1.1 Relations and Functions

YOU WILL NEED

- graphing calculator or graph paper

GOAL

Recognize functions in various representations.

INVESTIGATE the Math

Ang recorded the heights and shoe sizes of students in his class.

Shoe Size	Height (cm)
10	158
11.5	175
10	173
9	164
9	167
10	170
11	172
8	160
8	174
11	175
8	166
7.5	153
10	171
11	181
11	171
10	170

Shoe Size	Height (cm)
8	156
7.5	161
12	179
11	178
10.5	173
8.5	177
8	165
12	182
13	177
13	192
7.5	157
8.5	163
12	183
10	168
11	180

? Can you predict a person's height from his or her shoe size?

A. Plot the data, using shoe size as the **independent variable**. Describe the relationship shown in the scatter plot.

B. Use your plot to predict the height of a person with each shoe size.
i) 8 **ii)** 10 **iii)** 13

C. Use your plot to predict what shoe size corresponds to each height.
i) 153 cm **ii)** 173 cm **iii)** 177 cm

D. Draw a **line of good fit** on your plot. Write the equation of your line, and use it to determine the heights corresponding to the shoe sizes in part B. How are your results different from those in part B?

Tech *Support*

For help drawing a line of best fit on a graphing calculator, see Technical Appendix, B-11.

E. Describe the **domain** and **range** of the relationship between shoe size and height in Ang's class.

F. Explain why the **relation** plotted in part A is not a **function**.

G. Is the relation drawn in part D a function? Explain.

H. Which of the relations in parts A and D could be used to predict a single height for a given shoe size? Explain.

domain

the set of all values of the independent variable of a relation

range

the set of all values of the dependent variable of a relation

relation

a set of ordered pairs; values of the independent variable are paired with values of the dependent variable

function

a relation where each value of the independent variable corresponds with only one value of the dependent variable

Reflecting

I. How did the numbers in the table of values show that the relation was not a function?

J. How did the graph of the linear function you drew in part D differ from the graph of the relation you plotted in part A?

K. Explain why it is easier to use the linear function than the scatter plot of the actual data to predict height.

APPLY the Math

EXAMPLE 1 Representing functions in different ways

The ages and soccer practice days of four students are listed.

Student	Age	Soccer Practice Day
Craig	15	Tuesday
Magda	16	Tuesday
Stefani	15	Thursday
Amit	17	Saturday

Communication *Tip*

Use braces to list the values, or elements, in a set.

For example, the set of the first five even numbers is {2, 4, 6, 8, 10}.

For each of the given relations, state the domain and range and then determine whether or not the relations are functions.

a) students and the day for soccer practice

b) ages and the day for soccer practice

Jenny's Solution: Using Set Notation

a) {(Craig, Tuesday),
(Magda, Tuesday),
(Stefani, Thursday),
(Amit, Saturday)}

Domain = {Craig, Magda, Stefanie, Amit}

Range = {Tuesday, Thursday, Saturday}

I wrote the relation as a set of ordered pairs, (student's name, day for practice). I wrote the domain by listing the students' names—the independent variable, or first elements, in each ordered pair.

I listed the day for practice—the dependent variable, or second elements—to write the range.

Each element of the domain corresponds with only one element in the range, so the relation between students and their soccer practice day is a function. The first elements appear only once in the list of ordered pairs. No name is repeated.

Each student has only one practice day, so the relation is a function. In this case, if I know the student's name, I can predict his or her practice day.

b) {(15, Tuesday), (16, Tuesday), (15, Thursday), (17, Saturday)}

Domain = {15, 16, 17}

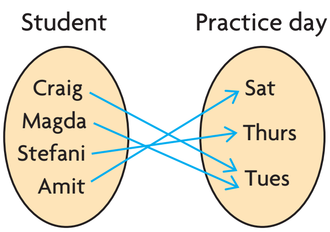
Range = {Tuesday, Thursday, Saturday}

15 in the domain corresponds with two different days in the range, so this relation is not a function.

I noticed that one 15-year-old practiced on Tuesday, but another practiced on Thursday, so I can't predict a practice day just by knowing the age. This is not a function.

Olivier's Solution: Using a Mapping Diagram

a)



I drew a diagram of the relation between students and soccer practice days by listing the student names in an oval and the days in another oval. Then I drew arrows to match the students with their practice days. The diagram is called a mapping diagram, since it *maps* the elements of the domain onto the elements of the range.

Domain = {Craig, Magda, Stefanie, Amit}

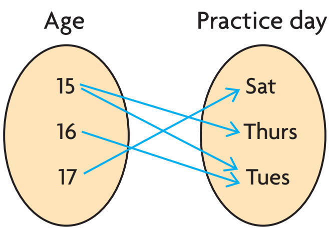
Range = {Tuesday, Thursday, Saturday}

The elements in the left oval are the values of the independent variable and make up the domain. The elements in the right oval are the values of the dependent variable and make up the range. I wrote the domain and range by listing what was in each oval.

Each element of the domain has only one corresponding element in the range, so the relation is a function.

The relation is a function because each student name has only one arrow leaving it.

b)



I drew another mapping diagram for the age and practice day relation. I matched the ages to the practice days.

Domain = {15, 16, 17}

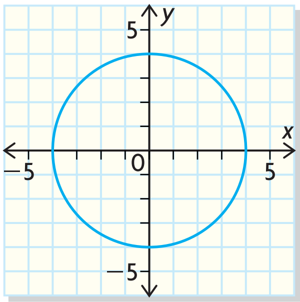
Range = {Tuesday, Thursday, Saturday}

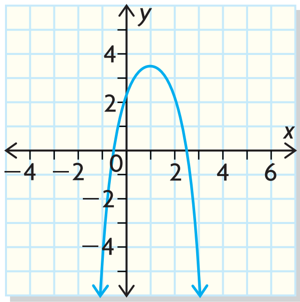
The value 15 of the independent variable, age, maps to two different values of the dependent variable, days. This relation is not a function.

Two arrows go from 15 to two different days. This cannot be a function. An element of the domain can't map to two elements in the range.

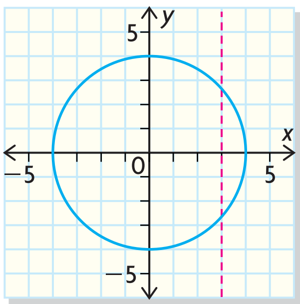
EXAMPLE 2 Selecting a strategy to recognize functions in graphs

Determine which of the following graphs are functions.

a) 

b) 

Ken's Solution

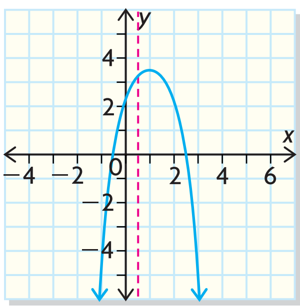
a) 

I used the **vertical-line test** to see how many points on the graph there were for each value of *x*.

An easy way to do this was to use a ruler to represent a vertical line and move it across the graph.

The ruler crossed the graph in two places everywhere except at the leftmost and rightmost ends of the circle. This showed that there are *x*-values in the domain of this relation that correspond to two *y*-values in the range.

At least one vertical line drawn on the graph intersects the graph at two points. This is not the graph of a function.

b) 

I used the vertical-line test again. Wherever I placed my ruler, the vertical line intersected the graph in only one place. This showed that each *x*-value in the domain corresponds with only one *y*-value in the range.

Any vertical line drawn on the graph intersects the graph at only one point. This is the graph of a function.

vertical-line test

if any vertical line intersects the graph of a relation more than once, then the relation is not a function

EXAMPLE 3 Using reasoning to recognize a function from an equation

Determine which equations represent functions.

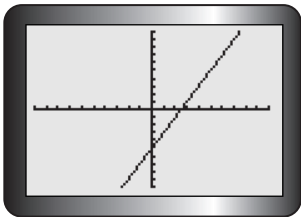
a) $y = 2x - 5$ **b)** $x^2 + y^2 = 9$ **c)** $y = 2x^2 - 3x + 1$

Keith's Solution: Using the Graph Defined by its Equation

a) This equation defines the graph of a linear function with a positive slope. Its graph is a straight line that increases from left to right.

I used my graphing calculator and entered $y = 2x - 5$. I graphed the function and checked it with the vertical-line test.

This graph passes the vertical-line test, showing that for each x-value in the domain there is only one y-value in the range. This is the graph of a function.

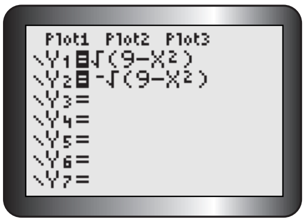


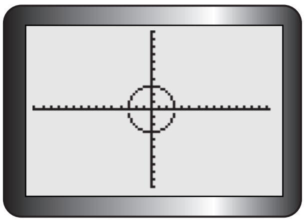
$y = 2x - 5$ is a function.

b) This equation defines the graph of a circle centred at (0, 0) with a radius of 3.

I used my graphing calculator and entered the upper half of the circle in Y1 and the lower half in Y2. Then I applied the vertical-line test to check.

This graph fails the vertical-line test, showing that there are x-values in the domain of this relation that correspond to two y-values in the range. This is not the graph of a function.

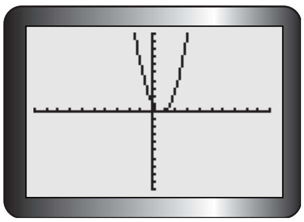




$x^2 + y^2 = 9$ is not a function.

c) This equation defines the graph of a parabola that opens upward.

I used my graphing calculator to enter $y = 2x^2 - 3x + 1$ and applied the vertical-line test to check.



This graph passes the vertical-line test, showing that for each x-value in the domain there is only one y-value in the range. This is the graph of a function.

$y = 2x^2 - 3x + 1$ is a function.

Mayda's Solution: Substituting Values

a) For any value of x, the equation $y = 2x - 5$ produces only one value of y. For example,

$y = 2(1) - 5 = -3$

This equation defines a function.

I substituted numbers for x in the equation.

No matter what number I substituted for x, I got only one answer for y when I doubled the number for x and then subtracted 5.

b) Substitute 0 for x in the equation

$$x^2 + y^2 = 9.$$

$$(0)^2 + y^2 = 9$$

$$y = 3 \text{ or } -3$$

I substituted 0 for x in the equation and solved for y. I used 0 because it's an easy value to calculate with. I got two values for y with $x = 0$.

There are two values for y when $x = 0$, so the equation defines a relation, but not a function.

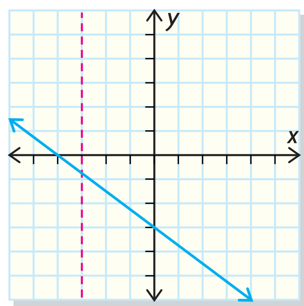
c) Every value of x gives only one value of y in the equation $y = 2x^2 - 3x + 1$.
This equation represents a function.

No matter what number I choose for x, I get only one number for y that satisfies the equation.

In Summary

Key Ideas

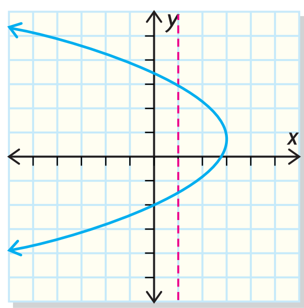
- A function is a relation in which each value of the independent variable corresponds with only one value of the dependent variable.
- Functions can be represented in various ways: in words, a table of values, a set of ordered pairs, a mapping diagram, a graph, or an equation.



A relation that is a function

Need To Know

- The domain of a relation or function is the set of all values of the independent variable. This is usually represented by the x-values on a coordinate grid.
- The range of a relation or function is the set of all values of the dependent variable. This is usually represented by the y-values on a coordinate grid.
- You can use the vertical-line test to check whether a graph represents a function. A graph represents a function if every vertical line intersects the graph in at most one point. This shows that there is only one element in the range for each element of the domain.
- You can recognize whether a relation is a function from its equation. If you can find even one value of x that gives more than one value of y when you substitute x into the equation, the relation is *not* a function. Linear relations, which have the general forms $y = mx + b$ or $Ax + By = C$ and whose graphs are straight lines, are all functions. Vertical lines are not functions but horizontal lines are. Quadratic relations, which have the general forms $y = ax^2 + bx + c$ or $y = a(x - h)^2 + k$ and whose graphs are parabolas, are also functions.



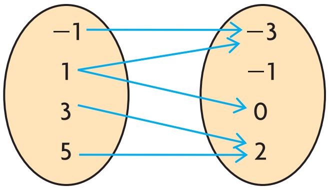
A relation that is not a function

CHECK Your Understanding

1. State which relations are functions. Explain.

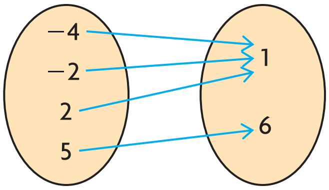
a) $\{(-5, 1), (-3, 2), (-1, 3), (1, 2)\}$

b)



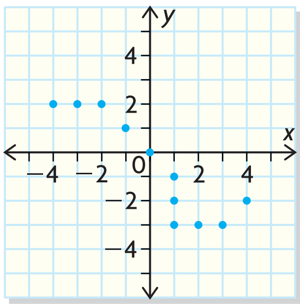
c) $\{(0, 4), (3, 5), (5, -2), (0, 1)\}$

d)

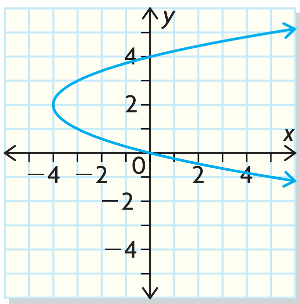


2. Use a ruler and the vertical-line test to determine which graphs are functions.

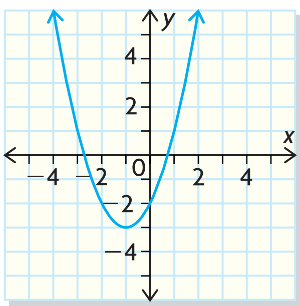
a)



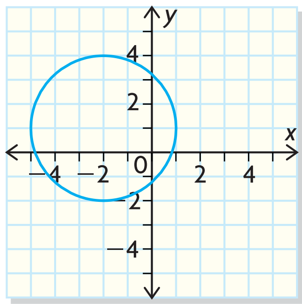
b)



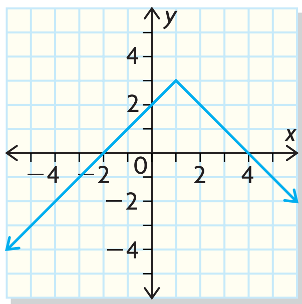
c)



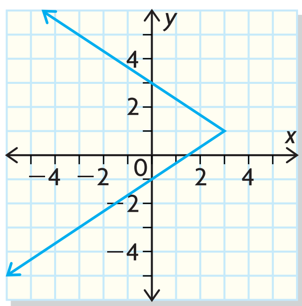
d)



e)



f)



3. Substitute -6 for x in each equation and solve for y. Use your results to explain why $y = x^2 - 5x$ is a function but $x = y^2 - 5y$ is not.

PRACTISING

4. The grades and numbers of credits for students are listed.

K

Student	Grade	Number of Credits
Barbara	10	8
Pierre	12	25
Kateri	11	15
Mandeep	11	18
Elly	10	16

a) Write a list of ordered pairs and create a mapping diagram for the relation between
- students and grades
- grades and numbers of credits
- students and numbers of credits

b) State the domain and range of each relation in part (a).

c) Which relations in part (a) are functions? Explain.

5. Graph the relations in question 4. Then use the vertical-line test to confirm your answers to part (c).

6. Describe the graphs of the relations $y = 3$ and $x = 3$. Are these relations functions? Explain.

7. Identify each type of relation and predict whether it is a function. Then graph each function and use the vertical-line test to determine whether your prediction was correct.

a) $y = 5 - 2x$ **c)** $y = -\frac{3}{4}(x + 3)^2 + 1$

b) $y = 2x^2 - 3$ **d)** $x^2 + y^2 = 25$

8. a) Substitute $x = 0$ into each equation and solve for y. Repeat for $x = -2$.

i) $3x + 4y = 5$ **iii)** $x^2 + y = 2$

ii) $x^2 + y^2 = 4$ **iv)** $x + y^2 = 0$

b) Which relations in part (a) appear to be functions?

c) How could you verify your answer to part (b)?

9. Determine which relations are functions.

a) $y = \sqrt{x + 2}$ **c)** $3x^2 - 4y^2 = 12$

b) $y = 2 - x$ **d)** $y = -3(x + 2)^2 - 4$

10. Use numeric and graphical representations to investigate whether the relation $x - y^2 = 2$ is a function. Explain your reasoning.

11. Determine which of the following relations are functions.

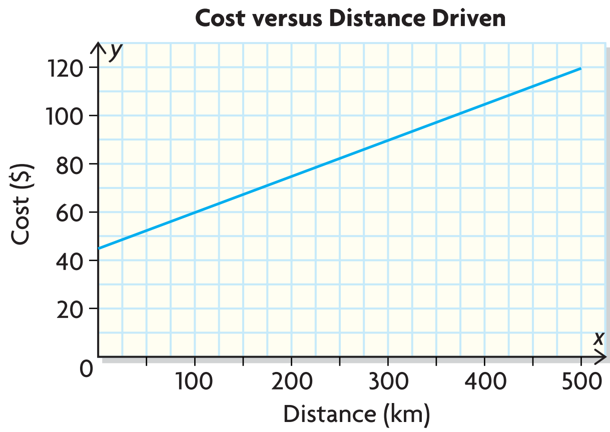
A

a) The relation between earnings and sales if Olwen earns $400 per week plus 5% commission on sales

b) The relation between distance and time if Bran walks at 5 km/h

c) The relation between students' ages and the number of credits earned

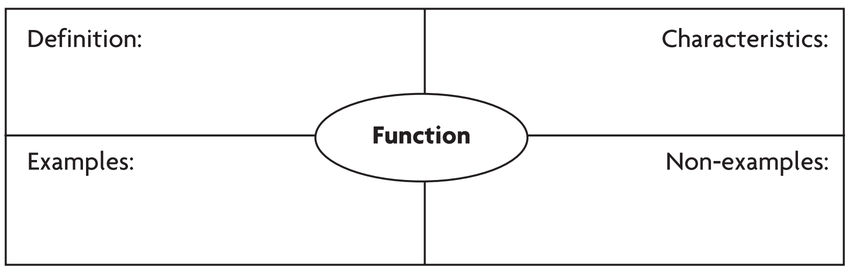
12. The cost of renting a car depends on the daily rental charge and the number of kilometres driven. A graph of cost versus the distance driven over a one-day period is shown.



a) What are the domain and range of this relation?
b) Explain why the domain and range have a lower limit.
c) Is the relation a function? Explain.

13. a) Sketch a graph of a function that has the set of integers as its domain and all integers less than 5 as its range.
b) Sketch a graph of a relation that is not a function and that has the set of real numbers less than or equal to 10 as its domain and all real numbers greater than -5 as its range.

14. Use a chart like the following to summarize what you have learned about functions.



Extending

15. A freight delivery company charges $4/kg for any order less than 100 kg and $3.50/kg for any order of at least 100 kg.
a) Why must this relation be a function?
b) What is the domain of this function? What is its range?
c) Graph the function.
d) What suggestions can you offer to the company for a better pricing structure? Support your answer.