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MATH323 - A2

Numbers from the 7th Edition

#	9.3	9.7	9.17	9.18	9.33	9.36	9.39	9.41	9.65	9.86	9.90	9.112
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9.3 Let $Y_1, Y_2, \ldots, Y_n$ denote a random sample from the uniform distribution on the interval $(\theta, \theta+1)$. Let

$$\hat{\theta}_1 = \overline{Y} - \frac{1}{2} \quad \text{and} \quad \hat{\theta}_2 = Y_{(n)} - \frac{n}{n+1}.$$

$\frac{1}{N}\sum_{i=1}^{n} Y_i$

a Show that both $\hat{\theta}_1$ and $\hat{\theta}_2$ are unbiased estimators of θ.

b Find the efficiency of $\hat{\theta}_1$ relative to $\hat{\theta}_2$.

$Var(Y) = \frac{(b-a)^2}{12}$

a) ① For $\hat{\theta}_1$

$E[\hat{\theta}_1] = E(\overline{Y} - \frac{1}{2}) = E[\overline{Y}] - \frac{1}{2}$

Since $Y_i \sim (\theta, \theta+1)$, we know: $E[Y_i] = \frac{\theta + (\theta+1)}{2} = \theta + \frac{1}{2}$

$E[\overline{Y}] = E[\frac{1}{n}\sum_{i=1}^{n} Y_i] = \frac{1}{n}\sum_{i=1}^{n} E[Y_i] = \frac{1}{n} \cdot n\,(\theta + \frac{1}{2}) = \theta + \frac{1}{2}$

$E[\hat{\theta}_1] = (\theta + \frac{1}{2}) - \frac{1}{2} = \theta$

⇒ $\hat{\theta}_1$ is an unbiased estimator of θ

② For $\hat{\theta}_2$ * determine $E[Y_{(n)}]$ where $Y_{(n)}$ is the maximum order stastic

→ find PDF of $Y_{(n)}$

→ CDF: $F_{Y_{(n)}}(y) = P(Y_{(n)} \leq y) = P(Y_1 \leq y)\,P(Y_2 \leq y) \ldots P(Y_n \leq y) = (y-\theta)^n$ for $\theta \leq y \leq \theta+1$

Since Y_i's are iid from $U(\theta, \theta+1)$ → $P(Y_i \leq y) = \frac{y-\theta}{1}$ for $\theta \leq y \leq \theta+1$

PDF: $f_{Y_{(n)}}(y) = \frac{d}{dy}(y-\theta)^n = n(y-\theta)^{-1}$, $\theta \leq y \leq \theta+1$

$E[Y_{(n)}] = \int_\theta^{\theta+1} y \cdot f_{Y_{(n)}}(y)\,dy = \int_\theta^{\theta+1} y \cdot n(y-\theta)^{-1}\,dy = \int_0^1 (\theta + t)\,n t^{n-1}\,dt = n\int_0^1 (\theta t^{n-1} + t^n)\,dt = n\theta\left[\frac{t^n}{n}\right]_0^1 + n\left[\frac{t^{n+1}}{n+1}\right]_0^1$

let $t = y - \theta$, so $dy = dt$

$= \theta \cdot 1 + \frac{n}{n+1} = \theta + \frac{n}{n+1}$

$E[\hat{\theta}_2] = E[Y_{(n)} - \frac{n}{n+1}] = (\theta + \frac{n}{n+1}) - \frac{n}{n+1} = \theta$

⇒ $\hat{\theta}_2$ is an unbiased estimator of θ

b) Efficiency $= \frac{Var(\hat{\theta}_2)}{Var(\hat{\theta}_1)}$ for uniform distubution $U(\theta, \theta+1)$, $Var(Y_i) = \frac{(1-0)^2}{12} = \frac{1}{12}$ ⇒ $Var(\overline{Y}) = \frac{1}{12n}$

$Var(\hat{\theta}_1) = Var(\overline{Y} - \frac{1}{2}) = Var(\overline{Y}) = \frac{1}{12n}$

* substracting constant doesn't affect variance

$Var(\hat{\theta}_2) = Var(Y_{(n)} - \frac{n}{n+1}) = Var(Y_{(n)}) = E[Y_{(n)}^2] - (E[Y_{(n)}])^2$

$t = y - \theta$, so that $dy = dt$

→ for $E[Y_{(n)}^2]$ → $\int_\theta^{\theta+1} y^2 f_{Y_{(n)}}(y)\,dy = \int_\theta^{\theta+1} y^2 n(y-\theta)^{n-1}\,dy = \int_0^1 (\theta+t)^2 n t^{n-1}\,dt = n\int_0^1 (\theta^2 + 2\theta t + t^2)\,t^{n-1}\,dt$

$= n\left[\theta^2 \int_0^1 t^{n-1}\,dt + 2\theta \int_0^1 t^n\,dt + \int_0^1 t^{n+1}\,dt\right] = n\left[\theta^2 \cdot \frac{1}{n} + 2\theta \cdot \frac{1}{n+1} + \frac{1}{n+2}\right]$

$= \theta^2 + \frac{2n\theta}{(n+1)} + \frac{n}{(n+2)}$

$\to$ for $E[Y_{(n)}]^2 \to \left(\theta + \frac{n}{n+1}\right)^2 = \theta^2 + \frac{2n\theta}{n+1} + \frac{n^2}{(n+1)^2}$

$$Var(\hat\theta_2) = E[Y^2_{(n)}] - (E[Y_{(n)}])^2 = \left(\cancel{\theta^2} + \cancel{\frac{2n\theta}{n+1}} + \frac{n}{n+2}\right) - \left(\cancel{\theta^2} + \cancel{\frac{2n\theta}{n+1}} + \frac{n^2}{(n+1)^2}\right) = \frac{n(n+1)^2 - n^2(n+2)}{(n+2)(n+1)^2} = \frac{n(n^2+2n+1) - n^2(n+2)}{(n+2)(n+1)^2}$$

$$= \frac{n^3 + 2n^2 + n - (n^3 + 2n^2)}{(n+2)(n+1)^2} = \frac{n}{(n+2)(n+1)^2}$$

9.7 Suppose that $Y_1, Y_2, \ldots, Y_n$ denote a random sample of size n from an exponential distribution with density function given by

$$f(y) = \begin{cases} (1/\theta)e^{-y/\theta}, & 0 < y, \\ 0, & \text{elsewhere.} \end{cases}$$

minimum order stat

In Exercise 8.19, we determined that $\hat\theta_1 = nY_{(1)}$ is an unbiased estimator of θ with $\text{MSE}(\hat\theta_1) = \theta^2$. Consider the estimator $\hat\theta_2 = \overline{Y}$ and find the efficiency of $\hat\theta_1$ relative to $\hat\theta_2$.

sample mean

$MSE(\hat\theta_1) = \theta^2$

Since $\hat\theta_1$ is unbiased $\to Var(\hat\theta_1) = \theta^2$

$$Var(\hat\theta_2) = Var(\bar Y) = Var\left[\sum_{i=1}^{n} \frac{Y_i}{n}\right] = \frac{1}{n^2} \cdot Var\left[\sum_{i=1}^{n} Y_i\right] = \frac{1}{n^2} \cdot n \cdot Var(Y_i) = \frac{\theta^2}{n}$$

$$\text{Efficiency} = \frac{Var(\hat\theta_2)}{Var(\hat\theta_1)} = \frac{\theta^2/n}{\theta^2} = \frac{1}{n}$$

9.17 Suppose that $X_1, X_2, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_n$ are independent random samples from populations with means μ_1 and μ_2 and variances σ_1^2 and σ_2^2, respectively. Show that $\overline{X} - \overline{Y}$ is a consistent estimator of $\mu_1 - \mu_2$.

By linearity of expectation: $E[\bar X - \bar Y] = E[\bar X] - E[\bar Y]$
$= \mu_1 - \mu_2$ (sample means)

$\Rightarrow E[\bar X - \bar Y]$ is an unbiased estimator of $\mu_1 - \mu_2$

Using independence property: $Var(\bar X - \bar Y) = Var(\bar X) + Var(\bar Y) = \frac{\sigma_1^2}{n} + \frac{\sigma_2^2}{n} = \frac{\sigma_1^2 + \sigma_2^2}{n}$

Checking consistency: $\lim_{n\to\infty} Var(\bar X - \bar Y) = \frac{\sigma_1^2 + \sigma_2^2}{n} = 0$

Since variance approaches 0 as $n \to \infty$, $\bar X - \bar Y$ is a consistent estimator

$\Rightarrow \bar X - \bar Y$ is a consistent estimator of $\mu_1 - \mu_2$

9.18 In Exercise 9.17, suppose that the populations are normally distributed with $\sigma_1^2 = \sigma_2^2 = \sigma^2$. Show that

$$\frac{\sum_{i=1}^{n}(X_i - \overline{X})^2 + \sum_{i=1}^{n}(Y_i - \overline{Y})^2}{2n-2} = S^2$$

is a consistent estimator of σ^2.

$E\left[\frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar X)^2\right] = \sigma^2 \quad \longrightarrow \quad E\left[\sum_{i=1}^{n}(X_i - \bar X)^2\right] = (n-1)\sigma^2$

$E\left[\frac{1}{n-1}\sum_{i=1}^{n}(Y_i - \bar Y)^2\right] = \sigma^2 \quad \longrightarrow \quad E\left[\sum_{i=1}^{n}(Y_i - \bar Y)^2\right] = (n-1)\sigma^2$

$$E[S^2] = E\left[\frac{\sum_{i=1}^{n}(X_i - \bar X)^2 + \sum_{i=1}^{n}(Y_i - \bar Y)^2}{2n-2}\right] = E\left[\frac{(n-1)\sigma^2 + (n-1)\sigma^2}{2n-2}\right] = \frac{\cancel{(2n-2)}\sigma^2}{\cancel{(2n-2)}} = \sigma^2$$

$\Rightarrow S^2$ is an unbiased estimator of σ^2

Prove consistency: $\lim_{n\to\infty} Var(S^2)$

$\lim_{n\to\infty} Var\left(\frac{1}{n}\sum_{i=1}^{n}(X_i - \bar X)^2\right) = 0$

$\lim_{n\to\infty} Var\left(\frac{1}{n}\sum_{i=1}^{n}(Y_i - \bar Y)^2\right) = 0$

$\left.\right\} \lim_{n\to\infty} Var(S^2) = 0$

$\Rightarrow S^2$ is a consistent estimator of σ^2

9.33 An experimenter wishes to compare the numbers of bacteria of types A and B in samples of water. A total of n independent water samples are taken, and counts are made for each sample. Let X_i denote the number of type A bacteria and Y_i denote the number of type B bacteria for sample i. Assume that the two bacteria types are sparsely distributed within a water sample so that $X_1, X_2, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_n$ can be considered independent random samples from Poisson distributions with means λ_1 and λ_2, respectively. Suggest an estimator of $\lambda_1/(\lambda_1+\lambda_2)$. What properties does your estimator have?

$X_1, X_2, \ldots, X_n \sim \text{Poisson}(\lambda_1)$ → counts nb type A bacteria
$Y_1, Y_2, \ldots, Y_n \sim \text{Poisson}(\lambda_2)$ → counts nb type B bacteria

Since λ_1 and λ_2 are means of respective Poisson distribution: $\hat{\lambda}_1 = \bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$, $\hat{\lambda}_2 = \bar{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i$

ratio $\frac{\lambda_1}{\lambda_1+\lambda_2}$ is $\frac{\bar{X}}{\bar{X}+\bar{Y}}$ ← Sample proportion estimator for type A bacteria relative to total

$\bar{X}$ and $\bar{Y}$ are unbiased estimators of λ_1 and λ_2
$\bar{X}/(\bar{X}+\bar{Y})$ is not exactly unbiased, but approx unbiased for large n

For consistency, by the law of large numbers: $\bar{X} \to \lambda_1$ and $\bar{X} \to \lambda_2$ as $n \to \infty$

$\Rightarrow \frac{\bar{X}}{\bar{X}+\bar{Y}} \longrightarrow \frac{\lambda_1}{\lambda_1+\lambda_2}$ ⇒ consistent estimator since estimator converges in probability to the true parameter

$\Rightarrow \frac{\bar{X}}{\bar{X}+\bar{Y}}$ is a consistent estimator of $\frac{\lambda_1}{\lambda_1+\lambda_2}$

9.36 Suppose that Y has a binomial distribution based on n trials and success probability p. Then $\hat{p}_n = Y/n$ is an unbiased estimator of p. Use Theorem 9.3 to prove that the distribution of $(\hat{p}_n - p)/\sqrt{\hat{p}_n\hat{q}_n/n}$ converges to a standard normal distribution. [*Hint:* Write Y as we did in Section 7.5.]

Since $Y \sim \text{Binomial}(n, p)$ → $E[Y] = np$, $\text{Var}(Y) = np(1-p) = npq$

$E[\hat{p}_n] = \frac{E[Y]}{n} = p$

$\text{Var}(\hat{p}_n) = \frac{\text{Var}(Y)}{n^2} = \frac{npq}{n^2} = \frac{pq}{n}$

By CLT, standardized form of $\hat{p}_n$ is: $Z_n = \frac{\hat{p}_n - p}{\sqrt{p(1-p)/n}} \sim N(0,1)$ for large n

$\Rightarrow$ Shows that $\frac{\hat{p}_n - p}{\sqrt{p(1-p)/n}} \to N(0,1)$ as $n \to \infty$

Replace p with $\hat{p}_n$ ← Sample-based variance estimate

$\Rightarrow \frac{\hat{p}_n - p}{\sqrt{\hat{p}_n\hat{q}_n/n}} \longrightarrow N(0,1)$

⇒ Proves that standardized form of $\hat{p}_n$ converges to a standard normal distribution

9.39 Let $Y_1, Y_2, \ldots, Y_n$ denote a random sample from a Poisson distribution with parameter λ. Show by conditioning that $\sum_{i=1}^n Y_i$ is sufficient for λ.

Since $Y_i \sim \text{Poisson}(\lambda) \longrightarrow P(Y_i = y_i) = \frac{e^{-\lambda}\lambda^{y_i}}{y_i!}$

Since $Y_1, Y_2, \ldots, Y_n$ are independent, their joint PMF is: $P(Y_1 = y_1, \ldots, Y_n = y_n) = \prod_{i=1}^{n} \frac{e^{-\lambda}\lambda^{y_i}}{y_i!} = e^{-n\lambda}\lambda^{\sum y_i}\prod_{i=1}^{n}\frac{1}{y_i!}$

Since sum of independent Poisson random variables is also Poisson $\longrightarrow T = \sum_{i=1}^{n} Y_i \sim \text{Poisson}(n\lambda)$

↳ PMF : $P(T=t) = \frac{e^{-n\lambda}(n\lambda)^t}{t!}$

$$P(Y_1 = y_1, \ldots, Y_n = y_n) = \frac{P(Y_1 = y_1, \ldots, Y_n = y_n)}{P(T=t)} = \frac{\cancel{e^{-n\lambda}}\lambda^t \prod_{i=1}^{n}\frac{1}{y_i!}}{\cancel{e^{-n\lambda}}(n\lambda)^t / t!} = \frac{t!}{\prod_{i=1}^{n} y_i! \cdot n^t} = \frac{t!}{(t - \sum_{i=1}^{n-1} y_i)!\,\prod_{i=1}^{n-1} y_i!}$$

↳ Doesn't depend on λ

$\Rightarrow T = \sum Y_i$ is a sufficient statistic for λ

9.41 Let $Y_1, Y_2, \ldots, Y_n$ denote a random sample from a Weibull distribution with known m and unknown α. (Refer to Exercise 6.26.) Show that $\sum_{i=1}^n Y_i^m$ is sufficient for α.

$= U$

↳ PDF: $f(y;\alpha, m) = \frac{m}{\alpha}\left(\frac{y}{\alpha}\right)^{m-1}\exp\left(-\left(\frac{y}{\alpha}\right)^m\right)$, $y > 0$

Since $Y_1, Y_2, \ldots, Y_n$ are independent, their joint PDF is:

↳ $L(\alpha) = \prod_{i=1}^{n} f(Y_i;\alpha, m)$

$= \prod_{i=1}^{n}\left[\frac{m}{\alpha}\left(\frac{y_i}{\alpha}\right)^{m-1}\exp\left(-\left(\frac{y_i}{\alpha}\right)^m\right)\right]$

$= \left(\frac{m}{\alpha}\right)^n\left(\prod_{i=1}^{n} y_i\right)^{m-1}\alpha^{-n(m-1)}\exp\left(-\sum_{i=1}^{n}\frac{y_i^m}{\alpha}\right)$

$= \alpha^{-nm}m^n\left(\prod_{i=1}^{n} y_i\right)^{m-1}\exp\left(\frac{\boxed{U}}{-\alpha}\right)$

$= \underbrace{\alpha^{-nm}\exp\left(\frac{U}{-\alpha}\right)}_{\text{depends on } \alpha} \times \underbrace{m^n\left(\prod_{i=1}^{n} y_i\right)^{m-1}}_{\emptyset \text{ depend on } \alpha}$

Factorization Theorem states that a statistic $T(Y)$ is sufficient for α if likelihood fct can be written as: $L(\alpha) = \underline{g(U,\alpha)}\ \underline{h(Y_1, \ldots, Y_n)}$, which is the case

$\Rightarrow U = \sum Y_i^m$ is a sufficient statistic of α

***9.65** In this exercise, we illustrate the direct use of the Rao–Blackwell theorem. Let $Y_1, Y_2, \ldots, Y_n$ be independent Bernoulli random variables with

$$p(y_i \mid p) = p^{y_i}(1-p)^{1-y_i}, \qquad y_i = 0, 1.$$

That is, $P(Y_i = 1) = p$ and $P(Y_i = 0) = 1 - p$. Find the MVUE of $p(1-p)$, which is a term in the variance of Y_i or $W = \sum_{i=1}^n Y_i$, by the following steps.

a Let

$$T = \begin{cases} 1, & \text{if } Y_1 = 1 \text{ and } Y_2 = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Show that $E(T) = p(1-p)$.

b Show that

$$P(T = 1 \mid W = w) = \frac{w(n-w)}{n(n-1)}.$$

c Show that

$$E(T \mid W) = \frac{n}{n-1}\left[\frac{W}{n}\left(1 - \frac{W}{n}\right)\right] = \frac{n}{n-1}\overline{Y}(1-\overline{Y})$$

and hence that $n\overline{Y}(1-\overline{Y})/(n-1)$ is the MVUE of $p(1-p)$.

a) Since T takes only two values (0 or 1) $\longrightarrow E(T) = P(T=1)$

T occurs when $Y_1 = 1$ and $Y_2 = 0 \rightarrow P(T=1) = P(Y_1 = 1, Y_2 = 0)$

Since Y_1 and Y_2 are independent Bernoulli random variables with success probability p:

→ $P(Y_1 = 1) = p$

→ $P(Y_2 = 0) = 1-p$

⇒ joint probability: $P(Y_1 = 1, Y_2 = 0) = P(Y_1 = 1)\,P(Y_2 = 0) = p(1-p) = E(T)$

b) $$P(T=1 \mid W=w) = \frac{P(T=1 \cap W=w)}{P(W=w)}$$

$$\begin{aligned} P(T=1 \cap W=w) &= P(Y_1 = 1, Y_2 = 0, W=w) \\ &= P(Y_1=1)\,P(Y_2=0)\,P\left(\sum_{i=3}^{n} Y_i = w-1\right) \\ &= p(1-p)\binom{n-2}{w-1} p^{w-1}(1-p)^{n-w-1} \\ &= \binom{n-2}{w-1} p^{w}(1-p)^{n-w} \end{aligned}$$

Probability for remaining $n-2$ observations, exactly $w-1$ of them are 1's

binomial distribution

$$P(T=1 \mid W=w) = \frac{P(T=1 \cap W=w)}{P(W=w)} = \frac{\binom{n-2}{w-1}\cancel{p^{w}(1-p)^{n-w}}}{\binom{n}{w}\cancel{p^{w}(1-p)^{n-w}}} = \frac{\frac{(n-2)!}{(w-1)!\,(n-w-1)!}}{\frac{n!}{w!\,(n-w)!}} = \frac{(n-2)!\,w!\,(n-w)!}{(w-1)!\,(n-w-1)!\,n!} = \frac{w(n-w)}{n(n-1)}$$

c) Since $P(T=1 \mid W=w) = \frac{w(n-w)}{n(n-1)} \longrightarrow E(T \mid W) = \frac{W(n-W)}{n(n-1)}$

$\sum_{i=1}^{n} Y_i$ (sum of independent trials)

$$\begin{aligned} &= \frac{(n\bar{Y})(n-\bar{Y})}{n(n-1)} \\ &= \frac{n^2\bar{Y}(1-\bar{Y})}{n(n-1)} \\ &= \frac{n}{n-1}\bar{Y}(1-\bar{Y}) \end{aligned}$$

→ T is an unbiased estimator of $p(1-p)$

→ W is sufficient statistic for p

→ By Rao-Blackwell theorem, the best estimator (MVUE) obtained by computing $E(T \mid W)$

⇒ $E(T \mid W) = \frac{n}{n-1}\bar{Y}(1-\bar{Y})$ is the MVUE of $p(1-p)$

9.86 Suppose that $X_1, X_2, \ldots, X_m$, representing yields per acre for corn variety A, constitute a random sample from a normal distribution with mean μ_1 and variance σ^2. Also, $Y_1\, Y_2, \ldots, Y_n$, representing yields for corn variety B, constitute a random sample from a normal distribution with mean μ_2 and variance σ^2. If the X's and Y's are independent, find the MLE for the common variance σ^2. Assume that μ_1 and μ_2 are unknown.

Two independent normal samples: → $X_1, X_2, \ldots, X_m \sim N(\mu_1, \sigma^2)$

$Y_1, Y_2, \ldots, Y_n \sim N(\mu_2, \sigma^2)$

$$\begin{aligned} L(\mu_1, \mu_2, \sigma^2) &= \prod_{i=1}^{m} f(X_i \mid \mu_1, \sigma^2) \times \prod_{j=1}^{n} f(Y_j \mid \mu_2, \sigma^2) \\ &= \prod_{i=1}^{m} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(X_i-\mu_1)^2}{2\sigma^2}\right) \times \prod_{j=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(Y_j-\mu_2)^2}{2\sigma^2}\right) \\ &= \frac{1}{(\sqrt{2\pi\sigma^2})^{m+n}} \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{m}(X_i-\mu_1)^2 - \frac{1}{2\sigma^2}\sum_{j=1}^{n}(Y_j-\mu_2)^2\right) \end{aligned}$$

taking the log →

$$\ell(\mu_1, \mu_2, \sigma^2) = -(m+n)\log(\sqrt{2\pi}) - \frac{(m+n)}{2}\log(\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{m}(X_i-\mu_1)^2 - \frac{1}{2\sigma^2}\sum_{j=1}^{n}(Y_j-\mu_2)^2$$

$$\frac{\partial \ell}{\partial \mu_1} = \frac{1}{\cancel{2}\sigma^2}\cdot\sum_{i=1}^{m}\cancel{2}(X_i-\mu_1) = \frac{1}{\sigma^2}\cdot\sum_{i=1}^{m}(X_i-\mu_1)$$

$$\frac{\partial \ell}{\partial \mu_2} = \frac{1}{2\sigma^2}\cdot\sum_{j=1}^{n}2(Y_j-\mu_2) = \frac{1}{\sigma^2}\cdot\sum_{j=1}^{n}(Y_j-\mu_2)$$

$$\frac{\partial \ell}{\partial \sigma^2} = \frac{-(m+n)}{2\sigma^2} + \frac{1}{2\sigma^4} \cdot \left(\sum_{i=1}^{m} (X_i - \mu_1)^2 + \sum_{j=1}^{n} (Y_j - \mu_2)^2 \right)$$

Calculate MLE:
* Equate partial derivative to 0
* $\frac{1}{\sigma^2} > 0$, $\hat{\mu}_1 = \bar{X}$ and $\hat{\mu}_2 = \bar{Y}$

$$\frac{-(m+n)}{2\sigma^2} + \frac{1}{2\sigma^4} \cdot \left(\sum_{i=1}^{m} (X_i - \hat{\mu}_2)^2 + \sum_{j=1}^{n} (Y_j - \bar{\mu}_2)^2 \right) = 0$$

$$\frac{1}{\sigma^2} \left(\sum_{i=1}^{m} (X_i - \hat{\mu}_2)^2 + \sum_{j=1}^{n} (Y_j - \bar{\mu}_2)^2 \right) = m+n$$

$$\hat{\sigma}^2 = \frac{m+n}{\left(\sum_{i=1}^{m} (X_i - \hat{\mu}_2)^2 + \sum_{j=1}^{n} (Y_j - \bar{\mu}_2)^2 \right)}$$

← required MLE of σ^2

9.90 A random sample of 100 men produced a total of 25 who favored a controversial local issue. An independent random sample of 100 women produced a total of 30 who favored the issue. Assume that p_M is the true underlying proportion of men who favor the issue and that p_W is the true underlying proportion of women who favor of the issue. If it actually is true that $p_W = p_M = p$, find the MLE of the common proportion p.

$Y_1 = 25$ and $Y_2 = 30$

$$L(p) = L(Y_1, Y_2 \mid p) = \binom{100}{Y_1} p^{Y_1} (1-p)^{100-Y_1} \cdot \binom{100}{Y_2} p^{Y_2} (1-p)^{100-Y_2}$$

$$= \binom{100}{Y_1} \cdot \binom{100}{Y_2} \cdot p^{Y_1+Y_2} (1-p)^{200-Y_1-Y_2}$$

$$\ell(p) = \log\left(\binom{100}{Y_1} \cdot \binom{100}{Y_2}\right) + (Y_1+Y_2)\log(p) + (200 - Y_1 - Y_2)\log(1-p)$$

$$\frac{d\ell}{dp} = 0 + \frac{Y_1+Y_2}{p} - \frac{200 - Y_1 - Y_2}{1-p}$$

Calculate MLE by setting partial der. to 0:

$$\frac{Y_1+Y_2}{p} = \frac{200 - Y_1 - Y_2}{1-p}$$

$$Y_1 + Y_2 - (Y_1+Y_2)\hat{p} = 200\hat{p} - (Y_1+Y_2)\hat{p}$$

$$200\hat{p} = Y_1 + Y_2$$

$$\hat{p} = \frac{Y_1+Y_2}{200} = \frac{25+30}{200} = 0.275$$

9.112 Let $Y_1, Y_2, \ldots, Y_n$ denote a random sample from a Poisson distribution with mean λ and define

$$W_n = \frac{\bar{Y} - \lambda}{\sqrt{\bar{Y}/n}}.$$

a Show that the distribution of W_n converges to a standard normal distribution.

b Use W_n and the result in part (a) to derive the formula for an approximate 95% confidence interval for λ.

a) Since $Y_i \sim$ Poisson (λ) → $E(Y_i) = \lambda$, $Var(Y_i) = \lambda$

By CLT, since Y_i iid from Poisson distribution → $\frac{\bar{Y} - \lambda}{\sqrt{\lambda/n}} \approx N(0,1)$ for large n

By law of large nb for $\bar{Y}$, $\frac{\bar{Y}}{\lambda} \to 1$ as $n \to \infty$

Since $\bar{Y}$ is consistent estimator of λ, substitute λ with $\bar{Y}$

$$\Rightarrow W_n = \frac{\bar{Y} - \lambda}{\sqrt{\bar{Y}/n}} \approx N(0,1) \longrightarrow N(0,1)$$

*W_n follows a standard normal distribution asymptotically

b) $P(-1.96 \le W_n \le 1.96) \approx 0.95$

$$P\left(-1.96 \le \frac{\bar{Y}-\lambda}{\sqrt{\bar{Y}/n}} \le 1.96\right) \approx 0.95$$

Solving for λ: $-1.96 \cdot \sqrt{\frac{\bar{Y}}{n}} \le \bar{Y} - \lambda \le 1.96 \cdot \sqrt{\frac{\bar{Y}}{n}}$

$$\bar{Y} - 1.96 \cdot \sqrt{\frac{\bar{Y}}{n}} \le \lambda \le \bar{Y} + 1.96 \cdot \sqrt{\frac{\bar{Y}}{n}}$$

$\Rightarrow$ Confidence interval $= \left(\bar{Y} - 1.96 \cdot \sqrt{\frac{\bar{Y}}{n}},\ \bar{Y} + 1.96 \cdot \sqrt{\frac{\bar{Y}}{n}}\right)$