}{4}\left[\frac{1}{2}\int(\cos 8t+1)\,dt+\frac{1}{2}\text{sen}\,4t+t\right]\right],$$

$$\Rightarrow \quad \int \text{sen}^6 t\cos^2 t\,dt = \frac{1}{16}\left[t-\frac{1}{3}\text{sen}^3 2t-\frac{1}{4}\left[\frac{1}{2}\left(\frac{1}{8}\text{sen}\,8t+t\right)+\frac{1}{2}\text{sen}\,4t+t\right]\right]+c,$$

$$\Rightarrow \quad \int \text{sen}^6 t\cos^2 t\,dt = \frac{1}{16}\left[t-\frac{1}{3}\text{sen}^3 2t-\frac{1}{64}\text{sen}\,8t-\frac{1}{8}t-\frac{1}{8}\text{sen}\,4t-\frac{1}{4}t\right]+c;$$

$$\therefore \quad \int \text{sen}^6 t\cos^2 t\,dt = \frac{1}{16}\left[\frac{5}{8}t-\frac{1}{3}\text{sen}^3 2t-\frac{1}{8}\text{sen}\,4t-\frac{1}{64}\text{sen}\,8t\right]+c.$$

Actividad complementaria II: *Soluciones*

Problema 1

$$\int sen^4 x\, dx$$

$$= \int (sen^2 x)^2\, dx = \int \left(\frac{1-\cos 2x}{2}\right)^2 dx = \int \left(\frac{1-2\cos(2x)+\cos^2(2x)}{4}\right) dx$$

$$= \frac{1}{4}\int dx - \frac{1}{2}\int \cos(2x)\, dx + \frac{1}{4}\int \cos^2(2x)\, dx$$

$$= \frac{1}{4}\int dx - \frac{1}{2}\int \cos(2x)\, dx + \frac{1}{4}\int \left(\frac{1+\cos 4x}{2}\right) dx$$

$$= \frac{1}{4}\int dx - \frac{1}{2}\int \cos(2x)\, dx + \frac{1}{8}\int dx + \frac{1}{8}\int \cos 4x\, dx$$

$$= \frac{3}{8}\int dx - \frac{1}{2}\int \cos(2x)\, dx + \frac{1}{8}\int \cos 4x\, dx$$

$sea\ u = 2x$; $\quad \frac{du}{dx} = 2$; $\quad \frac{du}{2} = dx$

$v = 4x$; $\quad \frac{dv}{dx} = 4$; $\quad \frac{dv}{4} = dx$

$$= \frac{3}{8}\int dx - \frac{1}{2}\int \cos u\, \frac{du}{2} + \frac{1}{8}\int \cos v\, \frac{dv}{4}$$

$$= \frac{3}{8}\int dx - \frac{1}{4}\int \cos u\, du + \frac{1}{32}\int \cos v\, dv$$

$$= \frac{3}{8}x - \frac{1}{4}sen\,(2x) + \frac{1}{32}sen\,(4x) + C$$

Problema 2

$$\int sen^2 3x\, cos^4 3x\, dx$$

$$= \int sen^2 3x\, cos^2 3x\, cos^2 3x dx = \int \left(\frac{1}{2}sen\, 6x\right)^2 \left(\frac{1+\cos 6x}{2}\right) dx$$

$$= \int \frac{1}{4} sen^2\, 6x \left(\frac{1+\cos 6x}{2}\right) dx$$

$$= \frac{1}{8}\int sen^2\,6x\,(1+\cos 6x)\,dx$$

$$= \frac{1}{8}\int sen^2\,6x dx + \frac{1}{8}\int sen^2\,6x\,cos6x\,dx$$

$$= \frac{1}{8}\int \left(\frac{1-\cos 12x}{2}\right)dx + \frac{1}{8}\int sen^2\,6x\cos 6x\,dx$$

$$= \frac{1}{8}\bullet\frac{1}{2}\int dx - \frac{1}{8}\bullet\frac{1}{2}\int \cos 12x\,dx + \frac{1}{8}\int sen^2\,6x\cos 6x\,dx$$

$sea\; u = 12x\,;\qquad \frac{du}{dx} = 12\,;\qquad \frac{du}{12} = dx$

$v = sen\,6x\,;\qquad \frac{dv}{dx} = \cos 6x\,(6)\,;\qquad \frac{dv}{6} = \cos 6x\,dx$

$$= \frac{1}{16}\,x - \frac{1}{16}\int \cos u \bullet \frac{du}{12} + \frac{1}{8}\int v^2 \bullet \frac{dv}{6}$$

$$= \frac{1}{16}\,x - \frac{1}{192}\,sen\,u + \frac{1}{48}\;\frac{v^3}{3} + C$$

$$= \frac{x}{16} - \frac{1}{192}\,sen\,12x + \frac{1}{144}\,sen^3\,6x + C$$

Problema 3

$$\int cos^{\,6}\,x\,dx$$

$$= \int (cos^2 x)^3\,dx = \int \left(\frac{1+\cos 2x}{2}\right)^3 dx = \int \left(\frac{1+3\cos 2x+3\,cos^2\,2x+\,cos^3\,2x}{8}\right)dx$$

$$= \frac{1}{8}\int dx + \frac{3}{8}\int \cos 2x\,dx + \frac{3}{8}\int cos^2\,2x\,dx + \frac{1}{8}\int cos^3\,2x\,dx$$

$$= \frac{1}{8}\int dx + \frac{3}{8}\int \cos 2x\,dx + \frac{3}{8}\int \left(\frac{1+\cos 4x}{2}\right)dx + \frac{1}{8}\int \cos 2x\,(cos^2\,2x)\,dx$$

$$= \frac{1}{8}\int dx + \frac{3}{8}\int \cos 2x\,dx + \frac{3}{16}\int dx + \frac{3}{16}\int \cos 4x\,dx + \frac{1}{8}\int \cos 2x\,(1-sen^2\,2x)\,dx$$

$$= \frac{5}{16}\int dx + \frac{3}{8}\int \cos 2x\,dx + \frac{3}{16}\int \cos 4x\,dx + \frac{1}{8}\int \cos 2x\,dx - \frac{1}{8}\int sen^2\,2x\cos 2x\,dx$$

$sea\; u = 2x\,;\qquad \frac{du}{2} = dx$

$v = 4x\,;\qquad \frac{dv}{4} = dx$

$w = sen\,2x\,;\qquad \frac{dw}{dx} = \cos 2x\,(2)\,;\qquad \frac{dw}{2} = \cos 2x\,dx$

$$= \frac{5\,x}{16} + \frac{4}{8}\int \cos u\ \frac{du}{2} + \frac{3}{16}\int \cos v\ \frac{dv}{4} - \frac{1}{8}\int w\ \frac{dw}{2}$$

$$= \frac{5\,x}{16} + \frac{1}{4}\,sen\,u + \frac{3}{64}\,sen\,v - \frac{1}{16}\frac{w^3}{3} + C$$

$$= \frac{5\,x}{16} + \frac{1}{4}\,sen\,2x + \frac{3}{64}\,sen\,4x - \frac{1}{48}\,sen^3\,2x + C$$

Problema 4

$$\int sen^4\,2x\,dx$$

$$= \int (sen^2\,2x)^2\,dx = \int \left(\frac{1-\cos 4x}{2}\right)^2 dx = \int \left(\frac{1 - 2\cos 4x + cos^2\,4x}{4}\right) dx$$

$$= \frac{1}{4}\int dx - \frac{1}{2}\int \cos 4x\,dx + \frac{1}{4}\int cos^2\,4x\,dx$$

$$= \frac{1}{4}\int dx - \frac{1}{2}\int \cos 4x\,dx + \frac{1}{4}\int \left(\frac{1+\cos 8x}{2}\right) dx$$

$$= \frac{1}{4}\int dx - \frac{1}{2}\int \cos 4x\,dx + \frac{1}{8}\int dx + \frac{1}{8}\int \cos 8x\,dx$$

$sea\ u = 4x\,;$ $\qquad \frac{du}{dx} = 4\,;$ $\qquad \frac{du}{4} = dx$

$v = 8x\,;$ $\qquad \frac{dv}{dx} = 8\,;$ $\qquad \frac{dv}{8} = dx$

$$= \frac{3\,x}{8} - \frac{1}{2}\int \cos u\ \frac{du}{4} + \frac{1}{8}\int \cos v\ \frac{dv}{8}$$

$$= \frac{3\,x}{8} - \frac{1}{8}\,sen\,u + \frac{1}{64}\,sen\,v + C$$

$$= \frac{3\,x}{8} - \frac{1}{8}\,sen\,4x + \frac{1}{64}\,sen\,8x + C$$

Problema 5

$$\int sen\,3x\cos 5x\,dx$$

$$= \int \frac{1}{2}\left[sen\,(3x-5x) + sen\,(3x+5x)\right] dx$$

$$= \frac{1}{2}\int (sen\,(-2x) + sen\,8x)\,dx$$

$$= -\frac{1}{2}\int sen\,(2x)\,dx + \frac{1}{2}\int sen\,8x\,dx$$

$sea\ u = 2x\,;\qquad \frac{du}{dx} = 2\,;\qquad \frac{du}{2} = dx$

$v = 8x\,;\qquad \frac{dv}{dx} = 8\,;\qquad \frac{dv}{8} = dx$

$= -\frac{1}{2}\int sen\,u\ \frac{du}{2} + \frac{1}{2}\int sen\,v\ \frac{dv}{8}$

$= -\frac{1}{4}\int sen\,u\,du + \frac{1}{16}\int sen\,v\,dv$

$= -\frac{1}{4}(-\cos u) + \frac{1}{16}(-\cos v) + C$

$= \frac{1}{4}\cos 2x - \frac{1}{16}\cos 8x + C$

Problema 6

$\int \frac{\tan\sqrt{3x}\ sec^2\sqrt{3x}}{\sqrt{3x}}\,dx$

$= \int \tan\sqrt{3x}\ sec^2\sqrt{3x} \bullet \frac{1}{\sqrt{3x}}\,dx$

$sec\,u = \tan\sqrt{3x}\,;\qquad \frac{du}{dx} = sec^2\sqrt{3x} \bullet \frac{3}{2\sqrt{3x}}\,;\qquad \frac{2\,du}{3} = sec^2\sqrt{3x} \bullet \frac{1}{\sqrt{3x}}\,dx$

$= \int u \bullet \frac{2\,du}{3} = \frac{2}{3}\int u\,du = \frac{2}{3}\frac{u^2}{2} + C = \frac{1}{3}\,tan^2\sqrt{3x} + C$

Problema 7

$\int cot^4\frac{x}{4}dx$

$= \int cot^2\frac{x}{4} \times cot^2\ \frac{x}{4}dx = \int cot^2\frac{x}{4}\left(csc^2\frac{x}{4} - 1\right)dx$

$= \int cot^2\frac{x}{4} \times\ csc^2\frac{x}{4}dx - \int cot^2\frac{x}{4}dx = \int cot^2\frac{x}{4}csc^2\ \frac{x}{4}dx - \int\left(csc^2\ \frac{x}{4} - 1\right)dx$

$sea\quad u = cot\frac{x}{4}\quad ;\quad \frac{du}{dx} = -csc^2\ \frac{x}{4}\left(\frac{1}{4}\right)\qquad ;\quad -4du = csc^2\ \frac{x}{4}dx$

$v = \frac{x}{4}\quad ;\quad \frac{dv}{dx} = \frac{1}{4}\quad ;\quad 4dv = dx$

$= \int u^2(-4du) - \int csc^2\ \frac{xdx}{4} + \int dx$

$= -4\int u^2du - 4\int csc^2v4dv + x\quad = -4\frac{u^3}{3} - 4(-cotv) + x + C\quad = -\frac{4}{3}cot^3\frac{x}{4} + 4\ cot\frac{x}{4} + x + C$

Problema 8

$$\int cot^6 2x csc^4 2x dx$$

$$= \int cot^6 2x csc^2 2x csc^2 2x dx = \int cot^6 2x\,(1 + cot^2 2x)\, csc^2 2x dx$$

$$= \int cot^6 2x csc^2 2x dx + \int cot^8 2x csc^2 2x dx$$

$$sea \quad u = cot 2x \quad ; \quad \frac{du}{dx} = -csc^2(2x)2 \quad ; \quad -\frac{du}{2} = csc^2 2x dx$$

$$= \int u^6 \times \left(-\frac{du}{2}\right) + \int u^8 \left(-\frac{du}{2}\right) = -\frac{1}{2}\frac{u^7}{7} - \frac{1}{2}\frac{u^9}{9} + C = -\frac{1}{14} cot^7 2x - \frac{1}{18} cot^9 2x + C$$

Problema 9

$$\int \frac{cos^2 \frac{x}{5}}{sen^4 \frac{x}{5}} dx$$

$$= \int \frac{cos^2 \frac{x}{5}}{sen^2 \frac{x}{5}} \times \frac{1}{sen^2 \frac{x}{5}} dx = \int cot^2 \frac{x}{5} \times csc^2 \frac{x}{5} dx$$

$$sea \quad u = cot \frac{x}{5} \quad ; \quad \frac{du}{dx} = -csc^2 \frac{x}{5}\left(\frac{1}{5}\right) \quad ; \quad -5du = csc^2 \frac{x}{5} dx$$

$$= \int u^2 \times (-5du) = -5 \int u^2 du = -5\frac{u^3}{3} + C = -\frac{3}{5} cot^3 \frac{x}{5} + C$$

Problema 10

$$\int \sqrt{tan^3 4x}\; sec^4 4x dx$$

$$= \int tan^{\frac{3}{2}} 4x sec^2 4x sec^2 4x dx = \int tan^{\frac{3}{2}} 4x(1 + tan^2 4x) sec^2 4x dx \quad = \int tan^{\frac{3}{2}} 4x sec^2 4x dx + \int tan^{\frac{7}{2}} 4x sec^2 4x dx$$

$$sea \quad u = tan 4x \quad ; \quad \frac{du}{dx} = sec^2 4x(4) \quad ; \quad \frac{du}{4} = sec^2 4x dx$$

$$= \int u^{\frac{3}{2}} \times \frac{du}{4} + \int u^{\frac{7}{2}} \times \frac{du}{4} \quad = \frac{1}{4}\int u^{\frac{2}{3}} du + \frac{1}{4}\int u^{\frac{7}{2}} du$$

$$= \frac{1}{4}\frac{u^{\frac{5}{2}}}{\frac{5}{2}} + \frac{1}{4}\frac{u^{\frac{9}{2}}}{\frac{9}{2}} + C$$

$$= \frac{1}{10}\sqrt{tan^5 4x} + \frac{1}{18}\sqrt{tan^9 4x} + C$$

Problema 11

$\int sec^6 x dx$

$$= \int sec^2 x(1 + tan^2 x)^2 = \int sec^2 x(1 + 2tan^2 x + tan^4 x)dx$$

$$= \int sec^2 x dx + 2\int sec^2 x tan^2 x dx + \int sec^2 x tan^4 x dx$$

$$sea \quad u = tanx \quad ; \quad \frac{du}{dx} = sec^2 x \quad ; \quad du = sec^2 x dx$$

$$= \int sec^2 x + 2\int u^2 \times du + \int u^4 du$$

$$= tanx + 2\frac{u^3}{3} + \frac{u^5}{5} + C$$

$$= tanx + \frac{2}{3}tan^3 x + \frac{1}{5}tan^5 x + C$$

Problema 12

$$\int cos^5 x sin^2 x dx$$

$$= \int sin^2 x(1 - 2sin^2 x + sin^4 x) \cos x dx$$

$$= \int sin^2 x \cos x dx - 2\int sin^4 x \cos x dx + \int sin^6 x dx$$

$$sea \quad u = sinx \quad ; \quad \frac{du}{dx} = cosx \quad ; \quad du = cosxdx$$

$$= \int u^2 du - 2\int u^4 du + \int u^6 du = \frac{u^3}{3} - 2\frac{u^5}{5} + \frac{u^7}{7} + c$$

$$= \frac{1}{3}sen^3 x - \frac{2}{5}sen^5 x + \frac{1}{7}sen^7 x + c$$

Problema 13

$$\int cos^2(3x)sin^3(3x)dx$$

$$= \int cos^2(3x)\left(1 - cos^2(3x)\right)\sin(3x)dx$$

$$= \int (cos^2(3x) - cos^4(3x)\sin(3x))dx$$

$$= \int cos^2(3x)\sin(3x)\,dx - \int cos^4(3x)\sin(3x)\,dx$$

$sea \quad u = \cos(3x) \quad ; \quad = \frac{du}{dx} = -\sin(3x)3 \quad ; du = -3\sin(3x)dx \quad ; \quad -\frac{du}{3} = \sin(3x)dx$

$$= \int u^2 \cdot -\frac{du}{3} - \int u^4 \cdot -\frac{du}{3}$$

$$= -\frac{1}{3}\int u^2 du + \frac{1}{3}\int u^4 du$$

$$= -\frac{1}{3}\frac{u^3}{3} + \frac{1}{3}\frac{u^5}{5} + c$$

$$= -\frac{1}{9}u^3 + \frac{1}{15}u^5 + c \quad = -\frac{1}{9}cos^3(3x) + \frac{1}{15}cos^5(3x) + c$$

Problema 14

$$\int \frac{\sqrt{sin^3(2x)}}{\sec(2x)}dx$$

$$= \int sin^{\frac{3}{2}}(2x) \cdot \frac{1}{\sec(2x)}dx = \int sin^{\frac{3}{2}}(2x)\cos(2x)dx$$

$sea \quad u = \sin(2x) \; ; \; \frac{du}{dx} = \cos(2x)(2) \; ; \quad \frac{du}{2} = \cos(2x)dx$

$$= \int u^{\frac{3}{2}} \cdot \frac{du}{2} = \frac{1}{2}\frac{u^{\frac{3}{2}+1}}{\frac{3}{2}+1} + c = \frac{1}{5}u^{\frac{5}{2}} + c = \frac{1}{5}\sqrt{sin^5(2x)} + c$$

Problema 15

$$\int sin^3\frac{x}{a}cos^3\frac{x}{a}dx$$

$$= \int \sin\frac{x}{a}\left(1 - cos^2\frac{x}{a}\right)cos^3\frac{x}{a}dx$$

$$= \int (\sin\frac{x}{a} - \sin\frac{x}{a}cos^2\frac{x}{a})\ cos^3\frac{x}{a}dx$$

$$= \int cos^3\frac{x}{a}\sin\frac{x}{a}dx - \int cos^5\frac{x}{a}\sin\frac{x}{a}dx$$

$sea \quad u = \cos\frac{x}{a} \quad ; \quad \frac{du}{dx} = -\sin\frac{x}{a}\left(\frac{1}{a}\right) \quad ; \quad du = -\frac{1}{a}\sin\frac{x}{a}dx \quad ; \quad -adu = \sin\frac{x}{a}dx$

$$= \int u^3(-adu) - \int u^5(-adu)$$

$$= -a\int u^3du + a\int u^5du$$

$$= -a\frac{u^4}{4} + \frac{au^6}{6} + c$$

$$= -\frac{acos^4\frac{x}{a}}{4} + \frac{acos^6\frac{x}{a}}{6} + c$$

$$= -\frac{a}{4}cos^4\ \frac{x}{a} + \frac{a}{6}cos^6\frac{x}{a} + c$$

Problema 16

$$\int sin^3 6xcos6xdx$$

$sea \quad u = sin6x \quad ; \quad \frac{du}{dx} = cos6x(6) \quad ; \quad \frac{du}{6}cos6xdx$

$$\int u^3 \cdot \frac{du}{6} = \frac{1}{6}\int u^3du \quad = \frac{1}{6}\frac{u^4}{4} = \frac{1}{24}sin^4 6x + c$$

Problema 17

$$\int sin^7 x dx = \int (sin^2 x)^3 sinx dx = \int (1 - cos^2 x)^3 sinx dx$$

$$= \int (1 - 3cos^2 x + 3cos^4 x - cos^6 x) sinx dx$$

$$= \int sinx dx - 3\int cos^2 x sinx dx + 3\int cos^4 x sinx dx - \int cos^6 x sinx dx$$

$$sea \quad u = cosx \quad ; \quad \frac{du}{dx} = -sinx \quad ; \quad -du = sinx dx$$

$$= \int sinx dx - 3\int u^2(-du) + 3\int u^4(-du) - \int u^6(-du)$$

$$= \int sinx dx + 3\int u^2 du - 3\int u^4 du + \int u^6 du$$

$$= \int sinx dx + 3\frac{u^3}{3} - 3\frac{u^5}{5} + \frac{u^7}{7} + c$$

$$= -\cos x + cos^3 x - \frac{3}{5} cos^5 x + \frac{1}{7} cos^7 x + c$$

INTEGRACIÓN POR PARTES.

ACTIVIDAD II.PROBLEMAS PROPUESTOS EN LA GUÍA II

PROBLEMAS RESUELTOS.

1. $\int x\cos x dx = \int (x)(senx) - \int senxdx = xsenx - \int -\cos x + c = xsenx + \cos x + c$

$u = x \qquad du = dx$

$dv = \cos x dx \qquad v = senx$

2.

$$\begin{aligned}
&\int x^2 sen\ x\ dx = u\ dv = uv - \int vdu \\
&= -x^2 \cos\ x - \int -\cos x \cdot 2x\ dx \\
&= -x^2 \cos\ x + 2\int x \cos\ x\ dx \\
&= -x^2 \cos\ x + 2\left[x\ sen\ x - \int sen\ x\ dx\right] \\
&= -x^2 \cos\ x + 2\left[x\ sen\ x - (-\cos x)\right] \\
&= -x^2 \cos\ x + 2x\ sen\ x + 2\cos x + c
\end{aligned}$$

$u = x^2$
$du = 2x\ dx$
$v = -\cos x$
$dv = sen\ x\ dx$

$u = x$
$dv = \cos x\ dx$
$du = dx$
$v = sen\ x$

3.

$$\begin{aligned}
&\int xe^x dx = xe^x - \int e^x dx \\
&= xe^x - e^x + c
\end{aligned}$$

$u = x$
$dv = e^x dx$
$du = dx$
$v = e^x$

4.

$$\begin{aligned}
&\int x^2 e^x dx = u\ dv = uv - \int vdu \\
&= x^2\ e^x - \int e^x \cdot 2x\ dx \\
&= x^2\ e^x - 2\int e^x x\ dx \\
&= x^2\ e^x - 2\left[xe^x - \int e^x dx\right] \\
&= x^2\ e^x - 2\ ex^x + 2\ e^x + c
\end{aligned}$$

$u = x^2$
$dv = e^x dx$
$du = 2x\ dx$
$v = e^x$

$u = x$
$dv = e^x dx$
$du = dx$
$v = e^x$

5. $$\int x^3 e^{x^2} dx = \int x^2 x e^{x^2} dx = \int w e^w \cdot \frac{dw}{2} = \frac{1}{2}\int w e^w dw = \frac{1}{2}\left[w e^w - \int e^w dw\right] = \frac{1}{2} w e^w - \frac{1}{2}\int e^w dw$$

u=w ; dv=e^w dw ; du=dw ; v=e^w

$w = x^2$
$dw = 2x dx$
$\frac{dw}{2} = x dx$

$$= \frac{1}{2} w e^w - \frac{1}{2} e^w + c = \frac{1}{2} x^2 e^{x^2} - \frac{1}{2} e^{x^2} + c$$

6. $$\int Ln\, x\, dx = x\, Ln\, x - \int x \frac{dx}{x} = x\, Ln\, x - \int dx = x\, Ln\, x - x + c$$

$u = Ln\, x$ $\quad du = \frac{dx}{x}$
$dv = dx$ $\quad v = x$

7.

$$\int x Ln\, x\, dx = x(Ln\, x - x) - \int x\, Ln\, x - x(dx)$$
$$= x^2 Lnx - x^2 - \int x\ Ln\, x\, dx + \int x\, dx$$

$$= \int x\, Lnx\, dx + \int x\, Ln\, x\, dx = x^2 Ln\, x - x^2 + \frac{x^2}{2}$$
$$= \int x\, Ln\, x\, dx = \frac{x^2 Ln\, x - x^2 + \frac{x^2}{2}}{2} = \frac{1}{2} x^2 \ln x - \frac{1}{4} x^2 + c$$

$u = x$
$dv = Ln\, x\, dx$
$du = dx$
$v = x\, Ln\, x - x$

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$$\int x^2 \cos x\, dx = u\, dv = uv - \int v du$$
$$= x^2 sen\ x - \int senx \cdot 2x\ dx$$
$$= x^2 sen\ x - 2\int x\ sen\ x\ dx$$
$$= x^2 sen\ x - 2\left[-x \cos x - \int -\cos x\ dx\right]$$
$$= x^2 sen\ x - 2\left[-x \cos x + \int \cos x\, dx\right]$$
$$= x^2 sen\ x - 2[-x \cos x + sen\, x]$$
$$= x^2 sen\ x + 2x \cos x - 2sen\, x + c$$

$u = x^2$
$dv = \cos x\, dx$
$du = 2x\, dx$
$v = sen\, x$

$u = x$
$dv = sen\, x\, dx$
$du = dx$
$v = -\cos x$

9.

$$\int x^3e^{2x}dx = u\,dv = uv - \int v du$$
$$= \frac{x^3e^{2x}}{2} - \int \frac{1}{2}e^{2x}\cdot 3x^2\,dx$$
$$= \frac{x^3e^{2x}}{2} - \frac{3}{2}\int x^2e^{2x}dx$$
$$= \frac{x^3e^{2x}}{2} - \frac{3}{2}\left[\frac{x^2e^{2x}}{2} - \int \frac{1}{2}e^{2x}2x\,dx\right]$$
$$= \frac{x^3e^{2x}}{2} - \frac{3}{2}\left[\frac{x^2e^{2x}}{2} - \int xe^{2x}dx\right]$$

$u = x^3$
$dv = e^{2x}dx$
$du = 3x^2dx$
$v = \frac{1}{2}e^{2x}$

$u = 2x$
$du = 2dx$
$\frac{du}{2} = dx$
$= \frac{1}{2}\int e^u du$
$= \frac{1}{2}e^u$

$$\int xe^{2x}dx = x\cdot\frac{1}{2}e^{2x} - \int\frac{1}{2}e^{2x}dx = \frac{xe^{2x}}{2} - \frac{1}{2}\int e^{2x}dx = \frac{xe^{2x}}{2} - \frac{1}{2}\left(\frac{1}{2}e^{2x}\right) = \frac{xe^{2x}}{2} - \frac{e^{2x}}{4} + c$$

$u = x$
$dv = e^{2x}dx$
$du = dx$
$v = \int dv = \frac{1}{2}e^{2x}$

$$\mathrm{v} = \int \mathrm{e}^{2x}\mathrm{dx} = \int \mathrm{e}^{\mathrm{u}}\,\frac{\mathrm{du}}{2} = \frac{1}{2}\int \mathrm{e}^{\mathrm{u}}\mathrm{du} = \frac{1}{2}\mathrm{e}^{\mathrm{u}} = \frac{1}{2}\mathrm{e}^{2x}$$

$u = 2x$
$du = 2dx$
$\frac{du}{2} = dx$

Finalmente la integral original se resuelve así:

$$\int x^3e^{2x}dx = \frac{x^3e^{2x}}{2} - \frac{3}{4}x^2e^{2x} + \frac{3}{4}xe^{2x} - \frac{3}{4}\int e^{2x}dx$$
$$= \frac{x^3e^{2x}}{2} - \frac{3}{4}x^2e^{2x} + \frac{3}{4}xe^{2x} - \frac{3}{4}\left(\frac{1}{2}e^{2x}x\right) + c$$
$$= \frac{1}{2}x^3e^{2x} - \frac{3}{4}x^2e^{2x} + \frac{3}{4}xe^{2x} - \frac{3}{8}xe^{2x}$$

10.

$$\int xe^{-x}dx =$$

$$x(-e^{-x}) - \int -e^{-x}dx = -xe^{-x} + \int e^{-x}dx = -xe^{-x} + (-e^{-x}) = -xe^{-x} - e^{-x} + c$$

$$u = x \quad dv = e^{-x}dx$$
$$du = dx \quad v = -e^{-x}$$

INTEGRALES DE POTENCIAS DE FUNCIONES TRIGONOMÉTRICAS.*PROBLEMAS ESPECIALES.*

PROBLEMA 1.

$$\int \frac{sen^{\frac{5}{2}}x\,dx}{cos^{\frac{11}{2}}x} = \int \sqrt{\frac{sen^5 x}{cos^{11}x}}\,dx = \int \frac{sen\,x\sqrt{sen\,x}}{cos^5 x\,\sqrt{cos x}}\,dx = \int \frac{sen\,x\sqrt{sen\,x}}{cosx cos^4\sqrt{cos x}} =$$

$$= \int tg\,x sec^4 x\sqrt{tgx}\,dx = \int tg^{\frac{3}{2}}sec^2 sec^2 x\,dx = \int tg^{\frac{3}{2}}(tg^2 + 1)sec^2 x\,dx =$$

$$\int tg^{\frac{7}{2}}xsec^2 x\,dx + \int tg^{\frac{3}{2}}sec^2 x\,dx = \int u^{\frac{7}{2}}du + \int u^{\frac{3}{2}}\,du =$$

$u = tg\,x$

$du = sec^2 x\,dx$

$$= \frac{u^{\frac{9}{2}}}{\frac{9}{2}} + \frac{u^{\frac{5}{2}}}{\frac{5}{2}} = \frac{2}{9}\sqrt{tg^9}x + \frac{2}{5}\sqrt{tg^5 x} + c$$

PROBLEMA 2.

$$\int \frac{dx}{sen^2 2x\,cos^4 2x} = \int csc^2 2x\,sec^4 2x\,dx = \int csc^2 u\,sec^4\frac{du}{2} = \frac{1}{2}\int csc^2 u\,sec^4 u\,du =$$

$$u = 2x \qquad du = 2dx \qquad \frac{du}{2} = dx$$

$$\frac{1}{2}\int csc^2 u(sec^2 u)^2 du = \frac{1}{2}\int csc^2 u(1 + tg^2 u)^2\,du = \frac{1}{2}\int csc^2(1 + 2tg^2 u + tg^4 u)du =$$

$$\frac{1}{2}\int csc^2 u du + \frac{1}{2}\int 2tg^2 u csc^2 u\,du + \frac{1}{2}\int csc^2 u\,tg^2 u\,du =$$

$$\frac{1}{2}(-ctg\,u)+\int\frac{sen^2u}{cos^2u}\cdot\frac{1}{sen^2u}\,du+\frac{1}{2}\int\frac{1}{sen^2}\cdot\frac{sen^4u}{cos^4u}du=$$

$$-\frac{1}{2}ctg\;u+\int\frac{du}{cos^2u}+\frac{1}{2}\int\frac{sen^2u}{cos^4u}du=$$

$$=-\frac{1}{2}ctg\;2x+\int sec^2u\;du+\frac{1}{2}\int\frac{sen^2}{cos^2u\,cos^2u}du$$

$$=-\frac{1}{2}ctg\;2x+tg\,u+\frac{1}{2}\int tg^2u\,sec^2u\,du$$

$$v=\text{tg}\,\text{u}\qquad dv=\sec^2\text{udu}$$

$$=\;-\frac{1}{2}ctg\;2x+tg2x+\frac{1}{2}\int v^2dv=-\frac{1}{2}ctg\;2x+tg\;2x+\frac{1}{2}\cdot\frac{v^3}{3}+c$$

$$=-\frac{1}{2}ctg\;2x+tg\;2x+\frac{1}{6}tg^32x+c$$

COMPROBACIÓN

$$d\left(-\frac{1}{2}ctg\;2x+tg\;2x+\frac{1}{6}tg^32x\right)=\left(-\frac{1}{2}\right)(-csc^22x)(2)+(sec^22x)(2)+\left(\frac{1}{6}\right)(3tg^22x)(sec^22x)(2)$$

$$=(csc^22x+2sec^22x+tg^22xsec^22x)dx=\left(\frac{1}{sen^22x}+\frac{2}{cos^22x}+\frac{sen^22x}{cos^22x}\cdot\frac{1}{cos^22x}\right)dx$$

$$=\left(\frac{1}{sen^22x}+\frac{2}{cos^22x}+\frac{sen^22x}{cos^42x}\right)dx=\left(\frac{cos^42x+2sen^22xcos^22x+sen^42x}{sen^22xcos^42x}\right)dx$$

$$=\left(\frac{sen^22xcos^22x}{sen^22xcos^42x}\right)dx=\frac{dx}{sen^22xcos^42x}$$

PROBLEMA 3.

$$\int sen^3\frac{x}{2}cos^3\frac{x}{2}dx=\int sen^3u\;cos^2u\;cosu\;2\,du=2\int sen^3u(1-sen^2u)\,cosu\;du=$$

$$u=\frac{x}{2}\quad u=\frac{dx}{2}\quad 2du=dx$$

$$=2\int sen^3u\;cos\;u\;\;du-2\int sen^5u\;cosu\;du$$

$$=\int sen^3\frac{x}{2}cos\frac{x}{2}dx-\int sen^5\frac{x}{2}cos\frac{x}{2}dx=$$

$$v=senu\qquad dv=cosu\;du$$

$=2\int v^3 dv - 2\int v^5 dv = 2\frac{v^4}{4} - \frac{2v^6}{6} = \frac{1}{2}sen^4\frac{x}{2} - \frac{1}{3}sen^6\frac{x}{2} + c$

COMPROBACIÓN

$$d\left(\frac{1}{2}sen^4\frac{x}{2} - \frac{1}{3}sen^6\frac{x}{2}\right) = \frac{1}{2}\cdot 4sen^3\frac{x}{2}cos\frac{x}{2}\cdot\frac{1}{2} - \frac{1}{3}\cdot 6sen^5\frac{x}{2}\,cos\frac{x}{2}\cdot\frac{1}{2}$$
$$= sen^3\frac{x}{2}cos\frac{x}{2} - sen^5\frac{x}{2}cos\frac{x}{2}$$

$$= sen^3\frac{x}{2}cos\frac{x}{2}\left(1 - sen^2\frac{x}{2}\right) = sen^3\frac{x}{2}cos\frac{x}{2}\left(cos^2\frac{x}{2}\right) = sen^3\frac{x}{2}cos^3\frac{x}{2} + c$$

PROBLEMA 4.

$$\int tg^3 5x\, sec^4 5x\, dx = \int tg^3 5x\;\, sec^2 5x\;\, sec^2 5x\, dx = \int tg^3 5x(1 + tg^2 5x)sec^2 5x =$$

$=\int tg^3 5x\, sec^2 5x\;\, dx + \int tg^5 5x sec^2 5x\, dx =$

$$u = tg5x$$

$du = sec^2 5x.5.dx$

$\frac{du}{5} = sec^2\; 5x\, dx$

$= \int u^3.\frac{du}{5} + \int u^5\frac{du}{5} = \int\frac{1}{5}.\frac{u^4}{4} + \frac{1}{5}\frac{u^6}{6} = \frac{u^4}{20} + \frac{u^6}{30} = \frac{tg^4 5x}{20} + \frac{tg^6 5x}{30} + c$

PROBLEMA 5.

$$\int\frac{sen^2x\, dx}{cos^6x} = \int\frac{sen^2x\, dx}{cos^2x\, cos^4x} = \int tg^2x\, sec^4x\, dx = \int tg^2x\, sec^2x sec^2x\, dx =$$

$=\int tg^2(1 + tg^2x)sec^2x\, dx = \int tg^2x sec^2x\, dx + \int tg^4x\, sec^2x\, dx =$

$u = tgx\;; du = sec^2x\, dx$

$=u^2 du + \int u^4 du = \frac{u^3}{3} + \frac{u^5}{5} = \frac{tg^3x}{3} + \frac{tg^5x}{5} + c$

PROBLEMA 6.

$$\int\left(\frac{tg\phi}{ctg\phi}\right)^3 d\phi = \int\frac{tg\theta\cdot tg^2\theta\, d\theta}{ctg^2\theta\cdot ctg\,\theta} = \int tg\theta\; tg^2\theta\; tg^2\; tg\theta\; d\theta = \int tg^6\theta\; d\theta = \int tg^4\theta\; tg^2\theta\; d\theta =$$

$$= \int (tg^2\theta)^2 tg^2\theta d\theta = \int (sec^2\theta - 1)^2 tg^2\theta d\theta = \int (sec^4\theta - 2sec^2\theta + 1)\, tg^2\theta d\theta$$

$$= \int tg^2\theta sec^2\theta sec^2\theta\, d\theta - 2\int tg^2\theta\, sec^2\theta\, d\theta + \int tg^2\theta\, d\theta$$

$v = tg\theta \;\; ; dv = sec^2\theta\, d\theta$

$$= \int tg^2\theta(1 + tg^2\theta)sec^2\theta\, d\theta - 2\int tg^2\theta\, sec^2\theta\, d\theta + \int (sec^2\theta - 1)d\theta$$

$$= \int tg^2\theta\, sec^2\theta d\theta + \int tg^4\theta\, sec^2\, d\theta - 2\int tg^2\theta sec^2\theta\, d\theta + \int sec^2\theta\, d\theta - \int d\theta$$

$$= \int tg^4\theta sec^2\theta\, d\theta - \int tg^2\theta sec^2\theta\, d\theta + \int sec^2\theta\, d\theta - \int d\theta = \int v^4 dv - \int v^2 dv + tg\theta - \theta + c$$

$$\frac{v^5}{5} - \frac{v^3}{3} + tg\,\theta - \theta + c = \frac{1}{5}tg^5\,\theta - \frac{1}{3}tg^3\,\theta + tg\,\theta\, - \theta + c$$

PROBLEMA7.

$$\int \frac{sen^5 y\, dy}{\sqrt{cos\, y}} = \int \frac{sen^4 sen\, y\, dy}{\sqrt{cos\, y}} = \int \frac{(1-cos^2)^2\, sen\, y\, dy}{\sqrt{cos\, y}} =$$

$$\int \frac{(1-2\,cos^2 y + cos^4)\, sen\, y\, dy}{\sqrt{cos\, y}} = \int \frac{sen\, y\, dy}{\sqrt{cos\, y}} - \int \frac{2\,cos^2 y\, sen\, y\, dy}{\sqrt{cos\, y}} + \int \frac{cos^4\, sen\, y\, dy}{\sqrt{cos\, y}} =$$

$$\int cos^{-\frac{1}{2}} y\, sen\, y\, dy - 2\int cos^{\frac{3}{2}} y\, seny\, dy + \int cos^{\frac{7}{2}} y\, seny\, dy =$$

$u = cosy \qquad du = -seny\, dy \qquad -du = seny\, dy$

$$= \int u^{-\frac{1}{2}}(-du) - 2\int u^{\frac{3}{2}}\,(-du) + \int u^{\frac{7}{2}}(-du)$$

$$= -\int u^{-\frac{1}{2}}\, du + 2\int u^{\frac{3}{2}} du - \int u^{\frac{7}{2}}\, du = -\frac{u^{\frac{1}{2}}}{\frac{1}{2}} + 2\cdot\frac{u^{\frac{5}{2}}}{\frac{5}{2}} - \frac{u^{\frac{9}{2}}}{\frac{9}{2}}$$

$$= -2\sqrt{u} + \frac{4}{5}\sqrt{u^5} - \frac{9}{2}\sqrt{u^9} = -2\sqrt{cosy} + \frac{4}{5}u^2\sqrt{u} - \frac{2}{9}u^4\sqrt{u} + c$$

$$= -\; 2\sqrt{cos\, y} + \frac{4}{5}u^2\sqrt{u} - \frac{2}{9}u^4\sqrt{u} + c$$

$$= -\; 2\sqrt{cosy} + \frac{4}{5}cos^2 y\sqrt{cosy} - \frac{2}{9}cos^4 y\sqrt{cosy} + c +$$

$$= -\; 2\sqrt{cosy}\,(1 - \frac{2}{5}cos^2 y + \frac{1}{9}cos^4 y) + c$$

PROBLEMA 8

$$\int \cot^2 x \csc^3 x dx$$

Solución -

$$\int \cot^2 x \csc^3 x dx = \int (\csc^2 x - 1)\csc^3 x dx = \int (\csc^5 x - \csc^3 x) dx = \int \csc^5 x dx - \int \csc^3 x dx \qquad (1)$$

- $$\int \csc^5 x dx = \int \csc^3 x \csc^2 x dx$$

sea

$$u = \csc^3 x, \Rightarrow du = -3\csc^3 x \cot x$$

$$dv = \csc^2 x dx, \Rightarrow v = -\cot x$$

de tal modo, que al hacer las sustituciones respectivas en la fórmula de integración por partes, queda:

$$\int \csc^5 x dx = -\cot x \csc^3 x - 3\int \cot^2 x \csc^3 x dx = -\cot x \csc^3 x - 3\int (\csc^2 x - 1)\csc^3 x dx,$$

$$\Rightarrow \quad \int \csc^5 x dx = -\cot x \csc^3 x - 3\int \csc^5 x dx + 3\int \csc^3 x dx \Leftrightarrow 4\int \csc^5 x dx = -\cot x \csc^3 x + 3\int \csc^3 x dx,$$

$$\Rightarrow \quad \int \csc^5 x dx = -\frac{1}{4}\cot x \csc^3 x + \frac{3}{4}\int \csc^3 x dx \qquad (2)$$

Sustituyendo (2) en (1), se obtiene:

$$\int \cot^2 x \csc^3 x dx = -\frac{1}{4}\cot x \csc^3 x + \frac{3}{4}\int \csc^3 x dx - \int \csc^3 x dx = -\frac{1}{4}\cot x \csc^3 x - \frac{1}{4}\int \csc^3 x dx \qquad (3)$$

- $$\int \csc^3 x dx = \int \csc x \csc^2 x dx$$

sea

$$u = \csc x, \Rightarrow du = -\csc x \cot x$$

$$dv = \csc^2 x dx, \Rightarrow v = -\cot x$$

de tal modo, que al hacer las sustituciones respectivas en la fórmula de integración por partes, queda:

$$\int \csc^3 x dx = -\cot x \csc x - \int \cot^2 x \csc x dx = -\cot x \csc x - \int (\csc^2 x - 1)\csc x dx = -\cot x \csc x - \int \csc^3 x dx + \int \csc x dx,$$

$$\Rightarrow \quad 2\int \csc^3 x dx = -\cot x \csc x + \ln|\csc x - \cot x| \Leftrightarrow \int \csc^3 x dx = -\frac{1}{2}\cot x \csc x + \frac{1}{2}\ln|\csc x - \cot x| \qquad (4)$$

sustituyendo (4) en (3) y agregando la constante de integración, se obtiene:

$$\int \cot^2 x \csc^3 x dx = -\frac{1}{4}\cot x \csc^3 x - \frac{1}{4}\left(-\frac{1}{2}\cot x \csc x + \frac{1}{2}\ln|\csc x - \cot x|\right) + C;$$

$$\therefore \quad \boxed{\int \cot^2 x \csc^3 x dx = -\frac{1}{4}\cot x \csc^3 x + \frac{1}{8}\cot x \csc x - \frac{1}{8}\ln|\csc x - \cot x| + C.}$$

INTEGRALES QUE SE RESUELVEN EMPLEANDO CAMBIO DE VARIABLE

PROBLEMA 1.

$$\int \frac{dx}{\sqrt[3]{x}+\sqrt{x}} =$$

Hacemos la sustitución :

$$u^6 = x$$

ya que " 6 " es el m.c.m de los índices de ambos radicales :2 y 3

$$u = x^{1/6}$$

$dx = 6u^5 du$; Además

$$\sqrt[3]{x} = u^2 \quad \sqrt{x} = u^3$$

$$\int \frac{dx}{\sqrt[3]{x}+\sqrt{x}} = \int \frac{6u^5 du}{u^2+u^3} = 6\int \frac{u^3 du}{1+u}$$

Hacemos la sustitución *t= u+1* y *u=t-1* entonces *du = dt*

$$= 6\int \frac{(t-1)^3 dt}{t} = 6\int \frac{\left(t^3-3t^2+3t-1\right)dt}{t}$$

$$= 6\int \left(t^2-3t+3-\frac{1}{t}\right)dt = 6\left(\frac{t^3}{3}-\frac{3t^2}{2}+3t-\ln|t|+c\right)$$

$$= 2(u+1)^3 - 9(u+1)^2 + 18(u+1) - 6\ln|u+1| + c$$

Por lo tanto:

$$\int \frac{dx}{\sqrt[3]{x}+\sqrt{x}} = 2\left(\sqrt[6]{x}+1\right)^3 - 9\left(\sqrt[6]{x}+1\right)^2 + 18\left(\sqrt[6]{x}+1\right) - 6\ln\left|\sqrt[6]{x}+1\right| + c$$

INTENTA REALIZAR LA COMPROBACIÓN ¡¡¡

PROBLEMA 2. ¡MUY DIFÍCIL!

$\int (x^3 + x^6)\sqrt[3]{x^3 + 2}\,\text{dx}$ Se factoriza x y se introduce bajo el radical :

$$= \int x(x^2 + x^5)\sqrt[3]{x^3 + 2}\,dx$$

$$= \int (x^2 + x^5)\sqrt[3]{x^3(x^3 + 2)}\,\text{dx} = \int (x^2 + x^5)\sqrt[3]{x^6 + 2x^3}\,\text{dx}$$

$$\text{u} = x^6 + 2x^3$$

$$\text{du} = (6x^5 + 6x^2)\text{dx}$$

$$= 6(x^5 + x^2)\,\text{dx}$$

$$\frac{du}{6} = (x^5 + x^2)\text{dx}$$

$$= \int u^{\frac{1}{3}} \cdot \frac{du}{6} = \frac{1}{6}\int u^{\frac{1}{3}}\,\text{du} = \frac{1}{6} \cdot \frac{u^{\frac{4}{3}}}{\frac{4}{3}} = \frac{1}{8}\sqrt[3]{(x^6 + 2x^3)^4} + \text{c}$$

$$= \frac{1}{8}\sqrt[3]{[x^3(x^3 + 2)]^4} = \frac{1}{8}\sqrt[3]{x^{12}(x^3 + 2)^4} = \frac{x^4}{8}\sqrt[3]{(x^3 + 2)^4} + c$$

$$= \boldsymbol{\frac{x^4}{8}(x^3 + 2)^{\frac{4}{3}} + c}$$

COMPROBACIÓN:

$$\text{d}\,\frac{x^4}{8}(x^3 + 2)^{\frac{4}{3}} = \frac{x^4}{8} \cdot \frac{4}{3}(x^3 + 2)^{\frac{1}{3}}(3x^2) + (x^3 + 2)^{\frac{4}{3}}\,\frac{1}{8} \cdot 4x^3$$

$$= \frac{x^6}{2}\sqrt[3]{x^3 + 2} + \frac{1}{2}x^3\sqrt[3]{(x^3 + 2)^4} = \frac{1}{2}x^3\sqrt[3]{x^3 + 2}\,(x^3 + x^3 + 2)$$

$$= \frac{1}{2}x^3\sqrt[3]{x^3 + 2}\,(2x^3 + 2) = (x^3 + 1)x^3\sqrt[3]{x^3 + 2}$$

$$= \boldsymbol{(x^6 + x^3)\sqrt[3]{x^3 + 2}}$$

INTEGRALES QUE SE RESUELVEN EMPLEANDO INTEGRACIÓN POR PARTES

PROBLEMA 1.

$$\int \left(5\sqrt{t}+2\right)^{-2} t^{-1/2}dt$$

Solución -

$$\int \left(5\sqrt{t}+2\right)^{-2} t^{-1/2}dt = \int \left(5t^{1/2}+2\right)^{-2} t^{-1/2}dt \quad (1)$$

Sea

$$u = 5t^{1/2}+2, \;\Rightarrow\; du = \left(\frac{1}{2}\times 5t^{1/2-1}+0\right)dt = \frac{5}{2}t^{-1/2}dt \;\Leftrightarrow\; t^{-1/2}dt = \frac{2}{5}du \quad (2)$$

$$\Rightarrow \quad \int \left(5\sqrt{t}+2\right)^{-2} t^{-1/2}dt = \frac{2}{5}\int u^{-2}du \qquad \{(2) \text{ en } (1)\},$$

$$\Rightarrow \quad \int \left(5\sqrt{t}+2\right)^{-2} t^{-1/2}dt = \frac{2}{5}\left(\frac{1}{-2+1}u^{-2+1}+c\right) = \frac{2}{5}\left(\frac{1}{-1}u^{-1}+c\right);$$

$$\therefore \quad \int \left(5\sqrt{t}+2\right)^{-2} t^{-1/2}dt = -\frac{2}{5}\left(5\sqrt{t}+2\right)^{-1}+C \qquad \left\{u = 5t^{1/2}+2 \;\text{ y }\; C = \frac{2}{5}c\right\}.$$

PROBLEMA 2.

$$\int \frac{e^{x-5}-e^{3+x}}{e^{1-x}}dx$$

Solución -

$$\int \frac{e^{x-5}-e^{3+x}}{e^{1-x}}dx = \int \left(\frac{e^{x-5}}{e^{1-x}}-\frac{e^{3+x}}{e^{1-x}}\right)dx = \int \left(e^{x-5-(1-x)}-e^{3+x-(1-x)}\right)dx =$$

$$\int \left(e^{2x-6}-e^{2x+2}\right)dx = \int e^{2x-6}dx - \int e^{2x+2}dx \quad (3)$$

Sea

$$\left.\begin{aligned} u &= 2x-5, \;\Rightarrow\; du = 2dx \;\Leftrightarrow\; dx = \frac{1}{2}du \\ v &= 2x+2, \;\Rightarrow\; dv = 2dx \;\Leftrightarrow\; dx = \frac{1}{2}dv \end{aligned}\right\} \quad (4)$$

sustituyendo (4) en (3), se obtiene: $\displaystyle\int \frac{e^{x-5}-e^{3+x}}{e^{1-x}}dx = \frac{1}{2}\int e^{u}du - \frac{1}{2}\int e^{v}dv = \frac{1}{2}e^{u} - \frac{1}{2}e^{v}+C$

$$\Leftrightarrow \int \frac{e^{x-5}-e^{3+x}}{e^{1-x}}dx \boxed{= \frac{1}{2}e^{2x-5} - \frac{1}{2}e^{2x+2}+C.}$$

PROBLEMA 3.

$$\int e^{\sqrt{x}}dx = 2\int \frac{\sqrt{x}}{2\sqrt{x}}e^{\sqrt{x}}dx = 2\int \sqrt{x}e^{\sqrt{x}}\frac{1}{2\sqrt{x}}dx$$

Sea $w = \sqrt{x}, \quad \Rightarrow \quad dw = \dfrac{1}{2\sqrt{x}}dx;$

$$\int e^{\sqrt{x}}dx = 2\int we^{w}dw \quad (1)$$

Ahora, sea:

$$u = w, \quad \Rightarrow \quad du = dw$$

$$dv = e^{w}dw, \quad \Rightarrow \quad v = e^{w}$$

Aplicando la fórmula de integración por partes, se obtiene:

$$\int we^{w}dw = we^{w} - \int e^{w}dw = we^{w} - e^{w} + c_1 \quad (2)$$

De tal manera que:

$$2\int we^{w}dw = 2(we^{w} - e^{w} + c_1) = 2we^{w} - 2e^{w} + c \quad (3)$$

Pero, $w = \sqrt{x}$; por lo tanto:

$$\int e^{\sqrt{x}}dx = 2\sqrt{x}e^{\sqrt{x}} - 2e^{\sqrt{x}} + c.$$

PROBLEMA 4.

$$\int \frac{\operatorname{sen} x dx}{1+\cos^2 x} \quad (1)$$

Sea

$$u = \cos x, \quad \Rightarrow \quad du = -\operatorname{sen} x dx \Leftrightarrow -du = \operatorname{sen} x dx \quad (2)$$

Sustiutuyendo (2) en (1), se obtiene:

$$\int \frac{-du}{1+u^2} = -\int \frac{du}{1+u^2} = \tan^{-1} u + C;$$

$$\therefore \quad \int \frac{\operatorname{sen} x dx}{1+\cos^2 x} = \tan^{-1}(\cos x) + C.$$

PROBLEMA 5.

- Demostrar la siguiente igualdad :

$$\int sen^n x dx = -\frac{sen^{n-1} x \cos x}{n} + \frac{n-1}{n}\int sen^{n-2} x dx$$

Solución:

$$\int sen^n x dx = \int sen^{n-1} x sen x dx$$

Proponiendo: u= $sen^{n-1} x$

Dv= $sen x dx$

$$\int sen^n x dx = -\cos x sen^{n-1} x + (n-1)\int \cos^2 x sen^{n-2} x dx$$

$$= -\cos x sen^{n-1} x + (n-1)\int sen^{n-2} x dx - (n-1)\int sen^n x dx$$

Agrupando se tiene:

$$\int sen^n x dx = -\frac{sen^{n-1} x \cos x}{n} + \frac{n-1}{n}\int sen^{n-2} x dx \quad \ldots\ldots$$ ***Así queda demostrado***

PROBLEMA 6.

$$\int e^{3x} Sen\frac{x}{3}dx = -3e^{3x} Cos\frac{x}{3} + 9\int e^{3x} Cos\frac{x}{3}dx$$

$$u = e^{3x} \qquad dv = Sen\frac{x}{3}dx \qquad ; \qquad u = e^{3x} \qquad dv = Cos\frac{x}{3}dx$$

$$du = 3e^{3x}dx \qquad v = -3Cos\frac{x}{3} \qquad ; du = 3e^{3x}dx \qquad v = 3Sen\frac{x}{3}$$

$$= -3e^{3x} Cos\frac{x}{3} + 27e^{3x} Sen\frac{x}{3} - 81\int e^{3x} Sen\frac{x}{3}dx$$

$$= \frac{3}{82}e^{3x}\left[9Sen\frac{x}{3} - Cos\frac{x}{3}\right] + C$$

PROBLEMA 7.

$$\int x^n \ln x dx =$$

$u = \ln x$	$du = \frac{dx}{x}$	$dv = x^n dx$	$v = \int x^n dx = \frac{x^{n+1}}{n+1}$

$$= \frac{x^{n+1}}{n+1}\ln x - \int \frac{x^{n+1}}{n+1}\left(\frac{dx}{x}\right)$$
$$= \frac{x^{n+1}}{n+1}\ln x - \frac{1}{n+1}\int x^{n+1}\left(\frac{dx}{x}\right)$$
$$= \frac{x^{n+1}}{n+1}\ln x - \frac{1}{n+1}\int \left(x^{n+1-1}\right)dx$$
$$= \frac{x^{n+1}}{n+1}\ln x - \frac{l}{n+1}\int x^n dx$$
$$= \frac{x^{n+1}}{n+1}\ln x - \frac{l}{n+1}\left(\frac{x^{n+1}}{n+1}\right) + c$$
$$= \frac{x^{n+1}}{n+1}\ln x - \frac{x^{n+1}}{(n+1)^2} + c$$
$$= \frac{x^{n+1}}{n+1}\left(\ln x - \frac{1}{n+1}\right) + c$$

PROBLEMA 8.

$$\int x\sqrt{1-x}\ dx =$$

Sea u= x ; du= dx

dv= $\sqrt{1-x}\ dx$; $\int dv = \int (1-x)^{1/2}$

w= $(1-x)$; $\frac{dw}{dx} = -1$; $dw = -dx$; $-dw = dx$

$$\int dv = \int w^{\frac{1}{2}}(-dw)$$

V= $-\int w^{\frac{1}{2}}\ dw$

V= $-\frac{w^{\frac{3}{2}}}{\frac{3}{2}} = -\frac{2}{3}w^{\frac{3}{2}} = -\frac{2}{3}(1-x)^{\frac{3}{2}}$

$$\int x\sqrt{1-x}\ dx = x\left(-\frac{2}{3}\ (1-x)^{\frac{3}{2}}\right) - \int \left(-\frac{2}{3}(1-x)^{\frac{3}{2}}\right)\ dx$$

$$= -\frac{2x}{3}(1-x)^{\frac{3}{2}} + \frac{2}{3}\int (1-x)^{\frac{3}{2}}\ dx$$

$$-\frac{2x}{3}(1-x)^{\frac{3}{2}} - \frac{2}{3}\frac{w^{\frac{5}{2}}}{\frac{5}{2}} + C$$

$$-\frac{2X}{3}(1-X)^{\frac{3}{2}} - \frac{4}{15}(1-X)^{\frac{5}{2}} + C$$

PROBLEMA 9

$$\int xarctanxdx$$

u= arctanx ; $\frac{du}{dx} = \frac{1}{1+x^2}$; $du = \frac{dx}{1+x^2}$

dv = xdx ; *v=* $\int dv = \int xdx = \frac{x^2}{2}$

$$\int xarctanxdx = arctan \times \frac{x^2}{2} - \int \frac{x^2}{2}\frac{dx}{1+x^2}$$

$$= arctanx\,\frac{x^2}{2} - \frac{1}{2}\int \frac{x^2}{1+x^2}dx$$

Haciendo la división:

$$= arctan\,\frac{x^2}{2} - \frac{1}{2}\int \left(1 - \frac{1}{x^2-1}\right)dx$$

$$= \frac{x^2}{2}arctanx - \frac{1}{2}\int dx + \frac{1}{2}\int \frac{dx}{x^2+1}$$

$$\frac{x^2}{2}arctanx - \frac{x}{2} + \frac{1}{2}\left[\frac{1}{1}arctanx\right] + C$$

$$= \frac{x^2}{2}arctanx - \frac{x}{2} + \frac{1}{2}arctanx + C$$

PROBLEMA 10.

$\int 3x^2 e^{-4x} dx =$

Sea u= $3x^2$; $du = 6xdx$

$dv = e^{-4x}dx$; $v = \int dv = \int e^{-4x}dx$

$w = -4x$; $\frac{dw}{dx} = -4$; $\frac{dw}{-4} = dx$

$= \int e^w \times \frac{dw}{-4} = -\frac{1}{4}\int e^w dw = -\frac{1}{4}e^w = -\frac{1}{4}e^{-4x}$

$\int 3x^2 e^{-4x}dx = 3x^2\left(-\frac{1}{4}e^{-4x}\right) - \int\left(-\frac{1}{4}e^{-4x}\right)6xdx$

$= -\frac{3}{4}x^2 e^{-4x} + \frac{2}{3}\int xe^{-4x}dx$

Integrando por partes

$u = x$; $du = dx$

$dv = e^{-4x}dx$; $v = -\frac{1}{4}e^{-4x}$

$= -\frac{3}{4}x^2 e^{-4x} + \frac{3}{2}\left[x\left(-\frac{1}{4}e^{-4x}\right) - \int -\frac{1}{4}e^{-4x}\,dx\right]$

$= -\frac{3}{4}x^2 e^{-4x} - \frac{3}{8}xe^{-4x} + \frac{3}{8}\int e^{-4x}dx$

$= -\frac{3}{4}x^2 e^{-4x} - \frac{3}{8}xe^{-4x} + \frac{3}{8}\left(-\frac{1}{4}e^{-4x}\right) + C$

$= -\frac{3}{4}x^2 e^{-4x} - \frac{3}{8}xe^{-4x} - \frac{3}{32}e^{-4x} + C$

$= \frac{1}{e^{4x}}\left[-\frac{3x^2}{4} - \frac{3x}{8} - \frac{3}{32}\right] + C$

PROBLEMA 11.

$$\int \frac{4x^2\,dx}{\sqrt{1-x}} =$$

$$= \int 4x^2(1-x)^{-\frac{1}{2}}dx = 4\int x^2(1-x)^{-\frac{1}{2}}dx$$

Sea $\quad u = x^2 \quad ; \quad du = 2xdx$

$$dv = (1-x)^{-\frac{1}{2}} \quad ; \quad v = \int dv = \int (1-x)^{-\frac{1}{2}} = -2(1-x)^{\frac{1}{2}}$$

$$\int \frac{4x^2}{\sqrt{1-x}}dx = 4\left[x^2\left(-2(1-x)^{\frac{1}{2}} - \int -2(1-x)^{\frac{1}{2}}(2xdx)\right)\right]$$

$$= 4\left[-2x^2(1-x)^{\frac{1}{2}} + 4\int (1-x)^{\frac{1}{2}}xdx\right]$$

$$= -8x^2(1-x)^{\frac{1}{2}} + 16\int x(1-x)^{\frac{1}{2}}dx$$

Integrando esta ultima por partes:

$$u = x \quad ; \quad du = dx$$

$$dv = (1-x)^{\frac{1}{2}}dx \quad ; \quad v = \int dv = \int (1-x)^{\frac{1}{2}}dx = -\frac{2}{3}(1-x)^{\frac{3}{2}}$$

$$= -8x^2(1-x)^{\frac{1}{2}} + 16\left[-\frac{2}{3}x(1-x)^{\frac{3}{2}} - \int -\frac{2}{3}(1-x)^{\frac{3}{2}}dx\right]$$

$$= -8x^2\sqrt{1-x} - \frac{32}{3}x(1-x)^{\frac{3}{2}} + \frac{32}{3}\int (1-x)^{\frac{3}{2}}dx$$

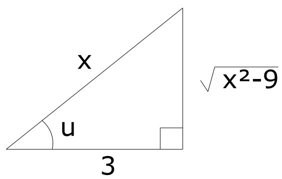
$$= -8x^2\sqrt{1-x} - \frac{32}{3}x(1-x)^{\frac{3}{2}} - \frac{64}{15}(1-x)^2\sqrt{1-x} + C$$

ACTIVIDAD III.PROBLEMAS PROPUESTOS

INTEGRALES QUE SE RESUELVEN EMPLEANDO *INTEGRACIÓN POR SUSTITUCIÓN TRIGONOMÉTRICA*

PROBLEMA 1.

$$\int \frac{5x\,dx}{\sqrt{x^2-9}} = 5\int \frac{x\,dx}{\sqrt{x^2-9}}$$



$$Secu = \frac{x}{3}$$

$$x = 3\,secu$$

$$dx = 3\,secu\;tgu\;du$$

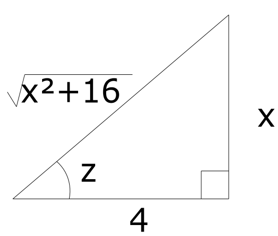
$$= 5\int \frac{3secu \cdot 3secutgu\,du}{\sqrt{9sec^2 - 9}} = 45\int \frac{sec^2u\,tgu\,du}{\sqrt{9(sec^2-1)}} = \frac{45}{3}\int \frac{sec^2utgu\,du}{\sqrt{sec^2u-1}}$$

$$= 15\int \frac{sec^2utgu\,du}{\sqrt{tg^2u}} = 15\int sec^2udu = 15tgu + c = 15\left(\frac{\sqrt{x^2-9}}{3}\right)+C$$

$$= 5\sqrt{x^2-9} + c$$

PROBLEMA 2

$$\int \frac{x^2dx}{x^2+16} =$$



$$tg\,z = \frac{x}{4}$$

$$x = 4\,tgz$$

$$dx = 4\,sec^2z\,dz$$

$$\int \frac{16tg^2z \cdot 4sec^2z\,dz}{16=tg^2z + 16} = \int \frac{16tg^2z \cdot 4sec^2z\,dz}{16(tg^2z+1)} = 4\int \frac{tg^2zsec^2z\,dz}{sec^2z} =$$

$$4\int tg^2z\,dz = 4\int (sec^2z-1)dz = 4\left[\int sec^2z\,dz - \int dz\right] = 4tg\,z - 4z + c$$

$$= x - 4arctg\,\frac{x}{4} + c$$

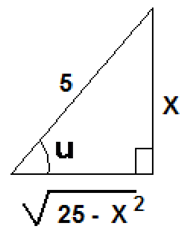
PROBLEMA 3

$$\int \frac{5dx}{\sqrt{25-x^2}} = 5\int \frac{5cosu\,du}{\sqrt{25-25sen^2u}} = 25\int \frac{cosu\,du}{\sqrt{25(1-sen^2u}} = 25\int \frac{cosu\,du}{5\sqrt{1-sen^2u}} = 5\int \frac{5cosu\,du}{\sqrt{25-25sen^2u}} =$$

$Sen\ u = \frac{x}{5}$

$x = 5\ senu$

$dx = 5\cos u\ du$



$$5\int \frac{cosu\,du}{\sqrt{25(1-sen^2u}} = 25\int \frac{cosu\,du}{5\sqrt{1-sen^2u}} = 5\int \frac{cosu\,du}{\sqrt{cos^2u}} = 5\int du = = 5u + c =$$ ***al llegar a ésta***

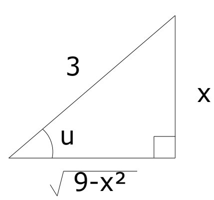
parte debemos pensar en quién es u ? *y al observar el triángulo comprendemos que* ***u*** *es el*

ángulo cuyo seno vale : $\frac{x}{5}$ *, lo cual se escribe:* ***arc sen*** $\frac{x}{5}$

∴ ***el resultado final es: 5 arcsen*** $\frac{x}{5}$***+c***

PROBLEMA 4

$$\int \frac{x^2\,dx}{\sqrt{9-x^2}} =$$



$Sen\ u = \frac{x}{3}$

$x = 3senu$

$dx = 3cosu\ du$

$$= \int \frac{9sen^2u\ \cdot\ 3cosu\,du}{\sqrt{9-\ 9sen^2u}} = \int \frac{9sen^2\ \cdot\ 3cosu\,du}{\sqrt{9(1-\ sen^2u)}} = \int \frac{9sen^2u\cdot\ 3cosu\,du}{3\sqrt{1-\ sen^2u}} =$$

$$9\int \frac{sen^2u\,cosu\,du}{\sqrt{cos^2u}} = 9\int sen^2u\,du = 9\int \frac{1}{2}(1-cos2u)\,du =$$

$$\frac{9}{2}\int du - \frac{9}{2}\int cos2u\ du$$

$$v = 2u$$

$$dv = 2du$$

$$du = \frac{dv}{2}$$

$$\frac{9}{2}u - \frac{9}{2}\cdot\frac{1}{2}\int cosv\,dv = \frac{9}{2}arc\,sen\frac{x}{3} - \frac{9}{4}senv + c$$

$$= \frac{9}{2}arc\,sen\frac{x}{3} - \frac{9}{4}sen\,2v + c = \frac{9}{2}arc\,sen\frac{x}{3} - \frac{9}{2}\cdot\frac{x}{3}\cdot\frac{\sqrt{9-x^2}}{3} + c$$

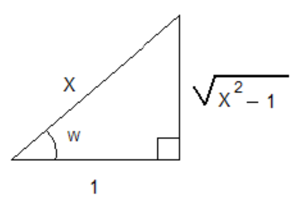
$$= \frac{9}{2}arc\,sen\frac{x}{3} - \frac{1}{2}x\sqrt{9-x^2} + c$$

PROBLEMA 5

Después de todos los problemas que hemos resuelto juntos estás obligado a resolverlo tú. Inténtalo y consíguelo !

PROBLEMA 6

$$\int\frac{dx}{x^2-1} =$$



$$Sec\,w = x$$

$$dx = secw\,tgw\,dw$$

$$= \int\frac{\mathbf{secw\ tgw\ dw}}{\mathbf{sec^2w - 1}} = \int\frac{\mathbf{secw\ tgw\ dw}}{\mathbf{tg^2w}} = \int\frac{\mathbf{secw\ dw}}{\mathbf{tg\ w}} = \int\frac{\frac{\mathbf{1}}{\mathbf{cos\ w}}\mathbf{dw}}{\frac{\mathbf{sen\ w}}{\mathbf{cos\ w}}}$$

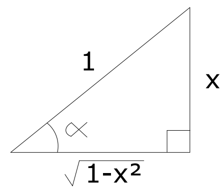
$$= \int csc\,w\,dw = ln|cscx - ctg\,x| + c$$

PROBLEMA 7

Después de todos los problemas que hemos resuelto juntos estás obligado a resolverlo tú. Inténtalo y consíguelo !

PROBLEMA 8

$$\int \frac{\sqrt{1-x^2}}{x}\,dx=$$



$\operatorname{sen}\alpha = \frac{x}{1}$; $x = \operatorname{sen}\alpha$

$dx = \cos\alpha\, d\alpha$

$$=\int \frac{\sqrt{1-sen^2\alpha}}{sen\,\alpha}\cos\alpha\, d\alpha \;=\int \frac{cos\alpha}{sen\,\alpha}\; cos\alpha\, d\alpha$$

$$=\int \frac{cos^2\alpha\, d\alpha}{sen\,\alpha} = \int \frac{1-sen^2\alpha}{sen\,\alpha}\,\alpha = \int \frac{1}{sen\,\alpha}\,d\alpha - \int sen\,\alpha\, d\alpha = \int \csc x\, dx - (-\cos\alpha)$$

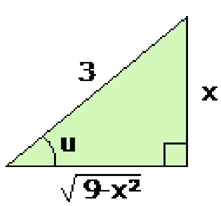
$$= Ln\left|\frac{1}{x} - \frac{\sqrt{1-x^2}}{x}\right| + \sqrt{1-x^2} + c \;=\; \boxed{Ln\left|\frac{1-\sqrt{1-x^2}}{x}\right| + \sqrt{1-x^2} + C}$$

PROBLEMA 9

Después de todos los problemas que hemos resuelto juntos estás obligado a resolverlo tú. Inténtalo y consíguelo !

PROBLEMA 10

$$\int \frac{x^2\,dx}{\sqrt{(9-x^2)^3}} = \int \frac{9\,sen^2 u \cdot 3\cos u\, du}{\sqrt{(9-9\,sen^2 u)^3}} = 27\int \frac{sen^2 u\, du \cdot \cos u\, du}{(9-9sen^2\,u)\sqrt{9-9\,sen^2\,u}} =$$



$\operatorname{sen} u = \frac{x}{3}$

$x = 3sen\, u$

$dx = 3\cos u\, du$

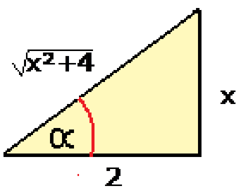
$$= 27\int \frac{sen^2\, u\, \cos u\, du}{9(1-sen^2\, u)\sqrt{9(1-sen^2 u)}} = \frac{27}{9\cdot 3}\int \frac{sen^2\, u \cos u\, du}{(1-sen^2 u)\sqrt{1-sen^2 u}} =$$

$$\int \frac{sen^2 \cos u\, du}{cos^2 u\sqrt{cos^2 u}} = \int \frac{sen^2\, u\, du}{cos^2\, u} = \int tg^2\, u\, du = \int (sec^2\, u - 1)du =$$

$$= \int sec^2\, u\, du - \int du = tg\, u - u + c = \;= \boxed{\frac{x}{\sqrt{9-x^2}} - arc\,sen\,\frac{x}{3} + c}$$

PROBLEMA 11

$$\int \frac{x^2\,dx}{(x^2+4)^2} = \int \frac{4tg^2\,\alpha \cdot 2\,sec^2\,\alpha\,d\alpha}{(4tg^2\,\alpha + 4)^2} = \int \frac{8tg^2\alpha\,sec^2\,\alpha\,d\alpha}{16(tg^2\,\alpha + 1)^2} = \frac{1}{2}\int \frac{tg^2\,\alpha\,sec^2\,\alpha\,d\alpha}{(sec^2\,\alpha)^2} =$$



$$\mathrm{tg}\,u = \frac{x}{2}$$

$$x = 2tg\,\alpha$$

$$dx = 2\,sec^2\,\alpha\,d\alpha$$

$$= \frac{1}{2}\int \frac{tg^2\,\alpha\,d\alpha}{sec^2\alpha} = \frac{1}{2}\int Sen^2\,\alpha\,d\alpha = \frac{1}{2}\int \frac{1}{2}\,(1-\cos 2\,\alpha)\,d\alpha =$$

$$= \frac{1}{4}\int d\alpha - \frac{1}{4}\int \cos 2\alpha\,d\alpha = \frac{1}{4}\,\alpha - \frac{1}{4}\int \cos u\,\frac{du}{2} \qquad u = 2 \quad du = 2d\alpha \;, \quad d\alpha = \frac{du}{2}$$

$$= \frac{1}{4}\,\alpha - \frac{1}{8}\int \cos u\,du = \frac{1}{4}\,\alpha - \frac{1}{8}\,sen\,u + c$$

$$= \frac{1}{4}\,\alpha - \frac{1}{8}\,sen\,2\alpha = \frac{1}{4}\,\alpha - \frac{1}{8}\,2sen\,\alpha\,cos\alpha + c$$

$$= \frac{1}{4}\,\alpha - \frac{1}{4}\,sen\,\alpha\,cos\alpha + c$$

$$= \frac{1}{4}\,arc\,tg\,\frac{x}{2} - \frac{1}{4}\left(\frac{x}{\sqrt{x^2+4}} \cdot \frac{2}{\sqrt{x^2+4}}\right) + c$$

$$= \frac{1}{4}\,arc\,tg\,\frac{x}{2} - \frac{x}{\sqrt{x^2+4}} + c$$

$$\boxed{= \frac{1}{4}\,\boldsymbol{arc\,tg}\,\frac{x}{2} - \frac{x}{2x^2+8} + \boldsymbol{c}}$$

Actividad Complementaria III. Resuelve las siguientes integrales indicando planteamientos ,operaciones y resultado.

1. $\int \frac{dx}{x^2\sqrt{4-x^2}}$

En este ejercicio la expresión dentro del radical es de la forma $a^2 - u^2$; por lo que la sustitución debe ser:

$$x = 2\,\text{sen}\,\theta, \qquad -\pi/2 < \theta < \pi/2$$

$$\Rightarrow \quad dx = 2\cos\theta\, d\theta$$

De tal manera que:

$$\int \frac{dx}{x^2\sqrt{4-x^2}} = \int \frac{2\cos\theta\, d\theta}{(2\,\text{sen}\,\theta)^2\sqrt{4-(2\,\text{sen}\,\theta)^2}} = \int \frac{2\cos\theta\, d\theta}{4\,\text{sen}^2\theta\sqrt{4-4\,\text{sen}^2\theta}} = \int \frac{\cos\theta\, d\theta}{2\,\text{sen}^2\theta\sqrt{4(1-\text{sen}^2\theta)}},$$

$$\Rightarrow \quad \int \frac{dx}{x^2\sqrt{4-x^2}} = \int \frac{\cos\theta\, d\theta}{2\,\text{sen}^2\theta\cdot 2\sqrt{\cos^2\theta}} = \int \frac{\cos\theta\, d\theta}{4\,\text{sen}^2\theta\cdot\cos\theta} = \frac{1}{4}\int \csc^2\theta\, d\theta = -\frac{1}{4}\cot\theta + c \qquad (1)$$

Sustituyendo estos valores en (1), se obtiene:

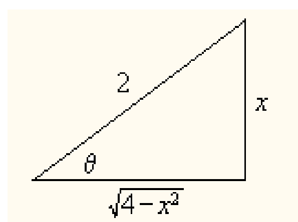
$$\int \frac{dx}{x^2\sqrt{4-x^2}} = -\frac{\sqrt{4-x^2}}{4x} + c.$$

Como $x = 2\,\text{sen}\,\theta$, entonces

$$\text{sen}\,\theta = \frac{x}{2}$$

Con estos datos, construimos el triángulo rectángulo que se observa en la figura de la derecha.
De la figura, se deduce que:

$$\cot\theta = \frac{\sqrt{4-x^2}}{x}$$



Sustituyendo estos valores en (1), se obtiene:

$$\int \frac{dx}{x^2\sqrt{4-x^2}} = -\frac{\sqrt{4-x^2}}{4x} + c.$$

2. $\displaystyle\int \frac{1}{(z^2-2z+5)^2}\,dz$

Solución:

$$\int \frac{1}{(z^2-2z+5)^2}\,dz = \int \frac{1}{(z^2-2z+1+4)^2}\,dz = \int \frac{1}{((z-1)^2+4)^2}\,dz$$

Sea

$$z-1 = 2\tan\theta, \quad \Rightarrow \quad dz = 2\sec^2\theta\, d\theta$$

De tal manera que:

$$\int \frac{1}{((z-1)^2+4)^2}\,dz = \int \frac{2\sec^2\theta\, d\theta}{(4\tan^2\theta+4)^2} = \int \frac{2\sec^2\theta\, d\theta}{(4(\tan^2\theta+1))^2} = \int \frac{2\sec^2\theta\, d\theta}{(4\sec^2\theta)^2} = \int \frac{2\sec^2\theta\, d\theta}{16\sec^4\theta},$$

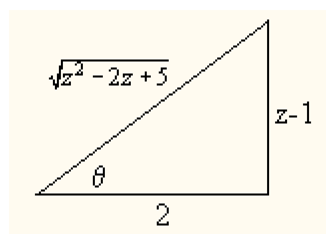
$$\Rightarrow \quad \int \frac{1}{((z-1)^2+4)^2}\,dz = \int \frac{d\theta}{8\sec^2\theta} = \frac{1}{8}\int \cos^2\theta\, d\theta = \frac{1}{16}\int (\cos 2\theta + 1)\, d\theta,$$

$$\Rightarrow \quad \int \frac{1}{((z-1)^2+4)^2}\,dz = \frac{1}{16}\cdot\frac{1}{2}\operatorname{sen} 2\theta + \frac{1}{16}\theta + c = \frac{1}{32}(2\operatorname{sen}\theta\cos\theta) + \frac{1}{16}\theta + c,$$

$$\Rightarrow \quad \int \frac{1}{(z^2-2z+5)^2}\,dz = \int \frac{1}{((z-1)^2+4)^2}\,dz = \frac{1}{16}(\operatorname{sen}\theta\cos\theta) + \frac{1}{16}\theta + c \qquad (1).$$

Como $z-1 = 2\tan\theta$, entonces

$$\tan\theta = \frac{z-1}{2} \Leftrightarrow \theta = \tan^{-1}\frac{z-1}{2}$$



($Fig.$1)

Con estos datos, construimos el triángulo rectángulo que se observa en la figura de la derecha.
De la figura, se deduce que:

$$\operatorname{sen}\theta = \frac{z-1}{\sqrt{z^2-2z+5}} \quad \text{y} \quad \cos\theta = \frac{2}{\sqrt{z^2-2z+5}}$$

Sustituyendo estos valores en (1), se obtiene:

$$\int \frac{1}{(z^2-2z+5)^2}\,dz = \frac{1}{16}\left(\frac{z-1}{\sqrt{z^2-2z+5}}\cdot\frac{2}{\sqrt{z^2-2z+5}}\right) + \frac{1}{16}\tan^{-1}\frac{z-1}{2} + c;$$

$$\therefore \quad \int \frac{1}{(z^2-2z+5)^2}\,dz = \frac{1}{8}\frac{(z-1)}{(z^2-2z+5)} + \frac{1}{16}\tan^{-1}\frac{z-1}{2} + c.$$

3. $\int \frac{1}{\sqrt{1-x^2}}\,dx$

Solución:

En este ejercicio la expresión dentro del radical es de la forma a^2-u^2; por lo que la sustitución debe ser:

$x = \operatorname{sen}\theta, \qquad -\pi/2 < \theta < \pi/2$

$\Rightarrow \quad dx = \cos\theta\, d\theta$

De tal manera que:

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \int \frac{\cos\theta\, d\theta}{\sqrt{1-\operatorname{sen}^2\theta}} = \int \frac{\cos\theta\, d\theta}{\sqrt{\cos^2\theta}} = \int \frac{\cos\theta\, d\theta}{\cos\theta} = \int d\theta = \theta + C$$

Como $x = \operatorname{sen}\theta \Leftrightarrow \theta = \operatorname{sen}^{-1}x$; concluimos que:

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \operatorname{sen}^{-1}x + c.$$

4. $\int \frac{\sqrt{25-x^2}}{x}\,dx$

Solución:

En este ejercicio la expresión dentro del radical es de la forma a^2-u^2; por lo que la sustitución debe ser:

$x = 5\operatorname{sen}\theta, \qquad -\pi/2 < \theta < \pi/2$

$\Rightarrow \quad dx = 5\cos\theta\, d\theta$

De tal manera que:

$$\int \frac{\sqrt{25-x^2}}{x}\,dx = \int \frac{\sqrt{25-(5\operatorname{sen}\theta)^2}}{5\operatorname{sen}\theta}5\cos\theta\, d\theta = \int \frac{\sqrt{25-25\operatorname{sen}^2\theta}}{\operatorname{sen}\theta}\cos\theta\, d\theta,$$

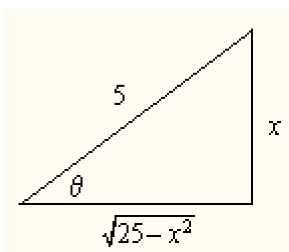
$$\Rightarrow \quad \int \frac{\sqrt{25-x^2}}{x}\,dx = \int \frac{\sqrt{25(1-\operatorname{sen}^2\theta)}}{\operatorname{sen}\theta}\cos\theta\, d\theta = \int \frac{5\cos\theta\sqrt{\cos^2\theta}}{\operatorname{sen}\theta}\,dx = \int \frac{5\cos^2\theta}{\operatorname{sen}\theta}\,d\theta,$$

$$\Rightarrow \quad \int \frac{\sqrt{25-x^2}}{x}\,dx = 5\int \frac{(1-\operatorname{sen}^2\theta)}{\operatorname{sen}\theta}\,d\theta = 5\left(\int \csc\theta\, d\theta - \int \operatorname{sen}\theta\, d\theta\right),$$

$$\Rightarrow \quad \int \frac{\sqrt{25-x^2}}{x}\,dx = 5\left(\ln\left|\csc\theta - \cot\theta\right| + \cos\theta\right) + c \qquad (1)$$

Como $x = 5\operatorname{sen}\theta$, entonces

$$\operatorname{sen}\theta = \frac{x}{5}$$



Con estos datos, construimos el triángulo rectángulo que se observa en la figura de la derecha.
De la figura, se deduce que:

$$\cos\theta = \frac{\sqrt{25-x^2}}{5}, \quad \cot\theta = \frac{\sqrt{25-x^2}}{x}, \quad \csc\theta = \frac{5}{x} \qquad (2)$$

Sustituyendo (2) en (1), se obtiene:

$$\int \frac{\sqrt{25-x^2}}{x}dx = 5\left(\ln\left|\frac{5}{x}-\frac{\sqrt{25-x^2}}{x}\right|+\frac{\sqrt{25-x^2}}{5}\right)+c;$$

$$\therefore \quad \int \frac{\sqrt{25-x^2}}{x}dx = 5\ln\left|\frac{5-\sqrt{25-x^2}}{x}\right|+\sqrt{25-x^2}+c.$$

5. $\int \sqrt{x^2+4}\,dx$

Solución:

$$\int \sqrt{x^2+4}\,dx \quad (1)$$

En este ejercicio la expresión dentro del radical es de la forma a^2+u^2; por lo que la sustitución debe ser:

$$\left.\begin{aligned} &x = 2\tan\theta \\ \Rightarrow \quad &dx = 2\sec^2\theta d\theta \end{aligned}\right\} \quad (2)$$

De tal manera que, al sustituir (2) en (1), se obtiene:

$$\int \sqrt{x^2+4}\,dx = \int \sqrt{(2\tan\theta)^2+4}\cdot 2\sec^2\theta d\theta = \int \sqrt{4\tan^2\theta+4}\cdot 2\sec^2\theta d\theta,$$

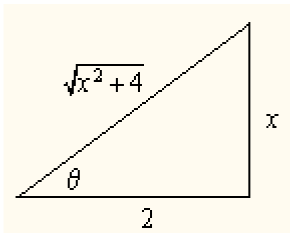
$$\Rightarrow \quad \int \sqrt{x^2+4}\,dx = \int \sqrt{4(\tan^2\theta+1)}\cdot 2\sec^2\theta d\theta = \int 2\sqrt{\sec^2\theta}\cdot 2\sec^2\theta d\theta = 4\int \sec^3\theta d\theta,$$

$$\Rightarrow \quad \int \sqrt{x^2+4}\,dx = 4\left(\frac{1}{2}\sec\theta\tan\theta+\frac{1}{2}\ln|\sec\theta+\tan\theta|\right)+c = 2\sec\theta\tan\theta+2\ln|\sec\theta+\tan\theta|+c,$$

$$\Rightarrow \quad \int \sqrt{x^2+4}\,dx = 2\sec\theta\tan\theta+\ln(\sec\theta+\tan\theta)^2+c \quad (3)$$

Como $x = 2\tan\theta$, entonces

$$\tan\theta = \frac{x}{2} \quad (4)$$



Con estos datos, construimos el triángulo rectángulo que se observa en la figura de la derecha.
De la figura, se deduce que:

$$\sec\theta = \frac{\sqrt{x^2+4}}{2} \quad (5)$$

Sustituyendo (4) y (5) en (3), se obtiene:

$$\int \sqrt{x^2+4}\,dx = 2\frac{\sqrt{x^2+4}}{2}\cdot\frac{x}{2}+\ln\left(\frac{\sqrt{x^2+4}}{2}+\frac{x}{2}\right)^2+c = \frac{x\sqrt{x^2+4}}{2}+2\ln\left(\frac{x+\sqrt{x^2+4}}{2}\right)^2+c.$$

6. $\int \frac{x}{1+x^4}dx$

Solución:

$$\int \frac{xdx}{1+x^4} \quad (1)$$

Sea

$$x^2 = \tan\theta \Leftrightarrow \theta = \tan^{-1}x^2, \;\Rightarrow\; xdx = \frac{1}{2}\sec^2\theta d\theta \quad (2)$$

Sustituyendo (2) en (1), se obtiene:

$$\int \frac{xdx}{1+x^4} = \frac{1}{2}\int \frac{\sec^2\theta d\theta}{1+\tan^2\theta} = \frac{1}{2}\int \frac{\sec^2\theta d\theta}{\sec^2\theta} = \frac{1}{2}\int d\theta = \frac{1}{2}\theta + c \quad (3)$$

Por último, sustituyendo $\theta = \tan^{-1}x^2$ en (3), se obtiene:

$$\int \frac{xdx}{1+x^4} = \frac{1}{2}\tan^{-1}x^2 + c.$$

7. $\int \frac{1}{x^2-1}dx$

Sea

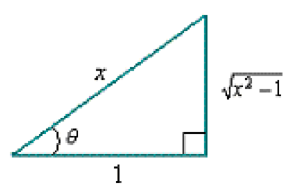
$$x = \sec\theta, \;\Rightarrow\; dx = \sec\theta\tan\theta d\theta \quad (2)$$

$$\int \frac{1}{x^2-1}dx = \int \frac{1}{\sec^2\theta - 1}\sec\theta\tan\theta d\theta \qquad \{(2) \text{ en } (1)\},$$

$$\Rightarrow \quad \int \frac{1}{x^2-1}dx = \int \frac{\sec\theta\tan\theta d\theta}{\tan^2\theta} = \int \frac{\sec\theta}{\tan\theta}d\theta = \int \csc\theta d\theta = \ln\left|\csc\theta - \cot\theta\right| + c \quad (3)$$

La figura de la derecha se construye a partir de la definición de

$\sec\theta = \frac{\text{hipotenusa}}{\text{cateto adyacente}}$ y del hecho de que $\sec\theta = x$.



A partir de dicha figura se deduce que:

$$\csc\theta = \frac{x}{\sqrt{x^2-1}}, \quad \cot\theta = \frac{1}{\sqrt{x^2-1}} \quad (4)$$

Finalmente, sustituyendo (4) en (3), se obtiene:

$$\int \frac{1}{x^2-1}dx = \ln\left|\frac{x}{\sqrt{x^2-1}} - \frac{1}{\sqrt{x^2-1}}\right| + c = \ln\left|\frac{x-1}{\sqrt{x^2-1}}\right| + c = \ln\sqrt{\frac{(x-1)^2}{x^2-1}} + c,$$

$$\Rightarrow \quad \int \frac{1}{x^2-1}dx = \ln\sqrt{\frac{(x-1)^2}{(x-1)(x+1)}} + c = \ln\sqrt{\frac{(x-1)}{(x+1)}} + c = \ln\left(\frac{x-1}{x+1}\right)^{1/2} + c;$$

$$\therefore \quad \int \frac{1}{x^2-1}dx = \frac{1}{2}\ln\left(\frac{x-1}{x+1}\right) + c.$$

ACTIVIDAD COMPLEMENTARIA IV.

INTEGRACIÓN DE FUNCIONES RACIONALES, POR FRACCIONES PARCIALES, CUANDO EL DENOMINADOR SÓLO TIENE FACTORES LINEALES

En los siguientes ejercicios, obtenga la integral indefinida:		
1. $\int \frac{x^2}{x^2+x-6}dx$	**2.** $\int \frac{5x-2}{x^2-4}dx$	**3.** $\int \frac{4x-2}{x^3-x^2-2x}dx$
4. $\int \frac{3x^2-x+1}{x^3-x^2}dx$	**5.** $\int \frac{5x^2-11x+5}{x^3-4x^2+5x-2}dx$	**6.** $\int \frac{6x^2-2x-1}{4x^3-x}dx$
7. $\int \frac{dP}{P-P^2}$		

Soluciones

1. $\int \frac{x^2}{x^2+x-6}dx$

Solución:

$$\frac{x^2}{x^2+x-6} = 1 - \frac{x-6}{x^2+x-6} \Leftrightarrow 1 - \frac{x-6}{(x+3)(x-2)}$$

{expresando el integrando en la forma: parte entera-fracción propia. Y factorizando el denominador}

De tal manera que:

$$\int \frac{x^2}{x^2+x-6}dx = \int\left(1 - \frac{x-6}{(x+3)(x-2)}\right)dx = x - \int \frac{x-6}{(x+3)(x-2)}dx + c_1 \qquad (♠)$$

Expresemos el integrando como una suma de fracciones parciales:

$$\frac{x-6}{(x+3)(x-2)} \equiv \frac{A}{x+3} + \frac{B}{x-2} \qquad (1)$$

Se multiplican ambos miembros de (1) por el mínimo común denominador $(x+3)(x-2)$, y se simplific

$$x-6 \equiv A(x-2) + B(x+3),$$

$$\Rightarrow \quad x-6 \equiv Ax - 2A + Bx + 3B \qquad \{\text{destruyendo paréntesis}\},$$

$$\Rightarrow \quad x-6 \equiv (A+B)x + (-2A+3B) \qquad \{\text{asociando de una forma adecuada}\} \qquad (2)$$

Como (2) es una identidad, los coeficientes del miembro izquierdo deben ser iguales a los coeficientes correspondientes del miembro derecho. De tal menera que:

$$A + B = 1 \qquad (3)$$

$$-2A + 3B = -6 \qquad (4)$$

Multiplicamos (3) por 2, y la ecuación resultante la sumamos con la (4):

$$\begin{array}{r} 2A + 2B = 2 \\ \underline{-2A + 3B = -6} \end{array}$$

$$5B = -4 \Leftrightarrow B = -\frac{4}{5} \qquad (5)$$

Sustituyendo (5) en (3) y operando aritméticamente, se obtiene:

$$A = \frac{9}{5} \qquad (6)$$

Sustituyendo (5), (6) en (1), se obtiene:

$$\frac{x-6}{(x+3)(x-2)} \equiv \frac{9}{5(x+3)} - \frac{4}{5(x-2)}$$

De tala manera que:

$$\int \frac{x-6}{(x+3)(x-2)}dx = \frac{9}{5}\int \frac{1}{x+3}dx - \frac{4}{5}\int \frac{1}{x-2}dx;$$

Sustituyendo (7) en (♠), se obtiene:

$$\int \frac{x^2}{x^2+x-6}dx = x - \left(\frac{9}{5}\ln(x+3) - \frac{4}{5}\ln(x-2) + c_2\right) + c_1;$$

$$\therefore \quad \int \frac{x^2}{x^2+x-6}dx = x - \frac{9}{5}\ln(x+3) + \frac{4}{5}\ln(x-2) + c.$$

2. $\displaystyle\int \frac{5x-2}{x^2-4}dx$

<u>Solución:</u>

$$\int \frac{5x-2}{x^2-4}dx = \int \frac{5x-2}{(x+2)(x-2)}dx \qquad \{\text{factorizando el denominador}\}$$

Expresamos el integrando como una suma de fracciones parciales:

$$\frac{5x-2}{(x+2)(x-2)} \equiv \frac{A}{x+2} + \frac{B}{x-2} \qquad (1)$$

Se multiplican ambos miembros de (1) por el mínimo común denominador $(x+2)(x-2)$, y se simplifica:

$$5x - 2 \equiv A(x-2) + B(x+2),$$

$$\Rightarrow \quad 5x - 2 \equiv Ax - 2A + Bx + 2B \qquad \{\text{destruyendo paréntesis}\},$$

$$\Rightarrow \quad 5x - 2 \equiv (A+B)x + (-2A + 2B) \qquad \{\text{asociando de una forma adecuada}\} \qquad (2)$$

Como (2) es una identidad, los coeficientes del miembro izquierdo deben ser iguales a los coeficientes correspondientes del miembro derecho. De tal menera que:

$$A + B = 5 \qquad (3)$$
$$-2A + 2B = -2 \qquad (4)$$

Multiplicamos (3) por 2, y la ecuación resultante la sumamos con la (4):

$$\begin{array}{r} 2A + 2B = 10 \\ \underline{-2A + 2B = -2} \\ 4B = 8 \Leftrightarrow B = 2 \quad (5) \end{array}$$

Sustituyendo (5) en (3) y operando aritméticamente, se obtiene:

$$A = 3 \qquad (6)$$

Sustituyendo (5), (6) en (1), se obtiene:

$$\frac{5x-2}{(x+2)(x-2)} \equiv \frac{3}{x+2} + \frac{2}{x-2}$$

De tala manera que:

$$\int \frac{5x-2}{x^2-4}dx = \int \frac{3}{x+2}dx + \int \frac{2}{x-2}dx;$$

$$\therefore \qquad \int \frac{5x-2}{x^2-4}dx = 3\ln(x+2) + 2\ln(x-2) + c.$$

3. $\int \frac{4x-2}{x^3 - x^2 - 2x}dx$

Solución:

$$\int \frac{4x-2}{x^3 - x^2 - 2x}dx = \int \frac{4x-2}{x(x-2)(x+1)}dx \qquad \{\text{factorizando el denominador}\}$$

Expresemos el integrando como una suma de fracciones parciales:

$$\frac{4x-2}{x(x-2)(x+1)} \equiv \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+1} \qquad (1)$$

Se multiplican ambos miembros de (1) por el mínimo común denominador $x(x-2)(x+1)$, y se simplifica:

$$4x - 2 \equiv A(x-2)(x+1) + Bx(x+1) + Cx(x-2),$$

$$\Rightarrow \quad 4x - 2 \equiv Ax^2 - Ax - 2A + Bx^2 + Bx + Cx^2 - 2Cx \qquad \{\text{destruyendo paréntesis}\},$$

$$\Rightarrow \quad 4x - 2 \equiv (A + B + C)x^2 + (-A + B - 2C)x + (-2A) \qquad \{\text{asociando de una forma adecuada}\} \quad (2)$$

Como (2) es una identidad, los coeficientes del miembro izquierdo deben ser iguales a los coeficientes correspondientes del miembro derecho. De tal menera que:

$$A + B + C = 0 \quad (3)$$
$$-A + B - 2C = 4 \quad (4)$$
$$-2A = -2 \Leftrightarrow A = 1 \quad (5)$$

Sustituyendo (5) en (4), como también (5) en (3) y operando aritméticamente, se obtiene:

$$B - 2C = 5 \quad (6)$$

$$B + C = -1 \quad (7)$$

Restando (6) de (7), se obtiene:

$$\begin{array}{r} B + C = -1 \\ -B + 2C = -5 \\ \hline 3C = -6 \Leftrightarrow C = -2 \quad (8) \end{array}$$

Sustituyendo (8) en (6), y operando aritméticamente, se obtiene:

$$B = 1 \quad (9)$$

Sustituyendo (5), (8) y (9) en (1), se obtiene:

$$\frac{4x-2}{x(x-2)(x+1)} \equiv \frac{1}{x} + \frac{1}{x-2} - \frac{2}{x+1}$$

De tal manera que:

$$\int \frac{4x-2}{x^3 - x^2 - 2x}\,dx = \int \frac{1}{x}\,dx + \int \frac{1}{x-2}\,dx - \int \frac{2}{x+1}\,dx;$$

$$\therefore \quad \int \frac{4x-2}{x^3 - x^2 - 2x}\,dx = \ln x + \ln(x-2) - 2\ln(x+1) + c.$$

4. $\int \frac{3x^2 - x + 1}{x^3 - x^2}\,dx$

Solución:

$$\int \frac{3x^2 - x + 1}{x^3 - x^2}\,dx = \int \frac{3x^2 - x + 1}{x^2(x-1)}\,dx \qquad \{\text{factorizando el denominador}\}$$

Expresemos el integrando como una suma de fracciones parciales:

$$\frac{3x^2 - x + 1}{x^2(x-1)} \equiv \frac{A}{x^2} + \frac{B}{x} + \frac{C}{x-1} \qquad (1)$$

Se multiplican ambos miembros de (1) por el mínimo común denominador $x^2(x-1)$, y se simplifica:

$$3x^2 - x + 1 \equiv A(x-1) + Bx(x-1) + Cx^2,$$

$$\Rightarrow \quad 3x^2 - x + 1 \equiv Ax - A + Bx^2 - Bx + Cx^2 \qquad \{\text{destruyendo paréntesis}\},$$

$$\Rightarrow \quad 3x^2 - x + 1 \equiv (B+C)x^2 + (A-B)x - A \qquad \{\text{asociando de una forma adecuada}\} \quad (2)$$

Como (2) es una identidad, los coeficientes del miembro izquierdo deben ser iguales a los coeficientes correspondientes del miembro derecho. De tal menera que:

$B + C = 3 \qquad (3)$

$A - B = -1 \qquad (4)$

$-A = 1 \Leftrightarrow A = -1 \qquad (5)$

Sustituyendo (5) en (4) y operando aritméticamente, se obtiene:

$B = 0 \qquad (6)$

Sustituyendo (6) en (3) y operando aritméticamente, se obtiene:

$C = 3 \qquad (7)$

Sustituyendo (5), (6) y (7) en (1), se obtiene:

$$\frac{3x^2 - x + 1}{x^2(x-1)} \equiv \frac{-1}{x^2} + \frac{0}{x} + \frac{3}{x-1}$$

De tal manera que:

$$\int \frac{3x^2 - x + 1}{x^3 - x^2} dx = -\int \frac{1}{x^2} dx + 3\int \frac{1}{x-1} dx;$$

$$\therefore \quad \int \frac{3x^2 - x + 1}{x^3 - x^2} dx = \frac{1}{x} + 3\ln|x-1| + c.$$

5. $\displaystyle\int \frac{5x^2 - 11x + 5}{x^3 - 4x^2 + 5x - 2} dx$

Solución:

$$\int \frac{5x^2 - 11x + 5}{x^3 - 4x^2 + 5x - 2} dx = \int \frac{5x^2 - 11x + 5}{(x-1)^2(x-2)} dx \qquad \{\text{factorizando el denominador}\}$$

Expresemos el integrando como una suma de fracciones parciales:

$$\frac{5x^2 - 11x + 5}{(x-1)^2(x-2)} \equiv \frac{A}{(x-1)^2} + \frac{B}{x-1} + \frac{C}{x-2} \qquad (1)$$

Se multiplican ambos miembros de (1) por el mínimo común denominador $x^2(x-1)$, y se simplifica:

$$5x^2 - 11x + 5 \equiv A(x-2) + B(x-1)(x-2) + C(x-1)^2 \qquad (2),$$

$$\Rightarrow \quad 5x^2 - 11x + 5 \equiv Ax - 2A + Bx^2 - 3Bx + 2B + Cx^2 - 2Cx + C \qquad \{\text{destruyendo paréntesis}\},$$

$$\Rightarrow \quad 5x^2 - 11x + 5 \equiv (B + C)x^2 + (A - 3B - 2C)x + (-2A + 2B + C)$$

$\{\text{asociando de una forma adecuada}\} \qquad (3)$

Como (3) es una identidad, los coeficientes del miembro izquierdo deben ser iguales a los coeficientes correspondientes del miembro derecho. De tal menera que:

$B + C = 5 \qquad (4)$

$A - 3B - 2C = -11 \qquad (5)$

$-2A + 2B + C = 5 \qquad (6)$

Si en (2) se sustituye la x por 2, se obtiene:

$$5(2)^2 - 11(2) + 5 \equiv A(2-2) + B(x-1)(2-2) + C(2-1)^2;$$

$\therefore \quad C = 3 \qquad (7)$

Sustituyendo (7) en (4) y operando aritméticamente, se obtiene:

$B = 2 \qquad (8)$

Sustituyendo (7), (8) en (5), y operando aritméticamente, se obtiene:

$A = 1 \qquad (9)$

Sustituyendo (7), (8) y (9) en (1), se obtiene:

$$\frac{5x^2 - 11x + 5}{(x-1)^2(x-2)} \equiv \frac{1}{(x-1)^2} + \frac{2}{x-1} + \frac{3}{x-2}$$

De tala manera que:

$$\int \frac{5x^2 - 11x + 5}{x^3 - 4x^2 + 5x - 2}dx = \int \frac{1}{(x-1)^2}dx + \int \frac{2}{x-1}dx + \int \frac{3}{x-2}dx;$$

$$\therefore \qquad \int \frac{5x^2 - 11x + 5}{x^3 - 4x^2 + 5x - 2}dx =$$

$$-\frac{1}{x-1} + 2ln|x-1| + 3ln|x-2| + c$$

6. $\int \frac{6x^2 - 2x - 1}{4x^3 - x}dx$

Solución:

$$\int \frac{6x^2 - 2x - 1}{4x^3 - x}dx = \int \frac{6x^2 - 2x - 1}{x(2x-1)(2x+1)}dx \qquad \{\text{factorizando el denominador}\}$$

Expresemos el integrando como una suma de fracciones parciales:

$$\frac{6x^2 - 2x - 1}{x(2x-1)(2x+1)} \equiv \frac{A}{x} + \frac{B}{2x-1} + \frac{C}{2x+1} \qquad (1)$$

Se multiplican ambos miembros de (1) por el mínimo común denominador $x(2x-1)(2x+1)$,

y se simplifica:

$$6x^2 - 2x - 1 \equiv A(2x-1)(2x+1) + Bx(2x+1) + Cx(2x-1) \qquad (2),$$

$$\Rightarrow \quad 6x^2 - 2x - 1 \equiv 4Ax^2 - A + 2Bx^2 + Bx + 2Cx^2 - Cx \qquad \{\text{destruyendo paréntesis}\},$$

$$\Rightarrow \quad 6x^2 - 2x - 1 \equiv (4A + 2B + 2C)x^2 + (B - C)x + (-A) \qquad \{\text{asociando de una forma adecuada}\} \qquad (3)$$

Como (3) es una identidad, los coeficientes del miembro izquierdo deben ser iguales a los coeficientes correspondientes del miembro derecho. De tal menera que:

$$4A + 2B + 2C = 6 \Leftrightarrow 2A + B + C = 3 \qquad (4)$$

$$B - C = -2 \qquad (5)$$

$$-A = -1 \Leftrightarrow A = 1 \qquad (6)$$

Sustituyendo (6) en (4) y operando aritméticamente, se obtiene:

$$B + C = 1 \qquad (7)$$

Sumando, término a término, (5) y (7), se obtiene:

$$\begin{array}{l} B - C = -2 \\ \underline{B + C = \ \ 1} \\ 2B \quad = -1 \Leftrightarrow B = -\frac{1}{2} \qquad (8) \end{array}$$

Sustituyendo (8) en (7), y operando aritméticamente, se obtiene:

$$C = \frac{3}{2} \qquad (9)$$

Sustituyendo (6), (8) y (9) en (1), se obtiene:

$$\frac{6x^2 - 2x - 1}{x(2x-1)(2x+1)} \equiv \frac{1}{x} - \frac{1}{2(2x-1)} + \frac{3}{2(2x+1)}$$

De tala manera que:

$$\int \frac{6x^2 - 2x - 1}{x(2x-1)(2x+1)}dx = \int \frac{1}{x}dx - \frac{1}{2}\int \frac{1}{2x-1}dx + \frac{3}{2}\int \frac{1}{2x+1}dx,$$

$$\Rightarrow \quad \int \frac{6x^2 - 2x - 1}{4x^3 - x}dx = \ln|x| - \frac{1}{2}\cdot\frac{1}{2}\ln|2x-1| + \frac{3}{2}\cdot\frac{1}{2}\ln|2x+1| + c;$$

$$\therefore \quad \int \frac{6x^2 - 2x - 1}{4x^3 - x}dx = \ln|x| - \frac{1}{4}\ln|2x-1| + \frac{3}{4}\ln|2x+1| + c.$$

7. $\int \frac{dP}{P-P^2}$

Solución:

$$\int \frac{dP}{P-P^2} = \int \frac{dP}{P(1-P)} \qquad \{\text{factorizando el denominador}\}$$

Expresemos el integrando como una suma de fracciones parciales:

$$\frac{1}{P(1-P)} \equiv \frac{A}{P} + \frac{B}{1-P} \qquad (1)$$

Se multiplican ambos miembros de (1) por el mínimo común denominador $P(1-P)$, y se simplifica:

$1 \equiv A(1-P) + BP \qquad (2),$

$\Rightarrow \quad 1 \equiv A - AP + BP \qquad \{\text{destruyendo paréntesis}\},$

$\Rightarrow \quad 1 \equiv (-A+B)P + (A) \qquad \{\text{asociando de una forma adecuada}\} \qquad (3)$

Como (3) es una identidad, los coeficientes del miembro izquierdo deben ser iguales a los coeficientes correspondientes del miembro derecho. De tal menera que:

$-A + B = 0 \qquad (4)$

$A = 1 \qquad (5)$

Sustituyendo (5) en (4) y operando aritméticamente, se obtiene:

$B = 1 \qquad (6)$

Sustituyendo (5) y (6) en (1), se obtiene:

$$\frac{1}{P(1-P)} \equiv \frac{1}{P} + \frac{1}{1-P}$$

De tala manera que:

$$\int \frac{dP}{P-P^2} = \int \frac{1}{P}dP + \int \frac{1}{1-P}dP \Leftrightarrow \int \frac{dP}{P-P^2} = \int \frac{1}{P}dP - \int \frac{1}{P-1}dP,$$

$$\therefore \quad \int \frac{dP}{P-P^2} = \ln P - \ln(P-1) + c.$$

Integración de funciones racionales, por fracciones parciales, cuando el denominador contiene factores cuadráticos

Ejercicios resueltos		
1. $\int \frac{1}{9x^4+x^2}dx$	**2.** $\int \frac{1}{x^3+x^2+x}dx$	**3.** $\int \frac{2x^3+9x}{(x^2+3)(x^2-2x+3)}dx$
4. $\int \frac{2x^2+3x+2}{x^3+4x^2+6x+4}dx$	**5.** $\int \frac{(5z^3-z^2+15z-10)dz}{(z^2-2z+5)^2}$	**6.** $\int \frac{3}{x^4+x^2+1}dx$

Soluciones

1. $\int \frac{1}{9x^4 + x^2} dx$

Solución:

$$\int \frac{1}{9x^4 + x^2} dx = \int \frac{1}{x^2(9x^2+1)} dx \qquad \{\text{factorizando el denominador}\}$$

Expresemos el integrando como una suma de fracciones parciales:

$$\frac{1}{x^2(9x^2+1)} \equiv \frac{A}{x^2} + \frac{B}{x} + \frac{Cx+D}{9x^2+1} \qquad (1),$$

$\Rightarrow \quad 1 \equiv A(9x^2+1) + Bx(9x^2+1) + (Cx+D)x^2$

{multiplicando cada miembro de la identidad por el mínimo común denominador},

$\Rightarrow \quad 1 \equiv 9Ax^2 + A + 9Bx^3 + Bx + Cx^3 + Dx^2 \qquad$ {destruyendo paréntesis},

$\Rightarrow \quad 1 \equiv (9B + C)x^3 + (9A + D)x^2 + Bx + A \qquad (2)$

Como (2) es una identidad, los coeficientes del miembro izquierdo deben ser iguales a los coeficientes correspondientes del miembro derecho. De tal manera que:

$9B + C = 0 \qquad (3)$

$9A + D = 0 \qquad (4)$

$B = 0 \qquad (5)$

$A = 1 \qquad (6)$

Sustituyendo (5) en (3) y efectuando las operaciones aritméticas, se obtiene:

$C = 0 \qquad (7)$

Sustituyendo (6) en (4) y efectuando las operaciones aritméticas, se obtiene:

$D = -9 \qquad (8)$

Sustituyendo (5), (6), (7) y (8) en (1), se obtiene:

$$\frac{1}{x^2(9x^2+1)} \equiv \frac{1}{x^2} + \frac{0}{x} + \frac{0x-9}{9x^2+1} = \frac{1}{x^2} - \frac{9}{9x^2+1}$$

De tal manera que:

$$\int \frac{1}{9x^4 + x^2} dx = \int \left(\frac{1}{x^2} - \frac{9}{9x^2+1} \right) dx = \int \frac{1}{x^2} dx - \int \frac{9dx}{9x^2+1};$$

$$\int \frac{1}{9x^4 + x^2} dx = -\frac{1}{x} - 3\tan^{-1} 3x + c.$$

2. $\int \frac{1}{x^3+x^2+x}dx$

Solución:

$$\int \frac{1}{x^3+x^2+x}dx = \int \frac{1}{x(x^2+x+1)}dx \qquad \{\text{factorizando el denominador}\}$$

Expresemos el integrando como una suma de fracciones parciales:

$$\frac{1}{x(x^2+x+1)} \equiv \frac{A}{x} + \frac{Bx+C}{x^2+x+1} \qquad (1),$$

$\Rightarrow \quad 1 \equiv A(x^2+x+1)+(Bx+C)x$

{multiplicando cada miembro de la identidad por el mínimo común denominador},

$\Rightarrow \quad 1 \equiv Ax^2 + Ax + A + Bx^2 + Cx \qquad$ {destruyendo paréntesis},

$\Rightarrow \quad 1 \equiv (A+B)x^2 + (A+C)x + A \qquad (2)$

Como (2) es una identidad, los coeficientes del miembro izquierdo deben ser iguales a los coeficientes correspondientes del miembro derecho. De tal manera que:

$A + B = 0 \quad (3)$

$A + C = 0 \quad (4)$

$A = 1 \quad (5)$

Sustituyendo (5) en (3) y efectuando las operaciones aritméticas, se obtiene:

$B = -1 \quad (6)$

Sustituyendo (5) en (4) y efectuando las operaciones aritméticas, se obtiene:

$C = -1 \quad (7)$

Sustituyendo (5), (6), y (7) en (1), se obtiene:

$$\frac{1}{x(x^2+x+1)} \equiv \frac{1}{x} - \frac{x+1}{x^2+x+1}$$

De tal manera que:

$$\int \frac{1}{x^3+x^2+x}dx = \int \frac{1}{x}dx - \int \frac{x+1}{x^2+x+1}dx = \ln x - \int \frac{x+1}{x^2+x+1}dx + c,$$

$$\Rightarrow \quad \int \frac{1}{x^3+x^2+x}dx = \ln x - \frac{1}{2}\int \frac{2x+1+1}{x^2+x+1}dx + c = \ln x - \frac{1}{2}\int \frac{2x+1}{x^2+x+1}dx - \frac{1}{2}\int \frac{1}{x^2+x+1}dx + c,$$

$$\Rightarrow \quad \int \frac{1}{x^3+x^2+x}dx = \ln x - \frac{1}{2}\ln(x^2+x+1) - \frac{1}{2}\cdot\frac{2\sqrt{3}}{3}\tan^{-1}\frac{\sqrt{3}(2x+1)}{3} + c;$$

$$\therefore \quad \int \frac{1}{x^3+x^2+x}dx = \ln x - \frac{1}{2}\ln(x^2+x+1) - \frac{\sqrt{3}}{3}\tan^{-1}\frac{\sqrt{3}(2x+1)}{3} + c.$$

3. $\int \frac{2x^3+9x}{(x^2+3)(x^2-2x+3)}dx$

Solución:

$$\int \frac{2x^3+9x}{(x^2+3)(x^2-2x+3)}dx$$

Expresemos el integrando como una suma de fracciones parciales:

$$\frac{2x^3+9x}{(x^2+3)(x^2-2x+3)} \equiv \frac{Ax+B}{x^2+3}+\frac{Cx+D}{x^2-2x+3} \quad (1),$$

$\Rightarrow \quad 2x^3+9x \equiv (Ax+B)(x^2-2x+3)+(Cx+D)(x^2+3)$

{multiplicando cada miembro de la identidad por el mínimo común denominador},

$\Rightarrow \quad 2x^3+9x \equiv Ax^3-2Ax^2+3Ax+Bx^2-2Bx+3B+Cx^3+3Cx+Dx^2+3D$

{destruyendo paréntesis},

$\Rightarrow \quad 2x^3+9x \equiv (A+C)x^3+(-2A+B+D)x^2+(3A-2B+3C)x+(3B+3D) \quad (2)$

Como (2) es una identidad, los coeficientes del miembro izquierdo deben ser iguales a los coeficientes correspondientes del miembro derecho. De tal manera que:

$A+C=2 \quad (3)$

$-2A+B+D=0 \quad (4)$

$3A-2B+3C=9 \quad (5)$

$3B+3D=0 \Leftrightarrow B+D=0 \quad (6)$

Sustituyendo (6) en (4) y efectuando las operaciones aritméticas, se obtiene:

$A=0 \quad (7)$

Sustituyendo (7) en (3) y efectuando las operaciones aritméticas, se obtiene:

$C=2 \quad (8)$

Sustituyendo (7) y (8)en (5) y efectuando las operaciones aritméticas, se obtiene:

$B=-3/2 \quad (9)$

Sustituyendo (9) en (6) y efectuando las operaciones aritméticas, se obtiene:

$D=3/2 \quad (10)$

Sustituyendo (7), (8), (9) y (10) en (1), se obtiene:

$$\frac{2x^3+9x}{(x^2+3)(x^2-2x+3)} \equiv -\frac{3}{2(x^2+3)}+\frac{4x+3}{2(x^2-2x+3)}$$

De tal manera que:

$$\int \frac{2x^3+9x}{(x^2+3)(x^2-2x+3)}dx = -\frac{3}{2}\int \frac{1}{x^2+3}dx+\frac{1}{2}\int \frac{4x+3}{x^2-2x+3}dx;$$

$$\Rightarrow \quad \int \frac{2x^3+9x}{(x^2+3)(x^2-2x+3)}dx = -\frac{3}{2}\left(\frac{\sqrt{3}}{3}\tan^{-1}\frac{\sqrt{3}x}{3}\right)+\frac{1}{2}\left(2\ln(x^2-2x+3)+\frac{7\sqrt{2}}{2}\tan^{-1}\frac{\sqrt{2}(x-1)}{2}\right);$$

$$\Rightarrow \quad \int \frac{2x^3+9x}{(x^2+3)(x^2-2x+3)}dx = -\frac{\sqrt{3}}{2}\tan^{-1}\frac{\sqrt{3}x}{3}+\ln(x^2-2x+3)+\frac{7\sqrt{2}}{4}\tan^{-1}\frac{\sqrt{2}(x-1)}{2}+c.$$

4. $\displaystyle\int \frac{2x^2+3x+2}{x^3+4x^2+6x+4}dx$

Solución:

$$\int \frac{2x^2+3x+2}{x^3+4x^2+6x+4}dx = \int \frac{2x^2+3x+2}{(x^2+2x+2)(x+2)}dx \qquad \{\text{factorizando el denominador}\}$$

Expresemos el integrando como una suma de fracciones parciales:

$$\frac{2x^2+3x+2}{(x^2+2x+2)(x+2)} \equiv \frac{Ax+B}{x^2+2x+2} + \frac{C}{x+2} \qquad (1),$$

$\Rightarrow \quad 2x^2+3x+2 \equiv (Ax+B)(x+2) + C(x^2+2x+2)$

{multiplicando cada miembro de la identidad por el mínimo común denominador},

$\Rightarrow \quad 2x^2+3x+2 \equiv Ax^2+2Ax+Bx+2B+Cx^2+2Cx+2C$

{destruyendo paréntesis},

$\Rightarrow \quad 2x^2+3x+2 \equiv (A+C)x^2+(2A+B+2C)x+(2B+2C) \qquad (2)$

Como (2) es una identidad, los coeficientes del miembro izquierdo deben ser iguales a los coeficientes correspondientes del miembro derecho. De tal manera que:

$$A+C=2 \Leftrightarrow 2A+2C=4 \qquad (3)$$

$$2A+B+2C=3 \Leftrightarrow (2A+2C)+B=3 \qquad (4)$$

$$2B+2C=2 \Leftrightarrow B+C=1 \qquad (5)$$

Sustituyendo (3) en (4) y efectuando las operaciones aritméticas, se obtiene:

$$B=-1 \qquad (6)$$

Sustituyendo (6) en (5) y efectuando las operaciones aritméticas, se obtiene:

$$C=2 \qquad (7)$$

Sustituyendo (7) en (3) y efectuando las operaciones aritméticas, se obtiene:

$$A=0 \qquad (8)$$

Sustituyendo (6), (7) y (8) en (1), se obtiene:

$$\frac{2x^2+3x+2}{(x^2+2x+2)(x+2)} \equiv -\frac{1}{x^2+2x+2} + \frac{2}{x+2}$$

De tal manera que:

$$\int \frac{2x^2+3x+2}{x^3+4x^2+6x+4}dx = -\int \frac{1}{x^2+2x+2}dx + \int \frac{2}{x+2}dx = -\int \frac{1}{(x^2+2x+1)+1}dx + 2\ln|x+2| + c,$$

$$\Rightarrow \quad \int \frac{2x^2+3x+2}{x^3+4x^2+6x+4}dx = -\int \frac{1}{(x+1)^2+1}dx + \ln(x+2)^2 + c;$$

$$\therefore \quad \int \frac{2x^2+3x+2}{x^3+4x^2+6x+4}dx = -\tan^{-1}(x+1) + \ln(x+2)^2 + c.$$

5. $\int \frac{(5z^3 - z^2 + 15z - 10)dz}{(z^2 - 2z + 5)^2}$

Solución:

Expresemos el integrando como una suma de fracciones parciales:

$$\frac{5z^3 - z^2 + 15z - 10}{(z^2 - 2z + 5)^2} \equiv \frac{Ax + B}{(z^2 - 2z + 5)^2} + \frac{Cx + D}{z^2 - 2z + 5} \qquad (1)$$

Se multiplican ambos miembros de (1) por el mínimo común denominador $(z^2 - 2z + 5)^2$, y se simplifica:

$$5z^3 - z^2 + 15z - 10 \equiv Az + B + (Cz + D)(z^2 - 2z + 5) \qquad (2),$$

$$\Rightarrow \quad 5z^3 - z^2 + 15z - 10 \equiv Az + B + Cz^3 - 2Cz^2 + 5Cz + Dz^2 - 2Dz + 5D \qquad \{\text{destruyendo paréntesis}\},$$

$$\Rightarrow \quad 5z^3 - z^2 + 15z - 10 \equiv Cz^3 + (-2C + D)z^2 + (A + 5C - 2D)z + (B + 5D)$$

{asociando de una forma adecuada} (3)

Como (3) es una identidad, los coeficientes del miembro izquierdo deben ser iguales a los coeficientes correspondientes del miembro derecho. De tal menera que:

$$C = 5 \qquad (4)$$

$$-2C + D = -1 \qquad (5)$$

$$A + 5C - 2D = 15 \qquad (6)$$

$$B + 5D = -10 \qquad (7)$$

Sustituyendo (4) en (5) y operando aritméticamente, se obtiene:

$$D = 9 \qquad (8)$$

Sustituyendo (8) en (7), y operando aritméticamente, se obtiene:

$$B = -55 \qquad (9)$$

Sustituyendo (4) y (8) en (6), y operando aritméticamente, se obtiene:

$$A = 8 \qquad (10)$$

Sustituyendo (4), (8), (9) y (10) en (1), se obtiene:

$$\frac{5z^3 - z^2 + 15z - 10}{(z^2 - 2z + 5)^2} \equiv \frac{8z - 55}{(z^2 - 2z + 5)^2} + \frac{5z + 9}{z^2 - 2z + 5}$$

De tal manera que:

$$\int \frac{(5z^3 - z^2 + 15z - 10)dz}{(z^2 - 2z + 5)^2} = \int \frac{8z - 55}{(z^2 - 2z + 5)^2} dz + \int \frac{5z + 9}{z^2 - 2z + 5} dz,$$

$$\Rightarrow \quad \int \frac{(5z^3 - z^2 + 15z - 10)dz}{(z^2 - 2z + 5)^2} = 4\int \frac{2z - 55/4}{(z^2 - 2z + 5)^2} dz + \frac{5}{2}\int \frac{2z + 18/5}{z^2 - 2z + 5} dz,$$

$$\Rightarrow \quad \int \frac{(5z^3 - z^2 + 15z - 10)dz}{(z^2 - 2z + 5)^2} = 4\int \frac{2z - 2 - 47/4}{(z^2 - 2z + 5)^2} dz + \frac{5}{2}\int \frac{2z - 2 + 28/5}{z^2 - 2z + 5} dz,$$

$$\Rightarrow \quad \int \frac{(5z^3 - z^2 + 15z - 10)dz}{(z^2 - 2z + 5)^2} = 4\int \frac{2z - 2}{(z^2 - 2z + 5)^2} dz - 47\int \frac{1}{(z^2 - 2z + 5)^2} dz + \frac{5}{2}\int \frac{2z - 2}{z^2 - 2z + 5} dz + 14\int \frac{1}{z^2 - 2z + 5} dz,$$

$$\Rightarrow \quad \int \frac{(5z^3 - z^2 + 15z - 10)dz}{(z^2 - 2z + 5)^2} = -\frac{4}{z^2 - 2z + 5} - 47\int \frac{1}{(z^2 - 2z + 5)^2} + \frac{5}{2}\ln|z^2 - 2z + 5| + 14\int \frac{1}{z^2 - 2z + 5}dz,$$

$$\Rightarrow \quad \int \frac{(5z^3 - z^2 + 15z - 10)dz}{(z^2 - 2z + 5)^2} = -\frac{4}{z^2 - 2z + 5} - 47\left(\frac{1}{8}\frac{(z-1)}{(z^2 - 2z + 5)} + \frac{1}{16}\tan^{-1}\frac{z-1}{2}\right) + \frac{5}{2}\ln|z^2 - 2z + 5|$$
$$+14 \cdot \frac{1}{2}\tan^{-1}\frac{z-1}{2} + c,$$

$$\Rightarrow \quad \int \frac{(5z^3 - z^2 + 15z - 10)dz}{(z^2 - 2z + 5)^2} = -\frac{4}{z^2 - 2z + 5} - \frac{47}{8}\frac{(z-1)}{(z^2 - 2z + 5)} - \frac{47}{16}\tan^{-1}\frac{z-1}{2} + \frac{5}{2}\ln|z^2 - 2z + 5|$$
$$+7\tan^{-1}\frac{z-1}{2} + c;$$

$$\therefore \quad \int \frac{(5z^3 - z^2 + 15z - 10)dz}{(z^2 - 2z + 5)^2} = \frac{-32 - 47(z-1)}{8(z^2 - 2z + 5)} + \frac{-47 + 112}{16}\tan^{-1}\frac{z-1}{2} + \frac{5}{2}\ln|z^2 - 2z + 5| + c;$$

$$\therefore \quad \int \frac{(5z^3 - z^2 + 15z - 10)dz}{(z^2 - 2z + 5)^2} = \frac{15 - 47z}{8(z^2 - 2z + 5)} + \frac{65}{16}\tan^{-1}\frac{z-1}{2} + \frac{5}{2}\ln|z^2 - 2z + 5| + c.$$

6. $\int \frac{3}{x^4 + x^2 + 1}dx$

Solución:

$$\int \frac{3}{x^4 + x^2 + 1}dx = \int \frac{3}{x^4 + 2x^2 + 1 - x^2}dx = \int \frac{3}{(x^2 + 1)^2 - x^2}dx = \int \frac{3}{(x^2 + x + 1)(x^2 - x + 1)}dx$$

Expresemos el integrando como una suma de fracciones parciales:

$$\frac{3}{(x^2 + x + 1)(x^2 - x + 1)} \equiv \frac{Ax + B}{x^2 + x + 1} + \frac{Cx + D}{x^2 - x + 1} \quad (1),$$

$$\Rightarrow \quad 3 \equiv (Ax + B)(x^2 - x + 1) + (Cx + D)(x^2 + x + 1)$$

{multiplicando cada miembro de la identidad por el mínimo común denominador},

$$\Rightarrow \quad 3 \equiv Ax^3 - Ax^2 + Ax + Bx^2 - Bx + B + Cx^3 + Cx^2 + Cx + Dx^2 + Dx + D$$

{destruyendo paréntesis},

$$\Rightarrow \quad 3 \equiv (A + C)x^3 + (-A + B + C + D)x^2 + (A - B + C + D)x + (B + D) \quad (2)$$

Como (2) es una identidad, los coeficientes del miembro izquierdo deben ser iguales a los coeficientes correspondientes del miembro derecho. De tal manera que:

$A + C = 0 \quad (3)$

$-A + B + C + D = 0 \quad (4)$

$A - B + C + D = 0 \quad (5)$

$B + D = 3 \quad (6)$

Sustituyendo (3) en (5) y efectuando las operaciones aritméticas, se obtiene:

$-B + D = 0 \quad (7)$

Sumando (6) y (7) y despejando, se obtiene:

$D = \frac{3}{2} \quad (8)$

Sustituyendo (8)en (6) y efectuando las operaciones aritméticas, se obtiene:

$$B = \frac{3}{2} \quad (9)$$

Sustituyendo (6) en (4) y efectuando las operaciones aritméticas, se obtiene:

$$A - C = 3 \quad (10)$$

Sumando (3) y (10) y despejando, se obtiene:

$$A = \frac{3}{2} \quad (11)$$

Sustituyendo (11)en (3) y efectuando las operaciones aritméticas, se obtiene:

$$C = -\frac{3}{2} \quad (9)$$

De tal manera que:

$$\frac{3}{(x^2+x+1)(x^2-x+1)} \equiv \frac{\frac{3}{2}x+\frac{3}{2}}{x^2+x+1} + \frac{-\frac{3}{2}x+\frac{3}{2}}{x^2-x+1} = \frac{3}{2}\left[\frac{x+1}{x^2+x+1} - \frac{x-1}{x^2-x+1}\right],$$

$$\Rightarrow \quad \int \frac{3}{x^4+x^2+1}dx = \frac{3}{2}\int\left[\frac{x+1}{x^2+x+1} - \frac{x-1}{x^2-x+1}\right]dx = \frac{3}{2}\left[\int\frac{x+1}{x^2+x+1}dx - \int\frac{x-1}{x^2-x+1}dx\right], \ \ldots$$

$$\therefore \quad \int \frac{3}{x^4+x^2+1}dx = \frac{3}{2}\left[\frac{\sqrt{3}}{3}\tan^{-1}\frac{2x+1}{\sqrt{3}} + \frac{1}{2}\ln(x^2+x+1) + \frac{\sqrt{3}}{3}\tan^{-1}\frac{2x-1}{\sqrt{3}} - \frac{1}{2}\ln(x^2-x+1)\right].$$

MÁS PROBLEMAS SOBRE FRACCIONES PARCIALES.

1) $\int \frac{dx}{x^2-3x-4}$

Caso 1- $x^2 - 3x - 4 = (x-4)(x+1)$

$$\frac{x}{x^2-3x-4} = \frac{A}{x-4} + \frac{B}{x+1}$$

$$(x-4)(x+1)\frac{x}{x^2-3x-4} = \frac{A(x-4)(x+1)}{(x-4)} + \frac{B(x-4)(x+1)}{(x+1)}$$

$$X = A(x+1) + B(x-4)$$

$$X = Ax + A + Bx - 4B$$

$$Como \ \ x = (A+B)x + A - 4B$$

De esta ecuación obtenemos el siguiente sistema: *A+B=1*

A-4B=0

Resolviendo este sistema obtenemos: $A=\frac{4}{5}$ y $B = \frac{1}{5}$

$$\int \frac{xdx}{-3x-4} = \int \left(\frac{A}{x-4} + \frac{B}{x-1}\right) dx = \int \left(\frac{\frac{4}{5}}{x-4} + \frac{\frac{1}{5}}{x-1}\right) dx = \frac{4}{5}\int \frac{dx}{x-4} + \frac{1}{5}\int \frac{dx}{x+1}$$

$$= \frac{4}{5}\ln|x-4| + \frac{1}{5}\ln|x+1| + C = \frac{1}{5}\ln|(x-4(x-4)^4| + \frac{1}{5}\ln|x+1| + C$$

2) $\displaystyle \int \frac{x^4 dx}{(1-x)^3}$

$$(1-x)^3 = 1 - 3x + 3x^2 - x^3$$

Efectuando la división

$$\frac{x^4}{(1-x)^3} = \quad \frac{x^4}{-x^3 + 3x^2 - 3x^2 + 1} = -x - 3 + \frac{6x^2 - 8x + 3}{(1-x)^3}$$

$$\int \frac{x^4}{(1-x)^3} = \int \left(-x - 3 + \frac{6x^2 - 8x + 3}{(1-x)^3}\right) dx = -\int xdx - 3\int dx + \int \left(\frac{6x^2 - 8x + 3}{(1-x)^3}\right) dx$$

Caso 2 $\qquad \dfrac{6x^2 - 8x + 3}{(1-x)^3} = \dfrac{A}{1-x} + \dfrac{B}{(1-x)^3} + \dfrac{C}{(1-x)^3}$

$$(1-x)^3 \frac{(6x^2 - 8x + 3)}{(1-x)^3} = \frac{A(1-x)^3}{(1-x)} + \frac{B(1-x)^2}{(1-x)} + \frac{C(1-x)^3}{(1-x)^3}$$

$$6x^2 - 8x + 3 = A(1-x)^2 + B(1-x) + C$$

$$6x^2 - 8x + 3 = A + 2A + Ax^2 + B - Bx + C$$

$$6x^2 - 8x + 3 = Ax^2 + (-2A - B)x + A + B + C$$

De ésta identidad obtenemos

A=6 I

-2A-B=-8 II

A+B+C=3 III

Resolviendo el sistema tenemos

A=6 ; B=-4 ; C=1

$$\int \frac{x^4}{(1-x)^3} = \int xdx - 3\int dx + \int \left(\frac{A}{(1-x)} + \frac{B}{(1-x)^2} + \frac{C}{(1-x)^3}\right) dx = -\int xdx + \int \frac{6x}{1-x} + \int \frac{-4dx}{(1-x)^2} + \frac{dx}{(1-x)^3}$$

$$= -\int xdx - 3\int dx + 6\int \frac{dx}{(1-x)} - 4\int (1-x)^{-2}\, dx + \int (1-x)^3\, dx$$

Sea u=1-x ; $\frac{du}{dx} = -1$; $-du = dx$

=-$\int xdx - 3\int dx + 6\int \frac{-du}{u} - 4\int u^{-2}(-du) + \int u^{-3}(-du)$

=-$\int xdx - 3\int dx - 6\int \frac{du}{u}$+4$\int u^{-2}du - \int u^{-3}du$

=-$\frac{x^2}{2} - 3x - 6\ln|1-x| - 4(1-x)^{-1} + \frac{(1-x)^{-2}}{-2} + c$

$$= -\frac{x^2}{2} - 3x - 6\ln|1-x| - \frac{4}{1-x} + \frac{1}{2(1-x)} + c$$

3) $\int \frac{x^4 - 2x^3 + 3x^2 - x + 3}{x^3 - 2x^2 + 3x}dx$

$x^4 - 2x^3 + 3x^2 - x + 3/x^3 - 2x^3 + 3x$

$$\int \frac{x^4 - 2x^3 + 3x^2 - x + 3}{x^3 - 2x^2 + 3x}dx = \int x + \frac{-x+3}{x^3 - 2x + 3x}$$

Caso 3 $x^3 - 2x^2 + 3x = x(x^2 - 2x + 3)$

$$\frac{-x+3}{x^3 - 2x^2 + 3x} = \frac{A}{x} + \frac{Bx + C}{x^2 - 2x^2 + 3}$$

$$x(x^2 - 2x + 3)\left(\frac{-x+3}{x^3 - 2x^2 + 3x}\right) = \frac{A(x)(x^2 - 2x + 3)}{x} + \frac{(Bx + C)(x^2 - 2x + 3)(x)}{(x^2 - 2x + 3)}$$

-x+3=A $(x^2 - 2x + 3) + Bx^2 + Cx$

-x+3=A$x^2 - 2Ax + 3A + Bx^2 + C$

-x+3=(A+B)$x^2 + (-2A + C)x + 3A$

De esta identidad obtenemos que

A+B= 0 **I** ***-2ª+C= -1*** **II** ***3A= 3*** **III**

Resolviendo el sistema

A= 1 , B = -1 , C = 1

$$\int \frac{x^4 - 2x^3 + 3x^2}{x^3 - 2x^2 + 3x}dx = \int xdx + \int \left(\frac{A}{x} + \frac{Bx + C}{x^2 - 2x + 3}\right)dx = \int xdx + \int \frac{dx}{x} + \int \frac{-x+1}{x^2 - 2x + 3}$$

Sea $u=x^2-2x+3$; $\frac{du}{dx}=2x-2$; $du=(2x-2)dx$

$=\frac{x^2}{2}+\ln|x|+\left(-\frac{1}{2}\right)\int\frac{-2(-x+1)}{x^2-2x+3}dx$

$=\frac{x^2}{2}+\ln|x|-\frac{1}{2}\ln|x^2-2x+3|+\ C$

$=\frac{x^2}{2}+\ln\left|\sqrt{x^2-2x+3}\right|+\ C$

4) $\int\frac{2x^3dx}{(x^2+1)^2}$

Caso IV.- $\frac{2x^3}{(x^2+1)^2}=\frac{Ax+B}{x^2+1}+\frac{Cx+D}{(x^2+1)^2}$

$$\frac{2x^3}{(x^2+1)^2}=\frac{(Ax+B)(x^2+1)}{(x^2+1)^2}+\frac{(Cx+D)}{(x^2+1)^2}$$

$2x^3=(Ax+B)\ (x^2+1)+Cx+D$

$2x^3=Ax^3+Ax+Bx^2+B+Cx+D$

$2x^3=Ax^3+Bx^2+(A+C)\ x+B+D$

De esta identidad tenemos

$A=2$ I

$B=0$ II

$A+C=0$ III

$B+D=0$ IV

Resolviendo el sistema

$A=2$; $B=0$; $c=-2$; $D=0$

$\therefore\ \int\frac{2x^3\,dx}{\left(x^2+1\right)^2}=\ \int\frac{2xdx}{(x^2+1)^2}-\int(x^2+1)^2(2x)dx$

Sea $u^{2}=(x^2+1)^2$ $u=x^2+1$; $\frac{du}{dx}=2x$; $du=2xdx$

$=\int\frac{du}{u}-\int u^{-2}du=\ln|u|-\frac{u^{-1}}{-1}+C=\ \ln|u|-\frac{u^{-1}}{-1}+C=\quad\ln|x^2+1|+\frac{1}{x^2+1}+C$

5) $\int \frac{x^2+3x-4}{x^2-2x-8}dx$

Realizando división: x^2+3x-4/x^2-2x-8

$\int \frac{x^2+3x-4}{x^2-2x-8}dx = \int \left(1+\frac{5x+4}{x^2-2x-8}\right)dx$

Caso 1 : $x^2-2x-8=(x-4)(x+2)$

$\int \frac{5x+4}{x^2-2x-8} = \frac{A}{x-4}+\frac{B}{x+2}$

5x+4 = A$(x+2)+B(x-4)$

5x+4 = Ax+2A+Bx - 4B

5x+4=(A+B) x + 2A-4B

De ésta identidad obtenemos el siguiente sistema

A+B = 5 $\quad \mathrm{I}$

2A-4B = $4 \quad \mathrm{II}$

Resolviendo el sistema obtenemos

A=4 ;B=1

$$\int \frac{x^2+3x-4}{x^2-2x-8}dx = \int dx + \int \left(\frac{4}{x-4}+\frac{1}{x+2}\right)dx = \int dx + 4\int \frac{dx}{x-4}+\frac{dx}{x+2}$$

$= x + 4\ln|x-4| + \ln|x+2| + C$

$= x + \ln|(x+2)(x-4)^4| + C$

6) $\int \frac{xdx}{(x-2)^2}$

$\frac{x}{(x-2)^2} = \frac{A}{(x-2)} + \frac{B}{(x-2)^2}$ *Multiplicando ambos miembros por* $(x-2)^2$ *eliminamos los denominadores y obtenemos :*

$(x-2)^2\frac{x}{(x-2)^2} = \frac{A(x-2)^2}{(x-2)} + \frac{B(x-2)^2}{(x-2)^2}$

X=A(x-2)+B = Ax-2A+B

De esta identidad tenemos que: $A=1$ & $-2A+B=0$

Resolviendo el sistema: $A=1$; $B=2$

$$\int \frac{xdx}{(x-2)^2} = \int \left(\frac{A}{(x-2)} + \frac{B}{(x-2)^2}\right) dx = \int \frac{dx}{x-2} + 2\int (x-2)^{-2} dx$$

Sea $u^{2=}(x-2)^2$; $u = x-2$; $\frac{du}{dx} = 1$ $du = dx$

$=\int \frac{du}{u} + 2\int u^{-3} du$

$=\ln|x-2| + 2\frac{u^{-1}}{-1} + C$

$=\ln|x-2| - \frac{2}{u} + C$

$$\ln|x-2| - \frac{2}{x-2} + C$$

7) $\int \frac{5x+8}{x^2+3x+2} dx$

Caso 1 : $(x+2)(x+1)$

$$\frac{5x+8}{x^2+3x+2} = \frac{A}{x+2} + \frac{B}{x+1}$$

$$(x+2)(x+1)\frac{5x+8}{x^2+3x+2} = \frac{A(x+2)(x+1)}{(x+2)} + \frac{B(x+2)(x+1)}{(x+1)}$$

$$5x+8 = A(x+1) + B(x+2)$$

$5x+8 = Ax+A+Bx+2B$

$$5x+8 = (A+B)x + A + 2B$$

De esta identidad tenemos:

$A+B=5$

$A+2B=8$

Resolviendo el sistema tenemos que $A=2$, $B=3$

$$\int \frac{5x+8}{x^2+3x+2} dx = \frac{A}{x+2} dx + \int \frac{B}{x+1} dx = 2\int \frac{dx}{x+2} + 3\int \frac{dx}{x+1}$$

$=2\ln|x+2| + 3\ln|x+1| + C$

8) $\int \frac{4x^2+6}{x^3+3}dx$ *Caso 3* $x^3+3=(x^2+3)$

$$\frac{4x^2+6}{x^3+3}=\frac{A}{x}+\frac{Bx+C}{x^2+3}$$

$$(x)(x^2+3)\frac{(4x^2+6)}{x^2+3x}=\frac{A(X)(x^2+3)}{x}+\frac{(Bx+C)(x)(x^2+3)}{(x^2+3)}$$

$$4x^2+6=A(x^2+3)+(Bx+C)x$$

$$4x^2+6=Ax^2+3A+Bx^2+Cx$$

$4x^2+6=$(A+B)$x^2+Cx+3A$

De esta identidad tenemos : A+B= 4 C= 0 3A=6

Resolviendo el sistema a=2 ,b=2 c=0

$$\int\frac{4x^2+6}{x^3+3}=\int\left(\frac{A}{x}+\frac{Bx+C}{x^2+3}\right)dx=\int\frac{2}{x}+\frac{2x+0}{x^3+3}dx=2\int\frac{dx}{x}+\int\frac{2xdx}{x^2+3}$$

$=2\ln|x|+\ln|x^2+3|+C = \ln|x^2(x^2+3)|+C$

9) $\int\frac{2t^2-8t-8}{t^3-2t^2+4t-8}dt = \int\frac{2t^2-8t-8}{(t-2)(t^2+4)}dt =$

$$2\int\frac{t^2-4t-4}{(t-2)(t^2+4)}dt=2\int\left[\frac{A}{(t-2)}+\frac{Bt+C}{(t^2+4)}\right]dt=2\int\left[\frac{-1}{t-2}+\frac{2t}{t^2+4}\right]dt=2\left[-\int\frac{dt}{t-2}+\int\frac{2tdt}{t^2+4}\right]=$$

$u=t^2+4 \quad du=2tdt$

$$2\left[-ln|t-2|+\int\frac{du}{u}\right]=-2\,ln|t-2|+2ln|t^2+4|+c$$

$$=-ln|t-2|^2+ln\left||t^2+4|\right|^2=ln\left(\frac{t^2+4}{t-2}\right)^2+c$$

$t^2-4t-4=A(t^2+4)+(Bt+C)(t-2)$
$t^2-4t-4=$ $At^2+4A+B\,t^2$+Ct-2C-2Bt
$t^2-4t-4=$ (A+B) t^2 + (C-2B) t +4 A-2C
DE ESTA IDENTIDAD OBTENEMOS EL SIGUIENTE SISTEMA:
A+B=2
C-2B=-4
4A-2C=-4
RESOLVIENDO EL SISTEMA : A = -1 , B= 1 – A = 2 , C=0

PROBLEMA DE CONCURSO

$$\int \frac{dx}{x^4+1}$$

Solución:

$$\int \frac{dx}{x^4+1} = \int \frac{1}{\left(x^2+\sqrt{2}x+1\right)\left(x^2-\sqrt{2}x+1\right)}dx \qquad \{\text{factorizando}\}$$

Expresemos el integrando como una suma de fracciones parciales:

$$\frac{1}{\left(x^2+\sqrt{2}x+1\right)\left(x^2-\sqrt{2}x+1\right)} \equiv \frac{Ax+B}{x^2+\sqrt{2}x+1} + \frac{Cx+D}{x^2-\sqrt{2}x+1} \qquad (1),$$

$$\Rightarrow \quad 1 \equiv (Ax+B)\left(x^2-\sqrt{2}x+1\right) + (Cx+D)\left(x^2+\sqrt{2}x+1\right)$$

{multiplicando cada miembro de la identidad por el mínimo común denominador},

$$\Rightarrow \quad 1 \equiv Ax^3 - \sqrt{2}Ax^2 + Ax + Bx^2 - \sqrt{2}Bx + B + Cx^3 + \sqrt{2}Cx^2 + Cx + Dx^2 + \sqrt{2}Dx + D$$

{destruyendo paréntesis},

$$\Rightarrow \quad 1 \equiv (A+C)x^3 + (-\sqrt{2}A + B + \sqrt{2}C + D)x^2 + (A - \sqrt{2}B + C + \sqrt{2}D)x + (B+D) \qquad (2)$$

Como (2) es una identidad, los coeficientes del miembro izquierdo deben ser iguales a los coeficientes correspondientes del miembro derecho. De tal manera que:

$$A + C = 0 \qquad (3)$$

$$-\sqrt{2}A + B + \sqrt{2}C + D = 0 \qquad (4)$$

$$A - \sqrt{2}B + C + \sqrt{2}D = 0 \qquad (5)$$

$$B + D = 1 \qquad (6)$$

Sustituyendo (3) en (5) y efectuando las operaciones aritméticas, se obtiene:

$$-B + D = 0 \qquad (7)$$

Sumando (6) y (7) y despejando, se obtiene:

$$D = \frac{1}{2} \qquad (8)$$

Sustituyendo (8) en (6) y efectuando las operaciones aritméticas, se obtiene:

$$B = \frac{1}{2} \qquad (9)$$

Sustituyendo (6) en (4) y efectuando las operaciones aritméticas, se obtiene:

$$A - C = \frac{\sqrt{2}}{2} \qquad (10)$$

Sumando (3) y (10) y despejando, se obtiene:

$$A = \frac{\sqrt{2}}{4} \qquad (11)$$

Sustituyendo (11) en (3) y efectuando las operaciones aritméticas, se obtiene:

$$C = -\frac{\sqrt{2}}{4} \qquad (9)$$

De tal manera que:

$$\frac{1}{\left(x^2+\sqrt{2}x+1\right)\left(x^2-\sqrt{2}x+1\right)} \equiv \frac{\frac{\sqrt{2}}{4}x+\frac{1}{2}}{x^2+\sqrt{2}x+1} + \frac{-\frac{\sqrt{2}}{4}x+\frac{1}{2}}{x^2-\sqrt{2}x+1},$$

$$\Rightarrow \quad \int \frac{1}{x^4+1}dx = \frac{\sqrt{2}}{8}\int\left[\frac{2x+2\sqrt{2}}{x^2+\sqrt{2}x+1} - \frac{2x-2\sqrt{2}}{x^2-\sqrt{2}x+1}\right]dx = \frac{\sqrt{2}}{8}\left[\int\frac{2x+2\sqrt{2}}{x^2+\sqrt{2}x+1}dx - \int\frac{2x-2\sqrt{2}}{x^2-\sqrt{2}x+1}dx\right],$$

$$\Rightarrow \quad \int \frac{1}{x^4+1}dx = \frac{\sqrt{2}}{8}\left[\int\frac{2x+\sqrt{2}+\sqrt{2}}{x^2+\sqrt{2}x+1}dx - \int\frac{2x-\sqrt{2}-\sqrt{2}}{x^2-\sqrt{2}x+1}dx\right],$$

$$\Rightarrow \quad \int \frac{1}{x^4+1}dx = \frac{\sqrt{2}}{8}\left[\int\frac{2x+\sqrt{2}}{x^2+\sqrt{2}x+1}dx + \int\frac{\sqrt{2}}{x^2+\sqrt{2}x+1} - \int\frac{2x-\sqrt{2}}{x^2-\sqrt{2}x+1}dx + \int\frac{\sqrt{2}}{x^2-\sqrt{2}x+1}dx\right],$$

$$= \frac{\sqrt{2}}{8}\left[\ln\left|x^2+\sqrt{2}x+1\right| + \int\frac{\sqrt{2}}{x^2+\sqrt{2}x+\frac{1}{2}+\frac{1}{2}} - \ln\left|x^2-\sqrt{2}x+1\right| - \int\frac{\sqrt{2}}{x^2-\sqrt{2}x+\frac{1}{2}+\frac{1}{2}}dx\right],$$

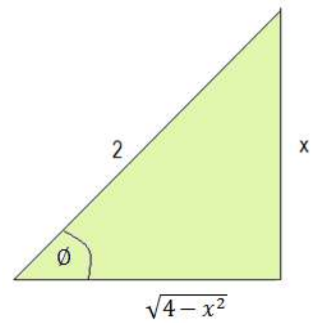
$$= \frac{\sqrt{2}}{8}\left[\ln\left|x^2+\sqrt{2}x+1\right| + \sqrt{2}\int\frac{1}{\left(x+\frac{\sqrt{2}}{2}\right)^2+\left(\frac{\sqrt{2}}{2}\right)^2} - \ln\left|x^2-\sqrt{2}x+1\right| - \sqrt{2}\int\frac{1}{\left(x-\frac{\sqrt{2}}{2}\right)^2+\left(\frac{\sqrt{2}}{2}\right)^2}dx\right],$$

$$= \frac{\sqrt{2}}{8}\left[\ln\left|x^2+\sqrt{2}x+1\right| + \sqrt{2}\cdot\sqrt{2}\tan^{-1}\left(\sqrt{2}\left(x+\frac{\sqrt{2}}{2}\right)\right) - \ln\left|x^2-\sqrt{2}x+1\right| - \sqrt{2}\cdot\sqrt{2}\tan^{-1}\left(\sqrt{2}\left(x-\frac{\sqrt{2}}{2}\right)\right)\right],$$

$$\therefore \quad \int\frac{1}{x^4+1}dx = \frac{\sqrt{2}}{8}\left[\ln\left|x^2+\sqrt{2}x+1\right| + 2\tan^{-1}\left(\sqrt{2}x+1\right) - \ln\left|x^2-\sqrt{2}x+1\right| - 2\tan^{-1}\left(\sqrt{2}x-1\right)\right]+c.$$

¡ MÁS PROBLEMAS DE INTEGRACIÓN POR SUSTITUCIÓN TRIGONOMÉTRICA!

P1) $\int \frac{x^2\,dx}{\sqrt{4-x^2}}$



$sea \quad sen\emptyset = \frac{x}{2} \quad ; \quad x = 2\,sen\emptyset \quad ; \quad x^2 = 4sen^2\emptyset$

$cos\emptyset = \frac{\sqrt{4-x^2}}{2} \quad ; \quad \sqrt{4-x^2} = 2cos\emptyset$

$\frac{dx}{d\emptyset} = 2\,cos\emptyset \quad ; \quad dx = 2cos\emptyset d\emptyset$

$$\int \frac{x^2dx}{\sqrt{4-x^2}} = \int \frac{4sen^2\emptyset 2cos\emptyset d\emptyset}{2cos\emptyset} = 4\int sen^2\emptyset d\emptyset = 4\int \left(\frac{1-cos2\emptyset}{2}\right) d\emptyset$$

$$= 2\int d\emptyset - 2\int cos2\emptyset d\emptyset$$

$sea \quad u = 2\emptyset \quad ; \quad \frac{du}{d\emptyset} = 2 \quad ; \quad \frac{du}{2} = d\emptyset$

$$= 2\int d\emptyset - 2\int cosu \times \frac{du}{2} = 2\int d\emptyset - \int cosudu = 2\emptyset - sen\,u + C$$

$$= 2\emptyset - sen2\emptyset + C$$

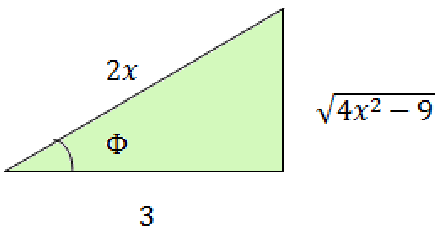
$$\emptyset = arcsen\,\frac{x}{2}$$

$y \quad sen2\emptyset = 2sen\emptyset cos\emptyset \quad (\textbf{identidad de angulos dobles})$

$$= 2\frac{x}{2} \times \frac{\sqrt{4-x^2}}{2} = \frac{x}{2}\sqrt{4-x^2}$$

$$\int \frac{x^2dx}{\sqrt{4-x^2}} = 2arcsen\frac{x}{2} - \frac{x\sqrt{4-x^2}}{2} + C$$

P2) $\int \frac{dx}{x^2\sqrt{4x^2 - 9}}$



$sec\Phi = \frac{2x}{3}$; $\quad x = \frac{3}{2}\, sec\Phi$; $\quad x^2 = \frac{9}{4}\, sec^2\Phi$

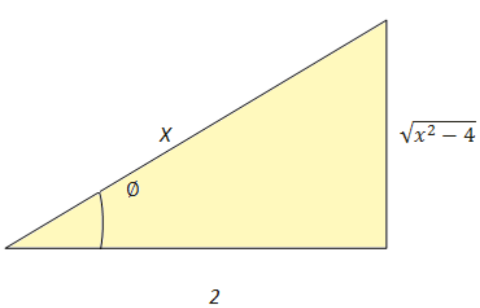
$\frac{dx}{d\Phi} = \frac{3}{2}\, sec\ \Phi tan\ \Phi$; $\quad dx = \frac{3}{2}\, sec\ \Phi tan\ \Phi\ d\Phi$

$\tan\Phi = \frac{\sqrt{4x^2 - 9}}{3}$; $\quad \sqrt{4x^2 - 9} = 3\tan\Phi$

$$\int \frac{dx}{x^2\sqrt{4x^2 - 9}} = \int \frac{\frac{3}{2} sec\ \Phi \tan\Phi\ d\Phi}{\frac{9}{4}\ sec^2\ \Phi 3\tan\Phi} = \frac{2}{9}\int \frac{d\Phi}{sec\ \Phi} = \frac{2}{9}\int \cos\Phi\ d\Phi = \frac{2}{9}\ sen\Phi + C$$

$como\ sen\Phi = \frac{\sqrt{4x^2 - 9}}{2x}$; $\quad \int \frac{dx}{x^2\sqrt{4x^2 - 9}} = \frac{2}{9}\frac{\sqrt{4x^2 - 9}}{2x} + C = \frac{\sqrt{4x^2 - 9}}{9x} + C$

P3) $\int \sqrt{x^2 - 4}dx =$



$sec\emptyset = \frac{x}{2}$; $x = 2sec\emptyset$

$$\frac{dx}{d\emptyset} = 2sec\emptyset tan\emptyset \ ; \ dx = 2sec\emptyset tan\emptyset \ d\emptyset$$

$$tan\emptyset = \frac{\sqrt{x^2-4}}{2} \ ; \ \sqrt{x^2-4} = 2tan\emptyset$$

$$\int \sqrt{x^2-4}dx = \int 2tan\emptyset 2sec\emptyset tan\emptyset d\emptyset = 4\int tan^2\emptyset sec\emptyset d\emptyset$$

$$= 4\int (sec^2\emptyset - 1)sec\emptyset d\emptyset = 4\int sece^3\emptyset d\emptyset - 4\int sec\emptyset d\emptyset$$

La integral de la secante cúbica ya fue resuelta en el tema de integración por partes

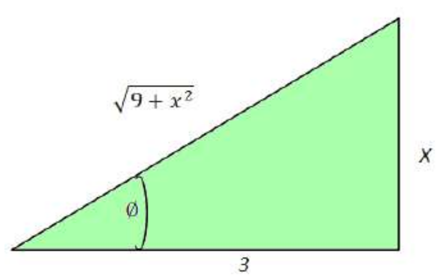
$$= 4\left[\frac{1}{2}sec\emptyset tan\emptyset + \frac{1}{2}ln \ |sec\emptyset + tan\emptyset|\right] - 4\ ln|sec\emptyset + tan\emptyset| + C$$

$$= [2sec\emptyset tan\emptyset + 2ln \ |sec\emptyset + tan\emptyset|] - 4\ ln|sec\emptyset + tan\emptyset| + C$$

$$= 2sec\emptyset tan\emptyset - 2ln \ |sec\emptyset + tan\emptyset| + c = 2\left(\frac{x}{2}\right)\frac{\sqrt{x^2-4}}{2} - 2ln\left|\frac{x}{2} + \frac{\sqrt{x^2-4}}{2}\right| + c =$$

$$= \frac{x\sqrt{x^2-4}}{2} - 2ln\left|\frac{x+\sqrt{x^2-4}}{2}\right| + c$$

P4) $$\int \frac{dx}{(9+x^2)^2} = \int \frac{dx}{(\sqrt{9+x^2})^4}$$



$$\tan\emptyset = \frac{x}{3} \quad ; \quad x = 3\tan\emptyset$$

$$\frac{dx}{d\emptyset} = 3sec^2\emptyset \quad ; \quad dx = 3sec^2\emptyset d\emptyset$$

$$sec\emptyset = \frac{\sqrt{9+x^2}}{3} \quad ; \quad \sqrt{9+x^2} = 3sec\emptyset$$

$$\int \frac{dx}{(\sqrt{9+x^2})^4} = \int \frac{3sec^2\emptyset d\emptyset}{(3\sec\emptyset))^4} = \int \frac{3\ d\emptyset}{81sec^2\emptyset} = \frac{1}{27}\int cos^2\emptyset d\emptyset = \frac{1}{27}\int\left(\frac{1+cos2\emptyset}{2}\right)d\emptyset$$

$$= \frac{1}{54}\int d\emptyset + \frac{1}{54}\int cos2\emptyset d\emptyset$$

$sea \quad u = 2\emptyset \quad ; \quad \frac{du}{d\emptyset} = 2 \quad ; \quad \frac{du}{2} = d\emptyset$

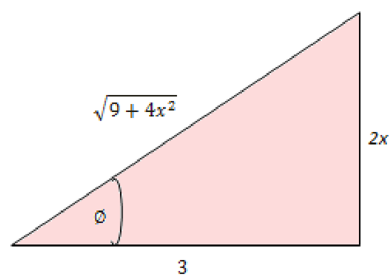
$$= \frac{1}{54}\emptyset + \frac{1}{54}\int cosu\left(\frac{du}{2}\right) = \frac{1}{54}\emptyset + \frac{1}{108}\int cosudu = \frac{1}{54}\emptyset + \frac{1}{108}sen2\emptyset + c$$

$como\ \emptyset = arctan\frac{x}{3}$ & $sen2\emptyset = 2sen\emptyset cos\emptyset$; $como\ sen\emptyset = \frac{x}{\sqrt{9+x^2}}$ & $cos\emptyset = \frac{3}{\sqrt{9+x^2}}$

$$sen2\emptyset = 2\frac{x}{\sqrt{9+x^2}} \cdot \frac{3}{\sqrt{9+x^2}} = \frac{6x}{9+x^2}$$

$$= \frac{1}{54}\ arctan\frac{x}{3} + \frac{1}{108}\ \frac{6x}{9+x^2} + c = \frac{1}{54}arctan\frac{x}{3} + \frac{x}{18(9+x^2)} + c$$

P5) $\int \frac{dx}{x\sqrt{9+4x^2}} =$



$tan\ \emptyset = \frac{2x}{3} \quad ; \quad x = \frac{3}{2}tan\ \emptyset$

$\frac{dx}{d\emptyset} = \frac{3}{2}sec^2\emptyset \quad ; \quad dx = \frac{3}{2}sec^2\ \emptyset d\emptyset$

$sec\ \emptyset = \frac{\sqrt{9+4x^2}}{3} \quad ; \quad \sqrt{9+4x^2} = 3\ sec\ \emptyset$

$$\int \frac{dx}{x\sqrt{9+4x^2}} = \int \frac{\frac{3}{2}sec^2\ \emptyset d\emptyset}{\frac{2}{2}tan\ \emptyset 3sec\emptyset} = \frac{1}{3}\int \frac{sec\emptyset}{tan\emptyset}d\emptyset = \frac{1}{3}\int \frac{\frac{1}{cos\emptyset}}{\frac{sen}{cos\emptyset}}d\emptyset = \frac{1}{3}\int \frac{1}{sen\emptyset}d\emptyset$$

$$= \frac{1}{3}\int csc\emptyset d\emptyset = \frac{1}{3}ln/\csc\emptyset - cot\emptyset/ + c$$

$como\ csc\ \emptyset = \frac{\sqrt{9+4x^2}}{2x}$ & $cot\ \emptyset = \frac{3}{2x}$

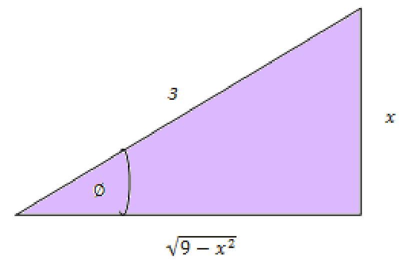
$$= \frac{1}{3}In\left[\frac{\sqrt{9+4x^2}}{2x} - \frac{3}{2x}\right] + c \quad = \frac{1}{3}In\left[\frac{\sqrt{9+4x^2}-3}{2x}\right] + c$$

P6) $\int \frac{dx}{x^2\sqrt{9-x^2}} =$

$sin\,\emptyset = \frac{x}{3}$; $x = 3\,sin\,\emptyset$; $x^2 = 9sin^2\,\emptyset$

$\frac{dx}{d\emptyset} = 3\,cos\,\emptyset$; $dx = 3\,cos\,\emptyset d\emptyset$

$cos\,\emptyset = \frac{\sqrt{9-x^2}}{x}$; $\sqrt{9-x^2} = 3cos\emptyset$



$$\int \frac{dx}{x^2\sqrt{9-x^2}} = \int \frac{3cos\emptyset d\emptyset}{9sen^2\emptyset 3cos\emptyset} = \frac{1}{9}\int csc^2\emptyset d\,\emptyset = \frac{1}{9}(-cot\emptyset) + c$$

$como\ cot\,\emptyset = \frac{\sqrt{9-x^2}}{x}$

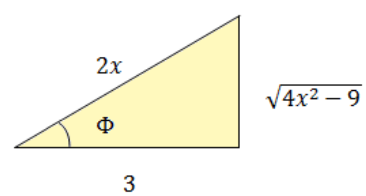
$= -\frac{1}{9}cot\emptyset + c \ = \ -\frac{1}{9}\frac{\sqrt{9-x^2}}{x} + C$

P7) $\int \frac{dx}{x^2\sqrt{4x^2-9}}$

$sec\Phi = \frac{2x}{3}$; $x = \frac{3}{2}\,sec\Phi$; $x^2 = \frac{9}{4}\,sec^2\Phi$

$\frac{dx}{d\Phi} = \frac{3}{2}\,sec\,\Phi tan\,\Phi$; $dx = \frac{3}{2}\,sec\,\Phi tan\,\Phi\,d\Phi$

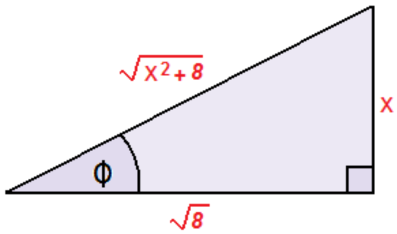
$\tan\Phi = \frac{\sqrt{4x^2-9}}{3}$; $\sqrt{4x^2-9} = 3\tan\Phi$



$$\int \frac{dx}{x^2\sqrt{4x^2-9}} = \int \frac{\frac{3}{2}sec\,\Phi\tan\Phi\,d\Phi}{\frac{9}{4}\,sec^2\,\Phi 3\tan\Phi} = \frac{2}{9}\int \frac{d\Phi}{sec\,\Phi} = \frac{2}{9}\int \cos\Phi\,d\Phi = \frac{2}{9}\,sen\Phi + C$$

$como\ sen\Phi = \frac{\sqrt{4x^2-9}}{2x}$; $\int \frac{dx}{x^2\sqrt{4x^2-9}} = \frac{2}{9}\frac{\sqrt{4x^2-9}}{2x} + C = \frac{\sqrt{4x^2-9}}{9x} + C$

P8) $\int \frac{x^2\,dx}{(x^2+8)^{\frac{3}{2}}} = \int \frac{x^2\,dx}{\left(\sqrt{x^2+8}\right)^3}$



$tan\emptyset = \frac{x}{\sqrt{8}} \quad ; \quad x = \sqrt{8}tan\emptyset \qquad x^2 = 8tan^2\emptyset$

$\frac{dx}{d\emptyset} = \sqrt{8}\,sec^2\emptyset \quad ; \quad dx = \sqrt{8}\,sec^2\emptyset d\emptyset$

$sec\emptyset = \frac{\sqrt{x^2+8}}{\sqrt{8}} \quad ; \quad \sqrt{x^2+8} = \sqrt{8}\,sec\emptyset$

$$\int \frac{x^2dx}{\left(\sqrt{x^2+8}\right)^3} = \int \frac{8tan^2\emptyset\sqrt{8}\,sec^2\emptyset d\emptyset}{\left(\sqrt{8}sec\emptyset\right)^3} = \int \frac{8tan^2\sqrt{8}\,sec^2\,\emptyset d\emptyset}{8\sqrt{8}\,sec^3\,\emptyset}$$

$$= \int \frac{tan^2}{sec\emptyset}d\emptyset = \int \left(\frac{sec^2\emptyset - 1}{sec\emptyset}\right)d\emptyset = \int \left(\frac{sec^2\emptyset}{sec\emptyset} - \frac{1}{sec\emptyset}\right)d\emptyset$$

$$= \int sec\emptyset d\emptyset - \int cos\emptyset d\emptyset = ln|sec\emptyset + tan\emptyset| - sen\emptyset + C$$

$como \quad sec\emptyset = \frac{\sqrt{x^2+8}}{\sqrt{8}} \quad ; \quad tan\emptyset = \frac{x}{8} \quad ; \quad sen\emptyset = \frac{x}{\sqrt{x^2+8}}$

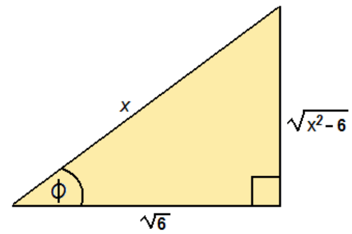
$$\int \frac{x^2dx}{\left(\sqrt{x^2+8}\right)^3} = ln\left|\frac{\sqrt{x^2+8}}{\sqrt{8}} + \frac{x}{\sqrt{8}}\right| - \frac{x}{\sqrt{x^2+8}} + C$$

$$= ln\left|\frac{\sqrt{x^2+8}+x}{\sqrt{8}}\right| - \frac{x}{\sqrt{x^2+8}} + C$$

$$= ln\left|x + \sqrt{x^2+8}\right| - ln\left|\sqrt{8}\right| - \frac{x}{\sqrt{x^2+8}} + C$$

$$= -\frac{x}{\sqrt{x^2+8}} + ln\left|x + \sqrt{x^2+8}\right| - ln\left|\sqrt{8}\right| + C = -\frac{x}{\sqrt{x^2+8}} + ln\left|x + \sqrt{x^2+8}\right| + C$$

P9) $\int \frac{x^2 dx}{\sqrt{x^2-6}}$



$sec\emptyset = \frac{x}{\sqrt{6}}$; $x = \sqrt{6}\, sec\emptyset$; $x^2 = 6sec^2\emptyset$

$\frac{dx}{d\emptyset} = \sqrt{6}\, sec\emptyset tan\emptyset$; $dx = \sqrt{6}\, sec\emptyset tan\emptyset d\emptyset$

$tan\emptyset = \frac{\sqrt{x^2-6}}{\sqrt{6}}$; $\sqrt{x^2-6} = \sqrt{6}\, tan\emptyset$

$$\int \frac{x^2 dx}{\sqrt{x^2-6}} = \int \frac{6sec^2\emptyset\sqrt{6}\, sec\emptyset tan\emptyset d\emptyset}{\sqrt{6}\, tan\emptyset} = 6\int sec^3\emptyset d\emptyset = 6\int sec\emptyset\, sec^2\emptyset d\emptyset$$

Integrando ésta última por partes :

$u = sec\emptyset$; $du = sec\emptyset tan\emptyset d\emptyset$

$dv = sec^2\emptyset d\emptyset$; $v = \int dv = \int sec^2\emptyset d\emptyset = tan\emptyset$

$$6\int sec^3\emptyset d\emptyset = 6\left[sec\emptyset d\emptyset - \int tan\emptyset(sec\emptyset tan\emptyset)d\emptyset\right]$$

$$6\int sec^3\emptyset d\emptyset = 6\left[sec\emptyset tan\emptyset - \int sec\emptyset tan^2\emptyset d\emptyset\right]$$

$$6\int sec^3\emptyset d\emptyset = 6\left[sec\emptyset tan\emptyset - \int sec\emptyset(sec^2\emptyset - 1)d\emptyset\right]$$

$$6\int sec^3\emptyset d\emptyset = 6\, sec\emptyset tan\emptyset - 6\int sec^3\emptyset d\emptyset + 6\int sec\emptyset d\emptyset$$

$$6\int sec^3\emptyset d\emptyset + 6\int sec^3\emptyset d\emptyset = 6sec\emptyset tan + 6\int sec\emptyset d\emptyset$$

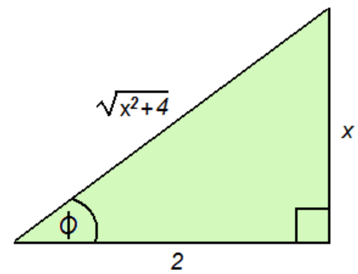
$$12\int sec^3\emptyset d\emptyset = 6\, sec\emptyset tan\emptyset + 6\, ln|sec\emptyset + tan\emptyset| + C$$

$$\int \mathbf{sec^3\emptyset d\emptyset} = = \frac{1}{2}\left(\frac{x}{\sqrt{6}} \times \frac{\sqrt{x^2-6}}{\sqrt{6}}\right) + \frac{1}{2}\, ln\left|\frac{x}{\sqrt{6}} + \frac{\sqrt{x^2-6}}{\sqrt{6}}\right| + C$$

$$= \frac{1}{12}x\sqrt{x^2-16} + \frac{1}{2}\, ln\left|\frac{x+\sqrt{x^2-6}}{\sqrt{6}}\right| + C$$

$$= \frac{1}{12}x\sqrt{x^2-16} + \frac{1}{2}\left[ln\left|x+\sqrt{x^2-6}\right| - ln|\sqrt{6}|\right] + C = \frac{1}{12}x\sqrt{x^2-16} + \frac{1}{2}\left[ln\left|x+\sqrt{x^2-6}\right|\right] + C$$

P10) $\int \frac{dx}{x\sqrt{x^2+4}}$



$tan\emptyset = \frac{x}{2} \quad ; \quad x = 2tan\emptyset$

$\frac{dx}{d\emptyset} = 2sec^2\emptyset \quad ; \quad dx = 2sec^2\emptyset d\emptyset$

$sec\emptyset = \frac{\sqrt{x^2+4}}{2} \quad ; \quad \sqrt{x^2+4} = 2sec\emptyset$

$$\int \frac{dx}{x\sqrt{x^2+4}} = \int \frac{2sec^2\emptyset d\emptyset}{2tan\emptyset 2sec\emptyset} = \frac{1}{2}\int \frac{sec\emptyset}{tan\emptyset} d\emptyset = \frac{1}{2}\int \frac{\frac{1}{cos\emptyset}}{\frac{sen\emptyset}{cos\emptyset}} d\emptyset$$

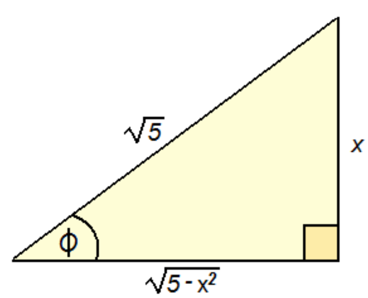
$$= \frac{1}{2}\int \frac{1}{sen\emptyset} d\emptyset = \frac{1}{2}\int csc\emptyset d\emptyset = \frac{1}{2}\,ln|csc\emptyset - cot\emptyset| + C$$

$como \quad csc\emptyset = \frac{\sqrt{x^2+4}}{x} \quad ; \quad cot\emptyset = \frac{x}{2}$

$$= \frac{1}{2}\,ln\left|\frac{\sqrt{x^2+4}}{x} - \frac{2}{x}\right| + C = \frac{1}{2}\,ln\left|\frac{\sqrt{x^2+4}-2}{x}\right| + C$$

P11) $\int \frac{dx}{x^2\sqrt{5-x^2}}$

$sen\emptyset = \frac{x}{\sqrt{5}} \quad ; \quad x = \sqrt{5}\,sen\emptyset \quad ; \quad x^2 = 5sen^2\emptyset$



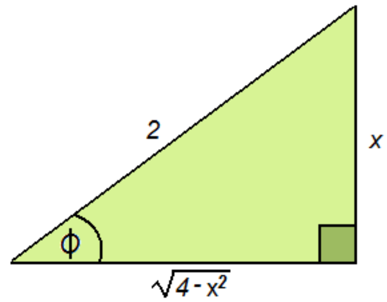
$\frac{dx}{d\emptyset} = \sqrt{5}\,cos\emptyset \quad ; \quad dx = \sqrt{5}\,cos\emptyset d\emptyset$

$cos\emptyset = \frac{\sqrt{5-x^2}}{\sqrt{5}} \quad ; \quad \sqrt{5-x^2} = \sqrt{5}\,cos\emptyset$

$$\int \frac{dx}{x^2\sqrt{5-x^2}} = \int \frac{\sqrt{5}\,cos\emptyset d\emptyset}{5sen^2\emptyset\sqrt{5}\,cos\emptyset} = \frac{1}{5}\int \frac{1}{sen^2\emptyset} d\emptyset = \frac{1}{5}\int csc^2\emptyset d\emptyset = \frac{1}{5}(-cot\emptyset) + C$$

$$como \quad cot\emptyset = \frac{\sqrt{5-x^2}}{x} \quad entonces: \int \frac{dx}{x^2\sqrt{5-x^2}} = -\frac{1}{5}\frac{\sqrt{5-x^2}}{x} + C$$

P12) $\int \frac{dx}{(4-x^2)^{\frac{3}{2}}} = \int \frac{dx}{\left(\sqrt{4-x^2}\right)^3}$



$sen\emptyset = \frac{x}{2} \quad ; \quad x = 2sen\emptyset$

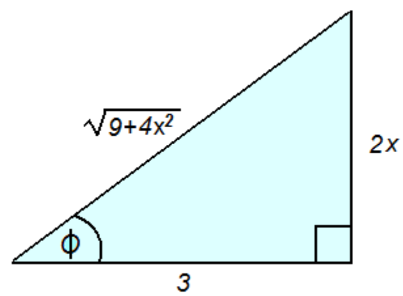
$\frac{dx}{d\emptyset} = 2cos\emptyset \;;\quad dx = 2\,cos\emptyset d\emptyset$

$cos\emptyset = \frac{\sqrt{4-x^2}}{2} \quad ; \quad \sqrt{4-x^2} = 2\,cos\emptyset$

$$\int \frac{2\,cos\emptyset d\emptyset}{(2cos\emptyset)^3} = \frac{1}{4}\int \frac{1}{cos^2\emptyset} d\emptyset = \frac{1}{4}\int sec^2\emptyset d\emptyset = \frac{1}{4}\,tan\emptyset + C$$

$como \quad tan\emptyset = \frac{x}{\sqrt{4-x^2}} = \frac{1}{4}\frac{x}{\sqrt{4-x^2}} + C$

P13) $\int \frac{dx}{x\sqrt{9+4x^2}}$



$tan\emptyset = \frac{2x}{3} \quad : \quad x = \frac{3}{2}tan\emptyset$

$\frac{dx}{d\emptyset} = \frac{3}{2}sec^2\emptyset \;;\quad dx = \frac{3}{2}sec^2\emptyset d\emptyset$

$sec\emptyset = \frac{\sqrt{9+4x^2}}{3} \quad ; \quad \sqrt{9+4x^2} = 3\,sec\emptyset$

$$\int \frac{dx}{x\sqrt{9+4x^2}} = \int \frac{\frac{3}{2}sec^2\emptyset d\emptyset}{\frac{3}{2}tan\emptyset 3sec\emptyset} = \frac{1}{3}\int \frac{sec\emptyset}{tan\emptyset}d\emptyset = \frac{1}{3}\int \frac{\frac{1}{cos\emptyset}}{\frac{sen\emptyset}{cos\emptyset}}d\emptyset = \frac{1}{3}\int \frac{1}{sen\emptyset}d\emptyset$$

$$= \frac{1}{3}\int csc\emptyset d\emptyset = \frac{1}{3}\,ln|csc\emptyset - cot\emptyset| + C$$

$como \quad csc\emptyset = \frac{\sqrt{9+4x^2}}{2x} \quad ; \quad cot\emptyset = \frac{3}{2x}$

$$= \frac{1}{3}\,ln\left|\frac{\sqrt{9+4x^2}}{2x} - \frac{3}{2x}\right| + C = \frac{1}{3}\,ln\left|\frac{\sqrt{9+4x^2}-3}{2x}\right| + C$$

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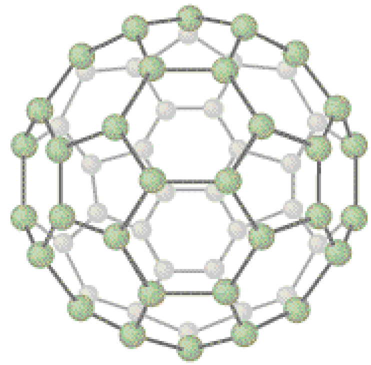
PÁGINAS ELECTRÓNICAS



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